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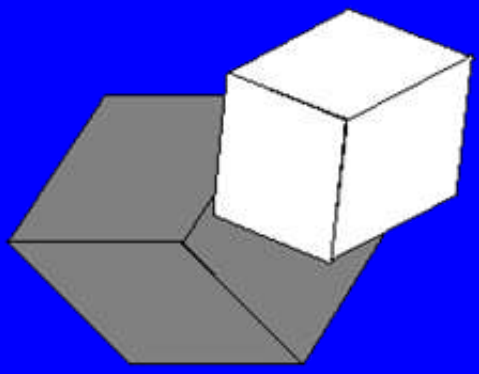
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New investigations on HX-groups and soft groups

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Abstract. The paper presents new results obtained on $HX-$groups introduced by Hongxing in [7] and then investigated by Corsini in his study on HX-hypergroups see [1]. We also present the results obtained by Corsini on the relationship between $HX-$groups and Soft groups. On the other hand we present some new results on these types of groups and define a new objects which we call Soft $HX-$groups. We give many examples to illustrate the notions introduced and explain their usefulness. To conclude we present some topics about the soft $HX-$groups that can be investigated.

Keywords: HX-group, soft groups.

1. Introduction

The notion of $HX-$groups has been introduced by some Chinese mathematicians like Mi Honghai and co [6], Li Honxing [7]. This notion has been revisited by Piergiulio Corsini where he made a link between $HX-$groups and hypergroups. In [3, 4, 5] he constructed some hyperstructures from $HX-$groups. Recently in his paper [1], he determined the hypergroups associated to the $HX-$group $\mathbb{Z}/n\mathbb{Z}$ and associated to the set of square matrices of order 2 with coefficients in $\mathbb{Z}/n\mathbb{Z}$. The notions of $HX-$groups and related topics are of the interest of many researchers worldwide.

The first part of the paper is devoted to the notion of $HX-$groups $\mathcal{G}$ with support a group G introduced by [7]. We investigate the relationship between the structure of $\mathcal{G}$ and G. We establish many interesting results. Among the results, we have been concerned by the extension of a given group morphism to the associated $HX-$structure.

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The second part of the paper is devoted to the soft groups. We establish some results and show that the conditions of the proposition 2.3 in [10] should be improved otherwise the result is not true.

The last part of the paper is concerned with the relationship between the notion of $HX-$groups and the soft groups. We give an example of soft $HX-$group when the support G is of order $p^m q^n$, for some values of the prime integers p and q.

Some works have been cited to enrich the bibliography and to give an overview of the state of the art.

2. Preliminaries

In this section we recall the principal definitions and some well known results related to our subject.

The following classical result will be used continually when the group is of finite order.

Proposition 2.1. *In the table of a finite group each row contains each element of the group exactly once and each column contains each element of the group exactly once.*

Definition 2.2. Let $(G,.)$ be a group and $\mathcal{G} \subset \mathcal{P}(G) \setminus \{\emptyset\}$. $\mathcal{G}$ is a $HX-$group if it is a group for the binary operation $*$ defined by:

$$\forall A, B \in \mathcal{G},\ A * B = \{a.b \mid a \in A,\ b \in B\}.$$

This means that $*$ is internal binary operation on $\mathcal{G}$, associative, $\exists E \in \mathcal{G}$ such that $A * E = E * A = A,\ \forall A \in \mathcal{G}$ and any element of $\mathcal{G}$ has a symmetric with respect to $*$.

In the sequel the group G will be called the support of $\mathcal{G}$.

Definition 2.3 ([4])**.** Let $\mathcal{G}$ be a $HX-$group with support G. If E and e are the identities of the groups $\mathcal{G}$ and G respectively. The group $\mathcal{G}$ is called regular if $e \in E$.

Proposition 2.4 ([7])**.** *The set E is a semigroup of the group G.*

Proposition 2.5 ([7])**.** *Let $\mathcal{G}$ be a $HX-$group with support G. If E is the identity of the group $\mathcal{G}$, then:*

1. $\forall\ A \in \mathcal{G}, |A| = |E|$.
2. $\forall A, B \in \mathcal{G}, A \cap B \neq \emptyset \implies |A \cap B| = |E|$.

Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ be the power set of U , and $A \subset E$.

Definition 2.6. A pair (A, f) is called a soft set over U, where f is a mapping given by

$$f : A \longrightarrow \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $f(a)$ may be considered as the subset of U of U of type a.

Definition 2.7. A soft set (A, f) over U is called a null soft set, denoted by $f_\emptyset$, if

$$\forall a \in A, f(a) = \emptyset.$$

Definition 2.8. A soft set (A, f) over U is called an absolute soft set, denoted by $\widetilde{f_A}$, if for all $a \in A, f(a) = U$.

Definition 2.9. The union of two soft sets (A, f), (B, g) over a common universe U is the soft set $h : E \longrightarrow \mathcal{P}(U)$ defined by

$$\forall x \in E, h(x) = \begin{cases} f(x), & \text{if } x \in A \setminus B \\ g(x), & \text{if } x \in B \setminus A \\ f(x) \cup g(x), & \text{if } x \in A \cap B \end{cases}.$$

Definition 2.10. The intersection of two soft sets (A, f), (B, g) over a common universe U the soft set $h : E \longrightarrow \mathcal{P}(U)$ defined by

$$\forall x \in E, h(x) = f(x) \cap g(x).$$

Definition 2.11. For a soft set (A, f) over U, the relative complement of (A^c, f^c) is defined by $f^c : E \longrightarrow \mathcal{P}(U)$, where $\forall a \in E, f^c(a) = U \setminus f(a)$.

Definition 2.12. A pair (A, f) is called a soft subgroup over a group U, where f is a mapping given by

$$f : A \longrightarrow \mathcal{P}(U)$$

if for all $x \in A$; $f(x)$ is a subgroup of U.

3. Main results

3.1 HX-groups

Proposition 3.1. *Let $\mathcal{G}$ be a $HX-$group. If E is its identity such that $|E| < \infty$, then E is a subgroup and therefore $\mathcal{G}$ is regular.*

Proof. To prove the proposition it suffices to prove that $e \in E$.

Let us set $E = \{a_1, a_2, \cdots, a_n\}$ since $E * E = E$, then for a fixed index $j \in \{1, 2, \cdots, n\}$, there exists $k \in \{1, 2, \cdots, n\}$ such that $a_j.a_k = a_j$ and then $a_j^{-1}.a_j.a_k = a_j^{-1}.a_j \iff a_k = e$. □

The following result is a consequence of the above Proposition and Proposition 2.5.

Corollary 3.2. *Let $\mathcal{G}$ be a $HX-$group of support G and E its identity.*

1. *If E is finite, two elements of $\mathcal{G}$ are disjoint and none of them excepted E is a subgroup of G.*
2. *If E is infinite, the intersection of two elements of $\mathcal{G}$ is also infinite.*

Proposition 3.3. *Let $\mathcal{G}$ be a $HX-$group. If E is finite, then $\forall a \in E,\ a^{-1} \in E$.*

Proof. Let us set $E = \{a_1, a_2, \cdots, a_n\}$ and let $a_i \in E$. Since $E * E = E$ and $e \in E$ then there exists $k \in \{1, 2, \cdots, n\}$ such that $a_i.a_k = e$. By the uniqueness of the symmetric element of a_i in G, even the group is not abelian. □

Corollary 3.4. *Let $\mathcal{G}$ be a $HX-$group. If E is finite, then it is a subgroup of G.*

Proof. the proof is a consequence of the above propositions and of the condition $E * E = E$. □

Example 3.5. Let $G = \mathbb{Z}_4$, the set $\mathcal{G} = \{\{0, 2\}, \{1, 3\}\}$ is a $HX-$group, its identity is the set $\{0, 2\}$ and the symmetric of $\{1, 3\}$ is $\{1, 3\}$ itself.

Lemma 3.6. *Let $\mathcal{G}$ be a $HX-$group. If G is finite and G has no element of order 2, then $|E|$ is of odd order.*

Proof. Since E is a subgroup of G, by Cauchy's theorem its order divides the order of G. If the order of E is even then 2 divides $|G|$ and then it contains an element of order 2, contradiction. □

Lemma 3.7. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$, then $\forall A \in \mathcal{G},\ |A| = 1$.*

Proof. Suppose that there exists in $\mathcal{G}$ an element A such that $|A| \geq 2$. Let a_1, a_2 two distinct elements of A. As $\mathcal{G}$ is a group $A^{-1} \in \mathcal{G}$, so it is non empty, it contains an element $c \in G$. On the other hand $A * A^{-1} = \{e\}$ so $a_1.c = e = a_2.c$ and then $a_1 = a_2$ contradiction. □

Example 3.8. From the example above and from the lemma, we can deduce that the only structure of $HX-$group on $\mathbb{Z}_4$ are

$$\begin{aligned}
\mathcal{G}_1 &= \{\{0\}, \{1\}, \{2\}, \{3\}\},\\
\mathcal{G}_2 &= \{\{0\}, \{2\}, \},\\
\mathcal{G}_3 &= \{\{0, 2\}, \{1, 3\}\}.
\end{aligned}$$

Proposition 3.9. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$ and H is a subgroup of G, then $\mathcal{H} = \{\{a\} \mid a \in H\}$ is a subgroup of $\mathcal{G}$.*

Proof. Although the proof is trivial, we give it as an illustration of our study.

1. Since $e \in H$, then $\{e\} \in \mathcal{H}$.
2. Let $\{a\}, \{b\} \in \mathcal{H}$, since H is a subgroup, $a, b \in H$ then $ab \in H$ and so $\{a\} * \{b\} = \{ab\} \in \mathcal{H}$.
3. Let $\{a\} \in \mathcal{H}$, then the element $a \in H$ so its symmetric a^{-1} is also an element of H. But $\{a\}^{-1} = \{a^{-1}\} \in \mathcal{H}$.

□

The converse of the above proposition is also true. More exactly;

Proposition 3.10. *Let $\mathcal{G}$ be a $HX-$group. If $E = \{e\}$, H a subset of G and $\mathcal{H} = \{\{a\} \mid a \in H \subset G\}$. If $\mathcal{H}$ is a subgroup of$\mathcal{G}$, the set H is then a subgroup of G.*

Proof. Note that since $E = \{e\}$, then for any $a \in G, \{a\}^{-1} = \{a^{-1}\}$.

1. Since $\mathcal{H}$ is a subgroup and $E = \{e\}$ then $e \in H$.
2. Let $a, b \in H$ then $\{a\}, \{b\}^{-1} \in \mathcal{H}$ and so $\{ab^{-1}\} = \{a\} * \{b\}^{-1} \in \mathcal{H}$. Finally $ab^{-1} \in H$.

□

Definition 3.11. A subgroup H of a group G is said to be closed with respect to the elements of the group $\mathcal{G}$ if for any expression of $x \in H$ as a product $x = a.b,\ a \in A,\ b \in B$ implies $a, b \in H$.

Proposition 3.12. *Let $\mathcal{G}$ be a $HX-$group with support G. If the elements of $\mathcal{G}$ form a cover of G and order of G is prime then $\mathcal{G} = \{G\}$ or $\mathcal{G} = \{\{a\} \mid a \in G\}$.*

Proof. From the proposition 2.5, the elements of $\mathcal{G}$ are disjoint and since they form a cover of G then $|G| = n|E|$ where $n = |\mathcal{G}|$. But $|G|$ is prime then $n \in \{1, |G|\}$ and the result follows. □

Proposition 3.13. *Let $\mathcal{G}$ be a $HX-$group. For any subgroup H of G closed with respect to the elements of the group $\mathcal{G}$ such that $E \subset H$ and $H \cap A \in \mathcal{G}$, $\forall A \in \mathcal{G}$, the set $\mathcal{H} = \{H \cap A \mid A \in \mathcal{G}\}$ is a subgroup of $\mathcal{G}$.*

Proposition 3.14 ([7])**.** *Let H be a subgroup of a group G, E an idempotent subset of G (i.e. $E^2 = E$). If for all $a \in H$, $aE = Ea$, then the set $\mathcal{G} = \{aE \mid a \in H\}$ is a $HX-$group with support the group G.*

In the proof it is shown that $H/kerf \approx \mathcal{G}$ where $f : H \longrightarrow \mathcal{G}$, $a \longmapsto aE$. We can then deduce the following corollary.

Corollary 3.15. *If E is a subgroup of G such that $H \cap E = \{e\}$, then $H \approx \mathcal{G}$.*

Proof. To proof the corollary it suffices to prove that f is injective. The conditions $H \cap E = \{e\}$ and $aE = E$ imply that $a = e$ and the mapping is injective. □

Example 3.16. Suppose that $|G| = p^m q^n,\ p,\ q$ distinct prime integers. Let E, H be respectively a p sylow and a q sylow subgroup of G. We have $E^2 = E$ and $E \cap H = \{e\}$. If $p^\alpha \equiv 1[q] \implies \alpha = 0$ then the $HX-$group $\mathcal{G} = \{aE \mid a \in H\}$ is isomorphic to the group H. This holds for example when $|G| = 5^2 \times 7^2,\ |E| = 7^2$ and $|H| = 5^2$.

Proof. 1. Since $E = H \cap E$ and $E \in \mathcal{G}$, then $E \in \mathcal{H}$.

2. Suppose that $A, B \in \mathcal{H}$, then there exist $A_1, B_1 \in \mathcal{G}$ such that $A = H \cap A_1,\ B = H \cap B_1$. It is easy to show that $A * B \subset H \cap (A_1 * B_1)$.

On the other hand, suppose that $x \in H \cap (A_1 * B_1)$ then there exist $a \in A_1,\ b \in B_1$ such that $x = a.b$. Since H is closed with respect to the elements of the group $\mathcal{G}$, the elements a, b are in H so $a.b \in (H \cap A_1) * (H \cap B_1)$ and then $H \cap (A_1 * B_1) \subset (H \cap A_1) * (H \cap B_1) = A * B$. The set $\mathcal{H}$ is then closed for $*$.

3. Let now $A \in \mathcal{H}$, then there exists $A_1 \in \mathcal{G}$ such that $A = H \cap A_1$. Let us prove that $A^{-1} = H \cap A_1^{-1}$.

Let $x \in A^{-1}$ then for all $y \in A,\ x.y \in E \subset H$. From $A \subset H$, we can deduce that $x \in H$. On the other hand since $H \cap A_1 \subset A_1$, then $(H \cap A_1)^{-1} \subset (A_1)^{-1}$ and so $x \in (A_1)^{-1}$. We can conclude that $A^{-1} \subset H \cap A_1^{-1}$.

Now let $x \in H \cap A_1^{-1}$ and $y \in A = H \cap A_1$. We then have $x, y \in H$, $x \in A_1^{-1}$ and $y \in A_1$ so $x.y \in H$ and $x.y \in E$ and then $x \in (H \cap A_1)^{-1} = A^{-1}$. Conclusion $H \cap A_1^{-1} \subset A^{-1}$.

□

Definition 3.17. A subgroup H of a group G is said to be strongly normal with respect to the $HX-$group $\mathcal{G}$ if for all $K \in \mathcal{G}$, for all $a \in G$, $x \in H \cap K \implies a.x.a^{-1} \in H \cap K,\ .$

Proposition 3.18. *Let the subgroup H as in the above proposition and suppose that there exists at least an element $K' \in \mathcal{G}$ such that $H \subset K'$ and for any element $K \in \mathcal{G}$ if $K \subset H \cap K'$, there exists $K'' \in \mathcal{G}$ such that $K = H \cap K''$. If it is normal then so is the subgroup $\mathcal{H}$.*

Proof. Let $A \in \mathcal{G}$ and $B \in \mathcal{H}$. and let us prove that $A * B * A^{-1} \in \mathcal{H}$. Take $x \in B, a \in A$ and $b \in A^{-1}$. The element $a.x = x'.a$ for some $x' \in H \cap K$ since H is strongly normal with respect to $\mathcal{G}$. Now the element $a.b \in A * A^{-1} = E$, then $a.x.b = x'.a.b \in B * E = B$. Consequently $A * B * A^{-1} \subset B$.

Let $x \in B,\ a \in A$, then for any $a \in A$ and $b \in A^{-1}$ we have $x = a.(a^{-1}.x.b^{-1})b$. But it is clear as shown above that $a^{-1}.x.b^{-1} \in B$. We can conclude that $A * B * A^{-1} \supset B$ and then $A * B * A^{-1} = B$ and $\mathcal{H}$ is normal in $\mathcal{G}$. □

Now, we want to extend a group morphism to the associated $HX-$group structures. More exactly we have the following

Proposition 3.19. *Let $f : G_1 \longrightarrow G_2$ be a group morphism. The mapping $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ defined by: $\widetilde{f}(A) = \{f(x) \mid x \in A\}$ is a group morphism, where $\mathcal{G}_1,\ \mathcal{G}_2$ are $HX-$groups with support G_1 and G_2 respectively.*

Proof. The proof is trivial. More exactly, let $A_1, A_2 \in \mathcal{G}$. We have

$$\begin{aligned}\widetilde{f}(A_1 * A_2) &= \{f(x) \mid x \in A_1 * A_2\} = \{f(x_1.x_2) \mid x_1 \in A_1, x_2 \in A_2\} \\ &= \{f(x_1).f(x_2) \mid x_1 \in A_1, x_2 \in A_2\} \\ &= \{f(x) \mid x \in A_1\} * \{f(x) \mid x \in A_2\} = \widetilde{f}(A_1) * \widetilde{f}(A_2).\end{aligned}$$

□

Proposition 3.20. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. we have the relation $\widetilde{f}(kerf) \subset ker\widetilde{f}$. The equality holds when $E = \{e\}$.*

Proof. Let $A = kerf$ then $ker\widetilde{f}(A) = \{f(x) \mid x \in A\} = \{e\} \subset E = ker\widetilde{f}$. The second assertion follows from the inclusion. □

Proposition 3.21. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If E_1 is finite and f is one-to-one then $\widetilde{f}$ is also one-to-one.*

Proof. Let us denote by E_1, E_2 the identities of $\mathcal{G}_1,\ \mathcal{G}_2$ respectively and suppose that $E_1 = \{a_1, a_2, ..., a_n\}$. Since $\widetilde{f}(E_1) = E_2$ and ff is one-to-one, necessary $E_2 = \{b_1, b_2, ..., b_n\}$ with distinct elements. Let $A \in \mathcal{G}$ such that $\widetilde{f}(A) = E_2$. If there exists $c \in A$ such that $c \notin E_1$ and $f(c) \in E_2$ then there exist two different element $c, a_i \in G$ such that $c \neq a_i,\ f(c) = f(a_i) = b_j$ contradiction. So $A \subset E_1$ and then $A = E_1$. □

Proposition 3.22. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If $G_2 \subset \bigcup_{A \in \mathcal{G}_2} A$ and $\widetilde{f}$ is onto, then f is also onto.*

Proof. Let $y \in G_2$. Since the elements of $\mathcal{G}_2$ form a cover of G_2, then there exists $B \in \mathcal{G}_2$ such that $y \in B$. The mapping $\widetilde{f}$ is onto, then there exists $A \in \mathcal{G}_1$ such that $\widetilde{f}(A) = B$ and then $y \in A = \{f(x) \mid x \in A\}$ so $y = f(x)$ for some $x \in A$ and the mapping f is onto. □

Proposition 3.23. *Let $f : G_1 \longrightarrow G_2$ be a group morphism and $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ as defined above. If the inverse image of any element of $\mathcal{G}_2$ is an element of $\mathcal{G}_1$ and f is onto then $\widetilde{f}$ is also onto.*

Proof. Let $B \in \mathcal{G}_2$. Since f is onto then $B \subset f(G_1)$ and then any element of B is the image of an element of G_1, so there exists $A \subset G_1$ such that $f(A) = B$. We deduce that A is the inverse image of B which is an element of $\mathcal{G}_2$ so it is an element of $\mathcal{G}_1$ and the proof follows. □

3.2 Soft groups

This section is devoted to some algebraic notions of Soft set theory which are different from others. We introduce and define some soft operations and also we establish a few interesting results in this context. However any soft (A, f) can be viewed as the soft set (E, f), where if $x \notin A, f(x) = \emptyset$, so all the soft sets will be denoted in the form (E, f) or simply f_E.

Let us start this subsection by an example of soft group.

Example 3.24. Let G be the additive group $(\mathbb{Z}/6\mathbb{Z}, +)$ for example and $E = \{1, 2, 3, 6\}$. We can parameterize the set of subgroups of $(\mathbb{Z}/6\mathbb{Z}, +)$ by their cardinality. More exactly let

$$\begin{array}{rccc} f: & E & \longrightarrow & \mathcal{P}(\mathbb{Z}/6\mathbb{Z}) \\ & 1 & \longmapsto & 6\mathbb{Z}/6\mathbb{Z} \\ & 2 & \longmapsto & 3\mathbb{Z}/6\mathbb{Z} \\ & 3 & \longmapsto & 2\mathbb{Z}/6\mathbb{Z} \\ & 6 & \longmapsto & \mathbb{Z}/6\mathbb{Z} \end{array}$$

the soft set (E, f) is a soft group.

Definition 3.25. Let $*$ be a binary operation on U. If f_E and g_E are two soft sets over U, then we define $f_E \star g_E$ as a soft set h_E defined by : $h : E \longrightarrow \mathcal{P}(U)$ such that $h(a) = \{x * y : x \in f(a) \ \ and \ \ y \in g(a)\}$.

We denote the collection of all soft sets over U with domain E by the symbol $S_E(U)$.

Proposition 3.26 ([2])**.** *The above operation $\star$ is a binary operation on $S_E(U)$.*

The following example shows that the Proposition 2.3 [10] is false, the problem is that the author at the end of his proof, affirm that $F_E \circ F_E^{-1} = I_E$ which is false as in the example. We adopt in this example the same notations as in [].

Example 3.27. Let $E = \{1, 2\}$ and U be the group $(\mathbb{Z}/3\mathbb{Z}, +)$ if $f_E(1) = \{0, 1\}$ then $f_E^{-1}(1) = \{0, 2\}$ so $(f_E(1) \circ f_E^{-1})(1) = \{x + y \mid x \in f_E(1) \ and \ -y \in f_E(1)\} = \{0, 1, 2\} \neq \{0\} = I_E(1)$.

In the following proposition which is easy to prove, we introduce a new definition of composition of soft sets such that the set of all soft sets over a group, becomes a group for this composition.

Proposition 3.28. *If $(U, *)$ is a group then $(S_E(U), \star)$ is also a group where $\star$ is defined by*

$$(f \star g)(a) = \begin{cases} \{x * y \mid x \in f(a) \ and \ -y \in g(a)\}, & if \ g \neq f^{-1} \\ \{e\}, & if \ g = f^{-1} \end{cases}.$$

Corollary 3.29. *Suppose that $(U,*)$ is a group and H is a subgroup of $(U,*)$, then the set*

$$\mathcal{H} = \{f : E \longrightarrow \mathcal{P}(H), \text{ the elements of } S_E(U) \text{ with values in } \mathcal{P}(H)\}$$

is a subgroup of $(S_E(U), \star)$.

The result follows directly from the Proposition 4, however we propose the following direct proof.

Proof. 1. H contains at least the neutral element e of U then $\{e\} \in \mathcal{P}(H)$ so the mapping $\mathcal{O} : E \longrightarrow \mathcal{P}(H),\ x \longmapsto \{e\}$ is an element of $\mathcal{H}$.

2. Let $f, g \in \mathcal{H}$ then for all $a \in E,\ (f \star g)(a) = \{x * y \mid x \in f(a),\ y \in g(a)\}$. But $x \in f(a),\ y \in g(a) \implies x \in H$ *and* $y \in H$ so the element $x * y \in H$ and then $\{x * y \mid x \in f(a),\ y \in g(a)\} \subset H$.

3. Let $f \in \mathcal{H}$ then for all $a \in E,\ (f^{-1})(a) = \{x^{-1} \mid x \in f(a)\} \subset H$ since the inverse x^{-1} of any x is also in H. The mapping f^{-1} is an element of $\mathcal{H}$.

□

Example 3.30. 1. Let $E = \mathbb{R}^*$ and $U = GL_n(\mathbb{R})$. For any $a \in \mathbb{R}^*$ let $f(a) = \{A \in U \mid detA = a\}$. The pair (E, f) is a soft set.

2. Suppose that H is a normal subgroup of $(GL_n(\mathbb{R}), \times)$ if the corresponding set $\mathcal{H}$ of the previous proposition is such that $\forall f \in \mathcal{H}$ and $\forall a \in \mathbb{R}^*$, $f(a)$ is closed by similarity, then it is a normal subgroup of $S_{\mathbb{R}^*}(GL_n(\mathbb{R}))$. Indeed suppose that $f \in \mathcal{H}, g \in GL_n(\mathbb{R})$ then for $a \in \mathbb{R}^*,\ (f \circ g)(a) = \{A \times P \mid A \in f(a), P \in g(a)\}$. But in $GL_n(\mathbb{R})$ if $A \times P = P \times A'$ then $A' = P^{-1} \times A \times P$ which still an element of $f(a)$ by our hypothesis on $\mathcal{H}$ and since H is normal, then $(f \circ g)(a) = \{A \times P \mid A \in f(a), P \in g(a)\} = (g \circ f)(a) = \{P \times A \mid A \in f(a), P \in g(a)\}$ for all $g \in GL_n(\mathbb{R})$ and $\mathcal{H}$ is then normal.

Proposition 3.31. *Suppose that $(U,*)$ is a group and H is a normal subgroup of $(U,*)$, if the set $\mathcal{H} = \{f : E \longrightarrow \mathcal{P}(H)\}$ is such that, for all $a \in E$ and for all $x \in H$ there exists $f \in \mathcal{H}$ such that $x \in f(a)$, then $\mathcal{H}$ is also a normal subgroup.*

Proof. If $f \in \mathcal{H}$ and $g \in S_E(U)$, then $(f \circ g)(a) = \{x * y \mid x \in f(a)$ *and* $y \in g(a)\}$. For any $x \in f(a) \subset H$ since H is normal then there exists $x' \in H$ such that $x * y = y * x'$. By our hypothesis as $x' \in H$ and $a \in E$ then there exists $f' \in \mathcal{H}$ such that $x' \in f'(a)$ so the set $(f \star g)(a)$ is included in the set $(g \star f')(a)$ for any $a \in E$ and then $\mathcal{H} \star g \subset g \star \mathcal{H}$, for all $g \in S_E(U)$. The proof of the other inclusion is similar. Finally we get

$$\forall g \in S_E(U), g \star \mathcal{H} = \mathcal{H} \star g$$

and the subgroup is then normal. □

Proposition 3.32. *Suppose that we are in the conditions of Proposition 3.30, then the following conditions are equivalent:*

1. $f \equiv g[\mathcal{H}]$,
2. $\forall a \in E,\ \forall x \in f(a),\ \forall y \in g(a), y \equiv x[H]$.

Proof. 1. Suppose that $f \equiv g[\mathcal{H}]$ then for all $a \in E,\ (f \star g^{-1})(a) \subset H$ and so $\{x * y^{-1} \mid x \in f(a), y \in g(a)\} \subset H$. the conclusion follows.

2. Suppose now that $\forall a \in E,\ \forall x \in f(a),\ \forall y \in g(a), y \equiv x[H]$, then $(f \star g^{-1})(a) = \{x * y^{-1} \mid x \in f(a), y \in g(a)\}$ is certainly included in H since any element $x * y^{-1} \in H,\ \forall x \in f(a),\ \forall y \in g(a)$ and then $f \equiv g[\mathcal{H}]$.

□

Proposition 3.33. *If (A, f) and (B, g) are soft groups such that (A, f) is normal, then:*

1. *(A, f^{-1}) is a soft group,*
2. *$(A \cap B, f \star g)$ is a soft group if and only if $A \cap B \neq \emptyset$.*

Proposition 3.34. *If the group U is of prime order then the only soft group of $S_E(U)$ are:*

1. $f : E \longrightarrow S_E(U); x \longmapsto f(x) = U$ *and*
2. $I : E \longrightarrow S_E(U); x \longmapsto f(x) = \{e\}$.

Proof. The proof follows from the classical theorem on groups, that is, the order of any subgroup divides the order of the group. □

3.3 Soft HX-group

The origin of this section is the paper [1] of Professor P. Corsini, where he studied certain types of HX-hypergroup.

Definition 3.35. Let $(G, *)$ be a group a soft HX-group is a soft set $f \in S_E(G)$ such that $f(E)$ is a group under the binary operation:

$$f(x) * f(y) = \{a * b,\ a \in f(x), b \in f(y)\}.$$

Definition 3.36. A soft set $f \in S_E(G)$ is said almost-surjective if for any subset $H \in \mathcal{P}^*(G)$ there exists $a \in E$ such that $f(a) \subset H$.

Lemma 3.37. *Let $f \in S_E(G)$ be an almost-surjective soft HX-group, on the set E, $\circ$ defined by:*

$$\forall x, y \in E, x \circ y = \{a \in E \mid f(a) \subset f(x) * f(y)\},$$

where $f(x) * f(y) = \{a * b,\ a \in f(x),\ b \in f(y)$, is a hyperoperation on E.

Proof. Since $\forall x \in E,\ f(x) \neq \emptyset$, then $f(x) * f(y) \neq \emptyset$ and is an element of $\mathcal{P}^*(G)$. f is almost-surjective then there is $a \in E$ such that $f(a) \subset f(x) * f(y)$ and then $x \circ y$ is a non empty subset of E and $\circ$ is then well defined. □

Lemma 3.38. *If the neutral element $e \in f(a),\ a \in E$, then for all $x \in E,\ x \in x \circ a$.*

Proof. If $e \in f(a)$ then $f(x) = e * f(x) \subset f(a) * f(x)$ and the conclusion follows. □

Proposition 3.39. *If $f \in S_E(G)$ is a surjective soft HX-group and $e \in f(x)$ for all $x \in E$, the set $(E, \circ)$ is a hypergroup.*

Proof. First if $x \in E$ and $F \subset E$ then we define

$$x \circ F = \bigcup_{a \in F} x \circ a.$$

1. Let $x, y, z \in G$ and $a \in x \circ (y \circ z)$. then there exists $b \in (y \circ z)$ such that $a \in x \circ b$ so $f(a) \subset f(x) * f(b)$. On the other hand $f(b) \subset f(y) * f(z)$ so $f(a) \subset f(x) * (f(y) * f(z)) = (f(x) * f(y)) * f(z)$. Since f is surjective there exists $c \in E$ such that $f(c) = f(x) * f(y)$ and then $f(a) \subset f(c) * f(z)$ which implies that $a \in c \circ z$. The equality $f(c) = f(x) * f(y)$ implies that $c \in x \circ y$ and then

$$a \in \bigcup_{\alpha \in x \circ y} \alpha \circ z.$$

And then $x \circ (y \circ z) \subset (x \circ y) \circ z$. The other inclusion is similar.

2. Let $a \in E$, then if $b \in E$, from the lemma $f(b) \subset f(a) * f(b)$ so $b \in a \circ b \subset \bigcup_{\alpha \in E} a \circ \alpha = a \circ E$ and the reproduction law follows.

□

Example 3.40 ([1])**.** Set $n = mq \in \mathbb{N}$ and define on the additive group $\mathbb{Z}/n\mathbb{Z}$ the family:

$A_0 = \{0, m, 2m, ..., (q-1)m\}$

$A_1 = \{1, 1+m, 1+2m, ..., 1+(q-1)m\}$

.

.

$A_{m-1} = \{m-1, 2m-1, 1+2m, ..., qm-1\}$.

If we define on the set $\mathcal{G} = \{A_0, A_1, ..., A_{m-1}\}$ the binary operation $\circ$ by:

$$A_i \circ A_j = \begin{cases} A_{i+j}, & \text{if } i+j \leq m-1 \\ A_{(m-1)-(i+j)}, & \text{if } i+j \geq m \end{cases},$$

$(\mathcal{G}, \circ)$ is a $HX-$group.

Now if we set $E = \mathbb{Z}/(m-1)\mathbb{Z}$ and $f(i) = A_i$, then $f(E) = \mathcal{G}$ and it defines a soft $HX-$group.

Remark 3.41. From the above example we note that if a prime p divides a natural n on $\mathbb{Z}/n\mathbb{Z}$, there exists a cyclic soft $HX-$group, defined as above by $f : \{0, 1, ..., p-1\} \longrightarrow \mathcal{G} = \{A_0, A_1, ..., A_{p-1}\}$. So there exists A_i which generates the group $\mathcal{G}$. What can we deduce for the mapping f in this case?

Since $\mathcal{G}$ is cyclic of prime order it is generated by A_1 for example and then $f(j) = A_1 + A_1 + ... + A_1,\ j\ times$ and so f is a group morphism.

From the above example we can deduce the following result.

Proposition 3.42. *Let G be a cyclic group of order $n = mq$ generated by a. Let for $i \leq m-1,\ A_i = \{a^i, a^{i+m}, ..., a^{(q-1)m+i}\}$. The set*

$$\mathcal{G} = \{A_i,\ 0 \leq i \leq m-1\},$$

endowed with the following binary operation $\circ$ is a $HX-$group.

$$A_i \circ A_j = \begin{cases} A_{i+j}, & \text{if } i+j \leq m-1 \\ A_{(m-1)-(i+j)}, & \text{if } i+j \geq m \end{cases}.$$

The structure $f : \mathbb{Z}/mq\mathbb{Z} \longrightarrow \mathcal{G}, i \longmapsto A_i$ is a soft $HX-$group.

Proof. The group G is then isomorphic to the group $\mathbb{Z}/mq\mathbb{Z}$. It suffices then to note that under this isomorphism the images of the sets A_i in the above example are exactly the sets A_i of the proposition. □

Proposition 3.43. *There exists a natural action of the group $\mathbb{Z}/m\mathbb{Z}$ on the group $\mathcal{G}$ of the above proposition given by $(g, A_i) \longmapsto A_{i+g}$ where the addition is performed modulo m.*

Lemma 3.44. *Let $f \in S_E(G)$ be a soft HX-group with support the group G. If $\mathcal{E}$ denotes the identity of $f(E)$, for any element $x \in E$,*

$$(f(x))^{-1} = \{a \in G \mid a.b \in \mathcal{E} \ \ and \ \ b \in f(x)\}.$$

Proof. Let $c \in (f(x))^{-1}$ and $b \in f(x)$ then $c.b \in (f(x))^{-1} * f(x) = \mathcal{E}$. So $c \in \{a \in G \mid a.b \in \mathcal{E} \ \ and \ \ b \in f(x)\}$.

Now let $c \in \{a \in G \mid a.b \in \mathcal{E} \ \ and \ \ b \in f(x)\}$. So $c.b \in \mathcal{E}$. From the uniqueness of the symmetric of b as an element of G and as $e \in \mathcal{E}$, $\exists e_1 \in \mathcal{E}$ such that $c.b = e_1$ and then $c = e_1.b^{-1} \in \mathcal{E} * (f(x))^{-1} = (f(x))^{-1}$. □

Proposition 3.45. *Suppose that $|G| = p^m q^n,\ p,\ q$ distinct prime integers. Let E, H be respectively a p sylow and a q sylow subgroup of G. We have $E^2 = E$ and $E \cap H = \{e\}$. If $p^\alpha \equiv 1[q] \implies \alpha = 0$ then the $HX-$group $\mathcal{G} = \{aE \mid a \in H\}$ is isomorphic to the group H and the mapping $f : H \longrightarrow \mathcal{G},\ a \longmapsto aE$ is a soft $HX-$group.*

Proof. The proof is trivial from the Example 3.15. □

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On weak δ- McCoy rings

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Abstract. Camillo, Kwak and Lee called a ring R right NC-McCoy if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ over R, $f(x)g(x) = 0$ implies $a_i c \in Nil(R)$ for some $c \in R - \{0\}$ and $0 \leq i \leq m$. For a derivation δ of a ring R, we in this paper introduce the weak δ- McCoy rings. When $\delta = 0$, this coincides with notation of a right NC-McCoy ring. Some properties of this generalization are established and connections of properties of a weak δ-McCoy ring R with $n \times n$ upper triangular $T_n(R, \sigma)$ and its polynomial ring $R[x]$, are investigated.
Keywords: McCoy rings, Skew polynomial ring, weak δ-McCoy rings.

1. Introduction

Throughout this note, all rings are associative with 1. Let R be a ring, δ be a derivation of R, that is δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x; \delta]$ the Ore extension whose elements are the polynomials over the ring R, the addition is defined as typical and the multiplication is defined as the relation $xa = ax + \delta(a)$, for any $a \in R$. We use $Nil(R)$, $Nil^*(R)$ and $M_n(R)$ for the nilradical, the upper nilradical (i.e., the sum of all nil two-sided ideals) and the n by n full matrix ring over R respectively.

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Use e_{ij} for the matrix with (i,j)-entry 1 and 0 for $i \neq j$. Let $C_{f(x)}$ denote the set of all coefficients of $f(x) \in R[x]$. By $\mathbb{Z}_n$ we mean the ring of integers module n.

Following Rege and Chhawchharia [1], a ring R is called Armendariz if polynomials $f(x) = a_0+a_1x+\cdots+a_nx^n$ and $g(x) = b_0+b_1x+\cdots+b_mx^m \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for each i and j. The name "Armendariz ring" is chosen because it is shown [2, Lemma 1] that reduced ring (that is a ring without nonzero nilpotent elements) satisfies the above condition. Nielsen [3], called a ring R *right McCoy* (resp., *left McCoy*) if, for any nonzero polynomials $f(x), g(x)$ over R, $f(x)g(x) = 0$ implies $f(x)c = 0$ (resp., $c'g(x) = 0$) for some $0 \neq c \in R$ (resp., $0 \neq c' \in R$). If a ring R is both left and right McCoy then it is called *McCoy*. By McCoy [4], commutative rings are McCoy rings. Clearly, Armendariz rings are McCoy but the converse does not hold by [1, Remark 4.3]. Following Camillo, Kwak and Lee [5], a ring R is called *right NC-McCoy* whenever, $f(x) = a_0+a_1x+\cdots+a_mx^m, g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ then $a_ic \in Nil(R)$ for some $0 \neq c \in R$. *Left NC-McCoy* rings are defined similarly. If a ring is both left and right NC-McCoy we say that the ring is *NC-McCoy* ring. The authors [10] introduced the notion of a ring with respect to a derivation δ of R. They defined a ring to be δ-McCoy if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x;\delta]$, $f(x)g(x) = 0$ implies that there exists $c \in R-\{0\}$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l\delta^{l-k}(c) = 0$ for $k = 0, 1, ..., m$.

We are motivated to introduce the notion of a *weak δ-McCoy* ring with respect to a derivation δ of R. This notion extends NC-McCoy rings. We do this by considering the NC-McCoy condition on polynomials in $R[x;\delta]$ instead of $R[x]$. This provides us with an opportunity to study NC-McCoy rings in a general setting, and several known results on NC-McCoy rings are obtained as corollaries.

2. Main results

We begin this section by the following definition and also we study properties of weak δ-McCoy rings.

Definition 2.1. *Let δ be a derivation of a ring R. The ring R is called* weak δ-McCoy *if for any nonzero polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x;\delta]$, $f(x)g(x) = 0$, implies that there exists $c \in R - \{0\}$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l\delta^{l-k}(c) \in Nil(R)$ for $k = 0, 1, ..., m$.*

It is clear that a ring R is right NC-McCoy if R is weak 0-McCoy, where 0 is the zero mapping. Also, it is clear that δ-McCoy rings are weak δ-McCoy, but the converse is not always true by the following example.

Example 2.2. Let $R = T_2(\mathbb{Z}_2)$ and the derivation $\delta : R \to R$ given by $\delta(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. If $f(x) = \sum_{i=0}^{m} A_ix^i$ and $g(x) = \sum_{j=0}^{n} B_jx^j$ be

nonzero polynomials in $R[x;\delta]$ such that $f(x)g(x)=0$, then

$$\sum_{l=k}^{m}\binom{l}{k}A_l\delta^{l-k}(e_{12})=\sum_{l=k}^{m}\binom{l}{k}A_le_{12}\in Nil(R)$$

for $k=0,1,...,m$. Therefore, R is weak δ-McCoy. But R is not δ-McCoy ring, because $(e_{11}+e_{12}+e_{12}x)(e_{12}+e_{22}+e_{12}x)=0$ in $R[x;\delta]$, and if $(e_{11}+e_{12}+e_{12}x)C=0$ for some $C\in R$, then $C=0$.

Let R_k be a ring, for each $k\in I$, δ_k a derivation of R_k and $R=\prod_{k\in I}R_k$. Then the map $\delta:R\to R$ defined by $\delta((a_k))=(\delta_k(a_k))$ is a derivation of R.

Proposition 2.3. *Let R_k be a ring with a derivation δ_k, where $k\in I$. If R_k is weak δ_k- McCoy, for each $k\in I$ then $R=\prod_{k\in I}R_k$ is weak δ-McCoy.*

Proof. Suppose that each R_k is weak δ_k- McCoy, for each $k\in I$ and $R=\prod_{k\in I}R_k$. Let $f(x)g(x)=0$ for some polynomials $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\delta]\backslash\{0\}$, where $a_i=(a_i^{(k)})$ and $b_j=(b_j^{(k)})$ are elements of the product ring R. Define $f_k(x)=\sum_{i=0}^{m}a_i^{(k)}x^i$ and $g_k(x)=\sum_{j=0}^{n}b_j^{(k)}x^j\in R[x;\delta_k]$. Since $f_k(x)g_k(x)=0$ and R_k is weak δ_k-McCoy ring, there exists $0\neq s_k\in R_k$ such that $\sum_{l=t}^{m}\binom{l}{t}a_l^{(k)}\delta_k^{l-t}(s_k)\in Nil(R_k)$. Thus,

$$\sum_{l=t}^{m}\binom{l}{t}(a_l^{(1)},\cdots,a_l^{(k)},\cdots)\delta^{l-t}(0,\cdots,s_k,0,\cdots)=$$

$$(0,\cdots,\sum_{l=t}^{m}\binom{l}{t}a_l^{(k)}\delta^{l-t}(s_k),0,\cdots)\in Nil(R).$$

Therefore, R is weak δ-McCoy. □

The converse of the above Proposition does not hold by the following example.

Example 2.4. Let R be any ring which is not weak 0-McCoy and consider $S=R\times\mathbb{Z}_4$. S is always weak 0-McCoy since if $f(t)\in S[t]$ is any polynomial, then one can take $c=(0,\bar{2})$ and then $f(t)c\in Nil(S[t])$.

Let I be an ideal and δ be a derivation of R. If $\delta(I)\subseteq I$, then $\overline{\delta}:R/I\to R/I$ defined by $\overline{\delta}(\overline{a})=\delta(a)+I$ for $a\in R$, is a derivation of the factor ring R/I, where $\overline{a}=a+I$. Now we have the following proposition.

Proposition 2.5. *Let δ be a derivation of a ring R and I be an ideal of R. If $\delta(I)\subseteq I$, $I\subseteq Nil(R)$ and R/I is weak $\overline{\delta}$-McCoy, then R is weak δ-McCoy.*

Proof. Let $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\delta]\backslash\{0\}$ such that $f(x)g(x)=0$. Then $(\sum_{i=0}^{m}\overline{a_i}x^i)(\sum_{j=0}^{n}\overline{b_j}x^j)=0$ in R/I. Thus there exists some positive integer n such that $(\sum_{l=k}^{m}\binom{l}{k}\overline{a}^l\overline{\delta}^{l-k}(\overline{s}))^n=0$ for some $s\in R-I$. Therefore $(\sum_{l=k}^{m}\binom{l}{k}a^l\delta^{l-k}(s))^n\in I\subseteq Nil(R)$. Thus, R is weak δ-McCoy. □

Now, we turn our attention to relationship between the weak δ-McCoy property of a ring R and its polynomial ring $R[x]$. Let δ be a derivation of a ring R. The map $\bar{\delta} : R[x] \to R[x]$ defined by $\bar{\delta}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \delta(a_i)x^i$ is a derivation of the polynomial ring $R[x]$, and clearly this map extends δ.

We say that R satisfies condition $(*)$, if for any $a, b \in R$ and $i \geq 0$, $ab \in Nil(R)$ implies that $a\delta^i(b) \in Nil(R)$.

Theorem 2.6. (1) *For a ring R and any derivation δ of R, if $R[x]$ is weak $\bar{\delta}$-McCoy, then R is weak δ-McCoy.*

(2) *Let R be a ring satisfying the condition $(*)$ and $Nil(R[x])$ is a subring of $R[x]$. If R is a weak δ-McCoy ring, then $R[x]$ is weak $\bar{\delta}$-McCoy.*

Proof. (1) Let $f(x)g(x) = 0$ for nonzero polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x, \delta]$. Set $f(y) = a_0 + a_1y + \cdots + a_my^m$ and $g(y) = b_0 + b_1y + \cdots + b_ny^n \in (R[x])[y, \bar{\delta}]$, where $R([x, \delta])[y]$ is the Ore extension of polynomials with an indeterminate y over $R[x]$. Then $f(y)$ and $g(y)$ are nonzero since $f(x)$ and $g(x)$ are nonzero. Moreover, $f(y)g(y) = 0$. So there exists a nonzero polynomial $c(x) = c_0 + c_1x + \cdots + c_tx^t \in R[x]$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l \bar{\delta}^{l-k}(c(x)) \in Nil(R[x])$ for $k = 0, \cdots, m$. Then $(\sum_{i=0}^{t} \sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_i))x^i \in Nil(R[x])$ for $k = 0, \cdots, m$. Since $c(x)$ is nonzero, there exists an integer p such that $c_p \neq 0$. Then $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c_p) \in Nil(R)$ for $k = 0, \cdots, m$. Therefore, R is weak δ- McCoy.

(2) Assume that R is weak δ-McCoy and $f(y)g(y) = 0$ for nonzero polynomials $f(y) = f_0 + f_1y + \cdots + f_my^m$ and $g(y) = g_0 + g_1y + \cdots + g_ny^n$ in $(R[x])[y]$. Take the positive integer t with $t = \sum_{i=0}^{m} deg(f_i) + \sum_{j=0}^{n} deg(g_j)$ where the degree of the zero polynomial is taken to be 0. Then $f(x^t)$ and $g(x^t)$ are nonzero polynomials in $R[x]$ and $f(x^t)g(x^t) = 0$, since the set of coefficients of $f(x^t)$ and $g(x^t)$ coincides with the set of coefficients of the f_i's and g_j's. Let $deg f(x^t) = s$. Since R is weak δ-McCoy, there exists a nonzero $c \in R$ such that $\sum_{l=k}^{s} \binom{l}{k} a_l \delta^{l-k}(c) \in Nil(R)$ for $0 \leq k \leq s$. Therefore,

$$\begin{aligned}
&a_s c \in Nil(R),\\
&a_{s-1}c + s a_s \delta(c) \in Nil(R),\\
&\vdots\\
&a_0 c + a_1\delta(c) + a_2\delta^2(c) + \cdots + a_s\delta^s(c) \in Nil(R).
\end{aligned}$$

Since $Nil(R[x])$ is a subring of $R[x]$, and R satisfies the condition $(*)$, then $ac \in Nil(R)$ for any $a \in C_{f_i(x)}$ and so $\sum_{l=k}^{m} \binom{l}{k} f_l \bar{\delta}^{l-k}(c) \in Nil(R)[x]$. On the other hand, for any $a \in Nil(R)$ and nonnegative integer t, ax^t is nilpotent. Thus $ax^t \in Nil(R[x])$, and so $Nil(R)[x] \subseteq Nil(R[x])$ as the latter is closed under addition. Thus $R[x]$ is weak $\bar{\delta}$-McCoy. □

Theorem 2.7. *Let R be a ring, e a central idempotent of R and δ be a derivation of R with $\delta(e) = 0$ for every $e^2 = e \in R$. Then R is weak δ- McCoy if and only if eR is weak δ- McCoy.*

Proof. Assume that R is a weak δ- McCoy ring. Consider $f(x) = \sum_{i=1}^{m} ea_i x^i$ and $g(x) = \sum_{i=1}^{n} eb_j x^j \in eR[x] \subseteq R[x]$ such that $f(x)g(x) = 0$. Since R is weak δ- McCoy, there exists $c \in R$ such that, $\sum_{l=k}^{m} \binom{l}{k} ea_l \delta^{l-k}(c) \in Nil(R)$ therefore, $\sum_{l=k}^{m} \binom{l}{k} ea_l \delta^{l-k}(ec) \in Nil(R)$. Consequently, eR is weak δ- McCoy. Conversely, let eR be a weak δ- McCoy, consider $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$ in $R[x;\delta]$ with $f(x)g(x) = 0$. Let $f_1(x) = ef(x)$ and $g_1(x) = eg(x)$. Then $f_1(x)g_1(x) = 0$, since eR is weak δ -McCoy, there exists c_1 in R such that $\sum_{l=k}^{m} \binom{l}{k} ea_l \delta^{l-k}(ec_1) \in Nil(R)$ for $0 \leq k \leq m$. If $c = ec_1$ then $\sum_{l=k}^{m} \binom{l}{k} a_l \delta^{l-k}(c) \in Nil(R)$ for $0 \leq k \leq m$. Thus R is weak δ-McCoy. □

Let δ be a derivation of a ring R and $M_n(R)$ be the $n \times n$ matrix over ring R and $\overline{\delta} : M_n(R) \to M_n(R)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. From Proposition 2.13 we may suspect that every $n \times n$ matrix ring over a ring R is weak $\bar{\delta}$-McCoy for any derivation δ on R. But the following example erase the possibility.

Example 2.8. Let R be a reduced ring with a derivation δ. Consider nonzero polynomials $f(x) = e_{11}+e_{12}x+e_{21}x^2+e_{22}x^3$ and $g(x) = -(e_{21}+e_{22})+(e_{11}+e_{12})x$ in $M_2(R)[x]$ with $f(x)g(x) = 0$. Assume to the contrary $M_2(R)$ is weak $\bar{\delta}$-McCoy. Then there exists nonzero $C = (c_{ij}) \in M_2(R)$ such that

$$\begin{aligned}
&(e_{22}C)^{n_1} = 0,\\
&(e_{21}C + 3e_{22}\bar{\delta}(C))^{n_2} = 0,\\
&(e_{12}C + 2e_{21}\bar{\delta}(C) + 3e_{22}\bar{\delta}(C))^{n_3} = 0,\\
&(e_{11}C + e_{12}\bar{\delta}(C) + e_{21}\bar{\delta}^2(C) + e_{22}\bar{\delta}^3(C))^{n_4} = 0
\end{aligned}$$

for some positive integers n_1, n_2, n_3, n_4, and so $c_{ij} = 0$ for any i, j by a simple computation, since R is reduced. This implies $C = 0$; which is a contradiction. Thus $M_2(R)$ is not weak $\overline{\delta}$-McCoy.

The next example shows that there exists a weak δ-McCoy ring R such that $R/J(R)$ is not weak $\bar{\delta}$-McCoy, where $J(R)$ is the radical Jacobson of R..

Example 2.9. Let R denote the localization of the ring $\mathbb{Z}$ of integers at the prime ideal $\langle 3\rangle$. Consider the quaternions $\mathbf{Q}$ over R, that is a free R-module with basis $\{1,\ i,\ j,\ k\}$ and multiplication satisfying $i^2 = j^2 = k^2 = -1,\ ij = k = -ji$. Then $\mathbf{Q}$ is a noncommutative domain with $J(\mathbf{Q}) = 3\mathbf{Q}$, and so is weak δ-McCoy. But $\mathbf{Q}/J(\mathbf{Q})$ is isomorphic to the 2-by-2 full matrix ring over $\mathbb{Z}_3$ and is not weak δ-McCoy by Example 2.8.

Although Example 2.8 shows that if R is a reduced ring, then $M_2(R)$ is not weak δ-McCoy, but we have the following.

Theorem 2.10. *Let R be a ring with derivation δ such that $\delta(Nil(R)) \subseteq Nil(R)$. Then:*

(1) *If R contains a nonzero nil one-sided ideal, then R is a weak δ-McCoy ring.*

(2) *Every ring R with $Nil^*(R) \neq 0$ is a weak δ-McCoy ring.*

(3) *If R contains a nonzero central nilpotent element, then the matrix ring over R ($M_n(R)$) is a weak $\overline{\delta}$-McCoy ring for $n \geq 2$.*

Proof. (1) If I is a nil one-sided ideal of R, then c in definition can be any nonzero element of I. Part (2) and (3) are trivial consequence of part (1). □

By using the same argument in the proof of [5, Proposition 11], we have the following.

Corollary 2.11. *Let R be a ring of bounded index with a nonzero nil two-sided ideal of bounded index. If δ is a derivation of R such that $\delta(Nil(R)) \subseteq Nil(R)$ and $(NilR)[x] = Nil(R[x])$ then both R and $R[x]$ are weak δ-McCoy $\left(weak\ \overline{\delta} - McCoy\right)$.*

Proof. Let I be a nonzero nil two-sided ideal of R. Since $0 \neq I \subseteq N^*(R)$, $N^*(R)$ containes a nonzero two-sided nilpotent ideal N of R by [[8], Lemma 5]. Then $N[x]$ is nonzero two-sided nilpotent ideal of $R[x]$. Since $\overline{\delta}(Nil(R[x])) \subseteq Nil(R[x])$ this implies that both R and $R[x]$ are weak δ- McCoy (weak $\overline{\delta} - McCoy$) rings, by Theorem 2.10. □

In [12], a ring R is called NI if $Nil^*(R) = Nil(R)$. Note that R is NI if and only if $Nil(R)$ forms a two sided ideal if and only if $R/Nil(R)$ is reduced. Any NI ring with a derivation δ such that $\delta(Nil(R)) \subseteq Nil(R)$ is weak δ-McCoy by Proposition 2.5. But the converse does not hold by the following example.

Example 2.12. Let R be a ring with derivation δ and nonzero central nilpotent element c such that $\delta(Nil(R)) \subseteq Nil(R)$. Then $M_n(R)$ $(n \geq 2)$ is a weak $\overline{\delta}$-McCoy ring by Proposition 2.10. However $M_n(R)$ can not be an NI ring as can be seen by the two nilpotent matrix units e_{12} and e_{21}.

Let R be a ring and σ denotes an endomorphism of R such that $\sigma(1) = 1$. In [9], the authors introduced *skew triangular matrix ring* a set of all triangular matrices with addition point-wise and a new multiplication subject to condition $e_{ij}r = \sigma^{j-i}(r)e_{ij}$. Therefore, $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + ... + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$. The derivation δ of R is extended to $\overline{\delta} : T_n(R, \sigma) \to T_n(R, \sigma)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$.

One can see that the map $\overline{\sigma} : R[x; \delta] \to R[x; \delta]$ defined by $\overline{\sigma}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \sigma(a_i)x^i$ is an endomorphism of the polynomial ring $R[x; \delta]$. Also the derivation δ of R is extended to $\overline{\delta} : T_n(R, \sigma) \to T_n(R, \sigma)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$,.

Proposition 2.13. *Let R be a ring, σ an endomorphism and δ a derivation of R such that $\delta\sigma = \sigma\delta$. Then $T_n(R, \sigma)$ is a weak $\overline{\delta}$-McCoy ring for $n \geq 2$.*

Proof. Let $f(x) = A_0 + A_1x + \cdots + A_px^p$ and $g(x) = B_0 + B_1x + \cdots + B_qx^q$ be elements of $T_n(R, \sigma)[x; \bar{\delta}]$ satisfying $f(x)g(x) = 0$. Then

$$\left(\sum_{l=k}^{p} \binom{l}{k} A_l \bar{\delta}^{(l-k)}(e_{1n})\right)^2 = 0$$

and the proof is complete. □

Proposition 2.14. *Let δ be a derivation of a ring R. Let S be a ring and $\phi : R \to S$ be a ring isomorphism. Then R is weak δ-McCoy if and only if S is weak $\phi\delta\phi^{-1}$-McCoy.*

Proof. Let $\alpha' = \varphi\alpha\varphi^{-1}$ and $\delta' = \varphi\delta\varphi^{-1}$. Since $\delta'(ab) = \varphi\delta(\varphi^{-1}(a)\ \varphi^{-1}(b)) = \varphi((\delta\varphi^{-1}(a)\varphi^{-1}(b) + \varphi^{-1}(a)(\delta\varphi^{-1}(b))) = \delta'(a)b + a\delta'(b)$, then δ' is a derivation of S. Suppose $a' = \varphi(a)$, for each $a \in R$. Therefore $p(x) = \sum_{i=0}^{m} a_ix^i$ and $q(x) = \sum_{j=0}^{n} b_jx^j$ are nonzero in $R[x; \delta]$ if and only if $p'(x) = \sum_{i=0}^{m} a'_ix^i$ and $q'(x) = \sum_{j=0}^{n} b'_jx^j$ are nonzero in $S[x; \delta']$. On the other hand, $p(x)q(x) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}\varphi(a_i\delta^{i-l}(b_{k-l})) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}a'_i\varphi(\varphi^{-1}\varphi\delta^{i-l}\varphi^{-1}\varphi(b_{k-l})) = 0$ iff $\sum_{l=0}^{k}\sum_{i=l}^{m}\binom{i}{l}a'_i\delta'^{i-l}(b'_{k-l}) = 0$ iff $p(x)q(x) = 0$ for $0 \leq k \leq m+n$. Also for some nonzero $c \in R$, $\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(c) \in Nil(R)$ iff for some positive integer n, $(\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(c))^n = 0$ iff $(\sum_{l=k}^{m}\binom{l}{k}\varphi(a_l)\varphi\delta^{l-k}\varphi^{-1}\varphi(c))^n = 0$ iff $(\sum_{l=k}^{m}\binom{l}{k}a'_l\delta'^{l-k}(c'))^n = 0$ iff $\sum_{l=k}^{m}\binom{l}{k}a'_l\delta'^{l-k}(c') \in Nil(S)$, for $0 \neq c' = \varphi(c) \in S$. Thus R is weak δ-McCoy if and only if S is weak $\varphi\delta\varphi^{-1}$-McCoy. □

Following Cohn [6], a ring R is called reversible if $ab = 0$ implies that $ba = 0$. Clearly, reduced rings are reversible. Moreover for any derivation δ, R is said to be δ-compatible if for each $a, b \in R$, $ab = 0$ implies that $a\delta(b) = 0$. The following lemma is appeared in [7].

Lemma 2.15. *Let R be a δ-compatible ring. If $ab = 0$, then $a\delta^m(b) = 0 = \delta^m(a)b$, for all positive integer m.*

Theorem 2.16. *Let R be a reversible ring. If R is δ-compatible then R is weak δ-McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ be nonzero polynomials in $R[x; \delta]$ such that $f(x)g(x) = 0$. We can assume $g(x)$ has minimum degree that satisfies $f(x)g(x) = 0$ and $b_1 \neq 0$. We show that $a_ib_j = 0$, for each i and j, and this implies $\sum_{l=k}^{m}\binom{l}{k}a_l\delta^{l-k}(b_1) = 0 \in Nil(R)$ and so R is weak δ-McCoy. Since $f(x)g(x) = 0$ and R is reversible, we have $a_mb_n = 0 = b_na_m$. So $b_nx^na_m = 0$, since R is δ-compatible. On the other hand, $f(x)g(x)a_m = f(x)(\sum_{j=0}^{n} b_jx^j)a_m = 0$. Thus $f(x)(b_0+...+b_{n-1}x^{m-1})a_m = 0$. Since the degree of $g(x)$ is minimum, we have $(b_0+...+b_{n-1}x^{n-1})a_m = 0$. So $b_ja_m = a_mb_j = 0$, for each $0 \leq j \leq n-1$, since R is reversible and δ-compatible. Hence $a_mx^mb_j = 0$, for $0 \leq j \leq n$, since R is δ-compatible. So $(a_0 + ... + a_{m-1}x^{m-1})g(x) = 0$,

and hence $a_{m-1}b_n = 0$. Therefore, $a_{m-1}b_n = b_n a_{m-1} = 0$. On the other hand, we have $f(x)g(x)a_{m-1} = 0$. Hence $f(x)(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since $b_n x^n a_{m-1} = 0$. Therefore we have $(b_0 + ... + b_{n-1}x^{n-1})a_{m-1} = 0$, since the degree of $g(x)$ is minimum, and so according to above $a_{m-1}b_j = b_j a_{m-1} = 0$, for all j. Continuing in this way, we get $a_i b_j = 0$, for each i and j, and the result follows. □

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let RS^{-1} be the localization of R at S. Then each derivation δ of R, extends to RS^{-1}, by setting $\overline{\delta}(rc^{-1}) = r\delta(c)^{-1}$, for each $r, c \in R$, with c regular. Now we have the following:

Theorem 2.17. *For a ring R and derivation δ of R, if R is weak δ-McCoy then RS^{-1} is weak $\overline{\delta}$-McCoy.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i c_i^{-1} x^i$ and $g(x) = \sum_{j=0}^{n} b_j d_j^{-1} x^j \in RS^{-1}[x; \overline{\delta}]$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = c^{-1}a_i'$ and $b_i d_i^{-1} = d^{-1}b_j'$ with c , d regular elements in R. So $f'(x)g'(x) = 0$ such that $f'(x) = \sum_{i=0}^{m} a_i' x^i$ and $g'(x) = \sum_{j=0}^{n} b_j' x^j$. Since R is weak δ -McCoy, there exists $0 \neq r \in R$ such that $\sum_{l=k}^{m} \binom{l}{k} a_l' \delta^{l-k}(r) \in Nil(R)$. Hence $\sum_{l=k}^{m} \binom{l}{k} a_l c_l^{-1} \overline{\delta}^{l-k}(r) \in Nil(RS^{-1})$ and so RS^{-1} is weak $\overline{\delta}$-McCoy. □

Corollary 2.18. *Let $R[x, \delta]$ be a weak δ-McCoy ring. Then $R[x; x^{-1}, \delta]$ is a weak δ-McCoy ring.*

Proof. It is directly follows from Proposition 2.17. Let $S = \{1, x, x^2, ...\}$, then clearly S is a multiplicatively closed subset of $R[x, \delta]$ and $R[x, x^{-1}, \delta] = S^{-1}R[x, \delta]$. □

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Numerical methods for solving Lane-Emden type differential equations by operational matrix of fractional derivative of modified generalized Laguerre polynomials

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Abstract. The present paper tries to elaborate on the application of operational matrix of derivative of modified generalized Laguerre polynomials for solving Lane–Emden type equations in astrophysics. Moreover, these equations were numerically solved by the help of this operational matrix. Furthermore, some representative instances were presented to indicate the capability, acceptability and logicality of the suggested methods.

Keywords: Lane–Emden type equations; operational matrix of derivative; modified generalized Laguerre polynomials; Caputo derivative; fractional calculus and astrophysics.

1. Introduction

Some of phenomena in mathematical physics and astrophysics are shaped by equations of type Lane-Emden as one of the most important equations in the category of second-order nonlinear ordinary differential equations (ODEs) [1, 2, 3, 4, 5].

Generalizing the idea of n-fold integration and integer-order differentiation leads to "fractional calculus" that has significant applications in diverse areas of engineering sciences and mathematical physics. In fact, fractional calculus as the theory of derivatives and integrals of a given principle or multiplex order can be fascinating and engaging for many researchers, as one of the most effective

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tools in fractional differential equations, to illustrate common characteristics of a variety of processes and materials, whereas such effects are ignored in the classical integer-order models. As the most important effective factor of fractional derivatives, one can consider the fractional derivatives more advantageous than its classical models in modeling electrical and mechanical characteristics of real materials in many fields [11].

As things are, for mathematical modeling of some physical phenomena, one may face the issue of solving varied kinds of fractional differential equations. Then, these equations play main roles in physics and several fields of engineering, as well as, mathematics. Since three decades ago, diverse operators have been investigated in some of papers on fractional calculus such as Erdlyi-Kober operators [13], Riemann-Liouville operators [12], Caputo operators [15], Weyl-Riesz operators [14] and Grnwald-Letnikov operators [16]. Moreover, the present researchers refer the reader to [17] in which the existence of definite solution and multi-positive solutions for nonlinear fractional differential equations are established [18, 19]. Also, in [6], the authors applied Legendre wavelet method for solving differential equations of Lane-Emden type.
It is noteworthy that the present paper is assigned to generalizing the explanation of Lane-Emden equations up to fractional order in the way as provided in the following equation:

$$\begin{gathered} D^{\tau}\omega(\xi) + \tfrac{\theta}{\xi^{\tau-\rho}} D^{\rho}\omega(\xi) + f(\xi,\omega) = g(\xi), \\ 0 < \xi \leq 1, \quad 0 \leq \theta, \quad 1 < \tau \leq 2, \quad 0 < \rho \leq 1, \end{gathered}$$

with the initial conditions (IC)

$$\omega(0) = A \quad , \quad \omega'(0) = B,$$

where A, B are constants, $f(\xi,\omega)$ is a real-valued function and $g \in C[0,1]$. As previously mentioned, the present researchers applied operational matrix of fractional derivative of modified generalized Laguerre polynomials (OMFDMGLPs) for solving Lane–Emden type equations. As it is shown, the simplicity of implementation of this method is very simple and the precision of answers is high. To this end, this paper is organized as: Section 2 represents definitions; in this section, the modified generalized Laguerre polynomials (MGLPs) and some attributes of fractional derivative are introduced. The OMDMGLPs of fractional derivative is presented in Section 3. In Section 4, the researchers implemented them on Lane–Emden equation. In Section 5, some models are discussed to illustrate the efficiency and precision of the method. Finally, Section 6 includes a conclusions of the obtained results and findings.

2. Mathematical preliminaries

2.1 Fractional derivative

To recall the requirements of the fractional calculus, the present researchers started with a definition. In the theory of integrals and derivatives of any order,

generalization and incorporation of two concepts (i.e., integer-order differentiation and n-fold integration) is called the fractional calculus [20, 21]. Some of mathematicians such as Grunwald-Letnikove and Riemann-Liouville's diversely introduced definitions for fractional integration and differentiation. They are not fruitful in our purpose since, for example, Riemann-Liouville has certain disadvantages in modeling real-world phenomena with fractional differential equations. In fact, the researchers used a changed fractional differential operator D^v proposed in Caputo's work on the theory of viscoelasticity [22].

Definition 2.1. *The Caputo fractional derivative is marked out as:*

$$D^v f(\xi) = \frac{1}{\Gamma(n-v)} \int_0^\xi \frac{f^{(n)}(\omega)}{(\xi-\omega)^{v+1-n}} d\omega, \qquad n-1 < v \leq n, n \in \mathbb{N}, \xi > 0$$

In that v is a positive real number as the order of the derivative and n is the smallest integer greater than v.

Note that [23]:

$$(1) \qquad D^v \xi^\beta = \begin{cases} 0, & for\ \beta \in \mathbb{N}_0\ and\ \beta < \lceil v \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} \xi^{\beta-v}, & for\ \beta \in \mathbb{N}_0, \beta \geq \lceil v \rceil\ \ or\ \beta \notin \mathbb{N}, \beta > \lfloor v \rfloor. \end{cases}$$

In this paper, the symbols $\lceil v \rceil$ and $\lfloor v \rfloor$(the ceiling and the floor functions) stand for the smallest integer greater than or equal to v and the largest integer less than or equal to v, respectively. In addition, the researchers utilized notations $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. It is noteworthy that the differential operator in the sense of Caputo for $v \in \mathbb{N}$ agrees with differential operator of an integer-order in the usual sense. The fractional differentiation in the sense of Caputo is a linear operation, as in the integer-order differentiation:

$$(2) \qquad D^v (\lambda f(\xi) + \mu g(\xi)) = \lambda D^v f(\xi) + \mu D^v g(\xi),$$

where λ and μ are constants.

2.2 MGLPs and properties ([19])

Let $\Lambda = (0, \infty)$ and $w^{(\alpha,\beta)}(\xi) = \xi^\alpha e^{-\beta\xi}$ be a weight function on Λ in the usual sense. Now, define:

$$L^2_{w^{(\alpha,\beta)}}(\Lambda) = \{v \mid v\ is\ measurable\ on\ \Lambda\ and\ \|v\|_{w^{(\alpha,\beta)}} < \infty\},$$

with the below inner product and norm:

$$(u, v)_{w^{(\alpha,\beta)}} = \int_\Lambda u(\xi)\, v(\xi)\, w^{(\alpha,\beta)}(\xi)\, d\xi, \qquad \|v\|_{w^{(\alpha,\beta)}} = (v, v)^{\frac{1}{2}}_{w^{(\alpha,\beta)}}.$$

Next, let $L_i^{(\alpha,\beta)}(\xi)$ be the MGLPs of degree i for $\alpha > -1$ and $\beta > 0$. Clearly, $L_i^{(\alpha,\beta)}(\xi)$ is explained by:

$$L_i^{(\alpha,\beta)}(\xi) = \frac{1}{i!} \xi^{-\alpha} e^{\beta\xi} \partial_\xi^i \left(\xi^{i+\alpha} e^{-\beta\xi}\right), \qquad i = 1, 2, \dots$$

For $\alpha > -1$ and $\beta > 0$, it is clear that

$$\partial_\xi L_i^{(\alpha,\beta)}(\xi) = -\beta L_{i-1}^{(\alpha+1,\beta)}(\xi),$$

$$L_{i+1}^{(\alpha,\beta)}(\xi) = \frac{1}{i+1}\left[(2i+\alpha+1-\beta\xi)\, L_i^{(\alpha,\beta)}(\xi) - (i+\alpha)\, L_{i-1}^{(\alpha,\beta)}(\xi)\right], \quad i = 1, 2, \ldots,$$

where $L_0^{(\alpha,\beta)}(\xi) = 1$ and $L_1^{(\alpha,\beta)}(\xi) = -\beta\ \xi + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}$.

The set of MGLPs is the $L^2_{w^{(\alpha,\beta)}}(\Lambda)$ -orthogonal system, i.e.

$$\int_0^\infty L_j^{(\alpha,\beta)}(\xi)\, L_k^{(\alpha,\beta)}(\xi) w^{(\alpha,\beta)}(\xi)\, d\xi = h_k \delta_{jk},$$

where δ_{jk} is the Kronecker function and $h_k = \frac{\Gamma(k+\alpha+1)}{\beta^{\alpha+1} k!}$. The MGLPs of degree i on the interval Λ is presented by:

$$L_i^{(\alpha,\beta)}(\xi) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)\beta^k}{\Gamma(k+\alpha+1)(i-k)!k!}\xi^k, \qquad i = 0, 1, \ldots, \tag{3}$$

where $L_i^{(\alpha,\beta)}(0) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)\Gamma(i+1)}$.

The special value:

$$D^q L_i^{(\alpha,\beta)}(0) = \frac{(-1)^q \beta^q \Gamma(i+\alpha+1)}{(i-q)!\Gamma(q+\alpha+1)}, \qquad i \geq q$$

can be of significant application later.

2.3 Operational matrix of fractional derivative of MGLPs in Caputo sense ([24])

Let $u \in L^2_{w^{(\alpha,\beta)}}(\Lambda)$, then $u(\xi)$ may be defined based on MGLP as:

$$u(\xi) = \sum_{\mathrm{j}=0}^{\infty} a_{\mathrm{j}} \mathrm{L}_{\mathrm{j}}^{(\xi,\beta)}(\xi),$$

$$a_{\mathrm{j}} = \frac{1}{\mathrm{h_k}} \int_0^\infty \mathrm{u}(\xi) \mathrm{L}_{\mathrm{j}}^{(\alpha,\beta)}(\xi) \mathrm{w}^{(\alpha,\beta)}(\xi)\, \mathrm{d}\xi, \quad \mathrm{j} = 0, 1, \ldots.$$

In specific uses, the MGLPs up to degree $N+1$ are noticed. Then, the present researchers have:

$$u_N(\xi) = a_{\mathrm{j}} \mathrm{L}_{\mathrm{j}}^{(\alpha,\beta)}(\xi) = \mathrm{C}^{\mathrm{T}} \mathbf{\Phi}(\xi),$$

where the MGLPs coefficient vector C and the MGLPs vector $\mathbf{\Phi}(\xi)$ are presented by:

$$C^T = [c_0, c_1, , c_N], \quad \mathbf{\Phi}(\xi) = \left[\mathrm{L}_0^{(\alpha,\beta)}(\xi), \mathrm{L}_1^{(\alpha,\beta)}(\xi), \ldots, \mathrm{L}_{\mathrm{N}}^{(\alpha,\beta)}(\xi)\right]^T,$$

Then, the derivative of the vector $\mathbf{\Phi}(\xi)$ can be uttered by the follows:

$$\frac{\mathrm{d}\mathbf{\Phi}(\xi)}{\mathrm{d}\xi}=\mathrm{D}^{(1)}\mathbf{\Phi}(\xi), \tag{4}$$

where $\mathrm{D}^{(1)}$ is the $(N+1)\times(N+1)$ operational matrix of derivative given by:

$$\mathrm{D}^{(1)}=-\beta\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}. \tag{5}$$

By using Eq. (4), it is clear that:

$$\frac{d^n\mathbf{\Phi}(\xi)}{d\xi^n}=\left(D^{(1)}\right)^n\mathbf{\Phi}(\xi), \tag{6}$$

where $n\in\mathbb{N}$ and the superscript in $\mathrm{D}^{(1)}$ give the meaning to matrix powers. Therefore,

$$D^{(n)}=\left(D^{(1)}\right)^n, \qquad n=1,2,\ldots.$$

Lemma 2.1. *Let* $\mathrm{L}_{\mathrm{i}}^{(\xi,\beta)}(\xi)$ *be a MGLPs. Then*

$$D^v L_j^{(\alpha,\beta)}(\xi)=0, \qquad i=0,1,\ldots,\lceil v\rceil-1, v>0.$$

Proof. By utilizing Eqs. (1) and (2) in Eq. (3), the lemma can be proved [24].

Theorem 2.1 ([24])**.** *Let* $\emptyset(\xi)$ *be the MGLPs vector explained in Eq. (3.3) and notice* $v>0$*, then*

$$D^v\mathbf{\Phi}(\xi)\simeq D^{(v)}\mathbf{\Phi}(\xi), \tag{7}$$

where $D^{(v)}$ *is the* $(N+1)\times(N+1)$ *OMDMGLPs of fractional derivatives of order* v *in Caputo sense, expressed as follows:*

$$D^{(v)}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 0\\ \Omega_v(\lceil v\rceil,0) & \Omega_v(\lceil v\rceil,1) & \Omega_v(\lceil v\rceil,2) & \cdots & \Omega_v(\lceil v\rceil,N)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \Omega_v(i,0) & \Omega_v(i,1) & \Omega_v(i,2) & \cdots & \Omega_v(i,N)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \Omega_v(N,0) & \Omega_v(N,1) & \Omega_v(N,2) & \cdots & \Omega_v(N,N) \end{pmatrix}, \tag{8}$$

where

$$\Omega_v(i,j)=\sum_{k=\lceil v\rceil}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}\beta^v j!\Gamma(i+\alpha+1)\,\Gamma(k-v+\alpha+l+1)}{(i-k)!\,(j-l)!\Gamma(k-v+1)\,\Gamma(k+\alpha+1)\,\Gamma(\alpha+l+1)}.$$

Notice that in $D^{(v)}$, the first $\lceil v\rceil$ rows are all zero.

Remark 2.1. In the case of $v=n\in\mathbb{N}$, Theorem 2.1 presents the same result as Eq. (6).

3. Applications of the OMFDMGLPs in Caputo sense

3.1 Solution of Lane-Emden type equations

Consider the Lane–Emden equation of the following form [25, 26]:

$$\omega''(\xi)+\frac{\theta}{\xi}\omega'(\xi)+f(\xi,\omega)=g(\xi),\qquad \theta,\xi\geq 0, \tag{9}$$

with IC:

$$\omega(0)=a,\quad \omega'(0)=0.$$

By approximating $\omega(\xi)$, $f(\xi,\omega)$ and $g(\xi)$ using MGLPs, we have:

$$\omega(\xi)\approx\sum_{i=0}^{N}c_i p_i(\xi)=C^T\boldsymbol{\Phi}(\xi). \tag{10}$$

$$f(\xi,\omega)\approx f\left(\xi,\ C^T\boldsymbol{\Phi}(\xi)\right)=H^T\boldsymbol{\Phi}(\xi),\qquad g(\xi)\approx\sum_{i=0}^{N}c_i\mathrm{p_i}(\xi)=G^T\boldsymbol{\Phi}(\xi),$$

where the unknowns are $C=[c_0,\ldots,c_N]^T$and $H=[h_0,\ldots,h_N]^T$. By applying (5), Eq. (9) can be as

$$C^T\boldsymbol{D}^{(2)}\boldsymbol{\Phi}(\xi)+\frac{\theta}{\xi}C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(\xi)+H^T\boldsymbol{\Phi}(\xi)\approx G^T\boldsymbol{\Phi}(\xi), \tag{11}$$

$$\omega(0)=C^T\boldsymbol{\Phi}(0)=d_0,\qquad \omega'(0)=C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(0)=d_1. \tag{12}$$

Eqs. (11) and (12) present two linear equations. Since the total unknowns for vector C in Eq. (10) is $(N+1)$, the present researchers collocate Eq. (11) in $(N-2)$ points ξ_p in the interval $[0,1]$ as

$$\xi_p=\frac{2p-1}{2(n+1)},\quad p=1,2,\ldots,n-2\ .$$

Then, we will have:

$$C^T\boldsymbol{D}^{(2)}\boldsymbol{\Phi}(\xi_i)+\frac{\theta}{\xi_i}C^T\boldsymbol{D}^{(1)}\boldsymbol{\Phi}(\xi_i)+H^T\boldsymbol{\Phi}(\xi_i)\approx G^T\boldsymbol{\Phi}(\xi_i). \tag{13}$$

For $i=1,2,\ldots,N-2$, now, the Eqs. (12) and (13) generate a system of $(N+1)$ nonlinear system of equations that can be solved by using known methods.

3.2 Solution of fractional differential equations of Lane–Emden type

Consider the fractional differential equations of Lane–Emden type

$$D^{\tau}\omega(\xi)+\frac{\theta}{\xi^{\tau-\rho}}D^{\rho}\omega(\xi)+f(\xi,\omega)=g(\xi),0<\xi\leq 1,\theta\geq 0,1<\tau\leq 2,0<\rho\leq 1, \tag{14}$$

with the IC:

$$\omega\left(0\right)=A\quad,\quad\omega'\left(0\right)=B, \tag{15}$$

where $A,\ B$ are constants, $f(\xi,\omega)$ is a continuous real-valued function and $g\left(\xi\right)\in C\left[0,1\right]$.

Approximating $\omega(\xi)$, $f(\xi,\omega)$ and $g(\xi)$ by MGLP, we have:

$$\omega(\xi)\approx\sum_{i=0}^{N}c_{i}p_{i}\left(\xi\right)=C^{T}\boldsymbol{\Phi}\left(\xi\right),$$

$$f\left(\xi,\omega\right)\approx f\left(\xi,\ C^{T}\boldsymbol{\Phi}\left(\xi\right)\right)=H^{T}\boldsymbol{\Phi}\left(\xi\right),\quad g\left(\xi\right)\approx\ \sum_{i=0}^{N}c_{i}\mathrm{p_{i}}\left(\xi\right)=G^{T}\boldsymbol{\Phi}\left(\xi\right), \tag{16}$$

where the unknowns are $C=[c_0,\ldots,c_N]^T$and $H=[h_0,\ldots,h_N]^T$. By using (7), Eq. (14) can be as

$$C^{T}\boldsymbol{D}^{(\tau)}\boldsymbol{\Phi}\left(\xi\right)+\frac{\theta}{\xi}C^{T}\boldsymbol{D}^{(\rho)}\boldsymbol{\Phi}\left(\xi\right)+H^{T}\boldsymbol{\Phi}\left(\xi\right)\approx G^{T}\boldsymbol{\Phi}\left(\xi\right). \tag{17}$$

The IC (15) is given by:

$$\omega\left(0\right)=C^{T}\boldsymbol{\Phi}\left(0\right)=d_{2},\qquad\omega'\left(0\right)=C^{T}\boldsymbol{D}^{(1)}\boldsymbol{\Phi}\left(0\right)=d_{3}. \tag{18}$$

Eqs. (17) and (18) present two linear system of equations. Since the total unknowns for vector C in Eq. (16) is $(N+1)$, the researchers collocate Eq. (17) in $(N-2)$ points ξ_i in the interval $[0,1]$ as,

$$\xi_{p}=\frac{2p-1}{2(n+1)},\qquad p=1,2,\ldots,n-2\ .$$

Then, the present researchers will have:

$$C^{T}\boldsymbol{D}^{(\tau)}\boldsymbol{\Phi}\left(\xi_{i}\right)+\frac{\theta}{\xi_{i}}C^{T}\boldsymbol{D}^{(\rho)}\boldsymbol{\Phi}\left(\xi_{i}\right)+H^{T}\boldsymbol{\Phi}\left(\xi_{i}\right)\approx G^{T}\boldsymbol{\Phi}\left(\xi_{i}\right). \tag{19}$$

For $i=1,2,\ldots,N-2$; now, Eqs. (18) and (19) generate a system of $(N+1)$ nonlinear equations that can be sove by known methods.

4. Illustrative examples

To illustrate the acceptability, logicality, and effectiveness of the suggested method, we solve some Lane–Emden type equations presented in Eqs. (9) and (14) numerically. It is noteworthy that all of the computations have been performed using Mathematica 9.

Example 4.1. Notice the following nonlinear Lane-Emden equation [8]:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + \omega^m = 0, \qquad 0 < \xi < 1,$$

The IC are $\omega(0) = 1, \quad \omega'(0) = 0$. For $m = 5$, this example has the exact solution $\omega(\xi) = (1 + \frac{\xi^2}{3})^{-\frac{1}{2}}$. The results of this example are tabulated in Table 1 and Fig. 1 for N= 8. In addition, the absolute errors diagram is shown in Fig. 2.

For $m = 3$, this example has the following form:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + \omega^3 = 0, \qquad 0 < \xi < 1,$$

with the IC

$$\omega(0) = 1, \quad \omega'(0) = 0.$$

Table 2 shows the approximation of $\omega(\xi)$ for the present method and exact values given by Horedt [29].

Table 1. Numerical results and exact solution for Example 4.1

ξ_i	$N = 8,\ \alpha = \beta = 1/2$	$N = 8,\ \alpha = \beta = 1$	Exact solution
0.0	1.	1.	1.
0.2	0.9933992680853407	0.9933992678561196	0.9933992677987828
0.4	0.9743547050475172	0.9743547050142785	0.9743547036924463
0.6	0.9449111844072604	0.9449111843377054	0.944911182523068
0.8	0.9078413117090349	0.9078413116788155	0.9078412990032037
1.0	0.8660300001529322	0.8660299995409204	0.8660254037844386

Table 2. Comparison between the numerical solutions and error obtained by the proposed methods and its numerical values [7].

ξ_i	Method of [8]	Present method	Exact value [29]	Error of [8]	Error at N=10
0.0	1.0000000	1.0000000	1.0000000	0.00	0.00
0.1	0.99833720	0.99833582	0.9983358	$1.40e-06$	$2.00e-08$
0.5	0.95984209	0.95983906	0.9598391	$2.99e-06$	$4.00e-08$
1.0	0.85505959	0.85505754	0.8550576	$1.99e-06$	$6.00e-08$

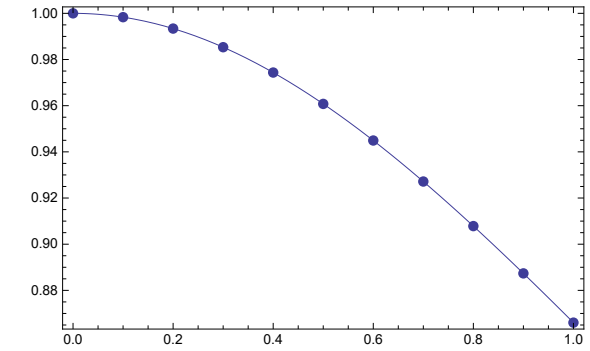


Figure 1: — Exact solution, . . . Approximate solution.

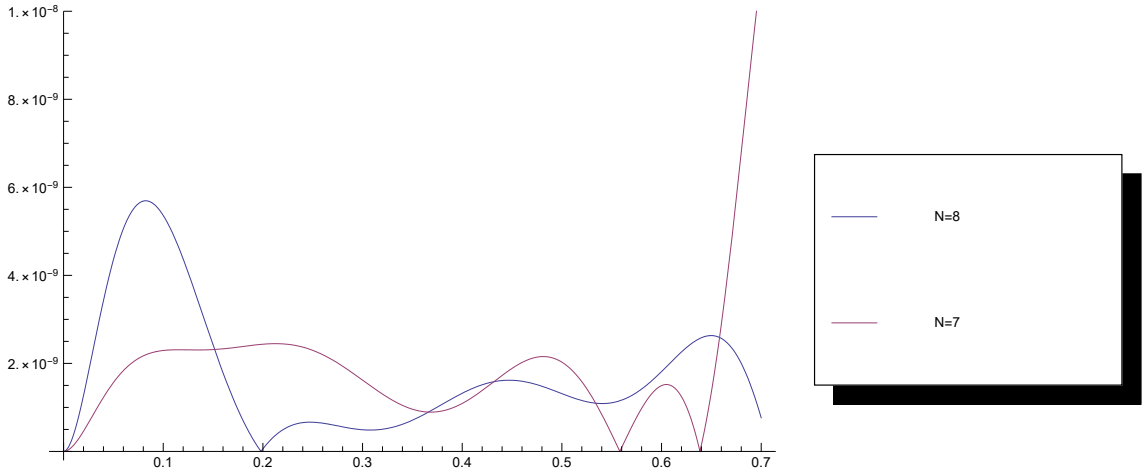


Figure 2: Graph of absolute errors for N = 7 and N = 8.

Example 4.2. The isothermal gas spheres are modeled by Davis (see [31]) as follows:

$$\omega''(\xi) + \frac{2}{\xi}\omega'(\xi) + e^{\omega(\xi)} = 0, \quad \xi \geq 0,$$

with the IC:

$$\omega(0) = 0, \quad \omega'(0) = 0.$$

The researchers clarified this instance by $N = 10$, which the results are tabulated in Table 3.

Table 3. Comparison of $\omega(\xi)$, with the present method and series of solutions offered by Wazwaz [27] and numerical values in [7] and [28], for isothermal gas sphere equation ($\alpha = \beta = 1$).

ξ_i	Method of [7]	Method of [27]	Method of [28]	Present method	Error of [7]	Error
0.0	0. 000000000	0. 000000000	0. 000000000	0. 000000000	0.00	0.00
0.1	−0. 0016664188	-0. 0016658339	-0. 0016655333	-0. 00166583385	5.85e -07	1.00e -10
0.2	-0. 0066539713	-0. 0066533671	−0. 0066533333	-0. 00665336709	6.04e -07	1.00e -11
0.5	-0. 0411545150	-0. 0411539568	-0. 04114583333	-0. 0411539572	5.58e -07	4.00e -10
1.0	-0. 1588281737	-0. 1588273537	-0. 1583333333	-0. 15882767809	8.20e -07	3.24e -07

Example 4.3. Consider the following equation [30]:

$$D^{\tau}\omega(\xi) + \frac{\theta}{\xi^{\tau-\rho}} D^{\rho}\omega(\xi) + \frac{1}{\xi^{\tau-2}}\omega(\xi) =$$

$$\left(2\left(\frac{\Gamma(3-\rho)+\theta(\Gamma(3-\tau))}{\Gamma(3-\rho)\Gamma(3-\tau)} + \frac{\xi^2}{2}\right) - 6\xi\left(\frac{\Gamma(4-\rho)+\theta(\Gamma(4-\tau))}{\Gamma(4-\rho)\Gamma(4-\tau)} + \frac{\xi^2}{6}\right)\right)\xi^{2-\tau},$$

where the exact solution is $\omega(\xi) = \xi^2 - \xi^3$ for $\tau = \frac{3}{2}, \rho = 1, \theta = 2$, and the IC are as follows:

$$\omega(0) = 0, \quad \omega'(0) = 0.$$

To compare the numerical results obtained by the proposed method with the exact solution, see Table 4.

Table 4: Comparing the numerical results and the exact solution for Example 4.3.

ξ_i	$N = 15,\ \alpha = 0, \beta = 1$	$N = 15,\ \alpha = 1/2, \beta = 1/2$	*Exact solution*
0.0	$1.033413307865862\times10^{-24}$	$3.7065625075664017\times10^{-24}$	0.
0.2	$1.487269876187146\times10^{-22}$	$6.110365821755565\times10^{-23}$	0.032
0.4	$3.435493502332913\times10^{-22}$	$1.6510946972972456\times10^{-22}$	0.096
0.6	$4.147499545115917\times10^{-22}$	$2.274652005311722\times10^{-22}$	0.144
0.8	$3.197835497234143\times10^{-22}$	$1.8155964929517072\times10^{-22}$	0.128

Example 4.4. Deem the equation [30]:

$$D^{\tau}\omega(\xi) + \frac{\theta}{\xi^{\tau-\rho}} D^{\rho}\omega(\xi) + \frac{1}{\xi^{\tau-2}}\omega(\xi) =$$

$$\left(6\xi\left(\frac{\Gamma(4-\rho)+\theta(\Gamma(4-\tau))}{\Gamma(4-\rho)\Gamma(4-\tau)}+\frac{\xi^2}{6}\right)-2\left(\frac{\Gamma(3-\rho)+\theta(\Gamma(3-\tau))}{\Gamma(3-\rho)\Gamma(3-\tau)}+\frac{\xi^2}{2}\right)\right)\xi^{2-\tau},$$

with the IC

$$\omega(0)=0\ ,\quad \omega'(0)=0,$$

and the exact solution $\omega(\xi)=\xi^3-\xi^2$, where $\tau=\frac{3}{2},\rho=\frac{1}{2},\ \theta=2$. To see absolute error for $N=10$, one can refer to Table 5.

Table 5: Absolute error for Example 4.4.

ξ_i	Error for $N=10,\ \alpha=0$ and $\beta=1$
0.0	1.93271×10^{-27}
0.2	0.032
0.4	0.096
0.6	0.144
0.8	0.128
1	6.01587×10^{-18}

Example 4.5. Consider the following equation

$$D^{\tau}\omega(\xi)+\frac{2}{\xi}\omega'(\xi)+\omega(\xi)^5=0.$$

The exact solution of this example for $\tau=2$ is $\omega(\xi)=(1+\frac{\xi^2}{3})^{-\frac{1}{2}}$. Fig. 3 illustrates approximation of $\omega(\xi)$ with $\tau=1.3$, $\tau=1.6$, $\tau=1.8$, and $\tau=2$, where the IC are $\omega(0)=1,\ \omega'(0)=0$. According to Fig. 3, it is clear that the proposed method can be suitable method for solving these equations.

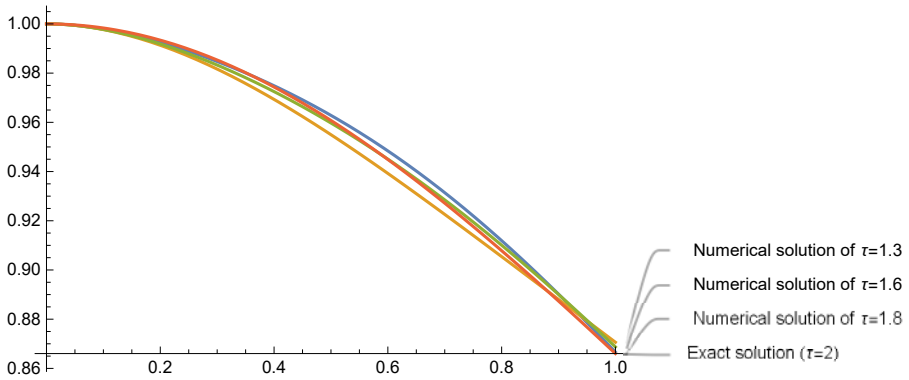


Figure 3: Numerical solutions for different values of τ for Example 4.5 with $N=15$.

5. Conclusions

In this paper, we have presented numerical methods for solving nonlinear singular differential equations of Lane-Emden type. Using the modified generalized Laguerre polynomials and the operational matrix of derivative of these polynomials, we converted these equations to a nonlinear system of equations that can be solved by the known methods. The numerical results show that the proposed method can be suitable method for solving these equations.

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Existence and exponential stability of second-order neutral stochastic functional differential equations with infinite delay and Poisson jumps

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Abstract. In this work, we study the existence and uniqueness of mild solutions to second-order neutral stochastic functional differential equations (NSFDEs) with infinite delay and Poisson jumps under global and local Carathéodory conditions by means of the successive approximation. The p-th moment exponential stability of mild solution to second-order NSFDEs with infinite delay and poisson jumps is also studied. Further, example is given to illustrate the proposed theory.

Keywords: Existence, stability, stochastic differential equations, infinite delay, Poisson jumps.

1. Introduction

In recent days, The study of stochastic differential equations (SDEs) has attracted the researchers because of its applicability to diverse fields [1, 30, 8, 11, 17, 21].

Stochastic systems depend on the present state and a period of past state as well, this system is said to be stochastic functional differential equation (SFDE). In many areas of science, there has been an increasing interest in the investigation of SFDEs incorporating memory or aftereffect i.e., there is the effect of infinite delay on state equations. The importance of SFDEs with infinite delay can be found in [4, 29, 14] and references there in. The development of the theory of functional differential equations with infinite delay depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space required a separate development of the theory [12]. The common phase space $\mathcal{B}$ is proposed by Hale and Kato in [9]. Kolmanovskii, in [14], introduced the NSFDE and its applications to chemical engineering and

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aero elasticity. Further, it has been drawn the attention of many researchers [3, 6, 10, 17, 18, 22, 23, 31] and references there in. In [5], the existence of the solutions of first order NSFDEs with delay and Poisson jumps are studied. It may be noted here that the mentioned works are confined to first-order systems.

To model the problems like mechanical vibrations, charge on a capacitor, condenser subjected to white noise excitation, second-order SDEs are more appropriate. With the advent of applications, second order SDEs, FSDEs, NSFDEs have attracted the focus of many researchers since last decade. The second-order damped functional stochastic evolution equations are studied by McKibben [19] and one can refer [16] for further works on this topic. Moreover, McKibben [20] established the existence and uniqueness of mild solutions for a class of second-order neutral stochastic evolution equations with finite delay. Balasubramaniam et al. [2] gave the sufficient conditions for the approximate controllability of the second-order neutral stochastic evolution equations with infinite delay. In [26], the authors established the asymptotic stability of second-order neutral stochastic differential equations using fixed point theorem.

The existence and uniqueness of solution of SFDEs with Poisson jumps are established in [15, 24]. Very recently, existence, uniqueness and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps are discussed in [25] and references there in. Further, Sakthivel et al. [27] presented the exponential stability of nonlinear second-order stochastic evolution equations with Poisson jumps by using a fixed point argument, Jiang et al.[13] discussed stability analysis for second-order stochastic neutral partial functional systems subject to infinite delays and impulses by the new integral inequality together with the stochastic analysis technique.

In this paper, inspired by the aforementioned works [13, 25, 23, 27], second-order NSFDEs with infinite delay and Poisson jumps is considered. To the best of our knowledge, there are no results on the existence of mild solutions of second-order NSFDEs with infinite delay and Poisson jumps under Carathéodory conditions in available literature. Motivated by the above attention, using Carathéodory conditions we aim to establish the existence and uniqueness of mild solutions to second-order NSFDEs with infinite delay and Poisson jumps in which the initial value belongs to the space $\mathcal{B}((-\infty, 0], H)$ (for more details refer section 2). Besides, the exponential stability in p-th moment of the considered NSFDEs with infinite delay and Poisson jumps is studied to obtain the required sufficient conditions.

The paper is structured as follows. In section 2, some preliminaries are presented. Existence and uniqueness of mild solutions are discussed in section 3. In section 4, the p-th moment exponential stability of mild solutions are presented. An example is provided in last section to illustrate the theory.

2. Preliminaries

In this section, some basic concepts that are useful for the development of our results are presented. For more details, the reader may refer to Da Prato and Zabczyk [6], Frattorini [7], Hale and Kato [9] and the references therein.

Let H and K be two real separable Hilbert spaces. $L(K, H)$ stands for the set of all bounded linear operators form K into H. We will use the notation $|.|$ and $\langle . \, , . \rangle$ to denote the norm and inner product for H and $L(K, H)$ respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space with an increasing right continuous family $\{\mathcal{F}_t\}_{t\geq 0}$ of complete sub $\sigma-$ algebras of $\mathcal{F}$.

Definition 2.1 ([9]). *$\mathcal{B}$ is a linear space of family of $\mathcal{F}_0$- measurable functions from $(-\infty, 0]$ into H endowed with a norm $\|.\|_{\mathcal{B}}$ which satisfies the following axioms:*

(A1) *If $x : (-\infty, T] \to H$ is continuous on $[0, T]$ and $x_0 \in \mathcal{B}$, then, for every $t \in [0, T]$, the following conditions hold.*

(1) $x_t \in \mathcal{B}$

(2) $|x(t)| \leq L\|x_t\|_{\mathcal{B}}$

(3) $\|x_t\|_{\mathcal{B}} \leq M(t) \sup_{0\leq s\leq t} |x(s)| + N(t)\|x_0\|_{\mathcal{B}}$

where $L > 0$ is a constant, $M, N : [0, \infty) \to [1, \infty)$ are continuous, $N(t)$ is locally bounded and L, M, N are independent of $x(.)$.

(A2) *The space $\mathcal{B}$ is complete.*

Remark 2.1. For convenience, the condition (3) in (A1) can be replaced by the following condition

$$\|x_t\|_{\mathcal{B}} \leq \sup_{0\leq s\leq t} |x(s)| + N\|x_0\|_{\mathcal{B}},$$

where $N = \sup_{0\leq s\leq t} |N(s)|$.

Let $Q \in L(K, H)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ (n = 1, 2, ...) are some nonnegative real numbers and $\{e_n\}$ (n = 1, 2, ...) is a complete orthonormal system $\{e_n\}$ in K. Then the above K valued stochastic process $w(t)$ is called a Q-wiener process. Let $\beta_n(t)$ $(n = 1, 2, ...)$ be a sequence of real valued one dimensional Brownian motion. Set

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t), \quad e \in K.$$

Let $\sigma \in L(K, H)$ and define

$$\|\sigma\|^2_{L_2^0} = Tr(\sigma Q \sigma^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\sigma e_n|^2.$$

If $|\sigma|^2_{L_2^0} < \infty$, then $\|\sigma\|$ is called a Q-Hilbert-Schmidt operator and $L_2^0(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators $\sigma : K \to H$. Let $p = p(t), t \geq 0$ be a stationary $\mathcal{F}_t-$ Poisson process with characteristic measure λ. $N(dt,du)$ denotes the Poisson counting measure associated with λ, i.e., $N(t,Z) = \sum_{t_1<s<t_2} I_Z(p(s))$ with a measurable set $Z \in \mathcal{B}(K-\{0\})$, which denotes the Borel $\sigma-$field of $K-\{0\}$. $\tilde{N}(dt,du) = N(dt,du) - dt\lambda(du)$ represents the compensated Poisson measure that is independent of $w(t)$.

The one parameter family $\{C(t) : t \in \mathbb{R}\} \subset L(H,H)$ satisfying

(i) $C(0) = I$,

(ii) $C(t)x$ is continuous in t on $\mathbb{R}$, for all $x \in H$,

(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$

is called a strongly continuous cosine family.

The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\} \subset L(H,H)$ is defined by $S(t)x = \int_0^t C(s)xds, t \in \mathbb{R}, x \in H$.

The generator $A : H \to H$ of $\{C(t) : t \in \mathbb{R}\}$ is given by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$ for all $x \in D(A) = \{x \in H : C(.)x \in C^2(\mathbb{R};H)\}$. It is well known that the infinitesimal generator A is closed, densely defined operator H. Such cosine and the corresponding sine families and their generators satisfy the following properties.

Lemma 2.1 ([7]). *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$. Then the following properties hold:*

(i) there exists $G^* \geq 1$ and $a \geq 0$ such that $\|C(t)\| \leq G^* e^{at}$ and therefore $\|S(t)\| \leq G^* e^{at}$

(ii) $A\int_s^u S(r)xdr = [C(u) - C(r)]x$ for all $0 \leq s \leq u < \infty$

(iii) there exists $G_1^* \geq 1$ such that $\|S(s) - S(u)\| \leq G_1^* \int_u^s e^{a|\theta|}d\theta$ for all $0 \leq u \leq s < \infty$.

Definition 2.2. *Denote by $\mathcal{M}^2((-\infty,T],H)$ the space of all H-valued continuous $\mathcal{F}_t$- adapted process $x = \{x(t)\}_{-\infty<t\leq T}$ such that*

(i) $x_0 = \phi \in \mathcal{B}$ and $x(t)$ is càdlàg on $[0,T]$;

(ii) define the norm $\|.\|_{\mathcal{M}}$ in $\mathcal{M}^2((-\infty,T],H)$ by

$$\|x\|^2_{\mathcal{M}} = E\|x_0\|^2_{\mathcal{B}} + E\int_0^T |x(t)|^2dt < \infty. \tag{1}$$

Then, $\mathcal{M}^2((-\infty,T],H)$ with the norm (1) is a Banach space. In the sequel, in case without confusion, we just use $\|.\|$ for the norm.

Lemma 2.2 ([6]). *For arbitrary $L_2^0(K,H)$- valued predictable process g such that*

$$\sup_{s\in[0,T]} E\left\|\int_0^s g(r)dw(r)\right\|^{2m} \leq (m(m-1))^m \left(\int_0^t (E\|g(s)\|_{L_2^0}^{2m})^{1/m}ds\right)^m, t\in[0,\infty).$$

Lemma 2.3 ([15]). *Let the space $M_\lambda^\nu([0,T]\times Z\times\Omega,H)$ $(\nu\geq 2)$ be the set of all random process $\rho(t,u)$ with values in H, predictable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that*

$$E\left(\int_0^t\int_Z \|\rho(s,u)\|^\nu \lambda(du)ds\right) < +\infty.$$

Suppose $\rho\in M_\lambda^2([0,t]\times Z\times\Omega,H)\cap M_\lambda^4([0,T]\times Z\times\Omega,H)$, then for any $t\in[0,T]$,

$$E\left[\sup_{s\in[0,t]}\left\|\int_0^s\int_Z S(s-s_1)\rho(s_1,u)\tilde{N}(ds_1,du)\right\|^2\right]$$
$$\leq C\left\{E\left(\int_0^t\int_Z\|\rho(s,u)\|^2\lambda(du)ds\right)+E\left(\int_0^t\int_Z\|\rho(s,u)\|^4\lambda(du)ds\right)^{\frac{1}{2}}\right\},$$

for some number $C>0$, dependent on $T>0$.

In this paper, we consider the second-order neutral stochastic functional differential equation with infinite delay and Poisson jumps of the form, for $t\in[0,T]$,

$$\begin{aligned} d[x'(t)-f_1(t,x_t)] &= [Ax(t)+f_2(t,x_t)]dt+f_3(t,x_t)dw(t)\\ &\quad+\int_Z f_4(t,x_t,u)\tilde{N}(dt,du)\\ x_0 &= \phi\in\mathcal{B},\\ x'(0) &= \xi \end{aligned} \tag{2}$$

where $A:D(A)\subset H\to H$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on H; $f_i\in[0,T]\times\mathcal{B}\to H(i=1,2)$, $f_3:[0,T]\times\mathcal{B}\to L_2^0(K,H)$ and $f_4:[0,T]\times\mathcal{B}\times(Z-\{0\})\to H$ are some suitable measurable mappings. The history $x_t:(-\infty,0]\to H, x_t(\theta)=x(t+\theta)$, for $t\geq 0$, belongs to the phase space $\mathcal{B}$. The initial data $\phi=\{\phi(t):-\infty<t\leq 0\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}-$ valued stochastic process with finite second moment and ξ ia an $\mathcal{F}_0-$ measurable H-valued random variable independent of the wiener process with finite second moment.

Definition 2.3. *A stochastic process $x:(-\infty,T]\to H$ is called a mild solution of (2) if*

(i) *$x(t)$ is $\mathcal{F}_t$- adapted and $\{x_t:t\in[0,T]\}$ is $\mathcal{B}-$valued;*

(ii) *$\int_0^T|x(s)|^2ds<\infty$, P-a.s.;*

(*iii*) *for each* $t \in [0,T]$, $x(t)$ *satisfies the following integral equation*

$$
\begin{aligned}
x(t) &= C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^t C(t-s)f_1(s,x_s)ds \\
&+ \int_0^t S(t-s)f_2(s,x_s)ds + \int_0^t S(t-s)f_3(s,x_s)dw(s) \\
&+ \int_0^t S(t-s)\int_Z f_4(s,x_s,u)\tilde{N}(ds,du)
\end{aligned} \tag{3}
$$

where $x_0(.) = \phi \in \mathcal{B}$.

In order to obtain existence and uniqueness of mild solutions to (2), we need the following assumptions:

(H1) the cosine family of operators $\{C(t) : t \in [0,T]\}$ on H and the corresponding sine family $\{S(t) : t \in [0,T]\}$ satisfy the conditions $\|C(t)\|^2 \leq K_1$ and $\|S(t)\|^2 \leq K_1, t \geq 0$ for a positive constant K_1;

(H2) f_i $(i = 1,2,3,4)$ satisfy the following conditions

(2a) there exists a function $\Gamma(t,v) : [0,T] \times [0,\infty) \to [0,\infty)$ such that

$$
\begin{aligned}
&E\int_0^t |f_i(s,\psi)|^2 ds + E\int_0^t |f_3(s,\psi)|^2 ds + E\int_0^t\int_Z |f_4(s,\psi,u)|^2 \lambda(du)ds \\
&\qquad + E\left(\int_0^t\int_Z |f_4(s,\psi,u)|^4 \lambda(du)ds\right)^{\frac{1}{2}} \\
&\leq \int_0^t \Gamma(s, E\|\psi\|_{\mathcal{B}}^2)ds
\end{aligned}
$$

for all $\psi \in \mathcal{B}$ and $t \in [0,T]$ $\quad (i = 1,2)$,

(2b) $\Gamma(t,v)$ is locally integrable in t for each fixed $v \in [0,\infty)$ and is continuous concave, and monotone nondecreasing in v for each fixed $t \in [0,T]$,

(2c) for any constant $M > 0$, the deterministic ordinary differential equation

$$\frac{dv}{dt} = M\Gamma(t,v), \quad 0 \leq t \leq T$$

has a global solution for any initial value v_0;

(H3) (3a) there exists a function $\Upsilon(t,v):[0,T]\times[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}E\int_0^t|f_i(s,\psi)-f_i(s,\varphi)|^2ds+E\int_0^t|f_3(s,\psi)-f_3(s,\varphi)|^2ds\\+E\int_0^t\int_Z|f_4(s,\psi,u)-f_4(s,\varphi,u)|^2\lambda(du)ds\\+E\left(\int_0^t\int_Z|f_4(s,\psi,u)-f_4(s,\varphi,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}\\\leq\int_0^t\Upsilon(s,E\|\psi-\varphi\|_{\mathcal{B}}^2)ds\end{aligned}$$

for all $\psi,\varphi\in\mathcal{B}$ and $t\in[0,T]\quad(i=1,2)$

(3b) $\Upsilon(t,v)$ is locally integrable in t for each fixed $v\in[0,\infty)$ and is continuous, nondecreasing and concave in v for each fixed $t\geq 0$. Moreover, $\Upsilon(t,0)=0$ and if a nonnegative continuous function $Y(t),0\leq t\leq T$ satisfies

$$Y(t)\leq D\int_0^t\Upsilon(s,Y(s))ds,\quad 0\leq t\leq T,$$

where $D>0$ is a positive constant, then $Y(t)\equiv 0$ for all $0\leq t\leq T$;

(H4) (the local condition)

(4a) for any integer $N>0$, there exists a function $\Upsilon_N(t,v):[0,T]\times[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}E\int_0^t|f_i(s,\psi)-f_i(s,\varphi)|^2ds+E\int_0^t|f_3(s,\psi)-f_3(s,\varphi)|^2ds\\+E\int_0^t\int_Z|f_4(s,\psi,u)-f_4(s,\varphi,u)|^2\lambda(du)ds\\+E\left(\int_0^t\int_Z|f_4(s,\psi,u)-f_4(s,\varphi,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}\\\leq\int_0^t\Upsilon_N(s,E\|\psi-\varphi\|_{\mathcal{B}}^2)ds\end{aligned}$$

for all $\psi,\varphi\in\mathcal{B}$ with $\|\psi\|_{\mathcal{B}},\|\varphi\|_{\mathcal{B}}\leq N$ and $t\in[0,T]\quad(i=1,2)$,

(4b) $\Upsilon_N(t,v)$ is locally integrable in t for each fixed $v\in[0,\infty)$ and is continuous, nondecreasing, and concave in v for each fixed $t\geq 0$. Moreover, $\Upsilon_N(t,0)=0$ and if a nonnegative continuous function $Y(t),0\leq t\leq T$ satisfies

$$Y(t)\leq D\int_0^t\Upsilon_N(s,Y(s))ds,\quad 0\leq t\leq T,$$

where $D>0$ is a positive constant, then $Y(t)\equiv 0$ for all $0\leq t\leq T$.

3. Existence and uniqueness

In this section, we prove the existence and uniqueness of mild solution of (2).

Let $x^0(t) = C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)], \quad t \in [0,T]$.

For each $n \geq 1$, the sequence of successive approximation is defined as follows

$$
\begin{aligned}
x^n(t) &= C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^t C(t-s) f_1(s, x_s^{n-1}) ds \\
&\quad + \int_0^t S(t-s) f_2(s, x_s^{n-1}) ds + \int_0^t S(t-s) f_3(s, x_s^{n-1}) dw(s) \\
&\quad + \int_0^t S(t-s) \int_Z f_4(s, x_s^{n-1}, u) \tilde{N}(ds, du), \quad t \in [0,T], \\
x^n(t) &= \phi(t), \quad -\infty < t \leq 0.
\end{aligned} \tag{4}
$$

Theorem 3.1. *If the assumptions (H1) - (H3) hold. Then, there exists a unique mild solution of* (2) *in space* $\mathcal{M}^2((-\infty, T], H)$.

Proof. The proof of this theorem is divided into the following three steps.

Step 1. Boundedness of $\{x^n(t); n \geq 0\}$ **in the space** $\mathcal{M}^2((-\infty, T], H)$**.**

$$
i.e., \quad E(\sup_{0 \leq s \leq t} |x^n(s)|^2) \leq u_t \leq u_T < \infty.
$$

It is obvious that $x^0(t) \in \mathcal{M}^2((-\infty, T], H)$ and now we prove that $x^n(t) \in \mathcal{M}^2((-\infty, T], H)$. From (4), using the Hölder inequality and the Doobs martingale inequality, we have

$$
\begin{aligned}
E\left(\sup_{0 \leq s \leq t} |x^n(s)|^2\right) &\leq 6K_1 E|\phi(0)|^2 + 12K_1 E|\xi|^2 + 12K_1 E|f_1(0,\phi)|^2 \\
&\quad + 6K_1 TE \int_0^t |f_1(s, x_s^{n-1})|^2 ds + 6K_1 TE \int_0^t |f_2(s, x_s^{n-1})|^2 ds \\
&\quad + 6K_1 E \int_0^t |f_3(s, x_s^{n-1})|^2 ds \\
&\quad + 6CE \int_0^t \int_Z |f_4(s, x_s^{n-1}, u)|^2 \lambda(du) ds \\
&\quad + 6CE \left(\int_0^t \int_Z |f_4(s, x_s^{n-1}, u)|^4 \lambda(du) ds\right)^{\frac{1}{2}} \\
&\leq K_2 + K_3 \int_0^t \Gamma(s, E\|x_s^{n-1}\|_{\mathcal{B}}^2) ds
\end{aligned}
$$

where $K_2 = 6K_1 E\|\phi\|_{\mathcal{B}}^2 + 12K_1 E|\xi|^2 + 12K_1 \Gamma(0, E\|\phi\|_{\mathcal{B}}^2), K_3 = 6(K_1(2T+1) + 2C)$. By Remark 2.1, we have

$$
E(\sup_{0 \leq s \leq t} |x^n(s)|^2) \leq K_2 + K_3 \int_0^t \Gamma(s, 2E(N^2\|\phi\|_{\mathcal{B}}^2 + \sup_{0 \leq r \leq s} |x^{n-1}(r)|^2)) ds \tag{5}
$$

For any $p \geq 1$, from (5), we get

$$\begin{aligned}
&\max_{1\leq n\leq p} E\left(\sup_{0\leq s\leq t}|x^n(s)|^2\right)\\
&\leq K_2 + K_3\int_0^t \Gamma\left(s, 2E\left(N^2\|\phi\|_{\mathcal{B}}^2 + |x^0(s)|^2 + \max_{1\leq n\leq p}\left(\sup_{0\leq r\leq s}|x^n(r)|\right)^2\right)\right)ds\\
&\leq K_2 + K_3\int_0^t \Gamma\left(s, \left(\frac{2}{3}K_2 + 2N^2E\|\phi\|_{\mathcal{B}}^2 + 2E\left(\max_{1\leq n\leq p}\left(\sup_{0\leq r\leq s}|x^n(r)|\right)^2\right)\right)\right)ds
\end{aligned}$$

or

$$\begin{aligned}
&\max_{1\leq n\leq p} E\left(\frac{2}{3}K_2 + 2N^2E\|\phi\|_{\mathcal{B}}^2 + 2\sup_{0\leq s\leq t}|x^n(s)|^2\right)\\
&\qquad\leq \frac{5}{3}K_2 + 2N^2E\|\phi\|_{\mathcal{B}}^2 + K_3\int_0^t \Gamma\Big(s, \Big(\frac{2}{3}K_2 + 2N^2E\|\phi\|_{\mathcal{B}}^2\\
&\qquad\quad + 2E\Big(\max_{1\leq n\leq p}\Big(\sup_{0\leq r\leq s}|x^n(r)|\Big)^2\Big)\Big)\Big)ds.
\end{aligned}$$

Using assumption (2c), it can be observed that u_t satisfies

$$u_t = \frac{5}{3}K_2 + 2N^2E\|\phi\|_{\mathcal{B}}^2 + K_3\int_0^t \Gamma(s, u_s)ds.$$

Since $E\|\phi\|_{\mathcal{B}}^2 < \infty$, we deduce that

$$\max_{1\leq n\leq p} E(\sup_{0\leq s\leq t}|x^n(s)|^2) \leq u_t \leq u_T < \infty. \tag{6}$$

Since p is arbitrary, we have

$$E|x^n(t)|^2 \leq u_T \quad \text{for all} \quad 0 \leq t \leq T, n \geq 1. \tag{7}$$

From Definition (2.4) and the above result, we obtain

$$\begin{aligned}
\|x^n\|^2 &= E\|x_0^n\|_{\mathcal{B}}^2 + E\int_0^T |x^n(t)|^2dt\\
&\leq E\|\phi\|_{\mathcal{B}}^2 + Tu_T < \infty,
\end{aligned}$$

which shows that the sequence $\{x^n(t), n \geq 0\}$ is bounded in $\mathcal{M}^2((-\infty, T], H)$.

Step 2. The sequence $\{x^n(t), n \geq 1\}$ is Cauchy. For $m, n \geq 0$ and $t \in [0, T]$, from (4), we get

$$E(\sup_{0\leq s\leq t}|x^{n+1}(s) - x^{m+1}(s)|^2)$$

$$
\begin{aligned}
&\leq 4K_1TE\int_0^t |f_1(s,x_s^n)-f_1(s,x_s^m)|^2ds\\
&\quad + 4K_1TE\int_0^t |f_2(s,x_s^n)-f_2(s,x_s^m)|^2ds\\
&\quad + 4K_1E\int_0^t |f_3(s,x_s^n)-f_3(s,x_s^m)|^2ds\\
&\quad + 4CE\int_0^t\int_Z |f_4(s,x_s^n)-f_4(s,x_s^m,u)|^2\lambda(du)ds\\
&\quad + 4CE\left(\int_0^t\int_Z |f_4(s,x_s^n)-f_4(s,x_s^m,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}\\
&\leq K_4\int_0^t \Upsilon(s,E\|x_s^n-x_s^m\|_{\mathcal{B}})ds, \quad \text{where} \quad K_4=4(K_1(2T+1)+2C)\\
&\leq K_4\int_0^t \Upsilon(s,E(\sup_{0\leq r\leq s}|x^n(r)-x^m(r)|^2))ds.
\end{aligned}
$$

Let $Y(t)=\lim_{n,m\to\infty}E(\sup_{0\leq s\leq t}|x^n(s)-x^m(s)|^2)$. From (6), (2b) and the Fatou's lemma, we obtain

$$Y(t)\leq K_4\int_0^t \Upsilon(s,Y(s))ds$$

by (3b), we get $Y(t)=0$, hence $\{x^n(t),n\geq 1\}$ is a Cauchy sequence in $\mathcal{M}^2$. As $n\to\infty$, using Borel-Cantelli lemma, $x^n(t)\to x(t)$ uniformly for $0\leq t\leq T$ and hence (4) tends to the solution $x(t)$ of (2), for all $-\infty<t\leq T$.

Step 3. Uniqueness of solutions of (2)**.** Suppose $x_1(t)$ and $x_2(t)$ are two solutions of (2). From step 1, we can see that $x_1(t),x_2(t)\in\mathcal{M}^2((-\infty,T],H)$. From step 2, it can be shown that

$$E(\sup_{0\leq s\leq t}|x_1(s)-x_2(s)|^2)\leq K_4\int_0^t \Upsilon(s,E(\sup_{0\leq r\leq s}|x_1(r)-x_2(r)|))ds.$$

or

$$Y(t)\leq K_4\int_0^t \Upsilon(s,Y(s))ds.$$

By assumption (3b), we obtain $Y(t)\equiv 0$, which implies that $x_1(t)=x_2(t)$ a.s. for any $t_0\leq t\leq T$. Therefore, for all $-\infty<t\leq T$, $x_1(t)=x_2(t)$ a.s. This completes the proof of the uniqueness of a solution. □

Next, we present the existence and uniqueness of mild solution of (2) with the local Carathéodory conditions.

Theorem 3.2. *Let* $(H1),(H2)$ *and* $(H4)$ *hold. Then there exists a unique mild solution of* (2) *in space* $\mathcal{M}^2((-\infty,T],H)$.

Proof. Let N be a positive integer and $T_0 \in (0,T)$. Let the sequence of the functions $f_i^N(t,v)(i=1,2,3,4)$ for $(t,v) \in [0,T_0] \times \mathcal{B}$ be defined as follows

$$f_i^N(t,v) = \begin{cases} f_i(t,v), & \text{if} \quad \|v\|_{\mathcal{B}} \leq N, \\ f_i\left(t, \frac{Nv}{\|v\|_{\mathcal{B}}}\right), & \text{if} \quad \|v\|_{\mathcal{B}} > N, \end{cases} (i=1,2,3)$$

$$f_4^N(t,v,u) = \begin{cases} f_4(t,v,u), & \text{if} \quad \|v\|_{\mathcal{B}} \leq N, \\ f_4\left(t, \frac{Nv}{\|v\|_{\mathcal{B}}}, u\right), & \text{if} \quad \|v\|_{\mathcal{B}} > N. \end{cases}$$

Then the function $f_i^N(t,v)$ $(i=1,2,3,4)$ satisfy the assumptions (H2) and (H4). As a consequence of Theorem 3.1, there exists a unique mild solution $x^N(t)$ and $x^{N+1}(t)$ to the following integral equations respectively;

$$\begin{aligned} x^N(t) &= C(t)\phi(0) + S(t)[\xi - f_1^N(0,\phi)] + \int_0^t C(t-s) f_1^N(s, x_s^N) ds \\ &+ \int_0^t S(t-s) f_2^N(s, x_s^N) ds + \int_0^t S(t-s) f_3^N(s, x_s^N) dw(s) \\ &+ \int_0^t S(t-s) \int_Z f_4^N(s, x_s^N, u) \tilde{N}(ds, du) \end{aligned} \tag{8}$$

$$\begin{aligned} x^{N+1}(t) &= C(t)\phi(0) + S(t)[\xi - f_1^{N+1}(0,\phi)] + \int_0^t C(t-s) f_1^{N+1}(s, x_s^{N+1}) ds \\ &+ \int_0^t S(t-s) f_2^{N+1}(s, x_s^{N+1}) ds + \int_0^t S(t-s) f_3^{N+1}(s, x_s^{N+1}) dw(s) \\ &+ \int_0^t S(t-s) \int_Z f_4^{N+1}(s, x_s^{N+1}, u) \tilde{N}(ds, du). \end{aligned} \tag{9}$$

Define the stopping times

$$\begin{aligned} \delta_N &= T_0 \wedge \inf\{t \in [0,T] : \|x_t^N\|_{\mathcal{B}} \geq N\} \\ \delta_{N+1} &= T_0 \wedge \inf\{t \in [0,T] : \|x_t^{N+1}\|_{\mathcal{B}} \geq N\} \\ \tau_N &= \delta_N \wedge \delta_{N+1}. \end{aligned}$$

From (8) and (9), we have

$$\begin{aligned} E(\sup_{0 \leq s \leq t \wedge \tau_N} &|x^{N+1}(s) - x^N(s)|^2) \\ &\leq 5K_1 E|f_1^N(0,\phi) - f_1^{N+1}(0,\phi)|^2 \\ &+ 5K_1 TE \int_0^{t \wedge \tau_N} |f_1^{N+1}(s, x_s^{N+1}) - f_1^N(s, x_s^N)|^2 ds \\ &+ 5K_1 TE \int_0^{t \wedge \tau_N} |f_2^{N+1}(s, x_s^{N+1}) - f_2^N(s, x_s^N)|^2 ds \end{aligned}$$

$$+ 5K_1E\int_0^{t\wedge\tau_N} |f_3^{N+1}(s,x_s^{N+1}) - f_3^N(s,x_s^N)|^2 ds$$
$$+ 5CE\int_0^{t\wedge\tau_N}\int_Z |f_4^{N+1}(s,x_s^{N+1},u) - f_4^N(s,x_s^N,u)|^2\lambda(du)ds$$
$$+ 5CE\left(\int_0^{t\wedge\tau_N}\int_Z |f_4^{N+1}(s,x_s^{N+1},u) - f_4^N(s,x_s^N,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}.$$

Clearly $f_1^N(0,\phi) = f_1^{N+1}(0,\phi)$ and since for $0 \le s \le \tau_N$

$$f_i^{N+1}(s,x_s^N) = f_i^N(s,x_s^N), \quad i = 1,2,3,$$
$$f_4^{N+1}(s,x_s^N,u) = f_4^N(s,x_s^N,u).$$

Note that $\|x_s^{N+1} - x_s^N\|_{\mathcal{B}}^2 \le \sup_{0\le v\le s}|x^{N+1}(v) - x^N(v)|^2$ and hence we obtain that

$$E(\sup_{0\le s\le t\wedge\tau_N}|x^{N+1}(s) - x^N(s)|^2)$$
$$\le 5K_1TE\int_0^{t\wedge\tau_N} |f_1^{N+1}(s,x_s^{N+1}) - f_1^{N+1}(s,x_s^N)|^2 ds$$
$$+ 5K_1TE\int_0^{t\wedge\tau_N} |f_2^{N+1}(s,x_s^{N+1}) - f_2^{N+1}(s,x_s^N)|^2 ds$$
$$+ 5K_1E\int_0^{t\wedge\tau_N} |f_3^{N+1}(s,x_s^{N+1}) - f_3^{N+1}(s,x_s^N)|^2 ds$$
$$+ 5CE\int_0^{t\wedge\tau_N}\int_Z |f_4^{N+1}(s,x_s^{N+1},u) - f_4^{N+1}(s,x_s^N,u)|^2\lambda(du)ds$$
$$+ 5CE\left(\int_0^{t\wedge\tau_N}\int_Z |f_4^{N+1}(s,x_s^{N+1},u) - f_4^{N+1}(s,x_s^N,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}.$$

Therefore, for all $0 \le t \le T_0$, employing assumption (H4), it follows that

$$E(\sup_{0\le s\le t}|x^{N+1}(s\wedge\tau_N) - x^N(s\wedge\tau_N)|^2)$$
$$\le 5(K_1(2T+1)+C)\int_0^t \Upsilon_{N+1}\left(s\wedge\tau_N, E\|x_{s\wedge\tau_N}^{N+1} - x_{s\wedge\tau_N}^N\|_{\mathcal{B}}^2\right) ds$$
$$\le 5(K_1(2T+1)+C)\int_0^t \Upsilon_{N+1}\Bigg(s\wedge\tau_N, E\Bigg(\sup_{0\le v\le s}|x^{N+1}(v\wedge\tau_N)$$
$$- x^N(v\wedge\tau_N)|^2\Bigg)\Bigg)ds.$$

By (4b), we get

$$E(\sup_{0\le s\le t}|x^{N+1}(s\wedge\tau_N) - x^N(s\wedge\tau_N)|^2) = 0.$$

Thus, for a.e., $\omega \in \Omega$, $x^{N+1}(t) = x^N(t)$ for $0 \le t \le T_0 \wedge \tau_N$. Note that for each $\omega \in \Omega$, there exists an $N_0(\omega) > 0$ such that $0 \le T_0 \le \tau_{N_0}$. Define $x(t)$ by

$$x(t) = x^{N_0}(t), \quad \text{for } t \in [0, T_0].$$

Since $x(t \wedge \tau_N) = x^N(t \wedge \tau_N)$, it holds that

$$\begin{aligned}
x(t \wedge \tau_N) &= C(t)\phi(0) + S(t)[\xi - f_1^N(0,\phi)] + \int_0^{t\wedge\tau_N} C(t-s) f_1^N(s, x_s^N) ds \\
&\quad + \int_0^{t\wedge\tau_N} S(t-s) f_2^N(s, x_s^N) ds + \int_0^{t\wedge\tau_N} S(t-s) f_3^N(s, x_s^N) dw(s) \\
&\quad + \int_0^{t\wedge\tau_N} S(t-s) \int_Z f_4^N(s, x_s^N, u) \tilde{N}(ds, du) \\
&= C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^{t\wedge\tau_N} C(t-s) f_1(s, x_s) ds \\
&\quad + \int_0^{t\wedge\tau_N} S(t-s) f_2(s, x_s) ds + \int_0^{t\wedge\tau_N} S(t-s) f_3(s, x_s) dw(s) \\
&\quad + \int_0^{t\wedge\tau_N} S(t-s) \int_Z f_4(s, x_s, u) \tilde{N}(ds, du).
\end{aligned}$$

Taking $N \to \infty$, for all $t \in [0, T]$, we have

$$\begin{aligned}
x(t) &= C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^t C(t-s) f_1(s, x_s) \\
&\quad + \int_0^t S(t-s) f_2(s, x_s) ds + \int_0^t S(t-s) f_3(s, x_s) dw(s). \\
&\quad + \int_0^t S(t-s) \int_Z f_4(s, x_s, u) \tilde{N}(ds, du).
\end{aligned}$$

□

4. Exponential stability

Definition 4.1. *The solution of integral equation* (3) *is said to be exponentially stable in p ($p \ge 2$) moment, if there exists a pair of positive constants $\mu > 0$ and $M_1 > 0$ such that*

$$E\|x(t)\|^p \le M_1 e^{-\mu t}, \quad t \ge 0, \quad p \ge 2.$$

In this section, we need the following assumptions to establish the exponential stability of (2).

(A1) The cosine family of operators $\{C(t) : t \le 0\}$ on H and the corresponding sine family $\{S(t) : t \le 0\}$ satisfy the conditions $\|C(t)\| \le Me^{-\beta t}$ and $\|S(t)\| \le Me^{-\alpha t}$, $t \ge 0$ for some constants $M \ge 1, \alpha > 0$ and $\beta > 0$.

(A2) There exist constants $\lambda_i > 0$ $(i = 1, 2, 3, 4)$ and a function $z : (-\infty, \infty) \to [0, \infty)$ with $\int_{-\infty}^{0} z(t)dt = 1$ and $\int_{-\infty}^{0} z(t)e^{-\rho t}dt < \infty$ $(\rho > 0)$ such that

$$
\begin{aligned}
\|f_i(t,x) - f_i(t,y)\| &\le \lambda_i \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta,\\
f_i(t,0) &= 0, i = 1, 2,\\
\|f_3(t,x) - f_3(t,y)\|_{L_2^0} &\le \lambda_3 \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta,\\
f_3(t,0) &= 0,\\
\int_Z \|f_4(t,x,u) - f_4(t,y,u)\|\lambda(du) &\le \lambda_4 \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta,\\
f_4(t,0,u) &= 0,\\
x, y \in \mathcal{B}, t &\ge 0.
\end{aligned} \tag{10}
$$

(A3) $6^{p-1}M^p\left[\beta^{-p}\lambda_1^p + \alpha^{-p}\lambda_2^p + \alpha^{-p/2}\lambda_3^p\left(\frac{2(p-1)}{p-2}\right)^{1-p/2}\left(\frac{p(p-1)}{4}\right)^{p/2} + \alpha^{-p}\lambda_4^p\right] < 1, \quad (p \ge 2).$

Lemma 4.1. *Let $L_1, L_2 \in (0, \rho]$ and assume that there exist some positive constants $C_i > 0$ (i=1,2,3,4) and a function $\hat{y} : (-\infty, \infty) \to [0, \infty)$ such that*

$$
\hat{y}(t) \le \begin{cases} C_1e^{-L_1t} + C_2e^{-L_2t} + C_3\int_0^t e^{-L_1(t-s)}\int_{-\infty}^{0} z(\theta)\hat{y}(s+\theta)d\theta ds \\ +C_4\int_0^t e^{-L_2(t-s)}\int_{-\infty}^{0} z(\theta)\hat{y}(s+\theta)d\theta ds, t \ge 0, \\ C_1e^{-L_1t} + C_2e^{-L_2t}, t \in (-\infty, 0], \end{cases} \tag{11}
$$

holds. If $\frac{C_3}{L_1} + \frac{C_4}{L_2} < 1$, then,

$$\hat{y}(t) \le M_2e^{-\mu t}, t \in (-\infty, \infty),$$

where $\mu \in (0, L_1 \wedge L_2)$ is a positive root of the algebra equation:

$$\left(\frac{C_3}{L_1 - \mu} + \frac{C_4}{L_2 - \mu}\right)\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta = 1$$

and

$$M_2 = \max\left\{\frac{C_1(L_1 - \mu)}{C_3\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, \frac{C_2(L_2 - \mu)}{C_4\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, C_1 + C_2\right\} > 0.$$

Proof. Let $F(\lambda) = \left(\frac{C_3}{L_1-\lambda} + \frac{C_4}{L_2-\lambda}\right)\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta - 1$, then it is obvious that there exists a positive constant $\mu \in (0, L_1 \wedge L_2)$ such that $F(\mu) = 0$.
For any $\epsilon > 0$ and Let

$$N_\epsilon = \max\left\{\frac{(C_1+\epsilon)(L_1-\mu)}{C_3\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, \frac{(C_2+\epsilon)(L_2-\mu)}{C_4\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, C_1+C_2\right\} > 0. \tag{12}$$

Now , in order to show this Lemma, we only claim that (11) implies

$$\hat{y}(t) \le N_\epsilon e^{-\mu t}, \quad t \in (-\infty, \infty). \tag{13}$$

Obviously, for any $t \in (-\infty, 0], (13)$ holds. Now we will prove (13) by the contradiction method. Assume that there exists a $t_1 > 0$ such that

$$\hat{y} < N_\epsilon e^{-\mu t}, \quad t \in (-\infty, t_1), \quad \hat{y}(t_1) = N_\epsilon e^{-\mu t_1}. \tag{14}$$

However, from (11)

$$\begin{aligned}
\hat{y}(t_1) &\le C_1 e^{-L_1 t_1} + C_2 e^{-L_2 t_1} + C_3 N_\epsilon \int_0^{t_1} e^{-L_1(t_1-s)} \int_{-\infty}^{0} z(\theta)e^{-\mu(s+\theta)}d\theta ds \\
&\quad + C_4 N_\epsilon \int_0^{t_1} e^{-L_2(t_1-s)} \int_{-\infty}^{0} z(\theta)e^{-\mu(s+\theta)}d\theta ds \\
&\le \left(C_1 - \frac{N_\epsilon C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)e^{-L_1 t_1} \\
&\quad + \left(C_2 - \frac{N_\epsilon C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)e^{-L_2 t_1} \\
&\quad + \left(\frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta + \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)N_\epsilon e^{-\mu t_1}
\end{aligned} \tag{15}$$

Note that $\mu \in (0, L_1 \wedge L_2)$, From (12), we obtain

$$\frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta + \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta = 1$$

and

$$\begin{aligned}
&C_1 - \frac{N_\epsilon C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta \\
&\le C_1 - \frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta \frac{(C_1+\epsilon)(L_1-\mu)}{C_3\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta} < 0 \\
&C_2 - \frac{N_\epsilon C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta \\
&\le C_2 - \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta \frac{(C_2+\epsilon)(L_2-\mu)}{C_4\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta} < 0.
\end{aligned}$$

Thus, (15) yields

$$\hat{y}(t_1) < N_\epsilon e^{-\mu t}, \tag{16}$$

which contradicts (14), that is, (13) holds. As $\epsilon > 0$ is arbitrarily small, in view of (13), it follows $\hat{y}(t) \leq M_2 e^{-\mu t}, t \geq 0$. □

Theorem 4.1. *Assume that* (*A*1)-(*A*3) *hold and* $\alpha, \beta \in (0, \rho]$, *then the mild solution of* (2) *is exponentially stable in p-th moment for* $p \geq 2$.

Proof. From (3), we have

$$\begin{aligned} E\|x(t)\|^p &\leq 6^{p-1}M^p E\|\phi\|^p e^{-\beta t} + 6^{p-1}M^p E\|\xi - f_1(0,\phi)\|^p e^{-\alpha t} \\ &\quad + 6^{p-1}E\left\|\int_0^t C(t-s)f_1(s,x_s)ds\right\|^p \\ &\quad + 6^{p-1}E\left\|\int_0^t S(t-s)f_2(s,x_s)ds\right\|^p \\ &\quad + 6^{p-1}E\left\|\int_0^t S(t-s)f_3(s,x_s)dw(s)\right\|^p \\ &\quad + 6^{p-1}E\left\|\int_0^t S(t-s)\int_Z f_4(s,x_s,u)\tilde{N}(ds,du)\right\|^p \\ &= 6^{p-1}\sum_{i=1}^{6}\Phi_i. \end{aligned} \tag{17}$$

From (A1), (A2) and the Hölder inequality, we have

$$\begin{aligned} \Phi_3 &\leq M^p\left(\int_0^t \left(e^{-\beta(t-s)(1-\frac{1}{p})}\right)^{\frac{p}{p-1}}ds\right)^{p-1}\int_0^t e^{-\beta(t-s)}E\|f_1(s,x_s)\|^p ds \\ &= M^p\left(\int_0^t e^{-\beta(t-s)}ds\right)^{p-1}\int_0^t e^{-\beta(t-s)}E\|f_1(s,x_s)\|^p ds \\ &\leq M^p\lambda_1^p\beta^{1-p}\int_0^t e^{-\beta(t-s)}E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\|d\theta\right)^p ds. \end{aligned}$$

Similarly, we have

$$\Phi_4 \leq M^p\lambda_2^p\alpha^{1-p}\int_0^t e^{-\alpha(t-s)}E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\|d\theta\right)^p ds.$$

By using the conditions (A1), (A2) and Lemma 2.2, it follows that

$$\begin{aligned} \Phi_5 &\leq M^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\int_0^t\left(e^{-\alpha p(t-s)}E\|f_3(s,x_s)\|^p_{L_2^0}\right)^{2/p}ds\right)^{p/2} \\ &= M^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\int_0^t e^{-2\alpha(t-s)}\left(E\|f_3(s,x_s)\|^p_{L_2^0}\right)^{2/p}ds\right)^{p/2} \end{aligned}$$

$$
\begin{aligned}
&\le M^p \left(\frac{p(p-1)}{4}\right)^{p/2} \left(\int_0^t \left(e^{-2\alpha(t-s)(1-\frac{1}{p})}\right)^{p/(p-2)} ds\right)^{(p/2)-1} \\
&\quad \times \int_0^t \left(e^{-2\alpha(t-s)(1/p)} \left(E\|f_3(s,x_s)\|^p_{L_2^0}\right)^{2/p}\right)^{p/2} ds \\
&\le M^p \left(\frac{p(p-1)}{4}\right)^{p/2} \left(\int_0^t e^{-2\alpha(t-s)((p-1)/(p-2))} ds\right)^{(p/2)-1} \\
&\quad \times \int_0^t e^{-\alpha(t-s)} E\|f_3(s,x_s)\|^p_{L_2^0} ds \\
&\le M^p \lambda_3^p \left(\frac{p(p-1)}{4}\right)^{p/2} \left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)} \\
&\quad \times \int_0^t e^{-\alpha(t-s)} E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\| d\theta\right)^p ds.
\end{aligned}
$$

From (A1)and (A2), we obtain

$$
\begin{aligned}
\Phi_6 &\le M^p \left(\int_0^t e^{-\alpha(t-s)} ds\right)^{p-1} \int_0^t e^{-\alpha(t-s)} \int_Z E\|f_4(s,x_s,u)\|^p \lambda(du) ds \\
&\le M^p \alpha^{1-p} \lambda_4^p \int_0^t e^{-\alpha(t-s)} E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\| d\theta\right)^p ds.
\end{aligned}
$$

These together with (17) yields

$$
\begin{aligned}
E\|x(t)\|^p \le\ & 6^{p-1} M^p E\|\phi\|^p e^{-\beta t} + 6^{p-1} M^p E\|\xi - f_1(0,\phi)\|^p e^{-\alpha t} \\
&+ 6^{p-1} M^p \beta^{1-p} \lambda_1^p \int_0^t e^{-\beta(t-s)} \int_{-\infty}^0 z(\theta) E\|x(s+\theta)\|^p d\theta ds \\
&+ 6^{p-1} M^p \alpha^{1-p} \lambda_2^p \int_0^t e^{-\alpha(t-s)} \int_{-\infty}^0 z(\theta) E\|x(s+\theta)\|^p d\theta ds \\
&+ 6^{p-1} M^p \lambda_3^p \left(\frac{p(p-1)}{4}\right)^{p/2} \left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)} \\
&\quad \times \int_0^t e^{-\alpha(t-s)} \int_{-\infty}^0 z(\theta) E\|x(s+\theta)\|^p d\theta ds \\
&+ 6^{p-1} M^p \alpha^{1-p} \lambda_4^p \int_0^t e^{-\alpha(t-s)} \int_{-\infty}^0 z(\theta) E\|x(s+\theta)\|^p d\theta ds.
\end{aligned}
$$

It can be easily verified that there exists two positive numbers $M' > 0$ and $M'' > 0$ such that $E\|x(t)\|^2 \le M' e^{-\beta t} + M'' e^{-\alpha t}$, for any $t \in (-\infty, 0]$. Let $\tilde{C}_1 = 6^{p-1} M^p E\|\phi\|^p, \tilde{C}_2 = 6^{p-1} M^p E\|\xi - f_1(0,\phi)\|^p, \tilde{C}_3 = 6^{p-1} M^p \beta^{1-p} \lambda_1^p, \tilde{C}_4 = 6^{p-1} M^p \alpha^{1-p} \lambda_2^p + 6^{p-1} M^p \lambda_3^p \left(\frac{p(p-1)}{4}\right)^{p/2} \left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)} + 6^{p-1} M^p \alpha^{1-p} \lambda_4^p$, if $\frac{\tilde{C}_3}{\beta} + \frac{\tilde{C}_4}{\alpha} < 1$, i.e., (A3) holds, then by using Lemma 4.1, we can obtain

$$
E\|x(t)\|^2 \le M_1 e^{-\mu t}, \quad t \in [0, \infty) \ \ (\mu \in (0, L_1 \wedge L_2)),
$$

where

$$M_1 = \max\Bigg\{6^{p-1}M^p(E\|\phi\|^p + E\|\xi - f_1(0,\phi)\|^p), 6^{p-1}M^p\beta^{-p}\lambda_1^p,$$
$$6^{p-1}M^p\left(\alpha^{-p}\lambda_2^p + \alpha^{-p/2}\lambda_3^p\left(\frac{p(p-1)}{2}\right)^{p/2}\left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)}\right.$$
$$\left.+\,\alpha^{-p}\lambda_4^p\right)\Bigg\} > 0.$$

□

5. Example

In this section, we present an example to illustrate the results obtained in previous sections. Let $H = L^2[0,\pi]$ with the norm $\|.\|$. And let $e_n = \sqrt{\frac{2}{\pi}}\sin(ny)$ $(n = 1, 2, ...)$ denote the complete orthonormal basis in H.

Let $w(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n$, $(\lambda_n > 0)$, where $\{\beta_n(t)\}$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space. $\tilde{N}(ds, du)$ is a compensated Poisson random measure on $[1,\infty)$ with parameter $\lambda(du)dt$. The Wiener process $w(t)$ is independent of $\tilde{N}(du)dt$.

Define the operator $A : H \to H$ by $(Ax)(y) = \frac{\partial^2}{\partial y^2}x(y)$ with the domain $D(A) = \{x \in H : x(0) = x(\pi)\}$. Then

$$Ax = \sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n, x \in D(A),$$

where $\{e_n : n \in N\}$ is the orthonormal set of eigen vectors of A corresponding to the eigen values $-n^2$ for $n \in N$. The operator $C(t)$ defined by

$$C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, e_n\rangle e_n, \quad t \in \mathbb{R},$$

from a cosine function on H, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x, e_n\rangle e_n, \quad t \in \mathbb{R},$$

for all $x \in H$, with $\|C(t)\| \le e^{-\pi^2 t}$ and $\|S(t)\| \le e^{-\pi^2 t}, t \ge 0$.

Now, we consider the following second-order neutral stochastic partial differential equations with infinite delay and Poisson jumps

$$\partial\left[\frac{\partial x(t,y)}{\partial t} - p_1 x(t+\theta, y)\right]$$
$$= \left[\frac{\partial^2 x(t,y)}{\partial y^2} + p_2 x(t+\theta, y)\right]\partial t + p_3 x(t+\theta, y)dw(t) \tag{18}$$

$$+\int_Z p_4 x(t+\theta,y)u\tilde{N}(dt,du),$$
$$t\in[0,T], y\in[0,\pi], \theta\in(-\infty,0),$$

subject to the conditions

$$x(t,y)=\phi(t,y), \quad -\infty<t\le 0, 0<y<\pi,$$
$$x(t,0)=x(t,\pi)=0, \quad 0\le t\le T,$$
$$\frac{\partial x(0,y)}{\partial t}=\xi(y), \quad 0<y<\pi,$$

where $\xi\in L^2[0,\pi], \phi\in\mathcal{B}$ and $\pi>0$, $(i=1,2,3,4)$. Define

$$f_1(t,x_t)=p_1x(t+\theta,y),$$
$$f_2(t,x_t)=p_2x(t+\theta,y),$$
$$f_3(t,x_t)=p_3x(t+\theta,y),$$
$$f_4(t,x_t,u)=p_4x(t+\theta,y)u,$$

where $\theta\in(-\infty,0]$.

The system (18) can be rewritten in the form of (2). Assume that f_i $(i=1,2,3,4)$ satisfy the conditions of Theorem 3.1 and Theorem 3.2. Then, the system (18) has a unique mild solution.

It is easy to see all the conditions are satisfied with $M=1,\alpha=\beta=\pi$. By virtue of Theorem 4.1, the mild solution of (18) is p-th moment exponentially stable provided that, $p\ge 2$,

$$6^{p-1}\left[\pi^{-p}p_1^p+\pi^{-p}p_2^p+\pi^{-p}p_4^p+\pi^{-p/2}p_3^p\left(\frac{2(p-1)}{p-2}\right)^{1-(p/2)}\left(\frac{p(p-1)}{4}\right)^{p/2}\right]<1.$$

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Production inventory model with exponential demand rate and exponentially declining deterioration

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Abstract. In this paper, a production inventory model with an exponential demand rate and exponentially declining deterioration is considered. The production rate of the model is expected to be proportional to the demand rate. The optimal total inventory cost per cycle, the optimal length of the cycle and optimal production length are determined. The proposed model has eight parameters. Numerical example of the proposed model is presented. Finally, the sensitivity analysis of the developed model is demonstrated.

Keywords: production inventory model, exponential demand rate, exponentially declining deterioration, optimal production cycle, total inventory cost per cycle.

1. Introduction

Inventory refers to goods or materials that are held by an organization for forthcoming usage. It is the common thread that ties all the functions and departments of the organization together. An inventory system is the set of policies and controls that monitors levels of inventory, so as to minimise the total inventory cost per cycle and to guarantee a smooth operation of the organisation. One main reason that an organisation maintains inventory that it is rarely possible to forecast / predict sale levels, production times, demand and usage requirements exactly. An inventory model can be either deterministic, probabilistic or imprecise, according to the predictability of demand involved. The demand of an inventory model depends upon many factors like market conditions, availability

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of substitutes etc., and hence, it is not under the control of the decision maker. If the demand is more than the supply, the shortage may arise and hence, the stock-out cost is incorporated.

In modelling inventory problem, two factors of the problem have been of growing interest to researchers, one being the deterioration of items and the other being the variation in the demand rate. Usually in most of all developed inventory models, it is assumed that the products during storage have endless life time. This means that an item once stocked will always remain in perfect condition and will be ideal to satisfy the customer's future requirement. But this assumption is not true for all types of products. There are many products such as vegetables, food products, fruits and pharmaceutical products in which deterioration occurs. After a certain period, the entire unsold lot will deteriorate completely and hence, they cannot be sold to customers. So, deterioration of physical goods is one of the important factors in the vendor-buyer system. Perishable inventory theory was developed in which products are to be deteriorated. Whitin [25] initiated and studied inventory of deteriorating fashion goods at the end of prescribed storage period. Ghare and Schrader [6] developed and analysed an inventory model for exponentially decaying perishable items. After then, many researchers have developed perishable inventory models. In such perishable models, the rate of deterioration is an important feature of consideration and these models were studied using exponential distribution and weibull distribution in general.

The economic order quantity (EOQ) model to assist organisations in minimizing total inventory costs was originated by Harris [9]. It balances inventory holding and setup costs and derives the optimal order quantity. The EOQ model is still applied in industry because of its simplicity. Covert and Philip [5] presented an inventory model for deteriorating items, in which the items deteriorate at the rate of weibull distribution function. Goswami and Chaudhuri [7] developed an inventory model for deteriorating items with finite production rate proportional and the time dependent demand rate. Two inventory models in which deterioration rate and demand rate are taken as a linear function of time were analysed by Bhunia and Maiti [2]. Liang-Yuh Ouyang et al. [12] proposed an EOQ inventory model for deteriorating items with exponential decreasing demand and partial backlogging. A deterministic inventory model for quadratic demand rate, quadratic production and weibull deterioration rate with shortages was developed by Kalam et al. [11]. Gour Chandra Mahata [8] constructed an order level inventory model for deteriorating items with instantaneous replenishment, exponential decay rate and a time varying linear demand without shortages under permissible delay in payments. Jhuma Bhowmick and Samanta [10] studied a continuous production inventory model for deteriorating items with shortages in which two different rates of production are considered.

An EOQ inventory mathematical model for deteriorating item having time dependent demand when delay in payment is permissible was constructed by Singh and Pattanayak [23]. Chandrasekhara and Ranganathan [4] discussed

the EOQ for an infinite horizon model when the demand increases exponentially under two levels of storage. An inventory model for deteriorating items with linear demand rate and traditional parameter of holding cost is linearly increasing functions of time was developed by Sachin et al. [18]. Shital and Raman [21] presented an EOQ model with demand rate is linear function of time, deterioration is two parameter weibull distribution and inventory holding cost is linear function of time under inflationary conditions with permissible delay in payments. An inventory model for deteriorating items with exponential time dependent demand rate was studied by Shukla et al. [22]. Palanivel and Uthayakumar [16] presented the economic production lot size model for determining the optimal production length and the optimal total cost for deteriorating items. Niketa et al. [15] analysed a deterministic inventory model with time dependent quadratic demand and time varying holding cost where units have expiry date.

An order level inventory model for deteriorating items with general ramp-type demand rate, partial backlogging of unsatisfied demand and time varying holding cost has been studied by Biplab and Karabi [3]. Ajay and Kuldeep [1] presented a continuous production control inventory model for deteriorating items under permissible delay in payments with variable demand rate and small deterioration rate. A production inventory model for exponential dependent demand and cost reduction delivery policy was proposed by Neeraj [14]. Sharmila and Uthayakumar [19] analysed the fuzzy inventory model for deteriorating items for power demand under fully backlogged conditions. Mishra et al. [13] investigated the inventory system for perishable items with time proportional deterioration rate, time dependent demand rate and linear holding cost with shortages. A deterministic inventory model for deteriorating items assuming exponentially declining demand was developed by Preeti and Malhotra [17]. Venkateswarlu and Reddy [24] proposed a production inventory model for deteriorating items which follow weibull distribution, variable holding cost and time dependent quadratic demand rate. A production inventory model for deteriorating items with price and inventory level dependent demand with different deterioration rates have been developed by Shital [20].

The rest of this article is arranged as follows: In Section 2., the assumptions and notations which are used throughout this article, are described. A production inventory model with exponential demand rate and exponentially declining deterioration items is considered in the Section 3 and the total inventory cost per cycle and the production cycle are optimized. Section 4 presents numerical example and sensitivity analysis. This is followed by conclusion in Section 5.

In the proposed model, we assume that the demand rate is an exponential function and the declining deterioration items follow exponential distribution. Because of the exponential nature, the demand is increasing smoothly which helps us to reduce the holding cost. The deterioration of the items is declining gradually because of exponential distribution which supports us to reduce the setup cost. The proposed model has an application in all organizations having

non-zero demand at the starting time of the production, smoothly increasing demand and gradually decreasing deterioration.

2. Assumptions and notations

In the proposed inventory model, the following assumptions and notations are being made.

2.1 Assumptions

A single item is considered over an infinite planning horizon.

The demand rate follows an exponential distribution and is given as

$$D(t) = ae^{bt},\ a > 0, 0 < b < 1$$

Production rate is proportional to the demand rate and is given as

$$P(t) = \lambda D(t), \lambda > 1.$$

The deterioration function follows an exponential distribution with probability density function,

$$f(t) = \begin{cases} \theta e^{-\theta t}, & t \geq 0 \\ 0, & t < 0 \end{cases}, (0 < \theta < 1)$$

The rate of deterioration, $h(t) = \theta$ is a constant fraction θ $(0 < \theta < 1)$ of the on hand inventory deteriorates per unit time. It is assumed that no repair or replacement of the deteriorated items takes place during a given cycle.

Shortages are not allowed.

Demand is non- zero at $t = 0$.

2.2 Notations

$D(t)$is the demand rate.

$P(t)$ is the production rate.

$h(t)$ is the rate of deterioration.

C_h is the holding cost per unit per unit time.

C_d is the deteriorating cost per unit per unit time.

C_o is the set up cost per cycle.

$I(t)$ is the inventory level at time t.

Q is the maximum inventory level.

T is the fixed duration of a production cycle.

T_1 is the time at which the maximum inventory level occurs.

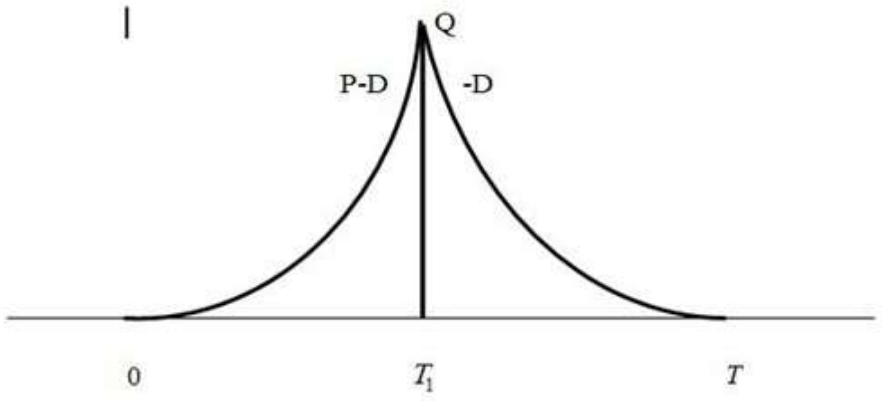
TC is the total inventory cost per cycle.

3. Model description

The proposed inventory model is developed on the basis of exponential market demand and production capacity of the organization with exponential decay. At the beginning, $t = 0$, the production starts with zero inventory. In this model, the production rate $P(t)$ is proportional to the demand rate $D(t)$. The demand exponentially decreases time to time and is given as $D(t) = ae^{bt}, a > 0, 0 < b < 1$. During the time $t = 0$ to $t = T_1$, due to the limited shelf–life and market demand, deterioration occurs. The inventory attains the maximum level Q at $t = T_1$ and the production is stopped at $t = T_1$.

Now, due to demand and deterioration from T_1 to T, the inventory level reduces and becomes zero at time $t = T$. The cycle repels itself after time T. Now, governing differential equation with boundary condition for the above said

Figure 1: The production inventory model



inventory model are given below:

$$\frac{d}{dt}I(t) + \theta I(t) = \begin{cases} (\lambda - 1)\, ae^{bt}, & 0 \le t \le T_1 \\ -ae^{bt}, & T_1 \le t \le T \end{cases} \tag{1}$$

with boundary conditions $I(0) = 0$, $I(T_1) = Q$ and $I(T) = 0$.
The solution of equation (1) is given below:

$$I(t) = \begin{cases} \frac{a(\lambda-1)}{b+\theta}\left(e^{bt} - e^{-\theta t}\right), & 0 \le t \le T_1 \\ \frac{a}{b+\theta}\left(e^{(b+\theta)T-\theta t} - e^{bt}\right), & T_1 \le t \le T. \end{cases} \tag{2}$$

Now, the sum of the holding cost, HC and the deteriorating cost, DC is given by

$$\begin{aligned} HC + DC &= C_h\left\{\int_0^T I(t)\,dt\right\} + C_d\left\{\int_0^T h(t)I(t)\,dt\right\} \\ &= C_h\left\{\int_0^T I(t)\,dt\right\} + C_d\left\{\int_0^T \theta I(t)\,dt\right\} \\ &= (C_h + \theta C_d)\left\{\int_0^T I(t)\,dt\right\} \\ &= (C_h + \theta C_d)\left\{\int_0^{T_1} \frac{a(\lambda-1)}{b+\theta}\left(e^{bt} - e^{-\theta t}\right)dt + \int_{T_1}^T \frac{a}{b+\theta}\left(e^{(b+\theta)T-\theta t} - e^{bt}\right)dt\right\} \\ &= (C_h + \theta C_d)\left\{\frac{a(\lambda-1)}{b+\theta}\left(\frac{e^{bt}}{b} + \frac{e^{-\theta t}}{\theta}\right)_0^{T_1} + \frac{a}{b+\theta}\left(\frac{e^{(b+\theta)T-\theta t}}{-\theta} - \frac{e^{bt}}{b}\right)_{T_1}^{T}\right\} \\ &= (C_h + \theta C_d)\left\{\begin{array}{l} \frac{a(\lambda-1)}{b+\theta}\left(\frac{e^{bT_1}-1}{b}\right) + \frac{a(\lambda-1)}{b+\theta}\left(\frac{e^{-\theta T_1}-1}{\theta}\right) \\ +\frac{a}{b+\theta}\left(\frac{e^{bT+\theta(T-T_1)}-e^{bT}}{\theta}\right) - \frac{a}{b+\theta}\left(\frac{e^{bT}-e^{bT_1}}{b}\right) \end{array}\right\}. \end{aligned}$$

Expanding the exponential function and neglecting the third and higher powers of b and θ, we get.

$$HC + DC = (C_h + \theta C_d)\left\{\begin{array}{l} \frac{a(\lambda-1)}{2}T_1^2 + \frac{a}{(b+\theta)}\{bT(T-T_1) \\ +\frac{\theta}{2}(T-T_1)^2 - \frac{b}{2}(T^2 - T_1^2)\} \end{array}\right\}. \tag{3}$$

Now, total inventory cost per cycle, TC = set up cost + inventory holding cost + deterioration cost, that is,

$$\begin{aligned} TC &= \frac{1}{T}\{C_o + HC + DC\} \\ &= \frac{C_o}{T} + \frac{(C_h + \theta C_d)}{T}\left\{\begin{array}{l} \frac{a(\lambda-1)}{2}T_1^2 + \frac{a}{(b+\theta)}\{bT(T-T_1) \\ +\frac{\theta}{2}(T-T_1)^2 - \frac{b}{2}(T^2 - T_1^2)\} \end{array}\right\}. \ (by(3)) \end{aligned} \tag{4}$$

Let us assume that $T_1 = pT$,where $0 < p < 1$. Now, the expression 4 becomes

$$TC = \frac{C_o}{T} + (C_h + \theta C_d)\left\{\begin{array}{l} \frac{a(\lambda-1)p^2}{2} + \frac{a}{(b+\theta)}\{b(1-p) \\ +\frac{\theta(1-p)^2}{2} - \frac{b(1-p^2)}{2}\} \end{array}\right\}T. \tag{5}$$

To find optimal values of T and TC. Now, since TC is continuous for all $T\ in\ (0,\infty)$ and $\frac{d^2}{dT^2}(TC) = \frac{2C_o}{T^3} > 0$, for all $T\ in\ (0,\infty)$, TC is convex on $(0,\infty)$.

Now, solving $\frac{d}{dT}(TC) = 0$, we obtain a critical point of TC, T^* where

$$T^* = \sqrt{\frac{2C_o(b+\theta)}{a(C_h+\theta C_d)\left\{(\lambda-1)p^2(b+\theta)+\{2b(1-p)+\theta(1-p)^2-b(1-p^2)\}\right\}}}$$

Now, since TC is convex on $(0,\infty)$ and $T^*$is a critical point of TC, TC attains its minimum value at T^* and $T^*$is the optimum value of T.

Now, the optimum of TC, $TC^*$is obtained from 5 by replacing T by $T^*$and the optimum of T_1, $T_1^*$is obtained from $T_1 = pT$ by replacing T by T^*

Remark 1. With the help of the parameter p, the length of the production period $(0, T_1)$can be modified according to the wish of the production department of the organization.

4. Numerical example and sensitivity analysis

Consider the production inventory model for exponential deteriorating items with exponential demand rate where the parameters are given as $C_o = 25$, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$, $b = 0.8$ and $p = 0.4$.

Now, using MATLAB, we obtain the optimum value of Tas $T^* = 0.1645$, the optimum value of T_1 as $T_1^* = 0.0658$ and the optimum total inventory cost as $TC^* = 303.9737$.

In the developed inventory model, the total cost per cycle TC is a real valued function of T in which eight model parameters are assumed to be static values. Therefore, the sensitivity of each one of the model parameters in the developed inventory model is examined.

Case (i). Set up cost $C_o$varies from 20 to 70 and the other parameters are fixed, that is, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$, $b = 0.8$ and $p = 0.4$. Using MATLAB, we obtain two graphs (Figure 2) for time verse set up cost and total inventory cost per cycle verse set up cost and from the Figure 2, we observe that T_1 , T and TC increases as C_0 increases.

Case (ii). Holding cost $C_h$varies from 30 to 70 and the other parameters are fixed, that is, $C_o = 25$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$, $b = 0.8$ and $p = 0.4$. Using MATLAB, we get two graphs (Figure 3) for time verse holding cost and total inventory cost per cycle verse holding cost and from the Figure 3, we observe that T_1 and T decreases and TC increases as C_h increases.

Case (iii). Deterioration cost C_d varies from 20 to 60 and the other parameters are fixed, that is, $C_o = 25$, $C_h = 30$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$, $b = 0.8$ and $p = 0.4$. Using MATLAB, we find two graphs (Figure 4) for time verse deteriorating cost and total inventory cost per cycle verse deteriorating cost and from the Figure 4, we observe that T_1 and T decreases and TC increases as C_d increases.

Figure 2: Set up cost

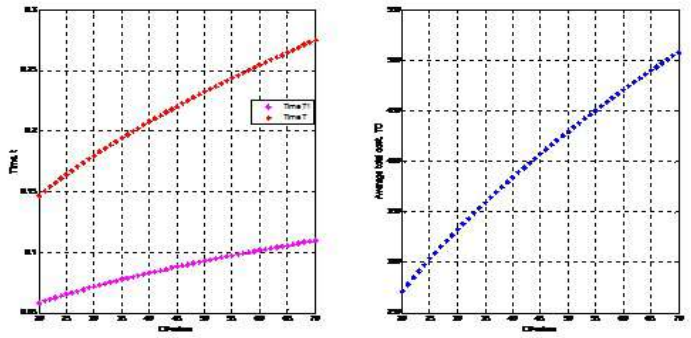
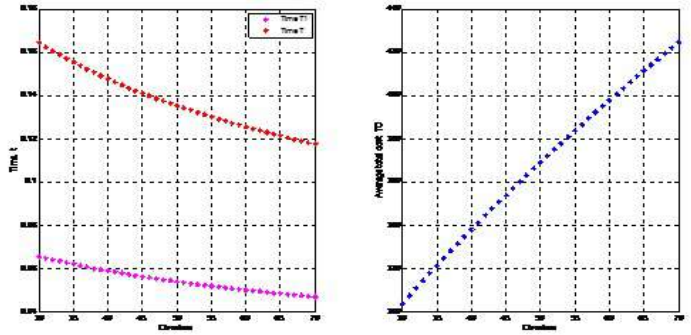


Figure 3: Holding cost



Case (iv). The rate of deterioration θ varies from 0.1 to 0.9 and the other parameters are fixed, that is, $C_o = 25$, $C_h = 30$, $C_d = 20$, $\lambda = 1.5$, $a = 100$, $b = 0.8$ and $p = 0.4$. Two graphs (Figure 5) for time verse rate of deterioration and total inventory cost per cycle verse rate of deterioration are obtained using MATAB and from the Figure 5, we observe that T_1 and T decreases and TC increases as θ increases.

Case (v). The production proportional λ varies from 1.2 to 5 and the other parameters are fixed, that is, $C_o = 25$, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $a = 100$, $b = 0.8$ and $p = 0.4$. Using MATLAB, two graphs (Figure 6) for time verse production proportional and total inventory cost per cycle verse production proportional are found and from the Figure 6, we observe that $T_1$and T decreases and TC increases as λ increases.

Case (vi). The demand constant a varies from 99 to 150 and the other parameters are fixed, that is, $C_o = 25$, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $b = 0.8$ and $p = 0.4$. Using MATLAB, two graphs (Figure 7) for time verse demand constant and total inventory cost per cycle verse demand constant are

Figure 4: Deterioration cost

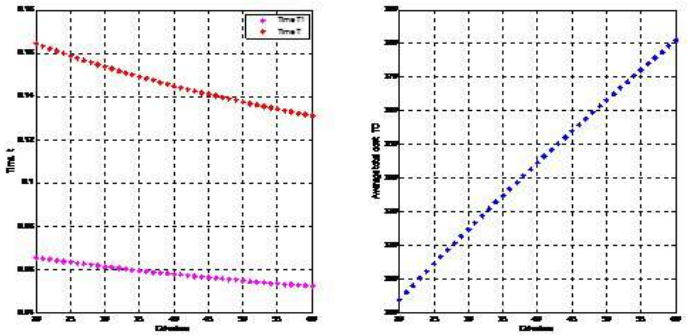
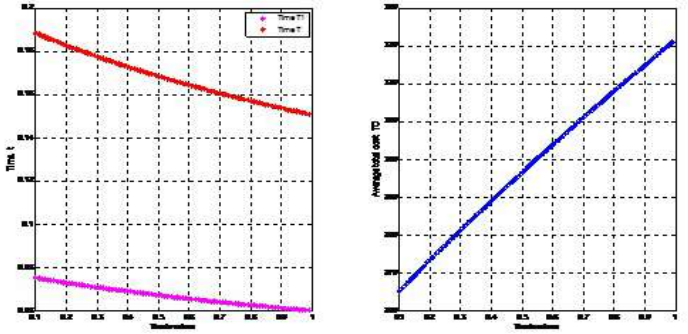


Figure 5: Rate of deterioration



got and from the Figure 7, we observe that T_1 and T decreases and TC increases as a, the demand coefficient, increases.

Case (vii). The exponential constant b in demand function varies from 0.1 to 0.9 and the other parameters are fixed, that is,$C_o = 25$, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$and $p = 0.4$. Using MATLAB, we obtain two graphs (Figure 8) for time verse exponential constant and total inventory cost per cycle verse exponential constant and from the Figure 8, we observe that T_1 , T and TC are constants as b, demand coefficient increases.

Case (viii). The time period constant p varies from 0.1 to 0.9 and the other parameters are fixed, $C_o = 25$, $C_h = 30$, $C_d = 20$, $\theta = 0.6$, $\lambda = 1.5$, $a = 100$and $b = 0.8$. Using MATLAB, we get two graphs (Figure 9) for time verse time period constant, p and total inventory cost per cycle verse time period constant, p and from the Figure 9, we observe that T_1 increases as p increases, T increases as p increases from 0.001 to 0.68 (nearly) and decreases as p increases from 0.68 (nearly) to 0.999 and TC decreases as p increases from 0.001 to 0.68 (nearly) and increases as p increases from 0.68 (nearly) to 0.999.

Figure 6: Production proportional

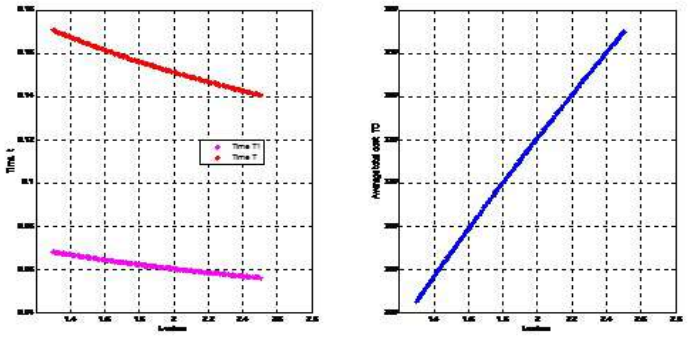
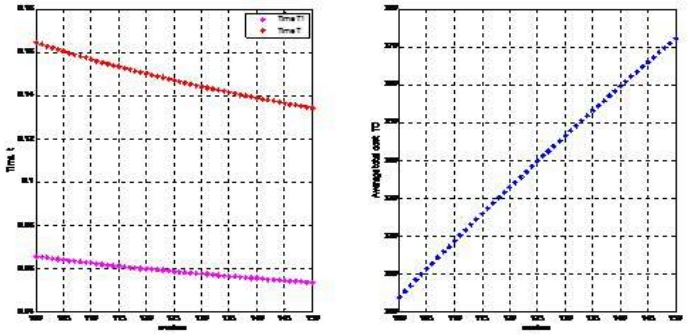


Figure 7: Demand constant



5. Conclusion

In this paper, a production inventory model for exponentially declining deterioration with an exponential demand rate is presented in which the production rate of the model is assumed to be proportional to the demand rate. The optimal total inventory cost per cycle and the optimal length of the cycle and optimal production length are determined. The developed model is illustrated with a numerical example. The proposed production inventory model having eight model parameters has been analysed sensitively with respect to each one of its parameters using MATLAB.

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Figure 8: Exponential constant

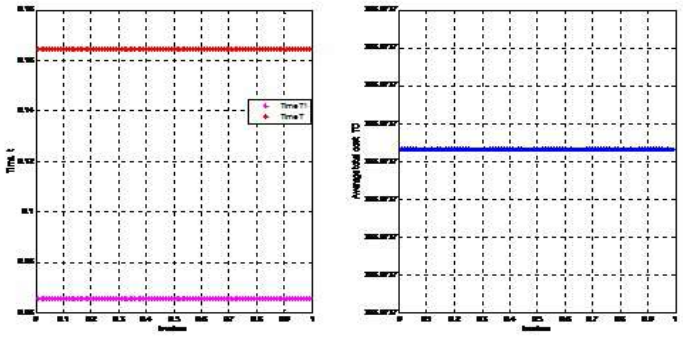
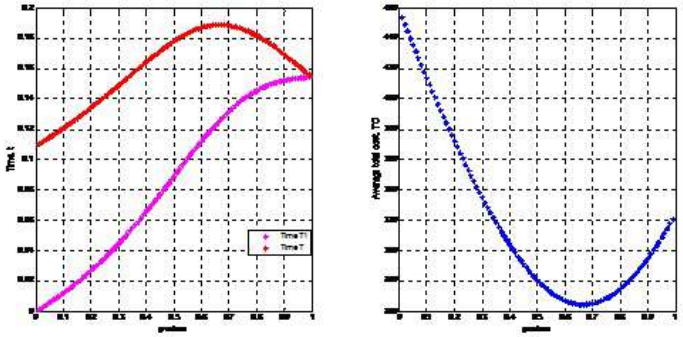


Figure 9: Time period constant



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On automatic surjectivity of some point spectrum preserving additive maps

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Abstract. Let X and Y be two infinite dimensional complex Banach spaces and let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map. We show that if the range of Φ contains the ideal of all finite rank operators of $B(Y)$ then Φ is a Jordan morphism. In the case where X is an infinite dimensional separable Hilbert space the map Φ is surjective.
Keywords: Linear preserver problem, point spectrum, rank one operators, operator algebra.

1. Introduction

The problem of characterizing linear or additive maps on matrix or operator algebras that leave invariant a given subset, function or relation defined on the underlying algebras represents one of the most active research areas; see for example [1], [2], [7], [8], [13], [14], [15] and the references therein. It has been proved by B. Aupetit in [2] that if $\Phi : \mathcal{M}_n(\mathbb{C}) \longrightarrow B$ is a spectrum preserving surjective additive map, (where B is a semi simple algebra of dimension $\leq n^2$) then either $\Phi(x) = uxu^{-1}$ or $\Phi(x) = ux^tu^{-1}$ for every $x \in \mathcal{M}_n(\mathbb{C})$ where u is an invertible element in B. He asked the question: what happens if such map is not surjective?

Many researchers are applied to generalize the result of B. Aupetit in the case of the algebras of linear bounded operators over complex Banach spaces but all theirs maps are supposed to be surjectives.

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In this paper we propose to have the same result of B. Aupetit always in the case of the algebras of linear bounded operators over complex Banach spaces but with weaker hypotheses such that Φ is an additive map preserving only the point spectrum and it's range contains the ideal of all finite rank operators.

Let X and Y are two infinite dimensional complex Banach spaces and $B(X)$ and $B(Y)$ are the algebras of linear bounded operators on X and Y respectively. For an operator $T \in B(X)$ we denote by $\sigma_p(T)$ the point spectrum of T which is the set $\{\lambda \in \mathbb{C}\lambda.I - T \text{ is not injective}\}$ where I is the identity of $B(X)$. It is said that a map $\Phi : B(X) \longrightarrow B(Y)$ preserves the point spectrum if $\sigma_p(\Phi(T)) = \sigma_p(T)$ for every $T \in B(X)$.

Every rank one operator in $B(X)$ can be written as $x \otimes f$ for some $x \in X - \{0\}$ and $f \in X^{'} - \{0\}$ and defined by $(x \otimes f)y = f(y)x$ for every $y \in X$. The operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ moreover $\sigma_p(x \otimes f) \subset \{0; f(x)\}$ (see [7] and [8]). The operator $x \otimes f + y \otimes g$ is of rank one if and only if either x and y are linearly dependent or f and g are linearly dependent for every $x, y \in X - \{0\}$ and $f, g \in X^{'} - \{0\}$. We denote by $F(X)$ and $F_1(X)$ the ideal of finite rank operators and the set of rank one operators in $B(X)$ respectively. The range of Φ will be denoted by $Im\Phi$ and the operator $\lambda.I$ by λ.

2. Main results

The aim of this work is to prove the following results:

Theorem 2.1. *Let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$. Then either*

i) $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$ where $A : X \longrightarrow Y$ is a bounded bijective map; or

ii) $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$ where $B : X^{'} \longrightarrow Y$ is a bounded bijective linear map.

Corollary 2.1. *Let X be an infinite dimensional separable Hilbert space and Y be an infinite dimensional Banach space. If $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$ then, Φ is surjective.*

Lemma 2.1. *Let $A \in B(X)$. We have $\sigma_p(T + A) \subset \sigma_p(T)$ for every $T \in F_1(X)$ if and only if $A = 0$.*

Proof. If $A = 0$, it is clear that $\sigma_p(T + A) \subset \sigma_p(T)$ for every $T \in F_1(X)$.

For the reverse implication we assume that $A \neq 0$. In the case where $A = I$ we take $x \in X$ and $f \in X^{'}$ such that $f(x) = 1$ and $T = x \otimes f$. We have $2 \in \sigma_p(T + A)$ but $\sigma_p(T) \subset \{0; 1\}$. In the case where $A \neq I$ there exist $x \in X$ such that $0 \neq Ax \neq x$. For $y = Ax$ there exists $f \in X^{'}$ such that $f(x) = 1$ and $f(y) \neq 0$. If we take the operator $T = (x - y) \otimes f$ we have $\sigma_p(T) \subset \{0; f(x - y)\}$ and $1 \in \sigma_p(T + A)$ but $f(x - y) \neq 1$. Then $\sigma_p(T + A) \not\subset \sigma_p(T)$.

Lemma 2.2. *let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map. Then Φ is injective.*

Proof. Let $A \in B(X)$ such that $\Phi(A) = 0$. We have $\sigma_p(T) = \sigma_p(\Phi(T)) = \sigma_p(\Phi(T) + \Phi(A)) = \sigma_p(T + A)$, for every $T \in F_1(X)$. By the lemma 2.1 we conclude that $A = 0$ and therefore Φ is injective.

Lemma 2.3. *let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$. Then $\Phi(\lambda) = \lambda$ for every $\lambda \in \mathbb{C}$.*

Proof. Let $S \in F_1(Y)$ and $T \in B(X)$ such that $\Phi(T) = S$. We have for every $\lambda \in \mathbb{C}$:

$$\begin{aligned} \sigma_p(S + \Phi(\lambda) - \lambda) &= \sigma_p((\Phi(T) + \Phi(\lambda)) - \lambda \\ &= \sigma_p(T + \lambda) - \lambda \\ &= \sigma_p(T) \\ &= \sigma_p(\Phi(T)) \\ &= \sigma_p(S). \end{aligned}$$

We apply the Lemma 2.1 to conclude that $\Phi(\lambda) = \lambda$.

Lemma 2.4. *Let R be non zero operator in $B(X)$. Then the following conditions are equivalent:*

i) R is of rank 1,

ii) $\sigma_p(T + R) \cap \sigma_p(T + 2R) \subset \sigma_p(T)$ for every $T \in F(X)$.

Proof. Let us prove that i) implies ii). Let $R \in B(X)$ be an operator of rank one. Suppose that there exists $T \in F(X)$ such that $\sigma_p(T + R) \cap \sigma_p(T + 2R) \not\subset \sigma_p(T)$. Then there exists $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective and $\lambda - T - R$ and $\lambda - T - 2R$ are not injective operators. Hence there exist two non zero vectors x and y such that $(\lambda - T)(x) = Rx$ and $(\lambda - T)(y) = 2R(y)$. It is clear that $R(x) \neq 0$ and $R(y) \neq 0$ and the vectors $R(x)$ and $R(y)$ are linearly dependent because R is of rank one. Let α a non zero complex number such that $R(x) = \alpha R(y)$. The vector $z = x - \frac{\alpha}{2}y$ satisfy $z \neq 0$ and $(\lambda - T)(z) = 0$ witch contradicts the fact that $\lambda - T$ is injective.

We prove now that ii) implies i). Assume that ii) is satisfied and $rankR > 1$. Suppose that there exists $x \in X$ such that x, Rx and R^2x are linearly independent. Define the operator S by:

$$\begin{aligned} Sx &= 3x - Rx \\ SRx &= 3Rx - 2R^2x \\ SR^2x &= x \\ Sz &= 0 \text{ pour } z \in L \end{aligned}$$

where L is a closed complement of the span of $\{x, Rx, R^2x\}$. We have $S \in B(X)$ and $3 \in \sigma_p(S+R) \cap \sigma_p(S+2R)$ but $3 \notin \sigma_p(S)$. Thus x, Rx, and R^2x are linearly dependent for every $x \in X$. So there exists a quadratic polynomial p such that $p(R) = 0$. This polynomial has one of the forms $p(t) = (t - \alpha)(t - \beta)$ or

$p(t) = (t-\alpha)^2$ or $p(t) = t(t-\alpha)$ or $p(t) = t^2$ where $\alpha \neq 0 \neq \beta \neq \alpha$. By the standard decomposition of algebraic operators and since $rankR > 1$ we can affirm that R has a finite dimensional invariant subspace W such that, R/W has a matrix representation

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

respectively. We consider a complement Z of W in X and a operator T such that $T/Z = 0$ and T/W has matrix representation

$$\begin{pmatrix} -3\alpha/2 & \beta/2 \\ \alpha/2 & -3\beta/2 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 \\ \alpha^2 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 2\alpha \\ 0 & -2\alpha & -2\alpha \end{pmatrix}$$
$$\text{or } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

respectively. We have $T \in F(X)$ and there exists $\lambda \in \sigma_p(T+R) \cap \sigma_p(T+2R)$ but $\lambda \notin \sigma_p(T)$, namely $\lambda = 0$ or $\lambda = \alpha + 1$ or $\lambda = 2\alpha$ or $\lambda = \sqrt{2}$.

Lemma 2.5. *Let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$ and $R \in B(X)$. Then $\Phi(R)$ is of rank 1 if and only if R is of rank 1.*

Proof. Let $R \in B(X)$. If R is of rank one we apply the Lemma 2.4 to have $\sigma_p(T+R) \cap \sigma_p(T+2R) \subset \sigma_p(T)$ for every $T \in F(X)$. Since Φ is additive and preserves the point spectrum we have $\sigma_p(\Phi(T)+\Phi(R)) \cap \sigma_p(\Phi(T)+2\Phi(R)) \subset \sigma_p(\Phi(T))$ for every $T \in F(X)$. By the hypothesis $F(Y) \subset Im\Phi$ we have $\sigma_p(S+\Phi(R)) \cap \sigma_p(S+2\Phi(R)) \subset \sigma_p(S)$ for every $S \in F(Y)$. We apply the Lemma 2.4 to conclude that $\Phi(R)$ is of rank one.

To prove the reverse case we assume that $\Phi(R)$ is of rank one and $T \in F(X)$. Since T is a finite sum of some rank one operators we have $\Phi(T) \in F(Y)$. we apply the Lemma 2.4 to get $\sigma_p(S+\Phi(R)) \cap \sigma_p(S+2\Phi(R)) \subset \sigma_p(S)$ for every $S \in F(Y)$. We take $S = \Phi(T)$ to have $\sigma_p(\Phi(T)+\Phi(R)) \cap \sigma_p(\Phi(T)+2\Phi(R)) \subset \sigma_p(\Phi(T))$. Since Φ preserves the point spectrum we have $\sigma_p(T+R) \cap \sigma_p(T+2R) \subset \sigma_p(T)$ and by the Lemma 2.4 we establish that R is of rank one.

Lemma 2.6. *Let $\Phi : B(X) \longrightarrow B(Y)$ be a point spectrum preserving additive map such that $F(Y) \subset Im\Phi$. Then Φ is 1-homogeneous over $F_1(X)$ (i.e: $\Phi(\lambda T) = \lambda\Phi(T)$ for every $\lambda \in \mathbb{C}$ and $T \in F_1(X)$).*

Proof. Every rank one operator can be written of the form $x \otimes f$ where $0 \neq x \in X$ and $0 \neq f \in X'$. So it suffices to proof that $\Phi(\alpha x \otimes f) = \alpha\Phi(x \otimes f)$ for every $\alpha \in \mathbb{C}$, $x \in X - \{0\}$ and $f \in X' - \{0\}$.

If $\alpha = 0$ or $\alpha = -1$ we have $\Phi(\alpha x \otimes f) = \alpha\Phi(x \otimes f)$ because Φ is additive.

If $\alpha \neq 0$ and $\alpha \neq -1$ we start by the case where $f(x) \neq 0$ in which case we can take $f(x) = 1$. By the Lemma 2.5 the operator $\Phi(x \otimes f)$ is of rank one so, there exist $u \in Y$ and $\varphi \in Y^{'}$ such that $\Phi(x \otimes f) = u \otimes \varphi$. Since Φ preserves the point spectrum we have $\varphi(u) = 1$. Let $v \in Y$ and $\psi \in Y^{'}$ such that $\varphi(v) = 0$ and $\psi(u) = 0$. The fact that $F(Y) \subset Im\Phi$ implies that there exist two vectors w and z in X and two linearly forms h and k in $X^{'}$ such that:

$$\Phi(w \otimes h) = v \otimes \varphi \text{ and } \Phi(z \otimes k) = u \otimes \psi.$$

The operator $\alpha x \otimes f$ is of rank one and by the Lemma 2.5 there exist $u_\alpha \in Y$ and $\varphi_\alpha \in Y^{'}$ such that $\Phi(\alpha x \otimes f) = u_\alpha \otimes \varphi_\alpha$. $\Phi(\alpha x \otimes f + x \otimes f) = u_\alpha \otimes \varphi_\alpha + u \otimes \varphi$ is an operator of rank one which implies that either the vectors u_α and u are linearly dependent or the forms φ_α and φ are linearly dependent. Suppose that φ_α and φ are linearly dependent, by absorbing a constant in the first term of the tensor product, we can take $\varphi_\alpha = \varphi$. We have $\Phi(x \otimes f + z \otimes k) = u \otimes \varphi + u \otimes \psi$ who is an operator of rank one and the Lemma 2.5 shows that $x \otimes f + z \otimes k$ is of rank one so, $\alpha x \otimes f + z \otimes k$ is also of rank one. By the Lemma 2.5 we have $\Phi(\alpha x \otimes f + z \otimes k) = u_\alpha \otimes \varphi + u \otimes \psi$ an operator of rank one. Since $\varphi(u) = 1$ and $\psi(u) = 0$ we can affirm that φ and ψ are linearly independent therefore, u_α and u are linearly dependent. Consequently $u_\alpha = \beta u$ for some $\beta \in \mathbb{C}^*$. Thus $\Phi(\alpha x \otimes f) = \beta u \otimes \varphi$ and $\alpha = \beta$ because Φ preserves the point spectrum.

In the case where u_α and u are linearly dependent, by absorbing a constant in the second term of the tensor product, we can take $u_\alpha = u$. The operator $\Phi(x \otimes f + w \otimes h) = u \times \varphi + v \otimes \varphi$ is of rank one and by the Lemma 2.5 we establish that $x \otimes f + w \otimes h$ is of rank one thus $\alpha x \otimes f + w \otimes h$ is also of rank one. We apply the Lemma 2.5 to show that $\Phi(\alpha x \otimes f + w \otimes h) = u \otimes \varphi_\alpha + v \otimes \varphi$ is rank one operator. Since $\varphi(u) = 1$ and $\varphi(v) = 0$ we can said that the vectors u and v are linearly independent therefore φ_α and φ are linearly dependent that is to say $\varphi_\alpha = \gamma\varphi$ for some $\gamma \in \mathbb{C}^*$. We have $\Phi(\alpha x \otimes f) = \gamma u \otimes \varphi$ and $\alpha = \gamma$ because Φ preserves the point spectrum. Finally $\Phi(\alpha x \otimes f) = \alpha\Phi(x \otimes f)$.

Now, if $f(x) = 0$ we can find $g \in X^{'}$ such that $g(x) = 1$. We have $\Phi(\alpha x \otimes f) = \Phi(\alpha x \otimes (f + g)) - \Phi(\alpha x \otimes g) = \alpha\Phi(x \otimes f)$.

3. Proof of main results

Proof of Theorem 2.1. The proof of this theorem follows the same plan of A. Jafarian and A.R. Sourour [1] in two steps but with different approaches especially in the second step.

In the first step we show that either there exist two bijective linear maps $A : X \longrightarrow Y$ and $C : X^{'} \longrightarrow Y^{'}$ such that:

$$\Phi(x \otimes f) = Ax \otimes Cf, \tag{1}$$

for every $x \in X$ and $f \in X^{'}$ or there exist two bijective linear maps $B : X^{'} \longrightarrow Y$ and $D : X \longrightarrow Y^{'}$such that:

$$\Phi(x \otimes f) = Bf \otimes Dx, \tag{2}$$

for every $x \in X$ and $f \in X'$.

In the second step we show that either $\Phi(T) = ATA^{-1}$ or $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$.

Step 1: The Lemmas 2.2, 2.5 and 2.6 which are specifics to our situation show that Φ is injective , $\Phi(F_1(X)) = F_1(Y)$ and Φ is $1-$homogeneous over $F_1(X)$ thus allow us to be in the same hypotheses of A.Jafarian and A.R.Sourour in [1] to affirm step 1. All that remains is to demonstrate that A, B, C and D are bounded. Let us begin with A and C, which are given by (1). Choose an operator T of finite rank in $B(X)$ and $\lambda \in \mathbb{C}^*$ with $\lambda \notin \sigma_p(T)$. For $x \in X - \{0\}$ and $f \in X' - \{0\}$ we have $\lambda \in \sigma_p(T+x\otimes f)$ if and only if $f((\lambda - T)^{-1}x) = 1$. The fact that Φ preserves the point spectrum we have $f((\lambda - T)^{-1}x) = Cf((\lambda - \Phi(T))^{-1}Ax)$. The closed graph Theorem shows that A and C are bounded. In the same way we can show that B and D are bounded.

Step 2: To establish this step A.Jafarian and A.R.Sourour have exploited with advantage the analytic properties of the resolvent which is related to the spectrum thing which can not be used here since we only consider the point spectrum and we give an algebraic proof.

First case : Let us put in the situation (1) and show that $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$. Fix $0 \neq x \in X$ and $0 \neq f \in X'$.

If T is injective we have $T(x) = y \neq 0$. Since Φ preserves the point spectrum we have,

$$f(x) = 1 \text{ if and only if } Cf(Ax) = 1. \tag{3}$$

We have $0 \in \sigma_p(T - y \otimes f) = \sigma_p(\Phi(T) - Ay \otimes Cf)$ so, there exists $z \in Y$ such that $\Phi(z) = Ay \otimes Cf(z)$ that is to say that $\Phi(T)(z) = Cf(z)Ay$. Show that z and Ax are linearly dependent in other wise there exists $g \in Y'$ such that $g(Ax) = 1$ and $g(z) = 0$. Since C is bijective there exists $f \in X'$ such that $Cf = g$. We have $Cf(Ax) = 1$ and by (3) we get $f(x) = 1$. Therefore $\Phi(T)(z) = Cf(z)Ay = g(z)Ay = 0$ from there $0 \in \sigma_p(\Phi(T)) = \sigma_p(T)$ which is absurd. Therefore z and Ax are linearly dependent and consequently we have for some suitable $\lambda \in \mathbb{C}^*$, $\Phi(T)(\lambda Ax) = Cf(\lambda Ax)Ay = \lambda Ay = \lambda ATx$. So $\Phi(T)A = AT$ and $\Phi(T) = ATA^{-1}$.

If T is not injective there exists $\lambda \in \mathbb{C}$ such that $T + \lambda$ is injective. The Lemma 2.4 indicate that $\Phi(\lambda) = \lambda$ and consequently $\Phi(T) = ATA^{-1}$.

Second case : Let us put in situation (2) and show that $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$. Fix $T \in B(X)$.

If T is injective we study the case where the range of T is dense in X and the case where is not. In the case where the range of T is dense in X we find $f \in X' - \{0\}$ and $x \in X$ such that $f(Tx) = 1$. For $y = Tx$ we have $0 \in \sigma_p(T - y \otimes T^*f) = \sigma_p(\Phi(T) - B(T^*f) \otimes Dy)$ and there exists $z \in Y$ such that:

$$\Phi(T)(z) = Dy(z)B(T^*f). \tag{4}$$

Show that z and Bf are linearly dependent in other wise there exists $g \in Y^{'}$ such that $g(z) = 0$ and $g(Bf) = 1$. Since the map D is bijective there exists $y \in Y$ such that $D(y) = g$ which implies that $D(y)(Bf) = 1$. Since Φ preserves the point spectrum we have $f(y) = 1$. The formula (4) implies that $\Phi(T)(z) = Dy(z)B(T^*f) = g(z)B(T^*f) = 0$ and consequently $0 \in \sigma_p(\Phi(T)) = \sigma_p(T)$ which is absurd. So, z and Bf are linearly dependent and for a suitable $\lambda \in \mathbb{C}^*$ we have $\Phi(T)(\lambda Bf) = D(y)(\lambda Bf)(BT^*f)$ which implies that $\Phi(T)(Bf) = BT^*f$ for every $f \in X^{'}$ and consequently $\Phi(T) = BT^*B^{-1}$. In the case where the range of T is not dense in X we can find a real number α such that $T + \alpha$ is surjective and we apply the forgoing and the fact that $\Phi(\alpha) = \alpha$ to deduce that $\Phi(T) = BT^*B^{-1}$.

If T is not injective there exists a non zero real number λ such that $T + \lambda$ is injective. Since $\Phi(\lambda) = \lambda$ we can affirm that $\Phi(T) = BT^*B^{-1}$. Finally $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$.

Proof of Corollary 2.1. The Theorem 2.1 shows that either $\Phi(T) = ATA^{-1}$ or $\Phi(T) = BT^*B^{-1}$ for every $T \in B(X)$ where $A : X \longrightarrow Y$ and $B : X^{'} \longrightarrow Y$ are bijective linear bounded maps. The fact that Φ preserves the point spectrum and $\Phi(T) = BT^*B^{-1}$ show that, T is injective if and only if T^* is injective which is not true if X is a separable Hilbert space of infinite dimension and T is a unilateral shift operator (see [12] problem 82). So the case $\Phi(T) = BT^*B^{-1}$ can not take place and we have $\Phi(T) = ATA^{-1}$ for every $T \in B(X)$. Consequently Φ is surjective.

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Solvability of certain groups of time varying artificial neurons

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Abstract. In the process of exploration structure of the most used artificial neural network-multilayer perceptron and functionality of artificial neuron, there were established structures of groups of artificial neurons. Currently the most interesting areas is the usage of time varying artificial neurons and their reflections in these algebraic structures. Using certain analogy with relations between structures based on certain groups of linear ordinary differential operators there is investigated access to new view point on these subjects. In this paper there is contained the solution of one classical problem-verification that the corresponding group of time is solvable.

Keywords: Neural network, linear ordinary differential operators, groups of neurons, solvable groups.

1. Introduction

Time dependence on neural networks is currently insufficiently explored. Yet, in biological systems this is an important aspect. Since forgetting or anticipation is as important as memorization. Most researchers have focused on memory as a time independent activity. However, if it was time independent as assumed in the connectionist models, healing, regrouping and creating new ideas in an unsupervised environment would not be possible. The biological nervous system's ability to process and produce spatio-temporal patterns of activity while interacting with real world environments is an attribute derived from the time dependent nature of neural spatial and temporal summation. These functional characteristics of neural activity, in conjunction with the principle of "learning" based on varying synaptic efficacy, have been incorporated into a "time dependent" neural network design. The behaviour of this network is responsive to the temporal relationship of stimulus "events" as well as the "spatial" intensity

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pattern of the stimulation. In addition, this network has the potential of generating time varying patterns of activity. In this article time dependence of neural networks is examined when looking at the role of algebraic structures based on neural networks, with emphasis on the feed-forward neural network.

A neuron also called an artificial or formal neuron is the basic stone of the mathematical model of any neural network. Its design and functionality are derived from observation of a biological neuron that is basic building block of biological neural networks (systems) which includes the brain, spinal cord and peripheral ganglia. In case of the artificial neuron the information comes into the body of the artificial neuron via inputs that are weighted (i.e. each input can be individually multiplied with a weight). The body of the artificial neuron then sums the weighted inputs, bias and "processes" the sum with a transfer function. At the end the artificial neuron passes the processed information via outputs.

Recall that in the framework of Artificial neural networks a perceptron is a network of simple neurons called perceptrons. The basic concept of a single perceptron was introduced by Rosenblatt in the year 1958. The perceptron computes a single output from multiple real-valued inputs by forming a linear combination according to its input weights and then possibly putting the output through some nonlinear activation function. As usually mathematically this can be written as:

$$y = \varphi\left(\sum_{i=1}^{n} w_i x_i + b\right) = \varphi(\vec{w}^{T}\vec{x} + b),$$

where $\vec{w} = (w_1, \ldots, w_n)$ denotes the vector of weights, $\vec{x} = (x_1, \ldots, x_n)$ is the vector of inputs, b is the bias and φ is the activation function.

This model explains the function of a neural network, but there is an aspect of memory. The simplest way to insert memory into the model is to add registers which store consecutive values of the model input (Mozer, 1993). These registers can be implemented as tapped delay lines, which are fed with new value at each time increment, and can store up to k past values of the input. This corresponds to making the delay coordinate reconstruction from the time series. Traditional way of using neural networks in TSP is to convert the temporal sequence into concatenated vector via a tapped delay line, and to feed the resulting vector as an input to a network. Using the tapped delay line delay line memory the time dimension is converted into spatial representation.

In this reconstruction, neural network approximates the mapping between the input vectors and the desired output. Often the output is one-step prediction of the time series. In this function approximation, the contextual information between the consecutive input vectors is lost. Also, the time difference between consecutive samples is implicitly assumed to be equal in all input vectors. In many cases, this reconstruction can capture the essential information from the sequence, allowing to build models to predict it. Especially, if the series is stationary, the reconstruction is often successful. When the series is

non-stationary, it can be difficult to determine a proper window length. Selecting the order of the delay coordinate embedding, or the length of the input vector, is usually carried out by using the model selection methods. However, the optimal window length for the model may vary in the time series, and thus the windowing should be adaptive. This architecture is known as Time delay neural network (TDNN). There are neural networks which employ the delay line memory include Time delay neural network (TDNN) (Waibel et al., 1989; Lang et al., 1990), Finite impulse response neural network (FIR NN) (Wan, 1993; Back and Tsoi, 1991), and Gamma memory neural network (de Vries and Principe, 1992). In the Gamma memory neural network, the memory is built from FIR filters, which include local feedback connections, thus making it a hybrid architecture, utilizing both delay lines and recurrent connections. For more information about artificial neural networks and artificial neurons there are sources to see like [1, 18, 20, 21, 22, 23, 24, 29, 33, 34, 36, 37, 38, 39, 41].

2. Linear differential operators of the n-th order and their groups

The substanding representative of the Brno school of differential equations František Neuman wrote in his paper [25]: "Algebraic, topological and geometrical tools together with the methods of the theory of dynamical systems and functional equations make possible to deal with problems concerning global properties of solutions by contrast to the previous local investigations and isolated results." Influence of mentioned ideas is a certain motivating factor of our investigations.

Thus, we consider linear ordinary differential operators of the form:

$$L_n \;=\; \sum_{k=0}^{n} p_k(x) D^k,$$

where $D_k = \frac{d^k}{dx^k}$, $p_k(x)$ is a continuous function on some open interval $J \subset \mathbb{R}$, $k = 0, 1, \ldots, n-1$, $p_n(x) \equiv 1$, i.e. $L_n(y) = 0$ which is a linear homogeneous ordinary differential equation of the form:

$$y^{(n)}(x) \;+\; \sum_{k=0}^{n} p_k(x) y^{(k)}(x) \;=\; 0.$$

By an ordered group we mean (as usually) a triad $(G, \cdot, \leq)$, where $(G, \cdot)$ is a group and $\leq$ is a reflexive, symmetrical and transitive binary relation on the set G such that for any triad $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$, $z \cdot x \leq z \cdot y$ is satisfied. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is the principal end generated by $a \in G$. To any element $a \in G$ there is assigned a pair of mappings $\lambda_a : G \to G, \rho_a : G \to G$, which are called a left translation, a right translation, respectively, determined by the element $a \in G$, i.e. $\lambda_a(x) = a \cdot x$, $\rho_a(x) = x \cdot a$.(Of course, in the case of a commutative group $(G, \cdot)$ we have $\lambda_a \equiv \rho_a$). Notice that

a group with an ordering $(G, \cdot, \leq)$ is an ordered group if and only if all its left and right translations $\lambda_a, \rho_a, a \in G$ are order-preserving mappings, i.e. isotone selfmaps of the ordered set $(G, \leq)$.

The following lemma which is crucial for further constructions is proved in [8, 9, 10] (the Czech version is proved in [9], pp. 146, 147).

Lemma 2.1. *Let $(G, \cdot, \leq)$ be an ordered group. Define a hyperoperation $* : G \times G \to \mathcal{P}(G)^*$ by*

$$a * b = [a, b)_{\leq} (= \{x \in G; a \cdot b \leq x\}),$$

for all pairs of elements $a, b \in G$. Then $(G, *)$ is a hypergroup which is commutative if and only if the group $(G, \cdot)$ is commutative.

Application of the above lemma and many new results are obtained in papers of Michal Novák. See at least titles [26, 27, 28].

To present following results we use similar notation published in [9]. So there $\mathbb{R}$ stands for the set of all reals, $J \subset \mathbb{R}$ is an open interval (bounded or unbounded) of real numbers, $\mathbb{C}^k(J)$ is the ring (with respect to usual addition and multiplication of functions) of all real functions with continuous derivatives up to the order $k \geq 0$ including. We write $\mathbb{C}(J)$ instead of $\mathbb{C}^0(J)$. For a positive integer $n \geq 2$ we denote by $\mathbb{A}_n$ the set of all linear homogeneous differential equations of the n-th order with continuous real coefficients on J, i.e.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0,$$

(cf. [8, 10], where $p_k \in \mathbb{C}(J)$, $k = 0, 1, \cdots, n-1$, $p_0(x) > 0$ for any $x \in J$ (this is not essential restriction). Denote $L(p_0, \cdots, p_{n-1}) : \mathbb{C}^n(J) \to \mathbb{C}^n(J)$ the above mentioned linear operator defined by

$$L(p_0, \cdots, p_{n-1})(y) = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y$$

and put

$$\mathbb{LA}_n(J) = \{L(p_0, \cdots, p_{n-1}); p_k \in \mathbb{C}(J),\, p_0 > 0\}.$$

Further $\mathbb{N}_0(n) = \{0, 1, \cdots, n-1\}$ and δ_{ij} stands for the Kronecker δ, $\overline{\delta_{ij}} = 1 - \delta_{ij}$. For any $m \in \mathbb{N}_0(n)$ we denote by $\mathbb{LA}_n(J)_m$ the set of all linear differential operators of the $n-$th order $L_0(p_0, \cdots, p_{n-1}) : \mathbb{C}^n(J) \to \mathbb{C}(J)$, where $p_k \in \mathbb{C}(J)$ for any $k \in \mathbb{N}_0(n)$, $p_m \in \mathbb{C}_+(J)$, (i.e. $p_m(x) > 0$ for each $x \in J$). Using the vector notation $\vec{p}(x) = (p_0(x), \cdots, p_{n-1}(x))$, $x \in J$ we can write $L_n(\vec{p}_0)y = y^{(n)} + (\vec{p}(x), (y, y', \cdots, y^{(n-1)}))$, (i.e. a scalar product).

We define a binary operation "$\circ_m$" and a binary relation "$\leq_m$ on the set $\mathbb{LA}_n(J)_m$ in this way:

For arbitrary pair $L(\vec{p})$, $L(\vec{q}) \in \mathbb{LA}_n(J)_m$, $\vec{p} = (p_0, \cdots, p_{n-1})$, $\vec{q} = (q_0, \cdots, q_{n-1})$ we put $L(\vec{p}) \circ_m L(\vec{q}) = L(\vec{u})$, $\vec{u} = (u_0, \cdots, u_{n-1})$, where

$$u_k(x) = p_m(x)q_k(x) + (1 - \delta_{km})p_k(x),\ x \in J$$

and $L(\vec{p}) \leq L(\vec{q})$ whenever $p_k(x) \leq q_k(x)$, $k \in \mathbb{N}_0(n)$, $p_m(x) = q_m(x)$, $x \in J$. Evidently, $(\mathbb{LA}_n(J)_m, \leq_m)$ is an ordered set.

In paper [11, 13] there is presented the sketch of the proof of the following lemma:

Lemma 2. *The triad* $(\mathbb{LA}_n(J)_m, \circ_m, \leq_m)$ *is an ordered (noncommutative) group.*

3. Group of artificial neurons

In what follows we will construct a group and hypergroup of artificial neurons using the above mentioned approach. So, recall the well-known mathematical description of a formal neuron:

Let $F : \mathbb{R} \to \mathbb{R}$ be a transfer function. Then the action of a neuron can be expressed as this model:

$$y(k) = F\left(\sum_{i=1}^{n} w_i(k)x_i(k) + b\right),$$

where $x_i(k)$ is input value in discrete time k where i goes from 0 to n, $w_i(k)$ is weight value in discrete time where i goes from 0 to m, b is bias, $y_i(k)$ is output value in discrete time k.

Notice that in some very special cases the transfer function F can be also linear. Transfer function defines the properties of artificial neuron and can be any mathematical function. Usually it is chosen on the basis of problem that artificial neuron (artificial neural network) needs to solve and in most cases it is taken (as mentioned above) from the following set of functions: step function, linear function and non-linear (sigmoid) function.

In what follows we will consider a certain generalization of classical artificial neurons mentioned above consisting in such a way that inputs x_i and weight w_i will be functions of an argument t belonging into a linearly ordered (tempus) set T with the least element 0. As the index set we use the set $\mathbb{C}(J)$ of all continuous functions defined on an open interval $J \subset \mathbb{R}$. So, denote by W the set of all non-negative functions $w : T \to \mathbb{R}$ forming a subsemiring of the ring of all real functions of one real variable $x : \mathbb{R} \to \mathbb{R}$. Denote by $Ne(\vec{w}_r) = Ne(w_{r1}, \ldots, w_{rn})$ for $r \in \mathbb{C}(J)$, $n \in \mathbb{N}$ the mapping

$$y_r(t) = \sum_{k=1}^{n} w_{r,k}(t)x_{r,k}(t) + b_r,$$

which will be called the artificial neuron with the bias $b_r \in \mathbb{R}$. By $\mathbb{AN}(T)$ we denote the collection of all such artificial neurons.

Neurons are usually denoted by capital letters X, Y or X_i, Y_i, nevertheless we use also notion $Ne(\vec{w})$, where $\vec{w} = (w_1, \ldots, w_n)$ is the vector of weights.

We suppose - for the sake of simplicity - that transfer functions (activation functions) φ, σ(or f) are the same for all neurons from the collection $\mathbb{AN}(T)$ or the role of this function plays the identity function $f(y) = y$.

Now, similarly as in the case of the collection of linear differential operators above, we will construct a group and hypergroup of artificial neurons.

Denote by δ_{ij} the so called Kronecker delta, $i, j \in \mathbb{N}$, i.e. $\delta_{ii} = \delta_{jj} = 1$ and $\delta_{ij} = 0$, whenever $i \neq j$.

Suppose $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in \mathbb{AN}(T)$, $r, s \in \mathbb{C}(J)$, $\vec{w}_r = (w_{r,1}, \ldots, w_{r,n})$, $\vec{w}_s = (w_{s,1}, \ldots, w_{s,n}), n \in \mathbb{N}$.

Let $m \in \mathbb{N}$, $1 \leq m \leq n$ be such an integer that $w_{r,m} > 0$. We define

$$Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) = Ne(\vec{w}_u),$$

where

$$\vec{w}_u = (w_{u,1}, \ldots, w_{u,n}) = (w_{u,1}(t), \ldots, w_{u,n}(t)),$$

$$\vec{w}_{u,k}(t) = w_{r,m}(t)w_{s,k}(t) + (1 - \delta_{m,k})w_{r,k}(t), t \in T$$

and, of course, the neuron $Ne(\vec{w}_u)$ is defined as the mapping:

$$y_u(t) = \sum_{k=1}^{n} w_k(t)x_k(t) + b_u,\ t \in T,\ b_u = b_r b_s.$$

Further for a pair $Ne(\vec{w}_r)$, $Ne(\vec{w}_s)$ of neurons from $\mathbb{AN}(T)$ we put $Ne(\vec{w}_r) \leq_m Ne(\vec{w}_s), \vec{w}_r{=}(w_{r,1}(t), \ldots, w_{r,n}(t)), \vec{w}_s{=}(w_{s,1}(t), \ldots, w_{s,n}(t))$ if $w_{r,k}(t) \leq w_{s,k}(t)$, $k \in \mathbb{N}$, $k \neq m$ and $w_{r,m}(t) = w_{s,m}(t)$, $t \in T$ and with the same bias. Evidently $(\mathbb{AN}(T), \leq_m)$ is an ordered set. A relationship (compatibility) of the binary operation "$\cdot$" and the ordering $\leq_m$ on $\mathbb{AN}(T)$ is given by this assertion analogical to the above one.

4. Solvable groups

Lemma 4.1. *The triad $(\mathbb{AN}(T), \cdot_m, \leq_m)$ (algebraic structure with an ordering) is a non-commutative ordered group.*

The sketch of the proof is contained in [13].

Denoting

$$\mathbb{AN}(T)_m = \{Ne(\vec{w}); \vec{w} = (w_1, \ldots, w_n),\ w_k \in \mathbb{C}(T),\ k = 1, \ldots, n,\ w_m(t) \equiv 1\},$$

we get the following assertion:

Proposition 4.1. *Let $T = [0, t_0) \subset \mathbb{R}$, $t_0 \in \mathbb{R} \cup \{\infty\}$. Then for any positive integer $n \in \mathbb{N}$, $n \geq 2$ and for any integer m such that $1 \leq m \leq n$ the semigroup $(\mathbb{AN}_1(T)_m, \cdot_m)$ is an invariant subgroup of the group $(\mathbb{AN}(T)_m, \cdot_m)$.*

The sketch of the proof is contained in [13].

Now recall some basic concepts from the theory of group series which will be needed in the following considerations. For more details see e.g. [4, 12, 13, 16, 17, 19]. A sequence of subgroups $H_i (i = 0, 1, ..., n)$ of a group G such that

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_{n-1} \lhd H_n = G \tag{1}$$

is said to be a subnormal series of the group G, the corresponding factor-groups $H_i/H_{i-1} (i = 1, 2, ..., n)$ are termed factors of the series $(\mathcal{H})$.

Definition 4.1 ([4, 11, 12, 19])**.** *If G is a group then its subnormal series*

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_{n-1} \lhd H_n = G$$

is said to be solvable, if all factors $H_i/H_{i-1} (i = 1, 2, ..., n)$ are commutative thus they are abelian groups. If a group G possesses at least one solvable subnormal series then G is called a solvable group.

Remark. Evidently, an arbitrary abelian group is solvable, however there exist solvable groups which are not abelian. In the literature one can find well-known results concerning solvable groups. Let us mention some of them: First of all it is famous Feit-Thompson Theorem or odd order Theorem which states that every finite group of odd order is solvable. It was proved by Walter Feit and John Griggs Thompson (1962, 1963)-[16, 17]. Further, every finite group of order $|G| \leq 100$ except 60 is solvable. (For this information and for other details both authors are very obliged to Professor Václav Havel.) Any p-group G, i.e. $|G| = p^n$ for some prime number p is solvable. If $K \lhd G$, then the group G is solvable if and only if both groups K and G/K are solvable. A group is solvable if and only if it possesses a subnormal series all factors of which are solvable. In particular a direct product of a finite number of solvable groups is a solvable group. Of course, any finite group G of the degree 5 and higher is not solvable. Other important information can be found in the literature [4, 12, 13, 16, 17, 19]. Recall that a subgroup G' of a group G generated by the set of commutators

$$[a, b] = a^{-1}b^{-1}ab$$

of all pairs $[a, b] \in G \times G$ is called the commutant of the group G. The commutant G'' of the group G' is called the second commutant of the group G. The usual notation is $G' = G^{(1)}$, $G'' = G^{(2)}$, etc. In general $G^{(n+1)} = (G^{(n)})'$. The commutant $G^{(n)}$ is also termed as the n-th derivative of the group G and the chain of commutants of the group G

$$G = G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \dots$$

is also called a derived chain of the group G.

Theorem 4.1 ([11], [30], Theorem 5., p.117, 2)**.** *Let G be a group. The following conditions are equivalent:*

(i) The group G is solvable.

(ii) There exists a positive integer n such that $G^{(n)} = 1$.

(iii) There exists a subnormal series of the group G, which is solvable.

Now we can use reverse approach. For reasons that will be apparent from the next interpretation, we will now introduce the following designation:

Example. We will present a concrete example. Let's name the structure $[\mathbb{C}(J)]^3 = \mathbb{C}(J)\times\mathbb{C}(J)\times\mathbb{C}(J)$ as a functional cube, then denote group $([\mathbb{C}(J)]_2^3, \circ_2)$, with elements $\vec{p}(x) \in [\mathbb{C}(J)]^3$, $\vec{p}(x) = (p_0(x), p_1(x), p_2(x))$, $\vec{q}(x) = (q_0(x), q_1(x), q_2(x))$, then

$$[\mathbb{C}(J)_2]^3 = \{\vec{p}(x) = (p_0(x), p_1(x), p_2(x)); p_k \in \mathbb{C}(J), p_2(x) \neq 0, x \in J\},$$

Define a binary operation $\circ_2 : [\mathbb{C}(J)_2]^3 \times [\mathbb{C}(J)_2]^3 \to [\mathbb{C}(J)_2]^3$ by:

$$\vec{p}(x) \circ_2 \vec{q}(x) = (p_0(x), p_1(x), p_2(x)) \circ_2 (q_0(x), q_1(x), q_2(x))$$

$$= \vec{w}(p_2(x)q_0(x) + p_0(x), p_2(x)q_1(x) + p_1(x), p_2(x)q_2(x)),$$

$$\text{then } (\vec{p}(x))^{-1} = (p_0(x), p_1(x), p_2(x))^{-1} = \left(-\frac{p_0(x)}{p_2(x)}, -\frac{p_1(x)}{p_2(x)}, \frac{1}{p_2(x)}\right) = \vec{v}(x).$$

These formulation with analogy of the operators have sufficiently general form showing the way to solving of the above mentioned problem in quite generality. Nevertheless, solving of that problem (i.e. the question of solvability of the group $(\mathbb{C}(J)_n, \circ_m)$ seems to be open up to now.

By the above considerations we have that $[\mathbb{C}(J)_2, \circ_2]^3$ is a non-commutative group with the neural element $\vec{e}(x) = (0, 0, 1)$ (here $\vec{e}(x) \circ_2 \vec{p}(x) = \vec{p}(x)$ for any element $\vec{p}(x) \in \mathbb{C}(J)_2]^3$. The inverse element to $\vec{p}(x) = (p_0(x), p_1(x), p_2(x))$ is given above. It is obvious that

$$\vec{p}(x) \circ_2 (\vec{p}(x))^{-1} = (p_0(x), p_1(x), p_2(x)) \circ_2 \left(-\frac{p_0(x)}{p_2(x)}, -\frac{p_1(x)}{p_2(x)}, \frac{1}{p_2(x)}\right)$$

$$= \left(-\frac{p_2(x)p_0(x)}{p_2(x)} + p_0(x), -\frac{p_2(x)p_1(x)}{p_2(x)} + p_1(x), \frac{p_2(x)}{p_2(x)}\right) = (0, 0, 1) = \vec{e}(x).$$

and similarly we get the neutral element from the product of the above vectors from the functional cube in the opposite order. Denote

$$[\mathbb{C}_{2C}(J)_2]^3 = \{(p_0(x), p_1(x), r); p_0, p_1 \in \mathbb{C}(J)\}, r \in R, r \neq 0$$

and

$$[\mathbb{C}_{11}(J)_2]^3 = \{(p_0(x), p_1(x), 1); p_0, p_1 \in \mathbb{C}(J)\}.$$

Then, we obtain

Theorem 4.2. *Let $J \subseteq \mathbb{R}$ be an open interval. The grupoids $([\mathbb{C}_{21}(J)_2]^3, \circ_2)$, $([\mathbb{C}_{2C}(J)_2]^3, \circ_2)$ are subgroups of the group $([\mathbb{C}(J)_2]^3, \circ_2)$ - the first from them is commutative - and we have*

$$([\mathbb{C}_{21}(J)_2]^3, \circ_2) \lhd ([\mathbb{C}_{2C}(J)_2]^3, \circ_2) \lhd ([\mathbb{C}(J)_2]^3, \circ_2).$$

Proof of Theorem 4.2. Evidently

$$(0,0,1) \in [\mathbb{C}_{21}(J)_2]^3 \subseteq [\mathbb{C}_{2C}(J)_2]^3.$$

If $(p_0(x), p_1(x), r)$, $(q_0(x), q_1(x), s) \in [\mathbb{C}_{2C}(J)_2, \circ_2]^3$ (with $r, s \in \mathbb{R}, r \neq 0 \neq s$) are vectors of continuous functions, then we have:

$$(p_0(x), p_1(x), r) \circ_2 (q_0(x), q_1(x), s)^{-1} = (p_0(x), p_1(x), r) \circ_2 \left(-\frac{q_0(x)}{s}, -\frac{q_1(x)}{s}, \frac{1}{s}\right)$$

$$= \left(p_0(x) - \frac{r}{s}q_0(x), p_1(x) - \frac{r}{s}q_1(x), \frac{r}{s}\right) \in [\mathbb{C}_{2C}(J)_2]^3.$$

Similarly

$$(p_0(x), p_1(x), 1) \circ_2 (q_0(x), q_1(x), 1) \in [\mathbb{C}_{2C}(J)_2]^3.$$

Thus $([\mathbb{C}_{21}(J)_2]^3, \circ_2)$, $([\mathbb{C}_{2C}(J)_2]^3, \circ_2)$ are subgroups of the group $([\mathbb{C}(J)_2]^3, \circ_2)$. Further,

$$(p_0(x), p_1(x), 1) \circ_2 (q_0(x), q_1(x), 1) = (p_0(x) + q_0(x), p_1(x) + q_1(x), 1)$$

$$= (q_0(x), q_1(x), 1) \circ_2 (p_0(x), p_1(x), 1),$$

for arbitrary vectors of continuous functions $(p_0(x), p_1(x), 1)$, $(q_0(x), q_1(x), 1) \in [\mathbb{C}_{21}(J)_2]^3$, hence the group $([\mathbb{C}_{11}(J)_2]^3, \circ_2)$ is commutative.

Now, for any vectors $(p_0(x), p_1(x), p_2(x)) \in [\mathbb{C}(J)_2]^3$ and $(q_0(x), q_1(x), r) \in [\mathbb{C}_{2C}(J)_2]^3$ we obtain:

$$\begin{aligned}
&(p_0(x), p_1(x), p_2(x))^{-1} \circ_2 (q_0(x), q_1(x), r) \circ_2 (p_0(x), p_1(x), p_2(x)) \\
&= \left(-\frac{p_0(x)}{p_2(x)}, -\frac{p_1(x)}{p_2(x)}, \frac{1}{p_2(x)}\right) \circ_2 (rp_0(x) + q_0(x), rp_1(x) + q_1(x), rp_2(x)) \\
&= (\varphi_0(x), \varphi_1(x), r) \text{ where } \varphi_k(x) = r(p_k(x) + q_k(x)),\ k = 0, 1.
\end{aligned}$$

Since $(\varphi_0(x), \varphi_1(x), r) \in [\mathbb{C}_{2C}(J)_2]^3$, we have

$$(p_0(x), p_1(x), p_2(x))^{-1} \circ_2 [\mathbb{C}_{2C}(J)_2]^3 \circ_2 (p_0(x), p_1(x), p_2(x)) \subseteq [\mathbb{C}_{2C}(J)_2]^3,$$

consequently $([\mathbb{C}_{2C}(J)_2]^3, \circ_2)$ is a normal subgroup of the group $([\mathbb{C}(J)_2]^3, \circ_2))$.

In a similar way we obtain that

$$(p_0(x), p_1(x), p_2(x))^{-1} \circ_2 [\mathbb{C}_{21}(J)_2]^3 \circ_2 (p_0(x), p_1(x), p_2(x)) \subseteq [\mathbb{C}_{21}(J)_2]^3$$

for an arbitrary vector $(p_0(x), p_1(x), r) \in [\mathbb{C}_{2C}(J)_2]^3$. Consequently

□ $$([\mathbb{C}_{21}(J)_2]^3, \circ_2) \lhd ([\mathbb{C}_{2C}(J)_2]^3, \circ_2) \lhd ([\mathbb{C}(J)_2]^3, \circ_2).$$

Theorem 4.3. *Let $J \subseteq \mathbb{R}$ be an open interval. The group $([\mathbb{C}(J)_2]^3, \circ_2)$ of all vectors of continuos functions is solvable.*

Proof of Theorem 4.3. Denote $G = ([\mathbb{C}(J)_2]^3, \circ_2)$. The first derivate $G^{'}$, i.e. the first commutant of the group G is its subgroup generated by all commutators of the form

$$(p_0(x), p_1(x), p_2(x))^{-1} \circ_2 (q_0(x), q_1(x), q_2(x))^{-1}$$
$$\circ_2 (p_0(x), p_1(x), p_2(x)) \circ_2 (q_0(x), q_1(x), q_2(x)),$$

where $(p_0(x), p_1(x), p_2(x)), (q_0(x), q_1(x), q_2(x)) \in [\mathbb{C}(J)_2]^3$. Since the above commutator

$$\begin{aligned}
&(\vec{p}(x))^{-1} \circ_2 (\vec{q}(x))^{-1} \circ_2 \vec{p}(x) \circ_2 \vec{q}(x) \\
&= \left(-\frac{p_0(x)q_2(x) + p_0(x)}{p_2(x)q_2(x)}, -\frac{p_1(x)q_2(x) + q_1(x)}{p_2(x)q_2(x)}, \frac{1}{p_2(x)q_2(x)} \right) \\
&\circ_2 (p_2(x)q_0(x) + p_0(x), p_2(x)q_1(x) + p_1(x), p_2(x)q_2(x)) \\
&= \left(\frac{q_0(x)(p_2(x) - 1) + p_0(x)(1 - q_2(x))}{p_2(x)q_2(x)}, \frac{q_1(x)(p_2(x) - 1) + p_1(x)(1 - q_2(x))}{p_2(x)q_2(x)}, 1 \right)
\end{aligned}$$

belongs into the group $([\mathbb{C}_{21}(J)_2]^3, \circ_2)$, we have that $G^{'}$ is a subgroup of the last group above.

Consider an arbitrary pair of vectors $(u_0(x), u_1(x), 1)$, $(v_0(x), v_1(x), 1) \in ([\mathbb{C}_{21}(J)_2]^3, \circ_2)$. The second derivative of G, i.e. the second commutant $G^{''}$ of the group $G = ([\mathbb{C}(J)_2]^3, \circ_2)$ is its subgroup generated by the set of all commutators

$$(u_0(x), u_1(x), 1)^{-1} \circ_2 (v_0(x), v_1(x), 1)^{-1} \circ_2 (u_0(x), u_1(x), 1) \circ_2 (v_0(x), v_1(x), 1)$$

$$= (-u_0(x), -u_1(x), 1) \circ_2 (-v_0(x), -v_1(x), 1) \circ_2 (u_0(x) + v_0(x), u_1(x) + v_1(x), 1) =$$

$$(-(u_0(x) + v_0(x)), -(u_1(x) + v_1(x)), 1) \circ_2 (u_0(x) + v_0(x), u_1(x) + v_1(x), 1) =$$

$$= (0, 0, 1)$$

hence $G^{''} = \{(0, 0, 1)\}$ which is trivial subgroup (formed by the unit only) of the group $G = ([\mathbb{C}(J)_2]^3, \circ_2)$. Consequently

$$\{(0, 0, 1)\} = G^{''} \subset G^{'} \subset G^{(0)} = G,$$

therefore the group $([\mathbb{C}(J)_2]^3, \circ_2)$. is solvable.

Notice that to decide whether the group $G^{'}$ is equal to $([\mathbb{C}_{11}(J)_2]^3, \circ_2)$ or not needs some other calculations. This consists in the question whether an arbitrary continuous function $f \in \mathbb{C}$ can be expressed in the form of coefficients of the operator

$$L^{-1}(\vec{p}(x)) \circ_2 L^{-1}(\vec{q}(x)) \circ_2 L(\vec{p}(x)) \circ_2 L(\vec{q}(x)).$$

□

Theorem 4.4. *The above defined group* $(\mathbb{AN}(T), \bullet_m)$ *of artificial neurons is solvable.*

Proof of Theorem 4.4. Let us denote $G = (\mathbb{AN}(T), \bullet_m)$. The first derivative G' of the group G, which means the first commutant of the group G is its subgroup generated by all commutators of the form

$$Ne^{-1}(\vec{w}_r) \bullet_m Ne^{-1}(\vec{w}_s) \bullet_m Ne(\vec{w}_r) \bullet_m Ne(\vec{w}_s),$$

where $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in \mathbb{AN}(T)$. Since the mentioned commutator is of the form

$$\begin{aligned}
&Ne\left(-\frac{w_{r,1}}{w_{r,m}}, \ldots, \frac{1}{w_{r,m}}, \ldots, -\frac{w_{r,n}}{w_{r,m}}\right) \bullet_m Ne\left(-\frac{w_{s,1}}{w_{s,m}}, \ldots, \frac{1}{w_{s,m}}, \ldots, -\frac{w_{s,n}}{w_{s,m}}\right) \bullet_m \\
&Ne\left(w_{r,m}w_{s,1} + w_{r,1}, \ldots, w_{r,m}w_{s,m}, \ldots, w_{r,m}w_{s,n} + w_{r,n}\right) \\
&= Ne\left(-\frac{w_{s,m}w_{r,1} + w_{s,1}}{w_{r,m}w_{s,m}}, \ldots, \frac{1}{w_{r,m}w_{s,m}}, \ldots, -\frac{w_{s,m}w_{r,n} + w_{s,n}}{w_{r,m}w_{s,m}}\right) \bullet_m \\
&Ne\left(w_{r,m}w_{s,1} + w_{r,1}, \ldots, w_{r,m}w_{s,m}, \ldots, w_{r,m}w_{s,n} + w_{r,n}\right) \\
&= Ne\left(\frac{(w_{r,m}-1)w_{s,1} + (1-w_{s,m})w_{r,1}}{w_{r,m}w_{s,m}}, \ldots, 1,\right. \\
&\qquad \left. \ldots, \frac{(w_{r,m}-1)w_{s,n} + (1-w_{s,m})w_{r,n}}{w_{r,m}w_{s,m}}\right) \\
&\in \mathbb{AN}_1(T)_m.
\end{aligned}$$

We obtain that the subgroups of the group $(\mathbb{AN}(T), \bullet_m)$ generated by all commutators of the above described form is just the group $(\mathbb{AN}_1(T)_m, \bullet_m)$. Indeed, suppose that k is a positive integer, $1 \leq k \leq n$, $k \neq m$ and $f \in \mathbb{C}(T)$ is an arbitrary function then choosing $w_{r,m}(t) = 1$, $w_{s,m}(t) = t^2 + 1$ and $w_{r,k}(t) = -\frac{f(t)}{t^2}(t^2+1)$, we obtain

$$\begin{aligned}
&\frac{1}{w_{r,m}(t)w_{s,m}(t)}\left((w_{r,m}(t)-1)w_{s,k}(t) + (1 - w_{s,m}(t)w_{r,k}(t)\right) \\
&= \frac{-t^2 w_{r,k}(t)}{t^2+1} = f(t),\ t \in T.
\end{aligned}$$

Thus an arbitrary continuous weight from the above used neuron can be expressed in the described form. Therefore $G' = (\mathbb{AN}_i(T)_m, \bullet_m)$. Furthermore consider an arbitrary pair of neurons

$$Ne(w_{u,1}, \ldots, 1, \ldots, w_{u,n}),\ Ne(w_{v,1}, \ldots, 1, \ldots, w_{v,n}) \in \mathbb{AN}_1(T)_m.$$

Under a consideration which is similar as the above one we have:

$$\begin{aligned}
&Ne^{-1}(w_{u,1}, \ldots, 1, \ldots, w_{u,n}) \bullet_m Ne^{-1}(w_{v,1}, \ldots, 1, \ldots, w_{v,n}) \bullet_m \\
&Ne(w_{u,1}, \ldots, 1, \ldots, w_{u,n}) \bullet_m Ne(w_{v,1}, \ldots, 1, \ldots, w_{v,n}) \\
&= Ne(-w_{u,1}, \ldots, 1, \ldots, -w_{u,n}) \bullet_m Ne(-w_{v,1}, \ldots, 1, \ldots, -w_{v,n}) \bullet_m
\end{aligned}$$

$$Ne(w_{u,1}+w_{v,1},\dots,1,\dots,w_{u,n}+w_{v,n})$$
$$= Ne(-(w_{u,1}+w_{v,1}),\dots,1,\dots,-(w_{u,n}+w_{v,n}))\bullet_m$$
$$Ne(w_{u,1}+w_{v,1},\dots,1,\dots,w_{u,n}+w_{v,n}) = Ne(0,\dots,0,1,0,\dots,0),$$

thus the second derivative G'' is the trivial group, which contains only the identity.

Therefore, we have obtained as the result

$$\{Ne(0,\dots,0,1,0,\dots,0)\} = G'' \subseteq G' = G^{(0)} = G = (\mathbb{AN}(T),\bullet_m),$$

hence the group $(\mathbb{AN}(T),\bullet_m)$ is solvable.

5. Conclusion

The treated group of artificial (formal) neurons is constructed on formal similarity with the group of ordinary linear n-th order operators which form left hand sides of corresponding linear homogeneous differential equations. In particular, using the theory of series of groups motivated by the Galois theory of solvability of algebraic equations and the modern theory of extensions of fields, we have obtained that the considered group of artificial neurons is solvable. It is to be noted that there exists the large and deep Galois theory of linear differential equations-contained e.g. in the book [31] and in other papers of authors of the just mentioned monograph, but results obtained in our contribution seems to be independent on the mentioned theory. Moreover, in paper [13] there is defined (so called) a differential time varying neuron, which can serve for construction of the solution space of certain differential equation. Investigation of its properties seems to be the task of a further analysis.

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Reciprocal sums of triple products of general second order recursion

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Abstract. By applying the method of generating function, the purpose of this paper is to give several summation of reciprocals related to triple product of general second order recurrence $\{W_{rn}\}$ for arbitrary positive integer r. As applications, some identities involving Fibonacci, Lucas numbers are obtained.
Keywords: second order recurrence, Fibonacci number, Lucas number, triple product, reciprocal.

1. Introduction

In the notation of Horadam [4], write

$$W_n = W_n(a, b; P, Q)$$

so that

$$W_n = PW_{n-1} - QW_{n-2}, \quad (W_0 = a,\ W_1 = b,\ n \geq 2), \tag{1}$$

where a, b, P and Q are integers, with $PQ \neq 0$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. Then it is easily to obtain the Binet formula [4]:

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

where $\alpha = \frac{P+\sqrt{P^2-4Q}}{2}$, $\beta = \frac{P-\sqrt{P^2-4Q}}{2}$, $A = b - \beta a$ and $B = b - \alpha a$. In particular, if $U_n = W_n(0, 1; P, Q), V_n = W_n(2, P; P, Q)$, then $\alpha^n - \beta^n = (\alpha - \beta)U_n, \alpha^n + \beta^n = V_n$.

In [1], Andre-Jeannin obtained the following series identities:

$$\sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}}, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left(\sum_{n=1}^{k} \frac{W_{n+1}}{W_n} - k\alpha \right), \tag{4}$$

where $P > 0$ and k and m are nonnegative integers. (3) and (4) are given by Good [3] with the case $Q = -1$. Brousseau [2] proved (4) for $W_n = F_n$. The Fibonacci sequence is a classical sequence generated from the question of the rabbits propagation. From the viewpoint of the rabbits propagationthe classical Fibonacci sequence is generalized in many forms and some general forms are obtained. If integers $F_0 = 0, F_1 = 1$, and $W_n(0, 1; 1, -1) = F_n = F_{n-1} + F_{n-2}, n \geq 2$ is called generalized Fibonacci number.

In [7], Mansour used the generating function techniques to get several summation of reciprocals related to generalized Fibonacci numbers.

Regarding taking l-th powers of terms in the sums, Xi [9] generalized the results of (3) and (4). For example, he derived the following infinitive reciprocal sums:

$$\begin{aligned}\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \Big[&(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \\ &\times\beta^{2(n-k)+m}\Big]^{l-1-i} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m \\ &-(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\Big]^i \\ = \ &\frac{(P^2 - 4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l}. \end{aligned} \tag{5}$$

In [6], Kilic and Irmak obtained the generalize the result of (5) regarding reciprocal sums of l-th powers of the terms with indices.

In [8], Xi obtained the summation of reciprocals related to of triple product of generalized Fibonacci sequences. By applying the method of generating function, the purpose of this paper is to give several summation of reciprocals related to triple product of general second order recurrence $\{W_{rn}\}$ for arbitrary positive integer r. As applications, some identities involving Fibonacci, Lucas numbers are obtained.

Throughout the paper, we assume that r, k, l, m and t are five positive integers with $t \geq k$.

2. Main result

In this section we consider both finite and infinite reciprocal sums of square of triple product of r-consecutive terms of sequence $\{W_n\}$. First of all, we give the following general second order recurrence $\{W_{rn}\}$ of [6] which are to be used later.

Let W_n be the nth term of sequence $\{W_n\}$ (see [5]) . Then for $n, r > 0$,

$$W_{rn} = Y_r W_{r(n-1)} - Z_r W_{r(n-2)} \tag{6}$$

where $Y_r = \alpha^r + \beta^r$ and $Z_r = (\alpha\beta)^r$.

Theorem 2.1. *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+1)}W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}}\right], \tag{7}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. Let $f(x) = \sum_{n=k}^{\infty} W_{rn}x^n$, $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$. From (6) we have

$$f(x) - W_{rk}x^k - W_{r(k+1)}x^{k+1} = Y_r x[f(x) - W_{rk}x^k] - Z_r x^2 f(x).$$

Hence, the following generating function is obtained:

$$f(x) = x^k \frac{W_{rk} + x[W_{r(k+1)} - Y_r W_{rk}]}{1 - Y_r x + Z_r x^2}.$$

Since $1 - Y_r x + Z_r x^2 = (1 - \alpha^r x)(1 - \beta^r x)$, we can decompose $f(x)$ into partial fractions:

$$f(x) = \frac{x^k}{\alpha^r - \beta^r}\left(\frac{\mu_{rk}}{1 - \alpha^r x} - \frac{\nu_{rk}}{1 - \beta^r x}\right).$$

Comparing the coefficients of x^n in both sides of above equation, we obtain that

$$W_{rn} = \frac{\mu_{rk}}{\alpha^r - \beta^r}\alpha^{r(n-k)} - \frac{\nu_{rk}}{\alpha^r - \beta^r}\beta^{r(n-k)}. \tag{8}$$

Let

$$T_{rn} = \frac{\beta^{r(n-k)}}{W_{rn}} = \frac{\beta^{r(n-k)}}{\frac{\mu_{rk}}{\alpha^r - \beta^r}\alpha^{r(n-k)} - \frac{\nu_{rk}}{\alpha^r - \beta^r}\beta^{r(n-k)}}. \tag{9}$$

Then

$$\begin{aligned} T_{rn} - T_{r(n+1)} &= \frac{\beta^{r(n-k)}}{W_{rn}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} \\ &= \frac{\beta^{r(n-k)}[W_{r(n+1)} - \beta^r W_{rn}]}{W_{rn}W_{r(n+1)}} \\ &= \frac{\mu_{rk}Q^{r(n-k)}}{W_{rn}W_{r(n+1)}} \end{aligned}$$

and so

$$T_{r(n+1)} - T_{r(n+2)} = \frac{\mu_{rk}Q^{r(n+1-k)}}{W_{r(n+1)}W_{r(n+2)}}.$$

Hence, we have

$$\begin{aligned}
&T_{rn} - T_{r(n+1)} - T_{r(n+1)} + T_{r(n+2)} \\
&= \frac{\beta^{r(n-k)}}{W_{rn}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} + \frac{\beta^{r(n+2-k)}}{W_{r(n+2)}} \\
&= \frac{\mu_{rk} Q^{r(n-k)}}{W_{rn} W_{r(n+1)}} - \frac{\mu_{rk} Q^{r(n+1-k)}}{W_{r(n+1)} W_{r(n+2)}} \\
&= \frac{\mu_{rk} Q^{r(n-k)}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] \\
&\qquad = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}}\right].
\end{aligned}$$

The proof of the theorem is completed. □

Theorem 2.2. *Let $P > 0$. Then*

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] \\
&\qquad = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}}\right], \qquad (10)
\end{aligned}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. By (7), we have

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] \\
&\qquad = \frac{Q^{rk}}{\mu_{rk}} \lim_{t\to\infty} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}}\right],
\end{aligned}$$

where the limiting process has been justified by

$$\left|\frac{\alpha}{\beta}\right| = \left|\frac{P + \sqrt{P^2 - 4Q}}{P - \sqrt{P^2 - 4Q}}\right| > 1, \quad \text{for} \quad P > 0,$$

and

$$\begin{aligned}
\lim_{t\to\infty} T_{rt} &= \lim_{t\to\infty} \frac{\beta^{r(t-k)}}{W_{rt}} \\
&= \lim_{t\to\infty} \frac{1}{\frac{W_{r(k+1)} - W_{rk}\beta^r}{\alpha^r - \beta^r} \left(\frac{\alpha}{\beta}\right)^{r(t-k)} - \frac{W_{r(k+1)} - W_{rk}\alpha^r}{\alpha^r - \beta^r}} = 0. \qquad (11)
\end{aligned}$$

The proof of the theorem is completed. □

Theorem 2.3. *Let $P > 0$. Then*

$$\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_{rk}\alpha^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rl)}\beta^{(rm)} \right] \right. \\
&\quad \left. + \nu_{rk}\beta^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)} \right] \right\} \\
&= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \left\{ \sum_{i=0}^{m-1} \left[\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right] \right. \\
&\quad \left. - \sum_{i=0}^{l-1} \left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} \right] \right\}, \qquad (12)
\end{aligned}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. From (9) and (8), we have

$$T_{rn} - T_{r(n+m)} = \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}},$$

$$T_{r(n+m)} - T_{r(n+m+l)} = \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}}.$$

Hence,

$$\begin{aligned}
&T_{rn} - T_{r(n+m)} - T_{r(n+m)} + T_{r(n+m+l)} \\
&\quad = \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}} - \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}} \\
&\quad = \frac{\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)^2 W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_{rk}\alpha^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} \right. \right. \\
&\quad \left. \left. - 2\alpha^{(rl)}\beta^{(rm)} \right] + \nu_{rk}\beta^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)} \right] \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_{rk}\alpha^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rl)}\beta^{(rm)} \right] \right. \\
&\quad \left. + \nu_{rk}\beta^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)} \right] \right\} \\
&= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \left\{ \sum_{i=0}^{m-1} \left[\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right] \right. \\
&\quad \left. - \sum_{i=0}^{l-1} \left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} \right] \right\}.
\end{aligned}$$

The proof of the theorem is completed. □

Theorem 2.4. *Let $P > 0$. Then*

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_{rk}\alpha^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm} \right] \right. \\
&\left. + \nu_{rk}\beta^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl} \right] \right\} \\
&= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \left[\sum_{i=0}^{m-1} \frac{\beta^{ri}}{W_{r(k+i)}} - \sum_{i=0}^{l-1} \frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} \right], \qquad (13)
\end{aligned}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. By (12) and (11), we have

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_{rk}\alpha^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm} \right] \right. \\
&\left. + \nu_{rk}\beta^{r(n+m-k)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl} \right] \right\} \\
&= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \lim_{t\to\infty} \left\{ \sum_{i=0}^{m-1} \left[\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right] \right. \\
&\left. - \sum_{i=0}^{l-1} \left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} \right] \right\}.
\end{aligned}$$

The theorem is proved. □

Corollary 2.5. *Let $P > 0$. Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left\{ \mu_r\alpha^{r(n+m-1)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm} \right] \right. \\
&\left. + \nu_r\beta^{r(n+m-1)} \left[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl} \right] \right\} \\
&= \frac{(\alpha^r - \beta^r)^2 Q^r}{\mu_r} \left[\sum_{i=0}^{m-1} \frac{\beta^{ri}}{W_{r(1+i)}} - \sum_{i=0}^{l-1} \frac{\beta^{r(m+i)}}{W_{r(1+m+i)}} \right],
\end{aligned}$$

where $\mu_r = W_{2r} - W_r\beta^r$, $\nu_r = W_{2r} - W_r\alpha^r$.

Proof. Take $k = 1$ in the identity (13), respectively. □

3. Some applications

In this section we can obtain some interesting identities involving Fibonacci, Lucas numbers by taking special values for a, b, P and Q.

3.1 Fibonacci numbers

We note that, $W_n(0,1;1,-1) = F_n$, the Fibonacci number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{rn}}{F_{rn}F_{r(n+m)}F_{r(n+m+l)}} &\left\{\alpha^{r(n+m)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\right]\right.\\ &\left.+\beta^{r(n+m)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\right]\right\}\\ &= \frac{5(-1)^r}{\alpha^r}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{F_{r(1+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{F_{r(1+m+i)}}\right].\end{aligned}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+4}F_{2n+8}}\left(\alpha^{2n+4}+\beta^{2n+4}\right)=\frac{151-48\sqrt{5}}{1512}.$$

3.2 Lucas numbers

We note that, $W_n(2,1;1,-1) = L_n$, the Lucas number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{rn}}{L_{rn}L_{r(n+m)}L_{r(n+m+l)}} &\left\{\alpha^{r(n+m)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\right]\right.\\ &\left.-\beta^{r(n+m)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\right]\right\}\\ &= \frac{(-1)^r}{\alpha^r}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{L_{r(1+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{L_{r(1+m+i)}}\right].\end{aligned}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}L_{2n+2}L_{2n+6}}\left[(15+\sqrt{5})\alpha^{2n+4}-(15-\sqrt{5})\beta^{2n+4}\right]=\frac{383\sqrt{5}}{5922}.$$

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On the dynamics of the generalized Hènon-Heiles of the galactic potential

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Abstract. The bifurcation of Liouville tori of a generalized Hènon-Heiles System (GHH) are determined. The phase portrait of separation functions of (GHH) are studied, and the classification of the singular points were found. Some figures are presented by using Poincaré surface section.

Keywords: Hamilton-Jacobi's equations, bifurcations of Liouville tori, topology of the level sets, momentum maps, periodic solution, elliptic functions, phase portrait, Poincaré surface-section.

1. Introduction

The integrable Hamiltonian system with n-degree of freedom possessing a periodic solutions and it has an n-invariant torus around these solution. The Hamiltonian Hènon-Heiles (HH) system [22] is one of the famous model systems which in general is not integrable. It represents the motion of a star in the rotating meridian plane of a galaxy in the neighborhood of a circular orbit or in the equatorial plane of a galaxy with axial symmetry. The problem can be integrated in the sense of Liouvill-Arnold [3] if there exist a single-valued, integral of the motion which are functionally independent and involution besides the integral of energy and the angular momentum integral. Hènon-Heiles introduced a two-dimensional axil-symmetric, potential admit a third isolating integral which is analytically and sufficiently complicated to give the trajectories which are far from trivial, this integral existed only for some values of the constant of energy, and in escape energy, the problem is not integrable.

The HH and GHH (Generalized Hènon-Heiles) has been studied in great numbers of scientific works, to get a complete picture of the motion. The Hamiltonian Toda system [30] is possessed another integral for the motion of three particles moving on a ring and, Ford et al. [17] gave the the same result nu-

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merically. The Hamiltonian can be reduced to the 2-D system by a canonical transformation [25], and the expansion of the system up to the third order terms coincided with HH system as shown in G. Contopoulos and C. Polymilis [9]. They expanded the exponential terms up to 10th order approximation. The periodic orbits of HH was studied by many authors: El-Sabaa and Sherief [11] classified and discussed nine classes of the main periodic orbits and its stability while Davies et al. [10] applied the monodromy method, for calculating the periodic trajectories, the monodromy method was computationally very efficient where the periodic orbits including a number of simple bifurcations. Ozaki and Kurosaki [27] found the periodic orbits and the bifurcation of it by calculating Poincaré surface-section and the residues of their orbits. Carrasco and Vidal [7] used small parameter in order to apply the average method of second order to proved the existence of different families of periodic solutions for some cases of parameters and characterized the stability of the family of periodic orbits. For the study of fractal HH, Barrio et al. [4] presented the appearance of different kinds of fractal structures in the paradigmatic HH Hamiltonian, and as the KAM (Kolmogrov, Arnold, Mozer) regime continues, the regular bounded region presents a fractal structure. For non integrability of HH, Holmes [24] used the Melnikov's method to prove that the all neighboring systems of HH type are non-integrable and estimate the width of the primary stochastic layer which is the cause of non-integrability.

The HH system was study too in quantum mechanics: Bixon and Jortner [5] considered the dynamics of wave packets of bound states in the nonlinear HH system, and explored the correspondence between classical and quantum dynamics. Akhiezer et al. [2] studied the chaos of charged particles in a crystal regarding HH potential. Bastida et al. [32] investigated the possibility of using oblique coordinates to determine the energy levels and wave functions of HH coupled oscillator systems.

The aim of this article is to study GHH the Painlevé property [20] for the problem is applied to show the identification of specific integrable cases. The topology of the level set is given and, the phase portrait of the separated function of the integrable cases are determined. The surface-section was introduced by Poincaré to get the regular motion in all integrable cases for different values of the constant of energy.

2. Painlevé analysis

The GHH Hamiltonian system is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + Ax^2 + By^2 + 2Dx^2y - \frac{2}{3}Cy^3). \tag{1}$$

Where the HH Hamiltonian can be written when $A = B = C = D = 1$. The GHH given a nonlinear dynamical system governed by a system of the equations:

$$\begin{aligned} &\ddot{x} + Ax + 2Dxy = 0, \\ &\ddot{y} + By - Dx^2 + Cy^2 = 0. \end{aligned} \tag{2}$$

By using Painlevé analysis, its required that a Laurent series of the solution, considering time as a complex variable, have only simple poles at all movable singularities, where these points are those that depend on the initial condition of the system. If the singular points independent to the constants of integration, then they are fixed singular points, and among these points, the branch and the essential singularities are said to be a critical points. When system satisfy Painlevé property (P-property), then they can be integrable and the solution expressed in Laurent expansion in neighborhood of the movable singular point. Many authors [21, 23, 6]have applied the Painlevé analysis to variety of the Hamiltonian dynamical system, and identified a considerable number of new integrable. Ablowitz et al. (1980) [1] announced an algorithm (ARS algorithm) to determine whether a non-linear system is integrable or not, this algorithm consists of the following steps:
The first step is to put

$$x \quad = a\tau^{\alpha}, y \quad = b\tau^{\beta}, \tau = t - t_0, \tag{3}$$

where α and β are integers and less than zero. So the system (2) becomes:

$$\begin{aligned} a\alpha(\alpha-1)\tau^{\alpha-2} &= -2Dab\tau^{\alpha+\beta}, \\ b\beta(\beta-1)\tau^{\beta-2} &= -Da^2(\tau^{2\alpha}) + Cb^2\tau^{2\beta}, \end{aligned} \tag{4}$$

and hance we can determine the leading order: we have two cases:

- *Case* (i) $\alpha = -2, a = \pm\frac{3}{D}\sqrt{2 + \frac{1}{\lambda}}, \beta = -2, b = \frac{6}{D}$.
- *Case* (ii) $\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 48\lambda}, a = arbitrary, \beta = -2, b = \frac{6}{C}$.

where $\lambda = D/C$.

The second step is determine the resonances, that is the power at which arbitrary constant of the solution can enter into the Laurent series expansion, so we have

$$\begin{aligned} x &= \pm\frac{3}{D}\sqrt{2 + 1/\lambda}t^{-2} + pt^{r-2}, \\ y &= -\frac{3}{D}t^{-2} + qt^{r-2}, \end{aligned} \tag{5}$$

where p and q are arbitrary parameters. Setting up the linear equations for p and q from the dominant balance terms in equation (2), and then the resonances are:

- for case (i)

$$r = -1, 6, \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24(1+\lambda)}, \tag{6}$$

- for case (ii)

$$r = -1, 0, 6, \pm\sqrt{1 - \frac{48}{\lambda}}. \tag{7}$$

The third step is to verified that sufficient number of arbitrary constants exist without the introduction of the movable critical points

According to [29], the arbitrary constants exist where the root $r = -1$ is represent the arbitrary of pole position, while the root $r = 0$ is associated to the arbitrary A. As a Painlevé criterion, the values of r must be integer. So, the problem is a Painlevé type and the integrable cases are:

1. $\lambda = -1, \frac{A}{B} = 1$
2. $\lambda = -\frac{1}{6}$ any A, B
3. $\lambda = \frac{1}{16}, \frac{A}{B} = \frac{1}{16}$

3. Topological analysis

In the first case, we recall that the Hamilton-Jacobi equation corresponding to the system (2) separates to u, v defined as [33]

$$x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v), \tag{8}$$

which the first case $\lambda = -1$, then p_x, p_y can be determined through the coordinates u, v as follows:

$$\begin{aligned} p_x &= \frac{1}{\sqrt{6}}(\sqrt{F_1(u) + F_2(v)}), \\ p_y &= \frac{1}{\sqrt{6}}(\sqrt{F_1(u) - F_2(v)}). \end{aligned} \tag{9}$$

Where $F_1(u)$,$F_2(v)$ denoted by the polynomials:

$$F_1(u) = f - 3h - \frac{3}{2}u^2 - u^3, \tag{10}$$

$$F_2(v) = -f - 3h - \frac{3}{2}v^2 + v^3, \tag{11}$$

and f is the constant of separation. So the Hamilton-Jacobi equations leads to

$$\frac{du}{\sqrt{F_1(u)}} - \frac{dv}{\sqrt{F_2(v)}} = \frac{1}{\sqrt{6}}\beta, \tag{12}$$

$$\frac{du}{\sqrt{F_1(u)}} + \frac{dv}{\sqrt{F_2(v)}} = \sqrt{\frac{3}{2}}(t_0 - t). \tag{13}$$

Therefore, the differential equations satisfied by u and v are

$$\sqrt{\frac{3}{2}}\int_{u_0}^{u}\frac{du}{\sqrt{F_1(u)}}=\int_{t_0}^{t}dt, \tag{14}$$

$$\sqrt{\frac{3}{2}}\int_{v_0}^{v}\frac{dv}{\sqrt{F_2(v)}}=\int_{t_0}^{t}dt. \tag{15}$$

In the second case where $\lambda=-\frac{1}{6}$, Wojciechowski [31]showed that the GHH can be separated shifted parabolic coordinates ξ,η such that

$$x^2=-4\xi\eta,\ y=\xi+\eta+(B-4A)/4, \tag{16}$$

and the Hamiltonian-Jacobi equation in this case

$$\begin{aligned}S=&-ht+\int[\frac{1}{2}(h-\frac{A(4A-B)^2}{32D})+\frac{k}{2}\xi-\frac{1}{2}D\xi^4+(\frac{3}{2}A-\frac{1}{4}B)\xi^3\\&-\frac{1}{2}(\frac{D}{2}(B(4A-B)+\frac{3}{2}(4A-B)^2))\xi^2]d\xi\\&+\int[\frac{1}{2}(h-\frac{A(4A-B)^2}{32D})-\frac{k}{2}\eta+\frac{1}{2}D\eta^4-(\frac{3}{2}A-\frac{1}{4}B)\eta^3\\&+\frac{1}{2}(\frac{D}{2}(B(4A-B)+\frac{3}{2}(4A-B)^2))\eta^2]d\eta,\end{aligned} \tag{17}$$

where h is the energy constant, k is the separation constant. And according to the equation of motion, we have the system of differential equations:

$$\frac{d\xi}{\sqrt{\xi P_1(\xi)}}-\frac{d\eta}{\sqrt{\eta P_2(\eta)}}=0, \tag{18}$$

$$\frac{\xi d\xi}{\sqrt{\xi P_1(\xi)}}+\frac{\eta d\eta}{\sqrt{\eta P_2(\eta)}}=-\frac{1}{2}dt. \tag{19}$$

these equations gives a separated functions

$$P(q)=16q^2-8(6A-B)q^3+(12A-B)(4A-B)q^2+(8h-A(4A-B)^2)q-k, \tag{20}$$

where q$\equiv(\xi,\eta)$

Fomenko [15] proposed a new approach in the qualitative theory of integrable Hamiltonian system, given the separation of the system, the determination of critical values of the energy momentum map boils down to the analysis of the discriminant surface of a polynomial. An Integrable Hamiltonian of a system has n-degrees of freedom always has a set of n-integrals of motion in n-involution consequently, the trajectory of the system is confined to n-dimensional manifold phase space. According to the Arnold Liouville theorem [**?**], for non-critical values of two constants of the problem, the regular level sets of a completely integrable Hamiltonian system consists of tori. Their number depends only upon the number and the location of the ovals of an associated Riemann surface.

We will study the topology of the first case ($\lambda = -1$) by using the Fomenko's classification theorem [16], this classification was studied in Jamil in the different cases in the rigid body problem [26], [18] . The second case ($\lambda = -\frac{1}{6}$) was studied by L.Gavrilov [19], which they are only separated of all integrable cases.

Let L_s be the real phase space topology of the system (9), where

$$L_s = [(x, y, p_x, p_y) \in R^4 : H = h, F = f] \subset R^4, \tag{21}$$

and let $\Sigma = \Sigma_1 \bigcup \Sigma_2$, where Σ_1 and Σ_2, are the discriminate locus of the polynomials $F_1(u)$ *and* $F_2(v)$, respectively:

$$\Sigma{=}\Sigma_1{\cup}\Sigma_2{=}[(h,f){\in}R^2/disc(F_1(u)){=}0]{\cup}[(h,f){\in}R^2/disc(F_2(v)){=}0]. \tag{22}$$

The set R^2/Σ consists of 16 connected component denoted by $D_i, i = 1, ..., 16$. Table 1 is presented the real roots of both $F_1(u)$, and $F_2(v)$ to find the ovals of $(\Gamma_1 : \omega_1 = \sqrt{F_1(u)}, \omega_2 = \sqrt{F_2(v)})$ to study the topological type of L_s (see Table 2).

To determine these ovals, it suffices to study the real roots of the polynomials $F_1(u)$, and $F_2(v)$ for different values of h and f. It is found that there exist exactly two "admissible" ovals whose projections on the u-plane and the v-plane which are denoted by $\triangle_1$ and $\triangle_2$, respectively as shown in Table 2.

To study the topological type of L_s, it is should be noted that the topological type of L_s depends only on passing the point $(h, f) \in R^2$ through Σ, for more details [12].

The topological type of L_s is either a torus as in $D_i, i = 1, 16$ or empty as in $D_i, i = 2, ..., 15$ (Liouville theorem [16]) as shown in Figure 1 and Table 2.

Table 1: The real roots of the polynomials $F_1(u)$ and $F_2(v)$ for $(h, f) \in R^2/\Sigma$

Domain	*Roots of* $F_1(u)$	*Roots of* $F_2(v)$
1	$u_1 < 0$	$v_1 < 0$
2	0	$v_1 < 0$
3	0	$v_1 < 0$
4	0	$v_1 < 0$
5	0	$v_1 < 0$
6	0	0
7	0	0
8	0	0
9	$u_1 < 0$	0
10	0	0
11	$u_1 < 0$	0
12	0	0
13	0	0
14	$u_1 < 0$	0
15	$u_1 < 0$	0
16	$u_1 < 0$	$v_1 > 0$

Table 2: Admissible ovals on diagram Σ.

Domain	$u - plane\ \triangle_1$	$v - plane\ \triangle_2$	*Topological type*
1	$[0, u_1]$	$[0, v_1]$	T
2	$\emptyset$	$[0, v_1]$	$\emptyset$
3	$\emptyset$	$[0, v_1]$	$\emptyset$
4	$\emptyset$	$[0, v_1]$	$\emptyset$
5	$\emptyset$	$[0, v_1]$	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$
7	$\emptyset$	$\emptyset$	$\emptyset$
8	$\emptyset$	$\emptyset$	$\emptyset$
9	$[0, u_1]$	$\emptyset$	T
10	$\emptyset$	$\emptyset$	$\emptyset$
11	$\emptyset$	$\emptyset$	$\emptyset$
12	$\emptyset$	$\emptyset$	$\emptyset$
13	$\emptyset$	$\emptyset$	$\emptyset$
14	$[0, u_1]$	$\emptyset$	$\emptyset$
15	$[0, u_1]$	$\emptyset$	$\emptyset$
16	$[0, u_1]$	$[0, v_1]$	T

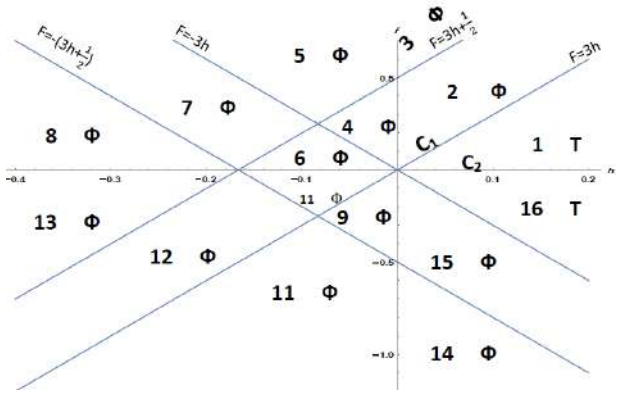


Figure 1: Diagram of bifurcation $\Sigma = \Sigma_1 \cup \Sigma_2$.

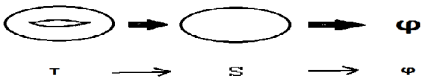


Figure 2: A torus T is contracted to the axial circle S and then vanishes.

A bifurcation of Liouville tori is due to the bifurcation of the polynomials $F_1(u)$ *and* $F_2(v)$ roots. In the present problem, there are two type of bifurcation as follows:

1. A torus contracts to circle and then vanished, as shown in Figure 2.
2. A symmetric bifurcation of one tori into one tori, as shown in Figure 3.

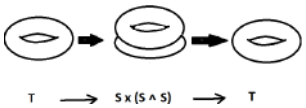


Figure 3: Bifurcation of Liouville tori, where $S \bigwedge S$ is union of two circles having exactly one common point.

Table 3 gives All generic bifurcations of Liouville tori of the system (9), Figures 4, and 5 show the bifurcations of the polynomials $F_1(u)$ and $F_2(v)$.

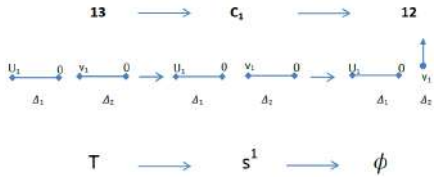


Figure 4: Correspondence between bifurcation of roots of polynomials $F_1(u)$ and $F_2(v)$ and bifurcation of invariant Liouville tori.

Domains	*The generic bifurcation*
$1 \to i, i = 2, ..., 15$	$T \to \emptyset$
$16 \to i, i = 2, ..., 15,$	$T \to \emptyset$
$1 \to 16$	$T \to T$

Table 3: Generic bifurcations of the level set L_s passing from domain i to domain j.

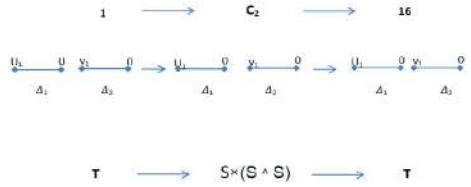


Figure 5: Correspondence between bifurcation of roots of polynomials $F_1(u)$ and $F_2(v)$ and bifurcation of invariant Liouville tori.

4. Periodic solutions

On the curve C_1 where $f = 3h$, we note that the tori T contracted to axial circle S and then vanish as shown in Figure 1, where the u parameter takes value in interval $[u_1, 0]$, and $v = \frac{-3}{2}$, then their exist a periodic solution on this curve as shown in the Table 4.

Domain	$u - plane \triangle_1$	$v - plane \triangle_2$	*Topological type*
C_1	$[u_1, 0]$	$[v_1, 0]$	S
C_2	$[u_1, 0]$	$[v_1, 0]$	$s \times (s \wedge s)$

Table 4: Topological type of L_s for $(h, f) \in B$.

By solving the second equation of (15), the function $F_1(u)$ is a polynomial of fourth degree and has three roots u_1, u_2, $andu_3$ such that

If $f > 3h$ or $f < 3h$, the real motion bounded by e_1, and e_2 such that $e_2 < u < e_1$.

Put

$$u = \frac{e_2 e_{31} - e_3 e_{21} sin^2(\phi)}{e_{31} - e_{21} sin^2(\phi)}, \tag{23}$$

where $e_{ij} = e_j - e_i$, and

$$du = \frac{2e_{21}e_{32} sin(\phi) cos(\phi)}{e_{31}(1 - k^2 sin^2(\phi))^2} d\phi, \tag{24}$$

where

$$k^2 = \frac{e_{21}}{e_{31}}, \tag{25}$$

the equations (15), become:

$$\int_{t_0}^{t} dt = \sqrt{\frac{3}{2}}\sqrt{\frac{4}{e_{31}}}\int_{\phi_0}^{\phi}\sqrt{\frac{d\phi}{1-k^2sin^2(\phi)}}. \tag{26}$$

Then, the solution is:

$$u = \frac{e_1e_{31} - e_3e_{21}sn^2(\sqrt{\frac{e_{31}}{6}}(t_0-t_1),k)}{e_{31} - e_{21}sn^2(\sqrt{\frac{e_{31}}{6}}(t_0-t_1),k)}. \tag{27}$$

Then, the period T of $F_1(u)$

$$T = g_1sn^{-1}(-1,k) = g_1K(k). \tag{28}$$

where

$$g_1 = \frac{\sqrt{6}}{\sqrt{e_{31}}} \tag{29}$$

and $K(k)$is complete elliptic function of the first kind.

5. Phase portrait of the separated functions

The aim of this section is to find topological interpretation of the trajectory of the second case by using the phase portrait as manner it was studied in [13], [14]

The phase portrait of the separated functions is:

$$\begin{aligned} P_q = {} & 16q^4 - 8(6A-B)q^3 - (12A-B)(4A-B)q^2 \\ & + A(4A-B)^2q - 8hq - k - 4qp^2. \end{aligned} \tag{30}$$

We first study the singular points of P_q. These points can be found from the equations

$$\begin{aligned} \frac{\partial P_q}{\partial p} &= -8qp = 0, \\ \frac{\partial P_q}{\partial q} &= 64q^3 - 24(6A-B)q^2 - 2(12A-B)(4A-B)q - A(B-4A) \\ & \quad - 8h - 4p^2 = 0, \end{aligned} \tag{31}$$

and hence, when $p = 0$, the second equation of (31) became

$$64q^3 - 24(6A-B)q^2 - 2(12A-B)(4A-B)q - A(B-4A) - 8h = 0, \tag{32}$$

and when $q = 0$, we get the two equations

$$p^2 = \frac{1}{4}(A(B-4A) + 8h), \tag{33}$$

then, from equation (31 and 32) we have the points $(\pm\sqrt{A(B-4A)+8h}, 0)$ and $(0, q_1)$, where q_1 the real root of the equation 31.

Now, we study the points in the following manner:

1. From equation (33) there are one singular point with p coordinate $(\pm\sqrt{A(B-4A)+8h}, 0)$ To get the type of the two points, put

$$q = y, \;\; p = \pm\sqrt{A(B-4A)+8h} + x, \tag{34}$$

in the function P_q, neglecting terms of degree greater than 2, then we have

$$P_q = -(12A-B)(4A-B)y^2 - 4(\pm\sqrt{A(B-4A)+8h})xy + A_0, \tag{35}$$

where A_0 contains the terms of zeros and first degree of x, y. These points are hyperbolic in two case:

1. $h > 0$ (Figure 6),
2. $h < 0$ (Figure 7),

where

$$\begin{bmatrix} \frac{\partial^2 P_q}{\partial x^2} & \frac{\partial^2 P_q}{\partial x \partial y} \\ \frac{\partial^2 P_q}{\partial x \partial y} & \frac{\partial^2 P_q}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{36}$$

2. From equation (32) there are one singular point with q coordinate $(0, q_1)$. To get the type of the point, put

$$q = q_1 + y, \;\; p = x, \tag{37}$$

in the function P_q, neglecting terms of degree greater than 2, then we have

$$P_q = [96q_1^2 - 24(6A-B)q_1 - (12A-B)(4A-B)]y^2 - 4q_1x^2 + A_0, \tag{38}$$

where A_0 contains the terms of zeros and first degree of x, y. This point is hyperbolic point in two case:

1. $h > 0$ (Figure 8),
2. $h < 0$ (Figure 9),

where

$$\begin{bmatrix} \frac{\partial^2 P_q}{\partial x^2} & \frac{\partial^2 P_q}{\partial x \partial y} \\ \frac{\partial^2 P_q}{\partial x \partial y} & \frac{\partial^2 P_q}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{39}$$

To study the phase portrait of the first case($\lambda = -1$) we have the separated function R:

$$R = p^2 + q^3 + \frac{3}{2}q^2 + 3h - f. \tag{40}$$

the singular points of this case are $(0, 0)$ and $(0, -1)$. These points are elliptic and hyperbolic respectively as they shown in Figure 10.

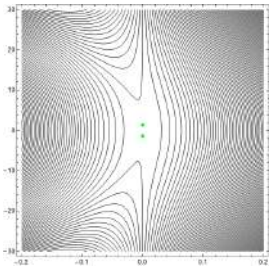


Figure 6: The two-hyperbolic point, where $h > 0$.

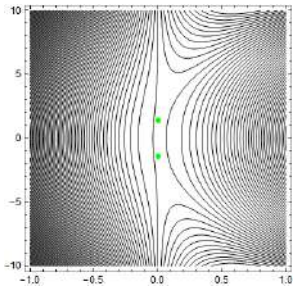


Figure 7: The two-hyperbolic point, where $h < 0$.

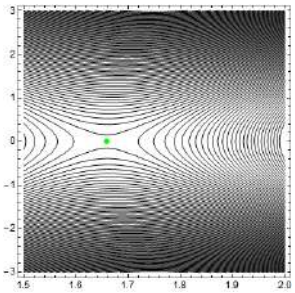


Figure 8: The hyperbolic point, where $h > 0$.

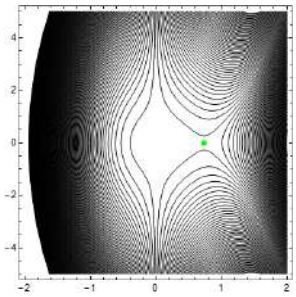


Figure 9: The hyperbolic point, where $h < 0$.

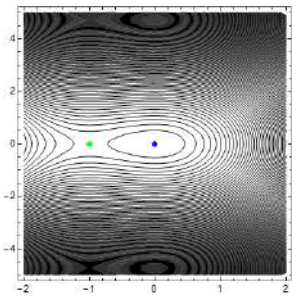


Figure 10: The one elliptic point, and the one hyperbolic point.

6. Poincaré surface section

To complete the picture of the present problem, we solve the GHH system numerically by using a surface section introducing by Poincaré [28]. The trajectory of the system may be treated in three dimension invariant $(x, \dot{x}, y, \dot{y})$, with h equals a constant. We find the Poincaré surface section $(x, \dot{x})$. If the regular lies on the smooth curves in the surface section then the system is integrable, and the regular orbits are called the invariant curves. If the system is non-integral

then a part of the curves seem to destroy and chaotic orbits appear instead, filling a stochastic region.

Putting $\dot{x} = \dot{y} = 0$ in (1), then we have the equipotential lines

$$V(x, y) = h. \tag{41}$$

Putting $\dot{x} = 0$ in integral energy equation, we have

$$\dot{y} = \sqrt{2(h - V(0, y))}. \tag{42}$$

Now, we deduce the surface-section in all integrable cases to confirm that the orbits of the motion lie on invariant tori and the motion is ordered:

1. The first case $\lambda = -1$, $\frac{A}{B} = 1$, the equipotential lines at different values of h are shown in Figure 11, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 12− 14.

2. The second case $\lambda = -\frac{1}{6}$ any A, B, the equipotential lines at different values of h are shown in Figure 15, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 16− 18.

3. The third case $\lambda = \frac{1}{16}$, $\frac{A}{B} = \frac{1}{16}$, the equipotential lines at different values of h are shown in Figure 19, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 20− 22.

4. Their are another case which GHH can be integrated, this case suggest by Chang et al. [8], this may exist for values of λ for which $\sqrt{1 - 48\lambda}$ = n/m with n and m two relatively prime integers. Then in this case the equipotential lines at different values of h are shown in Figure 23, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 24− 26.

7. Conclusion

We have studied the complete description of the real phase topology of the problem and concluded that the topological type of the the level set is torus, tours or empty set as shown in Table 2. And, the periodic solution of the problem is determined. It is separated which allows to construct the topological translation of the trajectory by using the phase portrait. The singular points of the separated functions which are elliptic or hyperbolic points are determined. The elliptic points in the figures are stable in the Lyapunov sense, because a small disturbance will result in a closed trajectory that surrounds it and along which the state of the system remains close to these points, which the hyperbolic points are unstable because any small disturbance will result in a trajectory on which the state of the system deviates more and more from these points as t goes to infinity.

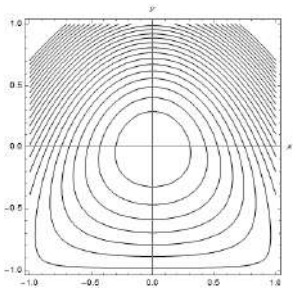


Figure 11: The equipotential line for $\lambda = -1$ with a different values of h.

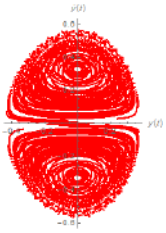


Figure 12: The invariant curves for $\lambda = -1$, and $h = 0.0833$.

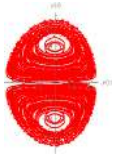


Figure 13: The invariant curves for $\lambda = -1$, and $h = 0.1667$.

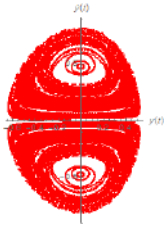


Figure 14: The invariant curves for $\lambda = -1$, and $h = 1$.

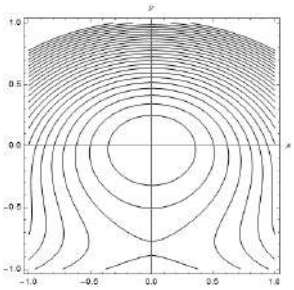


Figure 15: The equipotential line for $\lambda = -\frac{1}{6}$ with a different values of h.



Figure 16: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 0.0833$.



Figure 17: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 0.1667$.

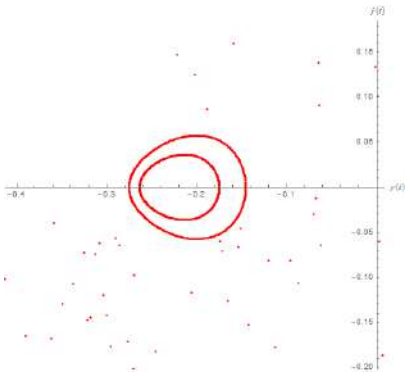


Figure 18: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 1$.

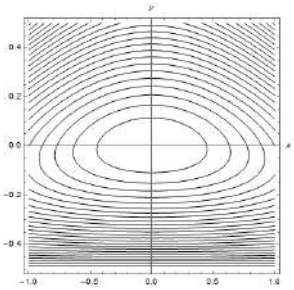


Figure 19: The equipotential line for$\lambda = -\frac{1}{16}$ with a different values of h.



Figure 20: The invariant curves for$\lambda = -\frac{1}{16}$, and $h = 0.0833$.

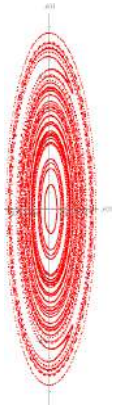


Figure 21: The invariant curves for $\lambda = -\frac{1}{16}$, and $h = 0.1667$.



Figure 22: The invariant curves for $\lambda = -\frac{1}{16}$, and $h = 1$.

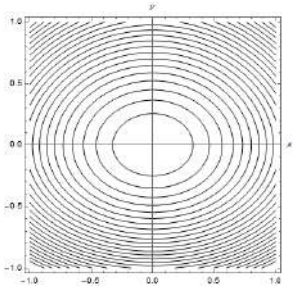


Figure 23: The equipotential line for $\lambda = \frac{1}{48}(1 - (\frac{n}{m})^2)$with a different values of h.

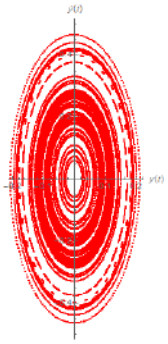


Figure 24: The invariant curves for $\lambda = \frac{1}{48}(1 - \frac{4}{9})$, and $h = 0.0833$.

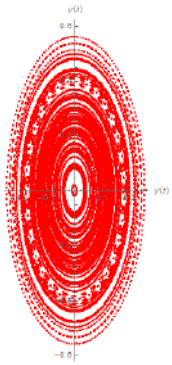


Figure 25: The invariant curves for $\lambda = \frac{1}{48}(1 - \frac{4}{9})$, and $h = 0.1667$.



Figure 26: The invariant curves for $\lambda = \frac{1}{48}(1 - \frac{4}{9})$, and $h = 1$.

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On groups acting on trees of ends >1

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Abstract. Call a group G to possess the property P if G is finitely generated and G is of end greater than one. That is e(G) >1. The main result of this paper is the following. A group G to possess the property P if and only if there exists a tree X such that G acts on X without inversions, the stabilize G_e for each edge e of X is finite, for each vertex v of X, $G_v \neq G$ and the quotient graph G/X for the action of G on X is finite. We prove the following:

(1) If the group G possesses the property P and H is a subgroup of G, then
   (i) if H is of finite index in G, then H possesses the property P,
   (ii) if H is finite and normal subgroup of G, then the quotient group of G over H possesses the property P.

(2) If G is a group acting on a tree X without inversions such that the stabilize G_v for each vertex v of X possesses the property P, $G_v \neq G$, the stabilizer of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then G to possesses the property P.

As an application, we show that if $A = \prod_{i\in I}^{*}(A_i; U_{ij} = U_{ji})$ is a tree product of the groups $A_i i \in$I with amalgamation subgroups $U_{ij}, i, j \in$I such that each A_i has the property P, U_{ij} is finite, $i, j \in$I and I is finite, then A has the property P. Furthermore, if G^* is the HNN group

$$G^* =< gen(G), t_i/rel(G), t_i a t_i^{-1} = \phi(a), a \in A_i, i \in I >$$

of basis G and associated pairs $(A_i, B_i), i \in I$ of isomorphic subgroups of G such that G has the property P, $A_i, i \in I$ is finite, and I is finite, then G^* has the property P.

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1. Introduction

the structures of group acting on trees with out inversions we refer the readers to [2], [11] and [12]. For the basic concepts of the ends of groups we refer the reader to [1], [2], [3], [13], [14], and [15]. In this paper we discuss and prove theorems connecting groups acting on trees without inversions and ends of groups. The main result of this paper is showing that if the stabilizers of the vertices of a tree are of ends greater than 1, then the group has an end greater than 1 providing that the stabilizers of the edges of the tree are finite, and the quotient graph of the action of group on a tree is finite. Then, we apply such result to tree product of groups and HNN extensions of groups. First, we begin a general background of groups acting on trees without inversions. A graph X consists of two disjoint sets V(x) (the set of vertices of X) and E(X) (the set of edges of X) with V(X) non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$ and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions $\partial_0\eta = \partial_1$ and $\partial_1\eta = \partial_0$ For simplicity, if $e \in E(X)$, then then we write $\partial_0(e) = O(e)$, $\partial_1(e) = t(e)$ and $\eta(e) = \overline{e}$. This implies that $O(\overline{e}) = t(e)$, $t(\overline{e}) = O(e)$ and $(\overline{\overline{e}} = e)$. We say that a group G acts on a graph X without inversions if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$ then we write g(x) for $(\phi(g(x))$ Thus, if $g \in G$ and $y \in E(x)$, then $g(O(y)) = O(g(y)), (g(t(y)) = t(g(y))), g(\overline{y}) = \overline{g(y)})$. If the group G acts on the graph X and $x \in X$ (x is a vertex or edge), then 1. The stabilizer of x, denoted G_x is defined to be the set $G_x = [g \in G : g(x) = x]$. It is clear that $G_x \leqslant G$ and if $x \in E(x)$ and $u \in [o(x), t(t)]$ then $G_{\overline{x}} = G_x$ and $G_x \leqslant G_u$ 2. the orbit of x denoted G(X) is defined to be the set $G_x = [g \in G : g(x) \in x]$. It is clear that G acts on the graph X without inversions if and only if $G(\overline{e}) \neq G(e)$. For any $e \in E(x)$. 3. The set of the orbits G/X of the action of G on X is defined as $G/X = [G(x) : x \in X]$ G/X forms a graph called the quotient graph for the action of G on X, where $V(G/X) = [G(V) : v \in V(x)]$, $E(G/X) = [G(e) : e \in E(x)]$ and if $e \in E(x)$ then $O(G(e)) = G(o(e)), t(G(e)) = G(t(e))$ and $\overline{G(e)} = G(\overline{e})$.

Definition 1 ([5]). *Let G be a group acting on a tree X without inversions, and let T and Y be two subtrees of X such that $T \subseteq Y$ and each edge of Y has at least one end in T. Assume that T and Y are satisfying the following:*

(i) *T contains exactly one vertex from each vertex orbit.*

(ii) *Y contains exactly one edge y(say) from edge orbit.*

The pair $(T; Y)$ is called a fundamental domain for the action of G on X. For the existence of fundamental domains.

Proposition 2.1. *The set of vertices V(T) of T is in one to one correspondence with the set of vertices V(X/G) of X/G and the set of edges E(Y) of Y is in*

one to one correspondence with the set of edges E (X/G) of X/G. For the rest of this section, G is a group acting on a tree X without inversions, and $(T; Y)$ is the fundamental domain for the action of G on X. We have the following definitions.

Definition 2. *For any vertex $v \in V(x)$ there exist a unique vertex denoted v^* of T and an element g (not necessarily unique) of G such that $g(v^*) = v$ That is $G(v^*) = v$, Moreover, if $v \in V(T)$ then $v^* = v$, and for each edge $y \in E(x)$, let $[y]$ be any element of G satisfying the following:*

(a) *if $o(y) \in V(T)$ then $[y]((t(y))^*)$ and $[y] = 1$ in case $y \in E(T)$.*

(b) *if $t(y) \in V(T)$ then $[y](o(y)) = (o(y)^*)$ and $[y] = [y]^{-1}$. Furthermore, let $+y$ be the edge if $o(y \in V(T))$ and $+y = [y](y)$ if $t(y) \in V(T)$ It is clear that $o(+y) = (o(y))^*$ and $G_{+y} \leqslant G_{(o(y))^*}$. If $y \in E(T)$,then $G_{+y} = G_y$.*

Definition 3. *If $g \in G$ is an element of G and $e \in E(Y)$ is an edge of Y, define $[g, e]$ to the pair $[g, e] = (gG_{+e}, e)$ Define $\hat{X}$ to be the set $\hat{X} = [[g, e] : g \in G, e \in E(Y)]$.*

2. The main result

There are a lot of ideas of introducing the concept of the ends of groups as shown in [1, p.17],[2, p. 124], [3], [13], [14], and [15, p. 171]. For example, in [2, p. 124], the concept of the ends of groups is illustrated as follows. The number of the ends of a group G, denoted $e(G)$ is defined as following. Let $Z_2 = [0, 1]$ be the ring of integers modulo 2 and $(G; Z_2)$ be the set of functions from G to Z_2. $(G; Z_2)$ forms a ring where for each $\alpha : G \to Z_2$, $(\alpha + \beta)(g) = \alpha(g) + \beta(g)$ and $(\alpha\beta)(g) = \alpha(g)\beta(g)$ for all $g \in G$. Let IG be the subring of the ring $(G; Z_2)$ consisting of all $\theta \in (G; Z2)$ defined $\theta(g(x)) = \theta(xg^{-1})$ for all $g, x \in G$. Let JG be the set of all complements of all prime ideals of the subring IG. the number of the elements of JG is called the number of the ends of G, and is denoted by $e(G)$.

In [3], the concept of the ends of groups is defined by means of the actions of groups on simplicial complexes. For further background information see [13].

The following are examples of ends of groups.

1. For any finitely generated group G, $e(G) \in [0, 1, 2, \infty]$ ([15], Corollary 5.9, p.176).

2. For a finite group G, $e(G) = 0$. ([2], Proposition 6.5, p.126).

3. If Z is the infinite cyclic group, and $G = Z^n$, $n \geqslant 2$, then $e(G) = 1$, ([15], Corollary 5.5, p.174).

4. $e(Z) = 2$, ([15], Theorem 5.12, p. 178).

5. If Z_2 is a cyclic group of order 2, and G is the free product $G = Z_2 * Z_2$, then $e(G) = 2$, ([1], Corollary, p. 21).

6. For a free group $F(X)$ of base X where $1 < |X| < \infty$ we have $e(F(X)) = \infty$, ([2], Example, p. 127).

In [1, Th. 3.1] and [14, Th. 4.A.6.5, and Th. 4.A.9], Cohen and Stallings showed that for a finitely generated G, $e(G) \geqslant 1$ if and only if G either G decomposes as a free product with amalgamation $G = *_H A_i, i \in I$ of the groups A_i, $i \in I$ with amalgamated subgroup H such that H and I are finite, $H \neq A_i$ for all $i \in I$, or is an HNN group, $G =< gen(K), t_1 ... t_n | rel(k), t_i L_i t^{-1} = M_i, i = 1...n >$ of basis K and associated pair (L_i, M_i) of isomorphic subgroups of G for all $i \in I$ such that L_i, M_i and I are finite for all $i \in I$. In fact, the free product with amalgamation group $G = *_H A_i, i \in I$ is equivalent to the tree product of groups as

$$A = \prod_{i \in I}^{*} (A_i; U_{ij} = U_{ji})$$

is a tree product of the groups $A_i, i \in$ with amalgamation subgroups $U_{ij}, i, j \in I$.

By working within the framework of groups acting on trees without inversions this theorem can be put in the following form.

Lemma 2.1. *A group G to possess the property P if and only if there exists a tree X such that G acts on X without inversions, the stabilize G_e for each edge e of X is finite, for each vertex v of X, and the quotient graph G/X for the action of G on X is finite.*

Proof. If $A = \prod_{i \in I}^{*} (A_i; U_{ij} = U_j)$ is a tree product of the groups $A_i, i \in$ with amalgamation subgroups $U_{ij}, i, j \in I$ then A acts on the tree X without inversions defined as follow: $V(X) = [(gA_i, i) : g \in A, i \in I]$ and $E(X) = [(gU_{ij}, ij) : g \in A, i, j \in I]$. If y is the edge $y = (gU_{ij}, ij)$, then $o(y) = (gA_i, i)$, $t(y) = (gA_j, j)$ and $\overline{y} = (gU_{ji}, ji)$. The group A acts on X as follows: let $f \in A$ Then $f(gA_i, i) = (fgA_i, i)$ and $f(gU_{ij}, ij) = (fgU_{ij}, ij)$. If v is the vertex $v = (gA_i, i) \in V(X)$ and y is the edge $(gU_{ij}, ij) \in E(X)$. Then the stabilizer of v is $A_v = gA_i g^{-1} \cong A_i$, a conjugate of A_i and the stabilizer of y is $A_y = gU_{ij} g^{-1} \cong U_{ij}$, a conjugate of U_{ij} The orbit of v is the set $A(v) = [(agA_i), i : a \in A, i \in I)]$ and the orbit of y is $A_y = [(agU_{ij}), ij : a \in A, i, j \in I)]$. If G is an HNN group $G =< gen(K), t_i ... t_n(k), t_i L_i t^{-1} = M_i, i = 1...n >$, of base K and of associated pairs (L_i, M_i) of isomorphic subgroups of $K, i \in I$, where $< gen(K) | rel(k)$ stands for any presentation of K, and $t_i L_i t_i^{-1}$ stands for the relations $t_i t_i^{-1} = \phi_i(x), x \in A_i$ Then G acts on the tree X without inversions defined as follow: $V(X) = [gK : g \in G]$ and $E(X) = [(gM_i, t_i), (gL_i, t_i^{-1})]$, where $g \in G$and $i \in I$. For the edges $(gM_i, t_{[}i])$ and $(gL_i, t_i^{-1}), i \in I$, defined $o(gM_i, t_i) = o(gL_i, t_i^{-1}) = gK$, $t(gM_i, t_i) = gt_i K$, $t(gL_i, t_i^{-1}) = gt_i^{-1} K$, and $\overline{(gM_i, t_i)} = (gt_i L_i, t_i^{-1})$ and $\overline{(gL_i, t_i^{-1})} = (gt_i^{-1} M_i, t_i)$. Let $f \in G$. Then for the vertex gG and the edges (gM_i, t_i) and (gL_i, t_i^{-1}) of X, define $f(gK) = fgK, f(gM_i, t_i) = (fgM_i, t_i)$ and $f(gL_i, t_i) = (fgL_i, t_i)$, The stabilizer of the vertex $v = gK$ is $K_v = gKg^{-1}$ a conjugate of G, the stabilizers of the edges (gB_i, t_i) and (gA_i, t_i) are $gM_i g^{-1}$ a conjugate of M_i and $gL_i g^{-1}$ a conjugate of L_i are finite for all $i \in I$ The orbits of $gG, (gB_i, t_i)$

and (gL_i, t_i^{-1}) are $[fK : f \in G]$ and $[(fM_i, t_i : f \in G)]$ Now if the group G acts on a tree X without inversions such that $G_v \neq G$ for any vertex $v \in V(X)$o f X, then by theorem of [5], there exists a fundamental domain $(T; Y)$ for the action of G on X, and [11, Th. 5.1], G has the presentation $G =< gen(G_v, y|rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y]y.y^{-1} = G_y) >$, where , m and y stand for edges of $E(Y)$ such that $m \in E(T)$, $o(y) \in V(T), t(y), \neq V(T)$. If $Y = T$ then G has the presentation $\prod^*_{v\in V(X)}(gen(G_v; rel(G_v), G_m = G_m)$, where $L_m = [m]^{-1}G_m[m]$ and $M_m = G_m$. It is clear that G is a tree product of the groups G_v with amalgamation subgroups L_m, M_m . If $Y \neq T$, then $G =< gen(G_v, y|rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y]y.y^{-1} = G_y) >$ is HNN extension group of base K, where $\prod^*_{v\in V(X)}(gen(G_v; rel(G_v), G_m = G_m)$ associated isomorphic subgroups L_m, M_m, and stable letters the edges $y \in E(X)$ of X such that $o(y) \in V(T) and t(y) \neq V(T)$. □

As a consequence of Lemma 2.1, we have the following proposition.

Proposition 2.1. *If the group G has the property P and H is a subgroup of G then:*

(1) *If H is of finite index, then H has the property P.*

(2) *If H is finite and normal, then the quotient group G/H has the property P.*

Proof. Since G has the property P, G is finitely generated, and by Lemma 2.1, exists a tree X such that G acts on X without inversions, the stabilize Ge for each edge e of X is finite and the quotient graph G/X for the action of G on X is finite. (1) Since G is finitely generated and H is of finite index in G, by the Reidemeister-Schreier subgroup theorem [6, Corollary 2.8, page 93], H is finitely generated. Then H acts on X by restriction. It is clear that the vertex stabilize He of the edge e of X satisfies $H_e = H \cap G_e$ Since G_e is finite, H_e is finite. Since H is of finite index in G, and G/X is finite , therefore by Lemma 7 of [11], the quotient graph H/X for the action of H on X is finite. Thus, H has the property P.

(2) Since G is finitely generated, it is clear that G/X is finitely generated. Let X^H be the set $X^H = [x \in X : H \leq G_x]$ By Proposition 4.3 of [12], X^H is a subtree of X and G/H acts on X^H without inversions, where if $g \in G$ and $x \in E(X^H)$, then such that $gH(x) = g(x)$. It is clear that the stabilizer of $x \in X^H$ under the action of G/H on X^H is $(G/H)_x = G_x/H$, where G_x, is the stabilizer of x under the action of G on X. Since stabilizer G_y of each edge y of X under the action of G on X is finite, therefore stabilizer $(G/H)_e = G_e/H$ of each edge $e \in E(X)$ under the action of G/H on X^H is finite. If $x \in V(V)$ and $(G/H)_x = G_x/H = G/H$ then $G_x = G$. Contradiction. Hence $(G/H)_x = G_x/H \neq G/H$ for any vertex $x \in V(V)$. It is clear that if $x \in X^H$, where x is a vertex or an edge, then the orbit $(G/H)(x)$ of x under the action of G/H on X^H is given by $(G/H)(x) = G(x)$ where G(x) is the orbit of x under the action of G on X. This implies that the quotient graph $(G/H)/X^H$ for the action of G/H

on X^H is given by $[(G/H)(x) = G(x) : x \in X] \subseteq G/X$. Since G/X is finite, therefore $(G/H)/X^H$ is finite. Consequently, the quotient group G/H has the property P. This completes the proof.

Remark. In Lemma 2.1, we note that for each vertex v of X, the stabilizer G_v acts on a tree v of one vertex and no edges. It is natural to ask if the tree X can be chosen so that the stabilizer G_v of each vertex v of X acts on a tree X_v of more than one vertex. In [10], Mahmood constructed a tree denoted $\tilde{X}$ on which G acts on $\tilde{X}$ without inversions such that the stabilize G_v for each vertex v of X,$G_v \neq G$ acts on a tree X_v such that X_v contains more than one vertex. The following theorem gives more details. Before we prove the theorem we introduce the following concept taken from [2]. Let H be a subgroup of a group G and let H act on a set X. Define $\equiv$ to be the relation on $G \times X$ defined as $(f, u) \equiv (g, v)$ if there exists $h \in H$ such that $f = gh$ and $u = h^{-1}(v)$ It is easy to show that $\equiv$ is an equivalence relation on $G \times X$ The equivalence class containing $(f; u)$ is denoted by $f \otimes_H u$. Thus, $f \otimes_H u = [(fh, h^{-1}) : h \in H]$ Define $G \otimes_H X$ to be the set $G \otimes_H X = [g \otimes_H x, x \in X]$.

The main result of this section is the following theorem.

Theorem 2.1. *If G is a group acting on a tree X without inversions such that the stabilize G_v for each vertex v of X has the property P,$G_v = G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then the group G has the property P.*

Proof. Since for each vertex $v \in V(X)$, G_v has the property P, G_v is finitely generated. Since the quotient graph G/X for the action of G on X is finite, by Lemma 4.4 of [11], G is finitely generated. Furthermore, by Lemma 2.1, exists a tree X_v such that G_v acts on X_v without inversions, the stabilizer $(G_v)_u$ for each edge u of X_v is finite and $(G_v)_w \neq G_v$ for each vertex w of X_v, and the quotient graph G_v/X_v for the action of G_v on X_v is finite. Let $(T; Y)$ be a fundamental domain for the action of G on X. Since the quotient graph G/H for the action of G on X is finite, T and Y are finite. By Lemma 4.4 of [6], G is generated by the generators of $G_v, v \in V(T)$ and by the elements $[y], y \in E(Y)$ By Theorem 3.4 of [11], there exists a tree denoted as $\tilde{X} = \hat{x} \bigcup_{v \in V(x)} (G \otimes_{G_y} X_v)$ where $\hat{x} = ([g : e] : g \in G, e \in E(Y))$ and $[g; e] = (gG_{+e}, e)$, $V(\tilde{X} = \bigcup_{v \in V(x)} (G \otimes_{G_y} V(X_v)))$ and $E(\tilde{X}) = \hat{x} \bigcup_{v \in V(x)} (G \otimes_{G_y} E(X_v))$ The ends and the inverse of the edge $G \otimes_{G_y} e$ are defined as follows: $t(g \otimes_{G_v} e) = fg \otimes_{G_V} t(e)$, $o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e)$ and $\overline{(g \otimes_{G_v} e)} = g \otimes_{G_v} \overline{e}$, where $t(e), o(e)$ and $\overline{e}$ are the ends and the inverse of the edge e in X_v. G acts on $\tilde{X}$ as follows: if $f, g \in G, y \in E(x), v \in V(T), e \in E(X_v)$ and $u \in V(X_v)$, then $f[g; y] = [fg; y]$, $f(g \otimes_{G_v} e) = fg \otimes_{G_V} e$ and $f(g \otimes_{G_u} e) = fg \otimes_{G_u} e$. If $g \in G$ and $e \in E(X)$,such that $g(1 \otimes_{G_v} e) = \overline{1 \otimes_{G_V} e} = 1 \otimes_{G_V} \overline{e}$, then $g \in G_v$ and $e \in E(X_v)$, $g(e) = \overline{e}$. Hence,G_v acts on X_v with inversions. This is a contradiction because G_v has the property P. This implies that G acts on $\tilde{X}$ without inversions. Now for $g \in G$

and $x \in X_v$, it is clear that the stabilizer $G_g \otimes_{G_v} x$ of the vertex $g \otimes_{G_v} x$ is $G_g \otimes_{G_v} x = g(G_v)_x g^{-1}$, where $(G_v)_x$ is the stabilizer of x under the action of G_v on X_v, Since $(G_v)_x$ is finite, therefore, $g \otimes_{G_v} x$ is finite. So the stabilizer of each vertex of $\tilde{X}$ under the action of G on $\tilde{X}$ is finite. Now we show that the quotient graph $G/\tilde{X}$ for the action of G on $\tilde{X}$ is finite. The fundamental domain $(T; Y)$ for the action of G on X induces a fundamental domain $(T_v; Y_v)$ for the action of G_v on X_v for every vertex v of T.For each edge $e \in E(Y)$, letv_e be a vertex $v_e \in V(T_{(o(e))^*})$, such that $G_{+e} \leqslant (G_{(o(e))^*})_{v_e}$, where $(G_{(o(e))^*})_{v_e}$ is the vertex stabilizer of the vertex v_e under the action of $(G_{(o(e))^*})$ on $(X_{(o(e))^*})$. Let $\hat{T} = [[1; e] : e \in E(T)]$, $\hat{Y} = [[1; e], [[e]; e] : e \in E(Y)]$, $\tilde{T} = \hat{T} \cup [\bigcup_{v \in V(T)}(1 \otimes_{G_v} T_v)]$, $\tilde{Y} = \hat{Y} \cup [\bigcup_{v \in V(T)}(1 \otimes_{G_v} Y_v)]$, where $[e]$ is the value of the edge e defined as in Definition 2 and $[[e], e]$ is defined as in Definition 3. Then, by Theorem 3.2 of [9], $(\tilde{T}; \tilde{Y})$ forms a fundamental domain for the action of G on $(\tilde{X}$. Since for each edge $e \in E(X)$, the stabilizer G_e is finite and $(T; Y)$ and, $(T_v; Y_v)$, $v \in V(T)$ are finite, then the fundamental domain $(\tilde{T}; \tilde{Y})$ is finite. This shows that the quotient graph G/X for the action of G on X is finite. Then, Lemma 2.1 implies that the group G has the property P. This completes the proof. □

We have the following corollaries of Theorem 2.1.

Corollary 2.1. *Let $A = \prod^*_{i \in I}(A_i; U_{ij} = U_{ji})$ be a tree product of the groups A_i of the groups $A_i, i \in I$ with amalgamation subgroups $U_{ij}, i, j \in I$ such that the group Ai has the property P for alli $\in I$, I is finite, and U_{ij} is finite for all $i, j \in I$. Then A has the property P.*

Corollary 2.2. *Let $A = *_c A_i, i \in I$ be the free product of the groups $A_i, i \in I$ with amalgamation subgroup C such that the group A_i has the property P for all $i \in I$, I is finite, and C is finite. Then the group A has the property P.*

Corollary 2.3. *Let G be the HNN group $G =< gen(K), t_i | rel(K), t_i L_i t i^{-1} = M_i, i \in I >$ of base K and of associated pairs (L_i, M_i) of isomorphic subgroups of G such that the group K has the property P, L_i and M_i are finite for all $i \in I$, and I is finite. Then the group G has the property P.*

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Bifurcation of subharmonic in Lassa fever epidemic model

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Abstract. Standard epidemiological theory and concepts such as the basic reproductive number R_0 no longer apply or enough to determine stability, and the implications for interventions that themselves may be periodic have not been formally examined. This study considers seasonal variation of rodent on the transmission dynamics of Lassa fever. Applying perturbation method, stable subharmonic bifurcation solutions of period n are proved to coexist simultaneously and an infinite number of stable subharmonic bifurcation solutions are established. This study suggests that transmission of Lassa fever can be curbed by fighting the proliferation of rodent especially during the wet season.

Keywords: seasonal variation, subharmonic bifurcation, stability, reproduction number.

1. Introduction

Infectious diseases pose a constant threat to human well. Every individual on the earth can be affected by a disease. Proper understanding of transmission mechanism of diseases caused by existing and new pathogens may facilitate devising prevention tools. Prevention tools against transmission, including drugs and vaccines need to be developed at the similar peace to that of the microbes. Seasonal variations in rainfall, temperature and resource availability are ubiquitous and can exert strong pressures on population dynamics. Usually, vector such as rodent, mosquito, snail etc are many during the raining season depending on the lengthen of rainfall.

Lassa fever is an acute viral illness that occurs in West Africa. The reservoir, or host, of Lassa virus is a rodent known as the "multimammate rat" of

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the genus Mastomys. According to the World Health Organization, 300,000 to 500,000 cases of Lassa fever and 5000 deaths occur yearly across West Africa [2]. The symptoms of Lassa fever are: flu-like illness characterized by fever, general weakness, cough, sore throat, headache, and abdominal pain. Depending on transmission intensity, the main complication of Lassa fever after recovery is deafness. Interestingly, recovered human has permanent immunity against Lassa virus and therefore remain recovered for life [7].

Now, we briefly give account of existing literatures on subharmonic bifurcation in SIR and SEIR model. Ira and Smith formulated a mathematical model that incur permanent immunity with seasonal variations in the contact rate and analysed rigorously. They conclude that random effects in the environment could perturb the state of the dynamics from the domain of attraction from one subharmonic to another.

From the idea of Ira and Smith model, Chow and Shaw [3] considered a piecewise linear second order ordinary dfferential equation which is topo- logically equivalent to the sine-Gordon equation. Their model was subjected to a time harmonic disturbance and the behaviour of the periodic solutions was examined. The results of their analysis revealed that that subharmonic motions with period n times that of the disturbance appear via saddle-node bifurcation for n = 1,2,3,...

The work by Ira and smith was extended by Kuznetsov and Piccardi [6], they investigated the bifurcation of the periodic solutions of SEIR and SIR epidemic models with sinusoidally varying contact rate. They demonstrated by numerical simulations that the parameter portrait of the SEIR model undergoes significant structural changes when the latent period is varied.

From another perspective, Schaffer and Bronnikova discussed the bifurcation structure of SEIR model subject to seasonality. Their results suggest that combination of phenomenological equation which admit to mathematical analysis and detailed simulation will prove a recipe for progress.

In a related work, Rachel et al studied how seasonal forcing influence the system when the unforced dynamic have either monotonic, oscillatory decay or stable cycle. Their results shows that the level of oscillation in the unforced system has a larger effect on the range of behaviour when the system is seasonally forced.

Gouhei and Kazuyuki [4] examined the impact of seasonal variation pattern on epidemic dynamics. Their results suggest that accurately estimated seasonal variability is necessary for better understanding the dynamics of infectious diseases.

To the best of our knowledge no work has been done on mathematical study of Lassa fever that incorporates seasonal variation of rodent to investigate the existence of subharmonic bifurcation.

2. Model formulation

The model presented here is an extension work by [8]. The model subdivides the total human population size at time t and discrete age a_i denoted by $N_h(t, a_i)$ with $i = 0, 1, 2, ..., L$ and a_L is the maximum age of humans in the population, into susceptible humans $S_h(t, a_i)$, exposed humans $E_h(t, a_i)$, infectious humans $I_h(t, a_i)$ and recovered humans $R_h(t, a_i)$. Hence we have $N_h(t, a_i) = S_h(t, a_i) + E_h(t, a_i) + I_h(t, a_i) + R_h(t, a_i)$. The term

$$\begin{aligned}\beta_f(t) = &\frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\ &+\frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\ &+\frac{(1+\theta(e_j)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\\ &+\frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\end{aligned}$$

denotes the rate at which the human hosts get infected by infectious human, rodent, inhalation of aerosol from urine or through reused needle when taking injection. The infectious human $I_h(t, a_i)$ gain individuals when exposed individuals becomes infectious and loses individual when they die $\mu_h(a_i)I_h(t, a_i)$, $\delta_h(a_i)I_h(t, a_i)$. After some time, exposed and infectious human recovers and moves to the recovered class $R_h(t, a_i)$. However recovered human has permanent immunity and never go back to susceptible class again. A loss of individuals is as a result of natural death $\mu_h R_h(t, a_i)$. It is assumed that $\theta(e_j) \propto \frac{C_v}{K_v}$ and $\theta(e_j)$ is generated from urine and faeces of infectious rodents, where C_v is the amount of virus in air and K_v is the saturation of virus in air. Similarly, $\kappa(a_i)$ is generated from blood of infectious individuals, $\kappa(a_i) \propto \frac{A_v}{S_v}$, where A_v is the amount of virus in needle and S_v is the saturation of virus in needle.

In similar manner, we subdivides the total rodent population size at time t and discrete age e_j denoted by $N_r(t, e_j)$ with $j = 0, 1, 2, ..., T$ and e_T is the maximum age of rodents in the population, into susceptible rodents $S_r(t, e_j)$, exposed rodents $E_r(t, e_j)$ and infectious rodents $I_r(t, e_j)$. Hence we have $N_r(t, e_j) = S_r(t, e_j) + E_r(t, e_j) + I_r(t, e_j)$. Susceptible rodent class $S_r(t, e_j)$ gain more individual into rodent population by input rate $\Lambda_r(e_j)(1+\phi_r(e_j)\cos 2\pi(t+T))$, while it loses rodents through natural death $\mu_r(e_j)S_r(t, e_j)$, hunting $\delta_r(e_j)S_r(t, e_j)$.

$$\begin{aligned}\gamma_f(t) = &\frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))S_r(t,e_j)I_r(t,e_j)}{N_r(t,e_j)}\\ &+\frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)}\end{aligned}$$

denote the rate at which the rodent get infected by infectious rodent or inhalation of aerosol from urine. Transmission of Lassa virus to susceptible rodents

occurs when they share unprotected storage of garbage, food stuff and water with infected rodents or from inhalation of aerosols from urine. When a susceptible rodent interact with infectious rodent, the virus enters the rodent with probability $\beta(e_j)$ and therefore the susceptible go to the exposed class $E_r(t, e_j)$. The exposed rodent then becomes infectious and enters the class $I_r(t, e_j)$ after a given time. . In this study, it is assumed that individuals who recovered from Lassa fever will never go back to susceptible class again (they remain recovered for life).

$$\begin{aligned}
&\frac{dS_h(t,a_i)}{dt}\\
&= \Lambda_h(a_i) - \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\
&+ \frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))}{N_h(t,a_i)} \times S_h(t,a_i)\\
&+ \frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)} - \mu_h(a_i)S_h(t,a_i),
\end{aligned} \tag{1}$$

$$\begin{aligned}
\frac{dE_h(t,a_i)}{dt} &= \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))S_h(t,a_i)I_h(t,a_i)}{N_h(t,a_i)}\\
&+ \frac{\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))S_h(t,a_i)I_r(t,e_j)}{N_h(t,a_i)}\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))}{N_h(t,a_i)} \times S_h(t,a_i)\\
&+ \frac{(1+\kappa(a_i)\cos(2\pi t+T))S_h(t,a_i)}{N_h(t,a_i)}\\
&- (\gamma(a_i)\alpha_1(a_i) + \epsilon_h(a_i) + \mu_h(a_i))E_h(t,a_i),
\end{aligned} \tag{2}$$

$$\frac{dI_h(t,a_i)}{dt} = \sum_{i=0}^{L} \epsilon_h(a_i)E_h(t,a_i) - (\psi(a_i)\alpha_2(a_i) + \mu_h(a_i) + \delta_h(a_i))I_h(t,a_i), \tag{3}$$

$$\frac{dR_h(t,a_i)}{dt} = \sum_{i=0}^{L} \gamma(a_i)\alpha_1(a_i)E_h(t,a_i) + \psi(a_i)\alpha_2(a_i)I_h(t,a_i) - \mu_h(a_i)R_h(t,a_i), \tag{4}$$

$$\begin{aligned}
\frac{dS_r(t,e_j)}{dt} &= \Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))\\
&+ \sum_{j=0}^{T} \frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))}{N_r(t,e_j)} \times S_r(t,e_j)I_r(t,e_j)\\
&+ \frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)} - (\mu_r(e_j) + \delta_r(e_j))S_r(t,e_j),
\end{aligned} \tag{5}$$

$$\frac{dE_r(t,e_j)}{dt} = \sum_{j=0}^{T} \frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))S_r(t,e_j)I_r(t,e_j)}{N_r(t,e_j)} + \frac{(1+\theta(e_j)\cos(2\pi t+T))S_r(t,e_j)}{N_r(t,e_j)} - (\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))E_r(t,e_j), \tag{6}$$

$$\frac{dI_r(t,e_j)}{dt} = \sum_{j=0}^{T} \epsilon_r(e_j)E_r(t,e_j) - (\mu_r(e_j)+\delta_r(e_j))I_r(t,e_j) \tag{7}$$

together with the initial conditions

$$\begin{aligned} &S_h(0,a_i) = S_{0h}(a_i), E_h(0,a_i) = E_{0h}(a_i),\\ &I_h(0,a_i) = I_{0h}(a_i), R_h(0,a_i) = R_{0h}(a_i),\\ &S_r(0,e_j) = S_{0r}(e_j), E_r(0,e_j) = E_{0r}(e_j), I_r(0,e_j) = I_{0r}(e_j). \end{aligned} \tag{8}$$

3. Model analysis

Here we present theoretical results and numerical simulation of the model presented above.

3.1 Existence and positivity of solutions

Lassa fever model governed by system (1)-(7) is mathematical well-posed in a feasible region $\mathcal{D}$ defined by $\mathcal{D} = \mathcal{D}_h \times \mathcal{D}_r \subset \mathcal{R}_+^4 \times \mathcal{R}_+^3$ where

$$\mathcal{D}_h = \left\{ S_h(t,a_i), E_h(t,a_i), I_h(t,a_i), R_h(t,a_i) \in \mathcal{R}_+^4 : N_h \leq \sum_{i=0}^{L} \frac{\Lambda_h(a_i)}{\mu_h(a_i)} \right\}$$

and

$$\mathcal{D}_r = \left\{ S_r(t,e_j), E_r(t,e_j), I_r(t,e_j), \in \mathcal{R}_+^3 : N_r \leq \sum_{j=0}^{T} \frac{\Lambda_r(e_j)(1+\phi_r(|\cos T|+|\sin T|))}{\mu_r(e_j)+\delta_r(e_j)} \right\}.$$

3.2 Disease-free equilibrium π_0

Disease-free equilibrium points are steady-state solutions where there is no Lassa fever infection. Thus, the disease-free equilibrium point, π_{01} for the Lassa fever model (1)-(7) implies that $S_h^*(a_i) \neq 0, E_h^*(a_i) = I_h^*(a_i) = 0, S_r^* \neq 0, E_r^* = I_r^* = 0$ and putting these into (1), (4) and (5) yields $R_h^*(a_i) = 0, S_h^*(a_i) = \frac{\Lambda_h(a_i)}{\mu_h(a_i)}$, and $S_r^*(e_j) = \frac{\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))}{\mu_r(e_j)+\delta_r(e_j)}$ respectively. Consequently, we obtain π_0 as

$$\pi_0 = \left(\frac{\Lambda_h(a_i)}{\mu_h(a_i)}, 0, 0, 0, \frac{\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T)}{\mu_r(e_j)+\delta_r(e_j)}, 0, 0 \right). \tag{9}$$

Using the next generation matrix technique as described in [7], we obtain the threshold parameter known as basic reproduction number, $\mathcal{R}_0$, as

$$\mathcal{R}_0(a,t)=\sum_{i=0}^{L}\frac{\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))\epsilon_h(a_i)}{(\gamma(a_i)\alpha_1(a_i)+\epsilon_h(a_i)+\mu_h(a_i))(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))}. \tag{10}$$

3.3 Endemic equilibrium point E_{e2}

Here, we shall show that the formulated model (1)-(7) has an endemic equilibrium point, E_e. The endemic equilibrium point is a positive steady state solution where the disease persists in the population.

Theorem 1. *Lassa fever model (1) - (7) has no endemic equilibrium when* $\mathcal{R}_0(a,t)<1$ *and a unique endemic equilibrium exist when* $\mathcal{R}_0(a,t)>1$.

Proof. Let $E_e=(S_h^{**},E_h^{**},I_h^{**},R_h^{**},S_r^{**},E_r^{**},I_r^{**})$ be a non-trivial equilibrium of system (1) -(2.7). That is, all component of E_e are positive. Then the Lassa fever model (1)-(7) at steady state becomes

$$\begin{aligned} I_r^{**}(e_j) &= \Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))\epsilon_r(e_j)\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)\\ &\quad -((1+\theta\cos(2\pi t+T))\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))\\ &\quad \times(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))+P, \end{aligned} \tag{11}$$

$$\begin{aligned} &E_r^{**}(e_j)\\ &=\sum_{j=0}^{T}\frac{\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)I_r^{**}(e_j)+(1+\theta(e_j)\cos(2\pi t+T))\Lambda_{0r}}{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))[\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))I_r^{**}(e_j)+E_{or}]} \end{aligned} \tag{12}$$

and

$$\begin{aligned} &S_r^{**}(e_j)\\ &=\frac{\Lambda_r^2(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))}{\sum_{j=0}^{T}[\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))I_r^{**}(e_j)+(1+\theta(e_j)\cos(2\pi t+T))+\Lambda_{0r}]r_d}, \end{aligned} \tag{13}$$

$$\begin{aligned} R_h^{**}(a_i) &= \frac{\sum_{i=0}^{L}(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\gamma(a_i)\alpha_1(a_i)I_h^{*}(a_i)}{\mu_h(a_i)\sum_{i=0}^{L}\epsilon_h(a_i)}\\ &\quad +\frac{\psi(a_i)\alpha_2(a_i)I_h^{**}(a_i)}{\mu_h(a_i)}, \end{aligned} \tag{14}$$

$$E_h^{**}(a_i) = \frac{I_h^{**}(a_i)(\psi(a_i)\alpha_2(a_i) + \mu_h(a_i) + \delta_h(a_i))}{\sum_{i=0}^{L} \epsilon_h(a_i)} \tag{15}$$

and

$$S_h^{**}(a_i) = \frac{I_h^{**}(a_i)\Lambda_h(a_i)(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))}{\sum_{i=0}^{L}\sum_{j=0}^{T}\mu_h(a_i)\epsilon_h(a_i)U}, \tag{16}$$

where

$$\begin{aligned} U = {} & [\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))\Lambda_{0r}\epsilon_r(e_j)\sigma_1(1+\beta(e_j)\cos(2\pi t+T)) \\ & -(1+\theta(e_j)\cos(2\pi t+T)) + \Lambda_{0r}\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))(\mu_r(e_j) \\ & +\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)) + \sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))P \\ & +\sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))I_h^{**}(a_i) \\ & +(1+\theta(e_j)\cos(2\pi t+T)) + (1+\kappa(a_i)\cos(2\pi t+T))], \end{aligned}$$

$$P = \frac{\sqrt{[(\theta_0+\Lambda_{0r})(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))-\Lambda_{0r}\epsilon_r(e_j)\beta_0]^2+M}}{2\beta_0(e_j)(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))},$$

$$\begin{aligned} & \Lambda_{0r} = \Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T)), \\ & \beta_0 = \sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)), \theta_0 = (1+\theta(e_j)\cos(2\pi t+T)), \\ & r_d = (\mu_r(e_j)+\delta_r(e_j)), E_{0r} = (1+\theta(e_j)\cos(2\pi t+T)) + \Lambda_{0r}, \end{aligned}$$

with

$$\begin{aligned} M = {} & 4\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))\Lambda_{0r}\epsilon_r(e_j) \\ & \times(1+\theta(e_j)\cos(2\pi t+T))(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)), \end{aligned}$$

$$K_1(I_h^{**})^2 + K_2 I_h^{**} + K_3 = 0, \tag{17}$$

where

$$\begin{aligned} K_1 = {} & \sigma_2(a_i)(1+\eta(a_i)\cos(2\pi t+T))\mu_h(a_i), \\ K_2 = {} & \sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T)(1+\beta(e_j)\cos(2\pi t+T))\epsilon_r(e_j)\mu_h(a_i)\Lambda_{0r} \\ & +\mu_h(a_i)\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))((1+\theta(e_j)\cos(2\pi t+T)) \\ & +\Lambda_{0r})(\mu_r(e_j)+\delta_r(e_j)) \times (\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)) \\ & +\mu_h(a_i)\sigma_1(e_j)(1+\rho(a_i)\cos(2\pi t+T))P \\ & +\mu_h(a_i)((1+\theta(e_j)\cos(2\pi t+T)) + (1+\kappa(a_i)\cos(2\pi t+T)) \\ & +\Lambda_h(a_i)\mu_h(a_i)(1-\mathcal{R}_0(a,t)), \\ K_3 = {} & d_1 - (d_2+d_3), \end{aligned}$$

with

$$d_1 = \frac{\begin{array}{ll}\mu_h(a_i)\epsilon_h(a_i)\Lambda_h(a_i)\sigma_1(e_j)(1 & +\rho(a_i)\cos(2\pi t+T))(\theta_0+\Lambda_{0r})(\mu_r(e_j)\\ & +\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))\end{array}}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))},$$

$$d_2 = \frac{\begin{array}{ll}\Lambda_h(a_i)H_0\Lambda_{0r}\epsilon_r(e_j)(1 & +\beta(e_j)\cos(2\pi t+T))\mu_h(a_i)\epsilon_h(a_i)\\ & +H_0\mu_h(a_i)\epsilon_h(a_i)\Lambda_h(a_i)P\end{array}}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))},$$

$$d_3 = \frac{\Lambda_h(a_i)\epsilon_h(a_i)\mu_h(a_i)((1+\theta(e_j)\cos(2\pi t+T))+(1+\kappa(a_i)\cos(2\pi t+T))}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))}$$

$H_0 = \sigma_2(a_i)(1+\rho(a_i)\cos(2\pi t+T))$. It is clearly seen that $K_1 > 0$. When $\mathcal{R}_0(a,t) < 1$, one sees that $K_2 > 0$. However, when $R_0(a,t) > 1$, then $K_2 < 0$ and endemic equilibrium exists. Finally, $K_3 < 0$ if $d_1 < (d_2 + d_3)$. □

3.4 Subharmonic bifurcation analysis

Here, we are interested in the case that the rodent birth and contact rate are periodic of period one year. For this reason it is convenient to take a year as our unit of time. With this convention, the human birth and death rate, $\Lambda_h(a_i)$, $\mu_h(a_i)$ and rodent birth and death rate $\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))$ and μ_r will be such that $\frac{1}{\Lambda_h(a_i)}$, $\frac{1}{\mu_h(a_i)}$,$\frac{1}{\Lambda_r(e_j)(1+\phi_r\cos(2\pi t+T))}$ and $\frac{1}{\mu_r(e_j)}$ are 50 years, the latent period prior to becoming infectious $\frac{1}{\mu_h(a_i)+\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)}$, $\frac{1}{\mu_r(e_j)+\delta_r(e_j)+\epsilon_r(e_j)}$ for human and mosquito are typically a few days to a week, the same is true for average infectious period $\frac{1}{\mu_h(a_i)+\delta(a_i)+\psi(a_i)\alpha_2(a_i)}$ and $\frac{1}{\delta_r(e_j)+\mu_r(e_j)}$. We exploit the fact that $\mu_h(a_i)$, $\Lambda_h(a_i)$, $\mu_r(e_j)$, $\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))$, $\frac{1}{\mu_h(a_i)+\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)}$, $\frac{1}{\mu_h(a_i)+\delta(a_i)+\psi(a_i)\alpha_2(a_i)}$, $\frac{1}{\mu_r(e_j)+\epsilon_r(e_j)}$ and $\frac{1}{\delta_r(e_j)+\mu_r(e_j)}$ are $0(10^{-2})$ by introducing a small parameters ε as follows:

$$\begin{aligned}
&I_h^{**}(a_i) = \Lambda_h(a_i)\\
&= \frac{I_h^{**}(a_i)\Lambda_h(a_i)(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\gamma^0\mathcal{R}_{02}^2(a,t)}{\mu_h(a_i)\epsilon_h(a_i)U(\mathcal{R}_{02}^2(a,t)-1)} = \varepsilon,\\
&\mu_h(a_i) = \frac{I_h^{**}(a_i)}{\mathcal{R}_{02}^2(a,t)-1} = \varepsilon,\\
&\epsilon_h(a_i)+\mu_h(a_i)+\gamma(a_i)\alpha_1(a_i)\\
&= \frac{I_h^{**}(a_i)\Lambda_h(a_i)\epsilon_h(a_i)}{(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))\mu_h(a_i)\mu_r(e_j)\mathcal{R}_{01}^2(a,t)} = \frac{\Delta_2}{\varepsilon},\\
&\psi(a_i)\alpha_2(a_i)+\delta_h(a_i)+\mu_h(a_i) = \frac{\Delta_3}{\varepsilon},
\end{aligned} \tag{18}$$

$$I_r^{**}(e_j)=\mu_r(e_j)=[\mu_r(e_j)+\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T)))I_r^{**}(e_)+\theta^0]\mu^0=\varepsilon,$$
$$\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)=\frac{\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)}{(1+\beta\cos(2\pi t+T))+E_{0r}}=\frac{\Delta_4}{\varepsilon},$$
$$\mu_r(e_j)+\delta_r(e_j)=\Lambda_r(e_j)(1+\phi_r(e_j)\cos(2\pi t+T))\epsilon(e_j)\sigma^0=\frac{\Delta_5}{\varepsilon},$$

where $\gamma^0=(\gamma(a_i)\alpha_1(a_i)+\mu_h(a_i)+\epsilon_h(a_i))$, $\theta^0=(1+\theta(e_j)\cos(2\pi t+T))+\Lambda_{0r}$, $\mu^0=\mu_r(e_j)+\delta_r(e_j)$, $\sigma^0=\sigma_1(e_j)(1+\beta(e_j)\cos(2\pi t+T))-(1+\theta(e_j)\cos(2\pi t+T)(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))+P$. New state variables (x,y,z,p,q,r) are introduced by setting

$$\begin{aligned}
&S_h(t,a_i)=S_h^{**}(a_i)(1+x^0), E_h(t,a_i)=E_h^{**}(a_i)(1+y^0),\\
&I_h(t,a_i)=I_h^{**}(a_i)(1+z^0),\\
&S_r(t,E_j)=S_r^{**}(e_j)(1+p^0), E_r(t,e_j)=E_r^{**}(e_j)(1+q^0),\\
&I_r(t,e_j))=I_r^{**}(e_j)(1+r^0).
\end{aligned}\tag{19}$$

Incorporating (18) and (19) in model (1)-(7) yield the following system of equations which has the property that when $\rho(a_i)=\eta(a_i)=\kappa(a_i)=\beta(e_j)=\theta(e_j)=0$ and $\phi_r=0$, the endemic equilibrium becomes $(x^0,y^0,z^0,p^0,q^0,r^0)=0$

$$\begin{aligned}
\dot{x}^0&=-\varepsilon[(\Theta+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))x^0\\
&\quad+(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))z^0\\
&\quad+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)\\
&\quad+(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)x^0z^0],\\
\dot{y}^0&=\frac{\Delta_{02}}{\varepsilon}[(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T)\\
&\quad+z^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))\\
&\quad+x^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))\\
&\quad+r^0x^0(1+(\rho(a_i)+\eta(a_i)+\kappa(a_i)\theta(e_j))\cos(2\pi t+T))-y^0],\\
\dot{z}^0&=\frac{\Delta_{03}}{\varepsilon}[y^0-z^0],\\
\dot{p}^0&=-\varepsilon[p^0+1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T)\\
&\quad+p^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))\\
&\quad+r^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))\\
&\quad+p^0z^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))],\\
\dot{q}^0&=\frac{\Delta_0}{\varepsilon}[(\beta(e_j)+\theta(e_j))\cos(2\pi t+T)\\
&\quad+p^0(1+(\beta(e_j)+\theta(e_j))\cos(2\pi t+T))
\end{aligned}\tag{20}$$

$$
\begin{aligned}
&+ r^0(1 + (\beta(e_j) + \theta(e_j))\cos(2\pi t + T)) \\
&+ p^0 z^0(1 + (\beta(e_j) + \theta(e_j))\cos(2\pi t + T)) - q^0], \\
\dot{r}^0 &= \frac{\Delta_{05}}{\varepsilon}[q^0 - r^0],
\end{aligned}
$$

where $\Theta = \frac{\mathcal{R}_0(a,t)}{\mathcal{R}_0(a,t)-1}$. Before proceeding further in analyzing the periodic system (20) we need information concerning the eigenvalues of the linearized system about the endemic steady state when $\rho(a_i) = \eta(a_i) = \kappa(a_i) = \beta(e_j) = \theta(e_j) = 0$. For $\rho(a_i) = \eta(a_i) = \kappa(a_i) = \beta(e_j) = \theta(e_j) = 0$ the endemic steady state is the origin and we have the following lemma.

Lemma 1. *The eigenvalues corresponding to the linearized system*

$$
\begin{pmatrix} x^0 \\ y^0 \\ z^0 \end{pmatrix}' = \begin{pmatrix} -\varepsilon\Theta & 0 & -\varepsilon \\ \frac{\Delta_{02}}{\varepsilon} & -\frac{\Delta_{02}}{\varepsilon} & \frac{\Delta_{02}}{\varepsilon} \\ 0 & \frac{\Delta_{03}}{\varepsilon} & -\frac{\Delta_{03}}{\varepsilon} \end{pmatrix} \begin{pmatrix} x^0 \\ y^0 \\ z^0 \end{pmatrix}
$$

are given by $\lambda_+, \lambda_-, \lambda_3$ *below:*$\lambda_\pm(\varepsilon) = \varepsilon r_1 \pm i\nu + 0(\varepsilon^2)$, *where*

$$
\begin{aligned}
\nu &= \sqrt{\frac{\Delta_{02}\Delta_{03}}{\Delta_2 + \Delta_3}}, \\
r_1 &= \frac{\Delta_{02}\Delta_{03} - (\Delta_{02} + \Delta_0 3)^2\eta}{2(\Delta_{02} + \Delta_{03})^2} < 0, \\
\lambda_3(\varepsilon) &= -\frac{(\Delta_{02} + \Delta_{03})}{\varepsilon} + 0(\varepsilon).
\end{aligned} \tag{21}
$$

We observed that the endemic steady state is asymptotically stable but the attraction is weak. There is a rapid relaxation onto a center manifold on which orbits slowly spiral into the origin. The underlying mechanism we will exploit is the weakly damped oscillation on the center manifold. to leading order, the frequency of this oscillation, ν depends on the reproduction rate $\mathcal{R}_0(a,t)$, the birth rate $\Lambda(a_i)$ and the sum of the exposed and infectious period.

Our goal at this point is to make change of variable in (20) when $\rho(a_i), \eta(a_i)$, $\kappa(a_i), \beta(e_j), \theta(e_j)$ are equal to zero. In addition, further analysis will require the assumption of a small perturbation in (2.20). Inspection of (2.20) shows that this will be the case only if $\frac{\rho(a_i)}{\varepsilon}, \frac{\eta(a_i)}{\varepsilon}, \frac{\kappa(a_i)}{\varepsilon}, \frac{\theta(e_j)}{\varepsilon}$ are small. With this remarks as motivation we proceed as follows. Let

$$
\begin{aligned}
&\bar{\rho}(a_i), \bar{\eta}(a_i), \bar{\kappa}(a_i), \bar{\theta}(e_j) = \frac{\rho(a_i)}{\varepsilon}, \frac{\eta(a_i)}{\varepsilon}, \frac{\kappa(a_i)}{\varepsilon}, \frac{\theta(e_j)}{\varepsilon}, \\
&\bar{x}^0 = v\left[\frac{x^0}{\varepsilon} - \frac{\varepsilon\Delta_{03}(z^0 - y^0)}{(\Delta_{02} + \Delta_{03})^2}\right], \\
&\bar{y}^0 = \frac{\Delta_{03}y^0 + \Delta_{02}z^0}{\Delta_{02} + \Delta_{03}}, \\
&\bar{z}^0 = z^0 - y^0,
\end{aligned} \tag{22}
$$

where ν is given in Lemma 1. While this transformation appears to mix up the various epidemiological classes, a glance at (20) indicates that one should expect $z^0 - y^0$ to be small, say order ε, hence (22) should take the particularly simple form

$$\begin{aligned} x^0 &= \frac{\varepsilon \bar{x}^0}{\nu} + 0(\varepsilon^3), \\ y^0 &= \bar{y}^0 + 0(\varepsilon), \\ z^0 &= \bar{y}^0 - 0(\varepsilon). \end{aligned} \tag{23}$$

Epidemiologically, equation (23) has the interpretation that the ratio of infectives to exposed individuals is $\frac{\psi(a_i)lpha_2(a_i)}{\epsilon_h(a_i)}$ to order ε.

Putting (22) in (20) and ignoring the bar over $\rho(a_i), \eta(a_i), \kappa(a_i), \beta(e_j), \theta(e_j)$ we obtain

$$\begin{aligned} (\bar{x}^0)' &= -\nu \bar{y}^0 + \varepsilon f_{01}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \\ (\bar{y}^0)' &= \nu \bar{x}^0 (1 + \bar{y}^0) + \frac{\nu \Delta_{03} \bar{x}^0 \bar{z}^0}{\Delta_{02} + \Delta_{03}} + (\nu)^2 (\rho(a_i) + \eta(a_i) + \kappa(a_i) + \theta(e_j)) \\ &\quad \times \cos(2\pi t + T) \left(1 + \bar{y}^0 + \frac{\Delta_{03} \bar{z}^0}{\Delta_{02} + \Delta_{03}}\right) \\ &\quad + \varepsilon f_{02}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \\ \varepsilon(\bar{z}^0)' &= -(\Delta_{02} + \Delta_{03}) \bar{z}^0 + \varepsilon f_{03}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t, \varepsilon, \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)), \end{aligned} \tag{24}$$

where

$$\begin{aligned} f_{01}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t+T, \pi, \varphi^0, \varepsilon) &= -\bar{x}^0 \left(\Theta - \frac{\Delta_{02}\Delta_{03}}{(\Delta_{02} + \Delta_{03})^2}\right) \\ &- \bar{x} \left(\bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02} + \Delta_{03})}\right) \left(1 - \frac{\Delta_{02}\Delta_{03}}{(\Delta_{02} + \Delta_{03})^2}\right) + 0(|\varepsilon| + \varphi^0|), \\ f_{02}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t+T, \pi, \varphi^0, \varepsilon) &= \frac{\nu^2 \Delta_{03} \bar{z}^0}{(\Delta_{02}+\Delta_{03})^2} \left(1 + \bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02}+\Delta_{03})}\right) + 0(|\varepsilon| + \varphi^0|), \\ f_{03}(\bar{x}^0, \bar{y}^0, \bar{z}^0, t+T, \pi, \varphi^0, \varepsilon) &= -\Delta_{02}\nu^{-1}\bar{x}^0 \left(1 + \bar{y}^0 - \frac{\Delta_{03}\bar{z}^0}{\Delta_{02}+\Delta_{03})}\right) + 0(|\varepsilon| + |\varphi^0|), \end{aligned}$$

where $\varphi^0 = \rho(a_i), \eta(a_i), \kappa(a_i), \theta(e_j)$ We will treat ε and φ^0 as small parameters in (24). Setting $\varepsilon = \varphi^0 = 0$ in (24) we obtain the reduced equations

$$\begin{aligned} (\bar{x}^0)' &= -\nu \bar{y}^0, \\ (\bar{y}^0)' &= \nu \bar{x}^0 (1 + \bar{y}^0), \\ \bar{z}^0 &= 0. \end{aligned} \tag{25}$$

Now,we show that the system (25) is conservative with first integral given by

$$V = (x^0)^2 + 2y^0 - 2\ln|1 + y^0|.$$

A system is conservative if there a function $V(x^0, y^0)$ such that $\frac{dV}{dt} = 0$ along the solution curves of x^0 and y^0.

$$\frac{dV}{dt} = \frac{dV}{dx^0}\frac{dx^0}{dt} + \frac{dV}{dy^0}\frac{dy^0}{dt},$$
$$\frac{dV}{dt} = 2x^0(-\nu y^0) + \left[2 - \frac{2}{1+y^0}\right](\nu x^0(1+y^0)),$$
$$\frac{dV}{dt} = -2x^0\nu y^0 + 2\nu x^0(1+y^0) - 2\nu x^0 = 0.$$

It is also interesting to observe that if we write $u = \ln(1+y^0)$ and use (4.2.41) to write a second order differential equation for u we obtain

$$y^0 = e^u - 1, u' = \frac{\nu x^0(1+y^0)}{1+y^0}, u' = \nu x^0, u'' + \nu^2(e^u - 1) = 0.$$

The essential feature the case when $\varepsilon = \varphi_h^0 = 0$ using (25) is that the origin is a center surrounded by periodic orbit with periods ranging between $\frac{2\pi}{\nu}$ near the origin to $+\infty$. In particular, for every integer $n, \frac{2\pi}{\nu} < \nu < \infty$, there is a periodic solution of (25), $(\bar{x}_n^0(t), \bar{y}_n^0(t))$ of at least period n. These solutions of the reduced equation may be excited by period one forcing at the correct amplitude. Hence, we are led to expect n-periodic solution of (23) near $\Gamma_{n1}^0 = (x^0, y^0, z^0) : (x^0, y^0, z^0) = (\bar{x}_n^0(t), \bar{y}_n^0(t), 0), 0 \le t \le n)$ at least for a suitable small values of ε and φ^0. Similar argument holds for $\varepsilon,\ \beta(e_j), \theta(e_j)$ and $\phi_r(e_j)$ in the rodent population. Thus, we have the following results:

Theorem 2. *Let $\bar{x}_n^0(t), \bar{y}_n^0(t) = \bar{x}_n^0(t+n), \bar{y}_n^0(t+n)$ denote a periodic solution of the reduced equation (25), where $n > \frac{2\pi}{\nu}$. Let*

$$\psi_2(a_i)\alpha_2(a_i) \equiv \nu^2 \int_0^n \bar{y}^0(t)\cos(2\pi t + T)dt \neq 0, \psi_2(a_i)\alpha_2(a_i) = \frac{-2r_1}{\nu}.$$

(Area interior to Γ_{n1}) and for $\epsilon_h(a_i)\varepsilon[0, n), |\varepsilon| \ll 1, |\varphi^0| \ll 1$, let

$$\mathbf{B}(\epsilon_h(a_i), \varepsilon, \varphi^0) = \psi_1(a_i)\alpha_2(a_i)\varepsilon + \psi_2(a_i)\alpha_2(a_i)\varphi^0\cos 2\pi\epsilon_h(a_i) + 0(|\varepsilon| + |\varphi^0)|)^2. \tag{26}$$

If $(\bar{x^0}'_n, \bar{y^0}'_n)$ spans the n- periodic of the variational equations of (25) about $(\bar{x}'_n, \bar{y^0}'_n)$, and $(\epsilon_h(a_i), \varepsilon, \varphi^0 = 0$, then equation (24) has an n-periodic solution $(\bar{x}^0, \bar{y}^0, \bar{z}^0)$ given by

$$\begin{aligned} \bar{x}^0(t) &= \bar{x}_n^0(t + \epsilon_h(a_i) + 0(|\varepsilon| + (1 + |\varphi^0|)), \\ \bar{y}^0(t) &= \bar{x}_n^0(t + \epsilon_h(a_i) + 0(|\varepsilon| + |\varphi^0|), \\ \bar{z}^0(t) &= -\frac{\varepsilon\Delta_{02}\bar{y^0}'(t + \epsilon_h(a_i))}{\nu^2(\Delta_{02} + \Delta_{03)}} + 0(|\varepsilon| + |\varphi^0|)^2. \end{aligned} \tag{27}$$

Table 2. Description of the model parameters

Definition	Parameter	Value
Recruitment term of humans	$\Lambda_h(a_i)$	0.038
Effective transmission rate in susceptible humans by infected rodents	$\rho(a_i)$	0.6
Interacting rate of reservoirs	$\sigma_1(e_j)$	0.8
Effective transmission rate in susceptible humans by infected humans	$\eta(a_i)$	0.6
Interacting rate of humans	$\sigma_2(a_i)$	0.56
Treatment rate of exposed humans	$\alpha_1(a_i)$	0.05
Treatment rate of infected humans	$\alpha_2(a_i)$	0.9
Effective contact rate between Lassa virus and humans or reservoirs	$\theta(e_j)$	0.022
Rate of inoculation	$\kappa(a_i)$	0.018
Progression rate of humans from the exposed state to the infectious state	$\epsilon_h(a_i)$	0.85
Diagnostic factor of exposed humans	$\gamma(a_i)$	0.9
proportion of effective treatment of infected humans	$\psi(a_i)$	0.45
Natural death rate of humans	$\mu_h(a_i)$	0.02
Progression rate of reservoirs from the exposed state to infectious state	$\epsilon_r(e_j))$	0.85
Effective transmission rate in susceptible reservoirs by infected reservoirs	$\beta(e_j)$	0.75
Disease induced death rate of humans	$\delta_h(e_j)$	0.2
Mortality of reservoirs due to hunting	$\delta_r(e_j)$	0.3
Natural death rate reservoirs	$\mu_r(e_j)$	0.6
Recruitment term of reservoirs	$\Lambda_r(e_j)$	0.56
Seasonal variation of rodent factor	$\phi_r(e_j)$	Assumed to vary

4. Numerical results

The model (1)-(7) is simulated using parameters in Table 1 to illustrate some of the theoretical results established in this study. Figures 1, and 2 show the impact of time harmonic disturbance for rodent birth and Lassa fever transmission to humans. It is noticed that rodent birth and Lassa fever transmission varies seasonally, depending on the influence of climatic variables.

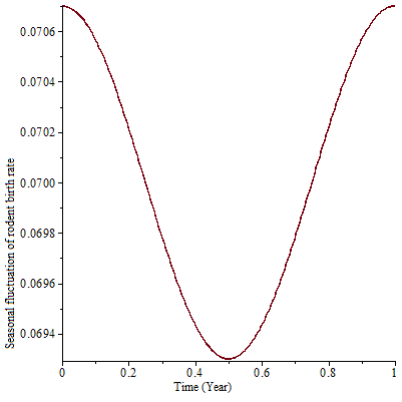


Figure 1: Seasonal fluctuation of rodents birth rate

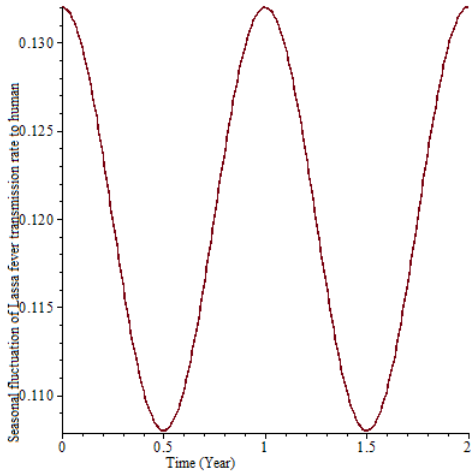


Figure 2: Seasonal fluctuation of Lassa fever transmission rate to humans

5. Conclusion

Seasonal change in the incidence of infectious diseases is a common phenomenon in both temperate and tropical climates. However, the mechanisms responsible for seasonal disease incidence, and the epidemiological consequences of seasonality, are poorly understood with rare exception. This study highlighted a compartmental modelling for the transmission dynamics of Lassa fever considering time harmonic disturbance for rodent (seasonal variation of rodent birth and transmission rate) with focus on two populations: Human and vector. The model was analysed for endemic equilibrium, the basic reproduction number $R_0(a,t)$ is obtained, and local stability of the non-trivial equilibrium was determined. Applying perturbation methods, stable subharmonic bifurcation solutions of period n are proved to coexist simultaneously and infinite number of stable subharmonic bifurcation solutions were established. Numerical simulations are carried out to investigate the behaviour of the system. This study suggests that intervention strategies needs to be deployed especially during the wet season. Also, every effort must be put in place by all concerned to prevent the virus infection by reducing the basic reproduction number, $R_0(a,t)$ until Lassa fever eradication is achieved.

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Characterizations and extensions of abelian rings

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Abstract. In this paper we investigate the abelian rings on their characterizations as well as their extensions and homomorphic properties. A new viewpoint on the characterization of abelian ring is given. Then it is proved that abelian property is inherited under series of extensions such as (skew) polynomial extension, (skew) power series extension, some matrix extensions, Dorroh extension as well as Nagata extension. In the end, we discuss the condition under which abelian property can be held by a homomorphism.

Keywords: abelian rings, idempotent, extension, endomorphism.

1. Introduction

Throughout this paper, all rings are associative rings with an identity. Let $E(R)$, $C(R)$ and $UC(R)$ denote the set of all idempotents, the center and the set of all central invertible element of R respectively. We call a ring R an abelian ring, if $E(R) \subset C(R)$. Obviously, a commutative ring is an abelian ring. Many authors have studied the judgement of abelian rings. In [1], Hum shows that reduced rings(i.e. a ring with no nonzero nilpotent) and semicommutative rings (i.e. ab=0 implies arb=0 for all a,b and R in it) are abelian rings; According to [2], Kim points out that Armendariz rings are abelian rings. In [3], Liu Zhongkui proved that $\alpha - rigid$ rings(i.e. rings with the condition $a\alpha(a) = 0$ implies a=0, in which α is an endomorphism of it) is reduced so that is abelian; In [4], Kim proves that a reversible ring (i.e. ab=0 implies ba=0) is a semicommutative ring. Then every reversible ring is abelian. In [5], Wang yonghui discusses the trivial extensions and some matrix extensions of abelian rings. Motivated above,

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we will make some further discussions on the characterizations and extensions of abelian rings.

2. Characterizations of abelian rings

Definition 2.1. A ring R is called abelian, if $E(R) \subset C(R)$.

To determine the abelian property of a ring R is very general in the research of abelian rings. We induce the following definitions.

Definition 2.2. Let $A(R) = \{a \in R \mid ea = a(1-e) \text{ for some } e \in E(R)\}$, $A_r(R,\sigma) = \{a \in R \mid ea = a(1-\sigma(e))\}$, $A_l(R,\sigma) = \{a \in R \mid ea = a(1-\sigma(e))\}$, where σ is an endomorphism of R.

Many properties of $A(R)$ could be found immediately such as $A(R)$ is nonempty since $0 \in A(R)$. For any nonnegative integer n, $A^{2n+1}(R) \subseteq A(R) = 0$, and $E(R) \subseteq C(A^{2n}(R))$, etc.

And it is well known and generally applied that R is abelian if and only if $eR(1-e)$, which is an inference of the next theorem:

Theorem 2.3. *Let R be a ring with an endomorphism σ. Then the following statements are equivalent:*

(1) *R is abelian and $\sigma(e) = e$ for any $e \in E(R)$;*
(2) *$eR(1-\sigma(e)) = 0$ for each $e \in E(R)$;*
(3) $A_r(R) = 0$;
(4) *for any $r \in R, e \in E(R)$, $er = r\sigma(e)$.*

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), (4) $\Rightarrow$ (1) Obviously.

It is suffice to proof (2) $\Rightarrow$ (3). For each $a \in A(R)$ there exists an $e \in E(R)$, such that $ea = a(1-\sigma(e)) = ea(1-\sigma(e)) = 0, (1-e)a = a\sigma(e) = (1-e)a\sigma(e) = 0$, thus $a = ea + (1-e)a = 0$ and $A_r(R) = 0$.

Corollary 2.4. *Let R be a ring with an endomorphism σ, and $A_r(R,\sigma) = 0$. Then $A(R) = 0$.*

While the converse is not true when $\sigma = 0$. This also indicates that the condition $\sigma(e) = e$ in Theorem 2.3 is not surplus.

Corollary 2.5. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*
(2) $eR(1-e) = 0$;
(3) $A(R) = 0$.

3. Extension properties of abelian rings

In this chapter, we will investigate some extension properties on abelian rings.

Let σ be an endomorphism of a ring R. Let $R[x,\sigma], R[[x,\sigma]]$ denote the skew polynomial ring and the skew power series ring, where $xa = \sigma(a)x$ for all $a \in R$, while the polynomial ring and power series ring over R is denoted by $R[x]$ and $R[[x]]$ respectively. Let σ^* denotes the endomorphism on $R[[x]]$ such that $\sigma^*(\sum_{j=0}^{\infty} a_j x^j) = \sum_{j=0}^{\infty} \sigma(a_j)x^j$. And in the next paragraph, $A_r(R[x,\sigma])(A_r(R[[x,\sigma]]))$ denotes $A_r(R[x,\sigma],\sigma^*)(A_r(R[[x,\sigma]],\sigma^*))$.

Lemma 3.1. *Let R be a ring , if $e(x) = \sum_{j=0}^{\infty} a_j x^j \in E(R[[x]])$, then $a_0 \in E(R)$ and $a_1 \in A(R)$.*

Proof. $e^2(x) = e(x)$, then $e_0^2 = e_0$ and $e_0e_1 + e_1e_0 = e_1$, thus the conclusion follows. More generally, if $e(x) = a_0 + \sum_{j=k}^{\infty} a_j x^j \in E(R[[x]])$, then $a_0 \in E(R)$ and $a_k \in A(R)$.

Theorem 3.2. *Let R be a ring with an endomorphism σ, then the following statements are equivalent:*

(1) *R is abelian and $\sigma(e) = e$ for any $e \in E(R)$;*
(2) *$E(R) = E(R[x,\sigma]) = E(R[[x,\sigma]])$;*
(3) *$R[x,\sigma]$ is abelian;*
(4) *$R[[x,\sigma]]$ is abelian;*
(5) *$A_r(R) = A_r(R[x,\sigma]) = A_r(R[[x,\sigma]])$.*

Proof. (1) $\Rightarrow$ (2) Clearly $E(R) \subseteq E(R[x,\sigma]) \subseteq E(R[[x,\sigma]])$, it is enough to show that $E(R[[x,\sigma]]) \subseteq E[R]$. for each $e(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x,\sigma]]$, by Lemma 3.1, $e_0 \in E(R)$, $e_1 \in A_r(R) = 0 = A(R)$, since R is abelian. By the same proceeding we have $a_2 = a_3 = ... = a_n = 0$.

(2) $\Rightarrow$ (4) For each $ef(x)(1-\sigma(e))$, where $f(x) \in R[[x,\sigma]]$, let $g(x) = e + ef(x)(1-\sigma(e))x$, it is easy to find that $g^2(x) = g(x) \in E(R) = E(R[[x,\sigma]])$, thus $ef(x)(1-\sigma(e)) = 0$ and $eR[x,\sigma](1-e) = 0$, therefore $R[[x,\sigma]]$ is abelian.

It is obvious (4) $\Rightarrow$ (3) $\Rightarrow$ (1), since it is well known that abelian property is closed by subset, and since $R[x,\sigma]$ is abelian, then for any $e \in E(R)$, $ex = xe = \sigma(e)x$, thus $\sigma(e) = e$.

(4) $\Rightarrow$ (5) $R[[x,\sigma]]$ is abelian, then $0 \subseteq A_r(R) \subseteq A_r(R[x,\sigma]) \subseteq A_r(R[[x,\sigma]]) = 0$, thus the conclusion follows.

(5) $\Rightarrow$ (1) For each $a \in A_r(R)$ such that $ea = a(1-\sigma(e))$, $ax \in A(R[[x,\sigma]]) \subseteq A(R[[x,\sigma]]) = A_r(R)$, thus $a = 0$, and $A_r(R) = 0$. Therefore R is abelian and $\sigma(e) = e$ for any $e \in E(R)$.

The next example will show that the condition $\sigma(e) = e$ is not surplus

Example 3.3. Let $R = \bigotimes_{i=0}^{\infty} Z_4$, where Z_4 denotes the ring of integers modulo 4, and let σ be an endomorphism of R such that $\sigma((a_0,a_1,a_2,...)) = (0,a_0,a_1,a_2,...), a_i \in Z_4$. Let $i \geq 1$ be an integer and e_i denote the element $(a_0,a_1,a_2,...)$ such that $a_{i-1} = 1$, while the others is 0. Obviously, $e_i \in E(R)$ and $\sigma(e_i) = e_{i+1}$ for each i.

Consider $f_k(x) = \sum_{i=0}^{k} e_i x^j$.

$$f_k^2(x) = \sum_{s=0}^{2k} \sum_{i+j=s} e_i \sigma^i(e_j) x^s = \sum_{s=0}^{2k} \sum_{i+j=s} e_i e_s x^s$$
$$= \sum_{s=0}^{k} e_s x^s = f_k(x) \in E([R,\sigma]) - E(R).$$

Similarly Let $f(x) = \sum_{i=0}^{\infty} e_i x^j$, we can also get $f^2 = f$ by the same proceeding. Thus $f(x) \in E(R[[x,\sigma]]) - E(R[x,\sigma])$. Choose $g(x) = (2e_0 + 3e_1)x$, $e_0 g(x) = 2e_0 x$ while $g(x)e_0 = 3e_1 x \neq e_0 g(x)$. So $R[x,\sigma]$, $R[[x,\sigma]]$ is not abelian.

When σ is an identical endomorphism, we can immediately get:

Corollary 3.4. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is abelian;*
(2) $E(R) = E(R[x]) = E(R[[x]])$;
(3) *$R[x]$ is abelian;*
(4) *$R[[x]]$ is abelian;*
(5) $A(R) = A(R[x]) = A(R[[x]])$.

Given a ring R and a $R-R-$bimodule M, the trivial extension of R by M is the ring $T(R,M) = \{(a,b) \mid a \in R, b \in R\}$ with the usual addition and the multiplication $(a_1,b_1)(a_2,b_2) = (a_1a_2, a_1b_2 + a_2b_1)$ where $a_i, b_i \in R$. Let $n(\geq 2)$ be an integer, $T_n(R)$ denotes $T(R, T_{n-1}(R))$. It is well known that $T_n(R)$ is isomorphic to the ring $R[x]/(x^n)$ which is also isomorphic to of all matrices

$$P_n(R) = \{ \begin{bmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{12} & \dots & a_{1n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a & a_{12} \\ 0 & 0 & 0 & \dots & a \end{bmatrix} \mid a, a_{ij} \in R \}.$$

By the same method we have:

Corollary 3.5. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is abelian;*
(2) $E(R) = E(R[x]/(x^n)) = E(T_n(R)) = E(P_n(R))$;
(3) *$R[x]/(x^n)$ is abelian;*
(4) *$T_n(R)$ is abelian;*
(5) *$P_n(R)$ is abelian;*
(6) $A(R) = A(R[x]/(x^n)) = A(T_n(R)) = A(P_n(R))$.

More generally, let

$$R_n = \{ \begin{bmatrix} a & a_{12} & \dots & a_{1n} \\ & a & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a \end{bmatrix} \mid a, a_{ij} \in R \},$$

and the abelian property of R can also be held by R_n.

Theorem 3.6. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*

(2) *Let $\sigma : R \to R_n$ be a homomorphism such that $\sigma(a) = aI$ where $I \in R_n$ is unit matrix. Then $\sigma(E(R)) = E(R_n)$;*

(3) *R_n is abelian;*

(4) $\sigma(A(R)) = A(R_n)$.

Proof. (1) $\Rightarrow$ (2) It is enough to show $E(R_n) \subseteq E(R)$. Since R is abelian, then $A(R) = 0$. Let $E = \begin{bmatrix} a & a_{12} & ... & a_{1n} \\ & a & ... & a_{2n} \\ & & \ddots & \vdots \\ & & & a \end{bmatrix} \in E(R_n)$. We have $a \in E(R)$, and $a_{12} = aa_{12} + a_{12}a$, then $a_{12} \in A(R) = 0$, then we have $a_{13} = aa_{13} + a_{13}a$, by the same proceeding we can get $a_{13} = 0$, repeating this process we will finally get $a_{12} = a_{13} = a_{14} = ... = a_{23} = ...a_{2n} = ... = a_{nn-1} = 0$, so the conclusion follows.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$, Let $A = \begin{bmatrix} 0 & 0 & ... & r \\ & 0 & ... & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$. Let $B = eI + eA(1 - e)$, then $B^2 = B \in E(R_n) = E(\sigma(R))$. Thus, $er(1 - e) = 0$, this indicate that for each $C \in R$, $eC(1-e) = 0$, therefore the conclusion follows.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) Obviously.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, let $A = \begin{bmatrix} 0 & 0 & ... & a \\ & 0 & ... & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix} \in E(R_n)$, clearly $A \in A(R_n) = \sigma(A(R))$, thus $a = 0$ and $A(R) = 0$. Therefore R is abelian.

The next example will show that R_n in Theorem 3.6 could not be replaced by $U_n(R) = \{ \begin{bmatrix} a_1 & * & \dots & * \\ & a_2 & \dots & * \\ & & \ddots & \vdots \\ & & & a_n \end{bmatrix} \mid a_i \in R\}$ when R is abelian.

Example 3.7. Let $R = Z$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(Z)$, it is clear that R is abelian and $A \in U_2(Z)$, while A could not commute with B, thus $U_n(R)$ is not abelian.

Proposition 3.8. Let R be a ring and $S = \{ \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in R\}$, then the following statements are equivalent:

(1) R is abelian;

(2) Let $\sigma : R \to S$ be a homomorphism such that $\sigma(a) = aI$ where $I \in R_n$ is unit matrix. $\sigma(E(R)) = E(S)$;

(3) S is abelian;

(4) $\sigma(A(R)) = A(S)$.

Proof. (1) $\Rightarrow$ (2) Let $E^2 = E = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix} \in R$. Then

$$\begin{bmatrix} a^2 & 0 & 0 \\ ba + ab & a^2 & ac + ca \\ 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & a & c \\ 0 & 0 & a \end{bmatrix},$$

thus $a \in E(R), b, c \in A(R) = 0$ Since R is abelian. Thus $E = aI \in \sigma(E(R))$.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$, and let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}$, let $B = eI + eA(1 - e)$, then $B^2 = B \in E(S) = E(\sigma(R))$. Thus $r = 0$, this indicates that for each $C \in S$, $eC(1 - e) = 0$ and therefore S is abelian.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, Let $A = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $A \in A(S) = A(\sigma(R))$, thus $a = 0$ and $A(R) = 0$. Therefore R is abelian.

Let R be a ring, and σ be a monomorphism of R. Recall that the Nagata extension of R by R and σ is the ring $N(R, R, \sigma) = \{(a, b) \mid a \in R, b \in R\}$ with the usual addition and multiplication $(a_1, b_1)(a_2, b_2) = (a_1a_2, \sigma(a_1)b_2 + a_2b_1)$ where $a_i, b_i \in R$.

Proposition 3.9. Let R be a ring, and let σ be a monomorphism of R. Then the following statements are equivalent:

(1) R is abelian, $\sigma(e) = e$ for any $e \in E(R)$;

(2) Let $\phi : R \to S$ be a homomorphism such that $\phi(a) = (a, 0)$. $\phi(E(R)) = E(N(R, R, \sigma))$;

(3) $N(R, R, \sigma)$ is abelian;

(4) $\phi(A_l(R)) = A_l(N(R, R, \sigma))$.

Proof. (1) $\Rightarrow$ (2) Since R is abelian, then $A(R) = 0$. For each $(a, b) \in E(R)$, we have $(a^2, \sigma(a)b + ab) = (a^2, 2ab) = (a, b)$, then $a \in E(R)$ and $b \in A(R) = 0$, thus $(a, b) = (a, 0) \in \phi(E(R))$. The conclusion follows.

(2) $\Rightarrow$ (3) For each $r \in R, e \in E(R)$ consider $(e, \sigma(e)r(1-e))$, we have $(e, \sigma(e)r(1-e))^2 = (e, \sigma(e)r(1-e)) \in E(N(R,R,\sigma)) = E(\phi(R))$, thus $\sigma(e)r(1-e) = 0$, therefore $N(R,R,\sigma)$ is abelian.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) Obviously.

(4) $\Rightarrow$ (1) For each $a \in A(R)$, $(0,a) \in A(N(R,R,\sigma)) \subseteq A_l(N(R,R,\sigma)) = \phi(A_l(R))$, thus $a = 0$ and $A_l(R) = 0$. Therefore R is ablian and $\sigma(e) = e$ for any $e \in E(R)$.

Let R be an algebra over a commutative ring S. The Dorroh extension of R by S is the ring $D(R,S) = \{(r,s) \mid r \in R, s \in S\}$ with usual addition and multiplication $(a,s)(b,t) = (ab + ta + sb, st)$ where $a, b \in$R and $s, t \in S$

Let R be a ring, and Consider $E_c(R) = \{a \in R \mid a^2 = ua$, for some $u \in UC(R)\}$, where $UC(R)$ denotes the subset of central invertible element of R. Clearly $E(R) \subseteq E_c(R)$.

Lemma 3.10. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is abelian;*

(2) *$E_c(R) \subseteq C(R)$.*

Proof. (2) $\Rightarrow$ (1) Obviously.

(1) $\Rightarrow$ (2) Since $(ua)^2 = u^2a^2 = ua$ and R is abelian, then $ua \in C(R)$, because $u \in UC(R)$, then $a \in C(R)$.

Proposition 3.11. Let R be a ring, which is an algebra over a commutative ring S. Then the following statements are equivalent:

(1) R is abelian;

(2) $D(R,S)$ is abelian.

Proof. Let (e,s) be an idempotent of $D(R,S)$, $(e,s)^2 = (e,s)$ and we have $(e^2 + 2se, s^2) = (e,s)$, then $e^2 = e(1-2s), s^2 = s$, by Lemma 3.10, $e \in C(R)$, thus $D(R,S)$ is abelian.

4. Abelian rings with homomorphisms

Let R and S be rings, $\phi : R \longrightarrow S$ and $\varphi : S \longrightarrow R$ are ring homomorphisms. At the end, We give the next Theorem on abelian properties between R and S.

Theorem 4.1. *Let R and S be rings, $\phi : R \longrightarrow S$ and $\varphi : S \longrightarrow R$ are ring homomorphisms such that $\phi\varphi = 1_s$, Then:*

(1) *If R is ablian, then so is S;*

(2) *If $\varphi\phi(E(S)) = E(R)$, and S is abelian, then so is R.*

Proof. (1) Clearly ϕ is an epimorphism. For each $r \in R$, $er(1-e) = \phi\varphi(er(1-e)) = 0$, since R is abelian. Thus S is abelian.

(2) Firstly we show $R = \varphi\phi(R) \oplus Ker\phi$. For each $a \in R$, $a = a - \varphi\phi(a) + \varphi\phi(a)$. Clearly $\varphi\phi(a) \in \varphi\phi(R)$ and $\phi(a - \varphi\phi(a)) = 0$ so that $a - \varphi\phi(a) \in$

$Ker\phi$. For every $b \in \varphi\phi(R) \cap Ker\phi$, there exists an element $c \in R$ such that $b = \varphi\phi(c)$ and $\phi(b) = \phi\varphi\phi(c) = \phi(c) = 0$, then $b = \varphi\phi(c) = \varphi(0) = 0$, thus $R = \varphi\phi(R) \oplus Ker\phi$.

Let $a = r - \varphi\phi(r) \in Ker\phi, b = \varphi\phi(r) \in \varphi\phi(R)$, clearly $r = a + b$, consider $c = e + er(1 - e) = e + ea(1 - e) + eb(1 - e) = e + ea(1 - e), c^2 = c$, since $\varphi\phi(R) = \varphi(S) \cong S$ is abelian. Thus $c = e+ea(1-e) \in E(R) = \varphi(E(S)) \in \varphi(S)$, and $ea(1 - e) = c - e \in \varphi(S)$. Note that $ea(1 - e) \in Ker\phi$, since $Ker\phi$ is an ideal of R. Therefore $ea(1 - a) = 0$, $er(1 - e) = 0$, then R is abelian.

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A new kind of $(0, 1, 2)$-interpolation

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Abstract. In this paper, we consider the existence, explicit representation and convergence of a new kind of $(0, 1, 2)$-interpolation on non-uniformly distributed set of nodes on the unit circle.
Keywords: Legendre polynomial, explicit representation, convergence.

1. Introduction

In 1960, O. Kiš [6] extended the study on the complex plane by considering $(0, 2)$, $(0, 1, 3)$ and $(0, 1, \ldots, r-2, r)$-interpolation cases in the roots of unity and showed the regularity for any $r \geq 2$. The Lagrange-Hermite interpolation process for rational functions was considered by J. F. Tarub [12]. After that J. Szabadós and A. K. Verma [10], made a study of $(0, 1, 2)$-interpolation in uniform metric on an arbitrary set of nodes and claimed that for arbitrary $f \in C[-1, 1]$, a uniquely determined polynomial of degree at most $3n - 1$, can not converge uniformly. The Pál-type interpolation with different kind of interpolatory conditions was considered by M. Lénárd [7].

In 2011, H. P. Dikshit [5] examined the regularity of $(0, m)$ and $(0, 1, \ldots, r-2, r)$-interpolation on the zeros of $(z^n + 1)(z - \tau)$, where τ is any complex number. After that, author[1] [1] consider the convergence of $(0, 1, 2)$-interpolation on the unit circle. The Lagrange-Hermite interpolation at the Jacobi zeros with its first $(r - 1)$ derivative at ± 1, was considered by G. Mastrianni, G. V. Milovanovic and I. Notarangelo [8]. Later on, the Lagrange-Hermite interpolating polynomial at some orthogonal polynomial with its first $(r - 1)$ derivative at 0, was considered by G. V. Milovanovic and G. Mastrianni and P. Paster [9]. Recently, author [1] [2] considered the convergence of Lagrange-Hermite interpolation on uniformly distributed zeros of the unit circle. Also, author[1] (with Varun) [3] considered the convergence of Lagrange-Hermite interpolation on the non-uniformly distributed zeros on the unit circle obtained the explicit repre-

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sentation and convergence of the interpolatory polynomial for the same. Also, authors [4] considered the convergence of modified Hermite interpolation on the unit circle.

These have motivated us to consider different type of interpolation on the unit circle. In this paper, we consider the non – uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)P_n'(x)$ onto the unit circle, where $P_n(x)$ stands for nth Legendre polynomial. We obtain the explicit forms of the interpolatory polynomials and establish a convergence theorem for the same. In section 2 we give some preliminaries and in section 3, we describe the problem and its existence. In section 4, we give the explicit formulae of the interpolatory polynomials. In section 5 and 6, estimation and convergence of interpolatory polynomials are given respectively.

2. Preliminaries

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n(x)$is

$$(1) \qquad \left(1-x^2\right) P_n^{''}(x) - 2xP_n'(x) + n(n+1) P_n(x) = 0,$$

$$(2) \qquad W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n \; P_n\left(\frac{1+z^2}{2z}\right) z^n,$$

$$(3) \qquad R_{4n-2}(z) = \prod_{k=1}^{4n-2} (z - t_k^*) = K_{1n} \; P_n\left(\frac{1+z^2}{2z}\right) P_n'\left(\frac{1+z^2}{2z}\right) z^{2n-1},$$

$$(4) \qquad R_{4n}(z) = \left(z^2-1\right) R_{4n-2}(z).$$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $R_{4n}(z)$ and $W(z)$ are respectively given as:

$$(5) \qquad l_k(z) = \frac{R_{4n}(z)}{\left(z - t_k^*\right) R_{4n}'\left(t_k^*\right)}, \qquad k = 0(1)4n-1,$$

$$(6) \qquad L_{1k}(z) = \frac{W(z)}{(z - z_k) W'(z_k)}, \qquad k = 1(1)2n.$$

We will also use the following results

$$(7) \qquad \begin{cases} For \; k = 1(1)2n, \\ R_{4n}'(t_k^*) = K_{1n} \frac{1}{2} \left(t_k^{*2} - 1\right)^2 \left\{P_n'(u_k^*)\right\}^2 t_k^{*2n-3}, \\ \text{For} \quad k = 2n+1, \ldots, 4n-2, \\ R_{4n}'(t_k^*) = \; K_{1n} \frac{1}{2} \left(t_k^{*2} - 1\right)^2 P_n(u_k^*) \, P_n^{''}(u_k^*) \; t_k^{*2n-3}. \end{cases}$$

We will also use the following well known inequalities:

For $-1 < x < 1$

$$\left(1-x^2\right)^{1/4}\left|P_n\left(x\right)\right| \leq \sqrt{\frac{2}{\pi}}n^{-1/2}, \tag{8}$$

$$\left(1-x^2\right)^{3/4}\left|P_n'\left(x\right)\right| \leq \sqrt{2}\, n^{1/2}\ , \tag{9}$$

$$\left(1-x^2\right)^{1/2}\left|P_n\left(x\right)\right| = O\left(n^{-1}\right)\ . \tag{10}$$

Let $x_k = \cos\theta_k\,(k = 1, 2, \ldots, n)$ are the zeros of n^{th} Legendre polynomial $P_n\left(x\right)$, with

$$1 > x_1 > x_2 >, \ldots, x_n > \ -1, \tag{11}$$

then

$$\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{12}$$

$$\left|P_n^{(s)}\left(x_k\right)\right| \sim \ k^{\ -s-1/2}\ n^{2s}, \qquad s = 0, 1, 2. \tag{13}$$

For more details one can see [11].

3. The problem and regularity

Let $T_n^* = \ \ Z_n \cup Z_n^{*}\ \cup \{-1, 1\}$, be obtained by projecting vertically the zeros of $\left(1-x^2\right) P_{\ n}\left(x\right) P_n'\left(x\right)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial,

$$Z_n = \left\{ \begin{array}{c} z_k = \cos\theta_k + i\sin\theta_k, \\ z_{n+k} = \overline{z_k}\ , \quad k = 1(1)n \end{array} \right\} \tag{14}$$

$$Z_n^* = \left\{ \begin{array}{c} z_k^* = \cos\varphi_k + i\sin\varphi_k, \\ z_{(n-1)+k}^* = \overline{z_k^*}\ , \quad k = 1\,(1)\,n-1 \end{array} \right\} \tag{15}$$

and

$$t_0^* = 1,\ t_{4n-1}^* = -1. \tag{16}$$

Here we are interested to determine the following polynomial $Q_{6n+3}\left(z\right)$ of degree $\leq 6n+3$ satisfying the conditions:

$$\begin{cases} Q_{6n+3}\left(t_k^*\right) = \alpha_k, & k = 1(1)4n-2, \\ \left[Q_{6n+3}\left(z\right)\right]'_{z=z_k} = \beta_k, & k = 1\,(1)\,2n, \\ \left[Q_{6n+3}\left(z\right)\right]^{(p)}_{z=t_k^*} = \gamma_k, & k = 0 \text{ and } 4n-1,\ \mathrm{p} = 0, 1, 2 \end{cases} \tag{17}$$

where α_k 's, β_k's and γ_k's are arbitrary complex constants.

Theorem A. *$Q_{6n+3}(z)$ is regular on T_n^*.*

Proof. It is sufficient, if we show that the unique solution of (17) is

$$Q_{6n+3}(z) \equiv 0, \tag{18}$$

when all data, $\alpha_k = \beta_k = \gamma_k = 0$.

In this case, we have

$$Q_{6n+3}(z) = \left(z^2-1\right)^2 R_{4n-2}(z)\ q(z), \tag{19}$$

where $q(z)$ is a polynomial of degree $\leq 2n+1$ and $R_{4n-2}(z)$ is defined in (3). Obviously,

$$Q_{6n+3}(t_k^*) = 0, \quad k = 1(1)4n-2 \tag{20}$$

$$[Q_{6n+3}(z)]^{(p)}_{z=t_k^*} = 0, \quad k = 0 \text{ and } 4n-1, \quad p = 0 \text{ and } 1. \tag{21}$$

From

$$[Q_{6n+3}(z)]'_{z=z_k} = 0, \quad k = 1(1)2n \tag{22}$$

we get

$$q(z_k) = 0. \tag{23}$$

Therefore, we have

$$q(z) = (a\ z+b)\, W(z), \tag{24}$$

where a and b arbitrary constants. From

$$[Q_{6n+3}(z)]''_{z=t_k^*} = 0, \quad k = 0,\ 4n-1 \tag{25}$$

we get

$$q(\pm 1) = 0 \tag{26}$$

using (26) in (24), we get

$$a = b = 0. \tag{27}$$

Hence, the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write $Q_{6n+3}(z)$ satisfying (17) as:

$$Q_{6n+3}(z) = \sum_{k=0}^{4n-1} \alpha_k \; D_{0,k}(z) + \sum_{k=1}^{2n} \beta_k \; D_{1,k}(z)$$
$$+ \beta_0 D_{1,0}(z) + \beta_{4n-1} D_{1,4n-1}(z) + \gamma_0 D_{2,0}(z) + \gamma_{4n-1} D_{2,4n-1}(z), \tag{28}$$

where $D_{0,k}(z)(k = 0(1)4n-1)$, $D_{1,k}(z)(k = 0(1)2n, 4n-1)$ and $D_{2,k}(z)(k = 0$ *and* $4n-1)$ are unique polynomials, each of degree at most $6n+3$ satisfying the condition:

For $k = 0\,(1)\,4n-1$

$$\begin{cases} D_{0,k}(t_j^*) = \delta_{jk}, j = 0(1)4n-1 \\ [D_{0,k}(z)]'_{z=z_j} = 0, j = 1(1)2n, \\ [D_{0,k}(z)]^{(p)}_{z=t_j^*} = 0, j = 0, 4n-1, p = 1, 2. \end{cases} \tag{29}$$

For $k = 1(1)2n$ and $k = 0, 4n-1$

$$\begin{cases} \begin{cases} \text{For } k = 1\,(1)\,2n \text{ and } k = 0, \quad 4n-1 \\ D_{1,k}\left(t_j^*\right) = 0, \qquad\qquad j = 1\,(1)\,4n-2 \\ [D_{1,k}\,(z)]'_{z=z_j} = \delta_{jk}, \qquad j = 1\,(1)\,2n, \end{cases} \\ [D_{1,k}\,(z)]'_{z=t_j^*} = \delta_{jk}, \qquad j,k = 0 \text{ and } 4n-1 \\ [D_{1,k}\,(z)]^{(p)}_{z=t_j^*} = 0, \quad k = 1(1)2n, \qquad j,k = 0 \text{ and } 4n-1,\ p = 0, 2. \end{cases} \tag{30}$$

For $k = 0$ and $4n-1$

$$\begin{cases} D_{2,k}(t_j^*) = 0, j = 1(1)4n-2 \\ [D_{2,k}(z)]'_{z=z_j} = 0, j = 1(1)2n, \\ [D_{2,k}(z)]^{(p)}_{z=t_j^*} = 0, j = 0,\ 4n-1, \quad p = 0, 1 \\ [D_{2,k}(z)]''_{z=t_j^*} = \delta_{jk}, j = 0, 4n-1, \end{cases} \tag{31}$$

Theorem 4.1. *For $k = 0$ and $4n-1$, we have*

$$D_{2,k}\,(z) = \left(z^2-1\right)^2 R_{4n-2}\,(z)\;\; W\,(z)\,\{c_{1k} + c_{2k}(z - t_k^*)\}\,, \tag{32}$$

where

$$c_{1k} = \frac{1}{8\; R_{4n-2}\left(t_k^*\right) W\left(t_k^*\right)}\;, \quad k = 0 \text{ and } 4n-1 \tag{33}$$

$$c_{2k} = (-1)^k \frac{1}{2} c_{1k}\;, \quad k = 0 \text{ and } 4n-1. \tag{34}$$

Proof. From equation (32), it is obvious

$$[D_{2,k}(z)]^{(p)}_{z=t_j^*} = 0, \quad j = 0, 4n-1 \quad \text{and} \quad p = 0, 1. \tag{35}$$

$$D_{2,k}\left(t_j^*\right) = 0, \quad j = 1\,(1)\,4n-2. \tag{36}$$

$$[D_{2,k}(z)]'_{z=z_j} = 0, \quad j = 1\,(1)\,2n, \tag{37}$$

From

$$[D_{2,k}(z)]''_{z=t_j^*} = \delta_{jk}, \quad j = 0,\ 4n-1 \tag{38}$$

we get (33)-(34).

Theorem 4.2. *For $k = 1\,(1)\,2n$ we have*

$$D_{1,k}(z) = b_k\left(z^2-1\right)^3 R_{4n-2}(z)\, L_{1k}(z), \tag{39}$$

where

$$b_k = \frac{1}{\left(z_k^2-1\right)^3 R'_{4n-2}(z_k)}. \tag{40}$$

For $k = 0, 4n-1$

$$D_{1,k}(z) = \frac{(z+t_k^*)^2 R_{4n}(z)\ W(z)}{4R'_{4n}\left(t_k^*\right)\quad W\left(t_k^*\right)} + \frac{\left(z^2-1\right) R_{4n}(z)\ W(z)}{4R'_{4n}\left(t_k^*\right)\quad W\left(t_k^*\right)} S_k(z), \tag{41}$$

where

$$S_k(z) = -\left[\frac{R''_{4n}\left(t_k^*\right)}{2R'_{4n}\left(t_k^*\right)} + \frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)} + t_k^*\right](z+t_k^*). \tag{42}$$

Proof. For $k = 1\,(1)\,2n$, from equation (39), it is obvious,

$$[D_{1,k}(z)]^{(p)}_{z=t_j^*} = 0, \quad j = 0, 4n-1 \quad \text{and} \quad p = 0, 1, 2. \tag{43}$$

$$D_{1,k}\left(t_j^*\right) = 0, \quad j = 1\,(1)\,4n-2. \tag{44}$$

For $j = 1\,(1)\,2n$,

$$D'_{1k}(z_j) = 0, \qquad \text{for} \qquad j \neq k \tag{45}$$

and for $j = k$, we get (40).

For $k = 0,\ 4n-1$, from equation (41), it is obvious

$$D_{1,k}\left(t_j^*\right) = 0, \quad j = 0\,(1)\,4n-1, \tag{46}$$

$$D'_{1k}(z_j) = 0, \quad j = 1\,(1)\,2n, \tag{47}$$

$$D'_{1k}\left(t_j^*\right) = \delta_{jk}, \quad j = 0, 4n-1. \tag{48}$$

From

$$D''_{1k}\left(t_j^*\right) = 0, \qquad j = 0 \text{ and } 4n-1 \tag{49}$$

we get (42).

Theorem 4.3. *For $k = 1\,(1)\,2n$ we have*

$$\begin{aligned} D_{0,k}(z) &= \frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2} L_{1k}(z)\, l_k(z) \\ &\cdot \left[1-(z-t_k^*)\left\{L'_{1k}(z_k) + l'_k(z_k) + \frac{4z_k}{\left(z_k^2-1\right)}\right\}\right]. \end{aligned} \tag{50}$$

For $k = 2n+1, \ldots, 4n-2$

$$D_{0,k}(z) = \frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2} \frac{W(z)}{W\left(t_k^*\right)} l_k(z). \tag{51}$$

For $k = 0$ and $4n-1$,

$$D_{0,k}(z) = \frac{(z+t_k^*)^3 R_{4n-2}(z)\, W(z)}{8\; R_{4n-2}\left(t_k^*\right) W\left(t_k^*\right)} + a_{1k} D_{1,k}(z) + a_{2k} D_{2,k}(z), \tag{52}$$

where

$$a_{1k} = -\left[\frac{3}{2} + \frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)} + \frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}\right] \tag{53}$$

$$\begin{aligned} a_{2k} = &-\left[\frac{3}{2}t_k^* + \frac{R''_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)} + 2\frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)} \frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)} + \frac{W''\left(t_k^*\right)}{W\left(t_k^*\right)}\right] \\ &- 3\left\{\frac{W'\left(t_k^*\right)}{W\left(t_k^*\right)} + \frac{R'_{4n-2}\left(t_k^*\right)}{R_{4n-2}\left(t_k^*\right)}\right\}. \end{aligned} \tag{54}$$

Proof. For $k = 1\,(1)\,2n$, let

$$D_{0,k}(z) = \frac{\left(z^2-1\right)^2}{\left(t_k^{*2}-1\right)^2} L_{1k}(z)\, l_k(z)\, h_k(z), \tag{55}$$

where $h_k(z)$ is a linear polynomial such that

$$h_k(z) = a_k + b_k\left(z - t_k^*\right). \tag{56}$$

From (55), obviously

$$D_{0,k}\left(t_j^*\right) = 0, \quad j = 0 \text{ and } 4n-1, \text{ for } j \neq k \tag{57}$$

and for $j = k$, we must have

$$h_k\left(t_k^*\right) = 1. \tag{58}$$

Again, from (55),

$$D'_{0k}\left(z_j\right) = 0,\ j = 1(1)2n,\ \text{for } j \neq k \tag{59}$$

and for $j = k$, we get

$$h'_k\left(z_k\right) = -\left\{L'_{1k}\left(z_k\right) + l'_k\left(z_k\right) + \frac{4z_k}{\left(z_k^2 - 1\right)}\right\}, \tag{60}$$

using (56), (58) and (60) in (55), we get (50).

From (51), for $k = 2n+1, \ldots, 4n-2$, obviously

$$D_{0,k}\left(t_j^*\right) = \delta_{jk},\quad j = 1(1)4n-2, \tag{61}$$

$$D'_{0,k}\left(z_j\right) = 0,\ j = 1(1)2n. \tag{62}$$

From (52), for $k = 0,\ 4n-1$

$$D_{0,k}\left(t_j^*\right) = 0,\quad j = 1\,(1)\,4n-2, \tag{63}$$

$$D_{0,k}\left(t_j^*\right) = \delta_{jk},\quad j = 0, 4n-1. \tag{64}$$

From

$$D'_{0k}\left(t_j^*\right) = 0,\quad j = 0 \text{ and } 4n-1, \tag{65}$$

we get (53).

From

$$D''_{0k}\left(t_j^*\right) = 0,\quad j = 0 \quad \text{and } 4n-1, \tag{66}$$

we get (54).

5. Estimation of fundamental polynomials

Lemma. *Let $D_{2,k}(z), D_{1,k}(z)$ and $D_{0,k}(z)$ be defined in section 4. Then for $|z| \leq 1$, we get*

$$|D_{2,k}(z)| \leq c\ n^{-2}\ ,\ k = 0 \text{ and } 4n-1 \tag{67}$$

$$|D_{1,k}(z)| \leq c\ n^{-1},\quad k = 0 \text{ and } 4n-1, \tag{68}$$

$$\sum_{k=1}^{2n} |D_{1,k}(z)| \leq c\ n^{-1}\log n\ , \tag{69}$$

$$|D_{0,k}(z)| \leq c,\quad k = 0 \text{ and } 4n-1, \tag{70}$$

$$\sum_{k=1}^{4n-2} |D_{0,k}(z)| \leq c\ \log n, \tag{71}$$

where c is a constant independent of n and z.

6. Convergence

Theorem B. *Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\beta_k's$ and $\gamma_k's$ be such that:*

$$|\beta_k| = O\left(n\ \omega\left(f, n^{-1}\right)\right), \qquad k = 1\,(1)\,2n, \quad k = 0 \text{ and } 4n-1, \tag{72}$$

$$|\gamma_k| = O\left(n^2 \omega\left(f, n^{-1}\right)\right), \qquad k = 0 \text{ and } 4n-1. \tag{73}$$

Then $\{Q_{6n+3}(z)\}$ be defined by

$$\begin{aligned} Q_{6n+3}(z) = &\sum_{k=0}^{4n-1} f\left(t_k^*\right) D_{0,k}(z) + \sum_{k=1}^{2n} \beta_k D_{1,k}(z) + \beta_0 D_{1,0}(z) \\ &+ \beta_{4n-1} D_{1,4n-1}(z) + \gamma_0 D_{2,0}(z) + \gamma_{4n-1} D_{2,4n-1}(z) \end{aligned} \tag{74}$$

satisfies the relation

$$|Q_{6n+3}(z) - f(z)| = O\left(\omega\left(f, n^{-1}\right) \log n\ \right), \tag{75}$$

where $\omega\left(f, n^{-1}\right)$ be the modulus of continuity of $f(z)$.

To prove the Theorem B, we shall need the followings:

Remark. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then there exist a polynomial $F_n(z)$ of degree $\leq 6n+3$ satisfying Jackson's inequality

$$|f(z) - F_n(z)| \leq c\ \omega\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 \leq \theta < 2\pi). \tag{76}$$

Also an inequality due to O. Kiš ([6]),

$$\left|F_n^{(m)}(z)\right| \leq c\ n^m\ \omega\left(f, n^{-1}\right)\ , \quad m \in I^+. \tag{77}$$

Proof. Since $Q_{6n+3}(z)$ be a uniquely determined polynomial of degree $\leq 6n+3$ and the polynomial $F_n(z)$ of degree $\leqslant 6n+3$ satisfying (76) and (77) can be expressed as

$$\begin{aligned} F_n(z) = &\sum_{k=0}^{4n-1} F_n\left(t_k^*\right) D_{0,k}(z) + \sum_{k=1}^{2n} F_n'(z_k)\, D_{1,k}(z) \\ &+ F_n'\left(t_0^*\right) D_{1,0}(z) + F_n'\left(t_{4n-1}^*\right) D_{1,4n-1}(z) \\ &+ F_n''\left(t_0^*\right) D_{2,0}(z) + F_n''\left(t_{4n-1}^*\right) D_{2,4n-1}(z). \end{aligned}$$

Then

$$\begin{aligned} &|Q_{6n+3}(z) - f(z)| \leq |Q_{6n+3}(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{4n-1} |f\left(t_k^*\right) - F_n\left(t_k^*\right)|\, |D_{0,k}(z)| + \sum_{k=1}^{2n} \left\{|\beta_k| + \left|F_n'(z_k)\right|\right\} |D_{1,k}(z)| \\ &+ \left\{|\beta_0| + \left|F_n'\left(t_0^*\right)\right|\right\} |D_{1,0}(z)| \left\{|\beta_{4n-1}| + \left|F_n'\left(t_{4n-1}^*\right)\right|\right\} |D_{1,4n-1}(z)| \\ &+ \left\{|\gamma_0| + \left|F_n''\left(t_0^*\right)\right|\right\} |D_{2,0}(z)| + \left\{|\gamma_{4n-1}| + \left|F_n''\left(t_{4n-1}^*\right)\right|\right\} |D_{2,4n-1}(z)| \\ &+ |F_n(z) - f(z)|, \end{aligned} \tag{78}$$

using (72)-(73), (76)-(77) and Lemma in section 5, we get (75).

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Some results and examples of the bi-f-harmonic maps

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Abstract. In this paper we present some results and examples of the bi-f-harmonic maps. In particular, we study the case of conformal maps between equidimensional manifolds. Examples are constructed when one of the factors is either Euclidean space.
Keywords: f-harmonic map, Bi-f-harmonic map, Conformal map.

1. Introduction

Harmonic maps $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds are the critical points of the energy functional

$$E(\phi) = \frac{1}{2}\int_D |d\phi|^2 dv_g,$$

for every compact domain $D \subset M$. The Euler-Lagrange equation associated to $E(\phi)$ is $\tau(\phi) = Tr_g \nabla d\phi = 0$, $\tau(\phi)$ is called the tension field of ϕ, one can refer to [5-8] for background on harmonic maps. The stress-energy tensor for a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ defined by (see [2]) $S(\phi) = e(\phi)g - \phi^* h$ and the relation between $S(\phi)$ and $\tau(\phi)$ is given by $\operatorname{div} S(\phi) = -h(\tau(\phi), d\phi)$. As the generalizations of harmonic maps, biharmonic maps are defined as follows. The map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is biharmonic if it is a critical point of the bi-energy functional defined by

$$E_2(\phi) = \frac{1}{2}\int_D |\tau(\phi)|^2 dv_g.$$

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The first variation formula for the bi-energy which is derived in [9] shows that the Euler-Lagrange equation for bi-energy is

$$\tau_2(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0.$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. In analogy with harmonic maps, Jiang in [9] has constructed for a map ϕ the stress bi-energy tensor defined by

$$S_2(\phi) = \left(\frac{-1}{2}\left|\tau(\phi)\right|^2 + \operatorname{div} h\left(\tau(\phi), d\phi\right)\right) g - 2symh\left(\nabla\tau(\phi), d\phi\right)$$

where $symh\left(\nabla\tau(\phi), d\phi\right)(X,Y) = \frac{1}{2}\left\{h\left(\nabla_X\tau(\phi), d\phi\left(Y\right)\right) + h\left(\nabla_Y\tau(\phi), d\phi\left(X\right)\right)\right\}$ for any $X, Y \in \Gamma\left(TM\right)$. The stress bi-energy tensor of ϕ satisfies the following relationship $\operatorname{div} S_2(\phi) = h\left(\tau_2(\phi), d\phi\right)$. Let $f \in C^{\infty}\left(M\right)$ be a positive function, the map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is said to be f-harmonic if it is a critical point of the f-energy functional:

$$E_f\left(\phi\right) = \frac{1}{2}\int_M f|d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is f-harmonic if it satisfies the associated Euler-Lagrange equations $\tau_f\left(\phi\right) = f\tau\left(\phi\right) + d\phi\left(\operatorname{grad} f\right) = f\left(\tau\left(\phi\right) + d\phi\left(\operatorname{grad}\ln f\right)\right) = 0$, $\tau_f\left(\phi\right)$ is called the f-tension field of ϕ. In the context of f-harmonic maps, the stress energy tensor $S_f\left(\phi\right)$ of ϕ associated to the f-energy functional $E_f\left(\phi\right)$ (which we call, the f-stress energy tensor of ϕ) is given by (see [1]) $S_f\left(\phi\right) = f\left(e\left(\phi\right)g - \phi^* h\right)$ and the relation between $S_f\left(\phi\right)$ and $\tau_f\left(\phi\right)$ is given by (see [3]) $\operatorname{div} S_f\left(\phi\right) = -h\left(\tau_f\left(\phi\right), d\phi\right)$. The map ϕ is said to be bi-f-harmonic if it is a critical point of the bi-f-energy functional: $E_{f,2}\left(\phi\right) = \frac{1}{2}\int_M \left|\tau_f(\phi)\right|^2 dv_g$. Equivalently, ϕ is bi-f-harmonic if it satisfies the associated Euler-Lagrange equations (see [13]):

$$\tau_{f,2}\left(\phi\right) = -Tr_g\nabla^{\phi} f\nabla^{\phi}\tau_f\left(\phi\right) - fTr_g R^N\left(\tau_f\left(\phi\right), d\phi\right)d\phi = 0, \tag{1}$$

$\tau_{f,2}(\phi)$ is called the bi-f-tension field of ϕ. Note that $Tr_g\nabla^{\phi} f\nabla^{\phi}\tau_f(\phi) = f(Tr_g(\nabla^{\phi})^2\tau_f(\phi) + \nabla_{\operatorname{grad}\ln f}\tau_f(\phi))$, then

$$\tau_{f,2}(\phi) = f(-Tr_g(\nabla^{\phi})^2\tau_f(\phi) - Tr_g R^N(\tau_f(\phi), d\phi)d\phi - \nabla_{gard\ln f}\tau_f(\phi)), \tag{2}$$

and ϕ is bi-f-harmonic if and only if (see [12],[14]),

$$Tr_g(\nabla^{\phi})^2\tau_f(\phi) + Tr_g R^N(\tau_f(\phi), d\phi)d\phi + \nabla_{\operatorname{grad}\ln f}\tau_f(\phi) = 0.$$

Following Jiang's notion, the stress bi-f-bienergy tensor of a ϕ is defined by (see [3])

$$S_{f,2}\left(\phi\right) = \left(\frac{1}{2}\left|\tau_f\left(\phi\right)\right|^2 + fTr_g h\left(\nabla\tau_f\left(\phi\right), d\phi\right)\right) g - 2f symh\left(\nabla\tau_f\left(\phi\right), d\phi\right),$$

where $Tr_g h(\nabla f\tau(\phi), d\phi) = h(\nabla^{\phi}_{e_i} f\tau(\phi), d\phi(e_i))$ (we sum over repeated indices). The stress bi-f-energy tensor of ϕ satisfies the following relationship div $S_{f,2}(\phi) = h(\tau_{f,2}(\phi), d\phi) + (Tr_g h(\nabla\tau_f(\phi), d\phi))df$. In [10], the author studied the f-harmonicity of some special maps from or into a doubly warped product manifold, he gives the conditions for f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps, furthermore, he investigates non-trivial f-harmonicity of the product of two harmonic maps. The authors in [15] gave a method to produce biharmonic maps and f-biharmonic maps from given biharmonic maps and they constructed many examples of biharmonic and f-biharmonic maps from the standard sphere S^2 and between two such spheres. In [14], the authors obtain the first variation formula of the bi-f-energy functional, they introduce the bi-f-harmonic maps, which are the natural generalization of biharmonic maps, and they study some properties of the bi-f-harmonic maps. The authors in [4] study a subclass of generalized f-harmonic maps called generalized f-harmonic morphisms which pull back local harmonic functions to local generalized f-harmonic functions. In [12], the authors studied the class of bi-f-harmonic maps and that of f-biharmonic maps from a conformal manifold of dimension ¿ 2 and they gave some results on nonexistence of proper bi-f-harmonic maps and f-biharmonic maps. In this paper, we give other constructions of bi-f-harmonic maps. In the first section, we give the relation between the bi-f-tension field and the bitension field and the relation between the stress bi-f-energy tensor and the stress bi-energy tensor of the smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ (Proposition 1 and Proposition 2). In this case we construct some examples of bi-f-harmonic maps and we study the case of the identity map. As a second result, we characterize the bi-f-biharmonicity of the conformal maps $\phi : (M^n, g) \longrightarrow (N^n, h)$ of dilation λ. We begin this section by the particular case where the function f is equal to the dilation λ (Theorem 2 and Theorem 3) and we give an example of bi-λ-harmonic map. We conclude this section by studying the general case (Theorem 4 and Theorem 5).

2. The main results of bi-f-harmonic maps

As a first result, if we consider $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $f \in C^{\infty}(M)$ be a positive function, the relation between the bi-f-tension and the bi-tension field of ϕ is given by the following proposition.

Proposition 1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $f \in C^{\infty}(M)$ be a positive function. Then*

$$\begin{aligned}\tau_{f,2}(\phi) &= f^2\tau_2(\phi) - 3f^2\nabla^{\phi}_{\operatorname{grad}\ln f}\tau(\phi) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi)\\ &- f^2(Tr_g(\nabla^{\phi})^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi)\\ &- 3f^2\nabla^{\phi}_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f).\end{aligned} \tag{3}$$

Proof of Proposition 1. Note that by a simple calculate, we have for any $V \in \Gamma\left(\phi^{-1}TN\right)$

$$Tr_g(\nabla^\phi)^2 fV = f(Tr_g(\nabla^\phi)^2 V + 2\nabla^\phi_{\operatorname{grad}\ln f}V + (\Delta \ln f + |\operatorname{grad}\ln f|^2)V). \tag{4}$$

By definition, the bi-f-tension field of ϕ is given by

$$\tau_{f,2}(\phi) = -fTr_g\left(\nabla^\phi\right)^2\tau_f(\phi) - fTr_g R^N(\tau_f(\phi), d\phi)\,d\phi - f\nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi), \tag{5}$$

where $\tau_f(\phi) = f(\tau(\phi) + d\phi(\operatorname{grad}\ln f))$.

Let us choose $(e_i)_{1\le i\le m}$ to be an orthonormal frame on M, let us start with the calculation of the first term $Tr_g(\nabla^\phi)^2\tau_f(\phi)$ of (5), we have $Tr_g(\nabla^\phi)^2\tau_f(\phi) = Tr_g(\nabla^\phi)^2 f\tau(\phi) + Tr_g(\nabla^\phi)^2 fd\phi(\operatorname{grad}\ln f)$. Using (4), we obtain $Tr_g(\nabla^\phi)^2 f\tau(\phi) = f(Tr_g(\nabla^\phi)^2\tau(\phi) + 2\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi)) + f(\Delta\ln f + |\operatorname{grad}\ln f|^2)\tau(\phi)$ and

$$\begin{aligned} Tr_g(\nabla^\phi)^2 fd\phi(\operatorname{grad}\ln f) &= f(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + 2\nabla^\phi_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f)) \\ &\quad + f(\Delta\ln f + |\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f). \end{aligned}$$

Its follow that

$$\begin{aligned} Tr_g(\nabla^\phi)^2\tau_f(\phi) &= fTr_g(\nabla^\phi)^2\tau(\phi) + fTr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) \\ &\quad + 2f\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) + 2f\nabla^\phi_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f) \\ &\quad + f(\Delta\ln f + |\operatorname{grad}\ln f|^2)(\tau(\phi) + d\phi(\operatorname{grad}\ln f)). \end{aligned} \tag{6}$$

For the term $Tr_g R^N(\tau_f(\phi), d\phi)d\phi$, it is easy to see that

$$\begin{aligned} Tr_g R^N(\tau_f(\phi), d\phi)d\phi &= fTr_g R^N(\tau(\phi), d\phi)d\phi \\ &\quad + fTr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi. \end{aligned} \tag{7}$$

Finally for the term $\nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi)$, we have

$$\begin{aligned} \nabla^\phi_{\operatorname{grad}\ln f}\tau_f(\phi) &= \nabla^\phi_{\operatorname{grad}\ln f}f\tau(\phi) + \nabla^\phi_{\operatorname{grad}\ln f}fd\phi(\operatorname{grad}\ln f) \\ &= f\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) + f|\operatorname{grad}\ln f|^2\tau(\phi) \\ &\quad + f\nabla^\phi_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f) + f|\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln f). \end{aligned} \tag{8}$$

If we replace (6), (7) and (8) in (5), we deduce that

$$\begin{aligned} \tau_{f,2}(\phi) &= f^2\tau_2(\phi) - 3f^2\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi) \\ &\quad - f^2(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi) \\ &\quad - 3f^2\nabla^\phi_{\operatorname{grad}\ln f}d\phi(\operatorname{grad}\ln f) - f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f). \end{aligned}$$

This completes the proof of Proposition 1.

Remark 1. Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. Then ϕ is bi-f-biharmonic if and only if

$$\begin{aligned}&\tau_2(\phi) - (Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi\\ &- 3\nabla^\phi_{\operatorname{grad}\ln f}\tau(\phi) - (\Delta \ln f + 2|\operatorname{grad}\ln f|^2)\tau(\phi)\\ &- 3\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) - (\Delta \ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f) = 0.\end{aligned}$$

In the case where ϕ is a harmonic map, we have the following result:

Remark 2. Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map and let $f \in C^\infty(M)$ be a positive function. Then ϕ is bi-f-biharmonic if and only if

$$\begin{aligned}&Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln f) + Tr_g R^N(d\phi(\operatorname{grad}\ln f), d\phi)d\phi\\ &+ 3\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln f) + (\Delta \ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f) = 0.\end{aligned}$$

We apply this remark to construct some examples of bi-f-harmonic maps.

Example 1. Let the projection $\phi : (\mathbb{R}^4, g_{\mathbb{R}^4}) \to (\mathbb{R}^3, g_{\mathbb{R}^3})$ defined by $\phi(t, x_2, x_3, x_4) = (t, x_2, x_3)$. We suppose that $\ln f = \alpha(t)$, then by Remark 2, the projection ϕ is bi-f-harmonic if and only if $\alpha''' + 4\alpha'\alpha'' + 2(\alpha')^3 = 0$. Let $\beta = \alpha'$, then the last expression becomes $\beta'' + 4\beta\beta' + 2\beta^3 = 0$. We deduce the special solutions of the form $\beta = \frac{a}{t}$ where $a \in \mathbb{R}$. Eliminating the trivial solution $a = 0$, we obtain $a = 1$ which gives us $f(t) = C|t|$ $(C > 0)$. In this case the projection ϕ is bi-f-harmonic maps, where $f(t) = C|t|$ $(C > 0)$.

Example 2. Let $\phi : (\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}}) \longrightarrow (\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}})$ be the Hopf map defined by $\phi(x, y) = (|x|^2 - |y|^2, 2x\overline{y}) \in \mathbb{R} \oplus K$, where $n = 2, 4$ or 8 and $x, y \in K \simeq \mathbb{R}^n$ with $K = C$ (complex numbers), H (quaternions), or O (octonions), respectively. Let us write a point $x, y \in \mathbb{R}^{2n}$ in the form $(x, y) = r(\cos\theta \cdot p, \sin\theta \cdot q), (r \in [0, +\infty), \theta \in [0, \frac{\pi}{2}], p, q \in S^{n-1})$ and those of $\mathbb{R}^{n+1}$ in the form $s(\cos t, \sin t \cdot w), (s \in [0, +\infty), \theta \in [0, \pi], w \in S^{n-1})$. The map ϕ takes the form $\phi(r\cos\theta \cdot p, r\sin\theta \cdot q) = (s\cos t, s\sin t \cdot p\overline{q})$, where $s = r^2$ and $t = 2\theta$. The metrics on $\mathbb{R}^{2n}$ and $\mathbb{R}^{n+1}$ have respectively the expressions $g_{\mathbb{R}^{2n}} = dr^2 + r^2 d\theta^2 + r^2\cos^2\theta \cdot g_{s^{n-1}} + r^2\sin^2\theta \cdot g_{s^{n-1}}$ and $g_{\mathbb{R}^{n+1}} = ds^2 + s^2 dt^2 + s^2\sin^2 t \cdot g_{s^{n-1}}$. An orthonormal basis of $\mathbb{R}^{2n}$ is given by $e_1 = \frac{\partial}{\partial r}, e_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, e_j = \frac{1}{r\cos\theta}\xi_j, e_k = \frac{1}{r\sin\theta}\xi_k$, where $\xi_j = r(\cos\theta \cdot X_j, 0), j = 3, \ldots, n+1$ and $\xi_k = r(0, \sin\theta \cdot X_k), k = n+2, \ldots, 2n$ where the vectors X_j and X_k are unit vectors tangent to the sphere S^{n-1}. We have $\nabla_{e_i} e_i = \frac{1-2n}{r}\frac{\partial}{\partial r} + \frac{n-1}{r^2}(\tan\theta - \cot\theta)\frac{\partial}{\partial\theta}$. Suppose that the function f depends only on r and set $\ln f = \alpha(r)$. Then by Remark 2, the map ϕ is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $r\beta'' + 4r\beta\beta' + (2n+1)\beta' + \frac{2n-1}{r}\beta + (2n+2)\beta^2 + 2r\beta^3 = 0$. We deduce the special solutions of the form $\beta = \frac{a}{r}$ where $a \in \mathbb{R}$. Eliminating the trivial solution $a = 0$, we obtain $a = -n+1$ which gives us $f(r) = \frac{C}{r^{n-1}}$ $(C > 0)$. In this case the Hopf map $\phi : (\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}}) \longrightarrow (\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}})$ is bi-f-harmonic map, where $f(r) = \frac{C}{r^{n-1}}$.

In particular, if we consider the identity map, we obtain:

Corollary 1. *The identity map $Id_M : (M^m, g) \longrightarrow (M^m, g)$ is bi-f-harmonic if and only if the function f satisfies the following equation*

$$\begin{aligned} &\operatorname{grad}\Delta \ln f + \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) \\ &+ (\Delta \ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 2Ricci(\operatorname{grad}\ln f) = 0. \end{aligned} \tag{9}$$

In the following we shall present an example of bi-f-harmonic.

Example 3. Let the identity map $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ when we suppose that $\ln f$ is radial ($\ln f = \alpha(r)$). A direct calculation gives $\operatorname{grad}\Delta \ln f = (\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha')\frac{\partial}{\partial r}$, $\operatorname{grad}(|\operatorname{grad}\ln f|^2) = 2\alpha'\alpha''\frac{\partial}{\partial r}$ and $\Delta \ln f = \alpha'' + \frac{n-1}{r}\alpha'$. Then by Corollary 1, we deduce that the map $Id_{\mathbb{R}^n\setminus\{0\}}$ is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $\beta'' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta + 4\beta\beta' + \frac{n-1}{r}\beta^2 + 2\beta^3 = 0$. Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), then $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n\setminus\{0\} \longrightarrow \mathbb{R}^n\setminus\{0\}$ is bi-f-harmonic if and only if $a = \frac{-n+5\pm\sqrt{(n-1)(n+7)}}{4}$. We obtain $f(r) = Cr^{\frac{-n+5\pm\sqrt{(n-1)(n+7)}}{4}}$ ($C > 0$) and in this case the identity map $Id_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is bi-f-harmonic.

Example 4. We consider $M = S^n$ with parameterization $x = (\cos s, \sin s\cdot y), s \in [0, \pi], y \in S^{n-1}$. An orthonormal basis of S^n is given by $e_1 = \frac{\partial}{\partial s}, e_i = (0, f_i), i = 2, \ldots, n$ where the vectors f_i are tangents to the sphere S^{n-1}. We suppose that $\ln f = \alpha(s)$. A direct calculation gives

$$\begin{aligned} &\operatorname{grad}\ln f = \operatorname{grad}\alpha = \alpha'\frac{\partial}{\partial s}, \\ &|\operatorname{grad}\ln f|^2 = |\operatorname{grad}\alpha|^2 = (\alpha')^2, \\ &\operatorname{grad}(|\operatorname{grad}\ln f|^2) = \operatorname{grad}(|\operatorname{grad}\alpha|^2) = 2\alpha'\alpha'', \\ &\Delta \ln f = \Delta\alpha = \alpha'' + (n-1)(\cot s)\alpha', \\ &\operatorname{grad}\Delta \ln f = \operatorname{grad}\Delta\alpha = (\alpha''' + (n-1)(\cot s)\alpha'' - (n-1)(1+\cot^2 s)\alpha')\frac{\partial}{\partial s} \end{aligned}$$

and $Ricci^{S^n}(\operatorname{grad}\ln f) = Ricci^{S^n}(\operatorname{grad}\alpha) = (n-1)\alpha'\frac{\partial}{\partial s}$. Then by Corollary 1, we deduce that the map Id_{S^n} is bi-f-harmonic if and only if the function $\beta = \alpha'$ satisfies the following differential equation $\beta'' + (n-1)\cot s\beta' + 4\beta\beta' + (n-1)(1-\cot^2 s)\beta + (n-1)\cot s\beta^2 + 2\beta^3 = 0$. For example, if $n = 1$ the map Id_{S^1} is bi-f-harmonic if and only if $\beta'' + 4\beta\beta' + 2\beta^3 = 0$. We find particular solutions of type $\beta = \frac{a}{s}$, we obtain $a = 1$ which gives us $f(s) = ks$ ($k > 0$) . Then the map Id_{S^1} is bi-f-harmonic, where $f(s) = ks$ ($k > 0$).

Now, for a smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ and a positive function $f \in C^\infty(M)$, we give a relation between the stress bi-f-energy tensor and the stress bi–energy tensor of ϕ.

3. The case of conformal maps

We study a conformal maps between manifolds of the same dimension $n \geq 3$. Note that by a result in [2], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exists a C^∞ function $\lambda : M \to \mathbb{R}_+^*$ such that for any $X, Y \in \Gamma(TM)$, $h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y)$. The function λ is called the dilation for the map ϕ. The tension field and the stress energy tensor for a conformal map are given by (see [2]):

Proposition 2. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ, we have $\tau(\phi) = (2-n)d\phi(\operatorname{grad} \ln \lambda)$, $S(\phi) = \frac{n-2}{2}\lambda^2 g$ and $\operatorname{div} S(\phi) = (n-2)\lambda^2 d \ln \lambda$.*

For a conformal map, we have the following theorem (see [12])

Theorem 1 ([12])**.** *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ, then for any function $\gamma \in C^\infty(M)$, we have*

$$\begin{aligned} Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\gamma) &= d\phi(\operatorname{grad}\Delta\gamma) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\gamma) \\ &\quad + d\phi(Ricci^M(\operatorname{grad}\gamma)) \\ &\quad + (\Delta \ln\lambda) d\phi(\operatorname{grad}\gamma) - 2(\Delta\gamma) d\phi(\operatorname{grad}\ln\lambda) \\ &\quad - (n-2) d\ln\lambda(\operatorname{grad}\gamma) d\phi(\operatorname{grad}\ln\lambda) \end{aligned} \tag{10}$$

and

$$\begin{aligned} Tr_g R^N(d\phi(\operatorname{grad}\gamma), d\phi) d\phi &= d\phi(Ricci^M(\operatorname{grad}\gamma)) \\ &\quad - (n-2) d\phi(\nabla_{\operatorname{grad}\gamma}\operatorname{grad}\ln\lambda) \\ &\quad - (\Delta\ln\lambda + (n-2)|\operatorname{grad}\ln\lambda|^2) d\phi(\operatorname{grad}\gamma) \\ &\quad + (n-2) d\ln\lambda(\operatorname{grad}\gamma) d\phi(\operatorname{grad}\ln\lambda) \end{aligned} \tag{11}$$

Remark 3. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ. The f-tension field of ϕ is given by $\tau_f(\phi) = f(\tau(\phi) + d\phi(\operatorname{grad}\ln f)) = f((2-n)d\phi(\operatorname{grad}\ln\lambda) + d\phi(\operatorname{grad}\ln f)) = f d\phi(\operatorname{grad}\ln f\lambda^{2-n})$. In particular if $f = \lambda$, the λ-tension field of ϕ is given by $\tau_\lambda(\phi) = (3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$.

In the first, we calculate bi-λ-tension field for a conformal map ϕ.

Theorem 2. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then the bi-λ-tension field of ϕ is given by $\tau_2(\phi) = (n-2)d\phi(H(\lambda))$ where*

$$\begin{aligned} H(\lambda) &= \operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) + 2Ricci^M(\operatorname{grad}\ln\lambda) \\ &\quad - ((n-7)|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\operatorname{grad}\ln\lambda. \end{aligned} \tag{12}$$

Proof of Theorem 2. By definition, the bi-λ-tension field is given by $\tau_{\lambda,2}(\phi) = \lambda(-Tr_g(\nabla^\phi)^2\tau_\lambda(\phi) - Tr_g R^N(\tau_\lambda(\phi), d\phi)d\phi - \nabla_{gard\ln\lambda}\tau_\lambda(\phi))$. The λ-tension field of the conformal map ϕ is given by $\tau_\lambda(\phi) = (3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$, it follows that

$$\begin{aligned}\tau_{\lambda,2}(\phi) &= (n-3)\lambda(Tr_g(\nabla^\phi)^2\lambda d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + Tr_g R^N(\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi + \nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda)).\end{aligned} \tag{13}$$

It is easy to see that

$$\begin{aligned}Tr_g(\nabla^\phi)^2\lambda d\phi(\operatorname{grad}\ln\lambda) &= \lambda Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + 2\lambda\nabla_{gard\ln\lambda}d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + \lambda(\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) + \lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{14}$$

By Theorem 1, we have

$$\begin{aligned}Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln\lambda) &= d\phi(\operatorname{grad}\Delta\ln\lambda)\\ &\quad + 2d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)) + d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\ &\quad - (\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) - (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{15}$$

For the term $\nabla_{gard\ln\lambda}d\phi(\operatorname{grad}\ln\lambda)$, we have (see [2])

$$\begin{aligned}\nabla_{\operatorname{grad}\ln\lambda}d\phi(\operatorname{grad}\ln\lambda) &= |\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda)\\ &\quad + \frac{1}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)).\end{aligned} \tag{16}$$

If we replace (15) and (16) in (14), we deduce that

$$\begin{aligned}Tr_g(\nabla^\phi)^2\lambda d\phi(\operatorname{grad}\ln\lambda) &= \lambda d\phi(\operatorname{grad}\Delta\ln\lambda) + 3\lambda d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\ &\quad - (n-5)\lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda) + \lambda d\phi(Ricci^M(\operatorname{grad}\ln\lambda)).\end{aligned} \tag{17}$$

Now, we will calculate $Tr_g R^N(d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi$. By Theorem 1, we obtain

$$\begin{aligned}Tr_g R^N(d\phi(\operatorname{grad}\ln\lambda), d\phi)d\phi &= d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\ &\quad - \frac{n-2}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)) - (\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{18}$$

To complete the proof, we will simplify the term $\nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda)$, a simple calculation gives

$$\begin{aligned}\nabla_{gard\ln\lambda}\lambda d\phi(\operatorname{grad}\ln\lambda) &= \frac{1}{2}\lambda d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\ &\quad + 2\lambda|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda).\end{aligned} \tag{19}$$

If we substitute (17), (18) and (19) in (13), we conclude that $\tau_{\lambda,2}(\phi) = (n-3)\lambda^2 d\phi(H(\lambda))$ where

$$\begin{aligned}H(\lambda) &= \operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) + 2Ricci^M(\operatorname{grad}\ln\lambda)\\ &\quad - ((n-7)|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\operatorname{grad}\ln\lambda.\end{aligned}$$

Corollary 2. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 4)$ to be a conformal map of dilation λ, then ϕ is bi-λ-harmonic if and only if*

$$\begin{aligned} &\operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) - (\Delta\ln\lambda)\operatorname{grad}\ln\lambda \\ &- (n-7)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln\lambda + 2Ricci^M(\operatorname{grad}\ln\lambda) = 0. \end{aligned} \tag{20}$$

In particular, we prove that the bi-λ-harmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial $(\ln\lambda = \alpha(r), r = |x| \text{and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$ is equivalent to an ordinary differential equation of the second order. More precisely, we have

Corollary 3. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ to be a conformal map of dilation λ when we suppose that $\ln\lambda$ is radial $(\ln\lambda = \alpha(r), r = |x| \text{and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$. Then ϕ is bi-λ-harmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:*

$$\beta'' - (n-8)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{n-1}{r}\beta^2 - (n-7)\beta^3 = 0. \tag{21}$$

Proof of Corollary 3. Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ to be a conformal map of dilation λ when we suppose that $\ln\lambda$ is radial $(\ln\lambda = \alpha(r), r = |x| \text{and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$. By Corollary 2, ϕ is bi-λ-harmonic if and only if

$$\begin{aligned} &\operatorname{grad}\Delta\ln\lambda - \frac{n-9}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) - (\Delta\ln\lambda)\operatorname{grad}\ln\lambda \\ &- (n-7)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln\lambda = 0. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} &\operatorname{grad}\ln\lambda = \alpha'\frac{\partial}{\partial r}, |\operatorname{grad}\ln\lambda|^2 = (\alpha')^2, \operatorname{grad}(|\operatorname{grad}\ln\lambda|^2) = 2\alpha'\alpha"\frac{\partial}{\partial r}, \\ &\Delta\ln\lambda = \alpha" + \frac{n-1}{r}\alpha', \end{aligned}$$

and $\operatorname{grad}\Delta\ln\lambda = (\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha')\frac{\partial}{\partial r}$. Hence, ϕ is bi-λ-harmonic if and only if the function α satisfies the following differential equation

$$\alpha''' - (n-8)\alpha'\alpha'' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' - \frac{n-1}{r}(\alpha')^2 - (n-7)(\alpha')^3 = 0.$$

Let $\beta = \alpha'$, the last equation is equivalent to the second-order differential equation $\beta'' - (n-8)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{n-1}{r}\beta^2 - (n-7)\beta^3 = 0$. As a consequence of the Corollary 3, we will present some remarks which we give a particular solutions of the equation (18) that allows us to construct a bi-λ-harmonic maps.

Remark 4. Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$. By (18), we deduce that $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 4)$ is bi-λ-harmonic if and only if a is a solution of the algebraic equation $(n-7)a^2 + 7a + 2n - 4 = 0$. This equation has real solutions if and only if $n \in \{4, 5, 6, 7, 8\}$.

1. If $n = 4$, we find $a = \frac{7+\sqrt{97}}{6}$ or $a = \frac{7-\sqrt{97}}{6}$, so $\lambda = Cr^{\frac{7+\sqrt{97}}{6}}$ or $\lambda = Cr^{\frac{7-\sqrt{97}}{6}}$ $(C > 0)$. Then, in this case any conformal map $\phi : (\mathbb{R}^4, g) \to (N^4, h)$ of dilation $\lambda = Cr^{\frac{7+\sqrt{97}}{6}}$ or $\lambda = Cr^{\frac{7-\sqrt{97}}{6}}$ is bi-λ-harmonic.
2. If $n = 5$, we find $a = \frac{7+\sqrt{97}}{4}$ or $a = \frac{7-\sqrt{97}}{4}$, so $\lambda = Cr^{\frac{7+\sqrt{97}}{4}}$ or $\lambda = Cr^{\frac{7-\sqrt{97}}{4}}$ $(C > 0)$. It follows that any conformal map $\phi : (\mathbb{R}^5, g) \to (N^5, h)$ of dilation $\lambda = Cr^{\frac{7+\sqrt{97}}{4}}$ or $\lambda = Cr^{\frac{7-\sqrt{97}}{4}}$ is bi-λ-harmonic.
3. If $n = 6$, we find $a = 8$ or $a = -1$, so $\lambda = Cr^8$ or $\lambda = Cr^{-1}$ $(C > 0)$. It follows that any conformal map $\phi : (\mathbb{R}^6, g) \to (N^6, h)$ of dilation $\lambda = Cr^8$ or $\lambda = Cr^{-1}$ is bi-λ-harmonic.
4. If $n = 7$, a$=\frac{-10}{7}$, so $\lambda = Cr^{\frac{-10}{7}}$ $(C > 0)$. Then, in this case any conformal map $\phi : (\mathbb{R}^7, g) \to (N^7, h)$ of dilation $\lambda = Cr^{\frac{-10}{7}}$ is bi-λ-harmonic.
5. If $n = 8$, we find $a = -3$ or $a = -4$, so $\lambda = Cr^{-3}$ or $\lambda = Cr^{-4}$ $(C > 0)$. It follows that any conformal map $\phi : (\mathbb{R}^8, g) \to (N^8, h)$ of dilation $\lambda = Cr^{-3}$ or $\lambda = Cr^{-4}$ is bi-λ-harmonic.

In a second result, wa calculate the stress bi-λ-energy tensor for a conformal map ϕ when we prove that $S_{\lambda,2}(\phi)$ depend only on the dilation.

Theorem 3. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then we have*

$$S_{\lambda,2}(\phi) = (3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)g - 2(3-n)\lambda^4(\nabla d\ln\lambda + (d\ln\lambda \otimes d\ln\lambda)), \tag{22}$$

and the trace of $S_{\lambda,2}(\phi)$ is given by

$$Tr_g S_{\lambda,2}(\phi) = (3-n)\lambda^4(\frac{n^2+n-4}{2}|\operatorname{grad}\ln\lambda|^2 + (n-2)\Delta\ln\lambda). \tag{23}$$

Proof of Theorem 3. By definition, the bi-λ-energy tensor of ϕ is given by:

$$S_{\lambda,2}(\phi)(X,Y) = (\frac{1}{2}|\tau_\lambda(\phi)|^2 + \lambda Tr_g h(\nabla\tau_\lambda(\phi), d\phi))g(X,Y) - 2\lambda symh(\nabla\tau_\lambda(\phi), d\phi)(X,Y), \tag{24}$$

where $\tau_\lambda(\phi) = (3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)$. Using the equations (2) et (4) of the Proposition 1, we have $\frac{1}{2}|\tau_\lambda(\phi)|^2 + \lambda Tr_g h(\nabla\tau_\lambda(\phi), d\phi) = \frac{1}{2}|(3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)|^2 + (3-n)\lambda Tr_g h(\nabla\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi)$.

A simple calculation gives $\frac{1}{2}|(3-n)\lambda d\phi(\operatorname{grad}\ln\lambda)|^2 = \frac{(3-n)^2}{2}\lambda^4|\operatorname{grad}\ln\lambda|^2$ and $Tr_g h(\nabla\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi) = (n+1)\lambda^3|\operatorname{grad}\ln\lambda|^2 + \lambda^3\Delta\ln\lambda$, then

$$\frac{1}{2}|\tau_\lambda(\phi)|^2 + \lambda Tr_g h(\nabla\tau_\lambda(\phi), d\phi) = (3-n)\lambda^4(\frac{n+5}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda). \tag{25}$$

Calculate now $symh(\nabla\tau_\lambda(\phi), d\phi)$, we have by definition for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}
&symh(\nabla\tau_\lambda(\phi), d\phi)(X,Y) = \frac{1}{2}(h(\nabla_X\tau_\lambda(\phi), d\phi(Y)) + h(\nabla_Y\tau_\lambda(\phi), d\phi(X)))\\
&= \frac{3-n}{2}(h(\nabla_X\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + h(\nabla_Y\lambda d\phi(\operatorname{grad}\ln\lambda), d\phi(X)))\\
&= \frac{3-n}{2}\lambda(h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + h(\nabla_Y d\phi(\operatorname{grad}\ln\lambda), d\phi(X)))\\
&+ \frac{3-n}{2}\lambda(X(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + Y(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(X))).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2(\nabla d\ln\lambda(X,Y) + |\operatorname{grad}\ln\lambda|^2 g(X,Y)),\\
&h(\nabla_Y d\phi(\operatorname{grad}\ln\lambda), d\phi(X)) = \lambda^2(\nabla d\ln\lambda(X,Y) + |\operatorname{grad}\ln\lambda|^2 g(X,Y)),\\
&X(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2 X(\ln\lambda)Y(\ln\lambda) = \lambda^2(d\ln\lambda\otimes d\ln\lambda)(X,Y)
\end{aligned}$$

and $Y(\ln\lambda)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(X))=\lambda^2 X(\ln\lambda)Y(\ln\lambda)=\lambda^2(d\ln\lambda\otimes d\ln\lambda)(X,Y)$. It follows that

$$(26)\quad symh(\nabla\tau_\lambda(\phi), d\phi) = (3-n)\lambda^3(\nabla d\ln\lambda + (d\ln\lambda\otimes d\ln\lambda) + |\operatorname{grad}\ln\lambda|^2 g)$$

If we substitute (25) and (26) in (24), we conclude that

$$S_{\lambda,2}(\phi)=(3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2+\Delta\ln\lambda)g-2(3-n)\lambda^4(\nabla d\ln\lambda+(d\ln\lambda\otimes d\ln\lambda)).$$

Calculate now the trace of bi-λ-energy tensor of ϕ. Let $(e_i)_{1\le i\le n}$ be an orthonormal frame on M, we have

$$\begin{aligned}
Tr_g S_{\lambda,2}(\phi) &= S_{\lambda,2}(\phi)(e_i,e_i)\\
&= (3-n)\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)g(e_i,e_i)\\
&- 2(3-n)\lambda^4(\nabla d\ln\lambda(e_i,e_i) + (d\ln\lambda\otimes d\ln\lambda)(e_i,e_i))\\
&= (3-n)n\lambda^4(\frac{n+1}{2}|\operatorname{grad}\ln\lambda|^2 + \Delta\ln\lambda)\\
&- 2(3-n)\lambda^4(\Delta\ln\lambda + |\operatorname{grad}\ln\lambda|^2).
\end{aligned}$$

Then $Tr_g S_{\lambda,2}(\phi) = (3-n)\lambda^4(\frac{n^2+n-4}{2}|\operatorname{grad}\ln\lambda|^2 + (n-2)\Delta\ln\lambda)$. By calculating the Laplacian of the function $\lambda^{\frac{n^2+n-4}{2(n-2)}}$ $((n \ge 4))$, we obtain immediately the following corollary

Corollary 4. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \ge 4)$ to be a conformal map of dilation λ, then the trace of $Tr_g S_{\lambda,2}(\phi)$ is zero if and only if the function $\lambda^{\frac{n^2+n-4}{2(n-2)}}$ is harmonic.*

Remark 5. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 4)$ to be a conformal map of dilation λ, we have $\operatorname{div} S_{\lambda,2}(\phi) = h(\tau_{\lambda,2}(\phi), d\phi) + (Tr_g h(\nabla\tau_\lambda(\phi), d\phi))d\lambda$. A simple calculation gives us $Tr_g h(\nabla\tau_\lambda(\phi), d\phi) = (3-n)\lambda^3(\Delta \ln\lambda + (n+1)|\operatorname{grad}\ln\lambda|^2)$. If we suppose that the map ϕ is bi-λ-harmonic, then $\operatorname{div} S_{\lambda,2}(\phi)$ is zero if and only if the function λ^{n+1} is harmonic.

Theorem 4. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then bi-f-tension field of ϕ is given by $\tau_{f,2}(\phi) = f^2 d\phi(H(\lambda, f))$ where*

$$
\begin{aligned}
H(\lambda, f) &= (n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&- (n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+ 2(n-2)Ricci^M(\operatorname{grad}\ln\lambda) + (n(\Delta\ln f) + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&- \operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&- (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&- 6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f - \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) - 2Ricci^M(\operatorname{grad}\ln f).
\end{aligned}
\tag{27}
$$

Proof of Theorem 4. The f-tension field of ϕ is defined by

$$\tau_f(\phi) = f d\phi(\operatorname{grad}\ln f\lambda^{2-n}),$$

then, the bi-f-tension field of ϕ is given by

$$
\begin{aligned}
\tau_{f,2}(\phi) &= -f(Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ fTr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi) - f\nabla^\phi_{gard\ln f} f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)).
\end{aligned}
$$

For the term $Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))$, we have by (4)

$$
\begin{aligned}
Tr_g(\nabla^\phi)^2 f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) &= fTr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ 2f\nabla^\phi_{\operatorname{grad}\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ f(\Delta\ln f + |\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))
\end{aligned}
$$

and it is easy to see that

$$
\begin{aligned}
\nabla^\phi_{gard\ln f} f d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) &= f\nabla^\phi_{gard\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ f|\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\tau_{f,2}(\phi) &= -f^2(Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&+ Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi)\\
&- f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))\\
&- 3f^2\nabla^\phi_{gard\ln f} d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))
\end{aligned}
\tag{28}
$$

We will study term by term the right-hand of (13). For the first term $Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi$, we have by (10) and (11)

$$\begin{aligned}
&Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi\\
&= d\phi(\operatorname{grad}\Delta\ln(\lambda^{2-n}f)) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln(\lambda^{2-n}f))\\
&- 2(\Delta\ln(\lambda^{2-n}f))d\phi(\operatorname{grad}\ln\lambda) - (n-2)d\phi(\nabla_{\operatorname{grad}\ln(\lambda^{2-n}f)}\operatorname{grad}\ln\lambda)\\
&- (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + 2d\phi(Ricci^M(\operatorname{grad}\ln(\lambda^{2-n}f))).
\end{aligned}$$

By using the fact that $\ln(\lambda^{2-n}f) = (2-n)\ln\lambda + \ln f$, we obtain

$$\begin{aligned}
&Tr_g(\nabla^\phi)^2 d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) + Tr_g R^N(d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)), d\phi)d\phi\\
&= (2-n)d\phi(\operatorname{grad}\Delta\ln\lambda) + \frac{(n-2)(n-6)}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\
&- 2(2-n)(\Delta\ln\lambda)d\phi(\operatorname{grad}\ln\lambda) + (n-2)^2|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln\lambda)\\
&+ 2(2-n)d\phi(Ricci^M(\operatorname{grad}\ln\lambda))\\
&+ d\phi(\operatorname{grad}\Delta\ln f) + 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f)\\
&- 2(\Delta\ln f)d\phi(\operatorname{grad}\ln\lambda) - (n-2)d\phi(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda)\\
&- (n-2)|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f) + 2d\phi(Ricci^M(\operatorname{grad}\ln f)).
\end{aligned} \tag{29}$$

Now, let us look at the last term $\nabla^\phi_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f))$; a simple calculation gives $\nabla^\phi_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = (2-n)\nabla^\phi_{gard\ln f}d\phi(\operatorname{grad}\ln\lambda) + \nabla^\phi_{gard\ln f}d\phi(\operatorname{grad}\ln f) = (2-n)(|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f) + d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda)) + 2d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f) - |\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln\lambda) + d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln f)$, which gives us

$$\begin{aligned}
&\nabla^\phi_{gard\ln f}d\phi(\operatorname{grad}\ln(\lambda^{2-n}f)) = (2-n)(|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f)\\
&+ d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda)) + 2d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f)\\
&- |\operatorname{grad}\ln f|^2 d\phi(\operatorname{grad}\ln\lambda) + \frac{1}{2}d\phi(\operatorname{grad}(|\operatorname{grad}\ln f|^2))
\end{aligned} \tag{30}$$

If we replace (29) and (30) in (28), we deduce that

$$\begin{aligned}
&\tau_{f,2}(\phi) = (n-2)f^2 d\phi(\operatorname{grad}\Delta\ln\lambda) - \frac{(n-2)(n-6)}{2}f^2 d\phi(\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2))\\
&- (n-2)f^2(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)d\phi(\operatorname{grad}\ln\lambda)\\
&+ 2(n-2)f^2 d\phi(Ricci^M(\operatorname{grad}\ln\lambda)) + f^2(n(\Delta\ln f)\\
&+ (2n-1)|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln\lambda)\\
&- f^2 d\phi(\operatorname{grad}\Delta\ln f) - 4d\phi(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f)\\
&+ 4(n-2)f^2|\operatorname{grad}\ln\lambda|^2 d\phi(\operatorname{grad}\ln f)\\
&- f^2(\Delta\ln f + 2|\operatorname{grad}\ln f|^2)d\phi(\operatorname{grad}\ln f) + 4(n-2)f^2 d\phi(\nabla_{gard\ln f}\operatorname{grad}\ln\lambda)
\end{aligned}$$

$$
\begin{aligned}
&- 6f^2 d\ln\lambda(\operatorname{grad}\ln f)d\phi(\operatorname{grad}\ln f) - \frac{3}{2}f^2 d\phi(\operatorname{grad}(|\operatorname{grad}\ln f|^2))\\
&- 2f^2 d\phi(Ricci^M(\operatorname{grad}\ln f)).
\end{aligned}
$$

Then $\tau_{f,2}(\phi) = f^2 d\phi(H(\lambda, f))$ where

$$
\begin{aligned}
H(\lambda,f) &= (n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&- (n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+ 2(n-2)Ricci^M(\operatorname{grad}\ln\lambda) + (n(\Delta\ln f) + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&- \operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&- (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&- 6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f - \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) - 2Ricci^M(\operatorname{grad}\ln f).
\end{aligned}
$$

Remark 6. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then ϕ is bi-f-harmonic if and only if

$$
\begin{aligned}
&(n-2)\operatorname{grad}\Delta\ln\lambda - \frac{(n-2)(n-6)}{2}\operatorname{grad}(|\operatorname{grad}\ln\lambda|^2)\\
&- (n-2)(2(\Delta\ln\lambda) + (n-2)|\operatorname{grad}\ln\lambda|^2)\operatorname{grad}\ln\lambda\\
&+2(n-2)Ricci^M(\operatorname{grad}\ln\lambda)+(n(\Delta\ln f)+(2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&- \operatorname{grad}\Delta\ln f - 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + 4(n-2)|\operatorname{grad}\ln\lambda|^2\operatorname{grad}\ln f\\
&- (\Delta\ln f + 2|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln f + 4(n-2)\nabla_{gard\ln f}\operatorname{grad}\ln\lambda\\
&-6d\ln\lambda(\operatorname{grad}\ln f)\operatorname{grad}\ln f-\frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2)-2Ricci^M(\operatorname{grad}\ln f)=0.
\end{aligned}
\tag{31}
$$

In particular, if the map ϕ is biharmonic, we obtain the following result.

Corollary 5. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal biharmonic map of dilation λ. Then ϕ is bi-f-harmonic if and only if*

$$
\begin{aligned}
&\operatorname{grad}\Delta\ln f + 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f - 4(n-2)\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda\\
&+ (\Delta\ln f + 2|\operatorname{grad}\ln f|^2 - 4(n-2)|\operatorname{grad}\ln\lambda|^2 + 6d\ln\lambda(\operatorname{grad}\ln f))\operatorname{grad}\ln f\\
&- (n\Delta\ln f + (2n-1)|\operatorname{grad}\ln f|^2)\operatorname{grad}\ln\lambda\\
&+ \frac{3}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2) + 2Ricci^M(\operatorname{grad}\ln f) = 0.
\end{aligned}
$$

By Corollary 5, we will construct an example of bi-f-harmonic maps.

Example 5. Let $\phi : \mathbb{R}^4\backslash\{0\} \longrightarrow \mathbb{R}^4\backslash\{0\}$ be the inversion defined by $\phi(x) = \frac{x}{|x|^2}$. ϕ is a conformal biharmonic map with dilation $\lambda = \frac{1}{r^2}(r = |x|)$. We suppose that $\ln f$ is radial ($\ln f = \alpha(r)$). Then by Corollary 5, we deduce that the map $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic if and only if the function α satisfies the following differential equation $\alpha''' + \frac{3}{r}\alpha'' - \frac{27}{r^2}\alpha' + 4\alpha'\alpha'' + \frac{5}{r}(\alpha')^2 + 2(\alpha')^3 = 0$.

Let $\beta = \alpha'$, this equation becomes $\beta'' + \frac{3}{r}\beta' - \frac{27}{r^2}\beta + 4\beta\beta' + \frac{5}{r}\beta^2 + 2\beta^3 = 0$. Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), then $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic if and only if $2a^2 + a - 28 = 0$. This equation has two solutions $a = -4$ and $a = \frac{7}{2}$.

1. For $a = -4$, we obtain $f(r) = Cr^{-4}$ and in this case the inversion $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is f-harmonic so bi-f-harmonic.

2. For $a = \frac{7}{2}$, we obtain $f(r) = Cr^{\frac{7}{2}}$; it follows that the inversion $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \setminus \{0\}$ is bi-f-harmonic (not f-harmonic).

Now, we calculate the stress bi-f-energy tensor for a conformal map ϕ when we prove that $S_{f,2}(\phi)$ depends only the dilation and the function f.

Theorem 5. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then we have*

$$\begin{aligned} S_{f,2}(\phi) &= f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2 + \frac{3}{2}|\operatorname{grad}\ln f|^2)g \\ &+ f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f) + (2-n)\Delta\ln\lambda + \Delta\ln f)g \\ &+ f^2\lambda^2(2(n-2)sym(d\ln\lambda\odot d\ln f) - 2(d\ln f)^2) \\ &- 2f^2\lambda^2((2-n)\nabla d\ln\lambda + \nabla d\ln f) \end{aligned} \tag{32}$$

and the trace of $S_{f,2}$ is given by

$$\begin{aligned} Tr_g S_{f,2}(\phi) &= -(n-2)^2\Delta\ln\lambda - \frac{n(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2 + (n-2)\Delta\ln f \\ &+ \frac{3n-4}{2}|\operatorname{grad}\ln f|^2 - (n-2)^2 d\ln\lambda(\operatorname{grad}\ln f). \end{aligned} \tag{33}$$

Proof of Theorem 5. By definition, the stress bi-f-energy tensor is given by: $S_{f,2}(\phi) = (\frac{1}{2}|\tau_f(\phi)|^2 + fTr_g h(\nabla\tau_f(\phi), d\phi))g - 2fsymh(\nabla\tau_f(\phi), d\phi)$. Since ϕ is a conformal map with dilation λ, we have $\tau_f(\phi) = (2-n)fd\phi(\operatorname{grad}\ln\lambda) + fd\phi(\operatorname{grad}\ln f)$ and

$$\begin{aligned} |\tau_f(\phi)|^2 &= (n-2)^2 f^2\lambda^2|\operatorname{grad}\ln\lambda|^2 + f^2\lambda^2|\operatorname{grad}\ln f|^2 \\ &+ 2(2-n)f^2\lambda^2 d\ln\lambda(\operatorname{grad}\ln f). \end{aligned} \tag{34}$$

Calculate the term $Tr_g h(\nabla\tau_f(\phi), d\phi)$, let $(e_i)_{1\le i\le n}$ be an orthonormal frame on M, we have

$$\begin{aligned} &Tr_g h(\nabla\tau_f(\phi), d\phi) = (2-n)h(\nabla^\phi_{e_i} fd\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) \\ &+ h(\nabla^\phi_{e_i} fd\phi(\operatorname{grad}\ln f), d\phi(e_i)) \\ &= (2-n)fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)h(e_i(f)d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) \\ &+ fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + h(e_i(f)d\phi(\operatorname{grad}\ln f), d\phi(e_i)) \\ &= (2-n)fh(\nabla^\phi_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)fh(d\phi(\operatorname{grad}\ln\lambda), d\phi(\operatorname{grad}\ln f)) \end{aligned}$$

$$
\begin{aligned}
&+ fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + fh(d\phi(\operatorname{grad}\ln f), d\phi(\operatorname{grad}\ln f))\\
&= (2-n)fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + (2-n)f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f)\\
&+ fh(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + f\lambda^2|\operatorname{grad}\ln f|^2.
\end{aligned}
$$

It is known that (see [2]) $h(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i))=h(\nabla d\phi(e_i, \operatorname{grad}\ln\lambda), d\phi(e_i))$ $+h(d\phi(\nabla_{e_i}\operatorname{grad}\ln\lambda), d\phi(e_i))=h(|\operatorname{grad}\ln\lambda|^2 d\phi(e_i), d\phi(e_i))+\lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln\lambda, e_i)$ $= n\lambda^2|\operatorname{grad}\ln\lambda|^2 + \lambda^2\Delta\ln\lambda$ and

$$
\begin{aligned}
&h(\nabla^{\phi}_{e_i} d\phi(\operatorname{grad}\ln f), d\phi(e_i)) = h(\nabla d\phi(e_i, \operatorname{grad}\ln f), d\phi(e_i))\\
&+ h(d\phi(\nabla_{e_i}\operatorname{grad}\ln f), d\phi(e_i))\\
&= h(e_i(\ln\lambda)d\phi(\operatorname{grad}\ln f), d\phi(e_i)) + h(d\ln\lambda(\operatorname{grad}\ln f)d\phi(e_i), d\phi(e_i))\\
&- h(e_i(\ln f)d\phi(\operatorname{grad}\ln\lambda), d\phi(e_i)) + \lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln f, e_i)\\
&= h(d\phi(\operatorname{grad}\ln f), d\phi(\operatorname{grad}\ln\lambda)) + d\ln\lambda(\operatorname{grad}\ln f)h(d\phi(e_i), d\phi(e_i))\\
&- h(d\phi(\operatorname{grad}\ln\lambda), d\phi(\operatorname{grad}\ln f)) + \lambda^2 g(\nabla_{e_i}\operatorname{grad}\ln f, e_i)\\
&= n\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) + \lambda^2\Delta\ln f.
\end{aligned}
$$

Which gives us

$$
\begin{aligned}
Tr_g h(\nabla\tau_f(\phi), d\phi) &= (2-n)nf\lambda^2|\operatorname{grad}\ln\lambda|^2 + (2-n)f\lambda^2\Delta\ln\lambda\\
&\quad + 2f\lambda^2 d\ln\lambda(\operatorname{grad}\ln f) + f\lambda^2\Delta\ln f + f\lambda^2|\operatorname{grad}\ln f|^2.
\end{aligned}
\tag{35}
$$

Now, let us look at the last term $symh(\nabla\tau_f(\phi), d\phi)$; we have by definition for any $X, Y \in \Gamma(TM)$

$$
symh(\nabla\tau_f(\phi), d\phi)(X,Y) = \frac{1}{2}(h(\nabla_X\tau_f(\phi), d\phi(Y)) + h(\nabla\tau_f(\phi), d\phi(X))).
$$

For the term $h(\nabla_X\tau_f(\phi), d\phi(Y))$, we have

$$
\begin{aligned}
&h(\nabla_X\tau_f(\phi), d\phi(Y)) = (2-n)h(\nabla_X f d\phi(\operatorname{grad}\ln\lambda), d\phi(Y))\\
&+ h(\nabla_X f d\phi(\operatorname{grad}\ln f), d\phi(Y))\\
&= (2-n)fh(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + (2-n)X(f)h(d\phi(\operatorname{grad}\ln\lambda), d\phi(Y))\\
&+ fh(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) + X(f)h(d\phi(\operatorname{grad}\ln f), d\phi(Y))\\
&= (2-n)fh(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) + (2-n)f\lambda^2 X(\ln f)Y(\ln\lambda)\\
&+ fh(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) + f\lambda^2 X(\ln f)Y(\ln f).
\end{aligned}
$$

Since ϕ is a conformal map, a simple calculate gives

$$
h(\nabla_X d\phi(\operatorname{grad}\ln\lambda), d\phi(Y)) = \lambda^2|\operatorname{grad}\ln\lambda|^2 g(X,Y) + \lambda^2\nabla d\ln\lambda(X,Y)
$$

and

$$
\begin{aligned}
h(\nabla_X d\phi(\operatorname{grad}\ln f), d\phi(Y)) &= \lambda^2 X(\ln\lambda)Y(\ln f) + \lambda^2 d\ln\lambda(\operatorname{grad}\ln f)g(X,Y)\\
&\quad - \lambda^2 X(\ln f)Y(\ln\lambda) + \lambda^2\nabla d\ln f(X,Y),
\end{aligned}
$$

then

$$h(\nabla_X\tau_f(\phi),d\phi(Y))=(2-n)f\lambda^2|\operatorname{grad}\ln\lambda|^2g(X,Y)+f\lambda^2d\ln\lambda(\operatorname{grad}\ln f)g(X,Y)$$
$$+(2-n)f\lambda^2\nabla d\ln\lambda(X,Y)+f\lambda^2X(\ln\lambda)Y(\ln f)+f\lambda^2\nabla d\ln f(X,Y)$$
$$-(n-1)f\lambda^2X(\ln f)Y(\ln\lambda)+f\lambda^2X(\ln f)Y(\ln f).$$

A similar calculation gives us

$$h(\nabla_Y\tau_f(\phi),d\phi(X))=(2-n)f\lambda^2|\operatorname{grad}\ln\lambda|^2g(X,Y)+f\lambda^2d\ln\lambda(\operatorname{grad}\ln f)g(X,Y)$$
$$+(2-n)f\lambda^2\nabla d\ln\lambda(X,Y)+f\lambda^2X(\ln f)Y(\ln\lambda)+f\lambda^2\nabla d\ln f(X,Y)$$
$$-(n-1)f\lambda^2X(\ln\lambda)Y(\ln f)+f\lambda^2X(\ln f)Y(\ln f).$$

It follows that

$$\begin{aligned}(36)\qquad &symh(\nabla\tau_f(\phi),d\phi)(X,Y)=(2-n)f\lambda^2|\operatorname{grad}\ln\lambda|^2g(X,Y)\\&+f\lambda^2d\ln\lambda(\operatorname{grad}\ln f)g(X,Y)\\&+(2-n)f\lambda^2\nabla d\ln\lambda(X,Y)+f\lambda^2\nabla d\ln f(X,Y)+f\lambda^2(d\ln f)^2(X,Y)\\&-(n-2)f\lambda^2sym(d\ln\lambda\odot d\ln f)(X,Y),\end{aligned}$$

where $(d\ln f)^2(X,Y)=d\ln f(X)d\ln f(Y)=X(\ln f)Y(\ln f)$ and

$$\begin{aligned}sym(d\ln\lambda\odot d\ln f)(X,Y)&=\frac{1}{2}((d\ln\lambda\otimes d\ln f)(X,Y)+(d\ln\lambda\otimes d\ln f)(Y,X))\\&=\frac{1}{2}(X(\ln\lambda)Y(\ln f)+X(\ln f)Y(\ln\lambda)).\end{aligned}$$

By (34), (35) and (36), we conclude that

$$\begin{aligned}S_{f,2}(\phi)(X,Y)&=f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3}{2}|\operatorname{grad}\ln f|^2)g(X,Y)\\&+f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f)+(2-n)\Delta\ln\lambda+\Delta\ln f)g(X,Y)\\&+f^2\lambda^2(2(n-2)sym(d\ln\lambda\odot d\ln f)-2(d\ln f)^2)(X,Y)\\&-2f^2\lambda^2((2-n)\nabla d\ln\lambda+\nabla d\ln f)(X,Y).\end{aligned}$$

It follows that

$$\begin{aligned}S_{f,2}(\phi)&=f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3}{2}|\operatorname{grad}\ln f|^2)g\\&+f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f)+(2-n)\Delta\ln\lambda+\Delta\ln f)g\\&+f^2\lambda^2(2(n-2)sym(d\ln\lambda\odot d\ln f)-2(d\ln f)^2)\\&-2f^2\lambda^2((2-n)\nabla d\ln\lambda+\nabla d\ln f).\end{aligned}$$

Calculate now the trace of stress bi-f-energy tensor. Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M, we have

$$\begin{aligned}
Tr_g S_{f,2}(\phi) &= S_{f,2}(\phi)(e_i,e_i)\\
&= f^2\lambda^2(-\frac{(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3}{2}|\operatorname{grad}\ln f|^2)g(e_i,e_i)\\
&+ f^2\lambda^2((2-n)d\ln\lambda(\operatorname{grad}\ln f)+(2-n)\Delta\ln\lambda+\Delta\ln f)g(e_i,e_i)\\
&+ f^2\lambda^2(2(n-2)sym(d\ln\lambda\odot d\ln f)-2(d\ln f)^2)(e_i,e_i)\\
&- 2f^2\lambda^2((2-n)\nabla d\ln\lambda+\nabla d\ln f)(e_i,e_i)\\
&= f^2\lambda^2(-\frac{n(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+\frac{3n}{2}|\operatorname{grad}\ln f|^2)\\
&+ f^2\lambda^2((2-n)nd\ln\lambda(\operatorname{grad}\ln f)+(2-n)n\Delta\ln\lambda+n\Delta\ln f)\\
&+ f^2\lambda^2(2(n-2)d\ln\lambda(\operatorname{grad}\ln f)-2|\operatorname{grad}\ln f|^2)\\
&- 2f^2\lambda^2((2-n)\Delta\ln\lambda+\Delta\ln f),
\end{aligned}$$

then

$$Tr_g S_{f,2}(\phi) = -(n-2)^2\Delta\ln\lambda-\frac{n(n-2)^2}{2}|\operatorname{grad}\ln\lambda|^2+(n-2)\Delta\ln f$$

$$+\frac{3n-4}{2}|\operatorname{grad}\ln f|^2-(n-2)^2 d\ln\lambda(\operatorname{grad}\ln f).$$

Remark 7. Let $\phi:(M^n,g)\to(N^n,h)$ to be a conformal map of dilation λ, we have $\operatorname{div}S_{f,2}(\phi)=h(\tau_{f,2}(\phi),d\phi)+(Tr_g h(\nabla\tau_f(\phi),d\phi))df$. A simple calculation gives us $Tr_g h(\nabla\tau_f(\phi),d\phi)=f\lambda^2((2-n)\Delta\ln\lambda+(2-n)n|\operatorname{grad}\ln\lambda|^2+\Delta\ln f+3d\ln\lambda(\operatorname{grad}\ln f))$. If we suppose that the map ϕ is bi-f-harmonic, then $\operatorname{div}S_{f,2}(\phi)$ is zero if and only if the functions λ and f satisfy the following equation $(2-n)\Delta\ln\lambda+\Delta\ln f+(2-n)n|\operatorname{grad}\ln\lambda|^2+3d\ln\lambda(\operatorname{grad}\ln f)=0$.

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Fixed point results for mappings satisfying Ciric and Hardy Roger type contractions

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Abstract. The aim of this paper is to establish some common fixed point results for generalized Ciric and Hardy Roger type contraction in ordered complete metric space. An example is constructed which shows the novelty of our results. Our results generalize and extend the results of Altun et. al (J. Funct. Spaces, Article ID 6759320, 2016).
Keywords: fixed point, Ciric contraction, complete ordered metric space, Hardy Roger contraction.

1. Introduction

Fixed point Theory has a wide range of applications in the different fields of analysis. The most important tool in fixed point theory is Banach contraction principle. Many authors obtained fixed point results in various metric spaces under certain contractive conditions (see [1]-[15]).

Let $S : W \to W$ be a mapping. A point $u \in W$ is called a fixed point of S, if $u = Su$. Fixed point theorems are used to find the solution of different mathematical models. Ran and Reurings [11] proved a fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. Nieto et. al. [10] extended the result in [11] for nondecreasing

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mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions.

In this paper, we obtained some common fixed point theorems for generalized Ciric and Hardy Roger type contraction endowed with ordered metric space. We start with some basic notions which will be needed in the sequel.

Definition 1.1 ([11]). Let (W, d) be a metric space. Then:

(i) A sequence $\{u_n\}$ in (W, d) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds a natural number n_0 such that for all $n, m \geq n_0$ we have $d(u_m, u_n) < \varepsilon$.

(ii) A sequence $\{u_n\}$ converges to u if $\lim_{n\to\infty} d(u_n, u) = 0$.

(iii) (W, d) is called complete if every Cauchy sequence in W converges to a point $u \in W$.

Definition 1.2 ([1]). Let $\psi \in \Psi$ and Ψ denotes the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) ψ is non-decreasing.

(Ψ_2) For all $t > 0$, we have $\mu_0(t) = \sum_{k=0}^{\infty} \psi^k(t) < \infty$. Where, ψ^k is the k^{th} iterate of ψ. The function $\psi \in \Psi$ is called comparison function.

Lemma 1.3 ([1]). Let $\psi \in \Psi$. Then:

(i) $\psi(t) < t$, for all $t > 0$,

(ii) $\psi(0) = 0$.

Definition 1.4 ([1]). Let W be a nonempty set. Then $\preceq$ is a partial order on W if:

(i) $u \preceq u$ for all $u \in W$.

(ii) $u \preceq v$ and $v \preceq u \Rightarrow u = v$ for all $(u, v) \in W \times W$.

(iii) $u \preceq v$ and $v \preceq w \Rightarrow u \preceq w$ for all $(u, v, w) \in W \times W \times W$.

Definition 1.5 ([3]). Let W be a nonempty set. Then $(W, \preceq, d)$ is called an ordered metric space if:

(i) d is a metric on W and (ii) $\preceq$ is a partial order on W.

Definition 1.6 ([1]). Let $S : W \to W$ be a function. Then S is level closed from left, if the set $levS_{\preceq} = \{u \in W : u \preceq Su\}$ is non-empty and closed.

2. Fixed point results for Ciric type contraction

In this section, we will prove fixed point result for generalized Ciric contraction in ordered metric space. Our result extend the result given in [1].

Theorem 2.1 Let $(W, \preceq, d)$ be an ordered complete metric space and $S, T : W \to W$ be the self mappings. Suppose that the following assertions hold:

(i) The operator $S : W \to W$ is level closed from left.

(ii) For every $u \in W$, we have $u \preceq Su \Longrightarrow Tu \succeq STu$,and $u \succeq Su \Longrightarrow Tu \preceq STu$.

(iii) There exists a function $\rho \in \Psi$ such that for every $(u, v) \in W \times W$, we have $d(Tu, Tv) \leq \rho(\max\{d(u, v), d(u, Tu), d(v, Tv)\})$, whenever $u \preceq Su$ and $v \succeq Sv$. Then S and T have a unique common fixed point.

Proof. Suppose that u_0 be the arbitrary element of $levS_{\preceq}$, that is, $u_0 \preceq Su_0$. From condition (ii), we have $u_1 \succeq Su_1$, where $u_1 = Tu_0$. Again from condition (ii), we have $u_2 \preceq Su_2$, where $u_2 = Tu_1$. Now, consider the Picard sequence $\{u_n\} \subset W$ define by $u_{n+1} = Tu_n$ where $n = 0, 1, 2, \cdots$. Continuing in this way, we get for even terms of sequence

$$u_{2n} \preceq Su_{2n}. \tag{2.1}$$

For odd terms of sequence, we have

$$u_{2n+1} \succeq Su_{2n+1}. \tag{2.2}$$

As inequalities (2.1) and (2.2) holds, so condition (iii) can be applied. Now, $d(u_{2n+1}, u_{2n+2}) = d(Tu_{2n}, Tu_{2n+1}) \leq \rho(\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, Tu_{2n+1}), d(u_{2n}, Tu_{2n})\}) = \rho(\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\})$. If $\max\{d(u_{2n+1}, u_{2n}), d(u_{2n+1}, u_{2n+2})\} = d(u_{2n+1}, u_{2n+2})$, then a contradiction arises. Therefore,

$$d(u_{2n+1}, u_{2n+2}) \leq \rho(d(u_{2n}, u_{2n+1})). \tag{2.3}$$

As inequalities (2.1) and (2.2) holds, so $d(u_{2n}, u_{2n+1}) = d(Tu_{2n-1}, Tu_{2n})$ where $n = 0, 1, 2, 3, ... \leq \rho(\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, Tu_{2n}), d(u_{2n-1}, Tu_{2n-1}))$ $= \rho(\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, u_{2n+1})\})$. If $\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, u_{2n+1})\} = d(u_{2n}, u_{2n+1})$, then a contradiction arises. Therefore, $d(u_{2n}, u_{2n+1}) \leq \rho(d(u_{2n-1}, u_{2n}))$. As ρ is non-decreasing, so

$$\rho(d(u_{2n}, u_{2n+1})) \leq \rho(\rho(d(u_{2n-1}, u_{2n}))). \tag{2.4}$$

Using inequality (2.4) in (2.3), so inequality (2.3) becomes $d(u_{2n+1}, u_{2n+2}) \leq \rho^2(d(u_{2n-1}, u_{2n}))$. Continuing in this way, we have

$$d(u_{2n+1}, u_{2n+2}) \leq \rho^{2n+1}(d(u_0, u_1)). \tag{2.5}$$

Similarly, we have

$$d(u_{2n}, u_{2n+1}) \leq \rho^{2n}(d(u_0, u_1)). \tag{2.6}$$

Combining inequalities (2.5) and (2.6) , we have

$$d(u_n, u_{n+1}) \leq \rho^n(d(u_0, u_1)). \tag{2.7}$$

Now, let $d(u_0,\ u_1) = 0$. This implies that $u_0 = u_1$. As $u_1 = Tu_0$, so

$$u_0 = Tu_0. \tag{2.8}$$

So, u_0 is a fixed point for T. Now, $u_0 = u_1 \succeq Su_1 = Su_0$. This implies that $u_0 \succeq Su_0$, which further implies

$$u_0 = Su_0. \tag{2.9}$$

From inequalities (2.8) and (2.9), u_0 is a common fixed point for S and T. Now, if $d(u_0, u_1) \neq 0$. Then we assumed that $\delta = d(u_0, u_1) > 0$. So from inequality (2.7), we have $d(u_n,\ u_{n+1}) \leq \rho^n(\delta)$, where $n = 0, 1, 2, \cdots$. For $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\sum_{k\geq n_0}^{\infty} \rho^k(\delta) < \epsilon$. Let $n, m \in \mathbb{N}$, such that $n + m > n \geq n_0$. Then

$$\begin{aligned} d(u_n, u_{n+m}) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+m-1}, u_{n+m}). \\ &\leq \rho^n(\delta) + \rho^{n+1}(\delta) + ... + \rho^{n+m-1}(\delta). \\ &= \sum_{i=n}^{n+m-1} \rho^i(\delta) \leq \sum_{k\geq n_0}^{\infty} \rho^k(\delta) < \epsilon. \end{aligned}$$

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in (W, d). So, there exists some $u^* \in W$ such that $\lim_{n\to\infty} d(u_n,\ u^*) = 0$. But, we know that $u_{2n} \in levS_{\preceq}$ where $n = 0, 1, 2, \cdots$. As $levS_{\preceq}$ is a closed set, and every closed set in a complete metric space is complete. Therefore $u^* \in levS_{\preceq}$. This implies that $u^* \preceq Su^*$. Now, we have

$$\begin{aligned} &d(u^*, Tu^*) \leq d(u^*, u_{2n+2}) + d(u_{2n+2}, Tu^*) \\ &\leq d(u^*, u_{2n+2}) + \rho(\max\{d(u_{2n+1}, u^*), d(u_{2n+1}, Tu_{2n+1}), d(u^*, Tu^*)\}). \end{aligned}$$

Letting $n \to \infty$, we have $d(u^*, Tu^*) \leq 0 + \rho(d(u^*, Tu^*))$. As $\rho(t) < t$, therefore $d(u^*, Tu^*) = 0$. This implies that $u^* = Tu^*$. As, $u^* \preceq Su^*$, so, from condition (ii) we have $u^* = Tu^* \succeq STu^* = Su^*$. This implies that $u^* = Su^*$. Hence, u^* is a common fixed point for S and T.

Uniqueness. Suppose u be the another fixed point for S and T. Then $Su = Tu = u$. As, $u \preceq u$, then $u \succeq u = Su \Rightarrow u \succeq Su$. Also $u^* \preceq Su^*$. Now,

$$\begin{aligned} &d(u^*, u) = d(Tu^*, Tu) \leq \rho(\max\{d(u^*, u), d(u^*, Tu^*), d(u, Tu)\}) \\ &d(u^*, u) \leq \rho d(u^*, u) \end{aligned}$$

As, $\rho(t) < t$ for all $t > 0$, so, $d(u^*, u) = 0$. Thus, u^* is unique common fixed point for S and T.

Example 2.2. Let $W = [0, \infty]$ and d be the metric on W defined by $d(u, v) = |u - v|, (u, v) \in W \times W$. Then (W, d) is a complete metric space. Let $\Re$ be a binary relation on W defined by $\Re = \{(u, u) : u \in W\} \cup \{(0, 2)\}$. Consider the partial order on W defined by $(u, v) \in W \times W, u \preceq v \Leftrightarrow (u, v) \in \Re$. Let us define the pair of mappings $T, S : W \to W$ by

$$Tu = \begin{cases} u, & \text{if } u \notin \{0, 2\} \\ 2, & \text{otherwise} \end{cases}, \quad Su = \begin{cases} 2, & \text{if } u \in [0, 2] \\ 1, & \text{if } u > 2. \end{cases}.$$

Observe that, $levS_{\preceq} = \{0, 2\}$, which is non-empty and closed. Therefore, the operator $S : W \to W$ is level closed from the left. Moreover, we have $\{u \in W : Su \preceq u\} = \{2\}$. In order to check the condition (ii) of Theorem 2.1 , let $u \in W$ be such that $u \preceq Su$; that is, $u \in \{0, 2\}$. If $u = 0$, then $Tu = T0 = 2$ and $STu = ST0 = S2 = 2$. Then $STu \preceq Tu$. If $u = 2$ then $Tu = T2$ and $STu = ST2 = S2 = 2$. Then $STu \preceq Tu$. Now, let $u \in W$ be such that $Su \preceq u$; that is $u = 2$. In this case, we have $STu = ST2 = S2 = 2$ and $Tu = T2 = 2$. Then $Tu \preceq STu$. Therefore, condition (ii) of Theorem 2.1 is satisfied. Now, let $(u, v) \in W \times W$ be such that $u \preceq Su$ and $Sv \preceq v$; that is, $u \in \{0, 2\}$ and $v = 2$. For $(u, v) = (0, 2)$, we have $d(Tu, Tv) = d(T0, T2) = d(2, 2) = 0 \leq \rho(2) = \rho(\max\{d(u, v), d(u, Tu), d(v, Tv)\})$. Now, for $(u, v) = (2, 2)$, $d(Tu, Tv) \leq d(T2, T2) \leq d(2, 2) \leq 0 \leq \rho(0) \leq \rho d(2, 2)$, for every $\rho \in \Psi$. Therefore, all conditions of Theorem 2.2 are satisfied and 2 is the common fixed point.

Now, we will prove fixed point results for generalized Hardy Roger contraction. Our result extend the result given in [1].

Theorem 2.3. Let $(W, \preceq, d)$ be an ordered complete metric space and $S, T : W \to W$ be the self mappings. Suppose that the following assertions hold:

(i)′ The operator $S : W \to W$ is level closed from left.

(ii)′ For every $u \in W$, we have $u \preceq Su \Longrightarrow Tu \succeq STu$, and $u \succeq Su \Longrightarrow Tu \preceq STu$.

(iii)′ There exist constants a, b, c such that $0 \leq a + 2b + 2c < 1$

$$d(Tu, Tv) \leq a(d(u, v)) + b[d(u, Tu) + d(v, Tv)] + c[d(u, Tv) + d(v, Tu)],$$

whenever $u \preceq Su$ and $v \succeq Sv$. Then S and T have a unique common fixed point.

Proof. Suppose that u_0 be the arbitrary element of $levS_{\preceq}$, that is, $u_0 \preceq Su_0$. From condition (ii), we have $u_1 \succeq Su_1$, where $u_1 = Tu_0$. Again from condition (ii), we have $u_2 \preceq Su_2$, where $u_2 = Tu_1$. Now, consider the Picard sequence $\{u_n\} \subset W$ define by $u_{n+1} = Tu_n$ where $n = 0, 1, 2, \cdots$. Continuing in this way we get, for even terms of sequence

$$u_{2n} \preceq Su_{2n}. \tag{2.10}$$

For odd terms of sequence, we have

$$u_{2n+1} \succeq Su_{2n+1.} \tag{2.11}$$

As a consequence, we have

$$\begin{aligned} d(u_{2n}, u_{2n+1}) &= d(Tu_{2n-1}, Tu_{2n}) \\ &\leq a(d(u_{2n-1}, u_{2n})) + b[d(u_{2n-1}, Tu_{2n-1}) + d(u_{2n}, Tu_{2n})] \\ &\quad + c[d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Tu_{2n-1})] \\ &\leq (a + b + c)d(u_{2n-1}, u_{2n}) + (b + c)\, d(u_{2n}, u_{2n+1}) \\ (1 - b - c)\, d(u_{2n}, u_{2n+1}) &\leq (a + b + c)d(u_{2n-1}, u_{2n}). \end{aligned}$$

Dividing by $(1-b-c)$ on both sides, we get

$$d(u_{2n},\ u_{2n+1}) \leq \frac{(a+b+c)}{(1-b-c)} d(u_{2n-1},\ u_{2n}). \tag{2.12}$$

Let

$$\xi = \frac{(a+b+c)}{(1-b-c)}. \tag{2.13}$$

Using inequality (2.13) in (2.12), then inequality (2.12) becomes

$$d(u_{2n}, u_{2n+1}) \leq \xi(d(u_{2n-1}, u_{2n})). \tag{2.14}$$

As, $0 < a + 2b + 2c < 1$ and $\xi \in [0, 1)$. So,

$$d(u_{2n-1}, u_{2n}) \leq \xi(d(u_{2n-2}, u_{2n-1})). \tag{2.15}$$

Using inequality (2.15) in (2.14), then inequality (2.14) becomes

$$d(u_{2n}, u_{2n+1}) \leq \xi^2(d(u_{2n-2}, u_{2n-1})).$$

Continuing in this way, we get

$$d(u_{2n}, u_{2n+1}) \leq \xi^{2n}(d(u_0, u_1)). \tag{2.16}$$

Similarly

$$d(u_{2n+1}, u_{2n+2}) \leq \xi^{2n+1}(d(u_0, u_1)). \tag{2.17}$$

Combining inequalities (2.16) and (2.17), we have

$$d(u_n, u_{n+1}) \leq \xi^n(d(u_0, u_1)). \tag{2.18}$$

Now, let $d(u_0, u_1) = 0$. This implies that $u_0 = u_1$. As $u_1 = Tu_0$, so

$$u_0 = Tu_0. \tag{2.19}$$

So u_0 is a fixed point for T. Now, $u_0 = u_1 \succeq Su_1 = Su_0$, therefore

$$u_0 = Su_0. \tag{2.20}$$

From inequalities (2.19) and (2.20), u_0 is a common fixed point for S and T. Now, if $d(u_0, u_1) \neq 0$. Then we suppose that $\eta = d(u_0, u_1) > 0$. So $d(u_n, u_{n+1}) \leq \xi^n(\eta), n = 0, 1, 2, 3, ...$ For $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\sum_{k \geq n_0}^{\infty} \xi^k(\eta) < \epsilon$. Let $n, m \in \mathbb{N}$, such that $n + m > n \geq n_0$. Then

$$\begin{aligned} d(u_n, u_{n+m}) &= d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+m-1}, u_{n+m}). \\ &\leq \xi^n(\eta) + \xi^{n+1}(\eta) + ... + \xi^{n+m-1}(\eta). \\ &= \sum_{i=n}^{n+m-1} \xi^i(\eta) \leq \sum_{k \geq n}^{\infty} \xi^k(\eta) < \epsilon. \end{aligned}$$

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in $(W,\ d)$. So, there exist some $u^* \in W$ such that $\lim_{n\to\infty} d(u_n, u^*) = 0$. By following similar steps as in previous theorem, we have $u^* \preceq Su^*$. Now,

$$\begin{aligned} d(u^*, Tu^*) &\leq d(u^*, u_{2n+2}) + d(u_{2n+2}, Tu^*) \\ &\leq d(u^*, u_{2n+2}) + a(d(u_{2n+1}, u^*)) + b[d(u_{2n+1}, Tu_{2n+1}) + d(u^*, Tu^*)] \\ &\quad + c[d(u_{2n+1}, Tu^*) + d(u^*, Tu_{2n+1})] \\ &\leq d(u^*, u_{2n+2}) + a(d(u_{2n+1}, u^*)) + b(d(u_{2n+1}, u_{2n+2})) + b(d(u^*, Tu^*)) \\ &\quad + c(d(u_{2n+1}, Tu^*)) + c(d(u^*, u_{2n+2})). \end{aligned}$$

If we take $\lim_{n\to\infty}$, then we obtain $(1 - b - c)\, d(u^*, Tu^*) \leq 0$. This implies that $u^* = Tu^*$. Now, by following similar steps as in previous theorem, we have $u^* = Su^*$. Hence, u^* is a common fixed point for S and T.

Uniqueness. Suppose r be the another common fixed point for S and T. Then $Sr = Tr = r$. As, $r \preceq r$, then $r \succeq r = Sr \Rightarrow r \succeq Sr$. Also $u^* \preceq Su^*$. Now,

$$\begin{aligned} d(u^*, r) &= d(Tu^*, Tr) \\ &\leq a(d(u^*, r)) + b[d(u^*, Tu^*) + d(r, Tr)] \\ &\quad + c[d(u^*, Tr) + d(r, Tu^*)] \\ &\leq a(d(u^*, r)) + b[d(u^*, u^*) + d(r, r)] \\ &\quad + c[d(u^*, r) + d(r, u^*)] \\ (1 - a - 2c)\, d(u^*, r) &\leq 0. \end{aligned}$$

But, $(1 - a - 2c) > 0$. So, $d(u^*, r) = 0$ or $u^* = r$. Hence, u^* is unique common fixed point for S and T

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A novel study on soft rough rings (ideals) over rings

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Abstract. In this paper, we investigate the relationship among rough sets, soft sets and rings. The concept of soft rough rings (ideals) of rings is introduced, which is an extended concept of rough rings (ideals) of rings. Further, we first put forward the concepts of C-soft sets and CC-soft sets over rings. Moreover, some new soft rough operations over rings are explored. In particular, lower and upper soft rough rings (ideals) over rings with another soft set are investigated.

Keywords: soft set, rough set, soft rough set, soft ideal, soft rough ring (ideal).

1. Introduction

The concept of rough set was introduced by Pawlak [25], a new mathematical approach to deal with uncertain knowledge, has recently received wide attention on the research clues in both of theory and applications. As far as known, an equivalence relation on set into disjoint classes and vice versa. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applied areas. Thus, some more general models have been proposed, such as [30, 31, 32]. Nowadays, rough set theory has been applied to many areas, such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [7, 11, 26]. Kuroki [19] proposed the concept of rough ideals in a semigroup. Davvaz [8, 9] applied this theory to rings. On the other hand, many researchers applied rough set theory to algebraic structures in many papers, such as [8, 10, 14].

Nowadays, the mathematical modelling and manipulating of various types of uncertainties has become an increasingly important issue in solving complicated problems arisings in a wide rang of areas such as economy, engineering,

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environmental science, medicine and social science. As far as known, there are several theories to describe uncertainty, for example, fuzzy set theory [27], rough set theory [25] and other mathematical tools. However, each of them has certain inherent limitations. Based on this reason, in 1999, Molodtsov [24] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, the research on soft sets is progressing rapidly. In 2003, Maji et al. [22] proposed some basic operations. Further, Ali et al. [2] revised some operations. In 2011, Ali [3] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [4, 5, 6, 23, 18]. In particular, Zhan and Zhu [28] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [15, 16].

Soft set theory and rough set theory are all mathematical tools to deal with uncertainty. Recently, Feng et al. [12, 13] provided a framework to combine rough sets with soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [20] investigated the relationship among soft sets, soft rough sets and topologies. In 2015, Zhan and Liu et al. [29] applied rough soft set theory to hemirings. In [21], Ma and Zhan put forth rough soft BCI-algebras by means of an ideal of BCI-algebra. In recent years, Shabir et al. [17] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough sets. The underlying concepts are very similar to Pawlak rough sets.

Based on the above idea, it is an interesting work to discuss further on this topic. This paper aims at providing a framework to combine soft sets, rough sets with rings all together, which propose the concept of soft rough rings (ideals). This paper is organized as follows: In Section 2, we recall some concepts and results on soft sets and rough sets. In Section 3, we study some new operations with respect to soft approximation spaces over rings. Further, the lower and upper soft rough rings (ideals) of rings are investigated in Section 4. In particular, in Section 5, we discuss soft rough rings (ideals) based on another soft set.

2. Preliminaries

In this section, we will review some basic notions relevant to soft sets and rough sets. Mododtsov [24] defined the notion of a soft set as follows: Let U be an initial universe and E be a set of parameters. The power set of U is denoted

by $\mathscr{P}(U)$. Throughout this paper, unless otherwise stated, R is always a ring.

Definition 2.1 ([24]). A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping.

For a soft set $\mathfrak{S} = (F, A)$, the set $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A).

Definition 2.2 ([12]). A soft set $\mathfrak{S} = (F, A)$ over U is called a full soft set if $\bigcup_{a \in A} F(a) = U$.

Definition 2.3 ([1]). Let $\mathfrak{S} = (F, A)$ be a soft set over R. Then

(i) (F, A) is called a soft ring over R if $F(x)$ is a subring of R for all $x \in \mathrm{Supp}(F, A)$.

(ii) (F, A) is called a soft ideal over R if $F(x)$ is an ideal of R for all $x \in \mathrm{Supp}(F, A)$.

Definition 2.4 ([25]). Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $R_*X = R^*X$; in the opposite case, i.e., if $R_*X - R^*X \neq \emptyset$, X is said to be a rough set, where two operators are defined as:

$$R_*X = \{x \in U | [x]_R \subseteq X\},$$

$$R^*X = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

Definition 2.5 ([13]). Let $\mathfrak{S} = (F, A)$ be a soft set over U. Then the pair $P = (U, \mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:

$$\underline{apr}_P(X) = \{x \in U | \exists a \in A[x \in F(a) \subseteq X]\},$$

$$\overline{apr}_P(X) = \{x \in U | \exists a \in A[x \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$.

Two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the lower and upper soft rough approximations of X in P, respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is said to be soft definible; otherwise, X is called a soft rough set. In what follows, we call it Feng-soft rough set.

In 2013, Shabir et al. [17] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough set as follows.

Definition 2.6 ([17])**.** Let (F, A) be a soft set over U and $\xi : U \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Then the pair (U, ξ) is called MS-approximation space and for any $X \subseteq U$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which two operators are defined as

$$\underline{X}_\xi = \{x \in X | \xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in U | \xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, then the X is said to be soft definable, otherwise, X is said to be a soft rough set. In what follows, we call it Shabir-soft rough set.

3. Soft rough sets over rings

In this section, we investigate some new operations and fundamental properties of soft rough sets over rings. Meanwhile, some examples are given. Firstly, we give the concept of soft rough sets over rings as follows.

Definition 3.1. Let (F, A) be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Then the pair (R, ξ) is called MS-approximation space and for any $X \subseteq R$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which are two operators are defined as

$$\underline{X}_\xi = \{x \in X | \xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in S | \xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, then X is said to be soft definable, otherwise, X is said to be soft rough set over R.

Remark 3.2. It follows from Definition 3.1 that for any $X \subseteq R$, we have $\underline{X}_\xi \subseteq X \subseteq \overline{X}_\xi$.

Now, we study some basic properties of lower and upper soft rough approximations of a subset X of R. In order to illustrate the roughness in X w.r.t. MS-approximation spaces over rings, we first introduce two special kinds of soft sets over rings.

Definition 3.3. Let $\mathfrak{S} = (F, A)$ be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over R if $\xi(a) = \xi(b)$ and $\xi(c) = \xi(d)$ imply $\xi(a + c) = \xi(b + d)$ and $\xi(ac) = \xi(bd)$ for all $a, b, c, d \in R$.

Definition 3.4. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over R if for all $c \in R$,

(i) $\xi(c) = \xi(x + y)$ for $x, y \in R$, there exist $a, b \in R$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = a + b$.

(ii) $\xi(c) = \xi(xy)$ for $x, y \in R$, there exist $a, b \in R$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = ab$.

Example 3.5. Let $R = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	a
c	0	0	a	a

Then R is a ring. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	0
e_2	0	0	0	1
e_3	1	1	1	0

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \{e_1, e_3\}$, $\xi(c) = \{e_2\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over R, because $\xi(0) = \xi(b)$ and $\xi(a) = \xi(b)$ but $\xi(0 + a) = \xi(a) \neq \xi(c) = \xi(a + b)$.

Example 3.6. We consider the ring R in Example 3.5. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 2.

Table 2 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \{e_1\}$, $\xi(b) = \xi(c) = \{e_1, e_2\}$. It is easy to check that $\mathfrak{S}$ is a C-soft set over R. Nevertheless, $\mathfrak{S}$ is not a CC-soft set over R.

Example 3.7. Let $S = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then R is a ring. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	0	0
e_3	0	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \{e_1, e_2\}$, $\xi(b) = \xi(c) = \{e_1, e_3\}$. Then we can check that $\mathfrak{S}$ is a CC-soft set over R.

Proposition 3.8. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi.$$

Proof. Let $c \in \overline{X}_\xi \cdot \overline{Y}_\xi$. Then $c = \sum_{i=1}^{n} x_i y_i$, where $x_i \in \overline{X}_\xi$ and $y_i \in \overline{Y}_\xi$. It follows from Definition 3.1 that $\xi(x_i) = \xi(x)$ and $\xi(y_i) = \xi(y)$ for some $x \in X$, $y \in Y$. Since $\mathfrak{S}$ is a C-soft set, $\xi(x_i y_i) = \xi(xy)$ and $\xi(\sum_{i=1}^{n} x_i y_i) = \xi(\sum_{i=1}^{n} xy)$ for some $\sum_{i=1}^{n} xy \in X \cdot Y$. Thus $c = \sum_{i=1}^{n} x_i y_i \in \overline{X \cdot Y}_\xi$, and so, $\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi$. □

Proposition 3.9. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi.$$

Proof. Let $c \in \overline{X}_\xi + \overline{Y}_\xi$, then $c = x_1 + y_1$, where $x_1 \in \overline{X}_\xi$ and $y_1 \in \overline{Y}_\xi$. It follows from Definition 3.1 that $\xi(x_1) = \xi(x)$ and $\xi(y_1) = \xi(y)$ for some $x \in X$, $y \in Y$. Since $\mathfrak{S}$ is a C-soft set, $\xi(x_1 + y_1) = \xi(x + y)$ for $x + y \in X + Y$. Hence $c = x_1 + y_1 \in \overline{X + Y}_\xi$. That is, $\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi$. □

The following example shows that the containment in Propositions 3.8 and 3.9 is proper.

Example 3.10. Consider the ring R in Example 3.5. $\mathfrak{S} = (F, A)$ is a soft set over R which is given by Table 4.

Table 4 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	0	0

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \xi(c) = \{e_1\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over R.

If we take $X = \{0, a\}$ and $Y = \{a, c\}$, then $\overline{X}_\xi = \{0, a, b, c\}$ and $\overline{Y}_\xi = \{0, a, b, c\}$, so $\overline{X}_\xi \cdot \overline{Y}_\xi = \{0, a\}$. Also we have, $\overline{X \cdot Y}_\xi = \overline{\{0, a\}}_\xi = \{0, a, b, c\}$. Thus $\overline{X}_\xi \cdot \overline{Y}_\xi \subsetneqq \overline{X \cdot Y}_\xi$. Similarly, we have $\overline{X}_\xi + \overline{Y}_\xi \subsetneqq \overline{X + Y}_\xi$.

If we strength the condition, we can obtain the following result:

Proposition 3.11. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi \cdot \overline{Y}_\xi = \overline{X \cdot Y}_\xi.$$

Proof. It follows from Proposition 3.8 that $\overline{X}_\xi \cdot \overline{Y}_\xi \subseteq \overline{X \cdot Y}_\xi$. Now let $c \in \overline{X \cdot Y}_\xi$. Thus $\xi(c) = \xi(\sum_{i=1}^n x_i y_i)$ for some $x_i \in X$ and $y_i \in Y$. Then there exist $a_i, b_i \in R$, such that $\xi(a_i) = \xi(x_i)$ and $\xi(b_i) = \xi(y_i)$ satisfying $c = \sum_{i=1}^n a_i b_i$ since $\mathfrak{S}$ is a CC-soft set over R. Thus $a_i \in \overline{X}_\xi$ and $b_i \in \overline{Y}_\xi$. Hence $c \in \overline{X}_\xi \cdot \overline{Y}_\xi$. Summing up the above arguments, $\overline{X}_\xi \cdot \overline{Y}_\xi = \overline{X \cdot Y}_\xi$. □

Proposition 3.12. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\overline{X}_\xi + \overline{Y}_\xi = \overline{X + Y}_\xi.$$

Proof. It follows from Proposition 3.9 that $\overline{X}_\xi + \overline{Y}_\xi \subseteq \overline{X + Y}_\xi$. Now let $c \in \overline{X + Y}_\xi$. Thus $\xi(c) = \xi(x + y)$ for some $x \in X$ and $y \in Y$. Then there exist $a, b \in R$, such that $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ satisfying $c = a + b$ since $\mathfrak{S}$ is a CC-soft set over R. Thus $a \in \overline{X}_\xi$ and $b \in \overline{Y}_\xi$. Hence $c \in \overline{X}_\xi + \overline{Y}_\xi$. Summing up the above arguments, $\overline{X}_\xi + \overline{Y}_\xi = \overline{X + Y}_\xi$. □

Next, we consider lower soft rough approximations over rings.

Proposition 3.13. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi$ is false, then there exists $c \in \underline{X}_\xi \cdot \underline{Y}_\xi$ but $c \notin \underline{X \cdot Y}_\xi$. Then $c = \sum_{i=1}^n a_i b_i$, where $a_i \in \underline{X}_\xi$ and $b_i \in \underline{Y}_\xi$, and so $\xi(a_i) \neq \xi(x_i)$ and $\xi(b_i) \neq \xi(y_i)$ for all $x_i \in X^c$ and $y_i \in Y^c$.

On the other hand, $c \notin \underline{X \cdot Y}_\xi$, then we may have the following two conditions:

(i) $c \notin X \cdot Y$, which contradicts with $c \in \underline{X}_\xi \cdot \underline{Y}_\xi \subseteq X \cdot Y$;

(ii) $c \in X \cdot Y$ and $\xi(c) = \xi(\sum_{i=1}^n x_i' y_i')$ for some $\sum_{i=1}^n x_i' y_i' \in (X \cdot Y)^c$. Thus $x_i' \in X^c$ or $y_i' \in Y^c$. In fact, if $x_i' \notin X^c$ and $y_i' \notin Y^c$, we have $\sum_{i=1}^n x_i' y_i' \in X \cdot Y$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over R, there exist $a_i', b_i' \in R$ such that $\xi(a_i') = \xi(x_i')$ and $\xi(b_i') = \xi(y_i')$ satisfying $\sum_{i=1}^n a_i' b_i' = c$, for some $x_i' \in X^c$ or $y_i' \in Y^c$. This is contradiction. Hence, $\underline{X}_\xi \cdot \underline{Y}_\xi \subseteq \underline{X \cdot Y}_\xi$. □

Proposition 3.14. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X, Y any two non-empty subsets of R. Then

$$\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X + Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X+Y}_\xi$ is false, then there exists $c \in \underline{X}_\xi + \underline{Y}_\xi$ but $c \notin \underline{X+Y}_\xi$. Then $c = a + b$, where $a \in \underline{X}_\xi$ and $b \in \underline{Y}_\xi$. Hence $\xi(a) \neq \xi(x)$ and $\xi(b) \neq \xi(y)$ for all $x \in X^c$ and $y \in Y^c$.

On the other hand, $c \notin \underline{X+Y}_\xi$, then we may have the following two conditions:

(i) $c \notin X + Y$, which contradicts with $c \in \underline{X}_\xi + \underline{Y}_\xi \subseteq X + Y$;

(ii) $c \in X + Y$ and $\xi(c) = \xi(x' + y')$ for some $x' + y' \in (X + Y)^c$. Thus $x' \in X^c$ or $y' \in Y^c$. In fact, if $x' \notin X^c$ and $y' \notin Y^c$, we have $x' + y' \in X + Y$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over R, there exist $a', b' \in R$, such that $\xi(a') = \xi(x')$ and $\xi(b') = \xi(y')$ satisfying $a' + b' = c$, for some $x' \in X^c$ or $y' \in Y^c$. This is contradiction. Hence $\underline{X}_\xi + \underline{Y}_\xi \subseteq \underline{X+Y}_\xi$. □

The following example shows that the containment in Propositions 3.13 and 3.14 are proper.

Example 3.15. Consider the ring S and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Then we know that $\mathfrak{S}$ is a CC-soft set over R.

If we take $X = \{a, b, c\}$ and $Y = \{0, b, c\}$, then $\underline{X}_\xi = \{b, c\}$ and $\underline{Y}_\xi = \{b, c\}$, so $\underline{X}_\xi \cdot \underline{Y}_\xi = \{b, c\}$. Also we have $\underline{X \cdot Y}_\xi = \underline{\{0, a\}}_\xi = \{0, a, b, c\}$, that is $\underline{X}_\xi \cdot \underline{Y}_\xi \subsetneqq \underline{X \cdot Y}_\xi$. Similarly, we have $\underline{X}_\xi + \underline{Y}_\xi \subsetneqq \underline{X+Y}_\xi$.

The following example shows that Propositions 3.13 and 3.14 are not true if $\mathfrak{S}$ is not a CC-soft set over R.

Example 3.16. Consider the ring R and the soft set $\mathfrak{S} = (F, A)$ in Example 3.5. Then we know that $\mathfrak{S}$ is not a CC-soft set over R. If we take $X = \{0, a, c\}$ and $Y = \{c\}$, then $\underline{X}_\xi = \{c\}$ and $\underline{Y}_\xi = \{c\}$. So $\underline{X}_\xi \cdot \underline{Y}_\xi = \{a\}$ and $\underline{X}_\xi + \underline{Y}_\xi = \{0\}$. Also we have $\underline{X \cdot Y}_\xi = \underline{\{0, a\}}_\xi = \emptyset$ and $\underline{X+Y}_\xi = \underline{\{0, b, c\}}_\xi = \emptyset$. So $\underline{X}_\xi \cdot \underline{Y}_\xi \nsubseteq \underline{X \cdot Y}_\xi$ and $\underline{X}_\xi + \underline{Y}_\xi \nsubseteq \underline{X+Y}_\xi$.

4. Characterizations of soft rough rings (ideals) of rings

In this section, we characterize soft rough rings (ideals) of rings. First, we give the notion of soft rough rings (ideals) as follows.

Definition 4.1. In Definition 3.1, if $\underline{X}_\xi \neq \overline{X}_\xi$, (i) X is called a lower (upper) soft rough ring (ideal) w.r.t. $\mathfrak{S}$ of R, if $\underline{X}_\xi$ ($\overline{X}_\xi$) is a subring (ideal) of R; (ii) X is called a soft rough ring (ideal) w.r.t. $\mathfrak{S}$ of R, if $\underline{X}_\xi$ and $\overline{X}_\xi$ are subrings (ideals) of R.

Example 4.2. Let $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be a ring of integers modulo 6 and $\mathfrak{S} = (F, A)$ be a soft set over R which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	0	1	1	1	0
e_2	0	1	1	0	1	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(3) = \{e_1\}$, $\xi(1) = \xi(5) = \{e_2\}$, $\xi(2) = \xi(4) = \{e_1, e_2\}$. It follows from Definition 4.1 that for $X = \{0, 1, 2, 3\} \subseteq R$, we have

$$\underline{X}_\xi = \{0, 3\} \text{ and } \overline{X}_\xi = \mathbb{Z}_6.$$

This shows that $\underline{X}_\xi$ and $\overline{X}_\xi$ are subrings of R. In other words, X is a soft rough ring of R.

Proposition 4.3. Let (R, ξ) be an MS-approximation space. If X and Y are lower soft rough rings (ideals) of R, then so is $X \cap Y$.

Proof. It follows from Definition 4.1 that $\underline{X}_\xi$ and $\underline{Y}_\xi$ are subrings (ideals) of R, so $\underline{X}_\xi \cap \underline{Y}_\xi$ is a subring (ideal) of R. It is easy to know that $\underline{X}_\xi \cap \underline{Y}_\xi = \underline{X \cap Y}_\xi$, so $\underline{X \cap Y}_\xi$ is also a subring (ideal) of R. Hence $X \cap Y$ is a lower soft rough ring (ideal) of R. □

In general, $X \cap Y$ is not an upper soft rough ring of R, if X and Y are upper soft rough rings of R. Actually we have the following example.

Example 4.4. Consider the ring R and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Now we define $X = \{0, c\}$ and $Y = \{a, c\}$, then $\overline{X}_\xi = \{0, a, b, c\}$ and $\overline{Y}_\xi = \{0, a, b, c\}$ are subrings of R. That is X and Y are upper soft rough rings of R. However, $\overline{X \cap Y}_\xi = \overline{\{c\}}_\xi = \{b, c\}$ is not an upper soft rough ring of R.

Finally, we study the lower and upper soft rough rings (ideals) of rings.

Theorem 4.5. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be a subring of R. Then X is an upper soft rough ring of R.

Proof. Since $X \subseteq \overline{X}_\xi$, $0 \in \overline{X}_\xi$. For all $m, n \in \overline{X}_\xi$, then $\xi(m) = \xi(x)$ and $\xi(n) = \xi(y)$ for some $x, y \in X$. It follows from $\xi(n) = \xi(y)$ that we have $\xi(-n) = \xi(-y)$. Since $\mathfrak{S}$ is a C-soft set and X is a subring of R, $\xi(m - n) = \xi(x - y)$ for $x - y \in X$, thus $m - n \in \overline{X}_\xi$. Similarly, $mn \in \overline{X}_\xi$. Hence, $\overline{X}_\xi$ is a subring of R, that is X is an upper soft rough ring of R. □

Theorem 4.6. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be a subring of R. Then X is a lower soft rough ring of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let $\underline{X}_\xi \neq \emptyset$, for all $m, n \in \underline{X}_\xi$. We suppose that $m \cdot n \notin \underline{X}_\xi$. Then we have $\xi(m) \neq \xi(x)$ for all $x \in X^c$ and $\xi(n) \neq \xi(y)$ for all $y \in X^c$.

On the other hand, $m \cdot n \notin \underline{X}_\xi$, then we may have the following two conditions:

(i) $m \cdot n \notin X$, which contradicts with $m \cdot n \in \underline{X}_\xi \cdot \underline{Y}_\xi \subseteq X \cdot X \subseteq X$;

(ii) $m \cdot n \in X$ and $\xi(x') = \xi(m \cdot n)$ for some $x' \in X^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set, there exist $x_1, y_1 \in R$ such that $\xi(m) = \xi(x_1)$ and $\xi(n) = \xi(y_1)$ satisfying $x_1 \cdot y_1 = x' \in X^c$. Thus, $x_1 \in X^c$ or $y_1 \in X^c$. In fact, if $x_1 \notin X^c$ and $y_1 \notin X^c$, we have $x_1 \cdot y_1 \in X \cdot X \subseteq X$, a contradiction. That is there exist

$x_1 \in X^c$ such that $\xi(m) = \xi(x_1)$ or $y_1 \in X^c$ such that $\xi(n) = \xi(y_1)$. This is contradiction to $m \cdot n \notin \underline{X}_\xi$. Thus $m \cdot n \in \underline{X}_\xi$. Similarly, we have $m - n \in \underline{X}_\xi$. This implies $\underline{X}_\xi$ is a subring of R, that is X is a lower soft rough ring of R. □

Theorem 4.7. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be an ideal of R. Then X is an upper soft rough ideal of R.

Proof. Let X be an ideal of R. It follows from Theorem 4.5 that $\overline{X}_\xi$ is a subring of R. If $r \in R$ and $s \in \overline{X}_\xi$, then $\xi(s) = \xi(x)$ for some $x \in X$. Since $\xi(r) = \xi(r)$ and $\mathfrak{S}$ is a C-soft set, $\xi(rs) = \xi(rx)$ for some $rx \in R \cdot X \subseteq X$, thus $rs \in \overline{X}_\xi$. Hence $\overline{X}_\xi$ is a left ideal of R. Similarly, we have $\overline{X}_\xi$ is a right ideal of R. So $\overline{X}_\xi$ is an ideal of R, that is X is an upper soft rough ideal of R. □

Theorem 4.8. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be an ideal of R. Then X is a lower soft rough ideal of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let X be an ideal of R. It follows from Theorem 4.6 that $\underline{X}_\xi$ is a subring of R. By Proposition 3.13, we have $R \cdot \underline{X}_\xi = \underline{R}_\xi \cdot \underline{R}_\xi \subseteq \underline{R \cdot X}_\xi \subseteq \underline{X}_\xi$. Similarly, $\underline{X}_\xi \cdot R \subseteq \underline{X}_\xi$. Therefore, $\underline{X}_\xi$ is an ideal of R, that is X is a lower soft rough ideal. □

The next example shows that the converse of Theorems 4.7 and 4.8 do not hold in general.

Example 4.9. We consider the ring R and soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Then $\mathfrak{S}$ is a CC-soft set over R.

Now we define $X = \{0, a, b\}$. So $\underline{X}_\xi = \{0, a\}$, $\overline{X}_\xi = \{0, a, b, c\}$. It is easy to know that $\overline{X}_\xi$ and $\underline{X}_\xi$ are ideals of R. However, X is not an ideal of R.

Theorem 4.10. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and X be a *bi*-ideal of R. Then X is an upper soft rough *bi*-ideal of R.

Proof. Let X be a *bi*-ideal of R. It follows from Theorem 4.7 that $\overline{X}_\xi$ is an ideal of R. It follows from Proposition 3.8 that $\overline{X}_\xi \cdot R \cdot \overline{X}_\xi = \overline{X}_\xi \cdot \overline{R}_\xi \cdot \overline{X}_\xi \subseteq \overline{X \cdot R \cdot X}_\xi \subseteq \overline{X}_\xi$. Hence $\overline{X}_\xi$ is a *bi*-ideal of R, that is X is an upper soft rough *bi*-ideal of R. □

Theorem 4.11. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and X be a *bi*-ideal of R. Then X is a lower soft rough *bi*-ideal of R if $\underline{X}_\xi \neq \emptyset$.

Proof. Let X be a *bi*-ideal of R. It follows from Theorem 4.8 that $\underline{X}_\xi$ is an ideal of R. It follows from Proposition 3.13, that $\underline{X}_\xi \cdot R \cdot \underline{X}_\xi = \underline{X}_\xi \cdot \underline{R}_\xi \cdot \underline{X}_\xi \subseteq \underline{X \cdot R \cdot X}_\xi \subseteq \underline{X}_\xi$. Hence $\underline{X}_\xi$ is a *bi*-ideal of R, that is X is a lower soft rough *bi*-ideal of R. □

5. Soft rough rings (ideals) with respect to another soft set

In this section, we investigate soft rough rings (ideals) based on another soft set.

Definition 5.1. Let $\mathfrak{S} = (F, A)$ be a soft set over R and $\xi : R \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over R. The lower and upper soft rough approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G,B)}_\xi = (\underline{G}_\xi, B)$ and $\overline{(G,B)}_\xi = (\overline{G}_\xi, B)$, respectively, which are two operators defined as

$$\underline{G(e)}_\xi = \{x \in G(e) | \xi(x) \neq \xi(y) \text{ for all } y \in S - G(e)\}$$

and

$$\overline{G(e)}_\xi = \{x \in S | \xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.
(i) If $\underline{(G,B)}_\xi = \overline{(G,B)}_\xi$, then $\mathfrak{T}$ is called definable.
(ii) If $\underline{(G,B)}_\xi \neq \overline{(G,B)}_\xi$ and $\underline{G(e)}_\xi$ ($\overline{G(e)}_\xi$) is a subring (ideal) of R for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) soft rough ring (ideal) with respect to $\mathfrak{S}$ over R. Moreover, $\mathfrak{T}$ is called a lower (upper) soft rough ring (ideal) with respect to $\mathfrak{S}$ over R if $\underline{G(e)}_\xi$ and $\overline{G(e)}_\xi$ are subrings (ideals) with respect to $\mathfrak{S}$ over R for all $e \in B$.

Example 5.2. We consider the ring R in Example 4.2, soft set $\mathfrak{S} = (F, A)$ over R which is given by Table 6.

Table 6 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	0	1	1	1	0
e_2	0	1	0	0	0	1
e_3	1	1	0	1	0	1

Then the mapping $\xi : R \to \mathscr{P}(A)$ of MS-approximation space (R, ξ) is given by $\xi(0) = \xi(3) = \{e_1, e_3\}$, $\xi(1) = \xi(5) = \{e_2, e_3\}$, $\xi(2) = \xi(4) = \{e_1\}$. Define a soft set $\mathfrak{T} = (G, B)$ as the following Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	1	1	1	0	0
e_2	1	0	0	1	0	0
e_3	1	1	1	0	0	0

By calculating, $\underline{G(e_1)}_\xi = \{0, 3\}$, $\overline{G(e_1)}_\xi = \{0, 1, 2, 3, 4, 5\}$, $\underline{G(e_2)}_\xi = \{0, 3\}$, $\overline{G(e_1)}_\xi = \{0, 3\}$, $\underline{G(e_3)}_\xi = \emptyset$, $\overline{G(e_3)}_\xi = \{0, 1, 2, 3, 4, 5\}$ It is easy to check that $\underline{(G,B)}_\xi$ and $\overline{(G,B)}_\xi$ are subrings of R for all $e \in B$. In other words, $\mathfrak{T} = (G, B)$ is a soft rough ring with respect to $\mathfrak{S}$ of R.

Definition 5.3. Let $\mathfrak{T} = (G, B)$ and $\mathfrak{I} = (H, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. The addition operation $+$ and a multiplication operation $\cdot$ of $\mathfrak{T} + \mathfrak{I}$ and $\mathfrak{T} \cdot \mathfrak{I}$ are defined as $\mathfrak{T} + \mathfrak{I} = (G, B) + (H, C) = (K, D)$ and $\mathfrak{T} \cdot \mathfrak{I} = (G, B) \cdot (H, C) = (L, D)$, where $K(a) = G(a) + H(a)$ and $L(a) = G(a) \cdot H(a)$ for all $a \in D$.

Proposition 5.4. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 + G_2, D)}_\xi$.

(2) $\overline{(G_1, B)}_\xi \cdot \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.8 and 3.9. □

If we strength the condition, we can obtain the following result.

Proposition 5.5. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi = \overline{(G_1 + G_2, D)}_\xi$.

(2) $\overline{(G_1, B)}_\xi \cdot \overline{(G_2, C)}_\xi = \overline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.11 and 3.12. □

Next, we consider the lower soft rough approximations over rings.

Proposition 5.6. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and (R, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over R with $D = B \cap C \neq \emptyset$. Then

(1) $\underline{(G_1, B)}_\xi + \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 + G_2, D)}_\xi$.

(2) $\underline{(G_1, B)}_\xi \cdot \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 \cdot G_2, D)}_\xi$.

Proof. The proof is similar to that of Propositions 3.13 and 3.14. □

Finally, we investigate the upper and lower soft rough rings (ideals) with respect to another soft set.

Theorem 5.7. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ring of R. Then $\mathfrak{T}$ is an upper soft rough ring of R.

Proof. The proof is similar to that of Theorem 4.5. □

Theorem 5.8. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ring of R. Then $\mathfrak{T}$ is a lower soft rough ring of R if $\underline{\mathfrak{T}_\xi} \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.6. □

Theorem 5.9. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ideal of R. Then $\mathfrak{T}$ is an upper soft rough ideal of R.

Proof. The proof is similar to that of Theorem 4.7. □

Theorem 5.10. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft ideal of R. Then $\mathfrak{T}$ is a lower soft rough ideal of S if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.8. □

Theorem 5.11. Let $\mathfrak{S} = (F, A)$ be a C-soft set over R and $\mathfrak{T} = (G, B)$ be a soft bi-ideal of R. Then $\mathfrak{T}$ is an upper soft rough bi-ideal of R.

Proof. The proof is similar to that of Theorem 4.10. □

Theorem 5.12. Let $\mathfrak{S} = (F, A)$ be a CC-soft set over R and $\mathfrak{T} = (G, B)$ be a soft bi-ideal of R. Then $\mathfrak{T}$ is a lower soft rough bi-ideal of R if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.11. □

6. Conclusion

In this paper, we initiate a novel soft rough set theory to rings and propose soft rough rings (ideals) over rings. We discuss some operational properties and algebraic structures of lower and upper soft rough approximations over rings. Besides, several examples are presented in order to investigate their characterizations. In particular, we discuss soft rough rings (ideals) based on another soft set.

As an extension of this work, the following problems maybe considered:

(1) Constructing soft rough sets to other algebras, such as hyperrings, hyperhemirings, BL-algebras and so on;

(2) Investigating decision making methods based on soft rough sets;

(4) Establishing soft rough sets to some applied some areas of applications, such as information sciences, intelligent systems and so on.

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The interval valued fuzzy graph associated with a Crisp graph

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Abstract. We define the interval valued fuzzy graph (IVFG) associated with a crisp graph based on the degrees of the nodes of the crisp graph and study its various properties. The nature of arcs of the IVFG associated with a crisp graph can be determined if the adjacency matrix of the crisp graph is given. We show that the IVFG associated with a regular graph is regular, totally regular, edge regular and totally edge regular, but the IVFG associated with a complete graph is not a complete IVFG. We prove that the IVFG associated with $C_n, n \geq 3$ is an interval valued fuzzy cycle and the IVFG associated with the wheel graph $W_n, n \geq 5$ is an interval valued fuzzy tree.

Keywords: interval valued fuzzy graph, Crisp graph, Regular graph

1. Introduction

Graphs are used to symbolize relations between objects. But uncertainty and vagueness happen to be significant elements in relations. To address this, Rosenfeld [14] developed the theory of fuzzy graphs giving membership degrees to nodes and arcs of crisp graphs. But there may be incidences in which one do not have an exact information about the choice of the membership degrees. In these situations, it is better to represent the membership degree by means of an interval. Replacing the membership degrees of nodes and arcs in fuzzy graphs by interval valued fuzzy sets such that they satisfy some particular condition,

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interval valued fuzzy graphs were defined. Thus IVFG provide a better explanation of vagueness and uncertainty within the particular interval than fuzzy graphs. Hongmei and Lianhua [4] introduced IVFGs generalizing fuzzy graphs.

We can associate any number of IVFGs with a particular crisp graph. This paper is an ernest effort to construct a particular type of IVFG from a crisp graph based on the degrees of the nodes of the crisp graph.Throughout our work, we consider only simple, connected, undirected graphs.

Definition 1.1 ([2]). *Let $G^\star = (V, E)$ be a crisp graph. Then an interval valued fuzzy graph (IVFG) G on $G^\star$ is defined as a pair $G = (A, B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an interval valued fuzzy set on V and $B = [\mu_B^-(xy), \mu_B^+(xy)]$ is an interval valued fuzzy set on E such that $\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y))$ for all $xy \in E$.*

If $\mu_B^-(xy) = min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_{B^+}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y))$ for all $x, y \in V$, then G is called a complete IVFG (CIVFG).

Definition 1.2 ([7]). *An IVFG G is said to be connected if any two nodes are joined by a path.*

Definition 1.3 ([5]). *An IVFG G is said to be regular (totally regular) if every node of G has the same degree (total degree).*

Definition 1.4 ([6]). *An IVFG G is said to be edge regular (totally edge regular) if every arc of G has the same edge degree (total edge degree).*

Definition 1.5 ([13]). *An IVFG G is said to be a complete product interval valued fuzzy graph (CPIVFG) if $\mu_B^-(xy) = \mu_A^-(x) \times \mu_A^-(y)$ and $\mu_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y)$ for all $x, y \in V$ where $\times$ denotes ordinary multiplication.*

Definition 1.6 ([12]). *Two arcs e_1 and e_2 are said to be comparable if their membership degrees are such that either $\mu_{B^-}(e_1) > \mu_{B^-}(e_2)$and $\mu_{B^+}(e_1) > \mu_{B^+}(e_2)$ or $\mu_{B^-}(e_1) < \mu_{B^-}(e_2)$ and $\mu_{B^+}(e_1) < \mu_{B^+}(e_2)$. They are said to be equal if their membership degrees are equal.*

Definition 1.7 ([11]). *A connected IVFG G is said to be strongly connected if there exists a strong path between every two nodes of G.*

Definition 1.8 ([12]). *An arc (u, v) in an IVFG G is said to be α strong if $\mu_{B^-}(u, v) > NCONN_{G-(u,v)}(u, v)$ and $\mu_{B^+}(u, v) > PCONN_{G-(u,v)}(u, v)$. It is said to be β strong if $\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$ and $\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v)$. And it is called a weak arc if*

$$\mu_{B^-}(u, v) < NCONN_{G-(u,v)}(u, v)$$

and $\mu_{B^+}(u, v) < PCONN_{G-(u,v)}(u, v)$

Some other concepts that are used in this work are provided in the Table 1. We use the following theorems in the subsequent discussions.

Notation	Concept
$O(G)$	Order of G [5]
S(G)	Size of G [5]
$d(u)$	Degree of a node u[5]
$td(u)$	Total degree of a node u[5]
$d(uv)$	Degree of an arc uv[5]
$td(uv)$	Total degree of an arc uv[5]
P	Path in an IVFG [7]
	Strong path in an IVFG [12]
$S_{\mu^-}(P)$, $S_{\mu^+}(P)$	Strength of the path P [7]
$(\mu_{B^-})^\infty$, $NCONN_G$	Maximum of the μ^- strength [7]
$(\mu_{B^+})^\infty$, $PCONN_G$	Maximum of the μ^+ strength [7]
	Interval valued fuzzy bridge[7]
	Interval valued fuzzy cutnode[7]
	Interval valued fuzzy tree[9]
	Interval valued fuzzy cycle[9]
	Isomorphism and co-weak isomorphism of IVFGs [2]
	Interval valued fuzzy line graph [1]
	Interval valued fuzzy node arc matrix [10]

Table 1: Major concepts related to interval valued fuzzy graphs

Theorem 1.1 ([11]). [*A*] *Let G be a connected IVFG such that its edges are either comparable or equal. Then G is strongly connected.*

Theorem 1.2 ([12]). [*B*] *Let G be an IVFG whose underlying crisp graph is not a tree. If every two arcs of G are equal,then all the arcs of G will be β strong.*

Theorem 1.3 ([12]). [*C*] *An arc (u, v) in an IVFG $G = (A, B)$ is a weak arc if and only if (u, v) is the unique weakest arc of atleast one cycle in G.*

Theorem 1.4 ([9]). [*D*] *Let $G = (A, B)$ be a connected IVFG on $G^* = (V, E)$, where G^* is not a tree. Then G is an IVFT if and only if G contains only α strong and weak arcs.*

Theorem 1.5 ([9]). [*E*] *Let $G = (A, B)$ be an IVFG on a cycle $G^* = (V, E)$. Then G is an IVFC if and only if G has atleast two β strong arcs.*

Theorem 1.6 ([8]). [*F*] *Let G be a non complete IVFG on $G^* = (V, E)$ with $|V| = n$ and $|E| = m$ Suppose that every two arcs are either comparable or equal and the s weaker arcs where $s = m - n + 1$ lies in s different cycles. Then those s arcs will be weak arcs and all the other $m - s$ arcs are α strong.*

Theorem 1.7 ([8]). [*G*] *Let $G = (A, B)$ be an Interval Valued Fuzzy Graph. Then an arc (u, v) of G is an IVF bridge if and only if it is α strong.*

Theorem 1.8 ([8]). [*H*] *If w is common node of atleast two α strong arcs, then w is an IVF cutnode.*

2. Construction of an interval valued fuzzy graph

Let $G^* = (V, E)$ be a simple connected graph with n nodes and m arcs. Let the nodes be labelled as $v_1, v_2, \ldots, v_n$. We can construct an IVFG from this crisp graph. The construction is as follows: Let v_i be any arbitrary node of G^*. Let $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1}, \mu_A^+(v_i) = \frac{d(v_i)}{n-1}$.

Now, let $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2}, \mu_B^+(v_iv_j) = \frac{[d(v_i)][d(v_j)]}{(n-1)^2}$ for all $(v_i, v_j) \in E$. Hence the membership degrees of the arcs are given according to the rule: The lower membership degree of an arc is the product of the lower membership degrees of the corresponding end nodes and the upper membership degree of an arc is the product of the upper membership degrees of the corresponding end nodes. The above construction yields an IVFG as proved below and it is called the IVFG associated with the crisp graph G^*. We use **A** and **B** as in **Definition** 1.1 in the following theorem.

Theorem 2.1. *Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$ and let $G = (A, B)$. Then, G is an Interval Valued Fuzzy Graph.*

Proof. Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$.

Consider $A = \{(v_i, [\mu_A^-(v_i), \mu_A^+(v_i)]) : v_i \in V\}$ and $B = \{(v_iv_j, [\mu_B^-(v_iv_j), \mu_B^+(v_iv_j)]) : v_iv_j \in E\}$. We prove that A is an IVFS on V and B is an IVFS on E satisfying the requirements of an Theorem.

A is an IVFS on V

From the above construction, we have, $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1}, \mu_A^+(v_i) = \frac{d(v_i)}{n-1}$.

Since G^* is connected, $d(v_i) \geq 1$.

$\therefore d(v_i) - 1 \geq 0$ and so clearly $\mu_A^-(v_i) = \frac{d(v_i)-1}{n-1} \geq 0$.

Clearly, $\mu_A^-(v_i) < \mu_A^+(v_i)$ since $d(v_i) - 1 < d(v_i)$.

Again, since $d(v_i) \leq n - 1$, $\mu_A^+(v_i) = \frac{d(v_i)}{n-1} \leq 1$

Thus, we have $0 \leq \mu_A^-(v_i) < \mu_A^+(v_i) \leq 1$. Hence A is an IVFS on V.

B is an IVFS on E

We have, $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2}, \mu_B^+(v_iv_j) = \frac{[d(v_i)][d(v_j)]}{(n-1)^2}$.

Since $d(v_i) - 1 \geq 0$ and $d(v_j) - 1 \geq 0$, $\mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2} \geq 0$.

And since, $d(v_i) - 1 < d(v_i)$, $\mu_B^-(v_iv_j) < \mu_B^+(v_iv_j)$.

Also, $d(v_i) \leq n-1$ and $d(v_j) \leq n-1$, implies that $d(v_i)d(v_j) \leq (n-1)^2$ and so $\mu_B^+(v_iv_j) \leq 1$. Thus, we have $0 \leq \mu_B^-(v_iv_j) < \mu_B^+(v_iv_j) \leq 1$. Hence B is an IVFS on E.

Now, we show that A and B satisfy the requirements of an Interval Valued Fuzzy Graph. We know that if x and y are any two real numbers in $[0, 1]$, then, $x.y \leq \min(x, y)$.

$\therefore \mu_B^-(v_iv_j) = \frac{[d(v_i)-1][d(v_j)-1]}{(n-1)^2} \leq \min(\frac{[d(v_i)-1]}{n-1}, \frac{[d(v_j)-1]}{n-1}) = \min(\mu_A^-(v_i), \mu_A^-(v_j))$ and $\mu_B^+(v_iv_j) = \frac{d(v_i)d(v_j)}{(n-1)^2} \leq \min(\frac{d(v_i)}{n-1}, \frac{d(v_j)}{n-1}) = \min(\mu_A^+(v_i), \mu_A^+(v_j))$. Thus, $G = (A, B)$ constructed as above is an IVFG. □

Example 2.1. *Figure 1 shows a crisp graph G^* and its associated Interval Valued Fuzzy Graph G. Using G, we can verify the above procedure.*

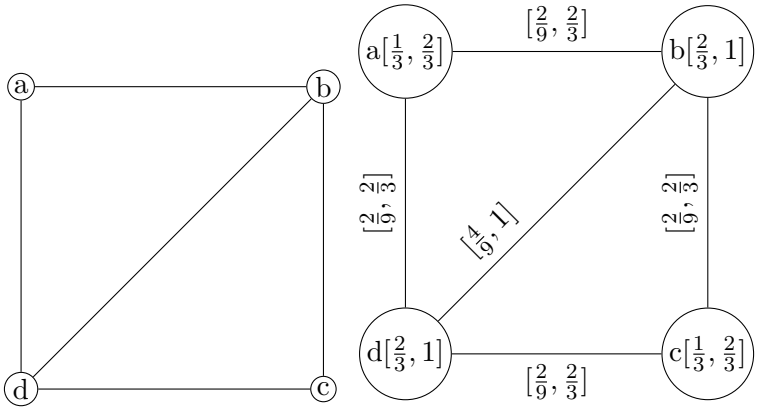


Figure 1: Example to illustrate the construction of the associated IVFG of a crisp graph

3. Main results

Theorem 3.1. *Let $G^* = (V, E)$ be a simple connected graph with $|V| = n$, $|E| = m$ and let $G = (A, B)$ be the Interval Valued Fuzzy Graph associated with G^*. Then $O(G) = [\frac{2m-n}{n-1}, \frac{2m}{n-1}]$*

Proof. Given that $G = (A, B)$ is the IVFG associated with the simple connected graph $G^* = (V, E)$ where $|V| = n$ and $|E| = m$.

$$\begin{aligned} O(G) = [O^-(G), O^+(G)] &= \left[\sum_{i=1}^{n} \mu_{A^-}(v_i), \sum_{i=1}^{n} \mu_{A^+}(v_i)\right] \\ &= \left[\sum_{i=1}^{n} \frac{d(v_i)-1}{n-1}, \sum_{i=1}^{n} \frac{d(v_i)}{n-1}\right] \\ &= \left[\frac{\sum_{i=1}^{n} d(v_i) - \sum_{i=1}^{n}(1)}{n-1}, \frac{\sum_{i=1}^{n} d(v_i)}{n-1}\right] \\ &= \left[\frac{2m-n}{n-1}, \frac{2m}{n-1}\right] \text{ since } \sum_{i=1}^{n} d(v_i) = 2m \end{aligned}$$

□

Theorem 3.2. *Let $G = (A, B)$ be the Interval Valued Fuzzy Graph associated with the simple connected graph $G^* = (V, E)$. Then $S(G) = [S^-(G), S^+(G)]$ where*

$$S^-(G) = \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + m}{(n-1)^2},$$

$$S^+(G) = \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j)}{(n-1)^2}.$$

Proof.

$$S^-(G) = \sum_{v_iv_j\in E, v_i\neq v_j} \mu_{B^-}(v_iv_j)$$

$$= \sum_{v_iv_j\in E, v_i\neq v_j} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right]$$

$$= \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + \sum_{1\le i<j\le n, v_iv_j\in E} (1)}{(n-1)^2}$$

$$= \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j) - \sum_{1\le i<j\le n, v_iv_j\in E} [d(v_i) + d(v_j)] + m}{(n-1)^2}.$$

Similarly, we can prove that $S^+(G) = \frac{\sum_{1\le i<j\le n, v_iv_j\in E} d(v_i)d(v_j)}{(n-1)^2}$. □

Theorem 3.3. *The Interval Valued Fuzzy Graph associated with a crisp graph is strongly connected.*

Proof. Let $G^* = (V, E)$ be a simple connected graph and let $G = (A, B)$ be the IVFG associated with G^*. It is obvious from the construction of G that any two arcs of G are either comparable or equal. Then, by Theorem (A), G is strongly connected. □

Theorem 3.4. *The nature of the arcs of the Interval Valued Fuzzy Graph G associated with G^* can be determined if the adjacency matrix of G^* is given.*

Proof. Let G^* be a crisp graph with vertices $v_1, v_2, \ldots, v_n$ and suppose $[A]$ be its adjacency matrix . The sum of the i^{th} row or column of matrix $[A]$ gives the degree of v_i. Once we know $d(v_i)$, we can write the interval valued fuzzy node arc matrix of G. Then by an algorithm in [10], we can determine the nature of arcs of G. □

Theorem 3.5. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a regular IVFG with degree $[\frac{k(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2}]$.*

Proof.

$$
\begin{aligned}
d_G(v_i) &= \left[d_G^-(v_i), d_G^+(v_i)\right] \\
&= \left[\sum_{v_iv_j\in E} \mu_{B^-}(v_iv_j), \sum_{v_iv_j\in E} \mu_{B^+}(v_iv_j)\right] \\
&= \left[\sum_{v_iv_j\in E} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right], \sum_{v_iv_j\in E} \left[\frac{d(v_i)}{n-1}\right]\left[\frac{d(v_j)}{n-1}\right]\right] \\
&= \left[\sum_{v_iv_j\in E} \left[\frac{k-1}{n-1}\right]\left[\frac{k-1}{n-1}\right], \sum_{v_iv_j\in E} \left[\frac{k}{n-1}\right]\left[\frac{k}{n-1}\right]\right] \\
&= \left[\frac{d(v_i)(k-1)^2}{(n-1)^2}, \frac{d(v_i)k^2}{(n-1)^2}\right] = \left[\frac{k(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2}\right].
\end{aligned}
$$

□

Theorem 3.6. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a totally regular Interval Valued Fuzzy Graph with total degree* $[\frac{(k-1)[k^2-k+(n-1)]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}]$

Proof.

$$
\begin{aligned}
td_G(v_i) &= [td_G^-(v_i), td_G^+(v_i)] \\
&= \left[\sum_{v_iv_j\in E} \mu_{B^-}(v_iv_j) + \mu_{A^-}(v_i), \sum_{v_iv_j\in E} \mu_{B^+}(v_iv_j) + \mu_{A^+}(v_i)\right] \\
&= \left[\sum_{v_iv_j\in E} \left[\frac{d(v_i)-1}{n-1}\right]\left[\frac{d(v_j)-1}{n-1}\right] + \frac{d(v_i)-1}{n-1}, \sum_{v_iv_j\in E} \left[\frac{d(v_i)}{n-1}\right]\left[\frac{d(v_j)}{n-1}\right] + \left[\frac{d(v_i)}{n-1}\right]\right] \\
&= \left[\sum_{v_iv_j\in E} [\frac{k-1}{n-1}][\frac{k-1}{n-1}\right] + \frac{k-1}{n-1}, \sum_{v_iv_j\in E} \left[\frac{k}{n-1}\right]\left[\frac{k}{n-1}\right] + \frac{k}{n-1}\right] \\
&= \left[\frac{d(v_i)(k-1)^2}{(n-1)^2} + \frac{k-1}{n-1}, \frac{d(v_i)k^2}{(n-1)^2} + \frac{k}{n-1}\right] \\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k-1}{n-1}, \frac{k^3}{(n-1)^2} + \frac{k}{n-1}\right] \\
&= \left[\frac{(k-1)[k(k-1)+n-1]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}\right] \\
&= \left[\frac{(k-1)[k^2-k+(n-1)]}{(n-1)^2}, \frac{k[k^2+(n-1)]}{(n-1)^2}\right].
\end{aligned}
$$

□

Theorem 3.7. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is an edge regular Interval Valued Fuzzy Graph with edge degree $[\frac{2(k-1)^3}{(n-1)^2}, \frac{2k^2(k-1)}{(n-1)^2}]$*

Proof.

$$\begin{aligned}
&d_G(v_iv_j) = [d^-(v_iv_j), d^+(v_iv_j)]\\
&= [d^-_G(v_i) + d^-_G(v_j) - 2\mu_{B^-}(v_iv_j), d^+_G(v_i) + d^+_G(v_j) - 2\mu_{B^+}(v_iv_j)]\\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k(k-1)^2}{(n-1)^2} - \frac{2(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2} + \frac{k^3}{(n-1)^2} - \frac{2k^2}{(n-1)^2}\right]\\
&= \left[\frac{2(k-1)^2(k-1)}{(n-1)^2}, \frac{2k^3}{(n-1)^2} - \frac{2k^2}{(n-1)^2}\right]\\
&= \left[\frac{2(k-1)^3}{(n-1)^2}, \frac{2k^2(k-1)}{(n-1)^2}\right].
\end{aligned}$$

□

Theorem 3.8. *The Interval Valued Fuzzy Graph G associated with a regular graph G^* of degree k is a totally edge regular Interval Valued Fuzzy Graph with total edge degree $[\frac{(k-1)^2(2k-1)}{(n-1)^2}, \frac{k^2(2k-1)}{(n-1)^2}]$*

Proof.

$$\begin{aligned}
&td_G(v_iv_j) = [td^-(v_iv_j), td^+(v_iv_j)]\\
&= [d^-_G(v_i) + d^-_G(v_j) - \mu_{B^-}(v_iv_j), d^+_G(v_i) + d^+_G(v_j) - \mu_{B^+}(v_iv_j)]\\
&= \left[\frac{k(k-1)^2}{(n-1)^2} + \frac{k(k-1)^2}{(n-1)^2} - \frac{(k-1)^2}{(n-1)^2}, \frac{k^3}{(n-1)^2} + \frac{k^3}{(n-1)^2} - \frac{k^2}{(n-1)^2}\right]\\
&= \left[\frac{2k(k-1)^2}{(n-1)^2} - \frac{(k-1)^2}{(n-1)^2}, \frac{2k^3}{(n-1)^2} - \frac{k^2}{(n-1)^2}\right]\\
&= \left[\frac{(k-1)^2(2k-1)}{(n-1)^2}, \frac{k^2(2k-1)}{(n-1)^2}\right].
\end{aligned}$$

□

Theorem 3.9. *If G^* is a k regular graph, then all the arcs of the associated Interval Valued Fuzzy Graph G are β strong arcs.*

Proof. Let G^* be a k regular graph and G be the associated IVFG. Then from the definition of G, $[\mu^-_B, \mu^+_B](v_iv_j) = [\frac{(k-1)^2}{(n-1)^2}, \frac{k^2}{(n-1)^2}]$ which is a constant for all arcs (v_iv_j). Hence by Theorem (B), all the arcs of G are β strong arcs. □

The following corollary is immediate if we use Theorem (D).

Corollary 3.1. *The Interval Valued Fuzzy Graph associated with a regular graph cannot be an IVF tree.*

Theorem 3.10. *The Interval Valued Fuzzy Graph associated with the complete graph $K_n, n \geq 2$ is not a CIVFG.*

Proof. Consider the complete graph $K_n, n \geq 2$.

Label the nodes as $v_1, v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with K_n.

Clearly, $\mu_A^-(v_i) = \frac{n-2}{n-1}, \mu_A^+(v_i) = \frac{n-1}{n-1} = 1$. Hence by the construction of G, $\mu_B^-(v_iv_j) = \frac{(n-2)^2}{(n-1)^2}, \mu_B^+(v_1v_i) = 1$. Clearly,$\mu_B^-(v_iv_j) \neq \min(\mu_A^-(v_i), \mu_A^-(v_j))$. Hence G cannot be a CIVFG. □

Theorem 3.11. *The Interval Valued Fuzzy Graph associated with the complete graph $K_n, n \geq 2$ is a complete PIVFG.*

Proof. Consider the complete graph $K_n = (V, E), n \geq 2$. Label the nodes as $v_1, v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with K_n. From the construction of G, it is clear that $\mu_B^-(v_iv_j) = \mu_A^-(v_i) \times \mu_A^-(v_j)$ and $\mu_B^+(v_iv_j) = \mu_A^+(v_i) \times \mu_A^+(v_j)$ for all $v_i, v_j \in V$ where $\times$ denotes ordinary multiplication. Hence by definition 1.5, G is a complete PIVFG. □

Theorem 3.12. *The Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$ is an IVF cycle.*

Proof. $C_n, n \geq 3$ is n-1 regular. Hence by Theorem 3.9, all arcs of C_n are β strong. Since $n \geq 3$, C_n contains at least two β strong arcs and hence by Theorem (E), $C_n, n \geq 3$ is an IVF cycle. □

In crisp graph theory, W_n does not contain any cutnode. But the IVFG associated with $W_n, n \geq 5$ contains an IVF cutnode which can be concluded from the following theorem.

Theorem 3.13. *The Interval Valued Fuzzy Graph associated with the wheel graph $W_n, n \geq 5$ is an IVFT.*

Proof. Consider the wheel graph $W_n, n \geq 5$.Label the central node as v_1 and the remaining nodes as $v_2, v_3, \ldots, v_n$. Let $G = (A, B)$ be the IVFG associated with W_n.

Clearly, $\mu_A^-(v_1) = \frac{n-2}{n-1}, \mu_A^+(v_1) = \frac{n-1}{n-1} = 1$ and $\mu_A^-(v_i) = \frac{2}{n-1}, \mu_A^+(v_i) = \frac{3}{n-1}$ where $i = 2, 3, \ldots, n$. Also, $\mu_B^-(v_1v_i) = \frac{2(n-2)}{(n-1)^2}, \mu_B^+(v_1v_i) = \frac{3}{n-1}$ where $i = 2, 3, \ldots, n$ and $\mu_B^-(v_nv_2) = \mu_B^-(v_iv_{i+1}) = \frac{4}{(n-1)^2}, \mu_B^+(v_nv_2) = \mu_B^+(v_iv_{i+1}) = \frac{9}{(n-1)^2}$ where $i = 2, 3, \ldots, n-1$.

Since $n \geq 5, \mu_B^-(v_iv_{i+1}) < \mu_B^-(v_1v_i) = \mu_B^-(v_1v_{i+1})$ and $\mu_B^+(v_iv_{i+1}) < \mu_B^+(v_1v_i) = \mu_B^+(v_1v_{i+1})$, where $i = 2, 3, \ldots, n-1$. Thus obviously, arc (v_i, v_{i+1}) is the

unique weakest arc of the cycle v_i, v_{i+1}, v_1, v_i. Hence by Theorem (C), arc (v_i, v_{i+1}) is a weak arc where $i = 2, 3, \ldots, n-1$. Also $\mu_B^-(v_n v_2) < \mu_B^-(v_1 v_2) = \mu_B^-(v_1 v_n)$ and $\mu_B^+(v_n v_2) < \mu_B^+(v_1 v_2) = \mu_B^+(v_1 v_n)$. So, arc (v_n, v_2) is the unique weakest arc of the cycle v_n, v_2, v_1, v_n. Hence again by Theorem (C) (v_n, v_2) is also a weak arc. Hence the arcs forming the outer cycle are all *weak* arcs. Thus, G satisfies all the requirements of theorem(F) and so by Theorem (F), all the remaining arcs are α strong. Hence by Theorem (D), the wheel graph $W_n, n \geq 5$ is an IVFT. □

Remark 3.1. By Theorem (G), all those α strong arcs are IVF bridges and by Theorem (H), the central node is an IVF cutnode.

In general,the IVFG associated with the line graph of a crisp graph G^* is not isomorphic to the IVF line graph of the IVFG associated with G^*. But as the following theorem shows, they are co-weak isomorphic in the case of $C_n, n \geq 3$.

Theorem 3.14. *The Interval Valued Fuzzy Graph associated with the line graph of $C_n, n \geq 3$ is co-weak isomorphic to the IVF line graph of the Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$.*

Proof. We have $C_n \cong L(C_n)$, the line graph of C_n.Let G be the IVFG associated with the line graph of $C_n, n \geq 3$. Clearly $\mu_A^-(v_i) = \frac{1}{n-1}, \mu_A^+(v_i) = \frac{2}{n-1}$ for every node v_i of G and $\mu_B^-(v_i v_j) = \frac{1}{(n-1)^2}, \mu_B^+(v_i v_j) = \frac{4}{(n-1)^2}$ for every arc (v_i, v_j) of G. Clearly G is isomorphic with the IVFG associated with C_n. Let G' be IVF line graph of the IVFG associated with C_n. Clearly, $\mu_A^-(u_i) = \frac{1}{(n-1)^2}, \mu_A^+(u_i) = \frac{4}{(n-1)^2}$ for every node u_i of G' and $\mu_B^-(u_i u_j) = \frac{1}{(n-1)^2}, \mu_B^+(u_i u_j) = \frac{4}{(n-1)^2}$ for every arc (u_i, u_j) of G'. Hence G is co-weak isomorphic to G'. Hence the theorem. □

4. Conclusion

In this paper, we have constructed an interval valued fuzzy graph from a crisp graph, G^* called the Interval Valued Fuzzy Graph associated with G^* based on the degrees of the nodes of the Interval Valued Fuzzy Graph. We proved that regularity is carried over to the so constructed Interval Valued Fuzzy Graph, but completeness is not carried over to. We have shown that every pair of nodes in the associated Interval Valued Fuzzy Graph is connected by a strong path. We also proved that the Interval Valued Fuzzy Graph associated with $C_n, n \geq 3$ is an IVF cycle and the Interval Valued Fuzzy Graph associated with the wheel graph $W_n, n \geq 5$ is an IVFT.

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A novel continuous genetic algorithm technique for the solution of partial differential equations

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Abstract. In this paper, a novel continuous genetic algorithm (CGA) approach is proposed for the solution of Laplace, Poisson, Helmholtz, and nonlinear partial differential equations (PDEs) due to their importance as they are encountered in a variety of mathematical and physical systems. The approach is formulated by firstly converting the equation into an algebraic equation using the finite difference scheme which approximates its partial derivatives. Then, the residual value of each interior node is calculated based on the nodal values generated by the CGA. The solution is finally obtained based on the minimization of the overall residual values and correspondingly maximizing the fitness value. Different test problems were considered in this work, which cover different boundary conditions including Dirichlet and Neumann boundary conditions. The solution obtained using the proposed CGA approach is compared with the exact solutions

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along with the finite difference and finite element methods. Numerical results show the strength, potentiality, and superiority of the proposed CGA approach.

Keywords: numerical methods, continuous genetic algorithms, partial differential equations, finite elements method, finite difference method.

1. Introduction

The exact solutions of PDEs are usually unavailable as reported by Morton et al. and others [28, 41]. Therefore, numerical approximated solutions of these equations are usually considered. In the modeling of natural phenomena, a crucial role is played by the study of PDEs of elliptic type as they arise in every field of science. Consequently, the desire to obtain accurate solutions of these equations has always a prominent place in the efforts of different scientists [44]. Due to their importance in science and engineering, PDEs are studied extensively by specialists and practitioners [48]. Numerical differentiation, on the other hand, plays an important role in solving ordinary differential equations (ODEs) and PDEs, which have been developed to derive useful difference formulas for numerical derivatives. Most techniques of numerical differentiation are classified into the following types [31]: finite difference type[16, 17, 34, 43], polynomial interpolation type [16, 17, 19, 20, 22, 30, 36, 38], [43, 45], operator type [47], lozenge diagrams [21], and undetermined coefficients [21, 8]. For a brief and interesting survey, the interested reader may consult [31]. In this paper, we will consider the solution of Laplace, Poisson, Helmholtz, and nonlinear PDEs. The Laplace and Poisson's equations have been considered by many scientists as it arises in a variety of mathematical and physical systems, ranging through fluid mechanics, heat conduction, electromagnetism, potential theory, solid mechanics, geometry, probability, number theory, etc. [27, 29, 33, 39, 49]. In the study of physical problems involving PDEs in both space and time, Helmholtz equation plays an important role as well. The equation arises when one is looking for mono-frequency or time-harmonic solutions to the wave equation [18]. Because of its relationship to the wave equation, the Helmholtz equation becomes important in the study of electromagnetic radiation, seismology, and acoustics [26, 50]. Despite the apparent simplicity of the underlying differential relations, nonlinear PDEs govern a great deal of complex phenomena of motion, reaction, diffusion, equilibrium, conservation, and more. In recent years, a variety of effective methods have been proposed for the solution of nonlinear PDEs as seen in the reaction-diffusion equations [51], lattice differential equations [15], and the heat equation [52]. Genetic algorithms (GAs) have also been applied to solve problems in computer science, computational physics, molecular chemistry, statistics and applied probability. Recently, they were applied for model order reduction [14], ARMA modeling [5], and system identification [4]. In 2002, Abo-Hammour developed continuous genetic algorithms (CGAs) as an efficient method for the solution of optimization problems [8]. This development has opened the doors for wide applications of the algorithm in the field of engineer-

ing and mathematics. The CGAs have been successfully applied in the solution of optimal control problems [6], collision-free Cartesian planning of robot manipulators [7], singular two-point boundary value problems [3], higher-order initial value problems [2], and nonlinear boundary value problems [1].

2. Problem formulation

A first-order partial differential equation with n independent variables has the general form

$$F(x_1, x_2, \cdots, x_n,\ u_{x_1}, u_{x_2}, \cdots, u_{x_n}) = 0, \tag{1}$$

where $u(x_1, x_2, \cdots, x_n)$ is the unknown function and F (...) is a given function.

A second-order nonlinear partial differential equation with two independent variables has the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2}$$

The classification of Equation (2) depends on the discriminate $\sigma = b^2 - ac$, where

$$a = \frac{\partial F}{\partial p}, b = \frac{1}{2}\frac{\partial F}{\partial q}, c = \frac{\partial F}{\partial r}, p = u_{xx}, q = u_{xy}, \text{ and } r = u_{yy}.$$

Thus, the classification is made such that $\sigma > 0$ is hyperbolic, $\sigma = 0$ is parabolic, and $\sigma < 0$ is elliptic [42].

In this paper, we will consider the linear PDEs, which involve Laplace equation, Poisson equation, and Helmholtz equation, and different types of nonlinear equations given as following:
Linear equations:

Laplace equation:

$$\nabla^2 u(x, y) = 0. \tag{3}$$

Poisson equation:

$$\nabla^2 u(x, y) = h(x, y). \tag{4}$$

Helmholtz equation:

$$\nabla^2 u(x, y) + g(x, y)u(x, y) = h(x, y), \tag{5}$$

For a general Nonlinear form, the equation may look like

$$(u^{n_0} + a_0)\nabla^2 u = a_1(x, y)u^{n_1} + a_2(x, y)u^{n_2} + a_3(x, y)u^{n_3} + a_4(x, y), \tag{6}$$

where a_0, n_i are real number for $i = 0, 1, 2, 3$, $a_i(x, y)$ are functions of x and y for $i = 1, 2, 3, 4$.

Since the exact solution for many partial differential equations usually does not exist, numerical methods for solving such equations are suggested. The most widely known numerical methods used for such purpose are the finite difference methods (FDMs) [9, 10] and finite element methods (FEMs).

The proposed genetic algorithm approach, for numerically approximating the solution, requires a mapping of the individual residual, R, into a fitness function, F, in order to convert the minimization problem of R into a maximization problem of F. For the Laplace equation, in order to find the fitness function, the finite difference formulas are applied to Equation (3) as following

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0, \tag{7}$$

where $u_{i,j} = u(x_i,\ y_j)$, h is the step size of the nodes x and k is the step size of the nodes y. Similarly, for the Poisson equation, the following is obtained

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = h_{i,j}, \tag{8}$$

where $h_{i,j} = h(x_i,\ y_j)$. Also, for the Helmholtz equation, the following is obtained

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + g_{i,j}u_{i,j} = h_{i,j}, \tag{9}$$

where $g_{i,j} = g(x_i,\ y_j)$. For the general nonlinear form, the fitness function can be obtained based on the observation that Equation (6) implies that

$$u_{i,j} = f(x_i, y_i, u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1}), \tag{10}$$

where

$$\begin{aligned} &f(x_i, y_i, u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1}) \\ &= \left(\frac{h^2k^2}{2h^2 + 2k^2 - h^2k^2g_{i,j}}\right)\left(\frac{u_{i-1,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j} + u_{i+1,j}}{k^2} - h_{i,j}\right). \end{aligned} \tag{11}$$

Then, the residual at each interior node is

$$r_{i,j} = u_{i,j} - f(x_i, y_i, u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1}). \tag{12}$$

Hence, the residual of the nonlinear equation in (6) is given by

$$\begin{aligned} r_{i,j} = u_{i,j}&\left(\frac{2}{h^2} + \frac{2}{k^2}\right)(u_{i,j}^{n_0} + a_0) + u_{i,j}^2 a_1(i,j)u_{i,j}^{n_1} + a_2(i,j)u_{i,j}^{n_2} + \\ &a_3(i,j)u_{i,j}^{n_3} + a_4(i,j) - (u_{i,j}^{n_0} + a_0)\left(\frac{u_{i-1,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j} + u_{i+1,j}}{k^2}\right). \end{aligned} \tag{13}$$

As a result, the overall residual at each parent can be written as

$$R = \left(\sum_{i=1}^{M_x} \sum_{j=1}^{M_y} r_{i,j}^2 \right)^{0.5}. \tag{14}$$

Thus, the suitable fitness function can then be given as

$$F = \frac{1}{1+R}. \tag{15}$$

3. Continuous genetic algorithm (CGA)

The main operators of CGA [23] have smooth characteristics and avoid sharp jumps in the parameter values. First, the random initialization using continuous surfaces reduce the possibility of highly oscillating values in the neighboring parameters. Second, the crossover operator(s) using continuous surfaces result in smooth transition in the parameter values. Third, the mutation process changes the value of the parameter in smooth transition in the parameter values. In summary, traditional GA type results in a step-function-like jump in the parameter values while CGA results in smooth transitions, for more details see [8].

The advantages of CGA over conventional GA when it is applied to problems with many parameters [8] are seen as following:

1. In traditional GA, the actual range of the nodal values should lie within the expected range during the evolution process; otherwise we will not be able to reach to the optimal solution. This is because the encoding/decoding processes in conventional GA require the range of values within which the solution should lie. To overcome this problem, we provide a wide range for the solution values. In CGA, we initialize the population to lie within some initial guesses and the GA will use the overall residual values throughout the evolution process.

2. There are no encoding/decoding processes in CGA. This means that the execution time of an optimization problem will be smaller in this case if both GA versions converge in approximately the same number of generations.

3. The memory requirements of traditional GA are higher than that of the CGA because the former uses genotype representation of the population individuals while the later utilizes only the phenotype data. This makes CGA more appropriate to problems of larger number of parameters.

The continuous genetic algorithm proposed in this work (which is generalization of [8]) consists of the following steps:

1. Initialization: The initial population is divided into four equal segments; each of them is generated using a different set of smooth functions to generate various initial populations. The first part of the initial population is generated using linear combination of one-dimensional Gaussian functions with respect to x and of one-dimensional Gaussian functions with respect to y, the second segment of the initial population, is generated using linear combination of one-dimensional tangent hyperbolic functions with respect to x and one-dimensional tangent hyperbolic functions with respect to y, the third part is generated using cosine function while the fourth one is sine function. The four functions are given as following

$$\begin{aligned} g_1(x_i,y_j)=&\frac{i}{i+j}\left[a_1(x_i)+(b_1(x_i)-a_1(x_i))\frac{j-1}{M_y}+c\exp\left(\frac{-(j-\mu_x)^2}{2\sigma_x^2}\right)\right]\\ &+\frac{j}{i+j}\left[a_2(y_j)+(b_2(y_j)-a_2(y_j))\frac{j-1}{M_x}+c\exp\left(\frac{-(i-\mu_y)^2}{2\sigma_y^2}\right)\right], \end{aligned} \tag{16}$$

$$\begin{aligned} g_2(x_i,y_j)=&\frac{i}{i+j}\left[a_1(x_i)+0.5(b_1(x_i)-a_1(x_i))\left(1+\tanh\left(\frac{j^d-\mu_x}{\sigma_x}\right)\right)\right]\\ &+\frac{j}{i+j}\left[a_2(y_j)+0.5(b_2(y_j)-a_2(y_j))\left(1+\tanh\left(\frac{i^d-\mu_y}{\sigma_y}\right)\right)\right], \end{aligned} \tag{17}$$

$$\begin{aligned} g_3(x_i,y_j)=&a_1(x_i)+(b_1(x_i)-a_1(x_i))\\ &\cdot\left[1-\cos\left(\pi d_1+\frac{\pi}{4}\left(c_1\tanh\left(\frac{2d\times j-\mu_x}{\sigma_x}\right)+1\right)\right)\right]\\ &+b_2(y_j)-(b_2(y_j)-a_2(y_j))\\ &\cdot\left[1-\cos\left(\pi d_1+\frac{\pi}{4}\left(c_1\tanh\left(\frac{2d\times i-\mu_y}{\sigma_y}\right)+1\right)\right)\right], \end{aligned} \tag{18}$$

$$\begin{aligned} g_4(x_i,y_j)=&a_1(x_i)+(b_1(x_i)-a_1(x_i))\\ &\sin\left(\pi d\left(\tanh\left(\frac{2d_2\times j-\mu_x}{\sigma_x}\right)+1\right)\right)\\ &+a_2(y_j)-(b_2(y_j)-a_2(y_j))\sin\left(\pi d\left(\tanh\left(\frac{2d_2\times i-\mu_y}{\sigma_y}\right)+1\right)\right), \end{aligned} \tag{19}$$

where $1\le i\le M_{x+1}$, $1\le j\le M_{y+1}$, M_x, M_y are the number of nodes of x and y respectively, c is a random number in the range [-0.5, 1], d is a random number in the range [0.3, 0.8], d_1 is a random number in the range [0.05, 0.06], d_2 is a random number in the range [0, 1], c_1 is a random number in the range [0.3, 1.8], $a_1(x_i)$, $a_2(y_j), b_1(x_i), b_2(y_j)$ represent the boundary values, μ_x is a random number in the range $[M_x/4, M_y/4]$, σ_x is random number in the range $[1, M_x/3]$, μ_y is a random number in the range $[M_y/4, 3M_y/4]$, σ_y is random

number in the range [1, $M_y/3$]. It is to be noted that μ_x and μ_y specify the center of the function while σ_x and $\sigma_y$specify the degree of dispersion of the curve.

The upper function, $g(x, y)$, is then rescaled within the range [*Pmin*, *Pmax*], where *Pmin*, *Pmax* represents the expected minimum and maximum initial nodal values.

The above functions are not suitable if all boundary values are equal, instead of that functions we can use suitable linear combination of $g_6(x, y)$ and $g_7(x, y)$ or we can use suitable linear combination of $g_8(x, y)$ and $g_9(x, y)$ where $g_6(x, y), ..., g_9(x, y)$ are

$$g_6(x_i, y_j) = a + d\left[1 - \cos\left(c\pi + \frac{cc\pi(j-1)}{M_y + 1}\right)\right], \tag{20}$$

$$g_7(x_i, y_j) = a + d\left[1 - \cos\left(c\pi + \frac{cc\pi(i-1)}{M_x + 1}\right)\right], \tag{21}$$

$$g_8(x_i, y_j) = a + d\sin\left(\frac{cc\pi(j-1)}{M_y + 1}\right), \tag{22}$$

$$g_9(x_i, y_j) = a + d\sin\left(\frac{cc\pi(i-1)}{M_x + 1}\right), \tag{23}$$

where $1 \le i \le M_x + 1$, $1 \le j \le M_y + 1$, M_x, M_y are the number of nodes of x and y respectively a is the boundary value, d is a random number in the range [-2, 2], c and cc are random numbers in the range [1.9, 2.1]. Note that we can rotate the above functions to reproduce new functions which give us more variety of functions. For illustration, Figure 1 shows the initialization at some given boundary conditions.

2. Evaluation: The fitness is calculated for each individual in the population.

3. Selection: We select Individuals from the current population according to their relative fitness, in order to receive more suitable copies.

4. Crossover: The way through which information is shared among the population. The crossover process combines the features of two parent individuals, i.e., P_1 and P_2, to form two children individuals, say C_1 and C_2 as given by the equations:

$$C_1(x_i, y_j) = W(x_i, y_j)P_1(x_i, y_j) + (1 - W(x_i, y_j))P_2(x_i, y_j), \tag{24}$$

$$C_2(x_i, y_j) = (1 - W(x_i, y_j))P_1(x_i, y_j) + W(x_i, y_j)P_2(x_i, y_j), \tag{25}$$

where W represents the mixing function in the range [0, 1]. Some types of weight functions were tested in this work are given by the equations:

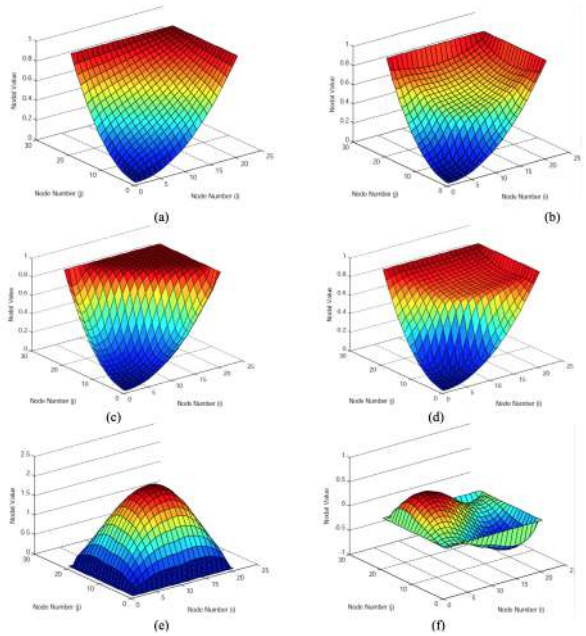


Figure 1: Two-dimensional initialization functions.

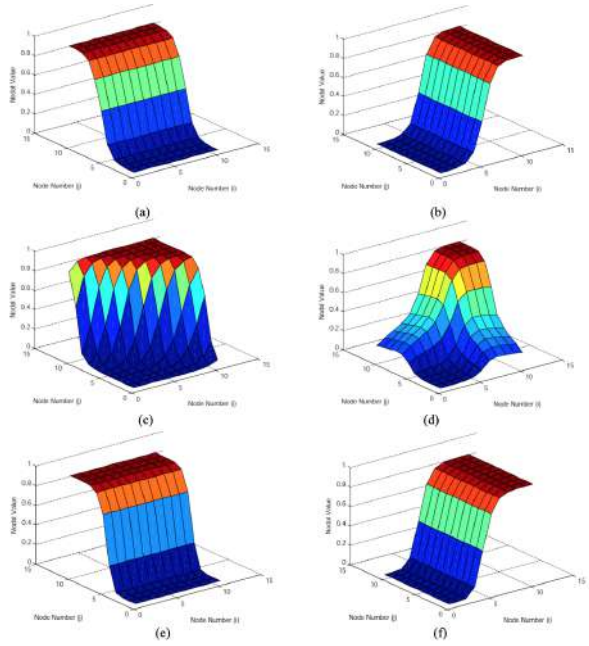


Figure 2: Two-dimensional crossover weighting functions.

$$W_1(x_i, y_j) = \frac{1}{2}\left[1 + \tanh\left(\frac{i-\mu_x}{\sigma_x}\right)\right], \tag{26}$$

$$W_2(x_i, y_j) = \frac{1}{2}\left[1 + \tanh\left(\frac{j-\mu_y}{\sigma_y}\right)\right], \tag{27}$$

$$W_3(x_i, y_j) = \frac{1}{2}\left[1 + \tanh\left(\frac{i-\mu_x}{\sigma_x} + \frac{j-\mu_y}{\sigma_y}\right)\right], \tag{28}$$

$$W_4(x_i, y_j) = (1+i)\frac{w_1(x,y)}{i+j+2} + (1+j)\frac{w_2(x,y)}{i+j+2}, \tag{29}$$

$$W_5(x_i, y_j) = 1 - \cos\left(\frac{\pi}{2}\tanh\left(\frac{i-\mu_x}{\sigma_x}\right)\right), \tag{30}$$

$$W_6(x_i, y_j) = 1 - \cos\left(\frac{\pi}{2}\tanh\left(\frac{j-\mu_y}{\sigma_y}\right)\right), \tag{31}$$

$$W_7(x_i, y_j) = 1 - \cos\left(\frac{\pi}{4}\left[\tanh\left(\frac{i-\mu_x}{\sigma_x} + \frac{j-\mu_y}{\sigma_y}\right) + 1\right]\right), \tag{32}$$

$$W_8(x_i, y_j) = (1+i)\frac{w_5(x,y)}{i+j+2} + (1+j)\frac{w_6(x,y)}{i+j+2}, \tag{33}$$

$$W_9(x_i, y_j) = \sin\left(\frac{\pi}{2}\tanh\left(\frac{i-\mu_x}{\sigma_x}\right)\right), \tag{34}$$

$$W_{10}(x_i, y_j) = \sin\left(\frac{\pi}{2}\tanh\left(\frac{j-\mu_y}{\sigma_y}\right)\right), \tag{35}$$

$$W_{11}(x_i, y_j) = \sin\left(\frac{\pi}{4}\left[\tanh\left(\frac{i-\mu_x}{\sigma_x} + \frac{j-\mu_y}{\sigma_y}\right) + 1\right]\right), \tag{36}$$

$$W_{12}(x_i, y_j) = (1+i)\frac{w_9(x_i,y_j)}{i+j+2} + (1+j)\frac{w_{10}(x_i,y_j)}{i+j+2}, \tag{37}$$

where $1 \le i \le M_x + 1$, $1 \le j \le M_y + 1$, M_x, M_y are the number of nodes of x and y respectively, σ_x and σ_y are chosen such that the tangent hyperbolic function achieves it's complete transition from -1 to 1 within the given ranges of both x and y, μ_x is a random number in the range $[M_x/3, 2M_x/3]$, μ_y is a random number in the range $[M_y/3,\ 2M_y/3]$, σ_x is random number in the range $[1,\ M_x/4]$, σ_y is random number in the range $[1,\ M_x/4]$. Figures 2 and [3] show some weighted functions and Figure 4 shows the crossover process between two random parents.

5. Mutation: The function of mutation is to make diversity within the population. The mutation process in CGA is governed by the following formulas:

$$C_m(x_i, y_j) = C(x_i, y_j) + d \times M(x_i, y_j), \tag{38}$$

where $C(x_i,y_j)$ represents the child produced through the crossover process, $C_m(x,\ y)$ is the mutated child and $M(x_i,y_j)$ is the mutation function, d represents a random number within the range $[-R_{ave},R_{ave}]$, R_{ave} is average overall

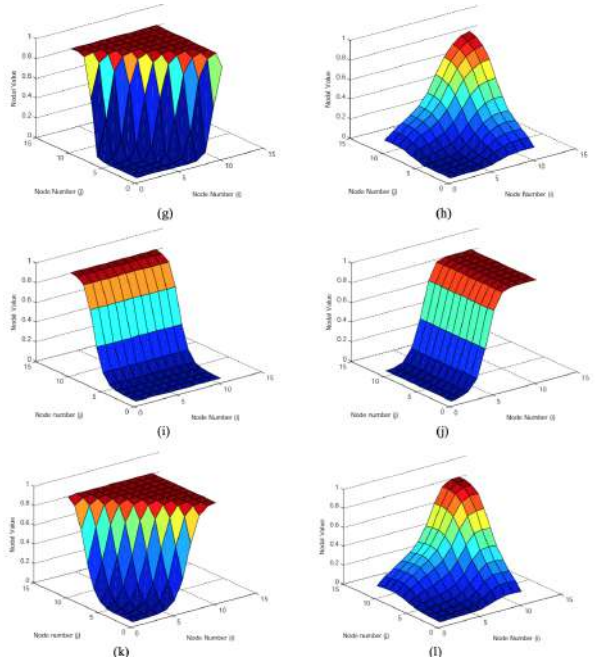


Figure 3: Two-dimensional crossover weighting functions.

residue of the population. The mutation functions are $M_1(x_i,y_j)$, linear combination of M_2 and M_3 or linear combination of M_4 and M_5 were M_1, M_2, M_3, M_4 and M_5 are

$$M_1(x_i, y_j) = \exp\left(-\frac{(i-\mu_x)^2}{2\sigma_x^2} - \frac{(j-\mu_y)^2}{2\sigma_y^2}\right), \tag{39}$$

$$M_2(x_i, y_j) = d\left(1 - \cos\left(\frac{(i-1)c\pi}{N-1}\right)\right), \tag{40}$$

$$M_3(x_i, y_j) = d\left(1 - \cos\left(\frac{(j-1)c\pi}{N-1}\right)\right), \tag{41}$$

$$M_4(x_i, y_j) = d\sin\left(\frac{(j-1)c\pi}{N-1}\right), \tag{42}$$

$$M_5(x_i, y_j) = d\sin\left(\frac{(i-1)c\pi}{N-1}\right), \tag{43}$$

where $1 \le i \le M_x + 1$, $1 \le j \le M_y + 1$, M_x, M_y are the number of nodes of x and y respectively, μ_x is a random number in the range $[M_x/3, 2M_x/3]$, y is a random number in the range $[M_y/3, 2M_y/3]$, σ_x is random number in the range $[1, M_x/4]$, σ_y is random number in the range $[1, M_x/4]$, d is a random number in the range [-2, 2], and c is a random number in the range [1.9, 2.1]. In mutation process, each individual child undergoes mutation with probability *Pmi*. However, for each child that should undergo mutation process, individual curves are mutated with probability *Pmc*. If *Pmi* value is set to 0.5 and *Pmc*

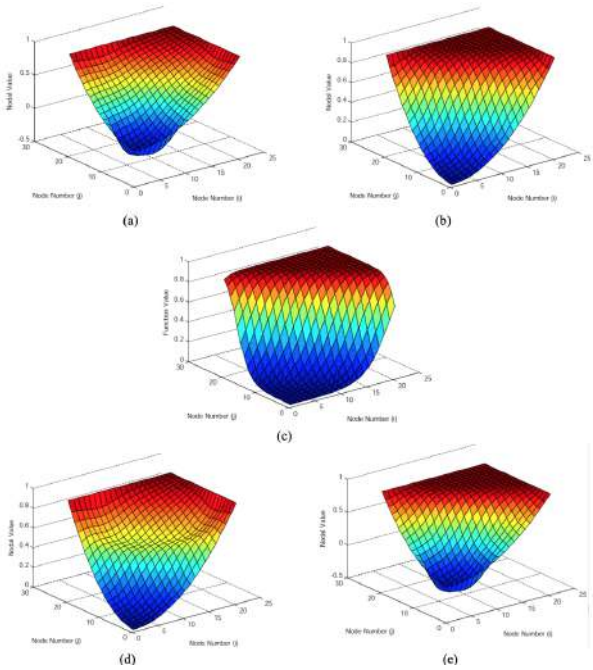


Figure 4: Crossover process: (a) first parent, (b) second parent, (c) crossover function, (d) first child and (e) second child.

value is set to 0.5, then one child out of two children is likely to be mutated, and within that child, $Nc/2$ of the solution curves are likely to be mutated [8]. Figure 5 shows the mutation process between two random parents.

6. Replacement: In the replacement process, to complete the life cycle of the population, the parent population may be replaced by the new one depending on the replacement scheme that we used.

7. Termination: This process is to terminate the CGA when the best fitness is obtained, i.e. the fitness reached to the recurred value, or when the number of generation reached to the maximum value.

4. Results and illustrations

In this section, we will consider four types of PDEs categorized as Laplace, Poisson, Helmholtz, and nonlinear PDE. The results will be compared for each method's solution in correspondence with the CGA proposed approach.

Problem 1: In this problem, we consider the following Laplace PDE with Neuman conditions [16]

$$\nabla^2 u = 0 \tag{44}$$

with boundary conditions

$$u(0,y) = y, \quad u_x(1,y) = y, \quad u_y(x,0) = x, \quad u_y(x,1) = x.$$

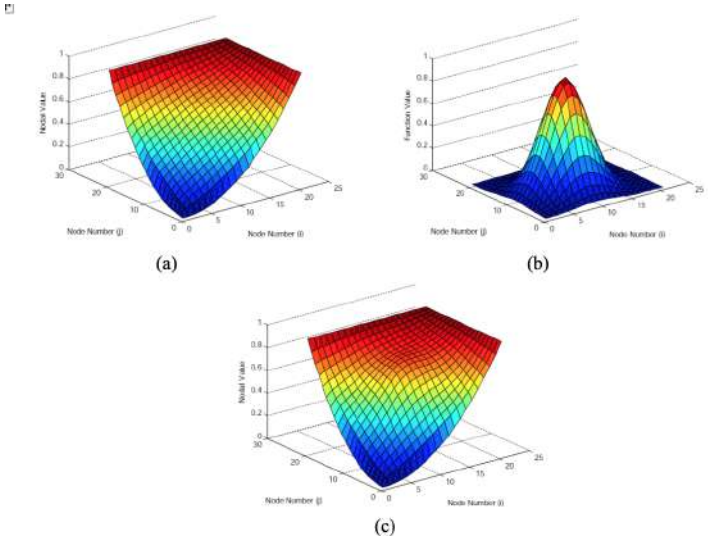


Figure 5: Crossover process: (a) first parent, (b) second parent, (c) crossover function, (d) first child and (e) second child.

Problem 2: Consider the Poisson's equation with Dirichlet and Neuman conditions

$$\nabla^2 u = (x^2 + y^2)\exp(xy) \tag{45}$$

with boundary conditions

$$u(0, y) = 0, \quad u_x(2, y) = \exp(2y), \quad u(x, 0) = 1, \quad u(x, 1) = \exp(x).$$

Problem 3: Consider the Helmholtz equation with mixed conditions

$$\nabla^2 u + xu(x, y) = (x + x^2)\exp(y) \tag{46}$$

with boundary conditions

$$u(0, y) = 0, \quad u_x(1, y) = \exp(y), \quad u(x, 0) = x, \quad u_y(x, 1) = x\exp(1).$$

Problem 4: Consider the semilinear elliptic PDE given by

$$\nabla^2 u - (u + 1)^2 = -1 - \exp(2(x + y)) \tag{47}$$

with boundary conditions $u(0, y) = \exp(y)$, $u(1, y) = \exp(y + 1)$, $u(x, 1) = \exp(1 + x)$, $u(x, 2) = \exp(x + 2)$ on $\partial\Omega$, where $\Omega = [0, 1] \times [1, 2]$

To investigate the performance of the proposed CGA method, we analyze each problem by showing the error whether the exact solution exists or not and showing the residual values for each problem, which in return, determines the suitability of the numerical solution for each problem.

Table 1: Error values using CGA for Problem 1

j\i	1	2	3	4	5	6	7	8	9	10	11
1	1.20E-4	1.40E-4	1.30E-4	1.10E-4	8.26E-5	6.77E-5	5.60E-5	5.37E-5	7.44E-5	1.20E-4	1.20E-4
2	1.30E-4	1.49E-4	1.23E-4	1.05E-4	8.49E-5	6.70E-5	5.22E-5	4.59E-5	5.34E-5	1.11E-4	1.18E-4
3	1.00E-4	9.95E-5	9.19E-5	8.53E-5	7.38E-5	6.48E-5	5.20E-5	4.72E-5	2.28E-5	4.43E-5	6.05E-5
4	5.60E-5	5.53E-5	5.87E-5	5.29E-5	5.28E-5	4.51E-5	4.32E-5	4.37E-5	3.19E-5	1.27E-5	1.97E-5
5	5.20E-6	5.92E-6	1.73E-5	2.32E-5	2.91E-5	2.40E-5	2.52E-5	2.57E-5	1.41E-5	3.60E-6	1.82E-6
6	5.43E-5	5.37E-5	3.81E-5	3.07E-5	1.55E-5	1.56E-5	1.07E-5	1.24E-5	1.20E-5	2.94E-5	2.78E-5
7	1.25E-4	1.24E-4	9.33E-5	8.02E-5	5.91E-5	5.29E-5	4.67E-5	4.29E-5	4.24E-5	4.74E-5	4.48E-5
8	1.96E-4	2.05E-4	1.60E-4	1.33E-4	1.10E-4	9.64E-5	8.67E-5	7.75E-5	6.79E-5	6.52E-5	6.14E-5
9	2.37E-4	2.43E-4	2.34E-4	1.79E-4	1.48E-4	1.29E-4	1.09E-4	9.10E-5	8.49E-5	7.49E-5	7.31E-5
10	2.47E-4	2.51E-4	2.75E-4	2.49E-4	1.96E-4	1.61E-4	1.30E-4	1.03E-4	8.42E-5	4.97E-5	6.70E-5
11	2.43E-4	2.48E-4	2.63E-4	2.49E-4	2.07E-4	1.71E-4	1.33E-4	1.04E-4	8.42E-5	6.88E-5	8.54E-5

Table 2: Residual values using CGA for Problem 1

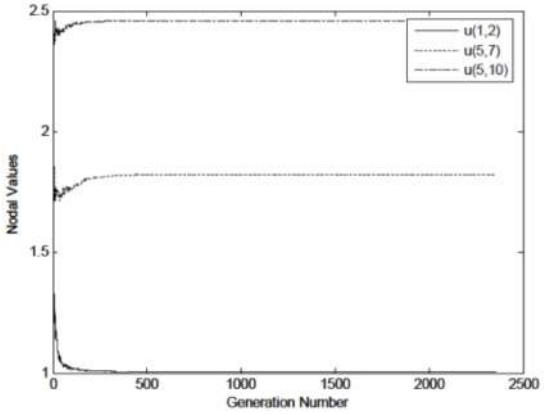
j\i	1	2	3	4	5	6	7	8	9	10	11
1	1.48E-5	3.18E-6	3.52E-6	7.08E-7	3.09E-6	4.86E-7	4.63E-7	1.81E-6	4.44E-6	1.57E-5	1.16E-6
2	3.75E-6	2.50E-5	5.15E-6	4.92E-6	2.90E-6	3.56E-7	3.06E-6	5.73E-6	1.01E-5	2.71E-5	1.84E-5
3	5.07E-6	2.81E-7	1.98E-7	4.52E-6	1.86E-6	5.25E-6	1.99E-7	6.11E-6	2.14E-5	7.40E-6	3.85E-6
4	1.40E-6	2.75E-7	4.36E-6	2.14E-6	2.53E-6	1.05E-6	1.67E-6	6.74E-6	8.53E-6	1.04E-5	1.36E-6
5	1.83E-6	1.09E-7	4.87E-6	6.04E-6	8.04E-6	3.04E-6	4.63E-6	8.00E-6	3.64E-6	2.50E-6	2.00E-6
6	2.51E-6	9.76E-7	1.97E-6	3.03E-6	3.52E-6	1.83E-6	1.65E-6	2.38E-6	5.52E-6	6.74E-6	1.40E-6
7	9.96E-9	5.16E-6	7.27E-6	1.28E-6	5.72E-6	1.53E-6	1.61E-6	1.86E-6	1.61E-7	1.93E-6	1.16E-6
8	3.19E-6	2.43E-5	6.67E-6	3.48E-7	1.56E-6	1.61E-6	4.18E-6	5.46E-6	3.66E-7	2.28E-6	6.32E-7
9	4.73E-6	1.16E-5	1.95E-5	1.20E-5	5.96E-6	6.30E-7	3.03E-8	2.67E-6	5.40E-6	6.68E-6	3.55E-6
10	1.17E-6	2.24E-6	2.58E-5	2.48E-5	4.11E-6	4.87E-6	3.56E-6	5.41E-7	3.76E-6	2.40E-5	2.52E-6
11	4.01E-6	4.09E-6	9.72E-7	6.83E-6	4.73E-6	4.93E-6	8.84E-7	1.46E-6	1.14E-6	1.51E-6	1.75E-5

In the first problem (Laplace equation type), we use 11 × 11 points, where x_0=0 and x_{11}=1, y_0=0 and y_{11}=1. As a result, the errors are shown in Table 1 while the residual values are in Table 2. Figure 6a shows the convergence at selected points and Figure 6b shows the maximum fitness. It is clear that the solution converges in about 1600 generations. In the second problem (Poisson equation type), we use 9 × 11 points, where x_0=0 and x_9=2, y_0=0 and y_{11}=1. As a result, the errors are shown in Table 3 while the residual values are in Table 4. Figure 7a shows the convergence at selected points and Figure 7b shows the maximum fitness. It is clear that the solution converges in about 500 generations. Observing the relations between the errors at each point and the residual values also shows a mutual correspondence.

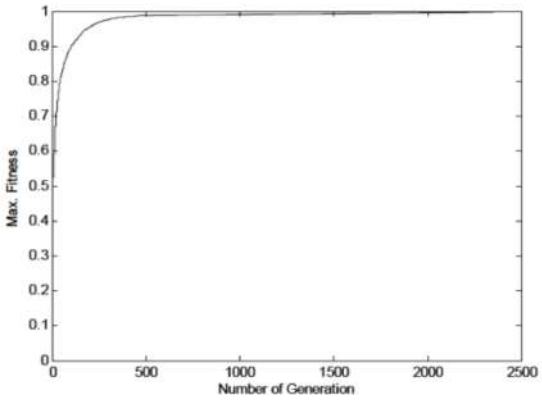
In the third problem (Helmholtz equation type), we use 11 × 11 points, where x_0=0 and x_{11}=1, $y_0 = 0$ and y_{11}=1. As a result, the errors are shown in Table 5 while the residual values are in Table 6. Figure 8a shows the convergence

Table 3: Error values using CGA for Problem 2

j\i	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	2.80E-4	3.30E-4	2.10E-4	1.80E-4	1.20E-4	2.00E-4	1.81E-4	2.80E-4	0
3	5.80E-4	6.40E-4	3.90E-4	3.30E-4	2.20E-4	4.40E-4	3.16E-4	4.10E-4	0
4	8.70E-4	8.30E-4	5.30E-4	4.30E-4	2.80E-4	6.00E-4	2.06E-5	5.30E-4	0
5	1.11E-3	9.60E-4	6.20E-4	4.60E-4	3.40E-4	6.10E-4	4.50E-4	7.10E-4	0
6	1.30E-3	1.01E-3	6.40E-4	4.60E-4	3.60E-4	5.90E-4	8.50E-4	9.10E-4	0
7	1.44E-3	1.01E-3	6.00E-4	4.40E-4	3.50E-4	5.60E-4	1.16E-3	1.06E-3	0
8	1.50E-3	9.20E-4	5.50E-4	3.80E-4	3.00E-4	5.30E-4	1.32E-3	1.13E-3	0
9	1.38E-3	7.70E-4	4.50E-4	3.20E-4	2.10E-4	4.90E-4	1.18E-3	1.04E-3	0
10	9.10E-4	4.50E-4	2.50E-4	2.10E-4	1.40E-4	3.40E-4	7.50E-4	7.20E-4	0
11	0	0	0	0	0	0	0	0	0



(a) Convergence

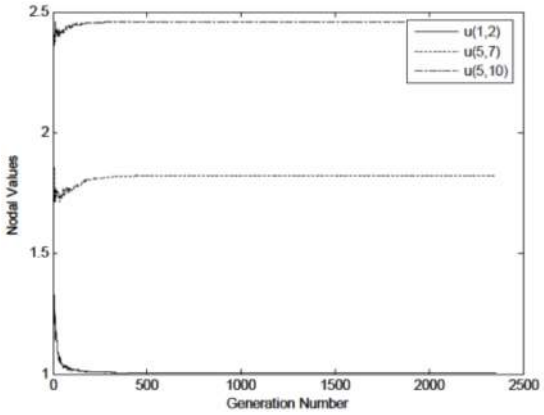


(b) Maximum Fitness

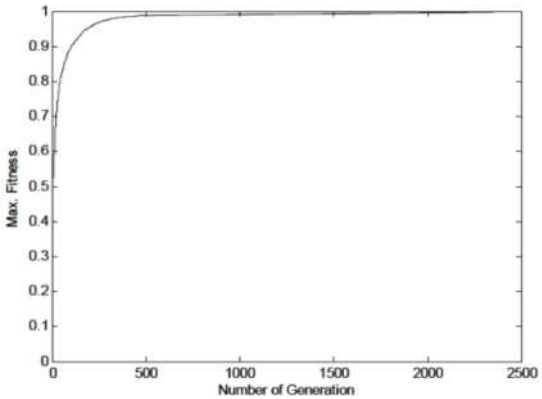
Figure 6: Convergence and Maximum Fitness for the first problem.

Table 4: Residual values using CGA for Problem 2

j\i	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	1.76E-5	2.46E-5	3.11E-6	1.21E-5	5.37E-7	6.61E-6	9.84E-5	7.24E-5	0
3	4.04E-7	6.72E-5	7.80E-6	2.93E-5	1.78E-5	8.75E-5	3.30E-4	3.57E-5	0
4	1.09E-5	4.86E-5	9.09E-6	2.49E-5	3.11E-5	1.14E-4	1.42E-4	1.35E-5	0
5	2.11E-5	4.28E-5	1.24E-5	1.22E-5	1.95E-5	2.88E-5	5.31E-5	4.59E-6	0
6	1.34E-5	3.08E-5	9.89E-6	3.25E-7	1.47E-5	1.35E-5	8.61E-6	8.68E-7	0
7	1.36E-5	3.13E-5	1.86E-5	4.88E-6	1.59E-5	5.88E-5	6.49E-5	1.41E-6	0
8	2.59E-5	7.36E-6	4.01E-6	1.38E-5	2.53E-5	6.36E-5	1.29E-4	3.29E-6	0
9	4.17E-5	3.65E-5	1.38E-5	1.79E-6	6.71E-5	2.58E-5	9.16E-5	1.28E-5	0
10	2.13E-5	2.58E-5	1.47E-5	1.07E-5	2.91E-5	1.12E-5	3.87E-5	1.16E-5	0
11	0	0	0	0	0	0	0	0	0



(a) Convergence



(b) Maximum Fitness

Figure 7: Convergence and Maximum Fitness for the second problem.

Table 5: Error values using CGA for Problem 3

j\i	1	2	3	4	5	6	7	8	9	10	11
1	1.75E-1	1.34E-1	3.00E-2	5.97E-2	1.17E-1	1.29E-1	1.04E-1	5.10E-2	2.74E-2	1.12E-1	1.47E-1
2	1.61E-1	1.22E-1	4.46E-2	5.54E-2	1.14E-1	1.28E-1	1.02E-1	4.75E-2	2.98E-2	1.06E-1	1.41E-1
3	1.55E-1	1.25E-1	5.49E-2	5.83E-2	1.20E-1	1.32E-1	1.03E-1	4.82E-2	3.05E-2	1.04E-1	1.34E-1
4	1.48E-1	1.19E-1	4.92E-2	4.88E-2	1.08E-1	1.22E-1	9.53E-2	4.09E-2	3.70E-2	9.30E-2	1.18E-1
5	1.35E-1	1.08E-1	4.03E-2	3.38E-2	8.54E-2	1.01E-1	7.97E-2	3.12E-2	3.73E-2	7.75E-2	9.62E-2
6	1.20E-1	9.49E-2	3.47E-2	1.70E-2	5.81E-2	7.67E-2	6.03E-2	2.23E-2	3.25E-2	6.00E-2	7.39E-2
7	1.02E-1	8.19E-2	3.32E-2	1.13E-3	3.13E-2	5.10E-2	4.13E-2	1.49E-2	2.37E-2	4.26E-2	5.28E-2
8	8.52E-2	6.97E-2	3.19E-2	1.24E-2	8.86E-3	2.89E-2	2.59E-2	1.02E-2	1.35E-2	2.71E-2	3.47E-2
9	7.10E-2	5.94E-2	2.96E-2	2.09E-2	3.14E-3	1.39E-2	1.62E-2	8.87E-3	4.45E-3	1.58E-2	2.16E-2
10	5.65E-2	5.16E-2	3.30E-2	2.12E-2	3.40E-3	1.27E-2	1.66E-2	1.07E-2	1.13E-3	1.12E-2	1.49E-2
11	4.81E-2	4.58E-2	3.90E-2	2.53E-2	6.09E-3	1.00E-2	1.54E-2	1.04E-2	5.96E-4	8.53E-3	1.12E-2

Table 6: Residual values using CGA for Problem 3

j\i	1	2	3	4	5	6	7	8	9	10	11
1	3.04E-2	1.91E-2	1.67E-2	7.73E-3	6.93E-3	4.70E-3	4.37E-3	5.68E-3	6.60E-3	8.87E-3	1.27E-2
2	1.37E-2	2.29E-3	3.67E-3	8.35E-4	3.96E-4	4.12E-4	1.70E-4	3.74E-4	1.03E-3	7.46E-4	9.81E-3
3	1.23E-2	7.05E-3	1.04E-2	9.72E-3	1.05E-2	8.21E-3	4.95E-3	5.39E-3	4.45E-3	5.30E-3	1.04E-2
4	1.23E-2	5.60E-3	5.63E-3	5.81E-3	7.86E-3	6.43E-3	5.55E-3	4.31E-3	3.81E-3	3.11E-3	6.00E-3
5	1.12E-2	4.31E-3	6.18E-3	2.77E-3	4.54E-3	4.03E-3	3.28E-3	2.27E-3	4.17E-3	1.86E-3	3.09E-3
6	9.15E-3	3.36E-3	3.30E-3	9.27E-4	2.13E-3	2.87E-3	2.51E-3	2.35E-3	3.55E-3	1.59E-3	1.59E-3
7	6.13E-3	2.60E-3	2.36E-3	1.26E-3	2.44E-3	2.35E-3	2.32E-3	2.20E-3	1.97E-3	1.77E-3	1.75E-3
8	3.87E-3	1.98E-3	3.12E-3	2.06E-3	4.29E-3	2.89E-3	2.71E-3	2.64E-3	2.23E-3	2.03E-3	2.14E-3
9	5.32E-3	2.44E-3	4.59E-3	4.03E-3	4.81E-3	5.43E-3	4.11E-3	3.39E-3	3.49E-3	2.70E-3	2.60E-3
10	4.48E-3	4.45E-4	2.03E-3	1.15E-3	1.21E-3	1.71E-3	1.29E-3	1.25E-3	9.98E-4	4.40E-4	1.97E-3
11	9.31E-3	5.27E-3	7.95E-3	5.17E-3	3.59E-3	2.49E-3	1.39E-3	1.29E-3	1.18E-3	2.21E-3	4.12E-3

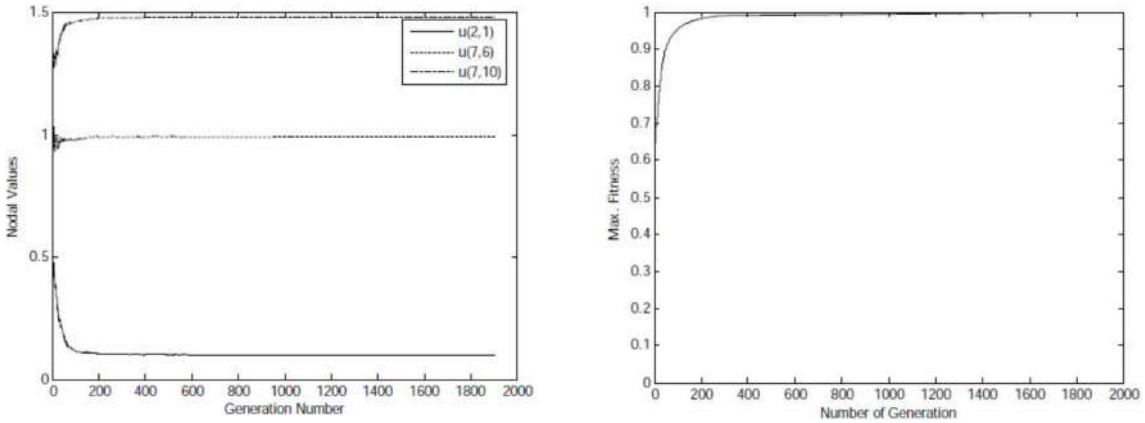
at selected points and Figure 8b shows the maximum fitness. It is clear that the solution converges in about 4000 generations.

In the fourth problem (nonlinear equation type), we use 11 $\times$ 11 points, where x_0=0 and x_{11}=1, y_0=1 and y_{11}=2. As a result, the errors are shown in Table 7 while the residual values are in Table 8. Figure 9a shows the convergence at selected points and Figure 9b shows the maximum fitness. It is clear that the solution converges in about 400 generations. Observing the relations between the errors at each point and the residual values, again shows a mutual correspondence.

Now, in order to investigate the potential of the proposed CGA approach, the previous four linear partial differential equation boundary value problems have also been solved using the FDM and FEM for a comparison purpose. Solutions of the FEM and FDM are summarized in Table 9 and Table 10 respectively, while the solutions of the CGA are summarized in Table 11.

Table 7: Error values using CGA for Problem 4

j\i	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	0	7.28E-5	1.20E-4	1.50E-4	1.70E-4	1.30E-4	7.84E-5	6.00E-5	2.20E-4	7.55E-5	0
3	0	1.20E-4	2.20E-4	2.70E-4	2.90E-4	2.80E-4	2.00E-4	1.41E-4	2.07E-4	3.10E-4	0
4	0	1.50E-4	2.70E-4	3.50E-4	3.90E-4	4.20E-4	4.00E-4	3.40E-4	1.91E-5	2.09E-5	0
5	0	1.60E-4	3.00E-4	3.90E-4	4.40E-4	4.60E-4	4.30E-4	3.50E-4	2.60E-4	1.50E-4	0
6	0	1.40E-4	3.00E-4	4.10E-4	4.60E-4	4.90E-4	4.80E-4	4.40E-4	3.40E-4	2.00E-4	0
7	0	1.20E-4	3.10E-4	4.10E-4	4.50E-4	4.90E-4	4.90E-4	4.60E-4	3.90E-4	2.50E-4	0
8	0	8.93E-5	3.47E-5	3.40E-4	3.90E-4	4.40E-4	4.40E-4	4.40E-4	3.70E-4	2.50E-4	0
9	0	4.20E-4	4.35E06	5.16E-5	3.20E-4	3.60E-4	3.80E-4	3.80E-4	3.40E-4	2.40E-4	0
10	0	8.74E-5	2.80E-4	7.00E-5	1.90E-4	2.30E-4	2.50E-4	2.50E-4	2.30E-4	1.70E-4	0
11	0	0	0	0	0	0	0	0	0	0	0

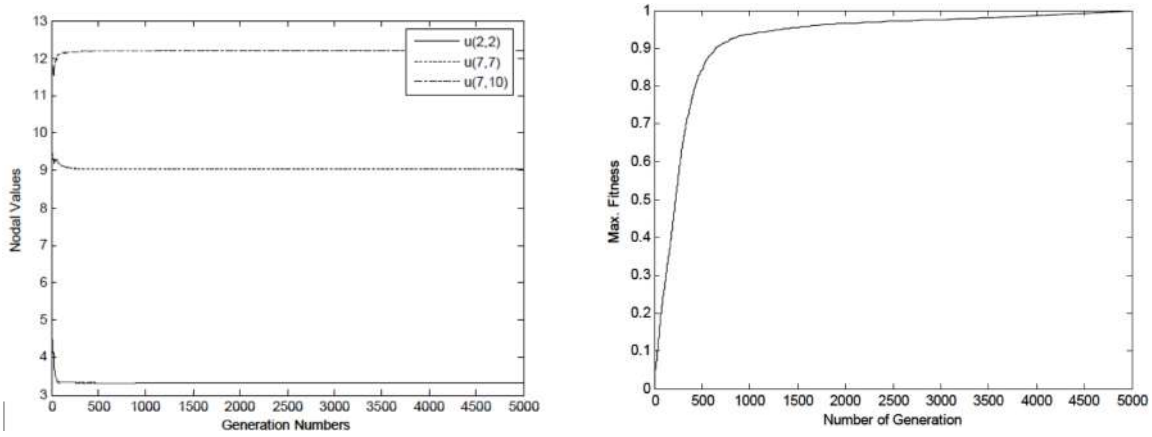


(a) Convergence (b) Maximum Fitness

Figure 8: Convergence and Maximum Fitness for the third problem.

Table 8: Residual values using CGA for Problem 4

j\i	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	0	4.15E-5	4.02E-5	2.22E-5	7.90E-5	2.63E-4	5.15E-4	1.62E-3	2.78E-3	1.42E-3	0
3	0	1.67E-5	8.76E-5	7.68E-5	4.30E-5	1.60E-4	5.31E-4	3.08E-3	5.65E-3	3.37E-3	0
4	0	4.62E-5	7.30E-5	9.17E-5	8.98E-5	4.40E-4	7.47E-4	2.47E-3	3.65E-3	1.72E-3	0
5	0	2.27E-5	1.14E-4	8.86E-5	1.40E-4	1.98E-4	2.35E-4	7.06E-4	6.47E-4	3.94E-4	0
6	0	1.80E-4	5.45E-5	7.30E-5	7.42E-5	1.07E-4	1.57E-4	1.28E-4	9.41E-5	2.79E-4	0
7	0	3.97E-4	9.49E-4	2.18E-4	1.30E-4	9.52E-5	1.15E-4	1.22E-4	1.95E-4	5.47E-5	0
8	0	1.99E-3	1.67E-3	1.20E-3	3.04E-4	1.05E-4	1.20E-4	1.50E-4	1.38E-4	4.32E-5	0
9	0	3.27E-3	1.98E-3	2.14E-3	4.00E-4	7.42E-5	1.06E-4	8.29E-5	2.76E-4	6.09E-5	0
10	0	9.08E-4	1.74E-3	7.81E-4	2.79E-4	6.72E-5	3.43E-5	2.53E-5	4.99E-5	5.19E-5	0
11	0	0	0	0	0	0	0	0	0	0	0



(a) Convergence (b) Maximum Fitness

Figure 9: Convergence and Maximum Fitness for the fourth problem.

Table 9: Numerical results using the FEM approach

#	Type	Average error	$\|\|Error\|\|_\infty$	$\|\|u\|\|_\infty$	$\|\|Exact\|\|_\infty$	Max. error	Min. error
problem 1	Laplace	2.19E-1	5.11	5.38	7.90	2.21E-1	2.16E-1
problem 2	Poisson	4.92E-4	3.22E-2	8.43E+1	8.43E+1	6.38E-3	1.04E-8
problem 3	Helmholtz	1.89E-4	5.26E-3	2.85E+1	28.51E+1	9.65E-4	4.49E-7
problem 4	Nonlinear	1.79E-2	1.50E-1	8.11E+1	8.12	4.99E-2	3.70E-3

Table 10: Numerical results using the FDM approach

#	Type	Average error	$\|\|Error\|\|_\infty$	$\|\|u\|\|_\infty$	$\|\|Exact\|\|_\infty$	Max. error	Min. error
problem 1	Laplace	2.30E-1	2.53	5.90	3.85	2.30E-1	2.30E-1
problem 2	Poisson	1.45E-2	2.17E-1	2.45E+1	2.44E+1	6.29E-2	7.58E-4
problem 3	Helmholtz	7.85E-3	1.42E-1	1.18E+1	1.18E+1	2.64E-2	6.81E-4
problem 4	Nonlinear	1.39E-1	1.18	7.33	7.26	2.88E-1	7.15E-2

As clearly seen from these tables, the proposed CGA approach outperforms the well known FEM and FDM in different places focusing on the average error.

5. Conclusion

In this paper, continuous genetic algorithms have been implemented for the solution of PDE's/PVB's given in the form of Laplace, Poisson, Helmholtz, and nonlinear. To investigate the performance of the proposed CGA technique, the four different problems were solved by the CGA and then compared their results with the results obtained by the FDM and FEM. Thus, it was concluded that the CGA performs its operation with different advantages; being a parallel type method (reduction of computational cost), requires a minimal problem knowledge, and being, in a sense, non-iterative technique. In addition to that, one can guess the suitability of the numerical solution to a given problem by the residual value; that is, whenever the residual value converges to zero the error converges to zero. Based on the analysis of the illustrative problems, it is shown clearly that the proposed CGA technique operates with a high potential and is suitable for solving different types of PDE's/PVB's with different advantages as presented.

Table 11: Numerical results using the CGA approach

#	Type	Average error	$\|\|Error\|\|_\infty$	$\|\|u\|\|_\infty$	$\|\|Exact\|\|_\infty$	Max. error	Min. error
problem 1	Laplace	9.16E-5	5.165E-6	1.23E-3	5.47E-5	2.75E-4	1.81E-6
problem 2	Poisson	6.03E-4	3.37E-5	5.77E-3	4.51E-4	1.49E-3	2.05E-5
problem 3	Helmholtz	3.50E-4	1.74E-5	4.01E-3	2.88E-4	9.25E-4	2.65E-6
problem 4	Nonlinear	2.72E-4	6.55E-4	2.58E-3	9.48E-3	4.88E-4	4.34E-6

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An error estimate of a nonmatching grids method for a biharmonic equation

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Abstract. Motivated by the work of Boulaaras and Haiour in [7], we provide a maximum norm analysis of Schwarz alternating method for biharmonic equation with repect to the mixed boundary condition, where an optimal error analysis each subdomain between the discrete Schwarz sequence and the continuous solution of bilaplace equations is established.

Keywords: maximum norm analysis, nonmatching grids method, Schwarz sequence, bilaplace differential equations.

1. Introduction

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping subdomains(see[1,2,5,6,9]). The solution is approximated by a infinite sequence of function which results from solving a sequence of stationary or evolutionary boundary value problems in each of the sub-domain. In this work, we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for bilaplace equation related to impulse control problem.

We consider the following elliptic equation : find $u \in H^2(\Omega)$ solution of

$$\begin{cases} \Delta^2 u + \alpha u = f, \quad \text{in } \Omega, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0 \text{ in } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ and α is a positive constant such that satisfies

$$\beta \leq \alpha, \quad \beta > 0. \tag{1.2}$$

f is a regular function.

Let $(.,.)_\Omega$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi \ > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

In [7] Boulaaras and Haiour provided a maximum norm analysis of a finite element Schwarz alternating method for a nonlinear parabolic partial differential equations on two overlapping subdomains with nonmatching grids. They considered a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Then according to Lipschitz assumption, they proved on each subdomain an optimal error estimate on uniform norm between the discrete Schwarz sequence and the exact solution of a nonlinear parabolic partial differential equations. In this paper, the same approach can be extended to other types as a linear parabolic partial differential equations see [2] and singularly perturbed advection-diffusion equations (see [11]) using the overlapping domain decomposition method, where we applied it in a full discrete (see [3,4,7]).

In [7], the authors studied the overlapping domain decomposition method combined with a finite element approximation for Laplace equuation, where an overlapping Schwarz method on nonmatching grids has been used on uniform norm of and they also proved the geometric convergence on every subdomain.

In this paper, similar to that in [7], we can extend the study to bilaplace equation with mixed boundary conditions, where we provide a maximum norm analysis of the finite element Schwarz alternating method of the presented problem on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one.

With respect to the stability analysis which has been given by our previous work in [7], we establish on each subdomain, an optimal error analysis between the discrete Schwarz sequence and the continuous solution of bilaplace equations. Also the geometric convergence is proved.

2. Nonlinear elliptic equation

We can reformulate (1.1) into the following variational equation by using Green's formula: find $u \in \mathcal{H}$ solution of

$$\begin{cases} \mathcal{A}(u,v) = (f,v)_\Omega + (\varphi,v)_{\Gamma_0}, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}(.,.)$ is the bilinear form defined as:

$$u, v \in H^2(\Omega) : \mathcal{A}(u,v) = (\Delta u, \Delta v)_\Omega + \rho\left(\partial_\eta u, \partial_\eta v\right)_\Gamma + \alpha(u,v). \tag{2.2}$$

Let $V_h^{(\varphi)}$ the discrete space defined as

$$V_h^{(\varphi)} = \{v \in \mathcal{H} \text{ such that } v_h \mid_K = P_1,\ k \in \tau_h, \frac{\partial v_h}{\partial \eta} = \pi_h \varphi \text{ in } \Gamma_0, \tag{2.3}$$
$$v_h = 0 \text{ in } \Gamma \backslash \Gamma_0\},$$

where Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. With the family τ_h is regular and quasi-uniform, and the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation, with P_1 Lagrangian polynomial of degree less than or equal to 1 and π_h is an interpolation operator on Γ_0. We consider r_h be the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(M_i)\, \varphi_i(x).$$

2.1 The Discrete Maximum Principle assumption (DMP)

We assume the matrices whose coefficients $\mathcal{A}(\varphi_i, \varphi_j)$ are M-matrix ([12] and [13]). For convenience in all the sequels, C will be a generic constant independent on h.It can be approximated the problem (1.1) by a following elliptic equation $v \in H^2(\Omega)$

$$\mathcal{A}(u,v) = (f,v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{2.4}$$

We discretize in space, i.e., we approach the space $\mathcal{H}$ by a space discretization of finite dimensional $V_h \subset \mathcal{H}$, we get the following of elliptic equation

$$\mathcal{A}(u_h, v_h) = (f, v_h)_\Omega + (\varphi, v_h)_{\Gamma_0}. \tag{2.5}$$

Theorem 1 ([7]). *Under suitable regularity of the solution of problem* (1.1), *there exists a constant C independent of h such that*

$$\|\zeta_h^\infty - \zeta\| \leq Ch^2 |\log h|. \tag{2.6}$$

Lemma 1 ([18]). *Let $w \in \mathcal{H} \cap C\left(\bar{\Omega}\right)$ satisfies $\mathcal{A}(w,\phi) + a(w,\phi) \geq 0$ for all nonnegative $\phi \in H^1(\Omega)$ and $w \geq 0$ on Γ, then $w \geq 0$ on $\bar{\Omega}$.*

Notation 1. $(f,\varphi); (\widetilde{f}, \widetilde{\varphi})$ be a pair of data and $\zeta = \partial(f,\varphi); \widetilde{\zeta} = \partial(\widetilde{f}, \widetilde{\varphi})$ the corresponding solutions to (2.4) .

Proposition 1. *Under the previous notation, we have*

$$\|\zeta_h - \zeta\|_{L^2(\Omega)} \leq \max\{\left(\frac{1}{\beta}\right) \left\|f - \widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi - \widetilde{\varphi}\|_{L\infty(\Omega)}\}. \tag{2.7}$$

Proof. First, putting

$$\mu = \max\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma)}\}, \tag{2.8}$$

then

$$\begin{cases}\widetilde{f} \leq f+\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)} \leq f+\left(\frac{a}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)} \\ \leq f+a\max\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma)}\} \leq f+a\mu.\end{cases}$$

So,

$$\mathcal{A}\left(\widetilde{\zeta},\phi\right) \leq \mathcal{A}\left(\zeta,\phi\right)+a\left(\mu,\phi\right), \text{ for all } \phi \geq 0,\ \phi \in \mathcal{H} \tag{2.9}$$

and thus

$$\mathcal{A}\left(\widetilde{\zeta},\phi\right) \leq \mathcal{A}\left(\zeta+\mu,\phi\right) = \left(f+a\mu,\phi\right).$$

On the other hand, we have

$$\zeta+\phi-\widetilde{\zeta} \geq 0 \text{ on } \Gamma_0. \tag{2.10}$$

So,

$$\mathcal{A}(\zeta+\phi-\widetilde{\zeta}) \geq 0. \tag{2.11}$$

By using the result of Lemma 1, we get

$$\widetilde{\zeta}+\phi-\zeta \geq 0 \text{ on } \overline{\Omega}. \tag{2.12}$$

Similarly, interchanging the roles of the couples (f,φ) and $(\widetilde{f},\widetilde{\varphi})$, we get

$$\widetilde{\zeta}+\phi-\zeta \geq 0 \text{ on } \overline{\Omega}, \tag{2.13}$$

which completes the proof. □

Remark 1. Proposition 1 stays true for the discrete case.

Lemma 2 ([18]). *Let $w \in V_h$ satisfy $\mathcal{A}(w,\phi_s) > 0$ for $s = 1,2...m(h)$ and $w \geq 0$ on Γ_0.then $w \geq 0$ on $(\overline{\Omega})$.*

Notation 2. (f,φ); $(\widetilde{f},\widetilde{\varphi})$ be a pair of data and $\zeta_h = \partial(f,\varphi); \widetilde{\zeta}_h = \partial(\widetilde{f},\widetilde{\varphi})$ the corresponding solutions to (**??**) .

Proposition 2. *Let DMP hold, we have*

$$\left\|\zeta_h-\widetilde{\zeta}_h\right\|_{L\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta}\right)\left\|f-\widetilde{f}\right\|_{L\infty(\Omega)}, \|\varphi-\widetilde{\varphi}\|_{L\infty(\Gamma_0)}\right\} \tag{2.14}$$

Proof. The proof is similar to that of the continuous case. □

3. Schwarz alternating methods for elliptic equation with bilaplace operator

We have the nonlinear elliptic problem with bilaplace operator: find $u \in H^2(\Omega)$ solution of

$$\begin{cases} \Delta^2 u + \alpha u = f, & \text{in } \Omega, \\ \Delta u + \rho \dfrac{\partial u}{\partial \eta} = 0, & \text{in } \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \varphi \text{ in } \Gamma_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

the weak formulation to be:

$$\mathcal{A}(u, v) = (f, v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{3.2}$$

We decompose Ω in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j; i \neq j$ is empty. Let

$$V_i^{(w_j)} = v \in \mathcal{H} \text{ such that } \ v = w_j \text{ on } \Gamma_i.$$

We associate with problem (3.2) the following system: find $(u_1, u_2) \in V_1 \times V_2$ solution to

$$\begin{cases} \mathcal{A}_1(u_1, v) + \alpha(u_1, v) = (f_1, v)_{\Omega 1} + (\varphi, v)_{\Gamma_1}, \\ \mathcal{A}_2(u_2, v) + \alpha(u_2, v) = (f_2, v)_{\Omega 2} + (\varphi, v)_{\Gamma_2}, \end{cases} \tag{3.3}$$

where

$$\mathcal{A}_i(u_i, v) = \int_{\Omega_i} \Delta u.\Delta v + \rho \partial_\eta u.\partial_\eta v dx \tag{3.4}$$

and

$$u_i = u/\Omega_i; i = 1, 2$$

3.1 The continuous Schwartz sequences

Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$\mathcal{A}(u_0, v) + \alpha(u_0, v) = (f, v). \tag{3.5}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in V_1^{\left(u_2^n\right)}$ solves of

$$\mathcal{A}_1(u_1^{n+1}, v) + \alpha(u_1, v) = (f_1, v), \tag{3.6}$$

where

$$f_1 = f(u_1)$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in V_2^{\left(u_1^{n+1}\right)}$ solves

$$\mathcal{A}_2(u_2^{n+1}, v) + \alpha(u_2, v) = (f_2, v), \quad f_2 = f(u_2) \tag{3.7}$$

Theorem 2 ([7]). *The sequences* $(u_{1h}^{n+1}); (u_{2h}^{n+1})$, $n \geq 0$ *produced by the Schwarz alternating method converge geometrically to a solution* u *of the elliptic obstacle problem. More precisely, there exist* $\lambda_1, \lambda_2 \in (0,1)$ *which depend on* (Ω_1, Γ_2) *and* (Ω_2, Γ_1) *such that for all* $n \geq 0$,

$$\sup_{\overline{\Omega}_1} \left|u_{1h} - u_1^{2n+1}\right| \leq \lambda_1^n \lambda_2^n \sup_{\Gamma_1} \left|u_{1h} - u_{1h}^0\right| \tag{3.8}$$

and

$$\sup_{\overline{\Omega}_2} \left|u_{2h} - u_2^{2n}\right| \leq \lambda_1^n \lambda_2^{n-1} \sup_{\Gamma_2} \left|u_{2h} - u_{2h}^0\right|. \tag{3.9}$$

3.2 The discrete Schwartz sequences

As we have defined before, for $i = 1, 2$, let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C(\Omega_i)$, we set

$$V_{hi}^{(w_j)} = \left\{ \ v \in \mathcal{H} \text{ such that } v = \phi \text{ on } \Gamma_{01} \cap \Gamma_{02};\ v = \pi_{h_i}(w) \text{ on } \Gamma_{0i},\ \right\}.$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.6) and (3.7) .

Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.5), we respectively, define by $u_{1h}^{n+1} \in V_{h1}^{\left(u_{2h}^n\right)}$ such that

$$\mathcal{A}_1(u_{1h}^{n+1}, v) = (f(u_{1h}^{n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0 \tag{3.10}$$

and $u_{2h}^{n+1} \in V_{h2}^{\left(u_{1h}^{n+1}\right)}$ such that

$$\mathcal{A}_2(u_{2h}^{n+1}, v) = (f(u_{2h}^{n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0. \tag{3.11}$$

3.3 Error analysis for the nonlinear elliptic equation

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

3.3.1 Two auxiliary Schwarz sequences

For $w_{2h}^0 = u_{2h}^0$, we define the sequences w_{1h}^{n+1} and w_{2h}^{n+1} such that $u_{1h}^{n+1} \in V_{h1}^{\left(u_2^n\right)}$ solves

$$\mathcal{A}_1(w_{1h}^{n+1}, v) = (f(u_{1h}^{n+1}), v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{3.12}$$

and $w_{2h}^{n+1} \in V_{2h}^{\left(u_{1h}^{n+1}\right)}$ solves

$$\mathcal{A}_2(w_{2h}^{n+1}, v) = (f(u_{2h}^{n+1}), v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{3.13}$$

respectively. It is then clear that w_{1h}^{n+1} and w_{2h}^{n+1} are the finite element approximation of u_1^{n+1} and u_2^{n+1} defined in (3.12), (3.13), respectively. Then, as $f(.)$ is continuous, $\left\|f\left(u_i^{n+1}\right)\right\|_\infty \leq C \left\|u_i^{n+1}\right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear elliptic problems, we have

$$\|u_i^n - u_{ih}^n\|_{L^\infty(\Omega_i)} \leq Ch^2 |\log h| \tag{3.14}$$

where C is a constant independent of both h and n.

Notation 3. From now on, we shall adopt the following notations: $|.|_1 = |.|_{L^\infty(\Gamma_1)}$, $|.|_2 = |.|_{L^\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L^\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L^\infty(\Gamma_2)}$, and we set $\pi_{h_1} = \pi_{h_2} = \pi_h$.

3.4 Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{n+1} = T_h u_{ih}^{n+1}, u_{ih}^{n+1} \in V_{h_i}^{\left(u_2^n\right)}, \tag{3.15}$$

where u_h is the solution of the problem (**??**) and the first iteration u_h^0 is solution of (3.5).

Lemma 3. *Let $\rho = \dfrac{\alpha}{\beta}$. Then, under assumption* (1.2)*, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \leq \frac{Ch^2 |\log h|}{1-\rho}, \quad i = 1, 2. \tag{3.16}$$

Proof. We know from standard error estimate on uniform norm for linear problem [17] that there exists a constant C independent of h such that

$$\left\|u^0 - u_h^0\right\|_{L=(\Omega)} \leq Ch^2 |\log h| . \tag{3.17}$$

Since $\dfrac{1}{2} < \rho < 1$, then $1 < \rho/(1-\rho)$ and

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq Ch^2 |\log h| \leq \frac{\rho Ch^2 |\log h|}{1-\rho}. \tag{3.18}$$

Let us now prove (3.16) by induction. Indeed for $n = 1$, using the result of Propsition1, we have in Ω_1

$$\begin{aligned}
\left\|u_1^1 - u_{1h}^1\right\|_1 &\leq \left\|u_1^1 - w_{1h}^1\right\|_1 + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \left\|w_1^1 - u_{1h}^1\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left|u_2^0 - u_{2h}^0\right|_1 \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_1^1\right) - f\left(u_{1h}^1\right)\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2 \\
&\leq Ch^2 \left|\log h\right| + \max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\}.
\end{aligned}$$

We distinguish between two cases

$$\max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.19}$$

or

$$\max\{\rho \left\|u_1^1 - u_{1h}^1\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.20}$$

(3.19) implies

$$\begin{cases} \left\|u_1^1 - u_{1h}1\right\|_1 \leq Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1, \\ \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^1 - u_{1h}^1\right\|_1, \end{cases}$$

then

$$\begin{cases} \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}. \\ \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho Ch^2 |\log h|}{1-\rho}. \end{cases}$$

(3.20) implies

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2,$$

so, by multiplying (3.20) by ρ we get

$$\rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{3.21}$$

So, $\rho\|u_1^1 - u_{1h}^1\|_1$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_2^0 - u_{2h}^0\|_2$ and $\|u_2^0 - u_{2h}^0\|_2$, this implies that

$$\rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{3.22}$$

or

$$\rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{3.23}$$

that is (3.22) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \tag{3.24}$$

and (3.23) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}. \tag{3.25}$$

It follows that only the case (3.22) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}, \tag{3.26}$$

then

$$\begin{aligned}\rho \left\|u_1^1 - u_{1h}^1\right\|_1 &\leq C h^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \leq \frac{C h^2 \left|\log h\right|}{1-\rho}.\end{aligned}$$

So, in both cases (3.19) and (3.20), we have

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{C h^2 \left|\log h\right|}{1-\rho}. \tag{3.27}$$

Similarly, we have in Ω_2

$$\begin{aligned}\left\|u_2^1 - u_{2h}^1\right\|_2 &\leq C h^2 \left|\log h\right| + \left\|w_2^1 - u_{2h}^1\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left|u_1^1 - u_{1h}^1\right|_2\} \\ &\leq C h^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|f\left(u_2^1\right) - f\left(u_{2h}^1\right)\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} \\ &\leq C h^2 \left|\log h\right| + \max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\}.\end{aligned}$$

So,

$$\max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} = \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \tag{3.28}$$

or

$$\max\{\rho \left\|u_2^1 - u_{2h}^1\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\} = \left\|u_1^1 - u_{1h}^1\right\|_1. \tag{3.29}$$

cases (3.28) implies

$$\begin{aligned}&\left\|u_2^1 - u_{2h}^1\right\|_2 \leq C h^2 \left|\log h\right| + \rho \left\|u_2^1 - u_{2h}^1\right\|_2, \\ &\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho \left\|u_2^1 - u_{2h}^1\right\|_2\end{aligned}$$

so,

$$\begin{cases} \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \\ \qquad\qquad \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \end{cases}$$

while case (3.29) implies

$$\begin{cases} \left\|u_2^1 - u_{2h}^1\right\|_2 \leq Ch^2 \left|\log h\right| + \left\|u_1^1 - u_{1h}^1\right\|_1 \\ \rho \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_1^1 - u_{1h}^1\right\|_1 . \end{cases} , \tag{3.30}$$

So, by multiplying (3.30) by ρ we get

$$\rho \left\|u_2^1 - u_{2h}^1\right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 . \tag{3.31}$$

Hence, $\rho\|u_2^1 - u_{2h}^1\|_2$ is bounded by both $\rho Ch^2|\log h| + \rho\|u_1^1 - u_{1h}^1\|_1$ and $\|u_1^1 - u_{1h}^1\|_1$, then

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \tag{3.32}$$

or

$$Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - u_{1h}^1\right\|_1 , \tag{3.33}$$

which (3.32) implies

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} < \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{3.34}$$

or (3.33) implies

$$\frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \left\|u_1^1 - u_{1h}^1\right\|_1 < \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{3.35}$$

Hence, (3.32) and (3.33) are true because they both coincide with (3.27). So, there is either a contradiction and thus case (3.28) is impossible or case (3.29) is possible only if

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \rho Ch^2 \left|\log h\right| + \rho \left\|u_1^1 - u_{1h}^1\right\|_1 , \tag{3.36}$$

that is

$$\left\|u_1^1 - u_{1h}^1\right\|_1 = \frac{\rho Ch^2 \left|\log h\right|}{1-\rho}, \tag{3.37}$$

thus

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq Ch^2 \left|\log h\right| + \left\|u_1^1 - u_{1h}^1\right\|_1 \leq Ch^2 \left|\log h\right| + \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \frac{Ch^2 \left|\log h\right|}{1-\rho},$$

that is, both cases (3.28) and (3.29) imply

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{3.38}$$

Now, let us assume that

$$\left\|u_2^n - u_{2h}^n\right\|_2 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{3.39}$$

and prove that

$$\begin{cases} \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho} \\ \left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho} \end{cases}$$

□

Theorem 3. *Let $h = \max(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_1 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho}, \quad \forall i = 1, 2. \tag{3.40}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\lambda = \lambda_1 \lambda_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned} \left\|u_1 - u_{1h}^{n+1}\right\|_1 &\leq \left\|u_1 - u_1^{n+1}\right\|_1 + \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \\ &\leq \lambda_1^n \lambda_2^n \left|u^0 - u\right|_1 + \frac{Ch^2 \left|\log h\right|}{1-\rho} \\ &\leq \lambda^{2n} \left|u^0 - u\right|_1 + \frac{Ch^2 \left|\log h\right|}{1-\rho}. \end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{3.41}$$

and thus

$$\left\|u_1 - u_{1h}^{n+1}\right\|_1 \leq Ch^2 + Ch^2 \left|\log h\right| \leq Ch^2 \left|\log h\right|,$$

which is the desired result. □

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SDIES: A Background subtraction method with sample dynamic indicator and edge similarity

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Abstract. Traditional sample-based methods are inefficient for detecting dynamic background and intermittent object motion. To solve the problem, a background subtraction method based on sample dynamic indicator and edge similarity is proposed. This method utilizes the standard deviation of recently observed decision distances as a background dynamic indicator and also uses the improved blink pixel estimator to control adaptive threshold feedback. Unlike the traditional sample-based methods, this method estimates the reference background at the initialization stage and distinguishes static objects from ghosts by calculating the edge similarity of foreground objects in reference background and current frame. So the existence property of foreground objects can suppress ghosts quickly. The evaluation results on the ChangeDetection 2012 dataset indicate that this method can be well adapted to the dynamic background and intermittent object motion. Besides, without paying too much attention to other scenario categories, the overall evaluation performance is comparable to most state-of-the-art methods.

Keywords: background subtraction, without neighborhood diffusion, edge similarity, sample dynamic.

1. Introduction

Motion foreground detection is the bottom task of intelligent video surveillance [1], visual object tracking [2] and optical motion capture [3]. Background subtraction (BS) is widely used to separate the motion objects called foreground and the static objects called background from video sequences. The main steps consist of background model initialization, foreground segmentation and background model update. Firstly, BS estimates a reference background by some historical frames. Then, a binary classification is generated by comparing the difference between the reference background and the current frame. Background model update makes the model to adapt well to the changes of video scenes. In the last two decades, a lot of BS methods have been proposed. Due to the existing typical challenges such as dynamic background, illumination change, camera jitter, shadows, etc., BS is still an active field of research. Significant progress has been made in BS methods and the following are some recent surveys [4, 5].

One of the most popular strategies for BS is probability-based method. Wren et al. [6] were the first to propose a single Gaussian distribution to present the color distribution of background pixels. Stauffer and Grimson [7, 8] proposed the Mixture of Gaussian Model (MoG), which was one of the most famous multimodal BS methods. MoG assumed that pixels obeying the normal distribution in time domain were background and could be used for updating background model and that other pixels that did not obey the normal distribution were foreground. While MoG could adapt to slightly dynamic background, as well as light change, it was unable to deal with highly dynamic background , as detailed in [11]. Varadarajan et al. [9, 10] proposed a Region-based Mixture of Gaussian Model (RMoG) framework, which took the neighboring pixels into account when constructing the model for the observed scenes. Besides, they

theoretically derived the update equations and extended the standard MoG more generally. However, the RMoG faces high computational costs.

Different from the idea of MoG, Barnich et al. [11] proposed a simpler and more efficient sample-based method named Visual Background Extractor (ViBe). Instead of estimating a reference background, this method kept several feature values (called samples or background samples) for the position of one pixel so that foreground segmentation and update tasks could make full use of the samples' information. ViBe was inspired by Sample Consensus (SACON) method [12, 13], which assumed that the pixels whose matching counts with samples exceed the threshold are background and other pixels are foreground. Moreover, ViBe creatively adopted neighborhood diffusion to suppress ghosts. Note that ViBe is easily implemented at high speed and is suitable for many scenarios. Researchers inspired by ViBe have proposed many improved methods [14, 15].

However, the fixed segmentation threshold of ViBe reduces the ability to adapt to the dynamic background. Droogenbroeck and Paquot [16] suggested an improved method named ViBe+. They followed the suggestion of [17] that ViBe should use a threshold associated with samples in the model for a better handling of camouflaged foreground. The half of standard deviation of samples was defined as the adaptive threshold. Instead of choosing the same threshold control strategy, Hofmann et al. [18] nominated the Pixel-Based Adaptive Segmenter (PBAS) method, which used average of currently observed minimal distance to estimate the dynamic degree. Threshold can achieve adaptive feedback in a liner way for static or dynamic region. However, the response time was relatively long. To counter this issue, Stcharles et al. [19] proposed Self-Balanced Sensitivity Segmenter (SuBSENSE) method, which introduced the blink pixel, a strictly-positive factor, to control feedback of segmentation threshold. Different from the above methods, Varghese and Sreelekha [20] pointed out that the adaptive minimal cardinality min can get better detection results. min is adapted for each pixel independently with background dynamic estimated. Although their methods can suppress dynamic noises, there is an over-segmentation problem when foreground objects pass through the dynamic region.

The neighborhood diffusion mechanism of ViBe can gradually eliminate ghosts, but the foreground object can be easily eaten-up slowly from the surrounding region since the low-speed motion and static object have similar characteristics with ghosts. ViBe+ calculated the gradient of the inner boundary of the background spot and that of the foreground spot. The neighborhood diffusion was restrained when the gradient value exceeded threshold. PBAS and SuBSENSE still utilized average of currently observed minimal distance to control update rate, which made sure foreground object shape is more complete. Note that the essence of these methods is to minimize the negative effects of the neighborhood diffusion mechanism. In general, neighborhood diffusion mechanism is not positive for intermittent motion objects.

Recently, some other complex BS methods are proposed, such as fuzzy background models [21, 22, 23], discriminative subspace learning models [24, 25], robust principle component analysis models [26, 27], sparse models [28, 29] and transform domain models [30, 31, 32]. Following the simple and practical principles, we still choose the framework of simple-based methods. In addition, selecting two or more features can obtain better detection results, though it also can increase computing costs. Thus, only a single color feature is selected. In addition, it is difficult for the sample-based methods to obtain the reference background directly. In our initialization phase, not only the regular background samples are built, the reference background is also accurately estimated. Our method can benefit from this reference background. Neighborhood update is the particular way of sample-based methods.

However, this mechanism has inherent defect in detecting intermittent motion objects. Our method does not select the neighborhood update. The edge similarity is utilized to determine whether the object is a foreground or a ghost, and then different updates are adopted to overcome the above defect. Some sample-based methods use adaptive feedback mechanism to control dynamic background noise which can easily cause over-segmentation. Our method utilizes the standard deviation of recently observed decision distances as the background dynamic indicator and improves blink pixel estimator. These guarantee the accurate shape of foreground objects while suppressing dynamic background.

This paper is organized as follow: Section 2 describes the initialization, reference background estimation, foreground segmentation and background update of the proposed method; Section 3 firstly sets universal parameters and then evaluates the detection quality of the proposed method from qualitative and quantitative aspects. Finally, section 4 summarizes the method.

2. Method description

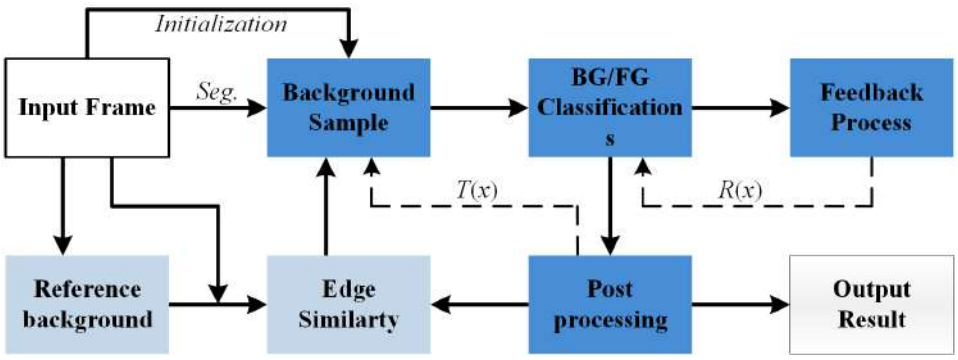


Figure 1: Block diagram of proposed SDIES method.

In general, there are two initialization operations for sample-based methods, such as selecting recently observed consecutive frames or repeatedly sampling

from neighborhood regions of pixels in the single frame. These operations can all give pixels multi-modal distribution. However, using single frame for initialization may cause the situation that foreground pixels exist in background samples. Moreover, variations among consecutive frames are often subtle. Therefore, in order to estimate the reference background, the traditional initialization method needs changes. In our method, we select discontinuous frames with fixed time intervals and increasing the sample intervals between each frame makes the difference between the current frame and the previous frame more obvious. Figure 1 illustrates the overall diagram of our method and Algorithm 1 describes our pseudocode in detail. The background model B is defined as

Algorithm 1. Pseudocode for SDIES

a. Use $(N+1)(P+1)$ frames construct background model
b. Estimate reference background by background model
c. Foreground segmentation
    for each$I(t)$ that $t > (N+1)(P+1)$
        F(x) = BG/FG classifications
    update background model with post-processing version of F(x)
    calculate edge similarity of foreground objects between reference background and $I(t)$
    if foreground objects are ghosts
        $com_b(x) = com_b(x) + 1$
        if $com_b(x)$ exceeds threshold
            update foreground objects into background model
d. Use 9*9 median filter of F(x) as final output

$$(1)\ B(x) = \{B_1(x), \cdots, B_N(x)\} \quad B_i(x) = I_{p(i-1)+1}(x) \quad i = 1, 2, \cdots, N.$$

where p is the fixed time interval, and N is background sample number.

$I_{p(i-1)+1}(x)$ is the current frame at $p(i-1)+1$ moment. After sampling period p, the current frame $I_{p(i-1)+1}(x)$ is added to the background model until the N frames are processed. In short, the time interval is extended from 1 to p. As mentioned, only color feature is used.

A. Reference Background Estimated

The traditional initialization ends at this stage. However, the reference background can be estimated with the existing background model. Our reference background estimation is inspired by Independent Multi-modal Background Subtraction (IMBS) [33], which holds that the background appears for a long time while the foreground is opposite. A pixel can be observed several times in the same position over a period of time, which indicates this pixel is more

likely to be the background. So, we create the reference background sample $RB(x) = RB_1(x), ..., RB_N(x)$ and count $C(x) = C_1(x), ..., C_N(x)$. The first element $B_1(x)$ of the background sample is inserted into the reference background $RB_1(x)$, then count $C_1(x) = 1$. Starting with the second element, $B_2(x)$ will be compared with $RB_i(x)$ as follows:

$$(2) \qquad \begin{cases} RB_i(x) = RB'_i(x), C_i(x) = C_i(x) + 1, & \text{if } \|RB_i(x) - B_i(x)\| \le R_{ini}, \\ RB_{i+1}(x) = B_i(x), & \text{otherwise}, \end{cases}$$

$$(3) \qquad RB_i(x) = \frac{B_i(x)C_i(x) + RB_i(x)}{C_i(x) + 1},$$

where $\|\cdot\|$ is the Euclidean distance and parameter R_{ini} is the distance threshold. If the distance difference between $RB_i(x)$ and $B_i(x)$ is small, then a fusion value is generated to replace the original background sample and corresponding counts adds 1. The specific fusion calculation process uses formula (3). On the contrary, $B_i(x)$ is inserted into the next position and the corresponding counts is set to 1. When N background samples are processed, there are many values in each reference background position. So, unimportant reference background values need to be filtered. Traversing the index of the maximum value in $C(x)$, then the pixel value at corresponding index in $RB(x)$ is treated as the reference background. Note that reference background and background sample are independent. Now our method can suppress ghosts effectively and that will be described in detail later.



(a) highway (b) PETS2006 (c) abandonedbox (d) fountain01

Figure 2: Reference background estimate on some CDnet2012 sequences. Even if there are foreground objects in the initial video sequence, our method can estimate the background accurately.

B. Foreground Detection

Our pixel-level foreground detection mechanism is inspired by [11]. It determines whether a coming pixel can be foreground or background based on the similarity with background samples as shown in formula (4)

$$(4) \qquad F(x) = \begin{cases} 1, & \text{if } \phi\{\|RB_i(x) - B_i(x)\| < R(x)\} < \phi_{\min}, \\ 0, & \text{otherwise}, \end{cases}$$

where $F(x) = 1$ implies foreground, $F(x) = 0$ implies background, $R(x)$ is the distance threshold, $\phi(\cdot)$ is the counts that distance difference between current pixel $I(x)$ and background sample $B_i(x)$ is close to $R(x)$. $\phi_{\min}$ is the minimum match counts threshold and we fixed $\phi_{\min} = 2$ like [11] for our method.

Setting a distance threshold R is a challenging issue for background subtraction. Different video scenes have different characteristics, and global R cannot adapt to the scenes changes [16]. An over-high or over-low threshold may lead to undesirable results. Naturally, designing the adaptive threshold adjustment strategies is a trend. As previously stated, the sample-based method is very sensitive to background samples reliability. Therefore, background dynamic degree is used as the basis for adjusting the threshold value. Similar to [18, 19], we use all the recently observed sample distance to estimate background dynamic. In order to achieve fast threshold response, the minimal distance (without considering whether the update is implemented or not) and the pixel classification state are observed. This strategy makes it possible for distance threshold R to make quick feedback. It is noted that R in static region, where foreground objects pass frequently, also presents a high value which makes it hard to detect the foreground object (canoe and boats sequences). To solve this problem, the blink pixels accumulator is adopted. Although the neighborhood update mechanism can suppress the ghost effect but it cannot distinguish the static foreground object from the ghost. Instead of neighborhood diffusion, our update could solve the main shortcoming of traditional sample-based methods and detect intermittent motion objects better. The detailed description is in the following subsection.

C. Feedback Control

Foreground detection is a binary classification task. The dynamic background has the same motion attribute as the foreground object. Without any distinction, the dynamic background will be detected as the foreground object falsely. So, the main idea is to measure the motion entropy of a single pixel position in a small time window based on model fidelity. The motion entropy of dynamic background is bigger than static background. In our case, instead of [18, 19] using the moving average of the recently observed minimal distance to measure background dynamics, we select the standard deviation $S(x)$ of mean of all the observed distances between $I(x)$ and $B(x)$ as follows

$$S(x) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(D_{mean_i}(x) - \frac{1}{N}\sum_{i=1}^{N} D_{mean_i}(x)\right)^2}, \tag{5}$$

where $S(x)$ is the standard deviation and $D_{mean}(x)$ is the mean of all the observed distances between $I(x)$ and $B(x)$. The background motion entropy depends on sample differences, that is, the dispersion scale. The standard deviation is more consistent with the cognition of statistical data and more accurate

representation of sample differences. In order to evaluate background dynamics, the threshold feedback is defined as follows:

$$R_t(x) = \begin{cases} R_{t-1}(x) + V(x), & \text{if } R_{t-1}(x) < 3S_t(x), \\ R_{t-1}(x) - \frac{1}{V(x)}, & \text{otherwise,} \end{cases} \tag{6}$$

where t is the index of the image frame and $V(x)$ is the blink pixel accumulator used to adjust the distance threshold, which is inspired by [19]. Since $S(x)$ in the dynamic regions become higher, the corresponding distance threshold $R(x)$ will gradually increase by Formula (6). In the static regions, the opposite result appears. The feedback mechanism suppressed the dynamic noise well, but there is an exception. Following the logic above, $S(x)$ in these static regions, where foreground objects pass frequently, also presents higher values. Due to the higher $S(x)$, foreground detection result may be incomplete or void. Therefore, PBAS avoids continuously adjusting threshold to evade this phenomenon. Feedback is only carried out when background sample is updated. As a result, feedback slows down in dynamic region so that many dynamic noises cannot be suppressed in time. To achieve fast response, our threshold adjustment mechanism is continuous. In other words, update and pixel classification will not be considered. To minimize the negative effect, we use the blink pixels proposed by [19] that current detection result $F_t(x)$ is different from the previous result $F_{t-1}(x)$. Meanwhile, a pixel-level 2D map accumulator is defined, note V, which can record blink pixels counts that captured by performing the XOR operation between $F_t(x)$ and previous frame $F_{t-1}(x)$. Owing to blink pixels, threshold feedback in this region, where foreground objects pass frequently, is weakened. SuBSENCE updates V using

$$v(x) = \begin{cases} V(x) + 1, & \text{if } XOR(t) = 1, \\ V(x) - 0.1, & \text{otherwise.} \end{cases} \tag{7}$$

Through experiments, we found there are two drawbacks in blink pixels: (i) the XOR operation is equivalent to frame difference, so borders of foreground objects will be falsely saved. SuBSENSE canceled these parts in XOR that intersect with the post-processed result of current frame. However, borders of the previous frame also remain in the XOR. (ii) Their blink pixels can fail in some sequences such as foreground objects that pass through the dynamic background (canoe and boats sequences), because blinking pixels accumulate all the time in the dynamic background region due to the largeness of $R(x)$. To make blink pixels more perfect, two improvement can be easily implemented: firstly, $F_t(x)$ and $F_{t-1}(x)$ with 5×5 median filters are used to capture blink pixels; secondly, these regions in XOR that intersect with the post-processed result of current frame and previous frame are canceled.

D. Ghost Suppression



(a) #963 (b) own (c) PBAS (d) SUNSENCE

Figure 3: Dynamic background detection on canoe sequence.

Sample-based methods usually adopt conservative update and neighborhood update with the $1/T$ probability. Conservative policy means that only background pixels are updated, however, the false foreground classification (ghost effect) will always exist. To address this problem, the sample-based methods utilize neighborhood update mechanism in which current pixels are diffused to neighborhood background samples. This mechanism cannot distinguish static foreground from ghost. As a price, static and low-speed foreground objects will be eaten up from the outside. Some methods try to control the different update rate to suppress the neighborhood diffusion, but the effect is not significant. Instead of suppressing neighborhood diffuse in a complex way, abandoning neighborhood updates might be a better solution. Without neighborhood update, other ghost suppressing method is considered.

Literature [34, 35] pointed out a fact that ghosts exist in background and does not exist in current frame, but the foreground is the opposite. Based on this fact, we can study the similarity among the foreground edges, the current frame edges and the background edges. Since the reference background is estimated at the initialization stage, properties of detected foreground objects can be categorized. If the foreground edges are more similar to the current frame edges, it is highly likely that the object exists in current frame, and vice versa. $Exist(k)$ is defined as follows:

$$(8) \qquad \begin{cases} 1, & \text{if } SE_C(k) - SE_B(k) > E_{th}, \\ 0, & \text{if } SE_B(k) - SE_C(k) > E_{th}, \end{cases}$$

where k is the $k-th$ connected component, $SE_C(k)$ are pixels sum of foreground edges intersected with current frame edges, $SE_B(k)$ are pixels sum of foreground edges intersected with reference background edges and E_{th} is fixed threshold. $Exist(k) = 1$ indicates that the object exists in current frame and $exist(k) = 0$ indicates that the object exists in reference background. Moreover, three counters $Comf$, $ComC$ and $ComB$ are constructed:

$$(9) \qquad Com_{f_t}(x) = \begin{cases} Com_{f_{t-i}}(x) + 1, & \text{if } F_p(x) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$Com_{C_t}(x) = \begin{cases} Com_{C_{t-1}}(x) + 1, & \text{if } Exist(x) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

$$Com_{B_t}(x) = \begin{cases} Com_{B_{t-1}}(x) + 1, & \text{if } Exist(x) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

where $F_p(x)$ is a post-processing version of $F(x)$, $Com_f(x)$ is number of pixels continuously classified as foreground, Com_B is number of objects continuously existing in reference background and Com_C is number of objects continuously existing in current frame. Our update mechanism proposed based on above description is

$$\begin{cases} B_i(x) = I(x), & \text{if } F_p(x) = 0, \\ B_i(x) = I(x), & \text{if } Com_f(x) \geq T_f \wedge Com_B(x) \geq T_B, \end{cases} \tag{12}$$

where T_f and T_B are fixed parameter and i is a random number. Conservative update is still implemented with the $1/T$ probability is still selected and the post-processing version is used as the update mask. Background noises are not added to background samples, which is positive for suppressing dynamic background. If a pixel is continuously detected as foreground and continuously exists in the background, it is immediately updated to the background samples. Note that the update rate T is a fixed parameter.



(a) background (b) current (c) 2700 (d) 2736

Figure 4: Intermittent object detection on abandonedbox sequence. Background object leaves the background region and ghost appears in the original position. When counters reach the own critical value, pixels in the ghost position are immediately updated to the background samples.

E. Post processing

Even if the adaptive feedback mechanism is adopted, the detection results may inevitably involve unwanted noises that reduce the accuracy of the results. So, the 99 median filter is used as the post-processing tool.

3. Experiments and results

To properly evaluate the performance of our background subtraction method, a benchmark dataset that contains as many video surveillance scenarios as possible is needed. Thanks to contribution of [36, 37], the ChangeDetection.net (CDnet) dataset offers diverse video scenes in real conditions with corresponding ground-truth masks. CDnet consists of 2012 version and 2014 version, including 11 categories: baseline, camera jitter, dynamic background, intermittent object motion, shadow, thermal, bad weather, low framerate, night videos, point-tilt-zoom (PTZ) and turbulence. Besides, 7 official metrics for evaluating methods are Recall (Re), Specificity (Sp), False Positive Rate (FPR), False Negative Rate (FNR), Percentage of Wrong Classifications (PWC), F-Measure (FM), and Precision (Pr). All the metrics are calculated by true positive (TP), false positive (FP), true negative (TN) and false negative (FN). For specific definitions, please refer to [36, 37]. In our evaluation process, FM is selected as the main evaluation indicator since it is generally accepted and widely used to evaluate the detection quality [19, 38, 39]. Our approach is primarily concerned with Dynamic background and intermittent object motion. So, only CDnet2012 dataset is selected.

A. Parameter Settings

As described above, the method needs to determine some parameters for adapting universal video scenes better. The number of samples N, distance thresholds R_{ini} (in reference background estimation phase), distance thresholds R (in foreground segmentation phase), update rate factor T, fixed time interval P, two critical values of counters T_f and T_B, and edge threshold E_{th} are all the parameters that should be discussed. It is noted that R can self-regulate by the adaptive feedback mechanism. So it is only necessary to determine lower bound and upper bound of R.

(a) $R_{ini} = 5$: This parameter is only used to estimate reference background. In the update phase, accurate reference background is helpful to obtain the correct background edges, otherwise, it will affect the edge similarity of foreground object. Considering that estimate reference background in CDnet dataset is not difficult, so we roughly determine that $R_{ini} = 5$. For more difficult dataset, $R_{ini} = 5$ can be modified by widely used background initialization performance indicators. Reader can refer to [19, 38, 39].

(b) $N = 30$: The number of background samples. In theory, more background samples are positive for samples multi-modal distribution. This positive effect can approximatively reach saturation when N increases. Note that much more samples only lead slight improvement for detection results.

(c) $P = 4$: The main purpose of increasing sampling interval is to increase the difference between background samples.

(d) $R_{lower} = 20$: The lower bound of decision threshold for foreground segmentation.

(e) $R_{upper} = 40$: The upper bound of decision threshold for foreground segmentation. Threshold feedback is determined by blinking pixels accumulation. Without upper bound, Re and FM are low.

(f) $T = 6$: Background update rate is fixed. Our method cancels neighborhood diffusion; static or low-speed foreground objects are not affected by high update probability.

(g) $T_f = 30$: Counter critical value.

(h) $T_b = 30$: Counter critical value.

(i) $E_{th} = 10$: Edge threshold determines edge similarity.

B. Experimental Results on CDnet2012

In this section, the method performance is evaluated in detail. Dynamic background and intermittent object motion have been targeted for particular attention. The complete evaluation results on dynamic background and intermittent object motion are respectively shown in Table 1 and Table 2.

Table 1. Evaluation results in dynamic background

	Re	***Sp***	***FPR***	***FNR***	***PWC***	***Pr***	***FM***
overpass	0.9381	0.9997	0.0003	0.0619	0.1116	0.9777	0.9575
fountain01	0.8640	0.9993	0.0007	0.1360	0.0778	0.5188	0.6483
fountain02	0.9537	0.9993	0.0007	0.0463	0.0758	0.7572	0.8441
fall	0.9435	0.9941	0.0059	0.0565	0.6759	0.7438	0.8318
boats	0.6167	0.9999	0.0001	0.3833	0.2507	0.9741	0.7553
canoe	0.9446	0.9991	0.0009	0.0554	0.2858	0.9739	0.9590
overall	0.8768	0.9986	0.0014	0.1232	0.2463	0.8243	0.8327

Table 2. Evaluation results in dynamic background

	Re	***Sp***	***FPR***	***FNR***	***PWC***	***Pr***	***FM***
aband.Box	0.9705	0.9934	0.0066	0.0295	0.7658	0.8821	0.9242
parking	0.4883	0.9842	0.0158	0.5117	5.4150	0.7214	0.5824
sofa	0.7031	0.9966	0.0034	0.2969	1.6257	0.9032	0.7907
streetLight	0.9157	0.9999	0.0001	0.0843	0.4158	0.9985	0.9553
tramstop	0.9762	0.9981	0.0019	0.0238	0.5789	0.9914	0.9837
wint.Dri.way	0.7448	0.9879	0.0121	0.2552	1.3922	0.3170	0.4447
overall	0.7998	0.9934	0.0067	0.2002	1.6989	0.8023	0.7802

Table 3. Complete results for proposed method on 2012 CDnet dataset

Category	***Re***	***Sp***	***FPR***	***FNR***	***PWC***	***Pr***	***FM***
baseline	0.9325	0.9962	0.0038	0.0675	0.6671	0.886	0.9077
camera jitter	0.7958	0.9757	0.0243	0.2043	2.9965	0.7165	0.7156
dynamic back.	0.8768	0.9986	0.0014	0.1232	0.2463	0.8243	0.8327
interm. obj. mot.	0.7998	0.9934	0.0067	0.2002	1.6989	0.8023	0.7802
shadow	0.9009	0.9891	0.0109	0.0991	1.4924	0.8090	0.8500
thermal	0.7228	0.9962	0.0038	0.2772	1.3445	0.9023	0.7814
overall	0.8381	0.9915	0.0085	0.1619	1.4076	0.8234	0.8113

Table 4. Compared with some state-of-the-art sample-based methods

	$\mathbf{FM}_{Overall}$	$\mathbf{FM}_{baseline}$	$\mathbf{FM}_{cam.jit.}$	$\mathbf{FM}_{dyn.bg}$	$\mathbf{FM}_{int.obj.mot}$	$\mathbf{FM}_{shadow}$	$\mathbf{FM}_{thermal}$
proposed	0.811	0.908	0.716	*0.833*	**0.780**	0.850	0.781
PAWCS [38]	**0.858**	0.940	*0.814*	**0.894**	*0.776*	0.891	**0.832**
SuBSENSE[19]	*0.826*	**0.950**	**0.815**	0.818	0.657	*0.899*	*0.817*
WeSamBE[39]	0.820	*0.941*	0.798	0.744	0.739	**0.900**	0.796
PBAS [18]	0.753	0.924	0.722	0.683	0.575	0.860	0.756
ViBe+ [16]	0.722	0.871	0.754	0.720	0.509	0.815	0.665

Note that bold entries indicate the best results and italics entries indicate the second-best results.

The SDIES adopts modified blink pixel and background dynamic indicator and the ability that suppresses dynamic background is significantly improved. Besides, F-measure increases about 13 over other methods (without PAWCS [38]). The post-processing version of F is used to capture blink pixels, which effectively solves the over-segmentation for boats and canoe sequences. In addition, the post-processing version of F is used as the update mask, which is positive for suppressing dynamic noise. This is the main reason that F-measure increases for dynamic background category.

The SDIES cancels neighborhood diffusion and uses three counters to realize conservative background update. That makes it possible to detect intermittent moving objects well and then F-measure increases about 20 over other methods. Neighborhood diffusion can suppress ghosts, but the process is slow. Our judging condition is that pixel is continuously detected as a ghost for 30 times. So, ideally, it only takes one second. This can be well represented on tramstop and abandonedbox sequences. The disadvantage of Neighborhood diffusion is

that it damages the completeness of static foreground objects. Fortunately, the neighborhood diffusion is canceled. There are fewer wrong pixel classifications while ensuring complete foreground objects. Rapid ghost suppression and complete foreground objects are the main reasons why F-measure increases for intermittent object motion category.

Besides, we selecte some state-of-the-art sample-based methods to compare with the proposed method. As shown in Table 4, F-measure is superior in dynamic background and intermittent object motion. In particular, FMinterm.obj.mo exceeds all other algorithms. However, the method does not only adapt these two scenarios; Table 3 indicates that the method can actually adapt all scenarios. Although the advantages in other scenarios are not so prominent, the SDIES meets the universal requirements. The subjective evaluation results are shown in Figure 5.

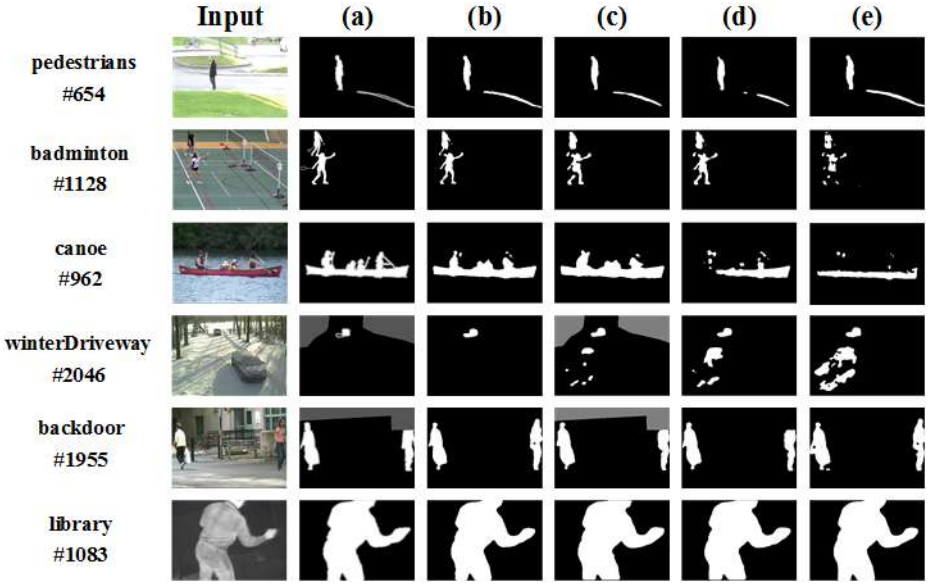


Figure 5: Foreground segmentation results on the CDnet2012 datasets. Column (a) are ground-truth, (b) are proposed results, (c) are PAWCS results [38], (d) are SuBSENSE results [35], (e) are PBASresults [18].

C. Processing Speed

The Code is written in MATLAB R2016b on PC with an Intel Core I5 CPU processor and 4GB memory. The average processing speed of SDIES is about 2 FPS. This code is the original version without any optimization, which could not meet the real-time requirements. In the future, the authors would like to try to turn it into C++ version.

4. Conclusion

The proposed method mainly focuses on dynamic background and intermittent object motion. Improved blinking pixels and sample dynamic indicators are used to better control adaptive threshold feedback. Instead of neighborhood update, using the estimated reference background to calculate the existence property of foreground objects could be the very way that effectively eliminates ghosts produced by intermittent object motion and ensures the complete foreground objects as well. It is noted that some scenes, such as those with hard shadow, complex background texture features and similar color features between background and foreground, have negative effect on edge similarity. In addition, adaptive threshold feedback can take high time complexity. In the future work, we devote to find a more reasonable measurement for edge similarity and try to optimize the code by C++ and parallel mode in order to meet the real-time requirements.

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A subordination result and integral mean for a class of analytic functions defined by q-differintegral operator

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Abstract. In this paper, we derive a subordination result and integral mean for certain class of analytic functions defined by means of a fractional q-differintegral operator $\Omega^{\delta}_{q,z} f(z)$.

Keywords: analytic functions, univalent functions, subordinating factor sequence, q-difference operator, Hadamard product (or convolution).

1. Introduction

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Also denote by $\mathcal{T}$, the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m \tag{2}$$

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which are univalent and normalized in $\mathcal{U}$. For $f \in \mathcal{A}$ and of the form (1) and $g(z) \in \mathcal{A}$ given by $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$, we define the convolution (or Hadamard product) $f * g$ of two power series f and g by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m.$$

The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [1, 5, 9, 13, 21].

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n = 0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha}; q)_n = \frac{\Gamma_q(\alpha + n)(1-q)^n}{\Gamma_q(\alpha)}, \ (n > 0), \tag{4}$$

where the q-gamma functions [9, 10] is defined by

$$\Gamma_q(x) = \frac{(q; q)_{\infty}(1-q)^{1-x}}{(q^x; q)_{\infty}} \ (0 < q < 1). \tag{5}$$

Note that, if $|q| < 1$, the q-shifted factorial (3), remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_{\infty} = \prod_{m=0}^{\infty} (1 - \alpha q^m).$$

Now, recall the following q-analogue definitions given by Gasper and Rahman [9]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [9])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)} \ (z \neq 0, \, q \neq 0) \tag{7}$$

and

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{8}$$

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{9}$$

we observe that the q-shifted fractional (3), reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Now, recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [17].

Definition 1.1. *The fractional q-integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order $\delta\,(\delta > 0)$ is defined by*

$$I^\delta_{q,z} = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-tq)_{1-\delta} f(t) d_q t, \tag{10}$$

where $f(z)$ is a analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q-binomial function defined by

$$\begin{aligned}(z-tq)_{\delta-1} &= z^{\delta-1}\prod_{m=0}^{\infty}\left[\frac{1-(\frac{tq}{z})q^m}{1-(\frac{tq}{z})q^\delta+m-1}\right] \\ &= z^\delta\, {}_1\phi_0\left[q^{-\delta+1};-;q,\frac{tq^\delta}{z}\right].\end{aligned} \tag{11}$$

According to Gasper and Rahman [9], the series ${}_1\phi_0[\delta;-;q,z]$ is single-valued when $|arg(z)| < \pi$. Therefore, the function $(z-tq)_{\delta-1}$ in (11), is single-valued when $|arg(\frac{-tq^\delta}{z}| < \pi$, $|tq^{\frac{\delta}{z}}| < 1$, and $|arg(z)| < \pi$.

Definition 1.2. *The fractional q-derivative operator $D^\delta_{q,z}$ of a $f(z)$ of order $\delta(0 \le \delta < 1)$ is defined by*

$$D^\delta_{q,z} f(z) = D_{q,z} I^{1-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\delta)} D_q \int_0^z (z-tq)_{-\delta} f(t) d_q t, \tag{12}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1.1 above.

Definition 1.3. *Under the hypotheses of Definition 1.2, the fractional q-derivative for the function $f(z)$ of order δ is defined by*

$$D^\delta_{q,z} f(z) = D^n_{q,z} I^{n-\delta}_{q,z} f(z), \tag{13}$$

where, $n-1 \le \delta < n$, $n \in \mathbb{N}_0$.

Now, we define a fractional q-differintegral operator $\Omega_{q,z}^{\delta} f(z)$ for the function $f(z)$ of the form (1), by

$$\begin{aligned} \Omega_q^{\delta} f(z) &= \Gamma_q(2-\delta) z^{\delta} D_{q,z}^{\delta} f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)} a_m z^m, \end{aligned} \tag{14}$$

where in $D_{q,z}^{\delta}$ (14), represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 < \delta < 2$. We note that $q \to 1^-$, the operator Ω_q^{δ} reduces the operator Ω^{δ} defined by Owa and Srivastava [14].

Recently, several authors investigated applications of fractional q-calculus operators by introducing certain new classes of functions which are analytic in the open disc, (see for example, [16, 18, 19, 27, 28]).

By making use of the concepts of fractional q- calculus, Ravikumar et al. [20] defined the following subclasses $\mathcal{S}_p^q(\alpha, \beta, \delta, b)$ of analytic function.

Definition 1.4. *For $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < \delta < 2$, $b \in \mathbb{C} - \{0\}$ and $0 < q < 1$, let $\mathcal{S}_p^q(\alpha, \beta, \delta, b)$ be the subclass of $\mathcal{A}$ consisting of functions of the form* (1) *and satisfying the analytic criterion*

$$\Re\left\{1 - \frac{2}{b} + \frac{2}{b}.\frac{zD_q(\Omega_q^{\delta} f(z))}{\Omega_q^{\delta} f(z)}\right\} > \beta\left|\frac{2}{b}.\frac{zD_q(\Omega_q^{\delta} f(z))}{\Omega_q^{\delta} f(z)} - \frac{2}{b}\right| + \alpha, \ z \in \mathcal{U}. \tag{15}$$

It can be seen that, the special cases of the class $\mathcal{TS}_p^q[\alpha, \beta, \delta, b]$ as $q \to 1^-$ and for different choices of the parameters we get the results obtained by Altintas and Owa [3], Bharthi, Parvtham and Swaminathan [6], Padamanabhan and Jayamala [15], Owa and Srivastava [14], Kim and Ronning [11].

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.5 (Subordination principle)**.** *Let $g(z)$ be analytic and univalent in $\mathcal{U}$. If $f(z)$ is analytic in $\mathcal{U}$, $f(0) = g(0)$, and $f(\mathcal{U}) \subset g(\mathcal{U})$, then we see that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, and we write $f(z) \prec g(z)$.*

Definition 1.6 (Subordinating factor sequence)**.** *A sequence $\{b_m\}_{m=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic , univalent and convex in $\mathcal{U}$, we have the subordination given by*

$$\sum_{m=2}^{\infty} b_m a_m z^n \prec f(z) \qquad (z \in \mathcal{U}, \ a_1 = 1). \tag{16}$$

Lemma 1.1 ([29])**.** *The sequence $\{b_m\}_{m=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{m=1}^{\infty} b_m z^m\right\} > 0, \qquad (z \in \mathcal{U}). \tag{17}$$

Lemma 1.2 ([20]). *If*

$$\sum_{m=2}^{\infty} [2(1+\beta)([m]_q - 1) + b(1-\alpha)]K_q(m,\delta)(\delta)\,|a_m| \leq b(1-\alpha), \tag{18}$$

then $f(z) \in \mathcal{S}_p^q(\alpha,\beta,\delta,b)$, *where,* $K_q(m,\delta) = \frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)}$, $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < \delta < 2$, $b > 0$ *and* $0 < q < 1$.

The result is sharp for the function

$$f_m(z) = z - \frac{b(1-\alpha)}{[2(1+\beta)([m]_q - 1) + b(1-\alpha)]K_q(m,\delta)(\delta)} z^m.$$

Let $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (18). We note that $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b) \subseteq \mathcal{S}_p^q(\alpha,\beta,\delta,b)$.

Unless otherwise mentioned, we assume throughout this paper that $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < \delta < 2$, $b > 0$ and $0 < q < 1$.

2. Subordination result

Employing the techniques used by Aouf et al. [2], Attiya [4], Frasin [7], Frasin et al. [8], Singh [25], Srivastava and Attiya [26], and others, we state and prove the following theorem.

Theorem 2.1. *Let the function* $f(z)$ *be defined by (1) be in the class* $\mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$. *Also let* $\mathcal{C}$ *denote the familiar class of functions* $f(z) \in \mathcal{A}$ *which are also univalent and convex in* $\mathcal{U}$. *Then*

$$\frac{(2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha) + (2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)]}(f*g)(z) \prec g(z), \tag{19}$$
$$(z \in \mathcal{U};, g \in \mathcal{C}),$$

and

$$Re(f(z)) > -\frac{b(1-\alpha) + (2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)}{(2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)}, (z \in \mathcal{U}). \tag{20}$$

The constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$ *is the best estimate.*

Proof. Let $f(z) \in \mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ and let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m \in \mathcal{C}$. Then

$$\frac{(2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha) + (2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)]}(f*g)(z)$$
$$= \frac{(2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha) + (2(1+\beta)q + b(1-\alpha))K_q(2,\delta)(\delta)]}\left(z + \sum_{m=2}^{\infty} a_m c_m z^m\right). \tag{21}$$

Thus, by Definition 1.6, the assertion of our theorem will hold if the sequence

$$\left\{\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}a_m\right\}_{m=1}^{\infty} \tag{22}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}a_m z^m\right\} > 0, \tag{23}$$
$(z \in \mathcal{U})$.

Now,

$$\begin{aligned}
&\operatorname{Re}\left\{1+\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\sum_{m=1}^{\infty}a_m z^m\right\}\\
=\ &\operatorname{Re}\left\{1+\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z+\right.\\
&+\frac{1}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\\
&\left.\cdot\sum_{m=2}^{\infty}(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)a_m z^m\right\}\\
\geq\ &1-\left\{\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r\right.\\
&-\frac{1}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}\\
&\left.\cdot\sum_{m=2}^{\infty}[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)a_m r^m\right\}
\end{aligned}$$

(because $[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)$ is increasing function of m, $(m \geq 2)$)

$$\begin{aligned}
&> 1-\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r\\
&-\frac{b(1-\alpha)}{b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}r>0, \qquad (|z|=r<1).
\end{aligned}$$

Thus (23) holds true in $\mathcal{U}$. This proves the inequality (19). The inequality (20) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{m=2}^{\infty} z^m$ in (19). To prove the sharpness of the constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$, we consider the function $f_2(z) \in \mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ given by

$$f_2(z) = z - \frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z^2.$$

Thus from the (19), we have

$$\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}f_2(z) \prec \frac{z}{1-z}. \tag{24}$$

It can easily verified that

$$\min\left\{\mathrm{Re}\left(\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}f_2(z)\right)\right\} = -\frac{1}{2}, \tag{25}$$

$(z \in \mathcal{U})$. This shows that the constant $\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{2[b(1-\alpha)+(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)]}$ is best possible. □

3. Integral means inequalities

Lemma 3.1 ([12]). *If the functions f and g are analytic in $\mathcal{U}$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\eta}d\theta \leq \int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta. \tag{26}$$

In [22], Silverman found that the function $F(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in [23] and settled in [24], that

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta \leq \int_0^{2\pi}\left|F(re^{i\theta})\right|^{\eta}d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [24], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of $\mathcal{T}$, where $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are starlike functions of order α and convex functions of order α, respectively.

Applying Lemma 3.1 and Lemma 1.2 , we prove the following result.

Theorem 3.1. *Suppose that $f \in \mathcal{S}_p^{*,q}(\alpha,\beta,\delta,b)$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi}|f(z)|^{\eta}\,d\theta \leq \int_0^{2\pi}|f_2(z)|^{\eta}\,d\theta, \qquad (\eta > 0). \tag{27}$$

Proof. For $f(z) = z - \sum_{m=2}^{\infty}|a_m|z^m$, (27) is equivalent to proving that

$$\int_0^{2\pi}\left|1-\sum_{m=2}^{\infty}|a_m|z^{m-1}\right|^{\eta}d\theta \leq \int_0^{2\pi}\left|1-\frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z\right|^{\eta}d\theta.$$

By Lemma 3.1, it suffices to show that

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1}\prec 1-\frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}z.$$

Letting

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1}=1-\frac{b(1-\alpha)}{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}w(z), \tag{28}$$

and using (1), we obtain $w(z)$ is analytic in $\mathcal{U}, w(0)=0$, and

$$\begin{aligned}|w(z)| &= \left|\sum_{m=2}^{\infty}\frac{(2(1+\beta)q+b(1-\alpha))K_q(2,\delta)(\delta)}{b(1-\alpha)}|a_m|z^{m-1}\right| \\ &\leq |z|\sum_{m=2}^{\infty}\frac{[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)}{b(1-\alpha)}|a_m| \\ &\leq |z|.\end{aligned}$$

This completes the proof by Theorem 3.1. □

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On the permanence and periodic solutions of a plankton system with impulses and diffusion

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Abstract. In this paper, given the sudden changes of external environment and seasonal variations of climate, we investigate the non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton model with periodic impulses and spatial diffusion. The sufficient conditions for ultimate boundedness of solutions and permanence of system are established by using theory of impulsive differential equations, comparison principle, upper-lower solution method and inequality techniques. Moreover, the existence and uniqueness of asymptotically stable periodic solution are studied with the help of auxiliary function. It is shown that the plankton populations will evolve periodically with time, provided that the system is permanent.

Keywords: marine ecosystem, toxin-producing phytoplankton, permanence, periodic solutions.

1. Introduction

In marine ecosystem, plankton lies at the basic trophic level of the entire food chain and therefore of all the food webs on Earth. Plankton can be primarily divided into autotrophic phytoplankton and grazing zooplankton. However, under eutrophication and suitable temperature, the red tide may outbreak due to the rapid growth of plankton and can make a huge negative impact on marine ecosystem, aquaculture and human health. Therefore it would be of great significance to using mathematical modeling to investigate the dynamic relationships between the phytoplankton and zooplankton populations, which can help to understand the occurrence mechanisms of red tide and further forecast the algae blooms. For more details, we refer to [1, 2, 3, 4].

Recent research has shown that toxin-producing phytoplankton plays an important role in the control of harmful algal blooms and can be used as a biocontrol agent [5, 6, 7]. Specifically, Chattopadhyay et al. [5] formulated a very

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general model by means of ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \dfrac{P}{K}\right) - \alpha f(P)Z, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \beta f(P)Z - \mu Z - \theta g(P)Z, \end{cases} \tag{1}$$

where $P(t)$ and $Z(t)$ are the densities of toxin-producing phytoplankton population and zooplankton population at time t, respectively. The functional responses $f(P)$ and $g(P)$ adopt Holling type I, II and III. Model (1) is related to field observations gathered in the Bay of Bengal. It is concluded that toxin-producing phytoplankton may terminate the planktonic blooms by decreasing the grazing pressure of zooplankton. It should be noted that there have been some meaningful results about the generalized cases of (1). For instance, Chattopadhyay et al. [8] later investigated the specific model with distributed delay and discrete delay respectively, and discussed the stability of equilibrium solution and existence of Hopf bifurcation. Wang et al. [9] also studied the Hopf-transcritical bifurcation for delayed model. Furthermore, Sharma et al. [10] also explored the effect of spatial diffusion on the stability of the plankton model. More related results can be found in [11, 12, 13, 14].

While the toxin-producing phytoplankton population is important in marine ecosystem, the role of non-toxic phytoplankton cannot be ignored. So, Chattopadhyay et al. [6] further proposed and analyzed the following non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton model:

$$\begin{cases} \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = rP_1\left(1 - \dfrac{P_1}{K}\right) - \alpha P_1 Z, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = sP_2\left(1 - \dfrac{P_2}{K}\right) - \dfrac{\theta P_2 Z}{\gamma + P_2}, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \beta P_1 Z - \mu Z - \dfrac{\theta_1 P_2 Z}{\gamma + P_2}, \end{cases} \tag{2}$$

where $P_1(t)$ is the concentration of the non-toxic phytoplankton at time t, $P_2(t)$ is the concentration of the toxin-producing phytoplankton population at time t, and $Z(t)$ is the concentration of the zooplankton population at time t.

In model (2), the authors have neglected the effect of the competition between different phytoplankton groups. But it is now well established that competition among different species may be a reason for stable coexistence [15]. Huisman and Weissing [16] also showed that multi-species competition may generate oscillations and chaos and that these fluctuations create opportunities for the coexistence of many species. Then Pal et al. [17] modified model (2) as

follows:

$$
\begin{cases}
\dfrac{\mathrm{d}P_1}{\mathrm{d}t} = rP_1\left(1-\dfrac{P_1+\alpha P_2}{K_1}\right) - \dfrac{\alpha_1 P_1 Z}{1+\beta_1 P_1+\beta_2 P_2}, \\
\dfrac{\mathrm{d}P_2}{\mathrm{d}t} = sP_2\left(1-\dfrac{P_2+\beta P_1}{K_2}\right) - \dfrac{\alpha_2 P_2 Z}{1+\beta_1 P_1+\beta_2 P_2}, \\
\dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{(\alpha_1' P_1 - \alpha_2' P_2)Z}{1+\beta_1 P_1+\beta_2 P_2} - \mu Z,
\end{cases} \tag{3}
$$

where α and β are the competition coefficients. It is shown that the high competition will result in planktonic bloom.

In reality, the plankton individual can freely float thanks to the wind and turbulence. So, some reaction-diffusion plankton ecological models have been proposed to simulate this phenomenon [18, 19]. Besides, the marine ecosystem is often deeply disturbed by human exploitation activities which might lead to the duration of abrupt changes [20, 21]. It is necessary to take into account this impulsive response. Recently, some research on the dynamics about impulsive and diffusive predator-prey models has appeared [22, 23, 24]. Nevertheless, there has been no result on the joint effects of spatial diffusion and impulsive response about the spatiotemporal dynamics of plankton system. Thus, motivated by the above work, in this paper, we mainly consider the following plankton system with periodic impulses and spatial diffusion:

$$
\begin{cases}
\dfrac{\partial P}{\partial t} = d_1\Delta P + r_1(t,x)P\left(1-\dfrac{P}{K_1(t,x)}\right) \\
\qquad -\alpha_1(t,x)PT - \dfrac{\beta_1(t,x)PZ}{a(t,x)+P}, & t\neq t_k, x\in\Omega, \\
\dfrac{\partial T}{\partial t} = d_2\Delta T + r_2(t,x)T\left(1-\dfrac{T}{K_2(t,x)}\right) \\
\qquad -\alpha_2(t,x)PT - \dfrac{\theta_1(t,x)TZ}{b(t,x)+T}, & t\neq t_k, x\in\Omega, \\
\dfrac{\partial Z}{\partial t} = d_3\Delta Z + \dfrac{\beta_2(t,x)PZ}{a(t,x)+P} \\
\qquad +\dfrac{\theta_2(t,x)TZ}{b(t,x)+T} - m(t,x)Z^2 - \rho(t,x)T^{\frac{2}{3}}Z, & t\neq t_k, x\in\Omega, \\
P(t_k^+,x) = P(t_k,x)f_k(x,P(t_k,x),T(t_k,x),Z(t_k,x)), & t=t_k, x\in\Omega, \\
T(t_k^+,x) = T(t_k,x)g_k(x,P(t_k,x),T(t_k,x),Z(t_k,x)), & t=t_k, x\in\Omega, \\
Z(t_k^+,x) = Z(t_k,x)h_k(x,P(t_k,x),T(t_k,x),Z(t_k,x)), & t=t_k, x\in\Omega, \\
\dfrac{\partial P}{\partial \nu} = \dfrac{\partial T}{\partial \nu} = \dfrac{\partial Z}{\partial \nu} = 0, & x\in\partial\Omega,
\end{cases} \tag{4}
$$

where $P(t,x)$, $T(t,x)$ and $Z(t,x)$ denote the densities of non-toxic phytoplankton, toxin-producing phytoplankton and zooplankton at time t and location x, respectively. The bounded space domain $\Omega\subset\mathbb{R}^n (n=1,2,3)$ has smooth boundary $\partial\Omega$, Δ is the Laplace operator, $\partial/\partial\nu$ is the outward normal derivative. In system (4), the toxic phytoplankton and non-toxic phytoplankton compete for

common survival resources, $d_i (i = 1, 2, 3)$ is the diffusivity of three plankton population respectively, the quadratic closure term is adopted to describe the mortality of zooplankton, the term $\rho T^{2/3} Z$ means that the toxic phytoplankton gathered in patches reduces the zooplankton's grazing.

For system (4), the nonnegative initial conditions are also satisfied:

$$P(0, x) = P_0(t) \geq 0, \quad T(0, x) = T_0(t) \geq 0, \quad Z(0, x) = Z_0(t) \geq 0, \quad x \in \partial\Omega.$$

The plankton populations are disturbed at time t_k, where $\{t_k\}$ is a real sequence, and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = +\infty$.

The main purpose of this paper is to investigate the permanence and periodic solutions of system (4). This paper is organized as follows. In Section 2, some basic assumptions and useful results are given. In Section 3, the ultimate boundedness of solutions and permanence of system (4) are explored. In Section 4, the existence of a unique periodic solution for system (4) is established. Finally, some conclusions are drawn in Section 5.

2. Basic assumptions and preliminaries

Considering the seasonal variation and cyclical factors of biological rhythm, we first state the basic assumptions as follows:

(H_1) The functions $r_i(t, x)$, $K_i(t, x)$, $\alpha_i(t, x)$, $\beta_i(t, x)$, $\theta_i(t, x)$, $a(t, x)$, $b(t, x)$, $m(t, x)$ and $(\rho(t, x))$ are bounded and positive-valued on $\mathbb{R} \times \bar{\Omega}$, continuously differentiable in t and x for $i = 1, 2$;

(H_2) The functions $f_k(x, P, T, Z)$, $g_k(x, P, T, Z)$ and $h_k(P, T, Z)$ are continuously differentiable in all arguments and positive for any $k \in \mathbb{N}^+$;

(H_3) The functions $r_i(t, x)$, $K_i(t, x)$, $\alpha_i(t, x)$, $\beta_i(t, x)$, $\theta_i(t, x)$, $a(t, x)$, $b(t, x)$, $m(t, x)$ and $(\rho(t, x))$ are $\omega-$periodic in t for $i = 1, 2$;

(H_4) There exists a positive integer p such that $t_{t+p} = t_k + \omega$ for any $k \in \mathbb{N}^+$;

(H_5) The function sequences $\{f_k\}$, $\{g_k\}$ and $\{h_k\}$ satisfy $f_{k+p}(x, P, T, Z) = f_k(x, P, T, Z)$, $g_{k+p}(x, P, T, Z) = g_k(x, P, T, Z)$ and $h_{k+p}(x, P, T, Z) = h_k(x, P, T, Z)$ for any $k \in \mathbb{N}^+$.

For convenience, we give the following notations:

$$G = \mathbb{R}^+ \times \Omega, \quad \bar{G} = \mathbb{R}^+ \times \bar{\Omega}, \quad \Gamma_k = \{(t, x) | \, t \in (t_{k-1}, t_k), x \in \Omega\},$$

$$\Gamma = \bigcup_{k\in\mathbb{N}^+} \Gamma_k, \quad \bar{\Gamma}_k = \{(t, x) | \, t \in (t_{k-1}, t_k), x \in \bar{\Omega}\}, \quad \bar{\Gamma} = \bigcup_{k\in\mathbb{N}^+} \bar{\Gamma}_k.$$

For a bounded function $\phi(t, x)$, we denote

$$\phi^L = \inf_{t>0, x\in\Omega} \phi(t, x), \quad \phi^M = \sup_{t>0, x\in\Omega} \phi(t, x).$$

Denote by Υ a class of functions $\phi : \bar{G} \to \mathbb{R}$ with the following properties:

(i) $\phi(t, x) \in C^{1,2}_{t,x}(\Gamma_k)$, $\phi(t, x) \in C^{1,1}_{t,x}(\bar{\Gamma}_k)$ for all $k \in \mathbb{N}^+$;

(ii) $\lim_{s\to t_k^-} \phi(s, x) = \phi(t_k, x)$, $\lim_{s\to t_k^+} \phi(s, x) = \phi(t_k^+, x)$ for all $k \in \mathbb{N}^+$ and $x \in \Omega$.

The vector-valued function $(P(t,x),T(t,x),Z(t,x)) \in \Upsilon \times \Upsilon \times \Upsilon$ is defined as the solution of system (4) if it satisfies all the equalities.

In the following, we restate some significant lemmas from [25, 26, 27] which will be useful in the proof of our main results.

Consider the impulsive logistic differential equation:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = az(b-z), & t \neq t_k, \\ z(t_k+0) = z(t_k)\lambda_k\left(z(t_k)\right), & k \in \mathbb{N}^+, \end{cases} \tag{5}$$

where $z \in \mathbb{R}^+$, a and b are positive constants, the strictly increasing sequence $\{t_k\}$ satisfies assumption (H_4). For any positive integer k, λ_k are continuous and positive-valued functions such that $\lambda_{k+p}(z) = \lambda_k(z)$ for all $z \in \mathbb{R}^+$. It can be obtained that $t_{k+1} - t_k \geq \theta = \min_{i=0,1,2,\cdots p}(t_{i+1} - t_i)$ by assumption (H_4).

Denote $A = \frac{b}{1-e^{-ab\theta}}$, $B = \max_{i=0,1,2,\cdots,p} \max_{z\in[0,A]} \lambda_k(z)$, $C = \max\{A, B\}$.

Lemma 2.1 ([25]). *Every solution $z(t) = z(t,0,z_0)$ of (5) satisfies $0 < z(t) < C$ for $t \geq 0$ and $z_0 > 0$.*

Lemma 2.2 ([26]). *Suppose that vector functions $v(t,x) = (v_1(t,x), \cdots, v_m(t,x))$ and $w(t,x) = (w_1(t,x), \cdots, w_m(t,x))$, $m \geq 1$ satisfy the following conditions:*

(i) they are of class C^2 in x, $x \in \Omega$ and of class C^1 in $(t,x) \in [a,b] \times \bar{\Omega}$, where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary;

(ii) $v_t - \mu\Delta v_t - g(t,x,v) \leq w_t - \mu\Delta w_t - g(t,x,w)$, where $(t,x) \in [a,b]\times\bar{\Omega}$ and $\mu = (\mu_1, \cdots, \mu_m) > 0$ (inequalities between vectors are satisfied coordinate-wise), vector function $g(t,x,u) = (g_1(t,x,u), \cdots, g_m(t,x,u))$ is continuously differentiable and quasi-monotonically increasing with respect to $\frac{\partial g_i(t,x,u_1,\cdots,u_m)}{\partial u_j} \geq 0$, $i,j = 1,2,\cdots m$, $i \neq j$;

(iii) $\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$, $(t,x) \in [a,b] \times \partial\Omega$.

Then $v(t,x) \leq w(t,x)$ for any $(t,x) \in [a,b] \times \bar{\Omega}$.

Lemma 2.3 ([27]). *Assume that ω and d are positive real numbers, a function $u(t,x)$ is continuous on $[0,\omega] \times \bar{\Omega}$, continuously differentiable in $x \in \bar{\Omega}$, with continuous derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial u}{\partial t}$, and $u(t,x)$ satisfies the following inequalities:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + c(t,x)u \geq 0, & (t,x) \in (0,\omega] \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & (t,x) \in (0,\omega] \times \partial\Omega, \\ u(0,x) \geq 0, & x \in \Omega, \end{cases}$$

where $c(t,x)$ is bounded on $(0,\omega]\times\Omega$. Then $u(t,x) \geq 0$ on $(0,\omega]\times\bar{\Omega}$. Moreover, $u(t,x)$ is strictly positive on $(0,\omega] \times \bar{\Omega}$ if $u(0,x) \not\equiv 0$.

Based on the above lemmas and upper-lower solution method, we can obtain the positive invariance of system (4), that is to say, the solutions of (4) always remain nonnegative.

Theorem 2.1. *Assume that* $(P_1(t), T_1(t), Z_1(t))$ *is the solution of impulsive ordinary differential system*

$$\begin{cases} \dfrac{dP(t)}{dt} = r_1(t)P(t)\left(1 - \dfrac{P(t)}{K_1(t)}\right), & t \neq t_k, \\ \dfrac{dT(t)}{dt} = r_2(t)T(t)\left(1 - \dfrac{T(t)}{K_2(t)}\right), & t \neq t_k, \\ \dfrac{dZ(t)}{dt} = \dfrac{\beta_2(t)P(t)Z(t)}{a(t)+P(t)} \\ \qquad + \dfrac{\theta_2(t)T(t)Z(t)}{b(t)+T(t)} - m(t)Z^2(t) - \rho(t)T^{\frac{2}{3}}(t)Z(t), & t \neq t_k, \\ P(t_k^+) = P(t_k)f_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots, \\ T(t_k^+) = T(t_k)g_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots, \\ Z(t_k^+) = Z(t_k)h_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \end{cases}$$

and $(P_2(t), T_2(t), Z_2(t))$ *is the solution of impulsive ordinary differential system*

$$\begin{cases} \dfrac{dP(t)}{dt} = r_1(t)P(t)\left(1 - \dfrac{P(t)}{K_1(t)}\right) \\ \qquad - \alpha_1(t)P(t)T(t) - \dfrac{\beta_1(t)P(t)Z(t)}{a(t)+P(t)}, & t \neq t_k, \\ \dfrac{dT(t)}{dt} = r_2(t)T(t)\left(1 - \dfrac{T(t)}{K_2(t)}\right) \\ \qquad - \alpha_2(t)P(t)T(t) - \dfrac{\theta_1(t)T(t)Z(t)}{b(t)+T(t)}, & t \neq t_k, \\ \dfrac{dZ(t)}{dt} = \dfrac{\beta_2(t)P(t)Z(t)}{a(t)+P(t)} \\ \qquad + \dfrac{\theta_2(t)T(t)Z(t)}{b(t)+T(t)} - m(t)Z^2(t) - \rho(t)T^{\frac{2}{3}}(t)Z(t), & t \neq t_k, \\ P(t_k^+) = P(t_k)f_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \\ T(t_k^+) = T(t_k)g_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots \\ Z(t_k^+) = Z(t_k)h_k(P(t_k), T(t_k), Z(t_k)), & k = 1, 2, \cdots. \end{cases}$$

If $0 < P_2(0) \leq P_0(x) \leq P_1(0)$, $0 < T_2(0) \leq T_0(x) \leq T_1(0)$ *and* $0 < Z_1(0) \leq Z_0(x) \leq Z_2(0)$, *then system (4) has the unique solution* $(P(t,x), T(t,x), Z(t,x))$ *and*

$$P_2(t) \leq P(t,x) \leq P_1(t), \quad T_2(t) \leq T(t,x) \leq T_1(t), \quad Z_1(t) \leq Z(t,x) \leq Z_2(t).$$

Proof. It is only need to verify that $(P_1(t), T_1(t), Z_2(t))$ and $(P_2(t), T_2(t), Z_1(t))$ are a pair of upper-lower solutions to system (4).

In fact, direct calculation can lead to $\frac{\partial P_1(t)}{\partial t} - d_1\Delta P_1(t) - r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) + \alpha_1(t)P_1(t)T_1(t) + \frac{\beta_1(t)P_1(t)Z_1(t)}{a(t)+P_1(t)} = r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) - r_1(t)P_1(t)(1 - \frac{P_1(t)}{K_1(t)}) + \alpha_1(t)P_1(t)T_1(t) + \frac{\beta_1(t)P_1(t)Z_1(t)}{a(t)+P_1(t)} \geq 0.$

Similarly, we also have

$$\begin{aligned}
&\frac{\partial T_1(t)}{\partial t} - d_2\Delta T_1(t) - r_2(t)T_1(t)\left(1 - \frac{T_1(t)}{K_2(t)}\right)\\
&+ \alpha_2(t)P_1(t)T_1(t) + \frac{\theta_1(t)T_1(t)Z_1(t)}{b(t)+T_1(t)} \geq 0,\\
&\frac{\partial P_2(t)}{\partial t} - d_1\Delta P_2(t) - r_1(t)P_2(t)\left(1 - \frac{P_2(t)}{K_1(t)}\right)\\
&+ \alpha_1(t)P_2(t)T_2(t) + \frac{\beta_1(t)P_2(t)Z_2(t)}{a(t)+P_2(t)} \leq 0,\\
&\frac{\partial T_2(t)}{\partial t} - d_2\Delta T_2(t) - r_2(t)T_2(t)\left(1 - \frac{T_2(t)}{K_2(t)}\right)\\
&+ \alpha_2(t)P_2(t)T_2(t) + \frac{\theta_1(t)T_2(t)Z_2(t)}{b(t)+T_2(t)} \leq 0,\\
&\frac{\partial Z_1(t)}{\partial t} - d_3\Delta Z_1(t) - \frac{\beta_2(t)P_1(t)Z_1(t)}{a(t)+P_1(t)}\\
&- \frac{\theta_2(t)T_1(t)Z_1(t)}{b(t)+T_1(t)} + m(t)Z_1^2(t) + \rho(t)T_1^{\frac{2}{3}}(t)Z_1(t) \leq 0,\\
&\frac{\partial Z_2(t)}{\partial t} - d_3\Delta Z_2(t) - \frac{\beta_2(t)P_2(t)Z_2(t)}{a(t)+P_2(t)}\\
&- \frac{\theta_2(t)T_2(t)Z_2(t)}{b(t)+T_2(t)} + m(t)Z_2^2(t) + \rho(t)T_2^{\frac{2}{3}}(t)Z_2(t) \geq 0.
\end{aligned}$$

It is seen that $(P_1(t), T_1(t), Z_2(t))$ and $(P_2(t), T_2(t), Z_1(t))$ are a pair of upper-lower solutions to system (4). This completes the proof.

Proposition 2.1. *System (4) is positively invariant.*

Proof. Set $P_2(t) = 0$, $T_2(t) = 0$, $Z_1(t) = 0$, $P_1(t) = \bar{P}(t)$, $T_1(t) = \bar{T}(t)$, $Z_2(t) = \bar{Z}(t)$, then $\big(\bar{P}(t), \bar{T}(t), \bar{Z}(t)\big)$ and $(0, 0, 0)$ are a pair of upper-lower solutions to system (4).

Thus, the result follows.

3. Permanence

Here, we first give the definitions of the ultimate boundedness of solutions and permanence of system (4).

Definition 3.1. *Solutions of system (4) are said to be ultimately bounded if there exist positive constants N_1, N_2 and N_3, such that for every solution $(P(t, x, P_0, T_0, Z_0)$,$T(t, x, P_0, T_0, Z_0)$,$Z(t, x, P_0, T_0, Z_0))$, there exists a moment of time $\bar{t} = \bar{t}(P_0, T_0, Z_0) > 0$ such that*

$$P(t, x, P_0, T_0, Z_0) \leq N_1, \quad T(t, x, P_0, T_0, Z_0) \leq N_2, \quad Z(t, x, P_0, T_0, Z_0) \leq N_3$$

for all $x \in \bar{\Omega}$ and $t \geq \bar{t}$.

Definition 3.2. *System* (4) *is said to be permanent if there exist positive constants* m_i *and* N_i, $i = 1, 2, 3$ *such that for every solution with non-negative initial functions that are not identically zero, there exists a moment of time* $\tilde{t} = \tilde{t}(t, x, P_0, T_0, Z_0)$ *such that*

$$\begin{aligned} m_1 \le P(t, x, P_0, T_0, Z_0) \le N_1, \\ m_2 \le T(t, x, P_0, T_0, Z_0) \le N_2, \\ m_3 \le Z(t, x, P_0, T_0, Z_0) \le N_3 \end{aligned} \tag{6}$$

for all $x \in \bar{\Omega}$ *and* $t \ge \tilde{t}$.

In the following, we first confirm the ultimate boundedness of solutions.

Theorem 3.1. *If the assumptions* (H_1)*-*(H_5) *are satisfied, and the following conditions hold:*

(i) *there exists a positive-valued function* $\eta_1(M)$ *such that* $f_k(x, P, T, Z) \le \eta_1(M_1)$ *for* $k \in \mathbb{N}^+$, $P \le M_1$, $T \ge 0$, $Z \ge 0$ *and* $x \in \Omega$*;*

(ii) *there exists a positive-valued function* $\eta_2(M)$ *such that* $g_k(x, P, T, Z) \le \eta_2(M_2)$ *for* $k \in \mathbb{N}^+$, $P \ge 0$, $T \le M_2$, $Z \ge 0$ *and* $x \in \Omega$*;*

(iii) *there exists a positive-valued function* $\eta_3(M)$ *such that* $h_k(x, P, T, Z) \le \eta_3(M_3)$ *for* $k \in \mathbb{N}^+$, $P \ge 0$, $T \ge 0$, $Z \le M_3$ *and* $x \in \Omega$.

Then all solutions of (4) *are ultimately bounded.*

Proof. Let $\bar{P}(t, x, P_0)$ be a solution of the equation

$$\frac{\partial \bar{P}}{\partial t} - d_1 \Delta \bar{P} - \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right) = 0. \tag{7}$$

From the inequality

$$\begin{aligned} 0 &= \frac{\partial P}{\partial t} - d_1 \Delta P - r_1(t, x) P \left(1 - \frac{P}{K_1(t, x)}\right) + \alpha_1(t, x) P T + \frac{\beta_1(t, x) P Z}{a(t, x) + P} \\ &\ge \frac{\partial P}{\partial t} - d_1 \Delta P - P\left(r_1^M - \frac{r_1^L}{K_1^M} P\right), \end{aligned}$$

we have

$$\begin{aligned} 0 &= \frac{\partial \bar{P}}{\partial t} - d_1 \Delta \bar{P} - \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right) \\ &\ge \frac{\partial P}{\partial t} - d_1 \Delta P - P\left(r_1^M - \frac{r_1^L}{K_1^M} P\right). \end{aligned}$$

Using Lemma 2.2, it can be concluded that $P(t, x, P_0, T_0, Z_0) \le \bar{P}(t, M_P)$, where $M_P \ge \max_{x \in \bar{\Omega}} = \|P_0(x)\|_C$. As $\bar{P}(t, M_P)$ is independent of the spatial variable x, function $\bar{P}(t, M_P)$ is the solution of ordinary differential equation

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}t} = \bar{P}\left(r_1^M - \frac{r_1^L}{K_1^M}\bar{P}\right).$$

Moreover, we also have

$$\begin{aligned}&\|P(t_k^+, x, P_0, T_0, Z_0)\|_C = \|P(t_k, x, P_0, T_0, Z_0) f_k(x, P(t_k, x, P_0, T_0, Z_0),\\ &T(t_k, x, P_0, T_0, Z_0), Z(t_k, x, P_0, T_0, Z_0))\|_C\\ &\le \bar{P}(t_k, M_P) \cdot \eta_1\left(\bar{P}(t_k, M_P)\right).\end{aligned}$$

According to Lemma 2.1, all solutions of impulsive differential equation

$$\begin{cases}\dfrac{\mathrm{d}\bar{P}}{\mathrm{d}x} = \bar{P}\left(r_1^M - \dfrac{r_1^L}{K_1^M}\bar{P}\right),\\ \bar{P}(t_k^+) = \bar{P}(t_k)\eta_1(\bar{P}(t_k))\end{cases}$$

are ultimately bounded. Then for the first equation of (4) and its corresponding impulsive condition, there exists a positive constant N_1, such that $P(t,x) \le N_1$ for $t > \bar{t}_1$.

With the same method, there exists a positive constant N_2, such that $T(t,x) \le N_2$ for $t > \bar{t}_2$; there exists a positive constant N_3, such that $Z(t,x) \le N_3$ for $t > \bar{t}_3$. This proves the theorem.

Theorem 3.2. *If the assumptions $(H_1) - (H_5)$ are satisfied, and the following conditions hold:*

(i) all solutions of (4) are ultimately bounded,that is, there exist positive constants N_i, $i = 1,2,3$ and a moment of time $\bar{t}_0$, such that $P(t,x) \le N_1$, $T(t,x) \le N_2$ and $Z(t,x) \le N_3$ for all $x \in \bar{\Omega}$ and $t > \bar{t}_0$;

(ii) the following inequalities

$$\sum_{k=1}^{p} \ln \inf_{x\in\Omega,(P,T,Z)\in S} f_k(x,P,T,Z) + \omega\left(r_1^L - \alpha_1^M N_2 - \frac{\beta_1^M N_3}{a^L}\right) > 0,$$

$$\sum_{k=1}^{p} \ln \inf_{x\in\Omega,(P,T,Z)\in S} g_k(x,P,T,Z) + \omega\left(r_2^L - \alpha_2^M N_1 - \frac{\beta_2^M N_3}{b^L}\right) > 0,$$

and

$$\sum_{k=1}^{p} \ln \inf_{x\in\Omega,(P,T,Z)\in S} h_k(x,P,T,Z) + \omega\left(\frac{\beta_2^L \sigma_1}{a^M + \sigma_1} + \frac{\theta_2^L \sigma_2}{b^M + \sigma_2} - \rho^M N_2^{\frac{2}{3}}\right) > 0$$

hold, where $S = \{(P,T,Z) : 0 < P \le N_1,\ 0 < T \le N_2\ 0 < Z \le N_3\}$, σ_1 and σ_2 are constants. Then system (4) is permanent.

Proof. Lemma 2.3 indicates that if initial functions $P_0(x)$, $T_0(x)$ and $Z_0(x)$ are nonnegative and not identically zero, then we have

$$P(t,x,P_0,T_0,Z_0) > 0, \quad T(t,x,P_0,T_0,Z_0) > 0, \quad Z(t,x,P_0,T_0,Z_0) > 0.$$

Without loss of generality, set

$$\min_{x\in\bar{\Omega}}P_0(x)=m_P>0,\quad \min_{x\in\bar{\Omega}}T_0(x)=m_T>0,\quad \min_{x\in\bar{\Omega}}Z_0(x)=m_Z>0.$$

From the inequality

$$\begin{aligned}0&=\frac{\partial P}{\partial t}-d_1\Delta P-r_1(t,x)P\left(1-\frac{P}{K_1(t,x)}\right)+\alpha_1(t,x)PT+\frac{\beta_1(t,x)PZ}{a(t,x)+P}\\&\le\frac{\partial P}{\partial t}-d_1\Delta P-P\left(r_1^L-\alpha_1^MN_2-\frac{\beta_1^MN_3}{a^L}-\frac{r_1^MP}{K_1^L}\right),\end{aligned}$$

we have

$$\begin{aligned}0&=\frac{\partial \hat{P}}{\partial t}-d_1\Delta \hat{P}-\hat{P}\left(r_1^L-\alpha_1^MN_2-\frac{\beta_1^MN_3}{a^L}-\frac{r_1^M\hat{P}}{K_1^L}\right)\\&\le\frac{\partial P}{\partial t}-d_1\Delta P-P\left(r_1^L-\alpha_1^MN_2-\frac{\beta_1^MN_3}{a^L}-\frac{r_1^MP}{K_1^L}\right).\end{aligned}$$

Afterwards, according to Lemma 2.2, we have $P(t,x,P_0,T_0,Z_0)\ge\hat{P}(t,m_P)$ for $0\le t\le t_1$. At the first impulsive moment of time t_1, it can be obtained

$$P(t_1^+,x,P_0,T_0,Z_0)\ge\hat{P}(t,m_P)\inf_{x\in\Omega,(P,T,Z)\in S}f_1(x,P,T,Z).$$

Thus, the solution of logistic differential equation with periodic impulses

$$\text{(8)}\qquad\begin{cases}\dfrac{\mathrm{d}\hat{P}}{\mathrm{d}t}=\hat{P}\left(r_1^L-\alpha_1^MN_2-\dfrac{\beta_1^MN_3}{a^L}-\dfrac{r_1^M\hat{P}}{K_1^L}\right),\\ \hat{P}(t_k^+)=\hat{P}(t_k)\inf\limits_{x\in\Omega,(P,T,Z)\in S}f_1(x,P,T,Z)\end{cases}$$

is a lower bound of aforementioned solution $P(t,x,P_0,T_0,Z_0)$.

By using the Theorem 2.1 in [28], equation (8) has the unique piece-wise continuous and strictly positive periodic solution $\hat{P}^*(t)$ such that each solution of (8) has the property $\lim_{t\to\infty}\hat{P}(t,m_P)=\hat{P}^*(t)$. Therefore, there exists a positive constant σ_1 such that for every solution $\hat{P}(t,m_P)$ of (8) we have $\hat{P}(t,m_P)\ge\sigma_1$ when $t>\hat{t}_1=\hat{t}_1(m_P)>0$. And because that $\hat{P}(t,m_P)$ is a lower bound of $P(t,x,P_0,T_0,Z_0)$, we can get $P(t,x,P_0,T_0,Z_0)\ge\sigma_1$ when $t>\hat{t}_1$.

Using the same procedures, there exists a positive constant σ_2 such that $T(t,x,P_0,T_0,Z_0)\ge\sigma_2$ when $t>\hat{t}_2$.

At last we consider the third equation of (4) and have

$$\begin{aligned}0&=\frac{\partial Z}{\partial t}-d_3\Delta Z-Z\left[\frac{\beta_2(t,x)P}{a(t,x)+P}+\frac{\theta_2(t,x)T}{b(t,x)+T}-m(t,x)Z-\rho(t,x)T^{\frac{2}{3}}\right]\\&\le\frac{\partial Z}{\partial t}-d_3\Delta Z-Z\left[\frac{\beta_2^L\sigma_1}{a^M+\sigma_1}+\frac{\theta_2^L\sigma_2}{b^M+\sigma_2}-\rho^MN_2^{\frac{2}{3}}-m^MZ\right].\end{aligned}$$

Repeating the previous steps will result that there exists a positive constant σ_3 such that $Z(t,x,P_0,T_0,Z_0)\ge\sigma_3$ when $t>\hat{t}_3$. Thus Theorem 3.2 holds.

4. The existence and uniqueness of periodic solutions

In this section, we mainly establish the existence and uniqueness of asymptotically stable solution for system (4) by constructing suitable auxiliary function and using inequality techniques.

Theorem 4.1. *If the assumptions* $(H_1)-(H_5)$ *are satisfied and system (4) is permanent, the following inequality*

$$\sum_{k=1}^{p} \ln L_k + \omega\lambda_M < 0$$

holds, where

$$L_k = \max_{P,T,Z\in\Xi,x\in\Omega} 2\left\{(f_k)^2+(g_k)^2+(h_k)^2+\left(N\frac{\partial f_k}{\partial P}\right)^2+\left(N\frac{\partial f_k}{\partial T}\right)^2+\left(N\frac{\partial f_k}{\partial Z}\right)^2 \right.$$
$$\left.+\left(N\frac{\partial g_k}{\partial P}\right)^2+\left(N\frac{\partial g_k}{\partial T}\right)^2+\left(N\frac{\partial g_k}{\partial Z}\right)^2+\left(N\frac{\partial h_k}{\partial P}\right)^2+\left(N\frac{\partial h_k}{\partial T}\right)^2+\left(N\frac{\partial h_k}{\partial Z}\right)^2\right\},$$

λ_M *is the maximal eigenvalue of the matrix*

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix},$$

where

$$E_{11} = 2\left[r_1^M - 2\sigma\frac{r_1^L}{K_1^M} - \alpha_1^L\sigma - \frac{a^L\beta_1^L\sigma}{\left(a^M+N\right)^2}\right],$$
$$E_{22} = 2\left[r_2^M - 2\sigma\frac{r_2^L}{K_2^M} - \alpha_2^L\sigma - \frac{b^L\theta_1^L\sigma}{\left(b^M+N\right)^2}\right],$$
$$E_{33} = 2\left[\frac{\beta_2^M N(a^M+N)}{\left(a^L+\sigma\right)^2} + \frac{\theta_2^M N(b^M+N)}{\left(b^L+\sigma\right)^2} - \rho^L\sigma^{\frac{2}{3}}\right],$$
$$E_{12} = E_{21} = N\left(\alpha_1^M+\alpha_2^M\right),$$
$$E_{13} = E_{31} = \frac{N\left(a^M\beta_1^M+\beta_1^M N+a^M\beta_2^M\right)}{\left(a^L+\sigma\right)^2},$$
$$E_{23} = E_{32} = \frac{N\left(b^M\theta_1^M+\theta_1^M N+b^M\theta_2^M\right)}{\left(b^L+\sigma\right)^2}.$$

Then system (4) has the unique globally asymptotically stable and strictly positive piecewise continuous $\omega-$*periodic solution.*

Proof. Due to the permanence of system (4), there exist positive constants N, σ, and a moment of time $\bar{t}$ such that every solution of (4) satisfies

$$(P(t,x),T(t,x),Z(t,x)) \in \Xi = \{(P,T,Z):\ \sigma \le P \le N, \sigma \le T \le N, \sigma \le Z \le N\}$$

for any $t > \bar{t}$ and with nonnegative initial conditions not identically equal to zero.

Assuming that $(P(t,x),T(t,x),Z(t,x))$ and $\big(\bar{P}(t,x),\bar{T}(t,x),\bar{Z}(t,x)\big)$ are two solutions of system (4), whose upper and lower bounds are σ and N respectively.

Construct the function

$$V(t) = \int_\Omega \big[P(t,x) - \bar{P}(t,x)\big]^2 \mathrm{d}x + \int_\Omega \big[T(t,x) - \bar{T}(t,x)\big]^2 \mathrm{d}x + \int_\Omega \big[Z(t,x) - \bar{Z}(t,x)\big]^2 \mathrm{d}x.$$

Calculating the derivative of $V(t)$ with respect to t leads to

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\int_\Omega \left(P-\bar{P}\right)\left(\frac{\partial P}{\partial t} - \frac{\partial \bar{P}}{\partial t}\right)\mathrm{d}x + 2\int_\Omega \left(T-\bar{T}\right)\left(\frac{\partial T}{\partial t} - \frac{\partial \bar{T}}{\partial t}\right)\mathrm{d}x \\
&\quad + 2\int_\Omega \left(Z-\bar{Z}\right)\left(\frac{\partial Z}{\partial t} - \frac{\partial \bar{Z}}{\partial t}\right)\mathrm{d}x \\
&\triangleq I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= 2d_1\int_\Omega \left(P-\bar{P}\right)\Delta\left(P-\bar{P}\right)\mathrm{d}x + 2d_2\int_\Omega \left(T-\bar{T}\right)\Delta\left(T-\bar{T}\right)\mathrm{d}x \\
&\quad + 2d_3\int_\Omega \left(Z-\bar{Z}\right)\Delta\left(Z-\bar{Z}\right)\mathrm{d}x \\
&= -2d_1\int_\Omega \nabla^2\left(P-\bar{P}\right)\mathrm{d}x - 2d_2\int_\Omega \nabla^2\left(T-\bar{T}\right)\mathrm{d}x - 2d_3\int_\Omega \nabla^2\left(Z-\bar{Z}\right)\mathrm{d}x \\
&\le -2d_1\int_\Omega \left|\nabla\left(P-\bar{P}\right)\right|^2\mathrm{d}x - 2d_2\int_\Omega \left|\nabla\left(T-\bar{T}\right)\right|^2\mathrm{d}x - 2d_3\int_\Omega \left|\nabla\left(Z-\bar{Z}\right)\right|^2\mathrm{d}x \\
&\le 0,
\end{aligned}$$

$$\begin{aligned}
I_2 &= 2\int_\Omega (P-\bar{P})\left[r_1P\left(1-\frac{P}{K_1}\right) - \alpha_1PT - \frac{\beta_1PZ}{a+P}\right. \\
&\quad \left. -r_1\bar{P}\left(1-\frac{\bar{P}}{K_2}\right) + \alpha_1\bar{P}\bar{T} + \frac{\beta_1\bar{P}\bar{Z}}{a+\bar{P}}\right]\mathrm{d}x \\
&= 2\int_\Omega (P-\bar{P})\left[r_1(P-\bar{P}) - \frac{r_1}{K_1}(P^2-\bar{P}^2) - \alpha_1PT + \alpha_1\bar{P}T + \alpha_1\bar{P}T + \alpha_1\bar{P}\bar{T}\right. \\
&\quad \left. + \frac{a\beta_1\bar{P}\bar{Z} + \beta_1P\bar{P}\bar{Z} - a\beta_1PZ - \beta_1P\bar{P}Z}{(a+P)(a+\bar{P})}\right]\mathrm{d}x \\
&= 2\int_\Omega \left[r_1(P-\bar{P})^2 - \frac{r_1}{K_1}(P+\bar{P})(P-\bar{P}) - \alpha_1T(P-\bar{P})^2 - \alpha_1\bar{P}(P-\bar{P})(T-\bar{T})\right. \\
&\quad \left. - \frac{a\beta_1(P-\bar{P})\left[\bar{Z}(P-\bar{P}) + P(Z-\bar{Z})\right] + \beta_1P\bar{P}(P-\bar{P})(Z-\bar{Z})}{(a+P)(a+\bar{P})}\right]\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
I_3 &= 2\int_\Omega (T-\bar{T})\left[r_2 T\left(1-\frac{T}{K_2}\right)-\alpha_2 PT-\frac{\theta_1 TZ}{b+T}\right.\\
&\quad \left.-r_2\bar{T}\left(1-\frac{\bar{T}}{K_2}\right)+\alpha_2\bar{P}\bar{T}+\frac{\theta_1\bar{T}\bar{Z}}{b+\bar{T}}\right]\mathrm{d}x\\
&= 2\int_\Omega \left[r_2(T-\bar{T})^2-\frac{r_2}{K_2}(T+\bar{T})(T-\bar{T})^2-\alpha_2 P(T-\bar{T})^2-\alpha_2\bar{T}(P-\bar{P})(T-\bar{T})\right.\\
&\quad \left.-\frac{b\theta_1(T-\bar{T})\left[\bar{Z}(T-\bar{T})+T(Z-\bar{Z})\right]+\theta_1 T\bar{T}(T-\bar{T})(Z-\bar{Z})}{(b+T)(b+\bar{T})}\right]\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= 2\int_\Omega (Z-\bar{Z})\left[\frac{\beta_2 PZ}{a+P}+\frac{\theta_2 TZ}{b+T}-mZ^2-\rho T^{\frac{2}{3}}Z\right.\\
&\quad \left.-\frac{\beta_2\bar{P}\bar{Z}}{a+\bar{P}}-\frac{\theta_2\bar{T}\bar{Z}}{b+\bar{T}}+m\bar{Z}^2+\rho\bar{T}^{\frac{2}{3}}\bar{Z}\right]\mathrm{d}x\\
&= 2\int_\Omega \left[\frac{a\beta_2(Z-\bar{Z})\left[\bar{Z}(P-\bar{P})+P(Z-\bar{Z})\right]+\beta_2 P\bar{P}(Z-\bar{Z})^2}{(a+P)(a+\bar{P})}-\rho T^{\frac{2}{3}}(Z-\bar{Z})^2\right.\\
&\quad \left.-\rho\bar{Z}(Z-\bar{Z})\left(T^{\frac{2}{3}}-\bar{T}^{\frac{2}{3}}\right)+\frac{b\theta_2(Z-\bar{Z})\left[\bar{Z}(T-\bar{T})+T(Z-\bar{Z})\right]+\theta_2 T\bar{T}(Z-\bar{Z})^2}{(b+T)(b+\bar{T})}\right]\mathrm{d}x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_2+I_3+I_4 \le\ & 2\int_\Omega (P-\bar{P})^2\left[r_1^M-2\sigma\frac{r_1^L}{K_1^M}-\alpha_1^L\sigma-\frac{a^L\beta_1^L\sigma}{(a^M+N)^2}\right]\mathrm{d}x\\
&+2\int_\Omega (T-\bar{T})^2\left[r_2^M-2\sigma\frac{r_2^L}{K_2^M}-\alpha_2^L\sigma-\frac{b^L\theta_1^L\sigma}{(b^M+N)^2}\right]\mathrm{d}x\\
&+2\int_\Omega (Z-\bar{Z})^2\left[\frac{\beta_2^M N(a^M+N)}{(a^L+\sigma)^2}+\frac{\theta_2^M N(b^M+N)}{(b^L+\sigma)^2}-\rho^L\sigma^{\frac{2}{3}}\right]\mathrm{d}x\\
&+2\int_\Omega \left|(P-\bar{P})(T-\bar{T})\right|\left[N\left(\alpha_1^M+\alpha_2^M\right)\right]\mathrm{d}x\\
&+2\int_\Omega \left|(P-\bar{P})(Z-\bar{Z})\right|\frac{N\left(a^M\beta_1^M+\beta_1^M N+a^M\beta_2^M\right)}{(a^L+\sigma)^2}\mathrm{d}x\\
&+2\int_\Omega \left|(T-\bar{T})(Z-\bar{Z})\right|\frac{N\left(b^M\theta_1^M+\theta_1^M N+b^M\theta_2^M\right)}{(b^L+\sigma)^2}\mathrm{d}x.
\end{aligned}
$$

From the conditions of this theorem, we have

$$
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le\lambda_M\int_\Omega\left[(P-\bar{P})^2+(T-\bar{T})^2+(Z-\bar{Z})^2\right]\mathrm{d}x.
$$

That is $\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le \lambda_M V(t)$, and $V(t_{k+1}) \le V(t_k)\exp\{\lambda_M(t_{k+1}-t_k)\}$. Thus,

$$\begin{aligned} V(t_{k+1}^+) \le & \int_\Omega \left[Pf_k(P,T,Z) - \bar{P}f_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & + \int_\Omega \left[Tg_k(P,T,Z) - \bar{T}g_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & + \int_\Omega \left[Zh_k(P,T,Z) - \bar{Z}h_k(\bar{P},\bar{T},\bar{Z})\right]^2 \mathrm{d}x \\ & \le L_{k+1}V(t_{k+1}) \le L_{k+1}\exp\left\{\lambda_M(t_{k+1}-t_k)\right\} V(t_k^+). \end{aligned}$$

Next, we estimate the variation of function $V(t)$ over one period. From

$$V(t+\omega) \le L^*V(t) = \prod_{k=1}^{p} L_k \exp\left\{\lambda_M \omega\right\} V(t),$$

then we have

$$L^* = \prod_{k=1}^{p} L_k \exp\left\{\lambda_M \omega\right\} < 1.$$

Therefore, $V(n\omega + s) \le L^*V(s) \to 0$ when $n \to \infty$, and

$$\lim_{t\to\infty} \left\|P(t,x) - \bar{P}(t,x)\right\| = \lim_{t\to\infty} \left\|T(t,x) - \bar{T}(t,x)\right\| = \lim_{t\to\infty} \left\|Z(t,x) - \bar{Z}(t,x)\right\| = 0,$$

where $\|\cdot\|$ is the norm of space $L_2(\Omega)$.

With the help of Theorem 9 in [25], the solutions of (4) are bounded in the space $C^{1+\nu}$. We have

$$\begin{aligned} & \lim_{t\to\infty} \sup_{x\in\Omega} \left\|P(t,x) - \bar{P}(t,x)\right\| \\ & = \lim_{t\to\infty} \sup_{x\in\Omega} \left\|T(t,x) - \bar{T}(t,x)\right\| = \lim_{t\to\infty} \sup_{x\in\Omega} \left\|Z(t,x) - \bar{Z}(t,x)\right\| = 0. \end{aligned}$$

Consider the sequence $(P(k\omega, x, P_0, T_0, Z_0), T(k\omega, x, P_0, T_0, Z_0), Z(k\omega, x, P_0, T_0, Z_0)) = w(k\omega, w_0), k \in \mathbb{N}^+$, again by the Theorem 9 in [25], this sequence is compact in the space $C(\bar{\Omega}) \times C(\bar{\Omega}) \times C(\bar{\Omega})$. Let $\bar{w}$ be a limit point of this sequence, $\lim_{n\to\infty} w(k_n\omega, w_0) = \bar{w}$, where $\lim_{n\to\infty} k_n = \infty$. Then $w(\omega, \bar{w}) = \bar{w}$. In fact, from $w(\omega, w(k_n\omega, w_0)) = w(k_n\omega, w(\omega, w_0))$ and $\lim_{n\to\infty}[w(k_n\omega, w(\omega, w_0)) - w(k_n\omega, w_0)] = 0$, we have

$$\begin{aligned} & \|w(\omega,\bar{w}) - \bar{w}\|_C \le \|w(\omega,\bar{w}) - w(\omega, w(k_n\omega, w_0))\|_C \\ & + \|w(\omega, w(k_n\omega, w_0)) - w(k_n\omega, w_0)\|_C \\ & + \|w(k_n\omega, w_0) - \bar{w}\|_C \to 0, \quad (n \to \infty). \end{aligned}$$

We claim that sequence $w(k\omega, w_0)$ has the unique limit point for all $k \in \mathbb{N}^+$. And if not, it is assumed that this sequence has two different limit points:

$$\lim_{n\to\infty} w(k_n\omega, w_0) = \bar{w}, \quad \lim_{n\to\infty} w(k_n\omega, w_0) = \tilde{w}.$$

The inequality $\|\bar{w} - \tilde{w}\|_C \le \|\bar{w} - w(k_n\omega, w_0)\|_C + \|w(k_n\omega, w_0 - \tilde{w}\|_C \to 0$ can deduce $\|\bar{w} - \tilde{w}\|_C$=0. Hence, $\bar{w} = \tilde{w}$. In conclusion, system (4) has a unique periodic solution and it is globally asymptotically stable. The proof is completed.

5. Conclusions

In this paper, we have proposed a reaction-diffusion plankton model with periodic impulses. By using the basic theory of impulsive differential equation, comparison principle, upper-lower solution method and inequality techniques, the sufficient conditions for ultimate boundedness of solutions and permanence of the system have been established. Theorem 4.1 manifests that the periodic solution is unique and globally asymptotically stable if the system is permanent. It is not difficult to find that, in real plankton ecosystem, various population numbers may vary periodically due to the comprehensive influence of seasonal variation and biological rhythm.

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The structure of a class of inverse residuated lattices

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Abstract. In this paper, we study some special inverse residuated lattices, namely, E-unitary inverse residuated chains. After giving some properties of such residuated lattices, we obtain a structure theorem for E-unitary inverse residuated chains.
Keywords: residuated lattice, $E-$unitary inverse semigroup, chain.

1. Introduction

Let $(\mathfrak{P}, \leq)$ be a poset. A (binary) operation $\circ$ is called *residuated* if there exist (binary) operations $\backslash$ and $/$ on $\mathfrak{P}$ such that

$$(\forall x, y \in \mathfrak{P})\ x \circ y \leq z \Longleftrightarrow y \leq x \backslash z \Longleftrightarrow x \leq z / y.$$

In this case, the operation pair $(\backslash, /)$ is called a *residuals* of the operation $\circ$. It is well known that an operation $\circ$ on the poset $(\mathfrak{P}, \leq)$ is residuated if and only if $\circ$ is order preserving in each argument and such that for all $a, b \in \mathfrak{P}$, both $\{p \in \mathfrak{P} \mid a \circ p \leq b\}$ and $\{q \in \mathfrak{P} \mid q \circ a \leq b\}$ contain the greatest element(denote by $a \backslash b$ and b/a, respectively). A *residuated lattice* is defined as an algebra $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$ satisfying the following conditions:

$(RL1)$ $(\mathfrak{L}, \wedge, \vee)$ is a lattice;
$(RL2)$ $(\mathfrak{L}, \circ)$ is a monoid with identity e; and
$(RL3)$ the operation pair $(\backslash, /)$ is a residuals of the operation $\circ$.

In this case, we call the lattice $(\mathfrak{L}, \wedge, \vee)$ and the semigroup $(\mathfrak{L}, \circ)$ the *lattice reduct* and the *semigroup reduct* of the residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$, respectively. Some time, residuated lattices are also called *residuated lattice-ordered monoids*. A residuated lattice is called an *inverse residuated lattice* if its semigroup reduct is an inverse semigroup. An inverse residuated lattice is called an *$E-$unitary inverse residuated lattice* if its semigroup reduct is an $E-$unitary inverse semigroup. An $E-$unitary inverse residuated lattice is called an *$E-$unitary inverse residuated chain* if its lattice reduct is a chain.

Motivated by the ideal lattices of rings with identity, residuated lattices are initially introduced by Ward and Dilworth in the late of 1930's [21]. Since that time, there has been substantial research regarding some specific classes

of residuated structures(see[4-16]). Recently, some authors investigated various types of residuated lattices from different perspectives . For example, Kühr in [10] considered the $\{\backslash, /, e\}$-subreducts of integral residuated lattices as Pseudo-BCK-algebras and used Pseudo-BCK-algebra theory to investigate integral residuated lattices. Residuated lattices can be considered as a class of partial ordered semigroups. We investigate idempotent residuated lattices from semigroup perspectives and obtain structure theorems and decomposition theorem for two class of idempotent residuated lattices in [5,6,7], which generalize [9, Theorem 20].

In the present paper we will use algebra theory of partial ordered semigroups to investigate E-unitary inverse residuated chains. In algebra theory of semigroups, E-unitary inverse semigroups are of the important kinds which are situated between groups and general semigroups. McAlister in [3] investigated E-unitary inverse semigroups. He obtained a structure theorem for E-unitary inverse semigroups. Saitô studied totally ordered E-unitary inverse semigroups in [2]. He described a structure theorem for totally ordered E-unitary inverse semigroups. In [1], Gomes, Giraldes and McAlister simplified Saitô's structure theorem for totally ordered E-unitary inverse semigroups by using McAlister's structure theorem for E-unitary inverse semigroups. We consider E-unitary inverse residuated chains as a class of totally ordered E-unitary inverse semigroups. The following question naturally arises: How can we characterize the structure of E-unitary inverse residuated chains? The purpose of this paper is to solve this question.

We proceed as follows: Section 2 presents some necessary notation and known results. In Section 3, we give some fundamental properties of E-unitary inverse residuated chains. In Section 4, we use the above results to give a solution to the question. We establish a structure theorem for E-unitary inverse residuated chains.

2. Preliminaries

In what follows, we shall use the notion and notation in [1,7]. Other undefined terms can be found in [17,18]. First we recall some of the basic facts about totally ordered E-unitary inverse semigroups.

An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. A semigroup S is called *regular* if all its elements are regular. A semigroup S is called *inverse* if S is regular and its idempotents commute. For brevity, we use E(S) to denote the set of idempotents of S. An inverse semigroup is called *E-unitary* if for all $e \in E(S)$ and $s \in S$, $es \in E(S)$ implies $s \in E(S)$. An inverse semigroup S is said to be a *partially ordered semigroup*, or to be *partially ordered*, if it admits a compatible partial ordering $\leq$; that is, $\leq$ is a partial order on the set S and for each $a, b \in S$, $a \leq b$ implies $xay \leq xby$ for all $x, y \in S^1$. A partially ordered inverse semigroup S is said to be *totally ordered*

or a *totally ordered inverse semigroup* if the imposed partial order is a total order; that is, if a, b are distinct elements of S then either $a < b$ or $b < a$.

A partially ordered set is said to be a *chain* if it is a totally ordered set. A semilattice S is called *tree* if for each $x \in S$, the principal ideal generated by x is a chain.

Suppose now that S is an E-unitary inverse semigroup and let $G = S/\sigma$ be its maximal group homomorphic image. Then S is isomorphic to a $P-$semigroup $P(G, \mathscr{X}, \mathscr{Y})$, where $(G, \mathscr{X}, \mathscr{Y})$ is a *McAlister triple* in the sense that $\mathscr{X}$ is a down directed partially ordered set, $\mathscr{Y}$ is a subsemilattice and order ideal of $\mathscr{X}$, and G acts, on the left, on $\mathscr{X}$, by order automorphisms, in such a way that $\mathscr{X} = G\mathscr{Y}$ and for $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$. Moreover, $\mathscr{Y}$ is isomorphic to the semilattice of idempotents of S. The elements of $P(G, \mathscr{X}, \mathscr{Y})$ are the pairs $(a, g) \in \mathscr{Y} \times G$ for which $g^{-1}a \in \mathscr{Y}$, and multiplication is defined by $(a, g)(b, h) = (a + gb, gh)$ where $+$ denotes greatest lower bound (when it exists) in the partially ordered set $\mathscr{Y}$. We always use $+$ for this operation to avoid confusion with $\vee$ and $\wedge$ when the semigroup in question is a totally ordered semigroup and use $\leq^*$ to denote the partial order on the set $\mathscr{X}$ following our convention of reserving $\leq$ for the imposed partial order. Saitô [20] has shown that in totally ordered E-unitary inverse semigroups the semilattice of idempotents must be a totally ordered semilattice and such a semilattice must be a binary tree. That is, it is a semilattice in which each principal ideal is a chain and in which there do not exist distinct incomparable elements a, b, c with $a \wedge b = b \wedge c = c \wedge a$.

We shall write $x \prec y (x \prec^* y)$ for the fact that x is covered by y; i.e., $x < y (x <^* y)$ and for every z, if $x \leq z \leq y (x \leq^* z \leq^* y)$, then $z = x$ or $z = y$. By Theorem 5.9.2 of [18] and Corollary 2.8 of [1], we have:

Theorem 2.1 ([1]). *Let G be a totally ordered group, $\mathscr{X}$ a tree, and $\mathscr{Y}$ an ideal of $\mathscr{X}$ which is a totally ordered binary tree. Suppose, further, that G acts by order automorphisms on $\mathscr{X}$ in such a way that $\mathscr{X} = G\mathscr{Y}$, for $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$ and if $a, b \in \mathscr{Y}, g \in G$ with $ga, gb \in \mathscr{Y}$ then $ga \leq gb$ if and only if $a \leq b$. Then, under the lexicographic ordering,*

$$(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b$$

$P(G, \mathscr{X}, \mathscr{Y})$ is a totally ordered E-unitary inverse semigroup. Conversely, any totally ordered E-unitary inverse semigroup is isomorphic to one of this form.

By a *lattice-ordered monoid*, we mean an ordered monoid $(M, \circ, \leq)$ in which the ordered set $(M, \leq)$ is still a lattice. Moreover, a lattice-ordered monoid is called *idempotent* if the monoid $(M, \circ)$ is a band. For the detail information on ordered semigroups, the reader is referred to references [16] and [18].

For our purpose, we need the following lemma.

Lemma 2.2 ([4]). *Let $(M, \circ, \leq)$ be a lattice-ordered idempotent monoid with identity e, and $a, b \in M$:*

(1) $a \wedge b \leq ab \leq a \vee b$.
(2) *If* $a, b \geq e$, *then* $ab = a \vee b$.
(3) *If* $a, b \leq e$, *then* $ab = a \wedge b$.
(4) *If* $a \leq e \leq ab$, *then* $ab = b$.
(5) *If* $ab \leq e \leq a$, *then* $ab = b$.

3. Fundamental properties

In this section, we will give some fundamental properties of E-unitary inverse residuated chains. We assume $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ is an E-unitary inverse residuated chain and $(e, 1)$ is an identity of $\mathfrak{L}$. By the proof of [17, Theorem 5.9.2], e is an identity of $\mathscr{Y}$ and 1 is an identity of G. Since $\mathfrak{L}$ is a totally ordered E-unitary inverse semigroup, by Theorem 2.1, $\mathscr{Y}$ is a binary tree and so $\mathscr{Y}$ is a chain with greatest element e with respect to $\leq^*$ by noting that e is an identity of $\mathscr{Y}$. We define $\mathscr{Y}^- = \{a \in \mathscr{Y} | a \leq e\}$, $\widetilde{\mathscr{Y}} = \{a \in \mathscr{Y} | (\exists b \in \mathscr{Y}^-) b \leq^* a\}$ and $A(\mathfrak{L}) = \{(a, 1) \in \mathfrak{L} | a \in \widetilde{\mathscr{Y}}\}$. We always denote by $\top$ the maximum element of $\mathscr{Y}$ with respect to the imposed ordering $\leq$ if it exists. We have the following results:

Proposition 3.1 ([8]). *Let* $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ *be an* E*-unitary inverse residuated chain. Then the following properties hold:*

(1) $\boldsymbol{A}(\mathfrak{L})$ *is a subalgebra of* $\mathfrak{L}$*;*

(2) *If* $\mathscr{Y} \backslash \widetilde{\mathscr{Y}} \neq \emptyset$, *then for every* $g \in G$, *there exists* $\bar{g} \in G$ *such that* $\bar{g} \prec g$ *and* $\top$ *exists; moreover, in this case,* $\top \leq^* a$ *for all* $a \in \mathscr{Y}$, $g\top = \top$ *for all* $g \in G$ *and* h *is central in* G *where* $h = \bar{1} \prec 1$;

(3) *If* $a \in \widetilde{\mathscr{Y}}$ *such that* $a > e$, *then there exists a unique element* $b \in \mathscr{Y}$ *such that* $b <^* a, b < e$ *and if* $c \in \mathscr{Y}$ *such that* $b <^* c <^* a$, *then* $c > e$*;*

(4) *If* $b \in \mathscr{Y}$ *such that* $b < e$, *then there exists a unique element* $a \in \mathscr{Y}$ *such that* $b <^* a, a \geq e$ *and if* $c \in \mathscr{Y}$ *such that* $b <^* c <^* a$, *then* $c < e$*;*

(5) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $a > b$, $(c, k) = (a, g) \backslash (b, h)$, *then* $c = \top$ *and* $k \prec g^{-1}h$*;*

(6) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $a \in \widetilde{\mathscr{Y}}$ *or* $a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $a \leq b$, $(c, k) = (a, g) \backslash (b, h)$, *then* $k = g^{-1}h$*;*

(7) *If* $(a, g), (b, h) \in \mathfrak{L}$ *such that* $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $hg^{-1}a > b$ *or* $hg^{-1}a \notin \mathscr{Y}$, $gh^{-1}b \notin \mathscr{Y}$ *and* $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$, $(d, l) = (b, h)/(a, g)$, *then* $d = \top$ *and* $l \prec hg^{-1}$*;*

(8) *If* $(a, g), (b, h) \in \mathfrak{L}$ *satisfying one of the following conditions:*

(a) $hg^{-1}a \notin \mathscr{Y}$, $gh^{-1}b \notin \mathscr{Y}$ *and* $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$;

(b) $hg^{-1}a \notin \mathscr{Y}$ *and* $gh^{-1}b \in \mathscr{Y}$;

(c) $hg^{-1}a \in \widetilde{\mathscr{Y}}$;

(d) $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ *and* $hg^{-1}a \leq b$*; and* $(d, l) = (b, h)/(a, g)$, *then* $l = hg^{-1}$.

Let $\mathfrak{L}$ be an inverse semigroup. The natural ordering $\leq_n$ on $\mathfrak{L}$ is given by for all $a, b \in \mathfrak{L}$, $a \leq_n b$ if and only if there exists $e \in E(\mathfrak{L})$ such that $a = eb$. Let

σ be the minimum group congruence on $\mathfrak{L}$. $\mathfrak{L}$ is called $F-$*inverse semigroup* if every $\sigma-$class contains a greatest element for $\leq_n$. By Exercise 5.11.36 of [18], an $F-$inverse semigroup contains an identity element and is $E-$unitary.

Proposition 3.2. *Let $\mathfrak{L}$ be a totally ordered inverse monoid. Then $\mathfrak{L}$ is an E-unitary semigroup if and only if $\mathfrak{L}$ is an F-inverse semigroup.*

Proof. We need only to verify the sufficiency because the necessity is obvious. Now assume $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ is a totally ordered E-unitary semigroup with an identity. Then by Theorem 2.1, $\mathscr{Y}$ is a principal ideal of $\mathscr{X}$ and $\mathscr{X}$ is a tree, hence by Exercise 5.38 of [18], $\mathfrak{L}$ is an F-inverse semigroup. □

Proposition 3.3. *Let $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ be a totally ordered E-unitary semigroup with an identity $(e, 1)$.*

(1) *For all $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$.*

(2) *If $\mathfrak{L}$ is a residuated lattice and $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$, then $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$.*

Proof. (1) Since $\mathscr{X}$ is a tree and $\mathscr{Y}$ is a principal ideal of $\mathfrak{L}$, $e+g^{-1}e, ge+e \in \mathscr{Y}$. Then $ge+e = g(e+g^{-1}e) \in g\mathscr{Y} \cap \mathscr{Y}$. Let $b \in g\mathscr{Y} \cap \mathscr{Y}$. Then there exists $c \in \mathscr{Y}$ such that $b = gc \in \mathscr{Y}$. Since $c \leq^* e$, $b = gc \leq^* ge$. Hence $b \leq^* ge + e$. Let $d \leq^* e + ge$. Since $e + g^{-1}e \in \mathscr{Y}$, $g^{-1}d \in \mathscr{Y}$. Then $d = g(g^{-1}d) \in g\mathscr{Y}$ and so $d \in g\mathscr{Y} \cap \mathscr{Y}$. Thus $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$.

(2) Suppose that $\mathfrak{L}$ is a residuated lattice and $g \in G$, $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$. If $e + ge \leq e$, then by (1), $g\mathscr{Y} \cap \mathscr{Y}$ is a principal ideal of $\mathscr{Y}$ with the greatest element $e + ge$ with respect to $\leq^*$, hence $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. If $e + ge > e$, we can claim that $e + ge \in \widetilde{\mathscr{Y}}$. Otherwise, if $e + ge \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, then for $b \in \mathscr{Y}$ such that $b <^* e + ge, b > e$, which implies $g\mathscr{Y} \cap \mathscr{Y}^- = \emptyset$, contrary to the hypothesis. Consequently, $e + ge \in \widetilde{\mathscr{Y}}$. By Proposition 3.1 (3), there exists a unique element $b \in \mathscr{Y}^-$ such that b is the maximum element in $g\mathscr{Y} \cap \mathscr{Y}^-$ for $\leq^*$. This means that $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. □

4. Construction

In this section, we will show how to construct E-unitary inverse residuated chains from some given specific ingredients.

Let $(G, \mathscr{X}, \mathscr{Y})$ be a McAlister triple such that G is a totally ordered group, $\mathscr{X}$ is a tree and $\mathscr{Y}$ is a principal ideal of $\mathscr{X}$ which is a totally ordered semilattice with an identity e. put $\mathfrak{L} = \{(a, g) \in \mathscr{Y} \times G | g^{-1}a \in \mathscr{Y}\}$. Let $\mathscr{Y}^- = \{a \in \mathscr{Y} | a \leq e\}$, $\widetilde{\mathscr{Y}} = \{a \in \mathscr{Y} | (\exists b \in \mathscr{Y}^-) b \leq^* a\}$ and $\widetilde{\mathscr{Y}}^+ = \{a \in \widetilde{\mathscr{Y}} | a \geq e\}$. Given two mappings

$$\varphi : \widetilde{\mathscr{Y}}^+ \backslash \{e\} \to \mathscr{Y}^-; \quad a \mapsto \varphi(a)$$

and

$$\psi : \mathscr{Y}^- \backslash \{e\} \to \widetilde{\mathscr{Y}}^+; \quad a \mapsto \psi(a).$$

We call $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ an $EIRC$-system if the following conditions hold:

$(E1)$ If $\mathscr{Y}\backslash\widetilde{\mathscr{Y}} \neq \emptyset$, then the greatest element $\top$ of $\mathscr{Y}$ exists with respect to $\leq$ and for every $g \in G$, there exists $\bar{g} \in G$ such that $\bar{g} \prec g$.

$(E2)$ For every $a \in Dom\varphi$, $\varphi(a) <^* a$.

$(E3)$ For every $a \in Dom\psi$, $a <^* \psi(a)$.

$(E4)$ If $a \in Dom\varphi, b \in \mathscr{Y}$ such that $\varphi(a) <^* b <^* a$, then $b > e$.

$(E5)$ If $a \in Dom\psi, b \in \mathscr{Y}$ such that $a <^* b <^* \psi(a)$, then $b < e$.

$(E6)$ If $a, b \in \mathscr{Y}, g \in G$ with $ga, gb \in \mathscr{Y}$, then $ga \leq gb$ if and only if $a \leq b$.

$(E7)$ If $g \in G$ such that $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$, then $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$.

Give an $EIRC-$system $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. We define a multiplication on $\mathfrak{L}$ by the rule that

$$(a, g) \circ (b, h) = (a + gb, gh).$$

By the proof of Theorem 5.9.2 of [18], we have the following result:

Lemma 4.1. *If $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC-$system, then $(\mathfrak{L}, \circ)$ is an E-unitary inverse semigroup with an identity $(e, 1)$.*

We define the lexicographic ordering on $\mathfrak{L}$ by

$$(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b.$$

By Theorem 2.1, we have the following lemma:

Lemma 4.2. *If $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC-$system, then $(\mathfrak{L}, \circ, \leq)$ is a totally ordered E-unitary inverse semigroup.*

Give an $EIRC-$system $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. Let $g \in G$. We always denote $\bar{g}$ the element k of G satisfying $k \prec g$, if it exists. If $g\mathscr{Y} \cap \mathscr{Y}^- \neq \emptyset$, then by (E7), $g\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$; in this case, we denote the largest element of $g\mathscr{Y} \cap \mathscr{Y}^-$ for $\leq^*$ by t_g. We define two division operations / and $\backslash$ on $\mathfrak{L}$ as follows: for $(a, g), (b, h) \in \mathfrak{L}$,

$$(b,h)/(a,g) = \begin{cases} (t_{hg^{-1}}, hg^{-1}) & \text{if } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \notin \mathscr{Y}, hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset \\ & \text{or } hg^{-1}a \leq^* b, hg^{-1}a < e, gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}; \\ (b, hg^{-1}) & \text{if } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \in \mathscr{Y} \\ & \text{or } b <^* h^{-1}ga \in \mathscr{Y} \\ & \text{or } hg^{-1}a = b \geq e; \\ (\psi(hg^{-1}a), hg^{-1}) & \text{if } hg^{-1}a \leq^* b, hg^{-1}a < e, gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}; \\ (\varphi(hg^{-1}a), hg^{-1}) & \text{if } hg^{-1}a \in \widetilde{\mathscr{Y}}, hg^{-1}a <^* b, hg^{-1}a > e; \\ (\top, \overline{hg^{-1}}) & \text{if } hg^{-1}a \in \mathscr{Y}\backslash\widetilde{\mathscr{Y}}, hg^{-1}a <^* b \\ & \text{or } h^{-1}ga \notin \mathscr{Y}, g^{-1}hb \notin \mathscr{Y}, hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset. \end{cases} \tag{4.1}$$

$$
\begin{aligned}
&(a,g)\backslash(b,h)\\
&\quad=\begin{cases}
(t_{g^{-1}h}, g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \notin \mathscr{Y};\\
& \text{or } a = b, a > e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \notin \mathscr{Y};\\
(\psi(g^{-1}a), g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \in \mathscr{Y}\\
& \text{or } a = b, a > e, g^{-1}a < e, h^{-1}g\psi(g^{-1}a) \in \mathscr{Y};\\
(g^{-1}a, g^{-1}h) & \text{if } a \leq^* b, a \leq e, g^{-1}a \geq e\\
& \text{or } a = b, a > e, g^{-1}a \geq e;\\
(g^{-1}\varphi(a), g^{-1}h) & \text{if } a \in \widetilde{\mathscr{Y}}, a <^* b, a > e,\\
(\top, \overline{g^{-1}h}) & \text{if } a \in \mathscr{Y}\backslash\widetilde{\mathscr{Y}}, a <^* b;\\
(g^{-1}b, g^{-1}h) & \text{if } b <^* a.
\end{cases}
\end{aligned}
\tag{4.2}
$$

We denote by $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ the above $(\mathfrak{L}, \wedge, \vee, \circ, \backslash, /, e)$.

Theorem 4.3. *If $(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC-$system, then $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an E-unitary inverse residuated chain.*

Proof. We only need to prove that $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ and $(a,g)\backslash(b,h) = \max\{(c,k) \in \mathfrak{L} | (a,g)(c,k) \leq (b,h)\}$ for all $(a,g), (b,h) \in \mathfrak{L}$. To show that $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g), (b,h) \in \mathfrak{L}$, we need to consider the following cases:

Case 1. $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$ or $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$. We need only to check the following subcases:

If $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$, then since $\mathscr{Y}$ is a totally ordered semilattice with an identity e, by Proposition 3.3, $hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$. Since $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, $gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$, which implies that $(t_{hg^{-1}}, hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $t_{hg^{-1}} <^* b$. Otherwise, if $b \leq^* t_{hg^{-1}}$, then $gh^{-1}b \leq^* gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$ by noting that $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}$, so $gh^{-1}b \in \mathscr{Y}$, contrary to $gh^{-1}b \notin \mathscr{Y}$. Thus $t_{hg^{-1}} <^* b$. Similarly, $gh^{-1}t_{hg^{-1}} <^* a$. We can claim that $t_{hg^{-1}} < b$; whereas, if $t_{hg^{-1}} \geq b$, then $b \leq e$, hence by Lemma 2.2(3), $t_{hg^{-1}} = t_{hg^{-1}} + b = t_{hg^{-1}} \wedge b = b$, contrary to $t_{hg^{-1}} <^* b$. Consequently, $t_{hg^{-1}} < b$. By equation (1), $(b,h)/(a,g) = (t_{hg^{-1}}, hg^{-1})$ and by the definition of $\circ$, $(t_{hg^{-1}}, hg^{-1}) \circ (a,g) = (t_{hg^{-1}} + hg^{-1}a, h) = (hg^{-1}(gh^{-1}t_{hg^{-1}} + a), h) = (t_{hg^{-1}}, h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$, $gh^{-1}c = k^{-1}c \in \mathscr{Y}$ and $c + hg^{-1}a \leq b$. Furthermore, we have $gh^{-1}c <^* a$. Otherwise, if $a \leq^* gh^{-1}c$, then $hg^{-1}a \leq^* hg^{-1}gh^{-1}c = c \in \mathscr{Y}$, hence $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. Thus $gh^{-1}c <^* a$, which implies that $c = hg^{-1}gh^{-1}c = hg^{-1}(gh^{-1}c + a) = c + hg^{-1}a \leq b$. Note that $gh^{-1}c = k^{-1}c \in \mathscr{Y}$ and $gh^{-1}b \notin \mathscr{Y}$, by similar arguments as in the proof of $gh^{-1}c <^* a$, we may prove $c <^* b$. Furthermore, $c < e$; whereas, if $c \geq e$, then

$b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Consequently, $c < e$. So $c \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ and $c \leq^* t_{hg^{-1}}$. Then by Lemma 2.2(3), $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. Thus $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$.

If $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$, then since $\mathscr{Y}$ is a totally ordered semilattice with an identity e, by Proposition 3.3, $hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$ is a principal ideal of $\mathscr{Y}^-$ and so $hg^{-1}a \leq^* t_{hg^{-1}}$. Since $t_{hg^{-1}} \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, $gh^{-1}t_{hg^{-1}} \in \mathscr{Y}$, which implies that $(t_{hg^{-1}}, hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $hg^{-1}a \leq b$. Otherwise, if $hg^{-1}a > b$, then $b < hg^{-1}a < e$, hence by Lemma 2.2(3), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \wedge b = b$, contrary to $hg^{-1}a > b$. We conclude that $hg^{-1}a \leq b$. By equation (4.1), $(b,h)/(a,g) = (t_{hg^{-1}}, hg^{-1})$ and by the definition of $\circ$, $(t_{hg^{-1}}, hg^{-1}) \circ (a,g) = (t_{hg^{-1}} + hg^{-1}a, h) = (hg^{-1}a, h) \leq b$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (t_{hg^{-1}}, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. If $c <^* hg^{-1}a$, then $c <^* b$ and $c = c + hg^{-1}a \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Consequently, $c < e$, which, together with $c = hg^{-1}gh^{-1}c = hg^{-1}(k^{-1}c) \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, derives $c \leq^* t_{hg^{-1}}$, so $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. Thus $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$. If $hg^{-1}a \leq^* c$, then we can claim that $c <^* \psi(hg^{-1})$; whereas, if $\psi(hg^{-1}) \leq^* c$, then $gh^{-1}\psi(hg^{-1}a) \leq^* gh^{-1}c = k^{-1}c \in \mathscr{Y}$, hence $gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}$, contrary to $gh^{-1}\psi(hg^{-1}a) \notin \mathscr{Y}$. We conclude that $hg^{-1}a \leq^* c <^* \psi(hg^{-1})$. Hence by $(E5)$, $c < e$, which together with $c \in hg^{-1}\mathscr{Y} \cap \mathscr{Y}^-$, derives $c = c + t_{hg^{-1}} = c \wedge t_{hg^{-1}} \leq t_{hg^{-1}}$. This implies that $(c,k) = (c, hg^{-1}) \leq (t_{hg^{-1}}, hg^{-1})$.

Case 2. $hg^{-1}a \notin \mathscr{Y}$ and $gh^{-1}b \in \mathscr{Y}$ or $b <^* hg^{-1}a \in \mathscr{Y}$ or $hg^{-1}a = b \geq e$. We need only to check the following subcases:

If $hg^{-1}a \notin \mathscr{Y}$ and $gh^{-1}b \in \mathscr{Y}$, then $(b, hg^{-1}) \in \mathfrak{L}$. We can claim that $gh^{-1}b <^* a$; whereas, if $a \leq^* gh^{-1}b$, then $hg^{-1}a \leq^* b \in \mathscr{Y}$, which implies that $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. Thus $gh^{-1}b <^* a$. By equation (4.1), $(b,h)/(a,g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a,g) = (b + hg^{-1}a, h) = (b,h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. Furthermore, we have $k^{-1}c = gh^{-1}c <^* a$. Otherwise, if $a <^* gh^{-1}c$, then $hg^{-1}a <^* c$, hence $hg^{-1}a \in \mathscr{Y}$, contrary to $hg^{-1}a \notin \mathscr{Y}$. We conclude that $gh^{-1}c <^* a$. This implies that $c <^* hg^{-1}a$, which derives that $c = c + hg^{-1}a \leq b$. Thereby, $(c,k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

If $b <^* hg^{-1}a \in \mathscr{Y}$, then $gh^{-1}b <^* a$ and so $gh^{-1}b \in \mathscr{Y}$, which implies that $(b, hg^{-1}) \in \mathfrak{L}$. By equation (4.1), $(b,h)/(a,g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a,g) = (b + hg^{-1}a, h) = (b,h) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. If $c \leq^* hg^{-1}a$, then $c = c + hg^{-1}a \leq b$ which implies that $(c,k) = (c, hg^{-1}) \leq (b, hg^{-1})$. If $hg^{-1}a <^* c$, then $hg^{-1}a =$

$c+hg^{-1}a \leq b$. We can claim that $b > e$; whereas, if $b \leq e$, then since $hg^{-1}a \leq b$, $hg^{-1}a \leq e$, hence by Lemma 2.2(3), $b = hg^{-1}a + b = hg^{-1}a \wedge b = hg^{-1}a$, contrary to $b <^* hg^{-1}a$. Consequently, $b > e$. Furthermore, we have $c < b$. Otherwise, if $c \geq b$, then $c > e$. Since $b <^* hg^{-1}a <^* c$, by Lemma 2.2(2), $b = b + c = b \vee c = c$, contrary to $b <^* c$. We conclude that $c < b$, which implies that $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

If $hg^{-1}a = b \geq e$, then $gh^{-1}b = a \in \mathscr{Y}$ and so $(b, hg^{-1}) \in \mathfrak{L}$. By equation (4.1), $(b, h)/(a, g) = (b, hg^{-1})$ and by the definition of $\circ$, $(b, hg^{-1}) \circ (a, g) = (b + hg^{-1}a, h) = (b, h)$. Let $(c, k) \in \mathfrak{L}$ such that $(c, k)(a, g) \leq (b, h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c, k) \leq (b, hg^{-1})$. If $kg < h$, then $k < hg^{-1}$, hence $(c, k) \leq (b, hg^{-1})$. If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. On one hand, if $c <^* hg^{-1}a$, then $c = c + hg^{-1}a \leq b$ and so $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$. On the other hand, if $b = hg^{-1}a <^* c$, then we can claim that $b > c$; whereas, if $b \leq c$, then since $b \geq e$, $c \geq e$, hence by Lemma 2.2(2), $b = b + c = b \vee c = c$, contrary to $b <^* c$. We conclude that $b > c$. This means that $(c, k) = (c, hg^{-1}) \leq (b, hg^{-1})$.

Case 3. $hg^{-1}a \leq^* b, hg^{-1}a < e$ and $gh^{-1}\psi(hg^{-1}a) \in \mathscr{Y}$. Then $(\psi(hg^{-1}a), hg^{-1}) \in \mathfrak{L}$ and by $(E3), hg^{-1}a <^* \psi(hg^{-1}a)$. Furthermore, we have $hg^{-1}a \leq b$. Otherwise, if $hg^{-1}a > b$, then $b < e$, hence by Lemma 2.2(3), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \wedge b = b$, contrary to $hg^{-1}a > b$. Consequently, $hg^{-1}a \leq b$. By equation (4.1), $(b, h)/(a, g) = (\psi(hg^{-1}a), hg^{-1})$ and by the definition of $\circ$, $(\psi(hg^{-1}a), hg^{-1}) \circ (a, g) = (\psi(hg^{-1}a) + hg^{-1}a, h) = (hg^{-1}a, h) \leq (b, h)$. Let $(c, k) \in \mathfrak{L}$ such that $(c, k)(a, g) \leq (b, h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c, k) \leq (\psi(hg^{-1}a), hg^{-1})$. We need only to check the following subcases:

If $kg < h$, then $k < hg^{-1}$, hence $(c, k) \leq (\psi(hg^{-1}a), hg^{-1})$.

If $kg = h$ and $c+ka \leq b$, then $k = hg^{-1}$. If $c <^* hg^{-1}a$, then $c = c+hg^{-1}a = c + ka \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$. Since $c <^* hg^{-1}a \leq^* b$, by Lemma 2.2(2), $c = c+b = c \vee b = b$, contrary to $c <^* b$. Thus $c < e$. Since by the definition of ψ, $\psi(hg^{-1}a) \geq e$, $c < \psi(hg^{-1}a)$. Hence $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$. If $hg^{-1}a \leq^* c <^* \psi(hg^{-1}a)$, then by $(E5)$, $c < e$ and so $c < \psi(hg^{-1}a)$. Hence $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$. If $\psi(hg^{-1}a) <^* c$, then we can claim that $c < \psi(hg^{-1}a)$; whereas, if $c \geq \psi(hg^{-1}a)$, then since $\psi(hg^{-1}a) \geq e$, $c \geq e$, hence by Lemma 2.2(2), $\psi(hg^{-1}a) = \psi(hg^{-1}a)+ c = \psi(hg^{-1}a) \vee c = c$, contrary to $\psi(hg^{-1}a) <^* c$. Consequently, $c < \psi(hg^{-1}a)$. This implies that $(c, k) = (c, hg^{-1}) \leq (\psi(hg^{-1}a), hg^{-1})$.

Case 4. $hg^{-1}a \in \widetilde{\mathscr{Y}}, hg^{-1}a <^* b$ and $hg^{-1}a > e$. Then by $(E2)$, $\varphi(hg^{-1}a) <^* hg^{-1}a <^* b$, hence $gh^{-1}\varphi(hg^{-1}a) <^* a$, so $gh^{-1}\varphi(hg^{-1}a) \in \mathscr{Y}$, which implies that $(\varphi(hg^{-1}a), hg^{-1}) \in \mathfrak{L}$. Furthermore, we have $\varphi(hg^{-1}a) < b$. Otherwise, if $\varphi(hg^{-1}a) \geq b$, then since by the definition of φ, $\varphi(hg^{-1}a) < e$, $b \leq \varphi(hg^{-1}a) < e$, hence by Lemma 2.2(3), $\varphi(hg^{-1}a) = \varphi(hg^{-1}a) + b = \varphi(hg^{-1}a) \wedge b = b$, contrary to $\varphi(hg^{-1}a) <^* b$. Thus $\varphi(hg^{-1}a) < b$. By equation (1), $(b, h)/(a, g) = (\varphi(hg^{-1}a), hg^{-1})$ and by the definition of $\circ$, $(\varphi(hg^{-1}a), hg^{-1}) \circ (a, g) = (\varphi(hg^{-1}a) + hg^{-1}a, h) = (\varphi(hg^{-1}a), h) \leq (b, h)$. Let $(c, k) \in \mathfrak{L}$ such

that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\varphi(hg^{-1}a), hg^{-1})$. We need only to check the following subcases:

If $kg < h$, then $k < hg^{-1}$, hence $(c,k) \leq (\varphi(hg^{-1}a), hg^{-1})$.

If $kg = h$ and $c + ka \leq b$, then $k = hg^{-1}$. We can claim that $hg^{-1}a >^* c$; whereas, if $hg^{-1}a \leq^* c$, then $hg^{-1}a = c + hg^{-1}a = c + ka \leq b$, which, together with $hg^{-1}a <^* b$ and $b > e$ by noting that $hg^{-1}a > e$, derives $hg^{-1}a = b + hg^{-1}a = b \vee hg^{-1}a = b$ by Lemma 2.2(2), contrary to $hg^{-1}a <^* b$. We conclude that $hg^{-1}a >^* c$. Furthermore, we have $c \leq^* \varphi(hg^{-1}a)$. Otherwise, if $\varphi(hg^{-1}a) <^* c <^* hg^{-1}a$, then by $(E4)$, $c > e$ and $c = c + hg^{-1}a \leq b$, hence $b > e$, which, together with $c <^* hg^{-1}a <^* b$, derives $c = c + b = c \vee b = b$ by Lemma 2.2(2), contrary to $c <^* b$. Consequently, $c \leq^* \varphi(hg^{-1}a)$. It follows that $c \leq^* \varphi(hg^{-1}a) <^* hg^{-1}a <^* b$ and so $c = c + hg^{-1}a = c + ka \leq b$. We can prove that $c < e$. Assume to the contrary that $c \geq e$. Then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Now, we conclude that $c < e$. Since $\varphi(hg^{-1}a) < e$, by Lemma 2.2(3), $c = c + \varphi(hg^{-1}a) = c \wedge \varphi(hg^{-1}a) \leq \varphi(hg^{-1}a)$, which implies that $(c,k) = (c, hg^{-1}) \leq (\varphi(hg^{-1}a), hg^{-1})$.

Case 5. $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ and $hg^{-1}a <^* b$ or $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$. We need only to check the following subcases:

If $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$ and $hg^{-1}a <^* b$, then by $(E1)$, $\top$ and $\overline{hg^{-1}}$ exist. By the proof of Case 5 of $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g), (b,h) \in \mathfrak{L}$, we have $g\top = \top$ for all $g \in G$ and $\top \leq^* d$ for all $d \in \mathscr{Y}$, hence $(\overline{hg^{-1}})^{-1}\top = \top \in \mathscr{Y}$, which implies that $(\top, \overline{hg^{-1}}) \in \mathfrak{L}$. Since $\overline{hg^{-1}} \prec hg^{-1}$, $\overline{hg^{-1}}g \prec h$. By equation (4.1), $(b,h)/(a,g) = (\top, \overline{hg^{-1}})$ and by the definition of $\circ$, $(\top, \overline{hg^{-1}}) \circ (a,g) = (\top + \overline{hg^{-1}}a, \overline{hg^{-1}}g) = (\top, \overline{hg^{-1}}g) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\top, \overline{hg^{-1}}g)$. Assume that $kg = h$ and $c + ka \leq b$. Then $k = hg^{-1}$ and $c + hg^{-1}a = c + ka \leq b$. If $c \leq^* hg^{-1}a$, then $c <^* b$ and $c = c + hg^{-1}a \leq b$. Since $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, $c \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, so $c > e$, which implies that $b > e$. By Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. If $hg^{-1}a <^* c$, then $hg^{-1}a = c + hg^{-1}a \leq b$. Since $hg^{-1}a \in \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$, $hg^{-1}a > e$ and so $b > e$. Note that $hg^{-1}a <^* b$, by Lemma 2.2(2), $hg^{-1}a = hg^{-1}a + b = hg^{-1}a \vee b = b$, contrary to $hg^{-1}a \leq^* b$. We conclude that $kg < h$. This implies that $k < hg^{-1}$ and so $(c,k) \leq (\top, \overline{hg^{-1}})$.

If $hg^{-1}a \notin \mathscr{Y}, gh^{-1}b \notin \mathscr{Y}$ and $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$, then $hg^{-1}\mathscr{Y} \cap \mathscr{Y} \subseteq \mathscr{Y} \backslash \widetilde{\mathscr{Y}}$. Note that $hg^{-1}\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$, $\mathscr{Y} \backslash \widetilde{\mathscr{Y}} \neq \emptyset$, which implies that $\top$ and $\overline{hg^{-1}}$ exist by $(E1)$. By the proof of Case 5 of $(b,h)/(a,g) = \max\{(c,k) \in \mathfrak{L} | (c,k)(a,g) \leq (b,h)\}$ for all $(a,g), (b,h) \in \mathfrak{L}$, we have $g\top = \top$ for all $g \in G$ and $\top \leq^* d$ for all $d \in \mathscr{Y}$, hence $(\overline{hg^{-1}})^{-1}\top = \top \in \mathscr{Y}$, which implies that $(\top, \overline{hg^{-1}}) \in \mathfrak{L}$. Since $\overline{hg^{-1}} \prec hg^{-1}$, $\overline{hg^{-1}}g \prec h$. By equation (4.1), $(b,h)/(a,g) = (\top, \overline{hg^{-1}})$ and by the definition of $\circ$, $(\top, \overline{hg^{-1}}) \circ (a,g) = (\top + \overline{hg^{-1}}a, \overline{hg^{-1}}g) = (\top, \overline{hg^{-1}}g) \leq (b,h)$. Let $(c,k) \in \mathfrak{L}$ such that $(c,k)(a,g) \leq (b,h)$. Then $kg < h$ or $kg = h$ and $c + ka \leq b$. We shall prove that $(c,k) \leq (\top, \overline{hg^{-1}}g)$. Assume that $kg = h$ and $c + ka \leq b$. Then $k = hg^{-1}$ and $c + hg^{-1}a = c + ka \leq b$. We can claim that $c <^* b$; whereas,

if $b \leq^* c$, then $gh^{-1}b \leq^* gh^{-1}c$, which together with $gh^{-1}c = k^{-1}c \in \mathscr{Y}$, derives $gh^{-1}b \in \mathscr{Y}$, contrary to $gh^{-1}b \notin \mathscr{Y}$. Consequently, $c <^* b$. Similarly, $k^{-1}c = gh^{-1}c <^* a$, which implies that $c = hg^{-1}(gh^{-1}c + a) = c + hg^{-1}a \leq b$. Furthermore, we have $c < e$. Otherwise, if $c \geq e$, then $b \geq c \geq e$, hence by Lemma 2.2(2), $c = c + b = c \vee b = b$, contrary to $c <^* b$. Thus $c < e$. This means that $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} \neq \emptyset$, contrary to $hg^{-1}\mathscr{Y} \cap \widetilde{\mathscr{Y}} = \emptyset$. We conclude that $kg < h$. This implies that $k < hg^{-1}$ and so $(c, k) \leq (\top, \overline{hg^{-1}})$.

Similarly, $(a, g)/(b, h) = \max\{(c, k) \in \mathfrak{L} | (a, g)(c, k) \leq (b, h)\}$ for all (a, g), $(b, h) \in \mathfrak{L}$. □

We will prove that any $E-$unitary inverse residuated chain is isomorphic to some $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$. In what follows, let $\mathfrak{L} = P(G, \mathscr{X}, \mathscr{Y})$ be an E-unitary inverse residuated chain. By Proposition 3.1(3) and (4), we can define two mappings φ and ϕ as follows:

$$\varphi : \widetilde{\mathscr{Y}}^+ \backslash \{e\} \to \mathscr{Y}^-; \quad a \mapsto \max\{b \in \mathscr{Y}^- | b <^* a\} \text{ for } \leq^*$$

and

$$\phi : \mathscr{Y}^- \backslash \{e\} \to \widetilde{\mathscr{Y}}^+; \quad b \mapsto \min\{a \in \widetilde{\mathscr{Y}}^+ | b <^* a\} \text{ for } \leq^* .$$

Lemma 4.4. *$(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an $EIRC$-system.*

Proof. By Proposition 3.1(2-4), conditions$(E1 - 5)$ hold. By Theorem 2.1, condition $(E6)$ holds. By Proposition 3.3(2), condition $(E7)$ holds. □

Theorem 4.5. *$\mathfrak{L}$ is equal to $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$.*

Proof. It is clear. □

By Theorems 4.3 and 4.5, we have the following theorem:

Theorem 4.6. *Let $(G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ be an $EIRC$-system. Then $EIRC(\mathfrak{L}; G, \mathscr{X}, \mathscr{Y}; \varphi, \psi)$ is an E-unitary inverse residuated chain. Conversely, any E-unitary inverse residuated chain can be constructed in this manner.*

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On soft closed graph and its characterizations

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Abstract. In this paper, we first introduce soft graph and soft closed graph of a soft mapping. Among other results, we obtain characterization of soft closed graph of identity soft mapping and characterization of soft-closedness of soft graphs using soft nets. We also give characterization of soft closure of a soft set in terms of convergence of soft net using soft points.

Keywords: soft topology, soft closed sets, soft graph, soft points, soft nets.

1. Introduction

In 1999, Molodtsov [7] introduced the theory of soft sets as a new mathematical tool for dealing with uncertainties. Topological structures of soft sets has been studied by some authors [11],[1],[10]. In 2011, Shabir and Naz [10] initiated the study of soft topological space. Consequently, they defined some basic notions of soft topological space such as soft open and closed sets, soft closure, soft separation axioms and established their several properties. Kharal and Ahmad [5], defined the notion of a soft mapping on soft classes and studied some properties of images and preimages of soft sets under soft mapping. Aygünoğlu and Aygün [1], introduced soft continuity of soft mappings and soft product topology. Nazmul and Samanta [8], studied the new concept of soft points and neighbourhood properties in a soft topological space. The purpose of this paper is to obtain new and significant results on soft closed graphs and soft continuity to further our understanding of this new and important mathematical discipline.

In section 2, we recall the basic concept related to soft topological space. In section 3, we define soft closed graph of soft mapping and study some charac-

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terizations of soft closed graph. We give characterization of soft closure of a set in terms of convergence of soft net by using notion of soft points.

2. Preliminaries

Zorlutuna, Min and Atmaca studied soft topological spaces in [11] and introduced the following important notions which we shall use in our results below. Let X be an initial universe set, $P(X)$ the power set of X and A a set of parameters. A pair F_A is called a *soft set* over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. The family of all soft sets F_A over X is denoted by $S(X, A)$. The complement of a soft set F_A is denoted by F_A^c and is defined by $(F_A)^c = (F^c, A)$ where $F^c : A \to P(X)$ is a mapping given by $F^c(a) = X - F(a)$, for all $a \in A$. F^c is said to be the *soft complement* function of F. $(F^c)^c$ is the same as F and $((F_A^c)^c = F_A$. A soft set F_A over X is said to be a *null soft set*, denoted by Φ_A , if for all $a \in A$, $F(a) = \phi$. A soft set F_A over X is said to be an *absolute soft set*, denoted by $\widetilde{X}$, if for all $a \in A$, $F(a) = X$. For two soft sets F_A and G_A over a common universe X, F_A is a *soft subset* of G_A if $\forall a \in A$, $F(a) \subseteq G(a)$ for all $a \in A$ and is denoted by $F_A \,\widetilde{\subseteq}\, G_A$.$F_A$ is said that to be a *soft superset* of G_A, if G_A is a soft subset of F_A and is denoted by $F_A \,\widetilde{\supseteq}\, G_A$. For two soft sets F_A and G_A over a common universe X, *union* of two soft sets of F_A and G_A is the soft set H_A, where $H(a) = F(a) \cup G(a)$ for all $a \in A$ and is denoted by $F_A \,\widetilde{\cup}\, G_A = H_A$. For two soft sets F_A and G_A over a common universe X, *intersection* of two soft sets of F_A and G_A is the soft set H_A, where $H(a) = F(a) \cap G(a)$ for all $a \in A$ and is denoted by $F_A \,\widetilde{\cap}\, G_A = H_A$.

Definition 2.1 ([11]). *Let τ be the collection of soft sets over X with the fixed set of parameters A. Then τ is said to be a soft topology on X, if:*

(i) Φ_A, $\widetilde{X}$ belong to τ,

(ii) the union of any number of soft sets in τ belongs to τ,

(iii) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, A) is called a soft topological space over X. The members of τ are called soft open sets. The soft complement of a soft open set is called a soft closed set in (X, τ, A). The family of all soft closed sets is denoted by τ^c.

Definition 2.2 ([11]). *Let (X, τ, A) be a soft topological space over X and (F, A) a soft set over X. Then the soft closure of F_A, denoted by $\overline{F_A}$, is the intersection of all soft closed supersets of F_A and $\overline{F_A}$ is the smallest soft closed set in (X, τ, A) which contains F_A.*

We shall also make use of the fundamental notion of soft points introduced in [8]. A soft set P_A over X is said to be a soft point, denoted by P_a^x, if for the element $a \in A$, $P(a) = \{x\}$ and $P(a') = \phi$, for all $a' \in A - \{a\}$. The soft point P_a^x is said to be in the soft set G_A, denoted by $P_a^x \tilde{\in} G_A$, if $x \in G(a)$. Two soft points $P_{a_1}^{x_1}$, $P_{a_2}^{x_2}$ are said to be equal if $a_1 = a_2$ and $x_1 = x_2$. Thus, $P_{a_1}^{x_1} \neq P_{a_2}^{x_2}$ $\Leftrightarrow a_1 \neq a_2$ or $x_1 \neq x_2$.

From now on, the family of all soft points over X will be denoted by SP(X,A).

Proposition 2.1 ([8]). *Let F_A,$G_A \in S(X,A)$ and $P_a^x \in SP(X,A)$ then we have,*

1. *$P_a^x \,\widetilde{\in}\, G_A$ if and only if $P_a^x \,\widetilde{\notin}\, G_A^c$.*
2. *$P_a^x \,\widetilde{\in}\, F_A \,\widetilde{\cup}\, G_A$ if and only if $P_a^x \,\widetilde{\in}\, F_A$ or $P_a^x \,\widetilde{\in}\, G_A$.*
3. *$P_a^x \,\widetilde{\in}\, F_A \,\widetilde{\cap}\, G_A$ if and only if $P_a^x \,\widetilde{\in}\, F_A$ and $P_a^x \,\widetilde{\in}\, G_A$.*
4. *$F_A \,\widetilde{\subseteq}\, G_A$ if and only if $P_a^x \,\widetilde{\in}\, F_A$ implies $P_a^x \,\widetilde{\in}\, G_A$.*

Definition 2.3 ([8]). *A soft set F_A in a soft topological space (X,τ,A) is called a soft neighbourhood of the soft point P_a^x if there exists a soft open set G_B such that $P_a^x \,\widetilde{\in}\, G_B \,\widetilde{\subseteq}\, F_A$.*

The soft neighbourhood system of soft point P_a^x, denoted by $\mathcal{N}_\tau(P_a^x)$, is the family of all its soft neighbourhoods.

Proposition 2.2 ([6]). *Let (X,τ,A) be a soft topological space. A point $P_a^x \,\widetilde{\in}\, \overline{F_A}$ if and only if each soft neighbourhood of P_a^x intersects F_A.*

We shall also make use of following definitions and propositions

Definition 2.4 ([5]). *Let $S(X,A)$ and $S(Y,B)$ be the families of all soft sets over X and Y respectively. Let $\varphi : X \to Y$ and $e : A \to B$ be two mappings. Then, the mapping (φ,e) is called a soft mapping X to Y, denoted $(\varphi,e) : S(X,A) \to S(Y,B)$, for which:*

(i) Let $F_A \in S(X,A)$. Then $(\varphi,e)(F_A)$ is the soft set over Y defined as follows:

$$(\varphi,e)(F_A)(b) = \begin{cases} \bigcup\limits_{a \in e^{-1}(b) \cap A} \varphi(F(a)), & \text{if } e^{-1}(b) \cap A \neq \phi, \\ \phi, & \text{otherwise,} \end{cases}$$

for all $b \in B$. $(\varphi,e)(F_A)$ is called soft image of a soft set F_A.

(ii) Let $G_A \in S(Y,B)$. Then $(\varphi,e)^{-1}(G_B)$ is the soft set over X defined as follows:

$$(\varphi,e)^{-1}(G_B)(a) = \begin{cases} \varphi(G(e((a))), & \text{if } e(a) \in B, \\ \phi, & \text{otherwise,} \end{cases}$$

for all $a \in A$. $(\varphi,e)^{-1}(G_B)$ is called soft inverse image of a soft set G_B.

Definition 2.5 ([3]). *Let $F_A \in S(X,A)$ and $G_B \in S(Y,B)$. The cartesian product $F_A \times G_B$ is defined by $H_{A\times B}$, where $H : A \times B \to P(X \times Y)$ and $H(a,b) = F(a) \times G(a)$ for all $(a,b) \in A \times B$.*

Definition 2.6 ([3]). *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. The soft topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$, where $\tau_1 \times \tau_2$ is the collection of all soft unions of elements of $\{F_A \times G_B \mid F_A \in \tau_1, G_B \in \tau_2\}$, is called soft product topological space over $X \times Y$.*

Definition 2.7 ([9]). *1. Let F_A be a soft set over X and Y be a nonempty subset of X. Then the sub-soft set of F_A over Y denoted by $({}^{Y}F_A)$ is defined as, ${}^{Y}F(a) = Y \cap F(a)$, for each $a \in A$.*

2. Let (X, τ, A) be a soft topological space over X and Y is a nonempty subset of X. Then $\tau_Y = \{({}^{Y}F_A) | F_A \in \tau\}$ is said to be soft relative topology on Y and (Y, τ_Y, A) is called a soft subspace of (X, τ, A).

Proposition 2.3 ([2]). *Let $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping and $P_a^x \in SP(X, A)$. Then $(\varphi, e)(P_a^x) = P_{e(a)}^{\varphi(x)} \in SP(Y, B)$.*

Proposition 2.4 ([2]). *Let $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping and $P_b^y \in SP(Y, B)$. If (φ, e) is bijective then $(\varphi, e)^{-1}(P_b^y) = P_{e^{-1}(b)}^{\varphi^{-1}(x)} \in SP(X, A)$.*

Proposition 2.5 ([10]). *Let (X, τ, A) be a soft topological space over X. Then the collection $\tau_a = \{F(a) : F_A \in \tau\}$ for each $a \in A$, defines a topology on X.*

3. Results

It is well known that the graph of a function $f : X \to Y$ is the collection of all ordered pair $(x, f(x))$ which is subset of $X \times Y$. In this section, we define soft graph of a soft mapping using the notion of soft points. Firstly, it must be a soft set in $X \times Y$ whose image of parameter (a, b) will be $\bigcup_x (\ x, \bigcup_a (\varphi, e)(P_a^x)(b))$. By Proposition 2.3, it can be easily seen that $(\varphi, e)(P_a^x)(b) = \varphi(x)$ when $b = e(a)$. Also, as every soft set is soft union of its soft points we get the following appropriate and simplified definition

Definition 3.1. *Let $S(X, A)$ and $S(Y, A)$ be the families of all soft sets over X and Y respectively and $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping. Then Soft graph of (φ, e) is a soft set $\mathbb{G}(\varphi, e)_{A \times B}$, where $\mathbb{G}(\varphi, e) : A \times B \to \mathbf{P}(X \times Y)$ is defined by*

$$\mathbb{G}(\varphi, e)(a, b) = \begin{cases} \mathbb{G}(\varphi), & \text{if } b = e(a), \\ \phi, & \text{if } b \neq e(a). \end{cases}$$

Here $\mathbb{G}(\varphi)$ is usual graph of the function φ.

Definition 3.2. *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. If $\mathbb{G}(\varphi, e)_{A \times B}$ is soft closed in the soft product topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$, then we say that (φ, e) is a soft mapping with soft closed graph.*

Remark 3.1. It is straight forward to see that for soft topological spaces (X, τ_1, A) and (Y, τ_2, B) if the soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has

a soft closed graph in soft product topological space $(X \times Y, \tau_1 \times \tau_2, A \times B)$ then the graph of $\varphi : X \to Y$ is also closed in the product topology on $X \times Y$. However, the converse is not true as shown in the example below

Example 3.1. Let $X = \{x, y, z\}$, $A = \{0, 1\}$ and $\tau = \{\Phi, \widetilde{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A)\}$ where

$$F_1(0) = \{x\}, F_1(1) = \{z\};$$
$$F_2(0) = \{y\}, F_2(1) = \{x\};$$
$$F_3(0) = \{z\}, F_3(1) = \{y\};$$
$$F_4(0) = \{x, y\}, F_4(1) = \{x, z\};$$
$$F_5(0) = \{x, z\}, F_5(1) = \{y, z\};$$
$$F_6(0) = \{y, z\}, F_6(1) = \{x, y\}.$$

Let $\varphi = 1_X$ and $e = 1_A$ and for each $a \in A$, (X, τ_a) is a Hausdorff space. Then, $\Delta(\varphi) = \{(x, x)|x \in X\}$ is closed subset of $X \times X$. Now $(\varphi, e) : S(X, A) \to S(Y, B)$ and soft graph of (φ, e), $\mathbb{G}(\varphi, e) : A \times A \to \mathbf{P}(X \times X)$ is defined by

$$\mathbb{G}(\varphi, e)(a, b) = \begin{cases} \Delta(\varphi), & \text{if } a = b \\ \phi, & \text{if } a \neq b \end{cases}$$

is not soft closed in $X \times X$.

We begin by giving characterization of soft closed graph of identity soft mapping

Theorem 3.1. *Let (X, τ, A) be a soft topological space. Let $1_X : X \to X$ and $1_A : A \to A$ be identity mappings. Then the soft mapping $(1_X, 1_A) : S(X, A) \to S(X, A)$ has a soft closed graph if and only if (X, τ, A) is soft Hausdorff space.*

Proof of Theorem 3.1. Let (X, τ, A) is soft Hausdorff space and soft graph of $(1_X, 1_A)$ is a soft set $\mathbb{G}(1_X, 1_A) : A \times A \to \mathbf{P}(X \times X)$ defined by

$$\mathbb{G}(1_X, 1_A)(a, b) = \begin{cases} \{(x, x)|x \in X\}, & \text{if } a = b, \\ \phi, & \text{if } a \neq b. \end{cases}$$

Assume that $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(1_X, 1_A)^c_{A \times A}$ implies $P^x_a \neq P^y_b$. Since (X, τ, A) is soft Hausdorff space, there exist soft open sets F_A and G_B such that $P^x_a \widetilde{\in} F_A$ and $P^y_b \widetilde{\in} G_A$ and $(F_A) \widetilde{\cap} (G_A) = \Phi$. Hence $P^{(x,y)}_{(a,b)} \widetilde{\in} F_A \times G_A \widetilde{\subseteq} G(1_X, 1_A)^c_{A \times A}$ where $F_A \times G_A$ is soft open set in $X \times X$ implies $\mathbb{G}(1_X, 1_A)^c$ is soft open and hence, $\mathbb{G}(1_X, 1_A)_{A \times A}$ is soft closed in $X \times X$.

Conversely, let P^x_a, $P^y_b \in$ SP(X,A) be any two distinct soft points such that $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(1_X, 1_A)^c_{A \times A}$ and $\mathbb{G}(1_X, 1_A)^c_{A \times A}$ is soft open in $X \times X$ then, there exist soft open sets M_A and N_A in X such that $P^{(x,y)}_{(a,b)} \widetilde{\in} M_A \times N_A \widetilde{\subseteq} \mathbb{G}(1_X, 1_A)^c_{A \times A}$. Therefore, $P^x_a \widetilde{\in} M_A$, $P^y_b \widetilde{\in} N_A$ and $(M_A) \widetilde{\cap} (N_A) = \Phi$.

Next, we need the characterization of soft closure in terms of convergence of soft nets to study soft graphs further. For this, we need the following

Definition 3.3 ([2]). *Let X be a set and $(D, \leq)$ be a directed set [4] where a directed set is a set with a relation $\leq$ (Throughout the paper, by $b \geq a$ we mean $a \leq b$) which is reflexive, transitive and upwards directive (where a set D is directed if for each $m, n \in D$, there is some $p \in D$ such that $p \geq m, p \geq n$). The function $T : D \to SP(X, A)$ defined by $T(\alpha) = P_{a_\alpha}^{x_\alpha}$ is called a soft net in X. For $\alpha \in D$, $T(\alpha)$ is denoted by T_α and hence a soft net is denoted by $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$.*

Example 3.2 ([2]). Since $(\mathcal{N}_\tau(P_a^x), \widetilde{\subseteq}^*)$ is a directed set (where the relation $\widetilde{\subseteq}^*$ is defined by $F_A \widetilde{\subseteq}^* G_B$ if and only if $G_B \widetilde{\subseteq} F_A$) the function $T : \mathcal{N}_\tau(P_a^x) \to SP(X, A)$ is a soft net $\{T_{F_A} \mid F_A \in \mathcal{N}_\tau(P_a^x)\}$ where $T_{F_A} = P_a^x \widetilde{\in} F_A$.

Remark 3.2. Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces and $(\varphi, e) : S(X, A) \to S(Y, B)$ be a soft mapping. Let $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in (X, τ_1, A) then $\{(\varphi, e)(T_\alpha) = (\varphi, e)(P_{a_\alpha}^{x_\alpha}) = P_{e(a_\alpha)}^{\varphi(x_\alpha)} \mid \alpha \in D\}$ is a soft net in (Y, τ_2, B) and $\{P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \mid \alpha \in D\}$ is a soft net in $(X \times Y, \tau_1 \times \tau_2, A \times B)$.

Definition 3.4 ([2]). *Let (X, τ, A) be a soft topological space, $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in X and $F_A \in S(X, A)$*

1. *The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called in F_A if $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$, for all $\alpha \in D$.*
2. *The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called eventually in F_A if there exists some $\alpha_0 \in D$ such that $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha_0$.*

Definition 3.5 ([2]). *A soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ in a soft topological space (X, τ, A) is said to converge to P_a^x, if it is eventually in every soft neighbourhood of P_a^x.*

Theorem 3.2. *$P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ where $P_a^x \in SP(X, A)$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$ where $P_b^y \in SP(X, A)$ if and only if $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$ where $P_{(a,b)}^{(x,y)} \in SP(X \times Y, A \times B)$.*

Proof of Theorem 3.2. Let $P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$. Assume $G_{A \times B} \in \mathcal{N}_\tau(P_{(a,b)}^{(x,y)})$ then there exist soft open sets F_A and H_B in X and Y respectively such that $P_{(a,b)}^{(x,y)} \widetilde{\in} F_A \times H_B \widetilde{\subseteq} G_{A \times B}$ which implies $P_a^x \widetilde{\in} F_A$ and $P_b^y \widetilde{\in} H_B$. As $P_{a_\alpha}^{x_\alpha} \longrightarrow P_a^x$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \longrightarrow P_b^y$ there exists $\alpha' \in D$ and $\alpha'' \in D$ such that $P_{a_\alpha}^{x_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha'$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \widetilde{\in} H_B$ for all $\alpha \geq \alpha''$. Let $\alpha^0 = max\{\alpha', \alpha''\}$. This implies $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \widetilde{\in} F_A \times H_B \widetilde{\subseteq} G_{A \times B}$ and hence $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$.

Conversely, let $P_{(a_\alpha, e(a_\alpha))}^{(x_\alpha, \varphi(x_\alpha))} \longrightarrow P_{(a,b)}^{(x,y)}$. Assume $F_A \in \mathcal{N}_{\tau_1}(P_a^x)$ and $G_B \in \mathcal{N}_{\tau_2}(P_b^y)$ then $F_A \times G_B \in \mathcal{N}_{\tau_1 \times \tau_2}(P_{(a,b)}^{(x,y)})$. Therefore, by assumption there exists

an index $\alpha' \in D$ such that $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \widetilde{\in} F_A \times G_B$ for all $\alpha \geq \alpha'$. This implies $P^{x_\alpha}_{a_\alpha} \widetilde{\in} F_A$ and $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \widetilde{\in} G_B$ for all $\alpha \geq \alpha'$ and hence, $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ and $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$.

The following theorem gives characterization of soft closure of a soft set in terms of convergence of its soft net by using soft points. It is important to note that sufficient part of the following theorem is not true when we use concept of points belongs to a soft set, rather than soft points and therefore, by using the notion of soft points we get better results and are able to characterize soft closure below.

Theorem 3.3. *Let (X, τ, A) be a soft topological space. Let $G_A \in S(X, A)$ be a soft set and $P^x_a \in SP(X, A)$. Then $P^x_a \widetilde{\in} \overline{G_A}$ if and only if there exists a soft net $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ in G_A i.e. $P^{x_\alpha}_{a_\alpha} \widetilde{\in} G_A$ for all $\alpha \in D$ such that $P^{x_\alpha}_{a_\alpha} \to P^x_a$.*

Proof of Theorem 3.3. Let $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ be a soft net in G_A such that $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$. Let $F_A \in \mathcal{N}_\tau(P^x_a)$. Now as $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$, there exists an index $\alpha' \in D$ such that $P^{x_\alpha}_{a_\alpha} \widetilde{\in} F_A$ for all $\alpha \geq \alpha'$. Thus, $P^{x_{\alpha'}}_{a_{\alpha'}} \widetilde{\in} F_A \widetilde{\cap} G_A$ i.e. $x_{\alpha'} \in F_A(a_{\alpha'}) \cap G_A(a_{\alpha'})$ and therefore we have $F_A \widetilde{\cap} G_A \neq \Phi$. Hence $P^x_a \widetilde{\in} \overline{G_A}$.

Conversely, let $P^x_a \in \overline{G_A}$. This implies every soft neighborhood of P^x_a intersects G_A, that is $H_A \widetilde{\cap} G_A \neq \Phi$ for every $H_A \in \mathcal{N}_\tau(P^x_a)$. So there exists an $a \in A$ such that $H_A(a) \cap G_A(a) \neq \phi$. Let $y_{H_A} \in H_A(a) \cap G_A(a)$ and so $P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\cap} G_A$. Now $(\mathcal{N}_\tau(P^x_a), \widetilde{\supseteq}^*)$ is a directed set (where the relation $\widetilde{\supseteq}^*$ is defined by $F_A \widetilde{\supseteq}^* G_B$ if and only if $F_A \widetilde{\subseteq} G_B$). Define a soft net, $T : \mathcal{N}_\tau(P^x_a) \to SP(X, A)$ as $T_{H_A} = P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\cap} G_A$. Now $P^{y_{H_A}}_a \widetilde{\in} G_A$ for every $H_A \in \mathcal{N}_\tau(P^x_a)$. This implies $P^{y_{H_A}}_a$ is a soft net in G_A. For proving $P^{y_{H_A}}_a \to P^x_a$, assume $F_A \in \mathcal{N}_\tau(P^x_a)$ and $H_A \widetilde{\supseteq}^* F_A$ such that $H_A \widetilde{\subseteq} F_A$. Therefore, $T_{H_A} = P^{y_{H_A}}_a \widetilde{\in} H_A \widetilde{\subseteq} F_A$ which implies $P^{y_{H_A}}_a \widetilde{\in} F_A$, for every $H_A \widetilde{\supseteq}^* F_A$. Hence $P^{y_{H_A}}_a \longrightarrow P^x_a$.

We are now able to give the following another characterization of soft closed graph of a soft mapping.

Theorem 3.4. *Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. A soft mapping $(\varphi, e) : S(X, A) \to S(Y, B)$ has a soft closed graph if and only if whenever $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ in X and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$ in Y then $P^y_b = (\varphi, e)(P^x_a) = P^{\varphi(x)}_{e(a)}$.*

Proof of Theorem 3.4. Suppose (φ, e) has a soft closed graph and $P^{x_\alpha}_{a_\alpha} \to P^x_a$ in X and $(\varphi, e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$ in Y. Then by Proposition 3.2, $P^{x_\alpha}_{a_\alpha} \to P^x_a$ in X and $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \to P^{(x,y)}_{(a,b)}$ where $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))} \widetilde{\in} \mathbb{G}(\varphi, e)_{A\times B}$, soft graph of (φ, e) which implies $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))}$ is a soft net in $\mathbb{G}(\varphi, e)_{A\times B}$ converging to $P^{(x,y)}_{(a,b)}$. Therefore by Theorem 3.3, $P^{(x,y)}_{(a,b)} \widetilde{\in} \overline{\mathbb{G}(\varphi, e)}_{A\times B}$. Since $\mathbb{G}(\varphi, e)_{A\times B}$ is soft closed, then $P^{(x,y)}_{(a,b)} \widetilde{\in} \mathbb{G}(\varphi, e)_{A\times B}$. Hence $b = e(a)$ and $y = \varphi(x)$. Therefore, $P^y_b = P^{\varphi(x)}_{e(a)} = (\varphi, e)(P^x_a)$.

Conversely, suppose $P^{x_\alpha}_{a_\alpha} \to P^x_a$ and $(\varphi,e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$ implies $P^y_b = (\varphi,e)(P^x_a) = P^{\varphi(x)}_{e(a)}$ and let $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} \overline{\mathbb{G}(\varphi,e)}_{A\times B}$. Then by Theorem 3.3, there exists a soft net $P^{(x_\alpha,\varphi(x_\alpha))}_{(a_\alpha,e(a_\alpha))}$ in $\mathbb{G}(\varphi,e)_{A\times B}$ converging to $P^{(x,y)}_{(a,b)}$ which implies $P^{x_\alpha}_{a_\alpha} \to P^x_a$ and $(\varphi,e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \to P^y_b$. Thus by assumption $P^y_b = P^{\varphi(x)}_{e(a)} = (\varphi,e)(P^x_a)$ implies $b = e(a)$ and $y = \varphi(x)$. Therefore, $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} \mathbb{G}(\varphi,e)_{A\times B}$ and hence $\mathbb{G}(\varphi,e)_{A\times B}$ is soft closed.

Remark 3.3. We note that the soft graph of the restriction of soft mapping with soft closed graph is also soft closed where restriction of soft mapping is defined in [12].

Theorem 3.5. *Let the soft mapping $(\varphi,e) : S(X,A) \to S(Y,B)$ has a soft closed graph and $M \subset X$, $N \subset Y$ such that $\varphi(M) \subset N$. Then $(\varphi,e)|_{S(M,A)} : S(M,A) \to S(Y,B)$ also has a soft closed graph.*

Proof of Theorem 3.5. Let $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ be a soft net in M such that $P^{x_\alpha}_{a_\alpha} \longrightarrow P^x_a$ where $P^x_a \in SP(M,A)$ and $\{(\varphi,e)(P^{x_\alpha}_{a_\alpha}) = P^{\varphi(x_\alpha)}_{e(a_\alpha)} \mid \alpha \in D\}$ be a soft net in N such that $P^{\varphi(x_\alpha)}_{e(a_\alpha)} \longrightarrow P^y_b$ where $P^y_b \in SP(N,B)$. Therefore, $\{P^{x_\alpha}_{a_\alpha} | \alpha \in D\}$ is a soft net in X and $\{P^{\varphi(x_\alpha)}_{e(a_\alpha)} \mid \alpha \in D\}$ is a soft net in Y. Since (φ,e) has a soft closed graph then by Theorem 3.4, $P^y_b = (\varphi,e)(P^x_a) = P^{\varphi(x)}_{e(a)} = (\varphi,e)|_{S(M,A)}(P^x_a)$.

Finally, in this section we give another characterization of soft closed graph using soft open sets.

Theorem 3.6. *The soft mapping $(\varphi,e) : S(X,A) \to S(Y,B)$ has a soft closed graph if and only if for every $P^x_a \in SP(X,A)$ and $P^y_b \in SP(Y,B)$ where $P^y_b \neq (\varphi,e)(P^x_a) = P^{\varphi(x)}_{e(a)}$, there exist soft open sets F_A in X where $P^x_a \mathrel{\widetilde{\in}} F_A$ and G_B in Y where $P^y_b \mathrel{\widetilde{\in}} G_B$ such that $(\varphi,e)(F_A) \mathrel{\widetilde{\cap}} G_B = \Phi$.*

Proof of Theorem 3.6. Suppose the given condition holds. To prove sufficiency it is enough to show that $\overline{\mathbb{G}(\varphi,e)}_{A\times B} \mathrel{\widetilde{\subseteq}} \mathbb{G}(\varphi,e)_{A\times B}$. Let us assume $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} \mathbb{G}(\varphi,e)^C_{A\times B}$ which implies either $b \neq e(a)$ or $y \neq \varphi(x)$. Then $P^y_b \neq (\varphi,e)(P^x_a) = P^{\varphi(x)}_{e(a)}$. Therefore by hypothesis, there exist soft open sets F_A in X where $P^x_a \mathrel{\widetilde{\in}} F_A$ and G_B in Y where $P^y_b \mathrel{\widetilde{\in}} G_B$ such that $(\varphi,e)(F_A) \mathrel{\widetilde{\cap}} G_B = \Phi$, which implies $(F_A \times G_B) \mathrel{\widetilde{\cap}} \mathbb{G}(\varphi,e)_{A\times B} = \Phi$, where $F_A \times G_B$ is soft open and $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} F_A \times G_B$. Hence $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} \overline{\mathbb{G}(\varphi,e)}^C_{A\times B}$ and so $\mathbb{G}(\varphi,e)_{A\times B}$ is soft closed.

Conversely, let $\mathbb{G}(\varphi,e)_{A\times B}$ be soft closed and $P^x_a \in SP(X,A)$ and $P^y_b \in SP(Y,B)$ where $P^y_b \neq P^{\varphi(x)}_{e(a)}$, Therefore either $b \neq e(a)$ or $y \neq \varphi(x)$, which implies $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\notin}} \mathbb{G}(\varphi,e)_{A\times B}$. As $\mathbb{G}(\varphi,e)_{A\times B}$ is soft closed, $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\notin}} \overline{\mathbb{G}(\varphi,e)}_{A\times B}$. Therefore there exist soft open sets F_A and G_B in X and Y respectively where $P^{(x,y)}_{(a,b)} \mathrel{\widetilde{\in}} F_A \times G_B$ such that $(F_A \times G_B) \mathrel{\widetilde{\cap}} \mathbb{G}(\varphi,e)_{A\times B} = \Phi$, which implies $(\varphi,e)(F_A) \mathrel{\widetilde{\cap}} G_B = \Phi$.

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On the complementary dual code over $\mathbb{F}_2 + u\mathbb{F}_2$

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Abstract. A linear complementary dual code (an LCD code) is a linear code, whose satisfies in $C \cap C^{\perp} = \{0\}$. A binary LCD code play an important role in armoring implementation against side-channel attacks and fault injection attacks. All non-binary LCD codes with characteristic 2 can be transformed into binary LCD codes by expansion. In this paper, we consider the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$, which is a ring with characteristic 2 and obtain some properties of LCD codes over this ring. Also we characterize all LCD free cycle codes over R.

Keywords: cyclic code, LCD code, dual code.

1. Introduction

Throughout this paper $\mathbb{F}_2$ is the field of order 2 and $R = \mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$. The ring R is a ring of characteristic two and has four elements $\{0, 1, u, \overline{u} = u+1\}$ such that $u^2 = 0$. This ring is a local ring with a maximal ideal $\{0, u\}$, that shares some good properties of $\mathbb{Z}_4$. A *linear code* C of length n is defined to be an additive submodule of the R-module R^n. Codes over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$

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where studied by Bachoc [1], Bonnecaze and Udaya [2], and Dougherty et al. [3].

The *Hamming weight* of a codeword $u = (u_1, u_2, \cdots, u_n)$, $w(u)$ is the number of nonzero entries in u. The *Hamming distance* of a linear code C is given by

$$d(C) = \min\{w(u) : u \in C, u \neq 0\}.$$

Suppose that C is a linear code of length n over $R = \mathbb{F}_2 + u\mathbb{F}_2$. we defined the *dual code* of C as

$$C^{\perp} = \{x' \in R^n; x'.x = 0, \forall x \in C\}.$$

With respect to inner product over R by $x.y = x_0y_0 + \cdots + x_{n-1}y_{n-1}$. A linear code with a *complementary-dual* (LCD code) was defined by Massy [5], is a linear code C satisfying $C \cap C^{\perp} = \{0\}$. In general for any linear code C over finite rings there is $(C^{\perp})^{\perp} \neq C$, but over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$, we have $(C^{\perp})^{\perp} = C$ [4].

Suppose that $f(x)$ is a monic polynomial of degree m with $f(0) = c \neq 0$, then we called $f^*(x) = c^{-1}x^m f(x^{-1})$ the monic *reciprocal* polynomial of $f(x)$. A monic polynomial $f(x)$ is called *reciprocal polynomial*, if $f(x) = f^*(x)$.

Let C be a linear code of length n over the ring R. A $k \times n$ matrix, G, over the ring R is called a *generator matrix* for C if the rows of G generate C and there is no proper subset of the rows of G generates C

Theorem 1.1 ([9])**.** *Suppose that $R = \mathbb{F}_2 + u\mathbb{F}_2$. Any nonzero R-linear code C is permutation-equivalent to a R-linear code with a generator matrix of the form*

$$\begin{bmatrix} I_{k_1} & A & B \\ 0 & uI_{k_2} & uD \end{bmatrix} \tag{1}$$

where I_{k_1},I_{k_2} denote the $k_1 \times k_1$ and $k_2 \times k_2$ are identity matrices, respectively A and D are F_2-matrices and B is a R-matrix. Then C is an abelian group of type $4^{k_1}2^{k_2}$, and C is a free R-module if and only if $k_2 = 0$.

The following theorem we stat a good results for finding the LCD code based on generating matrix.

Theorem 1.2 ([6])**.** *Let C be a free linear code over R with generator matrix G. Then C is an LCD code if and only if $k \times k$ matrix GG^T is invertible, where k is the number of rows of G.*

A linear code C over R is called *cyclic*, if it is invertible under a cyclic shift, i.e. , if $(x_0, x_1, \ldots, x_{n-1}, x_{n-2})$ is in C, then $(x_{n-1}, x_0, x_1, \ldots, x_{n-2}) \in C$. It is not difficult to see that every cyclic codes of length n over R are equivalent to an ideal of the ring $R_n = \frac{R[x]}{<x^n-1>}$.

Theorem 1.3 ([9])**.** *Let n be an odd positive integer, then every ideal of $R = \mathbb{F}_2 + u\mathbb{F}_2$ is principal.*

based on the Theorem 1.3, the factorization of $x^n - 1$ is important. This factorization over R_n is not unique , but if n is an odd positive integer we have the following theorem.

Theorem 1.4 ([2]). *Let n be an odd positive integer.*

If $x^n - 1 = f_1(x)f_2(x)\dots f_r(x)$, where f_i are basic irreducible and pairwise co-prime, then this factorization is unique. The factorization is obtained from factorization of the polynomial $x^n - 1$ over $\mathbb{F}_2$.

Theorem 1.5 ([2, 4]). *Suppose that C is a cyclic code of odd length n over R, then there are unique, monic polynomials $f(x), g(x), h(x)$ such that $C = (f(x)h(x), uf(x)g(x))$, where $x^n - 1 = f(x)g(x)h(x)$ and $|C| = 4^{deg(g(x))}2^{deg(h(x))}$.*

Corollary 1.1 ([4]). *Suppose $C = (f(x)h(x), uf(x)g(x))$ is a cyclic code of odd length n over R, where $f(x), g(x), h(x)$ are monic polynomials such that $x^n - 1 = f(x)g(x)h(x)$, then*

1. *If h(x)=1, then $C = (f(x)), |C| = 4^{n-deg(f(x))}$,*
2. *If g(x)=1, then $C = (uf(x)), |C| = 2^{n-deg(f(x))}$.*

Theorem 1.6 ([4]). *Suppose that $C = (f(x)h(x), uf(x)g(x))$ is a cyclic code of odd length n over R, where $f(x), g(x), h(x)$ are monic polynomials such that $x^n - 1 = f(x)g(x)h(x)$, and $|C| = 4^{deg(g(x))}2^{deg(h(x))}$. Then the dual of C is $C^\perp = (g^*(x)h^*(x), ug^*(x)f^*(x))$ and $|C^\perp| = 4^{deg(f(x))}2^{deg(h(x))}$, where $f^*(x), g^*(x)$ and $h^*(x)$ are reciprocal polynomials of $f(x), h(x)$ and $g(x)$, respectively.*

Corollary 1.2 ([4]). *Suppose that $C = (f(x)h(x), uf(x)g(x))$ is a cyclic code of odd length n over R, where $f(x), g(x), h(x)$ are monic polynomials such that $x^n - 1 = f(x)g(x)h(x)$ and $C^\perp = (g^*(x)h^*(x), ug^*(x)f^*(x))$, then*

1. *If $h(x) = 1$, then $C = (f(x))$ and $C^\perp = (g^*(x))$,*
2. *If $g(x) = 1$, then $C = (uf(x))$ and $C^\perp = (h^*(x), uf^*(x))$.*

In the rest of the paper we study the LCD free codes over $R = \mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$.

2. Main results

In this section we study free linear code over R.

Theorem 2.1. *Suppose that C is a free linear code over $R = \mathbb{F}_2 + u\mathbb{F}_2$ with generator matrix*

$$G = \begin{bmatrix} I_{k_1} & A & B_1 + uB_2 & A & B_1 + uB_2. \end{bmatrix} \tag{2}$$

Hence C is an LCD code.

Proof. We have $GG^T = I_{k_1}$. Hence C is an LCD code by Theorem 1.2. □

Theorem 2.2. *Suppose that C is a free linear code over $R = \mathbb{F}_2 + u\mathbb{F}_2$ with generator matrix $G = \begin{bmatrix} I_{k_1} & \alpha A & uB_1 \end{bmatrix}$. If $\frac{1}{\alpha^2}$ is not an eigenvalue of matrix A, then C is an LCD code.*

Proof. If C is not an LCD code, then $GG^T = I_{k_1} + \alpha^2 AA^T$ is not invertible. Hence there exists a vector v, such that $(I_{k_1} + \alpha^2 AA^T)v = 0$ and therefore $\alpha^2 AA^T v = -I_{k_1} v = v$. So $(\frac{1}{\alpha^2} I_{k_1} + AA^T)v = 0$ and we conclude that $\frac{1}{\alpha^2}$ is an eigenvalue. □

Lemma 2.1. *Let $C = (f(x))$ be a cyclic free code, where $f(x) \mid x^n - 1$. Suppose that $g(x) = \frac{x^n - 1}{f(x)}$. Cyclic code C is an LCD code if and only if $g.c.d(f(x), g^*(x)) = 1$.*

Proof. Let $k(x) = l.c.m(f(x), g^*(x))$. It is not difficult to see that $C \cap C^\perp = (k(x))$. Hence $C \cap C^\perp = 0$ if and only if $deg(k(x)) = n$. Since $deg(f(x)) = k$ and $deg(g^*(x)) = deg(g(x)) = n - k$, then $deg(k(x)) = n$ if and only if $gcd(f(x), g^*(x)) = 1$. □

The following theorem state that which polynomials are good as a generator for cyclic code.

Theorem 2.3. *Let $C = (f(x))$ be a free cyclic code over $R = \mathbb{F}_2 + u\mathbb{F}$ of length n with generator polynomial $f(x)$. Then C is an LCD code if and only if $f(x)$ is a self reciprocal polynomial.*

Proof. Suppose that $C = (f(x))$ is an LCD code and $x^n - 1 = f(x)g(x)$. Then $gcd(f(x), g^*(x)) = 1$ by Lemma 2.1. We have $f^*(x).g^*(x) = x^n - 1 = f(x)g(x)$ and therefore $f(x) \mid f^*(x).g^*(x)$. Hence $f(x) \mid f^*(x)$. So there exists a polynomial $l(x) \in F_2[x]$ such that $f^*(x) = l(x).f(x)$. We know that $degf^*(x) = degf(x)$, and so $l(x)$ is a constant and this fact implies that $f^*(x) = f(x)$.

For converse, if $f^*(x) \neq f(x)$, then $g.c.d(f(x), g^*(x)) \neq 1$. Hence C is not LCD code by Lemma 2.1. □

Based on the above results, there is a good relation between the factorization of $x^n - 1$ and free cyclic code over the ring R. This construction used for cyclic LCD codes over fields [7]. Here we stat this construction.

Let n be an odd positive integer. The *cyclotomic coset* of 2 (or 2-cyclotomic coset) module n containing i is defined by

$$C_i = \{i2^j (\mathrm{mod}\, n) \in \mathbb{Z}_n : j = 0, 1, 2, \cdots\}.$$

It is not difficult to see that cyclotomic cosets is a partition of $\mathbb{Z}_n$. A subset $\{t_1, t_2, \cdots, t_k\}$ is called a *complete set of representatives* of cyclotomic cosets of 2 module n if $C_{t_i} \cap C_{t_j} = \emptyset$ for any $1 \leq i < j \leq k$ and $\bigcup_{l=1}^{k} C_l = \mathbb{Z}_n$.

If α is a primitive element of field $\mathbb{F}_{2^n}$, then $\prod_{j \in C_i}(x - \alpha^j)$ is the minimal polynomial of α^j over $\mathbb{F}_2$. Now we have the following theorem.

Theorem 2.4. *Let n be an odd integer. Suppose that m is an positive integer such that $n \mid 2^m - 1$. Let α be a primitive element of $\mathbb{F}_{2^m}$ and let $M^{(j)}(x)$ be the minimal polynomial of α^j with respect to $\mathbb{F}_q$ and $\{t_1, t_2, \cdots, t_k\}$ is a complete set of representatives of cyclotomic cosets of 2 module n. Then*

$$x^n - 1 = \prod_{i=1}^{k} M^{(\frac{(2^n-1)t_i}{n})}(x).$$

A 2-cyclotomic coset, C_i, is called symmetric if $-i \in C_i$ and is called anti-symmetric, otherwise. Suppose that $\{t_1, t_2, \cdots, t_k\}$ is a complete set of representatives of cyclotomic cosets of 2 module n. Also suppose that for $1 \leq i \leq s$, 2-cyclotomic cosets C_i's are symmetric and for for $s+1 \leq i \leq s+l$, 2-cyclotomic cosets C_i's are anti-symmetric, where $s + 2l = k$. Hence we can decompose the polynomial $x^n - 1$ as follows

$$x^n - 1 = \prod_{i=1}^{s} M^{(\frac{(2^n-1)t_i}{n})}(x) \prod_{i=s+1}^{s+l} M^{(\frac{(2^n-1)t_i}{n})}(x) M^{(-\frac{(2^n-1)t_i}{n})}(x).$$

The following theorem give a good characterization of LCD code over $\mathbb{F}_2 + u\mathbb{F}_2$.

Theorem 2.5. *Let n be an odd prime and C be a cyclic free code over $\mathbb{F}_2 + u\mathbb{F}_2$ with generator $g(x)$. Hence C ia an LCD code if and only if*

$$g(x) = \prod_{i=1}^{s} (M^{(\frac{(2^n-1)t_i}{n})}(x))^{a_i} \prod_{i=s+1}^{s+l} (M^{(\frac{(2^n-1)t_i}{n})}(x) M^{(-\frac{(2^n-1)t_i}{n})}(x))^{b_i}.$$

where $a_i, b_i \in \{0, 1\}$

Proof. By Theorem 2.3, the cyclic free code C with generator $f(x)$ is an LCD code if and only if α is a root of $f(x)$ implies that α^{-1} is a root of $f(x)$. □

Theorem 2.6. *If $7|n$, then there exists an $[n, n-6, d]$ LCD cyclic code over the ring R with $d \leq 7$.*

Proof. Let $a = \frac{n}{7}$. We have $C_a = \{\frac{n}{7}, \frac{2n}{7}, \frac{4n}{7}\}$. Hence C_a is an anti-symmetric coset. Now consider the cyclic code, C, with the generator matrix $g(x)$, where

$$g(x) = M_a M_{-a}.$$

Since $deg(g(x)) = 6$ and weight of $g(x))$ is at most 7, the result is obtained. □

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A new interior-point method for $P_*(\kappa)$ linear complementarity problems based on a parameterized kernel function

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Abstract. In this paper, we propose a primal-dual interior-point method (IPM) for $P_*(\kappa)$ linear complementarity problems (LCPs) based on a new parameterized kernel function which is a generalization of the one presented by Bai et al. [A primal-dual interior-point method for linear optimization based on a new proximity function, Optim. method softw. (2002)]. A simple analysis shows that the iteration bound for large-update method obtained by Cho based on the original kernel function [A new large-update interior point algorithm for $P_*(\kappa)$ linear complementarity problems, J. Comput. Appl. Math. (2008)] is improved from $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ to $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon})$, and the small-update method has $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity. These are the currently best known complexity results for such methods. Some numerical results have been provided.

Keywords: interior-point method, $P_*(\kappa)$ LCPs, large-update method, parameterized kernel function.

1. Introduction

In this paper, we consider the following LCP in the standard form:

$$s = Mx + q, \quad xs = 0, \quad x \geq 0, \ s \geq 0. \tag{1}$$

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where $x, s, q \in R^n$, $M \in R^{n\times n}$, xs denotes the componentwise (Hadamard) product of vectors x and s.

Problem (1.1) is called a $P_*(\kappa)$ LCP if M is a $P_*(\kappa)$ matrix, i.e. there exists a nonnegative number κ such that

$$(1+4\kappa)\sum_{i\in J_+} x_i(Mx)_i + \sum_{i\in J_-} x_i(Mx)_i \geq 0,$$

where $J = \{1, 2, ..., n\}$, $J_+ = \{i \in J : x_i(Mx)_i \geq 0\}$ and $J_- = \{i \in J : x_i(Mx)_i < 0\}$. Furthermore, we define $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$. Note that the class P_* includes the class PSD of positive semi-definite matrices, and the class of P-matrices with all the principal minors positive.

Since the groundbreaking paper of Karmarkar in 1984 [1], some scholars and researchers are committed to the study of IPMs and a slew of results have been proposed (see [2, 3]). $P_*(\kappa)$ LCP is the largest class of LCPs which can be solved by interior-point methods (IPMs) in polynomial time. Indeed, the Karush-Kuhn-Tucker (KKT) optimality conditions for linear optimization (LO) and convex quadratic optimization can be formulated as LCPs, and LCPs are also closed related to variational inequalities. The basic theory, algorithms, and applications about LCPs can be referred in [3, 4, 5].

For a long time, most of the IPMs are designed based on the classical logarithmic barrier function. However, there is a gap between the theoretical behavior and practical computational efficiency of the algorithm: the large-update IPMs have better practical performance than the small-update IMPs, but theoretical result is relatively weaker. This gap was significantly reduced by Peng et al. who introduced the so-call self-regular kernel function and obtained the currently best known iteration bound of the large-update method for LO [6]. Subsequently, Bai et al. presented a series of eligible kernel functions and gave a scheme to analyze the complexity bound of the primal-dual IPMs for LO. They also obtained the best known iteration bound for large-update method [7]. Moreover, El Ghami et al. [8] first introduced a trigonometric kernel function, it is shown that in the IPMs based on this kernel function for large-update methods, the iteration bound is $O(n^{\frac{3}{4}}\log\frac{n}{\epsilon})$..

With the development of IPMs, the methods of solving LO based on a kernel function have been extended to some complementarity problems (CPs), which includes LCPs and nonlinear complementarity problems (NCPs), as well as some other important optimization problems such as semidefinite and second order cone optimization problems. For instance, Ji et al. proposed a large-update primal-dual IPM for $P_*(\kappa)$ LCPs based a new class kernel function [9], which is neither classical logarithmic nor self-regular function. Cai et al. introduced a new parametried kernel function with trigonometric barrier term [10], which yields a class of large-udate and small-update IPMs for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Li designed a large-update IPM for solving the $P_*(\kappa)$ nonlinear complementarity problem (NCP) based on a new class of parameterized kernel functions [11], the result coincides with the currently best known

iteration bounds for such methods. Achache et al. presented a full-Newton feasible step IPM for solving monotone horizontal linear complementarity problems [12]. For more studies about IPMs based on kernel functions please refer to [14, 13, 15, 16, 17, 18, 19].

The best known iteration bound for large-update IPMs was obtained so far based on some kernel functions in Table 1.

Table 1. Several known parameterized kernel functions

kernel function	parameter	iteration bound	References
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q-1}, q>1$	$q=\frac{1}{2}\log n$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[20],[21]
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q(q-1)}-\frac{q-1}{q}(t-1), q>1$	$q=\frac{1}{2}\log n$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[20],[21]
$\frac{t^2-1}{2}+\frac{1}{b}(e^{b(1-t)}-1),\ b\geq 0$	$b=O(\log n)$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[22]
$\frac{t^2-1}{2}-\int_1^t\left(\frac{e-1}{e^x-1}\right)^p dx,\ p\geq 1,$	$p=O(\log n)$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[17]
$\frac{t^2-1}{2}-\int_1^t e^{q(\frac{1}{\xi}-1)}d\xi,\ q\geq 1$	$q=O(\log n)$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[23]
$\frac{t^2-1}{2}+\frac{1}{q^2}\big(\frac{q}{t}-1\big)e^{q(\frac{1}{t}-1)}-\frac{q-1}{q^2}, q\geq 1$	$q=O(\log n)$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[24]
$\frac{t^2-1}{2}+\frac{q^{\frac{1}{t}-1}-1}{\log q},\ q>1$	$q=1+O(n)$	$O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$	[25]

Note that all kernel functions in Table 1 depend on a parameter. Moreover, the last three kernel functions in the above table are all generalized by the original kernel function when $q=1$, and the iteration bounds for large-update IPMs are reduced from $O\big(\sqrt{n}\log^2 n\log\frac{n}{\epsilon}\big)$ to $O\big(\sqrt{n}\log n\log\frac{n}{\epsilon}\big)$. In fact, as indicated in the recent studies about primal-dual IPMs based on kernel functions, the parameter not only represents a barrier degree of the kernel function, but also plays a role in improving the iteration complexity and practical computational efficiency of the algorithm.

Motivated by these work, in this paper we introduce a parameterized version of a kernel function that was previously presented by Bai et al. in 2002 [26]. This kernel function was also used to design a IPM for $P_*(\kappa)$ LCP proposed by cho [27], and the iteration bound for large-update method is $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$. As far as we know, no one has put forward its parameterized version. The new parameterized kernel function is not self-regular, but it is eligible. Based on this parameterized kernel function, we design a new primal-dual IPMs for $P_*(\kappa)$ LCP and obtain the best known iteration bounds for large-update and small-update method respectively, among which the result for large-update method is better than that of Cho.

The paper is organized as follows. In Section 2, we recall the basic concepts of the central path and search direction. We also give a generic IPM for $P_*(\kappa)$ LCP in Fig.1. New kernel function and its properties are introduced in Section 3. Section 4 is devoted to analyzing the convergence of the algorithm and deducing

the iteration bound of the algorithm. In section 5, we give some numerical results. Finally, concluding remarks are given in Section 6.

Some notations used throughout the paper are as below: R^n denotes the set of n-dimensional vectors, the set of n-dimensional nonnegative vectors and positive vectors are denoted as R^n_+ and R^n_{++}, respectively. For any vector $x \in R^n$, the ith component of vector x is denoted as x_i, x_{min} denotes the smallest ith component of vector x, $X := diag(x)$. Furthermore, e denotes n-dimensional vector of ones and E denotes identity matrix. $\|\cdot\|$ denotes the 2-norm of a vector. If $f : R_+ \to R_+$ and $x \in R^n_+$, then $f(x)$ denotes a vector in R^n_+ whose ith component is $f(x_i)$. The notation $f(x) = O(g(x))$ denotes $f(x) \leq cg(x)$ for some positive constant c, and $f(x) = \Theta(g(x))$ denotes $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for positive constants c_1 and c_2. The functions like f, g have a local meaning in this paper.

2. Preliminaries

It is well known that the basic idea of the primal-dual IPM is to replace the complementarity condition $xs = 0$ in (1) by the parameterized equation $xs = \mu e(\mu > 0)$, thus we have

$$s = Mx + q, \quad xs = \mu e, \quad x > 0,\ s > 0. \tag{2}$$

A solution of (2) exists if and only if LCP satisfies the *interior-point condition* (IPC), i.e., there exists (x^0, s^0) satisfying $s^0 = Mx^0 + q,\ x^0 > 0,\ s^0 > 0$. Without loss of generality, we can assume that the IPC is satisfied. Since M is a $P_*(\kappa)$ matrix and the IPC holds, then for each $\mu > 0$, a unique solution for parameterized system (2) exists. The solution of (2) is denoted as $(x(\mu), s(\mu))$, where $(x(\mu), s(\mu))$ is called the μ-center of the LCP. The set of μ-center gives a homotopy path, which is called the *central path* of the LCP. Under the above assumptions, if $\mu \to 0$, then the limit of the central path yields optimal solutions for LCP [28].

Let't define

$$d_x := \frac{\upsilon \Delta x}{x}, \qquad d_s := \frac{\upsilon \Delta s}{s}, \qquad \text{where } \upsilon := \sqrt{\frac{xs}{\mu}}. \tag{3}$$

The search direction of the algorithm is determined by Newton's method, and this yields the following scaled system:

$$\begin{cases} -\overline{M} d_x + d_s = 0, \\ d_x + d_s = \upsilon^{-1} - \upsilon, \end{cases} \tag{4}$$

where $\overline{M} := X^{\frac{1}{2}} S^{-\frac{1}{2}} M X^{\frac{1}{2}} S^{-\frac{1}{2}}$, $X = diag(x)$, $S = diag(s)$. A crucial observation is that the second equation in (4) is called the *scaled centering equation*, where $\upsilon - \upsilon^{-1}$ is exactly the negative gradient of the classical logarithmic barrier function, which is given by $\Psi_c(\upsilon) := \sum_{i=1}^n \left(\frac{\upsilon_i^2 - 1}{2} - log\ \upsilon_i \right)$.

In this paper, we replace the classical logarithmic barrier function $\Psi_c(\upsilon)$ by a new barrier function $\Psi(\upsilon)$. Therefore, the second equation in system (4) can be rewritten as $d_x + d_s = - \bigtriangledown \Psi(\upsilon)$. By using this result and (3), we have the following modified Newton system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ X\Delta s + S\Delta x = -\mu\upsilon \bigtriangledown \Psi(\upsilon). \end{cases} \tag{5}$$

The barrier function $\Psi(\upsilon)$ is determined by its kernel function $\psi(t)$ as below:

$$\Psi(\upsilon) := \sum_{i=1}^{n} \psi(\upsilon_i), \tag{6}$$

where a twice differentiable function $\psi(t) : R_{++} \to R_+$ is called a kernel function if $\psi(t)$ satisfies the following conditions:

$$\psi(1) = \psi'(1) = 0; \qquad \psi''(t) > 0, \quad \forall t > 0; \qquad \lim_{t \downarrow 0} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty.$$

Hence $\psi(t)$ can be completely determined by its second derivative as follows:

$$\psi(t) = \int_1^t \int_1^{\xi} \psi''(\zeta) d\zeta d\xi. \tag{7}$$

The system (5) uniquely defines a centering search direction $(\Delta x, \Delta s)$ [28]. By taking a step along the search direction, with the step size α defined by some line search rules. The new point (x_+, s_+) is then computed by

$$x_+ = x + \alpha\Delta x; \qquad s_+ = s + \alpha\Delta s. \tag{8}$$

A generic primal-dual algorithm for LCP is given below:

Generic Primal-Dual Algorithm for LCP

Input:
A threshold parameter $\tau > 0$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
begin
$x := x^0 > 0; s := s^0 > 0$; $x^0 s^0 = \mu^0 e$; $\mu = \mu^0$;
while $n\mu \geq \epsilon$ **do**
begin
$\mu := (1 - \theta)\mu$;
while $\Psi(\upsilon) > \tau$ **do**
begin
solve system (5) for Δx and Δs;
determine a step size α;
$x := x + \alpha\Delta x$; $s := s + \alpha\Delta s$;
end
end
end

Fig.1 Generic Algorithm

Remark 2.1. Usually, if $\tau = O(n)$ and $\theta = \Theta(1)$, the algorithm is called a *large-update* method. If $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, then we call the algorithm a *small-update* method.

3. New kernel function and its properties

Bai et al. introduced the following kernel function in [26]:

$$\psi_A(t) = \frac{t^2-1}{2} + \frac{(e-1)^2}{e(e^t-1)} - \frac{e-1}{e},\ t > 0. \tag{9}$$

In this paper, we parameterize it and obtain a new kernel functions as follows:

$$\psi(t) = \frac{t^2-1}{2} + \frac{(e-1)^{q+1}}{qe(e^t-1)^q} - \frac{e-1}{qe},\ t > 0,\ q \geq 1. \tag{10}$$

For simplicity of presentation, we define $M = \frac{(e-1)^{q+1}}{e}$. Some straightforward computations yield the first three derivatives of $\psi(t)$ with respect to t as below:

$$\psi'(t) = t - \frac{Me^t}{(e^t-1)^{q+1}}, \tag{11a}$$

$$\psi''(t) = 1 + \frac{M(qe^{2t}+e^t)}{(e^t-1)^{q+2}}, \tag{11b}$$

$$\psi'''(t) = -\frac{M[q^2e^{3t} + (3q+1)e^{2t} + e^t]}{(e^t-1)^{q+3}}. \tag{11c}$$

In the following convergence analysis of the method we use the norm-based proximity measure $\delta(\upsilon)$ defined by

$$\delta(\upsilon) := \frac{1}{2}\|\bigtriangledown\Psi(\upsilon)\| = \frac{1}{2}\|d_x + d_s\|. \tag{12}$$

It is known from the definition of the barrier function that the barrier function $\Psi(\upsilon)$ is strictly convex and the minimum value is obtained at $\upsilon = e$, thus we have $\Psi(\upsilon) = 0 \Leftrightarrow \delta(\upsilon) = 0 \Leftrightarrow \upsilon = e$.

Bai et al. gave five more conditions on the kernel function in the literature [7], namely,

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1; \tag{13a}$$

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1; \tag{13b}$$

$$\psi'''(t) < 0, \quad t > 0; \tag{13c}$$

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1; \tag{13d}$$

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 1,\ \beta > 1. \tag{13e}$$

Obviously, the condition (13a) is satisfied if $t \geq 1$, because then $\psi'(t) \geq 0$ and $\psi''(t) > 0$. Thus the condition (13a) holds for $t > 0$. Similar cases also

occur on conditions (13b) and (13d). Following [7], a kernel function $\psi(t)$ is called an eligible kernel function if $\psi(t)$ satisfies conditions (13a), (13c), (13d) and (13e). Moreover, If $\psi(t)$ satisfies (13b) and (13c), then $\psi(t)$ satisfies (13e). Now one easily checks that our kernel function $\psi(t)$ is eligible.

Theorem 3.1. *$\psi(t)$ is an eligible kernel function.*

Proof. Let $N = \frac{Me^t}{(e^t-1)^{q+1}},\ t > 0,\ q \geq 1$. To prove that the kernel function $\psi(t)$ is eligible, it is sufficient to prove that $\psi(t)$ satisfies (13a), (13b), (13c) and (13d).

(i) By using (11a) and (11b), we have

$$\begin{aligned} t\psi''(t) + \psi'(t) &= t\left[1 + \frac{N(qe^t+1)}{e^t-1}\right] + t - N \\ &= 2t + \frac{N[(qe^t+1)t - e^t + 1]}{e^t-1},\ t > 0,\ q \geq 1. \end{aligned}$$

Let $f(t) = (qe^t+1)t - e^t + 1$. One has $f'(t) = (q-1)e^t + tqe^t + 1 > 0,\ t > 0,\ q \geq 1$, which implies $f(t)$ is strictly monotonically increasing. Using that $f(t)$ is continuous, it follows that if $t > 0$, then $f(t) > f(0) = 0$ holds. In conclusion, we have $t\psi''(t) + \psi'(t) = 2t + \frac{Nf(t)}{e^t-1} > 0,\ t > 0,\ q \geq 1$.

(ii) From (11a) and (11b), we derive that

$$\begin{aligned} t\psi''(t) - \psi'(t) &= t\left[1 + \frac{N(qe^t+1)}{e^t-1}\right] - t + N \\ &= \frac{N(qe^t+1)t}{e^t-1} + N > 0,\ t > 0,\ q \geq 1. \end{aligned}$$

(iii) Due to (11c), we immediately get $\psi'''(t) < 0,\ t > 0,\ q \geq 1$.

(iv) Combining with (11a), (11b) and (11c), it is verified that

$$\begin{aligned} &2\psi''(t)^2 - \psi'(t)\psi'''(t) \\ &= 2\left[1 + \frac{N(qe^t+1)}{e^t-1}\right]^2 + (t-N)\frac{N\left[q^2e^{2t} + (3q+1)e^t + 1\right]}{(e^t-1)^2} \\ &\geq \frac{N^2[2(qe^t+1)^2 - (q^2e^{2t} + (3q+1)e^t + 1)]}{(e^t-1)^2} \\ &= \frac{N^2[q^2e^{2t} + (q-1)e^t + 1]}{(e^t-1)^2} > 0,\ t > 0, q \geq 1. \end{aligned}$$

This proves that $\psi(t)$ satisfies (13a), (13b), (13c) and (13d), the proof is completed. □

In the following lemma we list some formulas that are equivalent to (13a).

Lemma 3.2 ([21],Lemma 2.1.2)**.** *The following three formulas are equivalent if $\psi(t)$ is a twice differentiable function for $t > 0$:*

(i) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)),\ t_1,\ t_2 > 0;$

(ii) $\psi'(t) + t\psi''(t) \geq 0,\ t > 0;$

(iii) $\psi(e^{\xi})$ *is convex.*

In fact, a twice differentiable function is called *exponential convex* or *e-convex* if it satisfies the property described in Lemma 3.2, and this property has been proved to be essential in analyzing the convergence of the primal-dual IPMs based on kernel functions. It is clear that our kernel function $\psi(t)$ is e-convex by Theorem 3.1.

Define $\psi_b(t) := \frac{(e-1)^{q+1}}{qe(e^t-1)^q} - \frac{e-1}{qe},\ t > 0,\ q \geq 1$. One may easily verify that $\psi_b(t)$ is monotonically decreasing and $\psi_b'(t)$ is monotonically increasing for $t \in (0, \infty)$. We also have $\psi_b(1) = 0$ and $\psi_b'(1) = 1$.

The following lemmas give several crucial properties which are important in the analysis of the algorithm.

Lemma 3.3. *Let $\psi(t)$ be as defined in (10), we have*

(i) $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2,\ t > 0,\ q \geq 1;$

(ii) $\psi(t) \leq (q+1)(t-1)^2,\ t > 1,\ q \geq 1;$

(iii) $\delta(v) \geq \sqrt{\frac{\Psi}{2}},\ t > 0,\ q \geq 1.$

Proof. (i) By using (7) and $\psi''(t) \geq 1$, for $q \geq 1$ and $t > 0$, we have

$$\psi(t) = \int_1^t \int_1^{\xi} \psi''(\zeta) d\zeta d\xi \geq \int_1^t \int_1^{\xi} 1 d\zeta d\xi = \frac{1}{2}(t-1)^2$$

and

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^{\xi} \psi''(\zeta) d\zeta d\xi \leq \int_1^t \int_1^{\xi} \psi''(\xi)\psi''(\zeta) d\zeta d\xi \\ &= \int_1^t \psi''(\xi)\psi'(\xi) d\xi = \int_1^t \psi'(\xi) d\psi'(\xi) = \frac{1}{2}\psi'(t)^2.\end{aligned}$$

(ii) Using Taylor's theorem, $\psi(1) = \psi'(1) = 0$, $\psi''(1) = \frac{qe+1}{e-1}$, and $\psi'''(t) < 0,\ t > 0$, we have

$$\begin{aligned}\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{3!}\psi'''(\xi)(t-1)^3 \\ &= \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{3!}\psi'''(\xi)(t-1)^3 \\ &< \frac{qe+1}{2(e-1)}(t-1)^2 < (q+1)(t-1)^2,\ 1 < \xi < t,\ q \geq 1.\end{aligned}$$

(iii) From the right inequality of (i), we obtain

$$\Psi(v) = \sum_{i=1}^n \psi(v_i) \leq \frac{1}{2}\sum_{i=1}^n \psi'(v_i)^2 = \frac{1}{2}\| \bigtriangledown \Psi(v)\|^2 = 2\delta(v)^2,$$

which means $\delta(v) \geq \sqrt{\frac{\Psi}{2}}$. This completes the proof. □

Lemma 3.4. *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0, 1]$. Then*

$$\frac{1}{e^{\rho(s)} - 1} \leq \left(\frac{2s+1}{M}\right)^{\frac{1}{q+1}}, \ q \geq 1.$$

Proof. Putting $t = \rho(s)$, which implies $s = -\frac{1}{2}\psi'(t) = -\frac{1}{2}(t - \frac{Me^t}{(e^t-1)^{q+1}})$. Hence we may get

$$2s + 1 \geq 2s + t = \frac{Me^t}{(e^t - 1)^{q+1}} \geq \frac{M}{(e^t - 1)^{q+1}}, \ t \in (0, 1], \ q \geq 1.$$

This implies

$$\frac{1}{e^t - 1} = \frac{1}{e^{\rho(s)} - 1} \leq \left(\frac{2s+1}{M}\right)^{\frac{1}{q+1}}, \ t \in (0, 1], \ q \geq 1,$$

which proves the lemma. □

Lemma 3.5. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \in [1, \infty)$. Then $\sqrt{1 + 2s} \leq \varrho \leq 1 + \sqrt{2s}$.*

Proof. Let $t = \varrho(s)$, then we have $s = \psi(t) = \frac{t^2-1}{2} + \psi_b(t), \ q \geq 1, \ t \in [1, \infty)$. Since $\psi_b(t)$ is decreasing for $t > 1$ and $\psi_b(1) = 0$, one has $s = \psi(t) \leq \frac{t^2-1}{2}, \ t \in [1, \infty)$. This gives $\sqrt{1 + 2s} \leq \varrho(s)$. For the second inequality we use the result (i) in Lemma 3.3, thus we obtain $s = \psi(t) \geq \frac{1}{2}(t-1)^2$, therefore $\varrho(s) = t \leq 1+\sqrt{2s}$. This completes the proof of the lemma. □

4. Complexity analysis of the algorithm

Note that our algorithm consists of two parts: inner iteration and outer iteration. Every time before the outer iteration of the algorithm begins, just before the $\mu-$update with the factor $1 - \theta$, $0 < \theta < 1$, we have $\Psi(v) \leq \tau$. The vector v is divided by the factor $\sqrt{1 - \theta}$ in the outer iteration, which in general leads to an increase in the value of $\Psi(v)$. Then the algorithm starts executing the inner iterations if $\Psi(v) > \tau$, and the inner iterations will decrease the value of $\Psi(v)$. The algorithm returns to the outer iteration again when $\Psi(v) \leq \tau$. Repeat the above iteration process until μ is small enough, say, until $n\mu \leq \epsilon$, at this stage we have found an $\epsilon-$solution of the $P_*(\kappa)$ LCP.

4.1 Growth behavior

From the above analysis we conclude that the largest value of $\Psi(v)$ occur after the μ-update, just before the inner iteration begins. What we want is to find an

upper bound of $\Psi(\upsilon)$ to research the amount of decrease of the barrier function during an inner iteration. Due to the fact that $\psi(t)$ is eligible, and also [7], we have the following lemma.

Lemma 4.1. *Let ϱ be as defined in Lemma 3.5, and $\upsilon \in R^n_{++}$, $\beta \geq 1$, then*

$$\Psi(\beta\upsilon) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(\upsilon)}{n}\right)\right).$$

Define

$$L_\psi(n,\theta,\tau) := n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\tau}{n}\right)\right). \tag{14}$$

Obviously, if $\Psi(\upsilon) \leq \tau$ and $\beta = \frac{1}{\sqrt{1-\theta}}$, then $L_\psi(n,\theta,\tau)$ is an upper bound for $\Psi\left(\frac{\upsilon}{\sqrt{1-\theta}}\right)$ by using Lemma 4.1.

Lemma 4.2. *Using the notations of (14), if $\Psi(\upsilon) \leq \tau$, then we have*

(i) $L_\psi(n,\theta,\tau) \leq \frac{n\theta+2\sqrt{2n\tau}+2\tau}{2(1-\theta)}$;

(ii) $L_\psi(n,\theta,\tau) \leq \frac{(q+1)(\sqrt{n}\theta+\sqrt{2\tau})^2}{1-\theta}$.

Proof. (i) Since $\psi_b(t)$ is monotonically decreasing for $t \geq 1$, and $\psi_b(1) = 0$, we get

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t) \leq \frac{t^2-1}{2},\ t \geq 1,\ q \geq 1.$$

Since $\frac{1}{\sqrt{1-\theta}} > 1$ for $0 < \theta < 1$ and $\varrho(\frac{\tau}{n}) > 1$, we have $\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}} > 1$. Using above results, Lemma 3.5 and $\psi(t)$ is monotonically increasing for $t \in [1,\infty)$, we have

$$\begin{aligned} L_\psi(n,\theta,\tau) &= n\psi\left(\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right) \\ &\leq n\frac{\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right)^2 - 1}{2} = \frac{n\theta+2\sqrt{2n\tau}+2\tau}{2(1-\theta)}. \end{aligned}$$

(ii) From the result (ii) in Lemma 3.3 and Lemma 3.5, we deduce that

$$\begin{aligned} L_\psi(n,\theta,\tau) &= n\psi\left(\frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right) \\ &\leq n(q+1)\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}} - 1\right)^2 \\ &= n(q+1)\frac{\left(1+\sqrt{\frac{2\tau}{n}}-\sqrt{1-\theta}\right)^2}{1-\theta} \leq \frac{(q+1)\left(\sqrt{n}\theta+\sqrt{2\tau}\right)^2}{1-\theta}, \end{aligned}$$

where the last inequality is obtained by $1-\sqrt{1-\theta}=\frac{\theta}{1+\sqrt{1-\theta}}\leq\theta$. Hence the lemma is proved. □

Defining

$$L_{\psi 1}:=L_{\psi 1}(n,\theta,\tau)=\frac{n\theta+2\sqrt{2n\tau}+2\tau}{2(1-\theta)};$$
$$L_{\psi 2}:=L_{\psi 2}(n,\theta,\tau)=\frac{(q+1)\Big(\sqrt{n}\theta+\sqrt{2\tau}\Big)^2}{1-\theta}.$$

We will use $L_{\psi 1}$ and $L_{\psi 2}$ for the upper bounds of $\Psi(\upsilon)$ for large-update and small-update methods, respectively. From Remark 2.1, if we set $\tau=O(n)$ and $\theta=\Theta(1)$, then $L_{\psi 1}=O(n)$. If we choose $\tau=O(1)$ and $\theta=\Theta(\frac{1}{\sqrt{n}})$, then $L_{\psi 2}=O(q)$.

4.2 Decrease behavior and the choice of step size

This section serves to analyze the decreasing behavior of the barrier function and the choice of step size during an inner iteration. After a damped step we have

$$x_+=x+\alpha\Delta x=x(e+\alpha\frac{\Delta x}{x})=x(e+\frac{d_x}{\upsilon})=\frac{x}{\upsilon}(\upsilon+\alpha d_x);$$
$$s_+=s+\alpha\Delta s=s(e+\alpha\frac{\Delta s}{s})=s(e+\frac{d_s}{\upsilon})=\frac{s}{\upsilon}(\upsilon+\alpha d_s).$$

Thus we obtain

$$\upsilon_+=\sqrt{\frac{x_+s_+}{\mu}}=\sqrt{(\upsilon+\alpha d_x)(\upsilon+\alpha d_s)}. \tag{15}$$

Since ψ is e-convex, then one can get

$$\Psi(\upsilon_+)=\Psi(\sqrt{(\upsilon+\alpha d_x)(\upsilon+\alpha d_s)}\leq\frac{1}{2}\big(\Psi(\upsilon+\alpha d_x)+\Psi(\upsilon+\alpha d_s)\big). \tag{16}$$

Define

$$f(\alpha):=\Psi(\upsilon_+)-\Psi(\upsilon),$$

which denotes the amount of decrease of the barrier function after an inner iteration. An immediate consequence following from (16) is

$$f(\alpha)\leq f_1(\alpha):=\frac{1}{2}(\Psi(\upsilon+\alpha d_x)+\Psi(\upsilon+\alpha d_s))-\Psi(\upsilon).$$

The first derivative and the second derivative of $f_1(\alpha)$ are given as follows:

$$f_1'(\alpha)=\frac{1}{2}\sum_{i=1}^{n}(\psi'(\upsilon_i+\alpha d_{x_i})d_{x_i}+\psi'(\upsilon_i+\alpha d_{s_i})d_{s_i}); \tag{17}$$

$$f_1''(\alpha)=\frac{1}{2}\sum_{i=1}^{n}(\psi''(\upsilon_i+\alpha d_{x_i})d_{x_i}^2+\psi''(\upsilon_i+\alpha d_{s_i})d_{s_i}^2). \tag{18}$$

We thus have

$$f(0) = f_1(0) = 0; \quad f_1'(0) = \frac{1}{2}\nabla\Psi(\upsilon)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(\upsilon)^T\nabla\Psi(\upsilon) = -2\delta(\upsilon)^2.$$

Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (5), for $\Delta x, \Delta s \in R^n$ we have

$$(1+4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i + \sum_{i\in J_-}\Delta x_i\Delta s_i \geq 0.$$

where $J_+ = \{i \in J : \Delta x_i\Delta s_i \geq 0\}$ and $J_- = J - J_+$. Note that $d_x d_s = \frac{\upsilon^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}, \mu > 0$, we obtain

$$(1+4\kappa)\sum_{i\in J_+} d_{x_i}d_{s_i} + \sum_{i\in J_-} d_{x_i}d_{s_i} \geq 0.$$

With $\delta(\upsilon)$ as defined in (12), then we use the following notations:

$$\upsilon_1 := \upsilon_{\min}, \qquad \delta := \delta(\upsilon).$$

The following lemmas give the bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 4.3 ([27], Lemma 4.2)**.** $\sum_{i=1}^n (d_{x_i}^2 + d_{s_i}^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\sqrt{1+2\kappa}\delta$, *and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\delta$.

Lemma 4.4. $f_1''(\alpha) \leq 2(1+2\kappa)\delta^2\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)$.

Proof. Using Lemma 4.3, we have the following inequalities:

$$\upsilon_i + \alpha d_{x_i} \geq \upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}, \quad \upsilon_i + \alpha d_{s_i} \geq \upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa},$$

where $1 \leq i \leq n$. From the above result, Lemma 4.3, (18) and $\psi''(t)$ is monotonically decreasing, we deduce

$$\begin{aligned}
f_1''(\alpha) &= \frac{1}{2}\sum_{i=1}^n(\psi''(\upsilon_i + \alpha d_{x_i})d_{x_i}^2 + \psi''(\upsilon_i + \alpha d_{s_i})d_{s_i}^2) \\
&\leq \frac{1}{2}\sum_{i=1}^n\Big(\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)d_{x_i}^2 + \psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)d_{s_i}^2\Big) \\
&= \frac{1}{2}\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)\sum_{i=1}^n(d_{x_i}^2 + d_{s_i}^2) \\
&\leq \frac{1}{2}\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big)\big(\|d_x\|^2 + \|d_s\|^2\big) \\
&\leq 2(1+2\kappa)\delta^2\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big).
\end{aligned}$$

We get the desired result. □

Lemma 4.5. *If α satisfies the inequality*

$$-\psi'\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi'(\upsilon_1) \le \frac{2\delta}{\sqrt{1+2\kappa}}, \tag{19}$$

then $f_1'(\alpha) \le 0$.

Proof. By Lemma 4.4, we obtain

$$\begin{aligned}
f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\xi)d\xi \\
&\le -2\delta^2 + 2\delta^2(1+2\kappa)\int_0^\alpha \psi''\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big)d\xi \\
&= -2\delta^2 - \delta\sqrt{1+2\kappa}\int_0^\alpha \psi''\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big)d\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big) \\
&= -2\delta^2 - \sqrt{1+2\kappa}\delta\Big(\psi'\big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\big) - \psi'\big(\upsilon_1\big)\Big) \\
&\le -2\delta^2 + \sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}} \\
&= 0.
\end{aligned}$$

The lemma is proved. □

Lemma 4.6. *Using the notions of Lemma 3.4, if step size α satisfies (19), then the largest step size α is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\Big(\big(1 + \frac{1}{\sqrt{1+2\kappa}}\big)\delta\Big)\right). \tag{20}$$

Proof. Since $\psi''(t)$ is monotonically decreasing, the derivative of the left hand side in (19) with respect to υ_1 satisfies

$$-\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi''(\upsilon_1) < 0.$$

This implies that the left hand side in (19) is monotonically decreasing in υ_1. Therefore, for fixed δ, the smaller the value of υ_1, the smaller the value of α. By the definition of δ and $\Psi(\upsilon)$, we have

$$\delta = \frac{1}{2}\|\nabla\Psi(\upsilon)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\Psi(\upsilon))^2} \ge \frac{1}{2}|\psi'(\upsilon_1)| \ge -\frac{1}{2}\psi'(\upsilon_1).$$

Note that equality holds if and only if υ_1 is the only coordinate in υ which differs from 1 and $\upsilon_1 \le 1$, i.e., $\psi'(\upsilon_1) \le 0$. Hence the smallest step size α occurs when υ_1 satisfies

$$-\frac{1}{2}\psi'(\upsilon_1) = \delta. \tag{21}$$

In this case, we have

$$\upsilon_1 = \rho(\delta). \tag{22}$$

Since $\psi''(t) > 0$, the derivative with respect to α of the left hand side in (19) satisfies

$$2\delta\sqrt{1+2\kappa}\psi''\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) \geq 0,$$

which means the left hand side in (19) is monotone increasing with respect to α. Therefore, the largest step size satisfying (19) occurs when

$$-\psi'\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) + \psi'(\upsilon_1) = \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{23}$$

From (21), (23) can be written as

$$-\frac{1}{2}\psi'\big(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}\big) = \delta\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right).$$

By above equation and the definition of ρ, we obtain

$$\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa} = \rho\left(\delta\Big(1 + \frac{1}{\sqrt{1+2\kappa}}\Big)\right).$$

Hence, by (22), the largest step size α follows that

$$\bar{\alpha} = \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\Big(\Big(1 + \frac{1}{\sqrt{1+2\kappa}}\Big)\delta\Big)\right),$$

and the lemma is proved. □

Lemma 4.7. *Let ρ and $\bar{\alpha}$ be defined as in Lemma 4.6, then*

$$\bar{\alpha} \geq \tilde{\alpha} := \frac{1}{1+2\kappa}\frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}, \tag{24}$$

and we will use $\tilde{\alpha}$ as the default step size.

Proof. By the definition of ρ, we have

$$-\frac{1}{2}\psi'(\rho(\delta)) = \delta$$

Taking the derivative with respect to δ, we obtain

$$-\frac{1}{2}\psi''(\rho(\delta))\rho'(\delta) = 1.$$

Since $\psi''(\rho(\delta)) > 0$, we have

$$\rho'(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0,$$

which means that $\rho(\delta)$ is monotonically decreasing in δ. Using the above result, (20) and the fundamental theorem of calculus, we get

$$\begin{aligned}\bar{\alpha} &= \frac{1}{2\delta\sqrt{1+2\kappa}}\Big(\rho(\delta) - \rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)\Big)\\ &= \frac{1}{2\delta\sqrt{1+2\kappa}}\int_{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}^{\delta}\rho'(\xi)d\xi\\ &= \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}\frac{1}{\psi''(\rho(\xi))}d\xi\end{aligned}$$

We want to obtain a lower bound for $\bar{\alpha}$ by replacing the argument of the last integral with its minimal value. Hence we would like to know when the $\psi''(\rho(\xi))$ attains the maximum value for $\xi \in [\delta, (1+\frac{1}{\sqrt{1+2\kappa}})\delta]$. Since ψ'' and ρ are monotonically decreasing, we have

$$\rho(\xi) \le \rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)$$

and

$$\psi''(\rho(\xi)) \le \psi''\Big(\rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)\Big).$$

As a consequence, we obtain

$$\frac{1}{\psi''(\rho(\xi))} \le \frac{1}{\psi''\Big(\rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)\Big)}.$$

Therefore, we have

$$\begin{aligned}\bar{\alpha} &= \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}\frac{1}{\psi''(\rho(\xi))}d\xi\\ &\ge \frac{1}{\delta\sqrt{1+2\kappa}}\frac{1}{\psi''\Big(\rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)\Big)}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}d\xi\\ &= \frac{1}{1+2\kappa}\frac{1}{\psi''\Big(\rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)\Big)}\end{aligned}$$

The corollary is proved. □

Lemma 4.8. *If $\bar{\alpha}$ is defined as in Lemma 4.6, then*

$$\bar{\alpha} \ge \frac{1}{4(1+2\kappa)(2q+1)(4\delta+1)^{\frac{q+2}{q+1}}},\ q \ge 1.$$

Proof. Let $r = \rho\Big((1+\frac{1}{\sqrt{1+2\kappa}})\delta\Big)$. By using ρ defined in Lemma 3.4, we have

$$\frac{1}{e^{\rho\left((1+\frac{1}{\sqrt{1+2\kappa}})\delta\right)}-1} = \frac{1}{e^r-1} \leq \left(\frac{2\delta(1+\frac{1}{\sqrt{1+2\kappa}})+1}{M}\right)^{\frac{1}{q+1}},\ q \geq 1.$$

Combining the above results and (11b), through the simple calculation, we derive that

$$\begin{aligned}
\psi''(r) &= 1 + \frac{M(qe^{2r}+e^r)}{(e^r-1)^{q+2}} \\
&\leq 1 + \left(\frac{2\delta(1+\frac{1}{\sqrt{1+2\kappa}})+1}{M}\right)^{\frac{q+2}{q+1}} M(qe^{2r}+e^r) \\
&= 1 + M^{-\frac{1}{q+1}}\left(2\delta\big(1+\frac{1}{\sqrt{1+2\kappa}}\big)+1\right)^{\frac{q+2}{q+1}}(qe^{2r}+e^r) \\
&\leq 1 + (4\delta+1)^{\frac{q+2}{q+1}}(qe^{2r}+e^r) \\
&\leq 4(2q+1)(4\delta+1)^{\frac{q+2}{q+1}},\ q \geq 1.
\end{aligned}$$

From Lemma 4.6, we derive that

$$\begin{aligned}
\bar{\alpha} \geq \tilde{\alpha} &:= \frac{1}{1+2\kappa}\frac{1}{\psi''(\rho(1+\frac{1}{\sqrt{1+2\kappa}}\delta))} \\
&\geq \frac{1}{4(1+2\kappa)(2q+1)(4\delta+1)^{\frac{q+2}{q+1}}},\ q \geq 1.
\end{aligned}$$

This proves the lemma. □

Lemma 4.9 ([6], Lemma 12)**.** *Suppose that $h(t)$ is a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and $h(t)$ attains its global minimum at $t^* > 0$. If $h''(t)$ is monotonically increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2},\ 0 \leq t \leq t^*.$$

Lemma 4.10. *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2. \tag{25}$$

Proof. Define the univariate function h as below:

$$\begin{aligned}
&h(0) = f_1(0) = 0, h'(0) = f_1'(0) = -2\delta^2, \\
&h''(\alpha) = 2(1+2\kappa)\delta^2\psi''(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}).
\end{aligned}$$

According to Lemma 4.4, we have $f_1''(\alpha) \leq h''(\alpha)$, which implies $f_1'(\alpha) \leq h'(\alpha)$ and $f_1(\alpha) \leq h(\alpha)$. Since $\psi''(t) > 1$, $h''(\alpha) \geq 2(1+\kappa)\delta^2$. Thus $h(\alpha)$ is strongly

convex and $h(\alpha)$ attains its global minimum for some $\alpha^* > 0$. Taking $\alpha \leq \bar{\alpha}$, with $\bar{\alpha}$ as defined by (20), using the fundamental theorem of calculus, and Lemma 4.5, we may get

$$\begin{aligned} h'(\alpha) &= h'(0) + \int_0^\alpha h''(\xi)d\xi \\ &= -2\delta^2 + 2\delta^2(1+2\kappa)\int_0^\alpha \psi''(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}))d\xi \\ &= -2\delta^2 - \delta\sqrt{1+2\kappa}\Big(\psi'(\upsilon_1 - 2\alpha\delta\sqrt{1+2\kappa}) - \psi'(\upsilon_1)\Big) \\ &\leq -2\delta^2 + \delta\sqrt{1+2\kappa}\frac{2\delta}{\sqrt{1+2\kappa}} = 0 \end{aligned}$$

Since $h'''(\alpha) = -4(1+2\kappa)^{\frac{3}{2}}\delta^3\psi'''\Big(\upsilon_1 - 2\xi\delta\sqrt{1+2\kappa}\Big)$ and $\psi'''(t) < 0$, $h''(\alpha)$ is increasing in α. From Lemma 4.9, we may write

$$f_1(\alpha) \leq h(\alpha) \leq \frac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

Due to $f(\alpha) \leq f_1(\alpha)$, the proof is completed. □

Lemma 4.11. *One has*

$$(26)\ f(\tilde{\alpha}) \leq -\frac{1}{1+2\kappa}\frac{\delta^2}{\psi''(\rho(1+\frac{1}{\sqrt{1+2\kappa}}\delta))} \leq -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\Big(4+\frac{1}{\delta}\Big)^{\frac{q+2}{q+1}}}.$$

Proof. According to Lemmas 4.7, 4.8 and 4.10, we have

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2 \leq -\frac{1}{1+2\kappa}\frac{\delta^2}{\psi''(\rho(1+\frac{1}{\sqrt{1+2\kappa}}\delta))} \\ &\leq -\frac{\delta^2}{4(1+2\kappa)(2q+1)(4\delta+1)^{\frac{q+2}{q+1}}} \\ &\leq -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\Big(4+\frac{1}{\delta}\Big)^{\frac{q+2}{q+1}}}. \end{aligned}$$

Hence, the result of this lemma holds. □

4.3 Iteration complexity

Our aim in this section is to analyze the convergence of the algorithm. The question now is to count how many inner iterations are required to return to the situation where $\Psi(\upsilon) \leq \tau$. To investigate this, we define the value of $\Psi(\upsilon)$ after each $\mu-$update as Ψ_0, and the subsequent values during inner iterations as

$\Psi_\kappa,\ \kappa = 1, 2, \ldots, K$. Thus K is the number of iterations in the inner iteration after once $\mu-$update.

By using Lemmas 4.1, 4.2 and the definition of Ψ_0, we have $L_\Psi \geq \Psi_0 \geq \Psi \geq \tau$. In what follows we assume $L_\psi \geq \Psi_0 \geq \Psi \geq \tau \geq 2$. From Lemma 3.5, we deduce that $\delta(\upsilon) \geq \sqrt{\frac{\Psi}{2}} \geq 1$. Substitution into (26) gives

$$f(\tilde{\alpha}) \leq -\frac{\delta^{\frac{q}{q+1}}}{4(1+2\kappa)(2q+1)\Big(4+\frac{1}{\delta}\Big)^{\frac{q+2}{q+1}}} \leq -\frac{\Psi^{\frac{q}{2(q+1)}}}{56(1+2\kappa)(2q+1)}, \tag{27}$$

which implies

$$\Psi_{\kappa+1} \leq \Psi_\kappa - \frac{\Psi_\kappa^{\frac{q}{2(q+1)}}}{56(1+2\kappa)(2q+1)},\ \kappa = 0, 1, 2, \ldots, K-1. \tag{28}$$

To derive an upper bound for the total number of inner iterations in an outer iteration, we recall the following technical lemma, and its elementary proof please refer to [6].

Lemma 4.12. *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{\kappa+1} \leq t_\kappa - \beta t_\kappa^{1-\gamma},\ \kappa = 0, 1, 2, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lfloor \frac{t_0^\gamma}{\beta\gamma} \rfloor$.

Lemma 4.13. *The following inequality holds:*

$$K \leq 112(1+2\kappa)(2q+1)\Psi_0^{\frac{q+2}{2(q+1)}},\ q \geq 1.$$

Proof. Let $t_\kappa = \Psi_\kappa,\ \beta = \frac{1}{56(2q+1)},\ \gamma = \frac{q+2}{2(q+1)}$. Using Lemma 4.12 and substitution gives, we have

$$\begin{aligned} K \leq \frac{\Psi_0^\gamma}{\beta\gamma} &= \frac{112(1+2\kappa)(2q+1)(q+1)\Psi_0^{\frac{q+2}{2(q+1)}}}{q+2} \\ &\leq 112(1+2\kappa)(2q+1)\Psi_0^{\frac{q+2}{2(q+1)}},\ q \geq 1. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.13 gives an upper bound for the number of iterations in the inner iteration after once $\mu-$update. Multiplication the number K by the number of barrier parameter updates yields an upper bound for the total number of iterations [2], where the number of barrier parameter updates is bounded by $\frac{1}{\theta}\log\frac{n}{\epsilon}$. Thus we obtain an upper bound for the total number of iterations as follows:

$$K\frac{1}{\theta}\log\frac{n}{\epsilon} \leq \frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{n}{\epsilon}.$$

Recall that $L_\psi \geq \Psi_0$, and L_ψ is bounded by $L_{\psi 1}$ and $L_{\psi 2}$ in Lemma 4.2. Using $L_{\psi 1}$ and $L_{\psi 2}$ for the upper bounds of Ψ_0 for large-update and small-update methods, respectively, we immediately obtain the following theorem:

Theorem 4.14. *The total number of iterations required by the large-update and small-update methods are at most*

$$112\frac{(1+2\kappa)(2q+1)(n\theta+2\sqrt{2n\tau}+2\tau)^{\frac{q+2}{2(q+1)}}}{\theta[2(1-\theta)]^{\frac{q+2}{2(q+1)}}}\log\frac{n}{\epsilon} = O(q(1+2\kappa)n^{\frac{q+2}{2(q+1)}}\log\frac{n}{\epsilon})$$

and

$$\begin{aligned}&112\frac{(1+2\kappa)(2q+1)(q+1)^{\frac{q+2}{2(q+1)}}(\sqrt{n}\theta+\sqrt{2\tau})^{\frac{2(q+2)}{2(q+1)}}}{\theta(1-\theta)^{\frac{q+2}{2(q+1)}}}\log\frac{n}{\epsilon}\\ &= O(q^{\frac{3q+4}{2(q+1)}}(1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon}),\end{aligned}$$

respectively.

By choosing $q = O(\log n)$, we obtain the iteration bound for large-update method is $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon})$. If we choose $q = 1$, then small-update method have $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexities. These are curently the best known bound for such methods. It's worth noting that $\psi(t)$ is precisely the kernel function proposed by Bai et al. in [26] for $q = 1$, and the large-update method for $P_*(\kappa)$ LCP has $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ iteration complexities from Cho, which means the iteration bound for large-update method is improved based on our parameterized kernel function.

5. Numerical results

In this section, we give some numerical results. We present three test problems in subsection 5.1 to verify the effectiveness of our algorithm. Subsection 5.2 gives numerical results of large-update IPMs based on our kernel function and some other kernel functions, including classical logarithmic kernel function, trigonometric kernel function, parameterized kernel function with integral types, etc. The numerical result of the Mizuno-Todd-Ye (MTY) predictor-corrector algorithm are given in subsection 5.3.

In the implementation, we set threshold parameter $\tau = 2.5$, accuracy parameter $\epsilon = 10^{-6}$. The numerical results are obtained by using MATLAB 2012b. All numerical results include iteration numbers (iter), the duality gap (Gap) and the CPU time (CPU).

5.1 Test problem

Consider the following three $P_*(\kappa)$ LCPs. In fact, there has not been a polynomial algorithm for which we can calculate the value of the parameter κ for a

given $P_*(\kappa)$ matrix, the problem 5.1 is the only $P_*(\kappa)$ LCPs with $\kappa \neq 0$ ($\kappa = \frac{1}{4}$) we currently known. Therefore, we also test Problem 5.2 and Problem 5.3 with $\kappa = 0$ (monotone).

Problem 5.1 (Lee et al.'s example in [29])**.**

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x^0 = \begin{pmatrix} 0.4 \\ 0.45 \end{pmatrix}.$$

Problem 5.2 (Harker et al.'s example in [30])**.**

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Problem 5.3 (Geiger et al.'s example in [32])**.**

$$M = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ 0 & -1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

5.2 Numerical results for large-update IPMs

The best iteration numbers for Problem 5.1 based on our kernel function with different values of the parameters q and θ are stated in Table 2, where $q \in \{1, 2, 3, 5, 10\}$ and $\theta \in \{0.2, 0.4, 0.6, 0.8, 0.99\}$. We also give a series of numerical results in Table 3 to compare the practical computational efficiency of the methods based on different kernel functions and θ for Problem 5.1. Some kernel functions we used in our experiments are as follows:

$$\psi_{new}(t) = \frac{t^2-1}{2} + \frac{(e-1)^{q+1}}{qe(e^t-1)^q} - \frac{e-1}{qe}, \ q \geq 1,$$

$$\psi_1(t) = \frac{t^2-1}{2} - \log t, \qquad \psi_2(t) = \frac{t^2-1}{2} + \frac{t^{-1}-t}{2},$$

$$\psi_3(t) = \frac{t^2-1}{2} + \frac{6}{\pi}\tan(h(t)), \quad h(t) = \frac{\pi(1-t)}{4t+2},$$

$$\psi_4(t) = \frac{t^2-1}{2} - \int_1^t \Big(\frac{e-1}{e^x-1}\Big)^p dx, \ p \geq 1,$$

$$\psi_5(t) = \frac{t^2-1}{2} + \frac{1}{q^2}\big(\frac{q}{t}-1\big)e^{q(\frac{1}{t}-1)} - \frac{q-1}{q^2}, \ q \geq 1.$$

The first kernel function $\psi_{new}(t)$ is proposed in this paper, and it is exactly the kernel function presented by Bai et al. in [26] for $q = 1$. $\psi_1(t)$ is the classical logarithmic kernel function. $\psi_2(t)$ is a self-regular kernel function presented by Peng et al. in [31]. $\psi_3(t)$ is a trigonometric kernel function [8]. $\psi_4(t)$ [17] and $\psi_5(t)$ [24] are parameterized kernel functions, which obtain the best iteration complexity for $q = O(\log n)$.

Furthermore, the best iteration numbers for Problems 5.2 and 5.3 based on different kernel functions are stated in Table 4 and Table 5 when the scale of the problems changes. For the parameterized kernel functions $\psi_4(t)$, $\psi_5(t)$ and $\psi_{new}(t)$, we take the numerical results of the parameterized p or $q = 1, 2, 3$ respectively.

Table 2. Numerical results based on ψ_{new} for Problem 5.1

q		θ 0.2	0.4	0.6	0.8	0.99	Average
1	Iter	71	35	22	16	9	30.6
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0130	0.0119	0.0106	0.0105	0.0093	0.0111
2	Iter	71	34	22	16	9	30.4
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0131	0.0113	0.0110	0.0100	0.0091	0.0109
3	Iter	72	35	22	17	9	31
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0134	0.0121	0.0106	0.0102	0.0101	0.0113
5	Iter	73	37	24	19	12	33
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0127	0.0118	0.0104	0.0103	0.0102	0.0111
10	Iter	74	38	25	20	12	33.8
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0131	0.0120	0.0115	0.0115	0.0102	0.0117

From the numerical results of Tables 2, 3, 4, and 5, we can draw the following conclusions:

- The results in Tables 2 and 3 show that the larger θ is, the better iteration numbers will be.
- Note that the number of iteration results using $\psi_3(t)$ is efficient, but it consumes significantly more CPU time than other kernel functions, probably because its integral form leads to more complex operations.
- As the scale of the problems 5.2 and 5.3 increases, there is little or no increase in the number of iteration steps, which is precisely what is hoped for IPMs.
- Compared with the original kernel function, that is, when the parameter $q = 1$, the numerical result is obviously improved.
- From Tables 2-5, our kernel function has better numerical results than the other kernel functions.

Table 3. Numerical results based on different kernel functions for Problem 5.1

ψ_i		θ 0.2	0.4	0.6	0.8	0.99	Average
ψ_1	Iter	78	43	26	20	10	35.4
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0141	0.0106	0.0097	0.0099	0.0089	0.0106
ψ_2	Iter	75	38	24	19	12	33.6
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0127	0.0097	0.0095	0.0090	0.0089	0.0100
ψ_3	Iter	80	42	31	20	12	37
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0137	0.0111	0.0105	0.0097	0.0087	0.0119
ψ_4 p=1	Iter	71	35	22	16	9	30.6
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.9113	0.4736	0.3568	0.2980	0.2820	0.4643
ψ_4 p=2	Iter	72	36	22	17	9	31.2
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.8660	0.5065	0.3823	0.3163	0.2807	0.4704
ψ_4 p=3	Iter	72	36	23	18	11	32
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.9036	0.5200	0.4019	0.3366	0.2817	0.4888
ψ_5 q=1	Iter	71	35	22	16	9	30.6
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0121	0.0106	0.0102	0.0095	0.0091	0.0103
ψ_5 q=2	Iter	72	36	22	17	9	31.2
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0120	0.0111	0.0104	0.0100	0.0091	0.0105
ψ_5 q=3	Iter	72	36	23	18	12	32.2
	Gap	9.89e-07	7.26e-07	8.46e-07	2.02e-07	1.97e-08	
	CPU	0.0123	0.0106	0.0103	0.0102	0.0093	0.0105

Table 4. Numerical results based on different kernel functions for Problem 5.2

$\psi_i(\theta = 0.99)$		n=10			n=50			n=100			Average	
		Iter	Gap	CPU	Iter	Gap	CPU	Iter	Gap	CPU	Iter	CPU
ψ_1		22	1.32e-07	0.0112	26	1.67e-07	0.0607	28	1.33e-08	0.0379	25.3	0.0366
ψ_2		25	1.32e-07	0.0138	28	1.67e-07	0.0450	30	1.33e-08	0.0352	27.7	0.0313
ψ_3		27	1.32e-07	0.0156	26	1.67e-07	0.0680	33	1.33e-08	0.0493	28.7	0.0443
ψ_4	p=1	19	1.32e-07	1.1234	21	1.67e-07	5.2673	22	1.33e-08	11.3652	20.7	5.9186
	p=2	19	1.32e-07	1.1746	20	1.67e-07	5.5205	22	1.33e-08	11.7512	20.3	6.1488
	p=3	19	1.32e-07	1.1638	23	1.67e-07	5.8152	26	1.33e-08	12.5233	22.7	6.5008
ψ_5	q=1	18	1.32e-07	0.0131	22	1.67e-07	0.0293	24	1.33e-08	0.0419	21.3	0.0281
	q=2	19	1.32e-07	0.0138	23	1.67e-07	0.0356	26	1.33e-08	0.0461	22.7	0.0318
	q=3	19	1.32e-07	0.0132	22	1.67e-07	0.0214	25	1.33e-08	0.0388	22	0.0245
ψ_{new}	q=1	19	1.32e-07	0.0107	22	1.67e-07	0.0393	24	1.33e-08	0.0469	21.7	0.0323
	q=2	18	1.32e-07	0.0127	20	1.67e-07	0.0287	22	1.33e-08	0.0272	20	0.0229
	q=3	19	1.32e-07	0.0133	23	1.67e-07	0.0330	26	1.33e-08	0.0346	22.7	0.0270

Table 5. Numerical results based on different kernel functions for Problem 5.3

$\psi_i(\theta = 0.99)$		n=10			n=50			n=100			Average	
		Iter	Gap	CPU	Iter	Gap	CPU	Iter	Gap	CPU	Iter	CPU
ψ_1		21	1.20e-07	0.0110	25	5.20e-07	0.0401	27	1.02e-08	0.0374	24.3	0.0295
ψ_2		19	1.20e-07	0.0129	22	5.20e-07	0.0399	30	1.02e-08	0.0332	23.7	0.0287
ψ_3		22	1.20e-07	0.0147	25	5.20e-07	0.0592	33	1.02e-08	0.0455	26.3	0.0398
ψ_4	p=1	17	1.20e-07	1.1023	18	5.20e-07	4.6845	22	1.02e-08	9.5526	19	5.1131
	p=2	16	1.20e-07	1.1324	17	5.20e-07	4.5205	21	1.02e-08	9.8843	18	5.1791
	p=3	17	1.20e-07	1.1478	17	5.20e-07	5.0012	21	1.02e-08	10.1011	18.3	5.4167
ψ_5	q=1	17	1.20e-07	0.0118	17	5.20e-07	0.0205	22	1.02e-08	0.0336	18.7	0.0220
	q=2	16	1.20e-07	0.0110	17	5.20e-07	0.0201	22	1.02e-08	0.0321	18.3	0.0211
	q=3	17	1.20e-07	0.0122	17	5.20e-07	0.0194	21	1.02e-08	0.0277	18.3	0.0197
ψ_{new}	q=1	17	1.20e-07	0.0105	17	5.20e-07	0.0293	22	1.02e-08	0.0315	18.7	0.0238
	q=2	16	1.20e-07	0.0100	17	5.20e-07	0.0225	21	1.02e-08	0.0252	18	0.0192
	q=3	17	1.20e-07	0.0115	17	5.20e-07	0.0230	21	1.02e-08	0.0319	18.3	0.0221

5.3 Numerical results for MTY predictor-corrector method

In the previous subsection, the number results of the method based on the classical logarithmic kernel function have been given, that is, the so-called primal-dual path-following algorithm. The numerical results of our algorithm are better. Our goal in this subsection is to make comparison between our algorithm and MTY predictor-corrector algorithm in [33].

Table 6. Numerical results for MTY predictor-corrector method

Problem		our (q=2,θ=0.99)			MTY		
		Iter	Gap	CPU	Iter	Gap	CPU
5.1		9	1.97e-08	0.0093	49	9.22e-07	0.7532
5.2	n=10	18	1.32e-07	0.0127	21	9.81-09	0.6823
	n=50	20	1.67e-07	0.0287	24	4.08e-08	1.2595
	n=100	22	1.33e-08	0.0272	50	8.09e-07	7.4915
5.3	n=10	16	1.20e-07	0.0100	24	6.94e-07	0.6511
	n=50	17	5.20e-07	0.0225	37	5.45e-07	1.7485
	n=100	21	1.02e-08	0.0346	36	6.74e-07	5.4926

- Table 6 show that the iteration numbers and CPU time of our algorithm are significantly better than the MTY predictor-corrector method

These results imply that our kernel function is quite efficient and promising.

6. Concluding remarks

In this paper, a new parameterized kernel function is proposed, which is a kind of generalization of the one used in [26]. We apply it in the designing of a primal-dual IPM for $P_*(\kappa)$ LCP. The properties of the kernel function have been analyzed. As a result, the iteration bound in [27] is improved from $O((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ to $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon})$ for large-update method, the latter is

currently the best known iteration bound for such methods, and we also obtain the best iteration bound for small-update method. The numerical results show that the algorithm depend on the value of kernel function is effective.

Further research may focus on finding a kernel function which makes the iteration bound of the large-update method based on this kernel function improve to $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ (or even better), or prove that such a kernel function does not exist. Moreover, the extensions of the proposed kernel function in this paper to semidefinite optimization and second order cone optimization can be investigated.

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Some results for best coapproximation on Banach lattices

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Abstract. In this paper we introduce the concept of best coapproximation on Banach lattices with a strong unit. We study the existence problem of best coapproximation in these spaces. Also, we develop the theory of best coapproximation in quotient of Banach lattice spaces and discuss about the relationship between the coproximinal elements of a given space and its quotient space. Finally, we show that every lattice isomorphism is an coapproximation preserving operator.

Keywords: best coapproximation, Banach lattice spaces, downward set, quotient spaces.

1. Introduction and preliminaries

The theory of best approximation is an important topic in functional analysis. It is very extensive field which has many applications in mathematics and some other sciences. An interesting question in this field is: under what conditions on the set G does the point $g \in G$ nearest from point x in the spaces exist and this is unique? One of an important conditions on G is convexity. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets. Downward sets are nonconvex sets which play an important role in some parts of mathematical economics and game theory. Recently, many authors have been studied the notions of best approximation, best simultaneous approximation and farthest points for downward sets in Banach lattice spaces ([3],[5],[6],[9],[10],[11]). Another kind of approximation, called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [4]. The theory of best coapproximation is much less developed as compared to the theory of best approximation in abstract spaces. Some results on best coapproximation theory in metric spaces and linear normed spaces have obtained by P.L. Papini, T.D. Narang, and others ([1],[7],[8],[12],[13],[14],[15],[16]).

In this parer, we develop the theory of best coapproximation by elements of closed downward sets, in a Banach lattice with a strong unit. Also, we introduce the notion of best coapproximation in quotient Banach lattice spaces. We shall

determine some conditions which coproximinality can be transmitted to quotient spaces and vice versa. Finally, we obtain some conditions for maps under which preserve best coapproximation by downward subsets of Banach lattices.

A real vector space X is said to be an ordered vector space whenever it is equipped with an order relation $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, and transitive binary relation on X). A vector lattice space (or a Riesz space) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$, the $\sup\{x, y\}$ and the $\inf\{x, y\}$ both exist in X. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. Recall that a vector subspace W of a vector lattice space X is said to be a vector sublattice, whenever W is closed under the lattice operations of X, i.e., whenever for each pair $x, y \in W$ the vector $x \vee y$ and $x \wedge y$ (taken in X) belongs to W. A subset A of a vector lattice space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice space is referred to as an ideal. Note that from the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, it follows immediately that every ideal is a vector lattice subspace.

For any vector x in a vector lattice space define $x^+ := x \vee 0$, $x^- := x \wedge 0$ and $|x| := x \vee (-x)$. The element x^+ is called the positive part, x^- is called the negative part, and $|x|$ is called the absolute value of x. If X is an ordered vector space, then the set $X^+ = \{x \in X \ : \ x \geq 0\}$ is called a positive cone of X, and its members are called the positive elements of X. In a vector lattice space, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. For each $x \in X$ we define $x^\perp = \{y \in X : |x| \wedge |y| = 0\}$. It is easy to show that $x^\perp$ is a closed vector sublattice of X.

An element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda \mathbf{1}$. Then for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $|x| \leq \lambda \mathbf{1}$. Using $\mathbf{1}$ we can define a norm on X by

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\}. \tag{1}$$

Recall that a norm $\|.\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space. If a normed vector lattice space is also norm complete, then it is referred to as a Banach lattice. It is well known that X equipped with the norm (1) is a Banach lattice which is called a Banach lattice with strong unit $\mathbf{1}$.

The closed ball with center at x and radius r defined on Banach lattice X as follows:

$$B(x, r) = \{y \in X \ : \ \|y - x\| \leq r\} = \{y \in X \ : \ x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}. \tag{2}$$

Let A be an ideal in Banach lattice space X. We recall that the equivalence class determined by x in $\frac{X}{A}$ will be denoted by $\dot{x} = x + A$. In $\frac{X}{A}$ we introduce a relation $\dot{\leq}$ by letting $\dot{x} \dot{\leq} \dot{y}$ whenever there exist $x_1 \in \dot{x}$ (i.e., $x_1 - x \in A$) and $y_1 \in \dot{y}$ with $x_1 \leq y_1$. Clearly, $\frac{X}{A}$ under the relation $\dot{\leq}$ is an ordered vector space and it is easy to show that $\frac{X}{A}$ is a vector lattice space.

Let A be a closed ideal of a Banach lattice space X. Then the vector lattice space $\frac{X}{A}$ under the quotient norm

$$\|\dot{x}\| = \inf\{\|y\| : y \in \dot{x}\}, \tag{3}$$

is a Banach lattice space. In fact, the quotient vector space $\frac{X}{A}$ is itself a Banach lattice space.

A linear operator $T : X \longrightarrow Y$ between two lattice vector spaces X and Y is said to be a lattice (or Riesz) homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in X$. Note that every lattice homomorphism is a positive operator, i.e., it carries positive vectors to positive vectors. This is equivalent to saying that every lattice homomorphism is order preserving, i.e., $x \leq y$ implies that $T(x) \leq T(y)$, and also it is equivalent to

$$|T(x)| = T(|x|) \tag{4}$$

for all $x \in X$. It is important to note that the range of a lattice homomorphism is a lattice subspace. A lattice homomorphism which is in addition one-to-one is referred to as a lattice isomorphism. Clearly, the map $x \to \dot{x}$, from X to $\frac{X}{A}$, is a linear operator called the canonical projection of X onto $\frac{X}{A}$. The lattice homomorphisms are closely related to ideals. For every ideal A of a Banach lattice space X, the canonical projection of X onto the Banach lattice space $\frac{X}{A}$ is a lattice homomorphism (for more details see [2]).

Let X be a normed vector space. For a nonempty subset G of X and $x \in X$, define

$$d(x, G) = \inf_{g \in G} \|x - g\|.$$

A point $g_0 \in G$ is called a best approximation for $x \in X$ if

$$\|x - g_0\| = d(x, G).$$

If each $x \in X$ has at least one best approximation $g_0 \in G$, then G is called a proximinal subset of X. Let $G \subset X$. For $x \in X$, the set of all best approximations of x in G,is denote by $P_G(x)$; that is

$$P_G(x) = \{g \in G : \|x - g\| = d(x, G)\}.$$

A counterpart to best approximation in normed linear spaces is best coapproximation which is defined as follows:

Let W be a nonempty subset of a normed linear space X. An element $w_0 \in W$ is called a best coapproximation to $x \in X$ from W if for every $w \in W$,

$$\|w - w_0\| \leq \|x - w\|.$$

The set of all elements of best coapproximation to $x \in X$ from W is denoted by $R_W(x)$. If each $x \in X$ has at least one best coapproximation $w_0 \in W$, then W is called a coproximinal subset of X.

2. Best coapproximation by downward sets

In this section we show that a closed downward set is coproximinal and obtain some results on best coapproximation elements.

Definition 2.1. *A nonempty subset W of an ordered vector space X is called downward if*

$$(w \in W \ , \ x \leq w) \Longrightarrow x \in W.$$

A simple example of a downward set is a set of the form $\{y \in X \ : \ y \leq g\}$, where $g \in X$. For another example, let $f : X \longrightarrow \mathbb{R}$ be an increasing function, then its lower level sets $S_c(f) = \{x \in X : f(x) \leq c\}$ for all $c \in \mathbb{R}$, are downward.

Proposition 2.1. *Let W be a downward and coproximinal subset of Banach lattice space X and $w_1, w_2 \in R_W(x)$. Then $w_1 \wedge w_2 \in R_W(x)$.*

Proof. Since W is a downward subset of X, it follows that $w_1 \wedge w_2 \in W$. Due to the relations $|x \wedge y| \leq |x| \vee |y|$ and $x + y \wedge z = (x + y) \wedge (x + z)$, for each $w \in W$ we have

$$\begin{aligned} |w - w_1 \wedge w_2| &= |(w - w_1) \wedge (w - w_2)| \\ &\leq |w - w_1| \vee |w - w_2| \\ &\leq |x - w| \vee |x - w| = |x - w|. \end{aligned}$$

This implies that $\|w - w_1 \wedge w_2\| \leq \|x - w\|$ for all $w \in W$. Hence $w_1 \wedge w_2 \in R_W(x)$ and the proof is complete. □

Corollary 2.1. *Let W be a vector sublattice and $R_W(x)$ be a subspace of Banach lattice space X, then $R_W(x)$ is a vector sublattice of X.*

Proof. Similar to the proof of Proposition 2.1, we can prove that $w_1 \wedge w_2 \in R_W(x)$ and $w_1 \vee w_2 \in R_W(x)$. This yields that $R_W(x)$ is a vector sublattice of X. □

Theorem 2.1. *Let W be a nonempty subset of Banach lattice space X and $x \in X \setminus W$. Then $w_0 \in R_W(x)$ if and only if for every $w \in W$,*

$$w - r_w \boldsymbol{1} \leq w_0 \leq w + r_w \boldsymbol{1}$$

where $r_w = \|x - w\|$.

Proof. Due to (2) for all $w \in W$, we have $w_0 \in R_W(x) \iff \|w_0 - w\| \leq \|x - w\| = r_w \iff -r_w \mathbf{1} \leq w_0 - w \leq r_w \mathbf{1} \iff w - r_w \mathbf{1} \leq w_0 \leq w + r_w \mathbf{1}$. □

Theorem 2.2. *Let W be a closed downward subset of Banach lattice X. Then for each $x \in X$ such that $x \geq W$, W is a coproximinal subset of X.*

Proof. Let $x_0 \in X$ be arbitrary such that $x_0 \geq W$ and $r := d(x_0, W) = \inf_{w\in W} \|x_0 - w\| > 0$. Then, for each $\varepsilon > 0$ there exists $w_\varepsilon \in W$ such that $\|x_0 - w_\varepsilon\| < r + \varepsilon$. Now, by (1), we have

$$w_\varepsilon \geq x_0 - (r + \varepsilon)\mathbf{1}.$$

Set $w_0 = x_0 - r\mathbf{1}$. So we have $w_\varepsilon \geq x_0 - r\mathbf{1} - \varepsilon\mathbf{1} = w_0 - \varepsilon\mathbf{1}$. We claim that $w_0 \in R_W(x_0)$. Since W is a downward set and $w_\varepsilon \in W$, it follows that $w_0 - \varepsilon\mathbf{1} \in W$ for all $\varepsilon > 0$. Closedness of W implies that $w_0 \in W$. Now, suppose that $w \in W$ be arbitrary. In view of $\|x_0 - w\| = r_w \leq r + r_w$, $r \leq r_w$ and (1.2), we have

$$(r - r_w)\mathbf{1} \leq 0 \leq x_0 - w \leq (r + r_w)\mathbf{1}.$$

This implies that for each $w \in W$,

$$w - r_w\mathbf{1} \leq w_0 = x_0 - r\mathbf{1} \leq w + r_w\mathbf{1}.$$

Thus, by Theorem 2.1, $w_0 \in R_W(x_0)$ and so W is coproximinal. □

Example 2.3. Let $X = \mathbb{R}$ and $W = (-\infty, 0]$. Then W is a closed downward set in X and so is coproximinal and we have $R_W(1) = [-1, 0]$.

Let S be a vector sublattice of X, we define

$$\check{S} = \{x \in X : \|w\| \leq \|x - w\| \ \ \forall w \in S\} = R^{-1}(\{0\}).$$

Proposition 2.2. *Let S be a vector sublattice of Banach lattice X. Then we have $S \cap \check{S} = \{0\}$.*

Proof. Suppose that $x \in S \cap \check{S}$. Since $x \in \check{S}$ so by definition of $\check{S}$ for all $w \in S$ we have $\|w\| \leq \|x - w\|$. On the other hand, since $x \in S$, by replacing w by x we obtain $\|x\| \leq \|x - x\| = 0$. Therefore $x = 0$ and hence $S \cap \check{S} = \{0\}$. □

Proposition 2.3. *Let S be a vector sublattice of Banach lattice X. Then for all $w \in S$ we have $d(w, \check{S}) = \|w\|$.*

Proof. Suppose that $x \in \check{S}$. Then for all $w \in S$ we have $\|w\| \leq \|x - w\|$. Fixed $w \in S$ and so for all $x \in \check{S}$ we obtain

$$\|w\| \leq \inf_{x\in S\cap\check{S}} \|x - w\| = d(w, \check{S}) \leq \|w\|.$$

It follows that $d(w, \check{S}) = \|w\|$. □

Corollary 2.2. *Let S be a vector sublattice of Banach lattice X. Then for all $w \in S$ we have $0 \in P_{\check{S}}(w)$.*

Proposition 2.4. *Let X be a Banach lattice. Then for all $x \in X$ we have $d(x, x^\perp) = \|x\|$.*

Proof. Suppose that $x \in X$, then in view of the relation $|x| = |x - y + y| \leq |x - y| + |y|$ for all $y \in x^{\perp}$ we have

$$\begin{aligned} |x| &= |x| \wedge |x| \leq |x| \wedge (|x - y| + |y|) \\ &= (|x - y| \wedge |x|) + (|x| \wedge |y|) \\ &= |x - y| \wedge |x| \leq |x|. \end{aligned}$$

Hence $|x| = |x - y| \wedge |x|$ and so $|x| \leq |x - y|$. Therefore for all $y \in x^{\perp}$, we have $\|x\| \leq \|x - y\|$ which implies that $d(x, x^{\perp}) = \|x\|$. □

Corollary 2.3. *Let S be a vector sublattice of Banach lattice X. Then for all $x \in X$ we have $0 \in P_{x^{\perp}}(x)$.*

Theorem 2.4. *Let S be a coproximinal vector sublattice of Banach lattice X. If $\check{S}$ is a vector sublattice of X, then $\check{S}$ is proximinal.*

Proof. Suppose that S be coproximinal, $x \in X$ and $w_1 \in R_S(x)$. Then $w_2 = x - w_1 \in \check{S}$ and so $x = w_1 + w_2$ where $w_1 \in S$ and $w_2 \in \check{S}$. In view of Proposition 2.3, we have

$$\|x - w_2\| = \|w_1\| = d(w_1, \check{S}) = d(x - w_2, \check{S}) = d(x, \check{S}).$$

Thus, $w_2 \in P_{\check{S}}(x)$, and therefore $\check{S}$ is proximinal. □

3. Best coapproximation in quotient spaces

In this section we discuss about coproximality of sublattices in quotient of Banach lattice spaces.

Theorem 3.1. *Let A be an ideal in Banach lattice X. Then the following statements are equivalent:*

(i) A is coproximinal.

(ii) The canonical mapping $Q : X \longrightarrow \frac{X}{A}$ maps $\check{A}$ onto $\frac{X}{A}$, i.e. $Q(\check{A}) = \frac{X}{A}$.

Proof. (i)$\Longrightarrow$ (ii). Let $x + A \in \frac{X}{A}$ be an arbitrary element and $w_0 \in R_A(x)$. Then $x - w_0 \in \check{A}$ and we have

$$Q(x - w_0) = (x - w_0) + A = x + A.$$

(ii)$\Longrightarrow$ (i). Suppose that $x \in X$.Then $x + A \in \frac{X}{A} = Q(\check{A})$, i.e., $x + A = Q(y)$ where $y \in \check{A} = R_A^{-1}(0)$. So $x + A = y + A$ and $0 \in R_A(y)$. Thus, $x - y = w_0 \in A$ and $0 \in R_A(x - w_0)$. This implies that $x - w_0 \in \check{A}$ and therefore $w_0 \in R_A(x)$. Hence A is coproximinal. □

Theorem 3.2. *A subset W of a vector lattice space X is downward in X if and only if $\frac{W}{A}$ is downward in $\frac{X}{A}$.*

Proof. Suppose that W is downward set. Let $\dot{x}, \dot{y} \in \frac{W}{A}$ such that $\dot{x} \dot{\leq} \dot{y}$ and $\dot{y} \in \frac{W}{A}$. Since $\dot{x} \dot{\leq} \dot{y}$, it follows that there exist $x_1 \in \dot{x}$ and $y_1 \in \dot{y}$ such that $x_1 \leq y_1$. In view of $\dot{y_1} = \dot{y} \in \frac{W}{A}$ we have $y_1 \in W$. Since W is downward set so $x_1 \in W$.Therefore $\dot{x} = \dot{x_1} \in \frac{W}{A}$and hence $\frac{W}{A}$ is a downward set in $\frac{X}{A}$.

Conversely, let $\frac{W}{A}$ be a downward set. Assume that $x, y \in W$such that $x \leq y$ and $y \in W$, so $\dot{y} \in \frac{W}{A}$. Since the canonical map is order preserving we have $\dot{x} \dot{\leq} \dot{y}$. Now, since $\frac{W}{A}$ is a downward set, it follows that $\dot{x} = x + A \in \frac{W}{A}$ and hence $x \in W$. Therefore W is downward set. □

Corollary 3.1. *Let W be a closed and downward subset and A be an ideal of X. Then $\frac{W}{A}$is coproximinal in $\frac{X}{A}$.*

Proof. In view of Theorem 3.2, $\frac{W}{A}$ is a downward subset of $\frac{X}{A}$. By continuity of the canonical map $Q : X \longrightarrow \frac{X}{A}$ we have $\frac{W}{A}$ is closed. Therefore, due to Theorem 2.2, $\frac{W}{A}$is coproximinal in $\frac{X}{A}$. □

Theorem 3.3. *Let Sbe a sublattice and A be an closed ideal in Banach lattice X such that $A \subseteq S$. If $S$is coproximinal in X, then $\frac{S}{A}$ is a coproximinal sublattice of $\frac{X}{A}$.*

Proof. In view of (3), $\frac{S}{A}$ is a Banach sublattice of $\frac{X}{A}$. Now we assume that $w_0 \in R_W(x)$and $w \in S$. Then for each $a \in A$ we have

$$\begin{aligned} \|\dot{w_0} - \dot{w}\| &= \|(w_0 - w) + A\| \\ &\leq \|w_0 - (w - a)\| \\ &\leq \|x - (w - a)\| = \|(x - w) + a\|. \end{aligned}$$

So for each $\dot{w} \in \frac{S}{A}$ we have $\|\dot{w_0} - \dot{w}\| \leq \|\dot{x} - \dot{w}\|$and therefore $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$. Hence $\frac{S}{A}$ is coproximinalin $\frac{X}{A}$. □

Theorem 3.4. *Let S be a sublattice and A be an closed and proximinal ideal in Banach lattice X such that $A \subseteq S$. If $\frac{S}{A}$is coproximinal in $\frac{X}{A}$, then S is a coproximinal sublattice of X.*

Proof. Let $\dot{x} \in \frac{X}{A}$and $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$. Then for each $\dot{w} \in \frac{S}{A}$we have

$$\|\dot{w_0} - \dot{w}\| \leq \|\dot{x} - \dot{w}\|$$

or in the other word

$$\|(w_0 - w) + A\| \leq \|(x - w) + A\|.$$

Now, since A is proximinal in X, we conclude that there exist $a_0 \in A$ such that

$$\|(w_0 - w) + a_0\| = \|(w_0 - w) + A\| \leq \|(x - w) + A\| \leq \|x - w\|.$$

Then for each $w \in S$ we have

$$\|(w_0 + a_0) - w\| \leq \|x - w\|.$$

So $w_0 + a_0 \in R_S(x)$ and therefore S is coproximinal in X. □

Theorem 3.5. *Let S be a sublattice and A be an closed ideal in Banach lattice X such that $A \subseteq S$. If S is coproximinal in X, then we have*

$$Q(R_S(x)) \subseteq R_{\frac{S}{A}}(Q(x)),$$

In particular if A is proximinal in X, then

$$Q(R_S(x)) = R_{\frac{S}{A}}(Q(x)).$$

Proof. Due to Theorem 3.3, it is obvious that $Q(R_S(x)) \subseteq R_{\frac{S}{A}}(Q(x))$. Now, let A is proximinal, $\dot{x} \in \frac{X}{A}$ and $\dot{w_0} \in R_{\frac{S}{A}}(\dot{x})$.In view of the Theorem 3.4, there exist $a_0 \in A$ such that $w_0 + a_0 \in R_S(x)$. Now, we have $\dot{w_0} = Q(w_0 + a_0) \in Q(R_S(x))$. Hence $R_{\frac{S}{A}}(Q(x)) \subseteq Q(R_S(x))$ and the proof is complete. □

4. Coapproximation preserving operators

In this section we shall obtain characterization of coapproximation preserving maps on Banach lattices.

The following lemma characterizes the maps which preserve downwardness of downward subsets of vector lattices.

Lemma 4.1 ([9])**.** (1) *Let X and Y be two vector lattices and $T : X \longrightarrow Y$ be an injective positive operator, such that T^{-1} is a positive operator. Then W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.*

(2) *If $T : X \longrightarrow X$ is a positive operator and $f : X \longrightarrow \mathbb{R}$ is an increasing function, then $S_c(f \circ T) = \{x \in X : f \circ T(x) \leq c\}$ for all $c \in \mathbb{R}$ are downward.*

The next theorem which has been proved in [2], page 94, described necessary and sufficient conditions for an operator between two vector lattice spaces, that is a lattice isomorphism.

Theorem 4.1. *Assume that an operator $T : X \longrightarrow Y$ between two vector lattice spaces is one-to-one and onto. Then T is a lattice isomorphism if and only if T and T^{-1} are both positive operators.*

The following proposition which is a consequence of Theorem 4.1 and relations (1), (4), plays a key role in proving our results.

Proposition 4.1. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then, T is a norm isometry, i.e. $\|T(x)\| = \|x\|$ for all $x \in X$.*

Definition 4.1. *Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. A linear operator $T : X \longrightarrow Y$ is called a coapproximation preserving operator if for all downward sets W in X and all $x \in X$:*

(i) W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.

(ii) $T(R_W(x)) = R_{T(W)}(T(x))$.

Theorem 4.2. *Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Let $T : X \longrightarrow Y$ be an injective positive operator which T^{-1} is a positive operator and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then T is a coapproximation persevering operator.*

Proof. In view of the Proposition 4.1, we have $\|T(x)\| = \|x\|$, for all $x \in X$. Then for all $x \in X$ and all downward sets W of X and $w, w_0 \in W$,

$$\|w - w_0\| \leq \|x - w\| \Leftrightarrow \|T(w) - T(w_0)\| \leq \|T(x) - T(w)\|.$$

Therefore, $T(R_W(x)) = R_{T(W)}(T(x))$. Also by Lemma 4.1, W is downward subset of X if and only if $T(W)$ is a downward subset of Y. □

Corollary 4.1. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Every lattice isomorphism between X and Y is a coapproximation preserving operator.*

Proof. From Theorems 4.1 and 4.2, the result obtained. □

Example 4.3. Let X be a Banach lattice with strong unit $\mathbf{1}$ and $a \in X$. Define

$$T_a : X \longrightarrow X, \quad T_a(x) = a + x.$$

Then $T^{-1} = T_{-a}$. Suppose that W is a downward subset of X; then $T_a(W) = a + W$ is a downward subset of X, and if $T(W)$ is downward, then $W = T_{-a}(T(W))$ is downward. Also for all $w, w_0 \in W$ and $x \in X$ we have

$$\|T_a(w_0) - T_a(w)\| \leq \|T_a(x) - T_a(w)\| \Longleftrightarrow \|w_0 - w\| \leq \|x - w\|.$$

Then $T(R_W(x)) = R_{T(W)}(T(x))$ and therefore T_a is a coapproximation preserving operator on X, for all $a \in X$.

Theorem 4.4. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$ respectively. If $T : X \longrightarrow Y$ is a coapproximation preserving operator, then W is a downward and coproximinal subset of X if and only if $T(W)$ is a downward and coproximinal subset of Y.*

Corollary 4.2. *The canonical mapping $Q : X \longrightarrow \frac{X}{A}$ is a coapproximation preserving operator.*

Proof. From Theorems 3.2 and 3.5, it is obvious. □

Definition 4.2. *A subset G of the cone X^+ is called normal if*

$$(g \in G, x \in X^+, x \leq g) \Longrightarrow x \in G.$$

For example, if f is an increasing function defined on X^+, then its lower level sets $S_c^+(f) = \{x \in X^+ : f(x) \leq c\}$ for all c are normal.

If $T : X \longrightarrow Y$ is a map between vector lattices X and Y, we assume that $T^+ = T|_{X^+}$. Then for all $x \in X^+$,

$$T(x) = T^+(x). \tag{5}$$

Lemma 4.2. (1) *Let X and Y be vector lattices and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive. Then G is closed normal subset of X^+ if and only if $T^+(G)$ is a closed normal subset of Y^+.*

(2) *Let $T : X \longrightarrow X$ be a positive operator; then $S_c^+(f \circ T^+)$ are normal for all $c \in \mathbb{R}$.*

Theorem 4.5. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Let $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then for all normal subset G of $X^+$and for all $x \in X^+$,*

$$T^+(R_G(x)) = R_{T^+(G)}(T^+(x)).$$

That is, T^+ preserves coapproximation.

Proof. It is clear from (5), Lemma 4.2 and Theorem 4.2. □

Conclusion. In this paper, we obtained some results on best coapproximation by element of downward sets and showed that every closed downward sets are coproximinal. Also, we developed of the theory of best coapproximation in quotient of Banach lattice spaces and give a necessary and sufficient condition about the coproximinal elements of downward sets of a given space and its quotient space. Finally, We have presented coapproximation preserving operators on Banach lattices and have shown that every lattice isomorphism is an approximation preserving operator. In particular, these operators preserve coproximinal downward subsets of X and normal subsets of X^+. For the future research, study on the uniqness of best coapproximation elements by downward sets is recommended.

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On the stability of a nonmultiplicative type sum form functional equation

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Abstract. Our purpose is to obtain all possible general solutions of a sum form functional equation containing two unknown mappings and also discuss criteria for stability of the same.
Keywords: additive mapping, logarithmic mapping, entropy of type (α, β), stability.

1. Introduction

For $n = 1, 2, \ldots$; let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n); p_i \geq 0, i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all n-component discrete probability distributions. Let $\mathbb{R}$ denote the set of real numbers; I denote the unit closed interval $[0, 1]$, i.e. $I = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and I^* denote the interval $]0, 1]$, i.e. $I^* =]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

One of the captivating branches of research work in the field of functional equations with reference to information theory is to study those functional equations that are used to characterize various entropies. For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, the Shannon entropy [15] is defined as follows:

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{1.1}$$

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where H_n is a real valued mapping with domain Γ_n, $n = 1, 2, \ldots$ and convention $0 \log_2 0 := 0$ is adopted. The functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \tag{1.2}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ plays a vital role in characterizing Shannon entropies given by (1.1). Functional equation (1.2) was first observed by Chaundy and McLeod [3] with reference to some statistical thermodynamical problem.

Behara and Nath [2] provided a generalization of the Shannon entropy (1.1) by introducing the phenomenon of entropies of type (α, β). For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, an entropy of type (α, β) is defined as follows:

$$H_n^{(\alpha,\beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1} \left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta} \right), & \text{if } \alpha \neq \beta \\ -2^{\beta-1} \sum_{i=1}^{n} p_i^{\beta} \log_2 p_i, & \text{if } \alpha = \beta \end{cases} \tag{1.3}$$

where $H_n^{(\alpha,\beta)} : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$ and conventions $0^{\alpha} := 0$, $0^{\beta} := 0$, $1^{\alpha} := 1$, $1^{\beta} := 1$, $0^{\beta} \log_2 0 := 0$ are adopted. An axiomatic characterization of entropies of type (α, β) given by (1.3) leads to the study of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i), \tag{1.4}$$

where f is a real valued mapping with domain I; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$, $\alpha \neq 1$, $\beta \neq 1$, $0^{\alpha} := 0$, $0^{\beta} := 0$, $1^{\alpha} := 1$ and $1^{\beta} := 1$.

An extensive literature is available on entropies of type (α, β) which are characterized by functional equation (1.4). Many authors (Behara and Nath [2], Kannappan [6], [7]) studied the functional equation (1.4) and explored its solutions by presuming some conditions on the mapping $f : I \to \mathbb{R}$. Finally without presuming any regularity condition on the mapping $f : I \to \mathbb{R}$, Losonczi and Maksa [9] obtained the general solutions of (1.4) if $\alpha \neq 1$, $\beta \neq 1$ and $n \geq 3$, $m \geq 2$ be fixed integers. Recently few generalizations of (1.4) were considered by Nath and Singh [12], [13]; Singh and Dass [16] who obtained their general solutions for fixed integers $n \geq 3, m \geq 3$.

Also, motivated by the notion of stability Kocsis and Maksa [8] examined for the stability of the functional equation (1.4). For the problem of stability of functional equations we refer to the survey paper of Hyers and Rassias [5].

Intrigued by the functional equation (1.4), we intend to obtain the general solutions of

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h(p_i) = 0, \tag{1.5}$$

where $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ be fixed integers; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$, $\alpha \neq 1$, $\beta \neq 1$, $0^\alpha := 0$, $0^\beta := 0$, $1^\alpha := 1$ and $1^\beta := 1$. Also, for fixed integers $n \geq 3$, $m \geq 3$ we discuss the problem of stability of the nonmultiplicative type sum form functional equation (1.5) which is given along the following lines:

Let $n \geq 3$, $m \geq 3$ and $0 \leq \varepsilon \in \mathbb{R}$ be fixed. Find all mappings $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ satisfying the functional inequality

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i) \right| \leq \varepsilon \tag{1.6}$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$.

This paper is divided into four sections. In section 2, we present some auxiliary results which will be used in the subsequent sections. In section 3, we obtain the general solutions of the functional equation (1.5) for $n \geq 3$, $m \geq 2$ being fixed integers. In section 4, we examine the stability of the functional equation (1.5) for fixed integers $n \geq 3$, $m \geq 3$.

2. Auxiliary results

In this section we state few widely known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $x \in I$, $y \in I$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. Daróczy and Losonczi [4] proved that an additive mapping $a : I \to \mathbb{R}$ has a unique additive extension to the set of real numbers, i.e. there exists a unique mapping $A : \mathbb{R} \to \mathbb{R}$, additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic on I if it satisfies $\ell(0) = 0$ and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in I^*$, $y \in I^*$.

Result 2.1 ([10])**.** Let $n \geq 3$ be a fixed integer. Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$; c a given real constant. Then there exists an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = a_1(p) - \frac{1}{n} a_1(1) + \frac{c}{n}$ for all $p \in I$.

Result 2.2 ([9])**.** Let $n \geq 3$, $m \geq 2$ be fixed integers and $g : I \to \mathbb{R}$ be a mapping which satisfies the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} g(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} q_j^\beta \tag{A}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; β being a fixed positive real power with $0^\beta := 0$, $1^\beta := 1$ and $\beta \neq 1$. If $g(0) = 0$, then any general solution of (A) is of the form $g(p) = g(1)p^\beta + a_2(p)$ for all $p \in I$; $a_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_2(1) = 0$.

Result 2.3 ([9])**.** Let $H : I \to \mathbb{R}$ be a solution of the functional equation

$$\text{(B)} \qquad H(u)+H(v)=(u+v)^{\beta}\left[H\left(\frac{u}{u+v}\right)+H\left(\frac{v}{u+v}\right)\right]+\frac{u^{\beta}+v^{\beta}}{(u+v)^{\beta}}H(u+v)$$

where $u \geq 0$, $v \geq 0$, $0 < u+v \leq 1$; β being a fixed positive real power satisfying $0^{\beta} := 0$, $1^{\beta} := 1$ and $\beta \neq 1$. If $H(0) = H(1) = 0$, then $H(p) = p^{\beta}\ell(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping.

Result 2.4 ([11])**.** Let $n \geq 3$ be a fixed integer and ε be a fixed positive real number. Suppose a mapping $\psi : I \to \mathbb{R}$ satisfies the functional inequality $|\sum_{i=1}^{n} \psi(p_i)| \leq \varepsilon$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then, there exists an additive mapping $a_3 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $b : \mathbb{R} \to \mathbb{R}$ with $b(0) = 0$, $|b(p)| \leq 18\varepsilon$ such that $\psi(p) - \psi(0) = a_3(p) + b(p)$ for all $p \in I$.

Result 2.5 ([8, 17])**.** Let ε be a fixed positive real number. Suppose a mapping $H : I \to \mathbb{R}$ satisfies the functional inequality

$$\text{(C)} \qquad |H(pq) - p^{\beta}H(q) - q^{\beta}H(p)| \leq \varepsilon$$

for all $p \in I$, $q \in I$; β be a fixed positive real power such that $0^{\beta} := 0$, $1^{\beta} := 1$ and $\beta \neq 1$. Then any solution of (C) is of the form $H(p) = p^{\beta}\ell(p) + b_1(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping and $b_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping.

3. The general solution of functional equation (1.5)

Theorem 3.1. *Let $n \geq 3$, $m \geq 2$ be fixed integers and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation (1.5) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; α and β being fixed positive real powers such that $0^{\alpha} := 0$, $1^{\alpha} := 1$, $0^{\beta} := 0$, $1^{\beta} := 1$, $\alpha \neq 1$ and $\beta \neq 1$. Then, for all $p \in I$, any general solution (f, h) of (1.5) is of the form*

$$(\alpha_1) \qquad \left.\begin{array}{lll} \text{(i)} & f(p) = f(1)p^{\beta} + p^{\beta}\ell(p) + a_0(p), \quad a_0(1) = 0 & \text{if } \alpha = \beta \\ \text{(ii)} & h(p) = p^{\beta}\ell(p) + \bar{a}_0(p) + h(0), \quad \bar{a}_0(1) = -nh(0) & \end{array}\right\}$$

or

$$(\alpha_2) \qquad \left.\begin{array}{lll} \text{(i)} & f(p) = c_1'p^{\alpha} + c_2'p^{\beta} + a_1(p), \quad a_1(1) = 0 & \text{if } \alpha \neq \beta \\ \text{(ii)} & h(p) = c_2'(p^{\beta} - p^{\alpha}) + \bar{a}_1(p) + h(0), \quad \bar{a}_1(1) = -nh(0) & \end{array}\right\}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$, $\bar{a}_0 : \mathbb{R} \to \mathbb{R}$, $a_1 : \mathbb{R} \to \mathbb{R}$ and $\bar{a}_1 : \mathbb{R} \to \mathbb{R}$ are additive mappings; $\ell : I \to \mathbb{R}$ is a logarithmic mapping; c_i' $(i = 1, 2)$ are arbitrary real constants.

Proof. Substituting $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (1.5), we obtain

$$\sum_{i=1}^{n}\{f(p_i) - (f(1) + (m-1)f(0))p_i^{\alpha} - h(p_i)\} = n(1-m)f(0).$$

By Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$h(p) = f(p) - (f(1) + (m-1)f(0))p^{\alpha} - a(p) - f(0) + h(0) \tag{3.1}$$

with

$$a(1) = n(h(0) - mf(0)). \tag{3.2}$$

From equations (1.5), (3.1) and (3.2), we have

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j) - \sum_{i=1}^{n} p_i^{\alpha}\sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} f(p_i) \\ &\quad + (f(1) + (m-1)f(0))\sum_{i=1}^{n} p_i^{\alpha}\sum_{j=1}^{m} q_j^{\beta} - n(m-1)f(0)\sum_{j=1}^{m} q_j^{\beta} = 0. \end{aligned} \tag{3.3}$$

Now, choosing $p_1 = 1, p_2 = \cdots = p_n = 0$ in (3.3). We obtain $m(n-1)f(0)\big(1 - \sum_{j=1}^{m} q_j^{\beta}\big) = 0$. In view of our presumption that $n \geq 3$, $m \geq 2$, it follows that either $f(0) = 0$ or $1 - \sum_{j=1}^{m} q_j^{\beta}$ vanish identically on Γ_m. Suppose $1 - \sum_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. Then in particular for a probability distribution $\big(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\big) \in \Gamma_m$, we have $\big(\frac{1}{2}\big)^{\beta} = \frac{1}{2}$. This hold only if $\beta = 1$, whereas throughout the paper we have assumed β to be a fixed positive real power such that $\beta \neq 1$. Consequently we arrive at a contradiction and thus have $f(0) = 0$. Thus equation (3.3) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j) - \sum_{i=1}^{n} p_i^{\alpha}\sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta}\sum_{i=1}^{n} f(p_i) + f(1)\sum_{i=1}^{n} p_i^{\alpha}\sum_{j=1}^{m} q_j^{\beta} = 0. \tag{3.4}$$

By Result 2.1, there exists a mapping $A : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable such that

$$\sum_{j=1}^{m} f(pq_j) - p^{\alpha}\sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} q_j^{\beta} f(p) + f(1)p^{\alpha}\sum_{j=1}^{m} q_j^{\beta} = A(p; q_1, \ldots, q_m) \tag{3.5}$$

with $A(1; q_1, \ldots, q_m) = 0$. Let $x \in I$ and $(r_1, \ldots, r_m) \in \Gamma_m$. Now replacing $p = xr_t$, $t = 1, \ldots, m$ successively in (3.5); summing the resulting m equations so obtained and then substituting the expression $\sum_{t=1}^{m} f(xr_t)$ calculated from (3.5). We get

$$\begin{aligned} &\sum_{t=1}^{m}\sum_{j=1}^{m} f(xr_tq_j) - (f(x) - f(1)x^{\alpha})\sum_{t=1}^{m} r_t^{\beta}\sum_{j=1}^{m} q_j^{\beta} = A(x; q_1, \ldots, q_m) \\ &+ x^{\alpha}\left\{\sum_{t=1}^{m} r_t^{\alpha}\sum_{j=1}^{m} f(q_j) - f(1)\sum_{t=1}^{m} r_t^{\alpha}\sum_{j=1}^{m} q_j^{\beta} + \sum_{j=1}^{m} q_j^{\beta}\sum_{t=1}^{m} f(r_t)\right\} + A(x; r_1, \ldots, r_m)\sum_{j=1}^{m} q_j^{\beta}. \end{aligned}$$

The left hand side of the above equation is symmetric in r_t and q_j, $t = 1, \ldots, m$; $j = 1, \ldots, m$. So should be its right hand side. Thus, we obtain

$$\begin{aligned} A(x; q_1, \ldots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] - A(x; r_1, \ldots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \\ = x^{\alpha} \left\{ \sum_{t=1}^{m} f(r_t) \left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} \right) - \sum_{j=1}^{m} f(q_j) \left(\sum_{t=1}^{m} r_t^{\alpha} - \sum_{t=1}^{m} r_t^{\beta} \right) \right. \\ \left. + f(1) \left(\sum_{t=1}^{m} r_t^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - \sum_{j=1}^{m} q_j^{\alpha} \sum_{t=1}^{m} r_t^{\beta} \right) \right\}. \end{aligned} \tag{3.6}$$

Case 1. $\alpha = \beta$.

In this case, equation (3.6) reduces to

$$A(x; q_1, \ldots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] = A(x; r_1, \ldots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right].$$

As explained earlier in this section that $1 - \sum_{t=1}^{m} r_t^{\beta} \not\equiv 0$ for $\beta \neq 1$, $0 < \beta \in \mathbb{R}$, there exists a probability distribution $(r_1^*, \ldots, r_m^*) \in \Gamma_m$ so that $1 - \sum_{t=1}^{m} r_t^{*\beta} \neq 0$. As a consequence, we have

$$A(x; q_1, \ldots, q_m) = a_0(x) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \tag{3.7}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$ defined as $a_0(x) = A(x; r_1^*, \ldots, r_m^*) \left[1 - \sum_{t=1}^{m} r_t^{*\beta}\right]^{-1}$ is an additive mapping with $a_0(1) = 0$. Now, from equations (3.5) and (3.7), we get

$$\sum_{j=1}^{m} H(pq_j) - p^{\beta} \sum_{j=1}^{m} H(q_j) - \sum_{j=1}^{m} q_j^{\beta} H(p) = 0 \tag{3.8}$$

where $H : I \to \mathbb{R}$ is a mapping defined as

$$H(x) = f(x) - f(1)x^{\beta} - a_0(x) \tag{3.9}$$

for all $x \in I$. Replacing $x = 0$ and $x = 1$ respectively in (3.9) and using $f(0) = 0$; $a_0(1) = 0$, it follows that $H(0) = 0$ and $H(1) = 0$.

Let $u \geq 0$, $v \geq 0$, $0 < u + v \leq 1$, substituting $p = u + v$, $q_1 = \frac{u}{u+v}$, $q_2 = 1 - q_1$, $q_3 = \cdots = q_m = 0$ in equation (3.8) and using $H(0) = 0$; $H(1) = 0$ equation (B) follows. Hence by applying Result 2.3, $H(u) = u^{\beta}\ell(u)$ for all $u \in I$, where $\ell : I \to \mathbb{R}$ is a logarithmic mapping. Thus using this in (3.9) the solution (α_1)(i) holds. Furthermore, equations (3.1) and (α_1)(i) along with the facts that

$\alpha = \beta$ and $f(0) = 0$ yields the solution (α_1)(ii) by defining an additive mapping $\bar{a}_0 : \mathbb{R} \to \mathbb{R}$ as $\bar{a}_0(p) = a_0(p) - a(p)$ with $\bar{a}_0(1) = -nh(0)$.

Case 2. $\alpha \neq \beta$.

In this case, we may assume that $n \geq m$.

Choosing $p_{m+1} = \cdots = p_n = 0$ in (3.4) and using $f(0) = 0$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{m} f(p_i) - f(1) \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} \tag{3.10}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Clearly, left hand side of (3.10) does not change if we interchange p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$. Consequently the right hand side must remain same on interchanging p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$. As a result, we get

$$\begin{aligned} &\sum_{i=1}^{m} f(p_i) \left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} \right) - \sum_{j=1}^{m} f(q_j) \left(\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta} \right) \\ &\quad + f(1) \left(\sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\beta} - \sum_{j=1}^{m} q_j^{\alpha} \sum_{i=1}^{m} p_i^{\beta} \right) = 0. \end{aligned}$$

Now we show that for $\alpha \neq \beta$, $\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}$ does not vanish identically on Γ_m. On the contrary we assume that $\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. Then, if we choose $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{2}$, $q_3 = \cdots = q_m = 0$, it follows that $\left(\frac{1}{2}\right)^{\alpha} = \left(\frac{1}{2}\right)^{\beta}$ which is possible only when $\alpha = \beta$. Thus our assumption is wrong. Hence there exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ so that $\sum_{j=1}^{m} q_j^{*\alpha} - \sum_{j=1}^{m} q_j^{*\beta} \neq 0$. So from the above equation, it follows that

$$\sum_{i=1}^{m} f(p_i) = c_1 \left(\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta} \right) - c_2 \sum_{i=1}^{m} p_i^{\alpha} + c_3 \sum_{i=1}^{m} p_i^{\beta} \tag{3.11}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and $(p_1, \ldots, p_m) \in \Gamma_m$. Now, define a mapping $g : I \to \mathbb{R}$ as

$$g(x) = f(x) - c_1 x^{\alpha} + c_2 x^{\alpha} - c_3 x^{\beta} \tag{3.12}$$

for all $x \in I$. Clearly $g(0) = 0$. From equations (3.4), (3.11), (3.12) and performing necessary calculations, functional equation (A) follows. Hence by Result 2.2, $g(p) = g(1)p^{\beta} + a_1(p)$ for all $p \in I$, where $a_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_1(1) = 0$. Thus using this in equation (3.12), solution (α_2)(i) follows by assuming $c_1' = c_1 - c_2$ and $c_2' = c_3 - c_1$ as arbitrary real constants. Substituting (α_2)(i) in (3.1) and using the fact $f(0) = 0$, we obtain solution (α_2)(ii) by defining an additive mapping $\bar{a}_1 : \mathbb{R} \to \mathbb{R}$ as $\bar{a}_1(p) = a_1(p) - a(p)$ with $\bar{a}_1(1) = -nh(0)$. □

4. The stability of functional equation (1.5)

In this section, we prove:

Theorem 4.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers; ε be a nonnegative real constant and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional inequality* (1.6) *for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$; α and β be fixed positive real powers such that $0^\alpha := 0$, $0^\beta := 0$, $1^\alpha := 1$, $1^\beta := 1$, $\alpha \neq 1$, $\beta \neq 1$. Then, for all $p \in I$, either*

$$\text{(a)} \quad \left.\begin{array}{ll} \text{(i)} & f(p) = f(1)p^\beta + p^\beta \ell(p) + a_0^*(p) + b_0(p), \quad a_0^*(1) = 0 \\ \text{(ii)} & h(p) = p^\beta \ell(p) + \bar{a}_0^*(p) + \bar{b}_0(p) \end{array} \right\} \text{if } \alpha = \beta$$

or

$$\text{(b)} \quad \left.\begin{array}{ll} \text{(i)} & f(p) = c(p^\alpha - p^\beta) + a_1^*(p) + b_1(p) \\ \text{(ii)} & h(p) = c(p^\alpha - p^\beta) + \bar{a}_1^*(p) + \bar{b}_1(p) \end{array} \right\} \text{if } \alpha \neq \beta$$

where $a_0^* : \mathbb{R} \to \mathbb{R}$, $\bar{a}_0^* : \mathbb{R} \to \mathbb{R}$, $a_1^* : \mathbb{R} \to \mathbb{R}$ and $\bar{a}_1^* : \mathbb{R} \to \mathbb{R}$ are additive mappings; $b_0 : \mathbb{R} \to \mathbb{R}$, $\bar{b}_0 : \mathbb{R} \to \mathbb{R}$, $b_1 : \mathbb{R} \to \mathbb{R}$ and $\bar{b}_1 : \mathbb{R} \to \mathbb{R}$ are bounded mappings; $\ell : I \to \mathbb{R}$ is a logarithmic mapping and c is an arbitrary real constant.

Proof. Let us put $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (1.6). We obtain

$$\left| \sum_{i=1}^n f(p_i) + n(m-1)f(0) \sum_{i=1}^n p_i - (f(1) + (m-1)f(0)) \sum_{i=1}^n p_i^\alpha - \sum_{i=1}^n h(p_i) \right| \leq \varepsilon,$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.4, there exists an additive mapping $\bar{A}_1 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $\bar{B}_1 : \mathbb{R} \to \mathbb{R}$ with $\bar{B}_1(0) = 0$, such that

$$\begin{aligned} &f(p) + n(m-1)f(0)p - (f(1) + (m-1)f(0))p^\alpha - h(p) - f(0) + h(0) \\ &\quad = \bar{A}_1(p) + \bar{B}_1(p) \end{aligned} \tag{4.1}$$

for all $p \in I$. From equation (4.1), we can easily obtain the expression

$$h(p) = f(p) + A_1(p) + B_1(p) - (f(1) + (m-1)f(0))p^\alpha \tag{4.2}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as $A_1(p) = n(m-1)f(0)p - \bar{A}_1(p)$ and $B_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $B_1(p) = -f(0) + h(0) - \bar{B}_1(p)$. From equations (1.6) and (4.2), we have

$$\begin{aligned} &\sum_{j=1}^m \left| \sum_{i=1}^n f(p_i q_j) - \sum_{i=1}^n p_i^\alpha f(q_j) - q_j^\beta \right. \\ &\quad \left. \times \left[\sum_{i=1}^n f(p_i) + A_1(1) + \sum_{i=1}^n B_1(p_i) - (f(1) + (m-1)f(0)) \sum_{i=1}^n p_i^\alpha \right] \right| \leq \varepsilon. \end{aligned}$$

By Result 2.4, there exists a mapping $A_2 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable and a bounded mapping $B_2 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ with $B_2(p_1, \ldots, p_n; 0) = 0$, such that

$$\begin{aligned}
&\sum_{i=1}^{n} f(p_i q) - \sum_{i=1}^{n} p_i^{\alpha} f(q) - q^{\beta} \\
&\times \left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i) - (f(1) + (m-1)f(0)) \sum_{i=1}^{n} p_i^{\alpha} \right] \\
&- nf(0) + \sum_{i=1}^{n} p_i^{\alpha} f(0) = A_2(p_1, \ldots, p_n; q) + B_2(p_1, \ldots, p_n; q). \qquad (4.3)
\end{aligned}$$

Let $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Let us put $q = r_t x$, $t = 1, \ldots, n$ consecutively in (4.3); summing the resulting n equations so obtained and then substituting the value of the expression $\sum_{t=1}^{n} f(r_t x)$ calculated from (4.3). We get

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{t=1}^{n} f(p_i r_t x) - \big[f(x) - f(0) - x^{\beta}(f(1) \\
&+ (m-1)f(0)) \big] \sum_{i=1}^{n} p_i^{\alpha} \sum_{t=1}^{n} r_t^{\alpha} - n^2 f(0) \\
&= x^{\beta} \Bigg\{ \left[\sum_{t=1}^{n} f(r_t) + A_1(1) + \sum_{t=1}^{n} B_1(r_t) \right] \sum_{i=1}^{n} p_i^{\alpha} \\
&+ \left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i) \right] \qquad (4.4) \\
&\times \sum_{t=1}^{n} r_t^{\beta} - (f(1) + (m-1)f(0)) \sum_{i=1}^{n} p_i^{\alpha} \sum_{t=1}^{n} r_t^{\beta} \Bigg\} + A_2(p_1, \ldots, p_n; x) \\
&+ \sum_{t=1}^{n} B_2(p_1, \ldots, p_n; r_t x) + A_2(r_1, \ldots, r_n; x) \sum_{i=1}^{n} p_i^{\alpha} \\
&+ B_2(r_1, \ldots, r_n; x) \sum_{i=1}^{n} p_i^{\alpha}.
\end{aligned}$$

We observe that the left hand side of (4.4) is symmetric in $(p_1, \ldots, p_n) \in \Gamma_n$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Thus the right hand side of (4.4) must remain unchanged on interchanging p_i and r_t, $i = 1, \ldots, n$; $t = 1, \ldots, n$. Consequently, we have

$$\begin{aligned}
&A_2(p_1, \ldots, p_n; x) \left(1 - \sum_{t=1}^{n} r_t^{\alpha} \right) - A_2(r_1, \ldots, r_n; x) \left(1 - \sum_{i=1}^{n} p_i^{\alpha} \right) \\
&= \sum_{i=1}^{n} B_2(r_1, \ldots, r_n; p_i x) + B_2(p_1, \ldots, p_n; x) \sum_{t=1}^{n} r_t^{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{t=1}^{n} B_2(p_1,\dots,p_n;r_t x) - B_2(r_1,\dots,r_n;x)\sum_{i=1}^{n} p_i^{\alpha} \\
&+ x^{\beta}\left\{\left[\sum_{i=1}^{n} f(p_i) + A_1(1) + \sum_{i=1}^{n} B_1(p_i)\right]\left(\sum_{t=1}^{n} r_t^{\alpha} - \sum_{t=1}^{n} r_t^{\beta}\right)\right. \\
&- \left[\sum_{t=1}^{n} f(r_t) + A_1(1) + \sum_{t=1}^{n} B_1(r_t)\right]\left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta}\right) \\
&\left.-(f(1)+(m-1)f(0))\left[\sum_{i=1}^{n} p_i^{\beta}\sum_{t=1}^{n} r_t^{\alpha} - \sum_{t=1}^{n} r_t^{\beta}\sum_{i=1}^{n} p_i^{\alpha}\right]\right\}. \qquad (4.5)
\end{aligned}
$$

Case 1. $\alpha = \beta$.

In this case, equation (4.5) reduces to

$$
\begin{aligned}
&A_2(p_1,\dots,p_n;x)\left(1-\sum_{t=1}^{n} r_t^{\beta}\right) - A_2(r_1,\dots,r_n;x)\left(1-\sum_{i=1}^{n} p_i^{\beta}\right) \\
&= \sum_{i=1}^{n} B_2(r_1,\dots,r_n;p_i x) + B_2(p_1,\dots,p_n;x)\sum_{t=1}^{n} r_t^{\beta} \\
&\quad - \sum_{t=1}^{n} B_2(p_1,\dots,p_n;r_t x) - B_2(r_1,\dots,r_n;x)\sum_{i=1}^{n} p_i^{\beta}.
\end{aligned}
$$

For fixed $(p_1,\dots,p_n)\in\Gamma_n$ and $(r_1,\dots,r_n)\in\Gamma_n$, the left hand side of the above equation is additive in x but the right hand side is bounded on I. Thus the left hand side is linear (Aczél [1], Sahoo and Kannappan [14]). We obtain

$$
\begin{aligned}
&[A_2(p_1,\dots,p_n;x) - xA_2(p_1,\dots,p_n;1)]\left[1-\sum_{t=1}^{n} r_t^{\beta}\right] \\
&= [A_2(r_1,\dots,r_n;x) - xA_2(r_1,\dots,r_n;1)]\left[1-\sum_{i=1}^{n} p_i^{\beta}\right]. \qquad (4.6)
\end{aligned}
$$

As described in the previous section 3 that $1-\sum_{t=1}^{n} r_t^{\beta} \not\equiv 0$ for $\beta\neq 1$, $0<\beta\in\mathbb{R}$, there exists a probability distribution $(r_1^*,\dots,r_n^*)\in\Gamma_n$ such that $1-\sum_{t=1}^{n} r_t^{*\beta}\neq 0$. Using this fact in equation (4.6), we have

$$
A_2(p_1,\dots,p_n;x) = xA_2(p_1,\dots,p_n;1) + a_0^*(x)\left[1-\sum_{i=1}^{n} p_i^{\beta}\right] \qquad (4.7)
$$

where $a_0^*:\mathbb{R}\to\mathbb{R}$ defined as $a_0^*(x) = [A_2(r_1^*,\dots,r_n^*;x) - xA_2(r_1^*,\dots,r_n^*;1)] \times \left[1-\sum_{t=1}^{n} r_t^{*\beta}\right]^{-1}$ is an additive mapping with $a_0^*(1)=0$. Making use of the

fact that $\alpha = \beta$ and $1^\beta := 1$ in equation (4.3), we obtain the expression

$$\begin{aligned} A_2(p_1,\dots,p_n;1) =& -A_1(1) - \sum_{i=1}^{n} B_1(p_i) \\ &+ mf(0)\sum_{i=1}^{n} p_i^\beta - nf(0) - B_2(p_1,\dots,p_n;1). \end{aligned} \tag{4.8}$$

Equations (4.3), (4.7), (4.8) together with the fact that $\alpha = \beta$ and $a_0^*(1) = 0$, yield

$$\begin{aligned} &\sum_{i=1}^{n} H(p_i q) - \sum_{i=1}^{n} p_i^\beta H(q) - q^\beta \sum_{i=1}^{n} H(p_i) \\ &= q^\beta \left[nf(0) + A_1(1) + \sum_{i=1}^{n} B_1(p_i) \right] \\ &+ q\Bigg[- A_1(1) - \sum_{i=1}^{n} B_1(p_i) + mf(0)\sum_{i=1}^{n} p_i^\beta \\ &- nf(0) - B_2(p_1,\dots,p_n;1) \Bigg] + B_2(p_1,\dots,p_n;q), \end{aligned} \tag{4.9}$$

where $H : I \to \mathbb{R}$ is a mapping defined as

$$H(x) = f(x) - a_0^*(x) - f(0) - (f(1) + (m-1)f(0))x^\beta \tag{4.10}$$

for all $x \in I$. Clearly, from (4.10) $H(0) = 0$ follows. Since the right hand side of (4.9) is bounded, we apply Result 2.4 and using $H(0) = 0$, we have

$$H(pq) - p^\beta H(q) - q^\beta H(p) = A_3(p,q) + B_3(p,q) \tag{4.11}$$

for all $p \in I$ and $q \in I$; where $A_3 : \mathbb{R} \times I \to \mathbb{R}$ is a mapping additive in the first variable and $B_3 : \mathbb{R} \times I \to \mathbb{R}$ is a bounded mapping with $B_3(0,q) = 0$. Define a mapping $G : I \times I \to \mathbb{R}$ as

$$G(p,q) = H(pq) - p^\beta H(q) - q^\beta H(p) \tag{4.12}$$

for all $p \in I$ and $q \in I$. Making use of (4.12), we can easily verify that

$$G(r,pq) + r^\beta G(p,q) = G(rp,q) + q^\beta G(r,p). \tag{4.13}$$

From equations (4.11), (4.12) and (4.13), we have

$$\begin{aligned} &A_3(r,pq) - A_3(rp,q) - q^\beta A_3(r,p) \\ &= B_3(rp,q) + q^\beta B_3(r,p) - B_3(r,pq) - r^\beta A_3(p,q) - r^\beta B_3(p,q). \end{aligned} \tag{4.14}$$

The right hand side of (4.14) is bounded while its left hand side is an additive mapping in r so it must be linear (Acźel [1], Sahoo and Kannappan [14]). Hence

$$A_3(r,pq) - A_3(rp,q) - q^\beta A_3(r,p) = r[A_3(1,pq) - A_3(p,q) - q^\beta A_3(1,p)]. \tag{4.15}$$

Now equation (4.14) with $r = 1$, results in

$$A_3(1,pq) - q^\beta A_3(1,p) = q^\beta B_3(1,p) - B_3(1,pq). \tag{4.16}$$

From equations (4.14), (4.15) and (4.16), it follows that

$$\begin{aligned}(r^\beta - r)A_3(p,q) = {} & B_3(rp,q) + q^\beta B_3(r,p) - B_3(r,pq) - r^\beta B_3(p,q) \\ & - rq^\beta B_3(1,p) + rB_3(1,pq).\end{aligned}$$

In view of our assumption that $\beta \neq 1$, the above equation yields that the additive mapping $A_3(p,q)$ is bounded in p on I. Thus as argued earlier it follows that it must be linear (Acźel [1], Sahoo and Kannappan [14]). Hence $A_3(p,q) = pA_3(1,q)$ for all $p \in I$ and $q \in I$. However equation (4.16) with the replacement $p = 1$ yields that the mapping $q \to A_3(1,q)$ is bounded on I. As a consequence the mapping $A_3(p,q)$ is bounded on $I \times I$. Apparently, follows same for the mapping G on $I \times I$ and thus by applying Result 2.5 on (4.11), there exists a logarithmic mapping $\ell : I \to \mathbb{R}$ and a bounded mapping $B_4 : \mathbb{R} \to \mathbb{R}$ such that $H(p) = p^\beta \ell(p) + B_4(p)$ for all $p \in I$. Thus from (4.10), (a)(i) holds where bounded mapping $b_0 : \mathbb{R} \to \mathbb{R}$ is defined as $b_0(p) = B_4(p) + f(0) + (m-1)f(0)p^\beta$. From (4.2), (a)(i) and $\alpha = \beta$, we obtain solution (a)(ii) where an additive mapping $\bar{a}_0^* : \mathbb{R} \to \mathbb{R}$ is defined as $\bar{a}_0^*(p) = a_0^*(p) + A_1(p)$; a bounded mapping $\bar{b}_0 : \mathbb{R} \to \mathbb{R}$ is defined as $\bar{b}_0(p) = b_0(p) + B_1(p) - (m-1)f(0)p^\beta$.

Case 2. $\alpha \neq \beta$.

In this case, there is no loss of generality in assuming $n \geq m$.

Let us put $p_{m+1} = \cdots = p_n = 0$ in (1.6) and making use of (4.2), we have

$$\begin{aligned}&\Bigg| \sum_{i=1}^m \sum_{j=1}^m f(p_i q_j) - \sum_{i=1}^m p_i^\alpha \sum_{j=1}^m f(q_j) - \sum_{j=1}^m q_j^\beta \sum_{i=1}^m f(p_i) \\ &\quad + (f(1) + (m-1)f(0)) \sum_{i=1}^m p_i^\alpha \sum_{j=1}^m q_j^\beta + m(n-m)f(0) \\ &\quad - [(n-m)f(0) + A_1(1) + (n-m)B_1(0)] \sum_{j=1}^m q_j^\beta - \sum_{i=1}^m B_1(p_i) \sum_{j=1}^m q_j^\beta \Bigg| \leq \varepsilon,\end{aligned} \tag{4.17}$$

for all $(p_1,\dots,p_m)\in\Gamma_m$ and $(q_1,\dots,q_m)\in\Gamma_m$. Now on interchanging p_i and q_j, $i=1,\dots,m$; $j=1,\dots,m$ in functional inequality (4.17), we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{m}\sum_{j=1}^{m}f(p_iq_j)-\sum_{j=1}^{m}q_j^{\alpha}\sum_{i=1}^{m}f(p_i)-\sum_{i=1}^{m}p_i^{\beta}\sum_{j=1}^{m}f(q_j)\right.\\ &+(f(1)+(m-1)f(0))\sum_{j=1}^{m}q_j^{\alpha}\sum_{i=1}^{m}p_i^{\beta}+m(n-m)f(0)\\ &\left.-[(n-m)f(0)+A_1(1)+(n-m)B_1(0)]\sum_{i=1}^{m}p_i^{\beta}-\sum_{j=1}^{m}B_1(q_j)\sum_{i=1}^{m}p_i^{\beta}\right|\le\varepsilon\,.\end{aligned}\tag{4.18}$$

Now, applying triangle inequality to inequalities (4.17) and (4.18), we get

$$\begin{aligned}&\left|\left(\sum_{j=1}^{m}q_j^{\alpha}-\sum_{j=1}^{m}q_j^{\beta}\right)\sum_{i=1}^{m}f(p_i)-\left(\sum_{i=1}^{m}p_i^{\alpha}-\sum_{i=1}^{m}p_i^{\beta}\right)\sum_{j=1}^{m}f(q_j)\right.\\ &+(f(1)+(m-1)f(0))\left(\sum_{i=1}^{m}p_i^{\alpha}\sum_{j=1}^{m}q_j^{\beta}-\sum_{i=1}^{m}p_i^{\beta}\sum_{j=1}^{m}q_j^{\alpha}\right)\\ &+[(n-m)f(0)+A_1(1)+(n-m)B_1(0)]\left(\sum_{i=1}^{m}p_i^{\beta}-\sum_{j=1}^{m}q_j^{\beta}\right)\\ &\left.-\sum_{i=1}^{m}B_1(p_i)\sum_{j=1}^{m}q_j^{\beta}+\sum_{j=1}^{m}B_1(q_j)\sum_{i=1}^{m}p_i^{\beta}\right|\le 2\varepsilon.\end{aligned}$$

Now, as explained in the previous section 3 that for $\alpha\neq\beta$, $\sum_{j=1}^{m}q_j^{\alpha}-\sum_{j=1}^{m}q_j^{\beta}$ does not vanish identically on Γ_m. Hence we choose a probability distribution $(q_1^*,\dots,q_m^*)\in\Gamma_m$ such that $\sum_{j=1}^{m}q_j^{*\alpha}-\sum_{j=1}^{m}q_j^{*\beta}\neq 0$ and $\varepsilon'=2\varepsilon\left[\sum_{j=1}^{m}q_j^{*\alpha}-\sum_{j=1}^{m}q_j^{*\beta}\right]^{-1}$. Hence the above inequality reduces to

$$\left|\sum_{i=1}^{m}f(p_i)-c_1\sum_{i=1}^{m}p_i^{\alpha}+c_2\sum_{i=1}^{m}p_i^{\beta}-c_3-c_4\sum_{i=1}^{m}B_1(p_i)\right|\le\varepsilon',$$

where $c_1,c_2,c_3,c_4\in\mathbb{R}$ and $(p_1,\dots,p_m)\in\Gamma_m$. By Result 2.4, there exists an additive mapping $A_4:\mathbb{R}\to\mathbb{R}$ and a bounded mapping $B_5:\mathbb{R}\to\mathbb{R}$ with $B_5(0)=0$ satisfying

$$f(p)-c_1p^{\alpha}+c_2p^{\beta}-c_3p-c_4B_1(p)-f(0)+c_4B_1(0)=A_4(p)+B_5(p)$$

for all $p\in I$. Thus (b)(i) holds where $a_1^*:\mathbb{R}\to\mathbb{R}$ defined as $a_1^*(p)=A_4(p)+c_3p$, for all $p\in\mathbb{R}$ is an additive mapping; $b_1:\mathbb{R}\to\mathbb{R}$ defined as $b_1(p)=B_5(p)+$

$c_4B_1(p) - c_4B_1(0) + f(0) + (c_1 - c_2)p^\beta$, for all $p \in \mathbb{R}$ is a bounded mapping and $c = c_1 \in \mathbb{R}$. From equations (b)(i) and (4.2), (b)(ii) follows where $\bar{a}_1^* : \mathbb{R} \to \mathbb{R}$ defined as $\bar{a}_1^*(p) = a_1^*(p) + A_1(p)$, for all $p \in \mathbb{R}$ is an additive mapping and $\bar{b}_1 : \mathbb{R} \to \mathbb{R}$ defined as $\bar{b}_1(p) = b_1(p) + B_1(p) - (f(1) + (m-1)f(0))p^\alpha$, for all $p \in \mathbb{R}$ is a bounded mapping. □

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Oscillation of second-order nonlinear neutral dynamic equations with "Maxima" on time scales with nonpositive neutral term

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Abstract. In this paper, we establish some new oscillation criteria for second order nonlinear neutral delay dynamic equation of the form

$$(r(t)((m(t)y(t) - p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0$$

on a time scale $\mathbb{T}$. The present results not only generalize and extend some existing results but also can be applied to some of the oscillation problems that are not covered before. Finally, we give some examples to illustrate our main results.

Keywords: oscillation, maxima, neutral equation, delay equation, dynamic equation, time scale.

1. Introduction

The theory of time scales was introduced by Hilger [13] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary

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nonempty closed subset of the reals. The cases when time scale equals to the reals or to the integers, the obtained results represent the classical theories of differential and difference equations. Many other interesting time scales exist (e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ which has important applications in quantum theory, $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = H_n$ the set of harmonic numbers). For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [10, 11]. In the last few years, the qualitative theory of differential equations with "Maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real systems (see, for example [4,14]). The oscillatory behavior of solutions of differential equations with "Maxima" is discussed in [2, 5-9, 15, 17,18,20], and the references cited therein. In [2], the authors discussed the oscillatory behavior of the second order quasilinear neutral differential equation with "Maxima"

$$(r(t)((y(t)+p(t)y(\tau(t)))')^{\gamma})' + q(t)\max_{s\in[t-\delta,t]} y^{\beta}(s) = 0,\ t \geq t_0 \geq 0, \tag{1.1}$$

where $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt < \infty$. The authors in [17] introduced sufficient conditions for the oscillation of the equation

$$(r(t)((y(t)+p(t)y(\tau(t)))')^{\gamma})' + q(t)\max_{s\in[t-\delta,t]} y^{\gamma}(s) = 0,\ t \geq t_0 \geq 0, \tag{1.2}$$

where, $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt < \infty$. The oscillatory and asymptotic behavior of the second order neutral delay difference equation

$$\Delta(a_n\Delta(x_n + p_n x_{n-\tau})) + q_n \max_{[n-\delta,n]} x_n^{\gamma} = 0,\ \ n \in \mathbb{N}, \tag{1.3}$$

where $\sum_{n=n_0}^{\infty}\frac{1}{a_n} < \infty$ studied in [1]. Also, in [16] Qi Li et al. considered the differential equation

$$(r(t)((y(t)-p(t)y(\tau(t)))')^{\gamma})' + q(t)f(y(\delta(t))) = 0,\ t \geq t_0 > 0, \tag{1.4}$$

where, $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt = \infty$. The authors in [3, 12, 19] improved the results obtained in [16]. They introduced sufficient conditions for oscillation of all solutions of Eq. (1.4).

In this paper, we study the oscillation of the second order nonlinear neutral dynamic equation with "Maxima" of the form

$$(r(t)((m(t)y(t)-p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t)\max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0,\ t \in \mathbb{T}, \tag{1.5}$$

where $\mathbb{T}$ is a time scale. Throughout this paper, we consider the following hypotheses:

(A_1) γ and β_i, $i = 1, 2, ..., n$ are quotient of odd positive integers.

(A_2) $\tau(t), \delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$, $\delta_i(t) \leq t$, $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}(\delta_i(t)) = \infty, i = 1, 2, ..., n$.

(A_3) $r(t)$, $m(t)$, $p(t)$ and $q_i(t)$, $i = 1, 2, ..., n$ are real valued rd-continuous positive functions defined on $\mathbb{T}$, $\lim_{t\to\infty} p(t) = p_0$ and $\lim_{t\to\infty} m(t) = m_0 > 0$ and $m_0 > p_0$. Also, $m(t) > p(t)$.

Also, the following condition is taken into consideration:

$$\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty. \tag{1.6}$$

Now, defining

$$x(t) := m(t)y(t) - p(t)y(\tau(t)). \tag{1.7}$$

Eq. (1.5) reduces to

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0. \tag{1.8}$$

2. Preliminaries and Lemmas

Lemma 2.1. *Assume that conditions $(A_1) - (A_3)$ and (1.6) hold. If $y(t)$ is an eventually positive solution of Eq. (1.5), then $x(t)$ satisfies one of the following two cases:*

(c_1) $x(t) > 0$, $x^{\Delta}(t) > 0$ *and* $(r(t)((x^{\Delta}(t))^{\gamma}))^{\Delta} < 0$.

(c_2) $x(t) < 0$, $x^{\Delta}(t) > 0$ *and* $(r(t)((x^{\Delta}(t))^{\gamma}))^{\Delta} < 0$.

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Proof. Since, $y(t)$ is an eventually positive solution of Eq. (1.5), then, by (A_2) there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta_i(t)) > 0$, $i = 1, 2, ..., n$ for $t \geq t_1$. From Eq. (1.8),

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} = -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) < 0.$$

Hence, $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing and of one sign, therefore there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for $t \geq t_2$. If $x^{\Delta}(t) > 0$, then either (c_1) or (c_2) occurs. Now, we prove that $x^{\Delta}(t) < 0$ cannot occur. If $x^{\Delta}(t) < 0$, then for $t \geq t_2$,

$$r(t)(x^{\Delta}(t))^{\gamma} \leq -C < 0, \tag{2.1}$$

where $C := -r(t_2)(x^{\Delta}(t_2))^{\gamma} > 0$, integrating inequality (2.1) from t_2 to t and using condition (1.6), we get

$$x(t) \leq x(t_2) - C^{\frac{1}{\gamma}} \int_{t_2}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s \to -\infty \; as \; (t \to \infty).$$

Hence,

$$\lim_{t\to\infty} x(t) = -\infty.$$

Now, we consider the following two cases.
Case(1): If $y(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} y(t_k) = \infty$, where $y(t_k) = \max\{y(s) : t_0 \leq s \leq t_k\}$. For sufficiently large k, we have $\tau(t_k) > t_0$. Using condition A_2, we get

$$y(\tau(t_k)) = \max\{y(s) : t_0 \leq s \leq \tau(t_k)\} \leq \max\{y(s) : t_0 \leq s \leq t_k\} = y(t_k).$$

Hence, $x(t_k) = m(t_k)y(t_k) - p(t_k)y(\tau(t_k)) \geq m(t_k)y(t_k) - p(t_k)y(t_k) \geq m(t_k)(1 - \frac{p(t_k)}{m(t_k)})y(t_k) > 0$, which contradicts $\lim_{t\to\infty} x(t) = -\infty$.
Case(2): If $y(t)$ is bounded, then using condition (A_3), $x(t)$ is bounded. This contradicts $\lim_{t\to\infty} x(t) = -\infty$. Hence, $x^{\Delta}(t) < 0$ cannot occur.

Remark 2.1. From Eq. (1.8), we have

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} = -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s). \tag{2.2}$$

If (c_1) holds. From the definition of $x(t)$, we have

$$\min_{s\in[\delta_i(t),t]} (\frac{x^{\beta_i}(s)}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} (\frac{x^{\beta_i}(s)}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s), \; i = 1, 2, ..., n.$$

Since $x^{\Delta} > 0$, then

$$x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s), \; i = 1, 2, ..., n.$$

Hence,

$$-\sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}) \geq -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s). \tag{2.3}$$

From Eq. (2.2) and inequality (2.3), we get

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} \leq -\sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}). \tag{2.4}$$

Lemma 2.2. *The function $y(t)$ is a negative solution of Eq. (1.5) if and only if $-y(t)$ is a positive solution of the equation*

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \min_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0. \tag{2.5}$$

Proof. The assertion of Lemma 2.2 can be verified easily.

Define,

$$R(t) = r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s,$$
$$A(l,a) = \int_{a}^{l} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s$$

and $\eta(t,a) = \frac{A(\delta(t),a)}{A(\sigma(t),a)}$.

Lemma 2.3. *Assume that the conditions $(A_1) - (A_3)$ and (1.6) hold. If $y(t)$ is an eventually positive solution of Eq. (1.5) and (c_1) in Lemma 2.1 holds, then*

(1) *$x(t) \geq R(t)x^{\Delta}(t)$ for $t \geq t_1$.*

(2) *$x(\delta(t)) \geq \eta(t,t_2)x(\sigma(t))$ for $t \geq \delta(t) \geq t_2 \geq t_1$.*

Proof. Since (c_1) holds,
(1) for $t \geq t_1$, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} \frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}} \Delta s \geq \ r^{\frac{1}{\gamma}}(t)x^{\Delta}(t) \int_{t_1}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s = R(t)x^{\Delta}(t),$$

(2) for $t > t_2 \geq t_1$, we have

$$x(\sigma(t)) - x(\delta(t)) = \int_{\delta(t)}^{\sigma(t)} \frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}} \Delta s \leq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t)) \int_{\delta(t)}^{\sigma(t)} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s.$$

Therefore,

$$\frac{x(\sigma(t))}{x(\delta(t))} \leq 1 + \frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))} A(\sigma(t),\delta(t)). \tag{2.6}$$

Also,

$$x(\delta(t)) \geq x(\delta(t)) - x(t_2) = \int_{t_2}^{\delta(t)} \frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s$$
$$\geq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t)) \int_{t_2}^{\delta(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s.$$

Hence,

$$\frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))} \leq \frac{1}{A(\delta(t),t_2)}. \tag{2.7}$$

From inequalities (2.6) and (2.7), we get

$$\frac{x(\sigma(t))}{x(\delta(t))} \leq 1+\frac{A(\sigma(t),\delta(t))}{A(\delta(t),t_2)} = \frac{A(\sigma(t),\delta(t))+A(\delta(t),t_2)}{A(\delta(t),t_2)} = \frac{A(\sigma(t),t_2)}{A(\delta(t),t_2)}.$$

Hence,

$$x(\delta(t)) \geq \eta(t,t_2)x(\sigma(t)). \tag{2.8}$$

This completes the proof.

Remark 2.2. Without loss of generality we can deal only with the positive solution of Eq. (1.5), since the proof for the negative solution is similar.

3. Main results

Theorem 3.1. *Assume that conditions* $(A_1)-(A_3)$ *and* (1.6) *hold,* $\gamma \geq \beta_i$, $i=1,2,...,n$ *and there exists a positive rd-continuous* Δ*-differentiable function* $z(t)$ *such that for a constant* $b>0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q(u)-\frac{z_+^{\Delta}(u)}{A^{\gamma}(u,t_1)}]\Delta u=\infty, \tag{3.1}$$

where

$$Q(u)=z(\sigma(u))\sum_{i=1}^{n}\frac{q_i(u)\eta_i^{\beta_i}(u,t_2)}{b^{\gamma-\beta_i}(A(\sigma(u),t_1))^{\gamma-\beta_i}}\min_{s\in[\delta_i(u),u]}(\frac{1}{m^{\beta_i}(s)}),$$

$$A(\sigma(u),t_1)=\int_{t_1}^{\sigma(u)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s, z_+^{\Delta}(u)=\max\{z^{\Delta}(u),0\}$$

and $\eta_i(t,a)=\frac{A(\delta_i(t),a)}{A(\sigma(t),a)}$, $i=1,2,...,n$. *Then, every solution of Eq.* (1.5) *is either oscillatory or tends to zero.*

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5), then by $(A_1)-(A_3)$ there exists $t_1\in\mathbb{T}$ sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta_i(t))>0$, $i=1,2,...,n$ for $t\geq t_1$ and Lemma 2.1 holds. If (c_1) holds, define the function $w(t)$ by the Riccati substitution

$$w(t):=z(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \quad for \;\; t\geq t_1, \tag{3.2}$$

then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= z^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right) + z(\sigma(t))\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\Delta} \\ &= z^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right) \\ &+ z(\sigma(t))\left(\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}x^{\gamma}(t) - (x^{\gamma}(t))^{\Delta}r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right) \\ &= z^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right) \\ &+ z(\sigma(t))\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{x^{\gamma}(\sigma(t))} - z(\sigma(t))\frac{(x^{\gamma}(t))^{\Delta}r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \end{aligned} \tag{3.3}$$

Using $\gamma > 0$ and Keller's chain rule, we get

$$\begin{aligned} (x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1 [x(t) + h\mu(t)x^{\Delta}(t)]^{\gamma-1}dh\; x^{\Delta}(t) \\ &= \gamma\int_0^1 [(1-h)x + hx^{\sigma}]^{\gamma-1}dh\; x^{\Delta}(t). \end{aligned}$$

Hence,

$$(x^{\gamma}(t))^{\Delta} \geq \begin{cases} \gamma x^{\gamma-1}(t)x^{\Delta}(t) & \text{if } \gamma \geq 1, \\ \gamma (x^{\sigma})^{\gamma-1}(t)x^{\Delta}(t) & \text{if } \gamma < 1. \end{cases} \tag{3.4}$$

From inequalities (2.4), (3.3) and (3.4), we get

$$w^{\Delta}(t) \leq \begin{cases} z_+^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right) \\ -z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)x^{\beta_i}(\delta_i(t))}{x^{\gamma}(\sigma(t))}\min_{s\in[\delta_i(t),t]}\left(\frac{1}{m^{\beta_i}(s)}\right) \\ -z(\sigma(t))\dfrac{\gamma x^{\gamma-1}x^{\Delta}(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}, \text{ if } \gamma \geq 1, \\ z_+^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right) \\ -z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)x^{\beta_i}(\delta_i(t))}{x^{\gamma}(\sigma(t))}\min_{s\in[\delta_i(t),t]}\left(\frac{1}{m^{\beta_i}(s)}\right) \\ -z(\sigma(t))\dfrac{\gamma (x^{\sigma})^{\gamma-1}x^{\Delta}(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}, \text{ if } \gamma < 1. \end{cases} \tag{3.5}$$

Using Lemma 2.3 in inequality (3.5), we get

$$w^{\Delta}(t) \leq \frac{z_+^{\Delta}(t)}{A^{\gamma}(t,t_1)} - z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)\eta_i^{\beta_i}(t,t_2)}{x^{\gamma-\beta_i}(\sigma(t))}\min_{s\in[\delta_i(t),t]}\left(\frac{1}{m^{\beta_i}(s)}\right). \tag{3.6}$$

Since,

$$x(t)-x(t_1)=\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta(s)\leq r^{\frac{1}{\gamma}}(t_1)x^{\Delta}(t_1)A(t,t_1),$$

then, there exists $T\in[t_1,\infty)_{\mathbb{T}}$ and a suitable constant $b>0$ such that

$$x(t)\leq bA(t,t_1)\text{ for }t\in[T,\infty)_{\mathbb{T}}. \tag{3.7}$$

From inequalities (3.6) and (3.7), we have

$$w^{\Delta}(t)\leq\frac{z_+^{\Delta}(t)}{A^{\gamma}(t,t_1)}-z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)\eta_i^{\beta_i}(t,t_2)}{b^{\gamma-\beta_i}A^{\gamma-\beta_i}(\sigma(t),t_1)}\min_{s\in[\delta_i(t),t]}(\frac{1}{m^{\beta_i}(s)}). \tag{3.8}$$

Integrating from $t_3=\max\{t_1,t_2,T\}$ to t and taking the limit supremum of both sides as $t\to\infty$, we get a contradiction with condition (3.1).

If (c_2) holds, then $\lim_{t\to\infty}y(t)=0$ (the proof is similar to that of Lemma 2.2 [16]). This completes the proof.

Theorem 3.2. *Assume that conditions* $(A_1)-(A_3)$ *and* (1.6) *hold,* $\gamma\geq\beta_i$, $\gamma\geq1$, $i=1,2,...,n$ *and there exists a positive rd-continuous* Δ*-differentiable function* $z(t)$ *such that for a constant* $b>0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q_1(u)-\frac{r(u)(z_+^{\Delta}(u))^2}{4\gamma z(u)R^{\gamma-1}(u)}]\Delta u=\infty, \tag{3.9}$$

where

$$Q_1(u)=z(u)\sum_{i=1}^{n}\frac{q_i(u)\eta_i^{\beta_i}(u,t_2)}{b^{\gamma-\beta_i}(A(\sigma(u),t_1))^{\gamma-\beta_i}}\min_{s\in[\delta_i(u),u]}(\frac{1}{m^{\beta_i}(s)}),$$

$$A(\sigma(u),t_1)=\int_{t_1}^{\sigma(u)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s,\ z_+^{\Delta}(u)=\max\{z^{\Delta}(u),0\}$$

and $\eta_i(t,a)=\frac{A(\delta_i(t),a)}{A(\sigma(t),a)}$, $i=1,2,...,n$. *Then, every solution of Eq.* (1.5) *is either oscillatory or tends to zero.*

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5), then by $(A_1)-(A_3)$ there exists $t_1\in\mathbb{T}$ sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta_i(t))>0$, $i=1,2,...,n$ for $t\geq t_1$ and Lemma 2.1 holds. If (c_1) holds, define the function $w(t)$ by the Riccati substitution

$$w(t):=\frac{z(t)}{x^{\gamma}(t)}r(t)(x^{\Delta}(t))^{\gamma}\ \ for\ \ t\geq t_1, \tag{3.10}$$

then $w(t)>0$ and

$$\begin{aligned}w^{\Delta}(t)&=\frac{z(t)}{x^{\gamma}(t)}(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}\\&\quad+r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma}(\frac{z^{\Delta}(t)x^{\gamma}(t)-z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}).\end{aligned} \tag{3.11}$$

Using inequalities (2.4) and (3.4),

$$w^{\Delta}(t) \leq \frac{-z(t)}{x^{\gamma}(t)} \sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)})$$
$$+ \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{z(t)}{z(\sigma(t))} w^{\sigma} \gamma \frac{x^{\Delta}}{x}. \tag{3.12}$$

Using Lemma 2.3 and inequality (3.7), we get

$$w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{\gamma z(t)}{z(\sigma(t))} w^{\sigma} \frac{r(t)(x^{\Delta})^{\gamma} R^{\gamma-1}(t)}{r(t)x^{\gamma}}, \tag{3.13}$$

where,

$$Q_1(t) := \sum_{i=1}^{n} \frac{z(t) q_i(t) \eta_i^{\beta_i}(t,t_2)}{b^{\gamma-\beta_i}(A(\sigma(t),t_1))^{\gamma-\beta_i}} \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)})$$

Since $(r(t)(x^{\Delta}(t)^{\gamma})^{\Delta} < 0$ and $x^{\Delta}(t) > 0$,

$$w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{\gamma z(t)}{r(t) z^2(\sigma(t))} R^{\gamma-1}(t) (w^{\sigma})^2. \tag{3.14}$$

Hence,

$$w^{\Delta}(t) \leq -Q_1(t) + \frac{r(t)(z_+^{\Delta}(t))^2}{4\gamma z(t) R^{\gamma-1}(t)} \tag{3.15}$$

Integrating from $t_1$to t and taking the limit supremum of both sides as $t \to \infty$, we get a contradiction with condition (3.9).

If (c_2) holds, the proof is similar to that of Theorem 3.1. This completes the proof.

Define $D = \{(t,s) \in T^2 : t \geqslant s \geqslant 0\}$ and $\mathbb{H} = \{H(t,s) \in C_{rd}(\mathbb{D},\mathbb{R}) : H(t,t) = 0,\ H(t,s) > 0$ and H^{Δ_s} exists for $t > s \geq 0\}$.

Theorem 3.3. *Assume that conditions $(A_1) - (A_3)$ and (1.6) hold, $\gamma \geq \beta_i, \gamma \geq 1,\ i = 1,2,...,n$ and there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a function $H \in \mathbb{H}$ and constant $b > 0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,u) Q_1(u) - \frac{r(u) z^2(\sigma(u)) C^2(t,u)}{4\gamma z(u) R^{\gamma-1}(u) H(t,u)}] \Delta u = \infty, \tag{3.16}$$

where $Q_1(u)$ is the same as in Theorem 3.2 and

$$C(t,u) = [H^{\Delta_u}(t,u) + \frac{z_+^{\Delta}(u)}{z(\sigma(u))} H(t,u)].$$

Then, every solution of Eq. (1.5) is either oscillatory or tends to zero.

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5). Proceeding as in the proof of Theorem 3.2, we get

$$w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{\gamma z(t)}{r(t) z^2(\sigma(t))} R^{\gamma-1}(t) (w^{\sigma})^2.$$

For a function $H \in \mathbb{H}$, we have

$$\begin{aligned} \int_{t_2}^{t} H(t,s) Q_1(s) \Delta s \leq & - \int_{t_2}^{t} H(t,s) w^{\Delta}(s) \Delta s + \int_{t_2}^{t} \frac{z_+^{\Delta}(s)}{z(\sigma(s))} H(t,s) w^{\sigma}(s) \Delta s \\ & - \int_{t_2}^{t} \frac{\gamma z(s) R^{\gamma-1}(s)}{r(s) z^2(\sigma(s))} (w^{\sigma}(s))^2 H(t,s) \Delta s. \end{aligned} \tag{3.17}$$

Integrating by parts, we get

$$\begin{aligned} - \int_{t_2}^{t} H(t,s) w^{\Delta}(s) \Delta s &= -H(t,s) w(s) \mid_{t_2}^{t} + \int_{t_2}^{t} H^{\Delta_s}(t,s) w^{\sigma}(s) \Delta s \\ &= H(t,t_2) w(t_2) + \int_{t_2}^{t} H^{\Delta_s}(t,s) w^{\sigma}(s) \Delta s. \end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned} \int_{t_2}^{t} H(t,s) Q_1(s) \Delta s \leq & H(t,t_2) w(t_2) + \int_{t_2}^{t} [H^{\Delta_s}(t,s) + \frac{z_+^{\Delta}(s)}{z(\sigma(s))} H(t,s)] w^{\sigma}(s) \Delta s \\ & - \int_{t_2}^{t} \frac{\gamma z(s) R^{\gamma-1}(s)}{r(s) z^2(\sigma(s))} H(t,s) (w^{\sigma}(s))^2 \Delta s \\ & \leq H(t,t_2) w(t_2) + \int_{t_2}^{t} \frac{r(s) c^2(t,s) z^2(\sigma(s))}{4\gamma z(s) R^{\gamma-1}(s) H(t,s)} \Delta s. \end{aligned} \tag{3.19}$$

Taking the limit supremum of both sides as $t \to \infty$, we get a contradiction to condition (3.16). If (c_2) holds, the proof is similar to that of Theorem 3.1. This completes the proof.

Theorem 3.4. *Assume that conditions* $(A_1) - (A_3)$ *and* (1.6) *hold,* $\beta_i \geq \gamma$, $i = 1, 2, ..., n$ *and there exists a positive rd-continuous* Δ*-differentiable function* $z(t)$ *such that for a constant* $b > 0$,

$$\limsup_{t \to \infty} \int_{t_0}^{t} [Q_2(u) - \frac{z_+^{\Delta}(u)}{A^{\gamma}(u, t_1)}] \Delta u = \infty, \tag{3.20}$$

where

$$Q_2(u) = \sum_{i=1}^{n} z(\sigma(u)) q_i(u) \eta_i^{\beta_i}(u, t_2) b^{\beta_i - \gamma} \min_{s \in [\delta_i(u), u]} (\frac{1}{m^{\beta_i}(s)}),$$

$A(u, t_1) = \int_{t_1}^{u} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s$, $z_+^{\Delta}(u) = \max\{z^{\Delta}(u), 0\}$ *and* $\eta_i(t, a) = \frac{A(\delta_i(t), a)}{A(\sigma(t), a)}$, $i = 1, 2, ..., n$. *Then, every solution of Eq. (1.5) is either oscillatory or tends to zero.*

Proof. The proof is similar to that of Theorem 3.1.

4. Examples

To illustrate our results, we give some examples of second order neutral delay differential equations with "Maxima" which cannot be studied by the previous known criteria of oscillation.

Example 4.1. Consider the equation

$$(4.1) \quad ((((2+\frac{1}{t})y(t)-(1+\frac{1}{t^2})y(\frac{t}{2}))')^{\frac{5}{3}})'+\lambda\sqrt{t}\max_{s\in[\frac{t}{3},t]}y^{\frac{3}{5}}(s)=0,\ t\in[1,\infty).$$

Here, $\gamma=\frac{5}{3}$, $\beta=\frac{3}{5}$, $r(t)=1$, $m(t)=2+\frac{1}{t}$, $p(t)=1+\frac{1}{t^2}$, $q(t)=\lambda\sqrt{t}$, $\tau(t)=\frac{t}{2}$ and $\delta(t)=\frac{t}{3}$. Hence, $\gamma>\beta$, $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt=\infty$ and $m(t)>p(t)$.

Also,

$$A(\sigma(u),t_1)=A(u,t_1)=\int_{t_1}^{u}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt=u-t_1,$$

and

$$\eta(u,t_2)=\frac{u-3t_2}{3(u-t_2)}.$$

Choosing $z(t)=1$, then $z'=0$ and

$$Q(u)=\frac{\lambda u^{\frac{1}{2}}(u-3t_2)^{\frac{3}{5}}u^{\frac{3}{5}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}(u-t_2)^{\frac{3}{5}}(u-t_1)^{\frac{16}{15}}(2u+3)^{\frac{3}{5}}}>\frac{\lambda(u-3t_2)^{\frac{17}{10}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}u^{\frac{34}{15}}}.$$

Hence,

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q(u)-\frac{z'_+(u)}{A^{\gamma}(u,t_1)}]\Delta u>\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda(u-3t_2)^{\frac{17}{10}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}u^{\frac{34}{15}}}du=\infty,$$

if $\lambda>0$. Then, by Theorem 3.1 every solution of Eq. (4.1) is either oscillatory or tends to zero if $\lambda>0$.

Example 4.2. Consider the equation

$$(4.2) \quad (((y(t)-\frac{1}{2}y(t-1))')^{\frac{7}{5}})'+\max_{s\in[t-5,t]}y(s)+\max_{s\in[\frac{t}{2},t]}y^{\frac{1}{7}}(s)=0,\ t\in[1,\infty),$$

choosing $z(t)=1$, one can easily verify that condition (3.1) is satisfied. Then, by Theorem 3.1 every solution of Eq. (4.2) is either oscillatory or tends to zero.

Example 4.3. Consider the equation

$$(4.3) \quad (((3y(t)-(1+\frac{1}{t})y(t-2))')^{\frac{1}{7}})'+\lambda t\max_{s\in[t-3,t]}y(s)=0,\ t\in[1,\infty),$$

choosing $z(t)=1$, one can easily verify that "condition (3.25) is satised if $\lambda>0$. Then, by Theorem 3.4 every solution of Eq. (4.3) is either oscillatory or tends to zero if $\lambda>0$.

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On restrained hub number in graphs

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Abstract. In this paper, we study the restrained hub number $h_r(G)$ of a graph G. We characterize the class of all graphs for which $h_r(G) = 1$. Also the relationship between cut vertices and restrained hub number are presented. The restrained hub number of the corona of two graphs is determined.
Keywords: hub number, restrained hub number, corona of two graphs.

1. Introduction

Let $G = (V, E)$ be a finite and undirected graph without loops and multiple edges. And $G = (p, q)$ graph if its with p vertices and q edges. The degree of a vertex v in a graph G denoted by $deg(v)$, and $\delta(G)(\Delta(G))$ denotes the minimum (maximum) degree among the vertices of G, respectively [2]. An end vertex is a vertex of degree one, a clique of a graph is a maximal complete subgraph, a block of a graph is a maximal nonseparable subgraph. A star is a complete bipartite graph $K_{1,p-1}$, and denoted by S_p. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its

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incident edges is denoted by $G-v$. For $v \in V(G)$, the open neighbourhood of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$, for $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$, the closed neighbourhood $N[v] = N(v) \cup \{v\}$, and $N[S] = N(S) \cup S$. The contraction of a vertex x in G, denoted by G/x, is being the graph obtained by deleting x and putting a clique on the (open) neighbourhood of x, (note that, this operation does not create multiple edges, if two neighbours of x are already adjacent, then they remain simply adjacent). Graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively, their union, $G(V, E) = G_1 \cup G_2$ has as expected, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The corona $G \circ F$ of two graphs G and F is the graph obtained by taking one copy of G of order p and p copies of F, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of F. For every $v \in V(G)$, denote by F_v the copy of F whose vertices are attached one by one to the vertex v [1]. The distance $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest path connecting them. For a vertex v of G, the eccentricity of v is $e(v) = \max\{d(v, u), u \in V(G)\}$. See [2] for terminology and notations not defined here.

M. Walsh [13] introduced the theory of hub in 2006. A hub set in a graph G is a set H of vertices in G such that any two vertices in $V(G) \setminus H$ are connected by a path whose all internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. If for every pair of vertices in $V(G) \setminus H$ are also connected by a path whose all internal vertices lie in $V(G) \setminus H$, then H is called a restrained hub set and denoted by H_r. The restrained hub number $h_r(G)$ is the minimum cardinality of a restrained hub set in G [6]. For more details on the hub studies we refer to [3, 4, 5, 7, 8, 9, 10, 11, 12].

Imagine there is a graph representing an industrial city map, where each point on the map represents a building place, with an edge between two points if there's an easy walk from one to the other. Some buildings will be implemented as factories and others as transit stations. Where raw material can be transferred between any two factories by transit stations. Also, trader can move between any two non-adjacent factories to buy goods without passing any transit point, the goal is to make costs as cheaply as possible by converting as few buildings as possible into transit stations. Motivated by this along with the concept of restrained hub number and the great attention from researchers in the concept of hub number in graph, we try to develop the theory of hub by establish a new results on this theory. The following results will be useful in the proof of our results.

Theorem 1.1 ([6]). *Let G be any graph. Then the set H_r is restrained hub set if and only if G/H_r is complete, and $G[V(G) \setminus H_r]$ is connected.*

Theorem 1.2 ([6]). *Let G be a graph with at least one end vertex. Then $h_r(G) = p - 2$ if and only if there exists minimum restrained hub set not containing an end vertex.*

Lemma 1.1. *If uv is an edge of a connected graph G, then $|e(u) - e(v)| \leq 1$.*

2. Main results

Proposition 2.1. *Let G be a graph. Then $h_r(G) = h(G) = 0$ if and only if G is a complete graph.*

Proof. Let $h_r(G) = 0$, and let $\{x, y\} \subseteq V(G)$. Then $H_r = \phi$ is a minimum restrained hub set of G, so there exists $xy-$ path with all internal vertices in $H_r = \phi$. Thus x is adjacent to y. Therefore, G is complete. The converse is trivial. □

Note that: If H_r is a restrained hub set of G, then $H_r \cup \{v\}$ for some $v \in V(G)$, may not be a restrained hub set of G. For example if $G \cong C_5$, then $H_r = \{x, y\}$, where $xy \in E(G)$ is a restrained hub set of G, but $H_r \cup \{z\}$, where z not adjacent to both x and y, is not a restrained hub set of G.

Theorem 2.1. *For any graph G with connected subgraph $G[H_{cr}]$. H_{cr} is a connected restrained hub set of G if and only if G has the following structure:*

1. *$V(G) = M \cup N \cup H_{cr}$, where M, N and H_{cr} are disjoint.*
2. *Every vertex in N is not adjacent to any vertex in H_{cr}.*
3. *Every vertex in M is adjacent to some vertices in H_{cr}.*
4. *Every vertex in M is adjacent to every vertex in N.*
5. *$G[N]$ is complete graph.*
6. *$N \neq \phi$ or $G[M]$ is connected.*

Proof. Assume that H_{cr} is a connected restrained hub set of G, take $M = N(H_{cr})$ and $N = V(G) \backslash N[H_{cr}]$. Its clear that the sets H_{cr}, M and N are disjoint sets. And by definition of N and M the conditions 1 to 3 are satisfied . Since H_{cr} is a restrained hub set, then by Theorem 1.1, G/H_{cr} is a complete graph. Since the contraction of H_{cr} is unaffected by the adjacency of vertices in N, then clearly any vertex in N must adjacent to every vertex in $V(G) \setminus H_{cr}$, and that proves conditions 4 and 5. Now we will prove 6^{th} condition by contradiction. Let $N = \phi$ and $G[M]$ is disconnected. Then $G[V(G) \setminus H_{cr}] = G[M]$ is disconnected, and that contradicts Theorem 1.1, this completes the proof. The converse is trivial. □

Remark 2.1. Note that, H_c is a connected hub set if and only if H_c satisfied the conditions $1 - 5$ in the previous Theorem.

Corollary 2.1. *Let $G \ncong K_p$. Then $h_r(G) = 1$ if and only if there exists non cut vertex v, such that $G[V(G) \setminus N[v]]$ is complete and every vertex of $V(G) \setminus N[v]$ adjacent to every vertex of $N(v)$.*

Proposition 2.2. *Let G be a graph, and H be a minimum hub set of G. If there exists $v \in V(G)$ such that $N[v] \cap H = \phi$, then $h_r(G) = h(G)$.*

Proof. Let H be a minimum hub set of G, such that there exists $v \in V(G)$ with $N[v] \cap H = \phi$. Let $u \in V(G) \setminus H$, then there is $vu-$ path with all internal vertices in H, but v is not adjacent to any vertex in H, so v is adjacent to u. Therefore $V(G) \setminus H$ is connected, and by Theorem 1.1, H is a restrained hub set. Therefore, $h_r(G) \leq h(G)$. □

Lemma 2.1. *Let G be a disconnected graph with components $G_1, G_2, ..., G_n$. Then $h_r(G) = min\{h_k\}$, where $h_k = \sum_{i=1, i\neq k}^{n} |G_i| + h_r(G_k), k = 1, 2, ..., n$.*

Proof. Let $G_1, G_2, ..., G_n$ be the components of G, and let H_r be a minimum restrained hub set of G. If there is $\{x, y\} \subseteq (V(G) \setminus H_r)$ belongs to two different components of G, then $G[V(G)\setminus H_r]$ is disconnected, which contracts proposition 2.1. So $(V(G) \setminus H_r) \subseteq G_j$, for some $j = 1, 2, ..., n$. Hence any minimal restrained hub set H_r must contains all vertices from all components except one, and the vertices of any minimum restrained hub set of the remaining component. Thus $h_r(G) = min\{h_k\}$, where $h_k = \sum_{i=1, i\neq k}^{n} |G_i| + h_r(G_k), k = 1, 2, ..., n$. □

Corollary 2.2. *Let G be a disconnected graph. Then $h_r(G) = 1$ if and only if $\overline{G} \cong S_p$.*

Proof. Let $h_r(G) = 1$. Then by Lemma 2.1, G has two components only, one of them has just one vertex, and the second one is complete. Therefore $\overline{G} \cong S_p$. The converse is trivial. □

Proposition 2.3. *Let $G \cong K_{n_1, n_2, ..., n_k}$ with $k \geq 3$. Then*

$$h_r(G) = \begin{cases} 0, & \text{if } n_i = 1 \text{ for all } 1 \leq i \leq k \ ; \\ 1, & \text{if } n_i \leq 2 \text{ for some } 1 \leq i \leq k \ ; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $G = K_{n_1, n_2, ..., n_k}$ with $k \geq 3$. Then we consider the following cases:
Case 1. $n_i = 1$ for all $1 \leq i \leq k$. Then its clear that G is a complete graph, hence by Proposition 2.1, $h_r(G) = 0$.
Case 2. $n_i \leq 2$. For some $1 \leq i \leq k$, let v be any vertex in this part. Then its clear that G/v is complete and $G - v$ is connected. Thus by Theorem 1.1, $\{v\}$ is a restrained hub set, and by Proposition 2.1, $\{v\}$ it's minimum since G is not a complete graph, so $h_r(G) = 1$.
Case 3. $n_i > 2$ for all $1 \leq i \leq k$. For any vertex $x \in V(G)$, G/x is not complete because there is at least two vertices in the part containing x are not adjacent in G/x, hence $h_r(G) \geq 2$. Let u, v be two vertices in different parts. Then $G/\{u, v\}$ is complete and $G - \{u, v\}$ is connected, hence by Theorem 1.1, $\{u, v\}$ is a restrained hub set. □

Lemma 2.2. *Let G be a graph with degree sequence $\Delta, d_2, d_3, ..., d_n$, and let $v \in V(G)$ such that $deg(v) = \Delta = n - 1$. Then $h_r(G) \leq n - d_2$.*

Proof. Let $u \in V(G)$ such that $deq(u) = d_2$, and let $H_r = (V(G) \setminus N[u]) \cup \{v\}$. Then $V(G) \setminus H_r = N[u] \setminus \{v\}$, so every two non adjacent vertices $x, y \in (V(G) \setminus H_r)$ has two paths x, v, y and x, u, y with all internal vertices in H_r and in $V(G) \setminus H_r$. So H_r is a restrained hub set. Thus

$$h_r(G) \leq |(V(G) \setminus N[u]) \cup \{v\}| = n - (d_2 + 1) + 1 = n - d_2.$$

□

Theorem 2.2. *If A and B are two components of graph $G - x$ and H_r is a restrained hub set of G, then $A \subseteq H_r$ or $B \subseteq H_r$.*

Proof. Let G be a graph with a cut vertex x, A and B are two components of $G - x$. Then the following cases are considered.
Case 1. $x \in H_r$. Let $u \in A$ and $v \in B$, such that $\{u, v\} \nsubseteq H_r$. Then $G[V(G) \setminus H_r]$ is disconnected, and it has at least two components, a contradiction. So $A \subseteq H_r$ or $B \subseteq H_r$.
Case 2. $x \notin H_r$. Let $u \in A$ and $v \in B$, such that $\{u, v\} \nsubseteq H_r$. Since x is a cut vertex, it follows that x lies in every path between u and v. Hence there is no path between u and v with all internal vertices are in H_r, which is a contradiction. Thus $A \subseteq H_r$ or $B \subseteq H_r$. □

Corollary 2.3. *Let x be a cut vertex of graph G. If H_r is a restrained hub set, then H_r contains all components of $G - x$ except one.*

Proof. Suppose that x is a cut vertex in a graph G, H_r is a restrained hub set, and $G_1, G_2, ..., G_k$ be the components of $G - x$. Take $A = G_1$, $B = G_2$, then by Theorem 2.2, either $G_1 \subseteq H_r$ or $G_2 \subseteq H_r$. Now let $A \nsubseteq H_r$, and B be the next component that does not compered yet, continuo in the progress to reach the last component. Therefore, there is just one component $G_j \nsubseteq H_r$, for some $1 \leq j \leq k$. □

Corollary 2.4. *For any graph G, let C be a cut vertex set, and H_r is a restrained hub set. If $C \subseteq H_r$, or $C \cap H_r = \phi$, then H_r contains all components of $G[V(G) \setminus C]$ except one.*

Corollary 2.5. *Let G be a connected graph, and $A_i = \{v \in V(G) : e(v) = i, r(G) < i < d(G)\}$. If H_r is a restrained hub set such that $A_j \subseteq H_r$ or $A_j \cap H_r = \phi$, for some $r(G) < j < d(G)$, then $\bigcup_{k=i+1}^{d-1} A_k \subseteq H_r$ or $\bigcup_{k=r+1}^{i-1} A_k \subseteq H_r$.*

Proof. Let G be a connected graph, $A_i = \{v \in V(G) : e(v) = i, r(G) < i < d(G)\}$, and H_r be a restrained hub set such that $A_j \subseteq H_r$ or $A_j \cap H_r = \phi$, for some $r(G) < j < d(G)$. Now, take $x \in A_u$, and $y \in A_l$, where $l < i < u$. Then by Lemma 1.1, every $xy-$ path has at least one vertex from A_i, hence A_i is a cut set, moreover $\bigcup_{k=i+1}^{d-1} A_k$ and $\bigcup_{k=r+1}^{i-1} A_k$, are in two different components of $G[V(G) \setminus A_i]$. So by Corollary 2.4, $\bigcup_{k=i+1}^{d-1} A_k \subseteq H_r$ or $\bigcup_{k=r+1}^{i-1} A_k \subseteq H_r$. □

Corollary 2.6. *Let H_r be a restrained hub set of a graph G. Then $V(G) \setminus H_r$ lies in one block from blocks of G.*

Proof. Let G be any graph, H_r be a restrained hub set of G. By contradiction, suppose that A, B be two different blocks of G suppose that there exist two vertices x and y belongs to A, B, respectively. Let $u \in A$ be a cut vertex such that $d(y,u) \leq d(y,z)$ for all $z \in A$. Then x and y belongs to two different components of $G-u$, thus by Theorem 2.2, $x \in H_r$ or $y \in H_r$, and that is a contradiction. □

Theorem 2.3. *Let $G(n_1, q_1)$ and $F(n_2, q_2)$ be two graphs. Then*

$$h_r(G \circ F) = \begin{cases} n_1 n_2 + h_r(G), & \text{if } n_1 > h_r(G) + s + t; \\ (n_1 - 1)(n_2 + 1) + (1 - t), & \text{if } n_1 \leq h_r(G) + s + t. \end{cases}$$

Where $t = \lfloor \frac{\delta(C)+1}{s} \rfloor$, and C is a component of F with maximum order s, and with largest number of edges if there is more than one.

Proof. Let $G(n_1, q_1)$ and $F(n_2, q_2)$ be two graphs, and let the copies of F are $F_1, F_2, ..., F_{n_1}$ are incident to vertices $v_1, v_2, ..., v_{n_1}$ of graph G respectively. Then every vertex in $V(G)$ is a cut vertex of the graph $G \circ F$. Therefore, $(F_i \cup \{v_i\}), i = 1, 2, ...n_1$, are blocks of the graph $G \circ F$. Let H_r be a minimum restrained hub set of $G \circ F$, so by Corollary 2.6, $V(G \circ F) \setminus H_r$ lies in one block of $G \circ F$. If $(V(G \circ F) \setminus H_r) \subseteq G$. Then its clear that all paths between any two vertices in $V(G)$ are consists from vertices of $V(G)$ it self, so any minimum restrained hub set for G will not changed in $G \circ F$. Therefore, $h_r(G \circ F) = n_1 n_2 + h_r(G)$. But if $(V(G \circ F) \setminus H_r) \subseteq Fi \cup \{v_i\}$, for some $i = 1, 2, ..., n_1$, (say $F_1 \cup \{v_1\}$), then consider that $F_1 \cong \bigcup_{i=1}^{m} C_i$, where C_i are the components of F_1, let C be a component of F_1 with maximum order s, and with largest number of edges if there is more than one. Now we have to discuss the following cases:

Case 1. C is not complete graph. Then $H_r = (V(G \circ F) \setminus V(C))$ is a restrained hub set of $(G \circ F)$ and its minimum, since $v_1 + C$ not complete subgraph of $G \circ F$. Thus,

$$|H_r| = |(V(G \circ F) \setminus V(C))| = (n_1 - 1)(n_2 + 1) + 1 + (n_2 - s).$$

Therefore, $h_r(G \circ F) = \min\{n_1 n_2 + h_r(G), (n_1 - 1)(n_2 + 1) + 1 + (n_2 - s)\}$. But

$$\begin{aligned} (n_1 - 1)(n_2 + 1) + 1 + (n_2 - s) \leq n_1 n_2 + h_r(G) &\iff \\ n_1 n_2 + n_1 - n_2 - 1 + 1 + n_2 - s \leq n_1 n_2 + h_r(G) &\iff \\ n_1 \leq h_r(G) + s. & \end{aligned}$$

Therefore,

$$h_r(G \circ F) = \begin{cases} n_1 n_2 + h_r(G), & \text{if } n_1 > h_r(G) + s; \\ (n_1 - 1)(n_2 + 1) + 1, & \text{if } n_1 \leq h_r(G) + s. \end{cases}$$

Case 2. C is complete graph. Then $v_1 + C$ is complete subgraph of $G \circ F$, so $(V(G \circ F) \setminus \{C \cup v_1\}$ is a minimum restrained hub set of $G \circ F$. Thus,

$$h_r(G \circ F) = \min\{n_1 n_2 + h_r(G), (n_1 - 1)(n_2 + 1) + (n_2 - s)\}.$$

Therefore,

$$h_r(G \circ F) = \begin{cases} n_1 n_2 + h_r(G), & \text{if } n_1 > h_r(G) + s + 1; \\ (n_1 - 1)(n_2 + 1), & \text{if } n_1 \leq h_r(G) + s + 1. \end{cases}$$

The two formulas can be merged in one formula as the following:

Let $t = \lfloor \frac{\delta(C)+1}{s} \rfloor$. Then

$$h_r(G \circ F) = \begin{cases} n_1 n_2 + h_r(G), & \text{if } n_1 > h_r(G) + s + t; \\ (n_1 - 1)(n_2 + 1) + (1 - t), & \text{if } n_1 \leq h_r(G) + s + t. \end{cases}$$

□

Lemma 2.3. *Let G be a graph with at least two internal vertices, and let $F = G[V(G) - E_n(G)]$, where $E_n(G)$ is the set of all end vertices of G. Then $h_r(G) = h_r(F) + |E_n(G)|$.*

Proof. Let G be a graph has at least two internal vertices, $F = G[V(G) - E_n(G)]$, and let H_r be a minimum restrained hub set of F.

Claim: $S_r = H_r \cup E_n$ is a minimum restrained hub set of G. Its clear that S_r is a restrained hub set, since the vertices of $G - S_r$ are the same vertices of $F - H_r$. Now, we will show that S_r is minimum, let $S_r - v$ be a restrained hub set, either $v \in H_r$ or $v \in E_n(G)$. If $v \in H_r$, then $H_r - v$ is a restrained hub set of F and this contradicts the minimality of H_r. If $v \in E_n$ then by Theorem 1.2, $|S_r - v| = p - 2$, so $|S_r| = p - 1$ which is a contradiction. Therefore, S_r is a minimum restrained hub set of G, thus $h_r(G) = |H_r \cup E_n(G)| = h_r(F) + |E_n(G)|$. □

Theorem 2.4. *Let G be a graph, and let $A \subseteq V(G)$ such that $G[V(G)/A]$ is a tree, where $G[A]$ is a nontrivial connected subgraph of G. Then $h_r(G) = h_r(G[A]) + |V(G)| - |A|$.*

Proof. Let G be a graph, and let $A \subseteq V(G)$ such that $G[V(G)/A]$ is a tree, where $G[A]$ is a nontrivial connected subgraph of G. Then by Lemma 2.3,

$$\begin{aligned} h_r(G) &= h_r(G_1) + |E_n(G)| \\ &= h_r(G_2) + |E_n(G_1)| + |E_n(G)| \\ &= h_r(G_3) + |E_n(G_2)| + |E_n(G_1)| + |E_n(G)| \\ &\ldots \\ &= h_r(G[A]) + |V(G)| - |A| \end{aligned}$$

where $G_1 = G[V(G) - E_n(G)]$, $G_{i+1} = G[V(G_i) - E_n(G_i)]$, and $E_n(G_i)$ are the end vertices of G_i that not in A. □

Corollary 2.7. *For any non trivial tree T, $h_r(T) = p - 2$.*

Proof. Let T be any tree, take $A = \{x, y\}$, where $xy \in E(T)$. Then its clear that $G[V(T)/A]$ is a tree, hence by Theorem 2.4,

$$h_r(T) = h_r(T[A]) + |V(T)| - |A| = 0 + p - 2 = p - 2.$$

□

Corollary 2.8. *Let F be a forest of order p. Then*

$$h_r(F) = \begin{cases} p-1, & if F \cong N_p \\ p-2, & if F \ncong N_p. \end{cases}$$

Proof. Let F be any forest of order p, if $F \cong N_P$, then $h_r(F) = p - 1$, while if not, then F can written as, $F \cong \bigcup_{i=1}^{n} T_i, |T_i| = p_i, i = 1, 2, ..., n$, with at least one non trivial tree T_k. Therefore, by Lemma 2.1,

$$\begin{aligned} h_r(F) &= \sum_{i=1, i\neq k}^{n} p_i + h_r(T_k), \\ &= \sum_{i=1, i\neq k}^{n} p_i + (p_k - 2) \quad \text{(by Corollary 2.7)} \\ &= \sum_{i=1}^{n} p_i - 2 \\ &= p - 2. \end{aligned}$$

Thus, we get that $h_r(F) = \begin{cases} p-1, & if F \cong N_p \\ p-2, & if F \ncong N_p. \end{cases}$ □

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Two simple confidence intervals for the population coefficient of variation under the non-normal and skewed distributions

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Abstract. This paper presents two simple confidence intervals (CIs) for the population coefficient of variation (CV) in the case of non-normal distributions. The first is based on the Bonett [6] formula for calculating an approximate CI for the variance (σ^2) of the non-normal distributions and the other is based on the Niwitpong and Kirdwichai [29] formula for calculating an adjusted Bonett [6] confidence interval for the variance of the non-normal distributions. An extensive Monte-Carlo simulation study was conducted to compare the performance of the proposed CIs with the other existing CIs available in the literature. The simulation results showed that the proposed two simple confidence intervals perform well in terms of coverage probability and expected average width. The proposed two methods are illustrated using two real life data which reinforced the findings of the simulation study to some extent.

Keywords: confidence interval, coefficient of variation, non-normal distribution, coverage probability, expected average width.

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1. Introduction

The coefficient of variation (hereinafter termed CV) is a quantity describing the variation between results in different populations. It is a relative measure of dispersion based on mean and standard deviation [12]. The concept of CV was introduced by Pearson [16]and since then it has been employed in a wide variety of fields such as medicine, engineering, economic, and science (See for example [24, 25]). The population CV is defined as a ratio of the population standard deviation (σ) to the population mean (μ):

$$CV = \frac{\sigma}{\mu}, \quad \mu \neq 0. \tag{1}$$

In practice, the population CV is unknown and needs to be estimated. The typical sample estimate of the population CV is given as:

$$CV = \frac{S}{\bar{X}} \tag{2}$$

where S and $\bar{X}$ are the sample standard deviation and the sample mean, respectively.

Developing a CI for the population CV is a key problem; see, for example, Koopmans et al. [20], Miller [9], Sharma and Krishna [18], Vangel [22], Tian [19], Wong and Wu [2], Mahmoudvand and Hassani [26], Curto and Pinto [15], Banik and Kibria [28], Niwitpong [27], Albatineh et al. [3] etc. The classical CIs for CV are not robust to violations of normality [7]. In this study, we propose two simple approximate confidence intervals for the population CV in non-normal distributions. We compare the performance of the proposed confidence intervals with the existing confidence intervals in terms of coverage probability and expected average width. An extensive Monte-Carlo simulation study is conducted to compare the performance of the interval estimators. Since in real-life the data could be non-normal or skewed, we also conducted simulations under skewed distributions. Because of this study is related to the performance comparison of the available and proposed methods for population coefficient of variation (CV), we want to express that the proposed two methods performs well and are good competitors compared with the methods in the literature.

The organization of this paper is as follows: Section 2 details the literature review of the study. In Sections 3 and 4 we present the existing and proposed confidence intervals. Section 5 includes simulation technique and results that represent simulations for normal and skewed distributions. Two scenarios using real-life data illustrate the implementation of the methods in Section 6. Finally, some concluding remarks are presented in Section 7.

2. Literature review

There are several approaches and various techniques considered by many researchers available for constructing the CI for the population CV. McKay [4]

proposed a confidence interval (hereinafter termed CI) for the population CV based on the chi-square distribution; this confidence interval works well when $CV \leq 0.33$ [11]. Hendricks and Robey [33] studied the distribution of the sample CV when the random sample is drawn from a normal distribution. Later, Koopmans et al. [20], Iglewicz [5], and others described confidence interval (CI) for population CV for normal and lognormal distributions. Lehmann [10] proposed an exact method for the confidence interval of population CV. Vangel [22] proposed a new confidence interval for population CV also called a modified McKay's confidence interval.

Miller [9] discussed the approximate distribution of population CV and proposed the approximate CI for population CV in the case of a normal distribution. Sharma and Krishna[18] introduced the asymptotic distribution and CI of the reciprocal of the population CV, which does not require any assumptions to be made about the population distribution. Wong and Wu [2] obtained approximate CI for the population CV of normal and non-normal models. Verrill [30] reviewed the exact approach appropriate for normally distributed data.

The performance of CIs for population CV obtained by McKay's, Miller's, and Sharma-Krishna's methods was compared under the same simulation conditions by Ng [17]. Panichkitkosolkul [31] modified McKay's CI by replacing the sample CV with the maximum likelihood estimator for a normal distribution. Panichkitkosolkul [32] presented three CIs for the CV in a normal distribution with a known population mean where all three proposed confidence intervals performed well in terms of coverage probability and expected length.

Mahmoudvand and Hassani [26] proposed an approximately unbiased estimator for population CV in a normal distribution and used this estimator for constructing two approximate CIs for the population CV. Curto and Pinto[15] constructed the CI for the population CV when random variables are not independently and identically distributed. Banik and Kibria [28] reviewed several interval estimators for estimating the population CV and compared them to bootstrap interval estimators for various distributions.

Buntao and Niwitpong [23] also introduced an interval estimating the difference of the CV for lognormal and delta-lognormal distributions. Gulhar et al. [21] compared CIs for estimating the population CV based on parametric, nonparametric, and modified methods. Niwitpong [27] presented CIs for the population CV of log-normal distribution with restricted parameter space. Finally, Sangnawakij and Niwitpong [25] proposed CIs for the single CV and the difference between coefficients of variation in two-parameter exponential distributions.

3. The Confidence intervals for the population coefficient of variation

Let X_1, X_2, X_3, ..., X_n be an independently and identically distributed (iid) random sample of size n from a distribution with finite mean (μ) and variance

(σ^2). In the following, we shall describe some interval estimation procedures for the population CV. Since we will deal with and evaluate a large number of methods for constructing CIs for population CV, we now catalog all of them and their acronyms for easy reference as follows:

HR: Hendricks and Robey.
McK: McKay.
MMcK: Modified McKay.
Panich: Panichkitkosolkul.
Mill: Miller.
S&K: Sharma and Krishna.
C&P: Curto and Pinto.
GKA&A: Gulhar, Kibria, Albatineh and Ahmed.
MOV: MOVER.
M&H(I) and M&H(II): Two versions of Mahmoudvand & Hassani.
AK&A(I) and AK&A(II): Two versions of Abu-Shawiesh, Khurshid & Akyüz.

3.1 Hendricks and Robey Confidence Interval

Hendricks and Robey [33] developed following CI for the population CV based on a random sample from a normal population:

$$CI_{HR} = \left(\hat{CV} - t_{(n-1\,,\,\alpha/2)}\; S_{\hat{CV}} \;,\; \hat{CV} + t_{(n-1\,,\,\alpha/2)}\; S_{\hat{CV}} \right), \tag{3}$$

where $t_{(\alpha/2\,,\,n-1)}$ is the $100(1-\alpha/2)$ th percentile of the Student t-distribution with $(n-1)$ degrees of freedom, $\hat{CV} = s/\bar{X}$ is the sample CV and $S_{\hat{CV}} = (\hat{CV}/\sqrt{2n})$ is the standard error of $\hat{CV}$.

3.2 McKay's Confidence Interval

McKay [4] suggested the following CI for population CV:

$$CI_{McK} = \left(\left(\frac{s}{\bar{X}}\right) \sqrt{\left(\frac{\chi^2_{v\,,\,1-\alpha/2}}{v+1} - 1\right) \left(\frac{s}{\bar{X}}\right)^2 + \frac{\chi^2_{v\,,\,1-\alpha/2}}{v}}, \right.$$
$$\left. \left(\frac{s}{\bar{X}}\right) \sqrt{\left(\frac{\chi^2_{v\,,\,\alpha/2}}{v+1} - 1\right) \left(\frac{s}{\bar{X}}\right)^2 + \frac{\chi^2_{v\,,\,\alpha/2}}{v}} \right) \tag{4}$$

where $\chi^2_{v\,,\,1-\alpha/2}$ and $\chi^2_{v\,,\,\alpha/2}$ are $100(1-\alpha/2)$ th and $100(\alpha/2)$ th percentile of the chi-square distribution with $v = (n-1)$ degrees of freedom respectively.

3.3 Modified McKay's Confidence Interval

Vangel [22] modified the McKay's CI as:

$$CI_{MMcK} = \left(\left(\frac{s}{\bar{X}}\right) \sqrt{\left(\frac{\chi^2_{v,\,1-\alpha/2}+2}{v+1} - 1\right)\left(\frac{s}{\bar{X}}\right)^2 + \frac{\chi^2_{v,\,1-\alpha/2}}{v}}, \right.$$

$$\left. \left(\frac{s}{\bar{X}}\right) \sqrt{\left(\frac{\chi^2_{v,\,\alpha/2}+2}{v+1} - 1\right)\left(\frac{s}{\bar{X}}\right)^2 + \frac{\chi^2_{v,\,\alpha/2}}{v}} \right) \tag{5}$$

where $\chi^2_{v,\,1-\alpha/2}$ and $\chi^2_{v,\,\alpha/2}$are respectively the $100(1-\alpha/2)$ th and $100(\alpha/2)$ th percentile of the chi-square distribution with $v = (n-1)$ degrees of freedom. When data are normally distributed, CI_{McK} and CI_{MMcK} can be used very well in terms of coverage probability and average width. However, these confidence intervals cannot be used for the non-normal distributions.

3.4 Panichkitkosolkul's Confidence Interval

Panichkitkosolkul [31] improved the CI_{MMcK} obtained by Vangel [22]. Panichkitkosolkul's confidence interval is:

$$CI_{Panich} = \left(\tilde{k} \sqrt{\left(\frac{\chi^2_{v,\,1-\alpha/2}+2}{v+1} - 1\right)\left(\tilde{k}\right)^2 + \frac{\chi^2_{v,\,1-\alpha/2}}{v}}, \right.$$

$$\left. \tilde{k} \sqrt{\left(\frac{\chi^2_{v,\,\alpha/2}+2}{v+1} - 1\right)\left(\tilde{k}\right)^2 + \frac{\chi^2_{v,\,\alpha/2}}{v}} \right) \tag{6}$$

where $\chi^2_{v,\,1-\alpha/2}$ and $\chi^2_{v,\,\alpha/2}$ $100(1-\alpha/2)$ th and $100(\alpha/2)$ th percentile of the chi-square distribution with $v = (n-1)$ degrees of freedom respectively and $\tilde{k}$ is the maximum likelihood estimator defined as $\tilde{k} = \frac{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2}}{\sqrt{n}\,\bar{X}}$.

3.5 Miller's Confidence Interval

Miller [9] proposed a confidence interval based on the sample coefficient of variation ($\hat{CV}$) approximating an asymptotic normal distribution. Miller's method has the following confidence interval:

$$CI_{Mill} = \left(\left(\frac{S}{\bar{X}}\right) - Z_{\,1-\alpha/2} \sqrt{\frac{\left(S/\bar{X}\right)^2}{n-1}\left(0.5 + \left(\frac{S}{\bar{X}}\right)^2\right)}, \right.$$

$$\left. \left(\frac{S}{\bar{X}}\right) + Z_{\,1-\alpha/2} \sqrt{\frac{\left(S/\bar{X}\right)^2}{n-1}\left(0.5 + \left(\frac{S}{\bar{X}}\right)^2\right)} \right) \tag{7}$$

where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution.

3.6 Sharma & Krishna's Confidence Interval

Sharma and Krishna [18] proposed following CI for the population CV:

$$CI_{S\&K} = \left(\left[\left(\frac{\bar{X}}{S} \right) - \frac{\phi^{-1}_{(\alpha/2)}}{\sqrt{n}} \right]^{-1}, \left[\left(\frac{\bar{X}}{S} \right) + \frac{\phi^{-1}_{(\alpha/2)}}{\sqrt{n}} \right]^{-1} \right) \tag{8}$$

where ϕ is the cumulative standard normal distribution.

3.7 Curto and Pinto's Confidence Interval

Curto and Pinto [15] proposed a confidence interval for the population CV, given as follows:

$$CI_{C\&P} = \left(\left(\frac{S}{\bar{X}} \right) - Z_{1-\alpha/2} \sqrt{\frac{1}{n} \left(\left(\frac{S}{\bar{X}} \right)^4 + \frac{1}{2} \left(\frac{S}{\bar{X}} \right)^2 \right)}, \right.$$
$$\left. \left(\frac{S}{\bar{X}} \right) + Z_{1-\alpha/2} \sqrt{\frac{1}{n} \left(\left(\frac{S}{\bar{X}} \right)^4 + \frac{1}{2} \left(\frac{S}{\bar{X}} \right)^2 \right)} \right) \tag{9}$$

where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution.

3.8 Gulhar, Kibria, Albatineh and Ahmed's Confidence Interval

Gulhar et al. [21], engrossing the method for calculating the confidence interval for σ^2 proposed following CI for normal population CV

$$CI_{GKA\&A} = \left(\frac{\sqrt{n-1}\,\left(S/\bar{X}\right)}{\sqrt{\chi^2_{v,\,1-\alpha/2}}}, \frac{\sqrt{n-1}\,\left(S/\bar{X}\right)}{\sqrt{\chi^2_{v,\,\alpha/2}}} \right) \tag{10}$$

where $\chi^2_{v,\,1-\alpha/2}$ and $\chi^2_{v,\,\alpha/2}$are respectively the $100(1-\alpha/2)$ th and $100(\alpha/2)$ th percentile of the chi-square distribution with $v = (n-1)$ degrees of freedom respectively.

3.9 MOVER's Confidence Interval

The MOVER CI for population CV is described as [1]:

$$CI_{MOV} = \left(\left(\bar{X} - \sqrt{\max\left\{0, \bar{X}^2 + a\,c\,(a-2)\right\}} \right) \left(\frac{S}{c} \right), \right.$$
$$\left. \left(\bar{X} + \sqrt{\max\left\{0, \bar{X}^2 + b\,c\,(b-2)\right\}} \right) \left(\frac{S}{c} \right) \right) \tag{11}$$

where $a = \sqrt{(n-1)\Big/\chi^2_{(n-1)\,,\,1-\alpha/2}}$, $b = \sqrt{(n-1)\Big/\chi^2_{(n-1)\,,\,\alpha/2}}$ and $c = \bar{X}^2 - \frac{Z^2_{\alpha/2}\, s^2}{n}$.

3.10 Mahmoudvand & Hassani's Confidence Intervals

Mahmoudvand and Hassani [26] introduced two CIs for the population CV when data are normally distributed. By using the normal approximation, their first CI is given as follows:

$$CI_{M\&H(I)} = \left(\left(\frac{S}{\bar{X}}\right) \left(2 - c_n - Z_{\,1-\alpha/2}\sqrt{1-c_n^2}\right)^{-1}, \right.$$
$$\left. \left(\frac{S}{\bar{X}}\right) \left(2 - c_n + Z_{\,1-\alpha/2}\sqrt{1-c_n^2}\right)^{-1} \right) \tag{12}$$

where $c_n = \sqrt{\frac{2}{n-1}}\left[\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\right]$ and $Z_{\,1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution. Further they introduced an asymptotically unbiased point estimator as $\hat{\tau} = \frac{(S/\bar{X})}{2-c_n}$. The resulting CI based on $\hat{\tau}$ is

$$CI_{M\&H(II)} = \left(\hat{\tau} - \frac{\hat{\tau}}{2-c_n} Z_{\,1-\alpha/2}\sqrt{(1-c_n^2) + \frac{\hat{\tau}^{\,2}}{n}}, \right.$$
$$\left. \hat{\tau} + \frac{\hat{\tau}}{2-c_n} Z_{\,1-\alpha/2}\sqrt{(1-c_n^2) + \frac{\hat{\tau}^{\,2}}{n}} \right), \tag{13}$$

where c_n and $Z_{\,1-\alpha/2}$ are defined above.

4. The proposed simple confidence intervals for the population CV

In this section, we propose two approximate and simple confidence intervals for estimating the population CV in non-normal and skewed distributions. The first is based on the Bonett [6] formula for calculating an approximate CI for the variance (σ^2) of the non-normal distributions. The second proposed CI is based on the Niwitpong and Kirdwichai [29] formula for calculating an adjusted Bonett [6] confidence interval for the variance of the non-normal distributions. Let X_1, X_2, X_3, ... , X_n be continuous, iid random variables with $\sigma^2 = Var(X_i) > 0$, $E(X_i) = \mu$ and finite fourth moment, then the two proposed confidence interval methods can be derived as follows:

4.1 Abu-Shewiesh, Khurshid & Akyüz's (AK&A(I)) Confidence Interval

The Bonett [6] proposed $(1-\alpha)\,100\,\%$ CI for the population variance (σ^2) is given as follows:

$$CI_{Bonett} = \left(\exp\left\{\ln\left(cS^2\right) - Z_{\,1-\alpha/2}\, Se\right\}, \exp\left\{\ln\left(cS^2\right) + Z_{\,1-\alpha/2}\, Se\right\}\right) \tag{14}$$

where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ th percentile of the standard normal distribution, $c = \frac{n}{(n-Z_{1-\alpha/2})}$, $Se = c\,[\frac{\{\hat{\gamma}_4^* - \frac{(n-3)}{n}\}}{(n-1)}]^{1/2}$ and $\hat{\gamma}_4^* = \frac{n_0\tilde{\gamma}_4 + n\tilde{\gamma}_4}{n_0+n}$.

Here $\tilde{\gamma}_4$ is a prior estimate of γ_4 obtained from a larger sample of size n_0 and $\bar{\gamma}_4 = \frac{n\sum_{i=1}^{n}(X_i-m)^4}{\left[\sum_{i=1}^{n}(X_i-\hat{\mu})^2\right]^2}$, $\hat{\mu} = \bar{X}$, where m is a trimmed mean with trim-proportion of $1/\left\{\, 2\,(n-4)^{\,1/2}\,\right\}$. However, if $\tilde{\gamma}_4$ is not available then $\hat{\gamma}_4^* = \tilde{\gamma}_4$. From equation (14), we have:

$$\exp\left\{\ln\left(cS^2\right) - Z_{1-\alpha/2}\,Se\right\} \;<\; \sigma^2 \;<\; \exp\left\{\ln\left(cS^2\right) + Z_{1-\alpha/2}\,Se\right\}. \tag{15}$$

Assuming that $\mu \neq 0$, dividing CI obtained in equation (15) by μ^2results in

$$\frac{\exp\left\{\ln\left(cS^2\right) - Z_{1-\alpha/2}\,Se\right\}}{\mu^2} < \left(\frac{\sigma}{\mu}\right)^2 < \frac{\exp\left\{\ln\left(cS^2\right) + Z_{1-\alpha/2}\,Se\right\}}{\mu^2}. \tag{16}$$

Since μ is unknown, we can replace it by the unbiased estimator of μ which is $\hat{\mu} = \bar{X}$ resulting in:

$$\frac{\exp\left\{\ln\left(cS^2\right) - Z_{1-\alpha/2}\,Se\right\}}{\bar{X}^{\,2}} < CV^2 < \frac{\exp\left\{\ln\left(cS^2\right) + Z_{1-\alpha/2}\,Se\right\}}{\bar{X}^{\,2}}. \tag{17}$$

Taking the square root, in the CI of equation (17) we obtain

$$\frac{\sqrt{\exp\left\{\ln\left(cS^2\right) - Z_{1-\alpha/2}\,Se\right\}}}{\bar{X}} < CV < \frac{\sqrt{\exp\left\{\ln\left(cS^2\right) + Z_{1-\alpha/2}\,Se\right\}}}{\bar{X}} \tag{18}$$

which can be written as

$$CI_{AK\&A(I)} = \left(\frac{\sqrt{\exp\left\{\ln\left(cS^2\right) - Z_{1-\alpha/2}\,Se\right\}}}{\bar{X}}, \right. \\ \left. \frac{\sqrt{\exp\left\{\ln\left(cS^2\right) + Z_{1-\alpha/2}\,Se\right\}}}{\bar{X}}\right). \tag{19}$$

4.2 Abu-Shewiesh, Khurshid & Akyüz's (AK&A(II)) Confidence Interval

The Niwitpong and Kirdwichai [29] adjusted Bonett [6] CI for the population variance σ^2 and proposed the following:

$$CI_{Adjusted\,Bonett} = \left(\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2,n-1)}\,Se\right\}, \right. \\ \left. \exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2,n-1)}\,Se\right\}\right), \tag{20}$$

where $t_{(\alpha/2\,,n-1)}$ is the $100(1-\alpha/2)$ th percentile of the student t-distribution with $(n-1)$ degrees of freedom, where

$$c = \frac{n}{\left(n - t_{(\alpha/2,\,n-1)}\right)},\; Se = c\,\left[\frac{\left\{\hat{\gamma}_4^* - \frac{(n-3)}{n}\right\}}{(n-1)}\right]^{1/2} \text{ and } \hat{\gamma}_4^* = \frac{n_0\tilde{\gamma}_4 + n\tilde{\gamma}_4}{n_0+n}.$$

Here, $\tilde{\gamma}_4$ is a prior estimate of γ_4 obtained from a larger sample of size n_0 and $\gamma_4^{/} = \frac{n\sum_{i=1}^{n}(X_i - MD)^4}{\left[\sum_{i=1}^{n}(X_i-\hat{\mu})^2\right]^2}$, $\hat{\mu} = \bar{X}$, where MD is the median, which is a good estimator of μ when the data are from asymmetric and skewed leptokurtic distributions. However, if $\tilde{\gamma}_4$ is not available then $\gamma_4^* = \tilde{\gamma}_4$. From equation (20), we have

$$\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2\,,n-1)}\,Se\right\} < \sigma^2 < \exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2\,,n-1)}\,Se\right\}. \tag{21}$$

Assuming that $\mu \neq 0$, dividing this confidence interval by μ^2results in

$$\frac{\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2\,,n-1)}\,Se\right\}}{\mu^2} < \left(\frac{\sigma}{\mu}\right)^2 < \frac{\exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2\,,n-1)}\,Se\right\}}{\mu^2}. \tag{22}$$

Since μ is unknown, we can replace it by the unbiased estimator of μ which is $\hat{\mu} = \bar{X}$resulting in

$$\frac{\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2\,,n-1)}\,Se\right\}}{\bar{X}^{\,2}} < CV^2 < \frac{\exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2\,,n-1)}\,Se\right\}}{\bar{X}^{\,2}}. \tag{23}$$

Taking the square root in (23) we have

$$\frac{\sqrt{\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2\,,n-1)}\,Se\right\}}}{\bar{X}} < CV < \frac{\sqrt{\exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2\,,n-1)}\,Se\right\}}}{\bar{X}}. \tag{24}$$

That is

$$CI_{AK\&A(II)} = \left(\frac{\sqrt{\exp\left\{\ln\left(cS^2\right) - t_{(\alpha/2\,,n-1)}\,Se\right\}}}{\bar{X}}, \frac{\sqrt{\exp\left\{\ln\left(cS^2\right) + t_{(\alpha/2\,,n-1)}\,Se\right\}}}{\bar{X}}\right). \tag{25}$$

5. Simulation study

To judge the performance of the CIs discussed in Sections 3 and 4 a Monte-Carlo simulation under the same simulation conditions, was conducted using the statistical software MATLAB R2016a. The performance of the confidence intervals was considered for various CV values, sample sizes (n) and probability distributions.

5.1 The Simulation Technique

In this simulation, we study the behavior of small and large sample sizes by using $n = 15, 25, 50, 100$. If the sample size (n) is increased further, this will be examined by using a second simulation for a sample size of $n = 500$. The data were generated from: (i) a normal distribution with parameters, with a known population mean $\mu = 10$ and known population standard deviation $\sigma = 1, 3, 5$.
(ii) Chi-Square distribution with degrees of freedom (df) $\nu = 200, 22, 8$.
(iii) Gamma distribution with parameters $\alpha = 100, 11.11, 4$ and $\beta = 2$.

The number of simulation replications was $M = 50,000$ for each case. The coefficient of variation and type I error were considered as $CV = 0.10, 0.30, 0.50$ and $\alpha = 0.05$, respectively. The CV was calculated for each one of the three distributions by utilizing the characteristics provided in Table 1.

Table 1: The CV and skewness of data from Normal, Chi-square and Gamma distributions

Distribution	CV	Skewness
$N(\mu\,,\,\sigma^2)$	σ/μ	0
Chi-Square (ν)	$\sqrt{2/\nu}$	$2\sqrt{2/\nu}$
Gamma $(\alpha,\, 2)$	$1/\sqrt{\alpha}$	$2/\sqrt{\alpha}$

The estimated coverage probability (CP) and expected width (EW) for two-sided CIs for CV were obtained as follows [13]:

$$CP = \frac{\sum_{i=1}^{50,000} I\ (L_i \leq CV \leq U_i)}{50,000}$$

$$EW = \frac{\sum_{i=1}^{50,000} (U_i - L_i)}{50,000}$$

where $I = 1$ if $(L \leq CV \leq U)$, and $I = 0$, otherwise.

The simulated coverage probabilities and expected widths for confidence intervals using Normal, Chi-Square and Gamma distributions for grid of values are presented in Tables 2, 3 and 4 respectively, whilst the results for $n = 500$ being provided in Table 5. Each table gives results for the various sample sizes and CV values previously mentioned.

Table 2: Estimated coverage probabilities (CP) and expected widths (EW) of the 95% confidence interval using Normal distribution

Measure	Coefficient of Variation Confidence Intervals												
	HR	McK	MMcK	Panich	Mill	S&K	C&P	GKA&A	MOV	M&H(I)	M&H(II)	AK&A(I)	AK&A(II)
n = 15 , CV = 0.10													
CP	0.9225	0.9105	0.9061	0.8845	0.9126	0.2038	0.9061	0.9459	0.9488	0.9444	0.8948	0.9578	0.9742
EW	0.0770	0.0728	0.0711	0.0702	0.0736	0.0102	0.0711	0.0831	0.0840	0.0801	0.0704	0.0966	0.1099
n = 15 , CV = 0.30													
CP	0.9133	0.9091	0.9063	0.8844	0.9136	0.5370	0.9063	0.9287	0.9536	0.9280	0.8943	0.9453	0.9653
EW	0.2319	0.2398	0.2318	0.2283	0.2400	0.0951	0.2318	0.2502	0.2735	0.2412	0.2280	0.2911	0.3315
n = 15 , CV = 0.50													
CP	0.8872	0.9028	0.9032	0.8803	0.9090	0.7193	0.9032	0.8887	0.9536	0.8888	0.8913	0.9164	0.9423
EW	0.3913	0.4934	0.4489	0.4531	0.4647	0.2919	0.4489	0.4222	0.5387	0.4070	0.4379	0.4924	0.5601
n = 25 , CV = 0.10													
CP	0.9333	0.9272	0.9245	0.9113	0.9289	0.2066	0.9245	0.9465	0.9492	0.9455	0.9179	0.9420	0.9623
EW	0.0577	0.0562	0.0554	0.0550	0.0566	0.0078	0.0554	0.0604	0.0610	0.0591	0.0551	0.0629	0.0702
n = 25 , CV = 0.30													
CP	0.91934	0.9262	0.9237	0.9096	0.9282	0.5468	0.9237	0.9281	0.9504	0.9272	0.9167	0.9254	0.9494
EW	0.1738	0.1839	0.1801	0.1787	0.1838	0.0723	0.1801	0.1817	0.1981	0.1780	0.1784	0.1894	0.2113
n = 25 , CV = 0.50													
CP	0.8880	0.9209	0.9219	0.9070	0.9258	0.7284	0.9219	0.8891	0.9518	0.8881	0.914	0.8871	0.9188
EW	0.2920	0.3639	0.3446	0.3485	0.3517	0.2121	0.3446	0.3053	0.3819	0.2990	0.3397	0.3180	0.3545
n = 50 , CV = 0.10													
CP	0.9416	0.9395	0.9382	0.9314	0.9402	0.2116	0.9382	0.9479	0.9507	0.9479	0.9347	0.9432	0.9538
EW	0.0400	0.0396	0.0394	0.0392	0.0398	0.0055	0.0394	0.0408	0.0412	0.0404	0.0393	0.0417	0.0437
n = 50 , CV = 0.30													
CP	0.9242	0.9391	0.9373	0.9299	0.9394	0.5507	0.9373	0.9274	0.9497	0.9270	0.9336	0.9241	0.9351
EW	0.1202	0.1291	0.1276	0.1274	0.1289	0.0505	0.1276	0.1228	0.1337	0.1216	0.1270	0.1254	0.1315
n = 50 , CV = 0.50													
CP	0.8916	0.9416	0.9391	0.9336	0.9412	0.7380	0.9391	0.8907	0.9512	0.8910	0.9351	0.8900	0.9045
EW	0.2009	0.2504	0.2417	0.2454	0.2441	0.1437	0.2417	0.2054	0.2539	0.2033	0.2400	0.2097	0.2198
n = 100 , CV = 0.10													
CP	0.9423	0.9422	0.9415	0.9378	0.9427	0.2151	0.9415	0.9458	0.9483	0.9456	0.9399	0.9426	0.9495
EW	0.0279	0.0280	0.0279	0.0278	0.0280	0.0039	0.0279	0.0282	0.0285	0.0281	0.0278	0.0284	0.0292
n = 100 , CV = 0.30													
CP	0.9266	0.9448	0.9432	0.9409	0.9444	0.5525	0.9432	0.9286	0.9491	0.9286	0.9412	0.9247	0.9318
EW	0.0840	0.0909	0.0903	0.0903	0.0907	0.0355	0.0903	0.0849	0.0924	0.0845	0.0901	0.0853	0.0877
n = 100 , CV = 0.50													
CP	0.8911	0.9486	0.9446	0.9444	0.9456	0.7414	0.9446	0.8911	0.9515	0.8916	0.9429	0.8866	0.8967
EW	0.1403	0.1752	0.1704	0.1735	0.1713	0.0998	0.1704	0.1419	0.1746	0.1412	0.1698	0.1424	0.1465

Table 3: Estimated coverage probabilities (CP) and expected widths (EW) of the 95% confidence interval using Chi-Square distribution

Measure	Coefficient of Variation Confidence Intervals												
	HR	McK	MMcK	Panich	Mill	S&K	C&P	GKA&A	MOV	M&H(I)	M&H(II)	AK&A(I)	AK&A(II)
n = 15 , CV = 0.10													
CP	0.9247	0.9125	0.9080	0.8879	0.9152	0.2063	0.9080	0.9505	0.9533	0.9486	0.8973	0.9669	0.9775
EW	0.0768	0.0726	0.0710	0.0700	0.0735	0.0101	0.0710	0.0829	0.0838	0.0799	0.0702	0.1005	0.1109
n = 15 , CV = 0.30													
CP	0.9243	0.9199	0.9182	0.8950	0.9236	0.5663	0.9182	0.9473	0.9670	0.9451	0.9060	0.9778	0.9835
EW	0.2314	0.2387	0.2310	0.2274	0.2391	0.0941	0.2310	0.2496	0.2725	0.2406	0.2272	0.3339	0.3568
n = 15 , CV = 0.50													
CP	0.9154	0.9223	0.9227	0.8997	0.9293	0.7759	0.9227	0.9430	0.9764	0.9402	0.9113	0.9799	0.9870
EW	0.3808	0.4651	0.4296	0.4302	0.4447	0.2688	0.4296	0.4109	0.5129	0.3961	0.4194	0.6114	0.6555
n = 25 , CV = 0.10													
CP	0.93194	0.9264	0.9235	0.9104	0.9279	0.2136	0.9235	0.9473	0.9500	0.9462	0.9169	0.9615	0.9642
EW	0.0577	0.0561	0.0553	0.0549	0.0565	0.0078	0.0553	0.0603	0.0609	0.0590	0.0550	0.0697	0.0709
n = 25 , CV = 0.30													
CP	0.9296	0.9340	0.9325	0.9185	0.9363	0.5719	0.9325	0.9438	0.9622	0.9421	0.9249	0.9734	0.9737
EW	0.1737	0.1835	0.1798	0.1784	0.1835	0.0720	0.1798	0.1816	0.1978	0.1778	0.1781	0.2344	0.2324
n = 25 , CV = 0.50													
CP	0.9195	0.9389	0.9401	0.9259	0.9433	0.7836	0.9401	0.9368	0.9746	0.9349	0.9327	0.9775	0.9781
EW	0.2868	0.3524	0.3352	0.3380	0.3421	0.2023	0.3352	0.2999	0.3710	0.2937	0.3306	0.4359	0.4376
n = 50 , CV = 0.10													
CP	0.9412	0.9389	0.9375	0.9303	0.9397	0.2122	0.9375	0.9479	0.9504	0.9474	0.9340	0.9560	0.9550
EW	0.0399	0.0396	0.0393	0.0392	0.0397	0.0055	0.0393	0.0408	0.0412	0.0404	0.0392	0.0445	0.0443
n = 50 , CV = 0.30													
CP	0.9340	0.9459	0.9449	0.9374	0.9469	0.5751	0.9449	0.9423	0.9619	0.9419	0.9409	0.9699	0.9690
EW	0.1203	0.1293	0.1277	0.1275	0.1290	0.0506	0.1277	0.1230	0.1338	0.1217	0.1271	0.1503	0.1486
n = 50 , CV = 0.50													
CP	0.9215	0.9545	0.9542	0.9476	0.9559	0.7853	0.9542	0.9286	0.9724	0.9274	0.9500	0.9736	0.9721
EW	0.1992	0.2468	0.2385	0.2419	0.2410	0.1407	0.2385	0.2036	0.2505	0.2015	0.2369	0.2852	0.2894
n = 100 , CV = 0.10													
CP	0.9447	0.9447	0.9442	0.9407	0.9451	0.2182	0.9442	0.9482	0.9502	0.9479	0.9425	0.9546	0.9539
EW	0.0279	0.0280	0.0279	0.0278	0.0280	0.0039	0.0279	0.0282	0.0285	0.0281	0.0278	0.0298	0.0297
n = 100 , CV = 0.30													
CP	0.9383	0.9554	0.9544	0.9506	0.9554	0.5736	0.9544	0.9420	0.9607	0.9419	0.9521	0.9685	0.9691
EW	0.0843	0.0913	0.0906	0.0907	0.0911	0.0357	0.0906	0.0852	0.0927	0.0848	0.0904	0.1009	0.1016
n = 100 , CV = 0.50													
CP	0.9214	0.9621	0.9600	0.9588	0.9608	0.7865	0.9600	0.9246	0.9690	0.9248	0.9584	0.9734	0.9754
EW	0.1396	0.1737	0.1691	0.1721	0.1700	0.0986	0.1691	0.1411	0.1733	0.1404	0.1686	0.1937	0.2016

Table 4: Estimated coverage probabilities (CP) and expected widths (EW) of the 95% confidence interval using Gamma distribution

Measure	Coefficient of Variation Confidence Intervals												
	HR	McK	MMcK	Panich	Mill	S&K	C&P	GKA&A	MOV	M&H(I)	M&H(II)	AK&A(I)	AK&A(II)
n = 15 , CV = 0.10													
CP	0.9236	0.9124	0.9082	0.8870	0.9146	0.2077	0.9082	0.9501	0.9530	0.9474	0.8969	0.9659	0.9769
EW	0.0769	0.0726	0.0710	0.0701	0.0735	0.0101	0.0710	0.0829	0.0838	0.0799	0.0702	0.1006	0.1110
n = 15 , CV = 0.30													
CP	0.9205	0.9161	0.9142	0.8923	0.9202	0.5635	0.9142	0.9456	0.9646	0.9435	0.9029	0.9762	0.9836
EW	0.2298	0.2369	0.2292	0.2257	0.2372	0.0929	0.2292	0.2479	0.2703	0.2390	0.2254	0.3313	0.3545
n = 15 , CV = 0.50													
CP	0.9137	0.9204	0.9207	0.8972	0.9272	0.7739	0.9207	0.9395	0.9750	0.9366	0.9088	0.9788	0.9857
EW	0.3805	0.4647	0.4293	0.4299	0.4443	0.2686	0.4293	0.4105	0.5125	0.3957	0.4191	0.6090	0.6528
n = 25 , CV = 0.10													
CP	0.9335	0.9280	0.9255	0.9120	0.9295	0.2089	0.9255	0.9494	0.9518	0.9482	0.9186	0.9628	0.9659
EW	0.0576	0.0561	0.0553	0.0549	0.0564	0.0078	0.0553	0.0603	0.0609	0.0590	0.0550	0.0697	0.0709
n = 25 , CV = 0.30													
CP	0.9309	0.9350	0.9335	0.9192	0.9371	0.5691	0.9335	0.9453	0.9639	0.9448	0.9263	0.9746	0.9748
EW	0.1729	0.1826	0.1789	0.1776	0.1826	0.0714	0.1789	0.1808	0.1968	0.1771	0.1772	0.2328	0.2308
n = 25 , CV = 0.50													
CP	0.9177	0.9380	0.9390	0.9243	0.9427	0.7805	0.9390	0.9333	0.9744	0.9311	0.9316	0.9773	0.9779
EW	0.2869	0.3526	0.3354	0.3381	0.3423	0.2025	0.3354	0.3000	0.3712	0.2937	0.3307	0.4370	0.4388
n = 50 , CV = 0.10													
CP	0.9404	0.9381	0.9365	0.9294	0.9392	0.2150	0.9365	0.9479	0.9503	0.9473	0.9326	0.9549	0.9556
EW	0.0399	0.0396	0.0394	0.0392	0.0398	0.0055	0.0394	0.0408	0.0412	0.0404	0.0392	0.0444	0.0443
n = 50 , CV = 0.30													
CP	0.9363	0.9483	0.9468	0.9399	0.9489	0.5757	0.9468	0.9438	0.9614	0.9425	0.9430	0.9709	0.9696
EW	0.1198	0.1286	0.1271	0.1268	0.1284	0.0501	0.1271	0.1224	0.1331	0.1212	0.1265	0.1493	0.1477
n = 50 , CV = 0.50													
CP	0.9194	0.9537	0.9534	0.9464	0.9547	0.7870	0.9534	0.9277	0.9705	0.9267	0.9492	0.9732	0.9720
EW	0.1992	0.2468	0.2385	0.2419	0.2409	0.1407	0.2385	0.2036	0.2505	0.2015	0.2369	0.2851	0.2893
n = 100 , CV = 0.10													
CP	0.9459	0.9460	0.9453	0.9418	0.9465	0.2143	0.9453	0.9478	0.9503	0.9478	0.9438	0.9544	0.9543
EW	0.0279	0.0280	0.0279	0.0278	0.0280	0.0039	0.0279	0.0283	0.0285	0.0281	0.0278	0.0298	0.0297
n = 100 , CV = 0.30													
CP	0.9380	0.9539	0.9528	0.9492	0.9537	0.5744	0.9528	0.9409	0.9606	0.9405	0.9509	0.9680	0.9683
EW	0.0839	0.0908	0.0901	0.0902	0.0905	0.0353	0.0901	0.0848	0.0922	0.0844	0.0899	0.1003	0.1009
n = 100 , CV = 0.50													
CP	0.9211	0.9632	0.9614	0.9586	0.9624	0.7845	0.9614	0.9255	0.9707	0.9238	0.9587	0.9741	0.9758
EW	0.1396	0.1738	0.1692	0.1721	0.1700	0.0987	0.1692	0.1412	0.1733	0.1405	0.1686	0.1936	0.2015

5.2 The simulation results

In this section, the simulation results will be discussed by the distribution type.

5.3 Normal distribution results

In Table 2, it is seen that coverage probabilities and average widths of proposed confidence intervals for Normal Distribution when $CV = 0.10, 0.30, 0.50$ and $\alpha = 0.05$ are quite close to the nominal confidence level even in small sample sizes. The coverage probabilities of all confidence intervals except S&K confidence interval are close to the nominal confidence level for each value of the coefficient of variation. Proposed two methods have performed very well in terms of average widths. As the value of the coefficient of variation decreases, the average width of confidence intervals decreases. For large samples, i.e. $n = 500$, from Table 5 is seen that the coverage probabilities of proposed confidence intervals for Normal distribution are similar to the coverage probabilities of confidence intervals based on skewed distributions. The coverage probabilities of proposed confidence intervals are quite close to the nominal confidence level for all the sample sizes and coefficient of variations when $\alpha = 0.05$.

5.4 Non-normal (skewed) distributions results

The coverage probabilities of proposed confidence intervals based on Chi-square and Gamma distributions are quite close to the nominal confidence level for $\alpha = 0.05$. Average widths of the confidence intervals shrunk as the sample size increases. The S&K confidence interval has the narrowest average widths in all cases. The proposed confidence intervals performed as well as the other confidence intervals (Tables 3-4).

The proposed methods are robust, but slightly wider than the existing confidence intervals even in very small samples. This can be easily seen from the Figures (1-4).

6. Empirical applications

In this section, we consider two real-life data to illustrate the performance of the proposed simple CIs.

6.1 Example 1: Infants Weights Data

The first data set was obtained from Ziegler et al. [8] (cited in Ledolter and Hogg [14]).

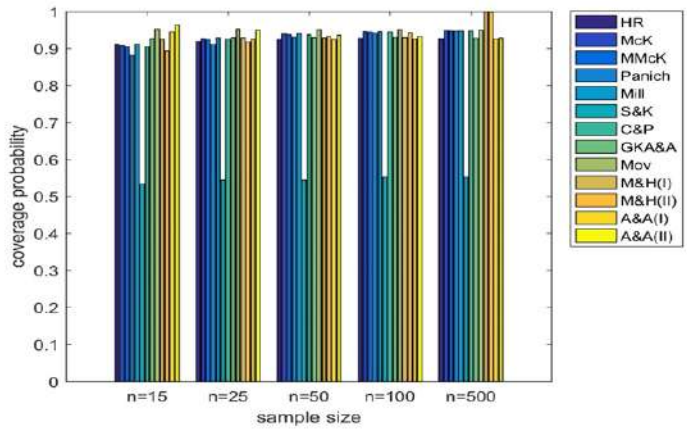


Figure 1: Sample size vs. coverage probability for normal distribution when CV = 0.30

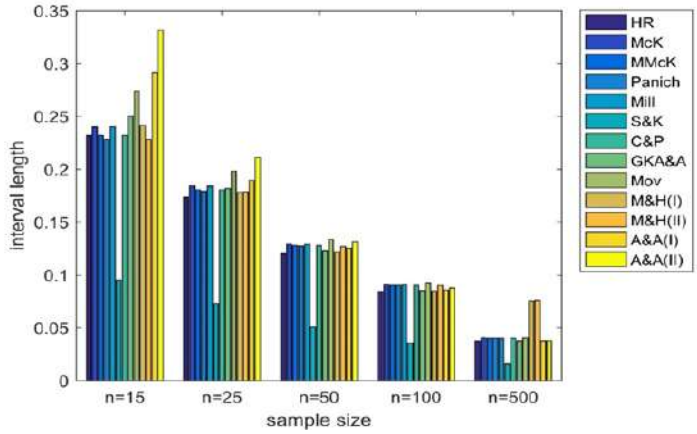


Figure 2: Sample Size vs. Interval Length for Normal Distribution when CV = 0.30

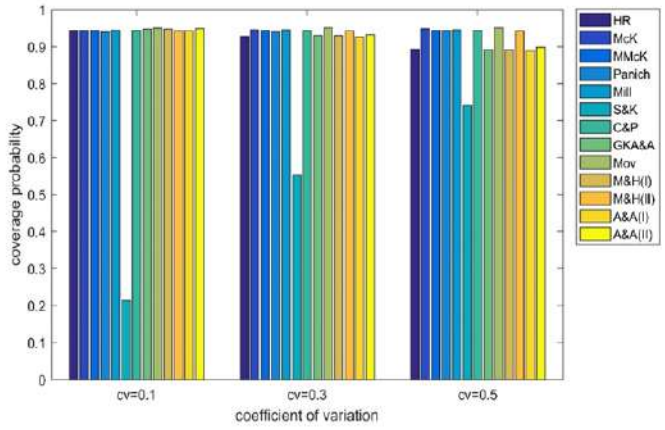


Figure 3: Coefficient of Variation vs. Coverage Probability for Normal Distribution when n = 100

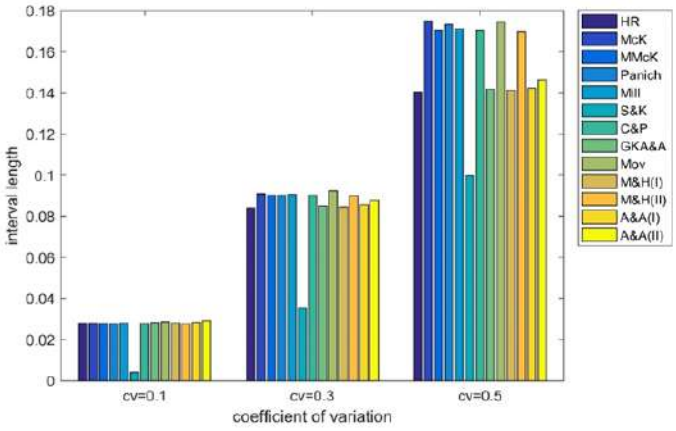


Figure 4: Coefficient of Variation vs. Interval Length for Normal Distribution when n = 100

Table 5: Estimated coverage probabilities (CP) and expected widths (EW) of the 95% confidence interval using Normal, Chi-Square and Gamma distributions (n=500)

	Coefficient of Variation Confidence Intervals												
	HR	McK	MMcK	Panich	Mill	S&K	C&P	GKA&A	MOV	M&H(I)	M&H(II)	AK&A(I)	AK&A(II)
Normal Distribution													
CV = 0.10													
CP	0.9445	0.9460	0.9459	0.9455	0.9460	0.2166	0.9459	0.9457	0.9481	0.9998	0.9997	0.9450	0.9462
EW	0.0124	0.0125	0.0125	0.0125	0.0125	0.0017	0.0125	0.0124	0.0125	0.0250	0.02476	0.0124	0.0125
CV = 0.30													
CP	0.9281	0.9502	0.9493	0.9491	0.9496	0.5536	0.9493	0.9288	0.9503	0.9997	0.9995	0.9274	0.9292
EW	0.0372	0.0405	0.0403	0.0405	0.0404	0.0158	0.0403	0.0373	0.0405	0.0752	0.0757	0.0373	0.0375
CV = 0.50													
CP	0.8888	0.9533	0.9485	0.9520	0.9488	0.7422	0.9485	0.8895	0.9502	0.9984	0.9987	0.8883	0.8906
EW	0.0621	0.0776	0.0759	0.0774	0.0760	0.0439	0.0759	0.0622	0.0763	0.1254	0.1310	0.0622	0.0626
Chi-Square Distribution													
CV = 0.10													
CP	0.9467	0.9485	0.9484	0.9474	0.9486	0.2148	0.9484	0.9483	0.95072	0.9999	0.99974	0.9537	0.9531
EW	0.0124	0.0125	0.0125	0.0125	0.0125	0.0017	0.0125	0.0124	0.0125	0.0250	0.0247	0.0128	0.0127
CV = 0.30													
CP	0.9387	0.9585	0.9578	0.9576	0.9579	0.5751	0.9578	0.9398	0.9595	0.9997	0.9998	0.9686	0.9725
EW	0.0374	0.0407	0.0406	0.0407	0.0406	0.0159	0.0406	0.0375	0.0408	0.0756	0.0761	0.0434	0.0446
CV = 0.50													
CP	0.9210	0.9698	0.9665	0.9690	0.9667	0.7875	0.9665	0.9214	0.96814	0.9994	0.9994	0.9791	0.9853
EW	0.0620	0.0775	0.0758	0.0773	0.0759	0.0438	0.0758	0.0622	0.0762	0.1254	0.1309	0.0846	0.0905
Gamma Distribution													
CV = 0.10													
CP	0.9474	0.9495	0.9493	0.9485	0.9496	0.2187	0.9493	0.9493	0.9514	0.9999	0.9997	0.9539	0.9535
EW	0.0124	0.0125	0.0125	0.0125	0.0125	0.0017	0.0125	0.0124	0.0125	0.0250	0.0247	0.0128	0.0128
CV = 0.30													
CP	0.9405	0.9588	0.9578	0.9577	0.9580	0.5793	0.9578	0.9412	0.9603	0.9998	0.9997	0.9689	0.9725
EW	0.0372	0.0405	0.0403	0.0404	0.0404	0.0157	0.0403	0.0373	0.0405	0.0752	0.0757	0.0431	0.0443
CV = 0.50													
CP	0.9208	0.9699	0.9666	0.9689	0.9668	0.7863	0.9666	0.9213	0.9687	0.9994	0.9995	0.9798	0.9860
EW	0.0620	0.0775	0.0758	0.0773	0.0759	0.0438	0.0758	0.0622	0.0762	0.1254	0.1309	0.0846	0.0905

Table 6: The 95% confidence intervals for the population coefficient of variation of the weights of one-month old infants

Method	Confidence Interval Limits		
	Lower Limit	Upper Limit	Width
HR	0.1132	0.1633	0.0501
McK	0.1130	0.1633	0.0503
MMcK	0.1133	0.1633	0.0500
Panich	0.1121	0.1620	0.0499
Mill	0.1130	0.1635	0.0505
S&K	0.1336	0.1432	0.0096
C&P	0.1133	0.1633	0.0500
GKA&A	0.1173	0.1683	0.0510
MOV	0.1169	0.1689	0.0520
M&H (I)	0.1169	0.1675	0.0506
M&H (II)	0.1127	0.1626	0.0499
AK&A (I)	0.1160	0.1703	0.0543
AK&A (II)	0.1156	0.1710	0.0554

The summary statistics for infants weight data are as follows: $n = 61$, $mean = 4431$, $median = 4368$ grams,$sd = 612.9$ grams,$cv = 0.1383$ and $skewness = -0.27$. The Kolmogorov-Smirnov (K-S) goodness-of-fit test with p-value greater than 0.05 suggested that the data follow a normal distribution. The resulting 95% CIs and their corresponding widths are calculated and reported in Table 6. From this table, we notice that the width of confidence intervals AK&A(I) and AK&A(II) are wider than all other methods but they are performs well and still very close to other widths.

6.2 Example 2: Postmortem Interval (PMI) data

The second data set was obtained from Banik and Kibria [28]. The data represents the postmortem interval (PMI) which is defined as the elapsed time between death and an autopsy. The summary statistics for PMI data are as follows: $n = 22$, $mean = 7.3$, $median = 6.15$, $sd = 3.185$, $cv = 0.4363$ and $skewness = 1.14$.According to Banik and Kibria [28], using the Kolmogorov-Smirnov (K-S) goodness-of-fit test, the PMI data follow a gamma distribution with shape parameter $\alpha = 5.25$, and scale parameter, $\beta = 1.39$. The resulting 95% CIs and corresponding widths are given in Table 7.

Table 7: The 95% confidence intervals for the population coefficient of variation of the postmortem interval (PMI)

Method	Confidence Interval Limits		
	Lower Limit	Upper Limit	Width
HR	0.2995	0.5730	0.2735
McK	0.2718	0.5863	0.3144
MMcK	0.2848	0.5877	0.3029
Panich	0.2727	0.5746	0.3018
Mill	0.2812	0.5913	0.3100
S&K	0.3690	0.5335	0.1645
C&P	0.2848	0.5877	0.3029
GKA&A	0.3356	0.6234	0.2878
MOV	0.3199	0.6586	0.3387
M&H (I)	0.3324	0.6134	0.2809
M&H (II)	0.2818	0.5805	0.2986
AK&A (I)	0.3006	0.6950	0.3944
AK&A (II)	0.2669	0.7875	0.5206

From this table, we notice that the width of confidence intervals AK&A(I) and AK&A(II) are still wider than all other methods but the method AK&A(I) performed better than the method AK&A(II). The results of the two numerical

examples given in this section shows that the two CIs proposed in this paper are good competitors compared with the other eleven confidence intervals studied.

7. Summary and concluding remarks

In this study, two simple approximate confidence interval methods namely, AK&A(I) and AK&A(II), were developed for the population coefficient of variation (CV). The study is related to the performance comparison of the available and proposed methods for population coefficient of variation. We express that the proposed two methods are good competitors compared with the other eleven confidence intervals studied in this paper. We clearly obtained that the coverage probabilities of proposed confidence intervals for all distributions when CV = 0.10, 0.30, 0.50 and $\alpha = 0.05$ are quite close to the nominal confidence level even in small sample sizes. Also, these confidence intervals performed as well as the other confidence intervals both coverage probability and average width. Furthermore, the proposed two methods are consistent to existing methods for normal distribution. However, for non-normal and skewed distributions, the average widths of the AK&A(I) confidence interval are tighter than the average widths of AK&A(II) confidence interval. Therefore, it is recommended to use the AK&A(I) confidence interval for skewed distributions. In addition to the simulation study, two real life data numerical examples are analyzed for illustrating the findings of the paper which reinforced the findings of the simulation study to some extent.

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Fixed point results via tri-simulation function

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Abstract. In this article, we introduce a new type of simulation functions involving three variables and utilize the same to unify several known contractions of the existing literature besides being general enough to yield new contractions. In doing so, we also introduce two new notions (namely: α-permissible and α-orbital permissible mappings) and investigate their relationship with some earlier relevant notions of the existing literature.

Keywords: simulation function, tri-simulation function, α-permissible mapping, α-orbital permissible mapping, α-admissible mapping.

1. Introduction

In 1922, Banach [1] proved that every contraction mapping S defined on a complete metric space (M, d) admits a unique fixed point. Since then, a multitude of contraction conditions enriching the natural contraction were introduced and utilized to generalize Banach celebrated principle (e.g. [2]-[4] and references therein). Of course, every new contraction gives rise yet another theorem which does require an independent proof as well. In 1997, Popa [5] initiated the idea of unifying several contraction conditions in one go. To accomplish this, he introduced the idea of the implicit function which is very extensively utilized in [6]-[11]. Another noted attempt to extend Banach contraction principle is essentially due to Wardowski [12] wherein the author initiated the idea of F-contractions which has been further studied in [13]-[15] besides several other

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ones. Last but not least, Khojasteh et al. [16] introduced the idea of simulation function which is also designed to unify several contractions.

In what follows, $\mathbb{R}_+$ stands for the set of non-negative real numbers whereas all other involved notions are used in their standard sense. For brevity, we write Su instead of writing $S(u)$. Further, for any u_0, $u_n := S^n u_0 := \underbrace{SS\dots S}_{n\text{-times}} u_0$.

The following definition is due to Khojasteh et al. [16].

Definition 1.1 ([16]). *Let $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a mapping. Then ξ is called a simulation function if it satisfies the following conditions:*

($\xi 1$) : $\xi(0,0)=0$;

($\xi 2$) : $\xi(y,x) < x - y$, *for all* $x, y > 0$;

($\xi 3$) : *if* $\{y_n\}$ *and* $\{x_n\}$ *are sequences in* $(0,\infty)$ *with* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, *then* $\limsup_{n\to\infty} \xi(y_n, x_n) < 0$.

Argoubi et al. [17] slightly sharpened Definition 1.1 by removing the condition ($\xi 1$). Moreover, to avoid the symmetry of the simulation function, Roldán-López-de-Hierro et al. [18] modified the condition ($\xi 3$) as follows:

($\xi\acute{3}$): if $\{y_n\}$ and $\{x_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$ and $y_n < x_n$, for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \xi(y_n, x_n) < 0$.

For examples of the simulation functions, we refer the readers to [16]-[19].

Definition 1.2 ([16]). *A self-mapping S on a metric space (M,d) is said to be $\mathcal{Z}$-contraction with respect to a simulation function ξ if the following condition is satisfied:*

$$\xi(d(Su,Sv), d(u,v)) \geq 0, \text{ for all } u, v \in M. \tag{1.1}$$

It is worth noting that the idea of simulation function is general enough to unify several contractions of the existing literature but this idea, in its earlier form, is not applicable to contractions involving variables beyond $d(Su,Sv)$ and $d(u,v)$ such as in the following result:

Theorem 1.1 ([4]). *Let (M,d) be a complete metric space and $S : M \to M$ a continuous mapping satisfying $\alpha(u,v)d(Su,Sv) \leq \psi\big(d(u,v)\big)$, for all $u, v \in M$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$ and $\alpha : M \times M \to \mathbb{R}_+$. Assume that the following two conditions hold:*

(*i*) *there exists $u_0 \in M$ such that $\alpha(u_0, Su_0) \geq 1$;*

(*ii*) *S is α-admissible i.e.,*

$$\alpha(u,v) \geq 1 \Rightarrow \alpha(Su,Sv) \geq 1, \text{ for all } u, v \in M.$$

Then S has a fixed point.

As already pointed out, Theorem 1.1 can not be covered by a simulation function. Very recently, Karapinar [19] and later Gubran et al. [20] were able to do so by embedding admissible function into a simulation function. However, the only flaw which continues to persist is that it forces the expression $\alpha(u,v)d(Su,Sv)$ to behave as one element whenever it occurs in the contraction inequality.

The aim of this article is to enlarge the class of simulation functions and overcome the shortcoming mentioned earlier so that the involved terms can occur independently. Specifically speaking, we introduce a new type of simulation functions in three variables and utilize the same to unify several known contractions of the existing literature besides being general enough to yield new contractions.

2. Tri-simulation function

Motivated by forgoing observations, we introduce a new simulation function involving three variables and hence call the same as tri-simulation which runs as follows:

Definition 2.1. *Let $T : \mathbb{R}_+^3 \to \mathbb{R}$ be a mapping. Then T is called a tri-simulation function if it satisfies the following conditions:*

$(T1)$: $T(z,y,x) < x - yz$, *for all* $x,y > 0, z \geq 0$;

$(T2)$: *if* $\{z_n\}$, $\{y_n\}$ *and* $\{x_n\}$ *are sequences in* $(0,\infty)$ *such that* $y_n < x_n$, *for all* $n \in \mathbb{N}$,$\lim_{n\to\infty} z_n \geq 1$ *and* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, *then*

$$\limsup_{n\to\infty} T(z_n, y_n, x_n) < 0.$$

Let us denote the set of all tri-simulation functions by $\mathcal{T}$. To substantiate the definition, let us furnish the following tri-simulation function examples:

Example 2.1. $T(z,y,x) = \lambda x - yz$, for all x,y and $z \in \mathbb{R}_+$, where $\lambda \in [0,1)$.

Example 2.2. $T(z,y,x) = \psi(x) - \phi(yz)$, for all x,y and $z \in \mathbb{R}_+$, where $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$.

Example 2.3. $T(z,y,x) = x - \psi(x) - yz$, for all x,y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi-continuous function such that $\psi(t) = 0$ if and only if $t = 0$.

Example 2.4. $T(z,y,x) = x\psi(x) - yz$, for all x,y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to [0,1)$ is such that $\lim_{t\to r^+} \psi(t) < 1$, for all $r > 0$.

Example 2.5. $T(z,y,x) = \frac{x}{x+1} - yz$, for all x,y and $z \in \mathbb{R}_+$.

Example 2.6. $T(z,y,x) = x - yz - \tau$, for all x,y and $z \in \mathbb{R}_+$, where $\tau > 0$.

Example 2.7. $T(z, y, x) = \psi(x) - z\phi(y)$, for all x, y and $z \in \mathbb{R}_+$, where $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$.

Example 2.8. $T(z, y, x) = x - \frac{f(y,x)}{g(y,x)}yz$, for all x, y and $z \in \mathbb{R}_+$, where $f, g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions w.r.t. each variable and such that $f(s, t) > g(s, t) > 0$, for all $s, t > 0$.

Example 2.9. $T(z, y, x) = \psi(x) - yz$, for all x, y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is Matkowski function i.e., non-decreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Observe that, $\psi(t) < t$, for all $t > 0$ (see [21], [22]).

Definition 2.2. *A self-mapping S on a metric space (M, d) is said to be $\alpha\mathcal{T}$-contraction with respect to $T \in \mathcal{T}$ if*

$$T(\alpha(u, v), d(Su, Sv), d(u, v)) \geq 0, \text{ for all } u, v \in M, \tag{2.1}$$

where $\alpha : M \times M \to \mathbb{R}_+$.

Remark 2.1. If S is an $\alpha\mathcal{T}$-contraction for some $T \in \mathcal{T}$, then (in view of the condition $(T1)$) we have

$$\alpha(u, v)d(Su, Sv) < d(u, v), \text{ for all distinct } u, v \in M. \tag{2.2}$$

3. α-admissibility

In this section, M stands for a non-empty set, $S : M \to M$ and $\alpha : M \times M \to \mathbb{R}_+$.

Popescu [3] relaxed the α-admissibility (see condition (ii) of Theorem 1.1) as follows:

Definition 3.1 ([3])**.** *S is called α-orbital admissible if for all $u, v \in M$,*

$$\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2v) \geq 1.$$

Karapinar et al. [2] introduced the following new and strong α-admissibility notion:

Definition 3.2 ([2])**.** *S is said to be triangular α-admissible if for all u, v and $w \in M$,*

(i) $\alpha(u, v) \geq 1 \Rightarrow \alpha(Su, Sv) \geq 1$*;*

(ii) $\{\alpha(u, w) \geq 1$ *and* $\alpha(w, v) \geq 1\} \Rightarrow \alpha(u, v) \geq 1$.

Later on, Popescu [3], enlarges the class of functions of Definition 3.2 as follows:

Definition 3.3 ([3])**.** *S is said to be triangular α-orbital admissible if for all $u, v \in M$,*

(*i*) $\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2u) \geq 1$;

(*ii*) $\{\alpha(u, v) \geq 1$ *and* $\alpha(v, Sv) \geq 1\} \Rightarrow \alpha(u, Sv) \geq 1$.

In the same continuation, we introduce the following definition:

Definition 3.4. *S is said to be α-permissible if for all $m \geq n \geq 1$ and $u, v \in M$,*

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(S^n u, S^m v) \geq 1.$$

A relatively weaker version of Definition 3.4 is the following:

Definition 3.5. *S is said to be α-orbital permissible if for all $m \geq n \geq 1$ and $u \in M$,*

$$\alpha(u, Su) \geq 1 \Rightarrow \alpha(S^n u, S^m u) \geq 1.$$

Example 3.1. Let $M = \{0, 1, 2, 3\}$. Define S and α as follows:

$$S(u) = \begin{cases} 1, & \text{if } u = 1; \\ (u+1) \mod 4, & \text{otherwise.} \end{cases}$$

and

$$\alpha(u, v) = \begin{cases} 1, & \text{if } \frac{u+v}{2} \text{ is an integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, $\alpha(0, 2) = 1$ but $\alpha(S0, S^2 2) = 0$ so that S is not α-permissible mapping. However, it is α-orbital permissible as $\alpha(1, S1) = \alpha(1, 1) = 1$ and $S^n 1 = 1$ for all $n \in \mathbb{N}$. Observe that S is vacuously α-orbitally permissible for other cases.

The forthcoming two remarks shed light on certain salient features of Definition 3.5 which we intend to utilize in our later discussions.

Remark 3.1. α-admissibility and α-orbital permissibility are independent. Recall that, the α-orbital permissible mapping in Example 3.1 is not α-admissible as $\alpha(1, 3) = 1$ but $\alpha(S1, S3) = 0$. The following example illustrates the reverse part of the claim:

Example 3.2. Let us take $M = \{0, 1, 2, 3\}$. Define S and α as follows:

$$S(u) = (u+1) \mod 4$$

and

$$\alpha(u, v) = \begin{cases} 0, & \text{if } u + v \text{ is an even number;} \\ 1, & \text{otherwise.} \end{cases}$$

By a routine calculation one can verify that S is α-admissible. However, it is not α-orbital permissible as $\alpha(1, S1) = 1$ but $\alpha(S1, S^3 1) = 0$.

Remark 3.2. Every triangular α-orbital admissible mapping is α-orbital permissible and the reverse implication need not to be true in general. Indeed, if there exists no $u \in M$ such that $\alpha(u, Su) \geq 1$, the assertion holds vacuously. Otherwise, Lemma 7 due to Karapinar et al. [2] (or Lemma 8 due to Popescu [3]) supports the assertion. On the other hand, S in the following example is α-orbital permissible but not triangular α-orbital admissible:

Example 3.3. Let $M = A \cup B$ where $A = \{0, 1\}$ and $B = \{2, 3\}$. Define S and α as follows:

$$S(u) = \begin{cases} 1, & \text{if } u = 0, \\ 0, & \text{if } u = 1, \\ 3, & \text{if } u = 2, \\ 2, & \text{if } u = 3. \end{cases}$$

and

$$\alpha(u, v) = \begin{cases} 1, & \text{if (both } u \text{ and } v \text{ are either in } A \text{ or in } B) \text{ or } (u, v) = (1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, $\alpha(1, S1) = 1$ and $\{\alpha(1, 2) = 1$ and $\alpha(2, S2) = 1\}$ but $\alpha(1, S2) = 0$.

The following diagram depicts the inter-relations amongst the six forgoing notions:

$$\begin{array}{ccc} \text{tringular } \alpha\text{-admissible} & \xrightarrow{implies} & \text{tringular } \alpha\text{-orbital admissible} \\ \downarrow implies & & \downarrow implies \\ \alpha\text{-admissibile} & \xrightarrow{implies} & \alpha\text{-orbital admissibile} \\ \uparrow implies & & \uparrow implies \\ \alpha\text{-permissibile} & \xrightarrow{implies} & \text{orbital } \alpha\text{-permissibile} \end{array}$$

4. Fixed point results

Our main result in this article runs as follows:

Theorem 4.1. *Let (M, d) be a complete metric space and $S : M \to M$ an $\alpha\mathcal{T}$-contraction under some tri-simulation function T. Suppose that*

(a) S is triangular α-permissible;

(b) there exists $u_0 \in M$ such that $\alpha(u_0, Su_0) \geq 1$;

(c) S is continuous.

Then S has a fixed point w (say) in M. Moreover, for every such u_0 in M, the Picard sequence $\{u_n := S^n u_0\}$ converges to w.

Proof. Choose $u_0 \in M$ such that $\alpha(u_0, u_1) \geq 1$. Then for all $m > n \geq 1$,

$$\alpha(u_n, u_m) \geq 1. \tag{4.1}$$

If $u_m = u_{m+1}$ for some $m \in \mathbb{N}$, then u_m is a fixed point of S so that we are done. Otherwise, let $d(u_n, u_{n+1}) > 0$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} 0 &\leq T(\alpha(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})) \\ &< d(u_n, u_{n+1}) - \alpha(u_n, u_{n+1}) d(u_{n+1}, u_{n+2}), \end{aligned} \tag{4.2}$$

which shows that $\{d(u_n, u_{n+1})\}$ is a strictly decreasing sequence of positive real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then on letting $n \to \infty$ on both sides of (4.2), we get $\lim_{n\to\infty} \alpha(u_n, u_{n+1}) = 1$. Moreover, by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})) < 0,$$

a contradiction. Therefore, for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{4.3}$$

Furthermore, we prove that $\{u_n\}$ is a bounded sequence. To establish the assertion, let on a contrary that $\{u_n\}$ is unbounded. Then, there exists a subsequence $\{u_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(u_{n_k}, u_{n_{k+1}}) > 1 \tag{4.4}$$

and

$$d(u_{n_k}, u_m) \leq 1, \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

Now, using triangular inequality, we have

$$\begin{aligned} 1 &< d(u_{n_k}, u_{n_{k+1}}) \\ &\leq d(u_{n_k}, u_{n_{k+1}-1}) + d(u_{n_{k+1}-1}, u_{n_{k+1}}) \\ &\leq 1 + d(u_{n_{k+1}-1}, u_{n_{k+1}}), \end{aligned}$$

which, on letting $k \to \infty$ in 4.3, gives rise

$$\lim_{k\to\infty} d(u_{n_k}, u_{n_{k+1}}) = 1.$$

By (2.2), we have

$$\alpha(u_{n_k-1}, u_{n_{k+1}-1}) d(u_{n_k}, u_{n_{k+1}}) < d(u_{n_k-1}, u_{n_{k+1}-1}).$$

Now, on using (4.1) and (4.4), we have

$$\begin{aligned} 1 &< \alpha(u_{n_k-1}, u_{n_{k+1}-1}) d(u_{n_k}, u_{n_{k+1}}) \\ &< d(u_{n_k-1}, u_{n_{k+1}-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + d(u_{n_k}, u_{n_{k+1}-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + 1, \end{aligned}$$

which, on letting $k \to \infty$ and using (4.3), gives

$$\lim_{k\to\infty} d(u_{n_k-1}, u_{n_{k+1}-1}) = \lim_{k\to\infty} \alpha(u_{n_k-1}, u_{n_{k+1}-1}) = 1.$$

Hence, on using $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_{n_k-1}, u_{n_{k+1}-1}), d(u_{n_k}, u_{n_{k+1}}), d(u_{n_k-1}, u_{n_{k+1}-1})) < 0,$$

which is a contradiction. Therefore, $\{u_n\}$ is a bounded sequence.

Now, let $c_n =: \sup\{d(u_i, u_j) : i, j \geq n\}$. Observe that, $\{c_n\}$ is decreasing sequence of non-negative real numbers which is bounded due to the boundedness of $\{u_n\}$. Therefore, there exists some $c \geq 0$ such that $\lim_{n\to\infty} c_n = c$. If $c \neq 0$, then by the definition of $\{c_n\}$, for every $k \in \mathbb{N}$ there exist m_k, n_k with $m_k > n_k \geq k$ such that

$$c_k - \frac{1}{k} \leq d(u_{m_k}, u_{n_k}) \leq c_k,$$

which gives rise

$$\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = c. \tag{4.5}$$

Owing to (4.1) and (2.2), we have

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &\leq \alpha(u_{n_k-1}, u_{m_k-1}) d(u_{n_k}, u_{m_k}) \\ &< d(u_{n_k-1}, u_{m_k-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_k-1}), \end{aligned}$$

which is (in view of (4.3) and (4.5)) on letting $k \to \infty$, gives rise

$$\lim_{k\to\infty} d(u_{n_k-1}, u_{m_k-1}) = c \text{ and } \lim_{k\to\infty} \alpha(u_{n_k-1}, u_{m_k-1}) = 1.$$

As S is $\alpha\mathcal{T}$-contraction w.r.t. T, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_{n_k-1}, u_{m_k-1}), d(u_{n_k}, u_{m_k}), d(u_{n_k-1}, u_{m_k-1})) < 0,$$

a contradiction which shows that $c = 0$. Consequently, $\{u_n\}$ is a Cauchy sequence. The completeness of M ensures the existence of some $w \in M$ such that

$$\lim_{n\to\infty} u_n = w.$$

Now, the continuity of S implies that

$$\lim_{n\to\infty} u_n = S \lim_{n\to\infty} u_{n-1} = Sw.$$

Now, the uniqueness of the limit concludes the proof. □

Theorem 4.2. *Theorem 4.1 remains true if one replaces condition* (c) *by the following: If* $\{u_n\}$ *is a sequence in* M *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $u_n \to w \in M$ *as* $n \to \infty$*, then there exists a subsequence* $\{u_{n(k)}\}$ *of* $\{u_n\}$ *such that* $\alpha(u_{n(k)}, w) \geq 1$ *for all* k*.*

Proof. On the lines of the proof of Theorem 4.1, we know that the Picard sequence $\{u_n\}$ based on u_0 converges to some $w \in M$. From (4.1) and the new condition (c), there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, w) \geq 1$ for all k. In view of (1.1), for all k, we get that

$$\begin{aligned} 0 &\leq T(\alpha(u_{n(k)}, w), d(Su_{n(k)}, Sw), d(u_{n(k)}, w)) \\ &= T(\alpha(u_{n(k)}, w), d(u_{n(k)+1}, Sw), d(u_{n(k)}, w)) \\ &< d(u_{n(k)}, w) - \alpha(u_{n(k)}, w) d(u_{n(k)+1}, Sw) \end{aligned}$$

so that

$$d(u_{n(k)+1}, Sw) = d(Su_{n(k)}, Sw) \leq \alpha(u_{n(k)}, w) d(Su_{n(k)}, Sw) \leq d(u_{n(k)}, w).$$

Letting $k \to \infty$, we have $d(w, Sw) = 0$. □

Observe that, every contraction condition (of the form (1.1)) obtained via a simulation function can be deduced using a suitable tri-simulation function. Indeed, for if the contraction condition obtained via a simulation function ξ, then $T(z, y, x) := \xi(x, yz)$ satisfies conditions $(T1)$ and $(T2)$. Further, inequality (2.1) remains true for $\alpha \equiv 1$.

Now, to exhibit the advantage of Theorem 4.1 (the same can be done for Theorem 4.2), we derive the following two consequences. We begin with the following result which remains a modified version of Theorem 1.1:

Corollary 4.1. *Let* (M, d) *be a complete metric space and* S *a self-mapping on* M *satisfying*

$$\alpha(u, v) d(Su, Sv) \leq \psi\big(d(u, v)\big), \text{ for all } u, v \in M,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing function such that* $\lim_{n\to\infty} \psi^n(t) = 0$*, for all* $t > 0$ *and* $\alpha : M \times M \to \mathbb{R}_+$*. If* S *satisfies conditions* (a) − (c) *of Theorem 4.1, then* S *has a fixed point.*

Proof. The proof follows directly in view of Example 2.9 and Theorem 4.1. □

Remark 4.1. The assumption $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$ in Corollary 4.1 is relatively weaker than the assumption $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$ assumed in Theorem 1.1.

Remark 4.2. In view of Remark 3.2, Corollary 4.1 remains weaker than the one deduced via Theorem 1.4 of Karapinar [19].

The following consequence can be proved in view of Theorem 4.1 and Example 2.7:

Corollary 4.2. *Let (M,d) be a complete metric space and S a self-mapping on M satisfying*

$$\alpha(u,v)\phi\big(d(Su,Sv)\big) \leq \psi\big(d(u,v)\big), \forall u,v \in M, \tag{4.6}$$

where $\phi,\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$ and $\alpha : M \times M \to \mathbb{R}_+$. If S satisfies conditions (a)–(c) *of Theorem 4.1, then S has a fixed point.*

Notice that, Corollary 4.2 can not be deduced using Theorem 1.4 of Karapinar [19] which also substantiate the utility of our extension carried out in this note.

Similarly, one can also derive an existence result corresponding to all other tri-simulation functions defined in this article or elsewhere.

Theorem 4.3. *The fixed point of S ensured by Theorem 4.1 (or Theorem 4.2) remains unique provided one of the following conditions hold:*

(*i*) $\alpha(u,v) \geq 1$ *for all* $u,v \in Fix(S) := \{x \in M : Sx = x\}$.

(*ii*) *S is α-permissible and for all $u,v \in M$ there exists $z \in M$ such that* $\alpha(u,z) \geq 1$ *and* $\alpha(v,z) \geq 1$.

Proof. Let u and v be two distinct fixed points of S. If the condition (i) is satisfied, then

$$\begin{aligned} 0 &\leq T(\alpha(u,v), d(Su,Sv), d(u,v)) \\ &= T(\alpha(u,v), d(u,v), d(u,v)) \\ &< d(u,v) - \alpha(u,v)d(u,v), \end{aligned}$$

a contradiction so that $u = v$ and we are done with the condition (i).

Alternately, if condition (ii) holds, there exists $w \in M$ such that $\alpha(u,w) \geq 1$ and $\alpha(v,w) \geq 1$. If w equals one of these two fixed points (say u), then, processing as in the case of condition (i), we can show that $w = v$ which is a contradiction. Thus, we assume that u, v and w are distinct points. Owing to the α-permissibility of S, we have $\alpha(u,w_n) \geq 1$ and $\alpha(v,w_n) \geq 1$, for all $n \geq 1$. Now, we assert that

$$\lim_{n\to\infty} w_n = u.$$

If $w_m = u$, for some $m \in \mathbb{N}$, then the assertion immediately follows. Otherwise, assume that $d(u,w_n) > 0$, for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} 0 &\leq T\big(\alpha(u_n,w_n), d(u_{n+1},w_{n+1}), d(u_n,w_n)\big) \\ &< d(u,w_n) - \alpha(u,w_n)d(u,w_{n+1}). \end{aligned}$$

That is $\{d(u, w_n)\}$ is a strictly decreasing sequence of non-negative real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T\big(\alpha(u, w_n), d(u, w_{n+1}), d(u, w_n)\big) < 0,$$

a contradiction which establishes the assertion. Likewise, we can show that

$$\lim_{n\to\infty} w_n = v.$$

Now, the uniqueness of the limit implies $u = v$ which concludes the proof. □

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Some remarks on fully stable gamma modules

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Abstract. In this work we study full stability in the gamma module theory. A gamma module M is fully stable, if for each gamma submodule N of M, $\theta(N) \subseteq N$ for each gamma homomorphism θ of N into M. Several properties and characterizations of this classes of gamma modules have been studied. The advantages of these characterizations have been considered. Finding some sources of full stability and discuss the direct sum in fully stable gamma modules, by the way we show that in fully stable gamma modules, each gamma submodule has a unique complement. Finally characterize full stability by some of their generalizations and relate with the (SIP) and (SSP) properties.

Keywords: gamma modules, fully stable gamma modules, uniserial gamma module, $I-$multiplication gamma modules, fully ds$-$(essential) stable gamma modules, the (SIP) and (SSP) properties.

1. Introduction

In 1964, N. Nobusawa gave, as a generalization of the concept of ring , the notion of gamma rings [6]. in 1966 W.E. Barnes generalized this concept and obtained some basic properties gamma rings [3].

Let R and Γ be two additive Abeline groups. R is called a $\Gamma-$ring if there is a mapping: $R \times \Gamma \times R \to R$, $(r, \alpha, \overline{r}) \to r\alpha\overline{r}$ such that the following hold for all $r_1, r_2, r_3 \in R$, $\alpha, \beta \in \Gamma$.

i. $(r_1 + r_2)\alpha r_3 = r_1\alpha r_3 + r_2\alpha r_3$

ii. $r_1(\alpha + \beta)r_2 = r_1\alpha r_2 + r_1\beta r_2$

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iii. $r_1\alpha(r_2 + r_3) = r_1\alpha r_2 + r_1\alpha r_3$

iv. $(r_1\alpha r_2)\beta r_3 = r_1\alpha(r_2\beta r_3)$.

A subset W of a $\Gamma-$ring R is called right (left) $\Gamma-$ideal of R if W is a subgroup of R and $W\Gamma R \subseteq W(RW \subseteq W)$, Where $R\Gamma W = \{W\alpha r | r \in R, \alpha \in \Gamma, w \in W\}$, W is called $\Gamma-$ideal if it is both right and left $\Gamma-$ideal of R,in particular $W\alpha R$ is called aright (left) $\alpha-$ideal of R Where $\alpha \in \Gamma$ if W is asubgroup of R and $W\alpha R(R\alpha W) \subseteq W$.

In 2010, R. Ameri, R. Sadeqhi extend the concept of modules to gamma modules [2]. Let R be a $\Gamma-$ring. A (left) $R_\Gamma-$module is an additive Abelian group M, if there exist a mapping: $R \times \Gamma \times M \to M$ $r\gamma m$ denote the image of (r, γ, m) such that the following hold for all $m, m_1, m_2 \in M$, $\gamma, \gamma_1, \gamma_2 \in R$ and $r, r_1, r_2 \in R$:

i. $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$

ii. $(r_1 + r_2)\gamma m = r1\gamma m + r_2\gamma m$

iii. $r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m_1$

iv. $r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m$.

A right $R_\Gamma-$module is defined a similar way. A left $R_\Gamma-$module M is unitary if there are element 1 in R and $\alpha_0 \in \Gamma$ such that $1\alpha_0 m = m$ for all m in M. For more detail of gamma module see [2].

In 1991, M. S. Abbas introduced and studied the notion of fully stable modules [1] Let M be an $R-$module. Asubmodule N of M is called stable if $f(N) \subseteq N$ for each homomorphism f of N into M. If each submodule of M is stable, then M is called fully stable.

In this work, we consider the notion of full stability in the category of gamma modules. A left $R_\Gamma-$module M is called fully stable if $f(N) \subseteq N$. for each $R_\Gamma-$submodule N of M and $R_\Gamma-$homomorphism $f : N \to M$ Let M be an $R_\Gamma-$module. For an arbitrary fixed α in Γ, a subset A of R and a subset W of M we define $\ell_R^\alpha(W) = \{r \in R | r\alpha W = 0\}$ and $\Upsilon_M^\alpha(A) = \{m \in M \mid A\alpha m = 0\}$.

We gave several properties and characterizations of this class of gamma modules a left $R_\Gamma-$module M is fully stable if and only if $\gamma_M^{\alpha_0}\left(\ell_R^{\alpha_0}(R\alpha_0 a)\right) = R\alpha_0 a$ for all a in M, that is fully stable $R_\Gamma-$modules are precisely those $R_\Gamma-$modules in which each α_0-cyclic $R_\Gamma-$submodule is a right annihilatar of some right α_0-ideal of R. with certain conditions this characterization is extended for α_0-finitely generated $R_\Gamma-$submodules. $I-$multiplication $R_\Gamma-$module and uniserial $R_\Gamma-$module with descending chain condition for α_0-cyclic $R_\Gamma-$submodules are fully stable.

By examples the full stability of the direct sum of fully stable $R_\Gamma-$modules being not true, and investigate conditions under which the full stability is closed under direct-sum. We see that in fully stable $R_\Gamma-$modules each direct summand

has a unique complement direct summand, this result motivate us to consider some generalizations of fully stable gamma module in fact we restrict the stability only on certain class of gamma submodule. By means of these generalizations we see that every fully stable gamma modules satisfies the (SIP) and (SSP) properties and every regular multiplication gamma modules is fully stable.

2. Fully stable gamma modules

Definition 2.1. *An $R_\Gamma-$submodule N of an $R_\Gamma-$module of M is called stable if $\theta(N) \subseteq N$ for each $R_\Gamma-$homomorphism θ of N into M. An $R_\Gamma-$module is called fully stable if all of its $R_\Gamma-$submodules are stable. A $\Gamma-$ring R is right (left) fully stable if R is a right (left) fully stable $R_\Gamma-$module.*

Let M be an $R_\Gamma-$module and an arbitrary fixed element. An $R_\Gamma-$submodule N is called $\alpha-$cyclic, if there exists an element $m \in M$ such that $N = R\alpha m$. Its an easy matter to see that an $R_\Gamma-$module M is fully stable if and only if each α_0-cyclic $R_\Gamma-$submodule of M is stable. Every $R_\Gamma-$submodule of fully stable $R_\Gamma-$module is fully stable.

Let M be an $R_\Gamma-$module, and N a non-empty subset of M. For arbitrary fixed $\alpha \in \Gamma$. N is called $R_\Gamma^\alpha-$submodule of M if N is a subgroup of M and $R\alpha N \subseteq N$. For a subset A of R and a subset W of M, we define $\ell_R^\alpha(W) = \{r \in R | r\alpha W = 0\}$ and $\Upsilon_M^\alpha(A) = \{m \in M | A\alpha m = 0\}$.

Then its an easy matter to see the following

Lemma 2.2. *Let M be an $R_\Gamma-$module with subset A and W of R and M respectively and α an arbitrary fixed element in Γ. Then:*

1. *$\ell_R^\alpha(W)$ is a left α or $\Gamma-$ideal of R.*
2. *If W is an $R_\Gamma^\alpha-$submodule of M, then $\ell_R^\alpha(W)$ is right $\alpha-$ideal of R.*
3. *If A is a right α or $\Gamma-$ideal of R, then $\Upsilon_M^\alpha(A)$ is an $R_\Gamma^\alpha-$submodule of M.*

If A is a right α or $\Gamma-$ideal of R, then clearly $\Upsilon_M^\alpha(A)$ is a stable $R_\Gamma^\alpha-$submodule of M. In the case that each α_0-cyclic $R_\Gamma-$submodule of M is of the form $\Upsilon_M^{\alpha_0}(I)$ for some right α_0-ideal of R, then M is fully stable. In the following we show that fully stable $R_\Gamma-$modules are precisely those $R_\Gamma-$modules in which each α_0-cyclic $R_\Gamma-$submoule is a right annihilator of some right α_0-ideal of R.

Theorem 2.3. *The following statements are equivalent for a an $R_\Gamma-$module M:*

1. *M is fully stable.*
2. *$X \subseteq Y$ for every $R_\Gamma-$submodules X and Y of M in which X is an $R_\Gamma-$homomorphic image of Y.*

3. For each a, b in M, $b \neq R\alpha_0 a$ implies that $\ell_R^{\alpha_0}(R\alpha_0 a) \nsubseteq \ell_R^{\alpha_0}(R\alpha_0 b)$.

4. $\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(R\alpha_0 a)\right) = R\alpha_0 a$, for all $a \in M$.

Proof. $(1 \to 2)$ let X and Y be an $R_\Gamma-$submodules of M and $\theta : Y \to X$ an $R_\Gamma-$epimorphism. Then $X = \theta(Y) \subseteq Y$.

$(2 \to 3)$Assume that there are a, b in M with $b \notin R\alpha_0 a$ and $\ell_R^{\alpha_0}(R\alpha_0 a) \subseteq \ell_R^{\alpha_0}(R\alpha_0 b)$, then $R\alpha_0 b$ is an $R_\Gamma-$homomorphic image of $R\alpha_0 a$ By (2) $R\alpha_0 b \subseteq R\alpha_0 a$, and hence $b \in R\alpha_0 a$ which is a contradiction.

$(3 \to 4)$ Assume that there exists $m \in \ell_{\mathrm{M}}^{\alpha_0}\left(\Upsilon_R^{\alpha_0}(R\alpha_0 a)\right)$ and $m \notin R\alpha_0 a$. By (3) $\ell_R^{\alpha_0}(R\alpha_0 a) \nsubseteq \ell_R^{\alpha_0}(R\alpha_0 m)$ and hence there is $s \in \ell_R^{\alpha_0}(R\alpha_0 a)$, and $s \notin \ell_R^{\alpha_0}(R\alpha_0 m)$ so $s\alpha_0 m = 0$ which is a contradiction.

Thus $\ell_{\mathrm{M}}^{\alpha_0}\left(\Upsilon_R^{\alpha_0}(R\alpha_0 a)\right) \subseteq R\alpha_0 a$. Thus the other inclusion is always true.

$(4 \to 1)$ its obvious. □

Corollary 2.4. *Let M be a fully stable $R_\Gamma-$module. Then distinct R_Γ-submodules of M are not isomorphic.*

Proof. Assume that there are distinct $R_\Gamma-$submodules X and Y of M and X is isomorphic to Y, then each of them is an $R_\Gamma-$homomorphic image of the other and hence $X \subseteq Y$ and $Y \subseteq X$ which is a contradiction. □

Example 2.5.

1. It is clear that every simple $R_\Gamma-$module is fully stable, but semisimple $R_\Gamma-$modules (in which every $R_\Gamma-$submodule is direct summand) may not be fully stable generally, for example, let R be a simple $\Gamma-$ring and $M = R \oplus R$ as $R_\Gamma-$module. Consider the $R_\Gamma-$submodule $N = R \oplus 0$ of M and define $\theta : N \to M$ by $\theta(r, 0) = (0, r)$ for all $r \in R$. Then $\theta(N) \nsubseteq N$.

2. Let R be a $\Gamma-$ring. Then every Abelian group G can be made into an $R_\Gamma-$module by defining $r\alpha g = 0$ for all $r \in R$, $\alpha \in \Gamma$ and $g \in G$. Then M is fully stable.

3. Let $R = \mathbb{Z}$ and $\Gamma = S$ be arbitrary subring of $\mathbb{Z}$. Then every arbitrary Abelian group M is $\mathbb{Z}_S-$module by define the mapping $(x, \overline{\alpha}, m) \to x\overline{\alpha}m$ for all $x \in \mathbb{Z}$, $\alpha \in S$ and $m \in M$. Consider the following cases:

   (a) If $M = \mathbb{Z}_{p^\infty}$ and $\theta : \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^\infty}$ an $R_\Gamma-$homomorphism, then the elements of $\mathbb{Z}_{p^k}$ are those elements of order less than or equal to p^k, hence $\theta(\mathbb{Z}_{p^k}) \subseteq \mathbb{Z}_{p^k}$. It follows that $\mathbb{Z}_{p^\infty}$ is a fully stable $\mathbb{Z}_S-$module.

   (b) Let $M = \mathbb{Z}$. For each non-zero elements $n, \overline{n} \in M$ with $\overline{n} \notin R\alpha_0 n$. It is always $\ell_{\mathbb{Z}}^{\alpha_0}(R\alpha_0 n) = \ell_{\mathbb{Z}}^{\alpha_0}(R\alpha_0 \overline{n}) = 0$, thus $\mathbb{Z}$ it is not fully stable $\mathbb{Z}_S-$module. By a similar way, we can see $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both not fully stable $\mathbb{Z}_S-$module.

4. Every fully stable $R-$module is fully stable $R_\Gamma-$module.

We will eventually attempt to take advantage of condition (4) in theorem (2.3). to begin our attempt we need the following proposition:

Proposition 2.6. *Let M be a fully stable $R_\Gamma-$module such that for each $x \in M$ and left $\Gamma-$ideal I of R, each $R_\Gamma-$homomorphism $\theta : I\alpha_0 x \to M$ can be extended to an $R_\Gamma-$homomorphism $\theta : R\alpha_0 x \to M$. If an $R_\Gamma-$submodule N of M satisfies the $\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(N)\right) = N$, then so does $N + R\alpha_0 x$.*

Proof. Set $A = \ell_R^{\alpha_0}(N)$ and $B = \ell_R^{\alpha_0}(R\alpha_0 x)$. Then by the hypothesis $\Upsilon_M^{\alpha_0}(A) = N$ and $\Upsilon_M^{\alpha_0}(B) = R\alpha_0 x$. Since $\ell_R^{\alpha_0}(N + R\alpha_0 x) = A \cap B$, it is enough to see that $\Upsilon_M^{\alpha_0}(A \cap B) \subseteq N + R\alpha_0 x$. Let $y \in \Upsilon_M^{\alpha_0}(A \cap B)$. Define the well defined $R_\Gamma-$homomorphism $\theta : A\alpha_0 x \to M$ by $\theta(a\alpha_0 x) = a\alpha_0 y$ for a$\in A$. Then there is an $R_\Gamma-$homomorphism $\overline{\theta} : R\alpha_0 x \to M$ which extends θ. Full stability of M implies that $\overline{\theta}(x) \in R\alpha_0 x$. For each $a \in A$, $a\alpha_0\overline{\theta}(x) = \overline{\theta}(a\alpha_0 x) = a\alpha_0 y$. Then $\overline{\theta}(x) - y \in \Upsilon_M^{\alpha_0}(A) = N$, so $y \in N + R\alpha_0 x$. □

Let M be an $R_\Gamma-$module and α be an arbitrary fixed element of Γ. An $R_\Gamma-$submodule N is called $\alpha-$finitely generated if there are $x_1, x_2, ..., x_n$ in M such that $N = R\alpha x_1 + R\alpha x_2 + \ldots + R\alpha x_n$. An $R_\Gamma-$module is called $\alpha-$Noetherian if each $R_\Gamma-$submodule is $\alpha-$finitely generated.

If M is an $R-$module, then M is Noetherian if and only if each submodule of M is $1-$finitely generated.

By using the principle of induction on the generators of an $\alpha-$finitely generated $R_\Gamma-$submodule with the help of Proposition 2.1 and Theorem 2.1 we have the following:

Theorem 2.7. *Let M be an $R_\Gamma-$module such that for x in M and left $\Gamma-$ideal I of R, each $R_\Gamma-$homomorphism of $I\alpha_0 x$ into M can be extended to an $R_\Gamma-$homomorphism of $R\alpha_0 x$ to M. Then M is fully stable if and only if each α_0-finitely generated $R_\Gamma-$submodule of M satisfies the double annihilator condition.*

Recall that an $R_\Gamma-$module M is quasi-injectiv, if for each $R_\Gamma-$submodule A of M, each $R_\Gamma-$homomorphism from A into M can be extended to an $R_\Gamma-$endomorphism of M [7].

The concept of quasi-injective $R_\Gamma-$module satisfies the condition of theorem (2.2), so we have the following, corollary

Corollary 2.8. *The following conditions are equivalent for quasi-injective α_0-Noetherian $R_\Gamma-$module M:*

1. *M is fully stable.*
2. *$\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(N)\right) = N$ for each $R_\Gamma-$submodule N of M.*

we have mentioned that an $R_\Gamma-$module M is fully stable if and only if for each m in M $R_\Gamma-$homomorphism $\theta : \mathrm{R}\alpha_0 m \to \mathrm{M}$, there exists $r \in R$ such that $\theta(m) = r\alpha_0 m$. We denote this condition by (B).

Proposition 2.9. *Let M be an $R_\Gamma-$module with $\ell_R^{\alpha_0}(N\cap K)=\ell_R^{\alpha_0}(N)+\ell_R^{\alpha_0}(K)$ for all α_0-finitely generated $R_\Gamma-$submodule N and K of M. Then M is fully stable if and only if condition (B) satisfied for α_0-finitely generated $R_\Gamma-$submodules.*

Proof. Assume that condition (B) holds for α_0-finitely generated R_Γ-submodule, in particular for α_0-cyclic hence by theorem (2.1) M is fully stable. Conversely. assume that M is fully stable. Let N be α_0-finitely generated $R_\Gamma-$submodule of M and $f:N\to M$ an $R_\Gamma-$homomorphism. Now $N=R\alpha_0x_1+R\alpha_0x_2+\ldots+R\alpha_0x_n$ for some $x_1,x_2,\ldots,x_n\in N$. We eventually use induction of the number of generators of N. For $n=1$, this is just theorem (2.1). Suppose that M satisfies condition (B) for all $R_\Gamma-$submodules α_0-generated by m elements for $m\leq n-1$. Then there are $r,s\in R$ such that $f(x)=r\alpha_0x$ for all $x\in R\alpha_0x_1+R\alpha_0x_2+\ldots+R\alpha_0x_{n-1}$ and $f(x')=s\alpha_0x'$ for all $x'\in R\alpha_0x_n$. For each $y\in(R\alpha_0x_1+R\alpha_0x_2+\ldots+R\alpha_0x_{n-1})\cap R\alpha_0x_n$, $f(y)=r\alpha_0y=s\alpha_0y$, then $r-s\in\ell_R^{\alpha_0}((R\alpha_0x_1+R\alpha_0x_2+\ldots+R\alpha_0x_{n-1})\cap R\alpha_0x_n)$. Then there are $u\in\ell_R^{\alpha_0}(R\alpha_0x_1+R\alpha_0x_2+\ldots+R\alpha_0x_{n-1})$ and $v\in\ell_R^{\alpha_0}(R\alpha_0x_n)$ such that $r-s=u+v$ put $t=r-u=s+v$. For each $z\in N, z=\sum_{i=1}^n r_i\alpha_0x_i$ for some $r_i\in R$.

$$\begin{aligned}
f(z)&=f\left(\sum_{i=1}^{n}r_i\alpha_0x_i\right)=f\left(\sum_{i=1}^{n-1}r_i\alpha_0x_i\right)+f(r_n\alpha_0x_n)\\
&=r\alpha_0\left(\sum_{i=1}^{n-1}r_i\alpha_0x_i\right)+s\alpha_0(r_n\alpha_0x_n)\\
&=r\alpha_0\left(\sum_{i=1}^{n-1}r_i\alpha_0x_i\right)-u\alpha_0\left(\sum_{i=1}^{n-1}r_i\alpha_0x_i\right)+s\alpha_0(r_n\alpha_0x_n)+v\alpha_0(r_n\alpha_0x_n)\\
&=(r-u)\alpha_0\left(\sum_{i=1}^{n-1}r_i\alpha_0x_i\right)+(s+v)\alpha_o(r_n\alpha_0x_n)=t\alpha_0z.
\end{aligned}$$

□

Corollary 2.10. *The following are equivalent for α_0-Noetherian $R_\Gamma-$module M.*

1. *M is fully stable.*
2. *Condition (B) holds for all $R_\Gamma-$submodules of M.*

In the following We prepare same sources of fully stable gamma modules. A left $\Gamma-$ideal A of a $\Gamma-$ring R is called idempotent if $A=A\Gamma A$ [5]. An $R_\Gamma-$module M is called $I-$multiplication if for each $R_\Gamma-$submodule N of M, there is an idempotent left $\Gamma-$ideal I of R such that $N=I\Gamma M$.

Proposition 2.11. *Every $I-$multiplication $R_\Gamma-$module is fully stable.*

Proof. Let M be $I-$multiplication $R_\Gamma-$module. For each $R_\Gamma-$submodule N of M and $R_\Gamma-$homomorphism $\theta : N \to M, N = I\Gamma M$ for some idempotent $\Gamma-$ideal I of R. Hence $\theta(N) = \theta(I\Gamma M) = \theta(I\Gamma I\Gamma M) = I\Gamma\theta(N) \subseteq I\Gamma M = N$. □

An $R_\Gamma-$module M is called uniserial if $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ for all two $R_\Gamma-$submodules N_1 and N_2 of M.

Proposition 2.12. *Every uniserial $R_\Gamma-$module M with descending chain condition for α_0-cyclic $R_\Gamma-$submodules is fully stable. In particular, if R is a finite $\Gamma-$ring in which each left $-$ideal is α_0-principle, then R is left fully stable.*

Proof. Assume that $a, b \in M$ with $\ell_R^{\alpha_0}(\mathrm{R}\alpha_O a) \subseteq \ell_R^{\alpha_0}(b)$ and $b \notin R\alpha_0 a$. Then $R\alpha_0 a \subseteq R\alpha_0 b$ such that $a = a\alpha_0 b$. Consider the descending chain condition $R\alpha_0(r\alpha_0 b) \supseteq R\alpha_0(r\alpha_0 r\alpha_0 b) \supseteq \ldots$ Then $R\alpha_0((r\alpha_0 r)^n\alpha_0 b) = R\alpha_0((r\alpha_0 r)^{n+1}\alpha_0 b)$ for same positive integer n and so $(r\alpha_0 r)^n\alpha_0 b = t\alpha_o(r\alpha_0 r)^{n+1}\alpha_0 b$ for some $t \in R$. Thus $(r\alpha_0 r)^{n-1}r\alpha_0 a = t\alpha_0(r\alpha_0 r)^n r\alpha_0 a$. Since $\ell_R^{\alpha_0}(R\alpha_0 a) \subseteq \ell_R^{\alpha_0}(R\alpha_0 b)$, then $(r\alpha_0 r)^{n-1}r\alpha_0 b = t\alpha_o(r\alpha_0 r)^n\alpha b$. By using a similar argument to obtain $r\alpha_0 b = t\alpha_0 r\alpha_0 r\alpha_0 b$, so $a = t\alpha_0 r\alpha_0 a$ a. Then $b = t\alpha_0 r\alpha_0 r\alpha_0 b = t\alpha_0 r\alpha_0 a \in R\alpha_0 a$ which is a contradiction. Thus M is fully stable. □

The above proposition shows in another way that $\mathbb{Z}^\infty$ is fully stable $\mathbb{Z}_S$-module for every subring S of $\mathbb{Z}$. The finite condition in the particular case is essential for example Z as $\mathbb{Z}_S-$module.

Corollary 2.13. *Let R be a $\Gamma-$ring with descending chain condition on α_0-principle left $\Gamma-$ideals. If M a uniserial $R_\Gamma-$module, then M is fully stable.*

Proof. It is enough to show that M satisfies the descending chain condition on α_0-cyclic. If $m_1, m_2 \in M$ with $R\alpha_0 m_2 \subseteq R\alpha_0 m_1$, then there is $r \in R$ such that $R\alpha_0 m_2 = R\alpha_0 r\alpha_0 m_1$. Thus every descending chain of α_0-cyclic $R_\Gamma-$submodule has the form $R\alpha_0 m \supseteq R\alpha_0 r\alpha_0 m \supseteq R\alpha_0 r_2\alpha_0 r_1\alpha_0 m \supseteq \ldots$

Consider the descending chain $R \supseteq R\alpha_0 r_1 \supseteq R\alpha_0 r_1\alpha_0 r_2 \supseteq \ldots$

By hypothesis there is a positive integer n such that $R\alpha_0 r_1\alpha_0 r_2 \ldots \alpha_0 r_n = R\alpha_0 r_1\alpha_0 r_2 \ldots \alpha_0 r_{n+1}$.

Hence M satisfies the descending chain condition for α_0-cyclic R_Γ-submodule . □

In the following we discuss the direct sum of fully stable gamma modules. First, let R be a fully stable left $\Gamma-$ring. Then $M = R \oplus R$ is a free $R_\Gamma-$module of rank two which is not fully stable.

Proposition 2.14. *Let $M = M_1 \oplus M_2$ where M_1, M_2 are fully stable $R_\Gamma-$modules. If $\ell_R^{\alpha_0}(M_1) + \ell_R^{\alpha_0}(M_2) = R$, then M is fully stable.*

Proof. Let K be an $R_\Gamma-$submodule of M. We show that $K = N_1 \oplus N_2$ for some $R_\Gamma-$submodules N_1 and N_2 of M_1 and M_2 respectively. If $k \in K$, then $k = x + y$ where $x\imath M_1$. and $y \in M_2$. Moreover there are $a \in \ell_R^{\alpha_0}(M_1)$ and

$b \in \ell_R^{\alpha_0}(M_2)$ with $a+b=1$. Let $N_1 = \ell_R^{\alpha_0}(M_2)\,\alpha_0 \mathrm{x}$ and $N_1 = \ell_R^{\alpha_0}(M_1)\,\alpha_0 \mathrm{y}$. Now $x = 1\alpha_0 x = (a+b)\,\alpha_0 x = b\alpha_0 x \in M_1$ and $y = a\alpha_0 y \in M_2$. Then $k = b\alpha_0 x + a\alpha_0 y \in N_1 \oplus N_2$. Thus $K \subseteq N_1 + N_2$. For other inclusion,let $W \in N1 + N2$, then $W = c\alpha_0 x + d\alpha_0 y$ for som $c \in \ell_R^{\alpha_0}(M_2)$ and $d \in \ell_R^{\alpha_0}(M_1)$. $W = c\alpha_O x + d\alpha_o x + c\alpha_o y + d\alpha_0 y = (c+d)\,\alpha_0\,(x+y) \in K$. Therefore $K = N_1 \oplus N_2$. Let $\theta : K \to M$ be an $\theta : K \to M$homomorphism, $J_1 : N_1 \to N_1 \oplus N_2$, $J_2 : N_2 \to N_1 \oplus N_2$ be the injection, and that $\pi_1 : M \to M_1$, $\pi_2 : M \to M_2$ be the projection. Put $\theta_1 = \pi_1 \circ \theta \circ J_1$ and $\theta_2 = \pi_2 \circ \theta \circ J_2$. Then $\theta = \theta_1 + \theta_2$ and $\theta(K) = \theta(N_1 \oplus N_2) = \theta_1(N_1) \oplus \theta_2(N_2) \subseteq N_1 \oplus N_2$ and or $(K) \subset K$. This shows that M is fully stable. □

Corollary 2.15. *Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where M_i are $R_\Gamma-$modules, $i = 1, 2, ..., n$ and $\ell_R^{\alpha_0}(M_i) + \cap_{i \neq i}\left(\ell_R^{\alpha_0}(M_j)\right) = R$. Then M is a fully stable $R_\Gamma-$module if and only if each M_i is fully stable.*

Lemma 2.16. *Let $M = A \oplus B$ where A and B are two $R_\Gamma-$submodules of M. If N is a stable $R_\Gamma-$submodule of M, then $N = (A \cap N) \oplus (B \cap N)$.*

Proof. Let $\pi_A : M \to A$ and $\pi_B : M \to B$ be the projections of M onto A and B respectively. Stability of N implies that $\pi_A(N) \subseteq A \cap N$ and $\pi_B(N) \subseteq B \cap N$. $N = \pi_A(N) \oplus \pi_B(N) \subseteq (A \cap N) \oplus (B \cap N)$.

If M is a left $R_\Gamma-$module, then $T = End_{R_\Gamma}(M) = \{\alpha |\ \propto$ is an R_Γ endomorphism of $M\}$ is an Abelian group with usual addition of functions. And T is $\Gamma-$ring with mapping $(f, \alpha, g) \to f\alpha g$, where $f\alpha g(m) = g(1\alpha f(m))$ for all $f, g \in \mathrm{T}, \alpha \in \Gamma$ and $m \in M$. Furthermore M is a right $T_\Gamma-$module by $(m, \alpha, f) \to m\alpha f$ where $m\alpha f = f(1\alpha m)$ for all $f \in \mathrm{T}, \alpha \in \Gamma$, and $m \in M$. □

Let R be a $\Gamma-$ring. An element r of R is called α_0-idempotent if $r = r\alpha_0 r$ Two $R_\Gamma-$endomorphisms f and g are called sum -1 orthogonal α_0-idempotent, if $f + g = 1_M$, $f\alpha_0 g = g\alpha_0 f = 0$ and f, g are α_0-idempotents. We note in the above lemma that π_A and π_B are sum -1 orthogonal α_0-idempotents. Generally we have the following:

Proposition 2.17. *Let M be an $R_\Gamma-$module. Then M is a direct sum of two $R_\Gamma-$submodules if and only if there exist sum -1 orthogonal α_0- idempotents in T.*

Proof. It is enough to assume that there are sum -1 α_0-idempotents fand g of M. Set $N = f(M)$ and $K = g(M)$, then $M = N + K$. Let $x \in N \cap K$. Then $x = f(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. $f(x) = g(x) = 0$ and hence $x = 0$. □

Theorem 2.18. *A direct summand $R_\Gamma-$submodule N of an $R_\Gamma-$module M has the property $M = N \oplus K = N \oplus K'$ implies $K = K'$ if and only if K is a stable in M.*

Proof. Assume that K is a stable $R_\Gamma-$submodule of M, then by lemma (2.16), $K = (K \cap N) \oplus (K \cap K') = K \cap K'$. Hence $K \subseteq K'$. Similarly $K' \subseteq K$ and hence $K = K'$.

Conversely, let $M = N \oplus K$, π_N and π_K the projections of M onto N and K respectively. If K is not stable then there is an $R_\Gamma-$homomorphism $\theta : K \to M$ with $\theta(K) \nsubseteq K$. We may extend θ to M by putting $\theta(n) = 0$ for each $n \in N$. Then $\theta\alpha_0\pi_K = \theta$ and $\theta\alpha_0\pi_N = 0$. The two $R_\Gamma-$endomorphisms $(\pi_N + \pi_N\alpha_0\theta)$ and $(\pi_K - \pi_N\alpha_0\theta)$ are sum -1 orthogonal α_0-idempotent of M By proposition (2.6), M is a direct sum of the $R_\Gamma-$submodules $(\pi_N + \pi_N\alpha_0\theta)(M)$ and $(\pi_K + \pi_N\alpha_0\theta)(M)$, but $(\pi_N + \pi_N\alpha_0\theta)(M) = N$ and $(\pi_K - \pi_N\alpha_0\theta)(M) \nsubseteq K$. Thus they define a direct sum decomposition $M = N \oplus K'$ where $K' \neq K$ which is a contradiction. □

One can restate theorem (2.3) as follows, a complement direct summand of a direct summand of an $R_\Gamma-$module M is unique if and only if it is stable in M. Hence we have the following corollary:

Corollary 2.19. *In a fully stable $R_\Gamma-$modules each direct summand has a unique complement direct summand.*

Now we turn to a unique complement of an arbitrary gamma submodules. First we say that an $R_\Gamma-$module M satisfies condition $(*)$ if for every pair of $R_\Gamma-$submodules N_1 and N_2 of M with $N_1 \cap N_2 = 0$, we have $Hom(N_1, N_2) = Hom(N_2, N_1) = 0$.

Proposition 2.20. *If an $R_\Gamma-$module M satisfies condition $(*)$, then every R_Γ-submodule of M has a unique complement.*

Proof. Let N be an $R_\Gamma-$submodule of M. Define $N' = \{m \in M | \forall n(\neq 0) \in N, (r, \alpha) \in R \times \Gamma, \exists (s, \beta) \in R \times \Gamma / S\beta n \neq 0, s\beta r\alpha m = 0\}$, we show that N' is the unique complement of N. First we show that N' is an $R_\Gamma-$submodule of M. Let $m_1, m_2 \in N', 0 \neq n \in N$ and $(r, \alpha) \in R \times \Gamma$. Then there exists $(s, \beta) \in R \times \Gamma$ such that $s\beta n \neq 0$ and $s\beta r\alpha m_1 = 0$. Since $m_2 \in N'$, for $0 \neq s\beta n \in N$ and $(s\beta r, \alpha) \in R \times \Gamma$, there exists $(t, \gamma) \in R \times \Gamma$ such that $t\gamma\,(s\beta n) \neq 0$ and $t\gamma\,(s\beta r)\,\alpha m_2 = 0$. Thus there exist $(t\gamma s, \beta) \in R \times \Gamma$ such that $(t\gamma s)\,\beta n \neq 0$ and $(t\gamma s)\,\beta r\alpha\,(m_1 + m_2) = 0$ and this shows that $m_1 + m_2 \in N'$. Its an easy matter to see that $R\Gamma N' \subseteq N'$. Its clear that $N \cap N' = 0$. Finally assume $N \cap W = 0$ for some $R_\Gamma-$submodule W of M. If $W \nsubseteq N'$, then there is $w \in W$ and $w \notin N'$, so there exist $0 \neq n \in N$ and $(a, \alpha) \in R \times \Gamma$ such that for any $(r, \beta) \in R \times \Gamma$, the condition $r\beta a\alpha w = 0$ implies that $r\beta n = 0$. This follows that $Hom\,(R\Gamma a\alpha w, R\Gamma n) \neq 0$, but $R\Gamma a\alpha w \cap N = 0$ which contradicts the condition $(*)$. Thus $W \subseteq N'$ and so N' is the unique complement of N. □

It is clear that fully stable $R_\Gamma-$module are satisfying the condition $(*)$, Then we have the following:

Corollary 2.21. *Every $R_\Gamma-$submodule of a fully stable $R_\Gamma-$module has a unique complement.*

3. Some generalizations

Let M be an $R_\Gamma-$module. An $R_\Gamma-$submodule N of M is called essential if N has nontrivial intersection with every nonzero $R_\Gamma-$submodule of M (see [7])

First we consider the following:

Definition 3.1. *An $R_\Gamma-$submodule N of M is called fully ess$-$stable if each essential $R_\Gamma-$submodule of M is stable.*

It's clear that in a semisimple $R_\Gamma-$module M, the only essential R_Γ-submodule is M itself. So every semisimple $R_\Gamma-$module is fully essential stable, but there are many semisimple $R_\Gamma-$modules which are not fully stable.

Lemma 3.2. *Each $R_\Gamma-$submodule of an $R_\Gamma-$module is direct summand of an essential $R_\Gamma-$submodule.*

Proof. Let M be an $R_\Gamma-$module and N an $R_\Gamma-$submodule of M. We shall see that $N \oplus N^c$ is essential in M. Let K be an $R_\Gamma-$submodule of M with $K \cap (N \oplus N^c) = 0$. Let $a = x + y \in N \cap (N^c \oplus K)$ where $x \in N^c$ and $y \in K$, so $y = a - x \in K \cap (N \oplus N^c) = 0$, so $a = x \in (N \oplus N^c) = 0$. By maximality of N^c, we have $N^c = N^c \oplus K$ and hence $K = 0$. □

Theorem 3.3. *An $R_\Gamma-$module M is fully stable if and only if M is fully ess$-$stable and satisfies the condition $(*)$.*

Proof. Let N be an $R_\Gamma-$submodule of M and $\theta : N \to M$ an R_Γ-homomorphism. θ can be extended to $\overline{\theta} : N \oplus N^c \to M$ in the susal way. By the hypothesis $\theta(N) = \overline{\theta}(N \oplus N^c) \subseteq N \oplus N^c$. $\pi_{N^c} \circ \theta : N \to N^c$ where $\pi_(N^c)$ is the projection of M onto N^c. By the condition $(*)$ we have $Hom(N, N^c) = 0$ and hence $\theta(N) \subseteq N$. □

Corollary 3.4. *Every semisimple $R_\Gamma-$module with condition $(*)$ is fully stable.*

We have proved in that fully stable $R_\Gamma-$module, each direct summand has a unique complement direct summand. In the following we consider some generalization of fully stable $R_\Gamma-$modules in which precisely each direct summand has a unique complement direct summand.

Definition 3.5. *An $R_\Gamma-$module M is called fully direct$-$summand stable (simply fully ds$-$stable) if every direct summand of M is stable. A $\Gamma-$ring R is called right (left) fully ds$-$stable if it is fully ds$-$stable right (left) $R_\Gamma-$module.*

It is clear that every fully stable $R_\Gamma-$module is fully ds$-$stable, but in general the converse is not true. An $R_\Gamma-$module M is uniform if each non$-$zero $R_\Gamma-$submodule has non$-$zero intersection with every non$-$zero $R_\Gamma-$submodule of M. Thus every uniform $R_\Gamma-$module is fully ds$-$stable, but in particular the $\mathbb{Z}_\mathbb{Z}-$module $\mathbb{Z}$ is fully ds$-$stable which is not fully stable.

Remark. 1. It is an easy matter to see that an R_Γ-module M is fully ds−stable if and only if each direct summand N is fully invariant ($f(N) \subseteq N$, for each R_Γ−endomorphism of M). Thus every duo R_Γ−module (in which every R_Γ−submodule is fully invariant) is fully ds−stable and hence we have the following implications:

Fully stable R_Γ−module $\Rightarrow$ duo R_Γ−module $\Rightarrow$ fully ds−stable R_Γ−module. Note that the $\mathbb{Z}_\mathbb{Z}$−module $\mathbb{Q}$ is fully ds−stable which is not duo.

2. By the above every commutative Γ−ring is fully ds−stable. Let $R = \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$ be the polynomial R−ring in two indeterminates x and y over $\mathbb{Z}_2$ modulo the ideal $\langle x^2, y^2\rangle$ and $\overline{R} = R/\langle x, y\rangle$. Since $\ell'_{\overline{R}}(\gamma_{\overline{R}}(\langle x\rangle)) = \ell'_{\overline{R}}(\langle \overline{\overline{x}}, \overline{\overline{y}}\rangle) = \langle \overline{\overline{x}}, \overline{\overline{y}}\rangle \neq \langle \overline{\overline{x}}\rangle$ where $\overline{\overline{x}}$ is an element of $\overline{R}$. Then $\overline{R}$ is not fully stable $\overline{R}$−ring while commutativity of $\overline{R}$ gives that $\overline{R}$ is fully ds−stable $\overline{R}$−ring.

The proof of the following proposition is as theorem (2.3), but replace that R_Γ−submodule by direct summand.

Proposition 3.6. *An R_Γ−module M is fully ds−stable if and only if for each direct summand K of M, there exists unique complement H of K in M such that $M = K \oplus H$.*

Proposition 3.7. *Let $M = N \oplus L$ be an R_Γ−module. Then N is fully invariant in M if and only if $Hom(N, L) = 0$.*

Proof. Let $\pi_L : M \to L$ be the projection of M onto L and θ an R_Γ-endomorphism of M. If $Hom(N, L) = 0$, then $\pi_L\theta$ must be trivial and hence $\theta(N) \subseteq N$. Conversely, let $w \in Hom(N, L)$. Then w can be extending to $\overline{w} : M \to M$ by $\overline{w}(n + \ell) = w(n)$. If N is fully invariant in M then for all $x \in M, w(x) = \overline{w}(x) \in N$. But $w(x) \in L$ and $N \cap L = 0$. Thus $Hom(N, L) = 0$. □

In the following we give a characterization of fully ds−stable gamma modules based on the above proposition and the fact that a direct summand is stable if and only if it is fully invariant.

Proposition 3.8. *An R_Γ−mdoule M is fully ds−stable if and only if for every decomposition $M = N \oplus L$, $Hom(N, L) = 0$.*

In the following we give some useful property of fully ds−stable gamma modules. We introduce the following:

Definition 3.9. *An R_Γ−module M is said to have the summand intersection property (SIP), if the intersection of any two direct summands of M is a direct summand.*

Theorem 3.10. *An R_Γ−module M has (SIP) if and only if for each decomposition $M = N \oplus K$ and R_Γ−homomorphism $\theta : N \to K$, $ker(\theta)$ is direct summand of M.*

Proof. Assume that M has (SIP), $M = N \oplus K$ and $\theta : N \to K$ an R_Γ-homomorphism. Let $L = \{x + \theta(x) | x \in N\}$. Since $r\alpha\,(x + \theta\,(x)) = r\alpha x + r\alpha\theta\,(x) = r\alpha x + \theta\,(r\alpha x)$ for all $r \in R$ and $\alpha \in \Gamma$, then L is an $R_\Gamma-$submodule of M. We claim that $M = L \oplus K$, let $x \in M$, there are $n \in N$ and $k \in K$ such that $x = n + k$. $x = n + f(n) - f(n) + k$, so $x \in L + K$ and hence $M = L + K$. Now, let $x \in L \cap K$. Then $x = n + f(n)$ for some $n \in N$ and hence $n = x - \theta\,(n) \in N \cap K = 0$. Thus $x = 0$ and hence $M = L \oplus K$. But M has (SIP), then $L \cap N$ is a direct summand of M. It is easily to show that $L \cap N = \ker(\theta)$, so $\ker(\theta)$ is a direct summand of M. Conversely, let N and K be two direct summands of M. Then $M = N \oplus N_1$ and $M = K \oplus K_1$ for some $R_\Gamma-$submodules N_1 and K_1 of M. Let $\pi_{N_1} : M \to N_1$ and $\pi_K : M \to K$ be the projections of M onto N_1 and K respectively. Define $h = (\pi_{N_1}\alpha\pi_K)\,|_N$. It is clear that h is defined from N onto N_1. By the hypothesis $\ker(h)$ is a direct summand of M. Notice that $\ker(h) = (N \cap K) \oplus (N \cap K_1)$. Now since $N \cap K$ is a direct summand of $\ker(h)$, then $N \cap K$ is a direct summand of M. □

Proposition 3.11. *Every fully ds−stable $R_\Gamma-$module has the (SIP) property.*

Proof. Assume that M is fully ds−stable $R_\Gamma-$module and let $M = A \oplus B$ be a decomposition of M. Proposition (3.3) implies that $Hom(A, B) = 0$, that is the kernel of every $R_\Gamma-$homomorphism from A to B is A which is a direct summand of M. So theorem (3.2) shows that M has the property (SIP).

The converse of proposition (3.4) may not be true, for example, let F be a field and $M = F \oplus F$ as F_F-module. It is clear that M is semisimple and hence has the property (SIP). But M is not fully ds−stable since let $S = \{(x, 0)\,|x \in F\}$ and $S' = \{(0, y)\,|y \in F\}$. Then $M = S \oplus S$", but S' is not stable in M.

We noticed that every fully stable gamma module is fully ds−stable and the converse is not true. in the following, we obtain a condition under which the converse is true. First recall that an $R_\Gamma-$module M is regular if for each $m \in M$, there are $\theta : M \to R$ and $\gamma \in \Gamma$ such that $m = \theta(m)\gamma m$ [7]. This is equivalent that every α_0-cyclic $R_\Gamma-$submodule of M is direct summand. □

Proposition 3.12. *Every regular fully ds−stable gamma module is fully stable.*

Proof. Let M be a regular fully ds−stable $R_\Gamma-$module and N be α_0-cyclic $R_\Gamma-$submodule of M. Regularity of M implies that N is a direct summand of M and hence stable. Thus M is fully stable. □

We have proved that every $I-$multiplication $R_\Gamma-$module is fully stable. An $R_\Gamma-$module M is called multiplication of for each $R_\Gamma-$submodule N of M, there is a $\Gamma-$ideal A of R such that $N = A\Gamma M$. [on multiplication $\Gamma-$modules]. Observe that the concepts of fully stable gamma modules and multiplication gamma modules are different, the $\mathbb{Z}_\mathbb{Z}-$module $\mathbb{Z}$ is multiplication which is not fully stable, while the $\mathbb{Z}_\mathbb{Z}-$module is fully stable which is not multiplication. In the following we have:

Proposition 3.13. *Every multiplication gamma module is fully ds−stable.*

Proof. Let M be a multiplication $R_\Gamma-$module, N a direct summand of M and $f : N \to M$ be a $R_\Gamma-$homomorphism. f can be extended to all M to an $R_\Gamma-$homomorphism $g : M \to M$ trivially. Since M is multiplication, then $N = A\Gamma M$ for some $\Gamma-$ideal A of R. Thus $f(N) = g(N) = g(A\Gamma M) = A\Gamma g(M) \subseteq A\Gamma M = N$. □

Corollary 3.14. *Every regular multiplication gamma module is fully stable.*

In the following part we consider the dual of the property (SIP) in gamma modules and its relation with full stability.

Definition 3.15. *An $R_\Gamma-$module M is said to have the sum summand property (SSP) if the sum of two direct summands of M is again a direct summand.*

Definition 3.16. *An $R_\Gamma-$module M is said to have:*

1. *(C_2) if any $R_\Gamma-$submodule of M which is isomorphic to a direct summand of M is itself direct summand.*

2. *(C_3) if for any two direct summands A and B with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.*

It is an easy matter of checking (as in the module theory) that if an $R_\Gamma-$module M has (C_2) then it has (C_3).

Theorem 3.17. *Every fully stable gamma module has the (SSP) property.*

Proof. Let M be a fully stable $R_\Gamma-$module and A, B be direct summands of M. By proposition (3.4), M has the (SIP) and hence $(A \cap B) \oplus L = M$ for some $R_\Gamma-$submodule L of M. By using gamma module law [7], we have $A = (A \cap B) \oplus (L \cap A)$ and $B = (A \cap B) \oplus (L \cap B)$ and hence $A + B = (A \cap B) + [(L \cap A) \oplus (L \cap B)]$

We claim that $(A \cap B) \cap [(L \cap A) \oplus (L \cap B)] = 0$

For if, $w \in (A \cap B) \cap [(L \cap A) \oplus (L \cap B)]$, then there exist $x \in L \cap A$ and $y \in L \cap A$ such that $w = x + y$. Now $y = w - x \in [(A \cap B) + (L \cap A)] \cap (L \cap B) \subseteq A \cap (L \cap B) = 0$. So $y = 0$ and $w = x \in (A \cap B) \cap (L \cap A) = A \cap B \cap L = 0$.

Thus $A + B = (A \cap B) \oplus [(L \cap A) \oplus (L \cap B)] = B \oplus (L \cap A)$.

Again since M has the SIP and L and A are direct summands of M, then $L \cap A$ is a direct summand of M. Corollary (2.4) implies that M has (C_2) and hence (C_3). It follows that $A + B = B \oplus (L \cap A)$ is a direct summand of M. Therefore, M has the (SSP). □

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On existence and uniqueness of best proximity points for proximal β-quasi contractive mappings on metric spaces

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Abstract. This paper focus on the study of the generalization of the best proximity point theorems for non-self contractions. In fact we propose two new theorems on the existence and the uniqueness of best proximity points for proximal β-quasi contractive mappings on metric spaces . The presented theorems extend and generalize the existence and the uniqueness of best proximity points for proximal contraction done by S.Basha and quasi-contraction mappings performed by Jleli and Samet.

Keywords: Best proximity points, Proximal β-Quasi Contractive Mappings on Metric Spaces.

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1. Introduction

The notion of best proximity point was proposed by Fan in [15] for non-self continuous mappings $T : A \to X$ where A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X. In fact, in metric space (X, d) he proved that there exists x such that $d(x, Tx) = d(Tx, A)$. Various extensions of Fan's theorem were established in the literature by Prolla [18], Reich [19], Singh and Sehgal [20].

In 2010, S. Bacha [6] extended the above definition to a pair of nonempty subsets (A, B) of a metric space (X, d) to introduce further extensions of Banach contraction principle by a best proximity theorem under the hypothesis that B is approximatively compact with respect to A.

Later on, several best proximity point results were derived (see eg. [7]-[9]). Best proximity point theorems for non-self set valued mappings have been obtained in [16] by M. Jleli and B. Samet, in the context of proximal orbital completeness condition which is weaker than the compactness condition. Best proximity point theorems serve for the generalization of fixed point theorems. In fact best proximity point becomes a fixed point in the self-mappings case.

Several extensions of non-self contractions for the existence of a best proximity points were studied in [13, 14, 1]. Moreover in [17, 2, 4] various best proximity theorems for some classes of non-self mappings were established.

Herein, we study the existence and the uniqueness of best proximity points for a novel class of non-self mappings. We show that the results obtained in [6, 16] are particular cases of our main result.

2. Preliminaries and definitions

Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. We consider the following notations:

$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$;
$d(x, B) := \inf\{d(x, b) : b \in B\}$;
$\Delta(A) := \sup\{d(a, b) : a, b \in A\}$;
$A_0 := \{a \in A :$ there exists $b \in B$ such that $d(a, b) = d(A, B)\}$;
$B_0 := \{b \in B :$ there exists $a \in A$ such that $d(a, b) = d(A, B)\}$.

Definition 2.1 ([6]). *Let $T : A \to B$ be a mapping. An element $x_* \in A$ is said to be a best proximity point of T if $d(x_*, Tx_*) = d(A, B)$.*

Definition 2.2 ([3]). *Let $\beta \in (0, +\infty)$. A $\beta-$comparison function is a map $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:*

(*i*) *φ is nondecreasing;*

(*ii*) *$\lim_{n\to\infty} \varphi_\beta^n(t) = 0$ for all $t > 0$, where φ_β^n denote the n^{th}-iterate of φ_β and $\varphi_\beta(t) = \varphi(\beta\, t)$;*

(*iii*) *there exists* $s \in (0, +\infty)$ *such that* $\sum_{n=1}^{\infty} \varphi_\beta^n(s) < \infty$.

(*iv*) $(id - \varphi_\beta) \circ \varphi_\beta(t) \leq \varphi_\beta \circ (id - \varphi_\beta)(t)$ *for all* $t \geq 0$, *where* $id : [0, +\infty) \to [0, +\infty)$ *is the identity function.*

The set of all $\beta-$*comparison functions* φ *satisfying (i)–(iv) will be denoted by* Φ_β.

Remark 2.1. Let $\alpha, \beta \in (0, +\infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

We recall some useful lemma concerning the comparison functions Φ_β.

Lemma 2.1 ([3])**.** *Let* $\beta \in (0, +\infty)$ *and* $\varphi \in \Phi_\beta$. *Then*

(*i*) φ_β *is nondecreasing;*

(*ii*) $\varphi_\beta(t) < t$ *for all* $t > 0$;

(*iii*) $\sum_{n=1}^{\infty} \varphi_\beta^n(t) < \infty$ *for all* $t > 0$;

(*iv*) $(id - \varphi_\beta) \circ \varphi_\beta^n(t) \leq \varphi_\beta^n \circ (id - \varphi_\beta)(t)$ *for all* $t \geq 0$ *and* $n \in \mathbb{N}_0$.

Definition 2.3 ([6])**.** *A mapping* $T : A \to B$ *is said to be a proximal contraction if there exists a nonnegative real number* $\alpha < 1$ *such that*

$$d(u, Tx) + d(Tx, Ty) + d(Ty, v) \leq \alpha d(x, y)$$

whenever, x *and* y *are distinct elements in* A *satisfying the condition that* $d(u, Tx) = d(A, B)$ *and* $d(v, Ty) = d(A, B)$ *for some* $u, v \in A$.

Definition 2.4 ([16])**.** *A non self mapping* $T : A \to B$ *is said to be a proximal quasi-contraction if there exists a number* $q \in [0, 1)$ *such that*

$$d(u, v) \leq q \max\{d(x, y), d(x, u), d(y, v), d(x, v), d(y, u)\}$$

whenever, $x, y, u, v \in A$ *satisfying the condition that* $d(u, Tx) = d(A, B)$ *and* $d(v, Ty) = d(A, B)$.

Lemma 2.2 ([16])**.** *Let* $T : A \to B$ *be a non self mapping. Suppose that the following conditions hold:*

(*i*) $A_0 \neq \emptyset$;

(*ii*) $T(A_0) \subseteq B_0$.

Then, for all $a \in A_0$, *there exists a sequence* $\{x_n\} \subset A_0$ *such that*

$$\begin{cases} x_0 = a \\ d(x_{n+1}, Tx_n) = d(A, B), \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{1}$$

Definition 2.5 ([16]). *Under the assumptions of Lemma 2.2, any sequence* $\{x_n\} \subset A_0$ *satisfying* (1) *is called a proximal Picard sequence associated to* $a \in A_0$.

For every $a \in A_0$, we denote $PP(a)$ the set of all proximal Picard sequences associated to a. Let $a \in A_0$ and $\{x_n\} \in PP(a)$. For all $(n,k) \in \mathbb{N}^2$, we define the following sets $\mathcal{O}(x_n,k) := \{x_n,\ldots,x_{n+k}\}$ and $\mathcal{O}(x_n,\infty) := \{x_k, k \geq n\}$

Definition 2.6 ([16]). *We say that* A_0 *is proximal* T*-orbitally complete if and only if every Cauchy sequence* $\{x_n\} \in PP(x_0)$ *for some* $x_0 \in A_0$*, converges to an element in* A_0.

Definition 2.7. *We say that* B *is approximatively compact with respect to* A *if and only if every sequence* $\{y_n\} \subset B$ *satisfying* $\lim_{n\to+\infty} d(x,y_n) = d(x,B)$ *for some* $x \in A$*, has a convergent subsequence.*

3. Main results and theorems

In [5] the following definition of Proximal β-quasi- contraction was introduced.

Definition 3.1 ([5]). *Let* $\beta \in (0,+\infty)$*. A non-self mapping* $T : A \to B$ *is said to be a proximal* β*-quasi-contraction if and only if there exist* $\varphi \in \Phi_\beta$ *and nonnegative numbers* $\alpha_0,\ldots,\alpha_4$ *such that:*

$$d(u,v) \leq \varphi(\max\{\alpha_0 d(x,y), \alpha_1 d(x,u), \alpha_2 d(y,v), \alpha_3 d(x,v), \alpha_4 d(y,u)\}).$$

For all $x,y,u,v \in A$ *satisfying,* $d(u,Tx) = d(A,B)$ *and* $d(v,Ty) = d(A,B)$.

In the case of self mappings, a proximal β-quasi-contraction is exactly a β-quasi contractive mapping introduced first in [3].

Definition 3.2 ([3]). *Let* (X,d) *be a non empty complete space. A self mapping* $T : X \to X$ *is called* β*-quasi contractive if there exist* $\beta > 0$ *and* $\varphi \in \Phi_\beta$ *such that* $d(Tx,Ty) \leq \varphi(M_T(x,y))$*, where,*

$$M_T(x,y) = \max\{\alpha_0 d(x,y), \alpha_1 d(x,Tx), \alpha_2 d(y,Ty), \alpha_3 d(x,Ty), \alpha_4 d(y,Tx)\},$$

for all $x,y \in X$ *with* $\alpha_k \geq 0$ *for* $k = 0,1,\cdots,4$.

Remark 3.1. (i) Definition 2.3 [Basha, [6]] follows from definition 3.1 by taking $\beta = \alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\varphi(t) = \alpha t$ for some $0 < \alpha < 1$.

(ii) Definition 2.4 [Samet, [16]] follows from definition 3.1 by taking $\beta = \alpha_0 = \cdots = \alpha_4 = 1$ and $\varphi(t) = qt$ for some $0 < q < 1$.

Our main results are given by the following best proximity point theorem.

Theorem 3.1. *Let (A, B) be a pair of subsets of a metric space (X, d). Let $T : A \to B$ be a given non-self mapping. Suppose that the following conditions hold:*

(*i*) $A_0 \neq \emptyset$;

(*ii*) A_0 *is proximal T-orbitally complete;*

(*iii*) $T(A_0) \subseteq B_0$;

(*iv*) *there exists $\beta \geq \max_{0\leq k\leq 4}\{\alpha_k\}$ such that T is a proximal β-quasi contraction.*

Furthermore, assume that one of the following conditions holds:

(*a*) φ *is continuous;*

(*b*) $\beta > \max\{\alpha_1, \alpha_4\}$;

Then, T has a unique best proximity point $x_* \in A_0$.

To prove Theorem 3.1, we need following preliminary lemmas.

Lemma 3.1. *Under the conditions of Theorem 3.1, let $\{x_n\} \in PP(x_0)$. Then for $(s, n) \in \mathbb{N} \times \mathbb{N}^*$ with $1 \leq i \leq j \leq n$, we have*

$$d(x_{s+i}, x_{s+j}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \leq \varphi_\beta(\Delta(\mathcal{O}(x_s, n))) \tag{2}$$

Proof. Since T is a proximal β-quasi contraction and since $d(x_{s+i}, Tx_{s+i-1}) = d(x_{s+j}, Tx_{s+j-1}) = d(A, B)$ then

$$\begin{aligned} d(x_{s+i}, x_{s+j}) \leq\ & \varphi(\max\{\alpha_0 d(x_{s+i-1}, x_{s+j-1}), \alpha_1 d(x_{s+i-1}, x_{s+i}), \\ & \alpha_2 d(x_{s+j-1}, x_{s+j}), \alpha_3 d(x_{s+i-1}, x_{s+j}), \alpha_4 d(x_{s+j-1}, x_{s+i})\}). \end{aligned}$$

Using the fact that φ is nondecreasing, we get

$$\begin{aligned} d(x_{s+i}, x_{s+j}) &\leq \varphi(\beta \max\{d(x_{s+i-1}, x_{s+j-1}), d(x_{s+i-1}, x_{s+i}), \\ &\quad d(x_{s+j-1}, x_{s+j}), d(x_{s+i-1}, x_{s+j}), d(x_{s+j-1}, x_{s+i})\}) \\ &\leq \varphi(\beta\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \\ &= \varphi_\beta(\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \\ &\leq \varphi_\beta(\Delta(\mathcal{O}(x_s, n))). \end{aligned}$$

□

Lemma 3.2. *Under the conditions of Theorem 3.1, let $\{x_n\} \in PP(x_0)$. Then, we have the following two assertions*

(*I*) *For all $(s, n) \in \mathbb{N} \times \mathbb{N}^*$, there exists $1 \leq j \leq n$ such that*

$$\Delta(\mathcal{O}(x_s, n)) = d(x_s, x_{s+j}). \tag{3}$$

(*II*)

$$(Id - \varphi_\beta)(\Delta(\mathcal{O}(x_0, n))) \leq d(x_0, x_1) \tag{4}$$

Proof. Using Lemma 3.1 and the property $\phi_\beta(t) < t$, we obtain that for $(s,n) \in \mathbb{N} \times \mathbb{N}^*$ with $1 \leq i \leq j \leq n$, $d(x_{s+i}, x_{s+j}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_s,n))) < \Delta(\mathcal{O}(x_s,n))$ Thus, $\Delta(\mathcal{O}(x_s,n)) \neq d(x_{s+i}, x_{s+j})$ when both $i,j \geq 1$. Therefore $\Delta(\mathcal{O}(x_s,n)) = d(x_{s+i}, x_{s+j})$ whenever one of i or j is equal to 0. Hence assertion (1) holds. Now, we prove the second assertion. Using the first assertion and Lemma 3.1, we have

$$\Delta(\mathcal{O}(x_0,n)) = d(x_0,x_j), \leq d(x_0,x_1) + d(x_1,x_j) \leq d(x_0,x_1) + \varphi_\beta(\Delta(\mathcal{O}(x_0,n))).$$

This proves the inequality (4). □

Lemma 3.3. *Under the conditions of Theorem 3.1, every sequence* $\{x_n\} \in PP(x_0)$ *is a Cauchy sequence.*

Proof. Let $(m,n) \in \mathbb{N}^2$ with $1 \leq n < m$. Using 2 we have

$$d(x_n,x_m) = d(x_{(n-1)+1}, x_{(n-1)+m-n+1}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{n-1}, m-n+1))). \tag{5}$$

On the other hand, from (3), we get $\Delta(\mathcal{O}(x_{n-1}, m-n+1)) = d(x_{n-1}, x_{n-1+j})$ for some $j \in \{1,\ldots,m-n+1\}$, and so equation (5) becomes

$$d(x_n,x_m) \leq \varphi_\beta(d(x_{n-1}, x_{n-1+j})). \tag{6}$$

Using Lemma 3.1 we obtain

$$d(x_{n-1}, x_{n-1+j}) = d(x_{(n-2)+1}, x_{(n-2)+j+1}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{n-2}, m-n+2))). \tag{7}$$

From (6) and (7), we obtain $d(x_n,x_m) \leq \varphi_\beta^2(\Delta(\mathcal{O}(x_{n-2}, m-n+2)))$. Continuing this process, by induction, we obtain that

$$d(x_n,x_m) \leq \varphi_\beta^n(\Delta(\mathcal{O}(x_0,m))). \tag{8}$$

Now by Lemma 3.2, we have $(Id - \varphi_\beta)(\Delta(\mathcal{O}(x_0,m))) \leq d(x_0,x_1)$. Applying the non-decreasing map $t \mapsto \sum_{\ell=0}^{p} \varphi_\beta^\ell(t)$ to both side of the previous inequality, we get

$$\sum_{\ell=0}^{p} \varphi_\beta^\ell \circ (id - \varphi_\beta)(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Using property (iv) of φ_β, we obtain

$$\sum_{\ell=0}^{p} (id - \varphi_\beta) \circ \varphi_\beta^\ell(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Which implies that

$$\Delta(\mathcal{O}(x_0,m)) - \varphi_\beta^{p+1}(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Hence, by letting $p \to \infty$ in the above inequality and using (ii) of definition 2.2, we obtain

$$\Delta(\mathcal{O}(x_0, m)) \leq \sum_{\ell=0}^{\infty} \varphi_\beta^\ell(d(x_0, x_1)), \quad \text{for all } m \in \mathbb{N}.$$

Taking φ_β^n for both sides of the above inequality and using (8) and the fact that φ_β^n is non-decreasing, we get

$$d(x_n, x_m) \leq \varphi_\beta^n(\Delta(\mathcal{O}(x_0, m))) \leq \varphi_\beta^n(\sum_{\ell=0}^{\infty} \varphi_\beta^\ell(d(x_0, x_1))).$$

Since, $\varphi_\beta^n(t) \to 0$ as $n \to +\infty$, we deduce that $\{x_n\}$ is a Cauchy sequence. □

Proof of Theorem 3.1. By Lemma 2.2 there is a sequence $\{x_n\} \subset A_0$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ and so by Lemma 3.3 the sequence is Cauchy. Since A_0 is proximal $T-$orbitally complete the sequence $\{x_n\}$ converge to $x^* \in A_0$. Since $T(A_0) \subseteq B_0$ there exists $u \in A_0$ such that $d(u, Tx^*) = d(A, B)$. Using (1) and the fact that T is β-proximal quasi-contraction we get

$$d(u, x_{n+1}) \leq \varphi\Big(\max\Big\{\begin{array}{c}\alpha_0 d(x^*, x_n), \alpha_1 d(x^*, u), \alpha_2 d(x_n, x_{n+1}),\\ \alpha_3 d(x^*, x_{n+1}), \alpha_4 d(x_n, u)\end{array}\Big\}\Big). \tag{9}$$

For simplicity, denote $\rho = d(u, x^*)$ and

$$X_n = \max\big\{\alpha_0 d(x^*, x_n), \alpha_1 d(x^*, u), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x^*, x_{n+1}), \alpha_4 d(x_n, u)\big\}.$$

Thus,

$$\lim_{n\to\infty} X_n = \max\{\alpha_1\rho, \alpha_4\rho\} = \max\{\alpha_1, \alpha_4\}\rho. \tag{10}$$

Now, we will prove that $\rho = 0$. Assume that $\rho > 0$. If φ is continuous, then by taking limit of (9) as $n \to \infty$ and using (10) we get

$$\rho \leq \varphi(\max\{\alpha_1, \alpha_4\}\rho) \leq \varphi(\beta\rho) = \varphi_\beta(\rho) < \rho$$

which is a contradiction. If $\beta > \max\{\alpha_1, \alpha_4\}$, then there exists $\varepsilon > 0$ and $N > 0$ such that for all $n > N$, we have

$$X_n < (\max\{\alpha_1, \alpha_4\} + \varepsilon)\rho \ \text{ and } \ \beta > \max\{\alpha_1, \alpha_4\} + \varepsilon.$$

Therefore,

$$\begin{aligned} d(u, x_{n+1}) &\leq \varphi(X_n) \\ &\leq \varphi((\max\{\alpha_1, \alpha_4\} + \varepsilon)\rho) = \varphi_\beta(\frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho) \\ &< \frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho < \rho. \end{aligned}$$

Thus, by letting $n \to \infty$, we get

$$\rho < \frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho < \rho,$$

which is a contradiction as well. Therefore, our claim holds. Assume that x^* and y^* are two distinct best proximity points of T on A_0 that is $d(x^*, Tx^*) = d(A, B) = d(y^*, Ty^*)$. Since T is a proximal β-quasi contraction, we obtain the following inequality $d(x^*, y^*) \leq \varphi(\max\{\alpha_0, \alpha_3, \alpha_4\}d(x^*, y^*)) \leq \varphi(\beta d(x^*, y^*)) = \varphi_\beta(d(x^*, y^*)) < d(x^*, y^*)$, which is a contradiction.

Theorem 3.2. *Let (A, B) be a pair of non empty closed subsets of a complete metric space (X, d). Further, suppose that A_0 and B_0 are non empty. Let $T : A \to B$ be a single valued mapping satisfying the following conditions:*

(*i*) *B is approximatively compact with respect to A;*

(*ii*) *$T(A_0) \subset B_0$;*

(*iii*) *there exists $\beta \geq \max_{0\leq k\leq 4}\{\alpha_k, 2\alpha_4\}$ such that T is a proximal β-quasi contraction.*

Moreover, assume that one of the following conditions holds:

(*a*) *φ is continuous;*

(*b*) *$\beta > \max\{\alpha_2, \alpha_3\}$;*

Then, T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$.

Let $x_0 \in A_0$. As before, we can find $x_{n+1} \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B).$$

for every positive integer n. If $x_n = x_{n+1}$ are equals for some non negative integer n, then nothing to prove. Thus, we assume that $x_n \neq x_{n+1}$ for every nonnegative integer n. Now we have the following claim

Claim. The sequence $\{x_n\}$ is Cauchy.

Proof of the claim. Since $d(x_{n+1}, Tx_n) = d(A, B)$ and $d(x_n, Tx_{n-1}) = d(A, B)$ and T is a proximal β-quasi contraction, we get

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\
&\quad \alpha_4 d(x_{n-1}, x_{n+1})\}) \\
&\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\
&\quad \alpha_4 d(x_{n-1}, x_n) + \alpha_4 d(x_n, x_{n+1})\}) \\
&\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\
&\quad 2\alpha_4 \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\}) \\
&\leq \varphi(\beta \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}).
\end{aligned}$$

If $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, then by Lemma 2.1,

$$d(x_{n+1}, x_n) \leq \varphi(\beta d(x_{n+1}, x_n)) = \varphi_\beta(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n)$$

which is a contradiction. So, for each $n \geq 1$ we have $d(x_{n-1}, x_n) > d(x_{n+1}, x_n)$ and so,

$$d(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta(d(x_n, x_{n-1})), \quad \forall n \geq 1.$$

Then, by induction we obtain that

$$d(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta^n(d(x_0, x_1)), \quad \forall n \geq 1.$$

Now, for $n < m$ and using triangle inequalities, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi_\beta^k(d(x_0, x_1)) \leq \sum_{k=1}^{\infty} \varphi_\beta^k(d(x_0, x_1)).$$

By Lemma 2.1, $\sum_{k=1}^{\infty} \varphi_\beta^k(d(x_0, x_1)) < \infty$, so for every $\epsilon > 0$ there exists $N > 0$ such that

$$\sum_{k=n}^{m-1} \varphi_\beta^k(d(x_0, x_1)) < \epsilon \quad \text{for all } m > n > N.$$

Thus, $d(x_n, x_m) < \epsilon$. This imply that $\{x_n\}$ is a Cauchy sequence in X.

To this end, as the space X is complete, the sequence $\{x_n\} \in A$ and converges to some element $x \in A$, (since A is closed). Furthermore,

$$\begin{aligned} d(A, B) \leq d(x, B) \quad &\leq \quad d(x, Tx_n) \leq d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= \quad d(x, x_{n+1}) + d(A, B) \\ &\leq \quad d(x, x_{n+1}) + d(x, B). \end{aligned} \tag{11}$$

Therefore, as $n \to +\infty$ in the above inequalities we get $d(x, Tx_n) \to d(x, B)$ and $d(x, B) = d(A, B)$ which implies that

$$d(x, Tx_n) \to d(x, B) = d(A, B) \tag{12}$$

Using B is approximatively compact with respect to A, it follows that there is a subsequence Tx_{n_k} which converges to an element $y \in B$. Thus, by (12)

$$d(x, y) = \lim_{k \to \infty} d(x, Tx_{n_k}) = d(x, B) = d(A, B) \tag{13}$$

So, $x \in A_0$. Since $T(A_0) \subset B_0$, there is $u \in A$ such that$d(u, Tx) = d(A, B)$. Since, $d(x_{n+1}, Tx_n) = d(A, B) = d(u, Tx)$, then by definition 3.1 we have

$$d(x_{n+1}, u) \leq \varphi\Big(\max \left\{ \begin{array}{c} \alpha_0 d(x, x_n), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x, u), \\ \alpha_3 d(x_n, u), \alpha_4 d(x, x_{n+1}) \end{array} \right\} \Big) \tag{14}$$

For simplicity, denote $\rho = d(u, x)$. We will prove that $\rho = 0$. Assume, by contradiction that $\rho > 0$. Now, If φ is continuous, then by letting $n \to +\infty$ on

the above inequality, we get $\rho \leq \varphi(\max\{\alpha_2, \alpha_3\}\rho) \leq \varphi(\beta\rho) = \varphi_\beta(\rho) < \rho$ which is a contradiction. Now consider $\beta > \max\{\alpha_2, \alpha_3\}$. Let

$$X_n = \max\{\alpha_0 d(x_n, x), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x, u), \alpha_3 d(x_n, u), \alpha_4 d(x, x_{n+1}))\}.$$

As $n \to +\infty$ in the above equality, then $X_n \to \max\{\alpha_2, \alpha_3\}\rho$. We show that $\rho = 0$. Suppose $\rho > 0$. Since $\beta > \max\{\alpha_2, \alpha_3\}$, then there exists $\epsilon > 0$ and $N > 0$ such that for all $n > N$

$$X_n < (\max\{\alpha_2, \alpha_3\} + \epsilon)\rho \text{ and } \beta > \max\{\alpha_2, \alpha_3\} + \epsilon.$$

Therefore,

$$\begin{aligned} d(u, x_{n+1}) &\leq \varphi(X_n) \\ &\leq \varphi((\max\{\alpha_2, \alpha_3\} + \epsilon)\rho) = \varphi_\beta(\tfrac{\max\{\alpha_2, \alpha_3\}+\epsilon}{\beta}\rho) \\ &\leq \tfrac{\max\{\alpha_2, \alpha_3\}+\epsilon}{\beta}\rho < \rho. \end{aligned}$$

Thus, by letting $n \to \infty$ we get

$$\rho \leq \frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta}\rho < \rho,$$

which is a contradiction as well. Hence, $u = x$ and therefore $d(x, Tx) = d(A, B)$, which implies that x is a best proximity point for the mapping T. For uniqueness of best proximity point, we proceed in a similar fashion as in the proof of the uniqueness in Theorem 3.1.

4. Consequences

The following result of [16] is a direct consequence of Theorem 3.1 by taking $\beta = \alpha_i = 1, i \in \{0, 1, 2, 3, 4\}$ and $\varphi(t) = qt$.

Corollary 4.1 ([16])**.** *Let (A, B) a pair subsets of a metric space (X, d). Let $T : A \to B$ be a giving mapping. Suppose that the following conditions hold:*

(*i*) $A_0 \neq \emptyset$;

(*ii*) A_0 *is proximal T-orbitally complete;*

(*iii*) $T(A_0) \subseteq B_0$;

(*iv*) T *is a proximal quasi contraction.*

Then, T has a unique best proximity point $x_* \in A_0$. Moreover, for any $x_0 \in A_0$, any sequence $\{x_n\} \in PP(x_0)$ converges to x_*.

Specializing $\beta = \alpha_i = 1, i \in \{0, 1, 2, 3, 4\}$ and $\varphi(t) = qt$ in Theorem 3.2, we obtain the following result of [16].

Corollary 4.2 ([16]). *Let (A, B) a pair of closed subsets of a complete metric space (X, d). Let $T : A \to B$ be a giving mapping. Suppose that the following conditions hold:*

(*i*) $A_0 \neq \emptyset$;

(*ii*) *B is approximatively compact with respect to A;*

(*iii*) $T(A_0) \subseteq B_0$;

(*iv*) *T is a proximal quasi contraction.*

Then, T has a unique best proximity point $x_* \in A_0$. Moreover, for any $x_0 \in A_0$, any sequence $\{x_n\} \in PP(x_0)$ converges to x_*.

Taking $\beta = \alpha_0 = 1$, $\alpha_i = 0, i \in \{1, 2, 3, 4\}$ and $\varphi(t) = \alpha t$, in Theorem 3.2 we obtain the following result of [6].

Corollary 4.3 ([6]). *Let X be a complete metric space. Let A and B be nonempty, closed subsets of X such that is approximatively compact with respect to A. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \to B$ be a single valued map satisfying the following conditions:*

(*i*) *$T(A_0)$ is contained in B_0;*

(*ii*) *T is a proximal contraction.*

Then, there exists a unique element x in A such that $d(x, Tx) = d(A, B)$.

The preceding results subsume the following fixed point theorem. First, as a consequence of Theorem 3.1, we obtain the following result:

Corollary 4.4. *Let A be a nonempty subsets of a metric space (X, d); Let $T : A \to A$ be a giving mapping. Assume that the following conditions hold:*

(*i*) *A is proximal T-orbitally complete;*

(*ii*) *there exists $\beta \geq \max_{0 \leq k \leq 4}\{\alpha_k\}$ such that T is a β-quasi contraction.*

Moreover, assume that one of the following conditions holds:

(*a*) *φ is continuous;*

(*b*) $\beta > \max\{\alpha_1, \alpha_4\}$.

Then, T has a unique fixed point in A.

Since, every set is approximatively compact with it self. As a consequence of Theorem 3.2, we obtain the following fixed point theorem

Corollary 4.5. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a giving mapping. Suppose there exists $\beta \geq \max_{0\leq k\leq 4}\{\alpha_k, 2\alpha_4\}$ such that T is a β-quasi contraction.*

Moreover, assume that one of the following conditions holds:

(*a*) *φ is continuous;*

(*b*) *$\beta > \max\{\alpha_2, \alpha_3\}$.*

Then, T has a unique fixed point in X.

Example 4.1. Consider the complete metric space $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$. Let $A = [0, 2]$ and $B = [3, 5]$. Also, let $T : A \to B$ be defined by $T(x) = 5 - x$. Then, it is easy to see that $d(A, B) = 1$ and $A_0 = \{2\}$, $B_0 = \{3\}$. Thus, $T(A_0) = T(\{2\}) = \{3\} = B_0$. It is clear that A_0 is proximal T-orbitally complete since the only sequence can be formed in A_0 is the constant sequence $x_n = \{2\}$ which is Cauchy sequence and converge to $2 \in A_0$.
Now we shall show that T is proximal β-quasi-contraction mapping with $\phi(t) = \frac{1}{10}t, \beta = 2$ and $\alpha_i = \frac{1}{3}$ for $i = 0, 1, 2, 3, 4$. Note that $\phi(t) = \frac{1}{10}t \in \Phi_2$ since $\phi_\beta t = \phi_2 t = \frac{2}{10}t = \frac{1}{5}t$.

As above the only $x, y, u, v \in A$ such that $d(u, Tx) = d(A, B) = 1 = d(v, Ty)$ is $x = y = u = v = 2 \in A$. Now

$$\begin{aligned} 0 = d(u, v) &= d(2, 2) \\ &\leq \frac{2}{10}\max\{\frac{1}{3}d(x, y), \frac{1}{3}d(x, u), \frac{1}{3}d(y, v), \frac{1}{3}d(x, v), \frac{1}{3}d(y, u)\} \\ &= \phi(\max\{\frac{1}{3}d(1, 1), \frac{1}{3}d(1, 1), \frac{1}{3}d(1, 1), \frac{1}{3}d(1, 1), \frac{1}{3}d(1, 1)\}) \\ &= \phi(\max\{0, 0, 0, 0, 0\}) \\ &= 0 \end{aligned}$$

So, T is a proximal β-quasi-contraction mapping with $\phi(t) = \frac{1}{10}t$. We deduce using our Theorem 3.1, that T has a unique best proximity point which is $x_* = 2$ in this example.

Finally, $\phi(t)$ is continuous mappings as well as $\beta > \max_{0\leq i\leq 4}\{\alpha_i\}$.

$$d(x_*, Tx_*) = d(2, 3) = 1 = d(A, B).$$

5. Conclusion

A novel class of non-self mappings is given in this paper. Under the proximal orbital completeness condition and compactness condition, we established the existence and uniqueness of best proximity points for such mappings. As a consequence of these theorems, we obtained some fixed point results for the self mapping case.

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A study on soft rough BCK-algebras in BCK-algebras

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Abstract. In this paper, the notion of soft rough BCK-algebras is introduced, which is an extended notion of subalgebras in BCK-algebras. Moreover, in order to illustrate the roughness in BCK-algebras with respect to MS-approximation spaces in BCK-algebras, we first introduce C-soft sets and CC-soft sets as two special kinds of soft sets in BCK-algebras. Some new soft rough operations in BCK-algebras are explored. In particular, lower and upper soft rough BCK-algebras with respect to another soft set are investigated.

Keywords: soft set, rough set, soft rough set, MSR-set, BCK-algebra.

1. Introduction

It is well known that, classical methods are not always successful in dealing with the problems in economy, engineering and social science, because of various types of uncertainties presented in these problems. As far as known that there are several theories to describe uncertainty, for example, fuzzy set theory [31], rough set theory [27] and other mathematical tools. However, the theories mentioned above have their limitations. In 1999, Molodtsov [24] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, research on soft sets is progressing rapidly. In 2003, Maji et al. [22] proposed some basic operations. Further, Ali et al. [1] revised some operations. In 2011, Ali [2] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications,

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such as [4, 5, 6, 23, 28]. In particular, Zhan and Zhu [34] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [16, 17].

The concept of rough set theory was first proposed by Pawlak [27] as an approach to deal with inexact and uncertain knowledge. It is well known that, an equivalence relation on set into disjoint classes and vice versa. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applied areas. Hence, some more general models have been proposed, such as [35, 36, 37]. In 2010, Herawan et al. [12] studied a rough set approach for selecting clustering attribute. In 2013, Ali et al. [3] investigated some properties of generalized rough sets. Nowadays, this theory has been applied to many fields, such as patter recognition, intelligent systems, machine learning, image processing, cognitive science, signal analysis and so on. On the other hand, many researchers applied this theory to algebraic structures in many papers, such as [7, 8, 15].

As far as known that research of t-norm based on logical systems has become increasingly more important in the field of logic. It is well known that BCK and BCI-algebras are two classes of logic algebras which were introduced by Imai and Iseki [13, 14]. These two classes of logical algebras have been investigated by many researchers, see [20, 21, 30]. Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras, MV-algebras and Boolean algebras are extensions of BCK-algebras, which shows that BCK/BCI-algebras are considerably general structures, and it means that this is an important topic on these two kinds of logical algebras.

Soft set theory and rough set theory are all mathematical tools to deal with uncertainty. In 2010, Feng et al. [9] provided a framework to combine rough sets with soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [18] investigated the relationship among soft sets, soft rough sets and topologies. In [19], Ma and Zhan put forth rough soft BCI-algebras by means of an ideal of BCI-algebra. In recent years, Shabir et al. [29] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough set, which is called an MSR-set. The underlying concepts are very similar to Pawlak rough sets.

Based on the above idea, in this paper, we provided a framework to combine rough sets, soft sets with BCK-algebras, the notion of soft rough BCK-algebras is introduced, and we show that this is an extended notion of subalgebras in BCK-algebras. This paper is organized as follows: In Section 2, we recall some concepts and results in BCK-algebras, soft sets and rough sets. In Section 3, we study some operations with respect to MS-approximation spaces and some new soft rough operations in BCK-algebras are explored. Further, lower and

upper soft rough BCK-algebras are investigated in Section 4. In particular, in Section 5, we discuss soft rough BCK-algebras based on a soft set.

2. Preliminaries

In this section, we review some basis notions about BCK-algebras, soft sets and rough sets.

An algebra $(X, *, 0)$ of type (2,0) is called a BCK-algebra [14] if it satisfies the following conditions:
(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.
(5) $0 * x = 0$,
For all $x, y, z \in X$.

In a BCK-algebra X, we can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$. In this paper, X is always a BCK-algebra.

A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of X is called an ideal of X, denoted by $I \lhd X$, if it satisfies: (1) $0 \in I$; (2) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Definition 2.1 ([24]). *A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping.*

For a soft set $\mathfrak{S} = (F, A)$, the set $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A).

Definition 2.2 ([10]). *A soft set $\mathfrak{S} = (F, A)$ over U is called a full soft set if $\bigcup_{a \in A} F(a) = U$.*

Definition 2.3 ([16, 17]). *Let $\mathfrak{S} = (F, A)$ be a soft set over X. Then*

(i) (F, A) is called a soft BCK-algebra over X if $F(x)$ is a subalgebra of X for all $x \in Supp(F, A)$,

(ii) (F, A) is called a soft ideal over X if $F(x)$ is an ideal of X for all $x \in Supp(F, A)$.

Definition 2.4 ([27]). *Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $\underline{R}(X) = \overline{R}(X)$; otherwise, i.e., if $\underline{R}(X) - \overline{R}(X) \neq \emptyset$, X is said to be a rough set, where the two operators are defined as:*

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},$$

$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Definition 2.5 ([9]). *Let $\mathfrak{S} = (F, A)$ be a soft set over U. Then the pair $P = (U, \mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:*

$$\underline{apr}_P(X) = \{u \in U : \exists a \in A[u \in F(a) \subseteq X]\}$$

$$\overline{apr}_P(X) = \{u \in U : \exists a \in A[u \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$.

Two sets $\underline{apr}_P(X)$ *and* $\overline{apr}_P(X)$ *called the lower and upper soft rough approximations of* X *in* P, *respectively. If* $\underline{apr}_P(X) = \overline{apr}_P(X)$, X *is said to be soft definable; otherwise* X *is called a soft rough set. In what follows, we call it Feng-soft rough set.*

In order to resolve theoretical and practical aspects, we usually require the soft set to be full in the above definition. If not, it is often limits the research value by means of Feng-soft rough sets, which can be found in the following example.

Example 2.6. Let $\mathfrak{S} = (F, A)$ be a soft set over U which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e_1	1	0	1	0	1	0	0
e_2	0	1	0	0	0	1	0
e_3	1	0	1	0	1	0	1

Assume that $P = (U, \mathfrak{S})$ is a soft approximation space, we can see that the $\mathfrak{S}$ is not full.

Then for $X = \{u_1, u_2, u_4, u_6\}$. It follows from Definition 2.5 that $\underline{apr}_P(X) = \{u_2, u_6\}$ and $\overline{apr}_P(X) = \{u_1, u_2, u_3, u_5, u_6, u_7\}$. It's just a shame that $X \not\subseteq \overline{apr}_P(X)$. In order to avoid this situations, in 2013, Shabir discuss another approach to soft rough sets as follows.

Definition 2.7 ([29])**.** *Let* (F, A) *be a soft set over* U *and* $\xi : U \to \mathscr{P}(A)$ *be a mapping defined as* $\xi(x) = \{a : x \in F(a)\}$. *Then the pair* (U, ξ) *is called* MS*-approximation space and for any* $X \subseteq U$, *the lower* MSR*-approximation and upper* MSR*-approximation of* X *are denoted by* $\underline{X}_\xi$ *and* $\overline{X}_\xi$, *respectively, which two operators are defined as*

$$\underline{X}_\xi = \{x \in X | \xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in U | \xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, *then* X *is said to be* MS*-definable, otherwise,* X *is said to be an* MSR*-set. In what follows, we call it Shabir-soft rough set.*

3. Soft rough sets in BCK-algebras

In this section, we investigate some operations and fundamental properties of soft rough sets in BCK-algebras. Meanwhile, some examples are given. Firstly, we give the concept of soft rough sets in BCK-algebras.

Definition 3.1. *Let (F, A) be a soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then the pair (X, ξ) is called MS-approximation space and for any $Y \subseteq X$, the lower MSR-approximation and upper MSR-approximation of Y are denoted by $\underline{Y}_\xi$ and $\overline{Y}_\xi$, respectively, which are two operators are defined as*

$$\underline{Y}_\xi = \{x \in Y | \xi(x) \neq \xi(y) \text{ for all } y \in Y^c\}$$

and

$$\overline{Y}_\xi = \{x \in X | \xi(x) = \xi(y) \text{ for some } y \in Y\}.$$

If $\underline{Y}_\xi = \overline{Y}_\xi$, then Y is said to be MS-definable, otherwise, Y is said to be an MSR-set over X.

Remark 3.2. It follows from Definition 3.1 that for any $Y \subseteq X$, we have $\underline{Y}_\xi \subseteq Y \subseteq \overline{Y}_\xi$.

Now, we study some basic properties of lower and upper MS-approximations of a subset Y of a BCK-algebra X. In order to illustrate the roughness in X w.r.t. MS-approximation spaces in BCK-algebras, we first introduce two special kinds of soft sets in BCK-algebras.

Definition 3.3. *Let $\mathfrak{S} = (F, A)$ be a soft set in X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over X if $\xi(a) = \xi(b)$ and $\xi(c) = \xi(d)$ imply $\xi(a * c) = \xi(b * d)$ for all $a, b, c, d \in X$.*

Example 3.4. Let $X = \{0, a, b\}$ be a BCK-algebra with the following Cayley Table 2.

Table 2 BCK-algebra X

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b
e_1	1	1	1
e_2	0	1	1
e_3	1	0	0

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \{e_1, e_3\}$, $\varphi(a) = \varphi(b) = \{e_1, e_2\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over X. In fact, $\varphi(a) = \varphi(b)$ and $\varphi(a) = \varphi(a)$ but $\varphi(a * a) = \varphi(0) \neq \varphi(b) = \varphi(b * a)$.

Example 3.5. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley Table 4.

Table 4 BCK-algebra X

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	1	1
e_3	0	0	1	1

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \{e_1, e_2\}$, $\varphi(b) = \varphi(c) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over X.

Let Y, Z be any two non-empty subsets in X. Denote $Y * Z = \{y * z | \forall y \in Y . z \in Z\}$.

Theorem 3.6. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X, Y and Z any two non-empty subsets in X. Then*

$$\overline{Y}_\xi * \overline{Z}_\xi \subseteq \overline{Y * Z}_\xi.$$

Proof. Let $d \in \overline{Y}_\xi \cdot \overline{Z}_\xi$. Then $d = a * b$, where $a \in \overline{Y}_\xi$ and $b \in \overline{Z}_\xi$. It follows from Definition 3.1 that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$ for some $y \in Y$, $z \in Z$. Since $\mathfrak{S}$ is a C-soft set, $\xi(a * b) = \xi(y * z)$ for some $y * z \in Y * Z$. Thus $d = a * b \in \overline{Y * Z}_\xi$. That is, $\overline{Y}_\xi * \overline{Z}_\xi \subseteq \overline{Y * Z}_\xi$.. □

The following example shows that the containment in Theorem 3.6 is proper.

Example 3.7. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley Table 6.

Table 6 BCK-algebra X

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	1	1
e_3	0	0	0	1

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \varphi(b) = \{e_1, e_2\}$, $\varphi(c) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over X.

If we take $Y = \{0, c\}$ and $Z = \{c\}$, then $\overline{Y}_\xi = \{0, a, b, c\}$ and $\overline{Z}_\xi = \{c\}$. So $\overline{Y}_\xi * \overline{Z}_\xi = \{0, c\}$. On the other hand, $\overline{Y * Z}_\xi = \{0, a, b, c\}$. Thus $\overline{Y}_\xi * \overline{Z}_\xi \subsetneqq \overline{Y * Z}_\xi$.

Next, we introduce the other kind of soft sets in BCK-algebras.

Definition 3.8. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over X if for all $c \in X$, $\xi(c) = \xi(x * y)$ for $x, y \in R$, there exist $a, b \in X$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = a * b$.*

Remark 3.9. (1) $\mathfrak{S}$ in Example 3.7 is a C-soft set over X, however, it is not a CC-soft set.

(2) $\mathfrak{S}$ in Example 3.5 is a CC-soft set over X.

If we strength the condition, we can obtain the following result.

Theorem 3.10. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and Y, Z any two non-empty subsets in X. Then*

$$\overline{Y}_\xi * \overline{Z}_\xi = \overline{Y * Z}_\xi.$$

Proof. It follows from Theorem 3.6 that we only need to show $\overline{Y * Z}_\xi \subseteq \overline{Y}_\xi * \overline{Z}_\xi$. Now let $c \in \overline{Y * Z}_\xi$. Thus $\xi(c) = \xi(y * z)$ for some $y \in Y$ and $z \in Z$. Then there exist $a, b \in X$, such that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$ satisfying $c = a * b$ since $\mathfrak{S}$ is a CC-soft set over X. Thus $a \in \overline{Y}_\xi$ and $b \in \overline{Z}_\xi$. Hence $c = a * b \in \overline{Y}_\xi * \overline{Z}_\xi$. So $\overline{Y}_\xi * \overline{Z}_\xi = \overline{Y * Z}_\xi$. □

Now, we consider lower MSR-approximations over BCK-algebras.

Theorem 3.11. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and Y, Z any two non-empty subsets in X. Then*

$$\underline{X}_\xi * \underline{Y}_\xi \subseteq \underline{X * Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi * \underline{Y}_\xi \subseteq \underline{X * Y}_\xi$ is false, then there exists $c \in \underline{Y}_\xi * \underline{Z}_\xi$ but $c \notin \underline{X * Y}_\xi$. Then $c = a * b$, where $a \in \underline{Y}_\xi$ and $b \in \underline{Z}_\xi$, and so $\xi(a) \neq \xi(y)$ and $\xi(b) \neq \xi(z)$ for all $y \in Y^c$ and $z \in Z^c$. $(\star)$

On the other hand, $c \notin \underline{X * Y}_\xi$, then we may have the following two conditions:

(i) $c \notin Y * Z$, which contradicts with $c \in \underline{Y}_\xi * \underline{Z}_\xi \subseteq Y \cdot Z$;

(ii) $c \in Y * Y$ and $\xi(c) = \xi(y' * z')$ for some $y' * z'_i \in (X * Y)^c$. Thus $y' \in Y^c$ or $z'_i \in Z^c$. In fact, if $y' \notin Y^c$ and $z'_i \notin Z^c$, we have $x' * y' \in Y * Z$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over X, there exist $a', b' \in X$ such that $\xi(a') = \xi(y')$ and $\xi(b') = \xi(z'_i)$ satisfying $a' * b' = c$, for some $y' \in Y^c$ or $z'_i \in Z^c$. This is contradiction with $(\star)$. Hence $\underline{Y}_\xi * \underline{Z}_\xi \subseteq \underline{Y * Z}_\xi$. □

If $\mathfrak{S}$ is not a CC-soft set over X, then Theorem 3.11 is not true. See the following example.

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley Table 8.

Table 8 BCK-algebra X

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	0	1	1	1	1
e_2	1	1	0	0	1
e_3	0	0	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in MS-approximation space (X, ξ) is given by $\xi(0) = \{e_2\}$, $\xi(a) = \{e_1, e_2\}$ $\xi(b) = \xi(c) = \{e_1, e_3\}$, $\xi(d) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is not a CC-soft set over X.

If we take $Y = \{0, b\}$ and $Z = \{b, d\}$, then $\underline{Y}_\xi = \{0\}$ and $\underline{Z}_\xi = \{c\}$. So $\underline{Y}_\xi * \underline{Z}_\xi = \{c\}$. On the other hand, $\underline{Y * Z}_\xi = \{0\}$. Thus $\underline{Y}_\xi * \underline{Z}_\xi \nsubseteq \underline{Y * Z}_\xi$.

The following example shows that the containment in Theorem 3.11 and is proper.

Example 3.13. Consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 3.5. Then we know that $\mathfrak{S}$ is a CC-soft set over X. If we take $Y = \{0, a, b\}$ and $Z = \{0, b, c\}$, then $\underline{Y}_\xi = \{0, a\}$ and $\underline{Z}_\xi = \{b, c\}$. So $\underline{Y}_\xi * \underline{Z}_\xi = \{b, c\}$. On the other hand, $\underline{Y * Z}_\xi = \{0, a, b, c\}$. Thus $\underline{Y}_\xi * \underline{Z}_\xi \subsetneq \underline{Y * Z}_\xi$.

4. Characterizations of MSR-BCK-algebras in BCK-algebras

In this section, we characterize MSR-BCK-algebras in BCK-algebras. First, we give the notion of MSR-BCK-algebras as follows.

Definition 4.1. *In Definition 3.1, if $\underline{Y}_\xi \neq \overline{Y}_\xi$, then*

(i) Y is called a lower (upper) MSR-BCK-algebra (ideal) w.r.t. $\mathfrak{S}$ of X if $\underline{Y}_\xi$ ($\overline{Y}_\xi$) is a subalgebra (ideal) of X;

(ii) Y is called a MSR-BCK-algebra (ideal) w.r.t. $\mathfrak{S}$ of X if $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are subalgebras (ideals) of X.

Example 4.2. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley Table 8.

Table 8 BCK-algebra X

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 9.

Table 9 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	0	1	0
e_2	0	1	0	1	0
e_3	1	1	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, ξ) is given by $\xi(0) = \{e_1, e_3\}$, $\xi(a) = \xi(c) = \{e_1, e_2, e_3\}$, $\xi(b) = \xi(d) = \{e_3\}$. It follows from Definition 3.1 that for a set $Y = \{0, b, c, d\}$, we have

$$\underline{Y}_\xi = \{0, b, d\} \text{ and } \overline{Y}_\xi = \{0, a, b, c, d\}.$$

It is easy to check that $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are subalgebras of X. That is Y is an MSR-BCK-algebra of X.

Example 4.3. Consider the BCK-algebra in Example 3.5. Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 10.

Table 13 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	1	1
e_3	1	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, ξ) is given by $\xi(0) = \xi(a) = \{e_1, e_3\}$, $\xi(b) = \xi(c) = \{e_2, e_3\}$. It follows from Definition 3.1 that for a set $Y = \{0, a, b\}$, we have

$$\underline{Y}_\xi = \{0, a\} \lhd X \text{ and } \overline{Y}_\xi = \{0, a, b, c\} \lhd X.$$

It is easy to check that $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are ideals of X. That is Y is a MSR-ideal of X.

Proposition 4.4. *Let (X, ξ) be an MS-approximation space. If Y and Z are lower MSR-BCK-algebras (ideals) over X, then so is $Y \cap Z$.*

Proof. It follows from Definition 4.1 that $\underline{Y}_\xi$ and $\underline{Z}_\xi$ are subalgebras (ideals) over X, so $\underline{Y}_\xi \cap \underline{Z}_\xi$ is a subalgebra (ideal) of X. It follows from Theorem 3 that in [29], we know that $\underline{X \cap Y}_\xi$ is also a subalgebra (ideal) of X. Thus $X \cap Y$ is a lower MSR-BCK-algebra (ideal) over X. □

In general, if Y and Z are upper MSR-BCK-algebras (ideals) over X, $Y \cap Z$ is not an upper MSR-BCK-algebras (ideals) over X. Actually, we have the following example.

Example 4.5. Consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 4.3. Let $Y = \{0, c\}$ and $Z = \{a, c\}$. Then $\overline{Y}_\xi = \overline{Z}_\xi = \{0, a, b, c\}$ are subalgebras of X. Thus Y and Z are upper MSR-BCK-algebras over X. However, $\overline{X \cap Y}_\xi = \{b, c\}$ is not a subalgebra over X.

Finally, we study the upper and lower MSR-BCK-algebras.

Theorem 4.6. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X. If Y is a subalgebra of X, then Y is an upper MSR-BCK-algebra over X.*

Proof. Let $a, b \in \overline{Y}_\xi$. Then there exist $y, z \in Y$ such that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$. Since $\mathfrak{S}$ is a C-soft set over X, $\xi(a * b) = \xi(y * z)$. Thus, $y * z \in Y$ since Y is a subalgebra of X. Hence $a * b \in \overline{Y}_\xi$. This means that $\overline{Y}_\xi$ is a subalgebra over X. So Y is an upper soft rough BCK-algebra over X. □

Theorem 4.7. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X. If Y is a subalgebra of X, then Y is a lower MSR-BCK-algebra over X when $\underline{Y}_\xi \neq \emptyset$.*

Proof. It follows from Theorem 3.11 that $\underline{Y}_\xi * \underline{Y}_\xi \subseteq \underline{Y * Y}_\xi$. Further, it follows from Theorem 3 that in [29], we know that $\underline{Y * Y}_\xi \subseteq \underline{Y}_\xi$ since $Y * Y \subseteq Y$. So $\underline{Y}_\xi * \underline{Y}_\xi \subseteq \underline{Y}_\xi$. Thus Y is a lower MSR-BCK-algebra over X. □

Remark 4.8. The above two theorems show that any MSR-BCK-algebra is a generalization of a subalgebra of BCK-algebras.

5. MSR-BCK-algebras with respect to another soft set

In this section, we investigate MSR-BCK-algebras based on another soft set.

Definition 5.1. *Let $\mathfrak{S} = (F, A)$ be a soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over X. The lower and upper MSR-approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G,B)_\xi} = (\underline{G_\xi}, B)$ and $\overline{(G,B)_\xi} = (\overline{G_\xi}, B)$, respectively, which are two operators defined as*

$$\underline{G(e)_\xi} = \{x \in G(e) | \xi(x) \neq \xi(y) \text{ for all } y \in X - G(e)\}$$

and

$$\overline{G(e)_\xi} = \{x \in X | \xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.

(i) If $\underline{(G,B)_\xi} = \overline{(G,B)_\xi}$, then $\mathfrak{T}$ is called definable.

(ii) If $\underline{(G,B)_\xi} \neq \overline{(G,B)_\xi}$ and $\overline{G(e)_\xi}$ ($\underline{G(e)_\xi}$) is a subalgebra (ideal) of X for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) MSR-BCK-algebra (ideal) with respect to $\mathfrak{S}$ over X. Moreover, $\mathfrak{T}$ is called a lower (upper) MSR-BCK-algebra (ideal) with respect to $\mathfrak{S}$ over X if $\overline{G(e)_\xi}$ and $\underline{G(e)_\xi}$ are subalgebras (ideals) with respect to $\mathfrak{S}$ over X for all $e \in B$.

Example 5.2. We consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ over X in Example 3.5. Define another soft set $\mathfrak{T} = (G, B)$ which is given by Table 14.

Table 14 Soft set $\mathfrak{T}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	0	1	0
e_3	1	1	0	0
e_4	1	1	0	0

By calculating, $\underline{G(e_1)_\xi} = \emptyset$, $\overline{G(e_1)_\xi} = \{0, a\}$, $\underline{G(e_2)_\xi} = \{0, a\}$, $\overline{G(e_2)_\xi} = \{0, a\}$, $\underline{G(e_3)_\xi} = \emptyset$, $\overline{G(e_3)_\xi} = \{0, a, b, c\}$, $\underline{G(e_4)_\xi} = \emptyset$, $\overline{G(e_3)_\xi} = \{0, a, b, c\}$.

It is easy to check that $\underline{(G,B)_\xi}$ and $\overline{(G,B)_\xi}$ are subalgebras and ideals of X for all $e \in B$. In other words, $\mathfrak{T} = (G, B)$ is a MSR-BCK-algebra and MSR-BCK-ideal with respect to $\mathfrak{S}$ over X.

Definition 5.3. *Let $\mathfrak{T} = (G, B)$ and $\mathfrak{I} = (H, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. The product operation $*$ is defined as $\mathfrak{T} * \mathfrak{I} = (G, B) * (H, C) = (K, D)$, where $K(a) = G(a) * H(a)$ for all $a \in D$.*

Theorem 5.4. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 * G_2, D)}_\xi,$$

where $(G_1 * G_2)(e) = G_1(e) * G_2(e)$ for all $e \in D$.

Proof. The proof is similar to the one of Theorem 3.6. □

The following example shows that the containment in Theorem 5.4 is proper.

Example 5.5. Consider the BCK-algebra X in Example 3.12 and define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 15.

Table 14 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	1	0	0
e_2	0	0	0	1	1
e_3	1	1	1	1	1
e_4	0	0	0	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in MS-approximation space (X, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \{e_1, e_3\}$, $\xi(c) = \xi(d) = \{e_2, e_3, e_4\}$. Then we can check that $\mathfrak{S} = (F, A)$ is a C-soft set over X. However, it is not a CC-soft set over X.

Define two soft sets $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ over X, where $B = \{e_1, e_2\}$ and $C = \{e_2, e_3\}$ with $B \cap C = \{e_2\}$ by $G_1(e_2) = \{c\}$ and $G_2(e_2) = \{c\}$. By calculating, $\overline{G_1(e_2)}_\xi = \{c, d\}$ and $\overline{G_2(e_2)}_\xi = \{c, d\}$. Thus, $\overline{G_1(e_2)}_\xi * \overline{G_1(e_2)}_\xi = \{0, a\}$. However, $G_1(e_2) * G_2(e_2) = \{0\}$, $\overline{G_1(e_2) * G_2(e_2)} = \overline{\{0\}} = \{0, a, b\}$. Thus $\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 * G_2, D)}_\xi$.

If we strength the condition, we can obtain the following result.

Theorem 5.6. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi = \overline{(G_1 * G_2, D)}_\xi.$$

Proof. The proof is similar to the one of Theorem 3.10. □

Theorem 5.7. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then $\underline{(G_1, B)}_\xi * \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 * G_2, D)}_\xi$.*

Proof. The proof is similar to the one of Theorem 3.11. □

Finally, we investigate the lower and upper MSR-BCK-algebras with respect to another soft set.

Theorem 5.8. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and and (X, ξ) be an MS-approximation space. If $\mathfrak{T} = (G, B)$ is a soft BCK-algebra over X. then $\mathfrak{T}$ is an upper MSR-BCK-algebra over X w.r.t. $\mathfrak{S}$.*

Proof. The proof is similar to the one of Theorem 4.6. □

Theorem 5.9. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and and (X, ξ) be an MS-approximation space. If $\mathfrak{T} = (G, B)$ is a soft BCK-algebra over X. then $\mathfrak{T}$ is a lower MSR-BCK-algebra over X w.r.t. $\mathfrak{S}$ when $\underline{\mathfrak{T}_\xi} \neq \emptyset$.*

Proof. The proof is similar to the one of Theorem 4.7. □

6. Conclusion

This paper is intended to apply soft rough set theory to BCK-algebras and proposes the notion of MSR-BCK-algebras. We discuss some operational properties and algebraic structures of lower and upper soft rough approximations in BCK-algebras. In particular, we discuss MSR-BCK-algebras based on another soft set.

We hope that our results given in this paper would constitute a foundation of studying other more complicated logic algebra structure in different areas.

As an extension of this work, the following topics maybe considered:

(1) Constructing soft rough sets to other algebras, such as groups, hyperrings, BL-algebras and so on;

(2) Investigating decision making methods based on soft rough sets;

(3) Establishing soft rough sets to some applied some areas of applications, such as information sciences, intelligent systems and so on.

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Generalization of locally cyclic and Condition (P) in Act-S

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Abstract. In this paper we introduce a new generalization of locally cyclic and Condition (P) in **Act**-S, called Condition (L_P). Then we give a classification of monoids over which Condition (L_P) implies other types of flatness and vice versa. Moreover we provide new equivalent conditions for (P) and strong flatness. This can help to have a better understanding of known conditions. Even more, it can help to provide a solution for open questions in the theory of acts over monoids in the future.
Keywords: Condition (P), Condition (L_P), S-act, locally cyclic.

1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to [11] for basic definitions and terminology relating to semigroups and acts over monoids. A right S-act, usually denoted by A_S, is a non-empty set A on which S acts unitarily from the right, that is to say $(as)t = a(st)$ and $a1 = a$ for every $a \in A, s, t \in S$, where 1 is the identity of S. The category of right S-acts with the mappings that preserve the S-action is denoted by $\mathbf{Act} - S$. Left S-act ${}_SA$ is defined dually. The study of flatness properties of S-acts began in the 1970s. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S, s, s' \in S$, $as = a's'$, implies that there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and

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$us = vs'$. Condition (P) plays an important role in the theory of S-acts. Many papers have been devoted to investigating this property. Normak [13] was the first to consider Condition (P) on its own. He proved that this condition does not imply pullback flatness, although the converse implication is true. We say A_S satisfies *Condition* (GP) if whenever $as = a't$ with $a, a' \in A_S, s, t \in S$, there exist $a'' \in A_S, u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u, a' = a''v$ and $us^n = vt^n$. A_S is said to satisfy *Condition* (E) if whenever $as = as'$ with $a \in A_S, s, s' \in S$, there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S is called *cyclic* if it is isomorphic to an act of the form S/ρ, where ρ is a right congruence on S. A_S is called *locally cyclic* if for all $a, a' \in A_S$ there exists $a'' \in A_S$ such that $a, a' \in a''S$. The monoid S is called right *reversible* if for all $s, t \in S$, there exist $u, v \in S$ such that $us = vt$. A right S-act A_S is called *decomposable* provided that there exist subacts $B_S, C_S \subset A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. Otherwise, A_S is called *indecomposable*.

2. Generalization of locally cyclic and Condition (P)

Definition 2.1. *The right S-act A_S satisfies Condition (L_P) if $aS \cap a'S \neq \emptyset$ for $a, a' \in A_S$, then there exists $a'' \in A_S$ such that $a, a' \in a''S$.*

Equivalently, we can say that A_S satisfies Condition (L_P) if $as = a't$ for $a, a' \in A_S, s, t \in S$, then there exist $a'' \in A_S, u, v \in S$ such that $a = a''u$ and $a' = a''v$.

Obviously in **Act**-S we have $(P) \Longrightarrow (GP) \Longrightarrow (L_P)$ and also *cyclic* $\Rightarrow$ *locally cyclic* $\Rightarrow$ *Condition*(L_P). The following examples show that these implications are strict.

Example 2.1. Let S be a monoid which is not right reversible. The one element right S-act Θ_S fails to satisfy Condition (GP) [1, Theorem 2.2], but evidently it satisfies Condition (L_P).

Example 2.2. Let $K \neq \{1\}$ be a proper right ideal of a monoid S. Then the right S-act S/K satisfies Condition (L_P) (because S/K is cyclic), but does not satisfy Condition (P) [11, III, 13.9].

Example 2.3. Let $\{P_i\}_{i \in I}$ be a family of acts so that each P_i satisfies Condition (L_P). Consider the right S-act $A_S = \coprod_{i \in I} P_i$ with $\mid I \mid > 1$. It can be easily checked that A_S is not locally cyclic, but it satisfies Condition (L_P).

Although it is clear that there are very many examples of acts which satisfy Condition (L_P), the following example shows that in general not all acts satisfy Condition (L_P).

Example 2.4. Let $S = (\mathbb{N}, .)$ and $A = \mathbb{Z} \setminus \{0, -1\}$. The equality $-2.3 = -3.2$ holds in A_S, but there is no $a \in A_S$ such that $-2, -3 \in aS$. So, A_S does not satisfy Condition (L_P).

Let I be a proper right ideal of a monoid S. Suppose that x, y and z are different symbols that do not belong to S. Consider the right S-act $A(I) = S \coprod^I S = ((S \setminus I) \times \{x, y\}) \cup (I \times \{z\})$ with the S-action defined by

$$(t, u)s = \begin{cases} (ts, u), & \text{if } ts \in S \setminus I, \\ (ts, z), & \text{if } ts \in I \end{cases}$$

where $u \in \{x, y\}$ and $(t, z)s = (ts, z)$.

Lemma 2.1. *For the right S-act $A(I)$ the following statements hold:*

1. *A(I) does not satisfy Condition (L_P).*
2. *A(I) is not locally cyclic.*
3. *A(I) satisfies Condition (E).*
4. *A(I) is indecomposable and is generated by exactly two elements.*

Proof. (1). The equality $(1, x)s = (1, y)s$ holds in $A(I)$ for $s \in I$. It can be easily checked that we cannot find $a \in A(I)$ and $s_1, s_2 \in S$ such that $(1, x) = as_1, (1, y) = as_2$. So $A(I)$ does not satisfy Condition (L_P).

(2). $(1, x)$ and $(1, y)$ are elements of A(I), but there is no cyclic subact of A(I) which contains these two elements.

(3). Note that $A(I) = (1, x)S \cup (1, y)S$. Obviously, $(1, x)S \cong S_S \cong (1, y)S$. So $A(I)$ is the union of two subacts both of which satisfy Condition (E). Then $A(I)$ satisfies Condition (E).

(4). If $A(I)$ is decomposable, then there exist $B_S, C_S \leq A(I)$ such that $A(I) = B_S \cup C_S, B_S \cap C_S = \emptyset$. Let $(1, x) \in B_S$. Since $A(I) = (1, x)S \cup (1, y)S$, hence $(1, y) \in C_S$. But $(1, x)S \cap (1, y)S = I \subseteq B_S \cap C_S$ which shows that $B_S \cap C_S \neq \emptyset$, a contradiction. Hence $A(I)$ is indecomposable. The equality $A(I) = (1, x)S \cup (1, y)S$ shows that $A(I)$ is generated by exactly two elements. □

Clearly every locally cyclic act is indecomposable, but by the previous lemma, we see that the converse is not true. Renshaw in [14] proved that a right S-act A_S satisfying Condition (P) is indecomposable if and only if it is locally cyclic. If one looks at the proof of this result, it is easy to see that it also holds for Condition (L_P). Indeed we have:

Proposition 2.1. *A right S-act A_S satisfying Condition (L_P) is indecomposable if and only if A_S is locally cyclic.*

Lemma 2.2. *Let A_S be a locally cyclic act. If $\{a_1, a_2, ..., a_n\}$ be a finite subset of A_S, then there exists $b \in A_S$ such that $a_i \in bS$ for each i $(1 \leq i \leq n)$.*

Proof. The result follows by induction. □

Corollary 2.1. *Every finitely generated right act satisfying Condition (L_P) is indecomposable if and only if A_S is cyclic.*

Lemma 2.3 ([11], I, 5.10)**.** *Every S-act A_S has a unique decomposition into indecomposable subacts.*

Theorem 2.1. *Every right S-act A_S that satisfies Condition (L_P) is a coproduct of locally cyclic subacts.*

Proof. Let A_S be a right S-act which satisfies Condition (L_P). By Lemma 2.3, A_S is a coproduct of indecomposable subacts i.e. $A_S = \coprod_{i\in I} A_i$ where A_i is indecomposable for every $i \in I$. Since A_S satisfies Condition (L_P), each A_i should satisfy this condition. Then by Proposition 2.1, A_i is locally cyclic for every $i \in I$ and we are done. □

By Theorem 2.1 and Lemma 2.2 we have the following corollary:

Corollary 2.2. *Every finitely generated right S-act satisfying Condition (L_P) is a coproduct of cyclic right S-acts.*

The above corollary is also valid for conditions which are stronger than Condition (L_P).

Corollary 2.3. *The following statements are equivalent:*

1. *Every locally cyclic S-act is a coproduct of cyclic right S-acts.*
2. *Every right S-act satisfying Condition (L_P) is a coproduct of cyclic right S-acts.*

Recall from [7] that A_S satisfies *Condition* (EP) if for all $a \in A, s, s' \in S, as = as'$ implies that there exist $a' \in A, u, v \in S$ such that $a = a'u = a'v$ and $us = vs'$. It is clear that $(P) \Rightarrow (EP)$ and $(E) \Rightarrow (EP)$. In [1] it was shown that the combination of Conditions (GP) and (E) is equivalent to strong flatness.

Theorem 2.2. *Let A_S be a right S-act. Then A_S satisfies Condition (P) if and only if it satisfies Conditions (L_P) and (EP).*

Proof. Necessity. Evidently Condition (P) implies both Conditions (L_P) and (EP).

Sufficiency. Suppose that A_S satisfies Conditions (L_P) and (EP) and let $as = a't$ for $a, a' \in A_S$ and $s, t \in S$. Applying Condition (L_P) to the last equation, we get $a'' \in A_S, u_1, v_1 \in S$ such that $a = a''u_1, a' = a''v_1$. Then we have $a''(u_1s) = a''(v_1t)$ and so Condition (EP) implies that there exist $\overline{a} \in A_S$, $w, w' \in S$ such that $a'' = \overline{a}w = \overline{a}w'$ and $w(u_1s) = w'(v_1t)$. If $u = wu_1$ and $v = w'v_1$, then $a = \overline{a}u, a' = \overline{a}v$ and $us = vt$, that is A_S satisfies Condition (P). □

Recall from [15] that an act A_S is strongly flat if and only if it satisfies both Conditions (P) and (E) . Since equalizer flatness implies Condition (E) and strong flatness implies equalizer flatness, we can say that strong flatness is equivalent to the combination of Condition (P) and equalizer flatness. Using Theorem 2.2, we have the following corollary:

Corollary 2.4. *For any right S-act A_S, the following statements are equivalent:*

1. *A_S is strongly flat.*
2. *A_S satisfies Conditions (E) and (L_P).*
3. *A_S is equalizer flat and satisfies Condition (L_P).*

Note that Condition (E) and strong flatness are equivalent for cyclic acts [11, III, 16.7]. Renshaw in [14, Theorem 3.9] proved that this statement is also valid for locally cyclic acts. The last corollary indicates that even for the right S-acts satisfying Condition (L_P), strong flatness and Condition (E) are equivalent.

Recall that the right S-act A_S satisfies Condition (E') if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = ut$. In [12, Proposition 2.17] it was proved that weak pullback flatness is equivalent to Conditions (P) and (E'). Using Theorem 2.2, we have the following corollary:

Corollary 2.5. *The right S-act A_S is weakly pullback flat if and only if it satisfies Conditions (L_P), (EP) and (E').*

3. Classification of monoids by Condition (L_P) of their acts

Recall from [8] that an act A_S satisfies Condition $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. An act A_S is called *strongly faithful* if for $s, t \in S$, the equality $as = at$ for some $a \in A_S$, implies that $s = t$. A_S is called *faithful* if for $s, t \in S$, the equality $as = at$ for all $a \in A_S$, implies that $s = t$. The right S-act A_S which was defined in Example 2.4, is strongly faithful. Thus, strong faithfulness (and regularity as well as Conditions (E), (EP), (E'), $(E'P)$ which are weaker) does not imply Condition (L_P).

Theorem 3.1. *The following statements are equivalent:*

1. *All right S-acts satisfy Condition (L_P).*
2. *All right S-acts satisfying Condition $(E'P)$, satisfy Condition (L_P).*
3. *All right S-acts satisfying Condition (EP), satisfy Condition (L_P).*
4. *All right S-acts satisfying Condition (E'), satisfy Condition (L_P).*
5. *All right S-acts satisfying Condition (E), satisfy Condition (L_P).*
6. *All right S-acts satisfying Condition (E), are strongly flat.*
7. *All faithful right S-acts satisfy Condition (L_P).*
8. *All indecomposable right S-acts satisfy Condition (L_P).*

9. S is a group.

Proof. Since we have $(E)\Rightarrow(E')\Rightarrow(E'P)$ and $(E)\Rightarrow(EP)\Rightarrow(E'P)$, the implications (1)⇒(2)⇒(4)⇒(5), (2)⇒(3)⇒(5), (1)⇒(7) and (1)⇒(8) are obvious.

(5) ⇔ (6). This is obvious by Corollary 2.4.

(5)⇒(9). Take $s \in S$ such that $sS \neq S$. Lemma 2.1 implies that $A(sS)$ satisfies Condition (E). By assumption, $A(sS)$ satisfies Condition (L_P) which is impossible by Lemma 2.1. Thus $sS = S$ for any $s \in S$ and so S is a group.

(9) ⇒ (1). Suppose that S is a group and A_S is an arbitrary right S-act. If $as = a't$, for $a, a' \in A_S$ and $s, t \in S$, then $a = a'ts^{-1}$ and $a' = a'.1$. Thus A_S satisfies Condition (L_P).

(7) ⇒ (9). Take $s \in S$ such that $sS \neq S$. It was mentioned that $A(sS) = (1, x)S \cup (1, y)S$ and $(1, x)S \cong S_S \cong (1, y)S$. Since S_S is faithful, $A(sS)$ has a faithful subact. Therefore, $A(sS)$ is faithful and satisfies Condition (L_P) by assumption, which is impossible by Lemma 2.1. Then for any $s \in S, sS = S$ which means that S is a group.

(8) ⇒ (9). Let $s \in S$ be such that $sS \neq S$. The right S-act $A(sS)$ is indecomposable by Lemma 2.1. So $A(sS)$ should satisfy Condition (L_P) and this is impossible. Hence $sS = S$ for any $s \in S$ which shows that S is a group. □

Notice that if $sS \neq S$ for $s \in S$, then $A(sS) = S \coprod^{sS} S = (1, x)S \cup (1, y)S$, and so the above theorem is also valid for finitely generated right S-acts as well as for right S-acts generated by exactly two elements.

Recall that the right S-act A_S is called *torsion free* if, for every right cancellable element $c \in S$ and every $a, b \in A_S$, $ac = bc$ implies $a = b$. The following example shows that Condition (L_P) does not imply torsion freeness.

Example 3.1. Let $S = (\mathbb{N}, .)$ and $K = 2\mathbb{N}$ which is a right ideal of S. Since S/K_S is cyclic, it satisfies Condition (L_P). But S/K_S is not torsion free by [11, III, 8.10]. Indeed, the element $2 \in 2\mathbb{N}$ is right cancellabe and we have $3.2 \in 2\mathbb{N}$ but $3 \notin 2\mathbb{N}$.

Now it is a natural question to ask: When does Condition (L_P) imply torsion freeness?

Theorem 3.2. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are torsion free.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are torsion free.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are torsion free.*
4. *All right cancellable elements are right invertible.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). By assumption any cyclic right S-act satisfies Condition (L_P), is torsion free, and so by [11, IV, 6.1] every right cancellable element is right invertible.

(4) ⇒ (1). If (4) holds, then by [11, IV, 6.1] all right S-acts are torsion free, and so any right S-act satisfying Condition (L_P) is torsion free. □

Recall from [12] that an act A_S satisfies Condition (PWP) if for all $a, a' \in A_S$ and $t \in S$, if $at = a't$, then there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and $ut = vt$. Condition (PWP_E) was defined in [4]. A right S-act A_S satisfies Condition (PWP_E) if whenever $a, a' \in A, s \in S$, and $as = a's$, there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue, a'f = a''vf, es = s = fs$ and $us = vs$.

Theorem 3.3. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (PWP_E).*
2. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (PWP_E).*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (PWP_E).*
4. *All right S-acts satisfying Condition (L_P) are principally weakly flat.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are principally weakly flat.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are principally weakly flat.*
7. *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (5) ⇒ (6) are obvious. By [4, Theorem 2.3] Condition (PWP_E) implies principal weak flatness, and so we have the implications (1) ⇒ (4) and (3) ⇒ (6).

(6) ⇒ (7). By assumption, all cyclic right S-acts satisfy Condition (L_P) and so are principally weakly flat. Then S is regular by [11, IV, 6.6].

(7) ⇒ (1). By [4, Theorem 3.1], all right S-acts satisfy Condition (PWP_E) and consequently all right S-acts satisfying Condition (L_P) satisfy Condition (PWP_E). □

Recall that a right S-act A_S satisfies Condition (P_E) if and only if for $a, a' \in A, u, v \in S$ and $au = a'v$ there exist $a'' \in A, s, t, e = e^2, f = f^2 \in S$ such that $ae = a''se, a'f = a''tf, eu = u, fv = v$ and $su = tv$ (see [9]).

Theorem 3.4. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (P_E).*
2. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (P_E).*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (P_E).*
4. *All right S-acts satisfying Condition (L_P) are weakly flat.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are weakly flat.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are weakly flat.*
7. *S is regular and satisfies Condition:*

   (R): For any elements $s, t \in S$, there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$ ($\rho(s,t)$ is the smallest right congruence on S containing (s,t)).

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. By [9, Theorem 2.3] Condition (P_E) implies weak flatness, and so we have the implications (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7). Since all cyclic right S-acts satisfy Condition (L_P), by assumption all cyclic right S-acts are weakly flat. Then by [11, IV, 7.5] S is regular and satisfies Condition (R).

(7) $\Rightarrow$ (1). By [11, IV, 7.5] all right S-acts are weakly flat. It follows from [9, Theorem 2.5] that all right S-acts satisfy Condition (P_E), and so all right S-acts satisfying Condition (L_P) satisfy Condition (P_E). □

Theorem 3.5. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (P).*
2. *All right S-acts satisfying Condition (L_P), satisfy Condition (EP).*
3. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (P).*
4. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (P).*
5. *S is a group or a group with a zero adjoined.*

Proof. (1) $\Leftrightarrow$ (2). Since $(P) \Leftrightarrow (L_P) \wedge (EP)$, it is clear.

The implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By assumption, all cyclic right S-acts satisfy Condition (L_P), and so satisfy Condition (P). Hence by [16, Theorem 2.1] $S = G^0$ or $S = G$, where G is a group.

(5) $\Rightarrow$ (1). If $S = G$, then by [13, Theorem 3.10] all right S-acts satisfy Condition (P). Therefore, all right S-acts satisfying Condition (L_P) satisfy Condition (P). Suppose that $S = G^0$ and let A_S be a right S-act which satisfies Condition (L_P). We show that A_S satisfies Condition (P). Take $a, a' \in A_S$ and $s, t \in S$ such that $as = a't$. There are three cases as follows:

Case 1: If $s, t \in G$, then we have $a = a'ts^{-1}, a' = a'1$ and $(ts^{-1})s = 1t$.

Case 2: If $t = 0, s \in G$, then $as = a'0$ implies that $(as)s^{-1} = (a'0)s^{-1}$. So $a = a'0, a' = a'1$ and $0s = 10$.

Case 3: If $s = t = 0$, then $a0 = a'0$. Since A_S satisfies Condition (L_P), there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and clearly $u0 = v0$. □

We deduce from the above theorem that whenever $S \neq G^0$ and $S \neq G$ (G is a group), then there exists at least one (finitely generated) right S-act that satisfies Condition (L_P) but does not satisfy Condition (P).

Theorem 3.6. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (PWP).*
2. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (PWP).*
3. *S is regular and $(\forall x, y, t \in S), (\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y)$.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (3). By assumption all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts satisfy Condition (PWP). Hence by [12, Proposition 3.9], the result follows.

(3) $\Rightarrow$ (1). If A_S is a finitely generated right S-act which satisfies Condition (L_P), then by Corollary 2.2, one can write $A_S = \coprod \alpha_i S$, where $1 \leq i \leq n (n \in \mathbb{N})$. On the other hand, by assumption and [12, Proposition 3.9], all cyclic right S-acts satisfy Condition (PWP). So each $\alpha_i S$ and consequently A_S satisfies Condition (PWP). □

Conditions (WP) was defined in [12]. A right S-act A_S satisfies Condition (WP) if and only if, for all elements $s, t \in S$, all homomorphisms $f :_S (Ss \cup St) \to_S S$, and all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'' \in A_S, u, v \in S, s', t' \in \{s, t\}$ such that $f(us') = f(vt'), a \otimes s = a'' \otimes us'$, and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes_S (Ss \cup St)$.

Theorem 3.7. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (WP).*
2. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (WP).*

3. *S is regular and* $(\forall x, y, t \in S), (\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y)$ *and for all* $s, t \in S$ *and all homomorphisms* $f : {}_S(Ss \cup St) \longrightarrow {}_SS$*, there exist* $u, v, z, w \in S$ *such that* $(uf(s) = vf(t) \wedge us\tau s \wedge vt\tau t) \vee (uf(s) = vf(s) \wedge u\tau 1 \wedge z\tau v \wedge zs = wt\tau t) \vee (uf(t) = vf(t) \wedge v\tau 1 \wedge w\tau u \wedge zs = wt\tau s)$*, where* $\tau = \rho(f(s), f(t))$.

Proof. By using of [12, Proposition 3.19], the proof is similar to Theorem 3.6. □

Recall from [5] that a right S-act A_S satisfies Condition (P') if for all $a, a' \in A_S, s, t, z \in S, as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vt$.

Theorem 3.8. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition* (L_P)*, satisfy Condition* (P').

2. *All right S-acts generated by at most two elements satisfying Condition* (L_P)*, satisfy Condition* (P').

3. *S is regular and* $(\forall x, y, z, t, t' \in S), (tz = t'z \Longrightarrow (\exists u, v \in S)(x\rho(xt, yt')u \wedge y\rho(xt, yt')v \wedge ut = vt'))$.

Proof. By using of [5, Theorem 3.3], the proof is similar to Theorem 3.6. □

Recall that a monoid S is called left collapsible if for every $s, s' \in S$, there exists $z \in S$ such that $zs = zs'$. S is called weakly left collapsible if for every $s, s', z \in S$, $sz = s'z$ implies the existence of $u \in S$ such that $us = us'$.

Example 3.2. Let S be a monoid which is not weakly left collapsible. Then Θ_S fails to satisfy Condition (E') but it satisfies Condition (L_P). So Condition (L_P) does not imply Condition (E') (as well as Condition (E), regularity, strong faithfulness , equalizer flatness which are stronger properties).

Theorem 3.9. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition* (L_P)*, satisfy Condition* (E').

2. *All finitely generated right S-acts satisfying Condition* (L_P)*, satisfy Condition* (E').

3. *All right S-acts generated by at most two elements satisfying Condition* (L_P)*, satisfy Condition* (E').

4. $(\forall s, t, z \in S), (sz = tz \Longrightarrow (\exists e \in E(S))(\rho(s, t) = ker\lambda_e))$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (4). By assumption all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts satisfy Condition (E'). Hence by [6, Theorem 2.5], the result follows.

(4) $\Rightarrow$ (1). By [6, Theorem 2.5], all right S-acts satisfy Condition (E'). Hence all right S-acts satisfying Condition (L_P), satisfy Condition (E'). □

Theorem 3.10. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are weakly pullback flat.*
2. *All right S-acts satisfying Condition (L_P), satisfy Conditions (EP) and (E').*
3. *All finitely generated right S-acts satisfying Condition (L_P) are weakly pullback flat.*
4. *All right S-acts generated by at most two elements satisfying Condition (L_P) are weakly pullback flat.*
5. *S is a group or $S = \{0, 1\}$.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4). It is obvious.

(1) $\Leftrightarrow$ (2). It is evident by Corollary 2.5.

(4) $\Rightarrow$ (5). By assumption, all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts are weakly pullback flat. Hence the statement (5) holds by [2, Proposition 25].

(5) $\Rightarrow$ (1). If S is a group, then by [2, Proposition 9] all right S-acts are weakly pullback flat and we are done. Now suppose that $S = \{0, 1\}$. Let A_S be a right S-act that satisfies Condition (L_P). It suffices to show that A_S satisfies Conditions (P) and (E'). Let $a, a' \in A_S$ and $s, t \in S$ be such that $as = a't$. To show that A_S satisfies Condition (P), we consider three cases:

Case 1: $s = t = 1$. Then we have $a = a'$, and so we can write $a = a'.1, a' = a'.1$ and $1.s = 1.t$.

Case 2: $s = 1, t = 0$. Then we have $a = a'.0, a' = a'.1$ and $0.s = 1.t$.

Case 3: $s = t = 0$. Since A_S satisfies Condition (L_P), the equality $a0 = a'0$ implies that there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a = a''v$ and clearly $u0 = v0$.

So A_S satisfies Condition (P) in each case. It remains to show that A_S satisfies Condition (E'). Since $S = \{0, 1\}$, by [13, Theorem 2.4] all cyclic right S-acts satisfy Condition (E), and consequently satisfy Condition (E'). □

Theorem 3.11. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are strongly flat.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are strongly flat.*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are strongly flat.*

4. *All right S-acts satisfying Condition (L_P) are equalizer flat.*

5. *All finitely generated right S-acts satisfying Condition (L_P) are equalizer flat.*

6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are equalizer flat.*

7. *All right S-acts satisfying Condition (L_P), satisfy Condition (E).*

8. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (E).*

9. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (E).*

10. *$S = \{0, 1\}$ or $S = \{1\}$.*

Proof. Implications (1) ⇒ (2) ⇒ (3), (4) ⇒ (5) ⇒ (6) and (7) ⇒ (8) ⇒ (9) are obvious.

Since we have strongly flat ⇒ equalizer flat ⇒ (E), the implications (1) ⇒ (4) ⇒ (7) and (3) ⇒ (6) ⇒ (9) are valid.

(9) ⇒ (10). Since all cyclic right S-acts satisfy Condition (L_P), then by assumption all cyclic right S-acts satisfy Condition (E). By [13, Theorem 2.4] we are done.

(10) ⇒ (1). If $S = \{1\}$, then all right S-acts are strongly flat and the result follows. Suppose that $S = \{0, 1\}$. Then by [13, Theorem 2.4] all cyclic right S-acts satisfy Condition (E) which shows that all right S-acts satisfy Condition (E). By Corollary 2.4 all right S-acts satisfying Condition (L_P), are strongly flat. □

Recall that a monoid S satisfies Condition (K) if, every left collapsible submonoid of S contains a left zero. The monoid S satisfies Condition (A) if every right S-act satisfies the ascending chain condition for cyclic subacts.

Theorem 3.12. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are projective.*

2. *All finitely generated right S-acts satisfying Condition (L_P) are projective.*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are projective.*

4. *$S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since projectivity implies strong flatness, by assumption all right S-acts generated by at most two elements satisfying Condition (L_P), are strongly flat. So by [16, Corollary 2.2], $S = \{1\}$ or $S = \{0, 1\}$.

(4) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are projective. Now suppose that $S = \{0, 1\}$. First, we show that the monoid S satisfies Condition (A). Assume that A_S is a right S-act and $\alpha_1 S \subseteq \alpha_2 S \subseteq ...$ be an ascending chain of cyclic subacts of A_S. Since $S = \{0, 1\}$, we have for all $i \in \mathbb{N}, 1 \leq | \alpha_i S | \leq 2$. So $| \alpha_1 S | \leq | \alpha_2 S | \leq ...$ and for all $i \in \mathbb{N}, \alpha_i S \subseteq \alpha_{i+1} S$, which shows that the above chain terminates. Thus the monoid S satisfies Condition (A). Obviously, the monoid S satisfies Condition (K), and so by [10, Theorem 1.1] and [10, Theorem 2.1], strong flatness implies projectivity. Since all right S-acts satisfying Condition (L_P) are strongly flat by Theorem 3.11, the result follows. □

Theorem 3.13. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are free.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are free.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are free.*
4. *All right S-acts satisfying Condition (L_P) are projective generator.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are projective generator.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are projective generator.*
7. *All right S-acts satisfying Condition (L_P) are generator.*
8. *All finitely generated right S-acts satisfying Condition (L_P) are generator.*
9. *All right S-acts generated by at most two elements satisfying Condition (L_P) are generator.*
10. *All right S-acts satisfying Condition (L_P) are strongly faithful.*
11. *All finitely generated right S-acts satisfying Condition (L_P) are strongly faithful.*
12. *All right S-acts generated by at most two elements satisfying Condition (L_P) are strongly faithful.*
13. *$S = \{1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12), are obvious.

Since *free* $\Rightarrow$ *projective generator* $\Rightarrow$ *generator*, the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (7) and (3) $\Rightarrow$ (6) $\Rightarrow$ (9) are easily obtained.

(12) $\Rightarrow$ (13). Since the one element act Θ_S satisfy Condition (L_P), by assumption Θ_S is strongly faithful. So $S = \{1\}$.

(13) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are free.

(13) $\Rightarrow$ (10). If $S = \{1\}$, then all right S-acts are strongly faithful.

(9) $\Rightarrow$ (13). By assumption Θ_S is generator. So there exists an epimorphism $\pi : \Theta_S \to S_S$ and this implies that $S = \{1\}$. □

An act A_S is said to be divisible, if $Ac = A$ for any left cancellable element $c \in S$.

Theorem 3.14. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are divisible.*
2. *All finitely generated right S-acts satisfying Condition (L_P), are divisible.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are divisible.*
4. *All left cancellable elements of S are left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since S_S is generated by at most two elements and also satisfies Condition (L_P), by [11, III, 2.2] all left cancellable elements of S are left invertible.

(4) $\Rightarrow$ (1). By [11, III, 2.2] every right S-act is divisible. So we are done. □

Theorem 3.15. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are principally weakly injective.*
2. *All finitely generated right S-acts satisfying Condition (L_P), are principally weakly injective.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are principally weakly injective.*
4. *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). If (3) holds, then all principal right ideals of S are principally weakly injective. By [11, IV, 1.6] S is regular.

(4) $\Rightarrow$ (1). It follows from [11, IV, 1.6]. □

Theorem 3.16. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are regular.*
2. *All finitely generated right S-acts satisfying Condition (L_P), are regular.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are regular.*
4. *$S = \{0\}$ or $S = \{0, 1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By assumption, all cyclic right S-acts are regular. By [3, Theorem 1.17] we get $S = \{0\}$ or $S = \{0, 1\}$.

(4) $\Rightarrow$ (1). It follows from [11, IV, 14.4]. □

For fixed elements $u, v \in S$, define a binary relation $P_{u,v}$ on S with $(x, y) \in P_{u,v} \iff ux = vy\,(x, y \in S)$. Also, for $s, t \in S$, let $\mu_{s,t} = ker\lambda_s \vee ker\lambda_t$.

Recall that for any right ideal I of S, Rees congruence ρ_I on S defined by $(x, y) \in \rho_I$ if $x = y$ or $x, y \in I$.

Recall that an act is called cofree if it is isomorphic to the act $X^S = \{f | f$ is a mapping from S to $X\}$ for some nonempty set X, where fs is defined by $fs(t) = f(st), t \in S$, for every $f \in X^S, s \in S$.

Theorem 3.17. *The following statements are equivalent:*

1. *All fg-weakly injective right S-acts satisfy Condition (L_P).*
2. *All weakly injective right S-acts satisfy Condition (L_P).*
3. *All injective right S-acts satisfy Condition (L_P).*
4. *All cofree right S-acts satisfy Condition (L_P).*
5. *For any $s, t \in S$, there exist $u, v \in S$ such that:*
   (i) $P_{u,v} \subseteq P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$.
   (ii) $ker\lambda_u \subseteq \rho_{sS}$.
   (iii) $ker\lambda_v \subseteq \rho_{tS}$.
   (iv) $ker\lambda_{us} \cup ker\lambda_{vt} \subseteq \mu_{s,t}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5). Suppose that $s, t \in S$ and let S_1 and S_2 be two sets such that $|S_1| = |S| = |S_2|$. Assume that $\alpha : S \to S_1$ and $\beta : S \to S_2$ are bijections. Let $X = S/\mu_{s,t} \sqcup S_1 \sqcup S_2$. Define the mappings $f, g : S \longrightarrow X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}}, & \text{if } x \in sS (x = sy) \\ \alpha(x), & \text{if } x \notin sS \end{cases}$$

and

$$g(x) = \begin{cases} [y]_{\mu_{s,t}}, & \text{if } x \in tS (x = ty) \\ \beta(x), & \text{if } x \notin tS \end{cases}$$

If $y_1, y_2 \in S$ are such that $sy_1 = sy_2$, then $(y_1, y_2) \in ker\lambda_s \subseteq \mu_{s,t}$ which implies that $f(sy_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = f(sy_2)$.

Similarly if $ty_1 = ty_2$ then $g(ty_1) = g(ty_2)$. So f and g are well-defined. Also $fs = gt$ by definition. Since X^S satisfies Condition (L_P), there exist $u, v \in S$ and the map $h : S \to X$ such that $g = hv$ and $f = hu$. Now, we show that the conditions (i) to (iv) are true.

(i) let $(l_1, l_2) \in P_{u,v}$, $l_1, l_2 \in S$. Then $ul_1 = vl_2$, and so

$$f(l_1) = hu(l_1) = h(ul_1) = h(vl_2) = hv(l_2) = g(l_2).$$

The equality $f(l_1) = g(l_2)$ implies that there exist $y_1, y_2 \in S$ such that $l_1 = sy_1, l_2 = ty_2, (y_1, y_2) \in \mu_{s,t}$. Hence $(l_1, l_2) \in P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$.

(ii) Assume that the condition (ii) does not hold. Then there exist $p_1, p_2 \in S$ such that $(p_1, p_2) \in ker\lambda_u \setminus \rho_{sS}$. It follows that

$$(\exists p_1, p_2 \in S)(up_1 = up_2 \wedge p_1 \neq p_2 \wedge (p_1 \notin sS \vee p_2 \notin sS)).$$

Hence $f(p_1) = hu(p_1) = h(up_1) = h(up_2) = hu(p_2) = f(p_2)$. Since $p_1 \neq p_2$ and α is injective, $p_1, p_2 \in sS$ which is a contradiction.

(iii) The argument is similar to (ii).

(iv) Suppose that $(l_1, l_2) \in ker\lambda_{us}$, $l_1, l_2 \in S$. So $usl_1 = usl_2$ and we have $f(sl_1) = hu(sl_1) = h(usl_1) = h(usl_2) = hu(sl_2) = f(sl_2)$. According to the definition of f, we have $sl_1 = sl_2$ or $[l_1]_{\mu_{s,t}} = [l_2]_{\mu_{s,t}}$. If $sl_1 = sl_2$ then $(l_1, l_2) \in ker\lambda_s \subseteq \mu_{s,t}$. If $[l_1]_{\mu_{s,t}} = [l_2]_{\mu_{s,t}}$ then $(l_1, l_2) \in \mu_{s,t}$. So $ker\lambda_{us} \subseteq \mu_{s,t}$. Similar argument shows that $ker\lambda_{vt} \subseteq \mu_{s,t}$. Consequently $ker\lambda_{us} \cup ker\lambda_{vt} \subseteq \mu_{s,t}$.

(5) $\Rightarrow$ (1). Suppose that A_S is fg-weakly injective and for $a, a' \in A_S, s, t \in S, as = a't$. By assumption there exist $u, v \in S$ such that the conditions (i) to (iv) hold. Define $\varphi : uS \cup vS \to A_S$ by

$$\varphi(x) = \begin{cases} ap, & \text{if } x \in uS (x = up), \\ a'r, & \text{if } x \in vS (x = vr), \end{cases}$$

for every $x \in uS \cup vS$. To show that φ is well defined, we consider three cases as follows:

Case 1. If there exist $p, r \in S$ such that $up = vr$, then by condition (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{1,s}, (y_1, y_2) \in \mu_{s,t}, (y_2, r) \in P_{t,1}$. Thus $p = sy_1, r = ty_2$ and $(y_1, y_2) \in (ker\lambda_s \vee ker\lambda_t)$. So, there exist $z_1, z_2, ..., z_n \in S$ such that

$$\begin{array}{llll} sy_1 = sz_1 & sz_2 = sz_3 & \dots & sz_{n-1} = sz_n \\ tz_1 = tz_2 & & & tz_n = ty_2. \end{array}$$

Now, we have $ap = asy_1 = asz_1 = a'tz_1 = a'tz_2 = ... = a'tz_n = a'ty_2 = a'r$.

Case 2. Assume that there exist $p_1, p_2 \in S$ such that $up_1 = up_2$. If $p_1 = p_2$, then $ap_1 = ap_2$. If $p_1 \neq p_2$, then by condition (ii), there exist $y_1', y_2' \in S$ such that $p_1 = sy_1'$ and $p_2 = sy_2'$. From $up_1 = up_2$ we have $usy_1' = usy_2'$ and so $(y_1', y_2') \in ker\lambda_{us} \subseteq \mu_{s,t}$. This means that there exist $z_1, z_2, ..., z_n \in S$ such that

$$\begin{aligned} sy_1' = sz_1 \qquad sz_2 = sz_3 \qquad \dots \qquad sz_{n-1} = sz_n \\ tz_1 = tz_2 \qquad\qquad\qquad\qquad tz_n = ty_2'. \end{aligned}$$

Hence $ap_1 = asy_1' = asz_1 = a'tz_1 = ... = a'tz_n = a'ty_2' = asy_2' = ap_2$.

Case 3. If there exist $r_1, r_2 \in S$ such that $vr_1 = vr_2$, then by a similar argument we get $a'r_1 = a'r_2$. Hence φ is well defined. Clearly φ is a homomorphism. Since A_S is fg-weakly injective, there exists a homomorphism $\psi : S_S \to A_S$ which extends φ. Put $a'' = \psi(1)$. Then $a = \varphi(u) = \psi(u) = \psi(1)u = a''u$ and $a' = \varphi(v) = \psi(v) = \psi(1)v = a''v$. Thus A_S satisfies Condition (L_P). □

Corollary 3.1. *If S is a commutative monoid, then all cofree S-acts satisfy Condition (L_P) if and only if S is a group.*

Proof. Let $s \in S$. By Theorem 3.17, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$ ($\mu_{s,s} = ker\lambda_s \vee ker\lambda_s = ker\lambda_s$). If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which implies that $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Commutativity of S implies that $uv = vu$ which yields $(v, u) \in P_{u,v}$ and so $u = v$. Hence $ker\lambda_u = P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$. Since $u.1 = u.1$, there exist $y_1, y_2 \in S$ such that $(1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, 1) \in P_{s,1}$. Hence $1 = sy_1 = sy_2$ and S is a group.

Conversely, suppose that S is a group. By Theorem 3.1, all right S-acts satisfy Condition (L_P). So all cofree right S-acts satisfy Condition (L_P). □

Corollary 3.2. *If S is a finite monoid, then all cofree S-acts satisfy Condition (L_P) if and only if S is a group.*

Proof. By Theorem 3.17, for any $s \in S$ there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{sS}$ and $ker\lambda_{us} \cup ker\lambda_{vs} \subseteq ker\lambda_s$.($\mu_{s,s} = ker\lambda_s \vee ker\lambda_s = ker\lambda_s$.) If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which means $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Suppose that $l_1, l_2 \in S$ are such that $ul_1 = ul_2$ and $l_1 \neq l_2$. Then there exist $y_1, y_2 \in S, l_1 = sy_1, l_2 = sy_2$ which implies that $usy_1 = usy_2$. The last equality implies that $(y_1, y_2) \in ker\lambda_{us} \subseteq ker\lambda_s$. So $sy_1 = sy_2$ that is, $l_1 = l_2$ which is a contradiction. Hence u is left cancellable. Let $S = \{1, x_1, x_2, ..., x_n\}$ (note that the elements of S are distinct). It is clear that $uS = \{u, ux_1, ux_2, ..., ux_n\} = S$. So $v \in uS$. If there exists $i \leq n$ such that $ux_i = v$, then $(x_i, 1) \in P_{u,v} \subseteq \Delta_S$ which implies that $x_i = 1$, a contradiction. Hence $v = u$ and by a similar argument to Corollary 3.1, we get that s has an inverse. Thus, S is a group.

The converse has been proved in Theorem 3.1. □

Corollary 3.3. *If S is an idempotent monoid, then all cofree S-acts satisfy Condition (L_P) if and only if $S = \{1\}$.*

Proof. Suppose that all cofree S-acts satisfy Condition (L_P). We claim that $S = \{1\}$. Assume that $S \neq \{1\}$. So there exists $e \in S \setminus \{1\}$. By Theorem 3.17, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,e} \circ ker\lambda_e \circ P_{e,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{eS}$. Obviously, $(u, 1) \in ker\lambda_u \subseteq \rho_{eS}$, and so $u = 1$ or there exist $y_1, y_2 \in S$ such that $u = ey_1$ and $1 = ey_2$. Since $e \neq 1$, we get that $u = 1$ and similarly $v = 1$. By a similar argument to the Corollary 3.1, e has a right inverse. Hence $e = 1$, which is a contradiction. Thus $S = \{1\}$ and we are done.

The converse is a part of the Theorem 3.1. □

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Magnetic effect of non-commutativity

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Abstract. In this paper we argue that 2D crystals emit electromagnetic radiation and exhibit extra electromagnetic effects, by virtue of the noncommutative nature of the 2D crystal space.
Keywords: electromagnetism, noncommutativity, extra effects.

1. Introduction

In an earlier communication [1] we had shown the effects of non–commutativity on Maxwell's equations starting from the usual relation

$$E^2 = p^2c^2 + m^2c^4. \tag{1}$$

Now, if we take natural units i.e.$\hbar = c = 1$ we can rewrite this equation as

$$E^2 = p^2 + m^2. \tag{2}$$

2. Non-commutative Maxwell's equation

Maxwell's equation in covariant format can be written as

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c} j_\nu. \tag{3}$$

This particular equation is an abridge form with $\mu = 1, 2, 3, 4$ and $\nu = 1, 2, 3, 4$. More fully

$$\partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{4\pi}{c} j_\nu, \tag{4}$$

$$\partial_\mu \partial_\mu A_\nu - \partial_\mu \partial_\nu A_\mu = \frac{4\pi}{c} j_\nu. \tag{5}$$

Let us consider a transformation [4, 6] in which

$$p_\mu p^\mu = D_{\mu\mu} - \Gamma^\mu_{\lambda\lambda}\partial_\mu, \tag{6}$$

$$p_\mu p^\nu = D_{\mu\nu} - \Gamma^\mu_{\lambda\lambda}\partial_\nu, \tag{7}$$

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where $p_\mu \equiv \partial_\mu$ and which is occurring due to the fact that space-time is non-commutative [4]. As we can see, the last term on the right side is zero then we return to continuous space-time. As shown in detail elsewhere the last term occurs below Compton wavelength when negative energies begins to dominate [4]. Now, we substitute (6) and (7) in equation (5) and try to analyse the physical behavior

$$(D_{\mu\mu} - \Gamma^{\mu}_{\lambda\lambda}\partial_\mu)A_\nu - (D_{\mu\nu} - \Gamma^{\mu}_{\lambda\lambda}\partial_\nu)A_\mu = \frac{4\pi}{c}j_\nu, \tag{8}$$

$$(D_{\mu\mu}A_\nu - D_{\mu\nu}A_\mu) = \frac{4\pi}{c}j_\nu + (\Gamma^{\mu}_{\lambda\lambda}\partial_\mu A_\nu - \Gamma^{\mu}_{\lambda\lambda}\partial_\nu A_\mu), \tag{9}$$

$$(D_{\mu\mu}A_\nu - D_{\mu\nu}A_\mu) = \frac{4\pi}{c}j_\nu + \Gamma^{\mu}_{\lambda\lambda}(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{10}$$

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c}j_\nu + \Gamma^{\mu}_{\lambda\lambda}F_{\mu\nu}, \tag{11}$$

or this can be written as

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c}j_\nu + \partial_\lambda \epsilon F_{\lambda\nu}, \tag{12}$$

where ε in (12) is a dimensionless number which is equal to one for our non-commutative case (6) and equals zero for the usual commutative case and has been introduced for clarity. In the usual commutative space and time $\varepsilon = 0$ and we get back the usual covariant Maxwell's equations given in (3). So now there is an additional term in the Maxwell equations. Now let us try to derive these equations for two dimensions. We will see that this is a non-trivial case. We get

$$\partial^1 F_{12} = \frac{4\pi}{c}j_2 + \partial_4 \varepsilon F_{42}, \tag{13}$$

$$\partial^2 F_{21} = \frac{4\pi}{c}j_1 + \partial_4 \varepsilon F_{41}, \tag{14}$$

$$\partial^4 F_{42} = \frac{4\pi}{c}j_2 + \partial_1 \varepsilon F_{12}, \tag{15}$$

$$\partial^4 F_{41} = \frac{4\pi}{c}j_1 + \partial_2 \varepsilon F_{21} \tag{16}$$

$$\partial^2 F_{24} = \frac{4\pi}{c}j_4 + \partial_1 \varepsilon F_{14}, \tag{17}$$

$$\partial^1 F_{14} = \frac{4\pi}{c}j_4 + \partial_2 \varepsilon F_{24}. \tag{18}$$

Now, from Covariant electrodynamics we know that $F_{\mu\nu}$ is the electromagnetic tensor and is given by [7] $\nu \rightarrow$

$$F_{\mu\nu} = \mu \downarrow \begin{pmatrix} 0 & -cB_z & cB_y & -E_x \\ cB_z & 0 & -cB_x & -E_y \\ -cB_y & cB_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}. \tag{19}$$

So, this electromagnetic tensor will give Maxwell's equations. We can see that the electromagnetic tensor is asymmetric. Now from the different components from (18)-(16) given by using equation (19) Taking $c = 1$ we get

$$-\frac{\partial B_z}{\partial x} = 4\pi j_y + \epsilon\frac{\partial E_y}{\partial t}, \tag{20}$$

$$\frac{\partial B_z}{\partial y} = 4\pi j_x + \epsilon\frac{\partial E_x}{\partial t}, \tag{21}$$

$$\frac{\partial E_y}{\partial t} = 4\pi j_y - \epsilon\frac{\partial B_z}{\partial x}, \tag{22}$$

$$\frac{\partial E_x}{\partial t} = 4\pi j_x + \epsilon\frac{\partial B_z}{\partial y}, \tag{23}$$

$$-\frac{\partial E_y}{\partial y} = 4\pi\frac{\partial \rho}{\partial t} - \epsilon\frac{\partial E_x}{\partial x}, \tag{24}$$

$$-\frac{\partial E_x}{\partial x} = 4\pi\frac{\partial \rho}{\partial t} - \epsilon\frac{\partial E_y}{\partial y}, \tag{25}$$

where E is for the extra term.

3. Results and discussions

So, from equations (20) to (25) we can see that there is an extra term in the Maxwell equations and if $\varepsilon = 0$ then we get the usual Maxwell's equations.

Thus, we can see that introduction of non-commutative space-time leads to extra effects in Maxwell's equations (20) to (25) which indicate an extra electromagnetic field.

In commutative space-time $\varepsilon = 0$ and the extra magnetic effects will disappear. To get more insight into the extra magnetic effect we observe an alternate way of expressing the non-commutativity above. This is with a minimum space-time extension which also leads to non-commutative geometry.

We have now a relation such that

$$[x, y] = l^2\Theta, \tag{26}$$

where x and y are coordinates and l is the minimum extension and Θ are matrices. Because of non-commutative geometry (26) it is known that there is the generation of a magnetic field which was shown independently by the author and Saito [8, 9], with the relation

$$Bl^2 = hc/e. \tag{27}$$

In other words, the non-commutativity of space-time generates the extra electromagnetic effects.

We specialize to the case of 2D crystals where also we have this situation. The result is that we can expect some additional radiation for example. This effect would be stronger than in the 3D case. We can crossover from 3D to 2D

by using the transformation [10, 11] which essentially replaces the velocity of light c with the Fermi velocity in crystals ([10]). (The remarkable thing about this theory is we are able to deduce theoretically the entire infinite sequence of Von Klitzing levels.)

So, we have the terms like the last transformation in the RHs of equations (20)-(25). These point to effects like extra radiation and the like.

4. Two dimensional graphene

The best example of the foregoing is the recent understanding of space-time in Graphene or any crystal [12] which approximates the above 2D scenario. The authors concluded that electron spin is attributed to the discreteness of space-time which resembles a chess board and this type of a discrete partition of space-time will lead to the generation of a magnetic field which may even affect the intrinsic behavior of the electron.

Here space-time is discrete and during the hopping of electrons there is change in spin direction which shows that the space is not smooth and this give rise to some sort of a magnetic field which can cause the change of spin in the electron during hopping. So we can see that above behavior of electron spin can be analogous to the above with appearance of a magnetic field in non-commutative space-time. The extra terms in the equations (20)-(25) correspond to additional electromagnetic effects including radiation.

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Generalizations of Simpson-type inequalities for relative semi-(h, α, m)-logarithmically convex mappings

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Abstract. By discovering an integral equality defined on relative convex set, we prove some new Simpson-type inequalities for mappings which have absolute values of the first derivatives which are relative semi-(h, α, m)-logarithmically convex. Some special cases are also considered.

Keywords: relative convex set, relative semi-(h, α, m)-logarithmically convex, Simpson's inequality.

1. Introduction

If $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on $[r_1, r_2]$ with $r_1, r_2 \in \mathcal{I}$ and $r_1 < r_2$, then one has

$$f\Big(\frac{r_1 + r_2}{2}\Big) \leq \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} f(x)\mathrm{d}x \leq \frac{f(r_1) + f(r_2)}{2}. \tag{1.1}$$

The following inequality is named in the literature as the Simpson's integral inequality:

$$\begin{aligned} &\left| \frac{1}{6}\left[f(r_1) + 4f\left(\frac{r_1 + r_2}{2}\right) + f(r_2)\right] - \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} f(t)\mathrm{d}t \right| \\ &\qquad \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (r_2 - r_1)^4, \end{aligned} \tag{1.2}$$

where $f : [r_1, r_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (r_1, r_2) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(r_1,r_2)}\left|f^{(4)}(t)\right| < \infty$.

For recent results with respect to (1.1) and (1.2), for example, see [1, 5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29] and the references mentioned in these papers.

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Let us consider a relative convex set M_φ. A set $M_\varphi \subseteq \mathbb{R}^n$ is named a relative convex set with respect to the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, if $t\varphi(x)+(1-t)\varphi(y) \in M_\varphi$ holds for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$. A function f is called relative convex on a relative convex set M_φ, if there exists a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq tf\big(\varphi(x)\big) + (1-t)f\big(\varphi(y)\big)$$

grips for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

A function $f : M_\varphi \to \mathbb{R}$ is called relative semi-convex on a relative convex set, if there exists a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

Let us now mention some definitions involving logarithmic convexity.

Definition 1.1 ([4]). *A function $f : M_\varphi \to \mathbb{R}^+$ is called relative semi-logarithmic convex on a relative convex set M_φ, if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq f(x)^t f(y)^{(1-t)}, \tag{1.3}$$

holds for every $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

Definition 1.2 ([3]). *Let $h : K \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative mapping, $h \neq 0$. A nonnegative function $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is named $(h-(\alpha,m))$-logarithmically convex, if the following inequality:*

$$f\big(tx + m(1-t)y\big) \leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{m\left(1-h^\alpha(t)\right)} \tag{1.4}$$

grips for all $x, y \in \mathcal{I}$ and $t \in [0,1]$ with some fixed $(\alpha, m) \in (0,1] \times (0,1]$.

Definition 1.3 ([2]). *A function $f : [0,b] \longrightarrow (0,\infty)$ with $b > 0$ is called (α, m)-logarithmically convex if the inequality*

$$f\big(tx + m(1-t)y\big) \leq \big[f(x)\big]^{t^\alpha} \big[f(y)\big]^{m(1-t^\alpha)} \tag{1.5}$$

holds for all $x, y \in [0,b]$, $t \in [0,1]$ with some fixed $(\alpha, m) \in (0,1] \times (0,1]$.

For $(\alpha, m) \in \{(\alpha, 1), (1, m), (1, 1)\}$, one obtains the following classes of functions: α-logarithmically convex, m-logarithmically convex and logarithmically convex, respectively. Some recent interesting and important results related to logarithmically convexity can be found in [2, 15, 16, 28].

Very recently, Du et al. in [6] established the following lemma and developed some Simpson type inequalities via extended (s, m)-convex mappings.

Lemma 1.1. *Let $f : \mathcal{I} \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable on $\mathcal{I}^0$ (interior of $\mathcal{I}$), where $a, b \in \mathcal{I}^0$ such that $0 < a < b$. For some fixed $m \in (0, 1]$, if $f' \in L^1([a, b])$ along with $k, t \in \mathbb{R}$, then for each $x \in [ma, b]$ the following identity holds:*

$$\begin{aligned} &tf(ma) + (1-k)f(b) + (k-t)f\left(\frac{b+ma}{2}\right) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \\ &= (b-ma)\bigg[\int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda \\ &\quad + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda\bigg]. \end{aligned} \tag{1.6}$$

Also, Zhou et al. [30] in 2017, introduced the concept of the relative semi-(α, m)-logarithmically convex functions and derived some Simpson-like type inequalities for such mappings by means of the following identity.

Lemma 1.2. *Let $f : \mathcal{I}^0 \subseteq \mathbb{R} \to (0, \infty)$ be a differentiable mapping on $\mathcal{I}^0$ and let $\varphi : \mathcal{I} \subseteq \mathbb{R} \to (0, \infty)$ be a continuous mapping along with $\varphi(a) < \varphi(b)$. If $f' \in L^1\big([\varphi(a), \varphi(b)]\big)$, then we have*

$$\begin{aligned} &\frac{1}{8}\bigg[f\big(\varphi(a)\big) + 6f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + f\big(\varphi(b)\big)\bigg] \\ &\quad - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x) \\ &= \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)} \\ &\quad \times \bigg\{\sqrt{\varphi(a)}\int_0^1 \Big(\frac{3}{4} - t\Big)\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\Big)\mathrm{d}t \\ &\quad + \sqrt{\varphi(b)}\int_0^1 \Big(\frac{1}{4} - t\Big)\Big[\frac{\varphi(a)}{\varphi(b)}\Big]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\Big)\mathrm{d}t\bigg\}. \end{aligned} \tag{1.7}$$

Motivated by the inspiring idea in [3, 6] and [30], we introduce the class of relative semi-(h, α, m)-logarithmically convex functions defined on relative convex set, and we explore new Simpson-type inequalities for such mappings. Some interesting special cases of our principal results are also investigated.

2. New definitions and a lemma

For convenience, we next take the symbol $M_\varphi = [\varphi(a), \varphi(b)]$ with $0 < \varphi(a) < \varphi(b) < \infty$ be a relative convex set.

Definition 2.1. *Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. A function $f : M_\varphi \to \mathbb{R}^+$ is named relative semi-(h, α, m)-logarithmically convex on a relative convex set M_φ, if there exists a mapping $\varphi : \mathbb{R} \to \mathbb{R}$ such that,*

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \leq \big[f(x)\big]^{h^\alpha(t)}\big[f(y)\big]^{m\left(1-h^\alpha(t)\right)}, \tag{2.1}$$

holds for every $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$*, along with some fixed* $(\alpha, m) \in (0,1] \times (0,1]$.

Clearly, when taking $h(t) = t$ in Definition 2.1, f becomes the relative semi-(α, m)-logarithmically convex function on M_φ, and when taking $h(t) = t^s$ in Definition 2.1, f becomes the relative semi-(s, α, m)-logarithmically convex function on M_φ.

Definition 2.2. *Let* $h : [0,1] \to \mathbb{R}$ *be a nonnegative function,* $h \neq 0$*. A function* $f : M_\varphi \to \mathbb{R}^+$ *is called relative semi-*(h, α)*-logarithmically convex on a relative convex set* M_φ*, if there exists a function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that,*

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{\left(1-h^\alpha(t)\right)}, \tag{2.2}$$

holds for all $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$*, together with some fixed* $\alpha \in (0,1]$.

If we take $t = \frac{1}{2}$, then the relative semi-(h, α)-logarithmically convex function reduces to

$$f\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \leq \big[f(x)\big]^{h^\alpha(\frac{1}{2})} \big[f(y)\big]^{\left(1-h^\alpha(\frac{1}{2})\right)}, \tag{2.3}$$

which is called Jensen relative semi-(h, α)-logarithmically convex function.

From Definition 2.2, it follows that

$$\begin{aligned} f\big(t\varphi(x) + (1-t)\varphi(y)\big) &\leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{\left(1-h^\alpha(t)\right)} \\ &\leq h^\alpha(t) f(x) + \big(1 - h^\alpha(t)\big) f(y), \end{aligned} \tag{2.4}$$

which reveals that every relative semi-(h, α)-logarithmically convex function is a relative semi-(h, α)-convex mapping, but the converse is not true.

Definition 2.3. *Let* $h : [0,1] \to \mathbb{R}$ *be a nonnegative function,* $h \neq 0$*. A function* $f : M_\varphi \to \mathbb{R}^+$ *is named relative semi-*(h, m)*-logarithmically convex on a relative convex set* M_φ*, if there exists a function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that,*

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \leq \big[f(x)\big]^{h(t)} \big[f(y)\big]^{m\left(1-h(t)\right)}, \tag{2.5}$$

holds for all $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$*, along with some fixed* $m \in (0,1]$.

Clearly, when taking $h(t) = t$ in Definition 2.3, f becomes the relative semi-m-logarithmically convex function on M_φ. Here, we provide an example to show that a function can be relative semi-m-logarithmically convex on $\mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} 1, & x > 0, \\ e^{-|x|}, & x \leq 0, \end{cases}$$

and the function $\varphi : \mathbb{R} \to \mathbb{R}$, where $\varphi(x) = -|x|$, for some fixed $m \in (0, 1]$.

However, it is neither convex nor logarithmically convex on $\mathbb{R}$. In fact, if we take $x = 1$, $y = -1$ and $t = \frac{1}{2}$, then we have

$$\begin{aligned} f(tx + (1-t)y) &= f(0) = 1 > tf(x) + (1-t)f(y) \\ &= \frac{1}{2}f(1) + \frac{1}{2}f(-1) = \frac{1}{2}\Big(1 + \frac{1}{e}\Big) \end{aligned}$$

and

$$f(tx + (1-t)y) = f(0) = 1 > [f(x)]^t[f(y)]^{1-t} = [f(1)]^{\frac{1}{2}}[f(-1)]^{\frac{1}{2}} = \sqrt{\frac{1}{e}}.$$

Remark 2.1. In Definition 2.1, 2.2 and 2.3, if we take φ is an identity mapping, then we have the definition of (h, α, m)-, (h, α)- and (h, m)-logarithmically convex mappings, respectively.

To demonstrate new Simpson-type inequalities via the relative semi-(h, α, m)-logarithmically convexity, we need the following lemma.

Lemma 2.1. *Let $f : \mathcal{I}^0 \subseteq \mathbb{R} \to (0, \infty)$ be a differentiable mapping on $\mathcal{I}^0$ and let $\varphi : \mathcal{I} \subseteq \mathbb{R} \to (0, \infty)$ be a continuous mapping with $\varphi(a) < \varphi(b)$. If $f' \in L^1\big([\varphi(a), \varphi(b)]\big)$ and $\lambda, k \in \mathbb{R}$, then for each $\varphi(x) \in [\varphi(a), \varphi(b)]$ we have*

$$\begin{aligned} &\triangle_{f,\varphi}(\lambda, k; a, b) \\ &= \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\bigg\{\int_0^{\frac{1}{2}}(t-\lambda)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t \\ &\quad + \int_{\frac{1}{2}}^1 (t-k)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t\bigg\}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &\triangle_{f,\varphi}(\lambda, k; a, b) \\ &:= \bigg[\lambda f\big(\varphi(a)\big) + (k-\lambda)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + (1-k)f\big(\varphi(b)\big)\bigg] \\ &\quad - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x). \end{aligned} \tag{2.7}$$

Proof. Integrating by parts, we have

$$\begin{aligned} &\Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\int_0^{\frac{1}{2}}(t-\lambda)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t \\ &= (t-\lambda)f\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t \\ &= \lambda f(\varphi(a)) + \Big(\frac{1}{2} - \lambda\Big)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) \\ &\quad - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\sqrt{\varphi(a)\varphi(b)}}\frac{f(\varphi(x))}{\varphi(x)}\mathrm{d}\varphi(x). \end{aligned} \tag{2.8}$$

Similarly, we have

$$\begin{aligned}
&\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\int_{\frac{1}{2}}^{1}(t-k)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^{t}\Big)\mathrm{d}t\\
&=(1-k)f(\varphi(b))+\Big(k-\frac{1}{2}\Big)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)\\
&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\sqrt{\varphi(a)\varphi(b)}}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}\mathrm{d}\varphi(x).
\end{aligned}\tag{2.9}$$

Lemma 2.1 is thus proved. □

Example 2.1. If we take $\varphi(x)=e^x$ and $f(x)=x$, then all the assumptions in Lemma 2.1 are satisfied. Clearly, the first part of the right-sided term of the equation (2.8) is

$$\begin{aligned}
&\Big(b-a\Big)e^a\int_0^{\frac{1}{2}}(t-\lambda)\Big[\frac{e^b}{e^a}\Big]^t f'\Big([e^a]^{1-t}[e^b]^t\Big)\mathrm{d}t\\
&=(t-\lambda)\Big([e^a]^{1-t}[e^b]^t\Big)\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}[e^a]^{1-t}[e^b]^t\mathrm{d}t\\
&=\lambda e^a+\Big(\frac{1}{2}-\lambda\Big)\Big(\sqrt{e^ae^b}\Big)-\frac{1}{b-a}\int_{e^a}^{\sqrt{e^ae^b}}\mathrm{d}e^x,
\end{aligned}$$

the second part of the right-sided term of the equation (2.9) is

$$\begin{aligned}
&(b-a)e^a\int_{\frac{1}{2}}^{1}(t-k)\Big[\frac{e^b}{e^a}\Big]^t f'\Big([e^a]^{1-t}[e^b]^t\Big)\mathrm{d}t\\
&=(1-k)e^b+\Big(k-\frac{1}{2}\Big)\Big(\sqrt{e^ae^b}\Big)-\frac{1}{b-a}\int_{\sqrt{e^ae^b}}^{e^b}\mathrm{d}e^x,
\end{aligned}$$

and the left-side term of the equation (2.7) is

$$\begin{aligned}
&\Big[\lambda f(\varphi(a))+(k-\lambda)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)+(1-k)f(\varphi(b))\Big]\\
&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\
&=\Big[\lambda e^a+(k-\lambda)\Big(\sqrt{e^ae^b}\Big)+(1-k)e^b\Big]-\frac{1}{b-a}\int_{e^a}^{e^b}\mathrm{d}e^x.
\end{aligned}$$

Corollary 2.1. *In Lemma 2.1, if we take φ is an identity mapping, $\lambda=\frac{1}{4}$ and $k=\frac{3}{4}$, then the equation (2.7) becomes to the following identity*

$$\frac{1}{4}\Big[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x$$

$$= \Big(\ln b - \ln a\Big)\Bigg\{\int_0^{\frac{1}{2}} \Big(t - \frac{1}{4}\Big)a^{1-t}b^t f'\Big(a^{1-t}b^t\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1 \Big(t - \frac{3}{4}\Big)a^{1-t}b^t f'\Big(a^{1-t}b^t\Big)\mathrm{d}t\Bigg\},$$

which is a new form of Lemma 2.1.

Corollary 2.2. *In Lemma 2.1, if we take φ is an identity mapping, $\lambda = \frac{1}{6}$ and $k = \frac{5}{6}$, then the equation* (2.7) *becomes to the following identity*

$$\begin{aligned}&\frac{1}{6}\Big[f(a) + 4f\Big(\sqrt{ab}\Big) + f(b)\Big] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x\\ &= \Big(\ln b - \ln a\Big)\Bigg\{\int_0^{\frac{1}{2}} \Big(t - \frac{1}{6}\Big)a^{1-t}b^t f'\Big(a^{1-t}b^t\Big)\mathrm{d}t\\ &\quad + \int_{\frac{1}{2}}^1 \Big(t - \frac{5}{6}\Big)a^{1-t}b^t f'\Big(a^{1-t}b^t\Big)\mathrm{d}t\Bigg\},\end{aligned}$$

which is another new form of Lemma 2.1.

3. Main results

Throughout this paper, we shall present our main results by assuming $\varphi : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ is a continuous positive homogeneous mapping with $\varphi(a) < \varphi(b)$, and $f : \mathcal{I}^0 \subseteq [0, \infty) \to (0, \infty)$ is a differentiable function on $\mathcal{I}^0$ such that $f' \in L^1\big([\varphi(a), \varphi(b)]\big)$.

We start the following results by using Lemma 2.1.

Theorem 3.1. *If $|f'(\varphi(x))|$ is increasing and relative semi-(h, α, m)-logarithmically convex function on $\big[0, \varphi\big(\frac{b}{m}\big)\big]$ with $(\alpha, m) \in (0, 1] \times (0, 1]$, $0 \le \lambda \le \frac{1}{2} \le k \le 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\triangle_{f,\varphi}(\lambda, k; a, b)\Big|\\ &\le \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\quad\times\Bigg\{\int_0^{\frac{1}{2}} |t - \lambda|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\\ &\quad + \int_{\frac{1}{2}}^1 |t - k|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\Bigg\}.\end{aligned} \tag{3.1}$$

Proof. By Lemma 2.1, Young inequality, monotonically increasing and relative semi-(h,α,m)-logarithmically convexity of $|f'(\varphi(x))|$, we get

$$
\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\Bigg\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|\mathrm{d}t\\
&\quad+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|\mathrm{d}t\Bigg\}\\
&\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\Bigg\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\big((1-t)\varphi(a)+t\varphi(b)\big)\Big|\mathrm{d}t\\
&\quad+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\big((1-t)\varphi(a)+t\varphi(b)\big)\Big|\mathrm{d}t\Bigg\}\\
&\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\Bigg\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'(a)\Big|^{h^\alpha(1-t)}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m\left(1-h^\alpha(1-t)\right)}\mathrm{d}t\\
&\quad+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'(a)\Big|^{h^\alpha(1-t)}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m\left(1-h^\alpha(1-t)\right)}\mathrm{d}t\Bigg\}\\
&= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\
&\quad\times\Bigg\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\\
&\quad+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\Bigg\}.
\end{aligned}
\tag{3.2}
$$

The proof is completed. □

If $|f'(\varphi(x))|^q$ is relative semi-(h,α,m)-logarithmically convex, then we deduce the following result.

Theorem 3.2. *If $|f'(\varphi(x))|^q$ for $q>1$ is increasing and relative semi-(h,α,m)-logarithmically convex function on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac12\le k\le1$, then the following inequality holds:*

$$
\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\
&\quad\times\Bigg\{\vartheta_1^{1-\frac1q}\Big(\lambda,\varphi(a),\varphi(b)\Big)\Bigg(\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{qh^\alpha(1-t)}\mathrm{d}t\Bigg)^{\frac1q}
\end{aligned}
\tag{3.3}
$$

$$+\vartheta_2^{1-\frac{1}{q}}\big(k,\varphi(a),\varphi(b)\big)\bigg(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\bigg]^{qh^\alpha(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\bigg\},$$

where

$$\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big):=\frac{\big(\frac{1}{2}-\lambda\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\lambda}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\lambda}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-1}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^2}\tag{3.4}$$

and

$$\begin{aligned}\vartheta_2\big(k,\varphi(a),\varphi(b)\big):=&\frac{\big(\frac{1}{2}-k\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}+(1-k)\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\ln\varphi(b)-\ln\varphi(a)}\\&+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{k}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^2}.\end{aligned}\tag{3.5}$$

Proof. Continuing from inequality (3.2) in the proof of Theorem 3.1, using the power mean inequality, we get

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\le&\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\&\times\bigg\{\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\times\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\bigg]^{qh^\alpha(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\\&+\bigg(\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\times\bigg(\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\bigg]^{qh^\alpha(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\bigg\}.\end{aligned}\tag{3.6}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, we have

$$\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t=\frac{\big(\frac{1}{2}-\lambda\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\lambda}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\lambda}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-1}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^2}\tag{3.7}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t=&\frac{\big(\frac{1}{2}-k\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}+(1-k)\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\ln\varphi(b)-\ln\varphi(a)}\\&+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{k}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^2}.\end{aligned}\tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we obtain the desired inequality (3.3). The proof is completed. □

A different method leads to the following result.

Theorem 3.3. *If $|f'(\varphi(x))|^q$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ is increasing and relative semi-(h,α,m)-logarithmically convex function on $\left[0,\varphi\left(\frac{b}{m}\right)\right]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, then the following inequality holds:*

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\chi_1^{\frac{1}{p}}(\lambda,p)\left(\int_0^{\frac{1}{2}}\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\left.+\chi_2^{\frac{1}{p}}(k,p)\left(\int_{\frac{1}{2}}^1\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{3.9}$$

where

$$\chi_1(\lambda,p):=\frac{\lambda^{p+1}+\left(\frac{1}{2}-\lambda\right)^{p+1}}{p+1}\tag{3.10}$$

and

$$\chi_2(k,p):=\frac{(1-k)^{p+1}+\left(k-\frac{1}{2}\right)^{p+1}}{p+1}.\tag{3.11}$$

Proof. Continuing from inequality (3.2) in the proof of Theorem 3.1, using Hölder's inequality, we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\left(\int_0^{\frac{1}{2}}\big|t-\lambda\big|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{1}{2}}^1\big|t-k\big|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.12}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, we have

$$\int_0^{\frac{1}{2}}\big|t-\lambda\big|^p\mathrm{d}t=\frac{\lambda^{p+1}+\left(\frac{1}{2}-\lambda\right)^{p+1}}{p+1}$$

and

$$\int_{\frac{1}{2}}^1\big|t-k\big|^p\mathrm{d}t=\frac{(1-k)^{p+1}+\left(k-\frac{1}{2}\right)^{p+1}}{p+1}.$$

The proof is completed. □

Next, we obtain new results related to the relative semi-(s,α,m)-logarithmically convexity.

Theorem 3.4. *If $|f'(\varphi(x))|$ is increasing and relative semi-(s,α,m)-logarithmically convex function in the first sense on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $\lambda+k=1$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, then the following inequality holds:*

$$\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M(\alpha,s,k,\lambda), \tag{3.13}$$

where

$$\eta_1=\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}, \tag{3.14}$$

$$\eta_2=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha s}}\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha s}}\eta_1 \tag{3.15}$$

and

$$M(\alpha,s,k,\lambda)=\begin{cases}\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big), & \eta_1=1,\\ \eta_1^{\alpha s}F_1(\eta_2,\alpha,s,k,\lambda), & 0<\eta_1<1,\\ \eta_1F_1(\eta_2,\alpha,s,k,\lambda), & \eta_1>1,\end{cases}$$

with

$$\begin{aligned}F_1(\eta_2,\alpha,s,k,\lambda):&=\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_2^{-\alpha st}\mathrm{d}t+\int_{\frac{1}{2}}^1\big|t-k\big|\eta_2^{-\alpha st}\mathrm{d}t\\&=\frac{(1-k)\eta_2^{-\alpha s}-\lambda}{-\alpha s\ln\eta_2}+\frac{2\eta_2^{-\alpha sk}+2\eta_2^{-\alpha s\lambda}-2\eta_2^{-\frac{\alpha s}{2}}-\eta_2^{-\alpha s}-1}{\big(\alpha s\ln\eta_2\big)^2}.\end{aligned}$$

Proof. For $h(t)=t^s$ in the inequality (3.2), we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\&\times\Bigg\{\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{(1-t)^{\alpha s}}\mathrm{d}t\\&+\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{(1-t)^{\alpha s}}\mathrm{d}t\Bigg\}.\end{aligned} \tag{3.16}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, let $\eta_1=\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}$ in (3.16), we have three cases as follows.

Case 1: For $\eta_1 = 1$, the inequality (3.16) becomes the form

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t+\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\Big\{\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big)+\vartheta_2\big(k,\varphi(a),\varphi(b)\big)\Big\}.\end{aligned}$$

Here, ϑ_1 and ϑ_2 are defined by (3.4) and (3.5), respectively.
Case 2: For $0<\eta_1<1$, we have $\eta_1^{(1-t)^{\alpha s}} \le \eta_1^{\alpha s(1-t)}$. Thus the inequality (3.16) becomes

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{\alpha s(1-t)}\mathrm{d}t+\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{\alpha s(1-t)}\mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1^{\alpha s}\cdot F_1(\eta_2,\alpha,s,k,\lambda).\end{aligned}$$

Case 3: For $\eta_1 > 1$, we have $\eta_1^{(1-t)^{\alpha s}} \le \eta_1{}^{(\alpha s(1-t)+1-\alpha s)} = \eta_1^{(1-\alpha st)}$. Thus the inequality (3.16) becomes

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{(1-\alpha st)}\mathrm{d}t+\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{(1-\alpha st)}\mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1\cdot F_1(\eta_2,\alpha,s,k,\lambda).\end{aligned}$$

The proof is completed. □

Corollary 3.1. *In Theorem 3.4, if φ is an identity mapping, then we obtain*

$$\begin{aligned}&\left|\Big[\lambda f(a)+(k-\lambda)f\Big(\sqrt{ab}\Big)+(1-k)f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right| \\ &\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M_1\big(\alpha,s,k,\lambda\big),\end{aligned}$$

where

$$\omega_1 = \Big(\frac{b}{a}\Big)^{-\frac{1}{\alpha s}}\eta_1$$

and

$$M_1(\alpha,s,k,\lambda)=\begin{cases}\vartheta_1\Big(\lambda,a,b\Big)+\vartheta_2\Big(k,a,b\Big), & \eta_1=1,\\ \eta_1^{\alpha s}F_2(\omega_1,\alpha,s,k,\lambda), & 0<\eta_1<1,\\ \eta_1 F_2(\omega_1,\alpha,s,k,\lambda), & \eta_1>1,\end{cases}$$

with

$$\begin{aligned}&F_2(\omega_1,\alpha,s,k,\lambda)\\&=\frac{(1-k)\omega_1^{-\alpha s}-\lambda}{-\alpha s\ln\omega_1}+\frac{2\omega_1^{-\alpha sk}+2\omega_1^{-\alpha s\lambda}-2\omega_1^{-\frac{\alpha s}{2}}-\omega_1^{-\alpha s}-1}{\big(\alpha s\ln\omega_1\big)^2}.\end{aligned}$$

Remark 3.1. In Corollary 3.1,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\\&\quad\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m M_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4}\Big),\end{aligned}$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\Big(\sqrt{ab}\Big)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\\&\quad\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m M_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6}\Big),\end{aligned}$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m M_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2}\Big).$$

Theorem 3.5. *If $|f'(\varphi(x))|^q$ for $q>1$ is increasing and relative semi-(s,α,m)-logarithmically convex function on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, then the following inequality holds:*

$$\left|\triangle_{f,\varphi}(\lambda,k;a,b)\right|\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{T}(\alpha,s,k,\lambda,q),\tag{3.17}$$

where

$$\mathcal{T}(\alpha,s,k,\lambda,q)=\begin{cases}\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big), & \eta_1=1,\\ \eta_1^{\alpha s}\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\mathcal{K}_1^{\frac{1}{q}}(\eta_3,\alpha,s,\lambda,q)\\ \quad+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\mathcal{K}_2^{\frac{1}{q}}(\eta_3,\alpha,s,k,q)\Big], & 0<\eta_1<1,\\ \eta_1\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\mathcal{K}_1^{\frac{1}{q}}(\eta_3,\alpha,s,\lambda,q)\\ \quad+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\mathcal{K}_2^{\frac{1}{q}}(\eta_3,\alpha,s,k,q)\Big], & \eta_1>1,\end{cases}$$

with

$$\mathcal{K}_1(\eta_3,\alpha,s,\lambda,q)=\frac{\left(\frac{1}{2}-\lambda\right)\eta_3^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sq\lambda}-\eta_3^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\eta_3)^2},$$

$$\mathcal{K}_2(\eta_3,\alpha,s,k,q)=\frac{\left(\frac{1}{2}-k\right)\eta_3^{-\frac{\alpha sq}{2}}+(1-k)\eta_3^{-\alpha sq}}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sqk}-\eta_3^{-\frac{\alpha sq}{2}}-\eta_3^{-\alpha sq}}{(\alpha sq\ln\eta_3)^2},$$

$$\eta_3=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha sq}}\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha sq}}\eta_1,$$

and $\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big)$, $\vartheta_2\big(k,\varphi(a),\varphi(b)\big)$ *are defined by* (3.4) *and* (3.5), *respectively.*

Proof. For $h(t)=t^s$ in the inequality (3.6), we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\&\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\left(\int_0^{\frac{1}{2}}\Big|t-\lambda\Big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\Big]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\mathrm{d}t\right.\\&\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\left(\int_{\frac{1}{2}}^1\Big|t-k\Big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\Big]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{3.18}$$

where ϑ_1 and ϑ_2 are defined by (3.4) and (3.5), respectively. When $0\le\lambda\le\frac{1}{2}\le k\le 1$, let $\eta_1=\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$ in (3.18), we have three cases as follows.

Case 1: For $\eta_1=1$, the inequality (3.18) becomes the form

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\&\times\left\{\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big)\right\}.\end{aligned}$$

Case 2: For $0<\eta_1<1$, we have $\eta_1^{q(1-t)^{\alpha s}}\le\eta_1^{q\alpha s(1-t)}$. Thus the inequality (3.18) becomes

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\&\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\left(\int_0^{\frac{1}{2}}\Big|t-\lambda\Big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{\alpha sq(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\&\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\left(\int_{\frac{1}{2}}^1\Big|t-k\Big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{\alpha sq(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\&=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1^{\alpha s}\end{aligned}$$

$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t\bigg)^{\frac{1}{q}}\right.$$
$$\left.+\,\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\bigg(\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t\bigg)^{\frac{1}{q}}\right\}.$$

Here

$$\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t=\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_3^{-\alpha sqt}\mathrm{d}t$$
$$=\frac{\big(\frac{1}{2}-\lambda\big)\eta_3^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sq\lambda}-\eta_3^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\eta_3)^2}$$

and

$$\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t=\int_{\frac{1}{2}}^1\big|t-k\big|\eta_3^{-\alpha sqt}\mathrm{d}t$$
$$=\frac{\big(\frac{1}{2}-k\big)\eta_3^{-\frac{\alpha sq}{2}}+(1-k)\eta_3^{-\alpha sq}}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sqk}-\eta_3^{-\frac{\alpha sq}{2}}-\eta_3^{-\alpha sq}}{(\alpha sq\ln\eta_3)^2}.$$

Case 3: For $\eta_1>1$, we have $\eta_1^{q(1-t)^{\alpha s}}\le\eta_1^{q(\alpha s(1-t)+1-\alpha s)}=\eta_1^{q(1-\alpha st)}$. Thus the inequality (3.18) becomes

$$\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m$$
$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{q(1-\alpha st)}\mathrm{d}t\bigg)^{\frac{1}{q}}\right.$$
$$\left.+\,\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\bigg(\int_{\frac{1}{2}}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{q(1-\alpha st)}\mathrm{d}t\bigg)^{\frac{1}{q}}\right\}$$
$$=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1$$
$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_3^{-\alpha sqt}\mathrm{d}t\bigg)^{\frac{1}{q}}\right.$$
$$\left.+\,\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\bigg(\int_{\frac{1}{2}}^1\big|t-k\big|\eta_3^{-\alpha sqt}\mathrm{d}t\bigg)^{\frac{1}{q}}\right\}.$$

The proof is completed. □

Corollary 3.2. *In Theorem 3.5, if φ is an identity mapping, then we obtain*

$$\left|\Big[\lambda f(a)+(k-\lambda)f\Big(\sqrt{ab}\Big)+(1-k)f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}_1\big(\alpha,s,k,\lambda,q\big),$$

where

$$\mathcal{T}_1(\alpha,s,k,\lambda,q)=\begin{cases}\vartheta_1(\lambda,a,b)+\vartheta_2(k,a,b), & \eta_1=1,\\ \eta_1^{\alpha s}\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,a,b\Big)\mathcal{K}_3^{\frac{1}{q}}(\omega_2,\alpha,s,\lambda,q)\\ +\vartheta_2^{1-\frac{1}{q}}\Big(k,a,b\Big)\mathcal{K}_4^{\frac{1}{q}}(\omega_2,\alpha,s,k,q)\Big], & 0<\eta_1<1,\\ \eta_1\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,a,b\Big)\mathcal{K}_3^{\frac{1}{q}}(\omega_2,\alpha,s,\lambda,q)\\ +\vartheta_2^{1-\frac{1}{q}}\Big(k,a,b\Big)\mathcal{K}_4^{\frac{1}{q}}(\omega_2,\alpha,s,k,q)\Big], & \eta_1>1,\end{cases}$$

with

$$\mathcal{K}_3(\omega_2,\alpha,s,\lambda,q)=\frac{\big(\frac{1}{2}-\lambda\big)\omega_2^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\omega_2}+\frac{2\omega_2^{-\alpha sq\lambda}-\omega_2^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\omega_2)^2},$$

$$\mathcal{K}_4(\omega_2,\alpha,s,k,q)=\frac{\big(\frac{1}{2}-k\big)\omega_2^{-\frac{\alpha sq}{2}}+(1-k)\omega_2^{-\alpha sq}}{-\alpha sq\ln\omega_2}+\frac{2\omega_2^{-\alpha sqk}-\omega_2^{-\frac{\alpha sq}{2}}-\omega_2^{-\alpha sq}}{(\alpha sq\ln\omega_2)^2}$$

and

$$\omega_2=\Big(\frac{b}{a}\Big)^{-\frac{1}{\alpha sq}}\eta_1. \tag{3.19}$$

Remark 3.2. In Corollary 3.2,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{4}\Big[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\\&\quad\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4},q\Big),\end{aligned}$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\\&\quad\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6},q\Big),\end{aligned}$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2},q\Big).$$

We get the following result by another approach.

Theorem 3.6. *If $|f'(\varphi(x))|^q$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ is increasing and relative semi-(s, α, m)-logarithmically convex function on $\left[0, \varphi\left(\frac{b}{m}\right)\right]$ along with $(\alpha, m) \in (0, 1] \times (0, 1]$, $0 \le \lambda \le \frac{1}{2} \le k \le 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \triangle_{f,\varphi}(\lambda, k; a, b) \right| \\ &\quad \le \Big(\ln \varphi(b) - \ln \varphi(a) \Big) \varphi(a) \left| f'\Big(\frac{b}{m}\Big) \right|^m \mathcal{L}(\alpha, s, k, \lambda, p, q), \end{aligned} \tag{3.20}$$

where

$$\mathcal{L}(\alpha, s, k, \lambda, p, q) = \begin{cases} \chi_1^{\frac{1}{p}}(\lambda, p) \left(\dfrac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}} - 1}{q\big(\ln \varphi(b) - \ln \varphi(a)\big)} \right)^{\frac{1}{q}} \\ \quad + \chi_2^{\frac{1}{p}}(k, p) \left(\dfrac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{q} - \left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}}{q\big(\ln \varphi(b) - \ln \varphi(a)\big)} \right)^{\frac{1}{q}}, & \eta_1 = 1, \\ \eta_1^{\alpha s} \mathcal{H}_1(\alpha, s, k, \lambda, p, q), & 0 < \eta_1 < 1, \\ \eta_1 \mathcal{H}_1(\alpha, s, k, \lambda, p, q), & \eta_1 > 1, \end{cases}$$

with

$$\mathcal{H}_1(\alpha, s, k, \lambda, p, q) = \chi_1^{\frac{1}{p}}(\lambda, p) \left(\frac{\eta_2^{-\frac{\alpha s q}{2}} - 1}{-\alpha s q \ln \eta_2} \right)^{\frac{1}{q}} + \chi_2^{\frac{1}{p}}(k, p) \left(\frac{\eta_2^{-\alpha s q} - \eta_2^{-\frac{\alpha s q}{2}}}{-\alpha s q \ln \eta_2} \right)^{\frac{1}{q}}.$$

Here, $\chi_1(\lambda, p)$, $\chi_2(k, p)$, η_1 and η_2 are defined by (3.10), (3.11), (3.14) *and* (3.15), *respectively.*

Proof. For $h(t) = t^s$ in the inequality (3.12), we have

$$\begin{aligned} &\left| \triangle_{f,\varphi}(\lambda, k; a, b) \right| \le \Big(\ln \varphi(b) - \ln \varphi(a) \Big) \varphi(a) \left| f'\Big(\frac{b}{m}\Big) \right|^m \\ &\times \left\{ \left(\frac{\lambda^{p+1} + \left(\frac{1}{2} - \lambda\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt} \left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m} \right]^{q(1-t)^{\alpha s}} \mathrm{d}t \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\frac{(1-k)^{p+1} + \left(k - \frac{1}{2}\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt} \left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m} \right]^{q(1-t)^{\alpha s}} \mathrm{d}t \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{3.21}$$

When $0 \le \lambda \le \frac{1}{2} \le k \le 1$, let $\eta_1 = \frac{|f'(a)|}{|f'(\frac{b}{m})|^m}$ in (3.21), we have three cases as follows.

Case 1: For $\eta_1 = 1$, the inequality (3.21) becomes the form

$$\begin{aligned} &\left| \triangle_{f,\varphi}(\lambda, k; a, b) \right| \le \Big(\ln \varphi(b) - \ln \varphi(a) \Big) \varphi(a) \left| f'\Big(\frac{b}{m}\Big) \right|^m \\ &\quad \times \left\{ \left(\frac{\lambda^{p+1} + \left(\frac{1}{2} - \lambda\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}} - 1}{q\big(\ln \varphi(b) - \ln \varphi(a)\big)} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{(1-k)^{p+1} + \left(k - \frac{1}{2}\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{q} - \left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}}{q\big(\ln \varphi(b) - \ln \varphi(a)\big)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Case 2: For $0 < \eta_1 < 1$, we have $\eta_1^{q(1-t)^{\alpha s}} \leq \eta_1^{q\alpha s(1-t)}$. Thus the inequality (3.21) becomes

$$\begin{aligned}
\Big| \triangle_{f,\varphi}(\lambda, k; a, b)\Big| &\leq \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\
&\times \left\{ \left(\frac{\lambda^{p+1} + \big(\frac{1}{2} - \lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{q\alpha s(1-t)} \mathrm{d}t \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{(1-k)^{p+1} + \big(k - \frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{q\alpha s(1-t)} \mathrm{d}t \right)^{\frac{1}{q}} \right\} \\
&= \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \eta_1^{\alpha s} \\
&\times \left\{ \left(\frac{\lambda^{p+1} + \big(\frac{1}{2} - \lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{-q\alpha st} \mathrm{d}t \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{(1-k)^{p+1} + \big(k - \frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{-q\alpha st} \mathrm{d}t \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Here

$$\int_0^{\frac{1}{2}} \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{-q\alpha st} \mathrm{d}t = \int_0^{\frac{1}{2}} \eta_2^{-q\alpha st} \mathrm{d}t = \frac{\eta_2^{-\frac{q\alpha s}{2}} - 1}{-q\alpha s \ln \eta_2}$$

and

$$\int_{\frac{1}{2}}^1 \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{-q\alpha st} \mathrm{d}t = \int_{\frac{1}{2}}^1 \eta_2^{-q\alpha st} \mathrm{d}t = \frac{\eta_2^{-q\alpha s} - \eta_2^{-\frac{q\alpha s}{2}}}{-q\alpha s \ln \eta_2}.$$

Case 3: For $\eta_1 < 1$, we have $\eta_1^{q(1-t)^{\alpha s}} \leq \eta_1^{q(\alpha s(1-t)+1-\alpha s)} = \eta_1^{q(1-\alpha st)}$. Thus the inequality (3.21) becomes

$$\begin{aligned}
\Big| \triangle_{f,\varphi}(\lambda, k; a, b)\Big| &\leq \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\
&\times \left\{ \left(\frac{\lambda^{p+1} + \big(\frac{1}{2} - \lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{q(1-\alpha st)} \mathrm{d}t \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{(1-k)^{p+1} + \big(k - \frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt} \eta_1^{q(1-\alpha st)} \mathrm{d}t \right)^{\frac{1}{q}} \right\} \\
&= \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \eta_1 \\
&\times \left\{ \left(\frac{\lambda^{p+1} + \big(\frac{1}{2} - \lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \eta_2^{-q\alpha st} \mathrm{d}t \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{(1-k)^{p+1} + \big(k - \frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \eta_2^{-q\alpha st} \mathrm{d}t \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

The proof is completed. □

Corollary 3.3. *In Theorem 3.6, if φ is an identity mapping, then we obtain*

$$\left|\left[\lambda f(a)+(k-\lambda)f\Big(\sqrt{ab}\Big)+(1-k)f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{L}_1\big(\alpha,s,k,\lambda,p,q\big),$$

where

$$\mathcal{L}_1(\alpha,s,k,\lambda,p,q)=\begin{cases}\chi_1^{\frac{1}{p}}(\lambda,p)\Big(\frac{[\frac{b}{a}]^{\frac{q}{2}}-1}{q(\ln b-\ln a)}\Big)^{\frac{1}{q}}\\ \quad+\chi_2^{\frac{1}{p}}(k,p)\Big(\frac{[\frac{b}{a}]^{q}-[\frac{b}{a}]^{\frac{q}{2}}}{q(\ln b-\ln a)}\Big)^{\frac{1}{q}}, & \eta_1=1,\\ \eta_1^{\alpha s}\mathcal{H}_2(\alpha,s,k,\lambda,p,q), & 0<\eta_1<1,\\ \eta_1\mathcal{H}_2(\alpha,s,k,\lambda,p,q), & \eta_1>1,\end{cases}$$

with

$$\mathcal{H}_2(\alpha,s,k,\lambda,p,q)=\chi_1^{\frac{1}{p}}(\lambda,p)\left(\frac{\omega_2^{-\frac{\alpha sq}{2}}-1}{-\alpha sq\ln\omega_2}\right)^{\frac{1}{q}}+\chi_2^{\frac{1}{p}}(k,p)\left(\frac{\omega_2^{-\alpha sq}-\omega_2^{-\frac{\alpha sq}{2}}}{-\alpha sq\ln\omega_2}\right)^{\frac{1}{q}}.$$

Here, η_1 and ω_2 are defined by (3.14) *and* (3.19)*, respectively.*

Remark 3.3. In Corollary 3.3,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\left|\frac{1}{4}\left[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4},p,q\Big),$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\Big(\sqrt{ab}\Big)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6},p,q\Big),$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2},p,q\Big).$$

Finally, we establish an inequality involving products of two relative semi-(h,α,m)-logarithmically convexity.

Theorem 3.7. *Assume that $f,g:\mathcal{I}^0\subseteq[0,\infty)\to(0,\infty)$ are differentiable mappings on $\mathcal{I}^0$ such that $f,g\in L^1\big([\varphi(a),\varphi(b)]\big)$. If $f(\varphi(x))$ is increasing and relative semi-(h,α,m_1)-logarithmically convex and $g(\varphi(x))$ is increasing and relative semi-(h,α,m_2)-logarithmically convex on $\big[0,\varphi\big(\frac{b}{m_i}\big)\big]$ along with $(\alpha,m_i)\in(0,1]\times(0,1]$ and $i=1,2$, then the following inequality holds:*

$$\begin{aligned}&\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\&\le\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{m_1}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{m_2}\\&\quad\times\int_0^1\left\{f(a)g(a)\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{-m_1}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{-m_2}\right\}^{h^\alpha(t)}\mathrm{d}t.\end{aligned}\tag{3.22}$$

Proof. Let $\varphi(x)=[\varphi(a)]^t[\varphi(b)]^{1-t}$ and using Young inequality, we get

$$\begin{aligned}&\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\&=\int_0^1 f\Big([\varphi(a)]^t[\varphi(b)]^{1-t}\Big)g\Big([\varphi(a)]^t[\varphi(b)]^{1-t}\Big)\mathrm{d}t\\&\le\int_0^1 f\Big(t\varphi(a)+(1-t)\varphi(b)\Big)g\Big(t\varphi(a)+(1-t)\varphi(b)\Big)\mathrm{d}t.\end{aligned}\tag{3.23}$$

Using the positive homogeneity of φ, the relative semi-(h,α,m)-logarithmically convexity of $f\big(\varphi(x)\big)$ and $g\big(\varphi(x)\big)$, we have

$$f\bigg(t\varphi(a)+m_1(1-t)\varphi\Big(\frac{b}{m_1}\Big)\bigg)\le\big[f(a)\big]^{h^\alpha(t)}\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{m_1\left(1-h^\alpha(t)\right)}\tag{3.24}$$

and

$$g\bigg(t\varphi(a)+m_2(1-t)\varphi\Big(\frac{b}{m_2}\Big)\bigg)\le\big[g(a)\big]^{h^\alpha(t)}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{m_2\left(1-h^\alpha(t)\right)}.\tag{3.25}$$

Using (3.24) and (3.25) in (3.23), we get

$$\begin{aligned}&\int_0^1 f\Big(t\varphi(a)+(1-t)\varphi(b)\Big)g\Big(t\varphi(a)+(1-t)\varphi(b)\Big)\mathrm{d}t\\&\le\int_0^1\big[f(a)\big]^{h^\alpha(t)}\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{m_1\left(1-h^\alpha(t)\right)}\big[g(a)\big]^{h^\alpha(t)}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{m_2\left(1-h^\alpha(t)\right)}\mathrm{d}t\\&=\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{m_1}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{m_2}\int_0^1\left\{f(a)g(a)\Big[f\Big(\frac{b}{m_1}\Big)\Big]^{-m_1}\Big[g\Big(\frac{b}{m_2}\Big)\Big]^{-m_2}\right\}^{h^\alpha(t)}\mathrm{d}t.\end{aligned}$$

□

Corollary 3.4. *If $h(t) = t$ in Theorem 3.7, then we have*

$$\frac{1}{\ln\varphi(b) - \ln\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} \frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)} \mathrm{d}\varphi(x) \le \left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1} \left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2} \Upsilon(\alpha),$$

where

$$\Upsilon(\alpha) = \begin{cases} 1, & \rho = 1, \\ \dfrac{\rho^{\alpha} - 1}{\alpha \ln \rho}, & 0 < \rho < 1, \\ \dfrac{\rho - \rho^{1-\alpha}}{\alpha \ln \rho}, & \rho > 1, \end{cases}$$

with

$$\rho = f(a)g(a)\left[f\Big(\frac{b}{m_1}\Big)\right]^{-m_1} \left[g\Big(\frac{b}{m_2}\Big)\right]^{-m_2},$$

which is the same result proved by Zhou et al. in [30].

4. Conclusion

Based on a new integral identity with multiple parameters, three main results of Simpson-type inequalities via relative semi-(h, α, m)-logarithmically convexity are established. For $h(t) = t^s$, more new multi-parameterized integral inequalities can be derived by means of relative semi-(s, α, m)-logarithmically convexity.

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Complete moment convergence for weighted sums of negatively orthant dependent random variables

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Abstract. The authors study the complete moment convergence for weighted sums of negatively orthant dependent random variables and obtain some new results. These results extend and improve the corresponding theorems of Chen [P. Chen., 2016, Complete Convergence and Strong Laws of Large Numbers For Weighted Sums of Negatively Orthant Dependent Random Variables, Acta Math. Hungar., 148 (1), 83-95], complete moment convergence, weighted sums, negatively orthant dependent random variables.
Keywords: complete moment convergence, weighted sums, L^p convergence, negatively orthant dependent.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. The limiting behavior of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ play important roles in some useful linear statistics, many authors studied the strong convergence for the weighted sums. We refer the reader to Cuzick (1995), Wu (1999), Bai and Cheng (2000), Sung (2001), Chen and Gan (2007), Cai (2008), Wu (2011), Zarei and Jabbari (2011), Sung (2011, 2012), Shen (2012), Chen and Sung (2014).

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

The concept of the complete convergence was introduced by Hsu and Robbins (1947).

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Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty, \quad \text{for some or all } \varepsilon > 0,$$

then the above result was called complete moment convergence by Chow (1988), which is the more general version of the complete convergence.

Joag-Dev and Proschan (1983) introduced the following negatively associated concept. A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinate-wise nondecreasing functions f on R^A and g on R^B,

$$\mathrm{Cov}(f(X_i, i \in A), g(Y_j, j \in B)) \leq 0$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The following concept was introduced by Lehmann (1966).

Definition 1.1. A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively orthant dependent if the following two inequalities hold,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i),$$

for all real numbers $x_1, \ldots, x_n$. An infinite family of random variables is negatively orthant dependent if every finite subfamily is negatively orthant dependent.

Note that negative association implies negative orthant dependence, but the converse does not hold.

Cai (2008) studied the complete convergence for weighted sums of identically distributed NA random variables, and obtained the results under the stronger moment condition $E\exp(h|X|^\gamma) < \infty$ for some $h > 0$. Sung (2011) obtained the following theorem, which improved the main results of Cai (2008) by replacing some much weaker moment conditions.

Theorem A. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n), \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \leq 2$. Then the following statements hold:

(i) If $\alpha > \gamma$, $E|X|^\alpha < \infty$, then

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq m\leq n} \Big| \sum_{i=1}^{m} a_{ni} X_i \Big| > b_n \varepsilon \Big) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.2}$$

(ii) If $\alpha = \gamma$, then $E|X|^\alpha \log(1 + |X|) < \infty$ implies (1.2).

(iii) If $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies (1.2).

For the negatively orthant dependent random variables, under the same moment condition as (1.2), there are some results about

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\Big| \sum_{i=1}^{n} a_{ni} X_i \Big| > b_n \varepsilon \Big) < \infty, \quad \text{for all } \varepsilon > 0, \tag{1.3}$$

(see Huang and Wang (2012), Shen (2013) and Zhang and Wang (2014)). However, (1.2) is stronger than (1.3). Recently, Chen and Sung (2016) proved (1.3) for the case $\alpha < \gamma$.

The main purpose of this article is to study the complete moment convergence, we shall extend and improve the main results of Chen (2016) by obtaining a much stronger conclusion under the same conditions. Using our method, we can easily get L^p convergence for negatively orthant dependent random variables.

Throughout this paper, the symbol C represents positive constants whose values may change from one place to another. For a finite set A the symbol $\sharp(A)$ denotes the number of elements in the set A.

2. Preliminaries

To prove our main result, the following lemmas are needed, Lemma 2.1 is the Rosenthal-type inequality for the sum of negatively orthant dependent random variables, and Lemma 2.2 is a Rosenthal-type inequality for maximum partial sums of negatively orthant dependent random variables, which are all crucial for the proof of our main results.

Lemma 2.1 (Asadian et al. 2006)**.** Let $\{X_n, n \geq 1\}$ be a sequence of negatively orthant dependent random variables with mean zero, and $E|X_n|^q < \infty$ for some $q \geq 2$ and all $n \geq 1$. Let $S_n = \sum_{i=1}^{n} X_i$, then,

$$|S_n|^q \leq C \Big\{ \sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2 \Big)^{q/2} \Big\} \tag{2.1}$$

where the constant C depends only on q.

Using Lemma (2.1) and Theorem 3 in Móricz (1976), it can be get the following Rosenthal type inequality for the maximum of partial sums of negatively orthant dependent random variables.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of negatively orthant dependent random variables with mean zero and $E|X_k|^q < \infty$ for some $q \geq 2$ and every $1 \leq k \leq n$. Then, there exists a positive constant C depends only on q such that

$$E \max_{1\leq m\leq n} |S_m|^q \leq C \left(\frac{\log 2n}{\log 2}\right)^q \left\{\sum_{k=1}^{n} E|X_k|^q + \Big(\sum_{k=1}^{n} EX_k^2\Big)^{q/2}\right\}. \tag{2.2}$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leq 2$. For all $n \geq 1$, let

$$A_n = \{1 \leq i \leq n : |a_{ni}| \leq n^{1/\alpha}(\log n)^{-t_1}\}, \tag{2.3}$$

where $t > 0$ will be specified later. Defined b_n as follows:

$$b_n = n^{1/\alpha}(\log n)^{1/\gamma}. \tag{2.4}$$

Lemma 2.3. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leq 2$, and X be a random variable with $E|X|^\gamma < \infty$, for $\gamma > \alpha$. Then, for $\beta > 0$ we have,

$$H =: \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i\in A_n} (\log n)^{q-\beta q+\alpha\beta} E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n(\log n)^{-\beta}) \leq CE|X|^\gamma,$$

where b_n is defined as (2.4).

Proof. Since $\alpha < \gamma$, we have,

$$\begin{aligned}
H &\leq \sum_{n=2}^{\infty} n^{-1} \sum_{i\in A_n} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\leq \sum_{n=2}^{\infty} n^{-1-\gamma/\alpha} \sum_{i\in A_n} (\log n)^{q-\beta q+\beta\gamma-1} E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\leq \sum_{n=2}^{\infty} n^{-2} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} \sum_{i\in A_n} |a_{ni}|^\alpha E|X|^\gamma \\
&\leq \sum_{n=2}^{\infty} n^{-1} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} E|X|^\gamma.
\end{aligned}$$

Hence $H \leq CE|X|^\gamma$ if we take t_1 large enough such that $t_1(\gamma-\alpha)+\beta q-\beta\gamma-q > 0$.

Lemma 2.4 Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$, and X be a random variable with $E|X|^\gamma < \infty$ for $\gamma > \alpha$. Then, for $q > \max\{\alpha, \gamma\}$, we have

$$\begin{aligned} L =: &\sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log n)^q \sum_{i \in A_n} \int_1^{\infty} t^{-q/\alpha} E|a_{ni}X|^q I(|a_{ni}X| \le b_n (\log n)^{-\beta}) \mathrm{d}t \\ &\le CE|X|^\gamma, \end{aligned}$$

where b_n is defined as (2.4).

Proof. Since $q > \max\{\alpha, \gamma\}$, we can get,

$$\begin{aligned} L &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log n)^q \sum_{i \in A_n} |a_{ni}|^q E|X|^q I(|a_{ni}X| \le b_n (\log n)^{-\beta}) \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log n)^{q - \beta q + \beta\gamma} \sum_{i \in A_n} |a_{ni}|^\gamma E|X|^\gamma I(|a_{ni}X| \le b_n (\log n)^{-\beta}) \\ &\le C \sum_{n=2}^{\infty} n^{\gamma/\alpha - 2} b_n^{-\gamma} (\log n)^{-t_1(\gamma-\alpha) + q - \beta q + \beta\gamma} \sum_{i \in A_n} |a_{ni}|^\alpha E|X|^\gamma I(|a_{ni}X| \le b_n (\log n)^{-\beta}) \\ &\le C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-t_1(\gamma-\alpha) + q - \beta q + \beta\gamma - 1} E|X|^\gamma. \end{aligned}$$

Hence, $L \le CE|X|^\gamma$ if we take t_1 large enough such that $t_1(\gamma - \alpha) + \beta q - \beta\gamma - q > 0$.

3. Main result

In this section, we state our main results and their proofs.

Theorem 3.1. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively orthant dependent random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$ for some $\gamma > \alpha$. Furthermore, suppose that $EX_i = 0$ when $1 < \alpha < 2$. If $E|X|^\gamma < \infty$, then

$$\sum_{n=2}^{\infty} n^{-1} E\left\{ b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| - \varepsilon \right\}_+^{\alpha} < \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

Proof. We may assume that $a_{ni} > 0$ and $\sum_{i=1}^{n} |a_{ni}|^\alpha \le n$. For any given $\varepsilon > 0$, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} n^{-1} E\left\{ b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| - \varepsilon \right\}_+^{\alpha} \\ &= \sum_{n=2}^{\infty} n^{-1} \int_0^{\infty} P\left(b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) \mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&\le \sum_{n=2}^{\infty} n^{-1} P\Big(\max_{1\le m\le n}\Big|\sum_{i=1}^{m} a_{ni}X_i\Big| > b_n\varepsilon\Big)\\
&+ \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\Big(\max_{1\le m\le n}\Big|\sum_{i=1}^{m} a_{ni}X_i\Big| > b_n t^{1/\alpha}\Big)\mathrm{d}t\\
&=: I_1 + I_2.
\end{aligned}$$

Therefore, to prove (3.1), it needs only to prove that $I_1 < \infty$ and $I_2 < \infty$. By Theorem 2.1 and Theorem 2.2 of Chen. P. (2016), we get directly $I_1 < \infty$. For all $t \ge 1$, we denote

$$\begin{aligned}
X_{ni}(1) &= a_{ni}X_i I(|a_{ni}X_i| \le b_n(\log n)^{-\beta}t^{1/\alpha})\\
&\quad + b_n(\log n)^{-\beta}t^{1/\alpha} I(a_{ni}X_i > b_n(\log n)^{-\beta}t^{1/\alpha})\\
&\quad - b_n(\log n)^{-\beta}t^{1/\alpha} I(a_{ni}X_i < -b_n(\log n)^{-\beta}t^{1/\alpha})\\
X_{ni}(2) &= \Big(a_{ni}X_i - b_n(\log n)^{-\beta}t^{1/\alpha}\Big) I(b_n(\log n)^{-\beta}t^{1/\alpha} < a_{ni}X_i) \le b_n\varepsilon t^{1/\alpha}/4N)\\
X_{ni}(3) &= \Big(a_{ni}X_i + b_n(\log n)^{-\beta}t^{1/\alpha}\Big) I(-b_n\varepsilon t^{1/\alpha}/4N \le a_{ni}X_i < -b_n(\log n)^{-\beta}t^{1/\alpha})\\
X_{ni}(4) &= \Big(a_{ni}X_i - b_n(\log n)^{-\beta}t^{1/\alpha}\Big) I(a_{ni}X_i > b_n\varepsilon t^{1/\alpha}/4N)\\
&\quad + \Big(a_{ni}X_i + b_n(\log n)^{-\beta}t^{1/\alpha}\Big) I(a_{ni}X_i < -b_n\varepsilon t^{1/\alpha}/4N)
\end{aligned}$$

Then,

$$\begin{aligned}
I_2 &\le \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\big(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(1)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t\\
&+ \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\big(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(2)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t\\
&+ \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\big(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(3)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t\\
&+ \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\big(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(4)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t\\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

To prove $J_1 < \infty$, we first show

$$\sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\Big|\sum_{i=1}^{m} EX_{ni}(1)\Big| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.2}$$

For $1 < \alpha \le 2$, by $EX_i = 0$, $\sum_{i=1}^{n}|a_{ni}|^{\alpha} \le n$, $|X_{ni}(1)| \le a_{ni}|X_i|$, we have

$$\sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\Big|\sum_{i=1}^{m} EX_{ni}(1)\Big|$$

$$
\begin{aligned}
&\le \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni} EX_i I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha})\right| \\
&+ \sup_{t\ge 1} (\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&= \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni} EX_i I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha})\right| \\
&+ \sup_{t\ge 1} (\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C(\log n)^{\beta(\alpha-1)} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| > b_n(\log n)^{-\beta}) \\
&\le CE|X|^{\alpha} (\log n)^{\beta(\alpha-1)-\alpha/\gamma} \to 0
\end{aligned}
$$

as $n \to \infty$, since $0 < \beta < 1/\gamma$ and $E|X|^{\alpha} \le (E|X|^{\gamma})^{\alpha/\gamma} < \infty$. For $0 < \alpha \le 1$, we have

$$
\begin{aligned}
&\sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} EX_{ni}(1)\right| \\
&\le \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&+ \sup_{t\ge 1} (\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le \sup_{t\ge 1} t^{-1} b_n^{-\alpha} (\log n)^{\beta(\alpha-1)} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&+ \sup_{t\ge 1} t^{-1} b_n^{-\alpha} (\log n)^{\beta(\alpha-1)} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C(\log n)^{\beta(\alpha-1)-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

From (3.2), we know that while n is sufficiently large,

$$
t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} EX_{ni}(1)\right| \le b_n t^{1/\alpha}/8. \tag{3.3}
$$

holds uniformly for $t \ge 1$.

Suppose $q > \max\{2, 2\gamma/\alpha\}$, by (3.3) and the Markov inequality, we get

$$\begin{aligned} J_1 &= \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n} \Big| \sum_{i=1}^{m} X_{ni}(1) \Big| > b_n \varepsilon t^{1/\alpha}/4 \Big) \mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n} \Big| \sum_{i=1}^{m} (X_{ni}(1) - EX_{ni}(1)) \Big| > b_n \varepsilon t^{1/\alpha}/8 \Big) \mathrm{d}t \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} E \max_{1\le m\le n} \Big| \sum_{i=1}^{m} (X_{ni}(1) - EX_{ni}(1)) \Big|^q \mathrm{d}t \end{aligned}$$

For $n \ge 1$ and $t > 0$, let A_n is defined as (2.3), and $B_n = \{1 \le i \le n : |a_{ni}| > n^{1/\alpha}(\log n)^{-t_1}\}$. By the fact that $n \ge \sum_{i=1}^{n} |a_{ni}|^\alpha \ge \sum_{i\in B_n} |a_{ni}|^\alpha \ge n(\log n)^{-t_1\alpha} \sharp B_n$ then

$$\sharp B_n \le (\log n)^{t_1\alpha}. \tag{3.4}$$

Then, By Lemma 2.2, we get that

$$\begin{aligned} J_1 &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} E \max_{1\le m\le n} \Big| \sum_{i\le m, i\in A_n} (X_{ni}(1) - EX_{ni}(1)) \Big|^q \\ &+ \Big| \sum_{i\le m, i\in B_n} (X_{ni}(1) - EX_{ni}(1)) \Big|^q \mathrm{d}t \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} \left\{ (\log n)^q \left(\sum_{i\in A_n} E|X_{ni}(1)|^q + \left(\sum_{i\in A_n} EX_{ni}^2(1) \right)^{q/2} \right) \right\} \mathrm{d}t \\ &+ C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} \left\{ (\log\log n)^q \left(\sum_{i\in B_n} E|X_{ni}(1)|^q + \left(\sum_{i\in B_n} EX_{ni}^2(1) \right)^{q/2} \right) \right\} \mathrm{d}t \\ &=: J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

Firstly, we will prove $J_{12} < \infty$ in two cases ($\alpha < \gamma < 2$ and $\gamma \ge 2$). For the case $\alpha < \gamma < 2$, since $q > \max\{2, 2\gamma/\alpha\}$, $|X_{ni}(1)| \le b_n(\log n)^{-\beta} t^{1/\alpha}$ and $|a_{ni}| \le n^{1/\alpha}(\log n)^{-t_1}$ for $i \in A_n$, we have,

$$\begin{aligned} J_{12} &= C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} (\log n)^q \left(\sum_{i\in A_n} EX_{ni}^2(1) \right)^{q/2} \mathrm{d}t \\ &\le C \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} \left(b_n^{-\gamma} t^{-\gamma/\alpha} (\log n)^{\beta\gamma+2-2\beta} \sum_{i\in A_n} |a_{ni}|^\gamma E|X|^\gamma \right)^{q/2} \mathrm{d}t \end{aligned}$$

$$\leq C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}\left(n^{-1}t^{-\gamma/\alpha}(\log n)^{-t_1(\gamma-\alpha)+\beta\gamma-2\beta+1}\sum_{i\in A_n}|a_{ni}|^{\alpha}E|X|^{\gamma}\right)^{q/2}\mathrm{d}t$$
$$\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)}(E|X|^{\gamma})^{q/2}\int_1^{\infty}t^{-\frac{q\gamma}{2\alpha}}\mathrm{d}t$$
$$\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)}(E|X|^{\gamma})^{q/2}.$$

Since $\alpha<\gamma$ and $E|X|^{\gamma}<\infty$, hence $J_{12}<\infty$ if we take t_1 large enough such that $\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)>1$. For the case $\gamma\geq 2$, since $|X_{ni}(1)|\leq|a_{ni}X_i|$, we can similarly have,

$$\begin{aligned}J_{12}&=C\sum_{n=2}^{\infty}n^{-1}b_n^{-q}\int_1^{\infty}t^{-q/\alpha}(\log n)^{q}\left(\sum_{i\in A_n}E(X_{ni}^2(1))\right)^{q/2}\mathrm{d}t\\&\leq C\sum_{n=2}^{\infty}n^{-1}b_n^{-q}(\log n)^{q}\int_1^{\infty}t^{-q/\alpha}\left(\sum_{i\in A_n}a_{ni}^2EX_i^2\right)^{q/2}\mathrm{d}t\\&\leq C(EX^2)^{q/2}\sum_{n=2}^{\infty}n^{-1}b_n^{-q}(\log n)^{q}\\&\cdot\int_1^{\infty}t^{-q/\alpha}\left(n^{-1+2/\alpha}(\log n)^{-t_1(2-\alpha)}\sum_{i\in A_n}|a_{ni}|^{\alpha}\right)^{q/2}\mathrm{d}t\\&\leq C(EX^{\gamma})^{q/\gamma}\sum_{n=2}^{\infty}n^{-1}(\log n)^{-t_1q(2-\alpha)/2-q/\gamma+q}.\end{aligned}$$

Hence, $J_{12}<\infty$ if we take t_1 large enough such that $t_1q(2-\alpha)/2+q/\gamma-q>1$. For J_{11}, we have,

$$\begin{aligned}J_{11}&=C\sum_{n=2}^{\infty}n^{-1}b_n^{-q}\int_1^{\infty}t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}E|X_{ni}(1)|^{q}\mathrm{d}t\\&\leq C\sum_{n=2}^{\infty}n^{-1}b_n^{-q}\int_1^{\infty}t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}|a_{ni}|^{q}E|X|^{q}I(|a_{ni}X|\\&\leq b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\&+C\sum_{n=2}^{\infty}n^{-1}\int_1^{\infty}(\log n)^{q-\beta q}\sum_{i\in A_n}P(|a_{ni}X|>b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\&=C\sum_{n=2}^{\infty}n^{-1}b_n^{-q}\int_1^{\infty}t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}|a_{ni}|^{q}E|X|^{q}I(|a_{ni}X|\\&\leq b_n(\log n)^{-\beta})\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} (\log n)^q \sum_{i\in A_n} |a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X| \\
&\le b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&+ C\sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} (\log n)^{q-\beta q} \sum_{i\in A_n} P(|a_{ni}X| > b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&=: J_{11}^{(1)} + J_{11}^{(2)} + J_{11}^{(3)}
\end{aligned}
$$

For $J_{11}^{(3)}$, by Markov inequality, we have,

$$
\begin{aligned}
J_{11}^{(3)} &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} \int_1^{\infty} t^{-\gamma/\alpha} \\
&\times \sum_{i\in A_n} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} \sum_{i\in A_n} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \int_1^{\infty} t^{-\gamma/\alpha} \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} \sum_{i\in A_n} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&= C\sum_{n=2}^{\infty} n^{-1-\gamma/\alpha} \sum_{i\in A_n} (\log n)^{q-\beta q+\beta\gamma-1} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-2} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} \sum_{i\in A_n} |a_{ni}|^{\alpha} E|X|^{\gamma} \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} E|X|^{\gamma}.
\end{aligned}
$$

Hence, $J_{11}^{(3)} \le CE|X|^{\gamma} < \infty$ if t_1 is large enough such that $t_1(\gamma-\alpha)+\beta q-\beta\gamma-q>0$. For $J_{11}^{(2)}$, by $q>2\ge\alpha$, we have

$$
\begin{aligned}
&\sum_{i\in A_n} \int_1^{\infty} t^{-q/\alpha} (\log n)^q |a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X| \\
&\le b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&\le \sum_{i\in A_n} \sum_{m=1}^{\infty} \int_m^{m+1} t^{-q/\alpha} (\log n)^q |a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X| \\
&\le b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&\le \sum_{i\in A_n} \sum_{m=1}^{\infty} m^{-q/\alpha} (\log n)^q |a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|
\end{aligned}
$$

$$
\begin{aligned}
&\le b_n(\log n)^{-\beta}(m+1)^{1/\alpha})\\
&\le \sum_{i\in A_n}\sum_{m=1}^{\infty} m^{-q/\alpha}\sum_{s=1}^{m}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha}<|a_{ni}X|\\
&\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})\\
&\le \sum_{i\in A_n}\sum_{s=1}^{\infty}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha}<|a_{ni}X|\\
&\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})\sum_{m=s}^{\infty} m^{-q/\alpha}\\
&\le \sum_{i\in A_n}\sum_{s=1}^{\infty} s^{-q/\alpha+1}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha}<|a_{ni}X|\\
&\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha}).
\end{aligned}
$$

Since $q>\alpha$, then,

$$
\begin{aligned}
J_{11}^{(2)} &\le C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in A_n}\sum_{s=1}^{\infty}(\log n)^{q+\beta\alpha-\beta q}b_n^{-\alpha}|a_{ni}|^{\alpha}\\
&\quad\times |E|X|^{\alpha}I(b_n(\log n)^{-\beta}s^{1/\alpha}<|a_{ni}X|\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})\\
&\le C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in A_n}(\log n)^{q+\beta\alpha-\beta q}b_n^{-\alpha}|a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X|>b_n(\log n)^{-\beta}).
\end{aligned}
$$

Then, by Lemma 2.3, we can get $J_{11}^{(2)}\le CE|X|^{\gamma}<\infty$. For $J_{11}^{(1)}$, by Lemma 2.4, we can get $J_{11}^{(1)}\le CE|X|^{\gamma}<\infty$. Consequently, we get $J_{11}<\infty$. For J_{13}, we have

$$
\begin{aligned}
J_{13} &= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q\sum_{i\in B_n}E|X_{ni}(1)|^q\mathrm{d}t\\
&= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i\in B_n}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q E|X|^q I(|a_{ni}X|\le b_n(\log n)^{-\beta})\mathrm{d}t\\
&+ C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i\in B_n}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q E|X|^q I(b_n(\log n)^{-\beta}<|a_{ni}X|\\
&\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\
&+ C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in B_n}\int_1^{\infty}(\log\log n)^q(\log n)^{-\beta q}P(|a_{ni}X|>b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\
&=: J_{13}^{(1)}+J_{13}^{(2)}+J_{13}^{(3)}.
\end{aligned}
$$

For $J_{13}^{(3)}$, by $q > 2\gamma/\alpha$ and $\alpha \le 2$, we have $q > \gamma$, then,

$$
\begin{aligned}
J_{13}^{(3)} &= C\sum_{n=2}^{\infty} n^{-1} \sum_{i\in B_n} \int_1^{\infty} (\log\log n)^q (\log n)^{-\beta q} P(|a_{ni}X| > b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i\in B_n} (\log\log n)^q (\log n)^{-\beta(q-\alpha)} E|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log\log n)^q (\log n)^{-\beta(q-\gamma)} b_n^{-\gamma} \sum_{i\in B_n} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-1-\gamma/\alpha} (\log\log n)^q (\log n)^{-\beta(q-\gamma)-1} \sum_{i\in B_n} E|a_{ni}X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log\log n)^q (\log n)^{-\beta(q-\gamma)-1} E|X|^{\gamma} < \infty
\end{aligned}
$$

For $J_{13}^{(1)}$, By $q > \gamma$, we have

$$
\begin{aligned}
J_{13}^{(1)} &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i\in B_n} \int_1^{\infty} t^{-q/\alpha} (\log\log n)^q E|X|^q I(|a_{ni}X| \le b_n(\log n)^{-\beta}) \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log\log n)^q \sum_{i\in B_n} |a_{ni}|^q E|X|^q I(|a_{ni}X| \le b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log\log n)^q (\log n)^{-\beta(q-\gamma)} \sum_{i\in B_n} |a_{ni}|^{\gamma} E|X|^{\gamma} I(|a_{ni}X| \le b_n(\log n)^{-\beta}) \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log n)^{-\beta(q-\gamma)-1} (\log\log n)^q E|X|^{\gamma} < \infty.
\end{aligned}
$$

For $J_{13}^{(2)}$, similar to the proof of $J_{11}^{(2)}$, by $q > 2 \ge \alpha$ and Lemma 2.2, we have

$$
\begin{aligned}
J_{13}^{(2)} &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i\in B_n} \int_1^{\infty} t^{-q/\alpha} (\log\log n)^q \\
&\quad \times E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X| \le b_n(\log n)^{-\beta} t^{1/\alpha}) \mathrm{d}t \\
&\le Cn^{-1} b_n^{-\alpha} (\log n)^{-\beta(q-\alpha)} (\log\log n)^q \sum_{i\in B_n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le Cn^{-1} b_n^{-\gamma} (\log n)^{-\beta(q-\gamma)} (\log\log n)^q \sum_{i\in B_n} |a_{ni}|^{\alpha} E|X|^{\gamma} I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\le Cn^{-1} (\log n)^{-\beta(q-\gamma)-1} (\log\log n)^q E|X|^{\gamma} < \infty.
\end{aligned}
$$

For J_{14}, we have

$$\begin{aligned}
J_{14} &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log\log n)^q \int_1^{\infty} t^{-q/\alpha} \left(\sum_{i\in B_n} EX_{ni}^2(1) \right)^{q/2} \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log\log n)^q \int_1^{\infty} \left(b_n^{-\alpha} t^{-1} (\log n)^{-\beta(2-\alpha)} \sum_{i\in B_n} |a_{ni}|^{\alpha} E|X|^{\alpha} \right)^{q/2} \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log\log n)^q \int_1^{\infty} t^{-q/2} \left((\log n)^{-\beta(2-\alpha)-\alpha/\gamma} E|X|^{\gamma} \right)^{q/2} \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} (\log n)^{-q\alpha/2\gamma - q\beta(2-\alpha)/2} (\log\log n)^q (E|X|^{\gamma})^{q\alpha/2\gamma}.
\end{aligned}$$

Then, $J_{14} \le (E|X|^{\gamma})^{q\alpha/2\gamma} < \infty$ for $q > 2\gamma/\alpha$ and $\alpha \le 2$.Above all, we have $J_1 < \infty$. For J_2, first note that $0 \le X_{ni}(2) \le b_n \varepsilon t^{1/\alpha}/4N$, then

$$\begin{aligned}
J_2 &= \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n} |\sum_{i=1}^{m} X_{ni}(2)| > b_n \varepsilon t^{1/\alpha}/4 \Big) \mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} \left(\sum_{i=1}^{n} P(a_{n,i} X_i > b_n (\log n)^{-\beta} t^{1/\alpha}) \right)^N \mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} \left(b_n^{-\gamma} (\log n)^{\beta\gamma} t^{-\gamma/\alpha} E|X|^{\gamma} \sum_{i=1}^{n} |a_{ni}|^{\gamma} \right)^N \mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{-1} (\log n)^{(\beta\gamma-1)N} \int_1^{\infty} t^{-N\gamma/\alpha} \mathrm{d}t.
\end{aligned}$$

Hence, $J_2 < \infty$ if we take N large enough such that $(1-\beta\gamma)N > 1$. Similar to the proof of J_2, we can get that $J_3 < \infty$ if we take N large enough such that $(1-\beta\gamma)N > 1$. For J_4, it can be get that,

$$\begin{aligned}
J_4 &= \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n} |\sum_{i=1}^{m} X_{ni}(4)| > b_n \varepsilon t^{1/\alpha}/4 \Big) \mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\sum_{i=1}^{n} |X_{ni}(4)| > b_n \varepsilon t^{1/\alpha}/4 \Big) \mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} b_n^{-q} \varepsilon^{-q} \int_1^{\infty} t^{-q/\alpha} E|X_{ni}(4)|^q \mathrm{d}t \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| > b_n \varepsilon/4N).
\end{aligned}$$

Hence, $J_4 \le CE|X|^{\gamma} < \infty$ by Lemma 2.3 of Wu et al. (2014). The proof is completed. □

Remark 3.1. Noting that

$$\sum_{n=2}^{\infty} n^{-1} E\left\{ b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| - \varepsilon \right\}_+^{\alpha}$$
$$= \int_0^{\infty} \sum_{n=2}^{\infty} n^{-1} P\left(b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right) \mathrm{d}t.$$

Therefore, Theorem 3.1 extends and improves the main results of Chen (2016).

Using the arguments similar to the proof of Theorem 3.1, we can get the following results about L^p convergence.

Theorem 3.2. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively orthant dependent random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > \alpha$. Furthermore, suppose that $EX_i = 0$ when $1 < \alpha < 2$. If $E|X|^{\gamma} < \infty$, then

$$\frac{1}{b_n} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| \xrightarrow{L^{\alpha}} 0.$$

Proof. Following the notation and arguments as Theorem 3.1, we have

$$E\left(\frac{1}{b_n} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right|\right)^{\alpha}$$
$$= b_n^{-\alpha} \int_0^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > t^{1/\alpha}\right) \mathrm{d}t$$
$$\le \varepsilon + b_n^{-\alpha} \int_{b_n^{\alpha}\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > t^{1/\alpha}\right) \mathrm{d}t$$
$$\le \varepsilon + b_n^{-\alpha} \int_{b_n^{\alpha}\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(1)\right| > t^{1/\alpha}\right) \mathrm{d}t$$
$$+ b_n^{-\alpha} \int_{b_n^{\alpha}\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(2)\right| > t^{1/\alpha}\right) \mathrm{d}t$$
$$+ b_n^{-\alpha} \int_{b_n^{\alpha}\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(3)\right| > t^{1/\alpha}\right) \mathrm{d}t$$
$$+ b_n^{-\alpha} \int_{b_n^{\alpha}\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(4)\right| > t^{1/\alpha}\right) \mathrm{d}t.$$

Similarly, to the proof of Theorem 3.1, we can get the proof, so we omit it.

4. Conclusion

In this paper, we have studied the complete moment convergence for negatively orthant dependent random variables. Our results strengthen the main results of Chen (2016) by obtaining a much stronger conclusion under the same conditions. Based on our method, we also investigated L^p convergence for the NOD sequence.

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On k-special R-implications

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Abstract. Triangular norm based implications play a significant role in many fields of mathematics and computer science. In this paper, the notion of k-special fuzzy implications is introduced for all $k \in [1, \infty[$. Then, it is shown that all k-special R-implications can be characterized by the k-Lipschitz continuity of the corresponding t-norms.

Keywords: fuzzy logics, fuzzy implications, triangular norms, k-Lipschitz continuity, k-special R-implications.

1. Introduction

Triangular norms (t-norms for short) and its related operations have various applications in many fields of mathematics, computer science and decision making [2,4,12,14,15]. Many existing results show that the connectives in fuzzy logic closely link to their counterparts in applications. In 1996, Hájek and Kohout uncovered the connections between implication connectives in fuzzy logic and implication quantifiers in the literature of GUHA data mining method in [3], the main result is that each special implication quantifier determines a special implication connective, and conversely each special implications connective is given by a special implicational quantifier. Recently, Emilia et. al. in [13] characterized all special R-implication connectives by 1-Lipschitz property of their corresponding t-norms. Inspired by those above, we introduce the k-speciality of R-implications, which can be characterized by k-Lipschitz continuity of the corresponding t-norms with $k \in [1, +\infty[$, which have been studied extensively in the framework of the controlling the output stability in the case of noisy inputs[5, 8, 9, 10]. By this mean, we can add some new insights into the fuzzy implications and investigate some properties of a subclass of BL-implications, which can be used to generate implication quantifiers in data mining in virtue of the close relationship between implication connectives and implication quantifiers.

1.1 Continuous Archimedean t-norms and their additive generators

For the basic notions and results on t-norms, we recommend references [6,7].

Lemma 1.1. *A t-norm T is continuous and Archimedean if and only if there is continuous strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$, and for all $(x,y) \in [0,1]^2, T(x,y) = f^{-1}(\min(f(x) + f(y), f(0)))$.*

Usually, the function f in Lemma 1.1 is called the additive generator of the t-norm T. It is known that all continuous Archimedean t-norms can be represented by their corresponding additive generators. Namely, a continuous t-norm T is characterized to be an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$, i.e.,

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}); & (x,y) \in]a_i, b_i[^2; \\ \min(x,y), & \text{else.} \end{cases} \tag{1}$$

with the index set $\mathcal{I}$ being countable and each T_i being a continuous Archimedean t-norm, which is isomorphic to Łukasiewicz or Product t-norm. According to [1,6,11], for any continuous Archimedean t-norm T with additive generator f, the R-implication I_T can also be represented by its additive generator f, i.e.,

$$I_T(x,y) = f^{-1}(\max(0, f(y) - f(x))). \tag{2}$$

Moreover, the R-implication of a continuous t-norm has related ordinal sum structure [1,2], i.e.,

$$I_T(x,y) = \begin{cases} g_i^{-1}(g_i(y) - g_i(x) + a_i); & (x,y) \in]a_i, b_i[^2, y < x; \\ I_{T_M}(x,y), & \text{else.} \end{cases} \tag{3}$$

where $T = (]a_i, b_i[, T_i)_{i\in\mathcal{I}}$ with the additive generator of T_i being f_i, and $g_i(x) = (b_i - a_i)f(\frac{x-a_i}{b_i-a_i}) + a_i$.

1.2 k-Lipschitz continuity of t-norms and k-convexity of its additive generators

Definition 1.1 ([8]). *(i) Let $f : [0,1] \to [0,+\infty[$ be a strictly decreasing function and $k \in [1,+\infty]$ be a real constant, then f is called k-convex, if*

$$f(x + k\varepsilon) - f(x) \le f(y + \varepsilon) - f(y)$$

holds for all $x \in [0,1[, y \in]0,1[$, with $x \le y$ and $\varepsilon \in]0, \min(1-y, \frac{1-x}{k})]$.

(ii) A t-norm is called k-Lipschitz, if there is a real constant $k \in [1,+\infty[$, such that

$$|T(x_1,y_1) - T(x_2,y_2)| \le k(|x_1 - x_2| + |y_1 - y_2|),$$

for all $(x_1,y_1), (x_2,y_2) \in [0,1]^2$.

The k-Lipschitz continuity [5, 8, 9] of t-norms was proposed in the text of stability of t-norms. Trivially, a t-norm is necessarily continuous if it is k-Lipschitz. In order to keep the paper self-contained, we list some related results here.

Lemma 1.2 ([8, 9])**.** (*i*) *A continuous Archimedean t-norm is k-Lipschitz if and only if its additive generator is k-convex;*

(*ii*) *A continuous t-norm is k-Lipschitz if and only if its summands* (*continuous Archimedean t-norms*) *are all k-Lipschitz.*

2. k-special R-implications

Usually, a binary operation $I : [0,1]^2 \to [0,1]$ is called fuzzy implication, if it holds that:

(i) $I(0,0) = I(0,1) = I(1,1) = 1, I(1,0) = 0$;

(ii) I is non-increasing w.r.t. the first argument and non-decreasing w.r.t. the second argument.

The prototypical example of fuzzy implication is the residual implication I_T (R-implication for short) of a left-continuous t-norm T, defined by $I_T(x,y) = \max\{t \in [0,1] | T(x,t) \leq y\}$. By the axioms of t-norms it can be concluded that $I_T(x,y) = 1$ for all $x \leq y$.

Definition 2.1. *Let $k \in [1,\infty[$, then a fuzzy implication I is called k-special if it follows that:$I(x,y) \leq I(x+\varepsilon, y+k\varepsilon)$, for all $(x,y) \in [0,1]^2, \varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$.*

Remark 2.1. If $k = 1$, then it is just the case (special fuzzy implications) discussed in [12], where a fuzzy implication I is called special if $I(x,y) \leq I(x+\varepsilon, y+\varepsilon)$for all $\varepsilon > 0$ with $\max(x+\varepsilon, y+\varepsilon) \leq 1$. Therefore, in the following we only focus on the cases $k \in]1,\infty[$.

Lemma 2.1. *If a fuzzy implication I is k-special, then I is l-special for all $l \geq k$.*

Proof. It is enough to consider the cases $l > k$. For all $\varepsilon > 0$, assume $(x,y) \in [0,1]^2$ and $x + \varepsilon \leq 1, y + k\varepsilon < y + l\varepsilon \leq 1$. Since I is k-special, then

$$I(x,y) \leq I(x+\varepsilon, y+k\varepsilon). \tag{4}$$

Since I is non-increasing w.r.t. the second argument, then

$$I(x+\varepsilon, y+k\varepsilon) \leq I(x+\varepsilon, y+l\varepsilon). \tag{5}$$

Therefore,$I(x,y) \leq I(x+\varepsilon, y+l\varepsilon)$ by the combination of (4) and (5). Thus I is l-special for all $l \geq k$. □

However, the converse is not true. That is to say, there may be a k-special fuzzy implication being not l-special, where $k > 1$ and $1 \leq l < k$(See Example 2.1). Moreover, by direct calculation, the Łuksiewicz implication I_{T_L} and Gödel implication I_{T_M}, defined by $I_{T_L}(x,y) = \max(1, 1-x+y)$ and

$$I_{T_M}(x,y) = \begin{cases} 1; & x \leq y \\ y, & \text{else.} \end{cases}$$

respectively, is k-special for all $k \geq 1$.

Theorem 2.1. *For any left-continuous t-norm T, if the R-implication I_T is k-special, then T is continuous.*

Proof. It is enough to check the right-continuity of T w.r.t. the first argument due to its commutativity. Assume that T is not right-continuous at a point $x \in [0,1[$. Trivially, the function $T(x,0) = 0$ is continuous. Thus there must be $z \in]0,1]$ and $d > 0$, such that $T(x,z) = y$, and for all $u > x, T(u,z) \geq T(x,z) + d = y + d$. In particular, $T(1,z) = z \geq y + d$.

In the following, we assume $z = I_T(x,y)$. In fact, $T(x,z) = y$ ensures $z \leq I_T(x,y)$, then $y = T(x,z) \leq T(x, I_T(x,y)) \leq y$. Thus $T(x, I_T(x,y)) = y$. Further, for all $u > x, T(u, I_T(x,y)) \geq T(u,z) \geq y + d$. Therefore, we can replace z with $I_T(x,y)$ whenever $z < I_T(x,y)$. Now let $\varepsilon = \min\{1-x, \frac{d}{2k}\}$, it follows that $T(x+\varepsilon, z) \geq y + d > y + k\varepsilon$, then $T(x+\varepsilon, z) > y + k\varepsilon$, which means that $I_T(x+\varepsilon, y+k\varepsilon) = \sup\{t \in [0,1] | T(x+\varepsilon, t) \leq y + k\varepsilon\} < z = I_T(x,y)$, which contradicts the k-speciality of I_T. □

Corollary 2.1. *k-special R-implications are BL-implications, i.e., R-implications of continuous t-norms.*

It must be mentioned that, not all BL-implications are special [13]. Furthermore, not all BL-implications are k-special. It can be checked similarly as [13] that, the R-implications of Schweizer-Sklar t-norms, i.e.,

$$T(x,y) = \begin{cases} \sqrt[p]{x^p + y^p - 1}, & x^p + y^p \geq 1; \\ 0, & \text{else.} \end{cases}$$

are not k-special whenever $p \neq 1$. Hence, k-special implications of left-continuous t-norms are a subclass of BL-implications. In other words, by Theorem 2.1, it holds that a left-continuous t-norm with its R-implication being k-special is necessarily continuous.

Theorem 2.2. *The R-implication I_T of a continuous Archimedean t-norm T is k-special if and only if the additive generator f of T is k-convex.*

Proof. "⇒": For all $(x,y) \in [0,1]^2$ and $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, there are the following cases to be discussed.

(i) If $x \leq y$, then it follows that $I_T(x,y) = I_T(x+\varepsilon, y+k\varepsilon) = 1$.

(ii) If $x > y$ with $x + \varepsilon \leq y + k\varepsilon$, then $f(y) - f(x) \geq 0 \geq f(y+k\varepsilon) - f(x+\varepsilon)$ by the non-increasingness of f. Thus $f(y + k\varepsilon) - f(y) \leq f(x + \varepsilon) - f(x)$.

(iii) If $x > y$ with $x + \varepsilon > y + k\varepsilon$, then $I_T(x,y) = f^{-1}(f(y) - f(x))$, $I_T(x + \varepsilon, y + k\varepsilon) = f^{-1}(f(y + k\varepsilon) - f(x + \varepsilon))$ by (2). With the k-speciality of I_T, $f^{-1}(f(y) - f(x)) \leq f^{-1}(f(y + k\varepsilon) - f(x + \varepsilon))$. It follows that $f(y) - f(x) \geq f(y + k\varepsilon) - f(x + \varepsilon)$, which means $f(y + k\varepsilon) - f(y) \leq f(x + \varepsilon) - f(x)$.

By all of the above, f is k-convex.

"⇐": For all $(x,y) \in [0,1]^2$ and $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, it can be gotten that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon) = 1$ for the case $x \leq y$ and the case $x > y$ with $x+\varepsilon \leq$

$y+k\varepsilon$. So it is enough to consider the case $x > y$ with $x+\varepsilon < y+k\varepsilon$, for which we have $I_T(x,y) = f^{-1}(f(y)-f(x))$, $I_T(x+\varepsilon, y+k\varepsilon) = f^{-1}(f(y+k\varepsilon)-f(x+\varepsilon))$. Moreover, the k-convexity of f ensures that $f(y)-f(x) \geq f(y+k\varepsilon)-f(x+\varepsilon)$, then $f^{-1}(f(y)-f(x)) \leq f^{-1}(f(y+k\varepsilon)-f(x+\varepsilon))$, i.e., $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$.

To sum up, I_T is k-special. □

By Lemma 1.2(i), we have the following result.

Corollary 2.2. *The R-implication of a continuous Archimedean t-norms T is k-special if and only if T is k-Lipschitz.*

Since the R-implications of continuous t-norms have ordinal sum structure, then we can characterize all k-special R-implications in the following.

Theorem 2.3. *The R-implication I_T of a continuous t-norm T is k-special if and only if the corresponding t-norm T is an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$, where each t-norm T_i is generated by a k-convex additive generator f_i. (Equivalently, each t-norm T_i is k-Lipschitz by Lemma 1.1(i).)*

Proof. "⇒": By the continuity of T, T can be represented as an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$ as (1), hence the R-implication I_T has the structure as (3). For each $i \in \mathcal{I}$ and $(x,y) \in [0,1]^2$ with all $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, assume $x > y$, then it can be classified into the following two cases.

(i) If $x+\varepsilon \leq y+k\varepsilon$, then $f_i(y)-f_i(x) \geq 0 \geq f_i(y+k\varepsilon)-f_i(x+\varepsilon)$ by the non-increasingness of f_i which follows that $f_i(y+k\varepsilon)-f_i(y) \leq f_i(x+\varepsilon)-f_i(x)$.

(ii) If $x+\varepsilon > y+k\varepsilon$, then $I_T(x(b_i-a_i)+a_i, y(b_i-a_i)+a_i) = g_i^{-1}(g_i(y(b_i-a_i)+a_i)-g_i(x(b_i-a_i)+a_i)+a_i) \leq I_T((x+\varepsilon)(b_i-a_i)+a_i, (y+k\varepsilon)(b_i-a_i)+a_i) = g_i^{-1}(g_i((y+k\varepsilon)(b_i-a_i)+a_i)-g_i((x+\varepsilon)(b_i-a_i)+a_i)+a_i)$ by (3) and the k-speciality of I_T. Thus we have $g_i(y(b_i-a_i)+a_i)-g_i(x(b_i-a_i)+a_i) \geq g_i((y+k\varepsilon)(b_i-a_i)+a_i)-g_i((x+\varepsilon)(b_i-a_i)+a_i)$, which implies $f_i(y)-f_i(x) \geq f_i(y+k\varepsilon)-f_i(x+\varepsilon)$, whence $f_i(y+k\varepsilon)-f_i(y) \leq f_i(x+\varepsilon)-f_i(x)$.

Then we get the k-convexity of each f_i.

"⇐": For the case $x \leq y$ and the case $x > y$ with $x+\varepsilon \leq y+k\varepsilon$ it holds that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon) = 1$, where $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$. For the case $x > y$ with $x+\varepsilon > y+k\varepsilon$, the proof can be divided into the following cases.

(i) If for each $i \in \mathcal{I}$, $(x,y) \notin]a_i, b_i[^2$ and $(x+\varepsilon, y+k\varepsilon) \notin]a_i, b_i[^2$, then $I_T(x,y) = I_{T_M}(x,y) = y \leq I_T(x+\varepsilon, y+k\varepsilon) = I_{T_M}(x+\varepsilon, y+k\varepsilon) = y+k\varepsilon$ by (3).

(ii) If $(x,y) \in]a_l, b_l[^2$ for some $l \in \mathcal{I}$ and $(x+\varepsilon, y+k\varepsilon) \notin]a_i, b_i[^2$ for each $i \in \mathcal{I}$, $I_T(x,y) \leq b_l$ and $I_T(x+\varepsilon, y+k\varepsilon) = I_{T_M}(x+\varepsilon, y+k\varepsilon) = y+k\varepsilon$. Since $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$ for $y+k\varepsilon \geq b_l$, it is enough to consider the case $y+k\varepsilon \in]a_l, b_l[$, for which it holds that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$. If not, there must be some x_0, y_0, ε_0 with $I_T(x_0,y_0) > I_T(x_0+\varepsilon_0, y_0+k\varepsilon_0) = y_0+k\varepsilon_0$, hence $T(x_0, y_0+k\varepsilon_0) \leq y_0$. Let $z_0 = I_T(x_0,y_0) = \max\{t \in [0,1] \mid T(x_0,t) \leq y_0\} > y_0+k\varepsilon_0$, then $T(x_0,z_0) \leq y_0$.

Moreover, $T(x_0, z_0) < y_0$. Indeed, if $T(x_0, z_0) = y_0$, then $y_0 \geq T(x_0, z_0) \geq T(x_0, y_0 + k\varepsilon_0) = y_0 \in]a_l, b_l[$, i.e., $T(x_0, z_0) = T(x_0, y_0 + k\varepsilon_0) = y_0 \in]a_l, b_l[$, which implies $g_l^{-1}(g_l(x_0) + g_l(z_0)) = g_l^{-1}(g_l(x_0) + g_l(y_0 + k\varepsilon_0)) = y_0$ with the fact that $g_l(x_0) + g_l(z_0), g_l(x_0) + g_l(y_0 + k\varepsilon_0) < g_l(a_l)$, thus $g_l(z_0) = g_l(y_0 + k\varepsilon_0)$, which contradicts the strictly decreasingness of g_l. So we have $T(x_0, y_0 + k\varepsilon_0) < y_0 < y_0 + k\varepsilon_0 = \min(x_0 + \varepsilon_0, y_0 + k\varepsilon_0) = T(x_0 + \varepsilon_0, y_0 + k\varepsilon_0)$. It follows that $T(x_0 + \varepsilon_0, y_0 + k\varepsilon_0) - T(x_0, y_0 + k\varepsilon_0) > k\varepsilon_0$ which contradicts the k-Lipschitz continuity of T implied by the k-convexity of each additive generator f_i with Lemma1.2(ii). To sum up, we have $I_T(x, y) \leq I_T(x + \varepsilon, y + k\varepsilon)$

(iii) If $(x, y) \in]a_i, b_i[^2$ and $(x + \varepsilon, y + k\varepsilon) \in]a_l, b_l[^2$for some $l, i \in \mathcal{I}$ with $i \neq l$, then $b_i \leq a_l$, whence $I_T(x, y) \leq b_i \leq a_l \leq I_T(x + \varepsilon, y + k\varepsilon)$.

(iv) If $(x, y) \in]a_i, b_i[^2$ and $(x + \varepsilon, y + k\varepsilon) \in]a_i, b_i[^2$ for some $i \in \mathcal{I}$, the k-speciality can be checked by the k-convexity of f_i.

(v) If $(x, y) \notin]a_i, b_i[^2$ for each $i \in \mathcal{I}$ and $(x + \varepsilon, y + k\varepsilon) \in]a_l, b_l[^2$for some $l \in \mathcal{I}$, then $y \leq a_l$, hence $I_T(x, y) = y \leq a_l \leq I_T(x + \varepsilon, y + k\varepsilon)$. □

Corollary 2.3. *An R-implication I_T of a left-continuous t-norm T is k-special if and only if T is k-Lipschitz.*

Proof. It can be easily checked by Theorem 2.1,Theorem 2.3 and Lemma 1.2. □

It is shown that if a t-norm T is k-Lipschitz, then T is l-Lipschitz for all $l \geq k$. Therefore Corollary 2.3 is consistent to Lemma 2.1.

There exists families of k-convex additive generators (see [5,8]). For a better understanding of k-speciality, we present an example of k-special R-implications.

Example 2.1. In virtue of the k-convex additive generators $t_k^{\frac{1}{2}}$ in [8], i.e., $t_k^{\frac{1}{2}}(x) = \min(1 - x, \frac{1}{2} + \frac{\frac{1}{2} - x}{k})$, with its corresponding t-norm $T_k^{\frac{1}{2}}$ being defined by

$$T_k^{\frac{1}{2}}(x, y) = \begin{cases} \max(0, x + y - k - \frac{1}{2}(k - 1)), & (x, y) \in [0, \frac{1}{2}]^2; \\ \max(0, x + ky - k), & (x, y) \in [0, \frac{1}{2}] \times]\frac{1}{2}, 1]; \\ \max(0, y + kx - k), & (x, y) \in]\frac{1}{2}, 1] \times [0, \frac{1}{2}]; \\ \max(0, k(x + y) - k - \frac{1}{2}(kn - 1)), & (x, y) \in]\frac{1}{2}, 1]^2, x + y \leq \frac{3}{2}; \\ x + y - 1, & (x, y) \in]\frac{1}{2}, 1]^2, x + y > \frac{3}{2} \end{cases}$$

where $k \in [1, +\infty[$. Then by direct calculation we can have a family of k-special R-implications I_k of the t-norms $T_k^{\frac{1}{2}}$, defined by

$$I_k^{\frac{1}{2}}(x, y) = \begin{cases} 1, & x \leq y; \\ 1 - \frac{x - y}{k}, & (x, y) \in [0, \frac{1}{2}[^2, x > y; \\ 1 - x + y, & (x, y) \in [\frac{1}{2}, 1]^2, x > y \\ \frac{y}{k} - x - \frac{1}{2k} + \frac{3}{2}, & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}[, x - \frac{y}{k} < 1 - \frac{1}{2k}, x > y; \\ y - kx + k, & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}[, x - \frac{y}{k} \geq 1 - \frac{1}{2k}, x > y; \end{cases}$$

Let $k = 2$, since $t_2^{\frac{1}{2}}$ is 2-convex, then $I_2^{\frac{1}{2}}$ is 2-special. However, $I_2^{\frac{1}{2}}$ is not 1-special (special). Indeed, let $x_0 = 0.3, y_0 = 0.1, \varepsilon_0 = 0.3$, then $I_2^{\frac{1}{2}}(x_0, y_0) = 0.9$ and $I_2^{\frac{1}{2}}(x_0 + \varepsilon_0, y_0 + \varepsilon_0) = 0.85$, hence $I_2^{\frac{1}{2}}(x_0, y_0) \leq I_2^{\frac{1}{2}}(x_0 + \varepsilon_0, y_0 + \varepsilon_0)$ does not hold.

3. Conclusion

The notion of k-special fuzzy implications is introduced and the k-special R-implications are characterized by k-Lipschitz continuity of the corresponding t-norms, also the k-convexity of the corresponding additive generators. We would expect the k-special implication connectives be applicable in generating new tools in (fuzzy) data analysis similar to that in [3].

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On skew GQC and skew QC codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$

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Abstract. In this paper, we study some algebraic structural properties of skew quasi cyclic codes and skew generalized quasi-cyclic codes over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. We discuss on Hermitian dual of these classes of codes over R. Then, we investigate on the generator polynomials and the parity-check polynomials of 1-generator skew QC codes and 1-generator skew GQC codes. Finally, we show that the Gray image of a skew quasi-cyclic code over R is a skew l-quasi-cyclic code of index 4 and the Gray image of skew GQC code over R is a skew GQC code of index 4.

Keywords: skew polynomial ring, skew QC code, skew GQC code, gray map, Hermitian dual.

1. Introduction

Algebraic coding theory is an area of discrete applied mathematics that is concerned with developing error-control codes and encoding/decoding procedures. Many areas of mathematics are used in coding theory, and we focus on the interplay between algebra and coding theory. Cyclic codes over finite fields have been studied since the late 1950 and play a significant role in the coding theory. In 1994 Hammons et al. [12] studied codes over $\mathbb{Z}_4$, then a lot of researches went towards studying codes over $\mathbb{Z}_4$. Bonnecaze and Udaya [5] have studied cyclic codes over the ring $F_2 + uF_2$; $u^2 = 0$. This ring is useful because it shares many properties of $\mathbb{Z}_4$. A complete structure of cyclic codes over $\mathbb{Z}_4$ of odd length has been given in [16].

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Recently, it has been shown that codes over finite rings are an important class of codes. Quasi-cyclic (QC) codes are a natural and remarkable generalization of cyclic codes. QC code of index l over finite ring, defined by the property that a cyclic shift of a codeword by l places is another codeword, generalize the class of cyclic codes ($l = 1$). Skew polynomial rings form an important class of non-commutative rings. Boucher, Geiselmann and Ulmer in [4] and [10] studied linear and cyclic codes over skew polynomial rings. Also skew QC codes are constructed in [1] with the property that $| < \theta > | = m$. The algebraic structure of generalized quasi-cyclic (GQC) codes over finite fields were introduced by Siap and Kulhan in [18]. Since then, some properties of these codes were studies by Esmaeili and Yari in [9]. In [7], Cao studied GQC code of arbitrary length over finite fields. Cao investigated structural properties of 1-generator GQC code. Also GQC codes over Galois rings were introduced by Cao.

In this paper we firstly study some basic properties of the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. Also we organize basic notations of skew QC and skew GQC codes. In Section 3 we discuss on the dual of skew QC and skew GQC codes. Also we obtain a necessary and sufficient condition on self-dual Hermitian skew QC codes and self-dual Hermitian skew GQC codes. Finitely we investigate the Gray image of skew QC codes and skew GQC codes over the ring R in Section 4.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of cardinality q. A k-dimensional vector subspace C of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$ is called a linear (n, k)-code over $\mathbb{F}_q$. A linear code C is called a cyclic code if whenever $(a_0, a_1, \ldots, a_{n-1}) \in C$, then $(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in C$. In order to convert the combinatorial structure of cyclic codes into an algebraic one, we consider the following correspondence:

$$\pi : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q[x]/(x^n - 1),$$
$$(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}.$$

The mapping π is a linear transformation of vector spaces over $\mathbb{F}_q$. Then a nonempty subset C of F_q^n is a cyclic code if and only if $\pi(C)$ is an ideal of $F_q[x]/(x^n - 1)$. We will sometimes identify $\mathbb{F}_q^n$ with $\mathbb{F}q[x]/(x^n - 1)$, and a vector $u = (u_0, u_1, \ldots, u_{n-1})$ with the polynomial $u(x) = \sum_{i=0}^{n-1} u_i x^i$.

The notion of skew cyclic codes introduced in [4] as a generalization of cyclic codes. Let θ be an automorphism of $\mathbb{F}_q$. A θ-cyclic code (or skew cyclic code) is a linear code C with the property that

$$(a_0, a_1, \ldots, a_{n-1}) \in C \Rightarrow (\theta(a_{n-1}), \theta(a_0), \ldots, \theta(a_{n-2})) \in C.$$

It is easy to check that cyclic codes correspond to the case where θ is the identity mapping.

Throughout this paper we consider the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. This ring is a Frobenius ring of characteristic 2. Since

$$u(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2) \not\subset v(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)$$

and

$$v(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2) \not\subset u(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)$$

so R is not a finite chained ring. Note that R is a local ring with the maximal ideal $I_{u,v} = \{0, u, v, u+v, uv, u+uv, v+uv, u+v+uv\}$, since $\overline{R} = \frac{R}{I_{u,v}} \cong \mathbb{F}_2$ and $R = \{0 + I_{u,v}\} \cup \{1 + I_{u,v}\}$. If we define the non-trivial ring homomorphism θ from R to R with $\theta(0) = 0, \theta(u) = v, \theta(1) = 1, \theta(uv) = vu, \theta(v) = u$, then $\theta^2(X) = X$ for any $X \in R$. This implies that θ is a ring automorphism of order 2. The Skew polynomial ring of R by θ is defined by

$$R[x;\theta] = \{a_0 + a_1x + \ldots + a_nx^n | a_i \in R, \forall i = 0, 1, 2, \ldots, n\},$$

where the coefficients are written on the left side of the variable x. The multiplication is defined by the basic rule $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$, and the addition is defined to be the usual addition rule of polynomials.

We can consider R as a natural extension of the ring $\mathbb{F}_2 + u\mathbb{F}_2$, so we can extend the definition of the Lee weight and Hamming weight from $\mathbb{F}_2 + u\mathbb{F}_2$ to the ring R.

Let w_l denote the Lee weight and w_H denote the Hamming weight for the binary codes. We set

$$w_L(x + uy + vz + uvt) = w_H(x + y + z + t, z + t, y + t, t) \quad \forall x, y, z, t \in \mathbb{F}_2.$$

The definition of the Lee weight and Hamming weight lead to a Gray map. By [20] we consider the following Gray map

$$\phi' : \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2 \to \mathbb{F}_2^4,$$
$$\phi'(x + uy + vz + uvt) = (x + y + z + t, z + t, y + t, t).$$

This map can be extended to R^n as follows:

$$\phi : ((\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)^n, Lee\ weight) \to (\mathbb{F}_2^{4n}, \text{Hamming weight}).$$

Now, let $R_m = \frac{R[x;\theta]}{<x^m - 1>}$ and let l be a positive integer. Then, the ring R_m^l is a left R_m-module where the scalar multiplication from left is defined by

$$f(x)(g_1(x), g_2(x), ..., g_l(x)) = (f(x)g_1(x), f(x)g_2(x), ..., f(x)g_l(x)),$$

for every $f(x) \in R[x;\theta]$ and $g_i(x) \in R_m$ for $i = 1, 2, \ldots, l$. Let C be a subset of R^{ml}. Consider the map ψ from R^{ml} to R_m^l given by $\psi(C) = (X_0(x), ..., X_{l-1}(x))$ where

$$X_j(x) = \sum_{i=0}^{m-1} r_{j,i}x^i \in \frac{R[x,\theta]}{< x^m - 1 >}, \qquad \forall j = 0, 1, 2, \ldots, l-1.$$

The map ψ is a one-to-one correspondence between the ring R^{ml} and R_m^l.

Definition 2.1. *A nonempty subset C of R^n is called a skew cyclic code of length n, if C satisfies the following conditions:*

1. *C is submodule of R^n; and*
2. *if $X = (r_0, r_1, ..., r_{n-1}) \in C$, then the skew cyclic shift $\sigma(X)$ is in C, that is $\sigma(X) = (\theta(r_{n-1}), \theta(r_0), ..., \theta(r_{n-2})) \in C$.*

Definition 2.2. *A subset C of R^n, where $n = ml$, is called a skew quasi-cyclic code of length n and index l (or a skew l-QC code) if C satisfies the following conditions:*

1. *C is a subspace of R^n; and*
2. *if $X = (r_{0,0}, r_{0.1}, \ldots, r_{0,l-1}, r_{1,0}, r_{1,1}, \ldots, r_{1,l-1}, \ldots, r_{m-1,0}, r_{m-1,1}, \ldots, r_{m-1,l-1})$ is a codeword in C, then $\tau_{\theta,m,l}(X) = (\theta(r_{m-1,0}), \theta(r_{m-1,1}), \ldots, \theta(r_{m-1,l-1}), \theta(r_{0,0}), \theta(r_{0,1}), \ldots, \theta(r_{0,l-1}), \ldots, \theta(r_{m-2,0}), \theta(r_{m-2,1}), \ldots, \theta(r_{m-2,l-1})) \in C$.*

Theorem 2.1. *A subset C of R^n, where $n = ml$, is a skew l-QC code of length n if and only if $\psi(C)$ is a left submodule of the ring R_m^l.*

Proof. Let C be a skew l-QC code and $c = (r_{0,0}, r_{0,1}, \ldots, r_{0,l-1}, r_{1,0}, r_{1,1}, \ldots, r_{1,l-1}, \ldots, r_{m-1,0}, r_{m-1,1}, \ldots, r_{m-1,l-1}) \in C$. Since $x^m = 1$ in R_m, we have

$$xX_j(x) = \sum_{i=0}^{m-1} \theta(r_{j,i}) x^{i+1} = \sum_{i=0}^{m-1} \theta(r_{j,i-1}) x^i \in R_m, \tag{1}$$

for any $j = 0, 1, \ldots, l-1$ and $i - 1 \in \{0, 1, \ldots, m-1\}$ by taking modulo m. Hence $(xX_0(x), xX_1(x), \ldots, xX_{l-1}(x)) \in \psi(C)$. Then, it follows from linearity that $f(x)\psi(c) \in \psi(C)$ for any $f(x) \in R_m$. Therefore, $\psi(C)$ is a left submodule of R_m^l.

Conversely, suppose that Y is a left R_m-submodule of R_m^l. We claim that $X = \psi^{-1}(Y) \in R^{ml}$ is a skew l-QC code. It is enough to show that C is closed under skew cyclic shift. Let $c = (r_{0,0}, r_{0.1}, \ldots, r_{0,l-1}, r_{1,0}, x_{1,1}, \ldots, r_{1,l-1}, \ldots, r_{m-1,0}, r_{m-1,1}, \ldots, r_{m-1,l-1})) \in X$. Then $\psi(c) \in Y$. By a similar argument like in (2.1), we see that $\psi(\tau_{\theta,m,l}(c)) = x\psi(c) \in Y$, so $\tau_{\theta,m,l} \in X$. Hence X is a skew l-QC code. □

Definition 2.3. *Let $m_1, m_2, \ldots, m_t$ be positive integers and $m = \sum_{i=1}^{t} m_i$. A subset C of R^m is called a skew generalized quasi-cyclic (skew GQC) code of block length $(m_1, m_2, \ldots, m_t)$ and length m if C satisfies in the following conditions:*

1. *C is a subspace of R^m,*
2. *if $X = (r_{0,0}, \ldots, r_{0,m_1-1}, r_{1,0}, \ldots, r_{1,m_2-1}, \ldots, r_{t-1,0}, \ldots, r_{t-1,m_t-1}) \in C$, then $\tau_{\theta,m}(X) = (\theta(r_{t-1,0}), \ldots, \theta(r_{t-1,m_t-1}), \theta(r_{0,0}), \ldots, \theta(r_{0,m_1-1}), \ldots, \theta(r_{t-2,0}), \ldots, \theta(r_{t-2,m_{t-1}-1})) \in C$.*

If $m_1{=}m_2{=}\ldots=m_t=m$, then a GQC code of block length $(m_1,m_2,\ldots,m_t)$ is a skew t-QC code of length mt over the ring R. Let m_i be as in Definition 2.3 and $R_i=\frac{R[x;\theta]}{<x^{m_i}-1>}$ for any $i=1,2,\ldots,t$. Assume that the multiplication is defined by

$$f'(x)(g'_1(x),g'_2(x),\ldots,g'_t(x))=(f'(x)g'_1(x),f'(x)g'_2(x),\ldots,f'(x)g'_t(x))$$

in which every $f'(x)g'_i(x)$ is considered $mod(x^{m_i}-1)$, $f'(x)\in R[x;\theta]$ and $g'_i(x)\in R_i$ for $i=1,2,\ldots,t$. In this case, the ring $R':=R_1\times R_2\times\ldots\times R_t$ is an $R[x;\theta]$-submodule.

Consider the map ψ' from R^m to R' defined by $\psi'(C)=(Y_0(x),\ldots,Y_{t-1}(x))$ where C is a subset of R^m and

$$Y_j(x)=\sum_{i=0}^{m_{j+1}-1} r'_{j,i}x^i\in\frac{R[x;\theta]}{<x^{m_{j+1}}-1>},$$

for any $j=0,1,...,t-1$.

The following is a similar result to Theorem 2.1 for skew GQC codes.

Theorem 2.2. *Let $m_1,m_2,\ldots,m_t$ be positive integers and $m=\sum_{i=1}^t m_i$. A subset C of R^m is a skew GQC code of block length $(m_1,m_2,\ldots,m_t)$ and length m if and only if $\psi'(C)$ is a left submodule of the ring R'.*

Proof. The proof is similar to that of Theorem 2.1. □

3. Dual of SKEW QC codes and skew GQC codes

Assume that C is a skew l-QC code of length $n(n=ml)$. Let $X=(x_{0,0},\ldots,x_{0,l-1},$ $x_{1,0},\ldots,x_{1,l-1},\ldots,x_{m-1,0},\ldots,x_{m-1,l-1})$ and $Y=(y_{0,0},\ldots,y_{0,l-1},$ $y_{1,0},\ldots,y_{1,l-1},\ldots,y_{m-1,0},\ldots,y_{m-1,l-1})$ be codewords in C. The Hermitian inner product is defined by

$$<X,Y>_H=\sum_{j=0}^{l-1}\sum_{i=0}^{s-1}x_{i,j}\theta(y_{i,j}),$$

where $x_{i,j}\in X$ and $y_{i,j}\in Y$. The codewords X,Y are called orthogonal Hermitian with respect to the Hermitian inner product if $<X,Y>_H=0$. The dual code $C^{\perp_H}$ with respect to Hermitian inner product of C is defined by

$$C^{\perp_H}=\{X\in R^n\ \mid\ <X,Y>_H=0\ ,\forall Y\in C\}.$$

A skew l-QC code is called Hermitian self-dual if $C=C^{\perp_H}$ and whenever $C\subseteq C^{\perp_H}$, we say that C is Hermitian self-orthogonal.

Theorem 3.1. *Let C be a skew l-QC code of length $n(n=ml)$ over R. Then, the Hermitian dual of C is also a skew l-QC code.*

Proof. Let $X = (x_{0,0}, \ldots, x_{0,l-1}, x_{1,0}, \ldots, x_{1,l-1}, \ldots, x_{m-1,0}, \ldots, x_{m-1,l-1}) \in C$ and $Y = (y_{0,0}, \ldots, y_{0,l-1}, y_{1,0}, \ldots, y_{1,l-1}, \ldots, y_{m-1,0}, \ldots, y_{m-1,l-1}) \in C^{\perp_H}$. Since C is skew l-QC code, so $(\theta(x_{m-1,0}), \ldots, \theta(x_{m-1,l-1}), \theta(x_{0,0}), \ldots, \theta(x_{0,l-1}), \ldots, \theta(x_{m-2,0}), \ldots, \theta(x_{m-2,l-1})) \in C$.

Continuing this way, we have $X^i = (\theta^i(x_{m-i,0}), \ldots, \theta^i(x_{m-i,l-1}), \theta^i(x_{m-i+1,0}), \ldots, \theta^i(x_{s-i+1,l-1}), \ldots, \theta^i(x_{s-i-1,0}), \ldots, \theta^i(x_{s-i-1,l-1})) \in C$, for any i=1, 2, ..., m. Hence $< X^i, Y >_H = 0$. If $i = m-1$, then $\theta^{m-1}(x_{i',j'}) = \theta(x_{i',j'})$ for any $x_{i',j'} \in X$. Therefore, $\theta(x_{1,0})y_{0,0} + \ldots + \theta(x_{1,l-1})y_{0,l-1} + \theta(x_{2,0})y_{1,0} + \ldots + \theta(x_{2,l-1})y_{1,l-1} + \ldots + \theta(x_{0,0})y_{s-1,0} + \ldots + \theta(x_{0,l-1})y_{m-1,l-1} = 0$. Note that θ is a ring automorphism of order 2. Hence, $x_{0,0}\theta(y_{m-1,0}) + \ldots + x_{0,l-1}\theta(y_{m-1,l-1}) + x_{1,0}\theta(y_{0,0}) + \ldots + x_{1,l-1}\theta(y_{0,l-1}) + \ldots + x_{m-1,0}\theta(y_{m-2,0}) + \ldots + x_{m-1,l-1}\theta(y_{m-2,l-1}) = 0$. This gives $(\theta(y_{m-1,0}), \ldots, \theta(y_{m-1,l-1}), \theta(y_{0,0}), \ldots, \theta(y_{0,l-1}), \ldots, \theta(y_{m-2,0}), \ldots, \theta(y_{m-2,l-1})) \in C^{\perp_H}$. Hence $C^{\perp_H}$ is also a skew QC code of index l and length n. □

We can define the Hermitian dual skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ in a similar way. Let C be a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$. Suppose that $X = (x_{0,0}, \ldots, x_{0,m_1-1}, x_{1,0}, \ldots, x_{1,m_2-1}, \ldots, x_{t-1,0}, \ldots, x_{t-1,m_t-1})$ and $Y = (y_{0,0}, \ldots, y_{0,m_1-1}, y_{1,0}, \ldots, y_{1,m_2-1}, \ldots, y_{t-1,0}, \ldots, y_{t-1,m_t-1})$ are two codewords in C. The Hermitian inner product for X and Y is defined as

$$< X, Y >_H = \sum_{j=0}^{m_1-1} x_{0,j}\theta(y_{0,j}) + \sum_{j=0}^{m_2-1} x_{1,j}\theta(y_{1,j}) + \ldots + \sum_{j=0}^{m_t-1} x_{t-1,j}\theta(y_{t-1,j}),$$

where $x_{i,j} \in X$, $y_{i,j} \in Y$ for each $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, m_{i+1} - 1$.

Theorem 3.2. *The Hermitian dual of a skew GQC code of block length $(m_1, m_2, ..., m_t)$ is a skew GQC code of block length $(m_1, m_2, ..., m_t)$.*

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.1. *Let $h(x) = \sum_{i=0}^{n} h_i x^i$ be a polynomial in the ring $R[x;\theta]$. The skew reciprocal polynomial of h is defined by*

$$h^*(x) = \sum_{i=0}^{n} x^{n-i} h_i = \sum_{i=0}^{n} \theta^i(h_{n-i}) x^i.$$

If $h(x) = h^*(x)$, then $h(x)$ is called a self-reciprocal polynomial.

In order to describe some properties of the skew reciprocal polynomial, we need the following morphism of rings:

$$\Theta : R[x;\theta] \longrightarrow R[x;\theta],$$
$$\sum_{i=0}^{n} a_i X^i \mapsto \sum_{i=0}^{n} \theta(a_i) X^i.$$

We recall the following useful Lemma from [3, Lemma 1].

Lemma 3.1. *Let f and g be skew polynomial in R and $n = deg(f)$. Then:*
1. $(fg)^* = \Theta^n(g^*)f^*$.
2. $(f^*)^* = \Theta^n(f)$.

We briefly state some facts regarding skew cyclic codes. We know that there exists a one-to-one correspondence between the codewords $(c_0, c_1, ..., c_{m-1})$ and polynomials $c(x) = c_0 + c_x + ... + c_{m-1}x^{m-1}$ in $R[x;\theta]$. Under this correspondence a skew cyclic code C of length n over the ring R can be considered as a principal ideal in R_m. Among all the generators of the ideal C, there is a unique monic one with minimal degree that divides $x^m - 1$. This polynomial is called the generator polynomial of the skew cyclic code C and we display it by $G(x)$. A polynomial $H(x)$ which satisfy in $H(x)G(x) = x^m - 1$ is called the check polynomial of C.

If C is a skew cyclic code with generator polynomial $G(x)$, then $C^{\perp_H}$ is a skew cyclic code with generator polynomial $H^*(x)$, where $H^*(x)$ is skew reciprocal polynomial of the check polynomial $H(x)$. Thus we obtain the following result.

Proposition 3.1. *A skew cyclic code C of length n is a skew cyclic Hermition self-dual if and only if $G(x) = H^*(x)$ where $G(x)$ is the generator polynomial of C, $H(x)$ is the check polynomial and $H^*(x)$ is the skew reciprocal polynomial of $H(x)$.*

By [2] we know that when n is an odd integer, $x^n - 1$ factors over $\mathbb{F}_2$ into pairwise coprime irreducible factors. We can consider the map $R[x;\theta] \to \mathbb{F}_2[x;\theta]$ and apply Hensel's Lemma. The factorization $x^n - 1$ in $\mathbb{F}_2[x;\theta]$ can be uniquely lifted to a factorization of $x^n - 1$ over R into pairwise coprime basic irreducible factors. Also the factorization of $x^n - 1$ over $\mathbb{F}_2$ is still valid over R. Therefore, if n is odd, then all factors of $x^n - 1$ in $R[x;\theta]$ are just its factors in $\mathbb{F}_2[x;\theta]$.

Let C be a skew l-QC code of length $n(n = ml)$ where $m = 2^a m'$ such that $(m', 2) = 1$ and a is a integer which dependent on m. By [17], one can write

$$x^{m'} - 1 = f_1 \dots f_s h_1 h_1^* \dots h_t h_t^*,$$

where h_j^* is the skew reciprocal polynomial of h_j for $j = 1, 2, \dots, t$ and f_i is skew self-reciprocal polynomial for any $i = 1, 2, \dots, s$. Also

$$(*) \qquad x^m - 1 = (x^{m'} - 1)^{2^a} = f_1^{2^a} \dots f_s^{2^a} h_1^{2^a} h_1^{*2^a} \dots h_t^{2^a} h_t^{*2^a}.$$

We continue this section to discuss on some structural properties of 1-generator skew QC codes and 1-generator skew GQC codes. Consider a skew l-QC code C of length $n(n = ml)$ over the ring R. Let $F(x) = (F_1(x), \dots, F_l(x)) \in (\frac{R[x;\theta]}{<x^m-1>})^l$ where $F_i(x) \in \frac{R[x;\theta]}{<x^m-1>}$. Then $R[x;\theta]F(x) := \{\alpha(x)F(x) | \alpha(x) \in R[x;\theta]\} = \{(\alpha(x)F_1(x), \alpha(x)F_2(x), \dots, \alpha(x)F_l(x)) | \alpha(x) \in R[x;\theta]\}$. is called a 1-generator skew l-QC code with generator $F(x)$. Note that for $l = 1$, a skew l-QC code over R is a skew cyclic code of length m over R. Define a well defined R-homomorphism ϕ_j from R^n onto R_m such that $\phi_j(F_1(x), \dots, F_l(x)) = F_j(x)$. Then $\phi_j(C)$ is a skew cyclic code of length m over R.

Theorem 3.3. *Suppose that C is a 1-generator skew l-QC code of length $n(n = ml)$ which is generated by $F(x) = (F_1(x), \ldots, F_l(x))$ such that $\phi_j(C)$ is a Hermitian self-dual code over the ring $\frac{R[x;\theta]}{<x^m-1>}$ for any $j = 1, 2, \ldots, l$. Then the generator polynomial of C is*

$$K(x).f_1^{(2^{a-1})} \ldots f_s^{(2^{a-1})} h_1^{l_1} h_1^{*2^a-l_1} \ldots h_t^{l_t} h_t^{*2^a-l_t},$$

where $K(x) \in R[x;\theta]$, $l_r = lcm\{\beta_{jr}\}$ and $0 \leq \beta_{jr} \leq 2^a$ for any $j = 1, 2, \ldots, l$ and $r = 1, 2, \ldots, t$.

Proof. Consider the homomorphism ϕ_j defined in the preceding paragraph. We know that $\phi_j(C)$ is a skew cyclic code of length m over the ring R_m for any $j = 1, 2, \ldots, l$. So $\phi_j(C)$ has a generator polynomial such as $G_j(x)$ such that $G_j(x)|x^m - 1$. By factorization $(*)$ for $x^m - 1$ we can write

$$G_j(x) = {f_1}^{\alpha_{j1}} \ldots {f_s}^{\alpha_{js}} {h_1}^{\beta_{j1}} {h_1^*}^{\eta_{j1}} \ldots {h_t}^{\beta_{jt}} {h_t^*}^{\eta_{jt}},$$

where $0 \leq \alpha_{ji} \leq 2^a$ and $0 \leq \beta_{jk}, \eta_{jk} \leq 2^a$ for any $i = 1, 2, \ldots, s$ and $k = 1, 2, \ldots, t$. The following polynomial is the check polynomial of $\phi_j(C)$: $H_j(x) = {f_1}^{2^a-\alpha_{j1}} \ldots {f_s}^{2^a-\alpha_{js}} h_1^{2^a-\beta_{j1}} {h_1^*}^{2^a-\eta_{j1}} \ldots h_t^{2^a-\beta_{jt}} {h_t^*}^{2^a-\eta_{jt}}$.

Assume that $deg(f_i^{2^a-\alpha_{ji}}) = n_i$, $deg(h_k^{2^a-\beta_{jk}}) = m_k$ and $deg({h_k^*}^{2^a-\eta_{jk}}) = m'_k$. We use Lemma 3.1 to obtain $H_j^*(x)$. Thus

$$\begin{aligned} H_j^*(x) = {} & \Theta^{\sum_{i=1}^{s} n_i + \sum_{j=1}^{t} m_j + \sum_{j=1}^{t-1} {m_j}'} ({h_{jt}^*}^{2^a-\eta_{jt}})^* \times \\ & \Theta^{\sum_{i=1}^{t} n_i + \sum_{j=1}^{t-1} m_j + \sum_{j=1}^{t-1} m'_j} (h_{jt}^{2^a-\beta_{jt}})^* \times \\ & \Theta^{\sum_{i=1}^{t} n_i + \sum_{j=1}^{t-1} m_j + \sum_{j=1}^{t-2} m'_j} ({h_{j,t-1}^*}^{2^a-\eta_{j,t-1}}) \times \\ & \vdots \\ & \Theta^{\eta_1} (f_{j1}^{2^a-\alpha_{j1}})^*. \end{aligned}$$

By proposition 3.1, a skew cyclic code generated by the polynomial generator $G_j(x)$ is skew Hermitian self-dual if and only if $G_j(x) = H_j^*(x)$. Therefore, $\alpha_{ji} = 2^a - \alpha_{ji}$ and $\eta_{jk} = 2^a - \beta_{jk}$. Thus

$$(**) \qquad G_j(x) = {f_1}^{2^{a-1}} \ldots {f_s}^{2^{a-1}} {h_1}^{\beta_{j1}} {h_1^*}^{2^a-\beta_{j1}} \ldots {h_t}^{\beta_{jt}} {h_t^*}^{2^a-\beta_{jt}}.$$

We know that $H_j(x)$ and $G_j(x)$ are the check polynomial and the generator polynomial for the skew cyclic code $\phi_j(C)$, respectively. Thus $H_j(x).G_j(x) = x^m - 1$. This implies that $H(x) = lcm\{H_1(x), H_2(x), \ldots, H_l(x)\}$ is the check polynomial of C. Let $G' = lcm\{G_j(x)\}_{j=1}^{l}$. Therefore,

$$(1) \qquad G'(x) = {f_1}^{2^{a-1}} \ldots {f_s}^{2^{a-1}} {h_1}^{l_1} {h_1^*}^{2^a-l_1} \ldots {h_t}^{l_t} {h_t^*}^{2^a-l_t},$$

where $l_r = lcm\{\beta_{jr}\}$ for each $r = 1, 2, \ldots, t$. If $H(x)$ and $G(x)$ are check polynomial and generator polynomial for C respectively, then, $G(x).H(x) = x^m - 1$. It is clear that $G(x)|G'(x)$. By (1), there exists a polynomial such as $K(x) \in R[x;\theta]$ and $G(x) = K(x).{f_1}^{2^{a-1}} \ldots {f_s}^{2^{a-1}} {h_1}^{l_1} {h_1^*}^{2^a-l_1} \ldots {h_t}^{l_t} {h_t^*}^{2^a-l_t}$. □

Consequently, by the factorization $(*)$, we can write

$$\frac{R[x;\theta]}{< x^m - 1 >} = (\bigoplus_{i=1}^{s} \frac{R[x;\theta]}{< f_i >}) \oplus (\bigoplus_{j=1}^{t} (\frac{R[x;\theta]}{< h_j >} \oplus \frac{R[x;\theta]}{< h_j^* >})). \tag{2}$$

Consider the following notations

$$G_i = \frac{R[x;\theta]}{< f_i >}, \quad H_j' = \frac{R[x;\theta]}{< h_j >}, \quad H_j'' = \frac{R[x;\theta]}{< h_j^* >}.$$

Let $X, Y \in R_m^l$ where $X = (X_1, \ldots, X_l)$ and $Y = (Y_1, \ldots, Y_l)$. By (2), we can write $X_i = (x_{i1}, \ldots, x_{is}, x'_{i1}, x''_{i1}, \ldots, x'_{it}, x''_{it})$ and $Y_i = (y_{i1}, \ldots, y_{it}, y'_{i1}, y''_{i1}, \ldots, y'_{it}, y''_{it})$ where $x_{ij}, y_{ij} \in G_j$, $x'_{ik}, y'_{ik} \in H'_k$ and $x''_{ik}, y''_{ik} \in H''_k$ for any $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, s$ and $k = 1, 2, \ldots, t$. Therefore,

$$< X, Y >_H = \sum_{j=0}^{l-1} \sum_{i=1}^{s} x_{ji}\theta(y_{ji}) + \sum_{j=0}^{l-1} \sum_{k=1}^{t} x'_{jk}\theta(y'_{jk}) + \sum_{j=0}^{l-1} \sum_{k=1}^{t} x''_{jk}\theta(y''_{jk}).$$

Clearly $< X, Y >_H = 0$ if and only if

$$\sum_{j=0}^{l-1} \sum_{i=1}^{s} x_{ji}\theta(y_{ji}) = 0, \quad \sum_{j=0}^{l-1} \sum_{k=1}^{t} x'_{jk}\theta(y'_{jk}) = 0, \quad \sum_{k=1}^{t} x''_{jk}\theta(y''_{jk}) = 0.$$

Thus we have the following result about characterization of skew Hermitian self-dual codes over $R[x;\theta]$.

Corollary 3.1. *A skew l-QC code of length $n(n = ml)$ is Hermitian self-dual if and only if $C = (\bigoplus_{i=1}^{s} C_i) \oplus (\bigoplus_{j=1}^{t} (C'_j \oplus C'^{\perp}_j))$ where C_i is a skew Hermitian self-dual code over G_i for any $i = 1, 2, \ldots, s$, C'_j is a linear code over H'_j and $C'^{\perp}_j$ is its Hermitian dual.*

Our next step is to introduce the generator polynomial and the check polynomial for skew GQC codes. Assume that C is a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ and length $m = \sum_{i=1}^{t}$. A 1-generator skew GQC code over R generated by $F(x) = (F_1(x), F_2(x), \ldots, F_t(x))$ where $F_i(x) \in \frac{R[x;\theta]}{< x^{m_i} - 1 >}$ is defined as

$$R[x;\theta]F(x) = \{\alpha(x)F_1(x), \ldots, \alpha(x)F_t(x) | \alpha(x) \in R[x;\theta]\}.$$

In a similar way for skew l-QC codes we can consider a well defined R-homomorphism ψ_i from R' to R_i given by $\psi_i(F(x)) = F_i(x)$. Then $\psi_i(C)$ is a skew cyclic code of length m_i and generated by $F_i(x)$ in R_i. Therefore, $\psi_i(x)$ has a generator polynomial such as $G_i(x)$ where $G_i(x)$ is a right divisor of $x^{m_i} - 1$. Therefore, $H_i(x) = \frac{x^{m_i}-1}{G_i(x)}$ is the check polynomial of $\psi_i(C)$. This implies that $H(x) = lcm\{H_i(x)\}_{i=1}^{t}$ is the check polynomial for C. Let C be a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$. We know that

$$x^{m_i} - 1 = f_{i1} \ldots f_{is} h_{i1} h_{i1}^* \ldots h_{ik} h_{ik}^*,$$

where f_{ir} is a skew self-reciprocal polynomial and $h^*_{ir'}$ is the skew reciprocal polynomial of $h_{ir'}$ for any $r = 1, 2, \ldots, s$ and $r' = 1, 2, \ldots, k$. In a similar way for skew l-QC codes we can write the following factorization for R_i.

$$R_i = \frac{R[x;\theta]}{< x^{m_i} - 1 >} = (\bigoplus_{r=1}^{s} \frac{R[x;\theta]}{< f_{ir} >}) \oplus (\bigoplus_{r'=1}^{k} (\frac{R[x;\theta]}{< h_{ir'} >} \oplus \frac{R[x;\theta]}{< h^*_{ir'} >})). \tag{3}$$

Let $X, Y \in R'$ where $X = (X_1, X_2, \ldots, X_t)$ and $Y = (Y_1, Y_2, \ldots, Y_t)$ such that $X_i, Y_i \in \frac{R[x;\theta]}{<x^{m_i}-1>}$ for each $i = 1, 2, \ldots, t$. By (3), X_i and Y_i are as follows:

$$X_i = (x_{i1}, \ldots, x_{is}, x'_{i1}, x''_{i1}, \ldots, x'_{it}, x''_{it})$$
$$Y_i = (y_{i1}, \ldots, y_{is}, y'_{i1}, y''_{i1}, \ldots, y'_{it}, y''_{it}),$$

where $x_{ir}, y_{ir} \in \frac{R[x;\theta]}{<f_{ir}>}$, $x'_{ir'}, y'_{ir'} \in \frac{R[x;\theta]}{<h_{ir'}>}$ and $x''_{ir'}, y''_{ir'} \in \frac{R[x;\theta]}{<h^*_{ir}>}$. Therefore,

$$< X, Y >_H = \sum_{i=1}^{t}\sum_{r=1}^{s} x_{i,r}\theta(y_{ir}) + \sum_{i=1}^{t}\sum_{r'=1}^{k} x'_{ir'}\theta(y'_{ir'}) + \sum_{i=1}^{t}\sum_{r'=1}^{k} x''_{ir'}\theta(y''_{ir'}).$$

Clearly $< X, Y >_H = 0$ if and only if

$$\sum_{i=1}^{t}\sum_{r=1}^{s} x_{i,r}\theta(y_{ir}) = 0, \sum_{i=1}^{t}\sum_{r'=1}^{k} x'_{ir'}\theta(y'_{ir'}) = 0$$

and $\sum_{i=1}^{t}\sum_{r'=1}^{k} x''_{ir'}\theta(y''_{ir'}) = 0$. Hence we have the following result.

Corollary 3.2. *A skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ is a self-dual Hermitian if and only if*

$$C = (\bigoplus_{r=1}^{s} C_r) \oplus (\bigoplus_{r'=1}^{k} (C'_{r'} \oplus C'^{\perp_H}_{r'})),$$

where C_i is a skew Hermitian self-dual over $\frac{R[x;\theta]}{<x^{m_i-1}>}$ for any $r = 1, 2, \ldots, s$ and $C'_{r'}$ is a linear code over $\frac{R[x;\theta]}{<h_{ir'}>}$ and $C'^{\perp_H}_{r'}$ is its dual Hermitian over $\frac{R[x;\theta]}{h^*_{ir'}}$.

4. Gray image of skew GQC codes and skew QC codes

Let $X = (X_0, X_1, \ldots, X_{4n-1}) = (X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) \in \mathbb{F}_2^{4n}$ where $X_{(i)} \in \mathbb{F}_2^n$ for all $i = 0, 1, 2, 3$. We define the following map

$$\Upsilon : \mathbb{F}_2^{4n} \to \mathbb{F}_2^{4n},$$
$$\Upsilon(X) = (\eta(X_{(0)}), \eta(X_{(1)}), \eta(X_{(2)}), \eta(X_{(3)})),$$

where η is a map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ with the property that

$$\eta(X_{(i)}) = (X_{i,n-1}, X_{i,0}, \ldots, X_{i,n-2}),$$

for every $X_{(i)} = (X_{i,0}, X_{i,1}, X_{i,n-1})$ where $X_{i,j} \in \mathbb{F}_2$ for $j = 0, 1, \ldots, n-1$. Also we consider the map λ from $\mathbb{F}^{4n}$ to $\mathbb{F}^{4n}$ given by $\lambda(X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) = (X_{(0)}, X_{(2)}, X_{(1)}, X_{(3)})$.

Definition 4.1. *A code C of length $4n$ over $\mathbb{F}_2$ is said l-QC code of index 4 if $\Upsilon(C) = C$.*

Proposition 4.1. *Assume σ_θ is the skew cyclic shift on R^n, Υ is as above and ϕ is the Gray map from R^n to $\mathbb{F}_2^{4n}$. Then $\phi\sigma_\theta = \lambda\Upsilon\phi$.*

Proof. Let $X_i = x_i + uy_i + vz_i + uvt_i$ be elements of R for each $i = 0, 1, 2, \ldots, n-1$. By definition of the Gray map we have $\phi(X_0, X_1, ..., X_{n-1}) = (x_0 + y_0 + z_0 + t_0, x_1 + y_1 + z_1 + t_1, \ldots, x_{n-1} + y_{n-1}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-1} + t_{n-1}, t_0, \ldots, t_{n-1})$. We apply Υ, so $\Upsilon(\phi(X_0, X_1, \cdot, X_{n-1})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$. If we apply λ, then $\lambda(\Upsilon(\phi(X_0, X_1, \ldots, X_{n-1})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, +t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$.

On the other hand, we have $\sigma_\theta(X_0, X_1, \ldots, X_{n-1}) = (\theta(X_{n-1}), \theta(X_0), \ldots, \theta(X_{n-2}))$. We apply ϕ, so $\phi(\sigma_\theta(X_0, X_1, \ldots, X_{n-1})) = \phi(\theta(X_{n-1}), \theta(X_0), \ldots, \theta(X_{n-2})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, \ldots, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$. □

Theorem 4.1. *Let C be a skew cyclic code of length n over the ring R. Then the Gray image of C is permutation equivalent to skew 4-QC code of index 4 and length $4n$ over $\mathbb{F}_2$.*

Proof. We know that $\sigma_\theta(C) = C$ and $\phi(\sigma_\theta(C)) = C$. By Proposition 4.1, $\phi(\sigma_\theta(C)) = \phi(C) = \lambda(\Upsilon(\phi(C)))$. So we can say that $\phi(C)$ is permutation equivalent to l-QC code of index 4 and length $4n$ over $\mathbb{F}_2$. □

Let Γ be a map from $\mathbb{F}_2^{4n}$ to $\mathbb{F}_2^{4n}$ given by $\Gamma(X) = (\xi(X_{(0)}), \xi(X_{(1)}), \xi(X_{(2)}), \xi(X_{(3)}))$, where ξ is the map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ given by

$$\xi(X_{(i)}) = ((X_{(i,m-1)}), (X_{(i,0)}), \ldots, (X_{(i,m-2)})),$$

for every $X_{(i)} = (X_{(i,0)}, \ldots, X_{(i,m-1)})$ where $X_{(i,j)} \in \mathbb{F}_2^l$ for all $j = 0, 1, \ldots, m-1$. A code of length $4n$ over $\mathbb{F}_2$ is called l-QC code of index 4 if $\Gamma(C) = C$.

Proposition 4.2. *Let $\tau_{\theta,m,l}$ be a skew QC shift on R^n and ϕ be the Gray map over R^n. Then $\phi\tau_{\theta,m,l} = \lambda\Gamma\phi$.*

Proof. The proof is similar to the proof of the Proposition 4.1. □

Theorem 4.2. *Let C be a skew l-QC cyclic code of length $n (n = ml)$. Then, C is permutation equivalent to l-QC code of index 4 and with length $4n$ over $\mathbb{F}_2$.*

Proof. Since C is a skew QC code, so $\tau_{\theta,m,l}(C) = C$. Also $\phi(\tau_{\theta,m,l}(C)) = \phi(C)$ and by the previous Proposition $\phi((\tau_{\theta,m,l}(C)) = \lambda\Upsilon\phi(C)$. This implies that $\phi(C)$ is permutation equivalent to l-QC code of index 4 and with length $4n$ over $\mathbb{F}_2$. □

We finish by investigating the above results for skew GQC codes. Suppose that $m = \sum_{k=1}^{t} m_k$ and $X = (X_0, X_1, \ldots, X_{4m-1}) = (X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) \in F_2^{4m}$ for $i = 0, 1, 2, 3$. Consider the map Γ' from $\mathbb{F}_2^{4m}$ to $\mathbb{F}_2^{4m}$ given by $\Gamma'(X) = (\xi'(X_{(0)}), \xi'(X_{(1)}), \xi'(X_{(2)}), \xi'(X_{(3)}))$, where ξ' is the map from $\mathbb{F}_2^m$ to $\mathbb{F}_2^m$ given by $\xi'(X_{(i)}) = (X_{(i,m-1)}, X_{(i,0)}, \ldots, X_{(i,m-2)})$, for any $X_{(i)} = (X_{(i,0)}, ..., X_{(i,m-1)})$ such that $X_{(i,j)} \in \mathbb{F}_2^{m_k}$ for $j = 0, 1, \ldots, m-1$. Also let λ' be a map from $4m$ to $4m$ given by $\lambda'(X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) = (X_{(0)}, X_{(2)}, X_{(1)}, X_{(3)})$.

Definition 4.2. *A code C of length $4m$ over F_2 is called GQC code of index 4 if $\Gamma'(C) = C$.*

By introduce the above maps and in similar ways of Proposition 4.1 and Theorem 4.1 we have the following results.

Theorem 4.3. *Let $\tau_{\theta,m}$ be the skew GQC shift on R^m where $m = \sum_{k=1}^{t} m_k$ and ϕ be the Gray map from R^m to F_2^{4m}. Then $\phi\tau_{\theta,m} = \lambda'\Gamma'\phi$.*

Theorem 4.4. *The Gray image a skew GQC code over R of length m and block length $(m_1, m_2, ..., m_t)$ is permutation equivalent to skew GQC code of index 4 and length $4m$ over F_2.*

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Event-triggered consensus control for the first-order multi-agent systems

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Abstract. In this paper, event-triggered consensus control problem for the first-order multi agent systems is studied. The control law updating considered is event-driven, which depends on the measurement error with state function norm and its neighboring agents. The proposed updating law is only triggered at its event time instants for each agent. The proposed event-based strategy doesn't require continuous monitoring the neighbors states that can lessen the rate of the control law updating, therefore, communication resources can be saved. The proposed control protocols ensure convergence to a ball focused on the average consensus, and provide a minimum interevent time to avoid Zeno behavior. Finally, a simulation example is shown to illustrate the theoretical result.

Keywords: event-triggered control, multi-agent systems (MASs), consensus.

1. Introduction

Multi-agent systems (MASs) have become an active research topic in control communities. MASs have been widely applied in practical fields, for example, intelligent robot design and traffic control. Average consensus is a basic research for multi-agent coordination that drives all agents to obtain the desired common goal relying solely on neighboring information. In [1], consensus analysis for single-integrator agents is first discussed. Nowadays, there are many excellent control strategies about consensus of MASs, for instance, continuous control [2] and periodic sampling control [3]. Consensus analysis has been widely investigated in Refs. [1-14] and many references therein.

Continuous communication among neighbor agents is commonly utilized for distributed consensus problem design. However, continuous control strategy requires that the control laws are updated uninterruptedly. This may result in large communication load and high frequencies of updating laws. Therefore, periodic sampling control strategy is introduced to reduce the rate of control updating. From the perspective of practical application, such control strategies must be implemented on a digital platform. Typically, control task is executed periodically. The measurements are carried out periodically based on a given constant sampling interval, which control law is updated in a fixed period. Although the traditional periodic sampling fashion is well-developed, it may not

be suitable for resource limited systems, because with the fixed sampling period setup, communication resource can be used even though it is not necessary from the performance perspectives. Therefore, an event-triggered method is more effective first proposed in [4, 5] by considering the limited resources for embedded processors. The author in [4] gives an event-triggered PID controller, where only discrete states are monitored so the CPU usage can be reduced. In [5], event based sampling is presented instead of periodic sampling, where control laws update only if the measured value exceeds a given threshold. Therefore, event-triggered control policy can be utilized for reducing the frequency of control updating, the numbers of re-computation and transmission. At the same time, it guarantees the desired performance so that event-based controls for MASs have been developed in the above papers.

Because of the advantage of event-triggered control, event-based consensus problems of MASs are discussed in [6]. But the triggering conditions require that agent i must monitor agent j's state continuously. In [7], an original control scheme for multi-agent coordination under event-triggered broadcasting is proposed. In this work, the state independent and exponentially declining threshold is introduced in the triggering condition. Then, continuous monitoring among neighbor agents can be averted by using this method. In [8], the distributed event-triggered control scheme was given on networked systems, whose subsystems are linear time-invariant and interconnected. In [9], event-based average consensus problem for linear system is investigated, where the state independent trigger conditions are introduced for reducing communication load. In [10], an event-based control scheme for general linear systems is provided, where the state independent trigger condition is utilized for saving communication resources. However, the trigger conditions used in [8-10] emphasize some state independent triggering function. The authors in [11-13] studied event-triggered control mechanisms for MASs. Here control updating relies on triggering condition that requires the state dependent threshold. Event-based consensus problem for distributed MASs is investigated by utilizing state-dependent trigger function in [14]. But the proof of the inequality (28) in [14] is incorrect. This motivates us to search the suitable triggering condition, which guarantees consensus of MASs.

In this paper, event-triggering control scheme will be planned for MASs. The control updating of each agent is decided by definitional events that rely on the states of neighboring agents and measurement error. The primary contribution is that one sufficient consensus condition is presented by using the state-dependent trigger function, and it avoids continuous communication between neighboring agents. By using this policy, the signals transmissions number is reduced, and then the inter-communication will decrease.

The rest is organized in the following. In section 2, we introduce some concepts on algebraic graph theory and statement of the problem. In section 3, we state the main results. The simulation result is presented to illustrate our analysis result in section 4. Finally, conclusions are given in section 5.

2. Preliminaries

2.1 Algebraic graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph of order N, which set of nodes $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ represents N agents, and set of edges $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ corresponds to the communication links between agents; v_i and v_j are called adjacent, if there exists an edge (v_i, v_j) between vertices v_i and v_j. In undirected graph, one has $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. The adjacency matrix $A = (a_{ij})$ is the symmetric matrix given by $a_{ij} = 1$, if $(v_i, v_j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. $\mathcal{G}$ is called connected, if there is a path between any two nodes in undirected graph $\mathcal{G}$. Given $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$, which denotes the indexes of agents that agent v_i can communicate. Let d_i be the cardinality of agent v_i's neighbor set $\mathcal{N}_i$. The degree D for an undirected graph $\mathcal{G}$ is a diagonal matrix $diag\{d_1, d_2, \cdots, d_N\}$. The Laplacian of $\mathcal{G}$ is the symmetric and positive semidefinite matrix $L = D - A$, i.e. $L = L^T > 0$. Note that the Laplacian matrix is zero average, i.e. $L\mathbf{1} = 0$ with $\mathbf{1}^T = [1, 1, \cdots, 1]$. The Laplacian matrix has a single zero eigenvalue for connected graph and corresponding eigenvector is the vector $\mathbf{1}$, which all entries equal to 1. We denote by $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$ the eigenvalues of L. If $\mathcal{G}$ is connected, then $\lambda_2(\mathcal{G}) > 0$.

2.2 System modelling and problem statement

By considering the system consists of N agents, we assume that each agent's dynamics follow a single integrator model:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{V}, \tag{1}$$

where, $x_i(t) \in R$ denotes the state of agent i. $u_i(t) \in R$ is the control input of the ith agent. As is known [1] that continuous distributed control law as follows:

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)). \tag{2}$$

Let's drive each agent to asymptotically converge average of the agents' initial conditions. Then, the closed-loop system can be written as follows:

$$\dot{x}(t) = Lx(t),$$

where $x^T(t) = (x_1^T(t), \cdots, x_N^T(t))$.

This paper will consider the continuous distributed control law with an event-based implementation, which is considered as:

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} (\hat{x}_i(t) - \hat{x}_j(t)). \tag{3}$$

While, in stack vector form $u(t) = -L\hat{x}(t)$, where $u^T(t) = (u_1^T(t), \cdots, u_N^T(t))$ and $\hat{x}^T(t) = (\hat{x}_1^T(t), \cdots, \hat{x}_N^T(t))$. In order to make sure that the agents' initial

states average is maintained throughout the evolution of the system, and the agent i does not use its actual state $x_i(t)$ but the last broadcast value $\hat{x}_i(t)$ with $\hat{x}_i(t) = x_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i)$, where $t_0^i, t_1^i, \cdots$ denotes the sequence of event triggers of agent i. The next event-triggering instant t_k^i is decided by triggering function $f_i(t, e_i(t), x_i(t)) > 0$, and it is shown as follows:

$$t_{k+1}^i = \inf\{t : t > t_k^i, f_i(t, e_i(t), x_i(t)) > 0\}, k = 0, 1, \cdots, \infty. \tag{4}$$

For each $i \in \mathcal{V}$ and $t \geq 0$, the measurement error is defined as

$$e_i(t) = \hat{x}_i(t) - x_i(t). \tag{5}$$

It can be expressed in compact form as

$$\dot{x}(t) = -\mathcal{L}\hat{x}(t) = -\mathcal{L}(x(t) + e(t)), \tag{6}$$

where $e^T(t) = [e_1^T(t), \cdots, e_N^T(t)]$.

Denote $a(t) = \frac{1}{N}\sum_{i=1}^N x_i(t)$ as the average of all agents' states. Then, the derivative of $a(t)$ is

$$\dot{a}(t) = \frac{1}{N}\sum_{i=1}^N \dot{x}_i(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}^T\dot{x}(t) = -\frac{1}{N}\sum_{i=1}^N \mathbf{1}^T\mathcal{L}\hat{x}(t) = 0.$$

Therefore, $a(t) = a(0) = a$ for all $t \geq 0$ and the state $x(t)$ can be decomposed as

$$x(t) = a\mathbf{1} + \delta(t), \tag{7}$$

where, $\delta(t)$ is the disagreement vector of the MASs, i.e. $\mathbf{1}^T\delta(t) = 0$. From (7) and $\delta(t) = x(t) - a\mathbf{1}$, it follows that

$$\dot{\delta}(t) = \dot{x}(t) = -\mathcal{L}(e(t) + x(t)) = -\mathcal{L}e(t) - \mathcal{L}\delta(t). \tag{8}$$

This means that

$$\delta(t) = e^{-\mathcal{L}t}\delta(0) - \int_0^t \mathcal{L}e(\tau)e^{-\mathcal{L}(t-\tau)}d\tau, \tag{9}$$

where, $\delta(0)$ means the initial value of $\delta(t)$.

The following lemma and definitions are introduced, which will be utilized in the proof of ours.

Lemma 1 ([7]). *Suppose $\mathcal{L}$ is the Laplacian matrix of an connected and undirected graph $\mathcal{G}$. Then, for $t \geq 0$ and all vectors $\nu \in R^n$ with $\mathbf{1}^T\nu = 0$, we have*

$$\| e^{-\mathcal{L}t}\nu \| \leq e^{-\lambda_2(\mathcal{G})t} \| \nu \| . \tag{10}$$

Definition 1 ([1])**.** MAS is called to achieve average consensus, if all agents' states converge to average of the initial states, i.e., and $\forall i \in \mathcal{V}$,

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{i\in\mathcal{V}} x_i(0), \tag{11}$$

where $x_i(0)$ is the initial state of agent i.

Definition 2. A hybrid system H is Zeno, if for some execution ε of H , there exists a finite constant τ_∞ such that $\lim_{i\to\infty}\tau_i = \sum_{i=0}^{\infty}(\tau_{i+1}-\tau_i) = \tau_\infty$. The execution ε is called a Zeno execution.

Remark 1. The definition of a Zeno execution results in the types of Zeno behavior. For an execution ε that is Zeno, ε is Chattering Zeno: If there exists a finite C such that $\tau_{i+1}-\tau_i > 0$ for all $i > C$.

Remark 2. The definition for the closed-loop system is following. Feedback occurs when outputs of a system are routed back as inputs as part of a chain of cause-and-effect that forms a circuit or loop. The system can then be said to feed back into itself.

3. Consensus analysis

Next, a sufficient consensus policy for MAS (1) is obtained in the following theorem, where the trigger condition only requires the discrete measurement of neighbor states.

Theorem 1. *Consider the MASs (1) with control rule (3) and event-triggering times is decided by (4). Suppose that the triggering function is devised as:*

$$f_i(t, e_i(t), x_i(t)) = \mid e_i(t) \mid -b_0\sqrt{\sum_{j\in\mathcal{N}_i} \mid \hat{x}_j(t) - \hat{x}_i(t) \mid^2}, \tag{12}$$

and $b_0 \in (0, \frac{\lambda_2(\mathcal{G})}{\sqrt{2}\left((\sqrt{\|L\|})^3+\lambda_2(\mathcal{G})\sqrt{\|L\|}\right)})$*. Then, for all initial conditions* $x(0) \in R^N$*, zeno behavior does not occur. Furthermore, the disagreement vector* $\delta(t)$ *for closed-loop system will exponential converge to a ball focus on the rendezvous with radius* Δ*, where,* $\Delta = \frac{\sqrt{2}b_0(\sqrt{\|\mathcal{L}\|})^3\sqrt{N}|a|}{\lambda_2(\mathcal{G})-\sqrt{2}b_0\sqrt{\|L\|}(\lambda_2(\mathcal{G})+\|\mathcal{L}\|)}$.

Proof. For the triggering condition (12), the upper bound of measurement error is shown as

$$\| e(t) \| = \sqrt{\sum_{i=1}^{N} \mid e_i(t) \mid^2} \le b_0\sqrt{\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} \mid \hat{x}_j(t) - \hat{x}_i(t) \mid^2}$$

$$
\begin{aligned}
&= b_0\sqrt{2\hat{x}^T(t)\mathcal{L}\hat{x}(t)} \le \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\,\| \hat{x}(t) \| \\
&\le \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}(\| e(t) + x(t) \|) \\
&\le \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\,\| e(t) \| + \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\,\| x(t) \|, \qquad (13)
\end{aligned}
$$

from which we can get

$$
\begin{aligned}
(1 - \sqrt{2}b_0\sqrt{\| \mathcal{L} \|})\,\| e(t) \| &\le \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\,\| x(t) \| \\
&\le \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}(\| \delta(t) \| + \sqrt{N} \mid a \mid) \\
&= \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\,\| \delta(t) \| + \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}\sqrt{N} \mid a \mid . \qquad (14)
\end{aligned}
$$

It follows that

$$
\| e(t) \| \le \frac{\sqrt{2}b_0\sqrt{\| \mathcal{L} \|}}{1 - \sqrt{2}b_0\sqrt{\| \mathcal{L} \|}}(\| \delta(t) \| + \sqrt{N} \mid a \mid), \qquad (15)
$$

where, $0 < b_0 < \frac{\lambda_2(\mathcal{G})}{\sqrt{2}\left((\sqrt{\|L\|})^3 + \lambda_2(\mathcal{G})\sqrt{\|L\|}\right)} < \frac{1}{\sqrt{2}\sqrt{\|L\|}}$.

Let $b_1 = \frac{\sqrt{2}b_0\sqrt{\|\mathcal{L}\|}}{1-\sqrt{2}b_0\sqrt{\|\mathcal{L}\|}}, b_2 = \frac{\sqrt{2}b_0\sqrt{\|\mathcal{L}\|}}{1-\sqrt{2}b_0\sqrt{\|\mathcal{L}\|}}\sqrt{N} \mid a \mid$. Rewrite the upper bound of measurement errors as:

$$
\| e(t) \| \le b_1 \| \delta(t) \| + b_2. \qquad (16)
$$

From Eq. (9), we can get that the disagreement vector is bounded by

$$
\begin{aligned}
\| \delta(t) \| &= \| e^{-\mathcal{L}t}\delta(0) \| + \int_0^t e^{-\mathcal{L}(t-\tau)} \| \mathcal{L}e(\tau) \| d\tau \\
&\le e^{-\lambda_2(\mathcal{G})t} \| \delta(0) \| + \int_0^t e^{-\lambda_2(\mathcal{G})(t-\tau)} \| \mathcal{L}e(\tau) \| d\tau \\
&\le e^{-\lambda_2(\mathcal{G})t} \| \delta(0) \| + \| \mathcal{L} \| \int_0^t e^{-\lambda_2(\mathcal{G})(t-\tau)} \| e(\tau) \| d\tau \\
&= e^{-\lambda_2(\mathcal{G})t} \| \delta(0) \| + b_2 \| \mathcal{L} \| \int_0^t e^{-\lambda_2(\mathcal{G})(t-\tau)} d\tau \\
&\quad + b_1 \| \mathcal{L} \| \int_0^t \| \delta(\tau) \| e^{-\lambda_2(\mathcal{G})(t-\tau)} d\tau. \qquad (17)
\end{aligned}
$$

Since $b_0 \in (0, \frac{\lambda_2(\mathcal{G})}{\sqrt{2}((\sqrt{\|L\|})^3 + \lambda_2(\mathcal{G})\sqrt{\|L\|})}) \subset (0, \frac{1}{\sqrt{2}\sqrt{\|\mathcal{L}\|}})$, we have $\frac{b_1\|\mathcal{L}\|}{\lambda_2(\mathcal{G})} < 1$. Therefore, there is a positive constant $\rho \in (0, \lambda_2(\mathcal{G}))$ such that $\frac{b_1\|\mathcal{L}\|}{\lambda_2(\mathcal{G})-\rho} < 1$. We can claim that

$$
\| \delta(t) \| \le \| \delta(0) \| e^{-\rho t} + \Delta, \qquad (18)
$$

where, $\Delta = \frac{\sqrt{2}b_0(\sqrt{\|\mathcal{L}\|})^3\sqrt{N}|a|}{\lambda_2(\mathcal{G})-\sqrt{2}b_0\sqrt{\|L\|}(\lambda_2(\mathcal{G})+\|\mathcal{L}\|)}$.

In order to prove (18), for any $\eta > 1$ and $t > 0$, we first show that the following inequality is true,

$$\| \delta(t) \| \leq \eta \| \delta(0) \| e^{-\rho t} + \Delta = v(t). \tag{19}$$

Otherwise, there is a $t^* > 0$ such that $\| \delta(t^*) \| = v(t^*)$ and $\| \delta(t) \| < v(t)$ as well as $t \in (0, t^*)$. From (17), we have:

$$\begin{aligned}
v(t^*) = \| \delta(t^*) \| &\leq e^{-\lambda_2(\mathcal{G})t^*} \| \delta(0) \| + b_2 \| \mathcal{L} \| \int_0^{t^*} e^{-\lambda_2(\mathcal{G})(t^*-\tau)} d\tau \\
&\quad + b_1 \| \mathcal{L} \| \int_0^{t^*} \| \delta(\tau) \| e^{-\lambda_2(\mathcal{G})(t^*-\tau)} d\tau \\
&= e^{-\lambda_2(\mathcal{G})t^*} \| \delta(0) \| + \frac{b_1 \eta \| \mathcal{L} \| \| \delta(0) \|}{\lambda_2(\mathcal{G}) - \rho} (e^{-\rho t^*} - e^{-\lambda_2(\mathcal{G})t^*}) \\
&\quad + \left(\frac{b_2 \| \mathcal{L} \|}{\lambda_2(\mathcal{G})} + \frac{b_1 \| \mathcal{L} \| \Delta}{\lambda_2(\mathcal{G})} \right) (1 - e^{-\lambda_2(\mathcal{G})t^*}) \\
&< \eta \| \delta(0) \| \left[e^{-\lambda_2(\mathcal{G})t^*} + \frac{b_1 \| \mathcal{L} \|}{\lambda_2(\mathcal{G}) - \rho} (e^{-\rho t^*} - e^{-\lambda_2(\mathcal{G})t^*}) \right] \\
&\quad + \left(\frac{b_2 \| \mathcal{L} \|}{\lambda_2(\mathcal{G})} + \frac{b_1 \| \mathcal{L} \| \Delta}{\lambda_2(\mathcal{G})} \right) (1 - e^{-\lambda_2(\mathcal{G})t^*}) \\
&< \eta \| \delta(0) \| e^{-\rho t^*} + \left(\frac{b_2 \| \mathcal{L} \|}{\lambda_2(\mathcal{G})} + \frac{b_1 \| \mathcal{L} \| \Delta}{\lambda_2(\mathcal{G})} \right) (1 - e^{-\lambda_2(\mathcal{G})t^*}) \\
&< \eta \| \delta(0) \| e^{-\rho t^*} + \frac{b_2 \| \mathcal{L} \| + b_1 \| \mathcal{L} \| \Delta}{\lambda_2(\mathcal{G})} \\
&= \eta \| \delta(0) \| e^{-\rho t^*} + \Delta = v(t^*).
\end{aligned} \tag{20}$$

Obviously, this is a contradiction which implies the inequality (19) holds for any $\eta > 1$. Then, by letting $\eta \to 1$, the inequality (18) holds. Since $b_0 \in (0, \frac{\lambda_2(\mathcal{G})}{\sqrt{2}\left((\sqrt{\|L\|})^3 + \lambda_2(\mathcal{G})\sqrt{\|L\|}\right)})$, we have

$$\lambda_2(\mathcal{G}) - \sqrt{2} b_0 \sqrt{\| L \|} (\lambda_2(\mathcal{G}) + \| \mathcal{L} \|) > 0.$$

This means that $\Delta > 0$. Consequently, $\| \delta(t) \|$ is exponentially converged to a ball with radius Δ as $t \to \infty$. Then, the state of MASs converge exponentially to a ball focuses on the average consensus.

To avoid the Zeno-behavior, we have to indicate that there exists a minimum interevent time τ, for $t \geq t_0$, $\hat{x}_i(t)$ and $\hat{x}_j(t)$ remain constant. Since $\dot{e}_i(t) = -\dot{x}_i(t)$, the evolution of the error is simply written as

$$e_i(t) = -(t - t_0) \sum_{j \in N_i} (\hat{x}_j(t) - \hat{x}_i(t)). \tag{21}$$

Then, when $f_i(t, e_i(t), x_i(t)) = 0$ occurs, we will find the time t^* at which a broadcast of agent i's state is triggered. If $\sum_{j \in N_i} (\hat{x}_j(t) - \hat{x}_i(t)) = 0$, which

implies $e_i(t) = 0$ for all $t \geq t_0$, no broadcast will ever happen ($t^* = \infty$). Hence, if $\sum_{j\in N_i}(\hat{x}_j(t)-\hat{x}_i(t)) \neq 0$, we can derive $(\sum_{j\in N_i}(\hat{x}_j(t)-\hat{x}_i(t)))^2 \neq 0$. By using (21), the trigger condition (12) prescribes a broadcast at the time $t^* \geq t_0$, which can satisfy that:

$$(t^* - t_0)^2\Big(\sum_{j\in N_i}(\hat{x}_j(t) - \hat{x}_i(t))\Big)^2 - b_0^2\sum_{j\in N_i}(\hat{x}_j(t) - \hat{x}_i(t))^2 = 0, \tag{22}$$

or, equivalently:

$$(t^* - t_0)^2 = \frac{b_0^2\sum_{j\in N_i}(\hat{x}_j(t) - \hat{x}_i(t))^2}{\Big(\sum_{j\in N_i}(\hat{x}_j(t) - \hat{x}_i(t))\Big)^2}. \tag{23}$$

For any $y_1, y_2, \cdots, y_p \in R$ and $p \in Z$, using the Cauchy-Schwarz inequility $(\sum_{k=1}^{p} y_k)^2 \leq p\sum_{k=1}^{p} y_k^2$, we get

$$\Big(\sum_{j\in N_i}(\hat{x}_i(t) - \hat{x}_j(t))\Big)^2 \leq \mid N_i \mid \sum_{j\in N_i}(\hat{x}_i(t) - \hat{x}_j(t))^2. \tag{24}$$

Therefore, we substitute Eq.(24) into Eq.(23) so that we can get the minimum interevent time by

$$\tau = t^* - t_0 \geq \frac{b_0}{\sqrt{\mid N_i \mid}} > 0.$$

From the above discussions, it can be easy to find that there exists a minimum inter-event time τ, which implies that the Zeno-behavior is avoided in agent i. This completes the proof.

Remark 1. The consensus analysis of a class of MASs with single-integrator is investigated in [14]. However, the proof of the inequality (28) in [14] is incorrect. By Eq. (27), the author in [14] gets:

$$\begin{aligned}&\frac{\lambda_2(\mathcal{G}) - \parallel \mathcal{L} \parallel (b_1+1)}{\lambda_2(\mathcal{G})}\int_0^t e^{-\lambda_2(\mathcal{G})(t-\tau)} \parallel \delta(\tau) \parallel d\tau \\ \leq\ & \frac{1}{\lambda_2(\mathcal{G})} \parallel \delta(t) \parallel - \Big(\frac{1}{\lambda_2(\mathcal{G})} \parallel \delta(0) \parallel + \frac{\parallel \mathcal{L} \parallel b_2}{\lambda_2^2(\mathcal{G})}\Big)e^{-\lambda_2(\mathcal{G})t} + \frac{\parallel \mathcal{L} \parallel b_2}{\lambda_2^2(\mathcal{G})}.\end{aligned}$$

Obviously, by the condition in [21],

$$\lambda_2(\mathcal{G}) - \parallel \mathcal{L} \parallel (b_1+1) = \frac{(1 - b_0\sqrt{\parallel \mathcal{L} \parallel})\lambda_2(\mathcal{G}) - \parallel \mathcal{L} \parallel}{(1 - b_0\sqrt{\parallel \mathcal{L} \parallel})\lambda_2(\mathcal{G})}.$$

Then, since $b_0 \in (0, \frac{1}{\sqrt{\|\mathcal{L}\|}})$ and $\lambda_2(\mathcal{G}) \leq \| \mathcal{L} \|$, it can be easy to get $\lambda_2(\mathcal{G}) - \| \mathcal{L} \| (b_1 + 1) < 0$. That implies

$$\begin{aligned}\int_0^t e^{-\lambda_2(\mathcal{G})(t-\tau)} \| \delta(\tau) \| d\tau \geq & \frac{\| \delta(t) \|}{\lambda_2(\mathcal{G}) - \| \mathcal{L} \| (b_1 + 1)} \\ & - \frac{\| \delta(0) \| \lambda_2(\mathcal{G}) + \| \mathcal{L} \| b_2}{\lambda_2(\mathcal{G}) \big(\lambda_2(\mathcal{G}) - \| \mathcal{L} \| (b_1 + 1)\big)} e^{-\lambda_2(\mathcal{G})t} \\ & + \frac{\| \mathcal{L} \| b_2}{\lambda_2(\mathcal{G}) \big(\lambda_2(\mathcal{G}) - \| \mathcal{L} \| (b_1 + 1)\big)}.\end{aligned}$$

Then, the inequality (28) is incorrect, which results that the following conclusions are wrong. In our paper, we correct this error and obtain the ideal result to guarantee the consensus of MASs. Our design is based on the evolution of the network divergence, and triggers are designed by using locally available information, which allows each agent to resolve when to broadcast its current state to neighbors.

Remark 2. The graph G is considered to be undirected in this paper. However, it can be easily found that Theorem 1 extends to strongly connected and balanced directed graphs.

4. Illustrative example

Now, an illustrative example is provided to show the validity of our approach. A MAS which consists of five agents is considered, and the system can be expressed as the form of (1).

Example. By considering a system of five agents, the Laplacian matrix is shown as:

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{25}$$

By using the control policy proposed in Theorem 1, Fig.1 depicts the communication topology between agents. Given $b_0 = 0.04 \in (0, 0.0475)$, by using the controller (3) and the triggering mechanism (12), Fig.2 shows the response of states of all agents, from which it can easy to find that five agents will achieve consensus.

Fig. 3-Fig. 7 show the evolution of error norm of each agent that embodies the distributed event-triggered strategy. When the error norm of each agent reach threshold $b_0 \sqrt{\sum_{j \in \mathcal{N}_i} | \hat{x}_j(t) - \hat{x}_i(t) |^2}$, the error norm of each agent is reset to 0.

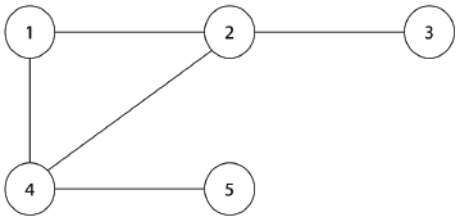


Figure 1: Communication graph $\mathcal{G}$ of the MASs

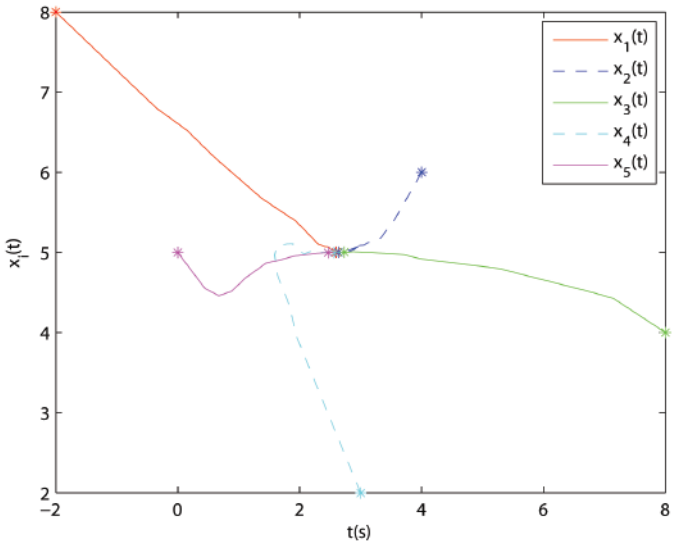


Figure 2: State trajectories of five agents with event-triggered control.

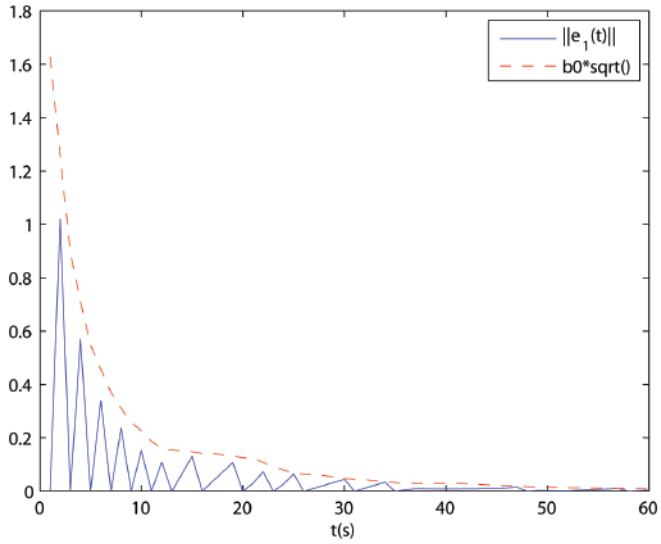


Figure 3: The evolution of $\| e_1(t) \|$ for agent 1

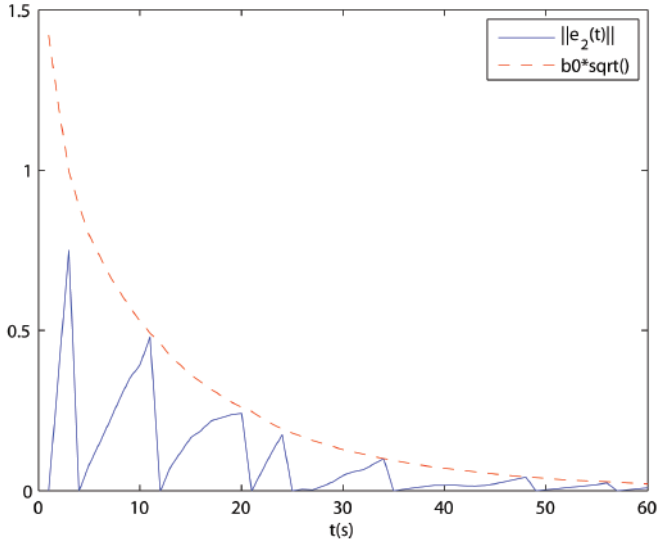


Figure 4: The evolution of $\| e_2(t) \|$ for agent 2

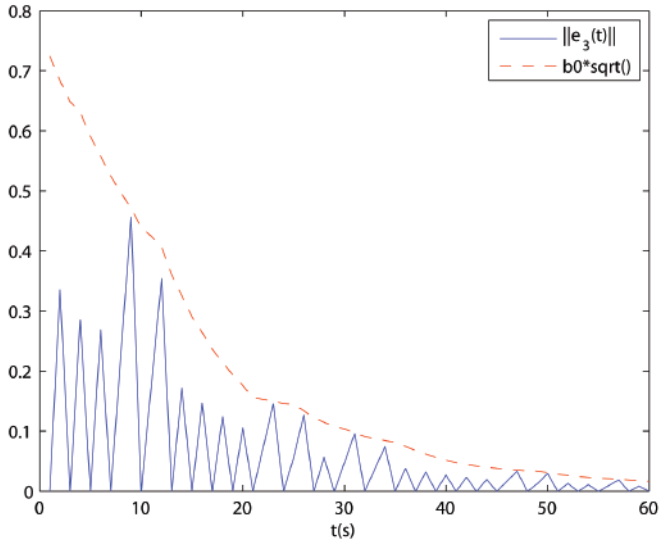


Figure 5: The evolution of $\| e_3(t) \|$ for agent 3

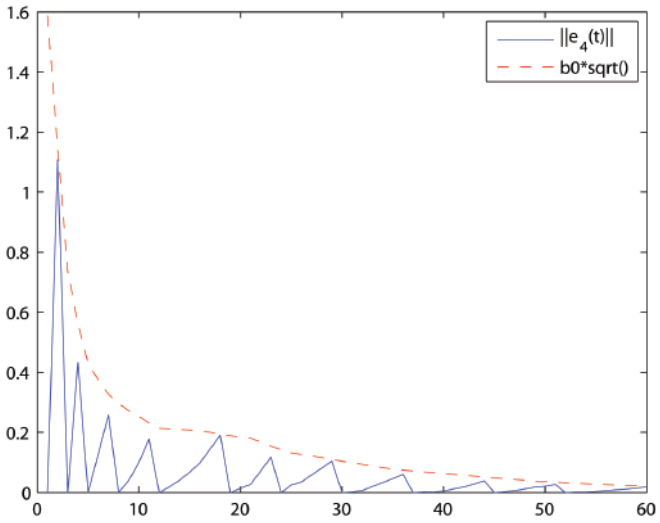


Figure 6: The evolution of $\| e_4(t) \|$ for agent 4

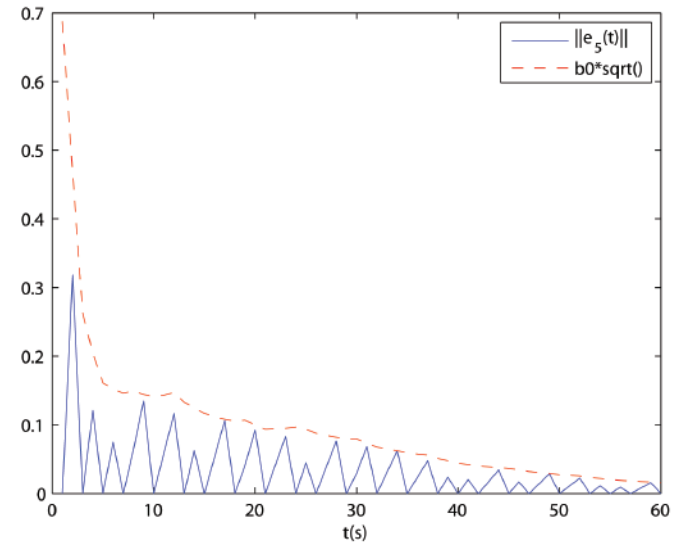


Figure 7: The evolution of $\| e_5(t) \|$ for agent 5

Fig.8 shows that event-triggered instants for each agent during $[0, 4.5]$. The distributed event-triggered scheme reduces significantly the frequency of signal transmission and control updating.

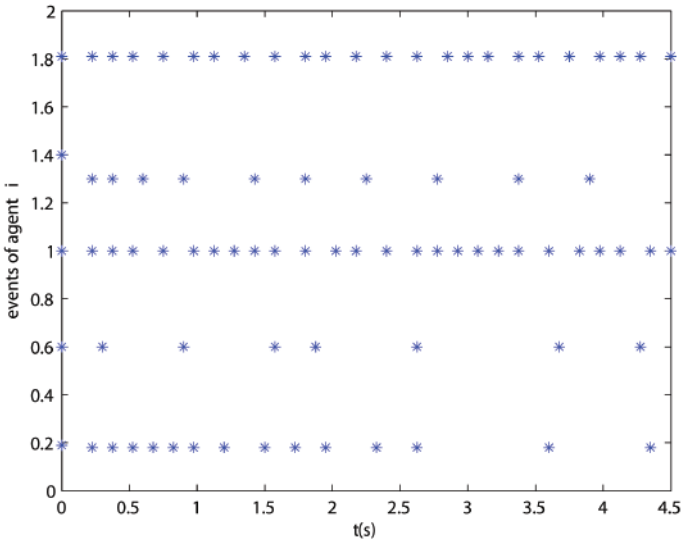


Figure 8: Event time of five agents with $b_0 = 0.04$.

5. Conclusion

This paper studied event-based consensus problem of distributed MASs by using the algebraic graph theory. One sufficient condition is presented to guarantee consensus on the trigger functions. The trigger condition avoids monitoring the states of each agents neighbors continuously. As the suggested event-triggered control policy does not require continuous local information, so it is more suitable for practical implementation. Finally, the simulation result was indicated to show the effectiveness of the proposed approach.

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Dunkl-Williams inequality for operators associated with r-angular distance

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Abstract. In this paper, We present several operator versions of the Dunkl-Williams inequality with respect to the r-angular distance for operators. We obtain refinements of some operator inequalities presented by Jiang and Zou.
Keywords: Dunkl-Williams inequality, R-angular distance, operator inequalities.

1. Introduction

Let B(H) be the algebra of all bounded linear operators acting on a complex Hilbert space H. For $A \in B(H)$, we denote the absolute value operator of A by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint operator of A. Throughout this paper, we assume that $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In 1964, Dunkl and Williams [2] showed that for any nonzero elements x, y in a normed linear space X,

$$\left\| \frac{x}{\|x\|} \right\| - \left\| \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}. \tag{1.1}$$

Pečarić and Rajić [9] gave the following refinement of (1.1): For any nonzero elements x, y in a normed linear space X,

$$\left\| \frac{x}{\|x\|} \right\| - \left\| \frac{y}{\|y\|} \right\| \leq \frac{(2\|x-y\|^2 + 2(\|x\| - \|y\|)^2)^{\frac{1}{2}}}{max\{\|x\|, \|y\|\}}. \tag{1.2}$$

Also they introduced an operator version of (1.2) by estimating $|A|A|^{-1} - B|B|^{-1}|$,

$$|A|A|^{-1} - B|B|^{-1}|^2 \leq |A|^{-1}(p|A-B|^2 + q(|A| - |B|^2)|A|^{-1}, \tag{1.3}$$

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where $A, B \in B(H)$ such that $|A|$ and $B|$ are invertible, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Saito and Tominaga [10] gave a generalization of the inequality(1.3) without additional assumptions related to inverse conditions. We would like to refer the reader to [4, 5, 6, 7, 8] and references therein for more information.

$$|(U - V)|A||^2 \leq p|A - B|^2 + q(|A| - |B|^2),$$

where $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Dadipour, Fujii and Moslehian [1] introduced operator versions of the Dunkl-Williams inequality with respect to the r-angular distance as a generalization of both the main result of Saito and Tominaga.

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2, \tag{1.4}$$

where $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$, $0 < r \leq 1$,$p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Recently, Jiang and Zou [11] gave refinements of some operator inequalities presented by Zou, He and Qaisar [12] as follows:

Theorem A. Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$. If $q > 0$, then

$$|(U-V)|A||^2+\frac{q}{p}|(1-p)(A-B)-V(|A|-|B|)|^2 \leq p|A-B|^2+q||A|-|B||^2. \tag{1.5}$$

If $q < 0$, then

$$|(U-V)|A||^2+\frac{q}{p}|(1-p)V(|A|-|B|)-(A-B)|^2 \leq q|A-B|^2+p||A|-|B||^2. \tag{1.6}$$

Theorem B. Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$. If $p < 0$, then

$$p(|A|-|B|)^2+q|A-B|^2-\frac{q}{p}|(1-p)V(|A|-|B|)-(A-B)|^2 \leq |(U-V)|A||^2. \tag{1.7}$$

If $0 < p < 1$, then

$$p|A-B|^2+q||A|-|B||^2-\frac{q}{p}|(1-p)(A-B)-V(|A|-|B|)|^2 \leq |(U-V)|A||^2. \tag{1.8}$$

If $1 < p \leq 2$, then

$$\begin{aligned} &p|A - B|^2 + q|V(|A| - |B|)|^2 - \frac{p+2}{2q}|(A - B) - (1 - q)V(|A| - |B|)|^2 \\ &\leq |(U - V)|A||^2. \end{aligned} \tag{1.9}$$

If $p > 2$, then

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{q+2}{2p}|(1-p)(A-B) - V(|A|-|B|)|^2$$
$$\leq |(U-V)|A||^2. \tag{1.10}$$

In this paper, We present several operator versions of the Dunkl-Williams inequality with respect to the r-angular distance for operators. We obtain refinements of some operator inequalities presented by Jiang and Zou [11].

2. Main results

In this section, we consider Dunkl-Williams inequality for operators as an application of the generalized parallelogram law of operators [3]:

$$|T_1 - T_2|^2 + \frac{1}{t}|tT_1 + T_2|^2 = (1+t)|T_1|^2 + (1+\frac{1}{t})|T_2|^2. \tag{2.1}$$

Applying the above generalized parallelogram law of operators, we can get the following lemma:

Lemma 2.1. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $0 \leq r \leq 1$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \tag{2.2}$$

and

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2$$
$$= q|A-B|^2 + p|V(|B|^r|A|^{1-r} - |B|)|^2 \tag{2.3}$$

Proof. Putting $T_1 = A - B, T_2 = V(|B|^r|A|^{1-r} - |B|), p = 1 + t$ and $q = 1 + \frac{1}{t}$ in (2.1), then we have

$$|(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2.$$

Note that

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 = |U|A| - V|B|^r|A|^{1-r}|^2 = |A - V|B|^r|A|^{1-r}|^2$$
$$= |(A-B) - V|B|^r|A|^{1-r} + B|^2 = |(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

The combination of above two equalities gives $|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 = p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2$.

By putting $T_1 = V(|B|^r|A|^{1-r} - |B|), T_2 = A - B, p = 1 + t$ and $q = 1 + \frac{1}{t}$ in (2.1), we can obtain the equality (2.3) in a similar way.

This completes the proof. □

Remark 2.2. Note that $p > 1$ implies $q > 1$, so the identity (2.2) gives immediately the inequality (1.4):

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2.$$

Theorem 2.3. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, $0 \leq r \leq 1$. If $q > 0$, then*

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.4}$$

If $q < 0$, then

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\ &\leq q|A - B|^2 + p||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.5}$$

Proof. If $q > 0$, since $V^*V \leq I$, we get

$$\begin{aligned} |V(|B|^r|A|^{1-r} - |B|)|^2 &= (|B|^r|A|^{1-r} - |B|)V^*V(|B|^r|A|^{1-r} - |B|) \\ &\leq ||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.6}$$

Combining with (2.2) and (2.6), we obtain

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2. \end{aligned}$$

Because $q < 0$ implies $0 < p < 1$, Combining with (2.3) and (2.6), we have

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\ &\leq q|A - B|^2 + p||B|^r|A|^{1-r} - |B||^2. \end{aligned}$$

Remark 2.4. Obviously, the inequalities (1.5) and (1.6) are special cases of the Theorem 2.2 when $r = 0$. Furthermore, inequality (2.4) is a refinement of inequality (1.4) when $p > 1$.

Next, we present some lower bounds for$|(U|A|^r - V|B|^r)|A|^{1-r}|^2$.

Theorem 2.5. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, $0 \leq r \leq 1$. Then the following statements hold:*

(i) *If* $p < 0$, *then*

$$\begin{aligned} &p||B|^r|A|^{1-r} - |B||^2 + q|A - B|^2 \\ &- \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.7}$$

(ii) *If* $0 < p < 1$, *then*

$$\begin{aligned} &q||B|^r|A|^{1-r} - |B||^2 + p|A-B|^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.8}$$

(iii) *If* $1 < p \le 2$ *and* $0 < \lambda < 1$, *then*

$$\begin{aligned} &q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2 \\ &- \frac{(p-2)\lambda + 2}{q}|(A - B) - (1 - q)V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.9}$$

(iv) *If* $p > 2$ *and* $0 < \lambda < 1$, *then*

$$\begin{aligned} &q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2 \\ &- \frac{(2-q)\lambda + q}{p}|(1 - p)(A - B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.10}$$

Proof. (i) Since $V^*V \le I$ and $p < 0$, we have

$$p|V(|B|^r|A|^{1-r} - |B|)|^2 \ge p||B|^r|A|^{1-r} - |B||^2$$

Combining with the above equality and (2.3), we obtain

$$\begin{aligned} &p||B|^r|A|^{1-r} - |B||^2 + q|A - B|^2 - \frac{q}{p}|(1 - p)V(|B|^r|A|^{1-r} - |B|) - (A - B)|^2 \\ &\le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned}$$

(ii) If $0 < p < 1$, then $q < 0$. So we have

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 \ge q||B|^r|A|^{1-r} - |B||^2.$$

Combining with the above equality and (2.2), we obtain

$$\begin{aligned} &q||B|^r|A|^{1-r} - |B||^2 + p|A - B|^2 - \frac{q}{p}|(1 - p)(A - B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\le |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned}$$

(iii) Note that

$$\frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 = \frac{p}{q}|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2$$

So, the equality (2.2) implies

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 = p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{q}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{p}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned} \tag{2.11}
$$

If $1 < p \leq 2$ and $0 < \lambda < 1$, then we have

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \geq p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{q}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2 \\
&= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 - \frac{(p-2)\lambda + 2}{q}|(A-B) \\
&- (1-q)V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

(iv) If $p > 2$ and $0 < \lambda < 1$, then $1 < q < 2$. By equality (2.11), we also have

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \geq p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2 \\
&= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{(2-q)\lambda + q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

Remark 2.6. It is easy to see that the Theorem B is a special case of the Theorem 2.5 when $r = 0$ and $\lambda = \frac{1}{2}$. In addition, inequalities (2.9) and (2.10) are refinements of inequalities (1.9) and (1.10) respectively when $r = 0$ and $\lambda > \frac{1}{2}$.

Remark 2.7. Since $p < 0$ implies $0 < q < 1$, $0 < p < 1$ implies $q < 0$, $1 < p \leq 2$ implies $q \geq 2$, and $p > 2$ implies $0 < q < 2$, combining with Theorems 2.3 and 2.5, we have the following results.

If $p < 0$, then

$$
\begin{aligned}
&p||B|^r|A|^{1-r} - |B||^2 + q|A-B|^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\
&\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2 \\
&\leq p|A-B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

If $0 < p < 1$, then

$$q||B|^r|A|^{1-r} - |B||^2 + p|A-B|^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq q|A-B|^2 + p||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2.$$

If $1 < p \leq 2$ and $0 < \lambda < 1$, then

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A-B|^2$$
$$- \frac{(p-2)\lambda + 2}{q}|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq p|A-B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

If $p > 2$ and $0 < \lambda < 1$, then

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A-B|^2$$
$$- \frac{(2-q)\lambda + q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq p|A-B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

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Legendre-adomian-homotopy analysis method for solving multi-term nonlinear differential equations of fractional order

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Abstract. In this article a modified approach using shifted Legendre-Adomian Homotopy analysis method is applied in order to solve multi-term fractional order nonlinear differential equations. This approach enable us to integrate the deformation equation in an easily manner, moreover the computations connected with the algorithm are so elementary than the standard Homotopy analysis method. The fractional derivatives are described in the Caputo's sense; the results of applying this procedure to the studied case showed that the proposed method is accurate and effective when it's compared with the usual Homotopy analysis method.

Keywords: fractional order differential equation (FDEs), shifted Legendre polynomials, homotopy analysis method (HAM).

1. Introduction

Differential equations stand out dramatically in the applications of physics, chemistry and even mathematical models of biological, social and even economic process. However, differential equations fail some times to formulate or describe like phenomena,so, to get best models, FDEs are appointed in state integer order ones [1], [2], [3], [4]. Fractional differential equations have been utilized in different area of science, engineering. applied mathematics et al. they are popularization of differential equations that exchange integral order derivatives by fractional order derivatives [5], [6]. Since the exact solutions of nonlinear FDE's are difficult to get it, therefore numerical and approximate approaches must be used, such as Adomian decomposition method [7], fractional differential

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transformation method [8], homotopy analysis method [9], polynomials method [10], variational iteration method [11], wavelet method [12], and B-Spline operational matrix method [13]. The feasible interest of the HAM has aroused of many authors in recent years. it has been successfully performed to handle different kinds of non-linear, non-homogenous equations and system of equations as well as problems in science and engineering, see for instance, [14], [15], [16], [17], [18], [19] , [20], [21], [22], [23], etc. Legendre polynomials (LPs) have a wide implementations to integer and fractional order ordinary, partial and integral equations such as [24], [25], [26], [27], [28], [29], [30]. In this paper a modified approach which combined the HAM and the orthogonal shifted Legendre polynomials which is so-called Legendre-Homotopy analysis method (LHAM) have been introduced for solving the following class of multi-term fractional order differential equations:

$$ {}^{c}D_{a}^{\alpha_M}u(x) = f(x, u(x), D_{a}^{\alpha_1}u(x), ..., D_{a}^{\alpha_{M-1}}u(x)) \tag{1} $$

We assume that $\alpha_M > \alpha_{M-1} > ... > \alpha_1 > 0, \alpha_i - \alpha_{i-1} \leq 1,\ M \in N$, and $\alpha_i \in Q$ for all i, subject to initial conditions:

$$ u^{(k)}(0) = {u_0}^{(k)} \quad k = 0, 1, ..., \lceil \alpha_M \rceil - 1, \tag{2} $$

where $\lceil \alpha_M \rceil$ is the least integer greater than or equal to α_M.

2. Fractional order derivatives and integrals

Here we review basic definitions and properties of derivatives and integrals of fractional order [31].

Definition 1. *The fractional integral operator in the Riemann-Liouville sense of order $\alpha > 0$ of function y is given by:*

$$ I_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1}\ y(\tau)\ d\tau, \alpha > 0, & \tau > 0, \\ y(t), & \alpha = 0, \end{cases} $$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2. *The fractional derivative operator in the Caputo sense of order $\alpha > 0$ of function y, is given as:*

$$ {}^{c}D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1}\ y^{(n)}(\tau)\ d\tau, & n-1 < \alpha \leq n, \\ \dfrac{d^n y(t)}{dt^n}, & \alpha = n, \end{cases} $$

where n is an integer.

Some properties of the fractional differential equations that we will use later.

2.1 Properties

(1) ${}^{\alpha}I_t t^{\beta} = (\Gamma(\beta+1)/\Gamma(\beta+\alpha+1))t^{\alpha+\beta}$, $\beta > -1, \alpha > 0$.

(2) ${}^{c}D_t^{\alpha} t^{\beta} = (\Gamma(\beta+1)/\Gamma(\beta-\alpha+1))t^{\beta-\alpha}, \beta > -1, \alpha > 0$.

(3) ${}^{c}D_t^{\alpha}(I_t^{\alpha} y(t)) = y(t)$.

(4) $I_t^{\alpha}({}^{c}D_t^{\alpha} y(t)) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0^+)}{k!} t^k, t \geq 0, n-1 < \alpha \leq n$.

3. Shifted Legendre polynomials

LPs which is denoted by $L_j(x)$ are outlined through the interval [-1,1] and satisfy the following recurrence relations:

(3) $L_{j+1}(x) = ((2j+1)/(j+1))xL_j(x) - (j/(j+1))L_{j-1}(x), \quad j = 1, 2, 3, ...,$

where $L_0(x) = 1$ and $L_1(x) = x$.

The shifted Legendre polynomials (SLPs) over the interval [0,1] are defined as $L_j^*(x) = L_j(2x-1)$. And satisfy the recurrence relation

(4) $L_{j+1}^*(x) = ((2j+1)/(j+1))(2x-1)L_j^*(x) - (j/(j+1))L_{j-1}^*(x), j = 1, 2, 3, ...,$

where $L_0^*(x) = 1$ and $L_1^*(x) = 2x - 1$.

Since LPs are orthogonal, therefore any function $f(x) \in L_2[0,1]$ can be decomposed in terms of SLPs, as

$$f(x) = \sum_{k=0}^{\infty} c_k L_k^*(x), \tag{5}$$

where $c_k's$ can be calculated, as

$$c_k = (2k+1) \int_0^1 f(x) L_k^*(x) dx. \tag{6}$$

4. The approach

In this section we shall construct a series solution corresponding to the multi-term FDEs of the form:

$${}^{c}D_a^{\alpha_M} u(x) = f(x, u(x), D_a^{\alpha_1} u(x), ..., D_a^{\alpha_{M-1}} u(x)), \quad x \in [a, b], \tag{7}$$

where $\alpha_M > \alpha_{M-1} > ... > \alpha_1 > 0, \alpha_i - \alpha_{i-1} \leq 1, M \in N$, and $\alpha_i \in Q$ for all i. Subject to the initial conditions:

$$u^{(k)}(0) = {u_0}^{(k)} \quad k = 0, 1, ..., \lceil \alpha_M \rceil - 1 \tag{8}$$

in vision of the homotopy technique, a zero order deformation equations can be defined using an embedding parameter $q \in [0,1]$ by $(1-q)Ł[\phi(x,q)-u_0(x)] = qhH(x)[{}^cD_a^{\alpha_M}\phi(x,q) - f(x,\phi(x,q),{}^cD_a^{\alpha_1}\phi(x,q),...,$

$$ {}^cD_a^{\alpha_{M-1}}\phi(x,q))], \tag{9}$$

where $h \neq 0$ and $H(x) \neq 0$ are an auxiliary parameter and function respectively, and Ł an auxiliary linear operator with property $Ł[u(x)] = 0$ when $u(x) = 0$. Its clear that when $q = 0$ and $q = 1$, we have $\phi(x,0) = u_0(x)$ and $\phi(x,1) = u(x)$.

In the proposed method an initial approximation $u_0(x)$ can be defined as:

$$u_0(x) = \sum_{j=0}^{m} a_{0j} L_j^*(x), \tag{10}$$

$$a_{0j} = (2j+1)\int_0^1 U_0(x) L_j^*(x)dx, \tag{11}$$

where $U_0(x)$ is the solution of the equation ${}^cD_a^{\alpha_M}u(x) = 0$, with respect to the conditions given by eq. (8).

The first order deformation equations can be expressed using shifted Legendre and Adomian polynomials, as:

$$\begin{aligned} Ł[u_1(x)] &= hH(x)[{}^cD_a^{\alpha_M}(\sum_{j=0}^{m} a_{0j}L_j^*(x)) \\ &- A_0(x,(\sum_{j=0}^{m} a_{0j}L_j^*(x)),{}^cD_a^{\alpha_1}(\sum_{j=0}^{m} a_{0j}L_j^*(x)),...,{}^cD_a^{\alpha_{M-1}}(\sum_{j=0}^{m} a_{0j}L_j^*(x))] \end{aligned} \tag{12}$$

and for $i = 1,2,3,,$ the deformation equations can be utilized, as:

$$\begin{aligned} Ł[u_{i+1}(x)] &= Ł[u_i(x)] + hH(x)[{}^cD_a^{\alpha_M}(\sum_{j=0}^{m} a_{ij}L_j^*(x)) \\ &- A_i(x,(\sum_{j=0}^{m} a_{ij}L_j^*(x)),D_a^{\alpha_1}(\sum_{j=0}^{m} a_{ij}L_j^*(x)),...,D_a^{\alpha_{M-1}}(\sum_{j=0}^{m} a_{ij}L_j^*(x))], \end{aligned} \tag{13}$$

where

$$a_{ij} = (2j+1)\int_0^1 u_i(x)L_j^*(x)dx \quad i = 1,2,3,..., \quad j = 0,1,2,...,m \tag{14}$$

and A_i are the Adomian polynomials of $x, u(x),{}^cD_a^{\alpha_1}u(x),...,{}^cD_a^{\alpha_{M-1}}u(x)$ defined by

$$\begin{aligned} &A_n(x,u(x),{}^cD_a^{\alpha_1}u(x),...,{}^cD_a^{\alpha_{M-1}}u(x)) \\ &= (1/n!)(d^n/d\lambda^n)[f(x,\sum_{k=0}^{n}\lambda^k u_k,\sum_{k=0}^{n}\lambda^k\ {}^cD_a^{\alpha_1}u_k,..., \\ &\sum_{k=0}^{n}\lambda^k\ {}^cD_a^{\alpha_{M-1}}u_k)]_{\lambda=0}, n = 0,1,2,... \end{aligned} \tag{15}$$

Finally, the exact solution $u(x)$ of problem (7)-(8) can be calculated as $u(x) = \sum_{i=0}^{\infty} u_i(x)$ and the n^{th} order analytical approximate solution can be calculated as:

$$u_n(x) = \sum_{i=0}^{n} u_i(x). \tag{16}$$

5. Illustrative examples

Example 1. Consider the following multi-term nonlinear higher-order non-homogenous FDEs,

$$\begin{aligned} &{}^cD_a^{2.2}u(x) + {}^c D_a^{1.25}u(x) + {}^c D_a^{0.75}u(x) + u(x)^3 \\ &= 2x^{0.8}/\Gamma(1.8) + 2x^{1.75}/\Gamma(2.75) + 2x^{2.25}/\Gamma(3.25) + (x^3/3)^3. \end{aligned} \tag{17}$$

Subject to:

$$u(0) = u^{(1)}(0) = u^{(2)}(0) = 0 \tag{18}$$

the closed form of problem (17)-(18) is $u(x) = x^3/3$.

The solution curves of problem (17)-(18) are plotted in Figure 1 using the proposed method, the exact solution and results were resulting from the usual HAM.

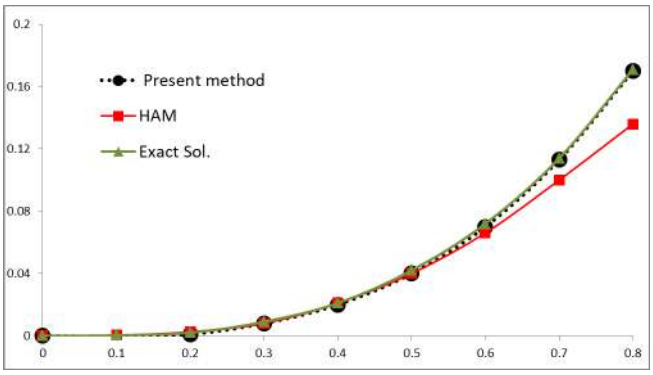


Figure 1: The proposed solution of problem (17)-(18)

Example 2. Consider the following linear nonhomogeneous FDE,

$${}^cD_a^2u(x) + {}^c D_a^{3/4}u(x) + u(x) = x^3 + 6x + 8.533x^{2.25}/\Gamma(0.25) \tag{19}$$

with

$$u(0) = u^{(1)}(0) = 0 \tag{20}$$

the closed form of problem (19)-(20) is $u(x) = x^3$.

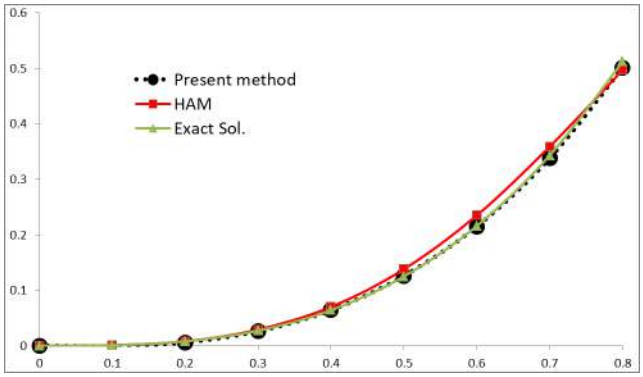


Figure 2: The proposed solution of problem (19)-(20)

The solution curves of problem (19)-(20) are plotted in Figure 2 using the proposed method, the exact solution and results were resulting from the usual HAM.

Example 3. Consider the following multi-term nonlinear non homogeneous FDE,

$$\begin{aligned} &{}^{c}D^{2}u(x) + 2\ D^{0.07621}u(x) + (1/2)\ D^{0.00196}u(x) + u(x)^{3} \\ &= 2x + 4x^{2.92379}/\Gamma(3.92379) + x^{2.99804}/\Gamma(3.99804) + x^{9}/27 \end{aligned} \tag{21}$$

with

$$u(0) = 0, \quad u^{(1)}(0) = 1 \tag{22}$$

the exact solution of problem (21)-(22) is $u(x) = (x^3)/3$.

The solution curves of problem (21)-(22) are plotted in Figure 3 using the proposed method, the exact solution and results were resulting from the usual HAM.

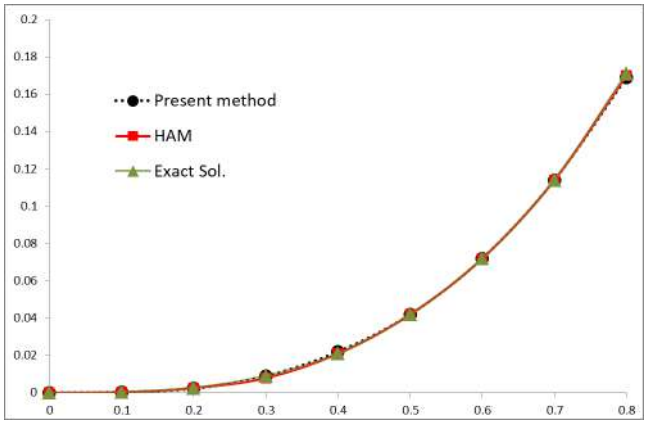


Figure 3: The proposed solution of problem (21)-(22)

6. Conclusions

In this work we have been used a modified approach consist of, the HAM, Adomian polynomials and the SLPs which is so called (SLAHAM) for solving a certain class of multi-term nonlinear FDEs. in the proposed method the deformation equations are easily integrable especially its decomposed the nonlinear term using Adomian polynomials therefore the computations connected with the algorithm are straightforward than the usual HAM. three examples are presented in order to illustrate the effectiveness and accuracy of the numerical scheme.

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A new study on rough soft lattices based on ideals

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Abstract. By using a novel relation θ_I based on an ideal I of a lattice L, which is a congruence relation, we investigate the roughness of soft lattices under this special ideal of L, such as rough soft sublattices, rough soft ideals and rough soft filters.
Keywords: lattice, ideal, filter, rough soft set, rough soft lattice (ideal, filter).

1. Introduction

In 1982, Pawlak [23] introduced the concept of rough set theory, a new mathematical approach to deal with uncertain knowledge, has recently received widespread attention on the research clues in both of theory and applications. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applications. Thus, some more general models have been proposed, such as [31, 32, 33]. Nowadays, rough set theory has been applied to many areas, such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [12, 11, 28]. In particular, many researchers applied this theory to algebraic structures. For example, Biswas and Nanda [4] proposed the notion of rough subgroups. Jun [16] studied roughness of ideals in BCK-algebras. Davvaz [9] studied roughness of rings. Xiao and Zhang [26] introduced the concept of rough prime ideals and rough fuzzy prime ideals in a semigroups. Liu [18] investigated the algebraic structures of generalized rough set theory. Davvaz and Mahdavipour [10] investigated rough modules. Rasouli and Davvaz [24] investigated roughness in MV-algebra. Ali et al. [3] investigated roughness in hemirings. Further, some authors applied rough set theory in a fuzzy algebraic system. For example, Kuroki [17] studied lower and upper

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approximations in a fuzzy group. In particular, Davvaz [9] constructed a t-level relation based on a fuzzy ideal and showed that $U(\mu, t)$ is a congruence relation on rings. In 2016, Zhan et al. [29] investigated roughness in n-ary semigroups based on fuzzy ideals.

In 1999, Molodtsov [21] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, the research on soft set theory is progressing rapidly. In 2003, Maji et al. [19] proposed some basic operations. Further, Ali et al. [1] revised some operations. In 2011, Ali [3] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [5, 6, 7, 14, 15, 20, 25]. Zhan and Zhu [30] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [14, 15]. In particular, in 2017, Zhu and Hu [34] investigated the relationship among soft sets, rough sets, fuzzy sets and lattices. The notion of soft rough fuzzy lattices (ideals, filters) over lattices was introduced. In 2018, Zhu and Hu [35] introduced the notion of Z-soft rough fuzzy BCI-algebras (ideals). Moreover, they studied roughness in BCI-algebras with respects to a Z-soft approximation space. Some new Z-soft rough fuzzy operations over BCI-algebras were explored. In particular, Z-lower and Z-upper soft rough fuzzy BCI-algebras (ideals) were investigated.

In mathematics, a lattice is a poset in which any two elements have a unique supremum and an infimum. We can also characterize lattices as algebraic structures satisfying certain axiomatic identities. Based on [9], in this paper, we consider the relationships between rough sets, soft sets and lattices. By using a special relation θ_I based on an ideal I of a lattice L, we present a definition of the lower and upper approximations of a subset of a lattice with respect to an ideal. Further, we characterize rough soft lattices (ideals, filters).

This paper is organized as follows. In Section 2, we recall some concepts and results on lattices, soft sets and rough sets. In Section 3, we study some operations of lower and upper approximations of lattices. Rough soft lattices (ideals, filters) based on ideals are investigated in Section 4.

2. Preliminaries

In this section, we recall some basic notions and results about lattices, soft sets and rough sets.

2.1 Lattices

A poset $(L, \leq)$ is called a lattice if it satisfies the condition that for any α, β in L, both $\alpha \vee \beta \in L$ and $\alpha \wedge \beta \in L$. L is said to be distributive if it satisfies the distributive law: $\forall x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 2.1 ([8]). Let L be a lattice and $\emptyset \subsetneq X \subseteq L$. Then X is a sublattice of L if $x, y \in L$, $x \vee y \in X$ and $x \wedge y \in X$.

Definition 2.2 ([8]). Let L be a lattice and $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if:

(1) $a, b \in I$ implies $x \vee y \in I$,

(2) $a \in L, b \in I$ and $a \leq b$ imply $a \in I$.

Definition 2.3 ([8]). Let L be a lattice and $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if:

(1) $a, b \in F$ implies $x \wedge y \in I$,

(2) $a \in L, b \in F$ and $a \geq b$ imply $a \in F$.

2.2 Soft sets

Definition 2.4 ([21]). A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \longrightarrow P(U)$ is a set-valued mapping.

For a soft set (F, A), the set $\text{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A). Hence, a null soft set is indeed a soft set with an empty support, and we say that a soft set (F, A) id non-null if $\text{Supp}(F, A) \neq \emptyset$.

Definition 2.5 ([13]). Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U. The inclusion symbol " $\subseteq$ " of $\mathfrak{S}$ and $\mathfrak{I}$ by $\mathfrak{S} \subseteq \mathfrak{I}$, is defined as:

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.6 ([1]). Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U. The inclusion symbol " $\subseteq$ " of $\mathfrak{S}$ and $\mathfrak{I}$ by $\mathfrak{S} \subseteq \mathfrak{I}$, is defined as:

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.7 ([1]). Let U be an initial universe set, E be the universe set of parameters and $A \subseteq E$.

(1) (N, A) is called a relative null soft set if $N(x) = \emptyset$.

(2) (W, A) is called a relative whole soft set if $W(x) = U$ for all $x \in A$.

(3) The relative complement of soft set $\mathfrak{S} = (F, A)$, denoted by $(F, A)^r$, is defined as the soft set (F^r, A), where $F^r(x) = U - F(x)$ for all $x \in A$.

Definition 2.8 ([1]). Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U.

(1) The restricted intersection of (F, A) and (G, B), denoted by $(F, A) \Cap (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$, and $H(x) = F(x) \cap G(x)$ for all $x \in C$.

(2) The extended intersection of (F, A) and (G, B), denoted by $(F, A)\widetilde{\cap}(G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

(3) The restricted union of (F, A) and (G, B), denoted by $(F, A) \Cup (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $H(x) = F(x) \cup G(x)$ for all $c \in C$.

(4) The extended union of (F, A) and (G, B), denoted by $(F, A)\widetilde{\cup}(G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

2.3 Rough sets

Definition 2.9 ([23]). Let ρ be an equivalence relation on the universe U and (U, ρ) be a Pawlak approximation space. A subset $A \subseteq U$ is called definable if $\underline{\rho}(A) = \overline{\rho}(A)$; in the opposite case, i.e., if $\underline{\rho}(A) \neq \overline{\rho}(A)$, A is said to be a rough set, where two operators are defined as:

$$\underline{\rho}(A) = \{x \in U | [x]_\rho \subseteq A\},$$

$$\overline{\rho}(A) = \{x \in U | [x]_\rho \cap A \neq \emptyset\}.$$

Definition 2.10 ([13]). Let (U, ρ) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (U, ρ) are denoted by $\underline{\rho}(\mathfrak{S}) = (\underline{F}, A)$ and $\overline{\rho}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with

$$\underline{F}(x) = \underline{\rho}(F(x)) = \{y \in U | [y]_\rho \subseteq F(x)\},$$

$$\overline{F}(x) = \overline{\rho}(F(x)) = \{y \in U | [y]_\rho \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

If $\underline{\rho}(\mathfrak{S}) = \overline{\rho}(\mathfrak{S})$, $\mathfrak{S}$ is called definable; otherwise, $\mathfrak{S}$ is called a rough soft set.

Definition 2.11 ([22]). Let (F, A) be a soft set over a lattice L. Then (F, A) is said to be a soft lattice (ideal, filter) over L if $F(x)$ is a sublattice (ideal, filter) of L, for all $x \in A$.

Theorem 2.12 ([27]). Let I be an ideal of a lattice L and define a relation θ_I on L by

for all $a, b \in L$, $a \equiv b(\text{mod } \theta_I) \Leftrightarrow \exists d \in I \ s.t. \ a \vee d = b \vee d$.

Then the following statement hold:

(1) θ_I is a join-congruence on L;

(2) θ_I is a congruence on L if and only L is distributive.

3. Rough soft lattices based on ideals

In this section, we introduce the concept of rough soft lattices based on ideals, and we investigate the operations of lower and upper rough approximations of soft sets over lattices.

An equivalence relation θ on L is called a joint-congruence and meet-congruence if, for all $a, b, c, d \in L$, $a \equiv b(\text{mod } \theta)$ and $c \equiv d(\text{mod } \theta)$ imply $a \vee c \equiv b \vee d(\text{mod } \theta)$ and $a \wedge c \equiv b \wedge d(\text{mod } \theta)$, respectively. θ is called congruence relation if it is both a joint-congruence and a meet-congruence. It is obvious that $\{x \vee y | x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \vee b]_\theta$, $\{x \wedge y | x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \wedge b]_\theta$. Let A and B be subsets of L. We define the join and meet of two non-empty subsets in a lattice as follows:

$$A \vee B = \{a \vee b | a \in A, b \in B\}, A \wedge B = \{a \wedge b | a \in A, b \in B\}.$$

In the following, L is always a distributive lattice.

Let I be an ideal of L. It follows from Theorem 2.12 that θ_I is a congruence on L. Therefore, when $U = L$ and ρ is the above equivalence relation, then we use (L, θ_I) instead of approximation space (U, ρ).

Definition 3.1. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over L. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (L, θ_I) are denoted by: $\underline{\theta_I}(\mathfrak{S}) = (\underline{F}_I, A)$ and $\overline{\theta_I}(\mathfrak{S}) = (\overline{F}_I, A)$, respectively, which are soft sets over L with

$$\underline{F}_I(x) = \underline{\theta_I}(F(x)) = \{y \in U | [y]_{\theta_I} \subseteq F(x)\},$$

$$\overline{F}_I(x) = \overline{\theta_I}(F(x)) = \{y \in U | [y]_{\theta_I} \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

(1) If $\underline{\theta_I}(\mathfrak{S}) = \overline{\theta_I}(\mathfrak{S})$, $\mathfrak{S}$ is called definable*;*

(2) If $\underline{\theta_I}(\mathfrak{S}) \neq \overline{\theta_I}(\mathfrak{S})$, $\underline{\theta_I}(\mathfrak{S})$ $(\overline{\theta_I}(\mathfrak{S}))$ is called a lower (upper) rough soft set. Moreover, $\mathfrak{S}$ is called a rough soft set.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ with $0 \leq a, b \leq c \leq 1$. Obviously, L is a distributive lattice. Let $I = \{0, a\}$ be an ideal of L. Then $[1]_{\theta_I} = \{1\}$, $[c]_{\theta_I} = \{b, c\}$, $[a]_{\theta_I} = \{0, a\}$. Define a soft set $\mathfrak{S} = (F, A)$ over L, $A = \{\alpha, \beta\}$, where $F(\alpha) = \{b, 1\}$, $F(\beta) = \{0, b, c\}$. It follows from Definition 3.1 that $\underline{F}_I(\alpha) = \{1\}$, $\underline{F}_I(\beta) = \{b, c\}$, $\overline{F}_I(\alpha) = \{1, b, c\}$, $\overline{F}_I(\beta) = \{0, a, b, c\}$.

Definition 3.3. In Definition 3.1, if $\underline{\theta_I}(\mathfrak{S}) \neq \overline{\theta_I}(\mathfrak{S})$, then $\mathfrak{S}$ is called a rough soft lattice (ideal, filter) if $\underline{F}_I(x)$ and $\overline{F}_I(x)$ are sublattices (ideals, filters) of L, for all $x \in A$.

Example 3.4. Consider Example 3.2. We define a soft set $\mathfrak{S} = (F, A)$ over L, $A = \{\alpha\}$, where $F(\alpha) = \{0, a\}$. It follows from Definition 3.1 that $\underline{F}_I(\alpha) = \{0, a\}$, $\underline{F}_I(\beta) = \{0, a\}$. It follows from Definition 2.2 that $\underline{F}_I(\alpha) = \{0, a\}$ and $\underline{F}_I(\beta) = \{0, a\}$ are ideals of L, for all $x \in A$. It follows from Definition 3.3 that $\mathfrak{S} = (F, A)$ is a rough soft ideal of L.

Proposition 3.5. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L. Then:

(1) $\underline{\theta_I}(\mathfrak{S}) \subseteq \mathfrak{S} \subseteq \overline{\theta_I}(\mathfrak{S})$;
(2) $\underline{\theta_I}((N, A)) = (N, A) = \overline{\theta_I}((N, A))$;
(3) $\underline{\theta_I}((W, A)) = (W, A) = \overline{\theta_I}((N, A))$;
(4) $\underline{\theta_I}(\underline{\theta_I}(\mathfrak{S})) = \underline{\theta_I}(\mathfrak{S})$;
(5) $\overline{\theta_I}(\overline{\theta_I}(\mathfrak{S})) = \overline{\theta_I}(\mathfrak{S})$;
(6) $\overline{\theta_I}(\underline{\theta_I}(\mathfrak{S})) = \underline{\theta_I}(\mathfrak{S})$;
(7) $\underline{\theta_I}(\overline{\theta_I}(\mathfrak{S})) = \overline{\theta_I}(\mathfrak{S})$;
(8) $\underline{\theta_I}(\mathfrak{S}) = (\overline{\theta_I}(\mathfrak{S}^r))^r$;
(9) $\overline{\theta_I}(\mathfrak{S}) = (\underline{\theta_I}(\mathfrak{S}^r))^r$.

Proposition 3.6. Let (L, θ_I) be a Pawlak approximation space, $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be soft lattices over L. Then:

(1) $\underline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I}) = \underline{\theta_I}(\mathfrak{S}) \Cap \underline{\theta_I}(\mathfrak{I})$;
(2) $\underline{\theta_I}(\mathfrak{S} \widetilde{\sqcap} \mathfrak{I}) = \underline{\theta_I}(\mathfrak{S}) \widetilde{\sqcap} \underline{\theta_I}(\mathfrak{I})$;
(3) $\overline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}) \Cap \underline{\theta_I}(\mathfrak{I})$;
(4) $\overline{\theta_I}(\mathfrak{S} \widetilde{\sqcap} \mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}) \widetilde{\sqcap} \overline{\theta_I}(\mathfrak{I})$;
(5) $\underline{\theta_I}(\mathfrak{S} \Cup \mathfrak{I}) \supseteq \underline{\theta_I}(\mathfrak{S}) \Cup \underline{\theta_I}(\mathfrak{I})$;
(6) $\underline{\theta_I}(\mathfrak{S} \widetilde{\sqcup} \mathfrak{I}) \supseteq \underline{\theta_I}(\mathfrak{S}) \widetilde{\sqcup} \underline{\theta_I}(\mathfrak{I})$;
(7) $\overline{\theta_I}(\mathfrak{S} \Cup \mathfrak{I}) = \overline{\theta_I}(\mathfrak{S}) \Cup \overline{\theta_I}(\mathfrak{I})$;
(8) $\overline{\theta_I}(\mathfrak{S} \widetilde{\sqcup} \mathfrak{I}) = \overline{\theta_I}(\mathfrak{S}) \widetilde{\sqcup} \overline{\theta_I}(\mathfrak{I})$;
(9) $\mathfrak{S} \subseteq \mathfrak{I} \Rightarrow \underline{\theta_I}(\mathfrak{S}) \subseteq \underline{\theta_I}(\mathfrak{I}), \overline{\theta_I}(\mathfrak{S}) \subseteq \overline{\theta_I}(\mathfrak{I})$;

Definition 3.7. Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be any two soft sets over L. Then:

(1) The $\widetilde{\vee}$ of $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$, denoted by $\mathfrak{S} \widetilde{\vee} \mathfrak{I} = (F \vee G, A \times B)$, where $(F \vee G)(x, y) = F(x) \vee G(y)$, for all $(x, y) \in A \times B$.

(2) The $\widetilde{\wedge}$ of $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$, denoted by $\mathfrak{S} \widetilde{\wedge} \mathfrak{I} = (F \wedge G, A \times B)$, where $(F \wedge G)(x, y) = F(x) \wedge G(y)$, for all $(x, y) \in A \times B$.

Theorem 3.8. Let I be an ideal of L, $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be any two non-null soft sets over L. Then:
(1) $\overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\vee}\mathfrak{I})$
(2) $\overline{\theta_I}(\mathfrak{S})\widetilde{\wedge}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\wedge}\mathfrak{I})$

Proof. (1) For $m \in \mathrm{Supp}(F, A)$, $n \in \mathrm{Supp}(G, B)$. Let $t \in \overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I})$. Then $t = p \vee q$, where $p \in \overline{\theta_I}(\mathfrak{S})$ and $q \in \overline{\theta_I}(\mathfrak{I})$. Hence, there exist $a, b \in L$ such that $a \in [p]_{\theta_I} \cap F(m)$, $b \in [q]_{\theta_I} \cap G(n)$. Since θ_I is a congruence on L, we have $a \vee b \in [p]_{\theta_I} \vee [q]_{\theta_I} \subseteq [p \vee q]_{\theta_I}$. On the other hand, since $a \vee b \in F(m) \vee G(n)$, we have $a \vee b \in [p \vee q]_{\theta_I} \cap (F(m) \vee G(n))$. This implies $p \vee q \in \overline{\theta_I}(F \vee G, A \times B))$, i.e., $t \in \overline{\theta_I}(F \vee G, A \times B)$. Therefore, $\overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\vee}\mathfrak{I})$.
(2) The proof is similar to that of (1). □

The following example shows that the containment in Theorem 3.8 (1) is proper.

Example 3.9. Consider Example 3.2. It is easy to check that $\overline{\theta_I}(F(\alpha))\widetilde{\vee}\overline{\theta_I}(F(\beta)) = \{1, b, c\}$ and $\overline{\theta_I}(F(\alpha)\widetilde{\vee}F(\beta)) = \{0, b, c, 1\}$. Thus, $\overline{\theta_I}(F(\alpha))\widetilde{\vee}\overline{\theta_I}(F(\beta)) \subsetneqq \overline{\theta_I}(F(\alpha)\widetilde{\vee}F(\beta))$.

In a similar way, we can check that the containment in Theorem 3.8 (2) is proper.

4. Characterizations of rough soft lattices (ideals, filters) of lattices based on ideals

In this section, we investigate rough soft lattices (ideals, filters) of lattices based on ideals.

Theorem 4.1. Let (L, θ_I) be a Pawlak approximation space, $\underline{\theta_I}(\mathfrak{S})$ and $\underline{\theta_I}(\mathfrak{I})$ be lower rough soft lattices (ideals, filters) over L. If $\mathfrak{S} \Cap \mathfrak{I}$ and $\mathfrak{S}\widetilde{\sqcap}\mathfrak{I}$ are non-null soft sets, then $\underline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I})$ and $\underline{\theta_I}(\mathfrak{S}\widetilde{\sqcap}\mathfrak{I})$ are lower rough soft lattices (ideals, filters) over L.

Proof. It follows from Proposition 3.6 (1) and (2). □

Theorem 4.2. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L. Then $\overline{\theta_I}(\mathfrak{I})$ is a soft lattice.

Proof. Let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft lattice over L, we have $F(x)$ is a sublattice of L for all $x \in \mathrm{Supp}(F, A)$. Thus, $a \vee b \in F(x)$ and $a \wedge b \in F(x)$. On the other hand, $a \vee b \in [m]_{\theta_I} \vee [n]_{\theta_I} \subseteq [m \vee n]_{\theta_I}$ and $a \wedge b \in [m]_{\theta_I} \wedge [n]_{\theta_I} \subseteq [m \wedge n]_{\theta_I}$. Hence, $a \vee b \in [m \vee n]_{\theta_I} \cap F(x)$ and $a \wedge b \in [m \wedge n]_{\theta_I} \cap F(x)$, which implies that $m \vee n \in \overline{\theta_I}(F(x))$ and $m \wedge n \in \overline{\theta_I}(F(x))$. Thus, for all $x \in \mathrm{Supp}(F, A)$, $\overline{\theta_I}(F(x))$ is a sublattice of L. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft lattice. □

Theorem 4.3. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L.

(1) If $\mathfrak{S} = (F, A)$ is a soft ideal of L, then $\overline{\theta_I}(\mathfrak{I})$ is a soft ideal of L.

(2) If $\mathfrak{S} = (F, A)$ is a soft filter of L, then $\overline{\theta_I}(\mathfrak{I})$ is a soft filter of L.

Proof. (1) For all $x \in \text{Supp}(F, A)$, let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft ideal of L, we have $F(x)$ is an ideal of L for all $x \in \text{Supp}(F, A)$. Thus $a \vee b \in F(x)$. On the other hand, $a \vee b \in [m]_{\theta_I} \vee [n]_{\theta_I} \subseteq [m \vee n]_{\theta_I}$. Hence, $a \vee b \in [m \vee n]_{\theta_I} \cap F(x)$, which implies that $m \vee n \in \overline{\theta_I}(F(x))$. Let $m \in L, n \in \overline{\theta_I}(F(x))$ and $m \leq n$. Then there exist $a \in [n]_{\theta_I} \cap F(x)$. Let $b \in [m]_{\theta_I}$. Then $a \wedge b \in [n]_{\theta_I} \wedge [m]_{\theta_I} \subseteq [n \wedge m]_{\theta_I} = [m]_{\theta_I}$. Since $a \wedge b \leq b$, we have $a \wedge b \in F(x)$. Hence, $[m]_{\theta_I} \cap F(x) \neq \emptyset$, which implies $m \in \overline{\theta_I}(F(x))$. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft ideal of L.

(2) For all $x \in \text{Supp}(F, A)$, let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft filter of L, we have $F(x)$ is a filter of L for all $x \in \text{Supp}(F, A)$. Thus $a \wedge b \in F(x)$. On the other hand, $a \wedge b \in [m]_{\theta_I} \wedge [n]_{\theta_I} \subseteq [m \wedge n]_{\theta_I}$. Hence, $a \wedge b \in [m \wedge n]_{\theta_I} \cap F(x)$, which implies that $m \wedge n \in \overline{\theta_I}(F(x))$. Let $m \in L, n \in \overline{\theta_I}(F(x))$ and $m \geq n$. Then there exist $a \in [n]_{\theta_I} \cap F(x)$. Let $b \in [m]_{\theta_I}$. Then $a \vee b \in [n]_{\theta_I} \vee [m]_{\theta_I} \subseteq [n \vee m]_{\theta_I} = [m]_{\theta_I}$. Since $a \vee b \geq b$, we have $a \vee b \in F(x)$. Hence, $[m]_{\theta_I} \cap F(x) \neq \emptyset$, which implies $m \in \overline{\theta_I}(F(x))$. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft filter of L. □

5. Conclusion

In this paper, we built up a connection between rough sets, soft sets and lattices. We introduced a new congruence relation induced by an ideal of a lattice, by using a novel congruence relation, we investigate the roughness of soft lattices under this special ideal of L, such as rough soft sublattices, rough soft ideals and rough soft filters.

As an extension of this work, the following problems maybe considered:

(I) Roughness of lattices based on fuzzy ideals;

(II) Rough prime ideals and rough fuzzy prime ideals based on fuzzy ideals;

(III) Rough sets induced by fuzzy ideal in lattices.

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Generalized Runge-Kutta integrators for solving fifth-order ordinary differential equations

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Abstract. In this paper, RKM integrators for solving special fifth-order ordinary differential equations (ODEs) have been generalized to solve two classes of general quasi-linear fifth-order ODEs which denoted by GRKM. The novel contribution of this work is the generalizing of RKM integrators by derivation two numerical methods for solving these classes of ODEs. The algebraic equations of order conditions for the proposed GRKM methods have derived up to the order-seventh using Taylor expansion approach. Based on these order conditions, two GRKM methods of fifth- and sixth-order with three- stages have derived. The numerical methods have been tested using two classes of problems in order to compare them with existing methods which show that the proposed integrators are less than existing RK methods in term of time complexity of function evaluations.

Keywords: Runge-Kutta method (RK), Runge-Kutta Nystrom method (RKN), direct Runge-Kutta method (RKD), Runge-Kutta-Mohammed method (RKM), RKT-RKTG- and RKTGG- and RKFD-method, fifth-order, ordinary differential equations, ODEs, Taylor expansion.

1. Introduction

The differential equations (DEs) are the most important tools in mathematical modelling. For examples, the phenomena of physics, fluid and heat flow, motion of objects, vibrations, chemical reactions and nuclear reactions have been modeled by systems of DEs. Many applications of DEs of different types, particularly ODEs of different orders, can be found in the mathematical modeling of real-life problems. Fourth-order DEs often arise in many fields of applied

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science such as mechanics, quantum chemistry, electronic and control engineering. Also, beam theory [12], fluid dynamics [2] and [13], ship dynamics [23] and neural networks [14]. Many numerical and analytical approximation for solving such DEs of various orders have studied in the literature. Most of the solutions of mathematical models of these applications should be approximated.

However, in the applications of DEs, the researchers used to developed the numerical and approximated methods for the solutions of type of DEs.

The methods of solutions of DEs are always not able to solve many types of DEs or they can solve some types of DEs indirectly. This reason make us to study the derivation of more direct numerical methods for this propose. In order to apply indirect numerical method to solve a ODE of higher-order, the ODE should be transformed into a system of first-order ODEs. Many researchers developed the family of RK methods for solving first- , second-, third-, fourth- , and fifth-order ODEs. The literature review of the methods of solutions of mathematical models which contains fifth-order BVPs [21], and [3]. For review of the methods of RK type, the first one-step method for solving first-order ODEs, was introduced by Runge in 1895. Also, Heun constructed one-step method in 1900, and Kutta formulated the general scheme of RK methods in 1901 [4]. [20] and [22] have derived Runge-Kutta-Nystrm (RKN) methods for solving second-order ODEs while the direct numerical methods with variable step-size have been derived by [6]. [19] has developed a singly diagonally implicit RKN method with periodical solutions while a symbolic derivation of order conditions of RKN method has introduced by [8].

For solving third-order ODEs, [15] and [24] have derived direct numerical methods of RK type for solving special third-order ODEs using one-step method with constant step-size, one-step direct integrators of RK type with variable step-size of orders 6(5), 5(4) and 4(3) have derived by [19] while [9] has constructed five-stage fourth-order Runge-Kutta type method for directly solving general third-order ODEs of the form $y''' = f(x, y, y', y'')$.

However, for solving special fourth- and fifth-order ODEs using one-step method with constant step-size, [17] and [11] have derived one-step direct numerical integrators of RK type with constant step-size while [15] derived one-step direct numerical integrators of RK type with constant step-size for solving special fifth-order ODEs. Finally, [10] have developed direct processes of explicit Runge-Kutta type (RKT) as solutions for any fourth-order ODEs of the structure $w''''(t) = f(x, w(t), w'(t), w''(t))$.

2. Preliminary

Here, we give the definition for a general quasi linear fifth-order ODE as the follows:

$$v^{(5)}(x) = f(x, v(x), v'(x), v''(x), v'''(x), v''''(x)), x \geq x_0, \tag{1}$$

Equation (1) has some special forms. In this paper, we have derived numerical methods of RK type for two of them.

2.1 Class one of quasi-linear fifth-order ODEs

The first class of general quasi-linear fifth-order ODE with no explicit dependence on the second-, third-, and fourth-derivatives $v^{(i)}(x)$, for $i = 2, 3, 4$ has the following form:

$$v^{(5)}(x) = f(x, v(x), v'(x)), \quad x \geq x_0, \tag{2}$$

with the initial conditions (ICs) $v^{(i)}(x_0) = \alpha^i$ where $\alpha^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_N^i]$ for $i = 1, 2, \ldots, 5$ where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, v(x) = [v_1(x), v_2(x), ..., v_N(x)]$ and

$$f(x, v(x), v'(x)) = [f_1(x, v_1(x), v_1'(x)), f_2(x, v_2(x), v_2'(x)), \ldots, f_N(x, v_N(x), v_N'(x))].$$

2.1.1 Proposed GRKM methods

The proposed GRKM integrator with s-stages for solving class one of general quasi-linear fifth-order ODEs can be written as follows:

$$z_{n+1} = z_n + hz_n' + \frac{h^2}{2!}z_n'' + \frac{h^3}{3!}z_n''' + \frac{h^4}{4!}z_n^{(4)} + h^5\sum_{i=1}^{s} b_i k_i, \tag{3}$$

$$z_{n+1}' = z_n' + hz_n'' + \frac{h^2}{2!}z_n''' + \frac{h^3}{3!}z_n^{(4)} + h^4\sum_{i=1}^{s} b_i' k_i, \tag{4}$$

$$z_{n+1}'' = z_n'' + hz_n''' + \frac{h^2}{2!}z_n^{(4)} + h^3\sum_{i=1}^{s} b_i'' k_i, \tag{5}$$

$$z_{n+1}''' = z_n''' + hz_n^{(4)} + h^2\sum_{i=1}^{s} b_i''' k_i, \tag{6}$$

$$z_{n+1}^{(4)} = z_n^{(4)} + h\sum_{i=1}^{s} b_i'''' k_i, \tag{7}$$

where

$$k_1 = f(x_n, z_n) \tag{8}$$

and

$$\begin{aligned} k_i = f(&x_n + c_i h, z_n + c_i hz_n' + c_i^2\frac{h^2}{2!}z_n'' + c_i^3\frac{h^3}{3!}z_n''' + c_i^4\frac{h^4}{4!}z_n^{(4)} + c_i^5\frac{h^5}{5!}z_n^{(5)} \\ &+h^6\sum_{m=1}^{i-1} a1_{im}k_m, z_n' + c_n hz_n'' + c_n^2\frac{h^2}{2!}z_n''' + c_i^3\frac{h^3}{3!}z_n'''' + h^4\sum_{l=1}^{i-1} a2_{im}k_m), \end{aligned} \tag{9}$$

for $i = 2, 3, \cdots, s$. The coefficients of the GRKM integrators can be formulated as : $b_i, b_i', b_i'', b_i''', b_i''''$ and $c_i, a1_{i,j}, a2_{i,j}$ for $i, j = 1, 2, \cdots, s$ are real. GRKM is an

explicit method if $a1_{i,j} = a2_{i,j} = 0$ for $i \leq j$ and otherwise GRKM is an implicit method. Using Butcher notation, GRKM coefficients can be expressed in the Table 1:

Table 1: The Butcher Tableau GRKM Method

c	A1
	A2
	b^T
	b'^T
	b''^T
	b'''^T
	b''''^T

2.2 Class two of general quasi-linear fifth-order ODEs

The second class of general quasi-linear fifth-order ODE with no explicit dependence on the third-, fourth-derivatives $v^{(i)}(x)$, for $i = 3, 4$ has following form:

$$v^{(5)}(x) = f(x, v(x), v'(x), v''(x)), \quad x \geq x_0, \tag{10}$$

with the initial conditions (ICs) $v^{(i)}(x_0) = \alpha^i$ where $\alpha^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_N^i]$ for $i = 0, 1, \ldots, 4$, $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, v(x) = [v_1(x), v_2(x), ..., v_N(x)]$ and $f(x, v(x), v'(x), v''(x)) = [f_1(x, v_1(x), v_1'(x), v_1''(x)), f_2(x, v_2(x), v_2'(x), v_2''(x)), \ldots, f_N(x, v_N(x), v_N'(x), v_N''(x))]$.

2.2.1 Proposed GRKM methods

The proposed GRKM integrator with s-stages for solving the class two of general quasi-linear fifth-order ODEs as equations (3)-(8) and the following equation:

$$\begin{aligned} k_i = f(&x_n + c_i h, z_n + c_i h z_n' + c_i^2 \frac{h^2}{2!} z_n'' + c_i^3 \frac{h^3}{3!} z_n''' + c_i^4 \frac{h^4}{4!} z_n^{(4)} \\ &+ c_i^5 \frac{h^5}{5!} z_n^{(5)} + h^6 \sum_{m=1}^{i-1} a1_{im} k_m, \\ z_n' + c_n h z_n'' + c_n^2 \frac{h^2}{2!} z_n''' + c_i^3 \frac{h^3}{3!} z_n'''' + h^4 \sum_{l=1}^{i-1} a2_{im} k_m, \\ z_n'' + c_n h z_n''' + c_n^2 \frac{h^2}{2!} z_n'''' + h^3 \sum_{l=1}^{i-1} a3_{im} k_m), \end{aligned} \tag{11}$$

for $i = 2, 3, \cdots, s$.

Here, the coefficients $a3_{i,j}$ for $i, j = 1, 2, \cdots, s$ are real. The additional condition of GRKM to be an explicit method if $a3_{i,j} = 0$ for $i \leq j$ and otherwise

GRKM is an implicit method. Using Butcher notation, GRKM coefficients can be expressed in Table 2:

Table 2: The Butcher Tableau GRKM Method

c	A1
	A2
	A3
	b^T
	b'^T
	b''^T
	b'''^T
	b''''^T

3. Derivation of order conditions

To determine the order conditions and then the coefficients of the first proposed numerical integrator indicated by equations (3)-(9), GRKM formula is expanded using the approach of Taylor's series expansion. This expansion, after some algebraic adaptations, is equated to the the solution which is proposed by Taylor expansion. From the direct expansion of the local truncation error we have found the general order conditions for the GRKM method based on the derivation of order conditions for the RK method presented in [7]. In the same manner, using equations (3)-(8) and Equation (11), we derived the following order conditions (12)-(18) of the proposed direct integrators of GRKM type using Maple software.

Order conditions for y

$$\sum b_i = \frac{1}{120}, \sum b_i c_i = \frac{1}{720}, \sum b_i c_i^2 = \frac{1}{2520}, \sum b_i c_i^3 = \frac{1}{840}. \tag{12}$$

Order conditions for y'

$$\begin{aligned} &\sum b'_i = \frac{1}{6}, \sum b'_i c_i = \frac{1}{24}, \sum b'_i c_i^2 = \frac{1}{120}, \\ &\sum b'_i c_i^3 = \frac{1}{630}, \sum \sum_{i,j=1}^{s} b'_i a_{ij} = \frac{1}{5040}, \sum b'_i c_i^4 = \frac{1}{210}. \end{aligned} \tag{13}$$

Order conditions for y''

$$\begin{aligned} &\sum b''_i = \frac{1}{6}, \sum b''_i c_i = \frac{1}{24}, \sum b''_i c_i^2 = \frac{1}{60}, \sum b''_i c_i^3 = \frac{1}{120}, \\ &\sum_{i,j=1}^{s} b''_i a_{ij} = \frac{1}{720}, \sum b''_i c_i^4 = \frac{1}{210}, \sum_{i,j=1}^{s} b''_i a_{ij} c_i = \frac{1}{5040}. \end{aligned} \tag{14}$$

Order conditions for y'''

$$\sum b_i''' = \frac{1}{2}, \sum b_i''' c_i = \frac{1}{6}\sum b_i''' c_i^2 = \frac{1}{12}\sum b_i''' c_i^3 = \frac{1}{20}\sum b_i''' c_i^4 = \frac{1}{30},$$

$$\sum_{i,j=1}^{s} b_i''' a_{ij} = \frac{1}{5040}, \sum b_i''' c_i^5 = \frac{1}{42}. \tag{15}$$

Order conditions for y''''

$$\sum b_i'''' = 1, \sum b_i'''' c_i = \frac{1}{2}, \sum b_i'''' c_i^2 = \frac{1}{3}, \sum b_i'''' c_i^3 = \frac{1}{4}, \sum b_i'''' c_i^4 = \frac{1}{5},$$

$$\sum b_i'''' c_i^5 = \frac{1}{6}, \tag{16}$$

$$a2_{21} = \frac{1}{125}(16\sqrt{6}(a2_{3,1} + a2_{32}) - 131(a2_{31} - a2_{32}) + \frac{16 - 3\sqrt{6}}{20}), \tag{17}$$

$$a3_{21} = \frac{85}{2000}\sqrt{6}, a3_{31} = \frac{1}{500}(\frac{51}{10}\sqrt{6} + \frac{33}{2500}), a3_{32} = \frac{51}{1250} + \frac{11}{1250}\sqrt{6}. \tag{18}$$

3.1 Derivation of the proposed methods

Using the algebraic order conditions (12)-(17), we get first proposed GRKM integrator for solving ODE in Equation (2) (see Table 3). However, adding the order condition (18) to obtain the second proposed GRKM integrator for solving ODE in Equation (10) (see Table 4)

Table 3: The Butcher Tableau GRKM Method

0	0		
$\frac{3}{5} - \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	0	
$\frac{3}{5} + \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	$-\frac{1}{2}$	0
	0		
	$\frac{1}{18}$	0	
	$\frac{1}{18}$	$\frac{-532}{5625} - \frac{133}{22500}\sqrt{6}$	0
	$-\frac{37}{360}$	$\frac{1}{18}$	$\frac{1}{18}$
	$\frac{17}{360} - \frac{7}{360}\sqrt{6}$	$\frac{1}{18}$	$-\frac{11}{180} + \frac{7}{360}\sqrt{6}$
	$\frac{1}{18}$	$\frac{1}{18} + \frac{1}{48}\sqrt{6}$	$\frac{1}{18} - \frac{1}{48}\sqrt{6}$
	$\frac{1}{9}$	$\frac{7}{36} + \frac{1}{18}\sqrt{6}$	$\frac{7}{36} - \frac{1}{18}\sqrt{6}$
	$\frac{1}{9}$	$\frac{4}{9} + \frac{1}{36}\sqrt{6}$	$\frac{4}{9} - \frac{1}{36}\sqrt{6}$

4. Implementations

In this section, two classes problems are tested. The numerical results are compared with classical methods when the same set of problems is reduced to a

Table 4: The Butcher Tableau GRKM Method

0	0		
$\frac{3}{5}-\frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	0	
$\frac{3}{5}+\frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	$-\frac{1}{2}$	0
	0		
	$\frac{12}{625}-\frac{3}{2500}\sqrt{6}$	0	
	$\frac{1}{2}$	$\frac{-1}{2}$	0
	0		
	$\frac{27}{500}-\frac{19}{100}\sqrt{6}$	0	
	$\frac{33}{2500}+\frac{51}{5000}\sqrt{6}$	$\frac{51}{1250}+\frac{11}{1250}\sqrt{6}$	0
	$-\frac{37}{360}$	$\frac{1}{18}$	$\frac{1}{18}$
	$\frac{17}{360}-\frac{7}{360}\sqrt{6}$	$\frac{1}{18}$	$-\frac{11}{180}+\frac{7}{360}\sqrt{6}$
	$\frac{1}{18}$	$\frac{1}{18}+\frac{1}{48}\sqrt{6}$	$\frac{1}{18}-\frac{1}{48}\sqrt{6}$
	$\frac{1}{9}$	$\frac{7}{36}+\frac{1}{18}\sqrt{6}$	$\frac{7}{36}-\frac{1}{18}\sqrt{6}$
	$\frac{1}{9}$	$\frac{4}{9}+\frac{1}{36}\sqrt{6}$	$\frac{4}{9}-\frac{1}{36}\sqrt{6}$

system of first-order equations and solved using existing RK of the same order [5].

4.1 First proposed method

In this subsection we have tested three problems involving $y'''' = f(x, y, y')$.

Example 4.1 (Homogenous ODE)**.**

$$y^{(5)}(t) = \cos(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0$. Exact solution: $y(t) = \sin(t), b = \pi$.

Example 4.2 (Linear ODE)**.**

$$y^{(5)}(t) = y'(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0$. Exact solution: $y(t) = \sin(t), b = \pi$.

Example 4.3 (Non-linear ODE)**.**

$$y^{(5)}(t) = -120y'^{3}(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = -6, y^{(4)}(0) = 24$. Exact solution: $y(t) = \frac{1}{1+t}, b = 0.1$.

4.2 Second proposed method

In this subsection we have tested three problems involving $y'''' = f(x, y, y', y'')$.

Example 4.4 (Homogenous ODE)**.** $y^{(5)}(t) = \cos(t),\ \ 0 < t \leq b.$
Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0.$
Exact solution: $y(t) = \sin(t), b = 1.$

Example 4.5 (Homogenous ODE)**.** $y^{(5)}(t) = y(t) + y'(t) + y''(t);,\ \ 0 < t \leq b.$
Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0.$
Exact solution: $y(t) = \sin(t), b = 0.1.$

Example 4.6 (Nonlinear ODE)**.** $y^{(5)}(t) = y^6(t) + y'^3(t) - 30y''^2(t),\ \ 0 < t \leq b.$
Initial conditions, $y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = -6, y^{(4)}(0) = 24.$
Exact solution: $y(t) = \frac{1}{1+t}, b = 10.$

5. Discussion and conclusion

In this study, RKM integrators have been generalized for solving two classes of general quasi-linear fifth-order ODEs. The contribution of this work is the generalizing of the integrators for derivation of two numerical methods for solving these classes which are denoted as GRKM. The objective of this work is to establish direct explicit integrators of RK type for solving two classes of fifth-order ODEs. For this purpose, we generalized the integrators RK, RKN, RKD, RKT, RKM and RKFD which are used for solving special first-, second-, third-, fourth- and fifth-order ODEs. We have generalized three-stages, fifth-order and four-stages sixth-order RKM methods. Numerical results using the constructed methods have been compared with existing methods, see Figure 1. They have clearly shown the advantage and the efficiency of the new methods. They are accurate as well-known existing methods. However, they are more efficient as they require less function evaluations. As such, these methods are more cost effective, in terms of computation time, than existing methods. The new methods agree very well with well-known existing methods in the literature and required less function evaluations. As such, these methods are more cost effective in terms of computation time than other existing methods. Hence, we can conclude that the new GRKM pairs are computationally very efficient in solving fifth-order ODEs.

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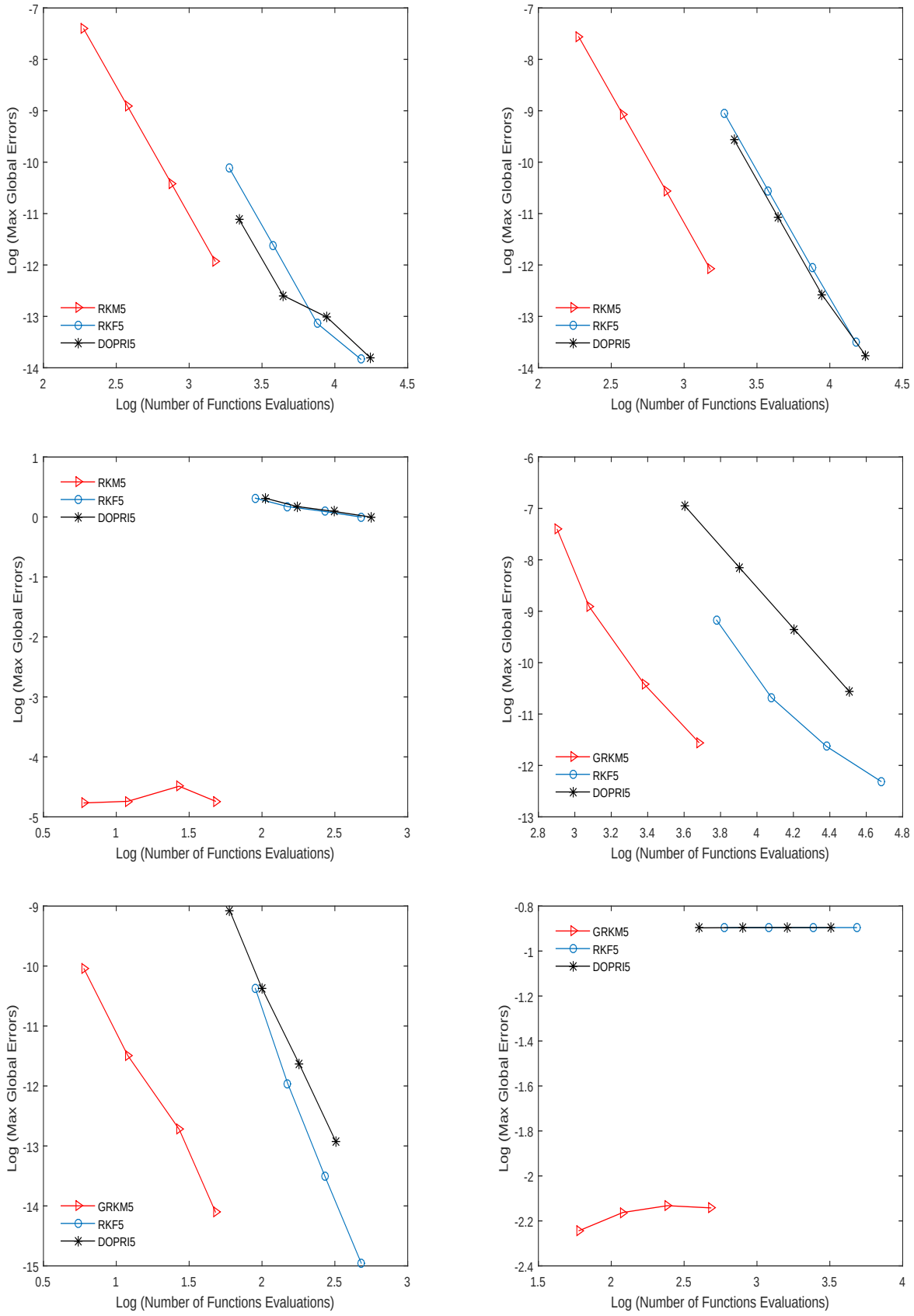


Figure 1: A comparison of numerical solutions for proposed with existing methods for examples (a) 4.1, (b) 4.2, (c) 4.3 and (d) 4.4, (e) 4.5 and (f) 4.6

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Semirings of $0, 1$-preserving endomorphisms of semilattices

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Abstract. In the paper, various endomorphism semirings of semilattices are investigated. In particular, various conditions are found under which these endomorphism semirings are simple.
Keywords: semiring, semimodule, semilattice, endomorphism.

1. Introduction

A semiring is an algebraic structure possessing two associative binary operations, usually denoted as addition and multiplication, where the addition is commutative and the multiplication distributes over the addition from both sides.

Let S be a semiring. An element $a \in S$ is *additively neutral (absorbing)* if $a + b = b$ ($a + b = a$) for every $b \in S$. We write $0_S \in S$ ($o_S \in S$) if S contains the uniquely determined additively neutral (absorbing) element 0_S (o_S). On the other hand, $0_S \notin S$ ($o_S \notin S$) means that S contains no element of that kind.

An element $a \in S$ is called *right multiplicatively absorbing* if $ba = a$ for every $b \in S$. The set of right multiplicatively absorbing elements will be denoted by

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$\underline{R}(S)$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is an ideal and, in fact, $\underline{R}(S)$ is the smallest right ideal of the semiring S.

The semiring S is called (*congruence-*)*simple* if S has just two congruences.

A commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ ($M \times S \to M$) is called *left* (*right*) *S-semimodule*. In the sequel, if not specified, semimodules are left S-semimodules.

A semimodule M is called *faithful* if for all $a, b \in S,\ a \neq b$, there is $x \in M$ with $ax \neq bx$, and M is called *simple* if there are just two congruence relations on M.

We are going to study various endomorphism semirings of some semilattices. Semirings of that type are additively idempotent, and therefore, throughout the paper all semirings and semimodules are assumed to be *additively idempotent*. It means that the respective additive semigroups are semilattices. Now, given a semilattice M ($= M(+)$), the basic order $\leq$ is defined on M by $x \leq y$ iff $x + y = y$. If $0_M \in M$ ($o_M \in M$) then 0_M (o_M) is the smallest (greatest) element of M. A non-empty subset A of M is an *ideal* if $A + M = A$. The ideal A is said to be *prime* if $M \setminus A$ is a subsemilattice of M.

If $o_M \in M$ ($0_M \in M$) then we will say that M satisfies (1β) ((2β)) if every infinite strictly increasing (decreasing) sequence $x_1 < x_2 < x_3 < \dots$ ($x_1 > x_2 > x_3 > \dots$) of elements from M is upwards (downwards) cofinal in $M \setminus \{o_M\}$ ($M \setminus \{0_M\}$).

2. Characteristic semimodules

Let S be a semiring, $|S| \geq 3$, and M be a (left S-)semimodule. We shall say that M is a *characteristic semimodule* if the following three conditions are satisfies:

(a) M is faithful and $|M| \geq 3$;

(b) $0_M, o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$;

(c) there is a mapping $\underline{\varepsilon} : N = M \setminus \{o_M\} \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x, y, z \in M$, $x \neq o_M$, $y \leq x$, $z \not\leq x$.

Throughout this section, let M be a characteristic semimodule. By [1, 2.2, 2.3], the semimodule M is simple, $\underline{\varepsilon}$ is an injective mapping of N into $R = \underline{R}(S)$, $|R| \geq 2$, $o_S \in S$ and $\underline{\varepsilon}(0_M) = o_S \in R$. By [1, 2.4], we have $RM = \{0_M, o_M\}$.

The semimodule M will be called *nearly critical* if for all $a, b \in R$, $a < b$, and $x \in K = M \setminus \{0_M, o_M\}$ there is at least one $c \in S$ such that $acx \neq bcx$ (then $acx = 0_M$ and $bcx = o_M$).

Lemma 2.1. *Assume that $\underline{\varepsilon}(N) = R$. Then M is nearly critical iff the following condition is satisfied:*

(3β) For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $c \in S$ such that $cw \leq v$ and $cw \not\leq u$.

Proof. It is easy. □

Lemma 2.2. *Assume that $R \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$ and that the set N has no minimal element. Then M is nearly critical iff (3β) is true.*

Proof. It is easy (take into account that for every $u \in N$ there is $v \in K$ with $u < v$). □

The semimodule M will be called *critical* if $Sx = M$ for every $x \in K$. Now, if M is critical then M is nearly critical and $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M are just all subsemimodules of M.

Lemma 2.3. *M is critical iff the following condition is satisfied:*

(4β) For all $x, y \in K$ there is $c \in S$ with $cx = y$.

Proof. It is obvious. □

Remark 2.4. (i) [2, 2.8] If the semiring S is simple then M is nearly critical. (ii) [2, 2.9] If the semiring S is simple, M is not critical and (1β) is true then both S and M are infinite, $o_N \in N$, $\underline{\varepsilon}(o_N) = 0_S \in S$, $P = M \setminus \{o_N\}$ is a subsemimodule of M and P is a critical semimodule. By 2.3, for all $x, y \in M \setminus \{0_M, o_N, o_M\}$ there is $c \in S$ with $cx = y$.

Lemma 2.5. *The right semimodule R_S is faithful.*

Proof. Let $a, b \in S$ be such that $ca = cb$ for every $c \in R$. If $x \in M$ is such that $ax \in N$ then $0_M = \underline{\varepsilon}(ax)ax = \underline{\varepsilon}(ax)bx$, and hence $bx \in N$ and $bx \leq ax$. Using symmetry, we conclude that $ax = bx$ for every $x \in M$. Since ${}_SM$ is faithful, we get $a = b$. □

Lemma 2.6. *If the right S-semimodule R_S is simple then the following condition is true:*

(5β) For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $c \in S$ such that $cw \leq v$, $cw \not\leq u$ and $cz \not\leq v$ for every $z \not\leq w$.

Proof. Using the right hand forms of [3, 4.10, 4.12(i)], we find $c \in S$ such that $\underline{\varepsilon}(v)c = \underline{\varepsilon}(w) < \underline{\varepsilon}(u)c$. □

Lemma 2.7. *Assume that the right S-semimodule R_S is simple. Then the condition (3β) is true and either $|R| = 2$ or the set $S \setminus \{o_S\}$ is not a subsemiring of S.*

Proof. (3β) follows immediately from (5β). If $S \setminus \{o_S\}$ is a subsemiring of S then the relation $((R \setminus \{o_S\}) \times (R \setminus \{o_S\}) \cup \mathrm{id}_R$ is a congruence of R_S. □

Lemma 2.8. *Assume that the right S-semimodule R_S is simple and that $0_S \in S$ (or $0_R \in R$). Then $0_S = 0_R$ and the following condition is satisfied:*

(6β) For all $u \in N \setminus \{o_N\}$ and $w \in K$ there is at least one $c \in S$ such that $cw \in N$, $cw \not\leq u$ and $cz = o_M$ for every $z \not\leq w$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $0_S c = \underline{\varepsilon}(w) < \underline{\varepsilon}(u)c$. □

Lemma 2.9. *If $o_N \in N$ then (5β) implies (6β).*

Proof. It is obvious. □

Lemma 2.10. *Assume that the right S-semimodule R_S is simple and that $0_S \in S$. Then the following condition is satisfied:*

(7β) For all $u \in N$ and $v \in K$, $u < v$, there is at least one $c \in S$ such that $cK \leq v$ and $cz \not\leq u$ for at least one $z \in K$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $\underline{\varepsilon}(v)c = 0_S < \underline{\varepsilon}(u)c$. □

Lemma 2.11. *If $o_N \in N$ then (5β) implies (7β).*

Proof. It is obvious. □

Lemma 2.12. *Assume that the right S-semimodule R_S is simple and that $0_S \in S$. Then the following condition is satisfied:*

(8β) For every $u \in N \setminus \{o_N\}$ there is at least one $c \in S$ such that $cK \subseteq N$ and $cz \not\leq u$ for at least one $z \in K$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $0_S c = 0_S < \underline{\varepsilon}(u)c$. □

Lemma 2.13. *If either $o_N \in N$ or the set N has no maximal element then (7β) implies (8β).*

Proof. It is obvious. □

Lemma 2.14. *Assume that (2β) is true and $\underline{\varepsilon}(N) = R$. Then the right S-semimodule R_S is simple iff (5β) is true and either $|R| = 2$ or $S \setminus \{o_S\}$ is not a subsemiring of S.*

Proof. If R_S is simple then we use 2.6 and 2.7. If $|R| = 2$ then R_S is simple anyway. Now, assume that (5β) is true and $S \setminus \{o_S\}$ is not a subsemiring of S. Taking into account the right hand form of [3,4.13], we have to show that for all $a, b, d \in R$, $a < b$, $d \neq o_S$, there is $c \in S$ with $ac = d < bc$. But $a, b, d \in \underline{\varepsilon}(N)$ and it remains to use (5β). □

Lemma 2.15. *Assume that (2β) is true, $0_S \in S$ and $R \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$. Then the right S-semimodule R_S is simple iff the conditions (5β), (6β), (7β) and (8β) are satisfied and either $|R| = 2$ or $S \setminus \{o_S\}$ is not a subsemiring of S.*

Proof. Using 2.6, 2.7, 2.8, 2.10 and 2.12, we proceed similarly as in the proof of 2.14. □

Remark 2.16. (cf. 2.15) If (1β) is true, $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ then $o_N \notin N$, $N+N = N$ and the set N has no maximal element. Now, (7β) implies (8β).

Lemma 2.17. *Assume that* (1β) *is true and the set* N *has no maximal element (equivalently,* $o_N \notin N$ *and* $N+N = N$*). If* $a \in S$ *then* $\underline{\varepsilon}(v) \leq a$ *for at least one* $v \in N$ *iff* $o_M \in aK$*.*

Proof. If $\underline{\varepsilon}(v) \leq a$ then $v < u$ for some $u \in K$ and $au = o_M$. Conversely, if $A \cap K \neq \emptyset$, where $A = \{\, x \in M \mid ax \neq o_M \,\}$, then $A + A = A$ and if $v = o_A \in A$ then $\underline{\varepsilon}(v) \leq a$. On the other hand, if $o_A \notin A$ then $A = N$ and $o_M \notin aK$, a contradiction. □

Lemma 2.18. *Assume that* (1β) *is true and* $N+N \neq N$*. If* $a \in S$ *then* $\underline{\varepsilon}(v) \leq a$ *for at least one* $v \in N$ *iff there is a maximal element* $w \in N$ *such that* $ax = o_M$ *for* $x \not\leq w$ *(then* $\underline{\varepsilon}(w) \leq a$*).*

Proof. Observing that there is no infinite strictly increasing sequence of elements in N, our result easily follows. □

3. Endomorphism semirings (a)

Throughout this section, let M $(= M(+))$ be a semilattice containing at least three elements and such that $0_M \in M$ and $o_M \in M$. The set $\underline{E}$ $(= \underline{E}_{0,1}(M))$ of endomorphisms of M that preserve the elements 0_M and o_M is a unitary semiring; we have $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$, and $1_{\underline{E}} = \mathrm{id}_M$ is the multiplicatively neutral element of $\underline{E}$.

The set $(\underline{D}_{0,1} =)$ $\underline{D} = \{\, f \in \underline{E} \mid f(M) = \{0_M, o_M\} \,\}$ is an ideal of the semiring $\underline{E}$ and the following result is obvious.

Proposition 3.1. (i) *If* $f \in \underline{D}$ *then the set* $\{\, x \in M \mid f(x) = o_M \,\}$ *is a prime ideal of the semilattice* M*.*
(ii) *If* A *is a prime ideal of* M*,* $f(A) = \{o_M\}$ *and* $f(M \setminus A) = \{0_M\}$ *then* $f \in \underline{D}$*.* □

For every $x \in N = M \setminus \{o_M\}$, the set $A_x = \{\, y \in M \mid y \not\leq x \,\}$ is a prime ideal of M and the corresponding endomorphism will be denoted by q_x. A_x is the principal prime ideal determined by x, $q_x(u) = 0_M$ for $u \leq x$ and $q)x(v) = o_M$ for $v \not\leq x$. We put $(\underline{B}_{0,1} =)$ $\underline{B} = \{\, q_x \mid x \in N \,\}$ and $(\underline{C}_{0,1} =)$ $\underline{C} = \{\, q_{x_1} + \cdots + q_{x_n} \mid n \geq 1, x_i \in N \,\}$. Clearly, $\underline{C}$ is just the subsemiring of $\underline{D}$ generated by the set $\underline{B}$ and we have the following results:

Proposition 3.2. $\underline{C} = \underline{B}$ *iff the ordered set* $M(\leq)$ *is a lattice.* □

Proposition 3.3. *Assume that the condition* (1β) *is true. Then* $\underline{D} = \underline{B}$ *if and only if either* $o_N \in N$ *or* $N + N \neq N$. □

Proposition 3.4. *If* $N+N = N$ *then* $\xi \in \underline{D}$, *where* $\xi(N) = \{0_M\}$ *and* $\xi(o_M) = o_M$. *If* $o_N \in N$ *then* $\xi = q_{o_N} \in \underline{B}$. □

Proposition 3.5. *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$. *Then:*
(i) $\underline{C} \subseteq E$.
(ii) $\underline{R}(E) = E \cap \underline{D}$.
(iii) *The right* E*-semimodule* $\underline{R}(E)_E$ *is faithful.*
(iv) $q_{0_M} = o_E \in E$.
(v) *If* $0_E \in E$ *then* $0_E \in \underline{R}(E)$ *and* $N + N = N$.
(vi) *If* $v \in N$ *then* $q_v = 0_E$ *if and only if* $v = o_N \in N$ *(then* $0_E = q_{o_N} = \xi$*).*
(vii) *If* $N + N = N$ *and* $\xi \in E$ *then* $\xi = 0_E$.
(viii) *If* $N + N = N$ *and* $\xi \notin E$ *then* $0_E \notin E$.
(ix) *If* $e \in E$ *is right multiplicatively neutral then* $e = 1_E = \mathrm{id}_M$. □

Proposition 3.6. *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$ *and* $|E| \geq 3$. *Then the (left* E*-)semimodule* ${}_E M$ *is characteristic (via* $\underline{\varepsilon}(x) = q_x$ *for every* $x \in N$*).* □

Proposition 3.7. [2, 1.1] *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$. *Then* E *is simple if and only if the right semimodule* $\underline{R}(E)_E$ *is simple and* $E \subseteq \underline{R}(E) + \underline{E}$. □

Proposition 3.8. *Let* E *be a simple subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$ *and* $|E| \geq 3$. *Then:*
(1) *For all* $u \in N = M \setminus \{o_M\}$ *and* $v, w \in K = M \setminus \{0_M, o_M\}$, $u < v$, *there is at least one* $f \in E$ *such that* $f(w) \leq v$, $f(w) \not\leq u$ *and* $f(z) \not\leq v$ *for every* $z \not\leq w$*;*
(2) *If* $0_E \in E$ (see 3.5) *then for all* $u \in N \setminus \{o_N\}$ *and* $w \in K$ *there is at least one* $f \in E$ *such that* $f(w) \in K$, $f(w) \not\leq u$ *and* $f(z) = o_M$ *for every* $z \not\leq w$*;*
(3) *If* $0_E \in E$ *then for all* $u \in N$ *and* $v \in K$, $u < v$, *there is at least one* $f \in E$ *such that* $f(N) \leq v$ *and* $f(z) \not\leq u$ *for at least one* $z \in K$*;*
(4) *If* $0_E \in E$ *then for every* $u \in N \setminus \{o_N\}$ *there is at least one* $f \in E$ *such that* $f(N) \subseteq N$ *and* $f(z) \not\leq u$ *for at least one* $z \in K$.

Proof. See 2.6, 2.8, 2.10 and 2.12. □

4. Endomorphism semirings (b)

Let M be a semilattice such that $|M| \geq 4$, $0_M \in M$, $o_M \in M$ and the condition (1β) is true (i.e., every infinite strictly increasing sequence of elements from M is upwards cofinal in $N = M \setminus \{o_M\}$). Let E be a subsemiring of $\underline{E}$ such that $\underline{B} \subseteq E$ (see the preceding section).

Proposition 4.1. (i) *The ordered set* $M(\leq)$ *is a lattice.*
(ii) $\underline{B} = \underline{C}$.

Proof. It is easy. □

Proposition 4.2. (i) *If $N + N \neq N$ then $o_N \notin N$, there is no infinite strictly increasing sequence of elements in M, $0_E \notin E$ and $\underline{R}(E) = \underline{B} = \underline{D}$.*
(ii) *If $o_N \in N$ then $N = N + N$, there is no infinite strictly increasing sequence of elements in M, $0_E = \xi = q_{o_N} \in E$ and $\underline{R}(E) = \underline{B} = \underline{D}$.*
(iii) *If $o_N \notin N = N + N$ and $\xi \notin E$ then the set N has no maximal element, $0_E \notin E$, $\xi \notin \underline{B}$, $\underline{R}(E) = \underline{B}$ and $\underline{D} = \underline{B} \cup \{\xi\}$.*
(iv) *If $o_N \notin N = N + N$ and $\xi \in E$ then the set N has no maximal element, $\xi = 0_E \in E \setminus \underline{B}$ and $\underline{R}(E) = \underline{D} = \underline{B} \cup \{\xi\}$.*

Proof. It is easy (use (1β), 3.3 and 3.5(ii),(v),(vi),(vii),(viii)). □

Proposition 4.3. *$0_E \in E$ if and only if either $o_N \in N$ (then $0_E = \xi = q_{o_N}$) or $o_N \notin N = N + N$ and $\xi \in E$ (then $0_E = \xi$).* □

Corollary 4.4. *$\underline{R}(E) = \underline{B} = \underline{\varepsilon}(N)$ except for the case when $o_N \notin N = N + N$ and $\xi \in E$. In that case, we have $\underline{R})E) = \underline{B} \cup \{\xi\} = \underline{D}$, $\xi \notin \underline{B}$ and $\underline{\varepsilon}(N) = \underline{B} = \underline{R}(E) \setminus \{0_E\}$.* □

Corollary 4.5. *$\underline{R}(E) = \underline{D}$ except for the case when $o_N \notin N = N + N$ and $\xi \notin E$. In that case, we have $\underline{R}(E) = \underline{B}$ and $\underline{D} = \underline{B} \cup \{\xi\}$.* □

Proposition 4.6. *The semimodule ${}_EM$ is nearly critical if and only if the following condition is satisfied:*

1. *For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $f \in E$ such that $f(w) \leq v$ and $f(w) \not\leq U$.*

Proof. Combine 2.1, 2.2 and 4.4. □

Proposition 4.7. *The semimodule ${}_EM$ is critical if and only if E operates transitively on the set K.*

Proof. This is obvious (see 2.3). □

Proposition 4.8. *Assume that the semiring E is simple, but E does not operate transitively on K. Then:*
(i) *E and M are infinite.*
(ii) *$o_N \in N$ and $M \setminus \{o_N\}$ is a subsemimodule of M.*
(iii) *E operates transitively on the set $K \setminus \{o_N\}$.*
(iv) *$E(x) = M \setminus \{o_N\}$ for every $x \in K \setminus \{o_N\}$.*
(v) *$M \setminus \{o_M\} \subseteq E(o_N)$.*

Proof. (i), (ii) and (iv) follow from 2.4(ii) and 4.7, and (iv) follows from (iii). As concerns (v), choose $v \in K \setminus \{o_N\}$. By 3.8(1), there is $f \in E$ with $0_M \neq f(o_N) \leq v$. Then $Ef(o_N) = M \setminus \{o_N\}$ by (iii). □

Proposition 4.9. (i) *If $N+N \neq N$ then $\underline{B}+\underline{E} = \{ f \in \underline{E} \mid$ there is a maximal element $w \in N$ such that $f(x) = o_M$ for $x \not\leq w \}$.*
(ii) *If $o_N \notin N = N+N$ then $\underline{B}+\underline{E} = \{ f \in \underline{E} \mid o_M \in f(K) \}$.*
(iii) *If $o_N \in N$ then $\underline{B}+\underline{E} = \underline{E}$.*
(iv) *If $\mathrm{id}_M \in \underline{B}+\underline{E}$ then $o_N \in N$.*

Proof. Use 2.17 and 2.18. □

5. Endomorphism semirings (c)

This section is an immediate continuation of the foregoing section. Here, moreover, we will assume that the condition (2β) is true (i.e., every infinite strictly decreasing sequence of elements from M is downwards cofinal in $L = M \setminus \{0_M\}$. Note that if E is simple then the conditions 3.8(1),(2),(3) and (4) are satisfied and, besides, E operates transitively on K or 4.8 takes place.

Theorem 5.1. *Assume that $0_E \notin E$* (see 4.2). *If the set $L = M \setminus \{0_M\}$ has at least one minimal element then the semiring E is simple if and only if $E \subseteq \underline{B}+\underline{E}$* (see 3.7, 4.8(i),(ii)) *and the condition* 4.6(1) *is satisfied. If the set L has no minimal element then the semiring E is simple if and only if $E \subseteq \underline{B}+\underline{E}$,* 4.6(1) *is true and, moreover, the following condition is satisfied:*

(1) *There are $f \in E$ and $v, w \in K$ such that $f(w) \neq o_M$ and $f(x) \not\leq v$ for every $x \in K$.*

Proof. First, let E be simple. By 3.7, the right semimodule $\underline{R}(E)_E$ is simple and we use 2.7 and 4.6 to show that 4.6(1) is true. Furthermore, $\underline{R}(E) = \underline{B}$ by 4.4 and $E \subseteq \underline{B}+\underline{E}$ follows from 3.7. Finally, by [2, 3.2.3(ii)], $o_E = q_v f$ for some $v \in K$ and $f \in E \setminus \{o_E\}$. That is, the condition (1) is true.

Now, we proceed conversely to show that E is simple. We have $E \subseteq \underline{R}(E)+\underline{E}$ and the semimodule ${}_E M$ is nearly critical by 4.6. It remains to use [2, 3.2.1, 3.2.2]. □

Theorem 5.2. *Assume that $o_N \in N$ (then $0_E = q_{o_N} \in E$). If the set L has at least one minimal element then the semiring E is simple if and only if the condition* 4.6(1) *is satisfied. If the set L has no minimal element then the semiring E is simple if and only if the conditions* 4.6(1) *and* 5.1(1) *are satisfied.*

Proof. Using [2, 3.2.5, 3.2.6], we proceed in the very same way as in the proof of 5.1. □

Theorem 5.3. *Assume that $0_E \in E$ and $o_N \notin N$* (see 4.3). *If the set L has at least one minimal element then the semiring E is simple if and only if the condition* 4.6(1) *is satisfied and, moreover, the following condition is satisfied as well:*

(1) *There are $f \in E$ and $v, w \in K$ such that $f(w) \neq 0_M$ and $f(x) \leq v$ for every $x \in K$.*

If the set L has no minimal element the the semiring E is simple if and only if the conditions (1), 4.6(1) *and* 5.1(1) *are satisfied.*

Proof. Using [2, 3.2.7 3.2.8], we proceed similarly as in the proof of 5.1. □

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A categorical approach to vitally dense monomorphisms of S-acts

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Abstract. In this paper we consider a new type of closure operators on acts over a commutative monoid, namely vital closure operator, to get the class of vitally dense monomorphisms derived from this closure operator and investigate injectivity and essentiality with respect to this class of monomorphisms. We study some categorical properties of vitally dense monomorphisms such as limits and colimits. It is proved that the three notions of essentiality for vitally dense monomorphisms are the same, and considering the relations between injectivity, retractivity, essentiality and injective envelopes, we show that injectivity well behaves in regard to such kind of monomorphisms of acts.

Keywords: S-act, vitally dense monomorphism, vitally dense injective, vitally dense essential.

1. Introduction and preliminaries

Closure operators are mostly applicable in topology and algebra (see [3]). Various kinds of closure operators can be found in many papers. For instance, Ebrahimi [5] considered ideal closure operators of M-sets. Also sequential closure operators of acts over semigroups were investigated in [9, 10, 11, 12]. Essentiality with respect to monomorphisms or an arbitrary class of morphisms of a category is a fundamental concept in algebra and category theory which is closely related to injectivity. In fact, injective objects are characterized and injective envelops are defined by means of essentiality (see, for example, [1, 2, 4, 13]).

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Here we introduce vital closure operators of acts over a commutative monoid to get the class of vitally dense monomorphisms arising from them. We study some elementary properties of vital closure operators and consider some categorical properties of vitally dense monomorphisms including limits and colimits. There are three different notions of essentiality with respect to a subclass $\mathcal{M}$ of monomorphisms in the literature. Moreover, Banaschewski [1] proposed three propositions mainly about the relations between injectivity, retractivity, essentiality and injective envelopes, all with respect to a class $\mathcal{M}$ of morphisms, and if they hold true in the given category, one says that $\mathcal{M}$-injectivity "well behaves". Here, considering $\mathcal{M}$ to be the class $\mathcal{M}_v$ of vitally dense monomorphisms of acts, we show that these three notions of essentiality are in fact equivalent and injectivity well behaves in regard to this class of monomorphisms.

In the following, we briefly give some definitions and preliminaries needed in the sequel.

Let S be a monoid. A (*right*) *S-act* is a non-empty set A together with a map $A \times S \to A, (a, s) \mapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. In this case, we say that A is an extension of B. An element θ in an S-act A is said to be a *zero* or *fixed element* if $\theta s = \theta$ for all $s \in S$. Let A and B be two S-acts. A mapping $f : A \to B$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. We denote the category of all S-acts and homomorphisms between them by **Act-S**. In this category, monomorphisms are exactly one-to-one homomorphisms and *epimorphisms* are exactly onto homomorphisms. A *congruence* ρ on an S-act A is an equivalence relation on A such that $a\rho a'$ implies that $as\rho a's$ for $a, a' \in A$ and $s \in S$. If ρ is a congruence on A, then the factor set $A/\rho = \{[a]_\rho : a \in A\}$ is an S-act, called the *factor act* of A by ρ, with the action given by $[a]_\rho s = [as]_\rho$ for $a \in A$ and $s \in S$.

Let A be an S-act. For any $H \subseteq A \times A$, the *congruence generated by H* is denoted as $\rho(H)$ which is the smallest congruence on A containing H. One can see that $x\rho(H)y$ if and only if either $x = y$ or there exist $s_1, s_2, \ldots, s_n \in S$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ such that $(a_i, b_i) \in H$ or $(b_i, a_i) \in H$ and

$$\begin{array}{ccccccc} x = a_1 s_1 & & b_2 s_2 = a_3 s_3 & & \cdots & & b_n s_n = y \\ & b_1 s_1 = a_2 s_2 & & & b_3 s_3 = a_4 s_4 & & \cdots \end{array}$$

Let $\{A_i \mid i \in I\}$ be a family of S-acts. The *product* of this family, denoted by $\prod_{i\in I} A_i$, is their cartesian product with the componentwise action. Also the *coproduct* $\coprod_{i\in I} A_i$ of this family is their disjoint union with natural action. For a family $\{A_i \mid i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number. For undefined terms and notations about S-acts, we refer to [6, 8].

2. Vital closure operator

In this section we introduce and study a closure operator on **Act-**S where S is a commutative monoid. This leads to the notion of vitally dense monomorphism for S-acts which is the subject of this study.

Let $\mathcal{C}$ be a category. Recall that a family $C = (C_B)_{B\in\mathcal{C}}$, with $C_B : \mathrm{Sub}B \to \mathrm{Sub}B$, taking any subobject $A \leq B$ to a subobject $C_B(A)$, is called a *closure operator* on $\mathcal{C}$ if it satisfies the following:

1. (*Extension*) $A \leq C_B(A)$,
2. (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,
3. (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Moreover, a closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Hereditary* if for every S-act B and $A_1 \leq A_2 \leq B$, $C_{A_2}(A_1) = C_B(A_1) \cap A_2$.

(c) *Additive* if for every S-act B, $C_B(A_1 \cup A_2) = C_B(A_1) \cup C_B(A_2)$.

(d) *Productive* if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B(A) = \prod_i C_{B_i}(A_i)$.

(e) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for every S-act B and $A \leq B$.

(f) *Discrete* if $C_B(A) = A$ for every S-act B and $A \leq B$.

Definition 2.1. For a commutative monoid S, a family $C^v = (C^v_B)_{B\in\mathbf{Act}-S}$ with $C^v_B : \mathrm{Sub}B \to \mathrm{Sub}B$ is defined as

$$C^v_B(A) = \{b \in B : bs \in A \ \text{ for some } \ s \in S\}$$

for any subact A of B.

It is easy to show that C^v is a closure operator on **Act-**S which is called a *vital closure operator*. Indeed, for any $b \in C^v_B(A)$ and $t \in S$, there exists $s \in S$ with $bs \in A$ and so $(bt)s = b(ts) = b(st) = (bs)t \in A$ by commutativity of S. Then $bt \in C^v_B(A)$, which means that $C^v_B(A)$ is a subact of B. The extension and monotonicity properties are clear. For continuity, take a homomorphism $f : B \to D$ and $b \in C^v_B(A)$. Then $bs \in A$ for some $s \in S$ and then $f(b)s = f(bs) \in f(A)$. Hence, $f(b) \in C^v_D(f(A))$, i.e. $f(C^v_B(A)) \leq C^v_D(f(A))$.

Clearly, if $A \leq B \leq D$, then $C^v_B(A) \leq C^v_D(A)$.

From now on, S stands for a commutative monoid.

A C^v*-dense* subact A of an S-act B is called *vitally dense* or v*-dense* for short, i.e. $C^v_B(A) = B$. A homomorphism $f : A \to B$ is said to be *vitally dense* or v*-dense* if $f(A)$ is a v-dense subact of B. The class of all v-dense monomorphisms of acts is denoted as $\mathcal{M}_v$.

We now prove some properties of this closure operator.

Theorem 2.2. *The closure operator C^v is weakly hereditary, hereditary, additive, idempotent and also discrete if S is a group. Moreover, C^v is productive whenever S is a finite commutative monoid.*

Proof. We just prove some parts of this result; the remainder are also straightforward. For idempotency, take any $b \in C^v_B(C^v_B(A))$ where $A \leq B$. Then $bs \in C^v_B(A)$ for some $s \in S$. This gives that there exists $t \in S$ such that $(bs)t \in A$. Thus $b(st) \in A$, which means that $b \in C^v_B(A)$. For hereditariness, let $A_1 \leq A_2 \leq B$ and $a \in C^v_{A_2}(A_1)$. Then $as \in A_1, a \in A_2$ for some $s \in S$. Thus $as \in A_1, a \in B$. Hence, $a \in C^v_B(A_1) \cap A_2$. Conversely, let $a \in C^v_B(A_1) \cap A_2$. Then $a \in A_2, as \in A_1$ for some $s \in S$. Thus $a \in C^v_{A_2}(A_1)$. For the last part, consider a family of subacts A_i of B_i and take $A = \prod_i A_i$ and $B = \prod_i B_i$. We must show that $C^v_B(A) = \prod_i C^v_{B_i}(A_i)$. Let $b = \{b_i\} \in \prod_i C^v_{B_i}(A_i)$. Then for any i, there exists $s_i \in S$ for which $b_i s_i \in A_i$. Since S is finite, so is the number of s_i's. Using commutativity of S, take the product of all s_i's, namely s. Then $b_i s \in A_i$ for each i and so $bs = \{b_i\}s = \{b_i s\} \in A$. This means that $\prod_i C^v_{B_i}(A_i) \leq C^v_B(A)$. The reverse inequality is clear. □

3. Categorical properties of vitally dense monomorphisms

In this section some categorical properties including limits and colimits with respect to the class $\mathcal{M}_v$ of all v-dense monomorphisms of acts derived from the vital closure operator are investigated.

An important property for a class $\mathcal{M}$ of morphisms of a category is being the composition closed, that is, the composition of any two morphisms of $\mathcal{M}$ belongs to $\mathcal{M}$. Also $\mathcal{M}$ is said to be *right* (*left*) *cancellable* if for monomorphisms f and g with $gf \in \mathcal{M}$ one has $g \in \mathcal{M}$ ($f \in \mathcal{M}$) (see, for example, [2]). In the following, we study these properties for the class $\mathcal{M}_v$.

Lemma 3.1. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two homomorphisms where g is a monomorphism. Then gf is a v-dense homomorphism if and only if so are f and g. In particular, $\mathcal{M}_v$ is closed under composition as well as right and left cancellable.*

Proof. Assume that gf is a v-dense homomorphism. We show that $f(A)$ and $g(B)$ are v-dense subacts of B and C, respectively. For any $b \in B$, $g(b) \in C$. Since $gf(A)$ is a v-dense subact of C, there exist $s \in S$ and $a \in A$ for which $g(b)s = gf(a)$. So $g(bs) = gf(a)$ and then $bs = f(a) \in f(A)$. Now let $c \in C$. Then there exists $s \in S$ with $cs \in gf(A)$ and so $cs \in g(B)$. Conversely, let f and g be v-dense homomorphisms. It must be shown that $gf(A)$ is a v-dense subact of C. Let $c \in C$. Since $g(B)$ is v-dense in C, there exist $s \in S$ and $b \in B$ such that $cs = g(b)$. Also, since $f(A)$ is v-dense in B, there exist $t \in S$ and $a \in A$ for which $bt = f(a)$. Now we have $c(st) = (cs)t = g(b)t = g(bt) = gf(a) \in gf(A)$, as desired. □

It is clear that any epimorphism is a v-dense homomorphism. So, using Lemma 3.1, the following is obtained.

Corollary 3.2. *The composition of a v-dense homomorphism with an epimorphism is a v-dense homomorphism.*

Proposition 3.3. *Let S be a finite monoid. Then the class $\mathcal{M}_v$ is closed under products and direct sums.*

Proof. Let $(f_i : A_i \to B_i)_{i\in I}$ be a family of v-dense monomorphisms. Consider the commutative diagram

$$\begin{array}{ccc} \prod_{i\in I} A_i & \xrightarrow{f} & \prod_{i\in I} B_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

We show that $f = (f_i)_{i\in I} : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is a v-dense monomorphism. Let $b = (b_i)_{i\in I} \in \prod_{i\in I} B_i$. Since each f_i is v-dense, there exists $s_i \in S$ for $i \in \{1, 2, \cdots, n\}$ such that $b_i s_i \in \mathrm{Im} f_i$. Now let $s := s_1 s_2 \cdots s_n$. We have $bs = (b_i s)_{i\in I} \in \mathrm{Im} f$. Hence, f is v-dense. It is obvious that f is a monomorphism. So $f \in \mathcal{M}_v$. For the second assertion, let $(f_i : A_i \to B_i)_{i\in I}$ be a family of v-dense monomorphisms where A_i's and B_i's have a unique fixed element 0. Then, using the first part, $f = \bigoplus_{i\in I} f_i = (\prod_{i\in I} f_i)|_{\bigoplus_{i\in I} A_i} : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$ is a v-dense monomorphism. More precisely, if $(b_i)_{i\in I} \in \bigoplus_{i\in I} B_i$, then there exists $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ and $s \in S$ with $f((a_i)_{i\in I}) = (b_i)_{i\in I} s$. But, $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$, because for all i with $b_i = 0$ we have $0 = b_i s = f_i(a_i)$, and so $a_i = 0$ since f_i is a monomorphism. □

Proposition 3.4. *The class $\mathcal{M}_v$ is closed under coproducts.*

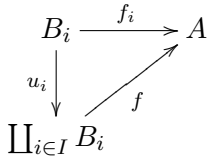
Proof. Consider the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u'_i} \\ \coprod_{i\in I} A_i & \xrightarrow{f} & \coprod_{i\in I} B_i \end{array}$$

in which $(f_i : A_i \to B_i)_{i\in I}$ is a family of v-dense monomorphisms. Let $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the homomorphism satisfying $f(u_i(a_i)) = u'_i f_i(a_i)$, for $a_i \in A_i$, which exists by the universal property of coproducts. In fact, $f(a_i, i) = (f_i(a_i), i)$. We claim that f is a v-dense monomorphism. Let $b \in \coprod_{i\in I} B_i$. Then $b \in B_i$ for some $i \in I$ and $b = u'_i(b_i)$. Since f_i is v-dense, there exists $a_i \in A_i$ and $s \in S$ for which $f_i(a_i) = bs$, and hence $bs = u'_i f_i(a_i) = f u_i(a_i) \in \mathrm{Im} f$, which means that f is v-dense. Also, f is a monomorphism since u'_i and f_i, $i \in I$ are monomorphisms. □

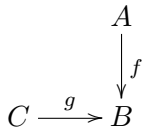
Proposition 3.5. *Let $(f_i : B_i \to A)_{i\in I}$ be a family of v-dense homomorphisms. Then $f : \coprod_{i\in I} B_i \to A$ is a v-dense homomorphism.*

Proof. Consider the diagram



where $f : \coprod_{i\in I} B_i \to A$ is the homomorphism obtained by the universal property of coproducts. For any $a \in A$, since f_i is v-dense, there exist $b_i \in B$ and $s \in S$ such that $f_i(b_i) = as$. So $as = f_i(b_i) = fu_i(b_i)$ and hence f is a v-dense homomorphism. □

Recall that the pullback of a given diagram

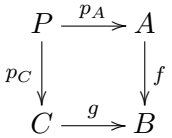


in **Act-**S is the subact $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ (if non-empty) of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps.

A class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the next result, we study this property for v-dense monomorphisms of acts.

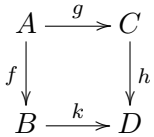
Proposition 3.6. *The class $\mathcal{M}_v$ is pullback stable.*

Proof. Consider the pullback diagram



where $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ and $p_C : P \to C$ and $p_A : P \to A$ are restrictions of the projection maps. Assume that $g \in \mathcal{M}_v$. We show that $p_A \in \mathcal{M}_v$. Let $a \in A$. Then $f(a) \in B$ and so it follows from the v-density of g that there exist $s \in S$ and $c \in C$ such that $g(c) = f(a)s = f(as)$. Hence, $(c, as) \in P$ and we have $as = p_A(c, as)$, which means that p_A is a v-dense homomorphism. □

For a class $\mathcal{M}$ of morphisms of a category, it is said that *pushouts transfer $\mathcal{M}$-morphisms* if for a pushout diagram



if $g \in \mathcal{M}$, then so is k.

The pushout of a given diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

in **Act-**S is the factor act $Q = (B \sqcup C)/\theta$, where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also the pushout maps are given as $h = \gamma u_C : C \to Q, k = \gamma u_B : B \to Q$, where $\gamma : B \sqcup C \to Q$ is the canonical epimorphism. Multiple pushouts in **Act-**S are constructed in a similar way.

Proposition 3.7. *In* **Act-**S*, pushouts transfer v-dense monomorphisms.*

Proof. Consider the pushout diagram (as described above)

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & Q \end{array}$$

where g is a v-dense monomorphism. We show that k is also a v-dense monomorphism. By [7], k is a monomorphism. So it remains to prove that k is v-dense. Let $[x]_\theta \in Q$. If $x = u_B(b)$ for some $b \in B$, then $[x]_\theta = k(b) \in \mathrm{Im}(k)$. If $x = u_C(c)$ for some $c \in C$, using that g is v-dense, $cs = g(a)$ for some $s \in S$ and $a \in A$ which implies that $[x]_\theta s \in \mathrm{Im}(k)$. □

For a class $\mathcal{M}$ of morphisms of a category, we say that multiple pushouts transfer $\mathcal{M}$-morphisms if in the multiple pushout $(Q, (B_i \xrightarrow{d_i'} Q)_{i\in I})$ of a family $\{d_i : A \to B_i \mid i \in I\}$ of $\mathcal{M}$-morphisms, $d_i' \in \mathcal{M}$, for every $i \in I$.

Similarly to the pushouts, we have the following result.

Proposition 3.8. *Multiple pushouts transfer v-dense monomorphisms.*

Proof. Let $\{d_i : A \to B_i \mid i \in I\}$ be a family of v-dense monomorphisms. Recall that the multiple pushout of this family is $(\coprod_{i\in I} B_i)/\theta$, where θ is the congruence on $\coprod_{i\in I} B_i$ generated by all pairs $H = \{(u_i d_i(a), u_j d_j(a)) : i, j \in I, a \in A\}$, where for each $i \in I$, $u_i : B_i \to \coprod_{i\in I} B_i$ is the i-th coproduct injection map. Also the multiple pushout maps are $d_i' = \gamma u_i : B_i \to (\coprod_{i\in I} B_i)/\theta$ where $\gamma : \coprod_{i\in I} B_i \to (\coprod_{i\in I} B_i)/\theta$ is the natural epimorphism. The same argument as in the proof of [10, Proposition 3.15] gives that d_i' is a monomorphism for each $i \in I$. To show that each $d_i' d_i$ (and hence each d_i' by Lemma 3.1) is v-dense, let

$b \in (\coprod_{i\in I} B_i)/\theta$. Then there exist $j \in I$ and $b_j \in B_j$ such that $b = [u_j(b_j)]_\theta$. Since d_j is v-dense, there exist $a \in A$ and $s \in S$ such that $d_j(a) = b_j s$. Now we get

$$bs = [u_j(b_j)]_\theta s = [u_j(b_j s)]_\theta = d'_j(b_j s) = d'_j d_j(a) = d'_i d_i(a) \in \mathrm{Im}(d'_i d_i),$$

as required. □

Definition 3.9. For a class $\mathcal{M}$ of morphisms of a category $\mathcal{A}$, we say that $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for each set indexed family $\{m_i : A \to A_i \mid i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all m_i's, that is, there are $d_i : A_i \to B$ with $d_i m_i = m$. In addition, if d_i's belong to $\mathcal{M}$, it is said that $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property.

Corollary 3.10. *The category* **Act-**S *has* $\mathcal{M}_v$*-bounds and* $\mathcal{M}_v$*-amalgamation property.*

Proof. Let $\{h_i : A \to B_i \mid i \in I\}$ be a set indexed family in $\mathcal{M}_v$ and $h : A \to B = (\coprod_{i\in I} B_i)/\theta$ be the multiple pushout of h_i's. Then h factors over all h_i's and is a v-dense monomorphism by Proposition 3.8. The second assertion also follows from Proposition 3.8. □

This following is devoted to the study of the behaviour of v-dense monomorphisms with respect to colimits and directed colimits. To this end, let us recall some preliminaries.

Let $\mathcal{A} : \mathbf{I} \to$ **Act-**S be a diagram in **Act-**S ($\mathbf{I}$ is a small category and $\mathcal{A}$ is a functor) determining the acts $\mathcal{A}(\alpha) = A_\alpha \in$ **Act-**S for $\alpha \in I = \mathrm{Obj}(\mathbf{I})$ and homomorphisms $g_{\alpha\beta} : A_\alpha \to A_\beta$ for $\alpha \to \beta$ in $\mathrm{Mor}(\mathbf{I})$. The *colimit* of this diagram is obtained as $\varinjlim_\alpha A_\alpha =: (\coprod_{\alpha\in I} A_\alpha)/\theta$, where θ is the congruence generated by

$$H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) : a_\alpha \in A_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}.$$

The colimit homomorphisms are $g_\alpha := \gamma_\theta u_\alpha : A_\alpha \to \varinjlim_\alpha A_\alpha$ where u_α's are the coproduct injection maps and γ_θ is the canonical epimorphism.

A *directed system* of S-acts and homomorphisms is a family $(B_\alpha)_{\alpha\in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha\le\beta\in I}$ of homomorphisms such that given $\alpha \le \beta \le \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, and also $g_{\alpha\alpha} = id$. Note that the *directed colimit* of a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\le\beta\in I})$ in **Act-**S is given as $\varinjlim_\alpha B_\alpha = (\coprod_{\alpha\in I} B_\alpha)/\rho$, where the congruence ρ is defined by $(b_\alpha, b_\beta) \in \rho$ if there exists $\delta \ge \alpha, \beta$ such that $u_\delta g_{\alpha\delta}(b_\alpha) = u_\delta g_{\beta\delta}(b_\beta)$, where u_δ's are injection maps of the coproduct.

Notice that the family $g_\alpha = \gamma_\rho u_\alpha : B_\alpha \to \varinjlim_{\alpha\in I} B_\alpha$ of homomorphisms satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \le \beta$, where $\gamma_\rho : \coprod_{\alpha\in I} B_\alpha \to \varinjlim_{\alpha\in I} B_\alpha$ is the canonical epimorphism.

Theorem 3.11. *The class of* v*-dense monomorphisms is closed under colimits.*

Proof. Assume that $\mathcal{A}, \mathcal{B} : \mathbf{I} \to$ **Act-**S be diagrams determining the S-acts A_α, B_α for $\alpha \in I = \mathrm{Obj}(\mathbf{I})$ and homomorphisms $g_{\alpha\beta} : A_\alpha \to A_\beta, \acute{g}_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta \in \mathrm{Mor}(\mathbf{I})$. Consider the colimits of these diagrams with the colimit homomorphisms $g_\alpha = \gamma_\theta u_\alpha : A_\alpha \to \varinjlim_\alpha A_\alpha = (\coprod_{\alpha\in I} A_\alpha)/\theta, \acute{g}_\alpha = \gamma_{\acute{\theta}}\acute{u}_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha = (\coprod_{\alpha\in I} B_\alpha)/\acute{\theta}$ where θ is the congruence generated by $H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) \mid a_\alpha \in A_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}$ and $\acute{\theta}$ is the congruence generated by $\acute{H} = \{(\acute{u}_\alpha(b_\alpha), \acute{u}_\beta \acute{g}_{\alpha\beta}(b_\alpha)) \mid b_\alpha \in B_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}$. Suppose that $\{f_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ is a family of v-dense monomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f_\alpha} & B_\alpha \\ {\scriptstyle g_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \acute{g}_{\alpha\beta}} \\ A_\beta & \xrightarrow{f_\beta} & B_\beta \end{array}$$

We show that $f = \varinjlim_\alpha f_\alpha$ with $f[u_\alpha(a_\alpha)]_\theta = [\acute{u}_\alpha f_\alpha(a_\alpha)]_{\acute{\theta}}$ is a v-dense monomorphism. We have the following commutative diagram:

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\quad f_\alpha \quad} & B_\alpha \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow{\scriptstyle \acute{g}_\alpha} \\ (\coprod_{\alpha\in I} A_\alpha)/\theta & \xrightarrow{f} & (\coprod_{\alpha\in I} B_\alpha)/\acute{\theta} \end{array}$$

Let $x = [\acute{u}_\alpha(b_\alpha)]_{\acute{\theta}} \in \varinjlim_\alpha B_\alpha$ for some $\alpha \in I$ and $b_\alpha \in B_\alpha$. Since f_α is v-dense, there exist $s \in S, a \in A$ such that $f_\alpha(a_\alpha) = b_\alpha s$. Hence, $xs = [\acute{u}_\alpha(b_\alpha s)]_{\acute{\theta}} = \acute{g}_\alpha(b_\alpha s) = \acute{g}_\alpha f_\alpha(a_\alpha) = f g_\alpha(a_\alpha) = f[u_\alpha(a_\alpha)]_\theta$. Also f is a monomorphism since so is f_α. □

Proposition 3.12. *The category* **Act-**S *has* $\mathcal{M}_v$*-directed colimits.*

Proof. Consider a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\le\beta\in I})$ of S-acts and homomorphisms, and the colimit homomorphisms $g_\alpha : B_\alpha \longrightarrow \varinjlim_\alpha B_\alpha$. Take v-dense monomorphisms $h_\alpha : A \longrightarrow B_\alpha, \alpha \in I$ with $g_{\alpha\beta}h_\alpha = h_\beta$ for $\alpha \le \beta \in I$. Let $h : A \longrightarrow \varinjlim_\alpha B_\alpha$ be the directed colimit of h_α for $\alpha \in I$. That is, $h = \varinjlim_\alpha h_\alpha = g_\alpha h_\alpha = g_\beta h_\beta = g_\gamma h_\gamma = \cdots$. Let $\alpha \in I$. Since h_α is a monomorphism, so is h. Now since g_α is an epimorphism, it is v-dense and then h is v-dense. □

Definition 3.13. For a class $\mathcal{M}$ of morphisms of a category $\mathcal{A}$, we say that $\mathcal{A}$ satisfies $\mathcal{M}$*-chain* condition if for any directed system $((A_\alpha)_{\alpha\in I}, (f_{\alpha\beta})_{\alpha\le\beta\in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha\in I}$ with $g_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 3.14. *The category* **Act-**S *fulfills the* $\mathcal{M}_v$*-chain condition.*

Proof. Take $A = \varinjlim_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be the colimit maps. Then, applying Proposition 3.12, the assertion holds. □

4. Vital dense essentiality and vital dense injectivity

In this section we proceed with the study of essentiality with respect to the class $\mathcal{M}_v$. We show that the three notions of essentiality introduced in [2] coincide for the class $\mathcal{M}_v$. We investigate the relations between v-dense essentiality, v-dense injectivity and v-dense absolute retractivity. It is proved that maximal v-dense essential extensions, minimal v-dense injective v-dense extensions and v-dense injective hulls of acts are equivalent and exist which will be called the v-dense injective envelopes. In other words, injectivity well behaves with respect to v-dense monomorphisms of acts in the sense of [1].

Recall that for a subclass $\mathcal{M}$ of the class $\mathcal{M}ono$ of monomorphisms of a category $\mathcal{A}$ and $A \xrightarrow{m} B \in \mathcal{M}$ one usually uses one of the following definitions to say that m is essential:

(1) $A \xrightarrow{m} B \xrightarrow{f} C \in \mathcal{M} \Rightarrow f \in \mathcal{M}$

(2) $A \xrightarrow{m} B \xrightarrow{f} C \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$ (essential monomorphism)

(3) $A \xrightarrow{m} B \xrightarrow{f} C \in \mathcal{M} \Rightarrow f \in \mathcal{M}ono$

Here, considering $\mathcal{A}$ to be the category **Act-**S and $\mathcal{M}$ to be the class $\mathcal{M}_v$ of all v-dense monomorphisms of acts, we show that the above types of essentialities are the same.

Theorem 4.1. *Let* $f : A \to B \in \mathcal{M}_v$*. Then the following are equivalent:*
(i) *Any homomorphism* $g : B \to C$ *for which* $gf \in \mathcal{M}_v$ *belongs to* $\mathcal{M}_v$*.*
(ii) *Any homomorphism* $g : B \to C$ *for which* $gf \in \mathcal{M}_v$ *is a monomorphism.*
(iii) *Any homomorphism* $g : B \to C$ *for which* gf *is a monomorphism is itself a monomorphism.*
(iv) *For every congruence* ρ *on* B*,* $\rho \cap (f(A) \times f(A)) = \Delta_{f(A)}$ *implies* $\rho = \Delta_B$*.*

Proof. The implication (i) ⇒ (ii) is clear.
(ii) ⇒ (iii) Let $g : B \to D$ be a homomorphism such that gf is a monomorphism. Since $gf : A \to g(B)$ is a v-dense monomorphism, by the assumption, $g : B \to g(B)$ and so g is a monomorphism.
(iii) ⇔ (iv) See Lemma 3.1.15 in [8].
(iv) ⇔ (i) Let gf be a v-dense monomorphism. By (iii) ⇔ (iv), g is a monomorphism. Now, using Lemma 3.1, g is a v-dense monomorphism. □

Definition 4.2. We say that a v-dense monomorphism $f : A \to B$ is a *v-dense essential monomorphism* or *$\mathcal{M}_v$-essential* if it satisfies any one of the equivalent conditions of Theorem 4.1.

It follows by Theorem 4.1 that:

Corollary 4.3. *A monomorphism f is $\mathcal{M}_v$-essential if and only if it is a v-dense as well as essential monomorphism.*

Lemma 4.4. *The composition of $\mathcal{M}_v$-essentials is an $\mathcal{M}_v$-essential.*

Proof. Let $f : A \to B$ and $g : B \to C$ be $\mathcal{M}_v$-essentials. We show that so is gf. Using Lemma 3.1, gf is a v-dense monomorphism. Suppose that $h : C \to D$ is a homomorphism for which $h(gf) = (hg)f$ is a monomorphism. It must be shown that h is a monomorphism. Since f is $\mathcal{M}_v$-essential, hg and hence h is a monomorphism because g is $\mathcal{M}_v$-essential. □

Lemma 4.5. *If gf is $\mathcal{M}_v$-essential and g is a monomorphism, then g is an $\mathcal{M}_v$-essential.*

Proof. Using Lemma 3.1, g is v-dense. In light of Corollary 4.3, it suffices to prove that g is essential. Let h be a homomorphism such that hg is a monomorphism. Then, since f is also a monomorphism, $h(gf) = (hg)f$ is a monomorphism. But gf is $\mathcal{M}_v$-essential and so h is a monomorphism. Thus g is $\mathcal{M}_v$-essential. □

Theorem 4.6. *The category* **Act-S** *fulfills Banaschewski's $\mathcal{M}_v$-condition, in the sense that for any $\mathcal{M}_v$-monomorphism $f : A \to B$ there exists a homomorphism $g : B \to C$ such that gf is $\mathcal{M}_v$-essential.*

Proof. Apply Zorn's lemma to the poset

$$\Sigma = \{\theta \in \mathrm{Con}(B) \mid A \xrightarrow{f} B \xrightarrow{\pi_\theta} B/\theta \in \mathcal{M}_v\}$$

with inclusion as the order. Then for a maximal element $\overline{\theta}$ of Σ, $\pi_{\overline{\theta}} f$ is a v-dense monomorphism. In view of Corollary 4.3, it remains to prove that $\pi_{\overline{\theta}} f$ is essential. Let $h : B/\theta \to D$ be a homomorphism such that $h\pi_{\overline{\theta}} f$ is a monomorphism. We show that h is a monomorphism. Let $h([b]_{\overline{\theta}}) = h([\acute{b}]_{\overline{\theta}})$, $[b]_{\overline{\theta}} \neq [\acute{b}]_{\overline{\theta}}$ for $[b]_{\overline{\theta}}, [\acute{b}]_{\overline{\theta}} \in B/\theta$. Define the following relation on B:

$$b\rho\acute{b} \quad \Leftrightarrow \quad [b]_{\overline{\theta}} \ (\ker h) \ [\acute{b}]_{\overline{\theta}}$$

It is clear that ρ is a congruence on B and $\overline{\theta} \subset \rho$. We show that $\pi_\rho f$ is a v-dense monomorphism. Since π_ρ is an epimorphism and f is v-dense, $\pi_\rho f$ is a v-dense homomorphism by Corollary 3.2. Suppose that $\pi_\rho f(a) = \pi_\rho f(\acute{a})$ for $a, \acute{a} \in A$. Then we have $(f(a), f(\acute{a})) \in \rho$ and so $h([f(a)]_{\overline{\theta}}) = h([f(\acute{a})]_{\overline{\theta}})$. This implies that $h\pi_{\overline{\theta}} f(a) = h\pi_{\overline{\theta}} f(\acute{a})$ and so $a = \acute{a}$ because $h\pi_{\overline{\theta}} f$ is a monomorphism. Hence, $\rho \in \Sigma$ which contradicts the maximality of $\overline{\theta}$. □

By a *v-dense retract* of an S-act B, we mean a v-dense subact A of B together with a homomorphism from B to A which maps A identically.

Remark 4.7. If A is a proper v-dense retract of B, then B is not a v-dense essential extension of A. Indeed, consider a retraction $g : B \to A$ and take $b \in B \setminus A$. Now $g(b) \in A$. Since $g(g(b)) = g(b)$, it follows that g is not a monomorphism whereas $g|_A$ is a monomorphism.

Let A be an S-act. Then A is said to be *v-dense injective* if it is injective with respect to all v-dense monomorphisms. Also A is called *v-dense absolute retract* if A is a v-dense retract of each of its v-dense extensions. Clearly, a v-dense retract of any v-dense injective S-act is v-dense injective.

By virtue of Proposition 3.7, Theorem 4.6 and [2, Theorem 3.6], the following result is obtained.

Theorem 4.8. *Let S be a monoid. Then the following are equivalent for each S-act A:*

(i) *A is v-dense injective.*

(ii) *A is v-dense absolute retract.*

(iii) *A has no proper v-dense essential extension.*

Definition 4.9. Let A be an S-act and E be a v-dense extension of A. Then E is called

(i) a *maximal v-dense essential extension* of A if E is a v-dense essential extension of A and not contained properly in any other v-dense essential extension of A.

(ii) a *minimal v-dense injective v-dense extension* of A if E is v-dense injective and contains no proper subact B which is a v-dense injective extension of A.

(iii) an *$\mathcal{M}_v$-injective hull* of A if E is v-dense injective as well as a v-dense essential extension of A.

Lemma 4.10. *If B is a v-dense essential extension of an S-act A and A is embedded into some (v-dense) injective S-act Q, then B can also be embedded into Q.*

Proof. Straightforward. □

As we shall see, the next result ensures the existence of v-dense injective envelopes of S-acts.

Theorem 4.11. *Every S-act has a maximal v-dense essential extension.*

Proof. Let A be an S-act and Q be an injective extension of A. By Lemma 4.10, we may assume that both A and all its v-dense essential extensions are subacts of Q. Let P be the set of all v-dense essential extensions of A. Consider P as a partially ordered set under inclusion. For any chain $(A_i)_{i\in I}$ in P, $\bigcup_{i\in I} A_i \in P$ is an upper bound. Using Zorn's lemma, P has a maximal element M which is in fact a maximal v-dense essential extension of A. □

Lemma 4.12. *If B is a maximal v-dense essential extension of an S-act A, then B is v-dense injective.*

Proof. In light of Theorem 4.8, it suffices to show that B has no proper v-dense essential extension. If there exists an S-act C such that $B \subset C$ and C is a v-dense essential extension of B, then C is a v-dense essential extension of A by Lemma 4.4 which contradicts the assumption. □

Theorem 4.13. *Let A be an S-act and E be a v-dense extension of A. The following are equivalent:*

(i) *E is a maximal v-dense essential extension of A.*

(ii) *minimal v-dense injective v-dense extension of A.*

(iii) *E is an $\mathcal{M}_v$-injective hull of A.*

Proof. (i) ⇒ (iii) Follows from Lemma 3.1, Proposition 3.7, Theorem 4.6 and [2, Theorem 3.8(iv)].

(iii) ⇒ (ii) Follows from Lemma 3.1 and [2, Theorem 3.8(v)].

(ii) ⇒ (i) Let E be a minimal v-dense injective v-dense extension of A and B be a maximal v-dense essential extension of A which exists by Theorem 4.11. Consider the diagram

$$\begin{array}{ccc} A & \overset{j}{\hookrightarrow} & B \\ {\scriptstyle i}\downarrow & & \\ E & & \end{array}$$

where i and j are the v-dense embeddings. Since E is v-dense injective, there exists a homomorphism $k : B \to E$ such that $kj = i$. Thus kj is a monomorphism. By v-dense essentiality of B, k is a monomorphism and so B is a v-dense subact of E by Lemma 3.1. Since B is a maximal v-dense essential extension of A, it is v-dense injective by Lemma 4.12. Therefore, $B = E$ by the minimality of E. □

An S-act E is called a *v-dense injective envelope* of an S-act A if it satisfies one of the equivalent conditions in Theorem 4.13.

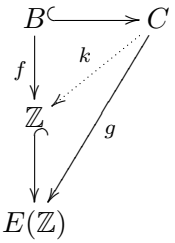
It is well-known that each S-act is embedded into an injective envelope. Here, using Theorems 4.11 and 4.13, we present a counterpart of this result for the vitally dense case.

Corollary 4.14. *For every S-act there exists a v-dense injective envelope unique up to isomorphism.*

Finally, the following example demonstrates that the notion of v-dense injective envelope is different from the usual notion of injective envelope for acts.

Example 4.15. Consider $\mathbb{N}$ and $\mathbb{Z}$ as $(\mathbb{N}_0, +)$-acts with usual addition as the actions. It is clear that $\mathbb{Z}$ is a v-dense extension of $\mathbb{N}$. Since $\mathbb{Z}$ has no fixed element, it is not injective and hence not an injective envelope of $\mathbb{N}$. However,

we claim that $\mathbb{Z}$ is a v-dense injective envelope of $\mathbb{N}$. First we show that $\mathbb{Z}$ is an essential extension of $\mathbb{N}$. Let $f : \mathbb{Z} \to A$ be a homomorphism such that $f|_{\mathbb{N}}$ is a monomorphism. Suppose that $f(z) = f(z')$ for $z, z' \in \mathbb{Z}$. There exist $n, n' \in \mathbb{N}_0$ such that $z + n, z' + n' \in \mathbb{N}$. Thus $z + (n + n'), z' + (n + n') \in \mathbb{N}$ and $f(z + (n + n')) = f(z' + (n + n'))$ so that $z + (n + n') = z' + (n + n')$ and then $z = z'$ which shows that f is a monomorphism. Now we prove that $\mathbb{Z}$ is v-dense injective. Suppose that B is a v-dense subact of C, $f : B \to \mathbb{Z}$ is a homomorphism and $E(\mathbb{Z})$ is an injective envelope of $\mathbb{Z}$. It follows from the injectivity of $E(\mathbb{Z})$ that there exists a homomorphism $g : C \to E(\mathbb{Z})$ such that the following diagram commutes:



We claim that $\mathrm{Im}g \subseteq \mathbb{Z}$. Indeed, for any $c \in C$, there exists $n \in \mathbb{N}_0$ such that $cn \in B$ since B is v-dense in C. Then $g(cn) = f(cn) \in \mathbb{Z}$ and there exists $x \in \mathbb{Z}$ such that $g(c)n = g(cn) = x + n$. This implies that $g(c) = x$, otherwise consider the homomorphism $h : E(\mathbb{Z}) \to E(\mathbb{Z})$ defined by $h(y) = yn$ for any $y \in E(\mathbb{Z})$. We have $h(g(c)) = g(c)n = x + n = h(x)$ and $g(c) \neq x$, which means that h is not a monomorphism. Since $E(\mathbb{Z})$ is an essential extension of $\mathbb{Z}$, $h|_{\mathbb{Z}}$ is not a monomorphism, which is a contradiction. Now taking $k := g : C \to \mathbb{Z}$, we have $k|_B = f$, which shows that $\mathbb{Z}$ is v-dense injective.

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On the condition number of integral equations in the elastic two-dimensional case using the cross multipole coefficients

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Abstract. The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified Green's function technique. In this work, we generalize a new criterion of optimality of the perturbed fundamental solution based on the minimization of the condition number of the modified integral equations using the cross multipole coefficients.
Keywords: cross multipole coefficients, modified Green's function, integral equations, linear elasticity, condition number.

1. Introduction

The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified Green's function technique, where the simple and cross multipole coefficients must satisfy some suitable and mild conditions [3]. Some criteria to determine an optimal choice for these multipole coefficients are developed recently. The first criterion is based on the minimization of the norm of the modified integral operator [2], [5], [11], [13], and [5], and motivated by enlarging the radius of convergence of the used numerical method (successive approximations). The second criterion is based on the minimization of the norm of the modified Green's function [9], and motivated by the minimization of the norm of the difference between the modified and exact Green's function. In [1], Argyropoulos et al. have presented another criterion based on the minimization of

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the condition number of the boundary integral equations describing the problem. In [8], we have developed a new criterion based on the minimization of the norm of the surface stress operator, or the norm of the modified traction operator using simple multipole coefficients, motivated by the minimization of the norm of the difference between the modified and exact kernel of the integral operator. In this work we generalize the same criterion developed by Argyropoulos et al. In [1], for the case of cross multipole coefficients. This paper is organized as follows, in the next section, we present the mathematical formulation of the exterior Neumann boundary value problem using the modified Green's function technique, in section 3 we present our main results which are the determination of the optimal choice of the simple and cross multipole coefficients minimizing the condition number of the modified boundary integral equations describing the problem.

2. Formulation of the problem

An exterior Neumann boundary value problem in two-dimensional elastic case can be described through a boundary integral equation of the form [3]:

$$\left(\frac{1}{2}I + \bar{K}_0^*\right)(\varphi)(p) = f(p), \ p \in \partial D, \tag{2.1}$$

where f is a Hölder continuous density, and the integral operator K_0 is defined as :

$$(K_0\varphi)(p) = \frac{1}{2\pi}\int_{\partial D} T_p G_0(p,\ q)\ \varphi(q)\ ds_q,\ p \in \partial D, \tag{2.2}$$

G_0 is the Green's function (fundamental solution), and T is the surface stress operator.

Using the modified Green's function technique, by introduce a regular solution [3], our boundary value problem can be described through a modified boundary integral equation of the form::

$$\left(\frac{1}{2}I + \bar{K}_1^*\right)(\varphi)(p) = f(p), \ p \in \partial D, \tag{2.3}$$

where the modified integral operator K_1 is defined as:

$$(K_1\varphi)(p) = \frac{1}{2\pi}\int_{\partial D} T_p G_1(p,\ q)\ \varphi(q)\ ds_q,\ p \in \partial D, \tag{2.4}$$

and the modified Green's function is written as:

$$\begin{aligned} G_1(p,\ Q) = \frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2} \Big[& F_m^{\sigma l}(P) \otimes \hat{F}_m^{\sigma l}(Q) + a_m^{\sigma l}\ F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q) \\ & + (-1)^{\sigma+l} b_m\ F_m^{\sigma l}(P) \otimes F_m^{(3-\sigma)(3-l)}(Q)\Big], \end{aligned} \tag{2.5}$$

where

$$F_m^{\sigma 1}(P) = \operatorname{grad}\left(H_m^1(kr_p)\ E_m^{\sigma}(\theta_p)\right), \tag{2.6}$$
$$F_m^{\sigma 2}(P) = rot\left(H_m^1(Kr_p)\ E_m^{\sigma}(\theta_p)\ \hat{e}_3\right).$$

$\hat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [3], and

$$E_m^{\sigma}(\theta_p) = \sqrt{\varepsilon_m}\begin{cases}\cos(m\theta_p), & \sigma = 1\\ \sin(m\theta_p), & \sigma = 2\end{cases}, \text{ with } \varepsilon_m = \begin{cases}1, & m = 0,\\ 2, & m > 0,\end{cases}$$

$a_m^{\sigma l}$ and b_m are the simple and cross multipole coefficients, which must satisfy the following conditions:

$$\bar{b}_m\left(a_m^{\sigma 1} + \frac{1}{2}\right) + b_m\left(\bar{a}_m^{\sigma 2} + \frac{1}{2}\right) = 0, \tag{2.7}$$

and

$$\left|a_m^{\sigma l} + \frac{1}{2}\right|^2 + |b_m|^2 - \frac{1}{4} < 0,\ \forall m = 0 : \infty \text{ and } \forall\sigma,\ l = 1,\ 2.$$

3. Main results

3.1 Case of the circle

As shown in [4], it is externally to get explicit results for the multipole coefficients which minimize the operator norms for arbitrary boundaries ∂D. Similar discussions are given in [1] for the condition number of integral equations in the elastic threedimensional case. Nevertheless, the special result for minimizing the condition number when ∂D is a circle serves as a guide to an explicit coefficient choice which leads to well conditioned integral equations for perturbations of circular domains. So we examine first the circular case, by considering the question of how to choose the simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m in the modified Green's function given by (2.5), minimizing the condition number of the modified boundary integral equations describing the problem (2.3). We introduce the operator:

$$M : L_2(\partial D) \longrightarrow L_2(\partial D) \quad / \ M = \frac{1}{2}I + \overline{K}_1^*, \tag{3.1}$$

and its L_2 adjoint M^*. It is well known that, the condition number which is given by the relation [4]:

$$Cond(M) = \|M\| \cdot \left\|M^{-1}\right\|$$

with respect to the L_2 norm can be expressed as:

$$Cond(M) = \sqrt{\frac{\lambda_{\max}^M}{\lambda_{\min}^M}}$$

where $\lambda_{\max}^{M}$ and $\lambda_{\min}^{M}$ denote respectively, the largest and the smallest spectral value of the self-adjoint operator M^*M.

Theorem 1. *There are uniquely defined simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m which minimize the condition number of the modified boundary integral equation* (2.3)*. These simple and cross multipole coefficients are given by the relations* (3.2) *and* (3.3)*:*

$$a_m^{\sigma l}.\frac{i}{4\mu K^2} = \frac{\overline{\beta}_m^{\sigma l}.g_m^{\sigma l} - \alpha_m^{(3-\sigma)(3-l)}.f_m^{\sigma l}}{\Delta_m^{\sigma l/}}, \tag{3.2}$$

$$(-1)^{\sigma+l}\, b_m.\frac{i}{4\mu K^2} = \frac{\beta_m^{\sigma l}.f_m^{\sigma l} - \alpha_m^{\sigma l}.g_m^{\sigma l}}{\Delta_m^{\sigma l/}}, \tag{3.3}$$

where

$$\Delta_m^{\sigma l/} = \alpha_m^{\sigma l}.\alpha_m^{(3-\sigma)(3-l)} - \left|\beta_m^{\sigma l}\right|^2, \tag{3.4}$$

$$\alpha_m^{\sigma l} = \left\|TF_m^{\sigma l}\right\|^2, \tag{3.5}$$

$$\beta_m^{\sigma l} = \left\langle TF_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)}\right\rangle, \tag{3.6}$$

$$f_m^{\sigma l} = \left\langle \overline{K}_0^* T\overline{F}_m^{\sigma l}, F_m^{\sigma l\perp}\right\rangle, \tag{3.7}$$

$$g_m^{\sigma l} = \left\langle \overline{K}_0^* T\overline{F}_m^{(3-\sigma)(3-l)}, F_m^{\sigma l\perp}\right\rangle. \tag{3.8}$$

Proof. It is easily proved in [10] that the following relations hold for a circle with radius a:

$$F_m^{\sigma 1}(p) = kH_m'(ka)\, P_m^{\sigma}(\theta_p) + \frac{m}{a}H_m(ka)Q_m^{\sigma}(\theta_p) \tag{3.9}$$

$$(-1)^{\sigma}F_m^{\sigma 2}(p) = \frac{m}{a}H_m(ka)P_m^{(3-\sigma)}(\theta_p) + kH_m'(ka)\, Q_m^{(3-\sigma)}(\theta_p), \tag{3.10}$$

where

$$P_m^{\sigma}(\theta_p) = E_m^{\sigma}(\theta_p)\,\overrightarrow{r} \quad \text{and} \quad Q_m^{\sigma}(\theta_p) = (-1)^{\sigma}E_m^{(3-\sigma)}(\theta_p)\,\overrightarrow{\theta}, \tag{3.11}$$

$$T_pG_1(p,q) = \left[T_pG_1(p,q)\right]^t. \tag{3.12}$$

In view of (3.12) we conclude that $K_1 = \overline{K}_1^*$.

Consider now the modified double layer potentials V_n^v and U_n^v with densities P_n^v and Q_n^v :

$$V_n^v(p) = \frac{1}{2\pi}\int_{R_q=a} T_pG_1(p,q).P_n^v(\theta_p).ds_q, \tag{3.13}$$

$$U_n^v(p) = \frac{1}{2\pi}\int_{R_q=a} T_pG_1(p,q).Q_n^v(\theta_q).ds_q. \tag{3.14}$$

Exploiting the orthogonality relations for the vectors P_n^v and Q_n^v, we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 1}(q).P_n^v(\theta_q).a.d\theta_q = kaH'_m(ka)\delta_{mn}\delta_{\sigma\nu}, \tag{3.15}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 2}(q).P_n^v(\theta_q).a.d\theta_q = (-1)^{\sigma} mH_m(Ka)\delta_{mn}\delta_{(3-\sigma)\nu}, \tag{3.16}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 1}(q).Q_n^v(\theta_q).a.d\theta_q = mH_m(ka)\delta_{mn}\delta_{\sigma\nu}, \tag{3.17}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 2}(q).Q_n^v(\theta_q).a.d\theta_q = (-1)^{(3-\nu)} Ka)H'_m(Ka)\delta_{mn}\delta_{\sigma(3-\nu)}. \tag{3.18}$$

Substituting the expressions (3.15) to (3.18) in (3.13) and (3.14), we obtain:

$$\begin{aligned} P_n^v(p) = {} & \tfrac{i}{4\mu K^2} kaH'_n(Ka) \begin{bmatrix} T\widehat{F}_n^{\nu 1}(p) + a_n^{\nu 1} TF_n^{\nu 1}(p) \\ +(-1)^{\nu+1} b_n TF_n^{(3-\nu)2}(p) \end{bmatrix} \\ & + \tfrac{i}{4\mu K^2}(-1)^{3-\nu} nH_n(Ka) \begin{bmatrix} T\widehat{F}_n^{(3-\nu)2}(p) + a_n^{(3-\nu)2} TF_n^{(3-\nu)2}(p) \\ +(-1)^{3-\nu} b_n TF_n^{\nu 1}(p) \end{bmatrix}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} U_n^v(p) = {} & \tfrac{i}{4\mu K^2} nH_n(ka) \begin{bmatrix} TF_n^{\nu 1}(p) + a_n^{\nu 1} TF_n^{\nu 1}(p) \\ +(-1)^{\nu+1} b_n TF_n^{(3-\nu)2}(p) \end{bmatrix} \\ & + \tfrac{i}{4\mu K^2}(-1)^{3-\nu} KaH'_n(Ka) \begin{bmatrix} T\widehat{F}_n^{(3-\nu)2}(p) + a_n^{(3-\nu)2} TF_n^{(3-\nu)2}(p) \\ +(-1)^{3-\nu} b_n TF_n^{\nu 1}(p) \end{bmatrix}. \end{aligned} \tag{3.20}$$

In order to find the eigenvalues of M, the following relation has to be satisfied:

$$MU(p) = \lambda U(p), \quad R_p = a. \tag{3.21}$$

Taking into account that $\{P_n^v, Q_n^v\}$ is a basis in $(L_2(\partial D))^2$, we can express $U(p)$ as a linear combination of these vectors, so:

$$U(p) = \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2} (\alpha_m^{\alpha} P_m^{\alpha}(p) + \beta_m^{\alpha} Q_m^{\alpha}(p)). \tag{3.22}$$

To calculate $MU(p)$, we must calculate:

$$MP_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) P_n^v(p), \text{ and } MQ_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) Q_n^v(p).$$

If we apply the Neumann boundary condition to V_n^v, we obtain [11]:

$$TV_n^v(p) = \left(\frac{1}{2}I + K_1\right) P_n^v(p) \qquad R_p = a, \tag{3.23}$$

using $K_1 = \overline{K}_1^*$, (3.23) becomes:

$$MP_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) P_n^v(p) = TV_n^\nu(p) \quad R_p = a. \tag{3.24}$$

In the same way, we can obtain:

$$MQ_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) Q_n^v(p) = TU_n^\nu(p) \quad R_p = a. \tag{3.25}$$

Exploiting the following relations [12]:

$$\begin{array}{c} TF_n^{v1}(p) = k^2\left(2\mu H_n''(ka) - \lambda H_n(ka)\right) P_n^v(\theta_p) \\ +\frac{2\mu n}{a}\left(kH_n'(ka) - \frac{H_n(ka)}{a}\right) Q_n^v(\theta_p), \end{array} \tag{3.26}$$

$$\begin{array}{c} TF_n^{v2}(p) = \mu K^2\left(2H_n''(Ka) + H_n(Ka)\right)(-1)^\nu P_n^{(3-v)}(\theta_p) \\ +\frac{2\mu n}{a}\left(KH_n'(Ka) - \frac{H_n(Ka)}{a}\right)(-1)^\nu Q_n^{(3-v)}(\theta_p), \end{array} \tag{3.27}$$

$$\begin{array}{c} T\widehat{F}_n^{v1}(p) = k^2\left(2\mu J_n''(ka) - \lambda J_n(ka)\right) P_n^v(\theta_p) \\ +\frac{2\mu n}{a}\left(kJ_n'(ka) - \frac{J_n(ka)}{a}\right) Q_n^v(\theta_p), \end{array} \tag{3.28}$$

$$\begin{array}{c} TF_n^{v2}(p) = \mu K^2\left(2J_n''(Ka) + J_n(Ka)\right)(-1)^\nu P_n^{(3-v)}(\theta_p) \\ +\frac{2\mu n}{a}\left(KJ_n'(Ka) - \frac{J_n(Ka)}{a}\right)(-1)^\nu Q_n^{(3-v)}(\theta_p), \end{array} \tag{3.29}$$

and using the notations:

$$\begin{array}{c} b_n^{v1} = k^2\left(2J_n''(ka) + J_n(ka)\right) + a_n^{v1}k^2\left(2H_n''(ka) + H_n(ka)\right) \\ +b_n\mu K^2\left(2H_n''(Ka) + H_n(Ka)\right), \end{array} \tag{3.30}$$

$$\begin{array}{c} b_n^{v2} = \mu K^2\left(2J_n''(Ka) + J_n(Ka)\right) + a_n^{(3-v)2}\mu K^2\left(2H_n''(Ka) + H_n(Ka)\right) \\ +b_n k^2\left(2\mu H_n''(ka) - \lambda H_n(ka)\right), \end{array} \tag{3.31}$$

$$\begin{array}{c} b_n^{v3} = \frac{2\mu n}{a}\left(kJ_n'(ka) - \frac{J_n(ka)}{a}\right) + a_n^{v1}\frac{2\mu n}{a}\left(kH_n'(ka) - \frac{H_n(ka)}{a}\right) \\ +b_n\frac{2\mu n}{a}\left(KH_n'(Ka) - \frac{H_n(Ka)}{a}\right), \end{array} \tag{3.32}$$

$$\begin{array}{c} b_n^{v4} = \frac{2\mu n}{a}\left(KJ_n'(Ka) - \frac{J_n(Ka)}{a}\right) + a_n^{(3-v)2}\frac{2\mu n}{a}\left(KH_n'(ka) - \frac{H_n(Ka)}{a}\right) \\ +b_n\frac{2\mu n}{a}\left(kH_n'(ka) - \frac{H_n(ka)}{a}\right), \end{array} \tag{3.33}$$

then, we obtain:

$$TV_n^v(p) = \frac{i}{4\mu K^2} kaH_n'(Ka)\left[b_n^{\nu 1}P_n^v(\theta_p) + b_n^{\nu 3}Q_n^v(\theta_p)\right] - \frac{i}{4\mu K^2} nH_n(Ka)\left[b_n^{\nu 2}P_n^v(\theta_p) + b_n^{\nu 4}Q_n^v(\theta_p)\right], \tag{3.34}$$

$$TU_n^v(p) = \frac{i}{4\mu K^2} nH_n(ka)\left[b_n^{\nu 1}P_n^v(\theta_p) + b_n^{\nu 3}Q_n^v(\theta_p)\right] - \frac{i}{4\mu K^2} KaH_n'(Ka)\left[b_n^{\nu 2}P_n^v(\theta_p) + b_n^{\nu 4}Q_n^v(\theta_p)\right]. \tag{3.35}$$

In view of (3.24), (3.25), (3.34), (3.35), and (3.21) leads to the equations:

$$\left(\lambda - \frac{i}{4\mu K^2}\left[kaH_n'(ka)b_n^{\nu 1} + nH_n(Ka)b_n^{\nu 2}\right]\alpha_n^v\right) - \frac{i}{4\mu K^2}\left[nH_n(ka)b_n^{\nu 1} + KaH_n'(Ka)b_n^{\nu 2}\right]\beta_n^v = 0, \tag{3.36}$$

$$-\frac{i}{4\mu K^2}\left[kaH_n'(ka)b_n^{\nu 3} + nH_n(Ka)b_n^{\nu 4}\right]\alpha_n^v + \left(\lambda - \frac{i}{4\mu K^2}\left[nH_n(ka)b_n^{\nu 3} + KaH_n'(Ka)b_n^{\nu 4}\right]\beta_n^v\right) = 0. \tag{3.37}$$

If we use the following notations:

$$\begin{aligned} A_n^{\nu 1} &= \left[kaH_n'(ka)b_n^{\nu 1} + nH_n(Ka)b_n^{\nu 2}\right], \\ A_n^{\nu 2} &= \left[kaH_n'(ka)b_n^{\nu 3} + nH_n(Ka)b_n^{\nu 4}\right], \\ A_n^{\nu 3} &= \left[nH_n(ka)b_n^{\nu 1} + KaH_n'(Ka)b+_n^{\nu 2}\right], \\ A_n^{\nu 4} &= \left[nH_n(ka)b_n^{\nu 3} + KaH_n'(Ka)b_n^{\nu 4}\right]. \end{aligned}$$

Then (3.36) and (3.37) can be rewritten as follows:

$$\left(\lambda - \frac{i}{4\mu K^2}A_n^{\nu 1}\alpha_n^{\nu}\right) - \frac{i}{4\mu K^2}A_n^{\nu 3}\beta_n^{\nu} = 0, \tag{3.38}$$

$$-\frac{i}{4\mu K^2}A_n^{\nu 2}\alpha_n^{\nu} + \left(\lambda - \frac{i}{4\mu K^2}A_n^{\nu 4}\beta_n^{\nu}\right) = 0. \tag{3.39}$$

In order to get a non-trivial solution of the above system, its determinant must vanish. So the eigenvalues must satisfy:

$$\lambda^2 - \frac{i}{4\mu K^2}\left(A_n^{\nu 1} + A_n^{\nu 4}\right)\lambda + \left(\frac{i}{4\mu K^2}\right)^2\left(A_n^{\nu 1}A_n^{\nu 4} - A_n^{\nu 2}A_n^{\nu 3}\right) = 0. \tag{3.40}$$

Obviously, in order to minimize the condition number we have to choose the multipole coefficients, in such a way that all eigenvalues become equal to 1. Then the condition number is 1. From (3.40), using the same technique developed in [1], in order that all the eigenvalues be equal to 1, we obtain the relations:

$$\left(\frac{i}{4\mu K^2}\right)^2\left(A_n^{\nu 1}A_n^{\nu 4} - A_n^{\nu 2}A_n^{\nu 3}\right) = e^{2i\theta_n^{\nu}}, \tag{3.41}$$

$$-\frac{i}{4\mu K^2}\left(A_n^{\nu 1}+A_n^{\nu 4}\right)=\rho_n^{\nu}e^{i\theta_n^{\nu}}, \tag{3.42}$$

where ρ_n^{ν} and θ_n^{ν} are arbitrary real numbers satisfying the inequalities:

$$0\leq\rho_n^{\nu}\leq 2 \text{ and } 0\leq\theta_n^{\nu}\leq 2\pi.$$

Indeed, using (3.41) and (3.42), (3.40) becomes:

$$\lambda^2+\rho_n^{\nu}e^{i\theta_n^{\nu}}\lambda+e^{2i\theta_n^{\nu}}=0. \tag{3.43}$$

Which admits the solutions:

$$\lambda_{1,2}=\frac{-\left(\rho_n^{\nu}e^{i\theta_n^{\nu}}\right)\pm ie^{i\theta_n^{\nu}}\sqrt{4-\left(\rho_n^{\nu}\right)^2}}{2}.$$

Note here that we have : $|\lambda_1|=|\lambda_2|=1$.

Obviously there are infinitely many choices of multipole coefficients $a_m^{\sigma l}$ and b_m which satisfy the imposed conditions. If we choose $a_m^{\sigma l}$ and b_m as the coefficients which minimize the norm of the modified integral operator [10], after some calculations we obtain:

$$A_n^{\nu 1}=A_n^{\nu 4}=-4\mu K^2 \text{and } A_n^{\nu 2}=A_n^{\nu 3}=0. \tag{3.44}$$

For these values (3.44) and for $\rho_n^{\nu}=2$, $\theta_n^{\nu}=\pi$, we find that (3.40) has a double root $\lambda=1$. □

The above choice of the multipole coefficients does not satisfy the inequalities (2.7) imposed on the coefficients by the uniqueness theorem [3]. But as in [10], it has been proved that with this choice, the norm of the modified integral operator is equal to zero. So the boundary integral equation is uniquely solvable.

3.2 Case of the perturbation of the circle

As in [8] and [10], we can consider a family of non-circular boundaries defined parametrically by the relation:

$$R_{\varepsilon}=a+\varepsilon\varphi\left(\theta_p\right), \quad 0\leq\theta_p\leq\pi, \tag{3.45}$$

where φ and $\frac{\partial\varphi}{\partial\theta}$ are all bounded. We will use the estimates for the multipole vectors [10]:

$$F_m^{\sigma l}(p_{\varepsilon})=F_m^{\sigma l}(p_a)+O(\varepsilon). \tag{3.46}$$

$$TF_m^{\sigma l}(p_{\varepsilon})=TF_m^{\sigma l}(p_a)+O(\varepsilon), \tag{3.47}$$

where p_ε is a point in the perturbed circle, while p_a describe points on the circle of radius a. In [10] it has been proved that the boundary integral operator K_1^ε is a perturbation of the boundary integral operator K_1^a defined on the circle:

$$K_1^\varepsilon = K_1^a + O(\varepsilon).$$

In view of this estimates it is straightforward that the eigenvalues of the perturbed operator M_ε are perturbations of the eigenvalues of the original operator M_a. So:

$$Cond(M_\varepsilon) = Cond(M_a) + O(\varepsilon) = 1 + O(\varepsilon).$$

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The weight hierarchy of Ham(r, q)

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Abstract. Calculation of the weight hierarchy for codes, is an attractive and applicable topic in coding theory and cryptography. In this paper, we obtain the weight hierarchy of Hamming codes over F_q. As a result of weight hierarchy of Hamming codes, we compute the weight hierarchy of simplex codes.
Keywords: generalized Hamming weight, weight hierarchy, MDS code, simplex code.

1. Introduction

In this paper, we denote $\mathbb{F}_q$ as a field of order q. Let C be a $[n, k, d]$ linear code over F_q, with parity check matrix, H. For a subspace D of C, the *support* of D is denoted by *supp*(D) and is defined as follows

$$\mathrm{supp}(D) = \{i : \exists(v_1, v_2, \cdots, v_n) \in D; v_i \neq 0\}.$$

The $r-th$ *generalized Hamming wight* for code C, denoted by $d_r(C)$, is defined as follows

$$d_r = d_r(c) = \min\{\|D\| : D \subset C, \dim(D) = r\},$$

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where $\|D\| = |\text{supp}(D)|$. It is not difficult to see that $d_1(C) = d(C)$.

The concept of generalized Hamming weights (GHWs) were introduced by Wei [7] in 1990. In [5], Ozarow and Wyner had introduced a linear coding scheme on the wire-tap channel of Type II in connection with Cryptography. Wei [7] has shown that the generalized Hamming weights completely characterize the performance of a linear code when it is used on the above channel.

An $[n, k, d]$ linear code ,C, over F_q is called *Maximum Distance Separable*, (MDS), code if $d = n - k + 1$ and we call that C as $r-th$ *Maximum Distance Separable*,($r-$MDS), code when $d_r(C) = n - k + r$. Also, we say that C is a *proper* r *-MDS*(or P_r-MDS) code, where C is an r-MDS code and it is not an $(r-1)$-MDS code.

One may find known theorems about MDS and r- MDS codes in [3, 6].

Consider the $r \times (q^r - 1/q - 1)$ matrix H, where each column of H is a nonzero vector of each 1-dimensional subspace of F_q^r. The linear code which H is a parity check matrix, is called Hamming code and it is denoted by $Ham(r, q)$. For $q = 2$, the weight hierarchy of code was found by Wei [7]. In this paper we calculate the weight hierarchy for any q, where q is a power of a prime.

Suppose that C is a linear code over F_q. The *dual code* of C, denoted by $C^\perp$, is the orthogonal complement of C.

Remember that two vectors x and y are orthogonal when the inner product of them is equal to 0. Also, for a nonempty subset D of F_q^n, the orthogonal complement of D, denoted by $D^\perp$, is defined as follows $D^\perp = \{x \in F_q^n : x.y = 0, \forall y \in D\}$.

The dual of $q-$ary Hamming code$Ham(r, q)$ is called a $q-$array simplex code. We denote it by $S(r, q)$ (see [4]).

2. Main result

In this section, we give the generalized Hamming weights of Hamming codes over $\mathbb{F}_q$. At first we stat some properties of generalized Hamming weights which help us to computing the generalized Hamming weights of Hamming codes.

Theorem 2.1 ([7]). *Suppose that C is an $[n, k, d]-$ code over F_q with parity check matrix H. Then:*

(a) $\{d_r(C) : 1 \le r \le k\} = \{1, 2, ..., n\} \setminus \{n + 1 - d_r(C^\perp) : 1 \le r \le n - k\}$,

(b) $1 \le d_1(C) < d_2(C) < ... < d_k(C) \le n$,

(c) If H_i's are columns of H for $1 \le i \le n$, then

$$d_r(C) = \min\{|X| : |X| - rank(< H_i : i \in X >) = r\}.$$

Theorem 2.2 ([1]). *If C is a linear $[n, k, d]$ code over F_q, then*

$$d_r \le n - k + r, 1 \le r \le k.$$

The following theorem shows a relation between the columns of H and $d(C)$.

Theorem 2.3 ([4]). *Let C be a linear code and H denote its parity-check matrix. Then the following statements are equivalent:*

(i) $d_1(C) = d$;

(ii) Any $d-1$ columns of H are linearly independent and H has d linearly dependent columns.

Theorem 2.4 ([1]). *Let C be an $[n, k, d]$ code over F_q and $C^\perp$ be an $[n, n-k, d^\perp]$ code. Then, we have:*

(a) If C is an r- MDS code over F_q, then C is an r_1- MDS code for all $r \le r_1 \le k$,

(b) If $d^\perp = 1$, then C is not a P_r-MDS code for any $1 \le r \le k$,

(c) If $d^\perp > 1$, then C is a $P_{k-d^\perp+2}$-MDS code.

The following theorem states the properties of $Ham(r, q)$

Theorem 2.5 ([4]). *Suppose that $Ham(r, q)$ is the Hamming code over the field $\mathbb{F}_q$. Then:*

(i) $Ham(r, q)$ is a $[(q^r-1)/(q-1), (q^r-1)/(q-1)-r, 3]$- code.

(ii) $Ham(r, q)$ is a perfect exactly single-error-correcting code.

Now we compute the generalized Hamming weights of Hamming codes over the field $\mathbb{F}_q$.

Theorem 2.6. *Let C be the Hamming code over the field $\mathbb{F}_q$ and $d_l(C)$ be the l-th generalized Hamming weight of C. Then*

$$d_l(C) = \begin{cases} l+2 & 1 \le l \le q-1, \\ l+3 & (T_2 =) q \le l \le q^2+q-2 (= T_3 - 3), \\ l+4 & (T_3 - 2 =) q^2+q-1 \le l \le q^3+q^2+q-3 (= T_4 - 4) \\ \vdots & \\ l+r & T_{r-1} - (r-2) \le l \le T_r - r \end{cases}$$

where $T_i = (q^i - 1)/(q-1)$

Proof. From Theorem 2.5, we know that $Ham(r, q)$ is a $[(q^r-1)/(q-1), (q^r-1)/(q-1)-r, 3]$ code. Hence $d = d_1 = 3$. By Theorems 2.1(b) and 2.2, we have $4 \le d_2 \le r+2$. For computing these generalized Hamming weights, we use Theorem 2.1(c). Let H denote the parity check matrix of $Ham(r, q)$ and H_i denote the i-th column of H. Since $d = 3$, by Theorem 2.4, every two columns of H are independent. Let H_t, H_s be two independent columns of H. Consider two columns H_s and H_t. The subspace $< H_s, H_t >$ has dimension two, and has $(q^2-1)/(q-1) = q+1$ subspaces of dimension 1. Except H_s and H_t, we have $q-1$ subspaces of dimension 1 in $< H_s, H_t >$. We denote these subspaces by $H_1, H_2, \cdots H_{q-1}$. For any $S \subseteq \{H_1, H_2, \cdots, H_{q-1}\}$, $< H_s, H_t >= < H_s, H_t, S >$. Hence by Theorem 2.1 (c), we have

$$d_l = l+2, 1 \le l \le (q-1).$$

For d_q we should use three independent columns of H (Note that since H is a matrix of rank r, then H has r independent columns). By Theorems 2.1(b) and 2.2, we have $q+2 \leq d_q \leq r+q$. We claim that $d_q = q+3$. Consider three independent columns H_{i_1}, H_{i_2} and H_{i_3}. The space generated by these three columns has $(q^3-1)/(q-1) = (q^2+q+1)$ subspaces of dimension 1. Except H_{i_1}, H_{i_2} and H_{i_3}, we have q^2+q-2 subspaces of dimension 1 in $< H_{i_1}, H_{i_2}, H_{i_3} >$. Call them $H_1, H_2, \cdots, H_{q^2+q-2}$. For any $S \subseteq \{H_1, H_2, \cdots, H_{q^2+q-2}\}$, we have $< H_{i_1}, H_{i_2}, H_{i_3} >= < H_{i_1}, H_{i_2}, H_{i_3}, S >$. Hence by Theorem 2.1 (c), we conclude that

$$d_l = l+3, (((q^2-1)/(q-1))-2) = q \leq l \leq q^2+q+1 = (((q^3-1)/(q-1))-3).$$

For the rest of prove, we continued this process and the result is obtained. □

Corollary 2.1. *The weight hierarchy for $S(r,q)$ is as follows* $\{q^{r-1}, q^{r-1}+q^{r-2}, q^{r-1}+q^{r-2}+q^{r-3}, ..., q^{r-1}+q^{r-2}+q^{r-3}+...+q, q^{r-1}+q^{r-2}+q^{r-3}+...+q+1\}$

Proof. Use Theorems 2.1 and 2.6. □

Corollary 2.2. *The code $Ham(r,q)$ is an P_s- MDS code for $s \geq ((q^r-1)/q-1) - q^{r-1} - (r-2)$.*

Let C be a code over $\mathbb{F}_q$. The *extended code* of C, which is denoted by $\bar{C}$, is defined as follows

$$\bar{C} = \{(c_1, c_2, ..., c_n, -\sum_{i=1}^{n} c_i) : (c_1, c_2, \cdots, c_n) \in C\}.$$

The code $\bar{C}$ is an $(n+1, M, \bar{d})$ code over F_q and $d \leq \bar{d} \leq d+1$. If C is linear, then $\overline{C}$ is linear too and if H is parity check matrix of C, then $\overline{H}$ the parity check matrix for $\overline{C}$, is as follows

$$\bar{H} = \left[\begin{array}{cccc|c} & & H & & 0 \\ - & - & - & - & - \\ 1 & 1 & 1 & \cdots & 1 \end{array}\right].$$

In the following we find generalized hamming weights for extended code of Hamming code over $GF(3)$.

Let $C = Ham(m,3)$ and $\bar{C} = \overline{Ham(m,3)}$. Let d_r and $\bar{d_r}$ denote the $r-$th generalized Hamming weight for C and $\bar{C}$, respectively. Also, H and $\bar{H}$ denote the parity check matrices for C and $\bar{C}$, respectively. Let H_i and $\bar{H}_i$ be the $i-$th columns of C and $\bar{C}$, respectively.

We know that

$$\bar{H} = \left[\begin{array}{cccc|c} & & H & & 0 \\ - & - & - & - & - \\ 1 & 1 & 1 & \cdots & 1 \end{array}\right].$$

The construction of $\bar{H}$ shows that it contains columns $(1,1,1,0,...,0)$ and $(1,0,1,0,...,0)$ and $(1,2,1,0,...,0)$. It is clear the element $\bar{x}$ whose coordinates are 1 in the corresponding columns to these vectors and 0 in other columns, is belonging to $\bar{C}$. Note that $wt(\bar{x}) = 3$, so by theorem (2.5) we conclude that $\bar{d_1} = 3$.

Remember that $\overline{d_2} = \min\{|X| : |X| - rank(< \bar{H}_i : i \in X >) = 2\}$. It is impossible to use $rank(< \bar{H}_i : i \in X >) = 2$ to calculate $\bar{d_2}$. For this, suppose that there exists $X = \{\bar{H}_{i_1}, \bar{H}_{i_2}, \bar{H}_{i_3}, \bar{H}_{i_4}\}$ such that $rank(< \bar{H}_{i_1}, \bar{H}_{i_2} >) = 2$. Now, $\bar{H}_{i_3}$ is a linear combination of $\bar{H}_{i_1}, \bar{H}_{i_2}$. Hence there are c_1, c_2 belonging to F_3 such that $c_1\bar{H}_{i_1} + c_2\bar{H}_{i_2} = \bar{H}_{i_3}$. Regarding that all the coordinates in the last row of $\bar{H}$ is 1, we have $c_1 + c_2 = 1$. Hence $c_1 = 1$ or $c_2 = 1$ and in particular $\bar{H}_{i_3} = \bar{H}_{i_1}$ or $\bar{H}_{i_3} = \bar{H}_{i_2}$. Then $H_{i_3} = H_{i_1}$ or $H_{i_3} = H_{i_2}$. This is a contradiction to the definition of H. So, we should use $rank3$ to calculate $\bar{d_2}$. Hence $\bar{d_2} = 5$.

Suppose that $rank(< \bar{H}_{j_1}, \bar{H}_{j_2}, \bar{H}_{j_3} >) = 3$. Now, we can add columns to these columns without changing the rank of this subspace. To calculate the number of these columns, we should solve the the equation $c_1\bar{H}_{j_1} + c_2\bar{H}_{j_2} + c_3\bar{H}_{j_3} = \bar{H}_{j_4}$ in particular we should solve the equation $c_1 + c_2 + c_3 = 1$ in F_3. The set

$$\{(2,2,0),(1,1,2),(2,0,2),(0,2,2),(1,2,1),(2,1,1)\}$$

contains the whole solutions of this equation. Then we can use $rank3$ to calculate the higher weights till $\bar{d_6}$. It is clear that $\bar{d_6} = 9$.

For $\bar{d_7}$, we should use $rank4$ and solve the equation $c_1 + c_2 + c_3 + c_4 = 1$ in F_3. This equation has 23 solution and we can use $rank4$ to calculate $\bar{d_{23}}$. By continuing this process, we can find the other values of $\bar{d_r}$'s.

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1-factorization of small regular graphs

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Abstract. It is a well-known conjecture that if a regular simple graph G of order $2n$ has degree $\Delta(G)$ satisfying $\Delta(G) \geq n$, then G is 1-factorizable. By the colour exchange theory, Cariolaro [J. London Math. Soc., 77 (2007), 387-404] proved the validity of this conjecture for regular graphs of even order at most 10. In this paper, we shall provide a slightly simple proof of this result.
Keywords: 1-factorization, regular graph.

1. Introduction

Graphs considered in this paper are finite simple graphs. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and the maximum degree of G, respectively. A graph G is *regular* if $\delta(G) = \Delta(G)$. A cycle is a connected 2-regular graph. A n-cycle is a cycle on n vertices, denoted by C_n. $K^+_{m,n-m}$ denotes the simple graph obtained from the complete bipartite graph $K_{m,n-m}$ by adding an edge between two vertices of degree $n-m$. If $E_1 \subset E(G)$, by $G - E_1$ we shall denote the graph obtained from G by deleting all the edges in E_1. The operation $+$ used in this paper to express the disjoint union of two graphs and $E_G[H_1 + H_2]$ is used to denote the edge cut of $V(H_1)$ and $V(H_2)$ in G, where $H_1 + H_2$ is a spanning subgraph of G. More terminologies and notations not defined here are all from [1].

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A *k-factor* of G is a k-regular spanning subgraph of G, and a *k-factorization* of G is a set F of edge-disjoint k-factors $F_1, F_2, \cdots, F_n$ such that $G = F_1 \cup F_2 \cup \cdots \cup F_n$ and in which each k-factor F_i is called a *decomposition factor* of G, where $1 \leq i \leq n$. G is said to be *k-factorizable* if it has a k-factorization. In fact, a 1-*factor* of G is just a perfect matching of G. Many colouring problems ask for a decomposition of a regular graph of even order into disjoint 1-factors. We call such a decomposition a 1-factorization. A well-known, longstanding conjecture (which first appeared in a paper by Chetwynd and Hilton [2], but which may go back to G.A. Dirac in the early 1950s) is as follows.

Conjecture 1.1. (1-*Factorization Conjecture*) *A regular graph G of order $2n$ satisfying $\Delta(G) \geq n$ is 1-factorizable.*

This conjecture considered very hard, once solved would have an important impact on graph theory as well as other branches of mathematics. Many experts and scholars have devoted to this study, yet the vast majority of works existing in the literature have concentrated on proving the truth of Conjecture 1.1 for nearly complete graphs [2, 5, 11, 12] and for graphs of relatively large degree [2, 3, 6, 9]. It is worth mentioning that there was a breakthrough in 1-factorization by Perkovic and Reed [10], who proved (by probabilistic methods) that Conjecture 1.1 is asymptotically true and an approximate version of which was recently put forward by Csaba et.al [4]. These results provide some evidence in support of Conjecture 1.1. However, it occurs to us that all the known methods for proving the existence of a 1- factorization are very hard to follow.

Deciding whether a given graph is 1-factorizable is a intractable problem in general, which is known to be NP-complete [8]. The difficulty is how to determine the decomposition form of 1- factorization. An obvious, but useful, fact is that if we have a collection of edge-disjoint 1-factors of G, say $\{F_1 \cup \cdots \cup F_k\}$, then G is 1-factorizable if $G-(F_1\cup\cdots\cup F_k)$ is 1-factorizable, where $1 \leq k < \Delta(G)$. In many cases, however, even if $G-(F_1\cup\cdots\cup F_k)$ is not 1-factorizable, G is also 1-factorizable. This suggests that the pre-existing collection $\{F_1\cup\cdots\cup F_k\}$ must contain some 1-factor F_j which is not a decomposition factor of G, where $1 \leq j \leq k$. We take L_3 (see Figure 1) for example. Three vertical bars constitute a 1-factor (denoted by F_1), but not a decomposition factor of L_3, that's because $L_3 - F_1 = C_3 + C_3$ is not 1-factorizable. Motivated by this, we investigated 1-factorization of small regular graphs and proved Conjecture 1.1 holds for regular graphs of even order at most 10, as follows.

Theorem 1.2. *Let G be a regular graph of order $2n$ satisfying $\Delta(G) \geq n$, where $n \leq 5$. Then G is 1-factorizable.*

This result has been proved by Cariolaro [7] who present a new theory on colour exchange and apply it to the class of regular graphs of even order at most 10. In this paper, we are aim to provide a slightly simple proof.

2. Preliminaries

This section we shall establish three useful theorems which are all from the book by Bondy and Murty [1]. The first one is a well-known theorem of Dirac.

Theorem 2.1. *Let G be a graph with $|V(G)| \geq 3$ and $\delta(G) \geq |V(G)|/2$. Then G is a hamiltonian graph.*

It is known that every k-regular bipartite graph with $k > 0$ has a perfect matching. In fact, we can further get the following stronger conclusion which is an exercise of chapter 5 in the book by Bondy and Murty [1].

Theorem 2.2. *For $k > 0$, each of the following holds.*
(1) *every k-regular bipartite graph is 1-factorizable.*
(2) *every $2k$-regular graph is 2-factorizable.*

Proof. (1) Let G be a k-regular bipartite graph. Then G has a perfect matching, say M_1. We consider the graph $G - M_1$ which is a $(k-1)$-regular bipartite graph. Let M_2 be a perfect matching of $G - M_1$. Next we consider the graph $G - M_1 - M_2$. And so it goes on, we get finally k edge-disjoint 1-factors $M_1, M_2, \cdots, M_k$ and $G = M_1 \cup M_2 \cup \cdots \cup M_k$. So G is 1-factorizable.

(2) Let G be a $2k$-regular graph and $V(G) = \{v_1, v_2, \cdots, v_v\}$. Then G admits an Euler tour C. Next we construct a simple bipartite graph G^* from C with bipartition (X, Y). Let $X = \{x_1, x_2, \cdots, x_v\}$ and $Y = \{y_1, y_2, \cdots, y_v\}$. For $i, j \in \{1, 2, \cdots, v\}$, $x_i y_j \in E(G^*)$ if and only if $v_i v_j \in E(C)$ and $v_i v_j$ is oriented from v_i to v_j. For any $u \in V(C)$, let $E^+(u)$ and $E^-(u)$ denote the edges with tail and head at u, respectively. Then $|E^+(u)| = |E^-(u)|$ since C is an Euler tour. Thus G^* is a k-regular bipartite graph. By (1), G^* is 1-factorizable. Note that each 1-factor of G^* is corresponding to a 2-factor of G. Thus G is 2-factorizable. □

Theorem 2.2 shows that every 2-regular graph without odd cycles is 1-factorizable and every 4-regular graph is 2-factorizable. For a 3-regular graph, there is an important result as follows.

Theorem 2.3. *Every 3-regular graph without cut edges has a perfect matching.*

3. Proof of Theorem 1.2

Let G be a regular graph of order $2n$ satisfying $\Delta(G) \geq n$, where $n \leq 5$. First of all, we make a significant claim.

Claim 1 G is a $\Delta(G)$-edge connected regular graph.

Proof of Claim 1. Let X be a nontrivial edge cut of G. We shall prove that $|X| \geq \Delta(G)$. Let G_1, G_2 be two nontrivial components of $G - X$ and $|V(G_1)| \leq |V(G_2)|$. Then $|V(G_1)| \leq n$ and $|X| = \Delta(G)|V(G_1)| - 2|E(G_1)|$. It follows that $|X| \geq \Delta(G)|V(G_1)| - |V(G_1)|(|V(G_1)| - 1)$. If $|X| < \Delta(G)$, then

$\Delta(G)|V(G_1)| - |V(G_1)|(|V(G_1)| - 1) < \Delta(G)$. It follows that $\Delta(G)(|V(G_1)| - 1) < |V(G_1)|(|V(G_1)| - 1)$. Note that $|V(G_1)| > 1$. Thus we get that $|V(G_1)| > \Delta(G) \geq n$, a contradiction. So $|X| \geq \Delta(G)$. This shows that G is a $\Delta(G)$-edge connected regular graph.

Next we shall handle each case of $n \leq 5$, respectively. It is trivil for $n = 1$. When $n \geq 2$, by Theorem 2.1, G is a hamiltonian graph. By the definition, G contains a hamiltonian cycle of order $2n$ as a spanning subgraph. So G must have two edge-disjoint perfect matchings. Let M be a perfect matching of G. If $\Delta(G - M) \geq n$ and $G - M$ is 1-factorizable, then G is also 1-factorizable. Hence it is sufficient to prove that G is 1-factorizable if G is a regular graph of order $2n$ satisfying $\Delta(G) = n$.

For $n = 2$, $|V(G)| = 4$ and $\Delta(G) = 2$. Then G is C_4 which is 1-factorizable.

For $n = 3$, $|V(G)| = 6$ and $\Delta(G) = 3$. If G is a 3-regular bipartite graph, by Theorem 2.2(1), G is 1-factorizable. Next we suppose that G contains some odd cycles. By Claim 1, G has no cut edges. Hence it follows that G has a perfect matching by Theorem 2.3. Let M be a perfect matching of G. Then $G - M = C_3 + C_3$. Thus G is isomorphic to the graph L_3 as shown in Figure 1. For the sake of narrative convenience, we use bold lines to mark a perfect matching of L_3 (the same below). It is easy to find that deleting the perfect matching we can get an even cycle C_6 which is 1-factorizable. Hence G is 1-factorizable.



L_3 H_1

Figure 1. Two graphs with odd cycles

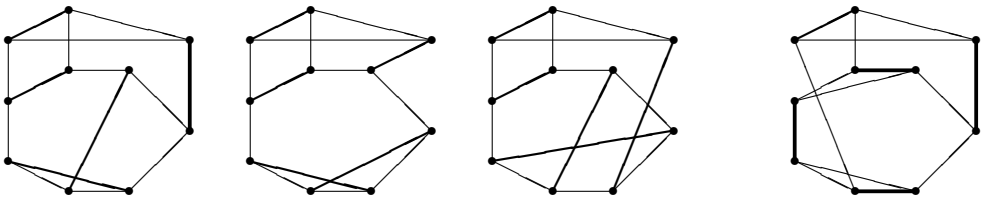
For $n = 4$, $|V(G)| = 8$ and $\Delta(G) = 4$. Let M_1 and M_2 be two perfect matchings of G and let $H = G - M_1 - M_2$. Then H is a 2-regular graph. If H is 1-factorizable, then G is also 1-factorizable, we are done. Next we suppose that H is not 1-factorizable. Then H is $C_3 + C_5$. This indicates that one of two perfect matchings, say M_1, is not a decomposition factor of G. Let $H_1 = H \cup M_1 = G - M_2$. Then H_1 is a 3-regular graph as shown in Figure 1. As above, it is easy to prove that H_1 is 1-factorizable. It follows that G is also 1-factorizable.

For $n = 5$, $|V(G)| = 10$ and $\Delta(G) = 5$. Let $M_1^{'}$ and $M_2^{'}$ be two perfect matchings of G and let $H^{'} = G - M_1^{'} - M_2^{'}$. Then $H^{'}$ is a 3-regular graph. If $H^{'}$ is 1-factorizable, then G is also 1-factorizable, we are done. Next we suppose that $H^{'}$ is not 1-factorizable.

Claim 2. $H^{'}$ has a perfect matching.

Proof of Claim 2. If $H^{'}$ has no cut edges, then by Theorem 2.3, $H^{'}$ has a perfect matching. Next we suppose that $H^{'}$ contains a cut edge e. Let R_1 and R_2 be two components of $H^{'} - e$. Note that $H^{'}$ is a 3-regular graph, there must be $|V(R_1)| = |V(R_2)| = 5$ and $R_1 \cong R_2 \cong K_{2,3}^{+}$. Let $e = uv$ and $u \in V(R_1)$, $v \in V(R_2)$. Note that $R_1 - u$ has a perfect matching $M_1^{''}$ and $R_2 - v$ has a perfect matching $M_2^{''}$. So $M_1^{''} \cup \{uv\} \cup M_2^{''}$ is a perfect matching of $H^{'}$.

Let $M_3^{'}$ be a perfect matching of $H^{'}$ and let $H^{''} = H^{'} - M_3^{'}$. Then $H^{''}$ is a 2-regular graph and contains just two odd cycles. For otherwise, $H^{'}$ is 1-factorizable, a contradiction. Thereby $H^{''}$ maybe $C_3 + C_7$, $C_3 + C_3 + C_4$ or $C_5 + C_5$.

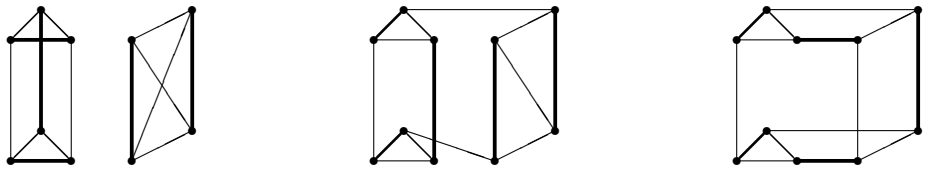


(a) (b) (c) (d)

Figure 2. Four types of $H^{'}$ with $H^{''} = C_3 + C_7$

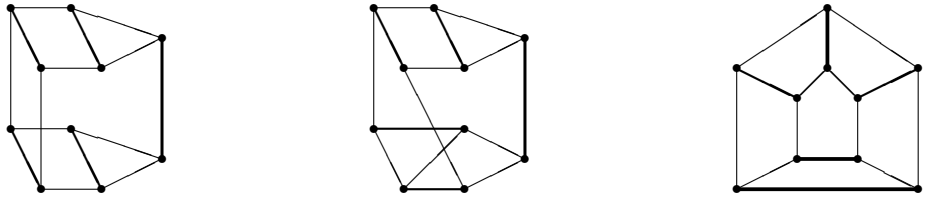
Claim 3 $H^{''} = C_5 + C_5$, and $H^{'}$ has one cut edge or $H^{'}$ is a Peterson graph (denoted by P_{10}).

Proof of Claim 3. If $H^{''} = C_3 + C_7$, then $H^{'}$ is isomorphic to one of the four graphs depicted in Figure 2. If $H^{''} = C_3 + C_3 + C_4$, then $H^{'}$ is isomorphic to one of the three graphs depicted in Figure 3. In both cases, we can prove that $H^{'}$ is 1-factorizable the same as above mentioned, a contradiction. So $H^{''} = C_5 + C_5$. If $H^{'}$ has no cut edge, then $|E_{H'}[C_5 + C_5]| = 3$ or 5. We list all possible cases of $H^{'}$ in Figure 4. Similarly, we can check that each graph in Figure 4 except for the Peterson graph is 1-factorizable. Hence we assume that $H^{'}$ has one cut edge or $H^{'}$ is a Peterson graph.

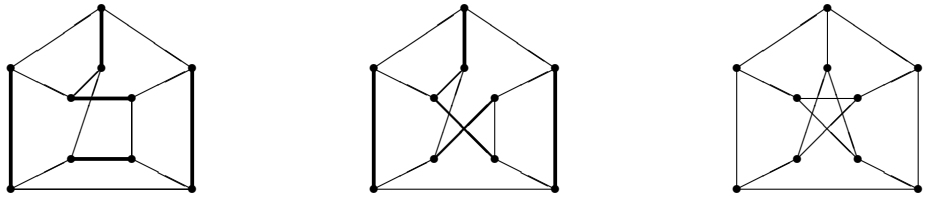


(a) (b) (c)

Figure 3. Three types of $H^{'}$ with $H^{''} = C_3 + C_3 + C_4$



(a) (b) (c)

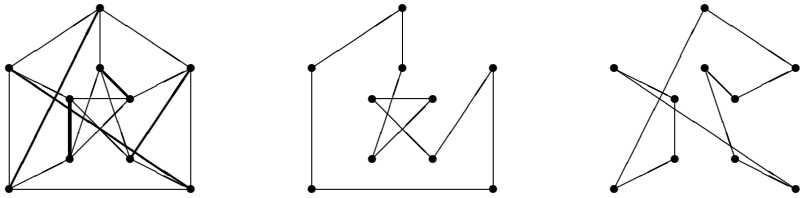


(d) (e) P_{10}

Figure 4. Six types of $H^{'}$ with $H^{''} = C_5 + C_5$

Finally, we consider the graph $H_i^{'} = H^{''} \cup M_i^{'}$ which is also a 3-regular graph, where $i \in \{1, 2\}$. Note that $G = H_1^{'} \cup M_2^{'} \cup M_3^{'}$ or $G = H_2^{'} \cup M_1^{'} \cup M_3^{'}$. If $H_1^{'}$ or $H_2^{'}$ is 1-factorizable, we are done. Therefore we suppose that both $H_1^{'}$ and $H_2^{'}$ are not 1-factorizable. As discussed in Claim 3, $H_i^{'}$ has one cut edge or $H_i^{'}$ is a Peterson graph. However, it is impossible that each of the three graphs $H^{'}$, $H_1^{'}$ and $H_2^{'}$ has one cut edge, that's because $|E_G[C_5 + C_5]| = 3$, contrary to Claim 1. We assume, without loss of generality, that $H^{'}$ is a Peterson graph and each of

$H_1^{'}$ and $H_2^{'}$ has one cut edge. Let $G^{'} = H^{'} \cup M_1^{'}$. Then $G^{'}$ is a 4-regular graph which is 2-factorizable follows from Theorem 2.2(2). Figure 5 illustrates $G^{'}$ (the bold lines form into $M_1^{'}$) and its two edge-disjoint 2-factors F_1 and F_2. It is clear that both F_1 and F_2 are 1-factorizable. It follows that $G^{'}$ is 1-factorizable. Note that $G = G^{'} \cup M_2^{'}$. It is obvious that G is also 1-factorizable. This completes our proof.



$G^{'} = H^{'} \cup M_1^{'}$ F_1 F_2

Figure 5. Graph $G^{'}$ and its two edge-disjoint 2-factors

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A new class of harmonic univalent functions associated with q-derivative defined by Hadamard product

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Abstract. In this paper, we introduce a class of harmonic univalent functions associated with q-derivative defined by Hadamard product. The object of the present paper is to determine coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class.
Keywords: harmonic univalent functions, q-derivative, extreme points.

1. Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [11]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.2}$$

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In [11] Clunie and Shell-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. Denote by V_H the subclass of S_H consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n, g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \qquad |b_1| < 1. \tag{1.3}$$

and $f(z)$ is the given by

$$f = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n}, \qquad |b_1| < 1. \tag{1.4}$$

For $0 < q < 1$, the q-derivative of a function $f \in$A is defined by (see [6], [7], [9], [16], [18], [25] and [26])

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & (z \neq 0) \\ f'(0), & (z = 0) \end{cases} \tag{1.5}$$

and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.5), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}, \tag{1.6}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (0 < q < 1). \tag{1.7}$$

As $q \longrightarrow 1^-$, $[n]_q \longrightarrow n$. For a function $h(z) = z^n$, we obtain

$$D_q(h(z)) = D_q(z^n) = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q \, z^{n-1},$$

$$\lim_{q \longrightarrow 1^-} (D_q(h(z))) = \lim_{q \longrightarrow 1^-} \left([n]_q \, z^{n-1} \right) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative. For functions $f(z) \in V_H$ given by given by (1.4) and $F \in V_H$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} \overline{|B_n| z^n}, \qquad A_n, B_n \geq 0, \tag{1.8}$$

we recall the Hadamard product (or convolution) of f and F by

$$\begin{aligned} (f * F)(z) &= (h * H)(z) + \overline{(g * G)(z)} \\ &= z + \sum_{n=2}^{\infty} |a_n| \, |A_n| z^n + \sum_{n=1}^{\infty} \overline{|b_n| \, |B_n| z^n}. \end{aligned} \tag{1.9}$$

For $0 < q < 1, 1 < \gamma \leq 2, 0 \leq \lambda \leq 1, A_n, B_n \geq 0$ and for all $z \in U$, let $\overline{S_{H,q}(F,\lambda,\gamma)}$ denote the family of harmonic functions $(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}$, where h and g are given by (1.2), H and G are given by (1.9) and satisfying the analytic criterion

$$\operatorname{Re}\left\{(1-\lambda)\frac{(f*F)(z)}{z} + \lambda D_q(f*F)(z)\right\} < \gamma. \tag{1.10}$$

Equivalently

$$\operatorname{Re}\left\{(1-\lambda)\frac{(h*H)(z) + \overline{(g*G)(z)}}{z} + \lambda\left[D_q(h*H)(z) + \overline{D_q(g*G)(z)}\right]\right\} < \gamma. \tag{1.11}$$

Let $\overline{S}_{H,q}(F,\lambda,\gamma)$ be the subclass of $S_{H,q}(F,\lambda,\gamma)$ consisting of functions $(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}$ given by (1.9).

We note that for suitable choices of F and λ we obtain the following subclasses:
(1) If we take $\lambda = 0$ and $F(z) = z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n + \sum_{n=1}^{\infty} \overline{\Psi_n(\alpha_1) z^n}$, where $\Psi_n(\alpha_1)$ is given by

$$\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1}\ldots(\alpha r)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_s)_{n-1}\ (n-1)!} \tag{1.12}$$

$(\alpha_i > 0, i = 1,\ldots,r, \beta_j, j = 1,\ldots,s; r \leq s+1; s, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,\ldots\},\ z \in U)$, the class $\lim_{q\longrightarrow 1^-} \overline{S}_H(z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n + \sum_{n=1}^{\infty} \overline{\Psi_n(\alpha_1) z^n}, 0, \gamma)$, where $\Psi_n(\alpha_1)$ is given by (1.12), reduces to the class $\overline{S}_{H_{r,s}}([\alpha_1]; \gamma)$ (see [8]).
(2) If we take $\lambda = 0$ and

$$F(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n}, (\eta \in \mathbb{N}_0, \mu \geq 0, l \geq 0),$$

the class $\lim_{q\longrightarrow 1^-} \overline{S_H}(z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n}, \lambda, \gamma)$ the class reduces to reduces to the class $\overline{SI^{\eta}}(\mu, l; \gamma)$ (see [21]).
(3) If we take $\lambda = 0$ and

$$F(z) = z + \sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta} z^n}, (0 \leq \mu < 1, \eta \in \mathbb{N}_0),$$

the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}(z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta}z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta}z^n},\gamma)$$

reduces to the class $\overline{R}_H(\eta,\mu,\gamma)$ (see [24]).
(4) If we take $\lambda=1$ and $F(z)=\frac{z}{1-z}+\overline{\frac{z}{1-z}}$, the class $\lim_{q\longrightarrow 1^-}\overline{S}_H(\frac{z}{1-z}+\overline{\frac{z}{1-z}},1,\gamma)$ reduces to the class $\overline{S}_H(\gamma)$ (see [13]).
(5) If we take $F(z)=z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)z^n}$, where $\Upsilon_n^m(\alpha_1,l,\mu)$ is given by

$$\Upsilon_n^m(\alpha_1,l,\mu)=\left[\frac{1+l+\mu(n-1)}{l+1}\Psi_n(\alpha_1)\right]^m, m\in\mathbb{N}_0,\mu\geq 0,l\geq 0 \tag{1.13}$$

and $\Psi_n(\alpha_1)$ is given by (1.12), the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}\left(z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)z^n+\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)z^n},\lambda,\gamma\right)$$

reduces to the class $\overline{R}_{H_{r,s}}([\alpha_1],\eta,\mu,l,\gamma)=$

$$\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{D_{\mu,l}^{m,r,s}f(z)}{z}+\lambda\left(D_{\mu,l}^{m,r,s}f(z)\right)'\right)<\gamma,\right.$$
$$\left.\mu,l\geq 0;0\leq\lambda\leq 1;r,s,m\in\mathbb{N}_0;1<\gamma\leq 2;z\in U\right\},$$

where $D_{\mu,l}^{m,r,s}f(z)$ is the modified extended multiplier Dziok-Srivastava operator, defined as follows:

$$\begin{aligned}I(\eta,\mu,l)f(z) &= I^{\eta}(\mu,l)h(z)+I^{\eta}(\mu,l)\overline{g(z)}\\ &= z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)|a_n|z^n+\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)|b_n|z^n}\end{aligned}$$

and the operator $D_{\mu,l}^{m,r,s}$ was introduced and studied by El-Ashwah et al. (see [15]).
(6) If we take

$$F(z)=z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n},$$

$(0\leq\mu<1,m\in\mathbb{N}_0)$, the class $\lim_{q\longrightarrow 1^-}\overline{S_H}(z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n},\lambda,\gamma)$ reduces to the class $\overline{R}_H(m,\mu,\lambda,\gamma)=$

$$\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{\Omega^m f(z)}{z}+\lambda\left(\Omega^m f(z)\right)'\right)<\gamma,\right.$$
$$\left.0\leq\mu<1;0\leq\lambda\leq 1;1<\gamma\leq 2;m\in\mathbb{N}_0;z\in U\right\},$$

the operator Ω^m was introduced and studied by Dixit and Porwal (see [14]).
(7) If we take

$$F(z)=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]^{\eta}C(n,\delta)z^n+(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1+\mu(n-1)]^{\eta}C(n,\delta)z^n},$$

where $C(n,\delta)=\binom{n+\delta+1}{\delta}$, $\mu>0,\eta,\delta\in\mathbb{N}_0$, the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}\left(z+\sum_{n=2}^{\infty}[1+\mu(n-1)]^{\eta}C(n,\delta)z^n+(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1+\mu(n-1)]^{\eta}C(n,\delta)z^n},\lambda,\gamma\right)$$

reduces to the class $\overline{S}_H(\mu,\eta,\delta,\lambda,\gamma)=$

$$\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{D(\eta,\delta,\mu)f(z)}{z}+\lambda\left(D(\eta,\delta,\mu)f(z)\right)'\right)<\gamma\right.$$
$$\left.\mu>0;0\le\lambda\le 1;\eta,\delta\in\mathbb{N}_0;1<\gamma\le 2;z\in U\right\},$$

where the operator $D(\eta,\delta,\mu)f(z)$ is defined as follows (see [12]):

$$\begin{aligned}D(\eta,\delta,\mu)f(z)&=D^{\eta}_{\delta,\mu}h(z)+D^{\eta}_{\delta,\mu}\overline{g(z)}\\&=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]^{\eta}C(n,\delta)|a_n|z^n\\&\quad+(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1+\mu(n-1)]^{\eta}C(n,\delta)|b_n|z^n},\end{aligned}$$

the operator $D^{\eta}_{\delta,\mu}$ was introduced and studied by Al-Shaqsi and Darus (see [5]).

Also we note that:
(1) If we take $\lambda=1$, the class $\overline{S}_{H,q}(F,1,\gamma)$ reduces to the class $\overline{S}_{H,q}(F,\gamma)=\{f\in\overline{S_H}:\operatorname{Re}(D_q(f*F)(z))<\gamma,0\le q<1,1<\gamma\le 2;z\in U\}$.
(2) If we take $F(z)=\frac{z}{1-z}+\overline{\frac{z}{1-z}}$, the class $\overline{S}_H(\frac{z}{1-z}+\overline{\frac{z}{1-z}},\lambda,\gamma)$ reduces to the class $\overline{S}_{H,q}(\lambda,\gamma)=\{f\in\overline{S_H}:\operatorname{Re}((1-\lambda)\frac{f(z)}{z}+\lambda D_qf(z))<\gamma,0<q<1,0\le\lambda\le 1;1<\gamma\le 2;z\in U\}$.
(3) If we take $F(z)=z+\sum_{n=2}^{\infty}\Gamma(\alpha_1,\beta_1,q,n)z^n+\sum_{n=1}^{\infty}\overline{\Gamma(\alpha_1,\beta_1,q,n)z^n}$, where $\Gamma(\alpha_1,\beta_1,q,n)$ is given by

$$\Gamma(\alpha_1,\beta_1,q,n)=\frac{(\alpha_1,q)_{n-1}\dots(\alpha_r,q)_{n-1}}{(q,q)_{n-1}(\beta_1,q)_{n-1}\dots(\beta_s,q)_{n-1}}\tag{1.14}$$

($\alpha_i>0,i=1,\dots,r$, $\beta_j,j=1,\dots,s$; $r=s+1;s,r\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,\dots\}$; $|q|<1,z\in U$), the class $\overline{S}_H(z+\sum_{n=2}^{\infty}\Gamma(\alpha_1,\beta_1,q,n)z^n+$

$\sum_{n=1}^{\infty}\overline{\Gamma(\alpha_1,\beta_1,q,n)z^n}$, $0,\gamma)$, where $\Gamma(\alpha_1,\beta_1,q,n)$ is given by (1.14), reduces to the class

$$\begin{aligned}&\overline{R}_H(q,\alpha_1,\beta_1,\lambda,\gamma)\\&=\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{H_s^r[\alpha_1,\beta_1,q]f(z)}{z}+\lambda D_q\left(H_s^r[\alpha_1,\beta_1,q]f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1;\alpha_i,\beta_j>0;0\le\lambda\le1;1<\gamma\le2;s,r\in\mathbb{N}_0;z\in U\right\},\end{aligned}$$

where $H_s^r[\alpha_1,\beta_1,q]f(z) = H_s^r(\alpha_1,\dots\alpha_r;\beta_1\dots\beta_s;q)f(z)$ and the operator $H_s^r[\alpha_1,\beta_1,q]$ was introduced and studied by Mohammed and Darus (see [20] and [1]), and $\lim_{q\longrightarrow1^-}H_s^r[\alpha_1,\beta_1,q]$ reduces to the operator $H_{q,s}(\alpha_1,\beta_1)$ is the modified Dziok-Srivastava operator (see [3] and [4]).

(4) If we take $F(z)=z+\sum_{n=2}^{\infty}\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n+\sum_{n=1}^{\infty}\overline{\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n}$ $(\eta>-1,)$, the class $\overline{S}_H(z+\sum_{n=2}^{\infty}\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n+\sum_{n=1}^{\infty}\overline{\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n},\lambda,\gamma)$ reduces to the class

$$\begin{aligned}\overline{S}_{H_q}(\eta,\lambda,\gamma)=&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{R_q^\eta f(z)}{z}+\lambda D_q\left(R_q^\eta f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1;\eta>-1;0\le\lambda\le1;1<\gamma\le2;z\in U\right\},\end{aligned}$$

the operator R_q^η was introduced and studied by Aldweby and Darus (see [2]),

(5) If we take $F(z)=z+\sum_{n=2}^{\infty}[n]_q^m\,z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[n]_q^m\,z^n}$ $(0<q<1,m\in\mathbb{N}_0)$, the class $\overline{S_H}(z+\sum_{n=2}^{\infty}[n]_q^m\,z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[n]_q^m\,z^n},m,\lambda,\gamma)$ reduces to the class

$$\begin{aligned}\overline{R}_H(q,m,\lambda,\gamma)=&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{D_q^m f(z)}{z}+\lambda D_q\left(D_q^m f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1,0\le\lambda\le1;1<\gamma\le2;m\in\mathbb{N}_0;z\in U\right\},\end{aligned}$$

the operator $D_q^m f(z)$ was introduced and studied by Govindaraj and Sivasubramanian [17] and Murugusundaramoorthy and Vijaya [23] (see also Jahangiri [19]).

(6) If we take $F(z)=z+\sum_{n=2}^{\infty}[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n}$ $(0<q<1,l,\mu\ge0,\eta\in\mathbb{N}_0)$, the class $\overline{S_H}(z+\sum_{n=2}^{\infty}[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n},\mu,\eta,\lambda,\gamma)$ reduces to the class $\overline{R}_H(q,l,\mu,\eta,\lambda,\gamma)=$

$$\begin{aligned}&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{I_q^\eta(\mu,l)f(z)}{z}+\lambda D_q\left(I_q^\eta(\mu,l)f(z)\right)\right)<\gamma\right.\\&\left.0<q<1;\eta\in\mathbb{N}_0;\mu,l\ge0;0\le\lambda\le1;1<\gamma\le2;z\in U\right\},\end{aligned}$$

where the operator $I_q^\eta(\mu,l)$ was introduced and studied by Wongsaijai and Sukantamala (see [28]) and $\lim_{q\longrightarrow1^-}I_q^\eta(\mu,l)$ reduces to the operator $I(\eta,\mu,l)$,

where $I(\eta,\mu,l)f(z)$ is the modified Catas operator (see [21]),

$$\begin{aligned} I(\eta,\mu,l)f(z) &= I^{\eta}(\mu,l)h(z)+\overline{I^{\eta}(\mu,l)g(z)} \\ &= z+\sum_{n=2}^{\infty}\left[\frac{1+l+\mu\left([n]_q-1\right)}{l+1}\right]^{\eta}|a_n|\,z^n \\ &\quad +(-1)^m\overline{\sum_{n=1}^{\infty}\left[\frac{1+l+\mu\left([n]_q-1\right)}{l+1}\right]^{\eta}|b_n|\,z^n}, \end{aligned}$$

where $I^{\eta}(\mu,l)$ is the extended multiplier transformation (see [10]).
(7) If we take $F(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n}$ $(0<q<1,\mu<2)$, the class

$$\overline{S_H}(z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n},\lambda,\gamma)$$

reduces to the class

$$\overline{R}_H(q,\mu,\lambda,\gamma)=\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{\Omega^{\mu}_{q,z}f(z)}{z}+\lambda D_q\left(\Omega^{\mu}_{q,z}f(z)\right)\right)<\gamma,\right.$$
$$\left.0<q<1;\mu<2;0\le\lambda\le1;1<\gamma\le2;z\in U\right\},$$

the operator $\Omega^m_{q,z}$ was introduced and studied by Murugusuurdaramoorthy et al. (see [22]) and $\lim_{q\longrightarrow1^-}\Omega^m_{q,z}f(z)$ reduces to the operator $\Omega^m_z f(z)$, where the operator $\Omega^m_z f(z)$ was introduced and studied by Owa and Srivastava, (see [27]).

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $1<\gamma\le2, 0\le\lambda\le1, A_n, B_n\ge0$ and $z\in U$.

Theorem 1. *Let $f=h+\overline{g}$ be such that $h(z)$ and $g(z)$ given by (1.2). Furthermore, let*

$$(2.1)\quad \sum_{n=2}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|A_n|\,|a_n|+\sum_{n=1}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|B_n|\,|b_n|\le\gamma-1.$$

where $a_1=1$ and $n(\gamma-1)\le[\lambda([n]_q-1)+1]|A_n|\le[\lambda([n]_q-1)+1]|B_n|$ for $n\ge2$, then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z)\in S_{H,q}(F,\lambda,\gamma)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{n=1}^{\infty} b_n\left(z_1^n-z_2^n\right)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n\left(z_1^n-z_2^n\right)}\right| \\ &> 1-\frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} \\ &\geq 1-\frac{\sum_{n=1}^{\infty} \frac{[\lambda([n]_q-1)+1]|B_n|}{\gamma-1}|b_n|}{1-\sum_{n=2}^{\infty} \frac{[\lambda([n]_q-1)+1]|A_n|}{\gamma-1}|a_n|} \geq 0,\end{aligned}$$

which proves univalence. Note that $f(z)$ is sense-preserving in U. This is because

$$\begin{aligned}\left|h'(z)\right| &\geq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\ &> 1-\sum_{n=2}^{\infty} n|a_n| \geq 1-\sum_{n=2}^{\infty} \frac{[\lambda(n-1)+1]|A_n|}{\gamma-1}|a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{[\lambda(n-1)+1]|B_n|}{\gamma-1}|b_n| \geq \sum_{n=1}^{\infty} n|b_n| \\ &> \sum_{n=1}^{\infty} n|b_n|\left|z^{n-1}\right| \geq \left|g'(z)\right|.\end{aligned}$$

Now, we will show that $f(z) \in S_H(F,\lambda,\gamma)$. We only need to show that if (2.1) holds then the condition (1.7) is satisfied. Using the fact that $Re\{w\} < \gamma$ if and only if $|w-1| < |w-(2\gamma-1)|$, it suffices to show that

$$\left|\frac{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q(h*H)(z)+\overline{D_q(g*G)(z)}\right]-1}{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q(h*H)(z)+\overline{D_q(g*G)(z)}\right]-(2\gamma-1)}\right| < 1.$$

We have

$$\begin{aligned}&\left|\frac{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda[D_q(h*H)(z)+\overline{D_q(g*G)(z)}]-1}{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda[D_q(h*H)(z)+\overline{D_q(g*G)(z)}]-(2\gamma-1)}\right| \\ &= \left|\frac{\sum_{n=2}^{\infty}[\lambda(n-1)+1]A_n a_n z^{n-1}+\sum_{n=1}^{\infty}[\lambda(n-1)+1]B_n\overline{b_n z^{n-1}}}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda(n-1)+1]A_n a_n z^{n-1}+\sum_{n=1}^{\infty}[\lambda(n-1)+1]B_n\overline{b_n z^{n-1}}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z|^{n-1}+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n||z|^{n-1}}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z|^{n-1}-\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n||z|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|-\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|},\end{aligned}$$

which is bounded above by 1 by using (2.1). This completes the proof of Theorem 1. The harmonic univalent functions of the form

$$f(z) = z+\sum_{n=2}^{\infty}\frac{\gamma-1}{[\lambda([n]_q-1)+1]|A_n|}x_n z^n+\sum_{n=1}^{\infty}\frac{\gamma-1}{[\lambda([n]_q-1)+1]|B_n|}\overline{y_n z^n}, \tag{2.2}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$, show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class $S_{H,q}(F,\lambda,\gamma)$ for all $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|\leq 1$ because coefficient inequality (2.1) holds.

Theorem 2. *A function $f(z)$ of the form* (1.3) *is in the class $\overline{S}_{H,q}(F,\lambda,\gamma)$ if and only if*

$$(2.3)\quad \sum_{n=2}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|A_n|\,|a_n|+\sum_{n=1}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|B_n|\,|b_n|\leq\gamma-1.$$

where $a_1=1$ and $n(\gamma-1)\leq[\lambda([n]_q-1)+1]|A_n|\leq[\lambda([n]_q-1)+1]|B_n|$ for $n\geq 2$.

Proof. Since $\overline{S}_{H,q}(F,\lambda,\gamma)\subset S_{H,q}(F,\lambda,\gamma)$, we only need to prove the "only if" part of this theorem. To this end , for functions $f(z)$ of the form (1.3), we notice that the condition

$$\operatorname{Re}\left\{(1-\lambda)\,\tfrac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q\left(h*H\right)(z)+D_q\overline{(g*G)\,(z)}\right]\right\}<\gamma,$$

is equivalent to

$$\operatorname{Re}\left\{1+\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|z^{n-1}+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|\overline{z^{n-1}}\right\}$$
$$\leq 1+\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z^{n-1}|+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|\overline{|z^{n-1}|}<\gamma.$$

Letting $z\to 1^-$ along the real axis, we obtain the inequality (2.3). This completes the proof of Theorem 2.

3. Distortion theorem

Theorem 3. *Let the function $f(z)$ defined by* (1.3) *belongs to the class $\overline{S}_{H,q}(F,\lambda,\gamma)$* and $(\lambda q+1)|A_2|\leq[\lambda([n]_q-1)+1]|A_n|\leq[\lambda([n]_q-1)+1]|B_n|$ for $n\geq 2$. *Then for $|z|=r<1$, we have*

$$\begin{aligned}&(1-|b_1|)\,r-\left(\tfrac{\gamma-1}{(\lambda q+1)|A_2|}-\tfrac{|B_1|}{(\lambda q+1)|A_2|}\,|b_1|\right)r^2\\ &\leq|f(z)|\leq(1+|b_1|)\,r+\left(\tfrac{\gamma-1}{(\lambda q+1)|A_2|}-\tfrac{|B_1|}{(\lambda q+1)|A_2|}\,|b_1|\right)r^2\end{aligned}\qquad(3.1)$$

for $|b_1|\leq\frac{\gamma-1}{|B_1|}$.The results are sharp with equality for the functions $f(z)$ defined by

$$(3.2)\qquad f(z)=z+|b_1|\,\overline{z}+\left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|}-\frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right)\overline{z}^2$$

and

$$f(z) = z - |b_1|\,\overline{z} - \left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|} - \frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right) z^2. \tag{3.3}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n},$$

then

$$\begin{aligned}
&|f(z)| \le (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n \le (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^2 \\
&= (1+|b_1|)r + \frac{\gamma-1}{(\lambda q+1)\,|A_2|} \sum_{n=2}^{\infty} \left(\frac{(\lambda q+1)\,|A_2|}{\gamma-1}\,|a_n| + \frac{(\lambda q+1)\,|A_2|}{\gamma-1}\,|b_n|\right) r^2 \\
&\le (1+|b_1|)r + \tfrac{\gamma-1}{(\lambda q+1)|A_2|} \sum_{n=2}^{\infty} \left(\tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1}\,|a_n| + \tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1}\,|b_n|\right) r^2 \\
&\le (1+|b_1|)r + \frac{\gamma-1}{(\lambda q+1)\,|A_2|}\left(1 - \frac{|B_1|}{\gamma-1}\,|b_1|\right) r^2 \\
&= (1+|b_1|)r + \left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|} - \frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right) r^2.
\end{aligned}$$

The functions $f(z)$ given by (3.2) and (3.3), respectively, for $|b_1| \le \frac{\gamma-1}{|B_1|}$ show that the bounds given in Theorem 3 are sharp.

4. Extreme points

Theorem 4. *Let $f(z)$ be given by* (1.3). *Then $f(z) \in \overline{S}_{H,q}(F,\lambda,\gamma)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} \left(\mu_n h_n(z) + \eta_n g_n(z)\right), \tag{4.1}$$

where $h_1(z) = z$,

$$h_n(z) = z + \frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|A_n|} z^n, \quad (n = 2,3,\ldots) \tag{4.2}$$

and

$$g_n(z) = z + \frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|B_n|} \overline{z^n}, \quad (n = 1,2,\ldots), \tag{4.3}$$

$\mu_n \geq 0, \eta_n \geq 0, \sum_{n=1}^{\infty}(\mu_n + \eta_n) = 1$. *In particular, the extreme points of the class* $\overline{S}_H(F, \lambda, \gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$, *respectively.*

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(\mu_n h_n(z) + \eta_n g_n(z)) \\ &= z + \sum_{n=2}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|A_n|}\mu_n z^n + \sum_{n=1}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|B_n|}\eta_n \overline{z^n}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right)+1\right]|A_n|}{\gamma-1}\left(\frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|A_n|}\mu_n\right) \\ &+ \sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right)+1\right]|B_n|}{\gamma-1}\left(\frac{\gamma-1}{\left[\lambda\left([n]_q - 1\right)+1\right]|B_n|}\eta_n\right) = \sum_{n=2}^{\infty}\mu_n + \sum_{n=1}^{\infty}\eta_n = 1 - \mu_1 \leq 1 \end{aligned}$$

and so $f(z) \in \overline{S}_{H,q}(F, \lambda\gamma)$.

Conversely, if $f(z) \in \overline{S}_{H,q}(F, \lambda, \gamma)$, then

$$|a_n| \leq \frac{\gamma - 1}{\left[\lambda\left([n]_q - 1\right) + 1\right]|A_n|}, \ (n \geq 2)$$

and

$$|b_n| \leq \frac{\gamma - 1}{\left[\lambda\left([n]_q - 1\right) + 1\right]|B_n|}, \ (n \geq 1).$$

Setting

$$\mu_n = \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|A_n|}{\gamma - 1}\ |a_n|, \quad (n = 2, 3, \ldots)$$

and

$$\eta_n = \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|B_n|}{\gamma - 1}\ |b_n|, \quad (n = 1, 2, \ldots).$$

Since $0 \leq \mu_n \leq 1$ $(n = 2, 3, \ldots)$ and $0 \leq \eta_n \leq 1$ $(n = 1, 2, \ldots)$, $\mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n + \sum_{n=1}^{\infty}\eta_n \geq 0$, then, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of the Theorem 4.

Now, we show that the class $\overline{S}_{H,q}(F, \lambda, \gamma)$ is closed under convex combinations of its members.

Theorem 5. *The class* $\overline{S}_{H,q}(F, \lambda, \gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{S}_{H,q}(F, \lambda, \gamma)$, where f_i is given by

$$f_i = z + \sum_{n=2}^{\infty}|a_{n_i}|\, z^n + \sum_{n=1}^{\infty}|b_{n_i}|\,\overline{z^n}.$$

Then, by using Theorem 2, we have

$$\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|A_n|}{\gamma - 1} |a_{n_i}| z^n + \sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|B_n|}{\gamma - 1} |b_{n_i}| \overline{z}^n \leq 1. \tag{4.4}$$

For $\sum_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \overline{z^n}. \tag{4.5}$$

Then by (4.4), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|A_n|}{\gamma - 1} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) + \sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|B_n|}{\gamma - 1} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|A_n|}{\gamma - 1} |a_{n_i}| + \sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right]|B_n|}{\gamma - 1} |b_{n_i}|\right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.2) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{S}_{H,q}(F, \lambda, \gamma)$. This completes the proof of the Theorem 5.

Remark 2. Specializing F, in the above results, we obtain the corresponding results for the corresponding classes $\overline{S}_{H,q}(F, \gamma), \overline{S}_{H,q}(\lambda, \gamma), \overline{R}_{H_{r,s}}(q, \alpha_1, \beta_1, \lambda, \gamma)$, $\overline{S}_{Hr}(\eta, \lambda, \gamma)$ $\overline{R}_H(q, m, \lambda, \gamma), \overline{R}_H(q, l, \mu, \eta, \lambda, \gamma)$ and $\overline{R_H}(q, \mu, \lambda, \gamma)$ defined in the introduction.

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On solving some classes of second order ODEs

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Abstract. In this paper, we introduce some analytical techniques to solve some classes of second order differential equations. Such classes of differential equations arise in describing some mathematical problems in Physics and Engineering. Moreover, these classes of differential equations generalize some very well known differential equations. For example, Chebyshev's Equation, Cauchy-Euler's Equation, and some of Hypergeometric differential Equations. Using one of these techniques we solve a special case of Lane-Emden Type of Differential Equations.

Keywords: Chebyshev's differential equation, hypergeometric differential equation, Cauchy-Euler's differential equation, exact second order differential equations, nonlinear second order differential equations, Lane-Emden type of differential equations.

1. Introduction

Solving differential equations has important impacts on Mathematics, sciences, and engineering. In [1]-[9], several techniques for solving differential equations are introduced. One of the most important applications in the calculus of variation is to maximize (minimize) the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx, \tag{1}$$

where $p(x)$ is a positive and differentiable function on some open interval $I \subset \mathbb{R}$, and $h(x)$ is a differentiable function. In fact, the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx \tag{2}$$

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attains its extreme values at a function $y(x) \in C^2(I)$, where I is an open interval in $\mathbb{R}$, when $y(x)$ satisfies the Euler's-Lagrange differential equation [12, 15]

$$\frac{\partial F}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x, y, y')\right) = 0, \tag{3}$$

where

$$F(x, y, y') := \sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}}. \tag{4}$$

Therefore, the problem of maximizing (minimizing) $Q[y]$ is reduced to solve the differential equation (3). i.e., to solve

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) = \frac{1}{2}h'(y(x)). \tag{5}$$

Hence, it is a matter to solve such differential equations. In the first part of this paper, we solve the following class of second order differential equation:

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0 \tag{6}$$

which generalizes the differential equation (5). The function $p(x)$ is positive and differentiable function on some open interval $I \subset \mathbb{R}$, and $f(\sqrt{p(x)}\, y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. In fact, equation (6) not only generalizes equation (5), but also it generalizes many of well known differential equation. For example,

1. the Chebyshev's Differential Equation, e.g., see, [12, 18],

$$(1 - x^2)y''(x) - x\, y'(x) + n^2 y(x) = 0, \;\; |x| < 1,$$

2. the Cauchy-Euler's Differential Equation, e.g., see, [13, 16],

$$ax^2 y''(x) + a\, x\, y'(x) + b\, y(x) = 0, \;\; x > 0,$$

3. the Nonlinear Chebyshev's Equation,

$$(1 - x^2)y''(x) + \left(\alpha\sqrt{1 - x^2} - x\right) y'(x) + f(y(x)) = 0, \quad |x| < 1,$$

4. the Hypergeometric Differential Equation, e.g., see, [11, 18],

$$x(1 - x)y''(x) + [c - (a + b + 1)x]\, y'(x) - a\, b\, y(x) = 0, \; 0 < x < 1$$

with c=1/2 and a=-b, and

5. the Nonlinear Hypergeometric Differential Equation,

$$x(1-x)y''(x)+\left(\frac{1}{2}-x+\alpha\sqrt{x(1-x)}\right)y'(x)+f(y(x))=0,\ 0<x<1.$$

Throughout the paper, the class of differential equations given in (6) is called by Chebyshev's-type of differential equations.

In the second part of this paper, we introduce an approach to solve the class of second order differential equations

$$a_2\left(f'(y)y''+(y')^2f''(y)\right)+a_1f'(y)y'+a_0f(y)=g(x), \tag{7}$$

where $a_0,\ a_1$ and a_2 are constants, and $f\in C^2(J)$, for some open interval $J\subset\mathbb{R}$. We also give an approach to solve the class of differential equation

$$p(x)\left(f'(y)y''+(y')^2f''(y)\right)+\frac{1}{2}p'(x)f'(y)y'+a_0f(y)=0, \tag{8}$$

where $p(x)$ is positive and differentiable function on some open interval $I\subset\mathbb{R}$, and $f\in C^2(J)$, for some open interval $J\subset\mathbb{R}$. Throughout the paper, these two classes of second order differential equations are called by $f-$types of second order differential equations.

In the third part of this paper, we introduce an approach to solve the following class of second order differential equations:

$$a_2(x,y,y')\left(f'(y)y''+f''(y)(y')^2\right)+a_1(x,y,y')(f'(y)y')+a_0(x,y,y')=0, \tag{9}$$

where $f(y)$ is an invertible function, and $f\in C^2(J)$ for some open interval $J\subset\mathbb{R}$. To solve this class of differential equations, we assume that

$$a_2\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)z''+a_1\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)z' \tag{10}$$

$$+a_0\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)=0 \tag{11}$$

is an exact differential equation. The differential equation (10) is exact if the conditions

$$\frac{\partial a_2}{\partial z}=\frac{\partial a_1}{\partial z'},\ \frac{\partial a_2}{\partial x}=\frac{\partial a_0}{\partial z'},\text{ and }\ \frac{\partial a_1}{\partial x}=\frac{\partial a_0}{\partial z}. \tag{12}$$

hold, see, [1, 7, 8, 9]. In this case, the first integral of (10) exists and is given by

$$\int_{x_0}^{x}a_0(\alpha,z,z')d\alpha+\int_{z_0}^{z}a_1(x_0,\beta,z')d\beta+\int_{z_0'}^{z'}a_2(x_0,z_0,\gamma)d\gamma=c.$$

Throughout the paper, this class of differential equations is called by $f-$type of second order differential equations that can be transformed to exact differential equations.

The layout of the paper: In the second section, we solve Chebyshev's-type of Second Order Differential Equations. In the third section, we solve the $f-$types of second order differential equations. In the fourth section, we solve the $f-$type of second order differential equation that can be transformed into exact equations. The fifth section has been devoted for the concluding remarks.

2. Solving Chebyshev's-type of second order differential equations

In this section, we present an approach to solve Chebyshev's-type of second order differential equations

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0, \tag{13}$$

where $p(x)$ is a positive and differentiable function on some open interval $I \in \mathbb{R}$, and $f(\sqrt{p(x)}$
$y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. The approach is described in the following theorem:

Theorem 2.1. *Assume that $p(x)$ be a positive and differentiable function on the open interval $I \subset \mathbb{R}$. Let $x_0 \in I$. Then*

$$t = \int_{x_0}^{x} \frac{d\xi}{\sqrt{p(\xi)}}$$

transforms the differential equation (13) *into the second order differential equation*

$$y''(t) + f(y(t), y'(t)) = 0. \tag{14}$$

Proof. Let

$$t = \int_{x_0}^{x} \frac{d\xi}{\sqrt{p(\xi)}}.$$

Since,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

Hence,

$$\frac{dy}{dt} = \sqrt{p(x)}\, \frac{dy}{dx}. \tag{15}$$

Therefore,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\sqrt{p(x)}\, \frac{dy}{dx}\right) = \frac{d}{dx}\left(\sqrt{p(x)}\, \frac{dy}{dx}\right)\frac{dx}{dt} = p(x)y'' + \frac{1}{2}p'(x)y'. \tag{16}$$

By substituting (15) and (16) in Equation (6), we get the result. □

Remark 2.1. The differential equation (14) is independent of variable t, and so, it can be solved by setting $\eta(t) = y'(t)$. Hence, it can be reduced to the following first order differential equation:

$$\eta \frac{d\eta}{dy} + f(y, \eta) = 0. \tag{17}$$

Particularly, if $f(\sqrt{p(x)}\, y', y) = f(y)$, then we have

$$\eta^2(t) = -2 \int^y f(\xi) d\xi + c.$$

Therefore,

$$y'(t) = \left(c - 2 \int^y f(\xi) d\xi \right)^{\frac{1}{2}},$$

where c is the integration constant.

Next, we present some illustrative examples.

Example 2.1. Consider the following special case of Lane-Emden type of differential equation:

$$y'' + \frac{n-1}{x} y' + x^{2(1-n)} f(y) = 0, \quad n > 2, \quad x > 0. \tag{18}$$

Generally, the Lane-Emden type of differential equations is given by

$$y'' + \frac{n-1}{x} y' + g(x) f(y) = 0, \quad n > 2, \quad x > 0. \tag{19}$$

This differential equation arises in mathematical physics and astrophysics to describe the temperature variations of a spherical gas cloud under the mutual attraction of its molecules and subject to classical thermodynamics law [17]. To solve (18), we multiply it by $x^{2(n-1)}$ to get

$$x^{2(n-1)} y'' + (n-1) x^{2n-3} y' + f(y) = 0, \quad n > 2, \quad x > 0. \tag{20}$$

Clearly, this equation takes the form of Eq. (13). Therefore it can be reduced to to the following simple differential equation:

$$y'' + f(y) = 0.$$

Example 2.2. Consider the nonlinear Chebyshev's differential equation

$$\begin{cases} (1 - x^2) y''(x) - x y'(x) + 4\sqrt{1 - x^2}\, y' y(x) = 0, \\ y(0) = \dfrac{1}{2}, \; y'(0) = -\dfrac{1}{2}. \end{cases} \tag{21}$$

Then

$$t = \int^x \frac{d\xi}{\sqrt{1 - \xi^2}} = \arcsin(x)$$

transforms (21) into

$$y''(t) + 4y'(t)y(t) = 0.$$

Set $\eta(t) = y'(t)$. The above equation becomes

$$\eta \frac{d\eta}{dy} + 4\eta y = 0.$$

The solution of this equation is $y(t) = \frac{1}{2(t+1)}$. Therefore,

$$y(x) = \frac{1}{2(\arcsin(x) + 1)}.$$

Example 2.3. Consider the initial value problem

$$\begin{cases} x^2y'' + xy' - 3y^2 = 0, \ x > 0, \\ y(1) = 2, \ y'(1) = 4. \end{cases} \tag{22}$$

Then

$$t = \int_1^x \frac{d\xi}{\xi} d\xi = \ln(x)$$

transforms the (22) into

$$\begin{cases} y'' - 3y^2 = 0, \\ y(0) = 2, \ y'(0) = 4. \end{cases} \tag{23}$$

The solution of the above differential equation is

$$y(t) = \frac{2}{(1-t)^2}.$$

Hence,

$$y(x) = \frac{2}{(1 - \ln(x))^2}.$$

Example 2.4. Consider the linear form of (6). i.e.,

$$\phi(x)y'' + \frac{1}{2}\phi'(x)y' + \lambda^2 y = 0, \tag{24}$$

where $\lambda > 0$, and $\phi(x)$ is a positive and differentiable function on some open interval $(a, b) \subset \mathbb{R}$. By applying the transformation

$$t = \int_{x_0}^x \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi$$

to equation (24), then it can be transformed into the following differential equation:

$$\frac{d^2y}{dt^2} + \lambda^2 y = 0.$$

The solution of this differential equation is

$$y(t) = C_1 \sin(\lambda t) + C_2 \cos(\lambda t).$$

Hence, the general solution of equation (24) is

$$y(x) = C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right).$$

Example 2.5. Consider the following second order linear differential equation [16, eq. 239, p. 335]:

$$4xy'' + 2y' + y = 0. \tag{25}$$

From the previous example, the general solution of this equation is given by

$$y(x) = C_1 \sin\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right) + C_2 \cos\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right)$$

and so,

$$y(x) = C_1 \sin\left(\sqrt{x}\right) + C_2 \cos\left(\sqrt{x}\right).$$

Remark 2.2. Consider the second order linear differential equation

$$(\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x) + \lambda y(x) = 0, \quad x \in (a, b), \tag{26}$$

and assume that $\phi(x)$ is a positive and differentiable function on the open interval $(a, b) \subset \mathbb{R}$. Moreover, assume that $\phi(a) = \phi(b) = 0$. Define the linear differential operator

$$L[y] := -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x)\right)) = \lambda y(x).$$

Then the boundary value problem

$$\begin{cases} L[y] = -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x)\right) = \lambda y(x), \quad a < x < b, \\ \phi(a) = \phi(b) = 0, \end{cases} \tag{27}$$

satisfies the Lagrange Identity $\int_a^b \phi L[\psi]dx = \int_a^b \psi L[\phi]dx$, where ϕ and ψ satisfy the above boundary value problem. Therefore, the operator $L[y]$ is self-adjoint. Hence, the boundary value problem (27) has an orthogonal set of eigenfunctions $\{\phi_n(x)\}_{n=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Since the above boundary value problem is a special case of (6). Then, by using the approach described in Theorem 2.1, it is easy to find its orthogonal set of eigenfunctions.

By using the same approach described in Theorem 2.1. We can solve the following class of second order linear differential equations:

$$[P(x)]^2 y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y(x) = 0, \tag{28}$$

where $P(x) > 0$, $P(x) \in C^1(a,b)$, and α and β are constants. In fact, the transformation

$$t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}, \tag{29}$$

where x_0, $x \in (a,b)$, transforms Eq. (28) into the following second order differential equation:

$$y''(t) + \alpha y'(t) + \beta y(t) = 0.$$

This differential equation is with constant coefficients can be solved by using the elementary techniques of solving second order differential equations. For illustration, we present the following examples:

Example 2.6. Consider the well-known Cauchy-Euler's Equation

$$x^2 y''(x) + (\alpha + 1)xy'(x) + \beta y = 0,\ x > 0.$$

Then $P(x) = x$, and the $t-$transformation is $t = \ln(x)$, which transforms the equation into

$$y''(x) + \alpha y'(x) + \beta y = 0$$

Example 2.7. Consider the Chebyshev's Equation

$$\left[1 - x^2\right] y''(x) - 2xy'(x) + n^2 y = 0,\ |x| < 1.$$

Then $P(x) = \sqrt{1 - x^2}$, and the $t-$transformation is $t = \sin^{-1}(x)$, which transforms the equation into

$$y''(x) + n^2 y = 0.$$

Using this transformation, the solution of Chebyshev's Equation is given by

$$y(x) = A\cos(n\sin^{-1}(x)) + B\sin(n\sin^{-1}(x)).$$

Example 2.8. Consider the Hypergeometric Equation

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 0,\ x \in (0,1). \tag{30}$$

Then $P(x) = \sqrt{x(1-x)}$, and the $t-$transformation is $t = \sin^{-1}(2x-1)$. This transforms the above equation into

$$y''(t) + a^2 y(t) = 0.$$

Hence, the solution of equation (30) is given by

$$y(x) = A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)).$$

For certain functions, $h(x) \in C(a,b)$, for some open interval $(a,b) \in \mathbb{R}$, we can solve the nonhomogeneous second order differential equation

$$[P(x)]^2 y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y = h(x). \tag{31}$$

Particularly, when $h(x)$ can be written in the form $H(t)$, where $t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}$. The following example shows this idea:

Example 2.9. Consider the nonhomogeneous differential equation

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 2x,\ x \in (0,1). \tag{32}$$

Then $P(x) = \sqrt{x(1-x)}$. The $t-$transformation is $t = \sin^{-1}(2x-1)$. This transforms the equation into

$$y''(t) + a^2 y(t) = 1 + \sin(t).$$

The solution of equation (32) is given by

$$y(x) = \begin{cases} A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)) + \dfrac{2x-1}{a^2-1} + \dfrac{1}{a^2}, \\ \text{if } \ a \neq \pm 1, \\ A\cos(\sin^{-1}(2x-1)) + B(2x-1) + \dfrac{1}{2}(1-2x)\sin^{-1}(2x-1) + 1, \\ \text{if } \ a = \pm 1. \end{cases}$$

3. Solving the $f-$types of second order differential equations

In this section, we solve the following class second order nonlinear differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{33}$$

where a_2, a_1 and a_0 are constants, and $f \in C^2(J)$, for some open interval $J \subset \mathbb{R}$. In this section, we also solve the following class of second order nonlinear differential equation:

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{34}$$

where $p(x)$ is a positive and differentiable function on the open interval $I \subset \mathbb{R}$, and $f \in C^2(J)$. To solve (33), let $z = f(y)$. Hence, $z' = f'(y)y'$, and $z'' = f'(y)y'' + (y')^2 f''(y)$. Substitute z, z' and z'' in equation (33), we get

$$a_2 z'' + a_1 z' + a_0 z = g(x), \tag{35}$$

Similarly, equation (34) becomes

$$p(x)z'' + \frac{1}{2}p'(x)z' + a_0 z = 0, \tag{36}$$

which is the linear form of (6). Therefore, it can be solved by using the technique described in Example 2.4. To illustrate the procedure of solving (33) and (34), we present the following examples:

Example 3.1. Consider Langumir Equation, with a slightly modification,

$$3yy'' + 3(y')^2 + 4yy' + y^2 = 1. \tag{37}$$

The original Langumir Equation is given by

$$3yy'' + (y')^2 + 4yy' + y^2 = 1$$

which originally appears in connection with the theory of current flow from hot cathode to an anode in a hight vacuum [10, 14]. To solve (37), we let $z = \frac{y^2}{2}$. Then $z' = yy'$ and $z'' = yy'' + (y')^2$. Hence, equation (37) becomes

$$3z'' + 4z' + 2z = 1.$$

The solution of this equation is

$$z(x) = e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + \frac{1}{2}.$$

Hence, the solution of (37) is given by

$$y^2 = 2e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + 1.$$

Example 3.2. Consider the initial value problem

$$\begin{cases} y'' + (y')^2 + 1 = (\cos\omega x)e^{-y}, \ \ \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

This problem is equivalent to

$$\begin{cases} \left(y'' + (y')^2\right)e^{y} + e^{y} = (\cos\omega x), \ \ \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

Let $z = e^y$. Then $z' = y'e^y$ and $z'' = y''e^y + (y')^2e^y$. By substituting z, z' and z'' in the above initial value problem, we get

$$\begin{cases} z'' + z = \cos\omega x, \ \ \omega \neq \pm 1, \\ z(0) = 1, \ z'(0) = 0. \end{cases}$$

The solution of this problem is $z(x) = \frac{1}{1-\omega^2}\left(\cos\omega x - \omega^2\cos x\right)$, $\omega \neq \pm 1$. Therefore, $y(x) = \ln\left(\frac{1}{1-\omega^2}\left(\cos\omega x - \omega^2\cos x\right)\right)$, $\omega \neq \pm 1$.

Example 3.3. Let $\phi(x)$ be a positive and differentiable function on the open interval $(a,b)\subset\mathbb{R}$, and consider the differential equation

$$\phi(x)\left(y''+(y')^2\right)+\frac{1}{2}\phi'(x)y'+\lambda=0.$$

By multiplying this equation by e^y, we get

$$\phi(x)\left(y''+(y')^2\right)e^y+\frac{1}{2}\phi'(x)y'e^y+\lambda e^y=0.$$

Let $z=e^y$. Then $z'=y'e^y$ and $z''=y''e^y+(y')^2e^y$. By substituting z, z' and z'' in the above differential equation, we get

$$\phi(x)z''+\frac{1}{2}\phi'(x)z'+\lambda z=0.$$

The solution of this equation is (see Example 2.4)

$$z(x)=C_1\sin\left(\lambda\int_{x_0}^{x}\frac{d\xi}{\sqrt{\phi(\xi)}}d\xi\right)+C_2\cos\left(\lambda\int_{x_0}^{x}\frac{d\xi}{\sqrt{\phi(\xi)}}d\xi\right).$$

Therefore,

$$y(x)=\ln\left[C_1\sin\left(\lambda\int_{x_0}^{x}\frac{d\xi}{\sqrt{\phi(\xi)}}d\xi\right)+C_2\cos\left(\lambda\int_{x_0}^{x}\frac{d\xi}{\sqrt{\phi(\xi)}}d\xi\right)\right].$$

4. Solving the f-type of second order differential equations that can be transformed into exact differential equations

In this section, we solve the following class of second order differential equations:

$$a_2(x,y,y')\left(f'(y)y''+f''(y)(y')^2\right)+a_1(x,y,y')(f'(y)y')+a_0(x,y,y')=0, \tag{38}$$

where $f(y)$ is invertible function and $f\in C^2(J)$, for some open interval $J\in\mathbb{R}$. To solve this class of differential equations, we let $z=f(y)$. Then $z'=f'(y)y'$ and $z''=f''(y)(y')^2+f'(y)y''$. Moreover, we let $y=f^{-1}(z)$. Then $y'=\frac{z'}{f'(f^{-1}(z))}$. Hence, equation (38) can be transformed into the following differential equation:

$$\begin{aligned}a_2\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)z''&+a_1\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)z'\\&+a_0\left(x,f^{-1}(z),\frac{z'}{f'\left(f^{-1}(z)\right)}\right)=0.\end{aligned} \tag{39}$$

Assuming that (39) is exact, i.e., the conditions (12) hold. Then its first integral exists, and so, it can reduced to a first order differential equation which in turn can be solved by using elementary techniques. To explain this procedure, we present the following illustrative examples:

Example 4.1. Consider the second order nonlinear differential equation

$$\begin{cases} e^y \left[y'' + (y')^2\right] + 12xe^{4y}y' + \left(3e^{4y} - 1\right) = 0, \\ y(0) = \ln 2,\ y'(0) = 0. \end{cases} \tag{40}$$

Let $z = e^y$. Then $z' = e^y y'$ and $z'' = e^y y'' + e^y (y')^2$. Hence, Eq. (40) becomes

$$\begin{cases} z'' + 12xz^3 z' + \left(3z^4 - 1\right) = 0, \\ z(0) = 2,\ z'(0) = 0. \end{cases} \tag{41}$$

Therefore, $a_2(x, z, z') = 1$, $a_1(x, z, z') = 12xz^3$, and $a_0(x, z, z') = \left(3z^4 - 1\right)$. In addition, we have

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'} = 0,\ \ \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'} = 0, \text{ and } \ \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z} = 12z^3, \tag{42}$$

Hence, equation (41) is exact differential equation. Hence, its first integral is given by

$$z' + 3xz^4 - x = 0.$$

For which an implicit solution can be obtained by separating the variables, and hence, $y(x) = \ln(z(x))$.

Remark 4.1. Assume that (39) is not exact. Then an integrating factor of (39) could be exist. Hence, it can be transformed into an exact differential equation [1]. To explain the procedure of solving (39), we present the following example:

Example 4.2. Consider the second order nonlinear differential equation

$$xe^y \left(2x + e^y\right)\left(y'' + (y')^2\right) + x\left(x + e^y\right)y' + \left(3x + e^y\right) = 0. \tag{43}$$

By multiplying this equation by e^y, we get

$$xe^{2y} \left(2x + e^y\right)\left(y'' + (y')^2\right) + x\left(x + e^y\right)e^y y' + e^y\left(3x + e^y\right) = 0. \tag{44}$$

Let $z = e^y$. Then $z' = e^y y'$ and $z'' = e^y y'' + e^y (y')^2$. Hence, by substituting z, z' and z'' in (44), we get

$$xz(2x + z)z'' + x(x + z)z' + z(3x + z) = 0. \tag{45}$$

This equation is not exact since $\frac{\partial a_2}{\partial z} = 2(x + z) \neq 0 = \frac{\partial a_1}{\partial z'}$. An integrating factor of this second order nonlinear differential equation exists, and it is given by $\mu(x, z) = \frac{1}{xz(2x+z)}$. Multiplying (45) by $\mu(x, z)$, we get

$$z'' + \frac{(x + z)}{z(2x + z)}z' + \frac{(3x + z)}{x(2x + z)} = 0. \tag{46}$$

Clearly,

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'} = 0, \quad \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'} = 0, \text{ and } \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z} = \frac{-1}{(2x+z)^2}. \tag{47}$$

Therefore, the differential equation (46) is exact, and its first integral is given by

$$c = z' + \ln\left(xz\sqrt{2x+z}\right). \tag{48}$$

This first order differential equation can be solved by using the elementary techniques of solving first order differential equations. Hence, $y(x) = \ln(z(x))$.

Finally, we consider the nonhomogeneous second order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x),$$

where $a_2(x) \neq 0$, $a_1(x)$, and $a_0(x)$ are differentiable functions on an open interval $(a, b) \subset \mathbb{R}$. This equation admits an integrating factor $\mu(x) = \frac{1}{a_2(x)}$ provided that $W(a_2, a_1)(x) = a_0(x)a_2(x)$, where $W(a_2, a_1)(x) = a_2(x)a_1'(x) - a_1(x)a_2'(x)$. For this case, we present the following example:

Example 4.3. consider the second order linear differential equation

$$e^x y'' + \cos x y' - (\cos x + \sin x)y = h(x).$$

By multiplying this equation by the integrating factor e^{-x}, we get

$$y'' + e^{-x}\cos x y' - e^{-x}(\cos x + \sin x)y = h(x)e^{-x}.$$

This equation can be written as

$$\frac{d}{dx}\left[y' + (e^{-x}\cos x)y\right] = h(x)e^{-x}$$

Hence, its first integral is given by

$$y' + (e^{-x}\cos x)y = \int^x h(\xi)e^{-\xi}d\xi + c_1$$

which can be solved by using the elementary techniques of solving first order differential equations.

5. Concluding remarks

In this paper, we solved some classes of second order differential equation. In fact, we solved the following classes of second order differential equations:

1. The Chebyshev's type of second order differential equations which is given by

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0, \quad x \in (a,b), \tag{49}$$

where $p(x)$ is a positive and differentiable function on the some open interval $I \subset \mathbb{R}$, and $f(\sqrt{p(x)}\, y'(x), y(x))$ is a continuous function on some domain $D \subset \mathbb{R}^2$.

2. The $f-$types of second order differential equations:

   a) the first type is given by the second order differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{50}$$

   where a_2, a_1 and a_0 are constants, and the function f is a second differentiable function of on some open interval $J \subset \mathbb{R}$, and

   b) the second type is given by the second order differential equation

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{51}$$

   where $p(x)$ is a positive and differentiable function on some open interval $I \subset \mathbb{R}$, and f is a second differentiable function on some open interval $J \subset \mathbb{R}$.

3. f-type of second order differential equations that can be transformed into exact second order Differential Equations. Particularly, the class of differential equations

$$a_2(x,y,y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x,y,y')(f'(y)y') + a_0(x,y,y') = 0, \tag{52}$$

   where the function $f(y)$ is an invertible function and is second differentiable function on some open interval $J \subset \mathbb{R}$.

In the paper, we also presented some illustrative examples.

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Generalized fractional integral inequalities for product of two convex functions

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Abstract. The aim of this paper is to generalize the results proved in [4] using generalized fractional integral. Some special cases are deduced from main results. Applying the techniques of our results, new results may be obtained during a similar manner for various operators.

Keywords: convex functions, Hermite-Hadamard inequalities, generalized fractional integrals.

1. Introduction

If $\mathcal{F}_1 : \mathbf{I} \to \mathbb{R}$ be a convex mapping on the interval $\mathbf{I}$ of real numbers and $\wp_1, \wp_2 \in \mathbf{I}$ with $\wp_1 \neq \wp_2$ we have the following double inequality well known in the literature:

$$\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)dx \leq \frac{\mathcal{F}_1(\wp_1)+\mathcal{F}_1(\wp_2)}{2}, \tag{1}$$

double inequality (1) also known as trapezium inequality.

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There are several refinements of the Hermite-Hadamard inequality on convex functions which are extensively investigated by variety of authors (e.g., [1, 2, 3, 5, 6, 7, 8, 11]).

In [4], Chen established two new Hermite-Hadamard inequalities via Riemann-Liouville fractional integral for product of convex functions as follows:

Theorem 1.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for Riemann-Liouville fractional integral holds:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^{\alpha}}\left[J^{\alpha}_{\wp_1}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+J^{\alpha}_{\wp_2}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ \leq & \left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)\mathbf{M}(\wp_1,\wp_2) \\ & +\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathbf{N}(\wp_1,\wp_2), \qquad (2) \\ & 2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ \leq & \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^{\alpha}}\left[I^{\alpha}_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ & +\mathbf{M}(\wp_1,\wp_2)\frac{\alpha}{(\alpha+1)(\alpha+2)} \\ & +\mathbf{N}(\wp_1,\wp_2)\left[\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}-\frac{1}{2}\right], \qquad (3) \end{aligned}$$

where $\mathbf{M}(\wp_1,\wp_2)=\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)$, *and* $\mathbf{N}(\wp_1,\wp_2)=\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1)$.

Some new integral inequalities involving two nonnegative and integrable functions that are associated with the Hermite Hadamard type also are obtained by several authors. In [12], B.G. Pachpatte projected some Hermite-Hadamard kind inequalities involving two *log*-convex functions. An identical result for s-convex functions is established by Kirmaci et al. in [11]. In [15], Sarikaya conferred some integral inequalities for two h-convex functions. For recent results and generalizations regarding Hermite-Hadamard type inequality for product of two functions see [16] and therefore the references given in that.

It is outstanding that Sarikaya and Ertuğral in [14] defined the following generalized fractional integral:

$$ {}_{\wp_1}I_{\varphi}\mathcal{F}_1(x)=\int_{\wp_1}^{x}\frac{\varphi\left(x-\jmath\right)}{x-\jmath}\mathcal{F}_1(\jmath)d\jmath, \quad x>\wp_1, \qquad (4) $$

$$ {}_{\wp_2}I_{\varphi}\mathcal{F}_1(x)=\int_{x}^{\wp_2}\frac{\varphi\left(\jmath-x\right)}{\jmath-x}\mathcal{F}_1(\jmath)d\jmath, \quad x<\wp_2 \qquad (5) $$

where the function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(\jmath)}{\jmath} d\jmath < \infty.$$

Remark 1.1. If we take $\varphi(\jmath) = \jmath$, $\varphi(\jmath) = \frac{\jmath^\alpha}{\Gamma(\alpha)}$, $\varphi(\jmath) = \frac{1}{k\Gamma_k(\alpha)}\jmath^{\frac{\alpha}{k}}$ and $\varphi(\jmath) = \jmath(x-\jmath)^{\alpha-1}$ in the operators (4) and (5), then we have the Riemann integral, Riemann-Liouville fractional integral [10], k-Riemann-Liouville fractional integral given by Mubeen and Habibullah in [13] and conformable fractional operators given by Khalil et al. in [9], respectively.

By applying generalized fractional integral, Sarıkaya and Ertuğral established following fascinating Hermite-Hadamard type integral inequalities in the same paper:

Theorem 1.2. *Let $\mathcal{F}_1 : [\wp_1, \wp_2] \to \mathbb{R}$ is convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_\varphi \mathcal{F}_1(\wp_2) + {}_{\wp_2-}I_\varphi \mathcal{F}_1(\wp_1)\right] \leq \frac{\mathcal{F}_1(\wp_1)+\mathcal{F}_1(\wp_2)}{2}, \tag{6}$$

where $\Lambda(1) = \int_0^1 \frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath} d\jmath$.

The main objective of our this paper is to generalize the proved results in [4] with help of generalized fractional integrals. We shall also discuss some of its special cases.

2. Main results

Theorem 2.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$\left[{}_{\wp_1+}I_\varphi^{\wp_2}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + {}_{\wp_2-}I_\varphi^{\wp_1}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \leq J_1\mathbf{M}(\wp_1,\wp_2) + J_2\mathbf{N}(\wp_1,\wp_2) \tag{7}$$

where

$$\begin{aligned} J_1 &= \int_0^1 \frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}(2\jmath^2 - 2\jmath + 1)d\jmath, \\ J_2 &= \int_0^1 \frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}(2\jmath - 2\jmath^2)d\jmath, \\ \mathbf{M}(\wp_1,\wp_2) &= \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2), \\ \mathbf{N}(\wp_1,\wp_2) &= \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1). \end{aligned}$$

Proof. Since $\mathcal{F}_1, \mathcal{F}_2$ are convex functions, then according to the definition of convex function we have

$$\begin{aligned} &\mathcal{F}_1(\wp_1 + \jmath(\wp_2-\wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2-\wp_1)) \\ &\leq \left((1-\jmath)\mathcal{F}_1(\wp_1) + \jmath\mathcal{F}_1(\wp_2)\right)\left((1-\jmath)\mathcal{F}_2(\wp_1) + \jmath\mathcal{F}_2(\wp_2)\right), \end{aligned} \tag{8}$$

and

$$\begin{aligned}&\mathcal{F}_1(\wp_2+\jmath(\wp_1-\wp_2))\mathcal{F}_2(\wp_2+\jmath(\wp_1-\wp_2))\\&\leq(\jmath\mathcal{F}_1(\wp_1)+(1-\jmath)\mathcal{F}_1(\wp_2))\,(\jmath\mathcal{F}_2(\wp_1)+(1-\jmath)\mathcal{F}_2(\wp_2))\,.\end{aligned}\tag{9}$$

Adding (8) and (9), we have

$$\begin{aligned}&\mathcal{F}_1(\wp_1+\jmath(\wp_2-\wp_1))\mathcal{F}_2(\wp_1+\jmath(\wp_2-\wp_1))\\&+\mathcal{F}_1(\wp_2+\jmath(\wp_1-\wp_2))\mathcal{F}_2(\wp_2+\jmath(\wp_1-\wp_2))\\&\leq((1-\jmath)\mathcal{F}_1(\wp_1)+\jmath\mathcal{F}_1(\wp_2))\,((1-\jmath)\mathcal{F}_2(\wp_1)+\jmath\mathcal{F}_2(\wp_2))\\&+(\jmath\mathcal{F}_1(\wp_1)+(1-\jmath)\mathcal{F}_1(\wp_2))\,(\jmath\mathcal{F}_2(\wp_1)+(1-\jmath)\mathcal{F}_2(\wp_2))\,.\end{aligned}$$

So, we get

$$\begin{aligned}&\mathcal{F}_1(\wp_1+\jmath(\wp_2-\wp_1))\mathcal{F}_2(\wp_1+\jmath(\wp_2-\wp_1))\\&+\mathcal{F}_1(\wp_2+\jmath(\wp_1-\wp_2))\mathcal{F}_2(\wp_2+\jmath(\wp_1-\wp_2))\\&\leq(2\jmath^2-2\jmath+1)\mathbf{M}(\wp_1,\wp_2)+2\jmath(1-\jmath)\mathbf{N}(\wp_1,\wp_2),\end{aligned}\tag{10}$$

Multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of inequality (10), then integrating the obtained inequality with respect to '$\jmath$' on $[0,1]$, we get our desired inequality (7). □

Remark 2.1. If we choose $\mathcal{F}_2(x)=1$ for all $x\in[\wp_1,\wp_2]$ in Theorem 2.1, then we obtain the second inequality of (6).

Remark 2.2. If we use $\varphi(\jmath)=\jmath$ in Theorem 2.1, we have a well known inequality for the product of two convex function, given by B. G. Pachpatte in [12], i.e.,

$$\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\leq\frac{1}{3}\mathbf{M}(\wp_1,\wp_2)+\frac{1}{6}\mathbf{N}(\wp_1,\wp_2),$$

$\mathbf{M}(\wp_1,\wp_2)$ and $\mathbf{N}(\wp_1,\wp_2)$ are defined in Theorem 2.1.

Remark 2.3. If we suppose $\varphi(\jmath)=\frac{\jmath^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, we have an inequality for the product of convex functions via Riemann-Liouville fractional integral given by Chen, F. in [4], i.e.,

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^\alpha}\left[J^\alpha_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+J^\alpha_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\&\leq\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)\mathbf{M}(\wp_1,\wp_2)\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathbf{N}(\wp_1,\wp_2),\end{aligned}$$

$\mathbf{M}(\wp_1,\wp_2)$ and $\mathbf{N}(\wp_1,\wp_2)$ are defined in Theorem 2.1.

Corollary 2.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for k-Riemann-Liouville fractional integral holds:*

$$\begin{aligned}
&\frac{\Gamma_k(\alpha)}{(\wp_2 - \wp_1)^{\frac{\alpha}{k}}} \left[I^{\alpha}_{\wp_{1+},k} \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + I^{\alpha}_{\wp_{2-},k} \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) \right] \\
&\leq \frac{k(\alpha^2 + \alpha k + 2k^2)}{\alpha(k+\alpha)(2k+\alpha)} \mathbf{M}(\wp_1, \wp_2) \\
&+ \frac{2k^2}{(k+\alpha)(2k+\alpha)} \mathbf{N}(\wp_1, \wp_2), \qquad (11)
\end{aligned}$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, we have our desired inequality (11). □

Theorem 2.2. . *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1 + \wp_2}{2}\right) \mathcal{F}_2\left(\frac{\wp_1 + \wp_2}{2}\right) \\
&\leq \frac{1}{4\Lambda(1)} \left[{}_{\wp_1+}I^{\wp_2}_{\varphi} \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + {}_{\wp_2-}I^{\wp_1}_{\varphi} \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) \right] \\
&+ \frac{1}{4\Lambda(1)} \left\{ J_2 \mathbf{M}(\wp_1, \wp_2) + J_1 \mathbf{N}(\wp_1, \wp_2) \right\}, \qquad (12)
\end{aligned}$$

where $J_1,\ J_2,\ \mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are same as used in Theorem 2.1.

Proof. Consider the left hand side of inequality (12) that can be written as

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1 + \wp_2}{2}\right) \mathcal{F}_2\left(\frac{\wp_1 + \wp_2}{2}\right) \\
&= \mathcal{F}_1\left(\frac{\jmath\wp_1 + (1-\jmath)\wp_2}{2} + \frac{(1-\jmath)\wp_1 + \jmath\wp_2}{2}\right) \qquad (13) \\
&\cdot \mathcal{F}_2\left(\frac{\jmath\wp_1 + (1-\jmath)\wp_2}{2} + \frac{(1-\jmath)\wp_1 + \jmath\wp_2}{2}\right) \\
&= \frac{1}{4} \left\{ (\mathcal{F}_1(\jmath\wp_1 + (1-\jmath)\wp_2) + \mathcal{F}_1((1-\jmath)\wp_1 + \jmath\wp_2)) \right. \\
&\left. \times (\mathcal{F}_2(\jmath\wp_1 + (1-\jmath)\wp_2)+) + \mathcal{F}_2((1-\jmath)\wp_1 + \jmath\wp_2)) \right\} \\
&= \frac{1}{4} \left\{ \mathcal{F}_1(\jmath\wp_1 + (1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1 + (1-\jmath)\wp_2) \right. \\
&+ \mathcal{F}_1((1-\jmath)\wp_1 + \jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1 + \jmath\wp_2) \\
&+ \mathcal{F}_1(\jmath\wp_1 + (1-\jmath)\wp_2)\mathcal{F}_2((1-\jmath)\wp_1 + \jmath\wp_2) \\
&\left. + \mathcal{F}_1((1-\jmath)\wp_1 + \jmath\wp_2)\mathcal{F}_2(\jmath\wp_1 + (1-\jmath)\wp_2) \right\}. \qquad (14)
\end{aligned}$$

Since $\mathcal{F}_1,\ \mathcal{F}_2$ are convex functions, so (14) can be written as

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{1}{4}\left\{\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\right.\\
&+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\\
&+(\jmath\mathcal{F}_1(\wp_1)+(1-\jmath)\mathcal{F}_1(\wp_2))((1-\jmath)\mathcal{F}_2(\wp_1)+\jmath\mathcal{F}_2(\wp_2))\\
&\left.+\ ((1-\jmath)\mathcal{F}_1(\wp_1)+\jmath\mathcal{F}_1(\wp_2))(\jmath\mathcal{F}_2(\wp_1)+(1-\jmath)\mathcal{F}_2(\wp_2))\right\}\\
&=\frac{1}{4}\left\{\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\right.\\
&+\ \mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\}\\
&+\frac{1}{4}\left\{2\jmath(1-\jmath)\left[\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)(\wp_2)\right]\right.\\
&\left.+\ (2\jmath^2-2\jmath+1)\left[\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1)\right]\right\}. \qquad (15)
\end{aligned}$$

Multiplying (15) with $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides and integrating the obtain inequality with respect to '$\jmath$' over $[0,1]$,then we have our required inequality (12). □

Remark 2.4. If we use $\varphi(\jmath)=\jmath$ in Theorem 2.2, we have a well known inequality for the product of two convex function, given by B. G. Pachpatte in [12], i.e.,

$$\begin{aligned}
2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \ &\leq\ \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
&\quad+\frac{1}{6}\mathbf{M}(\wp_1,\wp_2)+\frac{1}{3}\mathbf{N}(\wp_1,\wp_2),
\end{aligned}$$

where $\mathbf{M}(\wp_1,\wp_2)$ and $\mathbf{N}(\wp_1,\wp_2)$ are same as used in Theorem 2.1.

Remark 2.5. If we suppose $\varphi(\jmath)=\frac{\jmath^\alpha}{\Gamma(\alpha)}$ in Theorem 2.2, we have an inequality for the product of convex functions via Riemann-Liouville fractional integral given by Chen, F. in [4], i.e.,

$$\begin{aligned}
&2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq\frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^\alpha}\left[I^\alpha_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^\alpha_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&+\mathbf{M}(\wp_1,\wp_2)\frac{\alpha}{(\alpha+1)(\alpha+2)}\\
&+\mathbf{N}(\wp_1,\wp_2)\left[\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}-\frac{1}{2}\right].
\end{aligned}$$

Corollary 2.2. *If we choose $\mathcal{F}_2(x)=1$ for all $x\in[\wp_1,\wp_2]$ in Theorem 2.2, then we obtain the following inequality*

$$2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I^{\wp_2}_{\varphi}\mathcal{F}_1(\wp_2)+{}_{\wp_2-}I^{\wp_1}_{\varphi}\mathcal{F}_1(\wp_1)\right]+\frac{\mathcal{F}_1(\wp_1)+\mathcal{F}_1(\wp_2)}{2}.$$

Corollary 2.3. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for k-Reimann-Liouville fractinal integral holds:*

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{\alpha\Gamma_k(\alpha)}{4(\wp_2-\wp_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\wp_1+,k}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-,k}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&+\frac{\alpha}{4}\left[\frac{2k^2}{(k+\alpha)(2k+\alpha)}\mathbf{M}(\wp_1,\wp_2)+\frac{k(\alpha^2+\alpha k+2k^2)}{\alpha(k+\alpha)(2k+\alpha)}\mathbf{N}(\wp_1,\wp_2)\right], \qquad (16)
\end{aligned}$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.2, we have our desired inequality (16). □

Theorem 2.3. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for generalized fractional integral holds:*

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+{}_{\wp_2-}I_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \qquad (17)\\
&\leq \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}.
\end{aligned}$$

Proof. Since $\mathcal{F}_1,\ \mathcal{F}_2$ are convex functions, then by using the definition of convex function we have

$$\begin{aligned}
&2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&=2\mathcal{F}_1\left(\frac{x+y}{2}\right)\mathcal{F}_2\left(\frac{x+y}{2}\right)\leq \mathcal{F}_1(x)\mathcal{F}_2(x)+\mathcal{F}_1(y)\mathcal{F}_2(y), \qquad (18)
\end{aligned}$$

by setting $x = \wp_2 + \jmath(\wp_1 - \wp_2)$ and $y = \wp_1 + \jmath(\wp_2 - \wp_1)$ in18, we obtain

$$\begin{aligned}
&2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \mathcal{F}_1(\wp_2+\jmath(\wp_1-\wp_2))\mathcal{F}_2(\wp_2+\jmath(\wp_1-\wp_2))\\
&+\mathcal{F}_1(\wp_1+\jmath(\wp_2-\wp_1))\mathcal{F}_2(\wp_1+\jmath(\wp_2-\wp_1)). \qquad (19)
\end{aligned}$$

Now, multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of (19) and integrating resultant inequality with respect to '$\jmath$' over $[0, 1]$, we have

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+{}_{\wp_2-}I_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]. \qquad (20)
\end{aligned}$$

Now, we have to prove right part of our inequality (19), according to given condition our $\mathcal{F}_1$, $\mathcal{F}_2$ are convex so we have

$$\begin{aligned} &\mathcal{F}_1(\wp_2 + \jmath(\wp_1 - \wp_2))\mathcal{F}_2(\wp_2 + \jmath(\wp_1 - \wp_2)) \\ &+ \mathcal{F}_1(\wp_1 + \jmath(\wp_2 - \wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2 - \wp_1)) \\ &\leq \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2). \end{aligned} \tag{21}$$

Now, multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of (20) and integrating resultant inequality with respect to '$\jmath$' over $[0,1]$, we have

$$\begin{aligned} &[{}_{\wp_1+}I_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + {}_{\wp_2-}I_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)] \\ &\leq \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2), \end{aligned} \tag{22}$$

by relating (20) and (22), we have our desired inequality (17). □

Remark 2.6. If we choose $\mathcal{F}_2(x) = 1$ for all $x \in [\wp_1, \wp_2]$ in Theorem 2.3, then we obtain the inequality (6).

Corollary 2.4. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following integral inequalities holds:*

$$\begin{aligned} \mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) &\leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)\mathcal{F}_2(x)dx \\ &\leq \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \end{aligned} \tag{23}$$

Proof. By using the idea $\varphi(\jmath) = \jmath$ in Theorem 2.3 we get our required inequalities (23). □

Corollary 2.5. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for Reimann-Liouville fractional integral holds:*

$$\begin{aligned} &\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\leq \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^{\alpha}}\left[I^{\alpha}_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + I^{\alpha}_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ &\leq \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \end{aligned} \tag{24}$$

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.3 we have our desired inequalities (24). □

Corollary 2.6. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for k-Riemann-Liouville fractional*

integral holds:

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{\alpha\Gamma_k(\alpha)}{2(\wp_2-\wp_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\wp_1+,k}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-,k}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&\leq \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \qquad (25)
\end{aligned}$$

Proof. By using the idea $\varphi(\jmath)=\frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.3 we have our desired inequalities (25). □

Conclusion

In this paper, we derived inequalities of Hermite-Hadamard type for the product of two convex functions by using the generalized fractional integral. Interested reader can derive more inequalities of Hermite-Hadamard type with different approaches by using generalized fractional integral.

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M-bands wavelet multiresolution analysis of assets

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Abstract. Given that the predictability of financial assets is indispensable to optimize the allocation of the investors' portfolio, a large literature review was dedicated to the question of predictability. Indeed, different studies have examined the relationship between the expected returns and the financial and macroeconomic variables to determine the most predictive indicators, notably the impact of the variables fluctuations on the prediction of the expected returns. Consequently, this paper consists on investigating the impact of the fluctuations in the aggregate price-earnings ratio at different timescales on the stock returns by using financial data from the USA. The data frequency is quarterly from 1952 to 2011. By aggregating the price-earnings ratio via multiresolution wavelets analysis, the results of the estimation of the Vector Autoregressive Model (VAR) demonstrated that the cycles in the price-earnings ratio presented strong predictors for the stock returns at short and intermediate horizons.

Keywords: wavelets, decomposition, time series, financial ratios, multiscale analysis, VAR, stock returns.

1. Introduction

The predictability of the financial assets is an indispensable element to optimize the allocation of the investor's portfolio. Some studies approved the predictability of the returns and other studies denied it. For example, Fama and French (1989) approved this predictability as it could not be inconsistent with the ef-

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ficiency market theory, on the other hand, other researchers as Mankiw et al. (1989) who denied it due to the same reason of the efficiency market theory.

This research will adopt the hypothesis of the predictability according to the paper of (Fama and French, 1989). In this context, the researchers have studied the algebraic relations between the expected returns, the macroeconomic variables and the financial variables to predict the future returns and to get the most predictive financial variables. Fama and French (1989) and Lettau and Ludvigson(2001) demonstrated that the expected returns fluctuate with the economic cycle. Other studies have related the fluctuations of the expected returns to the macroeconomic variables and to the financial ones and they approved the existence of a relation between the logarithm of payout ratio and the excess returns ones (Campbell and Shiller, 1988). Also, Campbell(2000,2003) confirmed that the valorization ratios have a predictive capacity on the returns at long horizons. With approximating the total wealth by a linear combination of the log labor income and log observable assets, Lettau and Ludvigson (2001) created a new ratio called consumption-wealth ratio (the cay ratio), they approved that the cay ratio has a predictive capacity at the short horizons. Also, Lettau and Nieuwerburgh (2008) have approved that the aggregation of the price earnings-ratio has notable effects on the prediction of the future returns. Thus, according to the paper of Lettau and Nieuwerburgh (2008) and Boucher and Maillet (2011) adopted the idea of aggregating the price-earnings ratio(See also Boldrin et al., 1995; Boldrin et al., 1997; Boldrin et al., 1999;Boldrin et al., 2001;Rouwenhorst, 1995; Jermann, 1998 and Jermann and Quadrini, 2007)

The problematic was how to aggregate this ratio to extract the significant information. Two approaches have been most known; the first one is the spectral analysis which is used to distinguish the different frequency components (Hamilton, 1994) and the second one is the standard filters which allows the extraction of specific features such as seasonality (periodicity) and trends. Hodrick and Prescott(1997) (see also Genay et al., 2001 and the references sited there). Nevertheless, two major drawbacks were detected in the standard methods of the time series analyzing. The first one is the restrictive hypothesis like the stationarity of the series, or this is not always the case of the financial series. The second one is the pure frequency-domain representation of the data i.e. all information from the time-domain representation is not captured.

Recently, a new method became increasingly famous in the economics and finance, called the wavelets. The wavelets analysis provides a simultaneous representation of a data series in the frequency and time domains. In fact, this technic can be used to study the financial variables over time and frequency, and it avoids the drawbacks of the spectral analysis. Boucher and Maillet (2011) decomposed the price-earnings ratio, by using the dyadic wavelets (M-bands equals 2), to identify the source of fluctuations of the assets and thus to define the most predictive variables for the future returns. They identified a timescale for that price earnings ratio to isolate the stable movements from the fluctuations.

They concluded that the first-time scale component of the price-earnings ratio explains a large part of the one-quarter stock returns.

Bianchi and Tamoni (2016) defined the expected returns as the outcome of an updating process based on medium and long-term information, showed that accounting for the low- to mid-frequency information conveyed by standard predictors, allows to identify long-lasting effects in the dynamics of high-frequency expected returns. Bandi et al. (2017) worked on the predictability over short horizons by proposing a novel modeling framework in which predictability is specified as a property of low-frequency components of both excess markets returns and economic uncertainty. This research consists on studying the predictability of the expected returns at short-term. The used variables in this analysis are the ones provided in the paper of (Boucher and Maillet, 2011) and largely discussed in different researches. Taji and Gore (2014) studied the Adaptive Texture Segmentation by using the M-bands wavelets. Also Schrrer (2015) suggested an analysis of the high frequency time series by using wavelets. Kl and Uur (2018) used the Multiresolution method to examine the SP500 time series. Here, the M-bands wavelets analysis will be used were M-bands in 3, 4 to see if the obtained results of (Boucher and Maillet, 2011) depends on the number of bands or not. Using the M-bands with M¿2 allows the analysis of higher frequency components and more numerous ones. The rest of the paper is organized as follows. Section 2 explains the methodology used in the study. Section 3 presents the results of the study and section four concludes.

2. Methodology and data

The methodology section consists on providing a general overview about the wavelets, the multiresolution decomposition and of the Vector Autoregressive model (VAR).

2.1 Wavelets and multiresolution decomposition

Wavelets have been known as an extension of Fourier analysis. The wavelets analysis was developed and extended by authors such as Grossman and Morlet in 1985, Daubechies in 1990 and 1992 and Meyer in 1992. The success of the wavelets was due to a discretization method introduced in 1986 by Stphane Mallat and then developed by Meyer and Daubechies and others as continued method. (See Al Ani, 2016; Emeric and Hammadi, 2003; Quarta,2011;Conway and Frame, 2000 for further details)

As the name indicates, a wavelet is a small wave. The term "small" mainly explains in this context that the wave increases and decreases in a limited time support (Masset, 2008). Thus, the main feature of a wavelet is its compact support, i.e. the wavelet function is limited in both the time and frequency domains. In fact, wavelets can decompose the non-stationary series without any restrictive hypothesis with localization at time and frequency. Yves MEYER

defined the wavelets as following *"... unlike the Fourier series, the coefficients of a series of wavelets translate the properties of the functions in a simple, precise and reliable way, at least the properties that correspond to a discontinuity, an unplanned event."* Theoretically, a wavelet is simply a time function that follows a basic rule, known as the wavelet eligibility condition:

$$\mathbb{C}_\psi = \int_0^\infty \frac{|\Psi(f)|}{f} df < \infty \tag{1}$$

With

$$\Psi(f) = \int_{-\infty}^{+\infty} \psi(t) e^{-2\pi i f t} dt \tag{2}$$

Equation 2.2 presents Fourier transform. f is a frequency function for $\Psi(f)$, Ψ is called the mother wavelet. To ensure that $C_\psi < \infty$, the following conditions, related to the mother wavelet, must be imposed:

1. $\Psi(0) = 0$, or $\int_{-\infty}^{+\infty} \Psi(t) = 0$
2. $\int_{-\infty}^{+\infty} |\Psi(t)|^2 dt = 1$ (the energy unit).

Generally, wavelet transforms are classified into three categories; transformed into continuous wavelets, transformed into discrete wavelet and transformed into wavelets based on multiresolution analysis. In this analysis, the multiresolution decomposition analysis will be considered (for further details see (Boucher and Maillet, 2011) and the references stated there). Despite the problems of resolutions (time and frequency)[1] are the results of a similar phenomenon to the uncertainty's problem of Heisenberg and they exist independently from the used transformation, it is possible to analyze any signal by using an alternative approach named "multiresolution analysis". A multiresolution analysis allows a good time-resolution and a poor frequency-resolution for the high frequencies. In addition, a good frequency-resolution and a poor time-frequency for the low frequencies. This approach has a sense, especially when the real signal has components with high frequencies at the short-horizon and components with low frequencies at the long horizon. Mathematically, it is usually easier to present a development of signal on a countable group of wavelets.

This could be realized by varying the scale factor with dyadic way, by choosing $e = 2^j, j \in \mathbb{Z}$ $u = k2^j, j \in \mathbb{Z}$. In this context, a multiresolution analysis is defined as series of closed under- vectoral spaces of $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and verifying the following proprieties:

- $\forall j \in \mathbb{Z}, V_{j+1} \subset V_j$

1. Time resolution: How far two peaks in time can be separated from one another in the domain of transformation. Frequency resolution: How far two spectral components can be separated from one another in the field of transformation.

- $\forall j \in \mathbb{Z}, f(t) \in V_j \Leftrightarrow f(\frac{t}{2}) \in V_{j+1}$
- $\lim_{j\to\infty} V_j = U_{j\in\mathbb{Z}} V_j = 0$ and $\lim_{j\to-\infty} V_j = \overline{U_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
- $\exists \psi_0$ such as $\psi_0(t-n), n \in \mathbb{Z}$ is an orthonormal basis of V_0; ψ_0 is the scale function.

Thus, it could be confirmed that $2^{\frac{-j}{2}}\psi_0(\frac{t}{2^j}-k), k \in \mathbb{Z}$ constructs an orthonormal basis of V_j. The principal of multiresolution analysis of a signal f is the realizations of successive orthogonal projections of the signal in the spaces V_j which conducts to bigger approximations of f as j increases. The difference between two consecutive approximations represents the information "detail" or "cycle" which is lost in passing from a scale to another; this information is in the under-space W_j orthogonal to V_j with:$V_{j-1} = V_j \oplus W_j$. The existence of a wavelet $\psi_1 \in L^(\mathbb{R})$ is demonstrated with $2^{\frac{-j}{2}}\psi_1(\frac{t}{2^j}-k), k \in \mathbb{Z}$ is an orthonormal basis of W_j. The orthogonal wavelet decomposition of a signal f could be done in an efficient way by following the algorithm of (Mallat, 1998). Thus, the coefficients of approximations $(c_{j,0}[k])_{k\in\mathbb{Z}}$ and the details $(c_{j,1}[k])_{k\in\mathbb{Z}}$ are determined in each level of resolution and they are defined as:

$$\forall K \in \mathbb{Z}, c_{j,0}[k] = \left\langle f, \frac{1}{2^{j/2}}\psi_0(\frac{\cdot}{2^j}-k)\right\rangle \tag{3}$$

$$\forall K \in \mathbb{Z}, c_{j,1}[k] = \left\langle f, \frac{1}{2^{j/2}}\psi_1(\frac{\cdot}{2^j}-k)\right\rangle \tag{4}$$

$\langle ., . \rangle$ is the scalar product of $L^2(\mathbb{R})$.

2.2 Vector autoregressive model

The modeling part of this research is carried out in the context of Vector Autoregressive Model (VAR). This methodology has gained widespread use in empirical business cycles analysis, as it has proved to be a flexible and tractable way to analyze economic time series. Vector autoregression (VAR) models have been capable of describing the rich dynamic structure of the relationships between economic variables (Bjornland, 2000). The VAR models are usually presented through impulse responses (that measure the effects of the different shocks on the variables of study), and variance decomposition (which measures the relative importance of the different shocks to the variation in the different variables). (Sims, 1980) was the first who introduced VAR models as an alternative to the large-scale macroeconometric models. According to Sims, all variables appearing in the structural models could be argued to be endogenous. Economic theory place only weak restrictions on the reduced form coefficients and on which variables that should enter a reduced form model. Similar ideas had already been

put forward by (Liu, 1960), but the proposed solution by Sims was new. Sims suggested that empirical research should use small-scale models identified via a small number of constraints (Bjornland, 2000).

Sims argued that VARs provide a more systematic approach to imposing restrictions and could lead one to capture empirical regularities which remain hidden to standard procedures. In contrast, the results from policy exercises on large-scale macreconometric models are hard to compare and recreate and can easily be amended by their users with judgmental ex-post decisions (Bjornland, 2000). Before the VAR estimation, the unit roots using the augmented Dickey-Fuller (ADF) and KPSS tests [2] for all the variables will be tested. The VAR is often perceived as an alternative to the simultaneous equation method. It is a systems regression model in that there is more than one dependent variable. In the most basic bivariate example, where there are just two variables, then each of their current values will depend on combinations of the previous values of variables and error terms.

$$\begin{cases} y_t = \alpha_0 + \alpha_1 x_{t-1} + \beta_1 y_{t-1} + ... + \alpha_k x_{t-k} + \beta_k y_{t-k} + u_t \\ x_t = \chi_0 + \chi_1 x_{t-1} + \delta_1 y_{t-1} + ... + \chi_k x_{t-k} + \delta_k y_{t-k} + v_t \end{cases} \tag{5}$$

The number of lags included in the VAR depends on either the data (i.e. monthly data would require 12 lags) or the minimization of the Akaike or Schwarz-Bayesian criteria (maximizing in some textbooks depending on how the criteria are set up). In addition, it is assumed that the error term is not serially correlated. The system can be expanded to include any number of variables and is used extensively in the finance literature. VAR models have a number of advantages over univariate time series models, for instance, there is no need to specify which variables are exogenous and which endogenous, variables are endogenous. In addition, the issue of model identification does not occur when using a VAR. Providing there are no contemporaneous terms acting as regressors, OLS [3] can be used to estimate each equation individually, as the regressors are lagged so treated as pre-determined. In addition, VARs are often highly efficient at forecasting compared to traditional models.

A limitation of the VAR approach is that it must be estimated to low order systems and the effects of omitted variables will be in the residuals. This may lead to major distortions in the impulse responses, making them of little use for structural interpretations (Hendry, 1995), although the system may still be used for predictions (see e.g. (Hendry and Doornik, 1997), and the references stated there). Further, measurement errors or misspecifications of the model will also induce unexplained information left in the disturbance terms, making interpretations of the impulse responses even more difficult . (See also Bjornland,

2. To confirm the test results obtained from the ADF and PP tests, Kwiatkowski Phillips, Schmidt and Shins test (1992) (KPSS) is suggested to eliminate a possible low power against stationary near unit root processes which occurs in the ADF and PP tests.
3. OLS estimator,$\hat{beta} = (X'X)^{-}1X'Y$, Y dependent variable and X dependent variable.

2000; Hendry,1995;Lubrano,2007 for further details about VAR modelling). The following diagram presents this research framework.

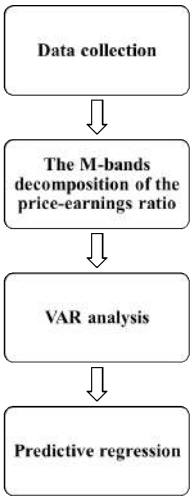


Figure 1: Research framework.

2.3 Dataset

The study sample is composed of financial data from the USA. The frequency is quarterly from the first quarter of the year 1952 to the earliest quarter of the year 2011. The data is extracted from different sources; the returns are from the CRSP (Center for Research in Security Prices). The Stock prices, the dividends per share and the quarterly earnings per share are available in the site of Standard and Poors and the site of Saint Louis bank. For the macro Financial and monetary indicators, the source is Federal Reflectance Database (FREDII). The input variables are described as following:
The dividend-earnings ratio or the dividend payout ratio (PER), defined as the ratio between the price of a stock and its earnings per share. Rancourt (2008) provided a full paper related to this variable.
Price-earnings ratio (PAY), calculated by dividing the dividend per share of earnings. This variable was considered in the paper of Lamont (1998) as a control variable.
The consumption-wealth ratio (CAY), defined by Lettau and Ludvigson (2001). It is the measure of short-term deviations from the long run cointegration relationship among the natural logarithm of consumption, labor income and aggregate wealth. The data related to cay variable is available at the official website of the authors.
Default spread (DEF), the default spread known also as a credit default swap spread or a credit spread. It is defined as the yield or return differential between long-term BAA corporate bonds and long-term AAA or U.S. Treasury bonds.

Elton et al. (2001) presented further details related to default spread. Different studies showed that the yield spread between BAA and AAA corporate bond spread can predict expected returns in stocks and bonds such as (Fama and Schwert, 1977; Keim and Stambaugh, 1986;Campbell,1987 and Fama and French, 1989 among others).
Term spread (TRM), defined as the difference between the 10-year Treasury bond yield and the 3-month Treasury bond yield. This variable was considered in the papers of Fama and French (1989) and Campbell (1987).
Detrended risk-free rate (RREL), is the stochastically detrended risk-free rate defined as the T-bill rate minus its last four-quarter average.
Inflation rate (INF), Campbell and Vuolteenaho(2004) demonstrated that the inflation rate is highly correlated with the price earnings ratio. In the following analysis part, the coefficients obtained from the decomposition of the price-earnings ratio by using Meyer wavelets will be used.[4]

3. Empirical findings

The aim of this section is to analyze the empirical results of this research; first, the M-bands decomposition of the price-earnings ratio will be provided (where the number of bands in 3,4) and then the relationship between the cycles of the price-earnings ratio and the macroeconomic variables by using VAR model will be investigated.

3.1 M-bands decomposition of the price-earnings ratio

To decompose the price-earnings ratio, the wavelet multiresolution analysis will be used. As shown in Figure 2, the x-axis presents the quarterly period 1952-2011. The y-axis presents the details and the approximation. The details show that the order of regular cycles of the price-earnings ratio is increasing with the level of decomposition. Thus, the wavelet decomposition allows the disaggregating of the variable in function of the degree of regularity. It could be explained otherwise, d1 presents the shorter term; d2 presents the short-term, d4 present the intermediate term and a4 presents the approximation. The variation of these cycles can be explained as follows:

- The cycles of the first decomposition d1 exhibit volatility during certain periods due to geopolitical, economic or financial conditions including the oil shocks of 1973, 1979 and 2008. Also, the Crash of October 1987 or "Black Monday", the attacks of September 2001, the Iraq war in 2003, the subprime crisis in 2007 and the banking and financial crisis of autumn 2008.

4. It was concluded that the results are independent from the choice of the type of wavelets. Results are not reported here.

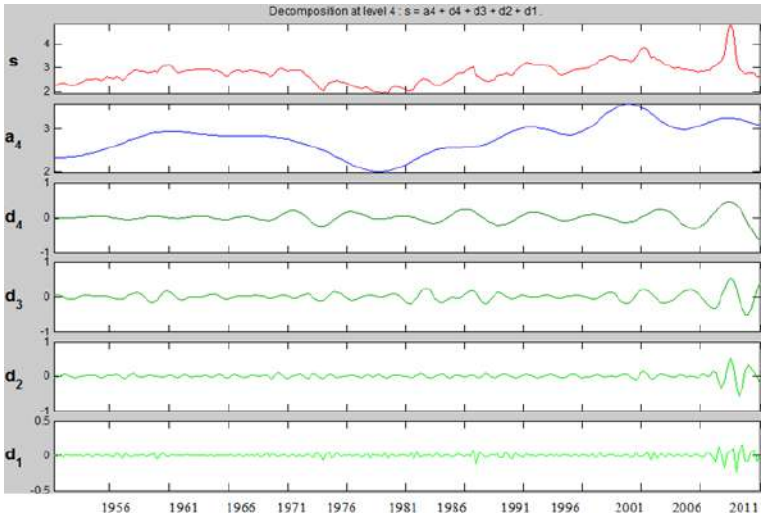


Figure 2: Decomposition of the price earning-ratio.

- The cycles of the details d2 and d3 show also similar events. They exhibit a common model regular than the d4.
- The a4 trend is very smooth with a stable phase almost of 20 years.

To approve if these cycles contain information about expected returns, an estimate of the autoregressive model VAR will be used with a lag of one quarter. The procedure will be detailed in the next section.

3.2 VAR analysis

Before carrying out the VAR model, the Granger causality test to examine the causality between the different variables will be provided.

3.2.1 VAR model

VAR model is especially useful for describing the dynamic behavior of economic and financial time series and forecasting. Thus, to examine the relationships between the cycles of the price-earnings ratio and the macroeconomic conditions, a model VAR was estimated twice by using the lagged variables. Following the Akaike criteria, the number of lag was 1 (which corresponds to the minimum Akaike value). As shown in (Table 2), the estimation of VAR model after the decomposition of price-earnings ratio (via 3-bands multiresolution), demonstrated that the first-decomposition (or detail d1) contains a distinguished information in short-term; the first scale decomposition has a strong ability to predict the variations in short-term. However, d1 appears related to the default spread and dividend payout ratio. The estimation of VAR model after including the decomposition of price-earnings ratio (via 4-bands) is presented in (Table 3),

the estimation resutls confirmed that the first cycle of decomposition of the ratio price-earnings capture the short-term variation (over one quarter) and still related to the default spread and to the dividend payout ratio. Therefore, it could be concluded that the variables defined in the literature do not appear at short horizons i.e. they have a low capacity for forecasting at short-term except the default spread and the dividend payout ratio However, their predictive power to capture the variations of returns over the intermediate-term appears at medium-term (more than one quarter).

3.2.2 Granger causality

As is well known, to carry out the Granger causality test, the series should be stationary. When the causal relations between the first decomposition d1 and the other macro-financial variables was examined, no causality between d1 and those variables (Table 4) was found. These results confirm that most of the macro-financial variables used in the literature don't capture the fluctuations at short-horizon but at intermediate-horizon (from few quarters to few years) and this is confirmed too by the presence of causality between some variables such as the default spread and the dividend-earnings ratio and the second decomposition of the price-earnings ratio (d2).

3.2.3 Predictive regression

To approve the predictive ability of price-earnings ratio after decomposition, univariate predictive regression and multivariate predictive regression have been made at the short-horizon (one quarter) (Table 5). The regressions contain an intercept and an autoregressive term at the order 1 (AR (1)) referring one quarter. The Newey-test for the correction of the t-student statistics was used. The first line of the table shows that the predictive ability of the price-earnings ratio is significant but not so important. In fact, it predicts just 1% from the returns of the next quarter. However, the first decomposition d1 has a predictive ability of 10%. So, using d1 instead of the PER gives 9% more to predict the future returns. The third line and the fourth one from the table of regression demonstrate that the predictive ability of the PER's ratio decreases with the augmentation of the resolution's level. The predictive regressions of the other variables such as the PAY and the default spread didn't demonstrate any effect on the prediction of the future returns. The predictions related to the CAY, RREL and TRM variables demonstrated a predictive ability of those variables but which is negligent compared to d1 and d2. For the multivariate predictive regressions, without including the price-earnings ratio, no predictive ability to get the future returns has been found. However, when the cycles of the PER were included, it appears that the cycles d1 and d2 have an important predictive capacity at the short horizon.

4. Conclusion and perspective

The paper results showed that the decomposition of the price-earnings ratio avoids the problem of non-stationarity. Moreover, the first decomposition scale-time (d1) of the price-earnings (PER) ratio demonstrated a strong capacity to predict the variations of the assets at short-horizon (which approves the results obtained by Boucher and Maillet (2011). The details d1, d2, d3, and d4 contain information which doesn't exist in the other variables defined in the literature. It was noted that the macroeconomic and the financial variables, such as the payout dividend ratio and the default spread, have acceptable predictive capacities at short horizon, but they show more predictive capacity at intermediate horizons. To improve this work, a lot of perspectives could be mentioned as including the decomposition of the dividends ratio and check the predictive capacity of his cycles as it has been considered as important as the price-earnings ratio.

5. Appendices

Table 1: Estimation of VAR model (after a 3-bonds decomposition)

		D1_1	D1_2	A1	D2_1	D2_2	A2	PAY	CAY	DEF	RREL	INF
D1_1(-1)	coef.	0.451522	0.227506	-0.712132	-0.026192	0.051286	-0.806962	-0.389208	-0.004373	-0.396572	0.359558	0.007037
	Std-error	(0.06160)	(0.26016)	(0.14312)	(0.01826)	(0.06784)	(0.41142)	(0.04112)	(0.00277)	(0.07104)	(0.26894)	(0.00303)
	t-statistic	[7.33017]	[0.87449]	[-4.97592]	[-1.43417]	[0.75597]	[-1.96142]	[-9.46498]	[-1.57606]	[-5.58204]	[1.33695]	[2.32549]
	p-value	0.0000		0.0000			0.0499	0.0000		0.0000		0.0237
D1_2(-1)	coef.	-0.001815	0.819551	-0.046118	-0.008993	0.001517	-0.136548	-0.008080	-2.47E-05	0.002431	0.034922	0.000643
	Std-error	(0.01001)	(0.04229)	(0.02327)	(0.00297)	(0.01103)	(0.06688)	(0.00668)	(0.00045)	(0.01155)	(0.04372)	(0.00049)
	t-statistic	[-0.18123]	[19.3784]	[-1.98226]	[-3.02908]	[0.13751]	[-2.04167]	[-1.20873]	[-0.05479]	[0.21050]	[0.79877]	[1.30786]
	p-value		0.0000	0.0476	0.0025		0.0413					
A1(-1)	coef.	-0.015004	-0.060076	0.858930	-0.013068	-0.046500	0.484274	-0.019228	0.000469	0.010832	0.028079	-0.000875
	Std-error	(0.01871)	(0.07904)	(0.04348)	(0.00555)	(0.02061)	(0.12499)	(0.01249)	(0.00084)	(0.02158)	(0.08171)	(0.00092)
	t-statistic	[-0.80176]	[-0.76010]	[19.7550]	[-2.35528]	[-2.25613]	[3.87450]	[-1.53915]	[0.55601]	[0.50186]	[0.34367]	[-0.95162]
	p-value			0.0000	0.0186	0.0242	0.0001					
D2_1(-1)	coef.	-0.133019	-1.092295	-0.855850	0.674840	0.066683	-1.089277	0.052499	0.018918	-0.049720	1.175235	-0.006197
	Std-error	(0.16948)	(0.71580)	(0.39376)	(0.05025)	(0.18666)	(1.13196)	(0.11314)	(0.00763)	(0.19547)	(0.73995)	(0.00833)
	t-statistic	[-0.78488]	[-1.52599]	[-2.17351]	[13.4302]	[0.35725]	[-0.96229]	[0.46403]	[2.47788]	[-0.25436]	[1.58826]	[-0.74422]
	p-value			0.0298	0.0000				0.0118			
D2_2(-1)	coef.	0.004768	-0.115373	-0.011727	0.003038	0.966746	-0.087157	-0.016796	-0.001772	-0.043976	-0.024513	-0.001547
	Std-error	(0.01817)	(0.07674)	(0.04221)	(0.00539)	(0.02001)	(0.12135)	(0.01213)	(0.00082)	(0.02096)	(0.07933)	(0.00089)
	t-statistic	[0.26245]	[-1.50350]	[-0.27780]	[0.56401]	[48.3121]	[-0.71822]	[-1.38481]	[-2.16564]	[-2.09859]	[-0.30901]	[-1.73273]
	p-value					0.0000			0.0300	0.0350		
A2(-1)	coef.	0.009684	0.016784	0.029960	0.002104	0.015080	0.807061	0.005730	-0.000218	0.003217	-0.010606	-0.000200
	Std-error	(0.00588)	(0.02483)	(0.01366)	(0.00174)	(0.00647)	(0.03927)	(0.00392)	(0.00026)	(0.00678)	(0.02567)	(0.00029)

Vector Autoregression Estimates

Sample (adjusted): 1952Q2 2010Q2

Included observations: 233 after adjustments

Standard errors in () & t-statistics in []

		D1_1	D1_2	A1	D2_1	D2_2	A2	PAY	CAY	DEF	RREL	INF
	Std-error	(0.00588)	(0.02483)	(0.01366)	(0.00174)	(0.00647)	(0.03927)	(0.00392)	(0.00026)	(0.00678)	(0.02567)	(0.00029)
	t-statistic	[1.64727]	[0.67596]	[2.19341]	[1.20693]	[2.32898]	[20.5538]	[1.45996]	[-0.82340]	[0.47445]	[-0.41319]	[-0.69233]
	p-value			0.0284		0.0199	0.0000					
PAY (-1)	coef.	0.151513	0.569953	0.473201	0.019467	0.004449	0.462579	0.979924	0.000424	0.033812	-0.187222	-0.003762
	Std-error	(0.04902)	(0.20703)	(0.11389)	(0.01453)	(0.05399)	(0.32739)	(0.03272)	(0.00221)	(0.05654)	(0.21401)	(0.00241)
	t-statistic	[3.09098]	[2.75303]	[4.15498]	[1.33950]	[0.08242]	[1.41291]	[29.9462]	[0.19185]	[0.59807]	[-0.87481]	[-1.56224]
	p-value	0.0020	0.0059	0.000				0.0000				
CAY (-1)	coef.	0.181601	-5.450554	0.279563	-0.117367	0.867025	-3.691627	0.231753	0.896889	-0.204919	0.284806	-0.071730
	Std-error	(0.65938)	(2.78492)	(1.53201)	(0.19550)	(0.72622)	(4.40408)	(0.44018)	(0.02970)	(0.76051)	(2.87891)	(0.03239)
	t-statistic	[0.27541]	[-1.95717]	[0.18248]	[-0.60035]	[1.19389]	[-0.83823]	[0.52649]	[30.1947]	[-0.26945]	[0.09893]	[-2.21427]
	p-value								0.0000			0.0272
DEF (-1)	coef.	0.072143	-0.022269	0.213889	-0.001579	-0.012180	-0.011899	-0.000838	-0.003021	0.831864	-0.221713	-0.005297
	Std-error	(0.03045)	(0.12863)	(0.07076)	(0.00903)	(0.03354)	(0.20341)	(0.02033)	(0.00137)	(0.03512)	(0.13297)	(0.00150)
	t-statistic	[2.36889]	[-0.17313]	[3.02284]	[-0.17488]	[-0.36314]	[-0.05850]	[-0.04122]	[-2.20225]	[23.6830]	[-1.66744]	[-3.54052]
	p-value	0.0179		0.0025					0.0276	0.0000		0.0003
RREL (-1)	coef.	-0.021381	0.024832	-0.063071	-0.006890	-0.012158	-0.055268	-0.003080	0.000204	0.032118	0.018134	0.003915
	Std-error	(0.01552)	(0.06557)	(0.03607)	(0.00460)	(0.01710)	(0.10369)	(0.01036)	(0.00070)	(0.01791)	(0.06778)	(0.00076)
	t-statistic	[-1.37724]	[0.37872]	[-1.74856]	[-1.49688]	[-0.71107]	[-0.53301]	[-0.29723]	[0.29141]	[1.79374]	[0.26754]	[5.13300]
	p-value											0.0000
INF (-1)	coef.	-0.237042	0.285119	-2.844377	0.006661	-0.089562	-2.709856	0.006615	0.015018	2.901643	1.676442	0.938542
	Std-error	(0.57999)	(2.44961)	(1.34755)	(0.17196)	(0.63878)	(3.87381)	(0.38718)	(0.02613)	(0.66894)	(2.53228)	(0.02849)
	t-statistic	[-0.40870]	[0.11639]	[-2.11078]	[0.03874]	[-0.14021]	[-0.69953]	[0.01708]	[0.57482]	[4.33769]	[0.66203]	[32.9383]

Vector Autoregression Estimates

Sample (adjusted): 1952Q2 2010Q2

Included observations: 233 after adjustments

Standard errors in () & t-statistics in []

		D1_1	D1_2	A1	D2_1	D2_2	A2	PAY	CAY	DEF	RREL	INF
	p-value			0.0349						0.0000		0.0000
C	coef.	-0.069966	0.547800	0.693010	0.071887	0.016901	1.285328	0.000824	0.004159	-0.100015	0.047553	0.017183
	Std-error	(0.14220)	(0.60059)	(0.33039)	(0.04216)	(0.15661)	(0.94977)	(0.09493)	(0.00641)	(0.16401)	(0.62086)	(0.00699)
	t-statistic	[-0.49203]	[0.91211]	[2.09756]	[1.70508]	[0.10791]	[1.35330]	[0.00868]	[0.64933]	[-0.60981]	[0.07659]	[2.45963]
	p-value			0.0360								0.0147

R-squared	0.382508	0.796053	0.933770	0.636466	0.947024	0.928666	0.884338	0.848571	0.868907	0.040117	0.929710
Adj. R-squared	0.351773	0.785902	0.930473	0.618372	0.944387	0.925115	0.878581	0.841033	0.862382	-0.007660	0.926212
Sum sq. resids	4.709703	84.01260	25.42369	0.414003	5.712838	210.1008	2.098884	0.009557	6.265022	89.77891	0.011367
S.E. equation	0.145982	0.616561	0.339174	0.043282	0.160779	0.975029	0.097454	0.006576	0.168370	0.637369	0.007172
F-statistic	12.44540	78.41966	283.2590	35.17467	359.1536	261.5538	153.6122	112.5842	133.1657	0.839663	265.7399
Log likelihood	123.9020	-211.7743	-72.52356	407.1725	101.4069	-318.5606	218.0595	846.2108	90.65789	-219.5080	826.0059
Akaike AIC	-0.960532	1.920810	0.725524	-3.392039	-0.767441	2.837430	-1.768751	-7.160608	-0.675175	1.987193	-6.987176
Schwarz SC	-0.782796	2.098546	0.903260	-3.214303	-0.589705	3.015166	-1.591015	-6.982872	-0.497439	2.164929	-6.809440

Table 2: Estimation of VAR model (after a 4-bonds decomposition)

		D1	D2	D3	A1	PAY	CAY	RREL	TRM	DEF	INF
D1(-1)	coef.	0.622264	0.005388	0.020981	-0.435114	-0.181114	-0.002241	0.093635	-0.058095	-0.197808	0.002905
	Std-error	(0.05006)	(0.01099)	(0.00807)	(0.09392)	(0.01836)	(0.00126)	(0.11847)	(0.15166)	(0.03128)	(0.00136)
	t-statistic	[12.4316]	[0.49042]	[2.59836]	[-4.63298]	[-9.86698]	[-1.77804]	[0.79038]	[-0.38307]	[-6.32338]	[2.14326]
	p-value	0.0000		0.0094	0.0000	0.0000				0.0000	0.0322
D2(-1)	coef.	0.096829	0.689666	0.053358	-0.257143	0.091788	0.002897	0.404787	-0.042290	0.265617	-0.004075
	Std-error	(0.20949)	(0.04598)	(0.03379)	(0.39306)	(0.07682)	(0.00527)	(0.49582)	(0.63473)	(0.13092)	(0.00567)
	t-statistic	[0.46221]	[15.0001]	[1.57891]	[-0.65421]	[1.19481]	[0.54934]	[0.81640]	[-0.06663]	[2.02881]	[-0.71852]
	p-value		0.0000							0.0426	
D3(-1)	coef.	-0.024764	-0.157602	0.724141	1.054133	0.151005	0.004616	-0.770168	1.285105	0.145387	-0.007444
	Std-error	(0.27932)	(0.06130)	(0.04506)	(0.52408)	(0.10243)	(0.00703)	(0.66109)	(0.84630)	(0.17456)	(0.00756)
	t-statistic	[-0.08866]	[-2.57088]	[16.0711]	[2.01140]	[1.47424]	[0.65641]	[-1.16500]	[1.51850]	[0.83286]	[-0.98430]
	p-value		0.0102	0.0000	0.0444						
A1(-1)	coef.	-0.002084	0.004345	0.000909	0.930373	-0.013678	-0.000206	-0.009252	-0.008118	0.004669	0.000242
	Std-error	(0.01540)	(0.00338)	(0.00248)	(0.02889)	(0.00565)	(0.00039)	(0.03644)	(0.04665)	(0.00962)	(0.00042)
	t-statistic	[-0.13538]	[1.28595]	[0.36581]	[32.2061]	[-2.42255]	[-0.53198]	[-0.25390]	[-0.17402]	[0.48528]	[0.58033]
	p-value				0.0000	0.0155					
PAY (-1)	coef.	0.213956	0.012174	-0.056182	0.716768	0.934418	0.000354	-0.156664	0.087650	-0.025471	-0.003339
	Std-error	(0.07701)	(0.01690)	(0.01242)	(0.14449)	(0.02824)	(0.00194)	(0.18226)	(0.23332)	(0.04813)	(0.00208)
	t-statistic	[2.77838]	[0.72033]	[-4.52261]	[4.96082]	[33.0895]	[0.18274]	[-0.85957]	[0.37566]	[-0.52925]	[-1.60144]
	p-value	0.0055		0.0000	0.0000	0.0000					
CAY (-1)	coef.	0.289964	0.140723	-0.058030	3.139056	0.858319	0.915426	-1.010917	6.763216	0.406904	-0.076494
	Std-error	(1.10777)	(0.24312)	(0.17870)	(2.07846)	(0.40623)	(0.02789)	(2.62183)	(3.35636)	(0.69230)	(0.02999)
	t-statistic	[0.26175]	[0.57881]	[-0.32474]	[1.51028]	[2.11291]	[32.8221]	[-0.38558]	[2.01504]	[0.58775]	[-2.55051]
	p-value					0.0347	0.0000		0.0440		0.0108
RREL (-1)	coef.	0.002557	-0.006993	-0.001043	-0.151754	-0.002472	0.000304	0.028421	0.366435	0.024616	0.004355
	Std-error	(0.03134)	(0.00688)	(0.00506)	(0.05881)	(0.01149)	(0.00079)	(0.07418)	(0.09496)	(0.01959)	(0.00085)
	t-statistic	[0.08158]	[-1.01661]	[-0.20624]	[-2.58059]	[-0.21509]	[0.38501]	[0.38314]	[3.85879]	[1.25676]	[5.13192]
	p-value				0.0099				0.0001		0.0000
TRM (-1)	coef.	-0.000384	-0.009627	-0.001929	-0.031492	-0.007214	4.94E-05	0.032578	0.787435	-0.016222	0.000349

		D1	D2	D3	A1	PAY	CAY	RREL	TRM	DEF	INF
	Std-error	(0.01679)	(0.00369)	(0.00271)	(0.03151)	(0.00616)	(0.00042)	(0.03974)	(0.05088)	(0.01049)	(0.00045)
	t-statistic	[-0.02285]	[-2.61240]	[-0.71208]	[-0.99957]	[-1.17147]	[0.11685]	[0.81972]	[15.4775]	[-1.54582]	[0.76850]
	p-value		0.0090						0.0000		
DEF (-1)	coef.	0.102748	0.002173	-0.002058	0.152938	-0.000677	-0.001787	-0.199414	0.359959	0.860669	-0.003848
	Std-error	(0.04738)	(0.01040)	(0.00764)	(0.08890)	(0.01738)	(0.00119)	(0.11215)	(0.14356)	(0.02961)	(0.00128)
	t-statistic	[2.16844]	[0.20900]	[-0.26921]	[1.72027]	[-0.03894]	[-1.49779]	[-1.77817]	[2.50730]	[29.0645]	[-2.99922]
	p-value	0.0302							0.0122	0.0000	0.0027
INF (-1)	coef.	-0.270133	-0.009270	-0.057422	-1.504212	-0.164514	0.016307	1.460967	-3.058649	2.463667	0.974772
	Std-error	(1.06623)	(0.23401)	(0.17200)	(2.00053)	(0.39099)	(0.02684)	(2.52353)	(3.23052)	(0.66634)	(0.02887)
	t-statistic	[-0.25335]	[-0.03961]	[-0.33385]	[-0.75191]	[-0.42076]	[0.60747]	[0.57894]	[-0.94680]	[3.69728]	[33.7676]
	p-value									0.0002	0.0000
C	coef.	0.081573	-0.017218	-0.046160	1.254865	0.113684	0.003438	0.073631	0.187818	-0.023335	-0.000617
	Std-error	(0.20432)	(0.04484)	(0.03296)	(0.38336)	(0.07493)	(0.00514)	(0.48358)	(0.61906)	(0.12769)	(0.00553)
	t-statistic	[0.39923]	[-0.38395]	[-1.40047]	[3.27331]	[1.51727]	[0.66825]	[0.15226]	[0.30339]	[-0.18274]	[-0.11162]
	p-value				0.0011						

R-squared	0.455682	0.627655	0.673736	0.915145	0.889930	0.851315	0.035878	0.600747	0.868995	0.927653
Adj. R-squared	0.431808	0.611324	0.659426	0.911424	0.885102	0.844794	-0.006408	0.583236	0.863249	0.924479
Sum sq. resids	16.09978	0.775489	0.418953	56.67694	2.164999	0.010206	90.18433	147.7948	6.288014	0.011801
S.E. equation	0.265731	0.058320	0.042866	0.498581	0.097445	0.006690	0.628925	0.805123	0.166069	0.007194
F-statistic	19.08724	38.43350	47.08207	245.8952	184.3402	130.5446	0.848457	34.30670	151.2386	292.3460
Log likelihood	-16.75620	345.6953	419.2751	-167.1543	223.0069	863.1979	-222.6609	-281.6900	95.59296	845.8395
Akaike AIC	0.232269	-2.800797	-3.416528	1.490831	-1.774117	-7.131363	1.955322	2.449289	-0.707891	-6.986105
Schwarz SC	0.392274	-2.640793	-3.256524	1.650836	-1.614112	-6.971359	2.115326	2.609293	-0.547886	-6.826100

Table 3: Granger causality

Tests of Granger causality by pairs
Sample: 1952Q1 2011Q4
Delay: 1

Null Hypothesis:	Obs.	F-Statistic	Prob.	Null Hypothesis:	Obs.	F-Statistic	Prob.
CAY does not Granger Cause A4	239	13.4976	0.0003	D4 does not Granger Cause D3	239	3.95479	0.0479
A4 does not Granger Cause CAY		0.37348	0.5417	D3 does not Granger Cause D4		1.44909	0.2299
D1 does not Granger Cause A4	239	0.00076	0.9781	DEF does not Granger Cause D3	239	7.49352	0.0067
A4 does not Granger Cause D1		0.00306	0.9559	D3 does not Granger Cause DEF		8.41002	0.0041
D2 does not Granger Cause A4	239	0.03759	0.8464	INF does not Granger Cause D3	239	0.33997	0.5604
A4 does not Granger Cause D2		0.00867	0.9259	D3 does not Granger Cause INF		1.26270	0.2623
D3 does not Granger Cause A4	239	0.09956	0.7526	PAY does not Granger Cause D3	239	10.0315	0.0017
A4 does not Granger Cause D3		0.08271	0.7739	D3 does not Granger Cause PAY		30.1070	1.E-07
D4 does not Granger Cause A4	239	8.73344	0.0034	PER does not Granger Cause D3	239	0.04116	0.8394
A4 does not Granger Cause D4		0.18646	0.6663	D3 does not Granger Cause PER		5.10839	0.0247
DEF does not Granger Cause A4	239	0.01338	0.9080	RREL does not Granger Cause D3	239	1.94303	0.1647
A4 does not Granger Cause DEF		0.85459	0.3562	D3 does not Granger Cause RREL		0.00295	0.9567
INF does not Granger Cause A4	239	1.47743	0.2254	TRM does not Granger Cause D3	239	4.66240	0.0318
A4 does not Granger Cause INF		4.39646	0.0371	D3 does not Granger Cause TRM		1.55106	0.2142
PAY does not Granger Cause A4	239	0.29295	0.5888	DEF does not Granger Cause D4	239	4.73804	0.0305

Tests of Granger causality by pairs
Sample: 1952Q1 2011Q4
Delay: 1

Null Hypothesis:	Obs.	F-Statistic	Prob.	Null Hypothesis:	Obs.	F-Statistic	Prob.
A4 does not Granger Cause PAY		8.3E-05	0.9927	D4 does not Granger Cause DEF		3.15058	0.0772
PER does not Granger Cause A4	239	3.09404	0.0799	INF does not Granger Cause D4	239	2.08720	0.1499
A4 does not Granger Cause PER		16.4743	7.E-05	D4 does not Granger Cause INF		0.21535	0.6430
RREL does not Granger Cause A4	239	0.09676	0.7560	PAY does not Granger Cause D4	239	20.8948	8.E-06
A4 does not Granger Cause RREL		0.32156	0.5712	D4 does not Granger Cause PAY		1.64974	0.2003
TRM does not Granger Cause A4	239	21.1850	7.E-06	PER does not Granger Cause D4	239	0.02341	0.8785
A4 does not Granger Cause TRM		0.05160	0.8205	D4 does not Granger Cause PER		0.04542	0.8314
D1 does not Granger Cause CAY	239	8.83628	0.0033	RREL does not Granger Cause D4	239	0.00044	0.9834
CAY does not Granger Cause D1		0.24510	0.6210	D4 does not Granger Cause RREL		1.36674	0.2436
D2 does not Granger Cause CAY	239	0.03556	0.8506	TRM does not Granger Cause D4	239	0.54263	0.4621
CAY does not Granger Cause D2		0.53357	0.4658	D4 does not Granger Cause TRM		1.03277	0.3105
D3 does not Granger Cause CAY	239	1.25861	0.2631	INF does not Granger Cause DEF	239	1.20379	0.2737
CAY does not Granger Cause D3		0.45915	0.4987	DEF does not Granger Cause INF		1.02397	0.3126
D4 does not Granger Cause CAY	239	3.51043	0.0622	PAY does not Granger Cause DEF	239	5.79461	0.0168
CAY does not Granger Cause D4		15.5673	0.0001	DEF does not Granger Cause PAY		0.11648	0.7332
DEF does not Granger Cause CAY	239	1.67459	0.1969	PER does not Granger Cause DEF	239	11.0063	0.0011
CAY does not Granger Cause DEF		0.00886	0.9251	DEF does not Granger Cause PER		11.1929	0.0010

Tests of Granger causality by pairs
Sample: 1952Q1 2011Q4
Delay: 1

Null Hypothesis:	Obs.	F-Statistic	Prob.	Null Hypothesis:	Obs.	F-Statistic	Prob.
INF does not Granger Cause CAY	239	0.00089	0.9763	RREL does not Granger Cause DEF	239	4.77307	0.0299
CAY does not Granger Cause INF		0.25709	0.6126	DEF does not Granger Cause RREL		3.13008	0.0782
PAY does not Granger Cause CAY	239	1.03334	0.3104	TRM does not Granger Cause DEF	239	11.9161	0.0007
CAY does not Granger Cause PAY		4.62580	0.0325	DEF does not Granger Cause TRM		1.99093	0.1596
PER does not Granger Cause CAY	239	1.52341	0.2183	PAY does not Granger Cause INF	239	2.19933	0.1394
CAY does not Granger Cause PER		9.90458	0.0019	INF does not Granger Cause PAY		0.89006	0.3464
RREL does not Granger Cause CAY	239	0.30500	0.5813	PER does not Granger Cause INF	239	5.55911	0.0192
CAY does not Granger Cause RREL		0.00249	0.9603	INF does not Granger Cause PER		0.89137	0.3461
TRM does not Granger Cause CAY	239	0.38469	0.5357	RREL does not Granger Cause INF	239	5.52006	0.0196
CAY does not Granger Cause TRM		4.52834	0.0344	INF does not Granger Cause RREL		0.46731	0.4949
D2 does not Granger Cause D1	239	0.00224	0.9623	TRM does not Granger Cause INF	239	4.21537	0.0412
D1 does not Granger Cause D2		0.57651	0.4484	INF does not Granger Cause TRM		0.35736	0.5505
D3 does not Granger Cause D1	239	0.04136	0.8390	PER does not Granger Cause PAY	239	11.9701	0.0006
D1 does not Granger Cause D3		0.00283	0.9576	PAY does not Granger Cause PER		13.9299	0.0002
D4 does not Granger Cause D1	239	0.10572	0.7454	RREL does not Granger Cause PAY	239	0.04121	0.8393
D1 does not Granger Cause D4		1.1E-05	0.9973	PAY does not Granger Cause RREL		1.38063	0.2412

Tests of Granger causality by pairs
Sample: 1952Q1 2011Q4
Delay: 1

Null Hypothesis:	Obs.	F-Statistic	Prob.	Null Hypothesis:	Obs.	F-Statistic	Prob.
DEF does not Granger Cause D1	239	0.75984	0.3843	TRM does not Granger Cause PAY	239	1.31066	0.2534
D1 does not Granger Cause DEF		0.00625	0.9370	PAY does not Granger Cause TRM		0.77129	0.3807
INF does not Granger Cause D1	239	0.00229	0.9618	RREL does not Granger Cause PER	239	6.20445	0.0134
D1 does not Granger Cause INF		1.75930	0.1860	PER does not Granger Cause RREL		0.34979	0.5548
PAY does not Granger Cause D1	239	0.76493	0.3827	TRM does not Granger Cause PER	239	0.26717	0.6057
D1 does not Granger Cause PAY		1.16587	0.2814	PER does not Granger Cause TRM		0.00108	0.9739
PER does not Granger Cause D1	239	0.00013	0.9910	TRM does not Granger Cause RREL	239	0.29065	0.5903
D1 does not Granger Cause PER		30.7299	8.E-08	RREL does not Granger Cause TRM		13.1531	0.0004
RREL does not Granger Cause D1	239	0.47075	0.4933	INF does not Granger Cause D2	239	0.02597	0.8721
D1 does not Granger Cause RREL		0.22222	0.6378	D2 does not Granger Cause INF		0.00917	0.9238
TRM does not Granger Cause D1	239	0.00735	0.9317	PAY does not Granger Cause D2	239	8.29409	0.0043
D1 does not Granger Cause TRM		0.25886	0.6114	D2 does not Granger Cause PAY		67.8245	1.E-14
D3 does not Granger Cause D2	239	4.04871	0.0453	PER does not Granger Cause D2	239	0.19506	0.6591
D2 does not Granger Cause D3		9.17764	0.0027	D2 does not Granger Cause PER		9.68642	0.0021
D4 does not Granger Cause D2	239	0.23363	0.6293	RREL does not Granger Cause D2	239	6.25818	0.0130
D2 does not Granger Cause D4		0.33883	0.5611	D2 does not Granger Cause RREL		2.77613	0.0970

Tests of Granger causality by pairs
Sample: 1952Q1 2011Q4
Delay: 1

Null Hypothesis:	Obs.	F-Statistic	Prob.	Null Hypothesis:	Obs.	F-Statistic	Prob.
DEF does not Granger Cause D2	239	7.67889	0.0060	TRM does not Granger Cause D2	239	0.74449	0.3891
D2 does not Granger Cause DEF		38.0151	3.E-09	D2 does not Granger Cause TRM		0.76242	0.3835

Table 4: Predictive regression of the returns

Univariate Predictive Regression

#		PER	D1	D2	D3	D4	A4	PAY	CAY	RREL	TRM	DEF	Adjusted R2
1	coef.	0.149969											
	t-statistic	2.249588											0.01
	p-value	0.0254											
2	coef.		0.191427										
	t-statistic		3.213660										0.1
	p-value		0.0015										
3	coef.			0.137164									
	t-statistic			2.082461									0.04
	p-value			0.0384									
4	coef.				0.145258								
	t-statistic				2.184514								0.01
	p-value				0.0299								
5	coef.					-0.000835							
	t-statistic					-0.051055							0.01
	p-value					0.9593							
6	coef.						-0.004774						
	t-statistic						-0.904432						0.01
	p-value						0.3667						
7	coef.							-0.007058					
	t-statistic							-0.709074					0.02
	p-value							0.4790					
8	coef.								0.148919				
	t-statistic								2.242868				0.013
	p-value								0.0258				
9	coef.									-0.005444			
	t-statistic									-2.059860			0.01
	p-value									0.0405			
10	coef.										0.003637		0.03

Univariate Predictive Regression

	t-statistic										2.298442		
	p-value										0.0224		
11	coef.											0.001288	
	t-statistic											0.173709	0.04
	p-value											0.8622	

Multivariate Predictive Regression

#		PER	D1	D2	D3	D4	A4	PAY	CAY	RREL	TRM	DEF	Adjusted R2
12	coef.	0.004157						-0.012689	-0.031856	-0.003082	0.003162	0.002280	
	t-statistic	0.682534						-1.328249	-0.280369	-1.095661	1.752707	0.403037	0.03
	p-value	0.4956						0.1854	0.7794	0.2744	0.0810	0.6873	
13	coef.		0.185977	0.073399	0.028922	0.021341	0.028922	-0.027233	0.0922	-0.0051	0.00198	0.0019	
	t-statistic		2.420076	3.893068	1.923321	1.191262	1.191262	-2.518882	0.9341	-2,0309	1.07503	0.3809	0.16
	p-value		0.0163	0.0001	0.0557	0.2348	0.4908	0.0125	0.3512	0.0434	0.2835	0.7036	

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A semi-partial isometries in Banach spaces

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Abstract. The purpose of this paper is to introduce and study some basic properties of the class of A-Semi partial isometries on Banach spaces.

Keywords: isometry, partial isometry, semi-partial isometry, semi-partial (m,p)-isometry.

1. Introduction

Let $\mathcal{X}$ denote a complex Banach space and $\mathcal{H}$ denote a complex Hilbert space. $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{B}(\mathcal{H})$) is the algebra of all bounded linear operators on $\mathcal{X}$ (resp. on $\mathcal{H}$). For every $T \in \mathcal{B}(\mathcal{X})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and its adjoint by T^*. A subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T (or T-invariant) if $T\mathcal{M} \subset \mathcal{M}$.

Partial isometries on Hilbert spaces $\mathcal{H}$ form an attractive and important class of operators. The definition is simple: An operator $T \in \mathcal{B}(\mathcal{H})$ is a partial isometry if and only if $TT^*T = T$ or equivalently, an operator T whose restriction to the orthogonal complement of its null-space is an isometry ([7]) i.e., $\|Tx\| = \|x\|$ for every $x \in \mathcal{N}(T)^\perp$, where $\mathcal{N}(T)^\perp$ is the orthogonal complement of $\mathcal{N}(T)$.

Partial isometries play a vital role in operator theory; they enter, for instance, in the theory of the polar decomposition of arbitrary operators. The concept of partial isometry can be defined in other equivalent ways.

The notion of A-partial isometries means bounded linear operators defined on a Hilbert space that behave (in some sense) as partial isometries when the semi-inner product induced by a positive (semidefinite) operator is considered. This concept was first introduced in [3] and studied also in [4, 14] where several results concerning projections and metrical properties were developed.

The theory of partial isometries on Banach spaces has been studied by several authors. For more details, see [10] and [11] where authors characterize the class of partial isometries on Banach spaces. A bounded linear operator T on a

Banach space $\mathcal{X}$ is called a semi-partial isometry if

$$\|Tx\| = d(x, \mathcal{N}(T)), \quad \text{for all } x \in X.$$

This class of operators, which is a natural generalization of partial isometries from Hilbert to general Banach spaces, contains in particular the class of partial isometries introduced by M. Mbekhta [10].

In this paper, we introduce the class of A-semi-partial isometries, which is a natural generalization of partial isometries from Hilbert and Banach spaces. This class of operators contains, among others, A-isometries,A- co-isometries, partial isometries (in the sense of M. Mbekhta). Our main goal here is to investigate and to classify this class of operators.

The minimum modulus of an operator $T \in \mathcal{B}(\mathcal{X})$ is defined by

$$\gamma(T) := \inf\{\|Tx\| : \quad x \in \mathcal{X}, \ \|x\| = 1\}.$$

Note that $\gamma(T) > 0$ if and only if T is injective and has closed range. (For more detail see [6]. The outline of the paper is as follows: Section 2 contains basic results on A-isometric and A-partial isometric operators on Banach spaces. In section 3 we study the concept of A-semi-partial isometries on Banach spaces and we investigate various structural properties of this class of operators. In the final section of the paper, we give some algebraic and analytical stability properties for the class of A-semi partial isometry.

2. A-isometric and A-partial isometric operators

An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be an isometry if $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$. In the sequel, we extend this term in a general way as follows : $T \in \mathcal{B}(\mathcal{X})$ is said to be isometry on a subspace $\mathcal{Y}$ of $\mathcal{X}$ if $\|Ty\| = \|y\|$ for all $y \in \mathcal{Y}$. $T \in \mathcal{B}(\mathcal{X})$ is said to be co-isometry if $T^* \in \mathcal{B}(\mathcal{X}^*)$ is an isometry and unitary if both T and T^* are isometries (T is both isometry and co-isometry).

Definition 2.1. *Let T and $A \in \mathcal{B}(\mathcal{X})$. T is said to be an A-isometry operator if*

$$\|TAx\| = \|Ax\|, \qquad \forall\, x \in \mathcal{X}.$$

Remark 2.1. (i) If $A = I_{\mathcal{X}}$ then an A-isometry is an isometry.

(ii) Every isometry is an A-isometry.

(iii) If $T(\mathcal{R}(A)) \subset \mathcal{R}(A)$ then, T is an A-isometry if and only if $T_{/\overline{\mathcal{R}(A)}}$ is an isometry.

(iv) If A is surjective then T is an A-isometry if and only if T is an isometry.

Proposition 2.1. *Let $T, S, A \in \mathcal{B}(\mathcal{X})$, the following statements hold:*

(1) *If T is invertible A-isometry such that $TA = AT$, then T^{-1} is an A-isometry.*

(2) *If T is an A-isometry and S is an A-isometry such that $AS = SA$, then TS is an A-isometry.*

(3) *If T is an isometry and S is an A-isometry, then TS is an A-isometry.*

Proof. (1) Since $\|TAx\| = \|Ax\|$, for all $x \in \mathcal{X}$ and $TA = AT$ it follows that

$$\|T^{-1}Ax\| = \|Ax\|, \quad x \in \mathcal{X}.$$

(2) By the assumptions we have, for all $x \in \mathcal{X}$

$$\|TSAx\| = \|TASx\| = \|ASx\| = \|SAx\| = \|Ax\|.$$

The statement (3) is obvious. □

The following example shows that the condition $TA = AT$ in the statement (1) is necessary.

Example 2.1. Let $T = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u, v)\|^2 := |u|^2 + |v|^2$. A simple computation shows that T is invertible A-isometry such that $TA \neq AT$ and moreover T^{-1} is not A-isometry.

Definition 2.2. *Let $\mathcal{X}$ be a Hilbert space and $A, T \in \mathcal{B}(\mathcal{H})$. T is said to be A-partial isometry if*

$$\|TAx\| = \|Ax\|, \qquad \forall\, x \in A^{-1}(\mathcal{N}(T)^{\perp}),$$

or, equivalently

$$\|Tz\| = \|z\|, \quad \forall\ z \in \mathcal{R}(A) \cap \mathcal{N}(T)^{\perp}.$$

3. A-semi partial isometries

This section is devoted to study of the class of A-semi partial isometries.

Definition 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$. T is said to be an A-semi -partial isometry if T satisfies*

$$\|TAx\| = d(Ax, \mathcal{N}(T)) \ , \forall\ x \in \mathcal{X},$$

or, equivalently, if

$$\|Ty\| = d(y, \mathcal{N}(T)) \ , \forall\ y \in \overline{\mathcal{R}(A)}.$$

Remark 3.1. (i) Let $T \in \mathcal{B}(\mathcal{X})$ then, T is a semi partial isometry if and only if T is $I_{\mathcal{X}}$-semi partial isometry.

(ii) If T is a semi partial isometry then, T is A-semi partial isometry for all A in $\mathcal{B}(\mathcal{X})$.

(iii) If $\mathcal{R}(A) \subset \mathcal{N}(T)$ then, T is A-semi partial isometry.

In the following example, we give an operator T that is A-semi-partial isometry but it is not a semi-partial isometry.

Example 3.1. If $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u, v)\|^2 := |u|^2 + |v|^2$, then T is an A-semi-partial isometry.

Indeed, we have $TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$\|TA \begin{pmatrix} u \\ v \end{pmatrix}\|^2 = \| \begin{pmatrix} u+v \\ u+v \end{pmatrix}\|^2 = 2|u+v|^2$. On the other hand

$$d(A \begin{pmatrix} u \\ v \end{pmatrix}, \mathcal{N}(T))^2 = d(\begin{pmatrix} 0 \\ \sqrt{2}(u+v) \end{pmatrix}, \mathcal{N}(T))^2 = 2|u+v|^2.$$

Hence $\|TA \begin{pmatrix} u \\ v \end{pmatrix}\| = d(A \begin{pmatrix} u \\ v \end{pmatrix}, \mathcal{N}(T))$ and so that T is an A-partial isometry as required.

Note that T is not a semi-partial isometry. Indeed, $\|Te_1\| = \frac{1}{\sqrt{2}} \neq \|e_1\| = d(e_1, \mathcal{N}(T))$ where $e_1 = (1, 0)$.

Proposition 3.1. *Let T, $A \in \mathcal{B}(\mathcal{X})$. The following statements hold:*

(i) if $\overline{\mathcal{R}(A)}$ is invariant by T and if T is an A-semi partial isometry then, T is a semi partial isometry on $\overline{\mathcal{R}(A)}$. In particular, if $\mathcal{R}(A)$ is dense in $\mathcal{X}$ and if T is an A-semi partial isometry then, T is a semi partial isometry;

(ii) if T is injective A-semi partial isometry and $\mathcal{R}(A)$ is T-invariant, then T is an isometry on $\overline{\mathcal{R}(A)}$;

(iii) if T is injective A-semi partial isometry and A is an isometry such that $TA = AT$, then T is an isometry.

Proof. (i) The restriction of T to $\overline{\mathcal{R}(A)}$ is a bounded operator on this space. Since T is an A-semi partial isometry then, by the second variant of the Definition 3.1, we have $\|Ty\| = d(y, \mathcal{N}(T)), \forall y \in \overline{\mathcal{R}(A)}$. Hence T is a semi partial isometry on $\overline{\mathcal{R}(A)}$.

If $\mathcal{R}(A)$ is dense in $\mathcal{X}$ then, $\overline{\mathcal{R}(A)}$ is the whole space $\mathcal{X}$ and hence the conclusion follows by applying the previous statement. (ii) Since T in injective A-semi partial isometry , it follows that $\|TAx\| = d(Ax, \mathcal{N}(T)) = \|Ax\|$, $x \in \mathcal{X}$ and hence $\|Ty\| = \|y\|$ for all $y \in \overline{\mathcal{R}(A)}$. Thus T is an isometry on $\overline{\mathcal{R}(A)}$. (iii) It is easy to check that $\|Tx\| = \|ATx\| = \|TAx\| = \|Ax\| = \|x\|$, $x \in \mathcal{X}$. □

Proposition 3.2. *Let $A, T \in \mathcal{B}(\mathcal{X})$ such that A is an isometry and $A(\mathcal{N}(TA)) = \mathcal{N}(T)$. Then the following statements are equivalent:*

(i) T is an A-semi partial isometry;

(ii) TA is a semi partial isometry.

Proof. Let x be in $\mathcal{X}$ we have $d(Ax, \mathcal{N}(T)) = \inf\{\|Ax - y\|, y \in \mathcal{N}(T)\}$. Since $A(\mathcal{N}(TA)) = \mathcal{N}(T)$, we get $d(Ax, \mathcal{N}(T)) = \inf\{\|Ax - Az\|, z \in \mathcal{N}(TA)\}$. Using the assumption that A is an isometry, the last equality becomes

$$d(Ax, \mathcal{N}(T)) = \inf\{\|x - z\|, \ z \in \mathcal{N}(TA)\} = d(x, \mathcal{N}(TA)).$$

Thus, we have

$$d(Ax, \mathcal{N}(T)) = d(x, \mathcal{N}(TA)), \ \forall x \in \mathcal{X}. \tag{3.1}$$

(3.1) gives us the desired equivalence. □

As a consequences of the Proposition 3.2, we get some corollaries:

Corollary 3.1. *Let A and T as in the Proposition 3.2 then, the following assertions are equivalent:*

(i) T is A-semi partial isometry;

(ii) $\gamma(TA) = \|TA\| = 1$;

(iii) TA is a contraction and $\gamma(TA) \geq 1$;

(iv) The operator $\widetilde{TA} : \mathcal{X}/\mathcal{N}(TA) \to \mathcal{X}$, given by $\widetilde{TA}\dot{x} = TAx$ for all $x \in \mathcal{X}$, is an isometry.

Proof. The assertion (i) is, by the Proposition 3.2, equivalent to that TA is semi partial isometry. Hence, we obtain the equivalence between (i)-(iv) by applying the Theorem 2.1 ([13]) to the operator TA. □

Corollary 3.2. *Let $A, T \in \mathcal{B}(\mathcal{X})$. If A is an invertible isometry, then T is A-semi partial isometry if and only if TA is a semi partial isometry.*

Proof. It is easy to check that A and T satisfy the conditions in the Proposition 3.2 and hence the conclusion follows by applying this proposition. □

Corollary 3.3. *Let $A, T \in \mathcal{B}(\mathcal{X})$. If A is an invertible isometry then, T is a semi-partial isometry if and only if TA^n is a semi-partial isometry for all $n \in \mathbb{N}$.*

Proof. By induction : To start, for $n = 1$, the result is an easy consequence of the statement (i) of the Proposition 3.1 combined with the Corollary 3.2. □

We give now some sufficient conditions for semi-partial isometry.

Proposition 3.3. *Let $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{N}(T) \subset \mathcal{R}(T)$. If T is T-semi partial isometry then, T is a semi partial isometry.*

Proof. It is obvious that $T(\mathcal{N}(T^2)) \subset \mathcal{N}(T)$. By the assumption $\mathcal{N}(T) \subset \mathcal{R}(T)$, we obtain the reverse inclusion and hence we get the equality $T(\mathcal{N}(T^2)) = \mathcal{N}(T)$. Now taking $A = T$, the Proposition 3.2 yields the desired result. □

Proposition 3.4. *Let $T \in \mathcal{B}(\mathcal{X})$. Then, T is a semi-partial isometry if and only if there is $A \in \mathcal{B}(\mathcal{X})$ the range of A is dense and T is A semi partial isometry.*

Proof. For the direct implication, Take $A = I_{\mathcal{X}}$. For the converse implication, use the second variant of the Definition 3.1 to conclude that T is a semi partial isometry □

The following proposition shows that the A-semi partial isometry is preserved when the parameter A runs over some specific class.

Proposition 3.5. *Let T, A_1 and A_2 be in $\mathcal{B}(\mathcal{X})$ such that assume that $TA_1 = TA_2$. Then T is A_1-semi partial isometry if and only if it is A_2-semi partial isometry.*

Proof. Let x be in $\mathcal{X}$. By the assumption $TA_1 = TA_2$ we have $\mathcal{R}(A_1 - A_2) \subset \mathcal{N}(T)$ and hence $A_1x - \mathcal{N}(T) = A_2x - \mathcal{N}(T)$. This implies $d(A_1x, \mathcal{N}(T)) = d(A_2x, \mathcal{N}(T))$. Consequently, $\|TA_1\| = d(A_1x, \mathcal{N}(T))$ if and only if $\|TA_2\| = d(A_2x, \mathcal{N}(T))$. This achieves the proof. □

Remark 3.2. The converse of the Proposition 3.5 is not true. Otherwise, by (ii) in the Remark 3.1applying to the identity, all operators will agree, which is possible only in the trivial case when $\mathcal{X} = \{0\}$.

If $\mathcal{H}$ is a Hilbert space, it is well know that $T \in \mathcal{B}(\mathcal{H})$ is a semi-partial isometry if and only if T is a partial isometry i.e; $\|Tx\| = \|x\| \ \forall\, x \in \mathcal{N}(T)^{\perp}$. The following theorem gives, under prerequisite condition, a necessary and sufficient condition for operator to be A-semi partial isometry in Hilbert space.

Theorem 3.1. *Let $T, A \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Assume that*

$$\mathcal{H} = A^{-1}(\mathcal{N}(T)) + A^{-1}(\mathcal{N}(T)^{\perp}). \tag{3.2}$$

The following statements are equivalent:

(i) T is an A-semi-partial isometry;

(ii) $\|TAx\| = \|Ax\|, \forall\, x \in A^{-1}\big(\mathcal{N}(T)^{\perp}\big)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Let $x \in \mathcal{X}, x = x_1 + x_2$ where $x_1 \in A^{-1}\big(\mathcal{N}(T)\big)(\Leftrightarrow Ax_1 \in \mathcal{N}(T))$ and $x_2 \in A^{-1}(\mathcal{N}(T)^{\perp})(\Leftrightarrow Ax_2 \in \mathcal{N}(T)^{\perp})$ and it follows that $\|TAx\| = \|Ax_2\| = d(Ax_2, \mathcal{N}(T))$. It is easy to see that for any $x \in \mathcal{X}$, $d(Ax, \mathcal{N}(T)) = d(Ax_2, \mathcal{N}(T))$.

We conclude that $\|TAx\| = d(Ax, \mathcal{N}(T)), \forall x \in \mathcal{X}$ and the proof of this implication is over. □

Remark 3.3. If A is invertible on a Hilbert space $\mathcal{H}$, the condition (3.2) is satisfied and hence T is A-semi partial isometry if and only if T is partial isometry.

It was observed in [13] that an operator $T \in \mathcal{B}(\mathcal{X})$ is a semi-partial isometry if and only if $T^* \in \mathcal{B}(\mathcal{X}^*)$ is a semi-partial isometry. This result fail to be true in general for A-semi-partial isometry as shown by the following example.

Example 3.2. If $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u, v)\|^2 := |u|^2 + |v|^2$.

We know that T is an A-semi-partial isometry,therefore T^* is not A^*-semi-partial isometry. Indeed, a simple computation shows that $T^*A^* = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$.

$$\|T^*A^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}\|^2 = 5 \quad \text{and} \quad d(A^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathcal{N}(T))^2 = 4.$$

Clearly, T^* is not A^*-semi-partial isometry.

4. Stability properties

We give in this section some algebraic and analytical stability properties for the A-semi partial isometry.

Proposition 4.1. *If T is A_1-semi partial isometry then, T is A_1A_2-semi partial isometry for any operator A_2.*

Proof. Applying the condition of A_1-semi partial isometry on the vector A_2x for an arbitrary vector$x$, we get $\|TA_1A_2x\| = d(A_1A_2x, \mathcal{N}(T))$. Thus T is A_1A_2-semi partial isometry. □

As a consequence of the Proposition 4.1 we obtain the following dynamic result:

Corollary 4.1. *If T is an A-semi partial isometry then, T is an A^n-semi-partial isometry for any integer $n \geq 1$.*

Proof. By induction, using Proposition 4.1. □

The following corollary is an immediate consequence of Corollary 4.1.

Corollary 4.2. *Let T be an A-semi-partial isometry. If A is idempotent ($A^2 = I_{\mathcal{X}}$) then, T is a semi-partial isometry.*

Proposition 4.2. *Let (A_n) be a sequence in $\mathcal{B}(\mathcal{X})$ which converges to A in the operator norm. If T is A_n-semi partial isometry for every n then, T is A-semi partial isometry.*

Proof. Let $x \in \mathcal{X}$. We have

$$\|TA_nx\| = d(A_nx, \mathcal{N}(T)), \quad \forall n \in \mathbb{N} \tag{4.1}$$

Since the left side of (4.1) tends to $\|TAx\|$ and the right side has as limit $d(Ax, \mathcal{N}(T))$, we obtain

$$\|TA\| = d\big(Ax, \mathcal{N}(T)\big).$$

Hence T is A-semi-partial isometry. □

Theorem 4.1. *Let $(T_n)_n$ be a sequence of semi partial isometry in $\mathcal{B}(\mathcal{H})$ such that $T_n \neq 0$ and T_n converges to $T(\neq 0)$ in norm (as $n \to \infty$). Suppose that there exists a positive integer m such that $\mathcal{N}(T) \subset \mathcal{N}(T_n)$ for every $n \geq m$, then T is a semi-partial isometry.*

Proof. Since $T_n \neq 0$ and T_n is a semi-partial isometry it follows by [13, Thoerem 2.1] that $\gamma(T_n) = \|T_n\| = 1$. As $T_n \to T$ in norm, we have $\|T_n\| \to \|T\|$ (as $n \to \infty$) and we get $\|T\| = 1$. Since by our assumption, $\mathcal{N}(T) \subset \mathcal{N}(T_n)$, for each $n \geq m$, it therefore follows from [9, Theorem 1.3] that $\gamma(T_n) \to \gamma(T)$ as $n \to \infty$ and so that $\gamma(T) = 1$. □

Theorem 4.2. *Let $(T_n)_n$ be a sequence of A-semi partial isometry in $\mathcal{B}(\mathcal{H})$ such that $T_n \neq 0$ and T_n converges to $T(\neq 0)$ in norm (as $n \to \infty$). Suppose that the following conditions are hold:*

(a) there exists a positive integer m such that $\mathcal{N}(TA) \subset \mathcal{N}(T_nA)$ for every $n \geq m$;

(b) $A(\mathcal{N}(T_nA)) = \mathcal{N}(T_n)$;

(c) $A(\mathcal{N}(TA)) = \mathcal{N}(T)$.

Then T is an A-semi-partial isometry.

Proof. Obviously, $T_nA \to TA$ as $n \to \infty$. By the condition (b) we know that T_nA is a semi-partial isometry (by Corollary 3.1). Now applying Theorem 4.1 we have that TA is a semi-partial isometry. The desired result follows from condition (c). □

Definition 4.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a linear subspace of $\mathcal{X}$. Let $\mathcal{Y}^\star$ be defined by*

$$\mathcal{Y}^\star = \{x \in \mathcal{X}, d(x, \mathcal{Y}) = \|x\|\}.$$

We say that $\mathcal{Y}$ is a $\star$-subspace if the following conditions hold:

(i)$_\star$ $\mathcal{Y}^\star$ is linear subspace of $\mathcal{X}$;

(ii)$_\star$ $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\star$.

Example 4.1. If $\mathcal{M}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$ then, $\mathcal{M}^\star = \mathcal{M}^\perp$. Hence $\mathcal{M}$ is $\star$-subspace.

Definition 4.2. *A Banach space $\mathcal{X}$ is said to be $\star$-Banach space, if any closed linear subspace of $\mathcal{X}$ is a $\star$-subspace.*

Example 4.2. Referring to the Example (0.1), any Hilbert space is a $\star$-Banach space.

Proposition 4.3. *Let $\mathcal{X}$ be a $\star$-Banach space, $\mathcal{Y}$ a linear closed subspace of $\mathcal{X}$ and $x \in \mathcal{X}$. Then, there exists a unique $y \in \mathcal{Y}$ such that $d(x, \mathcal{Y}) = \|x - y\|$, y is the unique vector in $\mathcal{Y}$ such that $x - y$ belongs to $\mathcal{Y}^\star$. y is called the $\star$-projection of x on $\mathcal{Y}$.*

Proof. The Proposition is an immediate consequence of the condition (ii). □

The following theorem gives a characterization of the semi partial isometry in the case of $\star$-Banach space.

Theorem 4.3. *Let $\mathcal{X}$ be a $\star$-Banach space and $T \in \mathcal{B}(\mathcal{X})$. Then, T is a semi partial isometry if and only if $\|Tx\| = \|x\|$, $\forall x \in \mathcal{N}(T)^\star$.*

Proof. For the direct implication, let $x \in \mathcal{N}(T)^\star$. Since T is a semi partial isometry, we have $\|Tx\| = d(x, \mathcal{N}(T))$. On the other hand, x satisfies

$d(x,\mathcal{N}(T)) = \|x\|$. Finally, we have $\|Tx\| = \|x\|$. Conversely, let $x \in \mathcal{X}$. By the Proposition 4.3, there is $y \in \mathcal{N}(T)$ such that $x - y \in \mathcal{N}(T)^\star$ and

$$d(x,\mathcal{N}(T)) = \|x - y\|. \tag{4.2}$$

By assumption, we have $\|T(x-y)\| = \|x-y\|$. Since $y \in \mathcal{N}(T)$, the last equality becomes

$$\|Tx\| = \|x - y\| \tag{4.3}$$

Taking (4.3) in (4.2), we get $\|Tx\| = d(x,\mathcal{N}(T))$. We can conclude that T is a semi partial isometry. □

Theorem 4.4. *Let $\mathcal{H}$ be a Hilbert space, S and T are two partial isometries on $\mathcal{H}$ such that $S(\mathcal{N}(TS)^\perp) \subset \mathcal{N}(T)^\perp$. Then, TS is a partial isometry.*

Proof. Let $x \in \mathcal{N}(TS)^\perp$. We have $Sx \in \mathcal{N}(T)^\perp$. Since T is a semi partial isometry, we get

$$\|TSx\| = \|Sx\|,$$

Since S is a partial isometry then it is a semi partial isometry and hence we have

$$\|TSx\| = d(x,\mathcal{N}(S)). \tag{4.4}$$

Hence, we obtain

$$\|TSx\| \leq \|x\|. \tag{4.5}$$

By the inclusion $\mathcal{N}(S) \subset \mathcal{N}(TS)$, we have $d(x,\mathcal{N}(S)) \geq d(x,\mathcal{N}(TS)$. By the assumption $x \in \mathcal{N}(TS)^\perp$, we have $d(x,\mathcal{N}(TS) = \|x\|$. Thus $d(x,\mathcal{N}(S)) \geq \|x\|$. Which gives, from (4.4),

$$\|TSx\| \geq \|x\|. \tag{4.6}$$

(4.5) and (4.6) show that $\|TSx\| = \|x\|$. Thus, we have proved that $\|TSx\| = \|x\|$, $\forall x \in \mathcal{N}(TS)^\perp$. Hence, TS is a partial isometry. □

In [2], A. Alahmari et al. introduced the concepts of semi-partial (m,p)-isometries an a generalization of (m,p)-isometries (see [1, 5, 8]. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be a semi partial (m,p)-isometry for some positive integer m and a real $p \geq 1$, if T satisfies

$$\sum_{1\leq k\leq m} (-1)^{m-k}\binom{m}{k}\|T^k x\|^p = (-1)^{m+1} d(x,\mathcal{N}(T))^p, \qquad \forall\, x \in \mathcal{X}.$$

Clearly, T a semi-partial (m,p)-isometry if and only if T is an (m,p)-isometry. Also, a semi-partial isometry corresponds here to a semi- partial $(1,2)$-isometry. In the following definition, we extend this concept to the concept of A-semi-partial-(m,p)-isometry.

Definition 4.3. *Let $A, T \in \mathcal{B}(\mathcal{X})$ and let m be a positive integer, $p \in [1, \infty)$. T is said to be A-semi-partial-(m,p)-isometry if*

$$\sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|T^k Ax\|^p = (-1)^{m+1} d(Ax, \mathcal{N}(T))^p, \qquad \forall\, x \in \mathcal{X}.$$

Remark 4.1. We make the following observations:

(1) If $A = I_{\mathcal{X}}$, then every A-semi-partial-(m,p)-isometry is a semi-partial-(m,p)-isometry.

(2) Every A-semi-partial-$(1,2)$-isometry is an A-semi-partial isometry (Definition 3.1).

(3) T is an A-semi partial-$(2,p)$- isometry if and only if

$$\|T^2 Ax\|^p - 2\|TAx\|^2 + d(Ax, \mathcal{N}(T))^p = 0, \qquad \forall\, x \in \mathcal{X}.$$

Proposition 4.4. Let $A, T \in \mathcal{B}(\mathcal{X})$ and let m be a positive integer, $p \in [1, \infty)$. Suppose that T is A-semi-partial-(m,p)-isometry. If V is isometry on $\mathcal{R}(T)$ with $VT = TV$ then VT is A-semi-partial-(m,p)-isometry.

Proof. Since V is isometry on $\mathcal{R}(T)$, we get

$$\|VTx\| = \|Tx\|, \ \forall\, x \in \mathcal{X} \ \text{ and } \ \mathcal{N}(VT) = \mathcal{N}(T).$$

This yields, by little calculation, taking in consideration that V commutes with T,

$$\|(VT)^k Ax\| = \|T^k Ax\|, \ 1 \leq k \leq m, \ x \in \mathcal{X}. \tag{4.7}$$

Now, since T is A-semi-partial-(m,p)-isometry, (4.7) implies

$$\begin{aligned} &\sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|(VT)^k Ax\|^p \\ &= (-1)^{m+1} d(Ax, \mathcal{N}(T))^p = (-1)^{m+1} d(Ax, \mathcal{N}(VT))^p, \qquad \forall\, x \in \mathcal{X}. \end{aligned}$$

Thus VT is A-semi-partial-(m,p)-isometry. □

It seems to be natural to enquire a detail study for the class of A-semi-partial-(m,p)-isometry. This study will be addressed in a forthcoming paper.

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Chain continuity for Zadeh's extension

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Abstract. This paper aims to study chain continuity for Zadeh's extension. More specially, the relations between (finite) chain continuity of discrete dynamical system and its induced fuzzy system have been discussed.

Keywords: Zadeh's extension, (finite) chain continuity, fuzzy, discrete dynamical system.

1. Introduction

Throughout this paper, let (X, d) be a compact metric space with metric d and $f : X \to X$ be continuous. A discrete dynamical system is a pair (X, f). For other notions and notations mentioned in this section, we refer to Section 2. A discrete dynamical system (X, f) can naturally induce its fuzzified counterpart and set-valued counterpart, i.e., fuzzy dynamical system $(\mathcal{F}(X), d_\infty)$ and set-valued dynamical system $(\mathcal{K}(X), d_H)$, respectively. The dynamics of set-valued dynamical system have been extensively studied and many elegant results have been obtained [1, 2, 3, 4, 5, and the references therein]. An initial work on the relations between dynamical properties of fuzzified and original system has been conducted[6]. Later the open question raised in the same article has been solved by Kupka[7]. From then on a systematic study on dynamics of induced fuzzy system has been developed [8, 9, 10, 11]. Meanwhile, the connections between original system and its induced fuzzy system attract lots of attention and quite a few inspiring results have been obtained [12, 13, 14, and the references therein].

On the other hand, the dynamics of the iteration of a continuous function can be very complicated. Hence it is natural to study the pseudo-orbits for a better understanding of true orbits. Along this line, the concept of chain continuity has been separately introduced in [15, 16]. Chain continuity is a pointwise notion and is generic, i.e., for certain spaces, most functions are chain continuous at most points. In this paper, the chain continuity for Zadeh's extension has been investigated. Below, basic notions are introduced in Section 2. Main results are presented in Section 3.

2. Preliminaries

In this section, we complete notations and recall some known definitions.

2.1 Basic concepts of dynamical systems

Let X be a compact metric space and $f : X \to X$ be continuous.

For $\delta > 0$, a *δ-pseudo-orbit* for f is a sequence in X such that $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{N}$.

A *δ-chain* is a finite *δ-pseudo-orbit* for f.

A point x is called a *chain continuity point* of f, or f is *chain continuous* at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that all δ-pseudo-orbits beginning δ close to x remains ϵ close to the points of the orbits of x, i.e., $d(x, x_0) < \delta$ and $d(f(x_i), x_{i+1}) < \delta$ imply $d(f^i(x), x_i) < \epsilon$ for $i = 1, 2, \cdots$.

A point x is called a *finite chain continuity point* of f, or f is *finite chain continuous* at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\{x_0, x_1, \cdots, x_m\}$ a δ-chain with $d(x, x_0) < \delta$ implies $d(f^i(x), x_i) < \delta$.

2.2 Metric space of fuzzy sets

Let $\mathcal{K}(X)$ be the class of all non-empty and compact subset of X. Define the ε-neighborhood of a nonempty subset A in X to be the set

$$U(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathcal{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 |\ A \subseteq U(B, \varepsilon)\},$$

The Hausdorff metric on $\mathcal{K}(X)$ is defined by letting

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

For a compact X, the topology generated by d_H coincides with the finite topology. In this case, the set of all finite subsets of X is dense in $\mathcal{K}(X)$.

Define $\mathcal{F}(X)$ as the class of all upper semicontinuous fuzzy sets $u : X \to [0, 1]$ such that $[u]_\alpha \in \mathcal{K}(X)$, where *$\alpha$-cuts* and the *support* of u are defined by

$$[u]_\alpha = \{x \in X | u(x) \geq \alpha\}, \alpha \in [0, 1]$$

and

$$supp(u) = \overline{\{x \in X | u(x) > 0\}},$$

respectively.

Moreover, for each $x \in X$, we denote $\hat{x}$ the characteristic function of x, it is clear that $\hat{x} \in \mathcal{F}(X)$ for all $x \in X$. Denote $\emptyset_X$ the *empty fuzzy set* ($\emptyset_X(x) = 0$ for all $x \in X$).

A *levelwise metric* d_∞ on $\mathcal{F}(X)$ is defined by

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha)$$

for all $u, v \in \mathcal{F}(X)$. It is well known that if (X,d) is complete, then $(\mathcal{F}(X), d_\infty)$ is also complete but is not compact and is not separable.

Lemma 2.1 ([6]). *Let $f : X \to X$ be continuous and $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ be its Zadeh's extension, then $[\hat{f}(u)]_\alpha = f([u]_\alpha)$.*

A fuzzy set u is *piecewise constant* if there exists a strictly decreasing sequence of closed subsets $\{C_1, C_2, \cdots, C_k\}$ of X and a strictly increasing sequence of real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_k\} \subseteq (0,1]$ such that $[u]_\alpha = C_{i+1}$, where $\alpha \in (\alpha_i, \alpha_{i+1}]$.

Lemma 2.2 ([7]). *For any $v \in \mathcal{F}(X)$ and $\varepsilon > 0$ there exists a piecewise constant $u \in \mathcal{F}(X)$ such that $d_\infty(u,v) < \varepsilon$, i.e., the set of piecewise constant fuzzy sets is dense in $\mathcal{F}(X)$.*

Denote by $\mathcal{SF}(X)$ the set of piecewise constant fuzzy sets.

3. Main results

Denote by $\mathcal{K}_F(X)$ the set of all finite subsets of X. Recall that for a compact X, the topology generated by the Hausdorff metric coincides with the finite topology. The following lemma illustrates the relation between finite chain continuity of (X,f) and its induced set-valued system.

Lemma 3.1. *Let (X,f) be a dynamical system and $(\mathcal{K}(X), \overline{f})$ be its induced set-valued system. Assume that $\mathcal{K}_F(X)$ is the set of all finite subsets of X. Then f is finite chain continuous if and only if $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$.*

Proof. Assume that f is finite chain continuous. To prove that $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$, let $\Omega = \{A_0, A_1, \cdots, A_k\}$ be a δ-chain in $\mathcal{K}_F(X)$ and $|A_i| = n_i$ for $i = 0, 1, \cdots, k$. Thus

$$d_H(\overline{f}(A_{i-1}), A_i) < \delta.$$

For each i take $A_i = \{a_{ij}\}$, where $j = 1, \cdots, n_k$. Fix any $j \in \{1, \cdots, n_k\}$. As $\{A_0, A_1, \cdots, A_k\}$ is a δ-chain, there exist $a_{0j} \in A_0$ and $a_{1j} \in A_1$ such that

$$d(f(a_{0j}), a_{1j}) < \delta.$$

Again, there is $a_{2j} \in A_2$ such that

$$d(f(a_{1j}), a_{2j}) < \delta.$$

By continuing this process, we obtain a sequence of δ-chains in X, say $\Omega_j = \{a_{0j}, a_{1j}, \cdots, a_{kj}\}$ for each $j \in \{1, \cdots, n_k\}$. On the other hand, since X is compact, there exists a finite subset $B = \{b_0, b_1, \cdots, b_k\}$ with $\bigcup_{b_i \in B} U(b_i, \delta) = X$. Thus, there is $B_i = \{b_{ij}\}_{j=1}^{n_k} \subseteq B$ such that

$$A_i \subset \bigcup_{j=1}^{n_k} U(b_{ij}, \delta) \qquad \text{and} \qquad A_i \cap U(b_{ij}, \delta) \neq \emptyset.$$

Hence, for each $a_{0j} \in \Omega_j$, there exists $b_{0j} \in B_0$ such that $d(a_{0j}, b_{0j}) < \delta$. Since f is finite chain continuous, we have $d(f^i(b_{0j}), a_{ij}) < \epsilon$.

Set $W = \{b_{0j}\}_{j=1}^{n_k}$. Thus $\{A_0, A_1, \cdots, A_k\}$ is a δ-chain in $\mathcal{K}_F(X)$ with $H(W, A_0) < \delta$ satisfying $d_H(\overline{f}^i(W), A_i) < \epsilon$. It follows that $\overline{f}$ is finite chain continuous.

Conversely, fin any $a \in X$. Let $\Lambda = \{x_0, x_1, \cdots, x_m\}$ be any δ-chain with $d(x_0, a) < \delta$ in X. Thus $\overline{\Lambda} = \{\{x_0\}, \{x_1\}, \cdots, \{x_m\}\}$ is a δ-chain in $\mathcal{K}_F(X)$ with $d_H(\{x_0\}, \{a\}) < \delta$. Since $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$, $d_H(\overline{f}^i\{a\}, \{x_i\}) < \epsilon$ holds, which yields $d(f^i(a), x_i) < \epsilon$. This completes the proof. □

Theorem 3.1. *Let $f : X \to X$ be continuous and $\hat{f}$ be the Zadeh's extension of f. Suppose that $\mathcal{SF}(X)$ is the set of piecewise constant fuzzy sets. If f is finite chain continuous then $\hat{f}$ is finite chain continuous on $\mathcal{SF}(X)$.*

Proof. Let $\delta > 0$ be given by finite chain continuity for f. Fix any $\omega \in \mathcal{SF}(X)$. Let $\widehat{\Lambda} = \{u_i \mid u_i \in \mathcal{SF}(X)\}_{i=0}^{m}$ be a δ-chain with $d_\infty(\omega, u_0) < \delta$ in $\mathcal{SF}(X)$. According to the definition of piecewise constant fuzzy set, for ω and each u_i, there exist decreasing sequences of closed subsets $\{A_j\}_{j=1}^{m}$, $\{B_j^i\}_{j=1}^{k}$, and increasing sequences of reals $\{\alpha_j\}_{j=1}^{m} \subseteq (0, 1]$, $\{\beta_j^i\}_{j=1}^{k} \subseteq (0, 1]$ such that

$$[\omega]_\alpha = A_{j+1}, \quad \text{where} \quad \alpha \in (\alpha_j, \alpha_{j+1}],$$

$$[u]_\beta = B_{j+1}^i, \quad \text{where} \quad \beta \in (\beta_j^i, \beta_{j+1}^i],$$

respectively. Note that $\{\alpha_j\}_{j=1}^{m}$ and $\{\beta_j^i\}_{j=1}^{k}$ are both increasing, thus there exists an increasing sequence $\{\lambda_j\}_{j=1}^{s} \subseteq (0, 1]$ which obtained by rearranging $\{\alpha_j\}_{j=1}^{m}$ and $\{\beta_j^i\}_{j=1}^{k}$, where $1 \leq s \leq m + k$. Therefore, one can find decreasing sequences of closed subsets $\{C_j\}_{j=1}^{m}$, $\{D_j^i\}_{j=1}^{k}$ with

$$[\omega]_\lambda = C_{j+1} \quad \text{and} \quad [u]_\lambda = D_{j+1}^i,$$

where $\lambda \in (\lambda_j, \lambda_{j+1}]$.

Since $\widehat{\Lambda} = \{u_i \mid u_i \in \mathcal{SF}(X)\}_{i=0}^{m}$ is a δ-chain with $d_\infty(\omega, u_0) < \delta$, it can be verified that $\{D_j^i\}_{j=1}^{k}$ is a δ-chain in $\mathcal{K}_F(X)$ satisfying $d_H(C_j, D_j^0) < \delta$. Combining the finite chain continuity of f with Lemma 3.1, $\overline{f}$ is finite chain continuous,

i.e., for any $\epsilon > 0$, $d_H(\overline{f}^i(C_j), D_j^i) < \epsilon$ holds. Consequently, it follows that

$$\begin{aligned} d_\infty(\hat{f}^i(\omega), u_i) &= \sup_{\alpha\in[0,1]} d_H([\hat{f}^i(\omega)]_\alpha, [u_i]_\alpha) \\ &= \sup_{\alpha\in[0,1]} d_H(f^i[\omega]_\alpha, [u_i]_\alpha) \\ &= d_H(\overline{f}^i(C_j), D_j^i) < \epsilon, \end{aligned}$$

which implies that $\hat{f}$ is finite chain continuous on $\mathcal{SF}(X)$. □

Theorem 3.2. *Let (X, f) be a dynamical system and $(\mathcal{F}(X), \hat{f})$ be its induced fuzzy system. If $\hat{f}$ is chain continuous then f is chain continuous.*

Proof. Fix any $x^* \in X$. For $\epsilon > 0$, take a δ-pseudo-orbit $\Omega = \{x_0, x_1, \cdots\}$ with $d(x^*, x_0) < \delta$ in X. Thus $\widehat{\Omega} = \{\hat{x}_0, \hat{x}_1, \cdots\}$ is a δ-pseudo-orbit with $d_\infty(\hat{x}^*, \hat{x}_0) < \delta$ in $\mathcal{F}(X)$. Since $\hat{f}$ is chain continuous, we have $d_\infty(\hat{f}^i(\hat{x}^*), \hat{x}_i) < \epsilon$ for $i = 0, 1, \cdots$. Hence

$$\begin{aligned} d_\infty(\hat{f}^i(\hat{x}^*), \hat{x}_i) &= \sup_{\alpha\in[0,1]} d_H([\hat{f}^i(\hat{x}^*)]_\alpha, [\hat{x}_i]_\alpha) \\ &= \sup_{\alpha\in[0,1]} d_H(f^i[(\hat{x}^*)]_\alpha, [\hat{x}_i]_\alpha) \\ &= d_H(\{f^i(x^*)\}, \{x_i\}) < \epsilon. \end{aligned}$$

It follows that $d(f^i(x^*), x_i) < \epsilon$ and consequently f is chain continuous. □

According to the Theorem 3.2, the following corollary is straightforward:

Corollary 3.1. *Let (X, f) be a dynamical system and $(\mathcal{F}(X), \hat{f})$ be its induced fuzzy system. If $\hat{f}$ is finite chain continuous then f is finite chain continuous.*

Remark 3.1. Since $\mathcal{F}(X)$ is not compact (see [8]) and it is not true that chain continuity is equivalent to the chain continuity on a dense subset (see [17]), finite chain continuity of $\hat{f}$ can not imply its own chain continuity. However, the completeness of $\mathcal{F}(X)$ lead us to the following theorem which is involved with a concept of contraction.

Definition 3.1. *Let $(\mathcal{F}(X), d_\infty)$ be a complete metric space. A function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is called a contraction if there exists $\lambda \in (0, 1)$ such that $d_\infty(\hat{f}(u), \hat{f}(v)) \leq \lambda d(x, y)$ holds for any $u, v \in \mathcal{F}(X)$.*

Theorem 3.3. *Let $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ be a contraction, then $\hat{f}$ is chain continuous.*

Proof. Fix $\epsilon > 0$ and take $\delta = \frac{1-\lambda}{4}\epsilon$. For $v \in \mathcal{F}(X)$, let $\{u_0, u_1, \cdots\}$ be a δ-pseudo-orbit with $d_\infty(v, u_0) < \delta$ in $\mathcal{F}(X)$. Define $U_n = U(u_n, \frac{\epsilon}{4})$ for $n \geq 1$.

Thus for any $\omega_n \in U_n$, it follows that

$$\begin{aligned} &d_\infty(\hat{f}(\omega_{n-1}), u_n) \le d_\infty(\hat{f}(\omega_{n-1}), \hat{f}(u_{n-1})) \\ &+d_\infty(\hat{f}(u_{n-1}), u_n) < \lambda d_\infty(\omega_{n-1}, u_{n-1}) + \delta \\ &< \frac{\lambda}{4}\epsilon + \frac{1-\lambda}{4}\epsilon = \frac{\epsilon}{4}, \end{aligned}$$

which implies $\hat{f}(U_{n-1}) \subset U_n$, and therefore $\hat{f}^n(U_0) \subset U_n$. Choose any $\omega_0 \in U_0$, $d_\infty(\hat{f}^n(\omega_0), u_n) < \frac{\epsilon}{4}$ holds.

On the other hand, note that

$$\begin{aligned} d_\infty(v, \omega_0) \quad &\le d_\infty(v, u_0) + d_\infty(u_0, \omega_0) \\ &< \delta + \frac{\epsilon}{4} = \frac{1-\lambda}{4}\epsilon + \frac{\epsilon}{4} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$d_\infty(\hat{f}^n(v), \hat{f}^n(\omega_0)) < \lambda^n d_\infty(v, \omega_0) < \lambda^n \frac{\epsilon}{2}.$$

Consequently,

$$\begin{aligned} &d_\infty(\hat{f}^n(v), u_n) \le d_\infty(\hat{f}^n(v), \hat{f}^n(\omega_0)) \\ &+d_\infty(\hat{f}^n(\omega_0), u_n) < \lambda^n \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon, \end{aligned}$$

which proves the chain continuity of $\hat{f}$. □

4. Conclusions and discussions

There is a difference between chain continuity and shadowing. Actually, chain continuity is a pointwise notion and shadowing is a global one. Both of the concepts are important tools for understanding the true orbits in the dynamical systems. In this paper, we investigate the relations between (finite) chain continuity of discrete dynamical system and its induced fuzzy system.

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Sumudu transform for solving some classes of fractional differential equations

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Abstract. Many authors recently, have demonstrated the usefulness of fractional calculus especially in the derivation of solutions for linear partial differential equations (PDEs) and ordinary differential equations (ODEs). In this paper, some properties of fractional Sumudu transform for solving fractional differential equations (FrDEs) have introduced. The approximated solutions of some classes of FrDEs using Sumudu transform method have studied. The objective of this work is to show the advantages of application of Sumudu transform method and the expansion of the coefficients of a binomial series for solving fractional differential equations.

Keywords: Sumudu transform, Laplace transform, fractional differential equations (FrDEs), binomial series, models, class.

1. Introduction

DEs are useful tools in mathematical models of life problems and applied mathematics. DEs have played very important role in different applications of mathematics for a long time and with the development of the computer. Thus, the investigation and the analysis of DEs had increase in applications leding to several mathematical problems; therefore, there are a lot of different techniques of transformations have proposed by authors in order to solve different types of DEs like: Laplace, Fourier, Mellin, Hankel transformations and Sumudu transform which is little known and not widely used in solving DEs yet. The single Sumudu transform (or Sumudu transform) was originally proposed by [1]

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for that functions of exponential order for solving DEs and control engineering problems and proved many interesting properties in t-domain and u-domain, also the properties and applications of Sumudu transform to ODEs have described by him. Among the integral transforms, Sumudu transform has units preserving properties and thus may be used to solve problems without resorting them to the frequency domain and this is one of many strength points of this new transform especially with the applications of problems with physical dimensions. The nonlinear PDEs that appears in many branches of physics, engineering and applied mathematics can not be described successfully, thus they turn out to describe them by models from fractional calculus. [2] derived formulas for the single Sumudu transform of partial derivatives and applied them in solving initial value problems (IVP) also, studied the applications of Sumudu transform in solving PDEs. The Sumudu transform of partial derivatives is derived by [2] and its applicability demonstrated using three different PDEs. [3] extended Sumudu transform to that functions of two variables. Using this extended definition, a function of two variables with emphasis on solutions to PDEs such as $f(x, y)$ is transformed to a function such as $F(u, v)$. [4] introduced the analytical investigations of the Sumudu transform and applications to integral production equations. Laplace transform is dual to Sumudu transform in solving mathematical problems. However, Sumudu transform may be used to solve mathematical problems without resorting to a new frequency domain. In the past two decades the subject of fractional calculus was widely investigated and has remarkably gained importance and popularity. [5] and [6] have studied the approximated solutions of FrDEs of Lane-emden type by method of collocation and least square method respectively. [7] extended the theory and the applications of Sumudu transform and used it to solve the FrDEs by direct integration methods. [8] studied and proved some of Sumudu transform properties and Laplace-Sumudu transforms duality and the complex inversion formula while [9] has developed an analytical methods for solving FrPDEs and extended Sumudu transform iterative method to solve a variety of time and space FrPDEs as well as systems of them. They demonstrated the utility of the method by finding the exact solutions to a large number of FrPDEs. [10] used Sumudu transform techniques to obtain the solutions of a cauchy problems for DEs with the Caputo fractional derivative and the solution of fractional Diffusion-Wave equation while [11] used Sumudu transform and variational iteration method (VIM) to approximate the solutions of FrDEs related to entropy, wavelets etc. [18] derived the approximated solutions to some homogeneous FrPDEs by applying the Laplace of the fractional derivative and the expansion of the coefficients of a binomial series. Lastly, several researchers studied fractional Sumudu transform in [12, 13, 14, 15, 16, 17]. In this paper, we derived a standard formulas for finding the approximated solutions to some classes of homogeneous FrPDEs by using the Sumudu fractional derivative and the expansion of the coefficients of binomial series.we solve the same equations solved in

[18]using Laplace transform but we use Sumudu transform instead and we get the same results.

2. Preliminary

In this section, we introduce preliminaries concepts and some definitions for this study.

Definition 2.1. *The Fractional Derivative of a Casual Function*

The fractional derivative of a casual function $f(t)$ is defined by [18] as follows:

$$\frac{d^\alpha}{dt^\alpha}f(t)=\begin{cases} f^{(n)}(t), & \text{if } \alpha=n\in N, \\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, & \text{if } n-1<\alpha<n, \end{cases} \tag{1}$$

where the Euler gamma function $\Gamma(.)$ is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, (R(z)>0).$$

Definition 2.2 (The Sumudu Transform of a Function)**.** *Let $f(t)$ be a real function defined on the domain $(0,\infty)$ the Sumudu transform of $f(t)$ is defined by [18] as follows:*

$$G(u)=S[f(t)]=\int_0^\infty e^{-t}f(ut)\,dt, u\in C$$

provided the integral exists for some u.

Definition 2.3 (The Mittag-Leffler Function)**.** *The Mittag-Leffler function [18] is defined by*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+\beta)}, \quad (z,\alpha,\beta\in C, R(\alpha>0).$$

Definition 2.4 (The Simplest Wright Function)**.** *The simplest Wright function is defined by [18] as follows:*

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\beta)}.\frac{z^k}{k!}, (z,\alpha,\beta\in C).$$

Definition 2.5 (The Riemann-Liouville Fractional Derivatives)**.** *The Riemann-Liouville fractional derivatives $D_{a+}^\alpha y$ and $D_{a-}^\alpha y$ of order $\alpha\in C(R(\alpha)\geqq 0)$ are defined by [18] as follows:*

$$(D_{a+}^\alpha y)(x)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x\frac{y(t)dt}{(x-t)^{\alpha-n+1}}; \qquad n=[R(\alpha)]+1; x>a$$

and

$$(D^{\alpha}_{b-}y)(x) = \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dx})^n \int_a^x \frac{y(t)dt}{(t-x)^{\alpha-n+1}}; \qquad n = [R(\alpha)]+1; x > b$$

respectively, where $[R(\alpha)]$ *means the integral part of* $R(\alpha)$.

Definition 2.6 (The Pochhammer Symbol)**.** *The Pochhammer symbol (or the shifted factorial, since* $(1)_n = n!$ *of* $n \in N_0 = \{0, 1, 2, ...\}$*) given by [18] as following:*

$$(\lambda)_n = \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\dots(\lambda+n-1), & (n \in N_0/\{0\}). \end{cases}$$

Definition 2.7 (The Binomial Coefficients)**.** *The binomial coefficients are defined by [18] as follows:*

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!},$$

where λ *and* n *are integers.*

Observe that 0!=1, then,

$$\binom{\lambda}{0} = \binom{\lambda}{\lambda} = 1$$

and

$$(1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda+r-1}{r} z^r.$$

2.1 Fractional Sumudu transform

Using the definition of fractional Sumudu transform, we can easily obtain the following operational formulas:

1. $S_\alpha\{f(at)\} = G_\alpha(au)$,
2. $S_\alpha\{f(t-b)\} = E_\alpha(-b^\alpha) = G_\alpha(u)$,
3. $S_\alpha\{E_\alpha(-c^\alpha t^\alpha)\} = \frac{1}{(1+cu)^\alpha} G_\alpha(\frac{u}{1+cu})$,
4. $S_\alpha\{\int_0^t f(t)(dt)^\alpha\} = u^\alpha \Gamma(1+\alpha) G_\alpha(u)$,
5. $S_\alpha\{f^\alpha(t)\} = \frac{G_\alpha(u) - \Gamma(1+\alpha)f(0)}{u^\alpha}$,
6. $S^2_\alpha\{f(at, bx)\} = G_\alpha(au) H_\alpha(bv)$,
7. $S^2_\alpha\{f(at)g(bx)\} = G^2_\alpha(au, bv)$,
8. $S^2_\alpha\{f(t-a, x-b)\} = E_\alpha(-(A+B)^\alpha) G^2_\alpha(au, bv)$,

9. $S_\alpha^2\{\partial_t^\alpha f(t,x)\} = \frac{G_\alpha^2(u,v)-\Gamma(1+\alpha)f(0,x)}{u^\alpha}$,

10. If one defines the convolution of order of the two function $f(t)$ and $g(t)$ by the expression

$$(f(x) * g(x))_\alpha = \int_0^x f(x-v)g(v)(dv)^\alpha,$$

then $S_\alpha\{(f(t) * g(t))_\alpha\} = u^\alpha G_\alpha(u)H_\alpha(u)$, where $G_\alpha(u) = S_\alpha\{f(t)\}$ and $H_\alpha(u) = S_\alpha\{g(t)\}$.

and for the detailed proof of the above properties.

Theorem 2.1. *Let $f(t)$ be a function, then*

$$S[D^\alpha f(t)](u) = u^{-\alpha}S[f(t)] - \sum_{k=0}^{n} u^{\alpha-k}f^{(k-1)}(0).$$

Proof. By the technique of integral transform and the Definition 2.2 of Sumudu transform, we have the following:

$$\begin{aligned}
S[D^\alpha f(t)](u) &= \int_0^\infty e^{-t}[D^\alpha f(ut)]dt, \\
&= \int_0^\infty e^{-t}\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(x)}{(ut-x)^{\alpha-n+1}}dxdt, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty\int_x^\infty e^{-t}\frac{f^{(n)}(x)}{(ut-x)^{\alpha-n+1}}dxdt, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty\int_x^\infty e^{-\frac{z+x}{u}}f^{(n)}(x)z^{n-\alpha+1}\frac{1}{u}dzdx, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}\int_x^\infty e^{-\frac{z}{u}}z^{n-\alpha+1}dzdx, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}u^{n-\alpha}\Gamma(n-\alpha)dx, \\
&= u^{n-\alpha}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}dx, \\
&= u^{n-\alpha}[\frac{Sf(t)}{u^n} - \frac{f(0)}{u^n} - \frac{f'(0)}{u^{n-1}}.....\frac{f^{(n-1)}(0)}{u^{n-\alpha}}],
\end{aligned}$$

Hence

$$S[D^\alpha f(t)](u) = u^{-\alpha}S[f(t)] - \sum_{k=0}^{n} u^{\alpha-k}f^{(k-1)}(0).$$

□

3. Method of solution of fractional differential equations

Throughout this section, consider $y(t)$ such that for some value of the parameter u, the Sumudu transform $G(u) = S(y(t))$ converges.

Theorem 3.1. *Let $o < \alpha < 1$ and $b \in R$. Then, the FrDE*

$$y^{\alpha}(t) - by(t) = 0, \quad t \geq 0. \tag{2}$$

with the initial condition $y(0) = c_0$ the solution is

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bu^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

Proof. Applying Sumudu transform for Equation (2)

$$\frac{G(u) - G(0)}{u^{\alpha}} - bG(u) = 0.$$

Then

$$G(u) = \frac{c_0}{1 - bu^{\alpha}}.$$

Since

$$\frac{1}{1 - bu^{\alpha}} = \sum_{k=0}^{\infty} (bu^{\alpha})^k.$$

So,

$$G(u) = c_0 \sum_{k=0}^{\infty} (bu^{\alpha})^k. \tag{3}$$

Taking inverse Sumudu transform for Equation (3)

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bu^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

□

Theorem 3.2. *Let $1 < \alpha < 2$ and $a, b \in R$. Then, the following FrDE*

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0. \tag{4}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k + r + 1)(-at^{\alpha-1})^r}{\Gamma((\alpha - 1)r + \alpha k + 1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k + r + 1)(-at^{\alpha-1})^r}{\Gamma((\alpha - 1)r + \alpha k + 2)r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{k+\alpha-1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k + r + 1)(-at^{\alpha-1})^r}{\Gamma((\alpha - 1)r + \alpha k + \alpha)r!}. \end{aligned}$$

Proof. Applying Sumudu transform for Equation (4)

$$\frac{G(u)-G(0)}{u^{-\alpha}}-\frac{G'(0)}{u^{\alpha-1}}+\frac{a(G(u)-G(0))}{u}+bG(u)=0.$$

That is,

$$G(u)(u^{-\alpha}+au^{-1}+b)=(c_0u^{-\alpha}+c_1u^{1-\alpha}+ac_0u^{-1}),$$
$$G(u)=\frac{c_0u^{-\alpha}+c_1u^{1-\alpha}+ac_0u^{-1}}{u^{-\alpha}+au^{-1}+b}.$$

Since

$$\begin{aligned}
\frac{1}{u^{-\alpha}+au^{-1}+b} &= \frac{u}{u^{1-\alpha}+a+bu},\\
&= \frac{u}{(u^{1-\alpha}+a)(1+\frac{bu}{u^{1-\alpha}+a})},\\
&= \frac{u}{u^{1-\alpha}+a}\sum_{k=0}^{\infty}(\frac{-bu}{u^{1-\alpha}}+a)^k,\\
&= \sum_{k=0}^{\infty}\frac{(-b)^ku^{\alpha k+\alpha}}{(1+au^{\alpha-1})^{k+1}},\\
&= \sum_{k=0}^{\infty}(-b)^ku^{\alpha k+\alpha}\sum_{r=o}^{\infty}(-au^{\alpha-1})^r\binom{k+r}{r},\\
&= \sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha}.
\end{aligned}$$

Then,

$$\begin{aligned}
G(u) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k}\\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+1}\\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha-1}. \qquad (5)
\end{aligned}$$

After taking inverse Sumudu transform for Equation (5), we obtain the solution

$$\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}\frac{(-b)^kt^{\alpha k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+1)r!}\\
&+ c_1\sum_{k=0}^{\infty}\frac{(-b)^kt^{\alpha k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+2)r!}\\
&+ ac_0\sum_{k=0}^{\infty}\frac{(-b)^kt^{\alpha k+\alpha-1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+\alpha)r!}
\end{aligned}$$

□

Example 3.1. If we let $\alpha = \frac{3}{2}, a = -1$ and $b = -2$ in Theorem 3.2, then, the following FrDE

$$y^{(\frac{3}{2})}(t) - y'(t) - 2y(t) = 0, \quad t \geq 0. \tag{6}$$

has a solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+2)r!} \\ &- c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+\frac{3}{2})r!}. \end{aligned} \tag{7}$$

[18] solved Equation (6) using Laplace transform with the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+2)r!} \\ &- c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+\frac{3}{2})r!}. \end{aligned} \tag{8}$$

Hence, the two solutions Equation (3.6) and (3.7) are identical

3.1 Remark

Let $1 < \alpha < 2$ and a ,b $\in R$. Then, the FrDE

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0.$$

If $a = 0$ then, the FrDE

$$y^{(\alpha)}(t) + by(t) = 0$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+1)} + c_1 t \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+2)}.$$

Theorem 3.3. *Let $1 < \alpha < 2$ and a ,$b \in R$. Then, the FrDE*

$$y^{(\alpha)}(t) + ay''(t) + by(t) = 0, \quad t \geq 0. \tag{9}$$

with the initial conditions $y(0) = c_0$ *and* $y'(0) = c_1$ *has the following solution*

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2)r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3)r!} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4)r!}. \end{aligned}$$

Proof. Taking Sumudu transform for Equation (9)

$$\frac{G(u) - G(u)}{u^2} - \frac{G'(0)}{u} - \frac{aG(u) - aG(0)}{u^\alpha} - \frac{G'(0)}{u^{\alpha-2}} + bG(u)$$

$$G(u)(u^{-2} + au^{-\alpha} + b) = c_0 u^{-2} + c_1 u^{-1} + ac_0 u^{-\alpha} + ac_1 u^{2-\alpha}.$$

Since,

$$\begin{aligned} \frac{1}{u^{-2} + au^{-\alpha} + b} &= \frac{u^\alpha}{u^{\alpha-2} + a + bu^\alpha}, \\ &= \frac{u^\alpha}{(u^{\alpha-2} + a)(1 + \frac{bu^\alpha}{u^{\alpha-a}})}, \\ &= \frac{u^\alpha}{(u^{\alpha-2} + a)} \sum_{k=0}^{\infty} (\frac{-bu^\alpha}{u^{\alpha-2} + a})^k, \\ &= \sum_{k=0}^{\infty} \frac{(-b)^k u^{k\alpha+\alpha}}{(u^{\alpha-2} + a)^{k+1}}, \\ &= \sum_{k=0}^{\infty} \frac{(-b)^k u^{2k+2}}{(1 + au^{2-\alpha})^{k+1}}, \\ &= \sum_{k=0}^{\infty} (-b)^k u^{2k+2} \sum_{r=0}^{\infty} (-au^{2-\alpha})^r \binom{k+r}{r}, \\ &= \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2}. \end{aligned}$$

Then

$$\begin{aligned} G(u) &= c_0 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k} \\ &+ c_1 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k+1} \\ &+ ac_0 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+2} \\ &+ ac_1 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+3}. \end{aligned} \tag{10}$$

Taking inverse Sumudu transform for Equation (10)

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r \frac{t^{(2-\alpha)r+2k}}{((2-\alpha)r+2k)!} \\ &+ c_1 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r \frac{t^{(2-\alpha)r+2k+1}}{((2-\alpha)r+2k+1)!} \\ &+ ac_0 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r \frac{t^{(2-\alpha)r+2k-\alpha+2}}{((2-\alpha)r+2k-\alpha+2)!} \\ &+ ac_1 \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r \frac{t^{(2-\alpha)r+2k-\alpha+3}}{((2-\alpha)r+2k-\alpha+3)!}. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2)r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3)r!} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4)r!}. \end{aligned}$$

□

Example 3.2. If we let $\alpha = \frac{3}{2}, a = \sqrt{3}$ and $b = 8$ in Theorem 3.2, then, the FrDE

$$y''(t) + ay^{(\alpha)} + by(t) = 0, \quad t \geq 0. \tag{11}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{12}$$

[18] solved Equation (11) using Laplace transform with the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{13}$$

Hence, the two solutions in equations (12) and (13) are identical

4. Conclusion

In this paper, Fractional Sumudu transform has been studied. The operational formulas of fractional Sumudu transform have been derived. Fractional Sumudu transform and the expansion of the coefficients of a binomial series used to solve some classes of FrDEs, which led to many interesting consequences. The approximated solutions of the problems using he new method agrees very well with the analytical solutions. As such, this method is more cost effective in terms of computation steps than other existing transform methods. Hence, we can conclude that the new method is computationally very efficient in solving some classes of FrDEs.

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Supra soft b-R_0 and supra soft $b - R_1$ spaces

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Abstract. The purpose of this paper is to introduce supra soft b-R_0 and supra soft b-R_1 spaces using supra soft b-open sets. Also we study several of their properties and characterizations in details. Furthermore, we investigate the relationships between these supra soft spaces and the relationships with some other supra soft spaces.

Keywords: supra soft b-open set, supra soft topological space, supra soft b-R0 space, supra soft b-R1 space.

1. Introduction

In 1999, Russian researcher Molodtsov [22] initiated the concept of soft sets as a new general mathematical tool for dealing with uncertain objects while modelling the problems with incomplete information in engineering, physics, computer science, economics, social sciences and medical sciences. In [23], Molodtsov applied soft sets successfully in directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability and theory of measurement. Maji et. al [20] applied soft sets in a multicriteria decision making problems. It is based on the notion of knowledge reduction of rough sets. They applied the technique of knowledge reduction to the information table induced by the soft set. In [21], they defined and studied several basic notions of soft set theory. In 2005, Pei and Miao [27] and Chen [4] improved the work of Maji et.al [20-21]. A. Kharal and B. Ahmad [19] defined and discussed the several properties of soft images and soft inverse images of soft sets. They also applied these notions to the problem of medical diagnosis in medical systems. Many researchers have contributed towards the algebraic structure of soft set theory [1-2],[3], [6], [9-19], [26], [29-30]. In 2011, Shabir and Naz [29] initiated the study of soft topological spaces. Also in 2011, S. Hussain and B. Ahmad [8] continued investigating the properties of soft open(closed), soft neighborhood and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary. Shabir et. al [29] and D. N. Georgiou et. al [6], defined and studied some soft separation axioms, soft -continuity and soft connectedness in soft spaces using (ordinary) points of a topological space X. Ittanagi [9] introduced the concept of soft bitopological space and studied some

types of soft separation axioms for soft bitopological spaces from his point of view. Recently, J. M. Mustafa [25], defined and studied supra soft b-compact and supra soft b-Lindelof spaces. Also, J. M. Mustafa [24], defined and studied b-R_0 and b-R_1 spaces. In this paper, we introduce the concepts of supra soft b-R_0 and supra soft b-R_1 spaces. Furthermore, we investigate the relationships between these supra sof spaces and the relatioships with some other supra soft spaces.

2. Preliminaries

In this section we will introduce necessary definitions and theorems for soft sets.

Definition 2.1 ([22]). Let X be an initial universe and E a set of parameters. Let $P(X)$ denote the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in E$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, E).

Definition 2.2 ([22]). The complement $\widetilde{X} - (F, E)$ of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$ where, $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$, for all $e \in E$.

Let us call F^c to be the soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F, E)^c)^c = (F, E)$.

Definition 2.3 ([32]). A soft set (F, E) over X is said to be a null soft set, denoted by $\widetilde{\phi}$, if for all $e \in E$, $F(e) = \phi$. Clearly, $(\widetilde{\phi}^c)^c = \widetilde{\phi}$.

Definition 2.4 ([32]). A soft set (F, E) over X is said to be an absolute soft set, denoted by $\widetilde{X}$, if for all $e \in E$, $F(e) = X$. Clearly, $\widetilde{X}^c = \widetilde{\phi}$.

Definition 2.5 ([21]). Let (F, E), (G, E) be two soft sets over X. Then:

(1) (F, E) is a soft subset of (G, E), denoted by $(F, E)\widetilde{\subseteq}(G, E)$, if $F(e) \subseteq G(e)$, $\forall e \in E$. In this case, (F, E) is said to be a soft subset of (G, E) and (G, E) is said to be a soft superset of (F, E).

(2) Two soft sets (F, E) and (G, E) over a common universe set X are said to be equal, denoted by $(F, E) = (G, E)$, if $F(e) = G(e)$, $\forall e \in E$.

(3) The union of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E)\widetilde{\cup}(G, E)$, is the soft set (H, E), where$H(e) = F(e) \cup G(e)$, for all $e \in E$.

(4) The intersection of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E)\widetilde{\cap}(G, E)$, is the soft set (M, E), where $M(e) = F(e) \cap G(e)$, for all $e \in E$.

Definition 2.6 ([29]). Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$.

Note that for any $x \in X$, $x \notin (F, E)$ if $x \notin F(e)$ for some $e \in E$.

Definition 2.7 ([29])**.** Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then τ is called a soft topology on X if

(1) $\widetilde{\phi}$, $\widetilde{X} \in \tau$.

(2) the union of any number of soft sets in τ belongs to τ.

(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.8 ([29])**.** Let (X, τ, E) be a soft topological space, and (F, E) be a soft set over X. If Y is a non empty subset of X, then the sub soft set of (F, E) over Y denoted by (F_Y, E), is defined as follows

$$F_Y(e) = Y \cap F(e) \text{ for all } e \in E.$$

Definition 2.9. Let (X, τ, E) be a soft topological space and Y be a non empty subset of X. Then, $\tau_Y = \{(F_Y, E) : (F, E) \in \tau\}$ is called the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).

Definition 2.10 ([19])**.** Let $SS(X)_E$ and $SS(Y)_K$ be families of soft sets on X and Y respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. We define a soft mapping $f_{pu} : SS(X)_E \to SS(Y)_K$ as:

(1) If $(F, E) \in SS(X)_E$, then, the image of (F, E) under f_{pu}, written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SS(Y)_K$ such that $f_{pu}(F, E)(k) = \cup\{u[F(e)] : e \in p^{-1}(k)\}$ if $p^{-1}(k) \neq \phi\phi$ if $p^{-1}(k) = \phi$, for all $k \in K$.

(2) If $(H, K) \in SS(Y)_K$, then the inverse image of (H, K) under f_{pu}, written as $f_{pu}^{-1}(H, K) = (f_{pu}^{-1}(H), E)$, is a soft set in $SS(X)_E$ such that:

$$f_{pu}^{-1}(H, K)(e) = u^{-1}[H(p(e))] \text{ for all } e \in E.$$

3. Supra soft b-open sets

Definition 3.1 ([5])**.** Let μ be a collection of soft sets over a universe X with a fixed set of parameters E. Then μ is called a supra soft topology on X with a fixed set E if:

(1) $\widetilde{\phi}$, $\widetilde{X} \in \mu$

(2) The union of any number of soft sets in μ belongs to μ.

(X, μ, E) is called supra soft topological space and the elements of μ are called supra soft open sets. The soft complement of any supra soft open set is called supra soft closed.

Definition 3.2. Let (X, μ, E) be a supra soft topological space and Y be a non empty subset of X. Then, $\mu_Y = \{(F_Y, E) : (F, E) \in \mu\}$ is called the supra soft relative topology on Y and (Y, μ_Y, E) is called a supra soft subspace of (X, μ, E).

Definition 3.3 ([5])**.** Let (X, μ, E) be a supra soft topological space and (F, E) be a soft set over X. Then the supra soft interior of (F, E) , denoted by $int^s(F, E)$ is the soft union of all supra soft open subsets of (F, E) . Clearly $int^s(F, E)$ is the largest supra soft open set over X which contained in (F, E).

Definition 3.4 ([5])**.** Let (X, μ, E) be a supra soft topological space and (F, E) be a soft set over X. Then the supra soft closure of (F, E) , denoted by $cl^s(F, E)$ is the soft intersection of all supra soft closed sets containing (F, E) . Clearly $cl^s(F, E)$ is the smallest supra soft closed set over X which contains (F, E).

Definition 3.5. A soft set (F, E) in a supra soft topological space X is called

(1) supra soft b-open (ssb-open) set if $(F, E)\widetilde{\subseteq}int^s(cl^s(F, E))\widetilde{\cup}cl^s(int^s(F, E))$.

(2) supra soft b-closed (ssb-closed) set if

$$(F, E)\widetilde{\supseteq}nt^s(cl^s(F, E))\widetilde{\cap}cl^s(int^s(F, E)).$$

We denote the family of all ssb-open (resp. ssb-closed) sets in a supra soft topological space X by SSBO(X) (resp. SSBC(X)).

Definition 3.6. Let (X, μ, E) be a supra soft topological space. Then a soft set (F, E) is called a supra soft b-neighborhood of x if there exists a supra soft b-open set (G, E) such that $x \in (G, E)\widetilde{\subseteq}(F, E)$

Definition 3.7. Let (X, μ, E) be a supra soft topological space and (F, E) be a soft set over X. Then the supra soft b-interior of (F, E) , denoted by $int^s_b(F, E)$ is the soft union of all supra soft b-open subsets of (F, E) . Clearly $int^s_b(F, E)$ is the largest supra soft b-open set over X which contained in (F, E).

Definition 3.8. Let (X, μ, E) be a supra soft topological space and (F, E) be a soft set over X. Then the supra soft b-closure of (F, E) , denoted by $cl^s_b(F, E)$ is the soft intersection of all supra soft b-closed sets containing (F, E) . Clearly $cl^s_b(F, E)$ is the smallest supra soft b-closed set over X which contains (F, E).

Definition 3.9. A soft mapping $f_{pu} : (X, \mu, E) \to (Y, \sigma, K)$ is said to be

(1) supra soft b-continuous (briefly ssb-continuous) if the inverse image of each supra soft open set of (Y, σ, K) is a supra soft b-open set in (X, μ, E).

(2) supra soft b-open if the image of each supra soft open set of (X, μ, E) is supra soft b -open set in (Y, σ, K).

(3) supra soft b-irresolute if the inverse image of each supra soft b-open set of (Y, σ, K) is a supra soft b-open set in (X, μ, E).

(4) supra soft b*-open if the image of each supra soft b-open set of (X, μ, E) is supra soft b-open set in (Y, σ, K).

(5) supra soft b*-closed if the image of each supra soft b-closed set of (X, μ, E) is supra soft b-closed set in (Y, σ, K).

4. Supra soft b-$\mathbf{R}_0$ and supra soft b-$\mathbf{R}_1$ spaces

Definition 4.1. Let (F,E) be a supra soft set over X. The supra soft b-kernel of (F,E) denoted by $Ker_b^s(F,E)$ is defined to be the set $Ker_b^s(F,E) = \widetilde{\cap}\{(G,E) \in SSBO(X) : (F,E)\widetilde{\subseteq}(G,E)\}$.

Definition 4.2. Let x be a point of a supra soft topological space (X,μ,E). The supra soft b-kernel of x, denoted by $Ker_b^s(\{x\})$ is defined to be the set $Ker_b^s(\{x\}) = \widetilde{\cap}\{(G,E) \in SSBO(X) : x \in (G,E)\}$.

Theorem 4.3. *Let (X,μ,E) be a supra soft topological space and $x \in X$. Then $Ker_b^s(F,E) = \{x \in X : cl_b^s(\{x\})\widetilde{\cap}(F,E) \neq \widetilde{\phi}\}$.*

Proof. To show that $Ker_b^s(F,E)\widetilde{\subseteq}\{x \in X : cl_b^s(\{x\})\widetilde{\cap}(F,E) \neq \widetilde{\phi}\}$ let $x \in Ker_b^s(F,E)$ and $cl_b^s(\{x\})\widetilde{\cap}(F,E) = \widetilde{\phi}$. Now $X - cl_b^s(\{x\})$ is a supra soft b-open set containing (F,E), but $x \notin \widetilde{X} - cl_b^s(\{x\})$ which is impossible since $x \in Ker_b^s(F,E)$. Consequently, $cl_b^s(\{x\})\widetilde{\cap}(F,E) \neq \widetilde{\phi}$. Now, let $cl_b^s(\{x\})\widetilde{\cap}(F,E) \neq \widetilde{\phi}$ and $x \notin Ker_b^s(F,E)$. Then, there exists a supra soft b-open set (G,E) containing (F,E) and $x \notin (G,E)$. Let $y \in cl_b^s(\{x\})$. Hence, (G,E) is a supra soft b-neighborhood of y with $x \notin (G,E)$. By this contradiction, $x \in Ker_b^s(F,E)$ and the claim. □

Definition 4.4. A supra soft topological space (X,μ,E) is said to be supra soft $b-R_0$ space if every supra soft b-open set contains the supra soft b-closure of each of its singletons.

Example 4.5. Let $X = \{h_1,h_2\}$ and $E = \{e_1,e_2\}$. Then the soft subsets over X are $(F_1,E) = \widetilde{\phi}$, $(F_2,E) = \widetilde{X}$, $(F_3,E) = \{(e_1,\phi),(e_2,X)\}$, $(F_4,E) = \{(e_1,\phi),(e_2,\{h_1\})\}$, $(F_5,E) = \{(e_1,\phi),(e_2,\{h_2\})\}$, $(F_6,E) = \{(e_1,X),(e_2,\phi)\}$, $(F_7,E) = \{(e_1,X),(e_2,\{h_1\})\}$, $(F_8,E) = \{(e_1,X),(e_2,\{h_2\})\}$, $(F_9,E) = \{(e_1,\{h_1\}),(e_2,\phi)\}$, $(F_{10},E)=\{(e_1,\{h_1\}),(e_2,X)\}$, $(F_{11},E) = \{(e_1,\{h_1\}),(e_2,\{h_1\})\}$, $(F_{12},E) = \{(e_1,\{h_1\}),(e_2,\{h_2\})\}$, $(F_{13},E) = \{(e_1,\{h_2\}),(e_2,\phi)\}$, $(F_{14},E) = \{(e_1,\{h_2\}),(e_2,X)\}$, $(F_{15},E) = \{(e_1,\{h_2\}),(e_2,\{h_1\})\}$ and $(F_{16},E)=\{(e_1,\{h_2\}),(e_2,\{h_2\})\}$.

Let $\mu = \{\widetilde{\phi},\widetilde{X},(F_{11},E),(F_{16},E)\}$. Then (X,μ,E) is a supra soft $b-R_0$ space.

Example 4.6. Let $X = \{h_1,h_2,h_3\}$ and $E = \{e\}$. Then the soft subsets over X are $(F_1,E) = \widetilde{\phi}$, $(F_2,E) = \widetilde{X}$, $(F_3,E) = \{(e,\{h_1\})\}$, $(F_4,E) = \{(e,\{h_2\})\}$, $(F_5,E) = \{(e,\{h_3\})\}$, $(F_6,E) = \{(e,\{h_1,h_2\})\}$, $(F_7,E) = \{(e,\{h_1,h_3\})\}$, $(F_8,E) = \{(e,\{h_2,h_3\})\}$.

Let $\mu = \{\widetilde{\phi},\widetilde{X},(F_3,E),(F_6,E)\}$. Then (X,μ,E) is not supra soft $b-R_0$ since (F_3,E) is a supra soft b-open set and $h_1 \in (F_3,E)$, but $cl_b^s(\{h_1\}) = \widetilde{X}\widetilde{\not\subseteq}(F_3,E)$.

Theorem 4.7. *Let (X,μ,E) be a supra soft topological space and $x \in X$. Then $y \in Ker_b^s(\{x\})$ if and only if $x \in cl_b^s(\{y\})$.*

Proof. Assume that $x \notin cl_b^s(\{y\})$ then $X - cl_b^s(\{y\})$ is a supra soft b-open set containing x but not y. Therefore we have $y \notin Ker_b^s(\{x\})$. The prove of the converse case can be done similarly. □

Theorem 4.8. *Let (X, μ, E) be a supra soft topological space and $x, y \in X$, then the following are equivalent:*
(1) $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$;
(2) $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$.

Proof. (1) $\rightarrow$ (2). Assume that $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$, then there exists a point z in X such that $z \in Ker_b^s(\{x\})$ and $z \notin Ker_b^s(\{y\})$ (or $z \in Ker_b^s(\{y\})$ and $z \notin Ker_b^s(\{x\})$). It follows from $z \in Ker_b^s(\{x\})$ that $x \in cl_b^s(\{z\})$. Since $z \notin Ker_b^s(\{y\})$, we have $y \notin cl_b^s(\{z\})$. Also, since $x \in cl_b^s(\{z\})$, $cl_b^s(\{x\}) \widetilde{\subseteq} cl_b^s(\{z\})$ and $\{y\} \widetilde{\cap} cl_b^s(\{x\}) = \widetilde{\phi}$. Therefore, $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. Now $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$ implies $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$.

(2) $\rightarrow$ (1): Suppose that $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. Then there exists a point z in X such that $z \in cl_b^s(\{x\})$ and $z \notin cl_b^s(\{y\})$. Then, there exists a supra soft b-open set containing z and therefore x but not y, i.e., $y \notin Ker_b^s(\{x\})$. Hence $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$. □

Theorem 4.9. *A supra soft topological space (X, μ, E) is supra soft $b - R_0$ if and only if for any x and y in X, $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$ implies $cl_b^s(\{x\}) \widetilde{\cap} cl_b^s(\{y\}) = \widetilde{\phi}$.*

Proof. $\Rightarrow$) Suppose that (X, μ, E) *is supra soft* $b - R_0$ and $x, y \in X$ such that $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. Then, there exists $z \in cl_b^s(\{x\})$ such that $z \notin cl_b^s(\{y\})$ (or $z \in cl_b^s(\{y\})$ such that $z \notin cl_b^s(\{x\})$). There exists $(G, E) \in SSBO(X)$ such that $y \notin (G, E)$ and $z \in (G, E)$; hence $x \in (G, E)$. Therefore, we have $x \notin cl_b^s(\{y\})$. Thus $x \in \widetilde{X} - cl_b^s(\{y\}) \in SSBO(X)$, which implies $cl_b^s(\{x\}) \widetilde{\subseteq} \widetilde{X} - cl_b^s(\{y\})$ and $cl_b^s(\{x\}) \widetilde{\cap} cl_b^s(\{y\}) = \widetilde{\phi}$. The prove for otherwise is similar.

$\Leftarrow$) Let $(G, E) \in SSBO(X)$ and let $x \in (G, E)$. We will show that $cl_b^s(\{x\}) \widetilde{\subseteq} (G, E)$. Let $y \notin (G, E)$, i.e., $y \in \widetilde{X} - (G, E)$. Then $x \neq y$ and $x \notin cl_b^s(\{y\})$. This shows that $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. By assumption, $cl_b^s(\{x\}) \widetilde{\cap} cl_b^s(\{y\}) = \widetilde{\phi}$. Hence $y \notin cl_b^s(\{x\})$ and therefore $cl_b^s(\{y\}) \widetilde{\subseteq} (G, E)$. □

Theorem 4.10. *A supra soft topological space (X, μ, E) is supra soft $b - R_0$ if and only if for any points x and y in X, $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$ implies $Ker_b^s(\{x\}) \widetilde{\cap} Ker_b^s(\{y\}) = \widetilde{\phi}$.*

Proof. $\Rightarrow$) Suppose that (X, μ, E) *is a supra soft* $b - R_0$ space. Thus by Theorem 4.8, for any points x and y in X if $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$ then $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. Now we prove that $Ker_b^s(\{x\}) \widetilde{\cap} Ker_b^s(\{y\}) = \widetilde{\phi}$. Assume that $z \in Ker_b^s(\{x\}) \widetilde{\cap} Ker_b^s(\{y\})$. By $z \in Ker_b^s(\{x\})$ and Theorem 4.7, it follows that $x \in cl_b^s(\{z\})$. Since $x \in cl_b^s(\{x\})$, by Theorem 4.9, $cl_b^s(\{x\}) = cl_b^s(\{z\})$. Similarly, we have $cl_b^s(\{y\}) = cl_b^s(\{z\}) = cl_b^s(\{x\})$. This is a contradiction. Therefore, we have $Ker_b^s(\{x\}) \widetilde{\cap} Ker_b^s(\{y\}) = \widetilde{\phi}$.

$\Leftarrow$) Let (X,μ,E) *be a supra soft* space such that for any points x and y in X, $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$ *implies* $Ker_b^s(\{x\})\widetilde{\cap}Ker_b^s(\{y\}) = \widetilde{\phi}$. If $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$, then by Theorem 4.8, $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$. Hence $Ker_b^s(\{x\})$ $\widetilde{\cap}Ker_b^s(\{y\}) = \widetilde{\phi}$ which implies $cl_b^s(\{x\})\widetilde{\cap}cl_b^s(\{y\}) = \widetilde{\phi}$. Because $z \in cl_b^s(\{x\})$ implies that $x \in Ker_b^s(\{z\})$. Therefore $Ker_b^s(\{x\})\widetilde{\cap}Ker_b^s(\{z\}) \neq \widetilde{\phi}$. By hypothesis, we have $Ker_b^s(\{x\}) = Ker_b^s(\{z\})$. Then $z \in cl_b^s(\{x\})\widetilde{\cap}cl_b^s(\{y\})$ implies that $Ker_b^s(\{x\}) = Ker_b^s(\{z\}) = Ker_b^s(\{y\})$. This is a contradiction. Hence $cl_b^s(\{x\})\widetilde{\cap}cl_b^s(\{y\}) = \widetilde{\phi}$. By Theorem 4.9, (X,μ,E) *is a supra soft* $b-R_0$ space. □

Theorem 4.11. *For a supra soft topological space (X,μ,E), the following properties are equivalent:*

(1) (X,μ,E) *is a supra soft* $b-R_0$ *space;*

(2) *For any* $(F,E) \neq \widetilde{\phi}$ *and* $(G,E) \in SSBO(X)$ *such that* $(F,E)\widetilde{\cap}(G,E) \neq \widetilde{\phi}$, *there exists* $(H,E) \in SSBC(X)$ *such that* $(F,E)\widetilde{\cap}(H,E) \neq \widetilde{\phi}$ *and* $(H,E)\widetilde{\subseteq}$ (G,E)*;*

(3) *If* $(G,E) \in SSBO(X)$, *then* $(G,E) = \widetilde{\cup}\{(F,E) \in SSBC(X) : (F,E)\widetilde{\subseteq}$ $(G,E)\}$*;*

(4) *If* $(F,E) \in SSBC(X)$, *then* $(F,E) = \widetilde{\cap}\{(G,E) \in SSBO(X) : (F,E)\widetilde{\subseteq}$ $(G,E)\}$*;*

(5) *For any* $x \in X, cl_b^s(\{x\})\widetilde{\subseteq}Ker_b^s(\{x\})$.

Proof. (1) $\rightarrow$ (2). Let $(F,E) \neq \widetilde{\phi}$ and $(G,E) \in SSBO(X)$ such that (F,E) $\widetilde{\cap}(G,E) \neq \widetilde{\phi}$. Then there exists $x \in (F,E)\widetilde{\cap}(G,E)$. Since $x \in (G,E) \in$ $SSBO(X)$, $cl_b^s(\{x\})\widetilde{\subseteq}(G,E)$. Let $(H,E) = cl_b^s(\{x\})$, then $(H,E) \in SSBC(X)$, $(H,E)\widetilde{\subseteq}(G,E)$ and $(F,E)\widetilde{\cap}(H,E) \neq \widetilde{\phi}$

(2) $\rightarrow$ (3). Let $(G,E) \in SSBO(X)$, then $\widetilde{\cup}\{(F,E) \in SSBC(X) : (F,E)\widetilde{\subseteq}$ $(G,E)\}\widetilde{\subseteq}(G,E)$. Let x be any point of (G,E). There exists $(F,E) \in SSBC(X)$ such that $x \in (F,E)$ and $(F,E)\widetilde{\subseteq}(G,E)$. Therefore, we have $x \in (F,E)\widetilde{\subseteq}\widetilde{\cup}\{(F,E)$ $\in SSBC(X) : (F,E)\widetilde{\subseteq}(G,E)\}$ and hence $(G,E) = \widetilde{\cup}\{(F,E) \in SSBC(X) :$ $(F,E)\widetilde{\subseteq}(G,E)\}$.

(3) $\rightarrow$ (4). This is obvious.

(4) $\rightarrow$ (5). Let x be any point of X and $y \notin Ker_b^s(\{x\})$. There exists $(H,E) \in SSBO(X)$ such that $x \in (H,E)$ and $y \notin (H,E)$; hence $cl_b^s(\{y\})\widetilde{\cap}(H,E)$ $= \widetilde{\phi}$. By (4)$(\widetilde{\cap}\{(G,E) \in SSBO(X) : cl_b^s(\{y\})\widetilde{\subseteq}(G,E)\})\widetilde{\cap}(H,E) = \widetilde{\phi}$. There exists $(G,E) \in SSBO(X)$ such that $x \notin (G,E)$ and $cl_b^s(\{y\})\widetilde{\subseteq}(G,E)$. Therefore, $cl_b^s(\{x\})\widetilde{\cap}(G,E) = \widetilde{\phi}$ and $y \notin cl_b^s(\{x\})$.

Consequently, we obtain $cl_b^s(\{x\})\widetilde{\subseteq}Ker_b^s(\{x\})$.

(5) $\rightarrow$ (1). Let $(G,E) \in SSBO(X)$ and $x \in (G,E)$. Suppose $y \in Ker_b^s(\{x\})$, then $x \in cl_b^s(\{y\})$ and $y \in (G,E)$. This implies that $cl_b^s(\{x\})\widetilde{\subseteq}Ker_b^s(\{x\})\widetilde{\subseteq}(G,E)$. Therefore, (X,μ,E) *is a supra soft* $b-R_0$ *space.* □

Corollary 4.12. *For a supra soft topological space (X,μ,E), the following are equivalent:*

(1) (X, μ, E) *is a supra soft* $b - R_0$ *space;*
(2) $cl_b^s(\{x\}) = Ker_b^s(\{x\})$ *for all* $x \in X$.

Proof. (1) $\rightarrow$ (2). Suppose that (X, μ, E) *is a supra soft* $b - R_0$ *space.* By Theorem 4.11, $cl_b^s(\{x\})\widetilde{\subseteq}Ker_b^s(\{x\})$ for each $x \in X$. Let $y \in Ker_b^s(\{x\})$, then $x \in cl_b^s(\{y\})$ and so $cl_b^s(\{x\}) = cl_b^s(\{y\})$. Therefore, $y \in cl_b^s(\{x\})$ and hence $Ker_b^s(\{x\})\widetilde{\subseteq}cl_b^s(\{x\})$. This shows that $cl_b^s(\{x\}) = Ker_b^s(\{x\})$.
(2) $\rightarrow$ (1). This is obvious by Theorem 4.10. □

Theorem 4.13. *For a supra soft topological space* (X, μ, E)*, the following properties are equivalents :*
(1) (X, μ, E) *is a supra soft* $b - R_0$ *space;*
(2) $x \in cl_b^s(\{y\})$ *if and only if* $y \in cl_b^s(\{x\})$*, for any points* x *and* y *in* X.

Proof. (1) $\rightarrow$ (2). Assume that (X, μ, E) *is a supra soft* $b - R_0$ space. Let $x \in cl_b^s(\{y\})$ and (G, E) be any supra soft b-open set such that $y \in (G, E)$. Now by hypothesis, $x \in (G, E)$. Therefore, every supra soft b-open set containing y contains x. Hence $y \in cl_b^s(\{x\})$.
(2) $\rightarrow$ (1). Let (H, E) be a supra soft b-open set and $x \in (H, E)$. If $y \notin (H, E)$, then $x \notin cl_b^s(\{y\})$ and hence $y \notin cl_b^s(\{x\})$. This implies that $cl_b^s(\{x\})\widetilde{\subseteq}(H, E)$. Hence (X, μ, E) *is a supra soft* $b - R_0$ space. □

Theorem 4.14. *For a supra soft topological space* (X, μ, E)*, the following properties are equivalent :*
(1) (X, μ, E) *is a supra soft* $b - R_0$ *space;*
(2) *If* (F, E) *is supra soft b-closed, then* $(F, E) = Ker_b^s(F, E)$*;*
(3) *If* (F, E) *is supra soft b-closed and* $x \in (F, E)$*, then* $Ker_b^s(\{x\})\widetilde{\subseteq}(F, E)$*;*
(4) *If* $x \in X$*, then* $Ker_b^s(\{x\})\widetilde{\subseteq}cl_b^s(\{x\})$.

Proof. (1) $\rightarrow$ (2). Let (F, E) be supra soft b-closed and $x \notin (F, E)$. Thus $\widetilde{X} - (F, E)$ is supra soft b-open and contains x. Since (X, μ, E) *is supra soft* $b - R_0$, $cl_b^s(\{x\})\widetilde{\subseteq}\widetilde{X} - (F, E)$. Thus $cl_b^s(\{x\})\widetilde{\cap}(F, E) = \widetilde{\phi}$ and by Theorem 4.3, $x \notin Ker_b^s((F, E))$. Therefore $Ker_b^s(F, E) = (F, E)$.
(2) $\rightarrow$ (3). In general, $(A, E)\widetilde{\subseteq}(B, E)$ implies $Ker_b^s(A, E)\widetilde{\subseteq}Ker_b^s(B, E)$. Therefore, it follows from (2) that $Ker_b^s(\{x\})\widetilde{\subseteq}Ker_b^s(F, E) = (F, E)$.
(3) $\rightarrow$ (4). Since $x \in cl_b^s(\{x\})$ and $cl_b^s(\{x\})$ is supra soft b-closed, by (3) $Ker_b^s(\{x\})\widetilde{\subseteq}cl_b^s(\{x\})$.
(4) $\rightarrow$ (1). We show the implication by using Theorem 4.11. Let $x \in cl_b^s(\{y\})$. Then by Theorem 4.7, $y \in Ker_b^s(\{x\})$. Since $x \in cl_b^s(\{x\})$ and $cl_b^s(\{x\})$ is supra soft b-closed, by (4) we obtain $y \in Ker_b^s(\{x\})\widetilde{\subseteq}cl_b^s(\{x\})$. Therefore, $x \in cl_b^s(\{y\})$ implies $y \in cl_b^s(\{x\})$. The converse is obvious and (X, μ, E) *is supra soft* $b - R_0$. □

Definition 4.15. *A* supra soft topological space (X, μ, E) *is said to be supra soft* $b - R_1$ *if for* x, y *in* X *with* $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$*, there exist disjoint supra soft b-open sets* (G, E) *and* (H, E) *such that* $cl_b^s(\{x\})\widetilde{\subseteq}(G, E)$ *and* $cl_b^s(\{y\})\widetilde{\subseteq}(H, E)$.

Theorem 4.16. *Every supra soft $b-R_1$ space is supra soft $b-R_0$.*

Proof. Let (X,μ,E) *be a supra soft* $b-R_1$ space and (G,E) be a supra soft b-open set such that $x \in (G,E)$. If $y \notin (G,E)$, then since $x \notin cl_b^s(\{y\})$, $cl_b^s(\{x\}) \neq cl_b^s(\{y\})$. Hence, there exists a supra soft b-open set (H,E) such that $cl_b^s(\{y\})\widetilde{\subseteq}(H,E)$ and $x \notin (H,E)$, which implies $y \notin cl_b^s(\{x\})$. Thus $cl_b^s(\{x\})\widetilde{\subseteq}(G,E)$. Therefore (X,μ,E) *is supra soft* $b-R_0$. □

Notice that the converse of the above theorem need not be true:

Example 4.17. Let $X=\{h_1,h_2,h_3\}$ and $E=\{e\}$. Then the soft subsets over X are $(F_1,E)=\widetilde{\phi}$, $(F_2,E)=\widetilde{X}$, $(F_3,E)=\{(e,\{h_1\})\}$, $(F_4,E)=\{(e,\{h_2\})\}$, $(F_5,E)=\{(e,\{h_3\})\}$, $(F_6,E)=\{(e,\{h_1,h_2\})\}$, $(F_7,E)=\{(e,\{h_1,h_3\})\}$, $(F_8,E)=\{(e,\{h_2,h_3\})\}$.

Let $\mu=\{\widetilde{\phi},\widetilde{X},(F_6,E),(F_7,E),(F_8,E\}\}$. Then (X,μ,E) is a supra soft $b-R_0$ space but not supra soft $b-R_1$.

Theorem 4.18. *A supra soft topological space (X,μ,E) is supra soft $b-R_1$ if and only if for $x,y \in X$, with $Ker_b^s(\{x\}) \neq Ker_b^s(\{y\})$, there exists disjoint supra soft b-open sets (G,E) and (H,E) such that $cl_b^s(\{x\})\widetilde{\subseteq}(G,E)$ and $cl_b^s(\{y\})\widetilde{\subseteq}(H,E)$.*

Proof. It follows from Theorem 4.7. □

Theorem 4.19. *In a supra soft topological space (X,μ,E), $\widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}=\widetilde{\phi}$ if and only if $Ker_b^s(\{x\}) \neq \widetilde{X}$ for every $x \in X$.*

Proof. $\Rightarrow$) Suppose that $\widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}=\widetilde{\phi}$. Assume that there is a point y in X such that $Ker_b^s(\{y\})=\widetilde{X}$. Then $y \notin (G,E)$, where (G,E) is some proper supra soft b-open subset of (X,μ,E). This implies that $y \in \widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}$. But this is a contradiction.

$\Leftarrow$) Now assume that $Ker_b^s(\{x\}) \neq \widetilde{X}$ for every $x \in X$. If there exists a point $y \in X$ such that $y \in \widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}$, then every supra soft b-open set containing y must contain every point of X. This implies that the space $\widetilde{X}$ is the unique supra soft b-open set containing y. Hence $Ker_b^s(\{x\})=\widetilde{X}$ which is a contradiction. Therefore, $\widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}=\widetilde{\phi}$. □

Theorem 4.20. *If $f_{pu} : (X,\mu,E) \to (Y,\sigma,K)$ is an injective supra soft b^*-closed function and $\widetilde{\cap}\{cl_b^s(\{x\}) : x \in X\}=\widetilde{\phi}$ then $\widetilde{\cap}\{cl_b^s(\{y\}) : y \in Y\}=\widetilde{\phi}$.*

Proof. Straightforward.

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Partially ordered objects in the topos of M set

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Abstract. In this paper first we introduce the concept of partially ordered objects and monotone morphisms in an arbitrary topos. Partially ordered objects are counterparts of partially ordered sets, commonly known as posets, in an arbitraty topos. We then study partially ordered objects in the topos M set. We will prove that partially ordered objects in M set is equivalent to the category M pos, of posets with compatible actions of a monoid M on them.

Keywords: partial order, monotone, topos, monoid, action.

1. Introduction

Partially ordered sets are among the most important mathematical structures. They have a wide range of applications in mathematics. In what follows we mention some of the most important usage of partially ordered sets.

Perhaps the best known use of the language of partially ordered sets is Zorn's Lemma. Proved independantly by Kazimierz Kuratowski in 1922 and Max Zorn in 1935, it states that every partially ordered set in which every chain has an upper bound has a maximal element [3].

Zorn's Lemma has many important and crucial results and consequences such as the existence of maximal ideals in rings, existence of basis for vector spaces and strong completeness of first-order logic.

So, according to what we mentioned above, it is evident that the study of posets is very important for mathematicians.

Introduced by Lawvere and Tierney, topos theory is a branch of mathematics which studies categories behaving similar to the category of sets and functions.

Each topos defines a frame work for practicing mathematics. Classical mathematics is the mathematics studied in the frame work defined by the category of sets and functions.

Every topos has an underlying logic which is a super intuitionistic one. The formulas valid in every topos are exactly intuitionistic valid formulas. Therefore, from the prespective of logic, practicing mathematics in a topos is the same as studying intuitionistic or constructivist mathematics. Constructivisim in mathematics has gained much attention recently. Besides its philosphical arguments

it is important because of its applications in computer science, for example see [8].

One line of research in topos theory is to recreat classical mathematics in an arbitrary topos. Acording to what we said previously, partially ordered sets have all sorts of applications in mathematics. Therefore it seems important to study these mathematical structures in a topos.

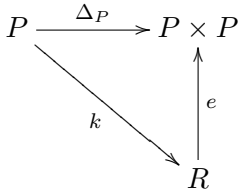
In the first section of this paper we introduce the concept of a partially ordered object which is actually due to MacLane and Moerdijk [10]. Next we define what we mean by a monotone morphism. It is easy to see that these form a subcategory of the underlying topos. In the other sections we study partially ordered objects in the topos $M\mathbf{Set}$, of sets with an action of the monoid M on them. We prove that a category of partially ordered objects and monotone morphisms in this topos is finitely complete and cartesian closed. Besides these properties we mention some other interesting properties, as well.

2. Partially ordered objects and monotone morphisms in a topos

Our aim in this section is to define partially ordered object (po-object) in an arbirary topos. Having in mind what a poset (partially ordered set) $(P, \leq)$ is, we give the following definitions. For more information regarding this definition see [10] page 199.

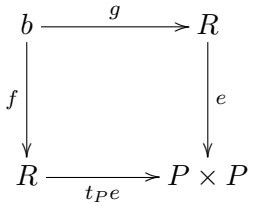
Definition 2.1. *Let P be an object of a topos $\mathcal{E}$. By a binary relation e on P we mean a subobject $e : R \rightarrowtail P \times P$ of $P \times P$. Now we have the following definitions:*

1. The binary relation $e : R \rightarrowtail P \times P$ on P is said to be reflexive whenever $\Delta_P : P \to P \times P$ factors through $e : R \rightarrowtail P \times P$. That is there exists a morphism $k : P \to R$ such that the



commutes. In other words, e is reflexive whenever $\Delta_P \leq e$.

2. We say that $e : R \to P \times P$ is anti-symmetric whenever $e \cap t_P e \leq \Delta_P$, where t_P is the twisting morphism which is $(Pr^2_P, Pr^1_P) : P \times P \to P \times P$ and Pr denotes the projection morphisms. That is, considering the pullback square



there exists a morphism $m : b \to P$ such that $\Delta_P m = eg = t_P ef$.

3. We say that $e : R \to P \times P$ is transitive whenever $(Pr_P^1 eq, Pr_P^2 ep) : C \to P \times P$ factors through e, where C is defined by the pullback square

$$\begin{array}{ccc} C & \xrightarrow{p} & R \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Pr_P^1 e} \\ R & \xrightarrow[Pr_P^2 e]{} & P \end{array}$$

4. A partially ordered object (po-object) in a topos $\mathcal{E}$ is a pair $(P, e :\leq_P \to P \times P)$, where P is an $\mathcal{E}$-object and e is a binary relation on P in the topos, which is reflexive, anti-symmetric, and transitive.

Here we fix some notations. For a po-object P we denote its order by $e_P :\leq_P \to P \times P$, and by $\sqsubseteq_P$ we mean the characterstic morphism of the relation on P.

Note that a monotone morphism $f : P \to Q$ between posets is a function satisfying

$$x \leq_P y \Rightarrow f(x) \leq_Q f(y).$$

Definition 2.2. *Let P and Q be two po-objests in a topos $\mathcal{E}$. An $\mathcal{E}$-morphism $f : P \to Q$ is said to be monotone whenever $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. That is,*

$$\begin{array}{ccc} P & \xrightarrow{(\sqsubseteq_P, \sqsubseteq_Q (f \times f))} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \Rightarrow} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

is commutative.

Remark 1. *Ω-axiom.* The Ω-axiom states that for each monomorphism $f : A \to B$ in a topos with subobject clasifier $\top : 1 \to \Omega$ there exists a unique morphism $\chi_f : B \to \Omega$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

is a pullback square.

Theorem 2.1. *Assume P and Q are po-objects. A morphism $f : P \to Q$ is monotone if and only if there exists a morphism $h :\leq_P \to \leq_Q$ such that*

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_p} & P \times P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times f} \\ \leq_q & \xrightarrow[e_Q]{} & Q \times Q \end{array}$$

is commutative.

Proof. Assume $f : P \to Q$ is monotone. That is, $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. Consider the diagram

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P,\sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \wedge} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

Since $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$, the outer square is a pullback. The lower square is the pullback defining $\wedge$. So, by the pullback lemma,

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P,\sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \end{array}$$

is also a pullback. Therefore, $\sqsubseteq_Q (f \times f)e_P = \sqsubseteq_P e_P = \top_{\leq_P}$. Now, consider the pullback square

$$\begin{array}{ccc} \leq_Q & \xrightarrow{e_Q} & Q \times Q \\ \downarrow & & \downarrow{\scriptstyle \sqsubseteq_Q} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

Since $\sqsubseteq_Q (f \times f)e_P = \top_{\leq_P}$, there exists a morphism $h :\leq_P \to \leq_Q$ such that $e_Q h = (f \times f)e_P$. For the converse, assume there exists a morphism $h :\leq_P \to \leq_Q$

such that

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times f} \\ \leq_Q & \xrightarrow{e_Q} & Q \times Q \end{array}$$

is commutative. Consider the square

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P,\sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \end{array}$$

One can see that the above is pullback. Now, consider the diagram

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P,\sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \wedge} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

Since the upper and lower squares are pullbacks, the whole square is pullback. Therefore, by Ω-axiom, we have $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. □

From now on we use this equivalent form of monotonicity. We have defined partially ordered objects and monotone morphisms in a topos. In the following sections we study the above definitions in the topos $M\mathbf{Set}$, of sets with actions of the monoid M on them (for $M\mathbf{Set}$ see for example [6]).

Before we go on any further note that an M-set, where M is monoid is a set X equipped with a function $\mu : M \times X \to X$ that satisfies (for the sake of simplicity $\mu(s, x)$ is denoted sx)

$$ex = x, \quad s(tx) = (st)x.$$

An action-preserving function between two M-sets X and Y is a function $f : X \to Y$ such that

$$f(sx) = sf(x).$$

3. Partially ordered objects in M**Set**

Now we study the partially orderd objects in the topos M**Set**. We will show that the category of po-objects in M**Set** is equivalent to the catgeory of M-posets. The category of M-posets has been the subject of major studies (for example see [1]).

Definition 3.1. *An M-poset is a poset with an action of the monoid M such that for $x, y \in P$ and $s \in M$ we have*

$$x \leq y \Rightarrow sx \leq sy.$$

An action-preserving function f between M-posets P and Q is said to be monotone whenever for each $x, y \in P$ we have

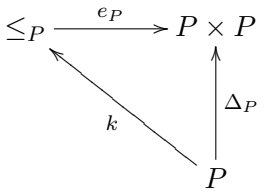
$$x \leq y \Rightarrow f(x) \leq f(y).$$

Let us fix a notation wich we use from now on. If $f : A \to B$ is a function and $C \subseteq A$ then by $f[C]$we mean the image of f restricted to the set C.

Theorem 3.1. *Let P be a po-object (as in Definition 2.1) in the topos M**Set** then $(P, e_P[\leq_P])$ is an M-poset.*

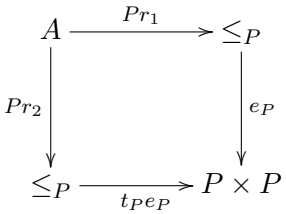
Proof. First of all it is easy to see that $e_P[\leq_P]$ is a sub M-set of $P \times P$.

To see that $e_P[\leq_P]$ is reflexive note that there exists an action-presereving map $k : P \to \leq_P$ such that



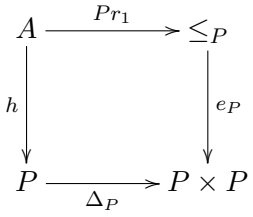
is commutative. Let $x \in P$, we want to show that $(x, x) \in e_P[\leq_P]$. We have $e_P k(x) \in e_P[\leq_P]$. That is $\Delta_P(x) \in e_P[\leq_P]$. So, $(x, x) \in e_P[\leq_P]$.

To see that $e_P[\leq_P]$ is anti-symmetric let $(x, y) \in e_P[\leq_P]$ and $(y, x) \in e_P[\leq_P]$. We want to prove that $x = y$. We have $(x, y) \in e_P[\leq_P] \cap t_P e_P[\leq_P]$. Let



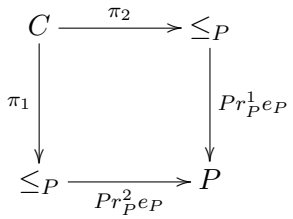
be a pullback square in M-**Set**. Note that $A = \{(a, b) \in \leq_P \times \leq_P : e_P(a) = t_P e_P(b)\}$ and Pr_1, Pr_2 are the usuall projection maps Now, by anti-symmetry

of e_P, there exists a morphism $h : A \to P$ such that

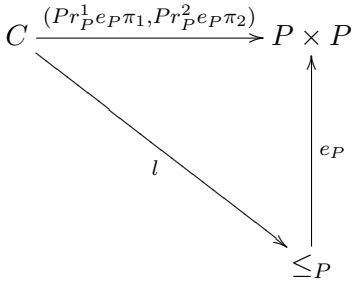


is commutative. Since $(x, y), (y, x) \in e_P[\leq_P]$, there exist $a, b \in \leq_P$ such that $e_P(a) = (x, y)$ and $e_P(b) = (y, x)$. So, $t_P e_P(b) = e_P(a)$. Hence $(a, b) \in A$. Therefore, $e_P Pr_1(a, b) = \Delta_P h(a, b)$. That is, $(x, y) = \Delta_P h(a, b)$. So, $x = y$.

The last step is to prove that $e_P[\leq_P]$ is transitive. Let $(x, y), (y, z) \in e_P[\leq_P]$. That is, there exist $a, b \in \leq_P$, such that $e_P(a) = (x, y)$ and $e_P(b) = (y, z)$. Thus $Pr^2_P e_P(a) = Pr^1_P e_P(b)$. Let



be a pullback in M-**Set**. We can consider C to be $\{(r, s) \in \leq_P \times \leq_P : Pr^1_P e_P(s) = Pr^2_P e_P(r)\}$. Then, $(a, b) \in C$. Since e_P is transitive, there exists a morphism $l : C \to \leq_P$ such that



is commutative. As a result $e_P l(a, b) = (x, z)$, and so $(x, z) \in e_P[\leq_P]$. □

Theorem 3.2. *Assume that P and Q are po-objects in M***Set** *and $f : P \to Q$ is monotone. Then $f : (P, e_P[\leq_P]) \to (Q, e_Q[\leq_Q])$ is a monotone action-preserving function between M-posets $(P, e_P[\leq_P])$ and $(Q, e_Q[\leq_Q])$.*

Proof. Since $f : P \to Q$ is monotone as a morphism between po-objects in $M\mathbf{Set}$, there exists a morphism $h :\leq_P \to \leq_Q$ such that

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times f} \\ \leq_Q & \xrightarrow[e_Q]{} & Q \times Q \end{array}$$

is commutative. We have to prove that

$$(x, y) \in e_P[\leq_P] \Rightarrow (f(x), f(y)) \in e_Q[\leq_Q].$$

So, let $(x, y) \in e_P[\leq_P]$. We have $(f(x), f(y)) = (f \times f)(x, y)$. Since $(x, y) \in e_P[\leq_P]$, there exists $a \in \leq_P$ such that $e_P(a) = (x, y)$. So, $(f(x), f(y)) = (f \times f)e_P(a)$. By the commutativity of the previous diagram, $(f(x), f(y)) = e_Q h(a) \in e_Q[\leq_Q]$. □

Let us denote the category of poset object and monotone maps of M-**Set** by **Po**M**Set**. By the above theorems, one can see that we have a functor $F :$ **Po**M**Set** $\to M$**Pos** which sends each po-object P to the M-poset $(P, e_P[\leq_P])$ and each monotone morphism $f : P \to Q$ between po-objects to itself as a monotone morphism between M-posets $(P, e_P[\leq_P])$ and $(Q, e_Q[\leq_Q])$.

Now if $(P, \leq_P)$ is an M-poset then one can easily see that $(P, i_{\leq_P})$ is a po-object in the topos M-**Set**, where $i_{\leq_P}$ is the inclusion function from $\leq_P$ to $P \times P$. It is also easy to see that an action-preserving monotone funtion between M-posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is monotone as a morphism between po-objects $(P, i_{\leq_P})$ and $(Q, i_{\leq_Q})$. So, we have a functor $G : M$**Pos** $\to$ **Po**M**Set** such that it sends each M-poset $(P, \leq_P)$ to the po-object $(P, i_{\leq_P})$, and each monotone function $f : (P, \leq_P) \to (Q, \leq_Q)$ between M-posets to itself considered as a morphism between po-objects (P, i_P) and (Q, i_Q).

Theorem 3.3. *The functors F and G constitute an equivalence between the categories* **Po**M**Set** *and* M**Pos**.

Proof. Let (P, e_P) be an object of **Po**M**Set**. One can see that

$$GF(P, e_P) = (P, i_{e_P[\leq_P]}).$$

One can see that $id_P : (P, e_P) \to (P, i_{e_P[\leq_P]})$ is a natural isomorphism.

On the other hand if $(P, \leq_P)$ is an M-poset then

$$FG(P, \leq_P) = (P, i_{\leq_p}[\leq_P]) = (P, \leq_P).$$

□

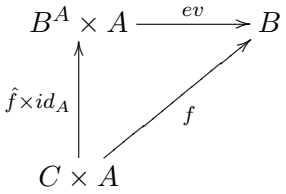
Now that we have proved **Po**M**Set** is equivalent to M**Pos**, we can study the categorical properties of **Po**M**Set** by investigating M**Pos**.

M**Pos** has been the subject of major studies, for example see [1], [2], [13]. However, we state some theorems which capture the categorical properties of M**Pos** completely.

Theorem 3.4. *The category* M**Pos** *is isomorphic to* $\mathbf{Pos}^M$.

Proof. For the proof of this theorem see [6]. □

Remark 2. Let $\mathcal{C}$ be a category and A and B objects of this category. The exponential object of A and B denoted by B^A is an object of $\mathcal{C}$ together with a morphism $ev : B^A \times A \to B$ such that for every morphism $f : C \times A \to B$ in $\mathcal{C}$ there exists a unique morphism $\hat{f} : C \to B^A$ making



commutative.

A category with finite products is said to be Cartesian closed whenever all exponential for each pair of objects exist.

Remark 3. Let P and Q be two M-sets then Q^P is the following set

$$\{f : M \times P \to Q : f \text{ is action-preserving}\}$$

equipped with an action such that for $s \in M$ and $f \in Q^P$ we have

$$sf : M \times P \to Q, \quad (m, x) \mapsto f(ms, x).$$

The M-set Q^P is the exponentiation of Q and P in M**Set**. If f is a member of Q^P and $s \in M$ then by f_s we mean the following function

$$f_s : P \to Q, \quad x \mapsto f(s, x).$$

Assuming that P and Q are two objects of M**Pos**, one can see that their product object is the M-set $P \times Q$ equipped with the point-wise order. Also, it can be seen that M**Pos** is a catesian closed category. The exponential object $[P \to Q]$ is the subobject of Q^P such that $f \in [P \to Q]$ if and only if for each $s \in M$ the function f_s is monotone.

4. Partial monotone morphism classifier

In this section we first describe the partial morphism classifier of the topos of M-sets using the language introduced in [5] and the process of [9]. After this we use the partiall morphism classifier to construct the partial monotone morphims classifier.

Remark 4. (a) Let A be an M-set, $X \in Sub(M \times A)$, and $s \in M$. We equip $Sub(M \times A)$ with the action $sX = \{(t, a) \in M \times A : (ts, a) \in X\}$. One can see that $Sub(M \times A)$ with this action is indeed an M-set.

(b) It is well-known that Ω^A is isomorphic to $Sub(M \times A)$. In fact, the isomorphism $\Lambda : \Omega^A \to Sub(M \times A)$ is given by $\Lambda(f) = f^{-1}(\{M\})$.

(c)If $X \in Sub(M \times A)$ then we denote it as $(X_m)_{m\in M}$, where $X_m = \{x \in A : (m, x) \in X\}$.

Now, we will use Kock and Wraith's construction to obtain the partial morphism classifier [9]. After this we equip this object with a partial order. We will prove that the resulting object can classify partial monotone morphisms with up M-sets as their domains. Note that an up M-subset of an M-poset is an upset of it which is also a sub M-set.

We define $\{\cdot\}_A : A \to Sub(M \times A)$ to be the exponential adjoint of $\delta_A : A \times A \to \Omega$, which is the characterstic morphism of $\Delta_A : A \to A \times A$. That is, $\delta_A(x, y) = \{s \in M : sx = sy\}$. Hence we have

$$\{\cdot\}_A : A \to Sub(M \times A), \quad x \to \{x\}_A,$$

where $\{x\}_A = \{(m, mx) : m \in M\}$. Equivalently we have $\{x\}_A = (\{mx\})_{m\in M}$. Thus, we have a monomorphism

$$(\{\cdot\}, id_A) : A \to Sub(M \times A) \times A, \quad x \mapsto (\{mx\}_{m\in M}, x).$$

Let us denote the characterstic morphism of this monomorphism by h. Therefore $h : Sub(M \times A) \times A \to \Omega$ is defined by

$$((X_m)_{m\in M}, x) \mapsto \{s \in M : \forall m \in M \quad X_{ms} = \{msx\}\}.$$

As a result, one can see that

$$h((X_m), x) = M \Leftrightarrow \forall m \in M \quad X_m = \{mx\}.$$

The morphism h has the exponential adjoint

$$\hat{h} : Sub(M \times A) \to Sub(M \times A),$$

which is given by

$$\hat{h}((X_t)_{t\in M}) = (\{x \in A : h(m(X_t)_{t\in M}, x) = M\})_{m\in M},$$

which is equal to

$$(\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\})_{m\in M}.$$

Definition 4.1. *We define $\tilde{A}$ to be the equalizer of $\hat{h}$ and $id_{Sub(M\times A)}$. One can see that $\tilde{A}$ is the set of all $(X_t)_{t\in M} \in Sub(M \times A)$ such that*

$$\forall m \in M, \quad \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} = X_m.$$

Theorem 4.1. *We have $(X_m)_{m\in M} \in \Omega^A$ is a member of $\tilde{A}$ if and only if for each $m \in M$, X_M is either empty or has exactly one element.*

Proof. Assume for $m \in M$, $X_m \neq \emptyset$. Thus

$$\exists x \in A \quad \forall t \in M \quad X_{tm} = \{tx\}.$$

So, for $t = e$ we have $X_m = \{x\}$, for some $x \in A$. Hence X_m has exactly one element.

To prove that $(X_m)_{m\in M}$ is a member of $\tilde{A}$, it is enough to show that for each $m \in M$, we have

$$X_m = \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\}.$$

We prove that if $X_m = \emptyset$, then

$$\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} = \emptyset.$$

Assume $X_m = \emptyset$. If

$$\exists x \in A \quad \forall t \in M \quad X_{tm} = \{tx\},$$

then, replacing t by e, we have $X_m = \{x\}$, for some $x \in A$, which is a contradiction. So, $\{x \in A : \forall t \in M \quad X_{tm} = \{tx\} = \emptyset\}$. If $X_m \neq \emptyset$, then by the hypothesis, there exists $x \in A$ such that $X_m = \{x\}$. Note that, since $(X_m)_{m\in M}$ is a member of Ω^A, for each $t, m \in M$ we have $tX_m = X_{tm}$. So, for each $t \in M$ we have $X_{tm} = tX_m = t\{x\} = \{tx\}$. Hence, x satisfies

$$\forall t \in M \quad X_{tm} = \{tx\}.$$

So,

$$X_m \subseteq \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\}.$$

Now assume a satisfies the property

$$\forall t \in M, \quad X_{tm} = \{ta\}.$$

Replacing t by e, we get $X_m = \{a\}$. So, $\{a\} = \{x\}$. That is $a \in X_m$. Therefore,

$$\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} \subseteq X_m.$$

□

Theorem 4.2. *Let P be an M-poset. We equip $\tilde{P}$ with the relation*

$$(X_t)_{t\in M} \quad \tilde{\leq} \quad (Y_t)_{t\in M} \Leftrightarrow \forall t \in M \quad \downarrow X_t \subseteq \downarrow Y_t.$$

Then $\tilde{P}$ is an M-poset

Proof. Obviously $\tilde{P}$ is a poset. So, we show that $\tilde{\leq}$ is compatible with the action of M. Let $(X_t)_{t\in M} \ \tilde{\leq} \ (Y_t)_{t\in M}$ and $m \in M$ then, we have

$$\forall t \in M \quad \downarrow X_t \subseteq \downarrow Y_t.$$

So,

$$\forall t \in M \quad \downarrow X_{tm} \subseteq \downarrow Y_{tm}.$$

Therefore, $(X_{tm})_{t\in M} \ \tilde{\leq} \ (Y_{tm})_{t\in M}$. That is $m(X_t)_{t\in M} \ \tilde{\leq} \ m(Y_t)_{t\in M}$. □

Theorem 4.3. *Let P be an M-poset. Then the function*

$$\eta_P : P \to \tilde{P}, \quad x \mapsto (\{mx\})_{m\in M}$$

is an action-preserving, monotone function.

Proof. To see that η_P is well-defined, let $m, t \in M$. We have $t\{mx\} = \{tmx\} \subseteq \{tmx\}$. To see that it is action-preserving, let $s \in M$. We have $s \cdot \eta_P(x) = (\{msx\})_{m\in M}$. On the other hand $\eta_P(sx) = (\{msx\})_{m\in M}$.

Next, we prove that η_P is monotone. Let $x \leq y$. Then $\downarrow \{mx\} \subseteq \downarrow \{my\}$, because $x \leq y$ implies $mx \leq my$. This gives that $(\{mx\})_{m\in M} \tilde{\leq} (\{my\})_{m\in M}$. □

Definition 4.2. *Let P be an M-poset. By an M-upset of P we mean a sub M-set F of P which is an upset with respect to the order of P. By a partial monotone morphism from an M-posets P to an M-poset Q we mean a morphism from an M-upset F of P to Q.*

Theorem 4.4. *Let P be an M-poset and $f : P \rightharpoonup Q$ be a partial monotone morphism such that $dom(f) = F$. Then $\hat{f} : P \to Q$ is monotone, where $\hat{f}$ is the unique morphism making the following square a pullback one in M-**Set**:*

$$\begin{array}{ccc} F & \xrightarrow{i_F} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ Q & \xrightarrow[\eta_Q]{} & \tilde{Q} \end{array}$$

Proof. First of all one can see that $\tilde{f}$ is defined by

$$\tilde{f} : P \to \tilde{Q}, \quad x \mapsto (X_m)_{m\in M},$$

such that for each $m \in M$ we have

$$X_m = \begin{cases} \{f(mx)\}, & \text{if } mx \in F \\ \emptyset, & \text{otherwise.} \end{cases}$$

To show that this morphism is monotone, let $x \leq y$ in P. We should prove that $\tilde{f}(x) \ \tilde{\leq} \ \tilde{f}(y)$. Let

$$\tilde{f}(x) = (X_m)_{m\in M}, \quad \tilde{f}(y) = (Y_m)_{m\in M}.$$

Then, we have to show that

$$\forall m \in M \quad \downarrow X_m \subseteq \downarrow Y_m.$$

If $X_m = \emptyset$, then obviously $\downarrow X_m \subseteq \downarrow Y_m$. If X_m is non-empty, then we have

$$X_m = \{f(mx)\}, \quad mx \in F.$$

But, since $mx \leq my$ and F is an upset, we have $my \in F$. So, $Y_m = \{f(my)\}$. Now, since f is monotone, we have $f(mx) \leq f(my)$. Hence $\downarrow X_m \subseteq \downarrow Y_m$. □

Note that the above diagram is obviously a pullback in $M\mathbf{Pos}$ and $\hat{f}$ is the only monotone morphism making into a pullback square.

5. Some adjoint situations

Let X be an M-set. Obviously $(X, =)$ is an M-poset. If $f : X \to Y$ is an action-preserving function, we have $f : (X, =) \to (Y, =)$ is monotone. So, we have a functor $F : M\mathbf{Set} \to M\mathbf{Pos}$ which sends each M-set X to $(X, =)$ and sends each action-preserving funtion to itself.

Let $(P, \leq)$ be an M-poset. We have a functor $G : M\mathbf{Pos} \to M\mathbf{Set}$ which forgets the order of each M-poset.

Theorem 5.1. *The functor F is the left adjoint of the functor G.*

Proof. Note that for an M-set X we have $GFX = X$. One can see that the natural transformation $\eta : id_{M\mathbf{Set}} \to GF$, with the X'th component being id_X, serves as the unit of this adjunction. □

Now we consider the obvious forgetful functor $R : M\mathbf{Pos} \to \mathbf{Pos}$. Define the functor $L : \mathbf{Pos} \to M\mathbf{Pos}$ such that for each poset P we have $L(P) = M \times P$, where the action on $M \times P$ is the product action, where P is considered as an M-set with the trivial action, and take the order on $M \times P$ to be the pointwise order, where M is considered as a poset with the equality order.

Theorem 5.2. *The functor L is the left adjoint of the functor R.*

Proof. The natural transformation $\eta : id_{\mathbf{Pos}} \to RF$ with the P'th component being

$$\eta_P \to M \times P, \quad x \mapsto (e, x),$$

is the unit of this adjunction. For $f : P \to RQ$ a monotone function between a poset P and the underlying poset of an M-poset Q the adjoint morphism $\hat{f} : LP \to Q$ sends (s, p) to $f(sp)$. □

Next is the zero functor $Z : M\mathbf{Pos} \to \mathbf{Pos}$ which sends each M-poset P to the poset of zero elements of P. For an action-preserving monotone map $f : P \to Q$, Zf is the restriction of f to the set ZP.

This functor has a left adjoint $T : \mathbf{Pos} \to M\mathbf{Pos}$, which sends a poset P to the M-poset P, where the action on P is trivial. Also T sends monotone maps to themselves considered as action-preserving monotone functions.

Theorem 5.3. *The functor Z is the right adjoint of the functor T.*

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Simple endomorphism semirings of semilattices

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Abstract. In the paper, various types of simple endomorphism semirings of semilattices are investigated.

Keywords: semiring, semimodule, semilattice, endomorphism.

1. Introduction

1.1 Semilattices. An (algebraic) semilattice M $(= M(+))$ is a commutative and idempotent semigroup (i.e., $x + x = x$, $x + y = y + x$ and $x + (y + z) = (x + y) + z$ for all $x, y, z \in M$). The basic relation of order is given on M by $x \leq y$ if and only if $x + y = y$. We have $x + y = \sup(x, y)$ for all $x, y \in M$.

An element $w \in M$ is the smallest (greatest, resp.) element of the ordered set $M(\leq)$ exactly when $w = 0_M$ ($w = o_M$, resp.) is the uniquely determined (additively) neutral (absorbing, resp.) element of the semilattice $M(+)$. If such an element exists in M, we simply write $0_M \in M$ ($o_M \in M$, resp.). If not, then we write $0_M \notin M$ ($o_M \notin M$, resp.). In the sequel, we will frequently use the

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following notation: $N = M \setminus \{o_M\}$, $L = M \setminus \{0_M\}$ and $K = M \setminus \{0_M, o_M\} = N \cap L$.

A non-empty subset A of M is an *ideal* of the semilattice M if $M + A \subseteq A$ (then $M + A = A$). An ideal A is called *prime* if the set $M \setminus A$ is a subsemilattice of M.

1.2 Semirings. A semiring S $(= S(=, \cdot))$ is an algebraic structure equipped with two associative binary operations (usually denoted as addition and multiplication), where the addition os commutative and the multiplication distributes over the addition.

A non-empty subset I of S is a *left* (*right*) *ideal* of S if $SI \cup (I + I) \subseteq I$ $(IS \cup (I + I) \subseteq I)$, I is an *ideal* if it is both left and right ideal, and I is a *bi-ideal* if $SI \cup IS \cup (S + I) \subseteq I$. In the latter case, the relation $(I + I) \cup \mathrm{id}_S$ is a congruence of the semiring S.

An element $a \in S$ is called *left (right) multiplicatively absorbing* if $ab = a$ $(ba = a)$ for every $b \in S$. The set of all right multiplicatively absorbing elements will be denoted by $\underline{R}(S)$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is an ideal of S and $\underline{R}(S) \subseteq I$ for every right ideal of S.

The semiring S is called (*congruence-*)*simple* if S has precisely two congruence relations.

1.3 Endomorphism semirings. If M $(= M(+))$ is a commutative semigroup then the set $(\underline{E}(M) =)$ $\underline{E}$ of all endomorphisms of M is a semiring via $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$. The semiring $\underline{E}$ is unitary, since the identity automorphism $\mathrm{id}_M = 1_{\underline{E}}$ is the (unique) multiplicatively neutral element of $\underline{E}$.

If M is a semilattice then the semiring $\underline{E}$ is additively idempotent. This means that the additive semigroup $\underline{E}(+)$ is a semilattice, too.

2. $0, 1$-preserving endomorphisms (a)

Let M be a semilattice such that $0_M, o_M \in M$ and $|M| \geq 4$. The set $\underline{E}_{0,1} = \{ f \in \underline{E} \mid f(0_M) = 0_M, f(o_M) = o_M \}$ is a unitary subsemiring of $\underline{E}$. The set $\underline{D}_{0,1} = \{ f \in \underline{E}_{0,1} \mid f(M) = \{0_M, o_M\} \}$ is an ideal of the semiring $\underline{E}_{0,1}$ and there is a one-to-one correspondence between prime ideals of the semilattice M and the endomorphisms from $\underline{D}_{0,1}$; if A is a prime ideal then the corresponding endomorphisms q_A is given by $q_A(A) = \{o_M\}$ and $q_A(M \setminus A) = \{0_M\}$. If $x \in N = M \setminus \{o_M\}$ then $A_x = \{ y \in M \mid y \not\leq x \}$ is just the prime ideal determined by x and the corresponding endomorphism will be denoted by q_x. We put $\underline{B}_{01} = \{ q_x \mid x \in N \}$ and $C_{0,1} = \{ q_{x_1} + \cdots + q_{x_n} \mid n \geq 1, x_i \in N \}$. Then $\underline{C}_{0,1}$ is just the subsemiring of $\underline{D}_{0,1}$ generated by the set $\underline{B}_{0,1}$ and one sees readily that $\underline{C}_{0,1} = \underline{B}_{0,1}$ iff the semilattice M is a lattice.

2.1 The one-element set $\{o_M\}$ is a prime ideal iff $N + N = N$. The corresponding endomorphism will be denoted by $\underline{\xi}$. We have $\underline{\xi}(N) = \{0_M\}$, $\underline{\xi} + f = f$ for every

$f \in \underline{E}_{0,1}$, and if $o_N \in N$ then $\underline{\xi} = q_{o_N}$. If $o_N \notin N$ then $\underline{\xi} \notin \underline{B}_{0,1}$. Anyway, if E is a subsemiring of $\underline{E}_{0,1}$ such that $\underline{\xi} \in E$ then $\underline{\xi} = 0_E$.

2.2 We have $q_{0_M} + f = q_{0_M}$ for every $f \in \underline{E}_{0,1}$. In particular, if E is a subsemiring of $|E_{0,1}$ such that $q_{0_M} \in E$ then $q_{0_M} = o_E$. Clearly, $q_{0_M} = q_L$, $L = M \setminus \{0_M\}$.

2.3 If E is a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ then $\underline{C}_{0,1} \subseteq E$, $o_E = q_{0_M} \in E$ and $\underline{R}(E) = E \cap \underline{D}_{0,1}$.

2.4 Assume that the ordered set $K(\leq)$ (or $L(\leq)$) is downwards directed (here $L = M \setminus \{0_M\}$ and $K = M \setminus \{0_M, o_M\} = N \cap L$). Then the set ${}^1\underline{F}_{0,1} = \{\, f \in \underline{E}_{0,1} \,|\, 0_M \in f(K) \,\}$ is a subsemiring of $\underline{E}_{0,1}$. We have $\underline{E}_{0,1}\, {}^1\underline{F}_{0,1} \subseteq {}^1\underline{F}_{0,1}$ and ${}^1\underline{F}_{0,1}$ is a left ideal of the semiring $\underline{E}_{0,1}$. Moreover, $\underline{B}_{0,1} \setminus \{q_{0_M}\} \subseteq {}^1\underline{F}_{0,1}$. More generally, if A is a prime ideal, $A \neq L$, then $q_A \in {}^1\underline{F}_{0,1}$. The set ${}^1\underline{F}'_{0,1} = {}^1\underline{F}_{0,1} \cup \{q_{0_M}\}$ is a left ideal of $\underline{E}_{0,1}$ and $\underline{D}_{0,1} \subseteq {}^1\underline{F}'0, 1$. Furthermore, ${}^2\underline{F}0, 1$ and ${}^2\underline{F}'_{0,1} = {}^2\underline{F}_{0,1} \cup \{q_{0_M}\}$ are subsemirings of $\underline{E}_{0,1}$, where ${}^2\underline{F}_{0,1}$ is the set of the endomorphisms $f \in \underline{E}_{0,1}$ such that the set $f(K)$ is downwards cofinal in H (i.e., for every $v \in K$ there is $u \in K$ with $f(u) \leq v$). Now, ${}^1\underline{F}_{0,1}$ is an ideal of ${}^2\underline{F}_{0,1}$, ${}^1\underline{F}'_{0,1}$ is an ideal of ${}^2\underline{F}'_{0,1}$ and the semirings ${}^2\underline{F}_{0,1}$ and ${}^2\underline{F}'_{0,1}$ are unitary.

2.5 Assume that the ordered set $K(\leq)$ (or $N(\leq)$) is upwards directed (equivalently, $N = N + N$). Then the set ${}^1\underline{G}_{0,1} = \{\, f \in \underline{E}_{0,1} \,|\, o_M \in f(K) \,\}$ is a subsemiring of $\underline{E}_{0,1}$. We have $\underline{E}_{0,1}\, {}^1\underline{G}_{0,1} \subseteq {}^1\underline{G}_{0,1}$, $\underline{E}_{0,1} + {}^1\underline{G}_{0,1} \subseteq {}^1\underline{G}_{0,1}$ and ${}^1\underline{G}_{0,1}$ is a left ideal of the semiring $\underline{E}_{0,1}$. Moreover, $\underline{D}_{0,1} \setminus \{\underline{\xi}\} \subseteq {}^1\underline{G}_{0,1}$ and we put ${}^1\underline{G}'_{0,1} = {}^1\underline{G}_{0,1} \cup \{\underline{\xi}\{$, so that $\underline{D}_{0,1} \subseteq {}^1\underline{G}'_{0,1}$.

Let ${}^2\underline{G}_{0,1}$ be the set of the endomorphisms $f \in \underline{E}_{0,1}$ such that the set $f(K)$ is upwards cofinal in K (i.e., for every $v \in K$ there is $u \in K$ with $v \leq f(u)$. It is easy to see that ${}^2\underline{G}_{0,1}$ and ${}^2\underline{G}'_{0,1} = {}^2\underline{G}_{0,1} \cup \{\underline{\xi}\}$ are subsemirings of $\underline{E}_{0,1}$. Besides, ${}^1\underline{G}_{0,1}$ is a bi-ideal of the semiring ${}^2\underline{G}_{0,1}$ and ${}^1\underline{G}'_{0,1}$ is an ideal of ${}^2\underline{G}'_{0,1}$, The semirings ${}^2\underline{G}_{0,1}$ and ${}^2\underline{G}'_{0,1}$ are unitary.

2.6 Let $\alpha = (u, v) \in K \times K$. Define f_α by $f_\alpha(0_M) = 0_M$, $f_\alpha(y) = v$ for $0_M < y \leq u$ and $f_\alpha(z) = o_M$ for $z \not\leq u$. Then $f_\alpha \in \underline{E}_{0,1}$, $f_\alpha(u) = v$, $f_\alpha(M) = \{0_m, v, o_<\}$ and $q_u \leq f_\alpha$. If $u = o_N$ then $f_\alpha(K) = \{v\}$. If $u \neq o_N$ then $f_\alpha(K) = \{v, o_M\}$, and if $N = N + N$ then $f_\alpha \in {}^1\underline{G}_{0,1}$ (see 2.5).

Assume that $N = N + N$ and define f'_α by $f'_\alpha(x) = 0_M$ for $x \leq u$, $f'_\alpha(y) = v$ for $y \in N$, $y \not\leq u$, and $f'_\alpha(o_M) = o_M$. Again, $f'_\alpha \in \underline{E}_{0,1}$, and if $u \neq o_N$ then $f'_\alpha(M) = \{0_M, v, o_M\}$. If $u \neq 0_M, o_N$ then then $f'_\alpha(K) = \{0_M, v\}$ (if $K(\leq)$ is downwards directed then $f'_\alpha \in^1 \underline{F}_{0,1}$ - see 2.4).

2.7 Let $\beta = (u, v, w) \in N \times K \times K$, $u < v$. Define g_β by $g_\beta(x) = 0_M$ for $x \leq u$, $g_\beta(y) = w$ for $y \leq v$, $y \not\leq u$, and $g_\beta(z) = o_M$ for $z \not\leq v$. Then $g_\beta \in \underline{E}_{0,1}$, $g_\beta(u) = 0_M$, $g_\beta(v) = w$, $g_\beta(M) = \{0_M, w, o_M\}$ and $q_v \leq g_\beta$. If $u = 0_M$ then $g_\beta = f_\alpha$, $\alpha = (v, w)$ (see 2.6). If $u \neq 0_M$ then $\{0_M, w\} \subseteq g_\beta(K)$ and $g_\beta(K) = \{0_M, w, o_M\}$ iff $v \neq o_N$. If $v \neq o_N$ then $\{o_M, w\} \subseteq g_\beta(K)$ and $g_\beta(K) = \{0_M, w, o_M\}$ iff $u \neq o_M$.

If $K(\leq)$ is downwards directed and $u \neq 0_M$ then $g_\beta \in {}^1\underline{F}_{0,1}$ (see 2.4). If $N = N + N$ and $w \neq o_N$ then $g_\beta \in {}^1\underline{G}_{0,1}$ (see 2.5).

2.8 Put $\underline{E}'_{0,1} = \underline{D}_{0,1} + \underline{E}_{0,1}$. If $N+N = N$ then $\underline{\xi} \in \underline{D}_{0,1}$ and we get $\underline{E}'_{0,1} = \underline{E}_{0,1}$. Assume, therefore, that $N+N \neq N$. Then $\mathrm{id_M} \notin \underline{\mathrm{E}}'_{0,1}$, and hence $\underline{E}'_{0,1}$ is a proper bi-ideal of the semiring $\underline{E}_{0,1}$ (in particular, the semiring $\underline{E}_{0,1}$ is not simple).

Lemma 2.8.1. *Let $f \in \underline{E}_{0,1}$, $B = \{ x \in M \mid f(x) = o + M \}$ and $C = M \setminus B$. Then:*
(i) *$f \in \underline{E}'_{0,1}$ iff there is a prime ideal A of M such that $A \subseteq B$.*
(ii) *If $f \in \underline{E}'_{0,1}$ and $A \neq \{o_M\}$ then $o_M \in f(K)$* (see 2.5).
(iii) *If $o_C \in C$ then $f \in \underline{E}'_{0,1}$.*

Proof. It is easy. □

2.9 Put $\underline{E}''_{0,1} = \underline{B}_{0,1} + \underline{E}_{0,1}$. If $o_N \in N$ then $\underline{\xi} = q_{o_N} \in \underline{B}_{0,1}$ and we have $\underline{E}''_{0,1} = \underline{E}_{0,1}$. Assume, therefore, that $o_N \notin N$. Then $\mathrm{id}_M \notin \underline{E}''_{0,1} \subseteq \underline{E}'_{0,1}$.

Lemma 2.9.1. *Let $f \in \underline{E}_{0,1}$, $B = \{ x \in M \mid f(x) = o_M \}$ and $C = M \setminus B$. Then:*
(i) *$f \in \underline{E}''_{0,1}$ iff there is $w \in N$ such that $\{ z \in M \mid z \not\leq w \} \subseteq B$.*
(ii) *If $f \in \underline{E}''_{0,1}$ and $q_u \leq f$ for $u \neq o_N$ then $o_M \in f(K)$* (see 2.5).
(iii) *If $o_C \in C$ then $f \in \underline{E}''_{0,1}$.*

Proof. It is easy. □

Proposition 2.10. ([1, 3.8]) *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and E is simple. Then:*
(1) *For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $f \in E$ such that $f(w) \leq v$, $f(w) \not\leq u$ and $f(z) \not\leq v$ for every $z \not\leq w$.*
[2] *If $0_E \in E$ (equivalently, $N + N = N$ and $\underline{\xi} \in E$) then for all $u \in N \setminus \{o_N\}$ and $w \in K$ there is at least one $f \in E$ such that $f(w) \in K$, $f(w) \not\leq u$ and $f(z) = o_M$ for every $z \not\leq w$.*
[3] *If $0_E \in E$ then for all $u \in N$ and $v \in K$, $u < v$, there is at least one $f \in E$ such that $f(K) \leq v$ and $f(z) \not\leq u$ for at least one $z \in K$.*
[4] *If $0_E \in E$ then for every $u \in N \setminus \{o_N\}$ there is at least one $f \in E$ such that $f(K) \subseteq N$ and $f(z) \not\leq u$ for at least one $z \in K$.*
[5] *$E \subseteq \underline{E}''_{0,1}$* (see 2.8).

Theorem 2.11. *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and $f_\alpha \in E$ for every $\alpha \in K \times K$* (see 2.6). *If $E \subseteq \underline{E}''_{0,1}$* (see 2.9) *then the semiring E is simple.*

Proof. Let $\varrho \neq \mathrm{id}_E$ be a congruence of the semiring E. Then there are $g, h \in E$ such that $g < h$, $(g, h) \in \varrho$, and there is $v \in K$ such that $g(v) < h(v)$. Let $u \in K$ and $\alpha = (u, v)$. Then $(q_u, q_{0_M}) = (q_{g(v)} g f_\alpha, q_{g(v)} h f_\alpha) \in \varrho$. It follows that $(q_x, q_{0_M}) \in \varrho$ for every $x \in N$. Finally, if $k, t \in E$ then there are $y, z \in N$ such that $q_y \leq k$ and $q_z \leq t$. Consequently, $(k, q_{0_M}) = (k + q_y, q_{0_M} + q_y) \in \varrho$, $t, q_{0_M}) = (t + q_z, q_{0_M} + q_z) \in \varrho$ and $(k, t) \in \varrho$. □

Theorem 2.12. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$, $f_\alpha \in E$ for every $\alpha = (o_N, v)$, $v \in K \setminus \{o_N\}$* (see 2.6)*, and $k(w) = o_N$ for some $k \in E$ and $w \in K \setminus \{o_N\}$. Then the semiring E is simple.*

Proof. Let $\varrho \neq \mathrm{id}_E$ be a congruence of E. Then there are $g, h \in E$ such that $g < h$ and $g(v) < h(v)$ for some $v \in K \setminus \{o_N\}$ (consider the endomorphisms gk, hk and put $v = w$ when necessary). Now, $(q_{o_N}, q_{0_M}) = (q_{g(v)} g f_\alpha, q_{g(v)} h f_\alpha) \in \varrho$, where $\alpha = (o_N, v)$, and it follows easily that $\varrho = E \times E$. □

Lemma 2.13. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{),1}$ such that $o_N \notin E(M \setminus \{o_N\})$. Define a relation $\underline{\tau}$ on E by $(f, g) \in \underline{\tau}$ iff $f \,|\, M \setminus \{o_N\} = g \,|\, M \setminus \{o_N\}$. Then:*
(i) *$\underline{\tau}$ is a congruence of the semiring E.*
(ii) *If $q_{0_M} \in E$ then $(f, q_{0_M}) \notin \underline{\tau}$ for every $f \in E$, $f \neq q_{0_M}$.*
(iii) *If $\underline{\xi} = q_{o_N} \in E$ and $(f, \underline{\xi}) \in \underline{\tau}$ then $f(N \setminus \{o_N\}) = \{0_M\}$.*
(iv) *$\underline{\tau} = \mathrm{id}_E$ iff for all $f, g \in E$, $f \neq g$, there is $v \in K \setminus \{o_N\}$ with $f(v) \neq g(v)$.*

Proof. It is easy. □

Theorem 2.14. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$, $f_\alpha \in E$ for every $\alpha = (o_N, v)$, $v \in K \setminus \{o_N\}$* (see 2.6) *and $o_N \notin E(M \setminus \{o_N\})$. Then the semiring E is simple if and only if for all $f, g \in E$, $f \neq g$, there is at least one $w \in K \setminus \{o_N\}$ with $f(w) \neq g(w)$.*

Proof. If E is simple then $\underline{\tau} = \mathrm{id}_E$ follows from 2.3(i), (ii) and hence 2.13 (iv) applies. Conversely, if $\underline{\tau} = \mathrm{id}_E$ then we proceed in a similar way as in the proof of 2.12 to show that E is simple. □

2.15 Let A be a prime ideal of the semilattice M. For every $v \in M$, define $f_{A,v}$ by $f_{A,v}(0_M) = 0_M$, $f_{A,v}(y) = v$ for $y \in L \setminus A$ and $f_{A,v}(A) = \{o_M\}$. Then $f_{A,v} \in \underline{E}_{0,1}$ and $f_{A,v} = \{0_M, v, o_M\}$, provided that $A \neq L = M \setminus \{0_M\}$. Of course, $q_A \leq f_{A,v}$, and so $f_{A,v} \in \underline{E}'_{0,1}$.

Put ${}^3\underline{F}_{0,1} = {}^2F'_{0,1} \setminus {}^1\underline{F}_{0,1}$. Then ${}^3\underline{F}_{0,1}$ is a unitary subsemiring of ${}^2\underline{F}'_{0,1}$ and we have ${}^3\underline{F}_{0,1} + {}^2\underline{F}'_{0,1} = {}^3\underline{F}_{0,1}$. The relation $({}^3\underline{F}_{0,1} \times {}^3\underline{F}_{0,1}) \cup ({}^1\underline{F}_{0,1} \times {}^1\underline{F}_{0,1})$

is a congruence of the semiring ${}^2\underline{F}'_{0,1}$ (use 2.4), and hence this semiring os not simple.

Theorem 2.16. ([2, 3.2]) *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and $f_{A,w} \in E$ for every prime ideal A of M and all $w \in K$* (see 2.15). *Then $\underline{D}_{0,1} \subseteq E$ and the semiring E is simple if and only if $E \subseteq \underline{E}'_{0,1}$* (see 2.8).

Proof. First, $q_w f_{A,w} = q_A$, and so $D_{0,1} \subseteq E$. The relation $\varsigma = (F \times F) \cup \mathrm{id}_E$, $F = E \cap \underline{E}'_{0,1}$, is a congruence of the semiring E (see 2.8) and $\varsigma \neq \mathrm{id}_E$, since $\underline{D}_{0,1} \subseteq F$. Thus $F = E$, provided that $\varsigma = E \times E$.

As concerns the converse implication, we proceed similarly as in the proof of 2.11 (just notice that $q_{g(v)} g f_{A,v} = q_A$ and $q_{g(v)} h f_{A,v} = q_{0_M}$). □

Corollary 2.17. ([2, 3.3]] *The semiring $\underline{E}_{0,1}$ is simple if and only if $N + N = N$.* □

Corollary 2.18. ([2, 3.3]) *The semiring $\underline{E}'_{0,1}$* (see 2.8) *is always simple.* □

Remark 2.19. (cf. 2.18) Let E be any subsemiring of $\underline{E}_{0,1}$ and $H = E \cap \underline{E}'_{0,1}$. If $H = \{f\}$ is a one-element set then f is a bi-absorbing element of the semiring E. If E is simple and $|H| \geq 2$ then $H = E$ and $E \subseteq \underline{E}'_{0,1}$.

Remark 2.20. Assume that $K(\leq)$ is downwards directed (see 2.4). It is easy to see that the relation $({}^1\underline{F}_{0,1} \times {}^1\underline{F}_{0,1}) \cup \mathrm{id}$ os a congruence of the semiring ${}^1\underline{F}'_{0,1}$. Consequently, this semiring is not simple (cf. 2.15).

Remark 2.21. Let $o_N \notin N = N + N$. Then $\underline{E}''_{0,1} \subseteq {}^1\underline{G}_{0,1}$. If ${}^1\underline{G}_{0,1} = \underline{E}''_{0,1}$ then the semiring ${}^1\underline{G}_{0,1}$ is simple (use 2.11).

Remark 2.22. Let $N = N + N$. We have $\underline{D}_{0,1} \subseteq {}^1\underline{G}_{0,1}$ and the relation $({}^1\underline{G}_{0,1} \times {}^1\underline{G}_{0,1}) \cup \mathrm{id}$ is a congruence of ${}^1\underline{G}'_{0,1}$ Therefore, the semiring ${}^1\underline{G}'_{0,1}$ is not simple. Similarly, the semiring ${}^2\underline{G}'_{0,1}$ is not simple.

3. $0, 1$-preserving endomorphisms (b)

The preceding section is immediately continued. Here, we will assume, moreover, that every infinite strictly increasing sequence $x_1 < x_2 < \dots$ of elements from M os *upwards cofinal* in K (or M). The following result is obvious.

Proposition 3.1. (i) *The ordered set $M(\leq)$ is a lattice.*
(ii) *$\underline{B}_{0,1} = \underline{C}_{0,1}$ and $\underline{E}''_{0,1}$ is a subsemiring of $\underline{E}_{0,1}$.*
(iii) *If $N + N \neq N$ then $o_N \notin N$, there is no infinite strictly increasing sequence of elements in M, $\underline{B}_{0,1} = \underline{D}_{0,1}$ and $\underline{E}'_{0,1} = \underline{E}''_{0,1}$.*
(iv) *If $o_N \in N$ then $N + N = N$, there is no strictly increasing sequence of elements in M, $B_{0,1} = \underline{D}_{0,1}$ and $\underline{E}'_{0,1} = \underline{E}''_{0,1} = \underline{E}_{0,1}$.*

(iv) *If* $o_N \notin N = N + N$ *then the set* N *has no maximal element,* $\underline{\xi} \notin \underline{B}_{0,1}$, $\underline{D}_{0,1} = \underline{B}_{0,1} \cup \{\underline{\xi}\}$, $\underline{E}'_{0,1} = \underline{E}_{0,1}$ *and* $\mathrm{id}_M \notin \underline{E}''_{0,1}$. □

Corollary 3.2. $\underline{D}_{0,1} = \underline{B}_{0,1}$ *except for the case when* $o_N \notin N = N + N$. *Then* $\underline{\xi} \notin \underline{B}_{0,1}$ *and* $\underline{D}_{0,1} = \underline{B}_{0,1} \cup \{\underline{\xi}\}$. □

Proposition 3.3. ([1, 4.7]) *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$, E *is simple and* E *does not operate transitively on* K. *Then:*
(i) *Both* E *and* M *are infinite.*
(ii) $o_N \in N$, $M \setminus \{o_N\}$ *is a subsemilattice of* M *and* $E(M \setminus \{o_N\}) \subseteq M \setminus \{o_N\}$.
(iii) E *operates transitively on the set* $K \setminus \{o_N\}$.
(iv) $E9x) = m \setminus \{o_N\}$ *for every* $x \in K \setminus \{o_N\}$.
(v) $M \setminus \{o_N\} \subseteq E(o_N)$.

Proposition 3.4. (i) *If* $N + N \neq N$ *then* $\underline{E}'_{0,1} = \underline{E}''_{0,1}$ *is just the set of the endomorphisms* $f \in \underline{E}_{0,1}$ *such that* $f(x) = o_M$ *for a maximal element* $w \in N$ *and all* $x \not\leq w$.
(ii) *If* $o_N \notin N = N + N$ *then* $\underline{E}''_{0,1} = {}^1\underline{G}_{0,1}$ (see 2.5) *and* $\underline{E}'_{0,1} = \underline{E}_{0,1}$.
(iii) *If* $o_N \in N$ *then* $\underline{E}'_{0,1} = \underline{E}''_{0,1} = \underline{E}_{0,1}$.

Proof. It is easy. □

Remark 3.5. Let $f \in \underline{E}_{0,1}$ be such that $f(M) = \{0_M, w, o_M\}$ is a three-element subsemilattice (we have $w \in K$). Put $A = \{\, x \in M \mid f(x) = 0_M \,\}$, $B = \{\, y \in M \mid f(y) = w \,\}$ and $C = \{\, z \in M \mid f(z) = o_M \,\}$. Then $A + A = A$, $B + (A \cup B) = B$ and $M + C = C$. Moreover, $u = o_A \in A$ and $A = \{\, x \in M \mid x \leq u \,\}$. If $v = o_{A \cup B} \in A \cup B$ then $v \in B$, $u < v$, $B = \{\, y \in M \mid y \leq v, y \not\leq u \,\}$ and $C = \}\, z \in M \mid z \not\leq v \,\}$. Thus $f = g_{(u,v,w)}$ in this case (see 2.7). On the other hand, if $o_{A \cup B} \notin A \cup B$ then $A \cup B = N$, $N + N = N$, $o_N \notin N$, $B = N \setminus A = \{\, y \in N \mid y \not\leq u \,\}$ and $C = \{o_M\}$. Thus $f = f'_{(u,w)}$ (see 2.6).

In the remaining part of this section, assume that every infinite strictly decreasing sequence $x_1 > x_2 > \dots$ of elements from M is *downwards cofinal* in K (or L).

Theorem 3.6. ([1, 5.1]) *Assume that* $o_N \notin N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \notin E$.
(i) *If the set* $L = M \setminus \{0_M\}$ *has at least one minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{E}''_{0,1}$ (see 3.4(i),(ii)) *and the following condition is satisfied:*

(ii1) *For all* $u \in N$ *and* $v, w \in K$, $u < v$, *there is ar least one* $f \in E$ *such that* $f(w) \leq v$ *and* $f(w) \not\leq u$.

(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{E}''_{0,1}$, *the condition* (i1) *is true and, besides, the following condition is satisfied:*

(ii1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq o_M$ *and* $f(x) \not\leq v$ *for every* $x \in K$. □

Theorem 3.7. ([1, 5.2]) *Assume that* $o_N \in N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$.
(i) *If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 3.6(i1) *is satisfied.*
(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if the conditions* 3.6(i1) *and* 3.6(ii1) *are satisfied.* □

Theorem 3.8. ([1, 5.3]) *Assume that* $o_N \notin N = N+N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \in E$.
(i) *If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 3.6(i1) *is satisfied and, moreover, the following condition is satisfied as well:*

(i1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq 0_M$ *and* $f(x) \leq v$ *for every* $x \in K$.

(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if the conditions* (i1), 3.6(i1) *and* 3.6(ii1) *are satisfied.* □

Corollary 3.9. *Let* E, F *be subsemirings of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \subseteq F$ *and* E *is simple.*
(i) *If* $o_N \in N$ *then* F *is simple.*
(ii) *If* $o_N \notin N = N + N$ *and* $\underline{\xi} \in E$ *then* F *is simple.*
(iii) *If* $o_N \notin N$ *and* $\underline{\xi} \notin F$ *then* F *is simple if and only if* $F \subseteq \underline{E}''_{0,1}$.
(iv) *If* $o_N \notin N = N + N$ *and* $\underline{\xi} \notin F$ *then* F *is simple if and only if the condition* 3.8(i1) *is true for* F. □

Proposition 3.10. *Assume that* $o_N \notin N = N + N$.
(i) $\underline{B}_{0,1} \subseteq {}^1\underline{G}_{0,1} = \underline{E}''0, 1$ *and the semiring* ${}^1\underline{G}_{0,1}$ *is simple.*
(ii) *If* E *is a simple subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \notin E$ *then* $E \subseteq {}^1\underline{G}_{0,1}$.
(iii) *If* E *is a subsemiring such that* ${}^1\underline{G}_{0,1} \subsetneq E \subseteq {}^2\underline{G}'_{0,1}$ *then* E *is not simple.*
(iv) *If* E *is a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \not\subseteq {}^2\underline{G}'_{0,1}$ *and* $\underline{\xi} \in E$ *then* E *satisfies* 3.8(i1).

Proof. (i) See 3.4(i) and 2.2.1.
(ii) Use 3.6.
(iii) If E is simple then $\underline{\xi} \in E$. Now, by 3.8, E satisfies 3.8(i1), and consequently $E \not\subseteq {}^2\underline{G}'_{0,1}$.
(iv) There is $\in E \setminus {}^2\underline{G}''_{0,1}$. Then there is $v \in K$ such that $v \not\leq f(x)$ for every $x \in N$ and it follows that $w = o_A \in A = f(N) \subseteq N$. Thus $f(N) \leq w$. □

Proposition 3.11. *Assume that* $o_N \in N$. *Then:*
(i) ${}^1\underline{G}_{0,1} = \{ f \in \underline{E}_{0,1} \mid f(o_N) = o_N \}$ *and* ${}^2\underline{G}_{0,1} = \{ f \in \underline{E}_{0,1} \mid f(o_N) \in$

$\{o_N, o_M\}\}$.
(ii) $\underline{B}_{0,1} \subseteq {}^1\underline{G}'_{0,1} \subseteq {}^2\underline{G}'_{0,1}$ *and these semirings are not simple.*
(iii) *If* E *is a subsemiring such that* $\underline{B}_{0,1} \subseteq E \subseteq {}^2\underline{G}'_{0,1}$ *then* E *is not simple.*
(iv) *If* E *is a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \not\subseteq {}^2\underline{G}'_{0,1}$ *then* E *satisfies* 3.8(i1).

Proof. (i) See 2.5.
(ii) and (iii). If E is simple then E satisfies 3.6(i1) (use 3.7) and there is $f \in E$ such that $0_M \neq f(o_N) < o_N$. Then $f \notin {}^2\underline{G}'_{0,1}$.
(iv) We proceed similarly as in the proof of 4.10(iv). □

Proposition 3.12. *The set* $\underline{A} = \underline{B}_{0,1} \cup \{f_\alpha \,|\, \alpha \in K \times K\}$ *is a subsemiring of* $\underline{E}_{0,1}$.

Proof. Easy to check directly. □

In the remaining part of this section, assume that the set $L = M \setminus \{0_M\}$ has at least one minimal element and denote by P the set of minimal elements.

Proposition 3.13. *The set* $\underline{A}' = \{\underline{B}_{0,1} \cup \{\, f(\alpha) \,|\, \alpha \in P \times K\}$ *is a subsemiring of* $\underline{A}$ (see 3.12).

Proof. Easy to check directly. □

Proposition 3.14. *Let* E *be a simple subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$. *If either* $o_N \notin N$ *or* M *is finite or* $o_N \in E(M \setminus \{o_N\})$ *then* E *operates transitively on* K.

Proof. See 3.3. □

Proposition 3.15. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* E *operates transitively on* K. *Then:*
(i) E *is simple, provided that either* $o_N \in N$, *or* $o_N \notin N$ *and* $\underline{\xi} \notin E \subseteq \underline{E}''_{0,1}$.
(ii) $\underline{A}' \subseteq E$.
(iii) *For all* $u, v \in K$ *there is* $u_1 \in N$ *such that* $u_1 < u$ *and* $g_{(u_1,u,v)} \in E$ (see 2.7 and 3.5).

Proof. (i) Clearly, 3.6(i1) is true and it remains to use 3.6(i) or 3.7(i).
(ii) We have $f_{(u,v)} = f + q_u$, where $u \in P$, $v \in K$, $f \in E$ and $f(u) = v$.
(iii) Choose $w \in P$ and find $g \in E$ with $g(u) = w$. Then there is $u_1 \in N$ such that $u_1 < u$ and $g_{(u_1,u,v)} = f_{(w,v)}g + q_u$. □

Lemma 3.16. *Let* E *be a subsemiring of* $\underline{A}$ *such that* $\underline{B}_{0,1} \subseteq E$. *Then* E *operates transitively on* K *if and only if* $E = \underline{A}$.

Proof. This is easy (see 3.12). □

Proposition 3.17. (i) *The semiring* $\underline{A}$ *is simple.*
(ii) *If* E *is a subsemiring of* $\underline{A}$ *such that* $\_B_{0,1} \subseteq E$ *and* E *is simple then* $o_N \in N$, $o_N \notin E(M \setminus \{o_N\})$, M *is infinite and* E *operates transitively on* $K \setminus \{o_N\}$.

Proof. Combine 3.14, 3.15, 3.16 and 3.3. □

Remark 3.18. Let E be a proper subsemiring of $\underline{A}$ such that $\underline{B}_{0,1} \subseteq E$ and E is simple (see 3.3 and 3.17). Then $o_N \in N$, $E = \underline{B}_{0,1} \cup \{ f_{(u,v)} \mid u, v \in K, v \neq o_N \}$, $o_{N_1} \in N_1 = M \setminus \{o_M, o_N\}$ and M is infinite. If E_1 is a simple subsemiring of E such that $\underline{B}_{0,1} \subseteq E_1$ then $E_1 = E$.

Proposition 3.19. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{A}' \subseteq E$. *Let* η *denote the congruence of* E *generated by the pairs* (q_u, q_{0_M}), $u \in P$. *If* $\varrho \neq \mathrm{id}_E$ *is a congruence of* E *then* $\eta \subseteq \varrho$. *IN particular, the semiring* E *is simple if and only if* $\eta = E \times E$.

Proof. First, if $(f, g) \in \varrho$ and $f(w) < g(w)$ for some $w \in P$ then $q_u = q_{f(w)} f f_\alpha$, $q_{0_M} = q_{f(w)} g f_\alpha$, $\alpha = (u, w)$, $u \in P$, and hence $(q_u, q_{0_M}) \in \varrho$ and $\eta \subseteq \varrho$. On the other hand, if $f(w) = g(w)$ for every pair $(f, g) \in \varrho$ then $f(wv) = f f_\beta(w) = g f_\beta(w) = g(v)$ for all $v \in K$, $\beta = (w, v)$, and hence $f = g$ and $\varrho = \mathrm{id}_E$. □

4. 0-preserving endomorphisms

Let M be a semilattice such that $0_M \in M$ and $|M| \geq 3$. The set $\underline{E}_0 = \{ f \in \underline{E} \mid f(0_M) = 0_M \}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism $\underline{\zeta}$, $\underline{\zeta}(M) = 0_M$, is the zero element of the semiring $\underline{E}_0$ (i.e., $\underline{\zeta}$ is additively neutral and multiplicatively absorbing). For every $f \in \underline{E}_0$, $f \neq \underline{\zeta}$, the set $A_f = \{ x \in M \mid f(x) \neq 0_M \}$ is a prime ideal of the semilattice M. For every prime ideal A of M and all $v \in M$, we have $q_{A,v} \in \underline{E}_0$, where $q_{A,v}(A) = \{v\}$ and $q_{A,v}(M \setminus A) = \{0_M\}$. Now, $q_{A,0_M} = \underline{\zeta}$, $q_{A_1,v_1} q_{A_2,v_2} = q_{A_2,v_1}$ if $v_2 \in A$, $q_{A_1,v_1} q_{A_2.v_2} = \underline{\xi}$ otherwise and the set $\underline{D}_0 = \{ q_{A_1,v_1} + \cdots + q_{A_n,v_n} \mid n \geq 1 \}$ is an ideal of the semiring $\underline{E}_0$.

For $u, v \in M$, $u \neq o_M$, put $q_{u,v} = q_{A,v}$, where $A = A_u = \{ x \in M \mid x \not\leq u \}$, and $q_{o_M,v} = \underline{\xi}$. Let $\underline{B}_0 = \{ q_{u,v} \mid u, v \in M \}$ and $\underline{C}_0 = \{ q_{u_1,v_1} + \cdots + q_{u_n,v_n} \mid n \geq 1 \}$. Then $\underline{B}_0 \subseteq \underline{C}_0 \subseteq \underline{D}_0$ and $\underline{C}_0$ is a left ideal of $\underline{E}_0$.

Theorem 4.1. *Let* E *be a subsemiring of* $\underline{E}_0$ *such that* $\underline{B}_0 \subseteq E$. *Then the semiring* E *is simple if and only if for every* $f \in E$ *there is an element* $w \in M$ *such that* $f(M) \leq w$.

Proof. The result follows immediately from [5, 2.6] (but see also [3, 3.6]). □

Corollary 4.2. ([3], [6]) *Put* $\underline{F}_0 = \{ f \in \underline{E}_0 \mid f(M) \leq w$ for some $w \in M \}$. *Then:*
(i) $\underline{F}_0$ *is an ideal of* $\underline{E}_0$ *and* $\underline{D}_0 \subseteq \underline{F}_0$.
(ii) *If* E *is a subsemiring of* $\underline{E}_0$ *such that* $\underline{B}_0 \subseteq E$ *then the semiring* E *is simple if and only if* $(\underline{C}_0 \subseteq)$ $E \subseteq \underline{F}_0$.

(iii) *The semirings* $\underline{C}_0$ *and* $\underline{F}_0$ *are simple.*
(iv) $\underline{F}_0 = \underline{E}_0$ *if and only if* $o_M \in M$ *(or, equivalently,* $\mathrm{id}_M \in \underline{F}_0$*).*
(v) $\underline{C}_0 = \underline{E}_0$) *if and only if* M *is a finite distributive lattice.* □

Remark 4.3. Let E be a subsemiring such that $\underline{B}_0 \subseteq E \subseteq \underline{F}_0$. If $e \in F$ is (left, right) multiplicatively neutral then $e = \mathrm{id}_M = 1_E.$, and hence $o_M \in M$ and $\underline{F}_0 = \underline{E}_0$.

Remark 4.4. $\underline{D}_0 = \underline{E}_0$ if and only if M is a finite distributive lattice.

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Application of anti-control strategy based on a modified washout filter controller

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Abstract. The problem of anti-control of Hopf bifurcation for Lü system is considered in this paper. A modified washout filter-aided dynamic feedback control law is introduced for the problem. The necessary conditions are presented in the controlled system, so that a certain bifurcation is created at equilibria with preferred stability. In addition, we find the control law can be applied to control the stability of the original bifurcated solution. Anti-control of chaotic attractor is also given. The direction of bifurcation and the stability of the bifurcating limit cycle are determined by the normal form theory and the center manifold theorem. Finally, some computer simulations are provided to illustrate the efficiency of the anti-control approach.
Keywords: Hopf bifurcation, limit cycle, Lü system, anti-control strategy.

1. Introduction

Bifurcation and chaos are well known for their complexity, and have been intensively studied in various fields such as physics, chemistry, biology, engineering and information. In recent years, control and anti-control of bifurcation have attracted the interest of many scholars. Bifurcation control means that a controller is designed to change the bifurcation characteristics and then some satisfactory dynamic behaviors are obtained [1, 2, 3, 4, 5, 6, 7, 8, 9]. Different from bifurcation control, an ideal bifurcation is generated artificially in a suitable location, which is called anti-control of bifurcation [10, 11]. Anti-control of Hopf bifurcation is regarded as a way to produce limit cycles in nonlinear dynamic systems [12, 13, 14]. In many practical applications, artificial bifurcation is required when it is beneficial and useful. Washout filter-based control strategy has been

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fully recognized in various fields. These include chaotic synchronization control [15], bifurcation control [16, 17], and anti-control of Hopf bifurcation [18, 19]. The main advantage of this controller is that the equilibrium structure of the original system can be guaranteed.

Lü system is described by Ref. [20] in the following:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where x, y, z are the state variables and a, b, c are real parameters. When the parameters are fixed as $a = 36, b = 3, c = 20$, system (1) is chaotic. For Lü system, most of the research articles mainly focus on the bifurcation analysis [21], bifurcation and chaos control via various controllers [22, 23] and chaos synchronization control [24]. As far as we know, anti-control of Hopf bifurcation for Lü system has not been found before.

In this paper, applying anti-control strategy to produce bifurcation in desired location in Lü system is discussed. In fact, we indicate a modified washout filter control law for generating limit cycles and ideal dynamic state. The rest of this paper is arranged as follows. In Section 2, the local stability of equilibria of the system is analyzed. In Section 3, a modified washout filter-aided controller is used to the model, and then the conditions of creating Hopf bifurcation at the equilibrium point are given. In Section 4, based on the normal form theory and the center manifold theorem, the direction of bifurcation and the stability of the bifurcating periodic solutions are studied. In Section 5, the results of numerical simulation show that the anti-control strategy is effective. Finally, it makes a summary in Section 6.

2. Local stability of the equilibria

The Jacobian matrix of system (1) at the point (x_0, y_0, z_0) is given by

$$J = \begin{pmatrix} -a & a & 0 \\ -z_0 & c & -x_0 \\ y_0 & x_0 & -b \end{pmatrix}. \tag{2}$$

The characteristic equation of system (1) is

$$\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0, \tag{3}$$

where

$$\begin{aligned} k_1 &= a + b - c, \\ k_2 &= x_0^2 + az_0 + ab - ac - bc, \\ k_3 &= ax_0^2 + ax_0y_0 + abz_0 - abc. \end{aligned}$$

For any parameter values, system (1) has equilibrium $S_0(0,0,0)$. For $bc>0$, it also has the symmetric equilibria $S_{\pm}(\pm\sqrt{bc},\pm\sqrt{bc},c)$. By the Routh-Hurwitz theorem, we get that:
(1) the equilibrium $S_0(0,0,0)$ is a saddle point, and is unstable for any $a,b,c\in(0,+\infty)$.
(2) for any $a+b>3c$, the symmetric equilibria $S_{\pm}(\pm\sqrt{bc},\pm\sqrt{bc},c)$ are stable.
(3) for any $a+b=3c$, the characteristic equation at equilibria $S_{\pm}(\pm\sqrt{bc},\pm\sqrt{bc},c)$ has a pair of pure imaginary eigenvalues $\lambda_{1,2}=\pm\mathrm{i}\sqrt{\frac{k_3}{k_1}}$.
(4) for any $a+b<3c$, the symmetric equilibria $S_{\pm}(\pm\sqrt{bc},\pm\sqrt{bc},c)$ are unstable.

3. Creation of Hopf bifurcations

In order to anti-control Hopf bifurcation effectively, the objective of this section is to design a control law to model (1), so that the system can create bifurcation at the location we need. A modified washout filters controller is added to the system, then the controlled system obtained in this paper is shown below:

$$\begin{cases}\dot{x}=a(y-x)+u,\\ \dot{y}=-xz+cy+u,\\ \dot{z}=xy-bz,\\ \dot{u}=m(x-y)+n(x-y)^3-du,\end{cases}\tag{4}$$

where u is a control input, and the real parameters m,n,d are control gains. The first two equations are chosen to control. Noticed that the equilibrium points of the original system are preserved during the control process, which is similar to the traditional washout filters controller. It is worth mentioning that the controller designed in this paper is different from the published controllers.

Next, the conditions of Hopf bifurcation at equilibrium point in the controlled system are discussed. To facilitate further discussion, we assume that $a,b,c\in(0,+\infty)$.

3.1 Existence of Hopf bifurcation at the steady state S_0

It is pointed out that if $a,b,c\in(0,+\infty)$, S_0 is a saddle point, which do not occur Hopf bifurcation in the original system(1). The following will realize Hopf bifurcation creating from S_0. The characteristic equation of the controlled model (4) at S_0 is

$$\lambda^4+l_1\lambda^3+l_2\lambda^2+l_3\lambda+l_4=0,\tag{5}$$

where

$$\begin{aligned}l_1&=a+b-c+d,\\ l_2&=ab-ac-bc+ad+bd-cd,\\ l_3&=-abc+abd-acd-bcd+cm,\end{aligned}$$

$$l_4 = -abcd + bcm.$$

Computing the following determinants:

$$\begin{aligned}
\Delta_1 &= l_1, \\
\Delta_2 &= \begin{vmatrix} l_1 & 1 \\ l_3 & l_2 \end{vmatrix} = l_1 l_2 - l_3, \\
\Delta_3 &= \begin{vmatrix} l_1 & 1 & 0 \\ l_3 & l_2 & l_1 \\ 0 & l_4 & l_3 \end{vmatrix} = l_3(l_1 l_2 - l_3) - l_1^2 l_4, \\
\Delta_4 &= \begin{vmatrix} l_1 & 1 & 0 & 0 \\ l_3 & l_2 & l_1 & 1 \\ 0 & l_4 & l_3 & l_2 \\ 0 & 0 & 0 & l_4 \end{vmatrix} = l_4 \Delta_3.
\end{aligned}$$

If $l_1 > 0, l_3 > 0, l_4 > 0$ and $l_3(l_1 l_2 - l_3) - l_1^2 l_4 > 0$, then $\Delta_i > 0 (i = 1, 2, 3, 4)$. Based on Routh-Hurwitz criteria, all roots of the characteristic equation have negative real parts. Thus, S_0 is locally asymptotically stable. If $l_3(l_1 l_2 - l_3) - l_1^2 l_4 \leq 0$, and $l_i > 0 (i = 1, 2, 3, 4)$, S_0 is unstable and non-hyperbolic. Choosing m as the bifurcation parameter. When

$$m = m_0 = \frac{-a^2 c + ac^2 + a^2 d - 2acd + c^2 d + ad^2 - cd^2}{c}, \tag{6}$$

the characteristic Eq.(5) has a pair of pure imaginary roots as follows:

$$\lambda_{1,2} = \pm \mathrm{i}\omega_0 = \pm \mathrm{i}\sqrt{ad - ac - cd} \quad (ad - ac - cd > 0). \tag{7}$$

The other two eigenvalues are

$$\lambda_3 = -b < 0, \tag{8}$$

$$\lambda_4 = -a + c - d < 0. \tag{9}$$

It is easy to verify that the following transversality condition holds:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda = \mathrm{i}\omega_0}) = \frac{c}{2(a^2 - 3ac + c^2 + 3ad - 3cd + d^2)} \neq 0. \tag{10}$$

So, according to Hopf bifurcation theory [25], the controlled model (4) occurs Hopf bifurcation at S_0.

3.2 Existence of Hopf bifurcation at the steady states S_+ and S_-

Since equilibrium S_+ and S_- are symmetric, we only consider the bifurcation at point S_+. The characteristic equation of the controlled model (4) at S_+ is

$$\lambda^4 + h_1 \lambda^3 + h_2 \lambda^2 + h_3 \lambda + h_4 = 0, \tag{11}$$

where

$$\begin{aligned} h_1 &= a + b - c + d, \\ h_2 &= ab + ad + bd - cd, \\ h_3 &= 2abc + abd, \\ h_4 &= 2abcd - 2bcm. \end{aligned}$$

As discussed above, choosing m as the bifurcation parameter, when

$$m = m_0 = \frac{2abcdh_1^2 - h_1h_2h_3 + h_3^2}{2bch_1^2}, \tag{12}$$

the characteristic Eq.(11) has a pair of pure imaginary roots:

$$\lambda_{1,2} = \pm \mathrm{i}\omega_0 = \pm \mathrm{i}\sqrt{\frac{h_3}{h_1}}, \tag{13}$$

and other two negative roots. The following transversality condition is also satisfied:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda=\mathrm{i}\omega_0}) = \frac{2bc(h_3 - 3h_1\omega_0^2)}{(h_3 - 3h_1\omega_0^2)^2 + (2h_2\omega_0 - 4\omega_0^3)^2} \neq 0. \tag{14}$$

Therefore, according to Hopf bifurcation theory [25], the controlled model (4) occurs Hopf bifurcation at S_+ . Section 2 shows that system (1) will undergo Hopf bifurcation at S_+ when $c = \frac{a+b}{3}$. So our control scheme also enlarges the parameter region in which Hopf bifurcation might occur at S_+.

4. Stability of the bifurcating limit cycle

By the normal form theory and the center manifold theorem [25], the direction of bifurcation and the stability of created limit cycle are determined in this section. Equilibrium S_0 is chosen as an example to discuss, and other equilibria can be similar. By the linear transform $(x, y, z, u)^T = P(X, Y, Z, U)^T$, where

$$P = \begin{pmatrix} \omega_0 & a-c & 0 & \frac{d}{c-d} \\ \omega_0 & a & 0 & -1 \\ 0 & 0 & 1 & 0 \\ (a-c)\omega_0 & -ac - \omega_0^2 & 0 & a+d \end{pmatrix}, \tag{15}$$

system (4) is transformed into the normal form

$$\begin{cases} \dot{X} = -\omega_0 Y + F_1(X, Y, Z, U), \\ \dot{Y} = \omega_0 X + F_2(X, Y, Z, U), \\ \dot{Z} = -bZ + F_3(X, Y, Z, U), \\ \dot{U} = (-a + c - d)U + F_4(X, Y, Z, U), \end{cases} \tag{16}$$

where F_1, F_2, F_3, F_4 are high order nonlinear functions of X, Y, Z, U. Because the expressions are too complex, it is omitted here. With the aid of detailed calculation, we get:

$$C_1(0) = \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \tag{17}$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{\alpha'(0)}, \tag{18}$$

$$\tau_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(0)\}}{\omega_0}, \tag{19}$$

$$\beta_2 = 2Re\{C_1(0)\}. \tag{20}$$

The expressions of $g_{11}, g_{02}, g_{20}, g_{21}$ can be found in Ref. [25]. Then we can get the following conclusions for the controlled model in accordance with the Ref. [25].

The properties of Hopf bifurcation in the controlled system (4) is determined by the parameters μ_2, β_2 and τ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical, and it is subcritical if $\mu_2 < 0$. Parameter β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$, the bifurcating periodic solutions is stable, and it is unstable if $\beta_2 > 0$. Parameter τ_2 determines the period of the bifurcating periodic solution. If $\tau_2 > 0$, the period increases and the period decreases if $\tau_2 < 0$.

5. Numerical analysis

In this section, some numerical simulations are presented to verify the above theoretical analyses. For the controlled model (4), at the equilibrium S_0, we choose $a = 2, b = 2, c = 1, d = 4$, then the bifurcation critical value is $m_0 = 18$. Choosing $n = 2$, we have results: $\mu_2 = 71.74987, \tau_2 = -1.06250, \beta_2 = -2.6574$. There exits a stable periodic orbit near the equilibrium point S_0. The bifurcation figure is shown in Fig. 1.

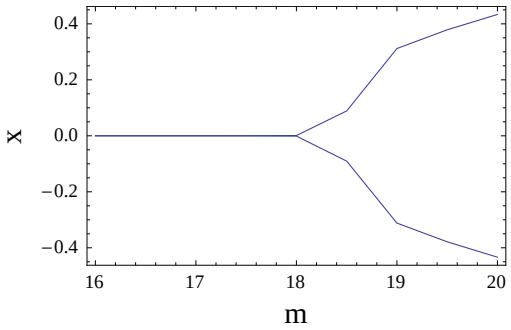


Figure 1: Bifurcation diagram of controlled model at the equilibrium S_0.

At the equilibrium S_+, as discussed in Section 2, S_+ is stable in the original system (1) when $c < \frac{a+b}{3}$. While in the controlled system (4), we choose $a = 1, b = 5, c = 1.5, d = 2$, the bifurcation critical value $m_0 = -0.60355$ is obtained. Then we have $\mu_2 = -0.02383, \tau_2 = 0.004163, \beta_2 = -0.01149$ when $n = 0.1$. This suggests that the controlled system (4) occur Hopf bifurcation at S_+ when $c < \frac{a+b}{3}$. The figure is shown in Fig. 2.

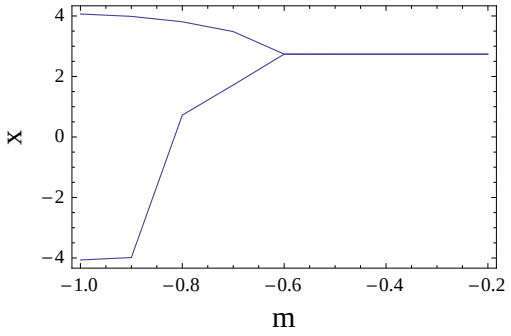
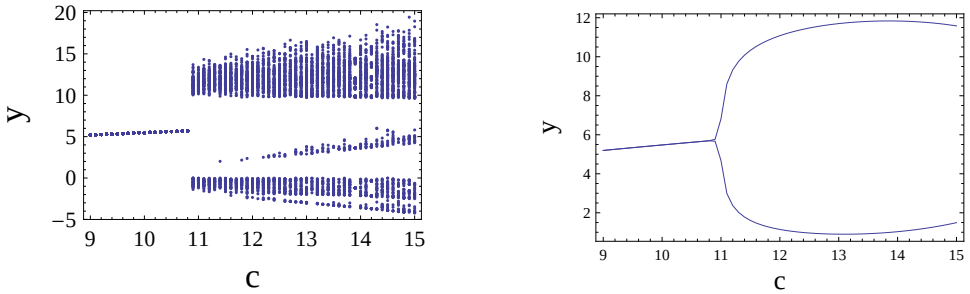


Figure 2: Bifurcation diagram of controlled model at the equilibrium S_+ when $c < \frac{a+b}{3}$.



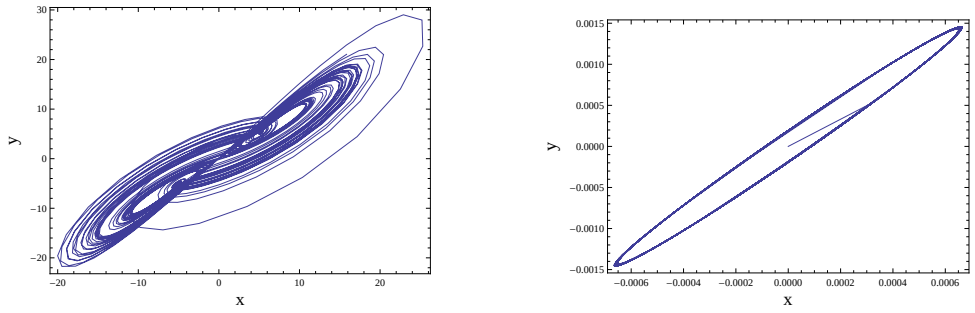
(a) (b)

Figure 3: Original and controlled model bifurcate at the equilibrium S_+.

Section 2 also points out that for any $c = \frac{a+b}{3}$, the characteristic equation at equilibrium S_+ has a pair of pure imaginary eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\sqrt{\frac{k_3}{k_1}}$. At this point, c is chosen as the bifurcation parameter. We choose $a = 30, b = 3$ for example, the original system (1) can occur Hopf bifurcation at $c_0 = 11$. After some calculations, the stability index $\beta_2 = 0.475766$ is obtained, which shows the bifurcating periodic solution is unstable. Now we will show the controller can change the stability of the existing bifurcation. In the controlled system (4), setting $a = 30, b = 3, d = 1, m = 0.01$, the bifurcation critical value is still at $c_0 = 11$. When $n = 4$, the stability index $\beta_2 = -0.0011313$ shows the bifurcating periodic solutions is stable. So the anti-control strategy in this paper

can not only enlarge the Hopf bifurcation parameter region, but also stabilize the existing bifurcation at S_+. The comparison of bifurcation figures are shown in Fig. 3.

Next, we are also interested in anti-control of the chaotic attractor. The original system is chaotic when $a = 36, b = 3, c = 20$. Fig. 4 shows the chaotic attractor and the limit cycle generated from the equilibrium when $d = 46, n = 0.1$ in the controlled system (4).



(a) (b)

Figure 4: Anti-control of the chaotic attractor.

6. Conclusions

This paper studies the problem of anti-control of Hopf bifurcation for chaotic Lü system. A modified washout filter-aided dynamic feedback control law is designed for the creation of Hopf bifurcations. That is, we succeed in generating limit cycle at equilibria through appropriate control parameters. Not only that, but we also achieve to stabilize an unstable bifurcated solution of the original system. The direction of bifurcation and the stability of created limit cycle are analyzed based on the normal form theory. It is worth mentioning that this controller has realized the stability control of an unstable Hopf bifurcation of the original system, which is rare in the existing literature. Simulation results indicate the effectiveness of the proposed approach. The anti-control strategy can also be applied to other chaotic systems.

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Edge irregularity strength of categorical product of two paths

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Abstract. For a simple graph $G = (V, E)$, a vertex labeling $\phi : V \to \{1, 2, \dots k\}$ is called k-labeling. The weight of an edge xy in G, denoted by $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, there is $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$.

In this paper, we determine the exact value of edge irregularity strength for categorical product of two paths.

Keywords: irregular assignment, irregularity strength, edge irregularity strength, categorical product of two graph, path.

1. Introduction

Labeled graphs are turning into an inexorably valuable group of Mathematical Models for an extensive variety of utilizations. While the subjective labelings of graph components have roused explore in assorted fields of human enquiry, for example, compromise in social brain science, electrical circuit hypothesis and vitality emergency, these labelings have prompted very multifaceted fields of use, for example, coding hypothesis issues, including the plan of good radar area codes, sync set codes; rocket direction codes and convolution codes with

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ideal autocorrelation properties. Labeled graphs have likewise been connected in deciding ambiguities in X-Ray crystallographic examination, to plan correspondence arrange tending to frameworks, in deciding ideal circuit designs, radio-Astronomy., and so on [1].

Give $G = (V, E)$ a chance to be the associated, basic and undirected diagram with vertex set V and edge set E. By a *labeling* we mean any mapping that conveys a set of diagram components to a set of numbers (normally positive whole numbers), called *labels*. On the off chance that the area is the vertex set or the edge set, the labelings are called individually *vertex labelings* or *edge labelings*. In the event that the space is $V \cup E$ then we call the labeling *total labeling*. Therefore, for an edge k-labeling $\phi : E(G) \to \{1, 2, \ldots, k\}$, the related weight of a vertex $x \in V(G)$ is

$$w_\phi(x) = \sum_{xy \in E(G)} \phi(xy),$$

where the addition is over all vertices y contiguous x, and for a total k-labeling $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$, the related weight of edge $xy \in E(G)$ is

$$wt_\phi(xy) = \phi(x) + \phi(xy) + \phi(y).$$

Chartrand *et al.* in [2] presented edge k-labeling ϕ of a graph G to such an extent that $w_\phi(x) \neq w_\phi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the smallest k for which G has an irregular assignment utilizing labels at generally k. This parameter has pulled in much consideration [3, 4, 5, 6].

Persuaded by the papers on the irregularity strength Bača *et al.* in [7] began to examine the $tes(G)$ and $tvs(G)$ a graph, an invariant similar to the es for total labelings. They characterize as:

Definition 1.1 ([7]). *For a graph $G = (V, E)$, a labeling $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an edge irregular total k-labeling of the graph G if for each two distinct edges xy and $x'y'$ of G the edge weights $wt_\phi(xy) = \phi(x) + \phi(xy) + \phi(y)$ and $wt_\phi(x'y') = \phi(x') + \phi(x'y') + \phi(y')$ are not equal. The total edge irregularity strength, $tes(G)$, is characterized as the smallest k for which G has an edge irregular total k-labeling.*

Definition 1.2 ([7]). *For a graph $G = (V, E)$, a labeling $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a vertex irregular total k-labeling of the graph G if for each two distinctive vertices x and y of G the vertex weights $wt_\phi(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$ and $wt_\phi(y) = \phi(y) + \sum_{xy \in E(G)} \phi(xy)$ are distinct. The total vertex irregularity strength, $tvs(G)$, is characterized as the smallest k for which G has a vertex irregular total k-labeling.*

The Ivančo and Jendroľ's figured the tes for complete graph and complete bipartite diagrams in [8], for the Cartesian product of two paths in [9], for the

Cartesian product of two cycles in [10], for corona product of a path with specific diagrams in [11], for huge thick graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [12], for the categorical product of two paths in [13] and for generalized Petersen diagram $P(n,k)$ in [14].

Some results on $tes(G)$ and $tvs(G)$ can be found in [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

Joining both past changes of their irregularity strength, Marzuki, Salman and Miller [28] presented a new irregular total k-labeling of a graph G called totally irregular total k-labeling, which is required to be in the mean time vertex irregular total and further more edge irregular total. They have given an upper bond and a lower bound for totally irregular total k-labeling, indicated by $ts(G)$. The most entire late review of graph labelings is [29, 30].

Motivated by all these irregular labeling concepts Ahmad *et al.* in [31] introduce a new labeling as follows:

Definition 1.3 ([31]). *A vertex k-labeling $\phi : V \to \{1, 2, \ldots, k\}$ is characterized to be an edge irregular k-labeling of the graph G if for each two distinct edges e and f there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The smallest k for which the diagram G has an edge irregular k-labeling is known as the edge irregularity strength of G, meant by $es(G)$.*

The following theorem proved in [31], establishes lower bound for the edge irregularity strength of a graph G.

Theorem 1.1 ([31]). *Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}.$$

In this paper, we determine the exact values of the edge irregularity strength of categorical product of two paths.

2. The categorical product of two paths

For numbers a and b let [a, b] be an interval of whole numbers x, $a \leq x \leq b$. In this area we manage a categorical product of two diagrams. The categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are neighbouring if and just if u, v are contiguous in G and u', v' are adjoining in H (see e.g. [32] or [33]). On the off chance that we consider graph G as the path P_n with:

$$V(P_n) = \{x_i : i \in [1, n]\},$$
$$E(P_n) = \{x_i x_{i+1} : i \in [1, n-1]\}$$

and graph H as the path P_m with

$$V(P_m) = \{y_j : j \in [1, m]\},$$
$$E(P_m) = \{y_j y_{j+1} : j \in [1, m-1]\}$$

then

$$V(P_n \times P_m) = \{(x_i, y_j) : i \in [1, n], j \in [1, m]\}$$

is the vertex set of $P_n \times P_m$ and

$$E(P_n \times P_m) = \{(x_i, y_j)(x_k, y_l) : i, k \in [1, n], j, l \in [1, m], |i - k| = 1, |j - l| = 1\}$$

is the edge set of $P_n \times P_m$. The categorical product of two paths $P_7 \times P_5$ is delineated in figure 1. It is easy to see that , $P_n \times P_m$ is the graph of order nm and size $2(n-1)(m-1)$. As the most extreme degree $\triangle(P_n \times P_m) = 4$ at that point from Theorem 1.1 it takes after that $es(P_n \times P_m) \geq (m-1)(n-1)+1$. The primary objective of this paper is to prove equality.

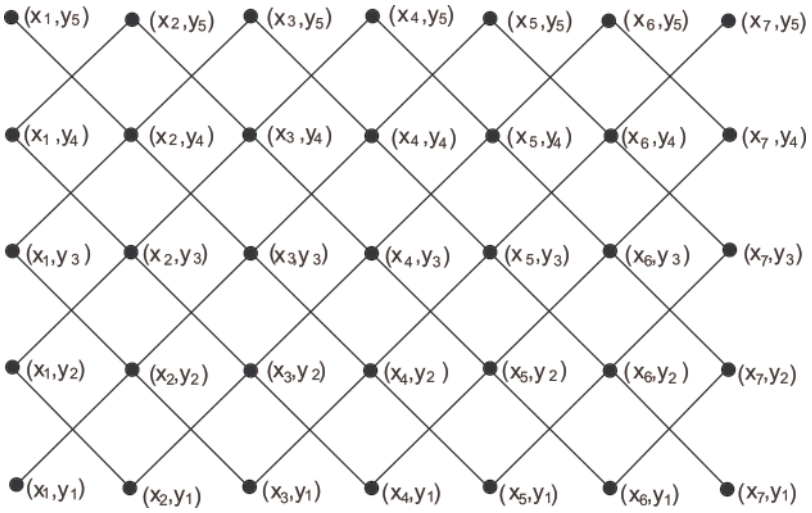


Figure 1: The Categorical Product of Two Paths $P_7 \times P_5$.

Theorem 2.1. *Let $m \geq 3$, odd and $n \geq 2$. Then*

$$es(P_n \times P_m) = (m-1)(n-1)+1.$$

Proof. Let $k = (m-1)(n-1)+1$. According to Theorem 1.1 it is enough to prove that $es(P_n \times P_m) \leq k$. For this we define a vertex labeling $\phi : V(P_n \times P_m) \to \{1, 2, \ldots, k\}$ as follows.

$$\phi((x_i, y_j)) = \begin{cases} k - n - \frac{m-1}{2} + i + \frac{j-1}{2}, & \text{for } i \text{ even and } j \text{ odd} \\ n - 2 + \frac{m-1}{2} + i + (2n-3)\frac{j-2}{2}, & \text{for } i \text{ odd and } j \text{ even} \\ i + \frac{j-1}{2}, & \text{for } i \text{ odd and } j \text{ odd} \\ i - 1 + \frac{j-2}{2}(2n-3), & \text{for } i \text{ even and } j \text{ even} \end{cases}$$

Now, we show that all the vertex labels are from 1 up to k for $1 \leq i \leq n, 1 \leq j \leq m$. i.e

$$1 \leq \phi((x_i, y_j)) \leq k$$

- For $1 \leq i \leq n$, even and $1 \leq j \leq m$, odd

$$\begin{aligned} \phi((x_i, y_j)) &= k - n - \frac{m-1}{2} + i + \frac{j-1}{2} \\ &\geq k - n - \frac{m-1}{2} + 2 = \frac{2k - 2n - m + 5}{2} \geq 1 \end{aligned}$$

and

$$\begin{aligned} \phi((x_i, y_j)) &= k - n - \frac{m-1}{2} + i + \frac{j-1}{2} \\ &\leq k - n - \frac{m-1}{2} + n + \frac{m-1}{2} = k \end{aligned}$$

For $1 \leq i \leq n$, odd and $1 \leq j \leq m$, even

$$\begin{aligned} \phi((x_i, y_j)) &= n - 2 + \frac{m-1}{2} + i + (2n-3)\frac{j-2}{2} \\ &\geq n - 2 + \frac{m-1}{2} + 1 \geq 1 \end{aligned}$$

and

$$\begin{aligned} \phi((x_i, y_j)) &= n - 2 + \frac{m-1}{2} + i + (2n-3)\frac{j-2}{2} \\ &\leq n - 2 + \frac{m-1}{2} + n + (2n-3)\frac{m-3}{2} \\ &= (m-1)(n-1) + 1 = k \end{aligned}$$

- For $1 \leq i \leq n$, odd and $1 \leq j \leq m$, odd

$$\begin{aligned} \phi((x_i, y_j)) &= i + \frac{j-1}{2} \geq 1, \\ \phi((x_i, y_j)) &= i + \frac{j-1}{2} \leq \frac{2n + m - 1}{2}. \end{aligned}$$

Now, we show that $\frac{2n+m-1}{2} \leq mn - n - m + 2$,

For $t \geq 1, n \geq 2$, we have

$$\begin{aligned} (2t-1)(n-1) - t &\geq 0, \\ 2tn - 3t - n + 1 &\geq 0, \\ (2t+1)n - n - (2t+1) + 2 &\geq n + t, \\ (2t+1)n - n - (2t+1) + 2 &\geq \frac{2n + 2t + 1 - 1}{2}. \end{aligned} \tag{1}$$

Sine m is odd, let $m = 2t + 1$ for all $t \geq 1$, then (1) becomes

$$mn - n - m + 2 \geq \frac{2n + m - 1}{2}.$$

- For $1 \leq i \leq n$, even and $1 \leq j \leq m$, even

$$\begin{aligned} \phi((x_i, y_j)) &= i + \frac{(2n-3)(j-2)}{2} \geq 1, \\ \phi((x_i, y_j)) &= i + \frac{(2n-3)(j-2)}{2} \leq \frac{2n + m - 4}{2}. \end{aligned}$$

Now, we show that $\frac{2n+m-4}{2} \leq mn - n - m + 2 = k$,

For $t \geq 1, n \geq 2$, we have

$$\begin{aligned} (4t-2)(n-1) - 2t + 3 &\geq 0, \\ 4tn - 6t - 2n + 5 &\geq 0, \\ 2tn - 2t + 1 &\geq \frac{2n + 2t + 1 - 4}{2}, \\ (2t+1)n - n - (2t+1) + 2 &\geq \frac{2n + 2t + 1 - 4}{2}. \end{aligned} \tag{2}$$

Sine m is odd, let $m = 2t + 1$ for all $t \geq 1$, then (2) becomes

$$mn - n - m + 2 \geq \frac{2n + m - 4}{2}.$$

Thus all the vertex labels satisfy $1 \leq \phi((x_i, y_j)) \leq k$.

Now, we define the weights on the edges for $1 \leq i \leq n, 1 \leq j \leq m$,

$$wt_\phi((x_i, y_{j+1})(x_{i+1}, y_j)) = \begin{cases} k - 1 + 2i + (n-1)(j-1), & \text{for } i \text{ odd and } j \text{ odd} \\ k + 2i + (n-1)(j-2), & \text{for } i \text{ even and } j \text{ even} \\ 2i + 1 + (j-2)(n-1), & \text{for } i \text{ odd and } j \text{ even} \\ 2i + (j-1)(n-1), & \text{for } i \text{ even and } j \text{ odd} \end{cases}$$

$$wt_\phi((x_i, y_j)(x_{i+1}, y_{j+1})) = \begin{cases} k - 1 + 2i + (n-1)(j-1), & \text{for } i \text{ even and } j \text{ odd} \\ k + 2i + (n-1)(j-2), & \text{for } i \text{ odd and } j \text{ even} \\ 2i + (j-1)(n-1), & \text{for } i \text{ odd and } j \text{ odd} \\ 2i + 1 + (j-2)(n-1), & \text{for } i \text{ even and } j \text{ even} \end{cases}$$

One can see that the set of edge weights form a sequence of consecutive integers $\{2, 3, 4, \ldots, 2k-1\}$. So the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ is an optimal edge irregular k-labeling. An illustration of Theorem 2 is depicted in Figure 2. In which number in black colour are vertex labels and number in blue colour shows the edge weights of the corresponding edge. This completes the proof. □

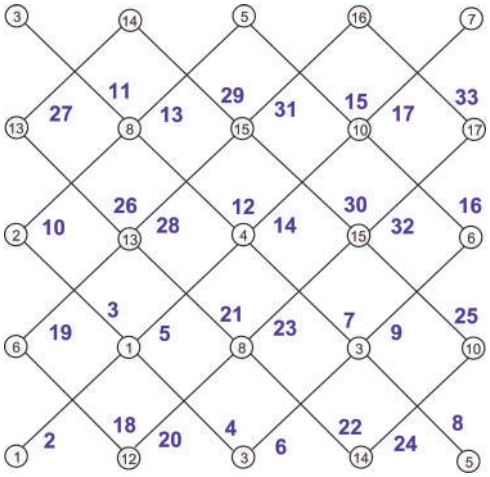


Figure 2: The Edge Irregular labeling of $P_5 \times P_5$.

Lemma 2.1. *Let $n \geq 2$. Then $es(P_n \times P_2) = n$.*

Proof. From Theorem 1.1 it follows that $es(P_n \times P_2) \geq n$. The existence of the optimal labeling proves the converse inequality.
Case 1. When n is odd

$$\phi((x_i, y_1)) = \begin{cases} \frac{i+1}{2}, & \text{for } 1 \leq i \leq n \text{ odd} \\ \frac{n-1+i}{2}, & \text{for } 1 \leq i \leq n \text{ even} \end{cases}$$

$$\phi((x_i, y_2)) = \begin{cases} \frac{n+i}{2}, & \text{for } 1 \leq i \leq n \text{ odd} \\ \frac{i}{2}, & \text{for } 1 \leq i \leq n \text{ even} \end{cases}$$

Case 2. When n is even

$$\phi((x_i, y_1)) = \begin{cases} \frac{i+1}{2}, & \text{for } 1 \leq i \leq n \text{ odd} \\ \frac{n+i}{2}, & \text{for } 1 \leq i \leq n \text{ even} \end{cases}$$

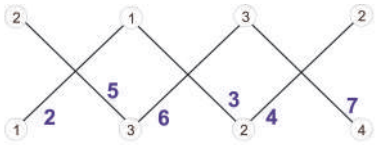
$$\phi((x_i, y_2)) = \begin{cases} \frac{n+i-1}{2}, & \text{for } 1 \leq i \leq n \text{ odd} \\ \frac{i}{2}, & \text{for } 1 \leq i \leq n \text{ even} \end{cases}$$

Now for all $n \geq 2$, the weight on the edges are define as follows.
$wt_\phi((x_i, y_1)(x_{i+1}, y_2)) = i + 1$, for $1 \leq i \leq n$, odd
$wt_\phi((x_{i+1}, y_1)(x_i, y_2)) = i + 1$, for $1 \leq i \leq n$,even
$wt_\phi((x_i, y_2)(x_{i+1}, y_1)) = n + i$, for $1 \leq i \leq n$,odd
$wt_\phi((x_i, y_1)(x_{i+1}, y_2)) = n + i$, for $1 \leq i \leq n$,even. □

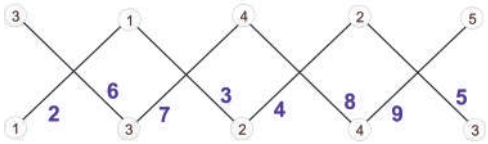
An illustration of Lemma 1 is depicted in Figure 3.

Theorem 2.2. *Let $m \geq 4$, even and $n \geq 3$. Then*

$$es(P_n \times P_m) = (m-1)(n-1) + 1.$$



(b) Edge irregular labeling of $P_4 \times P_2$



(a) Edge irregular labeling of $P_5 \times P_2$

Figure 3: The Edge Irregular labeling of $P_5 \times P_2$ and $P_4 \times P_2$.

Proof. Let $k = (m-1)(n-1)+1$. According to Theorem 1.1 it is enough to prove that $es(P_n \times P_m) \leq k$. For this we define a vertex labeling $\phi : V(P_n \times P_m) \to \{1, 2, \ldots, k\}$ as follows

$$\phi((x_i, y_j)) = \begin{cases} \frac{1}{2}[2i+j-3], & \text{for } i \text{ even and } j \text{ odd} \\ \frac{1}{2}[2i+(j-2)(2n-3)], & \text{for } i \text{ odd and } 1 \leq j \leq m-2 \text{ even} \\ \frac{1}{2}[2mn-3m-4n+10], & \text{for } i \equiv 3(mod4) \text{ and } j = m \\ \frac{1}{2}[2mn-3m-4n+8], & \text{for } i \equiv 1(mod4) \text{ and } j = m \end{cases}$$

Case 1. When n is odd

$$\phi((x_i, y_j)) = \begin{cases} k - \frac{1}{2}[(n-1)(m-j)-i+1], & \text{for } i \text{ odd and } j \text{ odd} \\ k - \frac{1}{2}[(n-1)(m-j+1)-i], & \text{for } i \text{ even and } j \text{ even} \end{cases}$$

Case 2. When n is even

$$\phi((x_i, y_j)) = \begin{cases} k - \frac{1}{2}[(n-1)(m-j)-i], & \text{for } i \text{ odd and } j \text{ odd} \\ k - \frac{1}{2}[(n-1)(m-j)+n-i], & \text{for } i \text{ even and } j \text{ even} \end{cases}$$

Now we show that all the vertex labels are from 1 up to k for $1 \leq i \leq n, 1 \leq j \leq m$. i.e

$$1 \leq \phi((x_i, y_j)) \leq k.$$

- For $1 \leq i \leq n$, even and $1 \leq j \leq m$, odd

$$\begin{aligned} \phi((x_i, y_j)) &= \frac{1}{2}[2i+j-3] \geq 1 \\ \phi((x_i, y_j)) &= \frac{1}{2}[2i+j-3] \leq \frac{1}{2}[2n+m-4] < k \end{aligned}$$

- For $1 \leq i \leq n$, odd and $1 \leq j \leq m-2$, even

$$\begin{aligned}
\phi((x_i, y_j)) &= \frac{1}{2}[2i + (j-2)(2n-3)] \geq 1 \\
\phi((x_i, y_j)) &= \frac{1}{2}[2i + (j-2)(2n-3)] \\
&\leq \frac{1}{2}[2n + (m-4)(2n-3)] \\
&= mn - n - m + 2 - \frac{1}{2}[4n + m - 8] \\
&= k - \frac{1}{2}[4n + m - 8] < k
\end{aligned}$$

- For $i \equiv 3(mod4)$ and $j = m$

$$\begin{aligned}
\phi((x_i, y_j)) &= \frac{1}{2}[2mn - 3m - 4n + 10] > 1 \\
\phi((x_i, y_j)) &= \frac{1}{2}[2mn - 3m - 4n + 10] \\
&= mn - n - m + 2 - \frac{1}{2}[2n + m - 6] \\
&= k - \frac{1}{2}[2n + m - 6] < k
\end{aligned}$$

- For $i \equiv 1(mod4)$ and $j = m$

$$\begin{aligned}
\phi((x_i, y_j)) &= \frac{1}{2}[2mn - 3m - 4n + 8] > 1 \\
\phi((x_i, y_j)) &= \frac{1}{2}[2mn - 3m - 4n + 8] \\
&= mn - n - m + 2 - \frac{1}{2}[2n + m - 4] \\
&= k - \frac{1}{2}[2n + m - 4] < k
\end{aligned}$$

- For n odd, $1 \leq i \leq n$, odd and $1 \leq j \leq m$, odd

$$\begin{aligned}
\phi((x_i, y_j)) &= k - \frac{1}{2}[(n-1)(m-j)] + \frac{i-1}{2} \\
&\geq k - \frac{1}{2}[(n-1)(m-1)] = \frac{1}{2}[mn - n - m + 3] > 1 \\
\phi((x_i, y_j)) &= k - \frac{1}{2}[(n-1)(m-j)] + \frac{i-1}{2} \\
&\leq k - \frac{n-1}{2} + \frac{n-1}{2} = k
\end{aligned}$$

- For n odd, $1 \leq i \leq n$, even and $1 \leq j \leq m$, even

$$\begin{aligned}\phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m+1-j)]+\frac{i}{2}\\ &\geq k-\frac{1}{2}[(n-1)(m-1)]+1=\frac{1}{2}[mn-n-m+5]>1\\ \phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m+1-j)]+\frac{i}{2}\\ &\leq k-\frac{n-1}{2}+\frac{n-1}{2}=k\end{aligned}$$

- For n even, $1 \leq i \leq n$, odd and $1 \leq j \leq m$, odd

$$\begin{aligned}\phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m-j)-1]+\frac{i-1}{2}\\ &\geq k-\frac{1}{2}[(n-1)(m-1)-1]=\frac{1}{2}[mn-n-m+4]>1\\ \phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m-j)-1]+\frac{i-1}{2}\\ &\leq k-\frac{n-1}{2}+\frac{n-1}{2}=k\end{aligned}$$

- For n even, $1 \leq i \leq n$, even and $1 \leq j \leq m$, even

$$\begin{aligned}\phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m-j)+n-2]+\frac{i-2}{2}\\ &\geq \frac{1}{2}[mn-n-m+4]>1\\ \phi((x_i,y_j)) &= k-\frac{1}{2}[(n-1)(m-j)+n-2]+\frac{i-2}{2}\\ &\leq k-\frac{n-2}{2}+\frac{n-2}{2}=k\end{aligned}$$

Thus, all the vertex labels satisfy $1 \leq \phi((x_i,y_j)) \leq k$.
Now, we define the weights on the edges for $1 \leq i \leq n, 1 \leq j \leq m$,

$$wt_\phi((x_i,y_{j+1})(x_{i+1},y_j)) = \begin{cases} 2i+j(n-1)-n+1, & \text{for } i \text{ odd} \\ & \text{and } 1 \leq j \leq m-2 \text{ odd} \\ m(n-1)-2n+4+i, & \text{for } i \equiv 3(mod 4) \\ & \text{and } j = m-1 \\ m(n-1)-2n+3+i, & \text{for } i \equiv 1(mod 4) \\ & \text{and } j = m-1 \end{cases}$$

$$wt_\phi((x_{i+1},y_j)(x_{i+2},y_{j+1})) = \begin{cases} 2i+j(n-1)-n+3, & \text{for } i \text{ odd} \\ & \text{and } 1 \leq j \leq m-2 \text{ odd} \\ m(n-1)-2n+3+i, & \text{for } i \equiv 3(mod 4) \\ & \text{and } j = m-1 \\ m(n-1)-2n+4+i, & \text{for } i \equiv 1(mod 4) \\ & \text{and } j = m-1 \end{cases}$$

- For $1 \leq i \leq n$, odd and $1 \leq j \leq m-2$, odd

$$wt_\phi((x_i, y_{j+1})(x_{i+1}, y_{j+2})) = 2i + j(n-1) - n + 2,$$
$$wt_\phi((x_{i+1}, y_{j+2})(x_{i+2}, y_{j+1})) = 2i + j(n-1) - n + 4,$$
$$wt_\phi((x_i, y_{j+2})(x_{i+1}, y_{j+1})) = 2k - (m-j-1)(n-1) + i.$$

- For $1 \leq i \leq n-1$, odd and $1 \leq j \leq m$, odd

$$wt_\phi((x_i, y_j)(x_{i+1}, y_{j+1})) = 2k - (m-j)(n-1) + i,$$

- For $1 \leq i \leq n-2$, odd and $1 \leq j \leq m$, odd

$$wt_\phi((x_{i+1}, y_{j+1})(x_{i+2}, y_j)) = 2k - (m-j)(n-1) + i + 1,$$

- For $1 \leq i \leq n-2$, odd and $1 \leq j \leq m-2$, odd

$$wt_\phi((x_{i+1}, y_{j+1})(x_{i+2}, y_{j+2})) = 2k - (m-j-1)(n-1) + i + 1.$$

It is easy to check that the set of edge weights form a sequence of integers

$$\{2, 3, 4, \ldots, mn - m - n + 2\} \cup \{mn - m - n + 4, mn - m - n + 5, \ldots, 2k\}.$$

So, the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ is an optimal edge irregular k-labeling. This completes the proof. □

3. Conclusion

This paper presented a new graph trademark, the edge irregularity strength, as a modification of the notable irregularity strength, add total edge irregularity strength and total vertex irregularity strength. In this paper we determined the exact an incentive for categorical product of two paths. Truth be told, it is by all accounts an extremely difficult issue to locate the correct an incentive for the edge irregularity strength of graphs. In future we are interested to compute the edge irregularity strength of categorical Product of cycle and path graph.

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A full-Newton step IIPM based on new search directions for $P_*(\kappa)$-LCP

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Abstract. In this paper, a new full-Newton step infeasible interior-point algorithm is proposed for solving $P_*(\kappa)$-linear complementarity problem. By using some new analytic tools, we show that the new algorithm is quadratically convergent with iteration complexity $O((1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon})$. This complexity matches the currently best known iteration bound for $P_*(\kappa)$-linear complementarity problem. Some computational results are provided as well.

Keywords: full-Newton step, infeasible interior-point algorithm, $P_*(\kappa)$-linear complementarity problem, iteration complexity.

1. Introduction

In this paper, we consider the standard form of linear complementarity problem (LCP)

$$(P) \qquad s = Mx + q, \quad xs = 0, \quad x \geq 0, \ s \geq 0,$$

where $M \in R^{n\times n}$ is a $P_*(\kappa)$-matrix, $q \in R^n$ and xs denotes the componentwise product (Hadamard product) of vectors x and s. LCP with M being a $P_*(\kappa)$-matrix is denoted by $P_*(\kappa)$-LCP. LCPs have many important applications in the

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field of mathematical programming and equilibrium problems. The interested readers can refer to the book [1].

Because $P_*(\kappa)$-matrix has good structure and $P_*(\kappa)$-LCP is an extension of monotone LCP (i.e., $P_*(0)$-LCP), the study on $P_*(\kappa)$-LCP has become a popular topic in the field of interior point methods (IPMs). In particular, extending feasible (IPMs) and infeasible interior-point methods (IIPMs) for linear optimization (LO) to $P_*(\kappa)$-LCP has been successful in many cases [2-6]. For example, Miao [3] extended the Mizuno-Todd-Ye (MTY) predictor-corrector algorithm to $P_*(\kappa)$-LCP with $O((\kappa+1)\sqrt{n}L)$ iteration complexity. Illes et al [6] gave a version of MTY predictor-corrector IPM for the $P_*(\kappa)$-LCP and obtained iteration complexity as $O((\kappa+1)^{\frac{3}{2}}\sqrt{n}L)$. For an overview these related results, we can refer to [7-14] and its references.

Recently, Mansouri et al [15] changed the definition of the feasibility step in [16] and showed that the iteration complexity enjoys $O(n\log\frac{\varepsilon}{n})$. Later on, Liu and Sun [17] proposed an IIPM that used full-Newton step, which was different from the algorithm given by Roos [16] in the definition of feasibility step, whereas the iteration bound essentially remained the same. It's worth noticing that the definition of the feasibility step in [15] is a special case of definition in [17]. Most recently, Mansouri et al [18] extended Roos's full-Newton IIPM [16] to LCP. Zhu et al [19] proposed a new IIPM for $P_*(\kappa)$-LCP, which is an extension of the work in [15].

Inspired by this series of work [15-19], in this paper we present a new full-Newton step IIPM for $P_*(\kappa)$-LCP. In our analysis, a specific feasibility step is introduced to induce the search directions, which is different from that of the algorithm in [15]. By using some new analytic tools, we show that the algorithm is quadratically convergent and the iteration complexity coincides with the currently best known one for IIPMs.

We use the following notation throughout the article: $\mathbf{R}^n_+$ denotes the nonnegative orthant of $\mathbf{R}^n$. x_{min} is the smallest component of $x=(x_1,x_2,...,x_n)^T\in\mathbf{R}^n$. $X=diag(x)$ denotes the diagonal matrix whose diagonal entries are the components of x. The all-one vector of length n is denoted by e, and $I=\{1,2,...,n\}$ is the index set. As usual, $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the 1-norm, 2-norm and infinity norm, respectively. Furthermore, the feasible set of $P_*(\kappa)$-LCP is denoted by

$$\mathcal{F}:=\{(x,s)\in\mathbf{R}^{2n}_+ : s=Mx+q\}.$$

Finally, the solution set of $P_*(\kappa)$-LCP is expressed by

$$\mathcal{F}^*:=\{(x^*,s^*)\in\mathcal{F} : (x^*)^Ts^*=0\}.$$

Throughout this paper it will be assumed that $\mathcal{F}^*$ is not empty, i.e., $P_*(\kappa)$-LCP has at least one solution. The proofs and technical lemmas are presented in Appendix.

2. Preliminaries

In case of the IIPM, we call (x, s) an ε-solution of $P_*(\kappa)$-LCP if the value of duality gap $x^T s$ and the norm of the residual vector $\|s - Mx - q\|$ do not exceed ε. We are now ready to present an IIPM that generates an ε-solution for $P_*(\kappa)$-LCP, if it exists.

2.1 The perturbed problems

As usual, we choose arbitrarily $(x^0,\ s^0) > 0$ as the initial point such that $(x^0)^T s^0 = \mu^0 e$ for some (positive) number μ^0. The initial value of the residual is denoted as

$$r^0 = s^0 - Mx^0 - q.$$

For any ν with $0 < \nu \leq 1$, we consider the perturbed problem

$$(P_\nu) \qquad s - Mx - q = \nu r^0, \quad (x, s) \geq 0.$$

Note that if $\nu = 1$ then $(x, s) = (x^0, s^0)$ yields a strictly feasible solution of (P_ν). We may conclude that if $\nu = 1$ then the problem (P_ν) has a strictly feasible solution, which means that the perturbed problem (P_ν) satisfies the interior-point condition (IPC).

Lemma 1. *If the original problem (P) is feasible, then the perturbed problem (P_ν) satisfies the IPC.*

Assuming that (P) is feasible and $0 < \nu \leq 1$, then Lemma 1 shows that the problem (P_ν) satisfies the IPC, and its central path exists. This means that the following system

$$(1) \qquad s - Mx - q = \nu r^0, \quad xs = \mu e, \quad x \geq 0, \quad s \geq 0,$$

has a unique solution for any $\mu > 0$, we donate this solution in the sequel as $(x(\mu, \nu), s(\mu, \nu))$. It is the μ-center of the perturbed problem (P_ν). In the following, we assume that the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$.

To measure the proximity of iterates (x, s) to the μ-center of the problem (P_ν), let us introduce a measures $\delta(x, s; \mu)$ as follows

$$(2) \qquad \delta(x, s; \mu) := \delta(\upsilon) := \frac{1}{2}\|\upsilon - \upsilon^{-1}\|, \quad \text{where} \quad \upsilon := \sqrt{\frac{xs}{\mu}}.$$

Now, we describe the main iteration of our algorithm. Assume that for some $\mu \in (0, \mu^0)$, we have (x, s) satisfying the feasibility condition (1) with $\nu = \frac{\mu}{\mu_0}$, and $x^T s \leq (n + \delta^2)\mu$, $\delta(x, s; \mu) \leq \tau$. For given $\theta \in (0, 1)$, we can find new iterates (x^+, s^+) that satisfy the perturbed problem (P_{ν^+}), with $\mu^+ := (1 - \theta)\mu$, $\nu^+ := \frac{\mu^+}{\mu^0}$, such that $(x^+)^T s^+ \leq (n + \delta^2)\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

More specifically, each main iteration consists of a feasibility step, a μ-update and a few centering steps. The feasibility step aims to get the iterates (x^f, s^f), which is strictly feasible for (P_{ν^+}) and close enough to its μ-center $(x(\mu^+, \nu^+), s(\mu^+, \nu^+))$. In fact, the feasibility step is designed in such a way that $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$. Since (x^f, s^f) is strictly feasible for (P_{ν^+}), by performing a few centering steps starting at (x^f, s^f) and targeting at the μ^+-center of (P_{ν^+}), it is easy to get iterates (x^+, s^+) that are strictly feasible for (P_{ν^+}) such that $\delta(x^+, s^+; \mu^+) \leq \tau$.

For the feasibility step, the system

$$M\Delta^f x - \Delta^f s = \theta\nu r^0, \quad s\Delta^f x + x\Delta^f s = \mu e - xs, \tag{3}$$

is used to define the search directions $\Delta^f x$ and $\Delta^f s$. After the feasibility step the iterates given by $x^f := x + \Delta^f x, s^f := s + \Delta^f s$. We conclude that after the feasibility step the iterates satisfy the system (1) with $\nu = \nu^+$. The hard part in the analysis will be to guarantee that (x^f, s^f) are positive and satisfy $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$.

In the centering steps, it starts at the iterates (x^f, s^f) and targets at the μ-centers of (P_{ν^+}). The search directions Δx and Δs are defined by the system $M\Delta x - \Delta s = 0, s\Delta x + x\Delta s = \mu e - xs$, which are the usual primal-dual Newton directions. After a centering step, the iterates are denoted as x^+ and s^+ which satisfy $(x^+)^T s^+ \leq (n + \delta^2)\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. More formal description of the algorithm is given in Fig. 1

Generic Full-Newton Step IIPM For $P_*(\kappa)$-LCP

Input:
A threshold parameter $\tau \in (0, 1)$; an accuracy parameter $\varepsilon > 0$;
barrier update parameter $\theta \in (0, 1)$;
a starting point $(x^0, s^0) > 0$ and $\mu^0 > 0$ such that $x^0 s^0 = \mu^0 e$;
begin
$x := x^0$; $s := s^0$; $\mu := \mu^0$;
while $\max(\mu(n + \delta^2), ||r^0||) \geq \varepsilon$ do
begin
feasibility step: $(x, s) := (x, s) + (\Delta^f x, \Delta^f s)$;
update of μ : $\mu := (1 - \theta)\mu$;
centering steps:
while $\delta(x, s; \mu) \geq \tau$ do
begin
$(x, s) := (x, s) + (\Delta x, \Delta s)$;
end
end
end

Fig. 1 Infeasible interior point algorithm

The required number of centering steps can be obtained by using Theorem A2 and Corollary A3. Indeed, supposing that $\delta = \delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, after k centering steps we will get iterates (x^+, s^+) that are still feasible for (P_{ν^+}) and satisfy $\delta(x^+, s^+; \mu^+) \leq (\frac{1}{\sqrt{2}})^{2^k} \frac{1}{1+4\kappa}$. From this one can deduce that $\delta(x^+, s^+; \mu^+) \leq \tau$ will be satisfied after at most $1+\lceil \log_2(\log_2 \frac{1}{\tau(1+4\kappa)}) \rceil$ centering steps.

In particular, in the current paper we change the definition for feasibility step by the following system

$$M\Delta^f x - \Delta^f s = \theta\nu r^0, \quad s\Delta^f x + x\Delta^f s = \beta xs, \tag{4}$$

where we first require $\beta > -1$, and we will give further requirement on β.

3. Analysis of the algorithm

Let (x, s) denote the starting point of an iteration and assume $\delta(x, s; \mu) \leq \tau$. Recall that this is certainly true at the start of the first iteration, because $\delta(x, s; \mu) = 0$.

3.1 Effect of the feasibility step

We already knew that the feasibility step generates new iterates (x^f, s^f) that satisfy the feasibility condition for (P_{ν^+}), except possibly the nonnegativity conditions. The crucial element in the analysis is to show that after the feasibility step $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, which guarantees the new iterates are within the region where the Newton process targeting at the μ^+-center of (P_{ν^+}) is quadratically convergent.

Defining

$$d_x^f := \frac{\upsilon\Delta^f x}{x}, \qquad d_s^f := \frac{\upsilon\Delta^f s}{s}, \qquad \text{where } \upsilon = \sqrt{\frac{xs}{\mu}}. \tag{5}$$

Then, the system (4) can be rewritten as

$$DMDd_x^f - d_s^f = \theta\nu\upsilon s^{-1}r^0, \qquad d_x^f + d_s^f = \beta\upsilon, \tag{6}$$

where $D := (S^{-1}X)^{\frac{1}{2}}, S := diag(s), X := diag(x)$. By using (4) and (5), we have

$$\begin{aligned} x^f s^f &= xs + (s\Delta^f x + x\Delta^f s) + \Delta^f x\Delta^f s \\ &= (1+\beta)xs + \Delta^f x\Delta^f s = (1+\beta)\mu\upsilon^2 + \mu d_x^f d_s^f. \end{aligned} \tag{7}$$

Lemma 2 (Lemma 5 in [17]). *The iterates (x^f, s^f) are strictly feasible if and only if $(1+\beta)\upsilon^2 + d_x^f d_s^f \geq 0$.*

Lemma 3 (Lemma 2.2 in [15]). *The iterates* (x^f, s^f) *are certainly strictly feasible if* $\|d_x^f\| < \frac{1}{\rho(\delta)}$ *and* $\|d_s^f\| < \frac{1}{\rho(\delta)}$, *where* $\rho(\delta) := \delta + \sqrt{1+\delta^2}$.

The proof of Lemma 3 in [15] makes clear that the elements of the vector v satisfy

$$\frac{1}{\rho(\delta)} \leq v_i \leq \rho(\delta), \qquad i = 1, ..., n. \tag{8}$$

In the following, let us denote

$$\omega_i := \omega_i(v) := \frac{1}{2}\sqrt{|d_{xi}^f|^2 + |d_{si}^f|^2} \text{ and } \omega := \omega(v) := \|(\omega_1, ..., \omega_n)\|.$$

We can get

$$\begin{aligned} (d_x^f)^T d_s^f &\leq \|d_x^f\|\|d_s^f\| \leq \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) \leq 2\omega^2, \\ \|d_{xi}^f d_{si}^f\| &= \|d_{xi}^f\|\|d_{si}^f\| \leq \frac{1}{2}(\|d_{xi}^f\|^2 + \|d_{si}^f\|^2) \leq 2\omega_i^2 \leq 2\omega^2, \quad 1 \leq i \leq n. \end{aligned}$$

Assuming $(1+\beta)v^2 + d_x^f d_s^f \geq 0$, which according to Lemma 2 holds if and only if the iterates (x^f, s^f) are strictly feasible, we proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. According to definition (2) it holds that

$$\delta(x^f, s^f; \mu^+) = \frac{1}{2}\|v^f - \frac{e}{v^f}\|, \qquad \text{where} \quad v^f = \sqrt{\frac{x^f s^f}{\mu^+}}. \tag{9}$$

For the convenience of analysis, in the sequel $\delta(x^f, s^f; \mu^+)$ and $\delta(x, s; \mu)$ will be denoted shortly by $\delta(v^f)$ and $\delta(v)$, respectively.

Lemma 4 (Lemma 7 in [17]). *Assuming* $(1+\beta)v^2 + d_x^f d_s^f > 0$, *one has*

$$4\delta^2(v^f) \leq \frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\omega^2\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)\omega^2\big)}.$$

Because we need to have $\delta(v^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, it follows from Lemma 4 that it suffices if

$$\frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\omega^2\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)\omega^2\big)} \leq \frac{2}{(1+4\kappa)^2}.$$

Now, let us define the function

$$\psi(t) = \frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2t}{1-\theta} + \frac{2(1-\theta)t\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)t\big)} - \frac{2}{(1+4\kappa)^2},$$

and the other two functions

$$\begin{aligned}\psi_1(t) &= \frac{4(1-\theta)\delta^2}{1+\beta}+\frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)}+\frac{2t}{1-\theta}-\frac{3}{2(1+4\kappa)^2},\\ \psi_2(t) &= \frac{2(1-\theta)t\rho^4(\delta)}{(1+\beta)\big((1+\beta)-2\rho^2(\delta)t\big)}-\frac{1}{2(1+4\kappa)^2}.\end{aligned}$$

Note that $\psi(t)=\psi_1(t)+\psi_2(t)$, and both $\psi_1(t)$ and $\psi_2(t)$ are monotone increasing to t. By applying Lemma A5, the root t^* of $\psi(t)$ satisfies: $t^*\geq\min\{t_1^*,t_2^*\}$, where t_1^*,t_2^* are respectively the roots of $\psi_1(t)=0,\psi_2(t)=0$, i.e.,

$$\begin{aligned}t_1^* &= \Big(\frac{3}{2(1+4\kappa)^2}-\frac{4(1-\theta)\delta^2}{1+\beta}-\frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)}\Big)\frac{(1-\theta)}{2},\\ t_2^* &= \frac{(1+\beta)^2}{4(1-\theta)(1+4\kappa)^2\rho^4(\delta)+2(1+\beta)\rho^2(\delta)}.\end{aligned}$$

At this stage, we decide to choose

$$0<\tau\leq\frac{1}{8(1+4\kappa)},\quad 0<\theta\leq\frac{\alpha}{2\sqrt{2n}(1+4\kappa)},\quad -\frac{1}{2}\leq\beta\leq 0,\quad \alpha\leq 1.$$

It is easy to check that $t_1^*\geq\frac{25}{64(1+4\kappa)^2}$ and $t_2^*\geq\frac{1}{57(1+4\kappa)^2}$. Furthermore, from the proof of Lemma A6, one can easily verify that if

$$\omega^2\leq\min\{\frac{25}{64(1+4\kappa)^2},\frac{1}{57(1+4\kappa)^2},\frac{2}{13}\}=\frac{1}{57(1+4\kappa)^2},\tag{10}$$

namely, $w<\frac{1}{\sqrt{57}(1+4\kappa)}$, then $\delta(v^f)\leq\frac{1}{\sqrt{2}(1+4\kappa)}$.

3.2 Upper bound for $\|d_x^f\|^2+\|d_s^f\|^2$

Now we are ready to derive an upper bound for $\|d_x^f\|^2+\|d_s^f\|^2$. We first give the following Lemma.

Lemma 5. *Let $x>0$ and $s>0$ be two n-dimensional vectors, and let $M\in R^{n\times n}$ be a $P_*(\kappa)$-matrix. Then the solution (u,z) of the linear system*

$$DMDu-z=a,\quad u+z=b,\tag{11}$$

satisfies the following relations

$$\|u\|\leq\sqrt{1+2\kappa}\|a+b\|,\tag{12}$$

$$\|u\|^2+\|z\|^2\leq\|b\|^2+2\kappa\|a+b\|^2+2\sqrt{1+2\kappa}\|a+b\|\cdot\|a\|,\tag{13}$$

where $D=(S^{-1}X)^{\frac{1}{2}}$.

According to Lemma 5, by replacing u, z, a and b with $d_x^f, d_z^f, \theta\nu\upsilon s^{-1}r^0$ and $\beta\upsilon$, respectively, the inequality (13) becomes

$$\begin{aligned}(14)\qquad \|d_x^f\|^2+\|d_s^f\|^2 &\leq \|\beta\upsilon\|^2+2\kappa\|\theta\nu\upsilon s^{-1}r^0+\beta\upsilon\|^2\\ &\quad +2\sqrt{1+2\kappa}\|\theta\nu\upsilon s^{-1}r^0+\beta\upsilon\|\|\theta\nu\upsilon s^{-1}r^0\|.\end{aligned}$$

To proceed, we have to specify our initial iterates (x^0, s^0). We assume that ρ_p and ρ_d satisfy

$$(15)\qquad \|x^*\|_\infty \leq \rho_p, \quad \max\{\|s^*\|_\infty, \rho_p\|Me\|_\infty, \|q\|_\infty\} \leq \rho_d,$$

for some $(x^*, s^*) \in \mathcal{F}^*$. As usual, we initialize the algorithm with $x^0 = \rho_p e, s^0 = \rho_d e, \mu^0 = \rho_p\rho_d$. For such starting points, we have

$$(16)\qquad \|\theta\nu\upsilon s^{-1}r^0\| \leq \frac{3\theta}{\rho_p\upsilon_{\min}}\|x\|_1.$$

By using (16) and (2) in (14), we get

$$\begin{aligned}\|d_x^f\|^2+\|d_s^f\|^2 &\leq \beta^2 n(1+2\delta)^2+2\kappa\Big(\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1+\beta\sqrt{n}(1+2\delta)\Big)^2\\ (17)\qquad &\quad +2\sqrt{1+2\kappa}\Big(\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1+\beta\sqrt{n}(1+2\delta)\Big)\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1.\end{aligned}$$

To specify a bound for $\|d_x^f\|^2+\|d_s^f\|^2$, we need an upper bound for $\|x\|_1$.

Lemma 6. (Lemma 4.5 in [19]) *Let (x, s) be feasible for the perturbed problem (P_ν), $\rho(\delta)$ as defined in Lemma 3. Then*

$$(18)\quad \|x\|_1 \leq (1+4\kappa)n\rho_p\big(2+\rho^2(\delta)\big), \quad \|s\|_1 \leq (1+4\kappa)n\rho_p\big(2+\rho^2(\delta)\big).$$

By substituting (18) into (17), we get

$$\begin{aligned}\|d_x^f\|^2+\|d_s^f\|^2 \leq f(\delta) &= \beta^2 n(1+2\delta)^2+2\kappa\Big(3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big)\\ &\quad +\beta\sqrt{n}(1+2\delta)\Big)^2+2\sqrt{1+2\kappa}\Big(3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big)\\ (19)\qquad &\quad +\beta\sqrt{n}(1+2\delta)\Big)3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big).\end{aligned}$$

We have found that $\delta(\upsilon^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$ holds if the inequality (10) is satisfied. Then by (19), inequality (10) holds if $f(\delta) \leq \frac{4}{57(1+4\kappa)^2}$ holds. To proceed, we give the values of θ and τ by the following Lemma.

Lemma 7. *Let $\delta = \delta(\upsilon)$ be given by (2), for $n \geq 2$, $\rho(\delta) = \delta+\sqrt{1+\delta^2}$ and $\delta \leq \tau$, if $\theta = \frac{1}{52n(1+4\kappa)^{\frac{5}{2}}}$, $\tau = \frac{1}{50(1+4\kappa)^{\frac{3}{2}}}$ and $-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$. Then $f(\delta) \leq \frac{4}{57(1+4\kappa)^2}$.*

3.3 Complexity analysis

In the previous sections, we have suggested that if at the start of an iteration the iterates satisfy $\delta(x, s; \mu) \leq \tau$, then after the feasibility step, the iterates satisfied $\delta(v^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$. Hence, after the feasibility step, the proposed algorithm at most

$$1 + \lceil \log_2(\log_2 \frac{1}{\tau(1+4\kappa)}) \rceil = 1 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}})) \rceil$$

centering steps are sufficient to get an iterate (x^+, s^+) that satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. Each main iteration consists of one feasibility step and at most

$$1 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}})) \rceil$$

centering steps. After the main iteration, both the norms of the residual vectors and the duality gap are reduced by the factor $1 - \theta$. So the total number of main iterations is bounded above by $\frac{1}{\theta} \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}$. If we take the value of θ as in Lemma 7, then the total number of inner iterations is bounded above by

$$52(2 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}})) \rceil)(1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}.$$

Theorem 2. *If problem (P) has optimal solution (x^*, s^*) such that $\|x^*\|_\infty \leq \rho_p$ and $\max\{\|s^*\|_\infty, \rho_p \|Me\|_\infty, \|q\|_\infty\} \leq \rho_d$, then after at most*

$$52(2 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}})) \rceil)(1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}$$

iterations the algorithm finds an ε-solution of $P_*(\kappa)$-LCP.

4. Numerical results

In this section, we present numerical results of the full-Newton step IIPM for $P_*(\kappa)$-LCP under the MatlabR2012a environment.

We consider the following problems. It should be pointed out that Problems 5.1-5.2 are $P_*(0)$-LCPs, which are important subclasses of $P_*(\kappa)$-LCP.

Problem 5.1 (Fathi's example in [7])**.**

$$M = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -3 \\ 1 \\ -1 \\ 5 \\ 4 \\ -1.5 \end{bmatrix}.$$

Problem 5.2 (Fathi's example in [7])

$$M = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}.$$

These problems are solved by applying the proposed algorithm. We take $\varepsilon = 10^{-5}$ as the accuracy parameter for all problems in the implementation. With an arbitrary positive point being initial point (x^0, s^0), the number of inner iterates (IIN) and main iterates (MIN) for combinations of the update parameters θ, value of β and the dimension of the problems 5.1-5.2 are shown in Tables 1-2, respectively. Here, we should notice that β lies in different interval for the problem with different dimension. In the following numerical results, we always choose β for three different values, which are the endpoint and midpoint of the interval corresponding to different dimension.

Table 1: The number of iterations for the problems 5.1.

x^0	s^0	DIM	β	θ	MIN	IIN
$(.514, \cdots, .514)$	$(.594, \cdots, .594)$	7	-.0222	.0027	4176	8354
			-.0111	.0027	4176	6266
			0	.0027	4176	4874

Table 2: The number of iterations for the problems 5.2.

x^0	s^0	DIM	β	θ	MIN	IIN
$(.29, .29)$	$(.729, .729)$	2	-.0416	.0096	1031	2062
			-.0208	.0096	1031	1374
			0	.0096	1031	1374
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	3	-.0340	.0064	1549	3098
			-.0170	.0064	1549	2065
			0	.0064	1549	1936
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	4	-.0294	.0048	2067	4134
			-.0147	.0048	2067	2756
			0	.0048	2067	2480
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	5	-.0263	.0038	2584	5168
			-.0132	.0038	2584	3876
			0	.0038	2584	3100
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	6	-.0240	.0032	3102	6204
			-.0120	.0032	3102	4653
			0	.0032	3102	3619

5. Concluding remarks

In this paper, we propose a full-Newton step IIPM based on new directions for $P_*(\kappa)$-LCP. The new directions are defined by equation $s\Delta^f x + x\Delta^f s = \beta xs$, where $-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$. The iteration bound coincides with the currently best known iteration bound of IIPMs for $P_*(\kappa)$-LCP.

For further research, this algorithm may be possible extended to the Cartesian $P_*(\kappa)$-LCP over symmetric cones.

Also, in this paper, the case $-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$ is discussed, a larger range of β will be explored.

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Some identities and generating functions of third-order recurrence relations

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Abstract. In this paper, we introduce a new generating functions for the product of Narayana numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers by making use of the symmetrizing endomorphism operators $\delta_{a_1 a_2}^k$ to the series $\sum_{n=0}^{\infty} S_n(E)\, a_1^n z^n$.
Keywords: Narayana numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, generating functions.

1. Introduction and notations

Many sequences of numbers can be defined by second-order recurrence relation, such as the Fibonacci sequence, Pell sequence, Lucas sequence, Jacobsthal and Jacobsthal-Lucas sequence. In particular, the Jacobsthal and the Jacobsthal-Lucas sequences are defined by [12]:

$$J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0,\ J_0 = 0,\ J_1 = 1,$$
$$j_{n+2} = j_{n+1} + 2j_n,\ n \geq 0,\ j_0 = 2,\ j_1 = 1,$$

respectively.

There are many studies about these two kinds of sequences, for examples. In [11], Horadam discussed the properties and has given the associated sequence with Jacobsthal numbers [12] and [13]. Moreover, in [15] Köken and Bozkurt have given some matrix properties of Jacobsthal and Jacobsthal-Lucas numbers.

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Consequently, Yilmaz and Bozkurt defined k-Jacobsthal numbers and described Binet's formula for the same [18].

In [10] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by Horadam [12] is expanded and extended to several identities for some of the higher order cases. Furthermore, the authors generalized the Jacobsthal recursion as

$$J_{n+r}^{(r)} = \sum_{s=1}^{r-1} J_{n+r-s}^{(r)} + 2J_n^{(r)},\ n \geq 0,$$

with initial conditions $J_0 = 0$ and $J_s = 1$ for $s = 1, ..., r-1$. For the n-th order-r Jacobsthal Lucas numbers $j_n^{(r)}$ we use the same recursion with initial conditions $j_s^{(r)} = j_s^{(r-1)}$ for $s = 1, ..., r-1$.

In particular, for $r = 3$, the third-order Jacobsthal numbers $\{J_n^{(r)}\}_{n\geq 0}$, are defined by recurrence relation:

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\quad n \geq 0,$$

with initial conditions $J_0^{(3)} = 0$ and $J_1^{(3)} = J_2^{(3)} = 1$ (see [10]).

Also the third-order Jacobsthal-Lucas numbers $\{j_n^{(3)}\}_{n\geq 0}$, is defined by using the following recurrence relation $j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\quad n \geq 0$, with $j_0^{(3)} = 2$, $j_1^{(3)} = 1$ and $j_2^{(3)} = 5$ (see [10]).

In this part, we define k-Fibonacci numbers and k-Jacobsthal, k-Jacobsthal Lucas numbers and Chebyshev polynomials of the third and fourth kinds.

Definition 1.1 ([8]). *For any positive real number k, the k-Fibonacci numbers, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 1$; $F_{k,1} = 1$.*

Definition 1.2 ([17]). *The Narayana numbers is defined as follows $b_{n+1} = b_n + b_{n-2}$, $n \geq 2$, $b_0 = 0$, $b_1 = 1$, $b_2 = 1$.*

Definition 1.3 ([14]). *For any positive real number k, the k-Jacobsthal numbers say $\{J_{k,n}\}_n$ is defined recurrently by $J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}$, for $n \geq 1$, with initial conditions $J_{k,0} = 0$; $J_{k,1} = 1$.*

Definition 1.4 ([14]). *For any positive real number k, the k-Jacobsthal -Lucas numbers say $\{j_{k,n}\}_n$ is defined recurrently by $j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$, for $n \geq 1$, with initial conditions $j_{k,0} = 2$, $j_{k,1} = k$.*

Definition 1.5 ([16]). *The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are respectively defined by the recurrence relations as follows:*

$$V_{n+2}(x) = 2xV_{n+1}(x) - V_n(x),\quad n \geq 0, V_0(x) = 1,\ V_1(x) = 2x - 1,$$
$$W_{n+2}(x) = 2xW_{n+1}(x) - W_n(x),\quad n \geq 0, W_0(x) = 1,\ W_1(x) = 2x + 1.$$

Let us now start at the following definition.

Definition 1.6. *Let E and A be any two alphabets, then we give $S_n(A-E)$ by the following form:*

$$\frac{\prod_{e\in E}(1-ez)}{\prod\limits_{a\in A}(1-az)}=\sum_{n=0}^{\infty}S_n(A-E)\,z^n, \tag{1.1}$$

with the condition $S_n(A-E)=0$ for $n<0$ (see [1]).

Equation (1.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty}S_n(A-E)\,z^n=\left(\sum_{n=0}^{\infty}S_n(A)\,z^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-E)\,z^n\right),$$

where

$$S_n(A-E)=\sum_{i=0}^{n}S_i(A)\,S_{n-i}(-E).$$

Remark 1.1. Taking $A=\{0,0,...,0\}$ in (1.1) gives

$$\sum_{n=0}^{\infty}S_n(-E)\,z^n=\prod_{e\in E}(1-ez). \tag{1.2}$$

Definition 1.7 ([5])**.** *Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{x_ix_{i+1}}f(x)=\frac{f(x_1,...,x_i,x_{i+1},...,x_n)-f(x_1,...,x_{i-1},x_{i+1},x_i,...,x_n)}{x_i-x_{i+1}}.$$

Definition 1.8. *Let n be positive integer and $E=\{e_1,e_2\}$ are set of given variables, then, the n-th symmetric function $S_n(e_1+e_2)$ is defined by*

$$S_n(E)=S_n(e_1+e_2)=\frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2},$$

with

$$\begin{aligned}S_0(E)&=S_0(e_1+e_2)=1,\\ S_1(E)&=S_1(e_1+e_2)=e_1+e_2,\\ S_2(E)&=S_2(e_1+e_2)=e_1^2+e_1e_2+e_2^2,\\ &\vdots\end{aligned}$$

Definition 1.9 ([5])**.** *Given an alphabet $E=\{e_1,e_2\}$, the symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{e_1e_2}f(e_1)=\frac{e_1^kf(e_1)-e_2^kf(e_2)}{e_1-e_2},\ k\in\mathbb{N}_0. \tag{1.3}$$

If $f(e_1)=e_1^n$, the operator (1.3) gives us

$$\delta^k_{e_1e_2}(e_1^n)=\frac{e_1^{k+n}-e_2^{k+n}}{e_1-e_2}=S_{k+n-1}(e_1+e_2).$$

2. Principal formulas

In this section, we provide a new theorem in order to derive some new generating functions of the products of some known numbers .

Theorem 2.1. *Let A and E be two alphabets, respectively, $\{e_1, e_2, e_3\}$ and $\{a_1, a_2\}$, then we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} S_n(E)\, S_{n+k}(A)\, z^n \\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^{k+1}(a_2^n)\, z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n\right)},\ k \in \mathbb{N}_0. \end{aligned} \tag{2.1}$$

Proof. Applying the operator $\delta_{a_1a_2}^k$ to the series $f(a_1z) = \sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n$, we have

$$\begin{aligned} \delta_{a_1a_2}^k f(a_1z) &= \delta_{a_1a_2}^k \left(\sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n\right) \\ &= \frac{\sum_{n=0}^{+\infty} S_n(E)a_1^{n+k+1}z^n - \sum_{n=0}^{+\infty} S_n(E)a_2^{n+k+1}z^n}{a_1 - a_2} \\ &= \sum_{n=0}^{+\infty} S_n(E)\frac{a_1^{n+k+1} - a_2^{n+k+1}}{a_1 - a_2}z^n \\ &= \sum_{n=0}^{\infty} S_n(E)\, S_{n+k}(A)\, z^n. \end{aligned}$$

On the other part, since $\sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n = \frac{a_1}{\prod_{e\in E}(1-ea_1z)}$ we have

$$\begin{aligned} \delta_{a_1a_2}^k f(a_1z) &= \delta_{a_1a_2}^k f\left(\frac{a_1}{\prod_{e\in E}(1-ea_1z)}\right) = \frac{\frac{a_1^{k+1}}{\prod_{e\in E}(1-ea_1z)} - \frac{a_2^{k+1}}{\prod_{e\in E}(1-ea_2z)}}{a_1 - a_2} \\ &= \frac{a_1^{k+1}\prod_{e\in E}(1-ea_2z) - a_2^{k+1}\prod_{e\in E}(1-ea_1z)}{(a_1-a_2)\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1-ea_2z)}. \end{aligned}$$

Using the fact that: $\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n = \prod_{e\in E}(1-ea_1z)$, then

$$\begin{aligned} \delta_{a_1a_2}^k f(a_1z) &= \frac{a_1^{k+1}\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n - a_2^{k+1}\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n}{(a_1-a_2)\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n\right)} \\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\frac{a_1^{k+1}a_2^n - a_2^{k+1}a_1^n}{a_1-a_2}z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n\right)} \\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^{k+1}(a_2^n)\, z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n\right)}. \end{aligned}$$

This completes the proof. □

3. On the symmetric functions of some numbers and polynomials

In this section, we now derive the new generating functions of the products of some known numbers. For the applications of generating functions of some known functions, we refer the reader to see the references [2, 3, 4, 6, 7, 8], we present some applications for the previous theorem if $k = 0, 1$.

3.1 The case of $E = \{e_1, e_2, 0\}, A = \{1, 0\}$

Lemma 3.1 ([5]). *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(e_1 + e_2) z^n = \frac{1}{(1 - e_1 z)(1 - e_2 z)}. \tag{3.1}$$

From (3.1) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(e_1 + e_2) z^n = \frac{z}{(1 - e_1 z)(1 - e_2 z)}. \tag{3.2}$$

By replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (3.1) and (3.2) and assuming that $e_1 - e_2 = x$ and $4e_1 e_2 = -1$, we obtain

$$\sum_{n=0}^{\infty} S_n(2e_1 + [-2e_2]) z^n = \frac{1}{1 - 2xz + z^2}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{z}{1 - 2xz + z^2}, \tag{3.4}$$

respectively.

Subtracting the equation (3.4) from (3.3), we obtain the following proposition and corollary.

Proposition 3.1. *For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} V_n(x)\ z^n = \frac{1 - z}{1 - 2xz + z^2}. \tag{3.5}$$

Corollary 3.1. *The following identity hold true:*

$$V_n(x) = S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2]).$$

By added the equation (3.3) to (3.4), we obtain the following proposition.

Proposition 3.2. *For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of the fourth kind is given by*

$$\sum_{n=0}^{\infty} W_n(x)\ z^n = \frac{1 + z}{1 - 2xz + z^2}. \tag{3.6}$$

By comparing the coefficients z^n on both sides of (3.6) , we have the following corollary.

Corollary 3.2. *The following identity hold true* $W_n(x) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2])$.

3.2 The case of $E = \{e_1, e_2, e_3\}, A = \{1, 0\}$

Lemma 3.2 ([9])**.** *Given an alphabet* $E = \{e_1, e_2, e_3\}$, *we have*

$$\sum_{n=0}^{\infty} S_n(E) z^n = \frac{1}{(1 - e_1 z)(1 - e_2 z)(1 - e_3 z)}. \tag{3.7}$$

From (3.7) *we can deduce*

$$\sum_{n=0}^{\infty} S_{n-1}(E) z^n = \frac{z}{(1 - e_1 z)(1 - e_2 z)(1 - e_3 z)}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E) z^n = \frac{z^2}{(1 - e_1 z)(1 - e_2 z)(1 - e_3 z)}, \tag{3.9}$$

with $(1 - e_1 z)(1 - e_2 z)(1 - e_3 z) = 1 - (e_1 + e_2 + e_3)z + (e_1 e_2 + e_1 e_3 + e_2 e_3)z^2 - e_1 e_2 e_3 z^3 = 1 + S_1(-E) z + S_2(-E) z^2 + S_3(-E) z^3$.

Setting $\begin{cases} S_1(-E) = -1, \\ S_2(-E) = 0, \\ S_3(-E) = -1, \end{cases}$ in (3.8) , we obtain the following proposition and corollary.

Proposition 3.3. *For* $n \in \mathbb{N}$, *the generating function of Narayana numbers is given by*

$$\sum_{n=0}^{\infty} b_n z^n = \frac{z}{1 - z - z^3}.$$

Corollary 3.3. *The following identity hold true:*

$$b_n = S_{n-1}(e_1 + e_2 + e_3).$$

Setting $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1 \\ S_2(-E) = -2 \end{cases}$ in (3.7) , (3.8) and (3.9) this gives

$$\sum_{n=0}^{\infty} S_{n-1}(E) z^n = \frac{z}{1 - z - z^2 - 2z^3}, \tag{3.11}$$

which represents a generating function of the third order Jacobsthal numbers, such that $J_n^{(3)} = S_{n-1}(e_1 + e_2 + e_3)$ and

$$\sum_{n=0}^{\infty} S_n(E) z^n = \frac{1}{1 - z - z^2 - 2z^3}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E) z^n = \frac{z^2}{1 - z - z^2 - 2z^3}, \tag{3.13}$$

which represents a new generating functions respectively.

Multiplying the equation (3.12) by 2 and added to (3.11) by (-1) and also added to (3.13) by 2, we obtain the following proposition and corollary.

Proposition 3.4. *For $n \in \mathbb{N}$, the generating function of the third order Jacobsthal-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} j_n^{(3)} z^n = \frac{2 - z + 2z^2}{1 - z - z^2 - 2z^3}.$$

Corollary 3.4. *The following identity hold true:*

$$j_n^{(3)} = 2S_n(e_1 + e_2 + e_3) - S_{n-1}(e_1 + e_2 + e_3) + 2S_{n-2}(e_1 + e_2 + e_3).$$

3.3 The case of $E = \{e_1, e_2, e_3\}$, $A = \{a_1, a_2\}$

Lemma 3.3 ([4])**.** *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(E) S_n(A) z^n = \frac{1 - a_1 a_2 S_2(-E) z^2 - a_1 a_2 (a_1 + a_2) S_3(-E) z^3}{\prod_{e \in E} (1 - e a_1 z) \prod_{e \in E} (1 - e a_2 z)}. \tag{3.14}$$

From (3.14) *we can deduce*

$$\begin{aligned} &\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(A) z^n \\ &= \frac{z - a_1 a_2 S_2(-E) z^3 - a_1 a_2 (a_1 + a_2) S_3(-E) z^4}{\prod_{e \in E} (1 - e a_1 z) \prod_{e \in E} (1 - e a_2 z)}. \end{aligned} \tag{3.15}$$

Lemma 3.4 ([9])**.** *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(E) S_{n+1}(A) z^n = \frac{(a_1 + a_2) + a_1 a_2 S_1(-E) z - a_1^2 a_2^2 S_3(-E) z^3}{\prod_{e \in E} (1 - e a_1 z) \prod_{e \in E} (1 - e a_2 z)}. \tag{3.16}$$

From (3.16) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(A)z^n = \frac{(a_1+a_2)z + a_1a_2S_1(-E)z^2 - a_1^2a_2^2S_3(-E)z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1-ea_2z)}. \tag{3.17}$$

By replacing a_2 by $(-a_2)$ in (3.14), (3.15) and (3.17) we obtain, respectively, the following relationships

$$\begin{aligned}&\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])z^n\\ &= \frac{1+a_1a_2S_2(-E)z^2 + a_1a_2(a_1-a_2)S_3(-E)z^3}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)},\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])z^n\\ &= \frac{z+a_1a_2S_2(-E)z^3 + a_1a_2(a_1-a_2)S_3(-E)z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)},\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])z^n\\ &= \frac{(a_1-a_2)z - a_1a_2\ S_1(-E)z^2 - a_1^2a_2^2S_3(-E)z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)},\end{aligned} \tag{3.20}$$

with

$$\begin{aligned}&\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)\\ &= 1-(a_1-a_2)(e_1+e_2+e_3)z + ((e_1e_2+e_1e_3+e_2e_3)(a_1-a_2)^2\\ &- a_1a_2((e_1+e_2+e_3)^2 - 2(e_1e_2+e_1e_3+e_2e_3)))z^2 - (e_1e_2e_3(a_1-a_2)^3\\ &- a_1a_2(a_1-a_2)((e_1e_2+e_1e_3+e_2e_3)(e_1+e_2+e_3) - 3e_1e_2e_3))z^3\\ &+ (-a_1a_2(a_1-a_2)^2e_1e_2e_3(e_1+e_2+e_3) + a_1^2a_2^2((e_1e_2+e_1e_3+e_2e_3)^2\\ &- 2e_1e_2e_3(e_1+e_2+e_3)))z^4 - e_1e_2e_3a_1^2a_2^2(a_1-a_2)(e_1e_2+e_1e_3+e_2e_3)z^5\\ &- e_1^2e_2^2e_3^2a_1^3a_2^3z^6.\end{aligned}$$

This case consists of three related parts.

Firstly, the substitutions $\begin{cases}S_1(-E)=-1\\ S_2(-E)=0\\ S_3(-E)=-1\end{cases}$ and $\begin{cases}a_1-a_2=k\\ a_1a_2=1\end{cases}$ in (3.20),

we obtain the following proposition and corollary.

Proposition 3.5. *For $n\in\mathbb{N}$, the new generating function of the product of Narayana numbers and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} b_nF_{k,n}z^n = \frac{kz+z^2+z^4}{1-kz-z^2-(k^3+3k)z^3-(k^2+2)z^4-z^6}.$$

Corollary 3.5. *The following identity hold true:*

$$b_n F_{k,n} = S_{n-1}(E)\, S_n(a_1 + [-a_2]).$$

Secondly, the substitutions $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = 0 \\ S_3(-E) = -1 \end{cases}$ and $\begin{cases} a_1 - a_2 = k \\ a_1 a_2 = 2 \end{cases}$ in (3.19),

we deduce the following proposition, corollary and theorem.

Proposition 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of Narayana numbers and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} b_n J_{k,n} z^n = \frac{z - 2kz^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}. \tag{3.22}$$

Corollary 3.6. *The following identity hold true:*

$$b_n J_{k,n} = S_{n-1}(E)\, S_{n-1}(a_1 + [-a_2]).$$

Theorem 3.1. *For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal-Luca numbers and Narayana numbers is given by*

$$\sum_{n=0}^{\infty} b_n j_{k,n} z^n = \frac{kz + 4z^2 + (2k^2 + 8)z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}. \tag{3.24}$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} b_n j_{k,n} z^n &= \sum_{n=0}^{\infty} S_{n-1}(E)\left(2S_n(a_1 + [-a_2]) - kS_{n-1} + [-a_2])\right) z^n \\ &= \sum_{n=0}^{\infty} 2S_{n-1}(E) S_n(a_1 + [-a_2]) z^n - k \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(a_1 + [-a_2]) z^n \\ &= 2\sum_{n=0}^{\infty} S_{n-1}(E) S_n(a_1 + [-a_2]) z^n - k \sum_{n=0}^{\infty} b_n J_{k,n} z^n. \end{aligned}$$

By using

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_n(a_1 + [-a_2]) z^n = \frac{kz + 2z^2 + 4z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}.$$

We obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b_n j_{k,n} z^n &= \frac{2kz + 4z^2 + 8z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6} \\ &- k\frac{z - 2kz^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6} \\ &= \frac{kz + 4z^2 + (2k^2 + 8)z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}. \end{aligned}$$

This completes the proof. □

Thirdly, the substitutions $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1 \\ S_3(-E) = -2 \end{cases}$ and $\begin{cases} a_1 - a_2 = 2 \\ a_1 a_2 = k \end{cases}$ in (3.19),

we deduce the following proposition, corollary and theorem.

Proposition 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of third order Jacobsthal numbers and k-Pell numbers numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n = \frac{z - kz^3 - 4kz^4}{1 - 2z - (4+3k)z^2 - (16+14k)z^3 + (-8k-3k^2)z^4 + 4k^2 z^5 - 4k^3 z^6}. \tag{3.25}$$

Corollary 3.7. *The following identity hold true:*

$$J_n^{(3)} P_{k,n} = S_{n-1}(E) S_{n-1}(a_1 + [-a_2]).$$

Theorem 3.2. *For $n \in \mathbb{N}$, the new generating function of the product of third order Jacobsthal- Lucas numbers and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n = \frac{z + 8z^2 + (16+7k) z^3 + 4kz^4 + 4k^2 z^5}{1 - 2z - (4+3k)z^2 - (16+14k)z^3 + (-8k-3k^2)z^4 + 4k^2 z^5 - 4k^3 z^6}. \tag{3.26}$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n = \sum_{n=0}^{\infty} \left(2S_n(E) - S_{n-1}(E) + 2S_{n-2}(E)\right) S_{n-1}(a_1 + [-a_2]) z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E) S_{n-1}(a_1 + [-a_2]) z^n - \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(a_1 + [-a_2]) z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E) S_{n-1}(a_1 + [-a_2]) z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E) \frac{a_1^n - (-a_2)^n}{a_1 + a_2} z^n - \sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E) \frac{a_1^n - (-a_2)^n}{a_1 + a_2} z^n \\
&= \frac{2}{a_1 + a_2} \left(\sum_{n=0}^{\infty} S_n(E) a_1^n z^n - \sum_{n=0}^{\infty} S_n(E) (-a_2)^n z^n \right) - \sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n \\
&+ \frac{2}{a_1 + a_2} \left(\sum_{n=0}^{\infty} S_{n-2}(E) a_1^n z^n - \sum_{n=0}^{\infty} S_{n-2}(E) (-a_2)^n z^n \right).
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} S_n\left(e_1+e_2+e_3\right) z^n = \frac{1}{1-z-z^2-2z^3},$$
$$\sum_{n=0}^{\infty} S_{n-2}\left(e_1+e_2+e_3\right) z^n = \frac{z^2}{1-z-z^2-2z^3},$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n \\
&= \frac{2}{a_1+a_2}\left(\frac{1}{1-a_1z-a_1^2z^2-2a_1^3z^3} - \frac{1}{1+a_2z-a_2^2z^2+2a_2^3z^3}\right) \\
&- \sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n + \frac{2}{a_1+a_2}\left(\frac{a_1^2z^2}{1-a_1z-a_1^2z^2-2a_1^3z^3}\right. \\
&\left. - \frac{a_2^2z^2}{1+a_2z-a_2^2z^2+2a_2^3z^3}\right) \\
&= \frac{2z+4z^2+4\left(4+k\right)z^3}{1-2z-(4+3k)z^2-(16+14k)z^3+(-8k-3k^2)z^4+4k^2z^5-4k^3z^6} \\
&- \frac{z-kz^3-4kz^4}{1-2z-(4+3k)z^2-(16+14k)z^3+(-8k-3k^2)z^4+4k^2z^5-4k^3z^6} \\
&+ \frac{4z^2+2kz^3+4k^2z^5}{1-2z-(4+3k)z^2-(16+14k)z^3+(-8k-3k^2)z^4+4k^2z^5-4k^3z^6} \\
&= \frac{z+8z^2+\left(16+7k\right)z^3+4kz^4+4k^2z^5}{1-2z-(4+3k)z^2-(16+14k)z^3+(-8k-3k^2)z^4+4k^2z^5-4k^3z^6}.
\end{aligned}$$

This completes the proof. □

Replacing a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.15) and (3.17), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1+[-2a_2]) z^n \\
&= \frac{z+4a_1a_2S_2\left(-E\right)z^3+8a_1a_2(a_1-a_2)S_3\left(-E\right)z^4}{\prod_{e\in E}(1-2ea_1z)\prod_{e\in E}(1+2ea_2z)}, \qquad (3.27)
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1+[-2a_2]) t^n \\
&= \frac{2\left(a_1-a_2\right)z-4a_1a_2S_1\left(-E\right)z^2-16a_1^2a_2^2S_3\left(-E\right)z^4}{\prod_{e\in E}(1-2ea_1z)\prod_{e\in E}(1+2ea_2z)}, \qquad (3.28)
\end{aligned}$$

respectively, with

$$\prod_{e\in E}(1-2ea_1z)\prod_{e\in E}(1+2ea_2z)$$

$$= 1 - 2(a_1 - a_2)(e_1 + e_2 + e_3)z + (4(e_1e_2 + e_1e_3 + e_2e_3)(a_1 - a_2)^2$$
$$- 4a_1a_2((e_1 + e_2 + e_3)^2 - 2(e_1e_2 + e_1e_3 + e_2e_3)))z^2 - (8e_1e_2e_3(a_1 - a_2)^3$$
$$- 8a_1a_2(a_1 - a_2)((e_1e_2 + e_1e_3 + e_2e_3)(e_1 + e_2 + e_3) - 3e_1e_2e_3))z^3$$
$$+ (-16a_1a_2(a_1 - a_2)^2e_1e_2e_3(e_1 + e_2 + e_3) + 16a_1^2a_2^2((e_1e_2 + e_1e_3 + e_2e_3)^2$$
$$- 2e_1e_2e_3(e_1 + e_2 + e_3)))z^4 - 32e_1e_2e_3a_1^2a_2^2(a_1 - a_2)(e_1e_2 + e_1e_3 + e_2e_3)z^5$$
$$- 64e_1^2e_2^2e_3^2a_1^3a_2^3z^6.$$

This case consists two related parts.

Firstly, by making the following restrictions: $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = 0 \\ S_3(-E) = -1 \end{cases}$ and

$\begin{cases} a_1 - a_2 = x \\ 4a_1a_2 = -1 \end{cases}$ in (3.27) and (3.28), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(2a_1 + [-2a_2])z^n$$
$$= \frac{z + 2xz^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}, \tag{3.29}$$
$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(2a_1 + [-2a_2])z^n$$
$$= \frac{2xz - z^2 + z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}. \tag{3.30}$$

which represents a new generating functions respectively.

We deduce the following theorem.

Theorem 3.3. *For $n \in \mathbb{N}$, the new generating function of the product of Narayana numbers and Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} b_n V_n(x) z^n = \frac{(2x - 1)z - z^2 + (1 - 2x)z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}.$$

Proof. We have

$$\sum_{n=0}^{\infty} b_n V_n(x) z^n = \sum_{n=0}^{\infty} S_{n-1}(E)(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]))z^n$$
$$= \sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2])z^n$$
$$- \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2])z^n,$$

by using the relationships (3.29) and (3.30), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} b_n V_n(x) z^n &= \frac{2xz - z^2 + z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6} \\ &- \frac{z + 2xz^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6} \\ &= \frac{(2x-1)z - z^2 + (1-2x)z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}.\end{aligned}$$

This completes the proof. □

Secondly, the substitutions $\begin{cases} S_1(E) = -1 \\ S_2(E) = -1 \\ S_3(E) = -2 \end{cases}$ and $\begin{cases} a_1 - a_2 = x \\ 4a_1a_2 = -1 \end{cases}$ in (3.27) and (3.28), give

$$\begin{aligned}&\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2]) z^n \\ &= \frac{z + z^3 + 4xz^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}, \qquad (3.31) \\ &\sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2]) z^n \\ &= \frac{2xz - z^2 + 2z^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}. \qquad (3.32)\end{aligned}$$

which represents a new generating functions respectively.

We deduce the following theorem.

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of the product of the third order Jacobsthal numbers and the Chebyshev polynomials of the third kind is given by*

$$\begin{aligned}&\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n \\ &= \frac{(2x-1)z - z^2 - z^3 + (2-4x)z^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.\end{aligned}$$

Proof. We have

$$\begin{aligned}&\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n = \sum_{n=0}^{\infty} S_{n-1}(E)(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]) z^n \\ &= \sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2]) z^n - \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2]) z^n.\end{aligned}$$

By using (3.31) and (3.32) we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n \\
&= \frac{2xz - z^2 + 2z^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&- \frac{z + z^3 + 4xz^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&= \frac{(2x-1)z - z^2 - z^3 + (2-4x)z^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of the third order Jacobsthal-Lucas numbers and the Chebyshev polynomials of the third kind is given by*

$$\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} V_n(x) z^n \\
&= \frac{2 - (2x+1)z + (1 - 8x + 8x^2)z^2 + (7 + 4x - 16x^2)z^3 + (4x-4)z^4 - 4z^5}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} j_n^{(3)} V_n(x) z^n &= \sum_{n=0}^{\infty} (2S_n(E) - S_{n-1}(E) + 2S_{n-2}(E))(S_n(2a_1 + [-2a_2]) \\
&- S_{n-1}(2a_1 + [-2a_2])z^n \\
&= \sum_{n=0}^{\infty} 2S_n(E)(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]) \\
&- \sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n + \sum_{n=0}^{\infty} 2S_{n-2}(E)(S_n(2a_1 + [-2a_2]) \\
&- S_{n-1}(2a_1 + [-2a_2])z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E) S_n(2a_1 + [-2a_2])z^n - \sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n \\
&- 2\sum_{n=0}^{\infty} S_n(E) S_{n-1}(2a_1 + [-2a_2])z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E)(S_n(2a_1 + [-2a_2])z^n \\
&- 2\sum_{n=0}^{\infty} S_{n-2}(E)(S_{n-1}(2a_1 + [-2a_2])z^n
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n - \sum_{n=0}^{\infty} J_n^{(3)} V_n(x)\, z^n \\
&- 2\sum_{n=0}^{\infty} S_n(E)\, \frac{(2a_1)^n-(-2a_2)^n}{2a_1+2a_2} z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E)\, \frac{(2a_1)^{n+1}-(-2a_2)^{n+1}}{2a_1+2a_2} z^n \\
&- 2\sum_{n=0}^{\infty} S_{n-2}(E)\, \frac{(2a_1)^n-(-2a_2)^n}{2a_1+2a_2} z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n \\
&- \frac{1}{a_1+a_2}\left(\sum_{n=0}^{\infty} S_n(E)\,(2a_1)^n z^n - \sum_{n=0}^{\infty} S_n(E)\,(-2a_2)^n z^n\right) \\
&- \sum_{n=0}^{\infty} J_n^{(3)} V_n(x)\, z^n + \frac{1}{a_1+a_2}\left(2a_1\sum_{n=0}^{\infty} S_{n-2}(E)\,(2a_1)^n z^n\right. \\
&+ \left. 2a_2\sum_{n=0}^{\infty} S_{n-2}(E)\,(-2a_2)^n z^n\right) \\
&- \frac{1}{a_1+a_2}\left(\sum_{n=0}^{\infty} S_{n-2}(E)\,(2a_1)^n z^n - \sum_{n=0}^{\infty} S_{n-2}(E)\,(-2a_2)^n z^n\right),
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{n=0}^{\infty} S_n(e_1+e_2+e_3)\, z^n &= \frac{1}{1-z-z^2-2z^3}, \\
\sum_{n=0}^{\infty} S_{n-2}(e_1+e_2+e_3)\, z^n &= \frac{z^2}{1-z-z^2-2z^3},
\end{aligned}
$$

and by using

$$
\begin{aligned}
&\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n \\
&= \frac{1+z^2+4xz^3}{1-2xz+(-4x^2+3)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} V_n(x)\, z^n \\
&= \frac{2+2z^2+8xz^3}{1-2xz+(-4x^2+3)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2z + 4xz^2 + 4\left(4x^2-1\right)z^3}{1 - 2xz + \left(-4x^2+3\right)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&- \frac{(2x-1)z - z^2 - z^3 + (2-4x)z^4}{1 - 2xz + \left(-4x^2+3\right)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&+ \frac{2(4x^2-1)z^2 - 4xz^3 - 2z^4}{1 - 2xz + \left(-4x^2+3\right)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&- \frac{4xz^2 - 2z^3 + 4z^5}{1 - 2xz + \left(-4x^2+3\right)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&= \frac{2 - (2x+1)z + (1 - 8x + 8x^2)z^2 + (7 + 4x - 16x^2)z^3 + (4x-4)z^4 - 4z^5}{1 - 2xz + \left(-4x^2+3\right)z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.
\end{aligned}
$$

So, the proof is completed. □

4. Conclusion

In this paper, by making use of Theorem 2.1, we have derived some new generating functions for the products of Narayana numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers and Chebyshev polynomials of the third and fourth kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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On a new class of derivations on residuated lattices

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Abstract. In this paper, as a generalization of a derivation in a residuated lattice, the notion of an f-derivation of a residuated lattice is proposed, and some related properties of isotone (resp. contractive) f-derivations and ideal f-derivations are investigated. The properties of principal ideal f-derivations are also investigated. We obtain that the fixed point set of principal ideal f-derivations and their implicative f-derivation are order isomorphism. Finally, by using the fixed point set of principal ideal f-derivations, we give a characterization of Heyting algebras.

Keywords: residuated lattice, derivation, f-derivation, principal ideal f-derivation.

1. Introduction

Derivation is helpful to the research of structure and property in algebraic systems, which was introduced from analytic theory. They were many authors who studied derivations in various algebraic structures. In 1957, Posner [12] proposed the notion of derivations in a prime ring $(R, +, -)$. In 2004, Jun and Xin [8] applied the notion of derivations to BCI-algebras. Based on [8], Zhan and Liu [18] introduced the notion of f-derivations of BCI-algebras. In 2008, Xin et al. [17] proposed the concept of a derivation on a lattice $(L, \wedge, \vee)$. They studied some properties of derivations and characterized modular and distributive lattices by some special derivations. Based on [17], Çeven and Özturk [4] introduced the notion of an f-derivation on a lattice. In 2010, Alshehri [1] introduced the concept of derivations in MV-algebra, and some related properties are investigated. In 2016, Xiao and Liu [16] introduced the notion of derivations for a quantale. In the same year, He et al. [7] introduced the concept of derivations in a residuated lattice, and then they characterized some special types of residuated lattices in terms of derivations. In 2018, Rachůnek and Šalounová [13] introduced the

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concept of derivations on non-commutative generalization of MV-algebras, and a complete description of all derivations on any non-commutative generalization of MV-algebras was given. In the same year, Liang et al. [9] introduced the notions of derivations on EQ-algebras and obtained several special types of them. Further, Wang et al. [14] introduced the notion of derivations of commutative multiplicative semilattices, they investigated related properties of some particular derivations and gave some characterizations of regular derivations in commutative multiplicative semilattices.

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In this paper, as a generalization of a derivation in a residuated lattice, the notion of an f-derivation is introduced, and some related properties are investigated. As important results of f-derivations on residuated lattices, fixed point sets are characterized by f-derivations under some conditions.

This paper is organized as follows. In Section 2, we recall some concepts and results on residuated lattices. In Section 3, we investigate and characterize some particular f-derivations in residuated lattices. Also, we discuss some properties of the fixed point set for ideal f-derivations and some operations of the kernel for f-derivations. In Section 4, we investigate principal ideal f-derivations and discuss their implicative f-derivations. In particular, we characterize Heyting algebras in terms of principal ideal f-derivations.

2. Preliminaries

In this section, we recall some basic notions and results about residuated lattices, which will be used in the further.

Definition 2.1 ([15])**.** A residuated lattice is an algebraic structure $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following conditions:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(L, \otimes, 1)$ is a commutative monoid,

(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$.

In what follows, we denote by L a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, unless otherwise specified.

For any $x \in L$ and a natural number n, we define $x' = x \rightarrow 0, x'' = (x')' = (x \rightarrow 0) \rightarrow 0, x^0 = 1$ and $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

Proposition 2.2 ([15])**.** For all $x, y, z, w \in L$, the following properties hold.

(1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$.

(2) $x \leq y$ if and only if $x \rightarrow y = 1$.

(3) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

(4) if $x \leq z$ and $y \leq w$ then $x \otimes y \leq z \otimes w$.

(5) $x \otimes y \leq x \wedge y$.

(6) $x \to (y \to z) = x \otimes y \to z = y \to (x \to z)$.
(7) $0' = 1, 1' = 0, x \leq x''$.
(8) $x \otimes y = 0$ if and only if $x \leq y'$.
(9) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

Definition 2.3 ([6]). L is called to be:
(1) divisible if $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$,
(2) idempotent if $x \otimes x = x$ for all $x \in L$.

In what follows, we recall the concept of Heyting algebras.

Definition 2.4 ([3]). A lattice $(L, \vee, \wedge)$ is called to be a Heyting algebra if for any $x, y \in L$, there exists $x \to y \in L$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$ for all $z \in L$.

Theorem 2.5 ([11]). Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice. Then, the following statements are equivalent:
(1) L is a Heyting algebra.
(2) $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$.

In what follows, we recall the notion of filters on residuated lattices, see [11].

Definition 2.6 ([11]). Let $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if:
(1) for any $x, y \in F$, $x \otimes y \in F$,
(2) for any $x, y \in L$, $x \leq y$ and $x \in F$ imply $y \in F$.

An equivalent definition for a filter in a residuated lattice L is: (1) $1 \in F$, (2) if $x, x \to y \in F$, then $y \in F$. Hence, a filter is also called a deductive system. We denote by $F(L)$ the set of all filters in $(L, \vee, \wedge, \otimes, \to, 0, 1)$.

For a nonempty subset W of L, we denote by $\langle W \rangle$ the filter generated by W. One can check that $\langle W \rangle = \{x \in L | x \geq x_1 \otimes x_2 \otimes \cdots \otimes x_n, x_i \in W, i = 1, 2 \cdots, n\}$.

Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice. We denote by $B(L)$ the set of all complement element of the lattice $(L, \wedge, \vee, 0, 1)$, see [2]. The set $B(L)$ is called the Boolean center of L. For any $t \in L$, $t \in B(L)$ if and only if $t \vee t' = 1$.

A mapping $f : L \longrightarrow L$ is called a homomorphism if it satisfies the following conditions:
(1) $f(0) = 0$, $f(1) = 1$,
(2) $f(x * y) = f(x) * f(y)$, for all $x, y \in L$ and $* \in \{\vee, \wedge, \otimes, \to\}$.
Moreover, f is called an isomorphism if it is bijective.

Proposition 2.7 ([10]). For any $t \in B(L)$, the following statements hold:
(1) $t \otimes t = t$.
(2) $t \otimes x = t \wedge x$ for any $x \in L$.

A nonempty subset I of L is called a lattice ideal of L if it satisfies: (1) for all $x, y \in I, x \vee y \in I$, (2) for all $x, y \in L$, if $x \in I$ and $y \leq x$, then $y \in I$, i.e., a lattice ideal of L is the notion of ideal in the underlying lattice $(L, \vee, \wedge)$.

For a nonempty subset A of L, the smallest lattice ideal containing A is called the lattice ideal generated by A. The lattice ideal generated by A will be denoted by $(A]$. In particular, if $A = \{a\}$, we write $(a]$ for $\{(a]\}$, $(a]$ is called a principal lattice ideal of L. It is easy to check that $(a] = \downarrow a = \{x \in L | x \leq a\}$, see [5].

At the end of this section, we recall the notion of a derivation in a residuated lattice [7] .

Definition 2.8 ([7])**.** A mapping $d : L \longrightarrow L$ is called a derivation on L if it satisfies the following conditions: for any $x, y \in L$,

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

3. f-derivations on residuated lattices

By means of the idea in [4], [7] and [18], in this section, as a generalization of a derivation on a residuated lattice, the notion of an f-derivation for a residuated lattice is introduced and some related properties are investigated. Firstly, we give the concept of an f-derivation in a residuated lattice as follows.

Definition 3.1. Let $f : L \longrightarrow L$ be a homomorphism. A mapping $d_f : L \longrightarrow L$ is called a multiplicative f-derivation on L if

$$d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)),$$

for any $x, y \in L$.

In what follows, unless otherwise stated, a multiplicative f-derivation is called an f-derivation on L. The pair (L, d_f) is said to be a residuated lattice with f-derivations.

Remark 3.2. It is obvious in Definition 3.1 that if f is an identity mapping, i.e., $f(x) = x$ for all $x \in L$, then d_f is a derivation on L. Under such a condition, f-derivation coincide with derivation, see Definition 2.8. Moreover, according to Definition 3.1, a mapping d_f on L is an f-derivation only when a homomorphism f satisfying the above equation. However, to obtain some results, d_f or f must satisfy some additional conditions in the following propositions and theorems.

In what follows, $f : L \longrightarrow L$ is always a homomorphism, unless otherwise specified. Next, we show two examples for f-derivations on residuated lattices.

Example 3.3. Let $t \in L$. Define a mapping $d_f : L \longrightarrow L$ by $d_f(x) = f(x) \otimes t$ for all $x \in L$. One can check that d_f is an f-derivation on L.

Example 3.4. Let $L = \{0, a, b, 1\}$ be a chain and operations $\otimes$ and $\rightarrow$ be defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that $L = \{0, a, b, 1\}$ is a residuated lattice. Now, we define a mapping $f : L \longrightarrow L$ as follows: for all $x \in L$,

$$f(x) = \begin{cases} 0, & x = 0, a, \\ b, & x = b, \\ 1, & x = 1. \end{cases}$$

Further, we define a mapping $d_f : L \longrightarrow L$ determined by f as follows: for all $x \in L$,

$$d_f(x) = \begin{cases} 0, & x = 0, a, \\ b, & x = b, 1. \end{cases}$$

One can check that d_f is an f-derivation on L.

Next, we present some properties of f-derivations in residuated lattices.

Proposition 3.5. Let d_f be an f-derivation on L. Then the following statements hold.

(1) $d_f(0) = 0$.

(2) $f(x) \otimes d_f(1) \leq d_f(x)$ for all $x \in L$.

(3) If $x \leq y'$, then $d_f(x) \leq f(y)'$ and $d_f(y) \leq f(x)'$ for all $x, y \in L$.

Proof. (1) It follows from Definition 3.1 that $d_f(0) = d_f(0 \otimes 0) = (d_f(0) \otimes f(0)) \vee (f(0) \otimes d_f(0)) = d_f(0) \otimes f(0)$. Since $f(0) = 0$, we have $d_f(0) = 0$.

(2) Let $x \in L$. Then we have $d_f(x) = d_f(x \otimes 1) = (d_f(x) \otimes f(1)) \vee (f(x) \otimes d_f(1)) = d_f(x) \vee (f(x) \otimes d_f(1))$, which implies $f(x) \otimes d_f(1) \leq d_f(x)$.

(3) For all $x, y \in L$, suppose that $x \leq y'$. It follows from Proposition 2.2 (8) that $x \otimes y = 0$. Then $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = 0$, which implies $d_f(x) \otimes f(y) = 0$ and $f(x) \otimes d_f(y) = 0$. Therefore, $d_f(x) \leq f(y)'$ and $d_f(y) \leq f(x)'$ for all $x, y \in L$. □

Proposition 3.6. Let d_f be an f-derivation on L. If $d_f(1) = 1$, then $f(x) \otimes f(y) \leq d_f(x \otimes y)$ for all $x, y \in L$.

Proof. It follows from Definition 3.1 that $d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. If $d_f(1) = 1$, then it follows from Proposition 3.5 (2) that $f(x) \leq d_f(x)$. Therefore, $f(x) \otimes f(y) \leq d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. □

In order to discuss some related properties of f-derivations, in what follows, we first introduce three kinds of special f-derivations.

Definition 3.7. Let d_f be an f-derivation on L. Then for all $x, y \in L$,

(1) if $x \leq y$ implies $d_f(x) \leq d_f(y)$, we call d_f an isotone f-derivation,

(2) if $d_f(x) \leq f(x)$, we call d_f a contractive f-derivation.

In particular, if d_f is both isotone and contractive, we call d_f an ideal f-derivation.

Example 3.8. Consider the Example 3.3. One can check that d_f is an ideal f-derivation on L.

Now, some properties of isotone f-derivations and contractive f-derivations are investigated, respectively, most of which are obtained from the adjoint pair in residuated lattices.

Proposition 3.9. Let d_f be an isotone f-derivation on L. Then the following statements hold.

(1) If $z \leq x \to y$, then $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$ for all $x, y, z \in L$.

(2) $f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$.

(3) $f(x) \leq d_f(y) \to d_f(x)$ and $f(y) \leq d_f(x) \to d_f(y)$ for all $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since d_f is an isotone f-derivation on L, we have $d_f(x \otimes z) \leq d_f(y)$. It follows from Definition 3.1 that $d_f(x \otimes z) = (d_f(x) \otimes f(z)) \vee (f(x) \otimes d_f(z))$. Thus, $(d_f(x) \otimes f(z)) \vee (f(x) \otimes d_f(z)) \leq d_f(y)$, which implies $d_f(x) \otimes f(z) \leq d_f(y)$ and $f(x) \otimes d_f(z) \leq d_f(y)$. Therefore, $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$.

(2) Since $x \otimes (x \to y) \leq y$ for all $x, y \in L$, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. It follows from Definition 3.1 that $d_f(x \otimes (x \to y)) = (d_f(x) \otimes f(x \to y)) \vee (f(x) \otimes d_f(x \to y))$, which implies $d_f(x) \otimes f(x \to y) \leq d_f(y)$ and $f(x) \otimes d_f(x \to y) \leq d_f(y)$. Therefore, $f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $d_f(x \otimes y) \leq d_f(x)$. It follows from Definition 3.1 that $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y))$. Thus, $f(x) \leq d_f(y) \to d_f(x)$. In a similar way, we have $f(y) \leq d_f(x) \to d_f(y)$. □

Proposition 3.10. Let d_f be a contractive f-derivation on L. Then the following statements hold.

(1) $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$ for all $x, y \in L$.

(2) If d_f is isotone, then $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$ for all $x, y \in L$.

(3) If $d_f(1) = 1$, then $d_f(x) = f(x)$ for all $x \in L$.

Proof. (1) Since d_f is a contractive f-derivation on L, we have $d_f(x) \otimes d_f(y) \leq f(x) \otimes d_f(y)$ and $d_f(x) \otimes d_f(y) \leq d_f(x) \otimes f(y)$ for all $x, y \in L$. Thus, for all $x, y \in L$

$$d_f(x) \otimes d_f(y) \leq (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = d_f(x \otimes y).$$

On the other hand, since $d_f(x) \otimes f(y) \leq d_f(x)$ and $f(x) \otimes d_f(y) \leq d_f(y)$, we have

$$d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) \leq d_f(x) \vee d_f(y).$$

Therefore, $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$.

(2) For all $x, y \in L$, since $x \otimes (x \to y) \leq y$ and d_f is isotone, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. It follows from the statement (1) that $d_f(x \to y) \otimes d_f(x) \leq d_f(x \otimes (x \to y))$, which implies $d_f(x \to y) \otimes d_f(x) \leq d_f(y)$, i.e., $d_f(x \to y) \leq d_f(x) \to d_f(y)$. On the other hand, since $d_f(y) \leq f(y)$, we have $d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$. Therefore, $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$.

(3) It follows from Proposition 3.5 (2) that $f(x) \otimes d_f(1) \leq d_f(x)$ for all $x \in L$. If $d_f(1) = 1$, then we have

$$f(x) = f(x) \otimes d_f(1) \leq d_f(x) \leq f(x),$$

which implies $d_f(x) = f(x)$ for all $x \in L$. □

Theorem 3.11. Let d_f be an f-derivation on L. For all $x, y \in L$, if d_f, f satisfy $d_f(x) \to d_f(y) = d_f(x) \to f(y)$, then d_f is an ideal f-derivation on L.

Proof. Let $d_f(x) \to d_f(y) = d_f(x) \to f(y)$ for all $x, y \in L$. Since $d_f(x) \otimes 1 \leq d_f(x)$, we have $1 \leq d_f(x) \to d_f(x) = d_f(x) \to f(x)$. Thus $d_f(x) \otimes 1 \leq f(x)$ for all $x \in L$, which implies d_f is contractive. On the other hand, let $x \leq y, x, y \in L$. Since f is a homomorphism, we have $f(x) \leq f(y)$. Thus $d_f(x) \otimes 1 \leq d_f(x) \leq f(x) \leq f(y)$, i.e., $d_f(x) \otimes 1 \leq f(y)$, which implies $1 \leq d_f(x) \to f(y) = d_f(x) \to d_f(y)$, i.e., $d_f(x) \otimes 1 \leq d_f(y)$. Thus, $d_f(x) \leq d_f(y)$, i.e., d_f is isotone. Therefore, d_f is an ideal f-derivation on L. □

Theorem 3.12. Let d_f be a contractive f-derivation on L, which satisfies $d_f(1) \in B(L)$. Then the following statements are equivalent.

(1) d_f is an ideal f-derivation on L.
(2) $d_f(x) \leq d_f(1)$ for all $x \in L$.
(3) $d_f(x) = d_f(1) \otimes f(x)$ for all $x \in L$.
(4) $d_f(x \wedge y) = d_f(x) \wedge d_f(y)$ for all $x, y \in L$.
(5) $d_f(x \vee y) = d_f(x) \vee d_f(y)$ for all $x, y \in L$.
(6) $d_f(x \otimes y) = d_f(x) \otimes d_f(y)$ for all $x, y \in L$.

Proof. (1) ⇒ (2) It is straightforward.

(2) ⇒ (3) Let $d_f(x) \leq d_f(1)$ for all $x \in L$. If $d_f(1) \in B(L)$, then it follows from Proposition 2.7 (2) that

$$d_f(x) = d_f(1) \wedge d_f(x) = d_f(1) \otimes d_f(x) \leq d_f(1) \otimes f(x).$$

On the other hand, it follows from Proposition 3.5 (2) that $f(x) \otimes d_f(1) \leq d_f(x)$. Therefore, $d_f(x) = d_f(1) \otimes f(x)$.

(3) ⇒ (4) Let $d_f(x) = d_f(1) \otimes f(x)$ for all $x \in L$. Then for all $x, y \in L$,

$$\begin{aligned} d_f(x \wedge y) &= d_f(1) \otimes f(x \wedge y) \\ &= d_f(1) \otimes (f(x) \wedge f(y)) \\ &= (d_f(1) \wedge f(x)) \wedge (d_f(1) \wedge f(y)) \\ &= (d_f(1) \otimes f(x)) \wedge (d_f(1) \otimes f(y)) \\ &= d_f(x) \wedge d_f(y). \end{aligned}$$

(4) ⇒ (1) Let $x \leq y, x, y \in L$. It follows from (4) that $d_f(x) = d_f(x \wedge y) = d_f(x) \wedge d_f(y)$, which implies $d_f(x) \leq d_f(y)$ for all $x, y \in L$. Since d_f is a contractive f-derivation, we have d_f is an ideal f-derivation on L.

(3) ⇒ (5) For all $x, y \in L$, it follows from (3) and Proposition 2.2 (9) that

$$\begin{aligned} d_f(x \vee y) &= d_f(1) \otimes f(x \vee y) \\ &= d_f(1) \otimes (f(x) \vee f(x)) \\ &= (d_f(1) \otimes (f(x))) \vee (d_f(1) \otimes f(x)) \\ &= d_f(x) \vee d_f(y). \end{aligned}$$

(5) ⇒ (1) Let $x, y \in L$ and $x \leq y$. It follows from (5) that $d_f(y) = d_f(x \vee y) = d_f(x) \vee d_f(y)$, which implies $d_f(x) \leq d_f(y)$, i.e., D is isotone. Since d_f is a contractive f-derivation, we have d_f is an ideal f-derivation on L.

(3) ⇒ (6) For all $x, y \in L$, it follows from (3) that

$$\begin{aligned} d_f(x \otimes y) &= d_f(1) \otimes f(x \otimes y) \\ &= d_f(1) \otimes (f(x) \otimes f(y)) \\ &= (d_f(1) \otimes f(x)) \otimes (d_f(1) \otimes f(x)) \\ &= d_f(x) \otimes d_f(y). \end{aligned}$$

(6) ⇒ (2) For all $x \in L$, it follows from (6) that

$$d_f(x) = d_f(x \otimes 1) = d_f(x) \otimes d_f(1) = d_f(x) \wedge d_f(1).$$

Thus, $d_f(x) \leq d_f(1)$. □

Let d_f be an f-derivation on L. Define a set $Fix_{d_f}(L) = \{x \in L | d_f(x) = f(x)\}$.

Now, we investigate some operations on $Fix_{d_f}(L)$.

Proposition 3.13. Let d_f be an f-derivation on L. If $x, y \in Fix_{d_f}(L)$, then $x \otimes y \in Fix_{d_f}(L)$.

Proof. Let $x, y \in Fix_{d_f}(L)$. Then $d_f(x) = f(x)$, $d_f(y) = f(y)$. It follows from Definition 3.1 that

$$\begin{aligned} d_f(x \otimes y) &= (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) \\ &= (f(x) \otimes f(y)) \vee (f(x) \otimes f(y)) \\ &= f(x) \otimes f(y) \\ &= f(x \otimes y). \end{aligned}$$

Therefore, $x \otimes y \in Fix_{d_f}(L)$. □

Proposition 3.14. Let d_f be an ideal f-derivation on L. If $x, y \in Fix_{d_f}(L)$, then $x \vee y \in Fix_{d_f}(L)$.

Proof. Let $x, y \in Fix_{d_f}(L)$. Then $d_f(x) = f(x)$, $d_f(y) = f(y)$. Thus

$$f(x \vee y) = f(x) \vee f(y) = d_f(x) \vee d_f(y) \leq d_f(x \vee y) \leq f(x \vee y),$$

which implies $d_f(x \vee y) = f(x \vee y)$. Therefore, $x \vee y \in Fix_{d_f}(L)$. □

Next, we discuss some properties of $Fix_{d_f}(L)$.

Theorem 3.15. Let d_f be an ideal f-derivation on L. For all $x, y \in L$, if L satisfies the divisibility condition $x \wedge y = x \otimes (x \to y)$, then $Fix_{d_f}(L)$ is a lattice ideal of L.

Proof. It follows from Proposition 3.14 that $Fix_{d_f}(L)$ is closed under $\vee$, i.e., for any $x, y \in Fix_{d_f}(L)$, $x \vee y \in Fix_{d_f}(L)$. Now let $x \in Fix_{d_f}(L)$, $y \in L$ and $y \leq x$. Since d_f is an ideal f-derivation on L, we have

$$\begin{aligned} d_f(y) &= d_f(x \wedge y) \\ &= d_f(x \otimes (x \to y)) \\ &= (d_f(x) \otimes f(x \to y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x) \otimes f(x \to y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x \otimes (x \to y))) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x \wedge y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= f(y) \vee (d_f(x \to y) \otimes f(x)), \end{aligned}$$

which implies $f(y) \leq d_f(y)$. Since $d_f(y) \leq f(y)$, we have $d_f(y) = f(y)$, i.e., $y \in Fix_{d_f}(L)$. Therefore, $Fix_{d_f}(L)$ is a lattice ideal of L. □

Let d_f be an f-derivation on L. Define a set $Kerd_f(L) = \{x \in L | d_f(x) = 0\}$. Now, we investigate some operations on $Kerd_f(L)$.

Proposition 3.16. Let d_f be an isotone f-derivation on L. If $x, y \in Kerd_f(L)$, then $x \wedge y \in Kerd_f(L)$.

Proof. Let $x, y \in Kerd_f(L)$. Then $d_f(x) = 0$ and $d_f(y) = 0$. Since d_f is an isotone f-derivation on L, we have $d_f(x \wedge y) \leq d_f(x) \wedge d_f(y) = 0$, i.e., $d_f(x \wedge y) = 0$. Therefore, $x \wedge y \in Kerd_f(L)$. □

Proposition 3.17. Let d_f be an f-derivation on L. If $x, y \in Kerd_f(L)$, then $x \otimes y \in Kerd_f(L)$.

Proof. Let $x, y \in Kerd_f(L)$. Then $d_f(x) = 0$ and $d_f(y) = 0$. It follows from Definition 3.1 that $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = 0$. Therefore, $x \otimes y \in Kerd_f(L)$. □

Proposition 3.18. Let d_f be an isotone f-derivation on L. If $x \leq y$ and $y \in Kerd_f(L)$, then $x \in Kerd_f(L)$.

Proof. Let $x \leq y$ and $y \in Kerd_f(L)$. Then $d_f(y) = 0$. Since d_f is an isotone f-derivation on L, we have $d_f(x) \leq d_f(y) = 0$, i.e., $d_f(x) = 0$. Therefore, $x \in Kerd_f(L)$. □

Finally, we introduce a special kind of an f-derivation on L.

Definition 3.19. Let d_f be an f-derivation on L. If $d_f(x \otimes y) = d_f(x) \otimes d_f(x)$ for all $x, y \in L$, then d_f is called a $\otimes$-f-derivation on L.

Example 3.20. Define an f-derivation by $d_f(x) = x$. One can check that d_f is a $\otimes$-f-derivation on L.

For a residuated lattice L, we define a set $Ide(L) = \{x \in L | x \otimes x = x\}$.

Theorem 3.21. Let d_f be a $\otimes$-f-derivation on L. If $d_f(x) \neq 0$ for all $x \in L$, then $d_f(Ide(L)) \subseteq Ide(L)$.

Proof. Let $y \in d_f(Ide(L))$. Then $y = d_f(x)$ for some $x \in Ide(L)$. Thus

$$y \otimes y = d_f(x) \otimes d_f(x) = d_f(x \otimes x) = d_f(x) = y.$$

Therefore, $y \in Ide(L)$, i.e., $d_f(Ide(L)) \subseteq Ide(L)$. □

4. Principal ideal f-derivations and their applications on residuated lattices

In this section, we investigate principal ideal f-derivations and discuss their applications. In particular, we characterize Heyting algebras in terms of principal ideal f-derivations.

In what follows, let $a \in L$. We define a mapping $d_{(f,a)} : L \longrightarrow L$ as follows: $d_{(f,a)}(x) = a \otimes f(x)$ for all $x \in L$.

Theorem 4.1. Let $a \in L$. Then the mapping $d_{(f,a)}$ is an ideal f-derivation on L.

Proof. Let $x, y \in L$. Then

$$\begin{aligned} d_{(f,a)}(x \otimes y) &= a \otimes f(x \otimes y) \\ &= (a \otimes f(x \otimes y)) \vee (a \otimes f(x \otimes y)) \\ &= (a \otimes f(x) \otimes f(y)) \vee (a \otimes f(x) \otimes f(y)) \\ &= ((a \otimes f(x)) \otimes f(y)) \vee (f(x) \otimes (a \otimes f(y))) \\ &= (d_{(f,a)}(x) \oplus f(y)) \vee (f(x) \otimes d_{(f,a)}(y)). \end{aligned}$$

Let $x \leq y$. Since f is a homomorphism, we have $f(x) \leq f(y)$. Thus $d_{(f,a)}(x) = a \otimes f(x) \leq a \otimes f(y) = d_{(f,a)}(y)$, which implies $d_{(f,a)}$ is isotone. Moreover, it follows from Proposition 2.1 (6) that $d_{(f,a)}(x) = a \otimes f(x) \leq f(x)$ for all $x \in L$, which implies $d_{(f,a)}$ is contractive. Therefore, $d_{(f,a)}$ is an ideal f-derivation on L. □

Remark 4.2. In Theorem 4.1, $d_{(f,a)}$ is called a principal ideal f-derivation on L.

Now, we discuss the properties of principal ideal f-derivations. First of all, we introduce a new derivation in residuated lattices.

Definition 4.3. A mapping $s_f : L \longrightarrow L$ is called an implicative f-derivation on L if it satisfies the following condition: for any $x, y \in L$,

$$s_f(x \to y) = (s_f(x) \to f(y)) \vee (f(x) \to s_f(y)).$$

Example 4.4. Let $a \in L$. Define a mapping $s_{(f,a)} : L \longrightarrow L$ by $s_{(f,a)}(x) = a \to f(x)$ for all $x \in L$. One can check that that $s_{(f,a)}$ is an implicative f-derivation on L.

In order to discuss the properties of principal ideal f-derivations, we recall the notion of Galois connection, see [3]. Let M and N be two posets. We say that the pair (d, s) of mappings: $d : M \to N$ and $s : N \to M$ is a Galois connection between M and N if d and s satisfy: (1) both d and s are isotone, (2) $n \leq d(m)$ if and only if $s(n) \leq m$ for all $m \in M, n \in N$.

Based on the notion of Galois connection, we introduce the notion of f-connection based on residuated lattices.

Definition 4.5. We say that the pair (d_f, s_f) of mappings: $d_f : L \longrightarrow L$ and $s_f : L \to L$ is an f-connection if d_f and s_f satisfy:
(1) both d_f and s_f are isotone,
(2) $d_f(m) \leq f(n)$ if and only if $f(m) \leq s_f(n)$ for all $m, n \in L$.

Example 4.6. For a given $a \in L$, consider the implicative f-derivation $s_{(f,a)}$ in Example 4.4 and define a mapping $d_{(f,a)} : L \longrightarrow L$ by $d_{(f,a)}(x) = a \otimes f(x)$ for all $x \in L$. One can check that the pair $(d_{(f,a)}, s_{(f,a)})$ is an f-connection.

Definition 4.7. Let d_f be a multiplicative f-derivation on L. The derivation d_f is called to be residuated if there exists an implicative f-derivation s_f on L such that a pair (d_f, s_f) forms an f-connection.

Remark 4.8. It follows from the notion of f-connections, we know that if the f-derivation d_f is residuated, then d_f and s_f must be isotone. Moreover, if the f-derivation d_f has an implicative f-derivation s_f, then s_f is unique. We denote this unique by d_f^*.

Next, we discuss some properties of d_f^*.

Proposition 4.9. Let d_f be a multiplicative f-derivation on L. Then the following statements hold.
(1) $d_f^*(1) = 1$.
(2) $d_f^*(x) \geq f(x)$ for all $x \in L$.
(3) $d_f^*(x) \vee d_f^*(y) \leq d_f^*(x \to y) \vee d_f^*(y \to x)$ for all $x, y \in L$.
(4) $d_f^*(x \wedge y) = d_f^*(x) \wedge d_f^*(y)$ for all $x, y \in L$.

Proof. (1) It follows from Definition 4.3 that

$$\begin{aligned} d_f^*(1) &= d_f^*(1 \to 1) \\ &= (d_f^*(1) \to f(1)) \vee (f(1) \to d_f^*(1)) \\ &= (d_f^*(1) \to 1) \vee (f(1) \to d_f^*(1)) \\ &= 1 \vee (f(1) \to d_f^*(1)) \\ &= 1. \end{aligned}$$

(2) Let $x \in L$. Then

$$\begin{aligned} d_f^*(x) &= d_f^*(1 \to x) \\ &= (d_f^*(1) \to f(x)) \vee (f(1) \to d_f^*(x)) \\ &= (1 \to f(x)) \vee (f(1) \to d_f^*(x)) \\ &= f(x) \vee (f(1) \to d_f^*(x)), \end{aligned}$$

which implies $d_f^*(x) \geq f(x)$.

(3) Let $x, y \in L$. Since $f(y) \leq d_f^*(x) \to f(y)$, $d_f^*(y) \leq f(x) \to d_f^*(y)$. It follows from Definition 4.3 and (2) that

$$\begin{aligned} d_f^*(y) &= d_f^*(y) \vee f(y) \\ &\leq (d_f^*(x) \to f(y)) \vee (f(x) \to d_f^*(y)) \\ &= d_f^*(x \to y). \end{aligned}$$

In a similar way, we have $d_f^*(x) \leq d_f^*(y \to x)$. Therefore, $d_f^*(x) \vee d_f^*(y) \leq d_f^*(x \to y) \vee d_f^*(y \to x)$.

(4) Since d_f^* is isotone, we have $d_f^*(x \wedge y) \leq d_f^*(x), d_f^*(y)$ for all $x, y \in L$. Hence, $d_f^*(x \wedge y)$ is a lower bound of $d_f^*(x)$ and $d_f^*(y)$. Now, let $f(n) \in L$ be any lower bound of $d_f^*(x)$ and $d_f^*(y)$. Then we have $f(n) \leq d_f^*(x)$ and $f(n) \leq d_f^*(y)$. It follows from the notion of f-connections that $d(n) \leq f(x)$, $d(n) \leq f(y)$, which implies $d(n) \leq f(x) \wedge f(y) = f(x \wedge y)$, i.e., $f(n) \leq d_f^*(x \wedge y)$. Thus, $d_f^*(x \wedge y)$ is the infimum of $d_f^*(x)$ and $d_f^*(y)$. Therefore, $d_f^*(x \wedge y) = d_f^*(x) \wedge d_f^*(y)$. □

In what follows, we denote by $Fix_{d_{(f,a)}}(L)$ the set of all fixed points of L for $d_{(f,a)}$ and $Fix_{s_{(f,a)}}(L)$ the set of all fixed points of L for $s_{(f,a)}$, respectively, i.e., $Fix_{d_{(f,a)}}(L) = \{x \in L | d_{(f,a)}(x) = f(x)\}$ and $Fix_{s_{(f,a)}}(L) = \{x \in L | s_{(f,a)}(x) = f(x)\}$.

In order to show the relationship between $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$, we first give the following lemma.

Lemma 4.10. For all $x, y \in L$, $x \to (x \otimes y) = y$ if and only if there exists $z \in L$ such that $x \to z = y$.

Proof. It can be obtained immediately from Proposition 2.2 (3). □

Theorem 4.11. If $f : L \to L$ is an isomorphism, then ordered sets $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$ are isomorphic.

Proof. For all $a \in L$, let $p : Fix_{d_{(f,a)}}(L) \to Fix_{d_{(f,a)}}(L)$ be defined by $p(x) = a \to f(x)$ for all $x \in Fix_{d_{(f,a)}}(L)$. Then it is easy to see that p is a mapping from $Fix_{d_{(f,a)}}(L)$ to $Fix_{d_{(f,a)}}(L)$, i.e., p is well defined.

Now, we complete the proof by three steps.

(1) First, we show that p is injective. Let $x, y \in Fix_{d_{(f,a)}}(L)$. Then $d_{(f,a)}(x) = f(x) = a \otimes f(x)$, $d_{(f,a)}(y) = f(y) = a \otimes f(x)$. Suppose that $p(x) = p(y)$. Then $a \to f(x) = a \to f(y)$. It follows from $a \otimes f(x) \leq f(x)$ that $f(x) \leq a \to f(x)$, then $f(x) \leq a \to f(y)$, i.e., $f(x) \otimes a \leq f(y)$. Therefore, $f(x) \leq f(y)$. In a similar way, we have $f(y) \leq f(x)$. Thus, $f(x) = f(y)$. Since f is isomorphism, we have $x = y$. Therefore, p is injective.

(2) Next, we show that p is surjective. It follows from Lemma 4.10 that $x \to (x \otimes y) = y$ if and only if there exists $z \in L$ such that $x \to z = y$ for all $x, y \in L$. Now, for all $x \in Fix_{\delta_a}(L)$, then $f(x) = d_{(f,a)}(x) = a \otimes f(x)$. Thus, $f(f(x)) = d_{(f,a)}(f(x))$. Therefore, $f(x) \in Fix_{\delta_a}(L)$. Moreover,

$$\begin{aligned} p(a \otimes f(x)) &= a \to (a \otimes f(x)) \\ &= a \to (a \otimes (a \to f(x))) \\ &= a \to f(x) \\ &= f(x). \end{aligned}$$

It follows from f is an isomorphism that g is surjective.

(3) Finally, let $x \leq y$. Then it is easy to know that $f(x) \leq f(y)$. Then

$$p(x) = a \to f(x) \leq a \to f(y) = p(y),$$

which implies p is order preserving. Further, the inverse mapping $p^{-1} : Fix_{s_{(f,a)}}(L) \to Fix_{d_{(f,a)}}(L)$ is also order preserving, where $p^{-1}(x) = a \to f(x)$ for all $x \in Fix_{s_{(f,a)}}(L)$.

Hence, p is an order isomorphism from $Fix_{d_{(f,a)}}(L)$ to $Fix_{s_{(f,a)}}(L)$. Therefore, ordered sets $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$ are isomorphic. □

Finally, we characterize Heyting algebra in terms of principal f-ideal derivations.

Theorem 4.12. Let $a \in L$ and $f(x) \leq x$. If L is a Heyting algebra, then $(a] \subseteq Fix_{d_{(f,a)}}(L)$.

Proof. Suppose that L is a Heyting algebra, we have $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. Taking $y = f(x)$, we have $x \otimes f(x) = x \wedge f(x) = f(x)$ for all $x \in L$. Since $d_{(f,a)}(x) = a \otimes f(x)$, we have $d_{(f,a)}(a) = a \otimes f(a) = f(a)$ for all $a \in L$, which implies $a \in Fix_{d_{(f,a)}}(L)$. Since L is a Heyting algebra, which satisfies $x \wedge y = x \otimes (x \to y)$, it follows from Theorem 3.15 that $Fix_{d_{(f,a)}}(L)$ is a lattice ideal of L, i.e., for all $x \in L$, if $x \leq a$, we have $x \in Fix_{d_{(f,a)}}(L)$, which implies $(a] \subseteq Fix_{d_{(f,a)}}(L)$. □

5. Conclusions

In this paper, as a generalization of a derivation in a residuated lattice, f-derivations is introduced, and some properties of isotone f-derivations, contractive f-derivations are discussed. The main conclusions in this paper and the further work to do are listed as follows.

(1) We investigated isotone (resp. contractive) f-derivations of residuated lattices and discussed some properties of the fixed point set for ideal f-derivations and some operations of the kernel for f-derivations.

(2) We obtained that the fixed point set of $d_{(f,a)}$ and $s_{(f,a)}$ are order isomorphism. In particular, by using the fixed point set of principal ideal f-derivations $d_{(f,a)}$, we gave a characterization of Heyting algebras.

As a further work, the following topics may be considered:

(1) Investigating some more properties related to f-derivations in residuated lattices.

(2) Studying f-derivations in EQ-algebras.

(3) Applying f-derivations to other algebras, such as skew residuated lattices, hyper MV-algebras and hyper BL-algebras and so on.

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Fuzzy Γ-ideals in regular and intra-regular Γ-AG-groupoids

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Abstract. This paper is devoted to the concepts of regular and intra-regular Γ-AG-groupoids. We investigate some characteristics of Γ-AG-groupoids by the properties of fuzzy Γ-quasi-ideals, fuzzy Γ-interior-ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals.

Keywords: Γ-AG-groupoids, regular and intra-regular Γ-AG-groupoids, fuzzy Γ-quasi-ideals, fuzzy Γ-bi-ideals and fuzzy Γ-interior-ideals.

1. Introduction

Abel-Grassmann's groupoid [1, 2], abbreviated as AG-groupoid, is a groupoid S whose elements satisfy the left invertive law $a(bc) = (cb)a$. This structure is also known as left almost semigroup [3, 4]. It is a non-associative structure which has wide applications in the theory of flocks and the theory of loops. Sen and Saha in [5, 6], defined Γ-semigroups as a generalization of semigroups.

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Γ-semigroups have been studied by a lot of mathematicians, for instance Dutta and Adhikari [7], Hila [8], Chinram [9], Sen et al., [5, 6, 10] and Seth [11]. After the introduction of fuzzy sets by Zadeh [12], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics as well as its application in many areas, the notion of fuzzy subgroups was introduced by Rosenfeld [13] and its structure was investigated. Several other mathematicians started the fuzzification of different branches of algebras [14, 15, 16, 17, 18]. The notion of fuzzy ideal in Γ-rings was first introduced by Jun and Lee [19]. They studied some preliminary properties of fuzzy ideals of Γ-rings. In [20], Dutta and Chanda studied the structures of fuzzy ideals of a -ring and characterized -field, Noetherian -ring with the help of fuzzy ideals via operator rings of Γ-ring. Jun [21] defined prime fuzzy ideal of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a prime fuzzy ideal.

We remark that the Γ-semigroup given in [5] by Sen and Saha may be called a one-sided Γ-semigroup. Later on in [7], Dutta and Adhikari introduced a both sided Γ-semigroup in which the operation $\Gamma \times S \times \Gamma \to \Gamma$ was also taken into consideration. In [22], Shah et al., introduced the notion of -AG-groupoids which is in fact a generalization of AG-groupoids and discussed some properties of Γ-ideals and bi-Γ-ideals in Γ-AG-groupoids. In continuation, Shah et al., studied M-systems in Γ-AG-groupoids [23]. Recently in [24], Shah, Rehman and Khan discussed fuzzy Γ-ideals, prime (res. semiprime) fuzzy Γ-ideals in Γ-AG-groupoids.

The aim of our present study is to characterize regular and intra-regular Γ-AG-groupoids by the properties of fuzzy Γ-quasi-ideals, fuzzy Γ-interior ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals.

2. Preliminaries

Throughout this study will denote a Γ-AG-groupoid unless otherwise stated. For the convenience of readers we recall certain definitions and results from [22, 24].

Definition 2.1 ([22]). *Let S and Γ be nonempty sets. We call S to be a Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \to S$, written as (a, γ, c) and denoted by $a\gamma c$ such that S satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.*

Definition 2.2 ([22]). *An element e of a Γ-AG-groupoid S is called left identity if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.*

Definition 2.3 ([22]). *A non-empty subset I of a Γ-AG-groupoid is called a left (right) Γ-ideal of S if $S\Gamma I \subseteq I (I\Gamma S \subseteq I)$. A non-empty subset I of a Γ-AG groupoid S is called a Γ-ideal if it is both a left and a right Γ-ideal of S.*

Definition 2.4 ([24]). *Let S be a Γ-AG-groupoid and $\phi \neq A \subseteq S$. Then the characteristic function χ_A of A is defined by*

$$\chi_A(x) = \begin{cases} 1, & if x \in A \\ 0, & if x \notin A \end{cases}$$

Definition 2.5 ([24]). *A function f from a non empty set X to the unit interval $[0,1]$ is called a fuzzy subset of S. For fuzzy subsets f, g of X, $f \subseteq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The fuzzy subsets $f \wedge g$ and $f \vee g$ of X are defined as*

$$(f \wedge g)(x) = f(x) \wedge g(x) = \min\{f(x), g(x)\},$$

$$(f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\} \text{ for all } x, y \in S.$$

Let f, g be any fuzzy subsets of S. We define the product $f\Gamma g$ of f and g as follows $(f\Gamma g)(x) = \sup\{\min\{f(y), g(z)\}$ if $\exists\, x, y \in S$ and $\gamma \in \Gamma$ such that $x = y\gamma z\}$ and $(f\Gamma g)(x) = 0$, if $x \neq y\gamma z$.

Definition 2.6 ([24]). *A fuzzy subset f of S is called a fuzzy subΓ-AG groupoid if $f(x\gamma y) \geq min\{f(x), f(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$ and f is called a fuzzy left (right) Γ-ideal of S if $f(x\gamma y) \geq f(y)(f(x\gamma y) \geq f(x))$ for all $x, y \in S$ and $\gamma \in \Gamma$. If f is both a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of S, then f is called a two-sided fuzzy Γ-ideal of S.*

Lemma 2.1 ([24]). *In a Γ-AG groupoid S with left identity e, every fuzzy right Γ-ideal of S is a fuzzy left Γ-ideal.*

Lemma 2.2 ([24]). *If S is a Γ-AG groupoid with left identity e and f, g are fuzzy Γ-ideals of S, then $f\Gamma g$ is a fuzzy Γ-ideal of S.*

Lemma 2.3 ([24]). *If S is a Γ-AG groupoid and f, g are fuzzy Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.*

Definition 2.7 ([24]). *A Γ-AG-groupoid S is said to be a regular Γ-AG-groupoid if for each a in S there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.*

3. Fuzzy ideals in regular Γ-AG-groupoids

We initiate with the following Lemma

Lemma 3.1. *In a regular Γ-AG-groupoid S, $f\Gamma g = f \cap g$, where f is a fuzzy right Γ-ideal and g is a fuzzy left Γ-ideal.*

Proof. Since $f\Gamma g \subseteq f \cap g$, we only show that $f \cap g \subseteq f\Gamma g$. If $a \in S$, then there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Now, $(f\Gamma g)(a) = \bigvee_{a=y\beta z} \{f(y) \wedge g(z)\} \geq f(a\alpha x) \wedge g(a) \geq f(a) \wedge g(a) = f(a) \wedge g(a) = (f \cap g)(a) \Rightarrow f \cap g \subseteq f\Gamma g$. Hence $f\Gamma g = f \cap g$. □

Lemma 3.2. *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is fuzzy Γ-idempotent.*

Proof. Let S be a regular Γ-AG-groupoid and f a fuzzy right Γ-ideal of S. Since $f\Gamma f\subseteq f$, we only show that $f\subseteq f\Gamma f$. If $a\in S$, then there exist x in S and $\alpha,\beta\in\Gamma$ such that $a=(a\alpha x)\beta a$.

Now, $(f\Gamma f)(a)=\bigvee_{a=y\gamma z}\{f(y)\wedge f(z)\}\geq f(a\alpha x)\wedge f(a)\geq f(a)\wedge f(a)=f(a)\Rightarrow f\subseteq f\Gamma f$. □

Definition 3.1. *Let A be a subset of Γ-AG-groupoid S. The characteristic function of A is denoted by C_A and is defined by*

$$C_A(a)=\begin{cases}1, \text{ if } a\in A\\ 0, \text{ otherwise}\end{cases}$$

We note that a Γ-AG-groupoid S can be considered as a fuzzy Γ-subset of itself and we write $S=C_S$, i.e., $S(x)=1$ for all $x\in S$.

Proposition 3.1. *If A and B are any non-empty subsets of Γ-AG-groupoids S, then following properties hold.*

(1) *If $A\subseteq B$, then $C_A\subseteq C_B$.*
(2) $C_A\Gamma C_B=C_{A\Gamma B}$.
(3) $C_A\cup C_B=C_{A\cup B}$.
(4) $C_A\cap C_B=C_{A\cap B}$.

Proof. (1) Let a be any element of S. Suppose $a\in A$, this implies $a\in B$. Thus $C_A(a)=1=C_B(a)$. This implies $C_A\subseteq C_B$. If $a\notin A$, then $a\notin B$. This implies $C_A(a)=0=C_B(a)$. Thus $C_A\subseteq C_B$.

(2) Let x be any element of S. Suppose $x\in A\Gamma B$. This implies $x=a\alpha b$ for some $a\in A$, $b\in B$ and $\alpha\in\Gamma$.

$(C_A\Gamma C_B)(x)=\bigvee_{x=y\gamma z}\{C_A(y)\wedge C_B(z)\}\geq C_A(a)\wedge C_B(b)=1\wedge 1=1=C_{A\Gamma B}(x)$.

Suppose $x\notin A\Gamma B$. This implies $x\neq a\alpha b$ for some $a\in A$, $b\in B$ and $\alpha\in\Gamma$.

$$(C_A\Gamma C_B)(x)=\bigvee_{x=y\gamma z}\{C_A(y)\wedge C_B(z)\}=0\wedge 0=0=C_{A\Gamma B}(x)$$

Hence $C_A\Gamma C_B=C_{A\Gamma B}$.

(3) Let a be any element of S. Suppose $a\in A\cup B$. Then there are three cases

(i) when $a\in A$ and $a\in B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=1\vee 1=1=C_{A\cup B}(a)$.

(ii) when $a\in A$ and $a\notin B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=1\vee 0=1=C_{A\cup B}(a)$.

(iii) when $a\notin A$ and $a\in B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=0\vee 1=1=C_{A\cup B}(a)$.

If $a\notin A\cup B$. This implies $a\notin A$ and $a\notin B$. This implies$(C_A\cup C_B)(a)=C_{A\cup B}(a)$. Hence in all cases $C_A\cup C_B=C_{A\cup B}$.

(4) Let a be any element of S. Suppose $a\in A\cap B$. This implies $a\in A$ and $a\in B$.

Now, $(C_A\cap C_B)(a)=C_A(a)\wedge C_B(a)=1\wedge 1=1=C_{A\cap B}(a)$. Suppose $a\notin A\cap B$. This implies $a\notin A$ and $a\notin B$. Now $(C_A\cap C_B)(a)=C_A(a)\wedge C_B(a)=0\wedge 0=0=C_{A\cap B}(a)$. Hence $C_A\cap C_B=C_{A\cap B}$. □

Definition 3.2. *A fuzzy Γ-subAG-groupoid f of a Γ-AG-groupoid S is called a fuzzy Γ-bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 3.3. *Every fuzzy right (two-sided) Γ-ideal of a Γ-AG-groupoid S is a fuzzy Γ-bi-ideal of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now, $f((x\alpha y)\beta z) = f(x\alpha y) \geq f(x)$ and $f((x\alpha y)\beta z) = f((z\alpha y)\beta x) \geq f(z\alpha y) \geq f(z)$.

It follows that $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$. Hence f is a fuzzy Γ-bi-ideal of S. Similarly it is fairly easy to prove that every two sided fuzzy Γ-ideal is also a fuzzy Γ-bi-ideal of S. □

Lemma 3.4. *If S is a regular Γ-AG-groupoid, then for every fuzzy Γ-bi-ideal f, $(f\Gamma S)\Gamma f = f$.*

Proof. Since $(f\Gamma S)\Gamma f \subseteq f$, so it follows that there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Now,

$$((f\Gamma S)\Gamma f)(a) = \bigvee_{a=y\gamma z} \{(f\Gamma S)(y) \wedge f(z)\} \geq (f\Gamma S)(a\alpha x) \wedge f(a)$$

$$= \bigvee_{a\alpha x = p\delta q} \{(f(p) \wedge S(q)\} \wedge f(a)\} \geq f(a) \wedge S(x) \wedge f(a) = 1 \wedge f(a) = f(a)$$

$$\Rightarrow f \subseteq (f\Gamma S)\Gamma f.$$

This implies $(f\Gamma S)\Gamma f = f$. □

Lemma 3.5. *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is a fuzzy Γ-ideal of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

$$f(a\gamma b) = f((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(b\beta a) \geq f(b). \Rightarrow f(a\gamma b) \geq f(b).$$

This implies f is a fuzzy left Γ-ideal and hence f is a fuzzy Γ-ideal of S. □

Definition 3.3. *A fuzzy subset f of Γ-AG-groupoid S is called a fuzzy Γ-interior ideal of S if $f((x\alpha y)\beta z) \geq f(y)$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 3.6. *Let S be a regular Γ-AG-groupoid. Then any non-empty fuzzy subset f of S is a fuzzy Γ-interior ideal of S if and only if f is a fuzzy Γ-ideal of S.*

Proof. Suppose f is a fuzzy Γ-interior ideal of S. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$,

$$f(a\gamma b) = f((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(a) \Rightarrow f(a\gamma b) \geq f(a).$$

It follows that f is a fuzzy Γ-right ideal of S and hence f is a fuzzy Γ-ideal of S. Conversely, let f be a fuzzy Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f((x\alpha y)\beta z) = f(x\alpha y) \geq f(y)$. Thus f is a fuzzy interior Γ-ideal of S. □

Remark. Fuzzy interior Γ-ideal and fuzzy Γ-ideal coincide if S is regular Γ-AG-groupoid.

Definition 3.4. *A fuzzy subset f of Γ-AG-groupoid S is called a fuzzy Γ-quasi-ideal of S if $(f\Gamma S) \cap (S\Gamma f) \subseteq f$.*

Proposition 3.2. *In a regular Γ-AG-groupoid S with left identity e, $(f\Gamma S) \cap (S\Gamma f) = f$ for every fuzzy right Γ-ideal f of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. This implies $(f\Gamma S) \cap (S\Gamma f) \subseteq f$, because every fuzzy right Γ-ideal is fuzzy Γ-quasi-ideal. Let $a \in S$. This implies there exist $x \in S$ and $\alpha, \beta \in \Gamma$, such that $a = (a\alpha x)\beta a$,

$$(f\Gamma S)(a) = \bigvee_{a=y\gamma z} \{f(y) \wedge S(z)\} \geq f(a\alpha x) \wedge S(a) = f(a\alpha x) \wedge 1 = f(a\alpha x) \geq f(a)$$

$$\Rightarrow f \subseteq f\Gamma S(S\Gamma f)(a) = \bigvee_{a=l\delta m} \{S(l) \wedge f(m)\}\}$$

$$\geq S(a\alpha x) \wedge f(a) = 1 \wedge f(a) = f(a) f \subseteq S\Gamma f \Rightarrow f \subseteq (f\Gamma S) \cap (S\Gamma f).$$

Hence $(f\Gamma S) \cap (S\Gamma f) = f$. □

Lemma 3.7. *In a Γ-AG-groupoid S with left identity e, every fuzzy Γ-quasi-ideal is a fuzzy Γ-bi-ideal of S.*

Proof. Let f be a fuzzy quasi-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. This implies $f((x\alpha y)\beta z) \geq ((f\Gamma S) \cap (S\Gamma f))((x\alpha y)\beta z) = (f\Gamma S)((x\alpha y)\beta z) \wedge (S\Gamma f)((x\alpha y\beta)z)$.

Hence

$$(S\Gamma f)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z = l\gamma m} \{S(l) \wedge f(m)\} = \bigvee_{(x\alpha y)\beta z = l\gamma m} \{1 \wedge f(m)\} \geq f(z)?$$

$$\Rightarrow (S\Gamma f)((x\alpha y)\beta z) \geq f(z).$$

$$(f\Gamma S)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z = p\gamma q} \{f(p) \wedge S(q)\} = \bigvee_{(x\alpha y)\beta z = p\gamma q} \{f(p) \wedge 1\}$$

$$(x\alpha y)\beta z = (x\alpha y)\beta(e\delta z) = (x\alpha e)\beta(y\delta z) \in (x\alpha e)\Gamma S = x\Gamma S.$$

This implies $(x\alpha y)\beta z \in x\Gamma S$, so $(x\alpha y)\beta z = x\eta r$ for some $r \in S$ and $\eta \in \Gamma$. This implies

$$(f\Gamma S)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z = x\eta r = pq} \{f(p) \wedge S(q)\} = \bigvee_{x\eta r = pq} \{f(p) \wedge 1\} \geq f(x)$$

$$\Rightarrow (f\Gamma S)((x\alpha y)\beta z) \geq f(x). f((x\alpha y)\beta z) \geq (f\Gamma S)((x\alpha y)\beta z) \wedge (S\Gamma f)((x\alpha y)\beta z)$$

$$\geq f(x) \wedge f(z).$$

This implies $f((x\alpha y)\beta z)\geq f(x)\wedge f(z)$. This implies f is a fuzzy Γ-bi-ideal of S. □

Theorem 3.1. *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*

(1) *S is regular.*

(2) *$f\cap g=f\Gamma g$ for every fuzzy right Γ-ideal f and every fuzzy left Γ-ideal g of S.*

(3) *$h=(h\Gamma S)\Gamma h$ for every fuzzy Γ-quasi-ideal h of S.*

Proof. (1) ⇒ (2) . Obvious

(1) ⇒ (3) . Let $a\in S$. This implies there exist elements x in S and α,β in Γ such that $a=(a\alpha x)\beta a$. Now,

$$\begin{aligned}&((h\Gamma S)\Gamma h)(a)=\bigvee\nolimits_{a=y\gamma z}\{(h\Gamma S)(y)\wedge h(z)\}\geq(h\Gamma S)(a\alpha x)\wedge h(a)\\&=(\bigvee\nolimits_{a\alpha x=l\delta m}\{h(l)\wedge S(m)\})\wedge h(a)\geq h(a)\wedge S(x)\wedge h(a)=h(a)\\&\Rightarrow h\subseteq(h\Gamma S)\Gamma h.\end{aligned}$$

Since every fuzzy Γ-quasi-ideal is fuzzy Γ-bi-ideal of S, this implies $(h\Gamma S)\Gamma h\subseteq h$. Hence it follows that $h=(h\Gamma S)\Gamma h$.

(3) ⇒ (2) . Let f be a fuzzy right Γ-ideal and g is a fuzzy left Γ-ideal of S. Since f and g are fuzzy Γ-quasi-ideals of S and intersection of any two fuzzy Γ-quasi-ideals of S is also a fuzzy Γ-quasi-ideal of S, $f\cap g$ is a fuzzy Γ-quasi-ideal of S. $f\cap g=((f\cap g)\Gamma S)\Gamma(f\cap g)\subseteq(f\Gamma S)\Gamma g\subseteq f\Gamma g\Rightarrow f\cap g\subseteq f\Gamma g$. Since $f\Gamma g\subseteq f\cap g$. Hence $f\Gamma g=f\cap g$.

(2) ⇒ (1) . Let $a\in S$. Since $S\Gamma a$ is a left Γ-ideal of S generated by a and $a\Gamma S\cup S\Gamma a$ is a right Γ-ideal of S containing a. This implies $C_{S\Gamma a}$ and $C_{a\Gamma S\cup S\Gamma a}$ are fuzzy Γ-left and fuzzy Γ-right ideals of S respectively and by hypothesis $C_{a\Gamma S\cup S\Gamma a}\cap C_{S\Gamma a}=C_{a\Gamma S\cup S\Gamma a}\Gamma C_{S\Gamma a}$.

So, by Proposition 1, we have $C_{(a\Gamma S\cup S\Gamma a)\cap S\Gamma a}=C_{(a\Gamma S\cup S\Gamma a)S\Gamma a}$. Consequently $(a\Gamma S\cup S\Gamma a)\cap S\Gamma a=(a\Gamma S\cup S\Gamma a)S\Gamma a$.

Now, as $a\in(a\Gamma S\cup S\Gamma a)\cap S\Gamma a$, so $a\in(a\Gamma S\cup S\Gamma a)S\Gamma a$. This implies a is regular and hence S is regular. □

Theorem 3.2. *For a Γ-AG-groupoid with left identity e, the following conditions are equivalent.*

(1) *S is regular.*

(2) *$f=(f\Gamma S)\Gamma f$ for every quasi-ideal f of S.*

(3) *$f=(f\Gamma S)\Gamma f$ for every bi-ideal f of S.*

Proof. (1) ⇒ (3) . Straightforward.

(3) ⇒ (2) . Because every fuzzy Γ-quasi -ideal is fuzzy Γ-bi-ideal.

(2) ⇒ (1). Obvious. □

Theorem 3.3. *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*
(1) *S is regular.*
(2) *$f\cap g=(f\Gamma g)\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy Γ-ideal g of S.*
(3) *$h\cap k=(h\Gamma k)\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy Γ-ideal k of S.*

Proof. (1) $\Rightarrow$ (3). If h is a fuzzy Γ-bi-ideal and k be a fuzzy Γ-ideal of S, then $(h\Gamma k)\Gamma h\subseteq(S\Gamma k)\Gamma S\subseteq k\Gamma S\subseteq k$, $(h\Gamma k)\Gamma h\subseteq(h\Gamma S)\Gamma h\subseteq h$.

It follows that $(h\Gamma k)\Gamma h\subseteq h\cap k$. Now,

$$((h\Gamma k)\Gamma h)(a)=\bigvee\nolimits_{a=l\eta m}\{(h\Gamma k)(l)\wedge h(m)\}\geq(h\Gamma k)(a\alpha((x\beta a)\alpha x))\wedge h(a)$$
$$=(\bigvee\nolimits_{a\alpha((x\beta a)\alpha x)=y\delta z}\{h(y)\wedge k(z)\})\wedge h(a)\geq h(a)\wedge k((x\beta a)\alpha x)\wedge h(a)$$
$$\geq h(a)\wedge k(a)=(h\cap k)(a)\Rightarrow h\cap k\subseteq(h\Gamma k)\Gamma h.$$

Consequently $h\cap k=(h\Gamma k)\Gamma h$.
(3) $\Rightarrow$ (2). Straightforward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of S.
(2) $\Rightarrow$ (1). Since S is fuzzy Γ-ideal, so $f\cap S=(f\Gamma S)\Gamma f$.

It follows that $f=(f\Gamma S)\Gamma f$, where f is a fuzzy Γ-quasi-ideal of S.

It follows that S is regular. □

Theorem 3.4. *For a Γ-AG-groupoid with left identity e, the following conditions are equivalent.*
(1) *S is regular.*
(2) *$f\cap g\subseteq g\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy right Γ-ideal g of S.*
(3) *$h\cap g\subseteq g\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy right Γ-ideal g of S.*
(4) *$k\cap g\subseteq g\Gamma k$ for every fuzzy generalized Γ-bi-ideal k and every fuzzy right Γ-ideal g of S.*

Proof. (1) $\Rightarrow$ (4). Let $a\in S$ and $\alpha,\beta\in\Gamma$, then by hypothesis, $a=(a\alpha x)\beta a$. Now, $(g\Gamma k)(a)=\bigvee_{a=l\gamma m}\{(g)(l)\wedge k(m)\}\geq g(a\alpha x)\wedge k(a)\geq g(a)\wedge k(a)=k(a)\wedge g(a)=(k\wedge g)(a)$. Consequently, $k\cap g\subseteq g\Gamma k$ for every fuzzy generalized Γ-bi-ideal k and every fuzzy right Γ-ideal g of S.
(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are straightforward.
(2) $\Rightarrow$ (1). Let f be a fuzzy right Γ-ideal and g be a fuzzy left Γ-ideal of S. Then clearly $g\Gamma f\subseteq f\cap g$. Also since every fuzzy left Γ- ideal is fuzzy Γ-quasi-ideal of S, so by hypothesis $f\cap g\subseteq g\Gamma f$. It follows that $f\cap g=g\Gamma f$ for every fuzzy right Γ-ideal f and g is a fuzzy left Γ-ideal of S. This implies that $g\cap f=g\Gamma f$. Hence S is regular. □

4. Fuzzy ideals in intra-regular Γ-AG-groupoids

Definition 4.1. *A Γ-AG-groupoid S is called intra-regular if for each $a \in S$, there exist x, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$.*

Lemma 4.1. *In an intra-regular Γ-AG-groupoid S, every fuzzy Γ-ideal is fuzzy Γ-idempotent.*

Proof. Let f be a any fuzzy Γ-ideal of S. Since $f\Gamma f \subseteq f$, we only show that $f \subseteq f\Gamma f$. Let $a \in S$. So, by definition $a = (x\alpha a)\beta(a\gamma y)$. Now, $(f\Gamma f)(a) = \bigvee_{a=y\delta z}\{f(y) \wedge f(z)\} \geq f(x\alpha a) \wedge f(a\gamma y) \geq f(a) \wedge f(a) = f(a) \Rightarrow f \subseteq f\Gamma f$. Hence $f = f\Gamma f$. □

Lemma 4.2. *If S is an intra-regular Γ-AG-groupoid S, then $g \cap f \subseteq f\Gamma g$ for every fuzzy left Γ-ideal f and for every fuzzy right Γ-ideal g of S.*

Proof. Let $a \in S$. This implies there exist x, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$. Now, $(f\Gamma g)(a) = \bigvee_{a=y\delta z}\{f(y) \wedge g(z)\} \geq f(x\alpha a) \wedge g(a\gamma y) \geq f(a) \wedge g(a) = g(a) \wedge f(a) = (g \cap f)(a) \Rightarrow g \cap f \subseteq f\Gamma g$. □

Lemma 4.3. *Every fuzzy right Γ-ideal of an intra-regular Γ-AG-groupoid S is a fuzzy Γ-ideal of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. Let $a, b \in S$. Then by definition, $a = (x\alpha a)\beta(a\gamma y)$. Now, $f(a\delta b) = f((x\alpha a)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(x\alpha a)) \geq f(b\beta(a\gamma y)) \geq f(b)$. This implies f is a fuzzy left Γ-ideal of S and consequently f is a fuzzy Γ-ideal of S. □

Theorem 4.1. *If S is an intra-regular Γ-AG-groupoid with left identity e, then any non-empty fuzzy subset f of S is a fuzzy Γ-interior ideal of S if and only if f is a fuzzy Γ-ideal of S.*

Proof. Suppose f is a fuzzy Γ-interior ideal of S. Let $a, b \in S$. Then, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$, such that $a = (x\alpha a)\beta(a\gamma y)$, $f(a\delta b) = f((a\alpha x)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(a\alpha x)) \geq f(a\gamma y) \geq f((e\eta a)\gamma y) \geq f(a) \Rightarrow f(a\delta b) \geq f(a)$. It follows that f is a fuzzy right Γ-ideal of S and hence f becomes a fuzzy Γ-ideal of S. Conversely, let f be a fuzzy Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f((x\alpha y)\beta z) = f(x\alpha y) \geq f(y)$. Thus f is a fuzzy interior Γ-ideal of S. □

Remark. In an intra-regular Γ-AG-groupoid S, fuzzy Γ-interior ideal and fuzzy Γ-ideal coincide.

Theorem 4.2. *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*

(1) *S is intra-regular.*

(2) *$f \cap g = (f\Gamma g)\Gamma f$ for every fuzzy Γ-quasi-ideal f and for every fuzzy Γ-ideal g of S.*

(3) *$h \cap k = (h\Gamma k)\Gamma h$ for every fuzzy Γ-bi-ideal h and for every fuzzy Γ-ideal k of S.*

Proof. (1) ⇒ (3). Let $x \in S$, so there exist $s,\ t \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = (s\alpha x)\beta(x\gamma t)$.

Consider

$$x\gamma t = ((s\alpha x)\beta(x\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma(e\eta t)$$
$$= (x\beta e)\gamma(((s\alpha x)\gamma t)\eta t) = (x\beta e)\gamma((t\gamma t)\eta(s\alpha x)) = (x\beta e)\gamma(((e\eta t)\gamma t)\eta(s\alpha x)$$
$$= (x\beta e)\gamma(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta(e\alpha x)) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta x)$$
$$= (x\beta((t\eta t)\gamma s))\gamma(e\eta x) = (x\beta((t\eta t)\gamma s))\gamma x$$

and $s\alpha x = s\alpha((s\alpha x)\beta(x\gamma t)) = s\alpha(x\beta((s\alpha x)\gamma t)) = (e\eta s)\alpha(x\beta((s\alpha x)\gamma t)) = (e\eta x)\beta(s\alpha((s\alpha x)\gamma t)) = x\beta(s\alpha((s\alpha x)\gamma t))$.

Now,

$$((h\Gamma k)\Gamma h)(x) = \bigvee_{x=y\gamma z} (h\Gamma k)(y) \wedge h(z) \geq (h\Gamma k)(s\alpha x) \wedge h(x\gamma t)$$
$$= (\bigvee_{s\alpha x = l\varepsilon m} h(l) \wedge k(m)) \wedge h((x\beta((t\eta t)\gamma s))\gamma x)$$
$$\geq h(x) \wedge k(s\alpha((s\alpha x)\gamma t)) \wedge h(x) \geq h(x) \wedge k(x) = h \cap k(x) \Rightarrow h \cap k \subseteq (h\Gamma k)\Gamma h.$$

Also, $(h\Gamma k)\Gamma h \subseteq (S\Gamma k)\Gamma S \subseteq k\Gamma S \subseteq k \Rightarrow (h\Gamma k)\Gamma h \subseteq k(h\Gamma k)\Gamma h \subseteq (h\Gamma S)\Gamma h \subseteq h \Rightarrow (h\Gamma k)\Gamma h \subseteq h \Rightarrow (h\Gamma k)\Gamma h \subseteq h \cap k$. Hence, $h \cap k = (h\Gamma k)\Gamma h$.

(3) ⇒ (2). Straightforward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of S.

(2) ⇒ (1). Let E be a fuzzy right Γ-ideal and J be a fuzzy two-sided Γ-ideal of S. Since every fuzzy right Γ-ideal of S is a fuzzy Γ-quasi-ideal of S. Then by hypothesis $E \cap J = (E\Gamma J)\Gamma E \subseteq (S\Gamma J)\,\Gamma E \subseteq J\Gamma E$. Since E is a fuzzy right Γ-ideal and J is also a fuzzy left Γ-ideal. Hence S is intra-regular. □

Theorem 4.3. *In a Γ-AG-groupoid S with left identity e, the following conditions are equivalent.*

(1) *S is intra-regular.*

(2) *$f \cap g \subseteq g\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy left Γ-ideal g of S.*

(3) *$h \cap g \subseteq g\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy left Γ-ideal g of S.*

Proof. (1) ⇒ (3). Let h be fuzzy Γ-bi-ideal and g be a fuzzy left Γ-ideal of S. Let $x \in S$. Then by definition $x = (s\alpha x)\beta(x\gamma t)$. Now,

$$x\gamma t = ((s\alpha x)\beta(x\gamma t))\delta t = (x\beta((s\alpha x)\gamma t))\delta t = (x\beta((s\alpha x)\gamma t))\delta(e\eta t)$$
$$= (x\beta e)\delta(((s\alpha x)\gamma t)\eta t) = (x\beta e)\delta((t\gamma t)\eta(s\alpha x)) = (x\beta e)\delta(((e\eta t)\gamma t)\eta(s\alpha x))$$
$$= (x\beta e)\delta(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\delta(((t\eta t)\gamma s)\eta(e\alpha x)) = (x\beta e)\delta(((t\eta t)\gamma s)\eta x)$$
$$= (x\beta((t\eta t)\gamma s))\delta(e\eta x) = (x\beta((t\eta t)\gamma s))\delta x.$$

Now,

$$(g\Gamma h)(x) = \bigvee_{x=a\mu b} \gamma(a) \wedge h(b) \geq g(s\alpha x) \wedge h((x\beta((t\eta t)\gamma s))\delta x)$$
$$\geq g(x) \wedge h(x) = h(x) \wedge g(x) = (h \cap g)(x) \Rightarrow h \cap g \subseteq g\Gamma h.$$

$(3) \Rightarrow (2)$. Obvious, because every fuzzy Γ-quasi-ideal of S is fuzzy Γ-bi-ideal of S.
$(2) \Rightarrow (1)$. Let R be a fuzzy right Γ-ideal of S and L be a fuzzy left Γ-ideal of S. Since every fuzzy right Γ-ideal of S is fuzzy Γ-quasi-ideal of S. Consequently R is a fuzzy Γ-quasi-ideal of S. So by hypothesis $R \cap L \subseteq L\Gamma R$. Hence S is intra-regular. □

5. Fuzzy idempotent ideals

Theorem 5.1. *In a Γ-AG-groupoid S with left identity e, the following conditions are equivalent.*
(1) *S is Regular and intra-regular.*
(2) *$f\Gamma f = f$ for all fuzzy Γ-bi-ideals of S.*
(3) *$f_1 \cap f_2 = (f_1\Gamma f_2) \cap (f_2\Gamma f_1)$ for all fuzzy Γ-bi-ideals f_1, f_2 of S.*

Proof. $(1) \Rightarrow (2)$. Let $x \in S$. Since S is regular, so there exist elements $a \in R$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha a)\beta x$, also S is intra-regular, so by definition $x = (s\alpha x)\beta(x\gamma t)$. Now,

$$\begin{aligned}
x &= (x\alpha a)\beta x = (x\alpha a)\beta((x\alpha a)\beta x) = ((s\alpha x)\beta(x\gamma t))\alpha(e\eta a)\\
&= ((s\alpha x)\beta e)\alpha((x\gamma t)\eta a) = ((e\alpha x)\beta s)\alpha((x\gamma t)\eta a) = (x\beta s)\alpha((x\gamma t)\eta a)\\
&= (x\beta s)\alpha((((x\alpha a)\beta x)\gamma t)\eta a) = (x\beta s)\alpha(((t\beta x)\gamma(x\alpha a))\eta a)\\
&= (x\beta s)\alpha((a\gamma(x\alpha a))\eta(t\beta x)) = (x\beta s)\alpha((x\gamma(a\alpha a))\eta(t\beta x))\\
&= (x\beta s)\alpha(((e\mu x)\gamma(a\alpha a))\eta(t\beta x)) = (x\beta s)\alpha((((a\alpha a)\mu x)\gamma e)\eta(t\beta x))\\
&= (x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta(e\beta x)) = (x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta x)\\
&= (x\beta s)\alpha(((t\mu x)\gamma(a\alpha a))\eta x) = (x\beta s)\alpha((((e\mu t)\mu x)\gamma(a\alpha a))\eta x)\\
&= (x\beta s)\alpha((((x\mu t)\mu e)\gamma(a\alpha a))\eta x) = (x\beta s)\alpha((((a\alpha a)\mu e)\gamma(x\mu t))\eta x)\\
&= (x\beta s)\alpha((((e\alpha a)\mu a)\gamma(x\mu t))\eta x) = (x\beta s)\alpha(((a\mu a)\gamma(x\mu t))\eta x)\\
&= (x\beta s)\alpha((x\gamma((a\mu a)\mu t))\eta x) = (x\beta s)\alpha((x\gamma z)\eta x), where z = (a\mu a)\mu t\\
&= (((x\gamma z)\eta x)\beta s)\alpha x = ((s\eta x)\beta(x\gamma z))\alpha x = (x\beta((s\eta x)\gamma z))\alpha x.
\end{aligned}$$

Now, consider

$$\begin{aligned}
(f\Gamma f)(x) &= \bigvee_{x=a\rho b} \{f(a) \wedge f(b)\} \geq f((x\beta((s\eta x)\gamma z))\alpha x) \wedge f((x\alpha a)\beta x)\\
&\geq (f(x) \wedge f(x) = f(x).
\end{aligned}$$

So, $f \subseteq f\Gamma f$. Also $f\Gamma f \subseteq f$. Hence it follows that $f = f\Gamma f$.
$(2) \Rightarrow (3)$. Let f_1, f_2 are fuzzy Γ-bi-ideals of S. Then obviously $f_1 \cap f_2$ is also a fuzzy Γ-bi-ideal of S. So by hypothesis, we have $f_1 \cap f_2 = (f_1 \cap f_2)\Gamma(f_1 \cap f_2) \subseteq f_1\Gamma f_2$. Also $f_1 \cap f_2 = (f_1 \cap f_2)\Gamma(f_1 \cap f_2) \subseteq f_2\Gamma f_1$. It follows that $f_1 \cap f_2 \subseteq (f_1\Gamma f_2) \cap (f_2\Gamma f_1)$. Now we claim that $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of S. For this, we show that

$((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) \subseteq (f_1\Gamma f_2)$,

$$((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) = ((f_1\Gamma f_2)\Gamma(S\Gamma S))\Gamma(f_1\Gamma f_2)$$
$$= ((f_1\Gamma S)\Gamma(f_2\Gamma S))\Gamma(f_1\Gamma f_2) = ((f_1\Gamma S)\Gamma f_1)((f_2\Gamma S)\Gamma f_2) \subseteq f_1\Gamma f_2.$$

Consequently, $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of S. Similarly $f_2\Gamma f_1$ is also a fuzzy Γ-bi-ideal of S. As intersection of fuzzy Γ-bi-ideals of S is also fuzzy Γ-bi-ideal of S, so $(f_1\Gamma f_2)\cap(f_2\Gamma f_1)$ is a fuzzy Γ-bi-ideal. Then, by hypothesis

$$(f_1\Gamma f_2)\cap(f_2\Gamma f_1) = ((f_1\Gamma f_2)\cap(f_2\Gamma f_1))\Gamma((f_1\Gamma f_2)\cap(f_2\Gamma f_1))$$
$$\subseteq (f_1\Gamma f_2)\Gamma(f_2\Gamma f_1) \subseteq (f_1\Gamma S)\Gamma(S\Gamma f_1) = ((S\Gamma f_1)\Gamma S)\Gamma f_1 = (((Se)\Gamma f_1)\Gamma S)\Gamma f_1$$
$$= (((f_1e)\Gamma S)\Gamma S)\Gamma f_1 = ((f_1\Gamma S)\Gamma S)\Gamma f_1 = ((S\Gamma S)\Gamma f_1)\Gamma f_1 = (S\Gamma f_1)\Gamma f_1$$
$$= ((Se)\Gamma f_1)\Gamma f_1 = ((f_1e)\Gamma S)\Gamma f_1 = (f_1\Gamma S)\Gamma f_1 \subseteq f_1.$$

Also, $(f_1\Gamma f_2)\cap(f_2\Gamma f_1) \subseteq f_2$, hence it follows that $(f_1\Gamma f_2)\cap(f_2\Gamma f_1) \subseteq f_1\cap f_2$. Hence $f_1\cap f_2 = (f_1\Gamma f_2)\cap(f_2\Gamma f_1)$.
(3) ⇒ (1). Let R be a fuzzy right Γ- ideal and L a fuzzy Γ-ideal of S. Then R and L are fuzzy Γ-bi-ideals of S. As every fuzzy right Γ-ideal and fuzzy two-sided Γ-ideal is a fuzzy Γ-bi-ideal of S. By hypothesis we have $R\cap L = (R\Gamma L)\cap(L\Gamma R)$. This implies $R\cap L \subseteq (R\Gamma L)\cap(L\Gamma R)$ and $R\cap L \subseteq L\Gamma R \Rightarrow R\cap L \subseteq R\Gamma L$. Since $R\Gamma L \subseteq R\cap L$ always true. Hence it follows that $R\cap L = R\Gamma L$ and $R\cap L \subseteq L\Gamma R$. Hence S is regular and intra-regular. □

Theorem 5.2. *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*
(1) *S is regular and intra-regular.*
(2) *Every fuzzy Γ-quasi-ideal of S is Γ-idempotent.*

Proof. (1) ⇒ (2) . Let g be a fuzzy Γ-quasi-ideal of S. Since $g\Gamma g\subseteq g$, so we only show that $g\subseteq g\Gamma g$. If $x\in S$, then we have by definition $x=(x\alpha a)\beta x$, also S is intra-regular, so there exist s, $t\in S$ and $\alpha,\beta,\gamma\in\Gamma$ such that $x=(s\alpha x)\beta(x\gamma t)$. Consider

$$x = (x\alpha a)\beta x = (((s\alpha x)\beta(x\gamma t))\alpha a)\beta x = ((a\beta(x\gamma t))\alpha(s\alpha x))\beta x$$
$$= ((x\beta(a\gamma t))\alpha(s\alpha x))\beta x = (((s\alpha x)\beta(a\gamma t))\alpha x)\beta x$$
$$= ((((e\eta s)\alpha x)\beta(a\gamma t))\alpha x)\beta x = ((((x\eta s)\alpha e)\beta(a\gamma t))\alpha x)\beta x$$
$$= ((((a\gamma t)\alpha e)\beta(x\eta s))\alpha x)\beta x = ((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x)\beta x.$$

Now, $(g\Gamma g)(x) = \bigvee_{x=y\mu z}\{g(y)\wedge g(z)\} \geq g((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x)\wedge g(x) \geq (g(x)\wedge g(x))\wedge g(x) = g(x) \Rightarrow g\subseteq g\Gamma g$.

Hence, it follows that $g=g\Gamma g$.
(2) ⇒ (1) If $a\in S$, then $S\Gamma a$ is a right Γ-ideal of S containing a. Since every right Γ-ideal is Γ-quasi ideal of S, so $S\Gamma a$ is a Γ-quasi-ideal of S. This implies $C_{S\Gamma a}$ a fuzzy Γ-quasi- ideal of S. Then by (2) , $C_{S\Gamma a}=C_{S\Gamma a}\Gamma C_{S\Gamma a}=C_{(S\Gamma a)(S\Gamma a)}$. It follows that $S\Gamma a=(S\Gamma a)(S\Gamma a)$. As $a\in S\Gamma a$, so $a\in(S\Gamma a)(S\Gamma a)$. Hence a is regular and intra-regular. Consequently S is regular and intra-regular. □

Theorem 5.3. *In a Γ-AG-groupoid S with left identity e, the following conditions are equivalent.*
(1) *S is regular and intra-regular.*
(2) *Every fuzzy Γ-quasi-ideal of S is Γ-idempotent.*
(3) *Every fuzzy Γ-bi-ideal of S is Γ-idempotent.*

Proof. $(1) \Rightarrow (3)$. Obvious.
$(3) \Rightarrow (2)$. Straightforward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of S.
$(2) \Rightarrow (1)$. Obvious. □

Theorem 5.4. *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*
(1) *S is regular and intra-regular.*
(2) *$f_1 \cap f_2 \subseteq f_1 \Gamma f_2$ for all fuzzy Γ-quasi-ideals f_1, f_2 of S.*
(3) *$f \cap g \subseteq f \Gamma g$ for every fuzzy Γ-quasi-ideal f and every fuzzy Γ-bi-ideal g of S.*
(4) *$g \cap f \subseteq g \Gamma f$ for every fuzzy Γ- bi-ideal g and every fuzzy Γ-quasi-ideal f of S.*
(5) *$g_1 \cap g_2 \subseteq g_1 \Gamma g_2$ for all fuzzy Γ-bi-ideals g_1, g_2 of S.*

Proof. $(1) \Rightarrow (5)$. Let g_1, g_2 are fuzzyΓ-bi-ideals of S. Then $g_1 \cap g_2$ is also a fuzzy Γ-bi-ideal of S. Also by Theorem 9, every fuzzy Γ-bi-ideal in S is Γ-idempotent. It follows that

$$g_1 \cap g_2 = (g_1 \cap g_2)\Gamma(g_1 \cap g_2) \subseteq g_1 \Gamma g_2.$$

$(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$. Straightforward.
$(2) \Rightarrow (1)$. Let R be a fuzzy right Γ- ideal and L is a fuzzy left Γ-ideal of S. Since every fuzzy right and fuzzy left Γ-ideal of S is fuzzy Γ-quasi-ideal of S. By hypothesis, we have $R \cap L \subseteq R\Gamma L$. Also it is always true that $R\Gamma L \subseteq R \cap L$. Thus $R \cap L = R\Gamma L$. Hence S is regular by Theorem 1. Again by (2), $R \cap L = L \cap R \subseteq L\Gamma R$. It follows that $R \cap L \subseteq L\Gamma R$. Hence S is intra-regular. □

6. Conclusion

Due to importance of the concept of non-associativity, it is a general prediction that in coming years, non-associativity will govern mathematics and applied sciences. If we look into literature, we can observe that the study of fuzzy sets was previously mostly restricted to associative algebraic structures, but on the other hand there are many sets which possess non-associative binary operations and we can dig out examples which investigate a non-associative structure in the context of fuzzy sets. In this paper we made an effort to use the theory of fuzzy sets to a non-associative algebraic structure (Γ-AG-groupoid). We introduced the notions of regular and intra-regular Γ-AG-groupoids. By defining fuzzy Γ-quasi-ideals, fuzzy Γ-interior ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals, we investigated various related properties. To extend this work, one could study the characterization of Γ-AG-groupoids in terms of fuzzy M-Systems.

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On locally F-semiregular and locally δ-semiregular modules

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Abstract. In this paper, we introduce locally $F-$ semiregular, locally $\delta-$ semiregular and locally finitely $\delta-$supplemented modules and investigate some properties of these new classes of modules. We prove that, if M is a self-projective module with the finite exchange property, then M is $\delta(M)-$semipotent. Finally, we show that the properties of $F-$semiregular, $\delta-$semiregular and $\delta-$supplemented modules which can be extended to their localizations.

Keywords: δ- small submodule, locally F-semiregular, locally δ-supplemented module, locally δ-semiregular.

1. Introduction

Unless stated otherwise, all rings are associative and have identity, all modules are unital left modules. Semiregular, supplemented, δ-semiregular, δ-supplemented module are constitute classes of modules that establishes nice homological, non-homological and algebraic properties. The concept of semiregular modules has been generalized to δ-semirgular module by Zhou [11].

In this paper, we construct the notions of locally δ-semiregular and locally δ-suppl-emented modules as generalizations of δ-semiregular and δ-supplemented modules, respectively. In section two, we investigate some properties and prove some new results on δ-semiregular modules. An $R-$module M is called $F-$semiregular module, if for every $x \in M$, there exists a decomposition $M = A \oplus B$ such that $A \subseteq Rx$ is projective and $B \cap Rx \subseteq F$. In third section, we define an element $x \in M$ is locally $F-$semiregular element with respect to a multiplicative closed system S in R if there exists $s \in S$ such that $\frac{x}{s}$ is F_S-semirgular in M_S . Therefore, a module M is said to be locally $F-$semiregular, if every element

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of M is locally $F-$semiregular. If R is a commutative ring with identity and P is a prime ideal of R, then the localization of R at P is local ring. We generalized this properties for some special types of modules among them we provide that, if M is an $F-$semiregular and $End\left(\left(\frac{M}{F}\right)_P\right)$ is commutative, then $\left(\frac{M}{F}\right)_P$ is local as an R_P-module, for any prime ideal P of R. Moreover, a module M is locally $F-$semiregular with respect to a multiplicative system S in R if and only if $\left(\frac{M}{F}\right)_S$ is a regular module and direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.

In section four, we investigate some properties of δ-semiregular modules that can be extended to their localizations. A submodule N of M is locally δ-small if and only if $X_P+N_P = M_P$, then there exists a projective semisimple submodule Y_P of N_P such that $X_P \oplus Y_P = M_P$ for every maximal ideal P of R. We prove that an $R-$module M is locally δ-semiregular module if and only if $\frac{M}{N}$ has a locally projective δ-cover, for every finitely generated submodule N of M. By finitely locally δ-supplemented module we mean, a module M such that every finitely generated submodule N_P of M_P has a supplemente submodule in M_P. Furthermore, every locally δ-semiregular module is locally finitely δ-supplemented module.

2. Results on δ-semiregular modules

In this section, we establish some results on endomorphism of a δ- semiregular module and providing some conditions under which the endomorphism of a δ-semiregular module would be δ- semiregular. Also, we investigate some new results on $U-$semipotent modules, where a module $M$is called $U-$ semipotent[6]if for every submodule $A$of Msuch that $A \nsubseteq U$, there exists a direct summand $B$of Msuch that $B \subseteq A$and $B \nsubseteq U$, for a submodule U of M.

Lemma 2.1. *Let M be an $R-$module and X be a submodule of M. If every submodule $N \ll_\delta M$ with $M = N + X$, then M is $X-$semipotent.*

Proof. Let $N \ll_\delta M$ with $M = N + X$, then by [11, lemma 2.1], there exists $Y \subseteq N$ such that $M = X \oplus Y$ and Y is projective semisimple $Y \nsubseteq X$ and hence M is an $X-$semipotent module. □

It is obvious that, every non-zero $N \ll_\delta M$ with $M = X + N$ is a $0-$potent.

Proposition 2.1. *If M is a self-projective module with the finite exchange property, then M is $\delta(M)-$semipotent.*

Proof. Let $N \leq M$ such that $N \nsubseteq \delta(M)$. Then there exist $x \in (N - \delta(M))$, this implies that $x \notin RadM$, since $RadM \subseteq \delta(M)$. So, there exists a maximal submodule K of M such that $M = Rx + K$. Since M has a finite exchange property, then exists a decomposition $M = P \oplus Q$ such that $P \leq Rx$ and $Q \leq K$. If $P \leq \delta(M)$, then $\frac{M}{P}$ is simple. Therefore, P is a maximal submodule and hence $P = K$. Also, $K = M$ which is a contradiction. Thus $P \nsubseteq \delta(M)$, so M is $\delta(M)-$semipotent. □

Proposition 2.2. *Let M be an $R-$module and $m \in M$. Then $Rm \ll_\delta M$ if and only if $m \in \delta(M)$.*

Proof. Suppose $Rm \ll_\delta M$, then Rm is a submodule of $\delta(M)$ and hence $m \in \delta(M)$.

Conversely, let $m \in \delta(M)$ and suppose that Rm is no δ-small, then there exists a maximal submodule N of M with $\frac{M}{N}$ is singular such that $m \notin N$ and hence $m \notin \delta(M)$. □

Proposition 2.3. *$\delta(M) = M$ if and only if all finitely generated submodules are δ-small.*

Proof. Suppose that $\delta(M) = M$ and N is a finitely generated submodule of M with $N + L = M$, where $\frac{M}{L}$ is singular. Since $N = Rx_1 + Rx_2 + \ldots + Rx_n$, where $x_i \in M = \delta(M)$, which implies that $Rx_i \ll_\delta M$ for each $i = 1, \ldots, n$. Therefore, $L = M$.

Conversely, if $m \in M$, then Rm is a finitely generated submodule of M and hence $Rm \ll_\delta M$. Thus $m \in \delta(M)$. □

Proposition 2.4. *Let $\delta(M) \ll_\delta M$. Then $\frac{M}{\delta(M)}$ has no nonzero δ-small submodule.*

Proof. Let $\frac{L}{\delta(M)}$ be a δ-small submodule of $\frac{M}{\delta(M)}$. Therefore, $L \ll_\delta M$ [9, Lemma 2.1], then $L \leq \delta(M)$ and hence $L = \delta(M)$. Whence $\frac{L}{\delta(M)}$ is a zero submodule of $\frac{M}{\delta(M)}$. □

Let A be a submodule of M. Then A is called a coclosed submodule of M if A/B is not small in M/B for any proper submodule B of A [7].

Proposition 2.5. *Let N be a coclosed submodule of an $R-$module M. Then $\delta(N) = N \cap \delta(M)$.*

Proof. Clearly, $\delta(N) \leq N \cap \delta(M)$. Let $x \in (N \cap \delta(M))$. Then $x \in N$ implies that $Rx \leq N$ and $x \in \delta(M)$. Hence $Rx \ll_\delta M$. Since N is coclosed submodule, then $Rx \ll_\delta N$, therefore $x \in \delta(N)$. □

An R-module M is called hollow if every submodule is small [2]. In the following Lemma, we give a result on hollow module.

Lemma 2.2. *Let M be a hollow module, which has a maximal submodule K with $\frac{M}{K}$ is singular. Then $\delta(M) = K$.*

Proposition 2.6. *Let M be a hollow module. Then $\delta(M) \neq M$ if and only if M is cyclic.*

Proof. Let M be a hollow module and $\delta(M) \neq M$. Then $\delta(M) \ll_\delta M$. Therefore, $\delta(M)$ is unique maximal submodule with $\frac{M}{\delta(M)}$ singular and simple. So, $\frac{M}{\delta(M)}$ is cyclic. Suppose $\frac{M}{\delta(M)} = Rm + \delta(M)$ for $m \in M$. We claim that $M = Rm$. Let $w \in M$. Then $(w + \delta(M)) \in \frac{M}{\delta(M)}$, therefore there is $r \in R$ such that $w + \delta(M) = r(m + \delta(M)) = rm + \delta(M)$, which implies that $w - rm = y$, for some $y \in \delta(M)$. Thus, $w = rm + y \in (Rm + \delta(M))$. Hence $M = Rm + \delta(M)$, then $M = Rm$.

Conversely, since M is cyclic, then every submodules of M is essential, so M contains a maximal essential submodule. Hence $\delta(M) \neq M$. □

Lemma 2.3. *Let M be a local module. Then $\delta(M) = RadM$.*

In the following proposition we give a characterization of local modules.

Proposition 2.7. *M is local if and only if $\delta(M)$ is a maximal and δ-small in M.*

Proof. Let $\delta(M)$ be a δ-small and maximal submodule in M. To show M is local, first we must show that $\delta(M)$ is unique maximal submodule of M. Suppose that L is another maximal submodule in M. Then $M = L + \delta(M)$, but $\delta(M) \ll_\delta M$, $\frac{M}{L}$ singular simple module, so $L = M$, which is a contradiction. Thus $\delta(M)$ is a unique maximal submodule in M. We claim that every proper submodule of M contained in $\delta(M)$. Let N be a proper submodule of M. Since if not $N + \delta(M) = M$ and $\frac{M}{N}$ is singular simple module, so we get $N = M$, which is a contradiction. Therefore, M is a local module. Conversely, is obvious. □

Theorem 2.1. *Let P be a projective module. Then the following are equivalent:*

1. *P has a δ-small and maximal submodule .*
2. *$\delta(P)$ is a δ-small and maximal submodule .*
3. *P is a local module.*
4. *$End(P)$ is a local ring.*
5. *Every maximal submodule in P is a δ-small submodule of P.*
6. *P is a projective δ-cover, for a simple module .*

Proof. $(1 \Rightarrow 2)$Let N be δ-small and maximal submodule of P. Then $N = \delta(P)$. Moreover,

$N \ll_\delta P$, this implies that $\delta(P) \ll_\delta N$. Hence $N = \delta(P)$.

$(2 \Rightarrow 3)$ P is a local module.

$(3 \Rightarrow 4)$ Since P is a local projective module, then $End(P)$ is a local ring.

$(4 \Rightarrow 5)$ Let N be a maximal submodule in P. We must show that $N \ll_\delta P$. Now since P is a projective module, then $End(P)$ is a local ring. Therefore P is a local module and hence P is hollow. Thus $N \ll P$ and hence $N \ll_\delta P$.

$(5 \Rightarrow 6)$ Since P is a projective module and $\delta(P) \neq P$, then P has a maximal submodule, say N. Now $\frac{P}{N}$ is a simple. Let $\pi{:}P \to \frac{P}{N}$ is a natural epimorphism. We have $ker\pi = N$ and $N \ll_\delta P$, then (P, π) is a projective δ-cover for $\frac{P}{N}$.

$(6 \Rightarrow 1)$ Let P be a projective δ-cover for a simple module , say M. So, there exists an epimorphism $g : P \to M$ such that $kerg \ll_\delta P$. We only have to show $kerg$ is a maximal submodule in P by First Isomorphism Theorem, $\frac{P}{kerg} \cong M$ and M is a simple module, then $\frac{P}{kerg}$ is also simple and $kerg$ is a maximal submodule in P. □

Proposition 2.8. *Let M be $\delta(M)-$semipotent. Then every indecomposable direct summand N of M with $N \nsubseteq \delta(M)$ is local.*

Proof. Suppose that N is indecomposable direct summand of M with $N \nsubseteq \delta(M)$. We claim that for every proper submodule K of N, $\frac{N}{K}$ is simple module and hence $K \leq RadN$. Assume not, K is not submodule of $RadN$, since $RadN = N \cap RadM$. We have $N \nsubseteq \delta(M)$, this implies that $N \nsubseteq RadM$ and hence $N \nsubseteq RadN$. Since M is $\delta(M)-$ semipotent, then there exists a summand X of M such that $X \nsubseteq \delta(M)$. Thus $X \nsubseteq RadM$ and hence X is a summand of N, but N is indecomposable, where $X = N = K$, which is a contradiction. Hence N local. □

Theorem 2.2. *Let U be a proper submodule of a module M. If M is indecoposable and $\delta(M) \ll_\delta M$, then the following are equivalent:*

1. *U respect every finitely generated submodule of M.*
2. *M is $U-$semipotent.*
3. *M is local.*

Proof. (**1 ⇒ 2**) Let A be a submodule of M with $A \nsubseteq U$**.** Suppose that $a \in (A - U)$, then $M = X \oplus Y$, where $X \leq Ra$ and $Y \cap Ra \leq U$.This implies that $Ra = X \oplus (Y \cap Ra)$, so $X \nsubseteq U$. Hence, M is $U-$semipotent.

(**2 ⇒ 3**) By (Proposition 2.7), M is local . Since $RadM$ is a maximal and $RadM \subseteq \delta(M)$, then $RadM = \delta(M)$. Hence $U \leq RadM = \delta(M)$.

Now, let $x \in \delta(M) - U$. Then there exists a direct summand $B \leq Rx$ and $B \nsubseteq U$. Since M local, then it is hollow and every hollow indecomposable, $B = 0$, which is a contradiction. Hence $U = \delta(M)$.

(**3 ⇒ 1**) Let N be a finitely generated submodule of M. If $N = M$ there is nothing to prove. Assume $N \neq M$, this implies $N \leq RadM$, since M is local, hence $N \leq \delta(M)$. Then $M = \{0\} \oplus M$, therefore $M \cap N = N \leq \delta(M)$. Hence $U = \delta(M)$ respects every finitely generated submodule of M. □

A submodule N of M is said to be PSD in M, if there exists a projective summand S of M such that $S \leq N$ and $M = S \oplus X$, whenever $N + X = M$, for any submodule X of M [10].

Lemma 2.4. *If M is δ-semiregular, then every finitely generated submodule N of M contains a PSD submodule of M.*

Proof. Let N be a finitely generated submodule of M. Then there exists a decomposition $M = P \oplus Q$ such that $P \subseteq N$ is projective and $N \cap Q$ is δ-small in Q. Now, suppose that $M = N \cap Q + X$, for a submodule X of M, then there exists a submodule S of $N \cap Q$ such that S is projective and $M = S \oplus X$. □

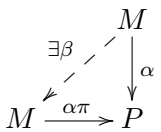
Proposition 2.9. *Let P be a projective module and N be a submodule of P. If $\frac{P}{N}$ has a projective δ-cover, then N contains a PSD submodule.*

Proof. Let $q : Q \to \frac{P}{N}$ be a projective δ-cover and $t : P \to \frac{P}{N}$ be the canonical epimorphism. Then there exists a decomposition $P = X \oplus Y$ such that $t|x : X \to \frac{P}{N}$ is projective δ-cover and $Y \subseteq kert = N$. Thus $X \cap N = kert|x \ll_\delta X$, since X is a direct summand of P, $X \cap N \ll_\delta P$, then we have $S = Y$ is a projective summand of P. □

Theorem 2.3. *Suppose that M is a module, where $E = End\ M$ and $A = \{\alpha \in E; \alpha(M) \ll_\delta M$ and $\frac{M}{(1-\alpha)M}$ is singular$\}$. Then $A \subseteq \delta(E)$, where M is direct projective. Moreover, if A is an essential, then E is δ-semiregular and $A = \delta(E)$ if and only if there exists a decomposition $M = P \oplus Q$ such that $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$, for every $\alpha \in E$.*

Proof. First to show $A \subseteq \delta(E)$, suppose $\alpha \in A$, then we have $\alpha(M) + (1-\alpha)(M) = M$, so $M = (1-\alpha)(M)$. Since M is a direct projective, then $1 - \alpha$ has an inverse in E. So, we get $\alpha \in J(E) \subseteq \delta(E)$. Suppose that E is δ-semiregular and $A = \delta(E)$. Let $\alpha \in E$. Then there exists $\pi^2 = \pi \in E\alpha$ such that $\alpha - \alpha\pi \in \delta(E) = A$, so $(\alpha - \alpha\pi)(M)$ is δ-small. Now, $M = \pi(M) \oplus (1-\pi)(M)$, where $\pi(M) \subseteq \alpha(M)$ and $\alpha(M) \cap (1-\pi)(M) = (\alpha - \alpha\pi)(M)$ is δ-small.

Conversely, suppose that there exists a decomposition $M = P \oplus Q$ such that $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$, for $\alpha \in E$. Let $\pi : M \to P$ be the projection map. Then $\pi(M) = P$ and $ker\pi = Q$. Since $\alpha : M \to P$, then $\alpha\pi : M \to P$ is an epimorphism and M is a direct projective, then the following diagram is commutative:



Then there exists $\beta \in E$ such that $\beta\alpha\pi = \pi$. Define $\tau = \pi\beta\alpha$. Then $\tau^2 = \tau \in \alpha(E)$, and $\tau(M) \subseteq \alpha(M)$, $ker\tau = ker\pi = Q$. Therefore,

$(\alpha - \alpha\tau)(M) = \alpha(1-\alpha)(M) \subseteq (\alpha(M) \cap Q)$, which is δ-small in M. So, $(\alpha - \alpha\tau)(M)$ is a δ-small in M and $\frac{M}{(1-\alpha+\alpha\tau)(M)}$ is singular. Therefore, $(\alpha - \alpha\tau) \in A \subseteq \delta(E)$, so α is δ-semiregular, this implies that E is semiregular. Since $\frac{E}{A}$ is regular, then A is essential maximal ideal, so $\delta(E) \subseteq A$. Hence $A = \delta(E)$. □

Theorem 2.4. *Let M be a δ-semiregular module. If every epimorphism in $EndM$ has a left inverse, $A = \{\gamma \in EndM; \gamma(M) \ll_\delta M\}$ and $G = \{\alpha \in EndM; \alpha(M)$ is Noetherian$\}$ are essential ideal in $EndM$, then G is δ-semiregul-ar.*

Proof. Let $E = EndM$ and $\alpha \in G$. Since $\alpha(M)$ is finitely generated, then there exists a decomposition $M = P \oplus Q$, where $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$. Let π be the projection mapping with $\pi(M) = P$ and $ker\pi = Q$. Since P is projective, then there exists $\beta : P \to M$ such that $\alpha\beta = \pi$. If we set, $\beta(Q) = 0$, then $\beta(M) \subseteq P \subseteq \alpha(M)$ is Noetherian, so $b \in G$. Moreover, $(\alpha - \alpha\pi)(M) \subseteq (\alpha(M) \cap ker\pi) = \alpha(M) \cap Q$, then $(\alpha - \alpha\pi) \in A \cap G$. We show that $\delta(G) = A \cap G$, since $\frac{G}{A \cap G}$ is regular, so $\delta(E) \subseteq (A \cap G)$. On the other hand, since each epimorphism in E has a left inverse, then $A \subseteq J(E) \subseteq \delta(E)$. So, $(A \cap G) \subseteq (\delta(E) \cap G) = \delta(G)$. Hence, $\delta(G) = A \cap G$ and this gives that G is δ-semiregular. □

Proposition 2.10. *Suppose that R is a δ-semiregular ring, if $ann\left(\frac{M}{N}\right)$ is finitely generated ideal, for every finitely generated submodules N of M. Then every faithful multiplication $R-$module is δ-semiregular module.*

Proof. Let N be a finitely generated submodule of M. Then $N = ann\left(\frac{M}{N}\right)M$, where $ann\left(\frac{M}{N}\right)$ is finitely generated ideal in R. Since R is δ-semiregular, so there exists a decomposition $R = A \oplus B$, such that $A \subseteq ann\left(\frac{M}{N}\right)$ is projective and $(ann\left(\frac{M}{N}\right) \cap B) \ll_\delta B$. Therefore, $M = AM \oplus BM$ and $AM \subseteq ann\left(\frac{M}{N}\right)M = N$, and $(BM \cap N) \ll_\delta BM$. □

Corollary 2.1. *Over a Noetherian δ-semiregular ring, every faithful multiplication module is δ-semiregular.*

3. locally $F-$semiregular modules

In this section, we construct a localization for $F-$semiregular module and investigate whether the properties of $F-$semiregular modules are extends under the localization. We start this section with the following definition.

Definition 3.1. *Suppose that M is an $R-$module, F is a submodule of M and S is a multiplicative closed system in R, then an element $x \in M$ is said to be locally $F-$semiregular with respect to S if $\frac{x}{s}$ is F_S-semiregular in M_S for some $s \in S$. If every element of M is locally $F-$semiregular, then M is called locally $F-$semiregular module with respect to S.*

The following is an example of locally $F-$semiregular in Z_{12}.

Example 3.1. *Consider Z_{12} as a $Z_{12}-$module and the closed set $S=\{1\ ,\ 11\}$, then $(Z_{12})_S=\left\{\frac{0}{1},\ \frac{1}{1}\ ,\ \frac{2}{1},\ldots,\frac{11}{1}\right\}$. It can be seen that $3\ \in\ Z_{12}$ is locally $F-$semiregular element in Z_{12}, where $F=\{0\ ,\ 3\ ,\ 6\ ,\ 9\}$, since $(Z_{12})_S=I_S\ \oplus\ J_S$, where $I=\{0\ ,\ 6\}$ and $J=\{0\ ,\ 4\ ,\ 8\}$, then $I_S=\left\{\frac{0}{1},\frac{6}{1}\right\}$, $J_S=\left\{\frac{0}{1},\ \frac{4}{1},\ \frac{8}{1}\right\}$, $I_S\ \subseteq\ \left\{\frac{0}{1},\frac{3}{1},\ \frac{6}{1},\ \frac{9}{1}\right\}$ is projective and $\left\{\frac{0}{1},\ \frac{3}{1},\ \frac{6}{1},\ \frac{9}{1}\right\}\ \cap\ J_S=\left\{\frac{0}{1}\right\}\ \subseteq\ F_S$, so $\frac{3}{1}$ is F_S-semiregular.*

If $\alpha\ \in\ M^*$, then $\alpha:M\to R$ is a homomorphism that induces a homomorphism $\alpha_S:M_S\to R_S$ which is defined by $\alpha_S\left(\frac{m}{s}\right)=\frac{\alpha(m)}{s}$, for every $m\ \in\ M$ and $s\ \in\ S$. This notation leads us to study the relations between $F-$semiregular, which is defined in [1] and locally $F-$semiregular.

Proposition 3.1. *Every $F-$semiregular element in M is locally $F-$semiregular.*

Proof. Suppose that x is $F-$semiregular, then there exists an $\alpha\ \in\ M^*$ such that $(\alpha(x))^2\ =\ \alpha(x)$ and $x-\alpha(x)\,x\ \in\ F$, therefore α_S, which is mentioned in above, exists. Suppose that s is an idempotent in S, then $\alpha_S\left(\frac{x}{s}\right)=\frac{\alpha(x)}{s}\ =\ \frac{(\alpha(x))^2}{ss}\ =\ \frac{\alpha(x)}{s}\frac{\alpha(x)}{s}\ =\ \alpha_S\left(\frac{x}{s}\right)\alpha_S\left(\frac{x}{s}\right)$, then $\left(\alpha_S\left(\frac{x}{s}\right)\right)^2\ =\ \alpha_S\left(\frac{x}{s}\right)$ and $\frac{x}{s}-\alpha_S\left(\frac{x}{s}\right).\frac{x}{s}\ =\ \frac{x}{s}-\frac{\alpha(x)}{s}.\frac{x}{s}\ =\ \frac{x}{s}-\frac{\alpha(x)x}{s^2}\ =\ \frac{s(x-\alpha(x)x)}{s}\ \in\ F_S$, this implies that $\frac{x}{s}$ is an F_S-semiregular. Hence x is locally $F-$semiregular. □

Theorem 3.1. *Suppose that there exists $\lambda\ \in\ M^*$ such that $\lambda(m)=e=e^2$ and $(1-e)\,m$ is a locally $F-$semiregular with respect to $s=s^2\ \in\ S^2$, then m is a locally $F-$semiregular.*

Proof. Since $(1-e)\,m$ is a locally $F-$semiregular, then there exists $s\ \in\ S$, such that $\frac{(1-e)m}{s}$ is F_S-semiregular in M_S . Therefore, there exists $\beta\ \in\ M_S^*$ such that $f\ =\ \beta\left(\frac{(1-e)m}{s}\right)$ is an idempotent and $\left(\frac{s}{s}-f\right)\frac{(1-e)m}{s}\ \in\ F_S$ and $\frac{e}{s}f\ =\ \frac{e}{s}\beta\left(\frac{(1-e)m}{s}\right)\ =\ \frac{e}{s}.\frac{1-e}{s}\beta\left(\frac{ms}{s}\right)\ =\ \frac{e-e^2}{s^2}\beta\left(\frac{ms}{s}\right)\ =\ \frac{0}{s}$. So, $\frac{e}{s}f\ =\ \frac{0}{s}$. Now, $g^2=\left(\frac{e}{s}+f-f\frac{e}{s}\right)^2=\frac{e}{s}+f-f\frac{e}{s}=g$ and $\left(\frac{s}{s}-\frac{e}{s}\right)\frac{m}{s}=\frac{(1-e)m}{s}$, then $\left(\frac{s}{s}-g\right)\frac{m}{s}=\left(\frac{s}{s}-\frac{e}{s}-f+f\frac{e}{s}\right)\frac{m}{s}\ =\ \left(\frac{s}{s}-f\right)\left(\frac{(1-e)m}{s}\right)\ \in\ F_S$. So, $\left(\frac{s}{s}-g\right)\frac{m}{s}\ \in\ F_S$. Let $\alpha=\lambda_S+\beta\left(\frac{m}{s}\right)\left(\frac{s}{s}-\frac{e}{s}\right)(\beta-\lambda_S)$, where $\lambda_S\left(\frac{m}{s}\right)=\frac{\lambda(m)}{s}$ for all $m\ \in\ M$. Then $\alpha\left(\frac{m}{s}\right)=g$. Thus, $\alpha\ \in\ M_S^*$ and $\left(\alpha\left(\frac{m}{s}\right)\right)^2=\alpha\left(\frac{m}{s}\right)$ and $\left(\frac{s}{s}-\alpha\left(\frac{m}{s}\right)\right)\frac{m}{s}\ \in\ F_S$. Hence $\frac{m}{s}$ is an F_S-semiregular. □

Proposition 3.2. *Suppose that $M=N\ \oplus\ K$ is an $R-$module, where N , K and F are submodules of M and S is a multiplicative system in R. If $m=n+k$, where $n\ \in\ F$ and k is a locally $F\ \cap\ K-$semiregular element in K, then m is a locally $F-$semiregular element in M.*

Proof. Let k be a locally $F\ \cap\ K-$semiregular in K. Then there exists $s\ \in\ S$ such that $\frac{k}{s}$ is $F_S\ \cap\ K_S-$semiregular. Therefore, there exists $\lambda:K_S\to R_S$

such that $\lambda\left(\frac{k}{s}\right) = e = e^2$ and $\left(\frac{s}{s} - e\right)\frac{k}{s} \in F_S \cap K_S$. Define $\alpha : M_S \to R_S$ by $\alpha(N_S) = 0$, then $\frac{m}{s} = \lambda\left(\frac{k}{s}\right) = e$ and $\left(\left(\frac{s}{s} - e\right)\frac{m}{s} = \left(\frac{s}{s} - e\right)\frac{n}{s} + \left(\frac{s}{s} - e\right)\frac{k}{s}\right) \in F_S$. So, $\left(\frac{s}{s} - e\right)\frac{m}{s} \in F_S$. Hence m is a locally $F-$semiregular element. □

Theorem 3.2. *If $M = N \oplus K$ is an $R-$module, where N, K and F are submodules of M and S is a multiplicative system in R, then n is a locally $F \cap N-$semiregular in N if and only if n is a locally $F-$semiregular element in M.*

Proof. Suppose that n is a locally $F \cap N-$semiregular element in N, then there exists $s \in S$ such that $\frac{n}{s}$ is an $F_S \cap N_S-$semiregular. We chose $\lambda : N_S \to R_S$, such that $\lambda\left(\frac{n}{s}\right) = e = e^2$ and $\left(\frac{s}{s} - e\right)\frac{n}{s} \in F_S \cap N_S$. Define $\alpha : M_S \to R_S$ by $\alpha\left(\frac{n}{s} + \frac{k}{t}\right) = \lambda\left(\frac{n}{s}\right)$, then $\alpha\left(\frac{n}{s}\right) = \lambda\left(\frac{n}{s}\right) = e$ and $\left(\frac{s}{s} - e\right)\frac{n}{s} \in F_S$, so $\frac{n}{s}$ is an F_S-semiregular element in M_S.

Conversely, let $n \in N$ be locally $F-$semiregular element in M. Then there exists $s \in S$ such that $\frac{n}{s}$ is an F_S-semiregular element in M_S. We chose $\gamma : M_S \to R_S$ such that $\gamma\left(\frac{n}{s}\right) = e = e^2$ and $\left(\frac{s}{s} - e\right)\frac{n}{s} \in F_S$. Define $\lambda = \gamma|_{N_S} : N_S \to R_S$, then $\lambda\left(\frac{n}{s}\right) = \gamma\left(\frac{n}{s}\right) = e$ and since $\frac{n}{s} \in N_S$, therefore $\left(\frac{s}{s} - e\right)\frac{n}{s} \in (F_S \cap N_S)$. So, $\frac{n}{s}$ is $F_S \cap N_S-$semiregular in N_S. Hence n is a locally $F \cap N-$semiregular element in N. □

Proposition 3.3. *The following are equivalent, for $x \in M$:*

1. *x is a locally $F-$semiregular with respect to S.*
2. *there exists $\alpha \in M_S^*$ and $s \in S$ such that $\left(\alpha\left(\frac{x}{s}\right)\right)^2 = \alpha\left(\frac{x}{s}\right)$ and $\left(\frac{x}{s} - \left(\alpha\left(\frac{x}{s}\right)\right)\frac{x}{s}\right) \in F_S$.*
3. *there exists a homomorphism $\gamma : M_S \to (Rx)_S$ such that $\gamma^2 = \gamma$; $\gamma(M_S)$ is projective and $\left(\frac{x}{s} - \gamma\left(\frac{x}{s}\right)\right) \in F_S$.*

Proof. Straight forward to [1, Proposition 2.2]. □

Theorem 3.3. *An element $x \in M$ is a locally $F-$semiregular with respect to S if and only if there exists a regular element $y \in (Rx)_S$ and $s \in S$ such that $\left(\frac{x}{s} - y\right) \in F_S$ and $(Rx)_S = R_S y \oplus R_S\left(\frac{x}{s} - y\right)$.*

Proof. It is similar to [1, Proposition 2.2]. □

Theorem 3.4. *Let F be a submodule of M such that F_S is a fully invariant in M_S, then the following are equivalent:*

1. *M is a locally $F-$semiregular with respect to S.*
2. *For every finitely generated submodule N_S of M_S there exists a homomorphism $\gamma : M_S \to N_S$ such that $\gamma^2 = \gamma$, $\gamma(M_S)$ is a projective $(I_S - \gamma)(N_S) \subseteq F_S$.*

3. *For any finitely generated submodule N_S of M_S, there exists a decomposition $M_S = A_S \oplus B_S$ such that A_S is a projective submodule of N_S and $(N \cap B)_S \subseteq F_S$.*

4. *For any finitely generated submodule N_S of M_S, then N_S can be written as $N_S = A_S \oplus K_S$, where A_S is a projective direct summand of M_S and $K_S \subseteq F_S$.*

Proof. It is similar to [1, Proposition 2.3]. □

Theorem 3.5. *If M is a locally $F-$semiregular module with respect to S, then every finitely generated submodule of $\left(\frac{M}{F}\right)_S$ is a direct summand.*

Proof. Let $\left(\frac{A}{F}\right)_S \subseteq \left(\frac{M}{F}\right)_S$ be a finitely generated submodule. Then there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$. Therefore, there exists a decomposition $M_S = C_S \oplus D_S$ where C_S and D_S are submodules of M_S such that $C_S \subseteq N_S$ is projective and $(N \cap D)_S \subseteq F_S$. Now, $N_S = C_S + (N \cap D)_S$ and $\left(\frac{A}{F}\right)_S = \left(\frac{N+F}{F}\right)_S = \left(\frac{C+F}{F}\right)_S$. Since $F_S = (F \cap D)_S \oplus (F \cap C)_S$ and $(D+F)_S \cap (C+F)_S = (D+(F \cap C))_S \cap (C+(F \cap D))_S = F_S$. Then $\left(\frac{C+F}{F}\right)_S \oplus \left(\frac{D+F}{F}\right)_S = \left(\frac{M}{F}\right)_S$. Hence $\left(\frac{A}{F}\right)_S$ is a direct summand of $\left(\frac{M}{F}\right)_S$. □

Corollary 3.1. *If M is a locally $F-$semiregular module with respect to S, then $\frac{M}{F}$ is a locally regular module with respect to S.*

Theorem 3.6. *If M is a locally $F-$semiregular with respect to S, then direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.*

Proof. Let $\left(\frac{M}{F}\right)_S = \left(\frac{A}{F}\right)_S \oplus \left(\frac{B}{F}\right)_S$, where A, B are submodules of M and $\left(\frac{A}{F}\right)_S$ is a finitely generated submodule of $\left(\frac{M}{F}\right)_S$. Then there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$. Therefore, there exists a decomposition $M_S = C_S \oplus D_S$ such that $C_S \subseteq N_S$ is projective and $(N \cap D)_S \subseteq F_S$. Then $M_S = C_S + B_S$, so by [5, Lemma 1.16], we can write $M_S = C_S \oplus Q_S$, where $Q_S \subseteq B_S$, then $\left(\frac{C+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$ and $\left(\frac{Q+F}{F}\right)_S = \left(\frac{B}{F}\right)_S$. □

Theorem 3.7. *Let M be a projective $R-$module an S a multiplicative closed set in R. If F is a submodule of M such that F_S is fully invariant, then M is locally $F-$semiregular with respect to S if and only if*

1. *$\frac{M}{F}$ is a locally regular with respect to S.*

2. *Direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.*

Proof. Let M be a locally $F-$semiregular with respect to S. Then by Corollary 3.1 and Theorem 3.6, (1) and (2) are hold respectively.

Conversely, suppose that (1) and (2) are hold and N_S is a finitely generated submodule of M_S, then $\left(\frac{N+F}{F}\right)_S$ is finitely generated submodule of $\left(\frac{M}{F}\right)_S$, so there exists a submodule B_S of M_S such that $F_S \subseteq B_S$ and $\left(\frac{N}{F}\right)_S = \left(\frac{N+F}{F}\right)_S \oplus \left(\frac{B}{F}\right)_S$, therefore by (2), we get $M_S = P_S \oplus Q_S$ with $P_S \subseteq N_S$, $\left(\frac{P+F}{F}\right)_S = \left(\frac{N+F}{F}\right)_S$ and $\left(\frac{B}{F}\right)_S = \left(\frac{Q+F}{F}\right)_S$. It is easy to see that $N_S = P_S + (N \cap F)_S$ and $F_S = (F \cap P)_S \oplus (F \cap Q)_S$, then $(Q \cap N)_S \subseteq F_S$. Hence M is a locally $F-$semiregular with respect to S. □

Proposition 3.4. *If M is a locally $F-$semiregular module with respect to S, then M_S is an F_S semipotent module.*

Proof. Let A_S be a submodule of M_S with $A_S \not\subseteq F_S$ and $\frac{a}{s} \in A_S \backslash F_S$, then a is a locally $F-$semiregular in M, so $M_S = X_S \oplus Y_S$, where $X_S \subseteq (Ra)_S$ is projective and $Y_S \cap (Ra)_S \subseteq F_S$. Now, $(Ra)_S = X_S \oplus (Y \cap Ra)_S$. Therefore, $X_S \not\subseteq F_S$. Hence M_S is F_S-semipotent. □

Theorem 3.8. *If M_S is a finitely generated projective and locally $F-$semiregular module with respect to S, then M_S is a F_S-potent module.*

Proof. By Proposition 3.4, M_S is F_S-semipotent, it remains to show F_S is a strongly lifting submodule of M_S. Since M_S is a finitely generated and projective, then M_S is Noetherian. Therefore, $\left(\frac{M}{F}\right)_S$ is Noetherian, this implues that every submodule of $\left(\frac{M}{F}\right)_S$ is finitely generated. If $M_S = \left(\frac{A+F}{F}\right)_S \oplus \left(\frac{B+F}{F}\right)_S$, then $\left(\frac{A+F}{F}\right)_S$ is finitely generated and there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N}{F}\right)_S = \left(\frac{A+F}{F}\right)_S$, therefore $M_S = C_S \oplus D_S$, where $C_S \subseteq N_S$ and $(D \cap N)_S \subseteq F_S$. Since $N_S = C_S \oplus (D \cap N)_S$, $M_S = (A+F)_S + B_S$, C_S is a summand of M_S and M_S is projective, then there exists a direct summand Q_S of M_S such that $M_S = C_S \oplus Q_S$ and $Q_S \subseteq F_S + B_S$. It is easy to see that $C_S \subseteq A_S$. So, $\left(\frac{C+F}{F}\right)_S = \left(\frac{A+F}{F}\right)_S$ and $\left(\frac{Q+F}{F}\right)_S = \left(\frac{B+F}{F}\right)_S$. Therefore, F_S is strongly lifting. □

Proposition 3.5. *If M is an $F-$semiregular module, then M is a locally $F-$semir-egular module with respect to any multiplicative closed sets in R.*

Proof. Let S be a multiplicative closed set in R and $x' \in M_S$. Then there exists $x \in M$ and $s \in S$ such that $x' = \frac{x}{s}$, $M = P \oplus Q$, $P \subseteq Rx$ is projective and $Rx \cap Q \subseteq F$. Therefore, $M_S = P_S \oplus Q_S$, $P_S \subseteq (Rx)_S = R_S x'$ is projective and $R_S x' \cap Q_S = (Rx \cap Q)_S \subseteq F_S$. Thus x' is F_S-semiregular. Hence M is a locally $F-$semiregular with respect to S. □

The converse of the above proposition is not true in general. For example, consider Z_{12} as $Z_{12}-$module and $F = 0$, then it is easy to see that Z_{12} is a

locally $F-$semiregular, but it is not $F-$semiregular. Now, we investigate some conditions for which the converse will be true.

Theorem 3.9. *Suppose that P is a maximal ideal in R, then a projective $R-$module M is $F-$semiregular if and only if it is locally $F-$semiregular with respect to $R\backslash P$.*

Proof. If M is $F-$semiregular, then M is a locally $F-$semiregular with respect to $R\backslash P$ by Proposition 3.5.

Conversely, suppose that M is locally $F-$semiregular with respect to $R\backslash P$ and N is a finitely generated submodule of M, then N_P is a finitely generated submodule of M_P and $N_P = A_P \oplus S_P$, where A_P is a projective direct summand of M_P and $S_P \subseteq F_P$. So, A is a projective direct summand of M and $S \subseteq F$. Hence M is $F-$semiregular. □

Theorem 3.10. *If every elements of S are unit, then M is an $F-$semiregular if and only if it is a locally $F-$semiregular with respect to S.*

Proof. By (Propositions 3.5 and 3.9). □

If R is a commutative ring with identity and P is a prime ideal of R, then the localization of R at P is local ring. In the following propositions we generalize this idea for some special classes of modules, before this we need the following theorem, which is mentioned in [3].

Theorem 3.11. *If M is a multiplication $R-$module and S is a multiplicative closed set in R, then M_S is a multiplication R_S-module.*

Proof. It is obvious. □

Theorem 3.12. *Suppose that P is a prime ideal in R and M is a multiplication $R-$module, then M_P is local as an R_P-module.*

Proof. Let M be a multiplication $R-$module. Then by Proposition 3.5, M_P is a multiplication R_P-module and every proper submodules of M_P contained in a maximal submodule. It remains to show that M_P has unique maximal submodule, if N_P and L_P are two maximal submodules of M_P, then $N_P = I_P M_P$ and $L_P = J_P M_P$, where I_P and J_P are maximal ideals in R_P, this implies that $I_P = J_P$, therefore $N_P = L_P$. Hence M_P is local. □

Corollary 3.2. *If M is a regular $R-$module and $End(M)$ is a commutative, then M_P is a local R_P-module, for every prime ideal P of R.*

Proposition 3.6. *If M is an $F-$semiregular and $End\left(\left(\frac{M}{F}\right)_P\right)$ is commutative, then $\left(\frac{M}{F}\right)_P$ is a local as an R_P-module, for any prime ideal P of R.*

Proof. Follows from corollary 3.2 and [4, Proposition 2.2]. □

Proposition 3.7. *Every semisimple projective R_P-module M is a locally F-semi-regular.*

Proof. Let $(Rx)_P$ be a cyclic submodule of M_P. Since M_P is semisimple, then $M_P = (Rx)_P \oplus B_P$ for some submodule $B_P$of M_P. Now, M_P is a projective module, therefore $(Rx)_P$ is also projective and $B_P \cap (Rx)_P = \{0\} \subseteq F_P$. Hence M_P is an F_P-semiregular module. □

Proposition 3.8. *Let M be a locally $F-$semiregular module and K_P a submodule of M_P such that $F_P \subseteq K_P$. Then K is a locally F_P-semiregular module.*

Proof. Let $(Rx)_P$ be a cyclic submodule of K_P, then $(Rx)_P$ is a finitely generated submodule of M_P, since M_P is semiregular, so there exists a decomposition $M_P = A_P \oplus B_P$, where A_P is a projective submodule of $(Rx)_P$ and $(Rx \cap B)_P \subseteq F_P$. We have $K_P = A_P \oplus (B \cap K)_P$ and $(Rx \cap B \cap K)_P = (Rx \cap B)_P \subseteq F_P$, therefore K_P is an F_P-semiregular. Hence K is a locally $F-$semiregular. □

According to the above proposition, if M is a locally $F-$semiregular module and F, K are submodules of M with $F \subseteq F_1$, then M is a locally F_1-semiregular and K is locally $(F \cap K)-$semiregular module.

Proposition 3.9. *Let $M = \oplus_{i \in I} M_i$ be a directsum of submodules of M_i. If M is a locally $F-$semiregular, then each M_i is a locally F_i-semiregular, where F is a submodule of M and $F_i = F \cap M_i$ for all $i \in I$.*

Proof. Let $x \in M_i$ for some $i \in I$. Since M is $F-$semiregular, then there exists a projective submodule A_P of $(Rx)_P$ such that $M_P = A_P \oplus B_P$ and $(Rx \cap)_P \subseteq F_P$ for some submodule B of M. Now, $(M_i)_P = A_P \oplus (B \cap M_i)_P$ and $(Rx)_P \cap (B \cap M_i)_P \subseteq (Rx)_P \cap B_P \subseteq (F \cap M_i)_P = (F_i)_P$. Therefore, $(M_i)_P$ is $(F_i)_P-$semiregular. Hence, M_i is a locally F_i-semiregular module. □

Proposition 3.10. *Let M_1 and M_2 be two $R-$modules such that $Ann\,(M_1) + Ann\,(M_2) = R$ and $M = M_1 \oplus M_2$. If M_1 is a locally F_1-semiregular and M_2 is a locally F_2-semirgular, then M is a locally $F_1 \oplus F_2-$semiregular.*

Proof. Let N be a finitely generated submodule of M. Since $Ann\,(M_1) + Ann\,(M_2) = R$, then $N = N_1 \oplus N_2$, where $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$. Since N is a finitely generated submodule, N_1 and N_2 are direct summands of N, then N_1 and N_2 are finitely generated, so there exists a decomposition $(M_i)_P = (A_i)_P \oplus (B_i)_P$, such that $(A_i)_P \subseteq N_i$ is projective and $(B_i \cap N_i)_P \subseteq (F_i)_P$. Now, $M_P = (A_1)_P \oplus (A_2)_P \oplus (B_1)_P \oplus (B_2)_P$, where $(A_1)_P \oplus (A_2)_P \subseteq (F_1)_P \oplus (F_2)_P$ is projective and $((B_1 \oplus B_2) \cap N)_P = (B_1 \cap N_1)_P \oplus (B_2 \cap N_2)_P \subseteq ((F_1)_P \oplus (F_2)_P)$. Hence M is a locally $F_1 \oplus F_2-$semiregular module. □

Theorem 3.13. *Let F_P be a fully invariant submodule of M_P. If M is a locally $F-$semiregular module, then*

1. *For all* $\frac{x}{p} \in M_P$, *there exists a regular element* $\frac{y}{t} \in M_P$ *such that* $(\frac{x}{p} - \frac{y}{t}) \in F_P$.
2. *Every submodule of* M_P *that is not contained in* F_P *contains a regular element not in* F_P.
3. $Rad(M_P) \subseteq F_P$ *and* $Z(M_P) \subseteq F_P$.

Proof.

1. Let $\frac{x}{p} \in M_P$. Since M is $F-$semiregular, then there exists $\alpha_P \in M_P^*$ such that $\left(\alpha_P\left(\frac{x}{p}\right)\right)^2 = \alpha_P\left(\frac{x}{p}\right)$ and $\left(\frac{x}{p} - \alpha_P\left(\frac{x}{p}\right)\frac{x}{p}\right) \in F_P$. If $\frac{y}{t} = \alpha_P\left(\frac{x}{p}\right)\frac{x}{p}$ then it is easy to show that $\frac{y}{t}$ is regular element and $(\left(\frac{x}{p} - \frac{y}{t}\right)) \in F_P$.
2. Assume that N_P is a submodule of M_P not contained in F_P, $\frac{x}{p} \in N_P$ and $\frac{x}{p} \notin F_P$. Since M_P is F_P-semiregular, then there exists a regular element $\frac{y}{t} \in (Rx)_P$ such that $(\frac{x}{p} - \frac{y}{t} \in F_P$. If $\frac{y}{t} \in F_P$, then $\frac{x}{p} \in F_P$ which is a contradiction. Thus $\frac{y}{p} \notin F_P$.
3. Let $\frac{x}{p} \in Rad(M_P)$. Since M_P is F_P-semiregular, then $(Rx)_P = A_P \oplus S_P$, where A_P is a projective direct summand of M_P and $S_P \subseteq F_P$. Since $(Rx)_P$ is a small in M_P and A_P is a direct summand of M_P, therefore $A_P = 0$. So, $(Rx)_P = S_P \subseteq F_P$, this implies that $Rad(M_P) \subseteq F_P$. The second part, can be proved in similar manner.

□

4. Locally $\delta-$semiregular modules

In this section we localized the concept of δ-small and investigate some properties of locally δ-small submodule. Also, we introduce the notion of locally δ-semiregular and locally δ-supplemented modules as the localizations for δ-semiregular δ-supplemented.

Definition 4.1. *A submodule* N *of* M *is said to be locally* δ*-small in* M *if* N_P *is* δ*-small submodule in* M_P *for every maximal ideal* P *of* R.

Every δ-small submodule of M is locally δ-small submodule. If N is any δ-small submodule of M and P is a maximal ideal of R. Suppose that $N_P + L_P = M_P$, then $N+L = M$, since L is δ-submodule of M, then there exists a projective semisimple submodule $Y \subseteq N$ such that $Y \oplus L = M$, $Y_P \subseteq M_P$ is projective semisimple submodule and $Y_P \oplus L_P = M_P$, so N is locally δ-small submodule. But the converse is not true in general. For example, consider that Z_6 as a $Z-$module, $N = \{0, 3\}$ and $K = \{0, 2, 4\}$, then $M = N + K$ and $\frac{M}{K}$ is singular. So, N is not δ-small. If P is a maximal ideal in Z, then

$P = \langle p \rangle$, where p is a prime number, then $p = 2$ or $p = 2n + 1$, $n \in N$. If $P = \langle 2 \rangle = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, then $Z - P = \{2n + 1 : n \in Z\}$. Therefore, $\frac{3}{2n+1} = \frac{3}{2m+1}$, since $1.(6n + 3 - 6m - 3) = 0$ and $\frac{3}{1} = \frac{3}{2n+1}$. Thus, $N_P = \left\{\frac{0}{1}, \frac{3}{1}\right\}$. Also, $3 \in (Z - P)$, then $\frac{0}{1} = \frac{2}{1} = \frac{2}{m} = \frac{4}{m} = \frac{4}{1}$, for all $m \in Z$, so $K_P = \{0\}$. If $P = \langle p \rangle$, where p is a prime, then $2 \in (Z - P)$, $\frac{3}{1} = \frac{0}{1} = \frac{0}{m} = \frac{3}{m}$, therefore $N_P = \{0\}$. In each case we conclude that N_P is δ-small.

Theorem 4.1. *Let N be a submodule of M. Then the following are equivalent:*

1. *N is a locally δ-small.*
2. *If $X_P + N_P = M_P$, then there exists a projective semisimple submodule Y_P of N_P such that $X_P \oplus Y_P = M_P$ for every maximal ideal P of R.*
3. *$X_P + N_P = M_P$ with $\left(\frac{M}{X}\right)_P$ is Goldie torsion, then $X_P = M_P$ for every maximal ideal P of R.*

Proof. Similar to [11, Lemma 2.1]. □

Lemma 4.1. *If A, B and K are submodules of M such that $A \subseteq B$, $M = A+K$ and $B \cap K$ is a locally δ-small in M, then $\frac{B}{A}$ is a locally δ-small in $\frac{M}{A}$.*

Proof. Follows from Theorem 4.1. □

Theorem 4.2. *Let M be an $R-$module. Then*

1. *For submodules N, K and L with $K \subseteq N$, then we have:*
2. *N is a locally δ-small in M if and only if K is a locally δ-small in M and $\frac{N}{K}$ is a locally δ-small in $\frac{M}{K}$.*
3. *$N + L$ is a locally δ-small in M if and only if N and L are locally δ-small in M.*
4. *If K is a locally δ-small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is a locally δ-small in N. In particular, if K is locally δ-mall in M, then K is a locally δ-small in N.*
5. *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2$ is a locally δ-small in $M_1 \oplus M_2$ if and only if K_1 is a locally δ-small in M_1 and K_2 is a locally δ-small in M_2.*

Proof. 1. (a) Assume that N is a locally $\delta-$small, this implies that $N_P \ll_\delta M_P$ for every maximal ideal P of R. We have $K_P \subseteq M_P$ and $N_P \ll_\delta M_P$. Hence $K_P \ll_\delta M_P$ by [11, Lemma 2.1]. Since $N_P \ll_\delta M_P$, $\frac{N_P}{K_P} \ll_\delta \frac{M_P}{K_P}$, therefore $\frac{N}{K}$ is a locally $\delta-$small.

Conversely, suppose that K is a locally $\delta-$small in M and $\frac{N}{K}$ is a locally $\delta-$small in $\frac{M}{K}$, since K is a locally $\delta-$small in M, this implies that

$K_P \ll_\delta M_P$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$, then $\frac{N_P}{K_P} \ll_\delta \frac{M_P}{K_P}$, hence $N_P \ll_\delta K_P$.

(b) Let $N+L$ be a locally $\delta-$small. Then $(N+L)_P \ll_\delta M_P$ and we have $(N+L)_P = N_P+L_P$, this implies that $N_P \ll_\delta M_P$ and $L_P \ll_\delta M_P$. For the converse direction the proof is similar.

2. Let K be a locally δ-small in M. Then $K_P \ll_\delta M_P$. Since $f : M \to N$ is a homomorphism, then $f_P : M_P \to N_P$ is a homomorphism, so $f_P(K_P) \ll_\delta N_P$.

3. Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is locally δ-small in $M_1 \oplus M_2$, this implies that $(K_1 \oplus K_2)_P \ll_\delta (M_1 \oplus M_2)_P$. Therefore, $K_{1P} \ll_\delta M_{1P}$ and $K_{2P} \ll_\delta M_{2P}$. Hence K_1 and K_2 are locally δ-small in M. The converse part is obvious.

□

Recall that if ϕ is that class of all singular simple modules. For a module M let $\delta(M) = \cap \{N \subseteq M; \frac{M}{N} \in \phi\} = \sum \{L \subseteq M; L \ll_\delta M\}$.

Proposition 4.1. *Let M be an $R-$module. Then $\delta(M_P) = (\delta(M))_P$ for every maximal ideal of R.*

Proof. It is obvious. □

Proposition 4.2. *Let A, B and C be submodules of an $R-$module M. If $M = A+B$, $B \subseteq C$ and $\frac{C}{B}$ is a locally δ-small in $\frac{M}{B}$, then $\frac{A \cap C}{A \cap B}$ is a locally δ-small in $\frac{M}{A \cap B}$.*

Proof. It is follows from **Theorem 4.2**. □

Theorem 4.3. *Let M be a module and $m \in M$. Then the following are equivalent:*

1. *mR is not locally δ-small in M.*
2. *There is a maximal submodule N_P of M_P such that $\frac{m}{s} \notin N_P$ and $\frac{M_P}{N_P}$ is singular, for every maximal ideal P in R.*

Proof. $(1 \Rightarrow 2)$ Suppose that mR is not locally δ-small in M, then it is not δ-small. Therefore, by [11, Lemma 2.2], there exists a maximal submodule N of M such that $m \notin N$, $\frac{M}{N}$ is a simple module, this implies that $Z\left(\frac{M}{N}\right) = \frac{M}{N}$. Then $\frac{M}{N}$ is singular. So, N_P is a maximal submodule of M_P such that $\frac{m}{s} \notin N_P$ and $\frac{M_P}{N_P}$ is singular for every maximal submodule P of R.

$(2 \Rightarrow 1)$ Suppose that there exists a maximal submodule N_P of M_P such that $\frac{m}{s} \notin N_P$ with $\frac{M_P}{N_P}$ is singular, then $\frac{m}{s}R_P + N_P = M_P$. Hence $\frac{m}{s}R_P$ is not δ-small in M_P, that is, mR is not locally δ-small in M. □

Theorem 4.4. *Let M be a module, K is a direct summand of M and $N \subseteq K$. Then N is locally δ-small in M if and only if N is a locally δ-small in K.*

Proof. Assume that N is a locally δ-small in K, then $N_P \ll_\delta K_P$ for every maximal ideal P in R, then $N_P \ll_\delta M_P$.

Conversely, suppose that $N_P \ll_\delta M_P$ and $X_P + N_P = K_P$, then $M_P = K_P \oplus Y_P = (X_P + N_P) \oplus Y_P = N_P + (X_P + Y_P)$. Therefore, there exists a projective semisimple submodule Z_P of N_P such that $M_P = Z_P \oplus (X_P + Y_P)$, then $K_P = (K \cap Z)_P \oplus X_P$ and $(K \cap Z)_P \subseteq N_P$ is projective semisimple, so N is a locally δ-small in K. □

Proposition 4.3. *Suppose that M is a module K, L and H are submodules of M. If L is a locally δ-small, then L is a locally δ-small in $K + H$.*

Proof. Obvious by Theorem 4.3. □

Definition 4.2. *M is called locally δ-semiregular if for every finitely generated submodule N of M, there is a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $N \cap B$ is locally $\delta-$small in B.*

Every regular, semiregular and δ-semiregular modules are locally δ-semiregular. But the converse is not true, in general (See example 3.2). If M is a projective module, then the converse is also true.

Theorem 4.5. *Let M be a module. Then the following are equivalent:*

1. *M is a locally δ-semiregular.*
2. *For every cyclic submodule N of M there exists a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $N \cap B$ is a locally δ-small in B.*
3. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ is projective and $(1-e)(N)$ is a locally δ-small in M.*

Proof. $(1 \Rightarrow 2)$ is obvious.
$(2 \Rightarrow 1)$ Let N be a finitely generated submodule of M with generators $x_1, \ldots, x_n$. If $n = 1$, then the result is hold, suppose that the result is hold, for every $(n-1)-$generated submodule of M. Since N is $n-$generated submodule of M. Then there exists an $(n-1)-$generated submodule A and a cyclic submodule B of M such that $N = A + B$. By our hypothesis there exists a decomposition $M = P_1 \oplus Q_1$ such that $P_1 \subseteq A$ is projective and $A \cap Q_1$ is locally δ-small in M. Let $h : M \to Q_1$ be the projection, since $P_1 \subseteq A \subseteq N$, then $N = P_1 \oplus h(N) = P_1 \oplus (h(A) + h(B))$ and $A = P_1 \oplus h(A)$, where $h(N) = N \cap Q_1$ and $h(A) = A \cap Q_1$. Since $h(B)$ is cyclic submodule of M, then there exists a decomposition $M = P_2 \oplus Q_2$ such that

$P_2 \subseteq h(B) \subseteq h(N) \subseteq Q$ and $h(B) \cap Q_2$ is a locally δ-small in M. So, $Q_1 = P_2 \oplus Q$, where $Q \subseteq Q_2$ and $h(B) \cap Q$ is a locally δ-small in M. Suppose that $P = P_1 \oplus P_2$, then $M = P \oplus Q$ and $P \subseteq N$ is projective. Now, to show $N \cap Q$ is a locally δ-small in M, let $\pi : Q_1 \to Q$ be the projection. Since $h(A)$ is a locally δ-small in M, then $\pi(h(A))$ is a locally δ-small in M therefore, $\pi(h(A)) + (h(B) \cap Q)$ is a locally δ-small in M. Since $P_2 \subseteq h(B)$, then $h(A)+h(B) = \pi(h(A))+P_2+(h(B) \cap Q)$. Now, $N \cap Q = h(N) \cap Q = (h(A) + h(B)) \cap Q = \pi(h(A))+P_2 \cap Q+h(B) \cap Q = \pi(h(A))+(h(B) \cap Q)$, which is a locally δ-small in M.

$(1 \Rightarrow 3)$ and $(3 \Rightarrow 1)$ are Obvious. □

According to Theorem 4.3, if M is an $R-$module, then an element $\frac{m}{s}$ of M_P is belonging to $\delta(M_P)$ if and only if $\frac{m}{s}$ R_P is a δ-small submodule of M, that is, $\delta(M_P) = \left\{\frac{m}{s} \in M_P; R_P\frac{m}{s} \ll_\delta M_P\right\}$. It can be seen that [8, Corollary 2.4] stated that a module M with $\delta(M) \ll_\delta M$ is a δ-semiregular module if and only if for every finitely generated submodule N of M, there exists an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ and $(1-e)(N) \subseteq \delta(M)$.

In the following proposition we provide the result for a module and its localization without using the condition $\delta(M) \ll_\delta M$ and $\delta(M)$ is a locally δ-small submodule in M.

Theorem 4.6. *For an $R-$module M, the following are equivalent:*

1. *M is a δ-semiregular module.*
2. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ is projective and $(1-e)(N) \subseteq \delta(M)$.*

Proof. It is obvious by Theorem 4.5. □

Theorem 4.7. *For an $R-$module M, the following are equivalent:*

1. *M is a Locally δ-semiregular module.*
2. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M_P)$ such that $e(M_P) \subseteq N_P$ is projective and $(1-e)(N_P) \subseteq \delta(M_P)$, for every maximal ideal P in R.*

Proof. It is obvious by Theorem 4.5. □

Theorem 4.8. *If M is a locally δ-semiregular module, then M_P is δ-simiregular, for every maximal ideal P of R.*

Proof. Let N' be a cyclic submodule of M_P, then there exists a cyclic submodule N of M such that $N' = N_P$. Therefore, there exists a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $(N \cap B)_P \ll_\delta M_P$, then $M_P = A_P \oplus B_P$ and $A_P \subseteq N'$ is projective and $N' \cap B_P \ll_\delta M_P$, so M_P is δ-semiregular. □

Corollary 4.1. *If M is a locally δ-semiregular module with $End\left(\frac{M}{\delta(M)}\right)$ is commutative, then $\frac{M_P}{\delta(M_P)}$ is local module over R_P.*

For a projective module the converse of the above theorem is true.

Theorem 4.9. *Suppose that M is a projective module with $\delta(M)$ is locally δ-small submodule in M. Then M is locally δ-semiregular if and only if $\frac{M_P}{\delta(M_P)}$ is regular and direct summands of $\frac{M_P}{\delta(M_P)}$ can be lifted to M_P.*

Proof. Clear by [5, Proposition 1.7] □

Definition 4.3. *A submodule N is called locally δ-supplemented in M if there exists a submodule L in M such that $M = N + L$ and $N \cap L$ is locally δ-small in L. A module M is called locally δ-supplemented if every submodule of M has a locally δ-supplement. Also M is said to be locally finitely δ-supplemented if every finitely generated submodule has a locally δ-supplemented.*

Lemma 4.2. *Every locally δ-semiregular module is a locally finitely δ-supplemented module.*

Proof. Let M be a locally δ-semiregular and U be a finitely generated submodule. Then there exists a decomposition $M = A \oplus B$, $A \subseteq U$ is projective and $U \cap B$ is a locally δ-small in B, therefore, $M = U + B$ and $U \cap B$ is a locally δ-small in B, so M is a locally finitely δ-supplemented. □

Definition 4.4. *A pair (T, π) is called a locally projective δ-cover of a module M, if T is projective and $\pi : T \to M$ is an epimorphism with $ker\pi$ is locally δ-small in P.*

Proposition 4.4. *Suppose that $M = M_1 \oplus M_2 \ldots \oplus M_n$ such that all $\pi_i : T_i \to M_i$, for $i = 1, \ldots, n$ are locally projective δ-covers. If $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n$, then $\pi = \oplus_{i=1}^{n} \pi_i : T \to M$ is a locally projective δ-cover.*

Proof. Since T_i is projective module, for $i = 1, \ldots, n$, then $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ is also projective module and every π_i is an epimorphism, so $\pi = \oplus_{i=1}^{n} \pi_i$ is an epimorphism. Now, $ker\pi = \oplus_{i=1}^{n} ker\pi_i$ and $ker\pi_i$ is a locally δ-small, for $i = 1, \ldots, n$, then $ker\pi$ is δ-small. Thus, (T, π) is a locally projective δ-cover of M. □

Theorem 4.10. *Let $\pi : T \to M$ be a locally projective δ-cover. If Q is projective and $q : Q \to M$ is an epimorphism, then there exists decompositions $T = A \oplus B$ and $Q = X \oplus Y$ such that:*

1. *$A \cong X$.*
2. *$\pi|_A : A \to M$ is a locally projective δ-cover.*
3. *$q|_X : X \to M$ is a locally projective δ-cover.*

4. B is a projective semisimple module, with $B \subseteq ker\pi$ and $Y \subseteq kerq$.

Proof. Since Q is projective, then there exists $h : Q \to T$ such that $q = \pi \circ h$. Thus, $\pi(h(Q)) = M = \pi(T)$ and so $T = h(Q) + ker\pi$. We have, $ker\pi$ is a locally δ-small in T. Then by Theorem 4.1 $T_P = A_P \oplus B_P$, where $A_P = h_P(Q_P)$ and B_P is projective semisimple with $B_P \subseteq (ker\pi)_P$. So, $T = A \oplus B$, B is projective semisimple with $B \subseteq ker\pi$ and $\pi|_A : A \to M$ is a projective δ-cover. Since $ker\pi|_A \subseteq ker\pi$ and $ker\pi$ is a locally δ-small, then $ker\pi|_A$ is also locally δ-small. Since A is projective then the homomorphism $h : Q \to A$ splits. So, there exists $g : A \to Q$ such that $h \circ g = I_A$. Therefore, $Q = X \oplus Y$ with $Y = kerh$ and $X = g(A)$. Thus, $X \cong A$ and $ker\pi|_A$ is locally δ-small in A, the we get $kerq|_X = g(ker\pi|_A)$ is locally δ-small in $g(A) = X$. Note that $q(X) = (\pi \circ h)(X) = (\pi \circ h)(X + Y) = (\pi \circ h)(Q) = q(Q) = M$. Thus $q|_X : X \to M$ is a locally projective δ-cover. □

Finally we give the following theorem.

Theorem 4.11. *Suppose that M is an $R-$module and N is a submodule of M. Then the following are equivalent:*

1. *$\frac{M}{N}$ has a locally projective δ-cover.*
2. *$M = M_1 \oplus M_2$, where $M_1 \subseteq N$ is a projective and $M_1 \cap N$ is a locally δ-samll in M.*

Proof. $(1 \Rightarrow 2)$ Let $q : Q \to \frac{M}{N}$ be a locally projective δ-cover and $\pi : M \to \frac{M}{N}$ the canonical mapping. Then there exists a decomposition $M = X \oplus Y$ such that $\pi|_X : X \to \frac{M}{N}$ is a locally projective δ-cover and $Y \subseteq ker\pi = N$. So, $X \cap N = ker\pi|_X$ which is locally δ-small in X. Therefore, $X \cap N$ is a locally δ-small in M.

$(2 \Rightarrow 1)$ Obvious. □

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Bounds on minimum distance for linear codes over $GF(q)$

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Abstract. Let $[n, k, d; q]$-codes be linear codes of length n, dimension k and minimum hamming distance d over $GF(q)$. Let $m_{47}(n, k)$ be the maximum possible minimum Hamming distance of a linear $[n, k, d; 47]$ - codes for given values of n and k. In this paper 21new linear codes over GF(47) are constructed, and a table of $m_{47}(n, k)k\leq$ 47, $n\leq$ 6267 is presented. First: we construct three Griesmer $[n, 3, d]_{47}-$ codes. Second: Also, a (k, r) - arc K corresponds to a projective $[k, n, d]_q$-code of length k, dimension n, and minimum distance $d = k - n$.

Keywords: bounds on $[n, k, d]q$-code, Griesmer codes, projective geometries, MDS-code, NMDS-code

1. Introduction

Let GF(q) denote the Galois field of q elements, and let $V(n, k)$ denote the vector space of all ordered n-tuples over $GF(q)$. A linear codes C of length n and dimension k over $GF(q)$ is a k-dimensional subspace of $V(n, q)$. Such a code is called $[n, k, d; q]$-code if its minimum Hamming distance is d.

A central problem in coding theory is that of optimizing one of the parameters n,k and d for given values of the other two. Two versions are:

Problem 1: Find $m_q(n, k)$, the largest value of d for which there exists an $[n, k, d; q]-$code.

Problem 2: Find $n_q(k, d)$, the smallest value of n for which there exists an $[n, k, d; q]$-code.

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A code which achieves one of these two values is called optimal.

For the case of linear codes over GF(47), problem 2 has been solved for $k \leq 3$ (see [11]). In this paper consider the next four dimensions. 21new linear codes over GF(47) are constructed, the nonexistence of sixteen linear codes is proved and a table of $m_q(n,k)k \leq 47, n \leq 6267$ is proved.

A number of researcher have studied the nonexistence of some (k,r)-arcs in $PG(2,q)$ have been proved by (Daskalov, 2004, 2008, 2013) in and (Najem, 2010), (Hirschfield, 2007), (Salam, 2018), (Khalied, 2013), when $q = 7, 17, 19, 23, 29, 31, 37, 41, 43$. Complete arcs have important connections with a number of other objects, see [2,3,4,5,6,7,8,9,10,12,13,14,15,16,19] and the references therein.

Definition 1.1. *A (k,r)-arc in PG(2,q) is a set K of k points, no r_+1 of which are collinear, but with at least one set of r points collinear. When $r = 2, a(k,2)$-arc is called a k-arc ([17]). The maximum size of a $(k,r)\_arc$ in $PG(2,q)$ is denoted by $m_r(2,q)$.*

Definition 1.2. *An $\{l,n\}-$blocking set S in $PG(2,q)$ is a set of I points such that every line of $PG(2,q)$ intersects S in at least n points, and there is a line intersecting S in exactly n points.*

Definition 1.3. *Let M be a set of point in any plane. An i-secant is a line meeting M in exactly I points. Define t_i as the number of i-secants to a set M.*

The t_i satisfy the next three Diophantine equations in any projective plane, which are known as the standard equations [5].

Definition 1.4 (Griesmer [18])**.** *The parameters of a linear $[n,k,d]_q$ code satisfy Griesmer bound $n \geq \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.*

Lemma 1.1. *For any set of k points in $PG(2,q)$ the following hold:*

1. $\sum_{i=0}^{q+1} t_i = q^2 + q + 1$,
2. $\sum_{i=1}^{q+1} it_i = k(q+1)$,
3. $\sum_{i=2}^{q+1} i(i-1)t_i = b(b-1)$.

Theorem 1.1. *Let K be a (k,r)- arc in $PG(2,q)$ where q is prime:*

1. *If $r \leq (q+1)/2$ then $m_r(2,q) \leq (r-1)q + 1$.*
2. *If $r \leq (q+3)/2$ then $m_r(2,q) \leq (r-1)q + r - (q+1)/2$.*

For $r = (q+3)/2$ the bound is achieved by the Barlotti construction [3]*. In this paper we shall prove that if $r > (q+3)/2$ then $m_r(2,q) \leq (r-1)q + r - (q+3)/2$ for $q = 47$.*

Theorem 1.2. *Let K be a (k,r)- arc in PG(2,q) with $r > (q+3)/2$ and q=47. Than $m_r(2,q) \leq (r-1)q + r - -(q+3)/2$.*

Proof. Finding a maximum $((r-1)q+r-(q+1)/2,r)\_arc$ is equivalent to finding a $q^2+q+1-(r-1)q-r+(q_+1)/2, q+1-r$ -blocking set B. Since:

$$q^2+q+1-(r-1)q-r+(q+1)/2=(q+1)(2q+3\_2r)/2.$$

Then B is a $(q+1)(2q+3\_2r)/2, q+1\_r-$ blocking set. □

Theorem 1.3. *Let B be an $\{l,n\}-$ blocking set in $PG(2,q)(q')$. Where $n=q+1-rl=n(q+1)+(q+1)/2$, then each point of B has exactly $(q+3)/2$ lines If through.*

Let $T=(q_+^2 1)(2q+3\_2r)/4(q+1-r)$ is a total number of $(q+1-r)-$secants.

Let t be the length of the longest secant for the blocking set B. If $t=q+1$ than B contains a line.

Theorem 1.4. *Let B be an $\{l,n\}-$-blocking set in $PG(2.q)$ that contains a line. If $(n-1,q)=1$, then: $B=1\geq q(n+1)$.*

It follows from this theorem that if $t=q+1$ then $B\geq q(q+2-r)$, which cannot occur since $r>(q+3)/2\leftrightarrow 0>1-r+(q+1)/2\rightarrow L=q^2+2q-rq>q^2+2q+1-rq-r+(q+1)/2=(q+1)(2q+3-2r)/2=B$.

If $3(q+1)/2-r<t\leq$ q then considering lines through a point on the longest secant but not in B. B must have at least $q(q+1-r)+t$ point. Now, we have a contradiction, since: $t>3(q+1)/2-r\rightarrow q(q+1--r)+t>q(q+1)-qr+3(q+1)/2-r=(q+1)(2q+3-2r)/2$.

Let $t_i=3\frac{48}{2}-r-i, i=0,1\ldots,\frac{q-1}{2}$.

Now, consider $(q+1-r)$-secants with the longest secant.

Let $t=t_i$. Suppose that p is a point on a t_i -secant L and P is not in B. Of the q other secants that P is on, let s of them be $(48-r)$- secants and $q-s$ be longer secants. Then:

$$B\geq 72+s(q+1-r)+(q--s)(q+2--r)=t_i+q(q+2-r)-s.$$

So, that $B-t_i=q(q+2-r)-q+i\geq q(q+2-r)-s$ making $s\geq q-i$, $T_q+1-r\geq t_i\frac{q_1}{2}+(q+1-ti)(q-i)=M_i$.

The inequality

$$M_i>T \tag{1}$$

is equivalent to $f(r)=2(q-2i+1)r^2-(3q^2-8iq+6q+4i^2\_8i+3)r+q^3+4q^2+3q-4iq^2+4i^2q-8iq+4i2-4i<0$.

It is clear that if $f((47+5)/2)<0$ and $f(47-1)<0$ for some values of I, then for these values condition (1) will be fulfilled for $r=(47+5)/2,\ldots,q-1$ and we shall have a contradiction

$$f(q+1/2)<0\leftrightarrow 2(q-3)i^2--(q-3)^2i-q^2+4q+5<0 \tag{2}$$

$$f(q-1)<0\leftrightarrow 8i^2-16i--q2+4q+5<0. \tag{3}$$

The solutions of inequalities (2) and (3) are as follows:
$i \in (\frac{(q-3)^2-\sqrt{q^4-4q^3\ -2q^2\ -52q+201}}{4\ (q-3)}.\ \frac{(q-3)^2+\sqrt{q^4-4q^3-2-52q+201}}{4(q-3)})$.
$i \in (1-\frac{1}{4}\sqrt{2q^2-8q+6}, 1+\frac{1}{4}\sqrt{2q^2-8q+6})$.
It is easy show that $1-\frac{1}{4}\sqrt{2q^2-8q+6}\ < 0$ and

$$\frac{(q-3)^2\ -\sqrt{q^4-4q^3-2q^2-52q+201}}{4(q-3)} < 0,$$

for $q \geq 7$.

Let

$$i_1 = 1+\frac{1}{4}\sqrt{2q^2-8q+6}, i_2 = \frac{(q-3)^2+\sqrt{q^4-4q^3-2q^2-52q+201}}{4(q-3)}v.$$

When $i_1 < i_2$, $q^4-4q^3-6q^2+4q+5 = (q-1)(q-5)(q+1)^2 \geq 0$ inequalities (2) and (3) are true when $0 \leq i < 1+\frac{1}{4}\sqrt{2q^2-8q+6}$.

Furthermore:

$$1+\frac{1}{4}\sqrt{2q^2-8q+6} \leq \frac{q-5}{2} \leftrightarrow (q-5)(q-19) \geq 0.$$

Hence $i_1 = 1+\frac{1}{4}\sqrt{2q^2-8q+6} \leq \frac{q-5}{2}$ for q ≥ 19.

Let $i = (q-3)/2$, in this case we shall use the standard equations for the set B. Let $x = t_{q-r+1}, y = t_{q-r+2}, z = t_{q-r+3}$, $a_1 = q-r+1, b1 = q-r+2, c_1 = q-r+3, a_2 = (q-r+1)(q-r), b_2 = (q-r+2)(q-r+1)$ and $c_2 = (q-r+3)(q-r+2)$.

Then the standard equation are $x+y+z = q^2+q+1, a_1x+b1y+c1z = (q+1)^2(q-r+3/2)$, $a_2x+b_2y+c2z = (q+1)(q-r+3/2)((q+1)(q-r+3/2)-1)$.

The unique solution of this system is $X = [q^2(4r-7)-4q(r^2-4r+3)+3]/8$, $Y = [4qr^2-4q(q+3)r+9q^2+12q+3]/4$, $Z = [q^2(4r-3)-4q(r-1)^2-1]/8$.

Let us consider the function $G(r) = 4qr^2-4q(q+3)r+9q^2+12q+3$.

This function increases when $(q+3)/2 < r < q$. therefore $q(r) \leq g(q-1) = -7q^2+28q+3 < 0$ if $q \geq 5$. So, $Y = t_{q-r+2} < 0$, a contradiction.

Let $i = (q-1)/2$. We shall use again the standard equation for the set B let $x = t_{q-r+1}, y = t_{q-r+2}, a = q-r+1, b = q-r+2, c = (q-r+1)$ and $d = (q-r+2)(q-r+1)$. Then the standard equation $Ax+by = (q+1)^2(q-r+3/2)$, $Cx+dy = (q+1)(q-r+3/2)((q+1)(q-r+3/2)-1)$. From the first two equations we obtain $y = -\frac{q+2q(1-r)-1}{2}, y = \frac{3q+2q(2-r)+1}{2}$. The difference between the left-hand side and right-hand side of the third equation now is $D(r) = qr^2-q(q+2)r+3q^2/4+q+1/4$.

The function D(r) is an increasing function when $(q+3)/2 < r < q$.

Hence, $D(q-1) = (-9q^2+16+1)/4 < 0$ when $q > (8+\sqrt{73})/9 \approx 1$ and so the system of standard equations no solutions.

On the basis of our observations up to now we can conclude that: $D_r(2,q) \leq (r-1)q+r-(q+3)/2$, $r > (q+3)/2$ and $7 \leq 17$.

If $q \geq 19$ then $i_1 \leq (q-5)/2$ (see(5)) and we have to check directly only the cases when $i_1 \leq i \leq \frac{q-5}{2}$. For $q = 47$ the results are as follow. Then $i_1 = 1 + \sqrt{253} \approx 16$. 96,(q-5) /2=21 and we check directly that inequality (1) is fulfilled for $i = 17, 18, 19, 20, 21$ and $r = 26, \ldots, 43$.

If $r = 44$ then $M21 = 1227$, $T = 1242$ and inequality (1) is not satisfied for $i = 21$, then $t_21 = 7$ and the respective system of standard equations in unknowns t_4, t_4, t_6 and t_7 has no solutions in nonnegative integers. More precisely, the standard equations are $t_4 + t_5 + t_6 + t_7 = 216$, $4t_4 + 5t_5 + 6t_6 + 7t_7 = 10368$, $12t_4 + 20t_5 + 30t_6 + 42t_7 = 46440$.

The solution of this system in term of t_7, $t_4 = -25380 - t_7, t_5 = 4168 + 3t_7, t_6 = -16092 - 3t_7$.

It follows now that $t_5 + t_6 = -11924$ and we have a contradiction.

If $r = 45$, then $M_{20} = 1268$, $M_{21} = 1230$, $T = 1288$ and inequality (1) is not satisfied for $i = 17, 18, 19, 20, 21$. If $i = 20$ then $t_{20} = 7$ and the respective system of standard equations in unknowns t_3, t_4, t_5, t_6 and t_7 has no solutions in nonnegative integers. More precisely, the standard equations are $t_3+t_4+t_5+t_6+t_7 = 168$, $3t_3+4t_4+5t_5+6t_6+7t_7 = 8064$, $6t_3+12t_4+20t_5+30t_6+42t_7 = 28056$.

The solution of this system in terms of t_6 and t_7 is $t_3 = -16548-t_6-3t_7, t_4 = 25872 + 3t_6 + 8t_7, t_5 = -9156 - 3t_6 - 6t7$, $t_4 + t_5 = -1192$ and we have a contradiction.

If $i = 21$ then $t_{21} = 6$ and the respective system of standard equations in unknowns t_3, t_4, t_5 and t_6, $t_3 + t_4 + t_5 + t_6 = 168$, $3t_3 + 4t_4 + 5t_5 + 6t_6 = 8064$, $6t_3 + 12t_4 + 20t_5 + 30t_6 = 28056$.

The solution of this system in terms of t_6, $t_3 = -16548 - t_6, t_4 = 2587 + 3t_6, t_5 = -9156 - 3t_6$ it follows now that $t_4 + t_5 = -6569$ and we have a contradiction.

If $r = 46$ then $M_{17} = 1377, M_{18} = 1344, M_{19} = 1309, M_{20} = 1272, M_{21} = 1233, T = 1380$ and inequality (1) is not satisfied for $i = 17, 18, 19, 20, 21$.

If i= 17 then t_{17}= 9 and the respective system of standard equations in unknowns t_2,t_3,t_4,t_5,t_6,t_7,t_8 and t_9 has no solutions in nonnegative integers. More precisely, the standard equations are $t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 = 120$, $2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 + 7t_7 + 8t_8 + 9t_9 = 5760$, $2t_2 + 6t_3 + 12t_4 + 20t_5 + 30t_6 + 42t_7 + 56t_8 + 72t_9 = 14280$

The solution of this system in terms of t_5, t_6, t_7, t_8, t_9 is $t_2 = -9420 - 5t_5 - 3t_6 - 6t_7 - 10t_8 - 15t_9, t_3 = 13560 + 3t_5 + 8t_6 + 15t_7 + 24t_8 + 35t_9, t_4 = -4020 - 3t_5 - 6t_6 - 10t_7 - 15t_8 - 12t_9$ it follows now that $t_3 + t_4 = -414$ and we have a contradiction.

If $i = 18$ then $t_{18} = 8$ and the respective system of standard equations in unknowns $t_2, t_3, t_4, t_5, t_6, t_7$ and t_8 has no solutions in nonnegative integers. More precisely, the standard equations are $t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 = 120$, $2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 + 7t_7 + 8t_8 = 5760$, $2t_2 + 6t_3 + 12t_4 + 20t_5 + 30t_6 + 42t_7 + 56t_8 = 14280$.

The solution of this system in terms of t_5, t_6, t_7, t_8 is $t_2 = -9420-5t_5-3t_6-6t_7-10t_8$, $t_3 = 13560+3t_5+8t_6+15t_7+24t_8$, $t_4 = -4020-3t_5-6t_6-10t_7-15t_8$.

It follows now that $t_3 + t_4 = -1060$ and we have a contradiction.

If $i = 19$ then $t_{19} = 7$ and the respective system of standard equations in unknowns t_2, t_3, t_4, t_5, t_6 and t_7 has no solutions in nonnegative integers. More precisely, the standard equations are $t_2 + t_3 + t_4 + t_5 + t_6 + t_7 = 120$, $2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 + 7t_7 = 5760$, $2t_2 + 6t_3 + 12t_4 + 20t_5 + 30t_6 + 42t_7 = 14280$.

The solution of this system in terms of t_5, t_6, t_7 is $t_2 = -9420 - 5t_5 - 3t_6 - 6t_7$, $t_3 = 13560 + 3t_5 + 8t_6 + 15t_7+$, $t_4 = -4020 - 3t_5 - 6t_6 - 10t_7$. It follow now that $t_3 + t_4 = -904$ and we have a contradictions.

If $i = 20$ then $t_{20} = 6$ and the respective system of standard equations in unknowns t_2, t_3, t_4, t_6 and $t6$ has no solutions in nonnegative integers More precisely, the standard equations are $t_2 + t_3 + t_4 + t_5 + t_6 = 120$, $2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 = 5760$, $2t_2 + 6t_3 + 12t_4 + 20t_5 + 30t_6 = 14280$.

The solution of this system in terms of t_5, t_6 is $t2 = -9420 - 5t_5 - 3t_6, t_3 = 13560 + 3t_5 + 8t_6, t_4 = -4020 - 3t_5 - 6t_6$.

It follow now that $t_3 + t_4 = -4770$ and we have a contradictions.

If $i = 21$ then $t_{21} = 5$ and the respective system of standard equations in unknowns t_2, t_3, t_4 and t_6 has no solutions in nonnegative integers More precisely, the standard equations are $t_2 + t_3 + t_4 + t_5 = 120$, $2t_2 + 3t_3 + 4t_4 + 5t_5 = 5760$, $2t_2 + 6t_3 + 12t_4 + 20t_5 = 14280$.

The solution of this system in terms of t_5 is $t2 = -9420 - 5t_5, t_3 = 1356 + 3t_5, t_4 = -4020 - 3t_5$. It follow now that $t_3 + t_4 = -2664$ and we have contradiction.

2. The related linear cods

Let $GF(q)$ denote the Galois field of q elements, end let $V(n, q)$ denote the vector space of all ordered n-tuples over $GF(q)$. A linear code C over $GF(q)$ of length n and dimension k is a k-dimensional subspace of $V(n, q)$. The vectors of C are called codewords. The Hamming distance between two codewords is defined to be number of coordinate places in which they differ. The minimum distance of a code is the smallest of the distances between distinct codewords. Such a code is called an $[n, k, d]_q$-code if its minimum Hamming distance is d. The code C is called projective if no two coordinate function on C are scalar multiples of each other.

Theorem 2.1. *There exists a projective $[n, 3, d]_q$ code if and only if there exists an $(n, n - d)-$ arc in PG(2,q).*

We know also that every Griesmer code with $d \leq q^{k-1}$ is projective [4]. Let $d = (r - i)q - (q + 1)/2$ and $(q + 3)/2 < r < q$. Since $[d] + [d/q^2] = (r - 1)q + r - (q + 1)/2$, using Theorems 2.2 and Theorem 2.1 we can formulate the following:

Corollary 2.1. *There do not exist Griesmer codes with parameters $[(r - 1)q + r - (q + 1)/2, 3, (r - 1)q - (q + 1)/2]_q$, where $(q + 3)/2 < r < q$ and $q = 47$.*

A central problem in coding theory is that of optimizing one of the parameters n, k and d for given values of the other two and q fixed.

Two versions are?

Problem 1: find $d_q(n,k)$, the largest value of d for which there exists $[n,k,d]_q$-code.

Problem 2: find $n_q(k,d)$, the smallest value of n for which there exists $[n,k,d]_q$-code.

The Griesmer bound states that $n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1}[\frac{d}{qj}]$. Codes with parameters $[g_q(k,d),k,d]_q$, are called Griesmer codes. There is a close relationship between $(n,r)-$ arcs in $PG(2.q)$ and $[n,3,d]_q-$codes, given by the next theorem [4].

25<r<47	$[n,3,d]_q$	$M_{q\text{-}}(n,k) \leq d$
26	$[1177,3,1151]_{47}$	$M_{47}(1177,3) \leq 1176$
27	$[1225,3,1198]_{47}$	$M_{47}(1225,3) \leq 1224$
28	$[1273,3,1245]_{47}$	$M_{47}(1273) \leq 1271$
29	$[1321,3,1295]_{47}$	$M_{47}(1321,3) \leq 1320$
30	$[1369,3,1339]_{47}$	$M_{47}(1369,3) \leq 1368$
31	$[1417,3,1386]_{47}$	$M_{47}(1417,3) \leq 1416$
32	$[1465,3,1433]_{47}$	$M_{47}(1465,3) \leq 1464$
33	$[1513,3,1480]_{47}$	$M_{47}(1513,3) \leq 1512$
34	$[1561,3,1527]_{47}$	$M_{47}(1561,3) \leq 1560$
35	$[1609,3,1575]_{47}$	$M_{47}(1609,3) \leq 1608$
36	$[1657,3,1621]_{47}$	$M_{47}(1657,3) \leq 1656$
37	$[1705,3,1668]_{47}$	$M_{47}(1705,3) \leq 1704$
38	$[1753,3,1715]_{47}$	$M_{47}(1753,3) \leq 1752$
39	$[1801,3,1762]_{47}$	$M_{47}(1801,3) \leq 1800$
40	$[1849,3,1809]_{47}$	$M_{47}(1849,3) \leq 1848$
41	$[1897,3,1856]_{47}$	$M_{47}(1897,3) \leq 1896$
42	$[1945,3,1903]_{47}$	$M_{47}(1945,3) \leq 1944$
43	$[1993,3,1950]_{47}$	$M_{47}(1993,3) \leq 1992$
44	$[2041,3,1997]_{47}$	$M_{47}(2041,3) \leq 2040$
45	$[2089,3,2044]_{47}$	$M_{47}(2089,3) \leq 2088$
46	$[2137,3,2091]_{47}$	$M_{47}(2137,3) \leq 2136$

3. Conclusions

1. We finding new largest Bounded:

   $M_r(2,47) \leq 2040$

   $M_r(2,47) \leq 2088$

   $M_r(2,47) \leq 2136$

2. Also we finding bounded of linear $[n,3,d]_q$ codes not exists

[2041,3,1997]47

[2089,3,2044]47

[2137,3,2091]47

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Graphs whose completely regular endomorphisms form a monoid

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Abstract. In this paper, we show that if the completely regular endomorphisms of $X + Y$ form a monoid, then the completely regular endomorphisms of X and Y form a monoid respectively. We give several approaches to construct new graphs whose completely regular endomorphisms form a monoid. In particular, we determine trees and the joins of trees whose completely regular endomorphisms form a monoid.
Keywords: endomorphism, monoid, completely regular, tree.

1. Introduction and preliminary concepts

As is known, endomorphism monoids of graphs are generalizations of automorphism groups of graphs. Endomorphism monoids of graphs have been studied for a long time and many interesting results concerning graphs and their endomorphism monoids have been obtained (cf. [3], [7-8], [12], [14-16]). The endomorphism monoids of graphs have valuable applications (cf. [10]) and are related to automata theory (cf. [11], [13]). Let X be a graph. Denote by $End(X)$ the set of all endomorphisms of X. It is known that $End(X)$ forms a monoid with respect to composition of mappings. We call $End(X)$ the endomorphism monoid of graph X. An element a of a semigroup S is said to be *completely regular* if $a = axa$ and $xa = ax$ hold for some $x \in S$. Let $f \in End(X)$. Then f is called a *completely regular endomorphism* of X if it is a completely regular element in $End(X)$. Denote by $cEnd(X)$ the set of all completely regular endomorphisms of graph X. For a monoid S, the composition of two completely regular elements of S is not completely regular in general. So it is natural to ask: Under what conditions does the set $cEnd(X)$ form a monoid for a graph X?

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However, it seems difficult to obtain a general answer to this question. Therefore a natural strategy for dealing with this question is to find various kinds of conditions for various kinds of graphs. In [9], completely regular endomorphisms of split graphs were characterized and the conditions under which the completely regular endomorphisms of split graphs form a monoid were given. In this paper, we show that if $cEnd(X+Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form monoids. We give several approaches to construct new graphs whose completely regular endomorphisms form monoids. In particular, we determine trees and the joins of trees whose completely regular endomorphisms form monoids.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in graph X, the edge joining x_1 and x_2 is denoted by $\{x_1, x_2\}$. Let $v \in V(X)$. Set $N(v) = \{x \in V(X) | \{x, v\} \in E(X)\}$. The cardinality of $N(v)$ is called the *degree* of v in X and is denoted by $d(v)$. A connected graph X is called a *tree* if and only if it has no cycle. A *path* with n vertices is written as P_n. If X has an u, v-path, then the *distance* from u to v, denoted by $d(u, v)$, is the least length of the u, v-path. The *diameter* of X is the greatest distance between any two vertices in X, denoted by $diam(X)$. Let X and Y be two graphs. The *join* of X and Y, denoted by $X + Y$, is a graph such that $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y) \cup \{\{a, b\} | a \in V(X), b \in V(Y)\}$.

Let X and Y be graphs. An adjacency preserving mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* from X to Y, i.e. for any $x_1, x_2 \in V(X)$, if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp. isomorphism) f from X to itself is called an *endomorphism* (resp. *automorphism*) of X. The sets of all endomorphisms and automorphisms of X are written as $End(X)$ and $Aut(X)$, respectively. Always, we denote an endomorphism f of a graph X with $V(X) = \{v_1, v_2, \cdots, v_n\}$ in the obvious sense as

$$f = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_{i_1} & v_{i_2} & \cdots & v_{i_n} \end{pmatrix}.$$

i.e. a mapping f from $V(X)$ to itself such that $f(v_1) = v_{i_1}$, $f(v_2) = v_{i_2}$, $\cdots$, $f(v_n) = v_{i_n}$.

A *proper coloring* of a graph X is a map from $V(X)$ into some finite set of colors such that no two adjacent vertices are assigned the same colors. If X can be properly colored with a set of k colors, then we say that X can be properly k-colored. The least value of k for which X can be properly k-colored is the *chromatic number* of X, and is denoted by $\chi(X)$. We know that if there is a homomorphism from X to Y, then $\chi(X) \leq \chi(Y)$. A *retraction* of a graph X is a homomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ of f to $V(Y)$ is the identity mapping on $V(Y)$. It is known that the idempotents of $End(X)$ are retractions of X. The set of all idempotents of $End(X)$ is denoted by $Idpt(X)$. Let $f \in End(X)$. A subgraph of X is called the *endomorphic image*

of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e. for any $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. The equivalence class of a under ρ_f is denoted by $[a]_{\rho_f}$.

We use the standard terminology and notation of semigroup theory as in [4,11] and of graph theory as in [2,5]. We list some known results which are used within this paper.

Lemma 1.1 ([16]). *Let X be a graph and let $f \in End(X)$. Then f is completely regular if and only if $f|_{I_f} \in Aut(I_f)$.*

Lemma 1.2 ([1]). *Let X be a bipartite graph. Then X is End-completely-regular if and only if X is one of K_1, K_2, P_3, $2K_1$, $2K_2$ and $K_1 \bigcup K_2$.*

Lemma 1.3 ([6]). *Let X and Y be two bipartite graphs. Then $X + Y$ is End-completely-regular if and only if one of them is End-completely-regular and the other is K_1 or K_2.*

Lemma 1.4 ([8]). *Let X and Y be two K_3-free graphs. If both of them are non-bipartite, then for any endomorphism f of $X + Y$, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(X) \subseteq Y$ and $f(Y) \subseteq X$.*

2. Trees whose completely regular endomorphisms form a monoid

In this section, we investigate the completely regular endomorphisms of a tree and give the conditions under which all completely regular endomorphisms of the tree form a monoid.

Lemma 2.1. *Let X be a graph and $f \in End(X)$. Then f is completely regular if and only if $f(a) \neq f(b)$ for any $a, b \in V(I_f)$ with $a \neq b$.*

Proof. This follows directly from Lemma 1.1. □

Lemma 2.2. *Let X be a graph and Y be a retract of X. If $cEnd(X)$ forms a monoid, then $cEnd(Y)$ forms a monoid.*

Proof. Let $g_1, g_2 \in cEnd(Y)$. We only need to prove $g_1 g_2 \in cEnd(Y)$. As Y is a retract of X, there exists a retraction h from X to Y such that the restriction $h|_Y$ of h to $V(Y)$ is the identity mapping on $V(Y)$. Then $g_1 h, g_2 h \in End(X)$. It is easy to check that $I_{g_1 h} = I_{g_1}$ and $I_{g_2 h} = I_{g_2}$. Let $a, b \in V(I_{g_1})$ be such that $a \neq b$. Then $g_1 h(a) = g_1(a)$ and $g_1 h(b) = g_1(b)$. Since g_1 is completely regular, by Lemma 2.1 $g_1(a) \neq g_1(b)$. Thus $g_1 h(a) \neq g_1 h(b)$. Hence $g_1 h \in cEnd(X)$. A similar argument will show that $g_2 h \in cEnd(X)$.

Denote $f = g_1 h g_2 h = g_1 g_2 h$. Then $f \in End(X)$. It is easy to check that $I_f = I_{g_1 g_2}$. Since $cEnd(X)$ forms a monoid, f is completely regular. Let $c, d \in V(I_{g_1 g_2})$ be such that $c \neq d$. Then $c, d \in V(I_f)$. Note that f is completely

regular. By Lemma 2.1, $f(c) \neq f(d)$. This means that $g_1g_2h(c) \neq g_1g_2h(d)$. Now $g_1g_2(c) = g_1g_2h(c)$ and $g_1g_2(d) = g_1g_2h(d)$. Thus $g_1g_2(c) \neq g_1g_2(d)$. By Lemma 2.1, $g_1g_2 \in cEnd(Y)$. □

Lemma 2.3. *Let X be a graph and Y be a retract of X. If $cEnd(Y)$ does not form a monoid, then $cEnd(X)$ does not form a monoid.*

Proof. This follows directly from Lemma 2.2. □

Now we start to seek the conditions for a tree T under which $cEnd(T)$ forms a monoid

Lemma 2.4. *Let T be a tree. If $diam(T) \geq 3$, then $cEnd(T)$ does not form a monoid.*

Proof. Let T be a tree. If $diam(T) \geq 3$, then P_4 (see Fig.1) is a retract of T. By Lemma 2.3, we only need to show that $cEnd(P_4)$ does not form a monoid. Let

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

Then $f \in Idpt(P_4)$, $g \in Aut(P_4)$. So $f, g \in cEnd(P4)$. Now

$$fg = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_2 & x_1 \end{pmatrix}.$$

It is not hard to see that $x_1, x_3 \in V(I_{fg})$. But $fg(x_1) = fg(x_3) = x_2$. By Lemma 2.1, fg is not completely regular. Hence $cEnd(P_4)$ does not form a monoid. □

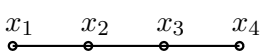


Fig.1 Graph P_4

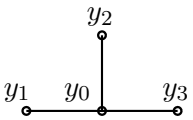


Fig.2 Graph S_3

Lemma 2.5. *Let T be a tree. If there exists $x \in V(T)$ such that $d(x) \geq 3$, then $cEnd(T)$ does not form a monoid.*

Proof. Let T be a tree. If there exists $x \in V(T)$ such that $d(x) \geq 3$, then the star S_3 (see Fig.2) is a retract of T. We only need to show that $cEnd(S_3)$ does not form a monoid. Let

$$f = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_1 & y_3 \end{pmatrix}.$$

and

$$g = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_1 & y_3 & y_3 \end{pmatrix}.$$

Then $f \in Aut(S_3)$, $g \in Idpt(S_3)$. So $f, g \in cEnd(S_3)$. Now

$$fg = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_3 & y_3 \end{pmatrix}.$$

It is easy to see that $y_2, y_3 \in V(I_{fg})$. But $fg(y_2) = fg(y_3) = y_3$. By Lemma 2.1, fg is not completely regular. Hence $cEnd(S_3)$ does not form a monoid. □

Theorem 2.1. *Let T be a tree. Then $cEnd(T)$ forms a monoid if and only if T is one of K_1, K_2 and P_3.*

Proof. Necessity. This follows from Lemmas 2.4 and 2.5.

Sufficiency. By Lemma 1.2, K_1, K_2 and P_3 are End-completely-regular. This means that $End(T) = cEnd(T)$ for $T \in \{K_1, K_2, P_3\}$. Therefore, $cEnd(T)$ forms a monoid. □

3. Graphs whose completely regular endomorphisms form a monoid

In this section, we show that if $cEnd(X+Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form a monoid. We give several approaches to construct new graphs whose completely regular endomorphisms form a monoid.

Lemma 3.1. *Let X and Y be two graphs. If $cEnd(X + Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form a monoid.*

Proof. Let $f_1, f_2 \in cEnd(X)$. Then $f_1 f_2 \in End(X)$. To prove that $cEnd(X)$ forms a monoid, we only need to prove that $f_1 f_2 \in cEnd(X)$. By Lemma 2.1, we only need to show that $f_1 f_2(x_1) \neq f_1 f_2(x_2)$ for any $x_1, x_2 \in I_{f_1 f_2}$ with $x_1 \neq x_2$. Define two mappings from $X + Y$ to itself by

$$F_1(x) = \begin{cases} f_1(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

and

$$F_2(x) = \begin{cases} f_2(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

Then $F_1, F_2 \in End(X+Y)$. It is not difficult to see that $I_{F_1} = I_{f_1}+Y$. Note that $F_1(x) = f_1(x) \in V(X)$ for any $x \in V(X)$ and $f_1 \in cEnd(X)$. Then $F_1|_{I_{f_1}} = f_1|_{I_{f_1}} \in Aut(I_{f_1})$. On the other hand, $F_1(x) = x$ for any $x \in V(Y)$. This means $F_1|_Y$ is an identity mapping. Then $F_1|_Y \in Aut(Y)$. Thus $F_1|_{I_{F_1}} \in Aut(I_{F_1})$. Hence $F_1 \in cEnd(X+Y)$. A similar argument will show that $F_2 \in cEnd(X+Y)$.

Since $cEnd(X+Y)$ forms a monoid, $F_1F_2 \in cEnd(X+Y)$. Furthermore,

$$F_1F_2(x) = \begin{cases} f_1f_2(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

Let $x_1, x_2 \in I_{f_1f_2}$ be such that $x_1 \neq x_2$. Then $x_1, x_2 \in I_{F_1F_2}$. By Lemma 2.1, $F_1F_2(x_1) \neq F_1F_2(x_2)$. Note that $F_1F_2(x_1) = f_1f_2(x_1)$ and $F_1F_2(x_2) = f_1f_2(x_2)$. Then $f_1f_2(x_1) \neq f_1f_2(x_2)$. By Lemma 2.1, $f_1f_2 \in cEnd(X)$. Hence $cEnd(X)$ forms a monoid. A similar argument will show that $cEnd(Y)$ forms a monoid. □

However, both $cEnd(X)$ and $cEnd(Y)$ form a monoid can not yield that $cEnd(X+Y)$ forms a monoid. In the following, we give an example in the range of trees.

Example 3.1. *Let $T_1 = T_2 = P_3$. By Theorem 2.1, both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid.*

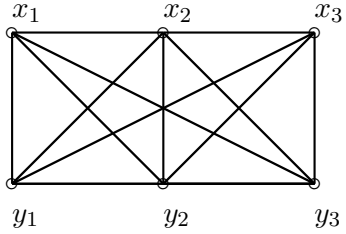


Fig.3 Graph $P_3 + P_3$

Let

$$f_1 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ x_3 & x_2 & x_3 & y_1 & y_2 & y_3 \end{pmatrix}.$$

Then $f_1 \in Aut(P_3+P_3)$ and $f_2 \in Idpt(P_3+P_3)$. Thus $f_1, f_2 \in cEnd(P_3+P_3)$. Now

$$f_1f_2 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ y_3 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}.$$

It is easy to check that $x_1, x_3 \in V(I_{f_1f_2})$. But $f_1f_2(x_1) = f_1f_2(x_3) = y_3$. By Lemma 2.1, f_1f_2 is not completely regular. Thus $cEnd(P_3 + P_3)$ does not form a monoid.

Theorem 3.1. *Let T_1 and T_2 be two trees. Then $cEnd(T_1+T_2)$ forms a monoid if and only if*

(1) *Both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid, and*

(2) $T_1 + T_2 \neq P_3 + P_3$.

Proof. Necessity. This follows from Lemma 3.1 and Example 3.1.

Sufficiency. If both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid and $T_1 + T_2 \neq P_3 + P_3$, then $T_1 + T_2 \in \{K_1 + K_1, K_1 + K_2, K_1 + P_3, K_2 + K_2, K_2 + P_3\}$. By Lemma 1.3, they are End-completely-regular. This means that $End(T_1 + T_2) = cEnd(T_1 + T_2)$. Therefore, $cEnd(T_1 + T_2)$ forms a monoid. □

In the following, we give some sufficient conditions under which $cEnd(X+Y)$ forms a monoid.

Lemma 3.2. *Let X and Y be two graphs such that both $cEnd(X)$ and $cEnd(Y)$ form monoids. If for any $f \in End(X + Y)$, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, then $cEnd(X + Y)$ forms a monoid.*

Proof. Let $f, g \in cEnd(X + Y)$. Then $fg \in End(X + Y)$. Denote $f_1 = f|_X$, $f_2 = f|_Y$, $g_1 = g|_X$ and $g_2 = g|_Y$. Note that $f(X) \subseteq X$ and $f(Y) \subseteq Y$ for any $f \in End(X + Y)$. Then $f_1, g_1 \in End(X)$, $f_2, g_2 \in End(Y)$ and $I_{fg} = I_{f_1g_1} + I_{f_2g_2}$. Since f is completely regular, $f_1(x_1) = f(x_1) \neq f(x_2) = f_1(x_2)$ for any $x_1, x_2 \in I_{f_1}$ and $f_2(y_1) = f(y_1) \neq f(y_2) = f_2(y_2)$ for any $y_1, y_2 \in I_{f_2}$. Hence $f_1 \in cEnd(X)$ and $f_2 \in cEnd(Y)$ by Lemma 2.1. Similarly, we have $g_1 \in cEnd(X)$ and $g_2 \in cEnd(Y)$. Note that $cEnd(X)$ and $cEnd(Y)$ form monoids. Then $f_1g_1 \in cEnd(X)$ and $f_2g_2 \in cEnd(Y)$.

Let $x, y \in V(I_{fg})$ with $x \neq y$. If $x, y \in V(I_{f_1g_1})$, then $fg(x) = f_1g_1(x) \neq f_1g_1(y) = fg(y)$. Similarly, if $x, y \in I_{f_2g_2}$, then $fg(x) \neq fg(y)$. If $x \in I_{f_1g_1}$ and $y \in I_{f_2g_2}$, then $fg(x) \in V(X)$ and $fg(y) \in V(Y)$. Clearly, $fg(x) \neq fg(y)$. By Lemma 2.1, $fg \in cEnd(X + Y)$. Therefore, $cEnd(X + Y)$ forms a monoid. □

Theorem 3.2. *Let X and Y be two graphs such that both $cEnd(X)$ and $cEnd(Y)$ form monoids. Then:*

(1) *If X and Y are two K_3-free non-bipartite graphs and the cores of X and Y are not isomorphic, then $cEnd(X + Y)$ forms a monoid.*

(2) *If X is a bipartite graph and Y is a K_3-free non-bipartite graph, then $cEnd(X + Y)$ forms a monoid.*

(3) *If X is a K_3-free non-bipartite graph and Y has at least one triangle with $\chi(Y) < \chi(X) + 1$, then $cEnd(X + Y)$ forms a monoid.*

Proof. (1) Let X and Y be two K_3-free non-bipartite graphs and $f \in End(X + Y)$. By Lemma 1.4, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(Y) \subseteq X$ and $f(X) \subseteq Y$. In the second case, the restriction $f|_X$ of f to $V(X)$ is a homomorphism from X to Y and the restriction $f|_Y$ of f to $V(Y)$ is a homomorphism from Y to X. Thus X and Y are homomorphically equivalent. Hence the cores of X and Y

are isomorphic. This is a contradiction. Therefore, $f(X) \subseteq X$ and $f(Y) \subseteq Y$. By Lemma 3.2, $cEnd(X+Y)$ forms a monoid.

(2) Let $f \in End(X+Y)$. We hope to prove that $f(X) \subseteq X$ and $f(Y) \subseteq Y$. There are two cases.

Case 1. Assume that $E(X) = \phi$. First, we claim that $f(X) \subseteq X$. Otherwise, there exists $x \in V(X)$ such that $f(x) \in Y$. As Y is non-bipartite, Y has an odd cycle. Note that every endomorphism image of an odd cycle must contain an odd cycle. Then $f(Y)$ contains an odd cycle. Note that $E(X) = \phi$. Then $f(Y)$ has an edge in Y, say $\{f(y_1), f(y_2)\}$. Thus $f(y_1), f(y_2), f(x)$ form a triangle in Y. This contradicts that Y is K_3-free.

Next, we prove $f(Y) \subseteq Y$. Otherwise, there exists $y \in V(Y)$ such that $f(y) \in V(X)$. Let $x \in V(X)$. Then $\{y, x\} \in E(X+Y)$. It follows that $\{f(y), f(x)\} \in E(X+Y)$. As $f(y), f(x) \in V(X)$, $\{f(y), f(x)\} \in E(X)$. This is impossible.

Case 2. Assume that $E(X) \neq \phi$. First, we prove that $f(X) \subseteq V(X)$. Suppose for contradiction that $f(X) \not\subseteq V(X)$. Then either $f(X) \subseteq V(Y)$, or there exist $x_1, x_2 \in V(X)$ such that $f(x_1) \in V(X)$ and $f(x_2) \in V(Y)$. In the first case, since X contains at least one edge, $f(X)$ contains at least one edge, say $\{a, b\}$. Now we claim that $f(Y) \subseteq V(X)$. Otherwise, there exists $y_0 \in V(Y)$ such that $f(y_0) \in V(Y)$. Then $a, b, f(y_0)$ form a triangle in Y. This contradicts that Y is K_3-free. Hence $f|_Y$ is a homomorphism from Y to X, and so $\chi(Y) \leq \chi(X)$. Note that $\chi(X) = 2$ and $\chi(Y) \geq 3$. This is a contradiction. In the second case, since Y has an odd cycle, $f(Y)$ also has an odd cycle. Thus $f(Y)$ either has an edge in X or has an edge in Y. If there exist $y_1, y_2 \in V(Y)$ such that $\{f(y_1), f(y_2)\} \in E(Y)$, then $f(x_2), f(y_1), f(y_2)$ form a triangle in Y. This contradicts that Y is K_3-free. If there exist $y_1, y_2 \in V(Y)$ such that $\{f(y_1), f(y_2)\} \in E(X)$, then $f(x_1), f(y_1), f(y_2)$ form a triangle in X. This is impossible since X is bipartite. Hence $f(X) \subseteq X$.

Next, we prove that $f(Y) \subseteq Y$. Otherwise, there exists $y \in V(Y)$ such that $f(y) \in V(X)$ and $f(y)$ is adjacent to every vertex of $f(X)$. Note that $f(X)$ contains at least one edge in X, say $\{f(x_1), f(x_2)\}$. Thus $f(x_1), f(x_2), f(y)$ form a triangle in X. This is a contradiction. Now the assertion follows directly from Lemma 3.2.

(3) We show that $f(Y) \not\subseteq X$. Otherwise, $f|_Y$ is a homomorphism from Y to X. Note that any homomorphism f maps a triangle to a triangle and Y contains at least one triangle. Then X must contain a triangle. This contradicts that X is K_3-free. Hence either $f(Y) \subseteq Y$, or there exist $y_1, y_2 \in V(Y)$ such that $f(y_1) \in V(Y)$ and $f(y_2) \in V(X)$.

In the second case, if $f(X) \subseteq V(X)$, then $f|_X$ is a homomorphism from X to itself. Hence $\chi(X) = \chi(I_{f|_X})$. On the other hand, $f(y_2)$ is adjacent to every vertex of $I_{f|_X}$. Hence $\chi(X) \geq \chi(I_{f|_X}) + 1$. This is a contradiction. If $f(X) \subseteq V(Y)$, then $f|_X$ is a homomorphism from X to Y. Note that $f(y_1)$ is adjacent to every vertex of $I_{f|_X}$. Then $\chi(Y) \geq \chi(I_{f|_X}) + 1 \geq \chi(X) + 1$. A contradiction. If there exist $x_1, x_2 \in V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$,

then both $f(X)$ and $f(Y)$ have no edge in X, otherwise, there exists a triangle in X. Denote $A = f(X) \cap V(X)$ and $B = f(Y) \cap V(X)$. Then the subgraph H of X induced by $A \cup B$ is bipartite. It is easy to see that I_f is a subgraph of $H + Y$. It follows that $\chi(I_f) \leq \chi(H + Y) = \chi(Y) + 2$. On the other hand, $\chi(X + Y) = \chi(X) + \chi(Y) \leq \chi(I_f)$. Hence $\chi(X) + \chi(Y) \leq \chi(Y) + 2$. This means that $\chi(X) \leq 2$. This is a contradiction since X is non-bipartite. Hence $f(Y) \subseteq Y$.

If $f(X) \not\subseteq X$, then there exists $x \in V(X)$ such that $f(x) \in V(Y)$. Clearly, $f(x)$ is adjacent to every vertex in $V(I_{f|_Y})$. Thus we have $\chi(Y) \geq \chi(I_{f|_Y}) + 1 \geq \chi(Y) + 1$. This is a contradiction. Hence $f(X) \subseteq X$. By Lemma 3.2, $cEnd(X + Y)$ forms a monoid. □

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Bochner-Martinelli type formula over the quaternionic Heisenberg group and the octonionic Heisenberg group

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Abstract. The tangential k-Cauchy-Fueter operator on the quaternionic Heisenberg group is counterpart of the tangential Cauchy-Riemann operator on the Heisenberg group in the theory of several complex variables. In this paper, we establish the Bochner-Martinelli type formula for tangential k-Cauchy-Fueter type operators over the quaternionic Heisenberg group and the octonionic Heisenberg group.

Keywords: the tangential k-Cauchy-Fueter operator, the Bochner-Martinelli formula, the quaternionic Heisenberg group, the octonionic Heisenberg group.

1. Introduction

The k-Cauchy-Fueter complex has been extensively studied (cf. [2,5,6,20,22] and references therein), which are used to investigate properties of k-regular functions (cf. [7, 20] and references therein), and 1-Cauchy-Fueter operator is exactly the usual Cauchy-Fueter operator in several quaternionic variables.

Quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ is the simplest model of quaternionic contact manifolds (cf. [3, 8, 9] and references therein for more information about quaternionic contact manifolds). A natural generalization of k-Cauchy-Fueter operator on the quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ is the tangential k-Cauchy-Fueter operator, which is quaternionic counterpart of the tangential Cauchy-Riemann operator on the Heisenberg group in the theory of several complex variables. More generally, we can also define this kinds of operators on the octonionic Heisenberg group.

The Bochner-Martinelli integral representation formula in several complex variables was appeared in the works of Martinelli and Bochner at the begin-

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ning of the 1940's. It have a lot of applications in several complex variables. See [11] for more details of the Bochner-Martinelli integral representation formula in several complex variables. The Bochner-Martinelli type formula for k-Cauchy-Fueter operator was obtained in [20]. H.-Y. Wang and G.-B. Ren found the explicit Bochner-Martinelli kernel [17]. Moreover they have established the Bochner-Martinelli formula for a regular type operator in octonionic space [16]. It was generalized to a massless field operator on $\mathbb{R}^6$ by Q.-Q. Kang and W. Wang [10]. Recently we have generalised Bochner-Martinelli formula to k-Cauchy-Fueter operator over $(4n+1)$-dimensional Heisenberg group [12].

In this paper, we will generalise the Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operators over the quaternionic Heisenberg group and the octonionic Heisneberg group.

In Section 2, we introduce the basic knowledge of quaternions, quaternionic Heisenberg group and octonions, octonionic Heisneberg group. We also give the definition of the tangential k-Cauchy-Fueter operator on quaternionic Heisenberg group and octionionic Heisenberg group, and their basic properties. In Section 3, we give the Bochner-Martinelli formula for tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group. In Section 4, we give the Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group.

2. Primaries

2.1 The quaternionic Heisenberg group and the tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group

The *quaternionic Heisenberg group* $\mathscr{H}_{\mathbb{H}}$ is $\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H}$ equipped with the multiplication given by

$$(y,s)\cdot(y',s') = \left(y+y', s+s'+2\mathrm{Im}\left(y\bar{y'}\right)\right),$$

where $y, y' \in \mathbb{H}^n$ and $s, s' \in \mathrm{Im}\,\mathbb{H}$. The norm of quaternionic Heisenberg group is defined by

$$\|(y,s)\| = \left(|y|^4 + |s|^2\right)^{\frac{1}{4}},$$

where $(y,s) = (y_1, \cdots, y_{4n}, s_1, s_2, s_3)$.

The multiplication of the quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ can be written in terms of real variables (cf. [19, (2.13)]) as

$$(y,s)\cdot(y',s') = \left(y+y', s_\beta + s'_\beta + 2\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B^{\beta}_{kj} y_{4l+k} y'_{4l+j}\right),$$

for $y, y' \in \mathbb{R}^{4n},\ s, s' \in \mathbb{R}^3,\ \beta = 1, 2, 3$, where B^β_{kj} is the (k, j)-th entry of the following matrices

$$B^1 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^2 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$B^3 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

satisfying the commutating relation of quaternions $(B^1)^2 = (B^2)^2 = (B^3)^2 = -\mathrm{id},\ B^1B^2 = B^3$. Then

$$Y_{4l+j} := \partial_{y_{4l+j}} + 2\sum_{\beta=1}^{3}\sum_{k=1}^{4} B^\beta_{kj} y_{4l+k}\partial_{s_\beta},$$

are the left invariant vector fields of $\mathscr{H}_{\mathbb{H}}$, whose brackets are

$$[Y_{4l+k}, Y_{4l+j}] = 4\sum_{\beta=1}^{3} B^\beta_{kj}\partial_{s_\beta}, \qquad \text{and} \qquad [Y_{4l+k}, Y_{4l'+j}] = 0 \quad \text{for } l \neq l',$$

where $l, l' = 0, 1, \cdots, n-1,\ j, k = 1, \cdots, 4$. We denote *SubLaplacian* $\Delta_{\mathbb{H}}$ *on the quaternionic Heisenberg group* by

$$\Delta_{\mathbb{H}} := -\sum_{k=1}^{4n} Y_k^2. \tag{2.1}$$

The standard $\mathbb{R}^3$-valued contact form of $\mathscr{H}_{\mathbb{H}}$ is

$$\Theta_{\mathbb{H}} := \mathrm{d}s - y \cdot \mathrm{d}\bar{y} + \mathrm{d}y \cdot \bar{y}.$$

If we write $\Theta_{\mathbb{H}} = (\theta_{\mathbb{H};1}, \theta_{\mathbb{H};2}, \theta_{\mathbb{H};3})$, then we have

$$\theta_{\mathbb{H};\beta} = \mathrm{d}s_\beta - 2\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B^\beta_{kj} y_{4l+k}\mathrm{d}y_{4l+j}, \quad \beta = 1, 2, 3,$$

(cf. [19, (2.20)]). Denote

$$(Z_{AA'}) := \begin{pmatrix} Y_1 + \mathbf{i}Y_2 & -Y_3 - \mathbf{i}Y_4 \\ Y_3 - \mathbf{i}Y_4 & Y_1 - \mathbf{i}Y_2 \\ \vdots & \vdots \\ Y_{4l+1} + \mathbf{i}Y_{4l+2} & -Y_{4l+3} - \mathbf{i}Y_{4l+4} \\ Y_{4l+3} - \mathbf{i}Y_{4l+4} & Y_{4l+1} - \mathbf{i}Y_{4l+2} \\ \vdots & \vdots \end{pmatrix}$$

they are complex left invariant vector fields, where $A = 0, 1, \cdots, 2n-1$, $A' = 0', 1'$. It is motivated by the embedding τ of quaternionic algebra $\mathbb{H}$ into $\mathfrak{gl}(2, \mathbb{C})$:

$$\tau(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}.$$

We will use matrices

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower primed indices, e.g. $Z_A^{A'} = \sum_{B'=0',1'} Z_{AB'}\varepsilon^{B'A'}$. Here $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. Then we get

$$\left(Z_A^{A'}\right) := \begin{pmatrix} -Y_3 - \mathbf{i}Y_4 & -Y_1 - \mathbf{i}Y_2 \\ Y_1 - \mathbf{i}Y_2 & -Y_3 + \mathbf{i}Y_4 \\ \vdots & \vdots \\ -Y_{4l+3} - \mathbf{i}Y_{4l+4} & -Y_{4l+1} - \mathbf{i}Y_{4l+2} \\ Y_{4l+1} - \mathbf{i}Y_{4l+2} & -Y_{4l+3} + \mathbf{i}Y_{4l+4} \\ \vdots & \vdots \end{pmatrix}. \tag{2.2}$$

We denote $\odot^p\mathbb{C}^2$ by the p-th symmetric power of $\mathbb{C}^2$. Its element is denoted by an 2^p-tuple $(f_{A'_1\cdots A'_p})$, $A'_1, \cdots, A'_p = 0', 1'$, which are invariant under permutations of subscripts. The *tangential k-Cauchy-Fueter operator* $\mathscr{D}_0^{(k)}$ *over* $\mathscr{H}_{\mathbb{H}}$ is given by

$$\left(\mathscr{D}_0^{(k)} f\right)_{AA'_2\cdots A'_k} := \sum_{A'_1=0',1'} Z_A^{A'_1} f_{A'_1A'_2\cdots A'_k},$$

for $f \in C^1(\Omega, \odot^k\mathbb{C}^2)$ and $k = 1, 2, \cdots$ (cf. [21, (2.19)]). We have isomorphisms

$$\odot^k\mathbb{C}^2 \cong \mathbb{C}^{k+1}, \qquad \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n} \cong \mathbb{C}^{2nk},$$

by identifying $f \in \odot^k\mathbb{C}^2$ and $F \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}$ with

$$f := \begin{pmatrix} f_{0'0'0'\cdots0'} \\ f_{1'0'0'\cdots0'} \\ f_{1'1'0'\cdots0'} \\ \vdots \\ f_{1'1'1'\cdots1'} \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} F_{0'0'\cdots0'0} \\ F_{0'0'\cdots0'1} \\ \vdots \\ F_{0'0'\cdots0'(2n-1)} \\ F_{1'0'\cdots0'0} \\ \vdots \\ F_{1'1'\cdots1'0} \\ \vdots \\ F_{1'1'\cdots1'(2n-1)} \end{pmatrix},$$

respectively. In the following of this paper we write $\mathscr{D}_0$ instead of $\mathscr{D}_0^{(k)}$ for simplicity. On a domain $\Omega \subset \mathscr{H}_{\mathbb{H}}$, denote the inner product

$$(u,v)_{\mathbb{H}} := \int_\Omega u \cdot \overline{v} \mathrm{d}V_{\mathbb{H}},$$

for $u, v \in L^2(\Omega, \mathbb{C})$, where

$$\mathrm{d}V_{\mathbb{H}} = \theta_{\mathbb{H};1} \wedge \theta_{\mathbb{H};2} \wedge \theta_{\mathbb{H};3} \wedge (\mathrm{d}\theta_{\mathbb{H};\beta})^{2n}, \quad \beta = 1,2,3$$

is the volume form on $\mathscr{H}_{\mathbb{H}}$. The inner product of $L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n})$ is defined as

$$\langle f, h \rangle_{\mathbb{H}} := \sum_{A=0}^{2n-1} \sum_{A'_2,\cdots,A'_k=0',1'} \left(f_{AA'_2\cdots A'_k}, h_{AA'_2\cdots A'_k} \right)_{\mathbb{H}}$$

for $f, h \in L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n})$. Now we list some properties of $Z_A^{A'}$'s and $\mathscr{D}_0$'s given in [14].

Proposition 2.1 ([14, Proposition 2.2]). *The formal adjoint operator of $Z_A^{A'}$ is*

$$\left(Z_A^{A'}\right)^* = \delta_{A'}^A, \quad \text{where} \quad \delta_{A'}^A := -\overline{Z_A^{A'}}.$$

Lemma 2.1 ([14, Lemma 2.3]). *For $f \in C_0^1(\mathscr{H}_{\mathbb{H}}, \odot^{k-1}\mathbb{O}^2 \otimes \mathbb{C}^{2n})$, we have*

$$(\mathscr{D}_0^* f)_{A'_1\cdots A'_k} = \sum_{A=0}^{2n-1} \delta_{(A'_1}^A f_{A'_2\cdots A'_k)A},$$

where $\mathscr{D}_0^*$ is the formal adjoint operator of $\mathscr{D}_0$.

Lemma 2.2 ([14, Lemma 2.4]). *For $A', B' = 0', 1'$, we have*

$$\sum_{A=0}^{2n-1} Z_A^{A'} \overline{Z_A^{B'}} = -\delta_{A'B'} \Delta_{\mathbb{H}},$$

where $\Delta_{\mathbb{H}}$ is the SubLaplacian defined in (2.1).

$\mathscr{D}_0^* \mathscr{D}_0$ is diagonal by the following proposition.

Proposition 2.2 ([14, Proposition 2.3]). *For $f \in C^2(\Omega, \odot^k \mathbb{C}^2)$, we have*

$$\mathscr{D}_0^* \mathscr{D}_0 f = \Delta_{\mathbb{H}} f.$$

2.2 Octonions, the octonionic Heisenberg group and the tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group

The octonionic algebra has a basis $\{e_j\}_{j=0}^{7}$ satisfying the relation $e_ie_0 = e_0e_i, i = 0, 1, \cdots, 7$, and

$$e_ie_j = -\delta_{ij} + \epsilon_{ijk}e_k, \tag{2.3}$$

$i, j = 1, \cdots, 7$. The constants ϵ_{ijk} in (2.3) are completely anti-symmetric in i, j, k, and equal the value $+1$ for $(i, j, k) \in \Omega$ (cf. e.g. [1, 18]), where

$$\Omega = \{(1,2,3),(2,4,6),(4,3,5),(3,6,7),(6,5,1),(5,7,2),(7,1,4)\}. \tag{2.4}$$

ϵ_{ijk} is nonzero only when (i, j, k) is a permutation of triples in Ω. So, $\epsilon_{ijk}e_k$ in (2.3) can be regarded as $\sum_{k=1}^{7}\epsilon_{ijk}e_k$, where there exists only one k such that $\epsilon_{ijk} \neq 0$. The octonionic algebra $\mathbb{O}$ is a non-associative algebra and octonionic multiplication is neither commutative nor associative, i.e.

$$(e_ie_j)e_k \neq e_i(e_je_k)$$

for some choice of i, j, k.

The octonionic Heisenberg group $\mathscr{H}_{\mathbb{O}}$ is $\mathbb{O} \oplus \mathrm{Im}\mathbb{O}$ equipped with multiplication given by

$$(x,t)\cdot(x',t') = \left(x'+x', t+t'+2\mathrm{Im}\left(x\bar{x'}\right)\right), \tag{2.5}$$

where $x, x' \in \mathbb{O}, t, t' \in \mathrm{Im}\ \mathbb{O}$. The multiplication of the octonionic Heisenberg group in terms of real variables can be written as (cf. eg. [15])

$$(x,t)\cdot(x',t') = \left(x'+x, t_\beta + t'_\beta + 2\sum_{k,j=1}^{8} E^{\beta}_{kj}x_kx'_j\right),$$

where $x = (x_1, \cdots, x_8) \in \mathbb{R}^8$, $t = (t_1, \cdots, t_7) \in \mathbb{R}^7$, and E^β are given by

$$E^\beta = \begin{pmatrix} 0 & -\nu_\beta \\ \nu_\beta^t & \varepsilon^\beta \end{pmatrix}, \quad \beta = 1, \cdots, 7.$$

Here ε^β are (7×7)-matrices with

$$\varepsilon^\beta_{kj} = \epsilon_{jk\beta},$$

and

$$\nu_\beta = (\nu_{\beta;1}, \cdots, \nu_{\beta;7}) \in \mathbb{R}^7, \quad \nu_{\beta;j} = \delta_{\beta j}.$$

Then E^β are anti-symmetric matrices. The norm of the octonionic Heisenberg group $\mathscr{H}_{\mathbb{O}}$ is defined by

$$\|(x,t)\| := (|x|^4 + |t|^2)^{\frac{1}{4}},$$

for $(x,t)\in\mathscr{H}_{\mathbb{O}}$. By definition,

$$X_j=\partial_{x_j}+2\sum_{\beta=1}^{7}\sum_{k=1}^{8}E_{kj}^{\beta}x_k\partial_{t_\beta},\quad j=1,\cdots,8,$$

are the left invariant vector fields on $\mathscr{H}_{\mathbb{O}}$, whose brackets are

$$[X_k,X_j]=4\sum_{\beta=1}^{7}E_{kj}^{\beta}\partial_{t_\beta},\tag{2.6}$$

for $k,j=1,\cdots,8$. We denote *SubLaplacian* $\Delta_{\mathbb{O}}$ *on the octonionic Heisenberg group* by

$$\Delta_{\mathbb{O}}:=-\sum_{k=1}^{8}X_k^2.\tag{2.7}$$

The standard $\mathbb{R}^7$-valued contact form of $\mathscr{H}_{\mathbb{O}}$ is

$$\Theta_{\mathbb{O}}:=\mathrm{d}t-x\cdot\mathrm{d}\bar{x}+\mathrm{d}x\cdot\bar{x}.$$

If we write $\Theta_{\mathbb{O}}=(\theta_{\mathbb{O};1},\cdots,\theta_{\mathbb{O};7})$, then we have

$$\theta_{\mathbb{O};\beta}=\mathrm{d}t_\beta-2\sum_{j,k=1}^{8}E_{kj}^{\beta}x_k\mathrm{d}x_j,$$

by direct calculation. On a domain $\Omega\subset\mathscr{H}_{\mathbb{O}}$, denote the inner product

$$(u,v)_{\mathbb{O}}:=\int_{\Omega}u\cdot\overline{v}\mathrm{d}V_{\mathbb{O}},$$

for $u,v\in L^2(\Omega,\mathbb{C})$, where

$$\mathrm{d}V_{\mathbb{O}}=\theta_{\mathbb{O};1}\wedge\cdots\wedge\theta_{\mathbb{H};7}\wedge\left(\mathrm{d}\theta_{\mathbb{H};\beta}\right)^4,\quad\beta=1,\cdots,7$$

is the volume form on $\mathscr{H}_{\mathbb{O}}$. The inner product of $L^2(\Omega,\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^4)$ is defined as

$$\langle f,h\rangle_{\mathbb{O}}:=\sum_{A=0}^{3}\sum_{A_2',\cdots,A_k'=0',1'}\left(f_{AA_2'\cdots A_k'},h_{AA_2'\cdots A_k'}\right)_{\mathbb{O}}$$

for $f,h\in L^2(\Omega,\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^4)$.

Define

$$\left(\nabla_A^{A'}\right):=\begin{pmatrix}-X_3-\mathbf{i}X_4 & -X_1-\mathbf{i}X_2\\ X_1-\mathbf{i}X_2 & -X_3+\mathbf{i}X_4\\ -X_7-\mathbf{i}X_8 & X_5-\mathbf{i}X_6\\ -X_5-\mathbf{i}X_6 & -X_7+\mathbf{i}X_8\end{pmatrix},\tag{2.8}$$

they are complex left invariant vector fields, for $A = 0, \cdots, 3, A' = 0', 1'$. It is a bit different from the traditional operator $Z_A^{A'}$ in (2.2) by take X_5 into $-X_5$, this comes from the technical difficult to establish the following Lemma 2.4 in the remaining case. Define

$$\left(\mathcal{D}_0^{(k)} f\right)_{AA'_2\cdots A'_k} := \sum_{A'_1=0',1'} \nabla_A^{A'_1} f_{A'_1A'_2\cdots A'_k},$$

for $f \in C^1(\Omega, \odot^k \mathbb{C}^2)$, we will still call it tangential k-Cauchy-Fueter type operator for simplicity. Although this difference, it is still interesting to study the function theory of this operator. Similar with the quaternionic case we will write $\mathcal{D}_0$ instead of $\mathcal{D}_0^{(k)}$ for simplicity.

Proposition 2.3. *The formal adjoint operator of* $\nabla_A^{A'}$ *is*

$$\left(\nabla_A^{A'}\right)^* = \delta'^A_{A'}, \quad \text{where} \quad \delta'^A_{A'} := -\overline{\nabla_A^{A'}}.$$

Lemma 2.3. *For* $f \in C_0^1(\mathscr{H}_{\mathbb{O}}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^4)$, *we have*

$$(\mathcal{D}_0^* f)_{A'_1\cdots A'_k} = \sum_{A=0}^{2n-1} \delta'^A_{(A'_1} f_{A'_2\cdots A'_k)A}, \tag{2.9}$$

where $\mathcal{D}_0^*$ *is the formal adjoint operator of* $\mathcal{D}_0$.

The proof of Proposition 2.3 and Lemma 2.3 are similar with Proposition 2.1 and Lemma 2.1 in the quaternionic case, respectively. We omit the details.

Lemma 2.4. *For* $A', B' = 0', 1'$, *we have*

$$\sum_{A=0}^{3} \overline{\nabla_A^{A'}} \nabla_A^{B'} = \sum_{A=0}^{3} \nabla_A^{B'} \overline{\nabla_A^{A'}} = -\delta_{A'B'} \Delta_{\mathbb{O}}, \tag{2.10}$$

where $\Delta_{\mathbb{O}}$ *is the SubLaplacian defined in* (2.7).

Proof. Note that for $A' = B' = 0'$, we have

$$\begin{aligned}
&-\sum_{l=0}^{1} \left(\overline{\nabla_{2l}^{0'}} \nabla_{2l}^{0'} + \overline{\nabla_{2l+1}^{0'}} \nabla_{2l+1}^{0'}\right) \\
=&(X_3 - \mathbf{i}X_4)(-X_3 - \mathbf{i}X_4) + (-X_1 - \mathbf{i}X_2)(X_1 - \mathbf{i}X_2) \\
&+ (X_7 - \mathbf{i}X_8)(-X_7 - \mathbf{i}X_8) + (X_5 - \mathbf{i}X_6)(-X_5 - \mathbf{i}X_6) \\
=&-\sum_{k=1}^{8} X_k^2 - \mathbf{i}[X_3, X_4] + \mathbf{i}[X_1, X_2] - \mathbf{i}[X_7, X_8] - \mathbf{i}[X_5, X_6] = \Delta_{\mathbb{O}},
\end{aligned}$$

by (2.6). For $A'=0', B'=1'$, we have

$$\begin{aligned}
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{0'}}\nabla_{2l}^{1'}+\overline{\nabla_{2l+1}^{0'}}\nabla_{2l+1}^{1'}\right)\\
=&(X_3-\mathbf{i}X_4)(-X_1-\mathbf{i}X_2)+(-X_1-\mathbf{i}X_2)(-X_3+\mathbf{i}X_4)\\
&+(X_7-\mathbf{i}X_8)(X_5-\mathbf{i}X_6)+(X_5-\mathbf{i}X_6)(-X_7+\mathbf{i}X_8)\\
=&[X_1,X_3]+[X_2,X_4]-[X_5,X_7]+[X_6,X_8]\\
&-\mathbf{i}[X_1,X_4]+\mathbf{i}[X_2,X_3]+\mathbf{i}[X_5,X_8]+\mathbf{i}[X_6,X_7]=0,
\end{aligned}$$

by (2.6). Similarly, (2.10) holds for $A'=1, B'=0'$ and $A'=B'=1'$ by

$$\begin{aligned}
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{1'}}\nabla_{2l}^{0'}+\overline{\nabla_{2l+1}^{1'}}\nabla_{2l+1}^{0'}\right)\\
=&-[X_1,X_3]-[X_2,X_4]+[X_5,X_7]-[X_6,X_8]\\
&-\mathbf{i}[X_1,X_4]+\mathbf{i}[X_2,X_3]+\mathbf{i}[X_5,X_8]+\mathbf{i}[X_6,X_7]=0,\\
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{1'}}\nabla_{2l}^{1'}+\overline{\nabla_{2l+1}^{1'}}\nabla_{2l+1}^{1'}\right)\\
=&\Delta_{\mathbb{O}}-\mathbf{i}[X_1,X_2]+\mathbf{i}[X_3,X_4]+\mathbf{i}[X_5,X_6]+\mathbf{i}[X_7,X_8]=\Delta_{\mathbb{O}}.
\end{aligned}$$

Similarly we have $\sum_{A=0}^{3}\nabla_A^{B'}\overline{\nabla_A^{A'}}=-\delta_{A'B'}\Delta_{\mathbb{O}}$. Then (2.10) follows. □

Similarly with the quaternionic case, $\mathcal{D}_0^*\mathcal{D}_0$ is also diagonal by the following proposition.

Proposition 2.4. *For $f\in C^2(\Omega,\odot^k\mathbb{C}^2)$, we have*

$$\mathcal{D}_0^*\mathcal{D}_0 f=\Delta_{\mathbb{O}}f.$$

Proof. Recall that for a $\otimes^k\mathbb{C}^2$-valued function $F_{A_1'\cdots A_k'}$ symmetric in $A_2'\cdots A_k'$, we have

$$F_{(A_1'\cdots A_k')}=\frac{1}{k}\left(F_{A_1'A_2'\cdots A_k'}+\cdots+F_{A_s'A_1'\cdots\widehat{A_s'}\cdots A_k'}+\cdots+F_{A_k'A_1'\cdots\widehat{A_k'}}\right),$$

by the definition of *symmetrisation* of primed indices

$$f_{\cdots(A_1'\cdots A_p')\cdots}:=\frac{1}{p!}\sum_{\sigma\in S_p}f_{\cdots A'_{\sigma(1)}\cdots A'_{\sigma(p)}\cdots}.$$

As usual, a hatnote means omittance of the corresponding index. Then for fixed $A'_1, \cdots, A'_k = 0', 1'$,

$$\begin{aligned}(\mathcal{D}_0^*\mathcal{D}_0 f)_{A'_1\cdots A'_k} &= \sum_A \delta'^A_{(A'_1}\left(\mathcal{D}_0 f\right)_{A'_2\cdots A'_k)A} = \frac{1}{k}\sum_{s=1}^k \delta'^A_{A'_s}\left(\mathcal{D}_0 f\right)_{\cdots\widehat{A'_s}\cdots A'_k A}\\ &= -\frac{1}{k}\sum_{s=1}^k\sum_{A,A'}\overline{\nabla_A^{A'_s}}\nabla_A^{A'} f_{A'\cdots\widehat{A'_s}\cdots A'_k}\\ &= \frac{1}{k}\sum_{s=1}^k\sum_{A'}\Delta_{\mathbb{O}} f_{A'\cdots\widehat{A'_s}\cdots A'_k}\delta_{A'_s A'} = \Delta_{\mathbb{O}} f_{A'_1\cdots A'_k},\end{aligned}$$

by using Lemma 2.4 and f symmetric in the primed indices, where $\mathcal{D}_0^*$ is given by (2.9). The proposition is proved. □

3. The Bochner-Martinelli formula for tangential k-Cauchy-Fueter operator over quaternionic Heisenberg group

Let 1-forms $\{\theta^1_{\mathbb{H}}, \cdots, \theta^{4n}_{\mathbb{H}}, \theta_{\mathbb{H};1}, \theta_{\mathbb{H};2}, \theta_{\mathbb{H};3}\}$ be the basis dual to $\{Y_1, \cdots, Y_{4n}, \partial_{s_1}, \partial_{s_2}, \partial_{s_3}\}$. It is known that

$$K(\eta) = \frac{C_Q}{\|\eta\|^{Q-2}}, \quad \eta = (y, s) \in \mathscr{H}_{\mathbb{H}},$$

is the fundamental solution to the SubLaplacian $\Delta_{\mathbb{H}}$ on the quaternionic Heisenberg group (cf. e.g. [13, Appendix]), where C_Q is a positive constant and $Q = 4n + 6$ is the homogeneous dimension of $\mathscr{H}_{\mathbb{H}}$. Denote $(4n + 2)$-forms

$$\mathrm{d}\sigma_A^{A'} := i_{Z_A^{A'}}\mathrm{d}V_{\mathbb{H}}, \tag{3.1}$$

for $A = 0, 1, \cdots, 2n - 1$, $A' = 0', 1'$, where i is the interior product. Define the *quaternionic Bochner-Martinelli kernel* to be a $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$-valued function

$$\mathcal{E}^A_{A'}(\eta) := \overline{Z_A^{A'}}K(\eta) \quad \text{on} \quad \mathscr{H}_{\mathbb{H}} \setminus (0, 0). \tag{3.2}$$

See [12, (1.16)] for Bochner-Martinelli kernel on $(4n+1)$-dim Heisenberg group. The following procedure of proving our main Theorem 3.1 is similar with which in the Heisenberg case [12].

Lemma 3.1. *Let Ω be a domain in $\mathscr{H}_{\mathbb{H}}$ and $w \in C^1(\Omega, \mathbb{C})$. Then we have*

$$\int_{\partial\Omega} w\mathrm{d}\sigma_A^{A'}(\eta) = \int_\Omega Z_A^{A'} w\mathrm{d}V_{\mathbb{H}}(\eta), \tag{3.3}$$

for $\eta \in \mathscr{H}_{\mathbb{H}}$.

Proof. Note that, for fixed $a, b \in \{1, 2, \cdots, 4n\}$,

$$\left(\sum_{j=1}^{4n} Y_j w \cdot \theta_{\mathbb{H}}^j\right) \wedge i_{c_1 Y_a + c_2 Y_b} \mathrm{d}V_{\mathbb{H}}(\eta) = (c_1 Y_a + c_2 Y_b)\, w \cdot \mathrm{d}V_{\mathbb{H}}(\eta).$$

Thus we have

$$\begin{aligned}\mathrm{d}\left(w \cdot \mathrm{d}\sigma_A^{A'}\right) =& \left(\sum_{j=1}^{4n} Y_j w \cdot \theta_{\mathbb{H}}^j + \sum_{\beta=1}^{3} \partial_{s_\beta} w \cdot \theta_{\mathbb{H};\beta}\right) \wedge i_{Z_A^{A'}} \mathrm{d}V_{\mathbb{H}}(\eta)\\ =& Z_A^{A'} w \cdot \mathrm{d}V_{\mathbb{H}}(\eta).\end{aligned}$$

Then (3.3) follows by Stokes formula. □

Lemma 3.2. *Let Ω be a domain in $\mathscr{H}_{\mathbb{H}}$ with smooth boundary. For $\Phi \in C\left(\bar{\Omega}, \odot^k \mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k \mathbb{C}^2\right)$ and a fixed point $\xi \in \mathscr{H}_{\mathbb{H}} \backslash \bar{\Omega}$, for any fixed $A_2', \cdots, A_k'$, we have*

$$\begin{aligned}&\sum_{A, A'} \int_{\partial\Omega} \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \Phi_{A' A_2' \cdots A_k'}(\eta) \mathrm{d}\sigma_A^{A'}(\eta)\\ =& \sum_A \int_\Omega \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \left(\mathscr{D}_0 \Phi(\eta)\right)_{A_2' \cdots A_k' A} \mathrm{d}V_{\mathbb{H}}(\eta).\end{aligned}$$

Proof. Applying Lemma 3.1 to $w(\eta) = \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \Phi_{A' A_2' \cdots A_k'}(\eta)$, we have

$$\begin{aligned}&\sum_{A, A'} \int_{\partial\Omega} \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \Phi_{A' A_2' \cdots A_k'}(\eta) \mathrm{d}\sigma_A^{A'}(\eta)\\ =& \sum_{A, A'} \int_\Omega \left[Z_A^{A'}\left(\overline{Z_A^{A_1'}} K(\xi^{-1}\eta)\right) \Phi_{A' A_2' \cdots A_k'}(\eta) + \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \cdot Z_A^{A'} \Phi_{A' A_2' \cdots A_k'}(\eta)\right] \mathrm{d}V_{\mathbb{H}}(\eta)\\ =& \sum_A \int_\Omega \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \left(\mathscr{D}_0 \Phi(\eta)\right)_{A A_2' \cdots A_k'} \mathrm{d}V_{\mathbb{H}}(\eta),\end{aligned}$$

by (3.2) and

$$\sum_A Z_A^{A'}\left(\overline{Z_A^{A_1'}} K(\xi^{-1}\eta)\right) = \sum_A \left(Z_A^{A'} \overline{Z_A^{A_1'}} K\right)(\xi^{-1}\eta) = -\delta_{A' A_1'} \Delta_{\mathbb{H}} K(\xi^{-1}\eta) = 0.$$

It holds by left invariance and Lemma2.2. □

Lemma 3.3. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{H}}$ with C^1 boundary and $\phi \in C\left(\bar{\Omega}, \odot^k \mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k \mathbb{C}^2\right)$. Then we have*

$$\sum_{A, A'} \int_{\partial\Omega} \mathcal{E}_{A_1'}^A(\xi^{-1}\eta) \phi_{A_2' \cdots A_k' A'}(\xi) \mathrm{d}\sigma_A^{A'}(\eta) = \begin{cases} N \cdot \phi_{A_1' A_2' \cdots A_k'}(\xi) & \text{if} \quad \xi \in \Omega, \\ 0 & \text{if} \quad \xi \notin \bar{\Omega}, \end{cases}$$

where

$$N = -C_Q (Q-2)(Q+2) \int_{\|(y,s)\| \le 1} |y|^2 \mathrm{d}V_{\mathbb{O}}(\eta). \tag{3.4}$$

Proof. For a fixed point $\xi \notin \bar{\Omega}$, apply Lemma 3.2 to $\Phi(\eta) \equiv \phi(\xi)$ we have

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}^A_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi)\mathrm{d}\sigma^{A'}_A(\eta) = 0,$$

by $\mathscr{D}_0\Phi(\eta) \equiv 0$. When $\xi \in \Omega$, let $B_{\mathbb{H}}(\xi,\epsilon) := \left\{\eta \in \mathscr{H}_{\mathbb{H}} : \|\xi^{-1}\eta\| \leq \epsilon\right\} \subset \Omega$ be the quaternionic Heisenberg ball with ϵ sufficient small. Applying Lemma 3.2 to $\Phi(\eta) = \phi(\xi)$ and $\Omega \setminus B_{\mathbb{H}}(\xi,\epsilon)$ we get

$$\sum_{A,A'} \int_{\partial(\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon))} \mathcal{E}^A_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi)\mathrm{d}\sigma^{A'}_A(\eta) = 0.$$

Therefore we have

$$\begin{aligned} &\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}^A_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi)\mathrm{d}\sigma^{A'}_A(\eta) \\ &= \sum_{A,A'} \int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^A_{A'_l}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi)\mathrm{d}\sigma^{A'}_A(\eta). \end{aligned} \tag{3.5}$$

Note that

$$\mathcal{E}^A_{A'}(\eta) = \overline{Z^{A'}_A}\frac{1}{\|\eta\|^{Q-2}} = -\frac{Q-2}{4}\frac{\overline{Z^{A'}_A}\|\eta\|^4}{\|\eta\|^{Q+2}}, \tag{3.6}$$

we have

$$\begin{aligned} &\sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^A_{A'_1}(\xi^{-1}\eta)\mathrm{d}\sigma^{A'}_A(\eta) \\ &= -\frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} Z^{A'}_A \overline{Z^{A'_1}_A}\|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ &= -\frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \sum_{A} Z^{A'}_A \overline{Z^{A'_1}_A}\|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ &= \frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \delta_{A'A'_1}\Delta_{\mathbb{H}}\|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ &= \frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \phi_{A'_1 A'_2\cdots A'_k}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \Delta_{\mathbb{H}}\|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta), \end{aligned} \tag{3.7}$$

by using (3.6), Lemma 2.2, Lemma3.1 and left invariance. Note that

$$Y_{4l+j}\|\eta\|^4 = 4|y|^2 y_{4l+j} + 4\sum_{\beta=1}^{3}\sum_{k=1}^{4} B^{\beta}_{kj} y_{4l+k} s_\beta, \tag{3.8}$$

we have

$$\begin{aligned}\Delta_{\mathbb{H}}\|\eta\|^4 =& -\sum_{l=0}^{n-1}\sum_{j=1}^{4}\left(8y_{4l+j}^2+4|y|^2+8\sum_{\beta=1}^{3}\sum_{k,k'=1}^{4}B_{k'j}^{\beta}B_{kj}^{\beta}y_{4l+k'}y_{4l+k}\right)\\ =& -4(Q+2)|y|^2,\end{aligned}\tag{3.9}$$

by using (3.8) and $\sum_{j=1}^{4}B_{kj}^{\beta}B_{jk'}^{\beta'}=(B^{\beta}B^{\beta'})_{kk'}$, antisymmetry for $B^{\beta}B^{\beta'}$ of $\beta\neq\beta'$ and $\left(B^{\beta}\right)^2=-\mathrm{id}$. As the Jacobi determinant of dilation $(y,s)\to(\epsilon y,\epsilon^2 s)$ is ϵ^Q, we have

$$\begin{aligned}&\frac{(Q-2)C_Q}{4\epsilon^{Q+2}}\int_{B_{\mathbb{H}}(\xi,\epsilon)}\Delta_{\mathbb{H}}\|\xi^{-1}\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\ =&\frac{(Q-2)C_Q}{4\epsilon^{Q+2}}\int_{B_{\mathbb{H}}(0,\epsilon)}\Delta_{\mathbb{H}}\|\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\ =&\frac{(Q-2)C_Q}{4}\int_{B_{\mathbb{H}}(0,1)}\Delta_{\mathbb{H}}\|\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\ =&-C_Q(Q-2)(Q+2)\int_{\|(y,s)\|\leq 1}|y|^2\mathrm{d}V_{\mathbb{H}}(\eta)=N,\end{aligned}$$

by (3.9). The lemma is proved. □

Now we can prove the Bochner-Martinelli type formula for tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group.

Theorem 3.1. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{H}}$ with C^1 boundary. For $\phi\in\left(\bar{\Omega},\odot^k\mathbb{C}^2\right)\cap\ C^1\left(\Omega,\odot^k\mathbb{C}^2\right)$, $\xi\in\Omega$ and $k=1,2,\cdots$, we have*

$$\begin{aligned}\phi_{A_1'A_2'\cdots A_k'}(\xi)=\frac{1}{N}\sum_{\substack{A=0,\cdots,2n-1\\A'=0',1'}}&\left\{\int_{\partial\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\right.\\ &\left.-\int_{\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{H}}(\eta)\right\},\end{aligned}$$

where N is given by (3.4).

Proof. For any fixed $\xi\in\Omega$, by Lemma 3.2 and the domain $\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon)$, we get

$$\begin{aligned}&\sum_{A,A'}\left\{\int_{\partial\Omega}-\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)}\right\}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\\ &=\sum_{A}\int_{\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon)}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{H}}(\eta).\end{aligned}$$

By (3.6) and (3.8), we have $\mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta) = O\left(\frac{1}{\|\xi^{-1}\eta\|^{Q-1}}\right)$. Since $Z^{A'_1}_{A}\phi(\eta)_{A'_1A'_2\cdots A'_k}$ is locally bounded, we find that

$$\lim_{\epsilon\to 0}\sum_{A}\int_{B_{\mathbb{H}}(\xi,\epsilon)}\mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A'_2\cdots A'_kA}\mathrm{d}V_{\mathbb{H}}(\eta) = 0.$$

On the other hand

$$\lim_{\epsilon\to 0}\sum_{A,A'}\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)}\mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\left(\phi_{A'_2\cdots A'_kA'}(\eta)-\phi_{A'_2\cdots A'_kA'}(\xi)\right)\mathrm{d}\sigma^{A'}_{A}(\eta) = 0,$$

by using $\left|\phi_{A'_2\cdots A'_kA'}(\eta)-\phi_{A'_2\cdots A'_kA'}(\xi)\right| \le \left\|\phi_{A'_2\cdots A'_kA'}\right\|_{C^1}|\xi-\eta|$. Then by Lemma 3.3, we get

$$\lim_{\epsilon\to 0}\sum_{A,A'}\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)}\mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_kA'}(\eta)\mathrm{d}\sigma^{A'}_{A}(\eta) = N\cdot\phi_{A'_1A'_2\cdots A'_k}(\xi).$$

The theorem is proved. □

4. The Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operator over octonionic Heisenberg group

It is known that

$$K'(\eta) = \frac{C_{Q'}}{\|\eta\|^{Q'-2}}, \quad \eta = (x,t)\in\mathscr{H}_{\mathbb{O}}, \tag{4.1}$$

is the fundamental solution to the SubLaplacian $\Delta_{\mathbb{O}}$ on the octonionic Heisenberg group (cf. e.g. [4]), where $C_{Q'}$ is a positive constant and $Q' = 22$ is the homogeneous dimension of $\mathscr{H}_{\mathbb{O}}$. Denote 14-forms

$$\mathrm{d}\sigma'^{A'}_{A} := i_{\nabla^{A'}_{A}}\mathrm{d}V_{\mathbb{O}}, \tag{4.2}$$

for $A = 0,\cdots,3$, $A' = 0',1'$, where i is the interior product. Define the *octonionic Bochner-Martinelli kernel* to be a $\mathbb{C}^2\otimes\mathbb{C}^4$-valued function

$$\mathcal{E}'^{A}_{A'}(\eta) := \overline{\nabla^{A'}_{A}}K'(\eta) \quad \text{on} \quad \mathscr{H}_{\mathbb{O}}\setminus(0,0). \tag{4.3}$$

Lemma 4.1. *For a domain Ω in $\mathscr{H}_{\mathbb{O}}$ and $w\in C^1(\Omega,\mathbb{C})$, we have*

$$\int_{\partial\Omega} w\mathrm{d}\sigma'^{A'}_{A}(\eta) = \int_{\Omega}\nabla^{A'}_{A}w\mathrm{d}V_{\mathbb{O}}(\eta), \tag{4.4}$$

for $\eta\in\mathscr{H}_{\mathbb{O}}$.

Lemma 4.2. *Let Ω be a domain in $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary. For $\Phi \in C\left(\bar{\Omega}, \odot^k \mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k \mathbb{C}^2\right)$ and a fixed point $\xi \in \mathscr{H}_{\mathbb{O}} \backslash \bar{\Omega}$, for any fixed $A_1', A_2', \cdots, A_k'$, we have*

$$\begin{aligned} &\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\Phi_{A'A_2'\cdots A_k'}(\eta)\mathrm{d}\sigma'^{A'}_{A}(\eta) \\ =&\sum_{A} \int_{\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\left(\mathcal{D}_0\Phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{O}}(\eta). \end{aligned}$$

The proof of Lemma 4.1 and 4.2 is just like Lemma 3.1 and 3.2, respectively. We omit the details.

Lemma 4.3. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary and $\phi \in C\left(\bar{\Omega}, \odot^k \mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k \mathbb{C}^2\right)$. Then we have*

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = \begin{cases} N' \cdot \phi_{A_1'A_2'\cdots A_k'}(\xi) & \text{if} \quad \xi \in \Omega, \\ 0 & \text{if} \quad \xi \notin \bar{\Omega}, \end{cases}$$

where

$$N' = -480 C_{Q'} \int_{\|(x,t)\| \le 1} |x|^2 \mathrm{d}V_{\mathbb{O}}(\eta). \tag{4.5}$$

Proof. For a fixed point $\xi \notin \bar{\Omega}$, applying Lemma 4.2 with $\Phi(\eta) \equiv \phi(\xi)$, we have

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = 0,$$

by $\mathcal{D}_0\Phi(\eta) \equiv 0$. When $\xi \in \Omega$, let $B_{\mathbb{O}}(\xi, \epsilon) := \left\{\eta \in \mathscr{H}_{\mathbb{O}} : \|\xi^{-1}\eta\| \le \epsilon\right\} \subset \Omega$ be the octionionic Heisenberg ball with ϵ sufficient small. Applying Lemma 4.2 to $\Phi(\eta) = \phi(\xi)$ and $\Omega \setminus B_{\mathbb{O}}(\xi, \epsilon)$ we get

$$\sum_{A,A'} \int_{\partial(\Omega \setminus B_{\mathbb{O}}(\xi,\epsilon))} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = 0,$$

i.e.

$$\begin{aligned} &\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) \\ =&\sum_{A,A'} \int_{\partial B_{\mathbb{O}}(\xi,\epsilon)} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta). \end{aligned} \tag{4.6}$$

Note that

$$\mathcal{E}'^{A}_{A'}(\eta) = \overline{\nabla^{A'}_{A}} \frac{C_{Q'}}{\|\eta\|^{Q'-2}} = -C_{Q'} \frac{Q'-2}{4} \frac{\overline{\nabla^{A'}_{A}}\|\eta\|^4}{\|\eta\|^{Q'+2}}, \tag{4.7}$$

we have

$$\sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{\partial B_{\mathbb{O}}(\xi,\epsilon)} \mathcal{E}'^{A}_{\ A'_1}(\xi^{-1}\eta) \mathrm{d}\sigma'^{A'}_{\ A}(\eta)$$
$$= \frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}} \phi_{A'_1 A'_2\cdots A'_k}(\xi) \int_{B_{\mathbb{O}}(\xi,\epsilon)} \Delta_{\mathbb{O}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta),$$

similar with (3.7). Note that

$$X_j \|\eta\|^4 = 4|x|^2 x_j + 4 \sum_{\beta=1}^{7} \sum_{k=1}^{8} E^{\beta}_{kj} x_k t_\beta, \tag{4.8}$$

we have

$$\begin{aligned} \Delta_{\mathbb{O}} \|\eta\|^4 &= -\sum_{j=1}^{8} \left(8x_j^2 + 4|x|^2 + 8 \sum_{\beta=1}^{7} \sum_{k,k'=1}^{8} E^{\beta}_{k'j} E^{\beta}_{kj} x_{k'} x_k \right) \\ &= -4(Q'+2)|x|^2, \end{aligned} \tag{4.9}$$

by using (4.8) and $\sum_{j=1}^{8} E^{\beta}_{kj} E^{\beta'}_{jk'} = (E^{\beta} E^{\beta'})_{kk'}$, antisymmetry for $E^{\beta} E^{\beta'}$ of $\beta \neq \beta'$ and $\left(E^{\beta}\right)^2 = -\mathrm{id}$ (cf. [15]). As the Jacobi determinant of dilation $(x,t) \to (\epsilon x, \epsilon^2 t)$ is $\epsilon^{Q'}$ and $\Delta_{\mathbb{O}}$ is left invariant, we have

$$\begin{aligned} \frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}} \int_{B_{\mathbb{O}}(\xi,\epsilon)} \Delta_{\mathbb{O}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta) &= \frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}} \int_{B_{\mathbb{O}}(0,\epsilon)} \Delta_{\mathbb{O}} \|\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta) \\ &= \frac{(Q'-2)C_{Q'}}{4} \int_{B_{\mathbb{O}}(0,1)} \Delta_{\mathbb{O}} \|\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta) \\ &= -480 C_{Q'} \int_{\|(x,t)\| \leq 1} |x|^2 \mathrm{d}V_{\mathbb{O}}(\eta) = N', \end{aligned}$$

by (4.9). The lemma is proved. □

Now we have the Bochner-Martinelli type formula for the tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group.

Theorem 4.1. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary. For $\phi \in \left(\bar{\Omega}, \odot^k \mathbb{C}^2\right) \cap\ C^1\left(\Omega, \odot^k \mathbb{C}^2\right)$, $\xi \in \Omega$ and $k = 1, 2, \cdots$, we have*

$$\begin{aligned} \phi_{A'_1 A'_2\cdots A'_k}(\xi) = \frac{1}{N'} \sum_{\substack{A=0,\cdots,3 \\ A'=0',1'}} \Bigg\{ &\int_{\partial\Omega} \mathcal{E}'^{A}_{\ A'_1}(\xi^{-1}\eta) \phi_{A'_2\cdots A'_k A'}(\eta) \mathrm{d}\sigma'^{A'}_{\ A}(\eta) \\ &- \int_{\Omega} \mathcal{E}'^{A}_{\ A'_1}(\xi^{-1}\eta) \left(\mathcal{D}_0 \phi(\eta)\right)_{A'_2\cdots A'_k A} \mathrm{d}V_{\mathbb{O}}(\eta) \Bigg\}, \end{aligned}$$

where N' is given by (4.5).

The proof is similar with the quaternionic case, we omit the detail.

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The number of the generating matrices of the subspaces which represent an F_pW_n-submodule where $F_p = GF(p)$ and W_n is the Weyl group of type B_n

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Abstract. In this paper, we will find the number of the generating matrices of the subspaces, which represent any ℓ-dimensional F_pW_n-submodule N of an m-dimensional F_pW_n-module M and apply this result on three samples of submodules.

Keywords: finite field $F_p = GF(p)$, Weyl group W_n of type B_n, group ring F_pW_n, F_pW_n-module, F_pW_n-submodule, pair of partitions (λ, μ) of a positive integer n, Specht polynomial, Specht module, generating matrix of a subspace, statistics theorems

1. The number of bases of an F_pW_n-module M and the number of the generating matrices of the vector space representing M

Throughout this paper let F_p be the finite field of order p, which is the Galois field $GF(p)$, W_n be the Weyl group of type B_n.

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Theorem 1.1. *There are exactly*

$$\frac{(p^m-1)\,(p^m-p)\ldots(p^m-p^{m-1})}{m!}$$

distinct F_p-bases of any m-dimensional F_pW_n-module M.

Proof. Let $B=\{b_1,b_2,\ldots,b_m\}$ be an F_p-basis of M and let $B'=\{b'_1,b'_2,\ldots,b'_m\}$ be any F_p-basis of M (B' and B are not necessarily distinct), then $b'_1,b'_2,\ldots,b'_m$ can be written as linear combinations of $b_1,b_2,\ldots,b_m$ as follows:

$$\begin{aligned}
b'_1 &= c_{11}b_1 + c_{1\,2}b_2 + \ldots + c_{1\,m}b_m\\
b'_2 &= c_{21}b_1 + c_{2\,2}b_2 + \ldots + c_{2\,m}b_m\\
&\vdots\\
b'_m &= c_{m1}b_1 + c_{m\,2}b_2 + \ldots + c_{m\,m}b_m
\end{aligned}$$

where $c_{ij}\in F_p$ for each $i=1,2,...,m$, and for each $j=1,2,...,m$ with the matrix $G=[c_{ij}]_{m\times m}$ is non-singular. b'_1 can be any nonzero element of M, which means that there are (p^m-1) ways to select b'_1 since $|M|=p^m$; b'_2 can be any element of M, which is not a multiple of b'_1 (i.e., $b'_2\neq k_{21}b'_1$ with $k_{21}\in F_p$), which means that there are (p^m-p) ways to select b'_2; b'_3 can be any element of M, which is not a linear combination of b'_1 and b'_2 (i.e., $b'_3\neq k_{31}b'_1+k_{32}b'_2$ with $k_{31},k_{32}\in F_p$), which means that there are (p^m-p^2) ways to select b'_3; ...; and b'_m can be any element of M, which is not a linear combination of $b'_1,b'_2,\ldots,b'_{m-1}$ (i.e., $b'_m\neq k_{m1}b'_1+k_{m2}b'_2+...+k_{m\,m-1}b'_{m-1}$ with $k_{m1},k_{m2},...,k_{m\,m-1}\in F_p$), which means that there are (p^m-p^{m-1}) ways to select b'_m.

Thus there are $(p^m-1)(p^m-p)\ldots(p^m-p^{m-1})$ ordered bases of M.

Therefore the number of bases of M is

$$\frac{(p^m-1)(p^m-p)\ldots(p^m-p^{m-1})}{m!},$$

since each basis can be written as $m!$ ordered bases. □

Corollary 1.1. *The number of generating matrices of F_p^m, which represent the F_pW_n-module M is $(p^m-1)(p^m-p)\ldots(p^m-p^{m-1})$.*

Proof. Let B and B' be the bases given in theorem 1.1, and let G be the matrix representing the basis B' with respect to B, then the first row r_1 of G represents the first element b'_1 of the basis B', the second row r_2 of G represents the second element b'_2 of the basis B',..., the m^{th} row r_m of G represents the m^{th} element b'_m of the basis B', and hence the matrix G is a generating matrix of F_p^m.

Thus there are (p^m-1) different vectors, which can be the first row of G, and there are (p^m-p) different vectors, which can be the second row of G,..., and there are (p^m-p^{m-1}) different vectors, which can be the m^{th} row of G. Therefore there are $(p^m-1)(p^m-p)\ldots(p^m-p^{m-1})$ generating matrices G of F_p^m, which represent M. □

2. Three examples of F_pW_n-submodules

Example 2.1. Let $M_{F_5}((2,2),(1))$ be the F_5W_5-module corresponding to the pair of partitions $((2,2),(1))$ of 5, which is generated over F_5W_5 by the monomial $x_1x_2^3x_3x_4^3$, that is $M_{F_5}((2,2),(1)) = F_5W_5\, x_1x_2^3x_3x_4^3$, and let $S_{F_5}((2,2),(1))$ is the Specht module corresponding to the pair of partitions $((2,2),(1))$. $S_{F_5}((2,2),(1))$ is the F_5W_5-submodule of $M_{F_5}((2,2),(1))$, which is generated over F_5W_5 by the Specht polynomial $f_{(5)}(Z_1^{((2,2),(1))}) = x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3$, that is $S_{F_5}((2,2),(1)) = F_5W_5f_{(5)}\ (Z_1^{((2,2),(1))})$ $= F_5W_5(x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3)$. Then:

First: $\dim_{F_5} M_{F_5}((2,2),(1)) = \frac{n!}{\lambda_1!.\lambda_2!.\mu_1!} = \frac{5!}{2!.2!.1!} = 30$ (see [1] p.7 and [3] p.14).

Second: $M_{F_5}((2,2),(1))$ has a basis $B = \{b_1, b_2, \ldots, b_{30}\}$, where:
$b_1 = x_1^3x_2^3x_3x_4$, $b_2 = x_1^3x_2x_3^3x_4$, $b_3 = x_1x_2^3x_3^3x_4$, $b_4 = x_1^3x_2x_3x_4^3$, $b_5 = x_1x_2^3x_3x_4^3$, $b_6 = x_1x_2x_3^3x_4^3$, $b_7 = x_1^3x_2^3x_3x_5$, $b_8 = x_1^3x_2x_3^3x_5$, $b_9 = x_1x_2^3x_3^3x_5$, $b_{10} = x_1^3x_2x_3x_5^3$, $b_{11} = x_1x_2^3x_3x_5^3$, $b_{12} = x_1x_2x_3^3x_5^3$, $b_{13} = x_1^3x_2^3x_4x_5$, $b_{14} = x_1^3x_2x_4^3x_5$, $b_{15} = x_1x_2^3x_4^3x_5$, $b_{16} = x_1^3x_2x_4x_5^3$, $b_{17} = x_1x_2^3x_4x_5^3$, $b_{18} = x_1x_2x_4^3x_5^3$, $b_{19} = x_1^3x_3^3x_4x_5$, $b_{20} = x_1^3x_3x_4^3x_5$, $b_{21} = x_1x_3^3x_4^3x_5$, $b_{22} = x_1^3x_3x_4x_5^3$, $b_{23} = x_1x_3^3x_4x_5^3$, $b_{24} = x_1x_3x_4^3x_5^3$, $b_{25} = x_2^3x_3^3x_4x_5$, $b_{26} = x_2^3x_3x_4^3x_5$, $b_{27} = x_2x_3^3x_4^3x_5$, $b_{28} = x_2^3x_3x_4x_5^3$, $b_{29} = x_2x_3^3x_4x_5^3$, $b_{30} = x_2x_3x_4^3x_5^3$.

Third: $\dim_{F_5} S_{F_5}((2,2),(1)) = \frac{5!}{3.2.2.1.1} = 10$ (see [1], p.21, [2], p.305, and [3], p.17).

Fourth: $S_{F_5}((2,2),(1))$ has an F_5-basis $D = \{d_1, d_2, \ldots, d_{10}\}$, where $d_1, d_2, \ldots,$ d_{10} are the following Specht polynomials:

$$
\begin{aligned}
d_1 &= x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3,\\
d_2 &= x_1^3x_2^3x_3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2x_3^3x_4^3,\\
d_3 &= x_1^3x_2x_3^3x_5 + 4x_1x_2^3x_3^3x_5 + 4x_1^3x_2x_3x_5^3 + x_1x_2^3x_3x_5^3,\\
d_4 &= x_1^3x_2^3x_3x_5 + 4x_1x_2^3x_3^3x_5 + 4x_1^3x_2x_3x_5^3 + x_1x_2x_3^3x_5^3,\\
d_5 &= x_1^3x_2x_4^3x_5 + 4x_1x_2^3x_4^3x_5 + 4x_1^3x_2x_4x_5^3 + x_1x_2^3x_4x_5^3,\\
d_6 &= x_1^3x_2^3x_4x_5 + 4x_1x_2^3x_4^3x_5 + 4x_1^3x_2x_4x_5^3 + x_1x_2x_4^3x_5^3,\\
d_7 &= x_1^3x_3x_4^3x_5 + 4x_1x_3^3x_4^3x_5 + 4x_1^3x_3x_4x_5^3 + x_1x_3^3x_4x_5^3,\\
d_8 &= x_1^3x_3^3x_4x_5 + 4x_1x_3^3x_4^3x_5 + 4x_1^3x_3x_4x_5^3 + x_1x_3x_4^3x_5^3,\\
d_9 &= x_2^3x_3x_4^3x_5 + 4x_2x_3^3x_4^3x_5 + 4x_2^3x_3x_4x_5^3 + x_2x_3^3x_4x_5^3,\\
d_{10} &= x_2^3x_3^3x_4x_5 + 4x_2x_3^3x_4^3x_5 + 4x_2^3x_3x_4x_5^3 + x_2x_3x_4^3x_5^3.
\end{aligned}
$$

The above polynomials $d_1,\ d_2, \ldots,\ d_{10}$ give the following generating matrix $\mu_{(5)}^{((2,2),(1))}$ for the subspace $V_{(5)}^{((2,2),(1))}$ (which represents the Specht module $S_{F_5}((2,2),(1))$) of the vector space F_5^{30} (which represents the module $M_{F_5}((2,2),$

(1))):

$$\mu_{(5)}^{((2,2),(1))} = \begin{bmatrix}
0&1&4&4&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&4&4&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&4&4&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&4&4&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&4&4&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&4&4&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
4&4&1&0&0&0&0&0&0&0\\
4&4&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&4&4&1&0\\
0&0&0&0&1&0&4&4&0&1
\end{bmatrix}$$

There are fifteen pairs of equal columns in the matrix $\mu_{(5)}^{((2,2),(1))}$.

Example 2.2. Let $M_{F_3}((2,2),(1))$ be the F_3W_5-module corresponding to the pair of partitions $((2,2),1)$ of 5, which is generated over F_3W_5 by the monomial $x_1x_2^3x_3x_4^3$, that is $M_{F_3}((2,2),(1)) = F_3W_5x_1x_2^3x_3x_4^3$, and let $N_{F_3}((2,2),(1))$ be the F_3W_5-submodule of $M_{F_3}((2,2),(1))$, which is generated over F_3W_5 by the symmetrized Specht polynomial $f_{(3)}[Z_1^{((2,2),(1))}] = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3$, that is $N_{F_3}((2,2),(1)) = F_3W_5f_{(3)}[Z_1^{((2,2),(1))}] = F_3W_5(x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3)$. Then:

First: $\dim_{F_3} M_{F_3}((2,2),(1)) = \frac{n!}{\lambda_1!.\lambda_2!.\mu_1!} = \frac{5!}{2!.2!.1!} = 30$ (see [1], p.7 and [3], p.14).

Second: $M_{F_3}((2,2),(1))$ has a basis $B = \{b_1, b_2, \ldots, b_{30}\}$, given in second of Example 2.1.

Third: $N_{F_3}((2,2),(1))$ has an F_3-basis $D = \{d_1, d_2, \ldots, d_5\}$, where $d_1, d_2, \ldots, d_5$ are the following symmetrized Specht polynomials:

$$\begin{aligned}
d_1 &= x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3,\\
d_2 &= x_1^3x_2^3x_3x_5 + x_1^3x_2x_3^3x_5 + x_1x_2^3x_3^3x_5 + x_1^3x_2x_3x_5^3 + x_1x_2^3x_3x_5^3 + x_1x_2x_3^3x_5^3,\\
d_3 &= x_1^3x_2^3x_4x_5 + x_1^3x_2x_4^3x_5 + x_1x_2^3x_4^3x_5 + x_1^3x_2x_4x_5^3 + x_1x_2^3x_4x_5^3 + x_1x_2x_4^3x_5^3,\\
d_4 &= x_1^3x_3^3x_4x_5 + x_1^3x_3x_4^3x_5 + x_1x_3^3x_4^3x_5 + x_1^3x_3x_4x_5^3 + x_1x_3^3x_4x_5^3 + x_1x_3x_4^3x_5^3,\\
d_5 &= x_2^3x_3^3x_4x_5 + x_2^3x_3x_4^3x_5 + x_2x_3^3x_4^3x_5 + x_2^3x_3x_4x_5^3 + x_2x_3^3x_4x_5^3 + x_2x_3x_4^3x_5^3.
\end{aligned}$$

Fourth: $\dim_{F_3} N_{F_3}((2,2),(1)) = 5$.
Fifth: The polynomials $d_1, d_2, \ldots, d_5$ given in Third above give the following generating matrix $\rho_{(3)}^{((2,2),(1))}$ for the subspace $U_{(3)}^{((2,2),(1))}$ (which represents the module $N_{F_3}((2,2),(1))$) of the vector space F_3^{30} (which represents the module $M_{F_3}((2,2),(1))$):

$$\rho_{(3)}^{((2,2),(1))} = \begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1 \end{bmatrix}$$

There are five of six equal columns in the matrix $\rho_{(3)}^{((2,2),(1))}$.

Example 2.3. Let $M_{F_5}((1,1,1),(2))$ be the F_5W_5-module corresponding to the pair of partitions $((1,1,1),(2))$ of 5, which is generated over F_5W_5 by the monomial $x_1x_2^3x_3^5$, that is $M_{F_5}((1,1,1),(2)) = F_5W_5x_1x_2^3x_3^5$, and let $N_{F_5}((1,1,1),(2))$ be the F_5W_5-submodule of $M_{F_5}((1,1,1),(2))$, which is generated over F_5W_5 by the symmetrized Specht polynomial $f_{(5)}[Z_1^{((1,1,1),(2))}] = 3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5$, that is $N_{F_5}((1,1,1),(2))$
$= F_5W_5f_{(5)}[Z_1^{((1,1,1),(2))}] = F_5W_5(3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5)$. Then:

First: $\dim_{F_5} M_{F_5}((1,1,1),(2)) = \frac{n!}{\lambda_1!.\lambda_2!.\lambda_3!.\mu_1!} = \frac{5!}{1!.1!.1!.2!} = 60$ (see [1], p.7 and [3], p.14).

Second: $M_{F_5}((1,1,1),(2))$ has a basis $B = \{b_1, b_2, \ldots, b_{60}\}$, where $b_1 = x_1^5x_2^3x_3$, $b_2 = x_1^3x_2^5x_3$, $b_3 = x_1^5x_2x_3^3$, $b_4 = x_1x_2^5x_3^3$, $b_5 = x_1^3x_2x_3^5$, $b_6 = x_1x_2^3x_3^5$, $b_7 = x_1^5x_2^3x_4$, $b_8 = x_1^3x_2^5x_4$, $b_9 = x_1^5x_2^3x_4$, $b_{10} = x_2^5x_3^3x_4$, $b_{11} = x_1^3x_3^5x_4$, $b_{12} = x_2^3x_3^5x_4$, $b_{13} = x_1^5x_2x_4^3$, $b_{14} = x_1x_2^5x_4^3$, $b_{15} = x_1^5x_3x_4^3$, $b_{16} = x_2^5x_3x_4^3$, $b_{17} = x_1x_3^5x_4^3$, $b_{18} = x_2x_3^5x_4^3$, $b_{19} = x_1^3x_2x_4^5$, $b_{20} = x_1x_2^3x_4^5$, $b_{21} = x_1^3x_3x_4^5$, $b_{22} = x_2^3x_3x_4^5$, $b_{23} = x_1x_3^3x_4^5$, $b_{24} = x_2x_3^3x_4^5$, $b_{25} = x_1^5x_2^3x_5$, $b_{26} = x_1^3x_2^5x_5$, $b_{27} = x_1^5x_3^3x_5$, $b_{28} = x_2^5x_3^3x_5$, $b_{29} = x_1^3x_3^5x_5$, $b_{30} = x_2^3x_3^5x_5$, $b_{31} = x_1^5x_4^3x_5$, $b_{32} = x_2^5x_4^3x_5$, $b_{33} = x_3^5x_4^3x_5$, $b_{34} = x_1^3x_4^5x_5$, $b_{35} = x_2^3x_4^5x_5$, $b_{36} = x_3^3x_4^5x_5$, $b_{37} = x_1^5x_2x_5^3$, $b_{38} = x_1x_2^5x_5^3$, $b_{39} = x_1^5x_3x_5^3$, $b_{40} = x_2^5x_3x_5^3$, $b_{41} = x_1x_3^5x_5^3$, $b_{42} = x_2x_3^5x_5^3$, $b_{43} = x_1^5x_4x_5^3$, $b_{44} = x_2^5x_4x_5^3$, $b_{45} = x_3^5x_4x_5^3$, $b_{46} = x_1x_4^5x_5^3$, $b_{47} = x_2x_4^5x_5^3$, $b_{48} = x_3x_4^5x_5^3$, $b_{49} = x_1^3x_2x_5^5$, $b_{50} = x_1x_2^3x_5^5$, $b_{51} = x_1^3x_3x_5^5$, $b_{52} = x_2^3x_3x_5^5$, $b_{53} = x_1x_3^3x_5^5$, $b_{54} = x_2x_3^3x_5^5$, $b_{55} = x_1^3x_4x_5^5$, $b_{56} = x_2^3x_4x_5^5$, $b_{57} = x_3^3x_4x_5^5$, $b_{58} = x_1x_4^3x_5^5$, $b_{59} = x_2x_4^3x_5^5$, $b_{60} = x_3x_4^3x_5^5$.

Third: $N_{F_5}((1,1,1),(2))$ has an F_5-basis $D = \{d_1, d_2, \ldots, d_{10}\}$, where $d_1, d_2, \ldots,$ d_{10} are the following symmetrized Specht polynomials:

$$
\begin{aligned}
d_1 &= 3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5,\\
d_2 &= 3x_1^5x_2^3x_4 + 2x_1^3x_2^5x_4 + 2x_1^5x_2x_4^3 + 3x_1x_2^5x_4^3 + 3x_1^3x_2x_4^5 + 2x_1x_2^3x_4^5,\\
d_3 &= 3x_1^5x_2^3x_5 + 2x_1^3x_2^5x_5 + 2x_1^5x_2x_5^3 + 3x_1x_2^5x_5^3 + 3x_1^3x_2x_5^5 + 2x_1x_2^3x_5^5,\\
d_4 &= 3x_1^5x_3^3x_4 + 2x_1^3x_3^5x_4 + 2x_1^5x_3x_4^3 + 3x_1x_3^5x_4^3 + 3x_1^3x_3x_4^5 + 2x_1x_3^3x_4^5,\\
d_5 &= 3x_1^5x_3^3x_5 + 2x_1^3x_3^5x_5 + 2x_1^5x_3x_5^3 + 3x_1x_3^5x_5^3 + 3x_1^3x_3x_5^5 + 2x_1x_3^3x_5^5,\\
d_6 &= 3x_1^5x_4^3x_5 + 2x_1^3x_4^5x_5 + 2x_1^5x_4x_5^3 + 3x_1x_4^5x_5^3 + 3x_1^3x_4x_5^5 + 2x_1x_4^3x_5^5,\\
d_7 &= 3x_2^5x_3^3x_4 + 2x_2^3x_3^5x_4 + 2x_2^5x_3x_4^3 + 3x_2x_3^5x_4^3 + 3x_2^3x_3x_4^5 + 2x_2x_3^3x_4^5,\\
d_8 &= 3x_2^5x_3^3x_5 + 2x_2^3x_3^5x_5 + 2x_2^5x_3x_5^3 + 3x_2x_3^5x_5^3 + 3x_2^3x_3x_5^5 + 2x_2x_3^3x_5^5,\\
d_9 &= 3x_2^5x_4^3x_5 + 2x_2^3x_4^5x_5 + 2x_2^5x_4x_5^3 + 3x_2x_4^5x_5^3 + 3x_2^3x_4x_5^5 + 2x_2x_4^3x_5^5,\\
d_{10} &= 3x_3^5x_4^3x_5 + 2x_3^3x_4^5x_5 + 2x_3^5x_4x_5^3 + 3x_3x_4^5x_5^3 + 3x_3^3x_4x_5^5 + 2x_3x_4^3x_5^5.
\end{aligned}
$$

Fourth: $\dim_{F_5} N_{F_5}((1,1,1),(2)) = 10$.

Fifth: The polynomials $d_1, d_2, \ldots, d_{10}$ given in third above give the following generating matrix $\rho_{(5)}^{((1,1,1),(2))}$ for the subspace $U_{(5)}^{((1,1,1),(2))}$ (which represents the submodule $N_{F_5}((1,1,1),(2))$) of the vector space F_5^{60} (which represents the module $M_{F_5}((1,1,1),(2))$):

$\rho_{(5)}^{((1,1,1),(2))}$

$$
= \begin{bmatrix}
3 & 2 & 2 & 3 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 \end{bmatrix}$$

There are twenty of triple equal columns in the matrix $\rho_{(5)}^{((1,1,1),(2))}$.

3. The number of generating matrices representing a submodule

Theorem 3.1. *Let N be an ℓ-dimensional F_pW_n-submodule of an m-dimensional F_pW_n-module M, U is an ℓ-dimensional subspace of F_p^m, where U represents N, and the vector space F_p^m represents M, and G is a generating matrix of U, which has $g_1, g_2, \ldots, g_s$ numbers of equal columns. Then the number of distinct generating matrices of all subspaces U that represent N is $\frac{m!}{g_1!.g_2!\ldots g_s!.\ell!}(p^\ell - 1)(p^\ell - p)\ldots(p^\ell - p^{\ell-1})$.*

Proof. Fix the order of the columns corresponding to the elements of a basis $B_M = \{b_1, b_2, \ldots, b_m\}$ of M. Then by corollary 1.1 the number of generating matrices of an ℓ-dimensional subspace U representing N is $(p^\ell-1)(p^\ell-p)\ldots(p^\ell-p^{\ell-1})$.
Changing the order of the columns corresponding to the elements of B_M, we get that the number of the permutations of the columns of G is $\frac{m!}{g_1!.g_2!\ldots g_s!.\ell!}$, since there are $g_1, g_2, ..., g_s$ numbers of equal columns and regarding that the changing of the order of the elements of a basis B_N of N (which implies the swapping of the columns representing any element of B_N with the columns representing another element of B_N) does not change the number of the generating matrices of U. Thus the number of the distincts generating matrices of all subspaces U that represent N is $\frac{m!}{g_1!.g_2!\ldots g_s!.\ell!}(p^\ell - 1)(p^\ell - p)\ldots(p^\ell - p^{\ell-1})$. □

Corollary 3.1. *The number of the distinct generating matrices of all the subspaces $V_{(5)}^{((2,2),(1))}$ of F_5^{30}, which represent the Specht module $S_{F_5}((2,2),(1))$ as an F_5W_5-submodule of the F_5W_5-module $M_{F_5}((2,2),(1))$ is $2^{302.7155187}$.*

Proof. The matrix $\mu_{(5)}^{((2,2),(1))}$ is a generating matrix of a subspace $V_{(5)}^{((2,2),(1))}$, which represents the Specht module $S_{F_5}((2,2),(1))$ for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \ldots, b_{30}\}$ given in example 2.1.

Since the matrix $\mu_{(5)}^{((2,2),(1))}$ has fifteen pairs of equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $V_{(5)}^{((2,2),(1))}$ that represent $S_{F_5}((2,2),(1))$ is $\frac{30!}{(2!)^{15}.10!}(5^{10}-1)(5^{10}-5)...(5^{10}-5^9) = 2^{70.91800623}.2^{231.7975125} = 2^{302.7155187}$. □

Corollary 3.2. *The number of the distinct generating matrices of all the subspaces* $U_{(3)}^{((2,2),(1))}$ *of* F_3^{30}*, which represent the* F_3W_5*-submodule* $N_{F_3}((2,2),(1))$ *of the* F_3W_5*-module* $M_{F_3}((2,2),(1))$ *is* $2^{77.24959681}$.

Proof. The matrix $\rho_{(3)}^{((2,2),(1))}$ is a generating matrix of a subspace $U_{(3)}^{((2,2),(1))}$, which represents the F_3W_5-submodule $N_{F_3}((2,2),(1))$, for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \ldots, b_{30}\}$ given in example 2.1.

Since the matrix $\rho_{(3)}^{((2,2),(1))}$ has five of six equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $U_{(3)}^{((2,2),(1))}$ that represent $N_{F_3}((2,2),(1))$ is $\frac{30!}{(6!)^5.10!}.(3^5-1)(3^5-3)...(3^5-3^4) = 2^{38.45874075}.2^{38.79085606} = 2^{77.24959681}$. □

Corollary 3.3. *The number of the distinct generating matrices of all the subspaces* $U_{(5)}^{((1,1,1),(2))}$ *of* F_5^{60}*, which represent the* F_5W_5*-submodule* $N_{F_5}((1,1,1),(2))$ *of the* F_5W_5*-module* $M_{F_5}((1,1,1),(2))$ *is* $2^{430.4401317}$.

Proof. The matrix $\rho_{(5)}^{((1,1,1),(2))}$ is a generating matrix of a subspace $U_{(5)}^{((1,1,1),(2))}$, which represents the F_5W_5-submodule $N_{F_5}((1,1,1),(2))$ for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \ldots, b_{60}\}$ given in Example 2.3.

Since the matrix $\rho_{(5)}^{((1,1,1),(2))}$ has twenty of triple equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $U_{(5)}^{((1,1,1),(2))}$ that represent $N_{F_5}((1,1,1),(2))$ is $\frac{60!}{(3!)^{20}.10!}(5^{10}-1)(5^{10}-5)...(5^{10}-5^9) = 2^{198.6426192}.2^{231.7975125} = 2^{430.4401317}$. □

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Upper bounds on deviations from the mean and the mean absolute deviation

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Abstract. In this paper, a number of upper bounds are introduced for the mean absolute deviation (MAD) from the mean when there is some information about the number of the data that are above or below the average. The upper bounds are compared with each other according to their tightness. A unified structure is found to be useful to prove all of the proposed upper bounds as well as the other existing upper bounds. In the path to formulate that structure, also, an upper bound is introduced for the individual deviations from the mean. The results are clarified and verified by a few examples.

Keywords: statistical dispersion, upper bound, partial knowledge, mean absolute deviation, individual deviations.

1. Introduction

Let $\{x_1, \dots, x_n\}$ be an arbitrary data-set, where $x_i \in \mathbb{R}$, and define $\mu_x = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, i.e., the (arithmetic) mean of the data, $x_{\min} = \min x_i$ and $x_{\max} = \max x_i$. Deviations from the mean are the distances of the data from their mean, i.e., $|x_i - \bar{x}|$, which is used in defining both the mean absolute deviation (MAD) and the standard deviation (SD). If $x_1 = x_2 = \cdots = x_n$, all deviations are zero and the data-set is trivial. Throughout the paper, *non-trivial* data-sets are considered in which at least one data is different from the others.

Three most common measures for statistical dispersion of a data-set $\{x_i\}$ are: (1) range: $r_x = x_{\max} - x_{\min}$; (2) MAD: $\rho_x = \frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|$; (3) SD: $\sigma_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$. For the MAD, we have

$$\rho_x = \frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}| \leqslant \frac{1}{n}\sum_{i=1}^{n}(x_{\max} - x_{\min}) = r_x. \tag{1}$$

For the variance and the SD, we can write

$$\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \leqslant \frac{1}{n}\sum_{i=1}^{n}(x_{\max} - x_{\min})^2 = r_x^2 \tag{2}$$

which means

$$\sigma_x \leqslant r_x. \tag{3}$$

Equations (1), (2), and (3), respectively, represent very simple upper bounds for MAD, variance, and SD of any arbitrary data-set $\{x_i\}$. Having an upper bound for a statistical measure is of analytical importance (e.g., to normalize the measure or to yield other useful inequalities) as well as practical importance (e.g., to verify the calculated measure for a given data-set). The tighter an upper bound, the more information it represents. Therefore, it is desirable to tighten the upper bounds as much as possible.

In Section 2, at first, some tight upper bounds are introduced for the individual deviations from the mean. Then, some upper bounds are introduced for the MAD which are tighter than (1). Simple examples are presented to clarify and justify the results after comparing the proposed upper bounds in Section 3.

Regarding the MAD as a statistical measure of dispersion, it is worth mentioning that some closed-form expressions have been introduced for the MAD in [3] for several probability distributions. The MAD can be used in real-world applications, e.g., to model the risk in a portfolio optimization problem, [4], [5]. The main benefit of using MAD-based optimization is that it allows for linear optimization which leads to a drastic reduction in the computational cost; See [10] to find more about this benefit and some others. The MAD is one of the measures that can be used to characterize the probability distributions; See [6] for example. The MAD per se may be considered as a random variable; See [7] where the probability distribution of the MAD and its estimation from data for some specific distributions are studied. Also, some other uses of MAD are introduced in [7], e.g., sample size determination for data with normal distribution.

2. Looking for appropriate upper bounds

Deviations from the mean of the data and their mean are considered in this section and a few somehow tight upper bounds are found for them.

2.1 Individual deviations from the mean

For every individual deviation from the mean we obviously know that $|x_i - \bar{x}| < x_{\max} - x_{\min} = r_x$. But let us introduce a tighter upper bound.

Theorem 2.1. *Let $\{x_1, \ldots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then:*

$$\begin{array}{lll} (i) & \frac{1}{n} r_x \leqslant |x_i - \bar{x}| \leqslant \frac{n-1}{n} r_x & \text{for} \quad x_i \in \{x_{\min}, x_{\max}\}; \\ (ii) & |x_i - \bar{x}| \leqslant \frac{n-2}{n} r_x & \text{for} \quad x_i \notin \{x_{\min}, x_{\max}\}. \end{array}$$

Proof. Let us sort $\{x_i\}$ in non-decreasing order and denote the ordered data-set by $\{y_j\}$. So,

$$y_1 \leqslant \ldots \leqslant y_j \leqslant \ldots \leqslant y_n, \tag{4}$$

where $y_1 = x_{\min}, y_n = x_{\max}$, and $r_y = r_x = y_n - y_1$.
Case (i): For $x_i = x_{\max}$, we should show that $\frac{1}{n} r_x \leqslant x_{\max} - \bar{x} \leqslant \frac{n-1}{n} r_x$ which is equivalent to $\frac{1}{n} r_y \leqslant y_n - \bar{y} \leqslant \frac{n-1}{n} r_y$. By expanding $\bar{y}$ as $\frac{1}{n}(y_1 + \sum_{j=2}^{n-1} y_j + y_n)$ we can write

$$y_n - \bar{y} = -\frac{1}{n} y_1 - \frac{1}{n} \sum_{j=2}^{n-1} y_j + (1 - \frac{1}{n}) y_n. \tag{5}$$

Using (4) we know that $(n-2)y_1 \leqslant \sum_{j=2}^{n-1} y_j \leqslant (n-2)y_n$. Therefore,

$$-\frac{1}{n} y_1 - \frac{n-2}{n} y_n + \frac{n-1}{n} y_n \leqslant y_n - \bar{y} \leqslant -\frac{1}{n} y_1 - \frac{n-2}{n} y_1 + \frac{n-1}{n} y_n$$

which leads to

$$\frac{1}{n} r_y \leqslant y_n - \bar{y} \leqslant \frac{n-1}{n} r_y. \tag{6}$$

Likewise, for $x_i = x_{\min}$, by writing

$$\bar{y} - y_1 = \frac{1-n}{n} y_1 + \frac{1}{n} \sum_{j=2}^{n-1} y_j + \frac{1}{n} y_n. \tag{7}$$

we obtain

$$\frac{1-n}{n} y_1 + \frac{n-2}{n} y_1 + \frac{1}{n} y_n \leqslant \bar{y} - y_1 \leqslant \frac{1-n}{n} y_1 + \frac{n-2}{n} y_n + \frac{1}{n} y_n$$

which leads to

$$\frac{1}{n} r_y \leqslant \bar{y} - y_1 \leqslant \frac{n-1}{n} r_y. \tag{8}$$

Case (ii): Now suppose that $x_i \in (x_{\min}, x_{\max})$ and x_i corresponds with y_j. So, for any $y_j \geqslant \bar{y}$ with $j \neq n$, we should show that $y_j - \bar{y} \leqslant \frac{n-2}{n} r_y$. To this end, we can expand $\bar{y}$ as $\frac{1}{n}(y_1 + \sum_{k=2, k\neq j}^{n-1} y_k + y_j + y_n)$ to write

$$y_j - \bar{y} = -\frac{1}{n} y_1 - \frac{1}{n} \sum_{k=2, k\neq j}^{n-1} y_k + (1 - \frac{1}{n}) y_j - \frac{1}{n} y_n. \tag{9}$$

Using (4) we know that $y_1 \leqslant y_j \leqslant y_n$ and $(n-3)y_1 \leqslant \sum_{k=2, k\neq j}^{n-1} y_k \leqslant (n-3)y_n$ which leads to

$$-\frac{n-2}{n} r_y \leqslant y_j - \bar{y} \leqslant \frac{n-2}{n} r_y. \tag{10}$$

Likewise, for any $y_j \leqslant \bar{y}$ with $j \neq 1$, we obtain

$$-\frac{n-2}{n} r_y \leqslant \bar{y} - y_j \leqslant \frac{n-2}{n} r_y \tag{11}$$

by the same expansion of $\bar{y}$. This completes the proof. □

Remark 1. Beside presenting an upper bound, item (i) of Theorem 2.1 presents a lower bound on $(x_{\max} - \bar{x})$ and $(\bar{x} - x_{\min})$.

Example 1. Consider an ascendingly-sorted data-set $\{y_1, y_2, y_3, y_4\}$. According to Theorem 2.1, we have $\frac{1}{4}(y_4 - y_1) \leqslant \bar{y} - y_1 \leqslant \frac{3}{4}(y_4 - y_1)$, etc.

2.2 Mean Absolute Deviation (MAD)

As presented in (1), we know that $\rho_x \leqslant r_x$. However, in this section, we are going to find some tighter upper bounds for MAD.

To this end, let us sort $\{x_i\}$ in non-decreasing order and denote the ordered data-set by $\{y_j\}$. So, in comparison with the mean value $\bar{y} = \bar{x}$, we will have a *lower* data segment $(y_j < \bar{y})$, an *upper* data segment $(\bar{y} < y_j)$, and a *boundary* data segment $(y_j = \bar{y})$. Hence

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_l < \bar{y} < y_{n-u+1} \leqslant \ldots \leqslant y_{n-1} \leqslant y_n, \tag{12}$$

where l and u are the number of data in the lower and upper data segments respectively. For a non-trivial data-set, l and u are in $[1, n-1]$ and

$$l + u \leqslant n. \tag{13}$$

Also, it is clear that $y_1 = x_{\min}, y_n = x_{\max}$, $r_x = r_y = y_n - y_1$, and $\rho_x = \rho_y = \frac{1}{n}\sum_{i=1}^{n} |y_i - \bar{y}|$.

It should be noticed that $n\bar{y} = \sum_{i=1}^{n} y_i$, so $\sum_{i=1}^{n}(y_i - \bar{y}) = 0$. For the lower data segment, $y_i - \bar{y} < 0$, and for the upper data segment, $y_i - \bar{y} > 0$. Defining two positive indices $L := \sum_{j=1}^{l}(\bar{y} - y_j)$ and $U := \sum_{j=n-u+1}^{n}(y_j - \bar{y})$, we have

$$L = U. \tag{14}$$

On the other hand, $n\rho_y = \sum_{j=1}^{l}(\bar{y} - y_j) + \sum_{j=n-u+1}^{n}(y_j - \bar{y}) = L + U$, Therefore,

$$\rho_y = \frac{2}{n}L = \frac{2}{n}U. \tag{15}$$

Lemma 2.1. *Let $\{y_j\}$ be a data-set in non-decreasing order. B is an upper bound of the MAD ρ_y, if $L \leqslant \frac{n}{2}B$ (or equivalently $U \leqslant \frac{n}{2}B$).*

Proof. The proof is obvious by (15). □

Let us define the arithmetic mean of the lower and upper data segments respectively as $\bar{y}_l := \sum_{j=1}^{l} y_j/l$ and $\bar{y}_u := \sum_{i=n-u+1}^{n} y_j/u$. Therefore

$$\begin{cases} L = l\bar{y} - \sum_{j=1}^{l} y_j = l\,(\bar{y} - \bar{y}_l)\,, \\ U = \sum_{j=n-u+1}^{n} y_j - u\bar{y} = u\,(\bar{y}_u - \bar{y}) \end{cases} . \tag{16}$$

Obviously, $y_1 \leqslant \bar{y}_l \leqslant \bar{y} \leqslant \bar{y}_u \leqslant y_n$.

Lemma 2.2. *Let $l, u, n \in \mathbb{N}$, such that $l + u \leqslant n$. Then,*

$$\frac{1}{l} + \frac{1}{u} \geqslant \frac{4}{n}. \tag{17}$$

Proof. We know that $u \leqslant n - l$. Thus, $\frac{1}{u} \geqslant \frac{1}{n-l}$, since $u \geqslant 1$ and $n - l \geqslant 1$. Therefore,

$$\frac{1}{l} + \frac{1}{u} \geqslant \frac{1}{l} + \frac{1}{n-l}$$

and the proof is complete since we know that $\frac{1}{l} + \frac{1}{n-l} \geqslant \frac{4}{n}$. □

Theorem 2.2. *Let $\{x_1, \ldots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then,*

$$\rho_x \leqslant \frac{1}{2} r_x. \tag{18}$$

Proof. We can equivalently prove $\rho_y \leqslant r_y/2$. By Lemma 2.1, it suffices to prove $L \leqslant nr_y/4$ or $U \leqslant nr_y/4$. Let us proceed by contradiction. Knowing $L = U$, suppose that L and U are greater than $nr_y/4$. By (16) we can write

$$\begin{cases} \bar{y} - \bar{y}_l > nr_y/4l, \\ \bar{y}_u - \bar{y} > nr_y/4u \end{cases}$$

which leads to $\bar{y}_u - \bar{y}_l > nr_y \left(\frac{1}{l} + \frac{1}{u}\right)/4$. Then, by Lemma 2.2, $\bar{y}_u - \bar{y}_l > r_y$ which is a contradiction. □

The upper bound presented by Theorem 2.2 is independent of the size of the data-set (n) and only uses the range (r_x). Even tighter upper bounds may be found for the MAD, when more information is available.

If the mean of the data is known in addition to $x_{\min}$ and $x_{\max}$, a tighter upper bound may be found. Considering the fact that the MAD is the arithmetic mean of the absolute value of individual deviations, we immediately find

$$\rho_x \leqslant \max\left(x_{\max} - \bar{x}, \bar{x} - x_{\min}\right) \tag{19}$$

which seems to represent an appropriate upper bound but a better one is presented in Theorem 2.3.

Lemma 2.3. *Let $\{x_1, \ldots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then,*

$$4lu \leqslant n^2. \tag{20}$$

Proof. Knowing $4lu \leqslant 4l(n-l)$ it suffices to prove $4l(n-l) \leqslant n^2$ which can be rewritten as $0 \leqslant n^2 - 4nl + 4l^2 = (n-2l)^2$. □

Theorem 2.3. *Let $\{x_1, \ldots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then,*

$$\rho_x \leqslant \sqrt{(x_{\max} - \bar{x})(\bar{x} - x_{\min})}. \tag{21}$$

Proof. By Lemma 2.1, it suffices to prove that either L or U is less than or equal to $n\sqrt{(y_n-\bar{y})(\bar{y}-y_1)}/2$. By contradiction and by (16) we can write

$$\begin{cases} \bar{y}-\bar{y}_l > n\sqrt{(y_n-\bar{y})(\bar{y}-y_1)}/2l, \\ \bar{y}_u-\bar{y} > n\sqrt{(y_n-\bar{y})(\bar{y}-y_1)}/2u \end{cases}$$

to result in

$$\frac{(\bar{y}_u-\bar{y})}{(y_n-\bar{y})}\frac{(\bar{y}-\bar{y}_l)}{(\bar{y}-y_1)} > n^2/4lu$$

which is a contradiction, since the left hand side is less than or equal to one, but by Lemma 2.3, the right hand side is greater than or equal to one. □

It is interesting to notice that the MAD and the SD are respectively the arithmetic mean and the quadratic mean of the absolute value of individual deviations. Thus, knowing the fact that the quadratic mean is always greater than or equal to the arithmetic mean, *every upper bound for the SD is an upper bound for the MAD too.* In [2], an upper bound that uses the mean of the data-set (known as Bhatia-Davis) is obtained for the SD which is the same as (21) and is tighter than (3), since $(x_{\max}-\bar{x})(\bar{x}-x_{\min}) < (x_{\max}-x_{\min})^2$. It is worth mentioning that the Bhatia-Davis upper bound can be used directly when $\bar{x}$ is known. If $\bar{x}$ is not known, $(x_{\max}-\bar{x})(\bar{x}-x_{\min})$ should take its maximum value which happens when $x_{\max}-\bar{x}=\bar{x}-x_{\min}$ or $\bar{x}=(x_{\max}+x_{\min})/2$. Therefore, we obtain

$$\sigma_x \leqslant r_x/2, \tag{22}$$

which represents the Popoviciu's upper bound for SD, introduced in [8]. Popoviciu's upper bound is looser than Bhatia-Davis (because of not using the value of $\bar{x}$) but tighter than (3) (while using the same amount of information). Besides, note that the Bhatia-Davis upper bound is indeed the geometric mean of $(x_{\max}-\bar{x})$ and $(\bar{x}-x_{\min})$. The harmonic mean of $(x_{\max}-\bar{x})$ and $(\bar{x}-x_{\min})$, represented in (23), too, is an upper bound for MAD which has been introduced firstly in [1] and used in [9]. Theorem 2.4 presents a new straightforward proof for it.

Theorem 2.4. *Let $\{x_1,\ldots,x_n\}$ be a non-trivial data-set, where $x_i\in\mathbb{R}$. Then,*

$$\rho_x \leqslant 2(x_{\max}-\bar{x})(\bar{x}-x_{\min})/r_x. \tag{23}$$

Proof. According to Lemma 2.1, it is sufficient to prove that $n(y_n-\bar{y})(\bar{y}-y_1)/r_y$ is an upper bound for L and U. We will proceed by contradiction. So, suppose that $L=U$ is greater than $n(y_n-\bar{y})(\bar{y}-y_1)/r_y$. Then, knowing $r_y>0$, by (16), we can write

$$\begin{cases} l(\bar{y}-\bar{y}_l)r_y > n(y_n-\bar{y})(\bar{y}-y_1), \\ u(\bar{y}_u-\bar{y})r_y > n(y_n-\bar{y})(\bar{y}-y_1) \end{cases}$$

which can be rewritten as

$$\begin{cases} l(\bar{y}-y_1)r_y + l(y_1-\bar{y}_l)r_y > n(y_n-\bar{y})(\bar{y}-y_1), \\ u(\bar{y}_u-y_n)r_y + u(y_n-\bar{y})r_y > n(y_n-\bar{y})(\bar{y}-y_1) \end{cases} . \tag{24}$$

On one hand, we know that $y_1 - \bar{y}_l < 0$ and $\bar{y}_u - y_n < 0$. Therefore, (24) can be reduced to

$$\begin{cases} l(\bar{y}-y_1)r_y > n(y_n-\bar{y})(\bar{y}-y_1), \\ u(y_n-\bar{y})r_y > n(y_n-\bar{y})(\bar{y}-y_1) \end{cases} . \tag{25}$$

On the other hand, we know that $\bar{y} - y_1 > 0$ and $y_n - \bar{y} > 0$. Therefore, (25) can be reduced to

$$\begin{cases} lr_y > n(y_n-\bar{y}), \\ ur_y > n(\bar{y}-y_1) \end{cases} . \tag{26}$$

Summing the inequalities (26) leads to $l + u > n$ which is a contradiction and the proof is complete. □

In what follows we are going to introduce some tighter upper bounds for the MAD when some information is available about the number of the data below or above the average.

Lemma 2.4. *Let s, l, u, n be natural numbers such that $l + u \leqslant n$ and s belongs to $[\min(l, n-l, u, n-u), \max(l, n-l, u, n-u)]$. Then,*

$$lu \leqslant \min\big(l(n-l), u(n-u)\big) \leqslant s(n-s). \tag{27}$$

Proof. The left inequality is correct, since l and u are positive, $u \leqslant n - l$, and $l \leqslant n - u$. For the right inequality, let us define $f(s) := s(n-s) = -s^2 + ns$ which is a parabolic function of s as depicted in Fig. 1. Obviously, $f(s) > \min\big(f(l), f(u)\big)$ for every s in the specified interval, since f is a convex function. This completes the proof. □

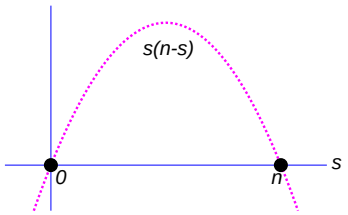


Figure 1: $f(s) = s(n-s)$

Theorem 2.5. *Let $\{x_1, \dots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then,*

$$\rho_x \leqslant \frac{\sqrt{lu}}{n} r_x \leqslant \frac{1}{n} \min\left(\sqrt{l(n-l)}, \sqrt{u(n-u)}\right) r_x \leqslant \frac{\sqrt{s(n-s)}}{n} r_x. \tag{28}$$

Proof. Regarding the left inequality, according to Lemma 2.1, it suffices to prove $L \leqslant r_y\sqrt{lu}/2$ or $U \leqslant r_y\sqrt{lu}/2$. We can prove it by contradiction. Knowing $L = U$, suppose that L and U are greater than $r_y\sqrt{lu}/2$. By (16) we can write

$$\begin{cases} \bar{y} - \bar{y}_l > r_y\sqrt{\frac{u}{l}}/2, \\ \bar{y}_u - \bar{y} > r_y\sqrt{\frac{l}{u}}/2 \end{cases}$$

which leads to $\bar{y}_u - \bar{y}_l > r_y(\sqrt{\frac{u}{l}} + \sqrt{\frac{l}{u}})/2 \geqslant r_y$ which is a contradiction. The middle and the right inequality are obtained directly from Lemma 2.4. □

Lemma 2.5. *Let $l, u, n \in \mathbb{N}$, such that $l + u \leqslant n$. Then,*

$$\frac{lu}{l+u} \leqslant \frac{1}{n} \min\big(l(n-l), u(n-u)\big) \leqslant \frac{s(n-s)}{n}. \tag{29}$$

Proof. First consider the left inequality. We know that $l \geq 0$ and $l + u \leq n$. So, $0 \leq -l(l + u - n)$. By adding nu to both sides we have $nu \leq -l^2 - lu + nl + nu = (n-l)(l+u)$. By multiplying l to both sides and rearranging we have $\frac{lu}{l+u} \leq \frac{l(n-l)}{n}$. Likewise, we derive $\frac{lu}{l+u} \leq \frac{u(n-u)}{n}$ and the left inequality is proved. The proof of the right inequality is the same as the proof of the right inequality in Lemma 2.4. □

Theorem 2.6. *Let $\{x_1, \dots, x_n\}$ be a non-trivial data-set, where $x_i \in \mathbb{R}$. Then,*

$$\rho_x \leqslant \frac{2lu}{l+u}.\frac{r_x}{n} \leqslant \frac{2}{n^2} \min\left(l(n-l), u(n-u)\right) r_x \leqslant \frac{2s(n-s)}{n^2} r_x, \tag{30}$$

for every $s \in [\min(l, n-l, u, n-u), \max(l, n-l, u, n-u)]$.

Proof. First consider the left inequality. According to Lemma 2.1, it suffices to prove $L \leqslant \frac{lu}{l+u} r_y$ or $U \leqslant \frac{lu}{l+u} r_y$. Let us proceed by contradiction. Having $L = U$, suppose that L and U are greater than $\frac{lu}{l+u} r_y$. So, by (16) we have

$$\begin{cases} \bar{y} - \bar{y}_l > \frac{u}{l+u} r_y, \\ \bar{y}_u - \bar{y} > \frac{l}{l+u} r_y \end{cases}$$

which leads to $\bar{y}_u - \bar{y}_l > r_y$ which is a contradiction. The middle and the right inequality are obtained directly from Lemma 2.5. □

Table 1: Summary of the upper bounds and their required information

Label	Upper Bound	Required Information	Reference
B_0	r_x	r_x	(obvious)
B_1	$r_x/2$	r_x	(18) [8]
B_2	$\sqrt{(x_{max}-\bar{x})(\bar{x}-x_{\min})}$	$x_{\min}, x_{\max}, \bar{x}$	(21) [2]
B_3	$2(x_{\max}-\bar{x})(\bar{x}-x_{\min})/r_x$	$x_{\min}, x_{\max}, \bar{x}$	(23) [1]
B_4	$\frac{\sqrt{s(n-s)}}{n} r_x$	r_x, n, s	(28)
B_5^l	$\frac{\sqrt{l(n-l)}}{n} r_x$	r_x, n, l	(28)
B_5^u	$\frac{\sqrt{u(n-u)}}{n} r_x$	r_x, n, u	(28)
B_5	$\min\left(B_5^l, B_5^u\right)$	r_x, n, l, u	(28)
B_6	$\frac{\sqrt{lu}}{n} r_x$	r_x, n, l, u	(28)
B_7	$2\frac{s(n-s)}{n^2} r_x$	r_x, n, s	(30)
B_8^l	$2\frac{l(n-l)}{n^2} r_x$	r_x, n, l	(30)
B_8^u	$2\frac{u(n-u)}{n^2} r_x$	r_x, n, u	(30)
B_8	$\min\left(B_8^l, B_8^u\right)$	r_x, n, l, u	(30)
B_9	$2lur_x/(l+u)n$	r_x, n, l, u	(30)

3. Summary and examples

Let us assign a label to each upper bound for ease of reference. Table 1 shows the assigned labels along with the information required to calculate the upper bounds.

It is obvious that $B_1 \leq B_0$. Also, as B_1, B_2 and B_3 can be considered respectively as arithmetic, geometric, and harmonic means of $(x_{\max}-\bar{x})$ and $(\bar{x}-x_{\min})$, it is obvious that $B_3 \leq B_2 \leq B_1$.

Note that by $0 \leqslant n^2 - 4sn + 4s^2 = (n-2s)^2$, we have $4s(n-s) \leqslant n^2$ and then $\frac{2}{n}\sqrt{s(n-s)} \leqslant 1$. So, we can write

$$2\frac{s(n-s)}{n^2} \leqslant \frac{\sqrt{s(n-s)}}{n}, \tag{31}$$

which leads to $B_7 \leqslant B_4$, $B_8^l \leqslant B_5^l$, $B_8^u \leqslant B_5^u$, and $B_8 \leqslant B_5$.

Furthermore, for a fixed n, both $2\frac{s(n-s)}{n^2}$ and $\frac{\sqrt{s(n-s)}}{n}$ take their maximum value when s is the nearest integer to $n/2$, So, we can write

$$\frac{\sqrt{s(n-s)}}{n} \leqslant \frac{1}{2} \tag{32}$$

which means $B_4 \leqslant B_1$ and $B_7 \leqslant B_1$.

We can categorize the upper bounds with respect to the required information, as shown in Table 2. Also in this table, the best upper bound is introduced for each information group.

Table 2: Information groups and suitable upper bounds for MAD

Group Label	Available Information	Possible Bounds	Tightest Bound
I_1	r_x	B_0,B_1	B_1
I_2	$x_{\min}, x_{\max}, \bar{x}$	B_2,B_3	B_3
I_3^s	r_x, n, s	B_4,B_7	B_7
I_3^l	r_x, n, l	B_5^l,B_8^l	B_8^l
I_3^u	r_x, n, u	B_5^u,B_8^u	B_8^u
$I_3^{l,u}$	r_x, n, l, u	B_5,B_6,B_8,B_9	B_9

Table 3: The actual MAD versus some upper bounds for each data-set in Example 2: B_1, B_2, and B_3 respectively from [8], [2], and [1]; B_5, B_6, B_8, and B_9 proposed in this paper.

Set	l	u	MAD	B0	B1	B2	B3	B5	B6	B8	B9
A	50	50	5.000	10	5	5.000	5.000	5.000	5.000	5.000	5.000
B	75	25	3.750	10	5	4.330	3.750	4.330	4.330	3.750	3.750
C	90	10	1.800	10	5	3.000	1.800	3.000	3.000	1.800	1.800
D	99	1	0.198	10	5	0.995	0.198	0.995	0.995	0.198	0.198
E	25	25	2.500	10	5	5.000	5.000	4.330	2.500	3.750	2.500
F	10	10	1.000	10	5	5.000	5.000	3.000	1.000	1.800	1.000
G	1	1	0.100	10	5	5.000	5.000	0.995	0.100	0.198	0.100

Example 2. Consider the following data-sets, all with $n = 100$, $x_{\min} = 0$, and $x_{\max} = 100$:

Data-set A Fifty "0" and fifty "10", i.e., $\{0 \times 50, 10 \times 50\}$.

Data-set B Seventy five "0" and twenty five "10', i.e., $\{0 \times 75, 10 \times 25\}$.

Data-set C Ninety "0" and ten "10", i.e., $\{0 \times 90, 10 \times 10\}$.

Data-set D Ninety nine "0" and one "10", i.e., $\{0 \times 99, 10 \times 1\}$.

Data-set E Twenty five "0", Fifty "5", and twenty five "10", i.e., $\{0 \times 25, 5 \times 50, 10 \times 25\}$.

Data-set F Ten "0", eighty "5", and ten "10", i.e., $\{0 \times 10, 5 \times 80, 10 \times 10\}$.

Data-set G One "0", ninety eight "5", and one "10", i.e., $\{0 \times 1, 5 \times 98, 10 \times 1\}$.

The actual MAD for each data-set is compared to the proposed upper bounds in Table 3. The upper bounds B_4 and B_7 are not present in Table 3 because they depend on s and s is not unique in general.

Example 3. Let $\{x_i\}$ be a confidential data-set which means that we are not aware of exactly what data does it contain. Only a limited amount of information is available to us such as $x_{\min} = 0$, $x_{\max} = 10$, and $n = 100$ in addition to only one of the following statements:

a. We know Nothing else.

b. We know that the mean of the data is 7.0.

c. We know that exactly 50 data are above average.

d. We know about 40 data instances that they are not above average.

e. We know that exactly 30 data are below average.

f. We know that exactly 50 data are above average and exactly 30 data are below average.

Let us find an appropriate upper bound for the MAD, i.e., $\rho \leqslant B$. Choosing the best upper bound for each case we have:

a. $B_1 = r_x/2 = 5.000$.

b. $B_3 = 2(x_{\max} - \bar{x})(\bar{x} - x_{\min})/r_x = 4.2$.

c. $B_8^u = 2u(n-u)r_x/n^2 = 5.000$.

d. $B_7 = 2s(n-s)r_x/n^2 = 4.800$.

e. $B_8^l = 2l(n-l)r_x/n^2 = 4.200$.

f. $B_9 = 2lur_x/n(l+u) = 3.750$.

Finally, note that the results of this paper can be applied to any data-set with finite number of elements and the probability distribution of the data is not needed to be known.

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First reformulated Zagreb index of four graph operations

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Abstract. A generalization of classical Zagreb indices of chemical graph theory were introduced in 2004, which is called the reformulated Zagreb indices. The first reformulated Zagreb index EM_1 of any graph G is the sum of the squares of degree of edges. In this paper, we compute the first reformulated Zagreb index of four operations of graphs.
Keywords: Zagreb indices, reformulated Zagreb indices, degree, subdivision of graph, operation on graph.

1. Introduction

The physical and chemical properties of a compound depends on its structure, and topology of a structure can be easily studied with the help of graph theory. The chemical structure can be described graphically, in which edges denotes

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chemical bonding of atoms, where as vertices denote atoms. In 1965, Crum Brown and Fraser presented the first quantitative structure activity relationships (QSAR) in [3]. In this work only simple connected graphs are under consideration. For a graph G we denote vertex set by $V(G)$ and edge set by $E(G)$. The degree of a vertex $v \in V(G)$ is the number of vertices incident to G and the degree of an edge e in G is defined as if $e = uv$, where $u, v \in V(G)$ then $d(e) = d(u) + d(v) - 2$. Topological index is a numerical quantity which gives information about molecular structure. In the description of a chemical structure, topological index is one of the most powerful and useful tool. The oldest topological index is the Wiener index, which was introduced in 1947. Later on many topological indices were introduced, for example Wiener index Hyper-Wiener index, Estrada index, Randíc index, Zagreb index, and Szeged index. There are many intense applications of topological indices of a molecular graphs, for further details of applications see [9, 10, 15, 21, 22, 23, 24].

In 1972, one of the oldest graph invariant was introduced by Gutman and N. Trinajstič called Zagreb index, for more details of Zagreb index see [13]. They examined the dependence of total π-electron energy on molecular structure and it was further studied in [12]. For a molecular graph, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} d_G(u) + d_G(v),$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

They were further studied in [1, 2, 5, 6, 11, 14, 17, 18]. An edge of graph G corresponds to a vertex of its line graph $L(G)$. Motivated by this fact, in 2004 Miličević et al. introduced the reformulated Zagreb indices in [19], which is a generalization of classical Zagreb indices of chemical graph theory. They defined Zagreb indices in term of edge degree instead of vertex degree as:

$$EM_1(G) = \sum_{e \in E(G)} d_G^2(e),$$

$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f).$$

where if $e = uv$, where $u, v \in V(G)$ then $d(e) = d(u) + d(v) - 2$ and $e \sim f$ means edges e and f are adjacent, that is, they shares a common end vertex. Different mathematical properties of reformulated Zagreb indices have been studied in [25]. In [16], Ilić et al.,establish further mathematical properties of the reformulated Zagreb indices. Recently, the upper and lower bounds on $EM_1(G)$ and $EM_2(G)$ were presented in [25, 16, 7]; Su et al. [20] characterize the extremal graph properties on $EM_1(G)$ with respect to given connectivity. The motivation to study reformulated Zagreb indices is the property that the reformulated

Zagreb index of a graph G is equals to the Zagreb index of corresponding line graph $L(G)$. Where, line graph of a graph G is a graph whose vertices are the edges of G, that is, $V(L(G)) = E(G)$ and two vertices in line graph of G are connected if they are adjacent edges in G. In other words, $EM_1(G) = M_1(L(G))$ and $EM_2(G) = M_2(L(G))$.

In this work, we considered four operations on graphs, which were introduced in [8] and we compute the first reformulated Zagreb index for them. While first and second Zagrab indices of these operations are given in [8]. It is worth recalling those operations here. Let $G_1 \times G_2$ denotes the cartesian product of the graphs G_1 and G_2. Then $G_1 \times G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $G_1 \times G_2$ forms an edge if and only if $u_1 = u_2$ and v_1v_2 is an edge in G_2 or $v_1 = v_2$ and u_1u_2 is an edge in G_1.

For a simple connected graph G, the subdivision graph is denoted by $S(G)$ and formulated by inserting an additional vertex in each edge of G. In other words, each edge of G is replaced by path of length 2. Other three related graphs can be defined as follows:

1. For a simple connected graph G, $R(G)$ is be formulated by adding a new vertex corresponding to every edge of G, then joining each new vertex to the end vertices of the corresponding edge.

2. For a simple connected graph G, $Q(G)$ is be formulated by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on edges of G.

3. For a simple connected graph G, $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. The graphs $T(G)$ is called the total graph of G.

For more details on these operations we refer the reader to [4].

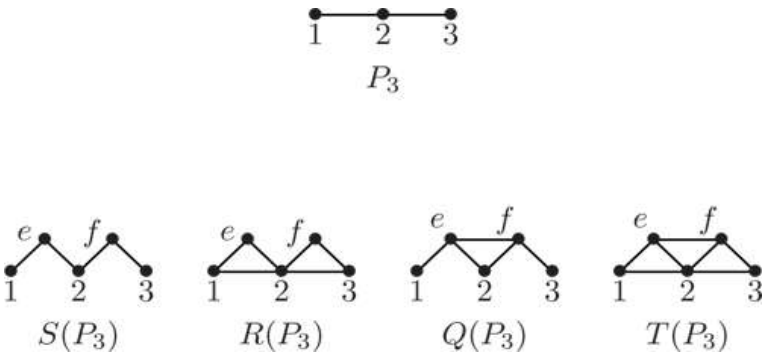


Figure 1: Four Operations on P_3.

In Figure 1, we have the examples of all four operations discussed above, where P_3 represent the path graph on 3 vertices. In general, a path graph on vertex set $\{v_1, v_2, \ldots, v_n\}$ is a graph whose edges are $\{v_i, v_{i+1}\}$, where $1 \leq i \leq n-1$. It is denoted by P_n.

Suppose that G_1 and G_2 are two connected graphs. Based on these operations above, Eliasi and Taeri [9] introduced four new operations on these graphs in the following: Let $F \in S, R, Q, T$. The F-sum of G_1 and G_2, denoted by $G_1 +_F G_2$ is defined by $F(G_1) \times G_2 - E^*$, where $E^* = \{(u, v_1)(u, v_2) \in E(F(G_1) \times (G_2)) : \ u \in V(F(G_1)) - V(G_1), \ v_1v_2 \in E(G_2)\}$ i.e., $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \bigcup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))]$.

2. The first reformulated Zagreb index of F-sums of graphs

In this section we compute the first reformulated Zagreb index of four operations of graphs discussed earlier. First of all we consider the case when the operation is S. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_S G_2$ is shown in Figure 2. Where $G_1 +_S G_2 = P_4 +_S P_2$ is defined by $S(P_4) \times G_{P_2} - E^*$, where $E^* = \{(u, v_1)(u, v_2) \in E(S(P_4) \times (P_2)) : \ u \in V(S(P_4)) - V(P_4), \ v_1v_2 \in E(P_2)\}$ i.e., $P_4 +_S P_2$ is a graph with the set of vertices $V(P_4 +_S P_2) = (V(P_4) \bigcup E(P_4)) \times V(P_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $P_4 +_S P_2$ are adjacent if and only if $[u_1 = v_1 \in V(P_4)$ and $u_2v_2 \in E(P_2)]$ or $[u_2 = v_2 \in V(P_2)$ and $u_1v_1 \in E(S(P_4))]$.

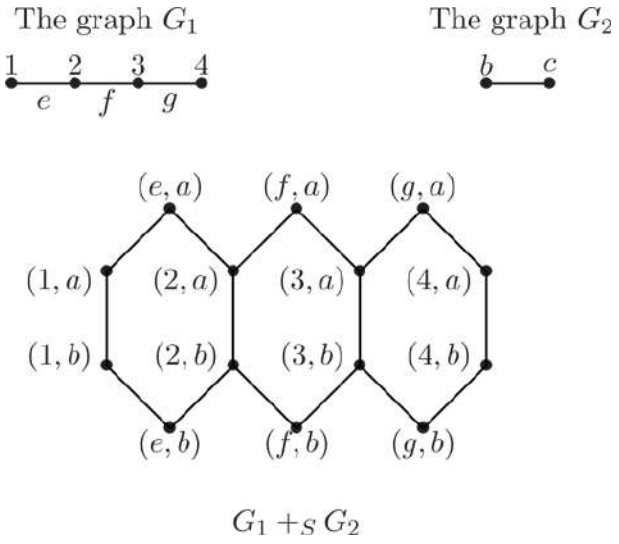


Figure 2: Graph G_1, G_2 and $G_1 +_S G_2$.

In the following theorem we compute $EM_1(G_1 +_S G_2)$.

Theorem 2.1. *Let G_1 and G_2 be two connected graphs. Then $EM_1(G_1 +_S G_2) = 8e_2M_1(G_1)+10e_1M_1(G_2)+n_1EM_1(G_2)+n_2EM_1(S(G_1))-16e_1e_2$, where $n_i = |V(G_i)|$ and $e_i = |E(G_i)|, i = 1, 2$.*

Proof. Let $d(u, v) = d_{G_1+_SG_2}(u, v)$ and

$$d((u_1, v_1)(u_2, v_2)) = d_{G_1+_SG_2}((u_1, v_1)(u_2, v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1+_SG_2$ respectively

$$\begin{aligned}
EM_1(G_1 +_S G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_SG_2)} d^2((u_1, v_1)(u_2, v_2)) \\
&= S_1 + S_2 \\
\text{where, } S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u, v_1)(u, v_2)) \\
\text{and } S_2 &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} d^2((u_1, v)(u_2, v)) \\
S1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u, v_1) + d(u, v_2) - 2]^2 \\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [2d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\
&= 4e_2M_1(G_1) + n_1EM_1(G_2) + 8e_1M_1(G_2) - 16e_1e_2. \qquad (1)
\end{aligned}$$

In the similar way,

$$\begin{aligned}
S_2 &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d(u_1, v) + d(u_2, v) - 2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d_{S(G_1)}(u_1) + d_{(G_2)}(v) + d_{S(G_1)}(u_2) - 2]^2.
\end{aligned}$$

Since $u_1 \in V(G_1)$, $u_2 \in V(S(G_1) - V(G_1))$ and $|E(S(G_1))| = 2|E(G_1)|$. So,

$$S_2 = 2e_1M_1(G_2) + 4e_2M_1(S(G_1)) - 16e_1e_2 + n_2EM_1(S(G_1)). \qquad (2)$$

Note that $M_1(S(G_1)) = M_1(G_1) + 4e_1$. So finally from (1) and (2), we have:

$$\begin{aligned}
EM_1(G_1 +_S G_2) &= 8e_2M_1(G_1) + 10e_1M_1(G_2) + n_1EM_1(G_2) \\
&\quad +n_2EM_1(S(G_1)) - 16e_1e_2. \qquad (3)
\end{aligned}$$

□

Let G be a graph, then $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Then for two connected graphs G_1 and G_2, $G_1+_RG_2$

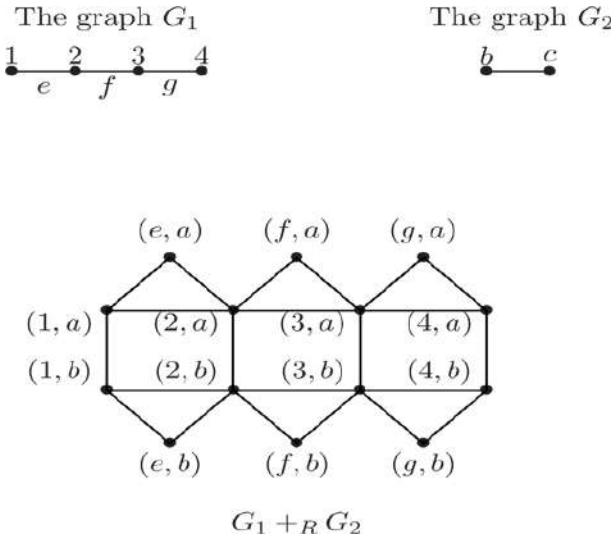


Figure 3: Graph G_1, G_2 and $G_1 +_R G_2$.

is defined with set of vertices $V(G_1 +_R G_2) = (V(G_1) \bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_R G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(R(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_R G_2$ is shown in Figure 3.

In the following theorem we compute $EM_1(G_1 +_R G_2)$.

Theorem 2.2. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1 +_R G_2) = 40e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + 4n_2EM_1(G_1) + 4n_2M_3(G_1) + 8n_2M_1(G_1) + 12n_2e_1 - 48e_1e_2$.

Proof. Let

$$d(u, v) = d_{G_1+_RG_2}(u, v) \text{ and } d((u_1, v_1)(u_2, v_2)) = d_{G_1+_RG_2}((u_1, v_1)(u_2, v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1+_RG_2$ respectively.

$$EM_1(G_1 +_R G_2) = \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_RG_2)} d^2((u_1, v_1)(u_2, v_2)) = S_1 + S_2,$$

where $S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u, v_1)(u, v_2))$ and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(R(G_1))} d^2((u_1, v)(u_2, v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u,v_1)+d(u,v_2)-2]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [4d_{G_1}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)-2]^2 \\ &= 16e_2M_1(G_1)+n_1EM_1(G_2)+16e_1M_1(G_2)-32e_1e_2. \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1,u_2\in V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &= S_1^{'}+S_2^{'} \end{aligned}$$

Note that $(i)u_1,u_2 \in V(G_1)$ and $u_1u_2 \in E(R(G_1))$ if and only if $u_1u_2 \in E(G_1)$, $(ii)d_{R(G_1)}(u_1) = 2d_{G_1}(u_1)$ for $u_1 \in V(G_1)$, and $d_{R(G_1)}(u_2) = 2$ for $u_2 \in V(R(G_1)) - V(G_1)$, we have

$$\begin{aligned} S_1^{'} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1,u_2\in V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1,u_2\in V(G_1)}} [2d_{G_2}(v)+d_{R(G_1)}(u_1)+d_{R(G_1)}(u_2)-2]^2 \\ &= 4n_2 \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1,u_2\in V(G_1)}} [d_{G_1}(u_1)+d_{G_1}(u_2)-1]^2 \\ &\quad +4e_1M_1(G_2)+16e_2M_1(G_1)-16e_1e_2 \\ &= 4n_2 \sum_{\substack{u_1u_2\in E(G_1),\\ u_1,u_2\in V(G_1)}} [(d_{G_1}(u_1)+d_{G_1}(u_2)-2)^2+4e_1M_1(G_2)+16e_2M_1(G_1) \\ &\quad -16e_1e_2+2(d_{G_1}(u_1)+d_{G_1}(u_2)-2)+1] \\ &= 4e_1M_1(G_2)+16e_2M_1(G_1)-16e_1e_2+4n_2EM_1(G_1) \\ &\quad +8n_2M_1(G_1)-12n_2e_1. \end{aligned}$$

and

$$\begin{aligned} S_2' &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [d_{G_2}(v)+d_{R(G_1)}(u_1)+d_{R(G_1)}(u_2)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [2d_{G_1}(u_1)+d_{G_2}(v)]^2 \\ &= 4n_2M_3(G_1)+2e_1M_1(G_2)+8e_2M_1(G_1). \end{aligned}$$

Therefore, $EM_1(G_1 +_R G_2) = 40e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + 4n_2EM_1(G_1) + 4n_2M_3(G_1) + 8n_2M_1(G_1) + 12n_2e_1 - 48e_1e_2$. □

Let G be a graph, then $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on edges of G. For two connected graphs G_1 and G_2, $G_1 +_Q G_2$ is defined with set of vertices $V(G_1 +_Q G_2) = (V(G_1) \bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_Q G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(Q(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_Q G_2$ is shown in Figure 4

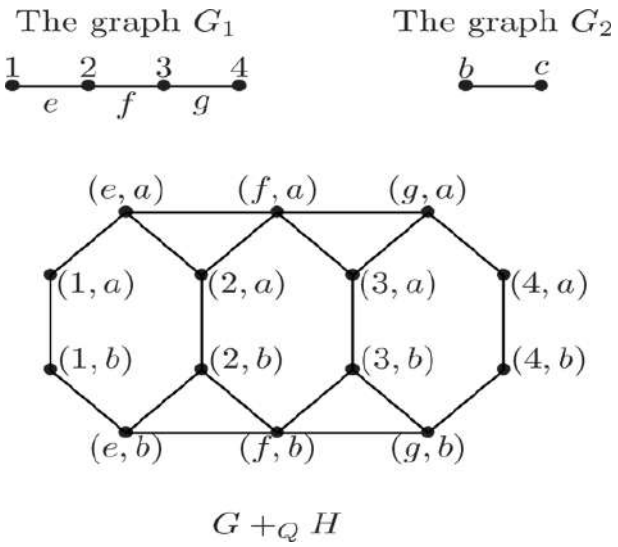


Figure 4: Graph G_1, G_2 and $G_1 +_Q G_2$.

In the following theorem we compute $EM_1(G_1 +_Q G_2)$.

Theorem 2.3. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1+_QG_2) = 16e_2M_1(G_1) + 10e_1M_1(G_2) - 32e_1e_2 + n_1EM_1(G_2) + n_2EM_1(Q(G_1))$.

Proof. Let

$$d(u,v) = d_{G_1+_QG_2}(u,v) \text{ and } d((u_1,v_1)(u_2,v_2)) = d_{G_1+_QG_2}((u_1,v_1)(u_2,v_2))$$

be the degrees of a vertex (u,v) and an edge $(u_1,v_1)(u_2,v_2)$ in the graph $G_1 +_Q G_2$, respectively

$$EM_1(G_1 +_Q G_2) = \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_QG_2)} d^2((u_1,v_1)(u_2,v_2)) = S_1 + S_2,$$

where $S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u,v_1)(u,v_2))$ and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(Q(G_1))} d^2((u_1,v)(u_2,v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u,v_1) + d(u,v_2) - 2]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [2d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\ &= 4e_2M_1(G_1) + 8e_1M_1(G_2) - 16e_1e_2 + n_1EM_1(G_2). \end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &= S_1' + S_2'. \end{aligned}$$

Then,

$$\begin{aligned} S_1' &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} [d(u_1,v) + d(u_2,v) - 2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} [d_{G_2}(v) + d_{Q(G_1)}(u_1) \\ &\qquad + d_{Q(G_1)}(u_2) - 2]^2. \end{aligned} \tag{5}$$

Note that $d_{T(G_1)}(u_2) = d_{G_1}(w_i) + d_{G_1}(w_j)$ for $u_2 \in V(T(G_1)) - V(G_1)$, where u_2 is the vertex inserted into the edge w_iw_j of G_1, we have

$$\begin{aligned} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d_{T(G_1)}(u_2) &= 2 \sum_{w_iw_j\in E(G_1)} [d_{G_1}(w_i) + d_{G_1}(w_j)] \\ &= 2M_1(G_1). \end{aligned}$$

So,

$$\begin{aligned} S_1' &= 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} (d_{Q(G_1)}(u_1) + d_{Q(G_1)}(u_2) - 2)^2 \\ &= 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 + n_2EM_1(Q(G_1)) \\ &- \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} [d(u_1,v) + d(u_2,v) - 2]^2. \end{aligned}$$

Hence,

$$S_2 = 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 + n_2EM_1(Q(G_1)) \tag{6}$$

So, finally from (4) and (6), we have:

$$\begin{aligned} EM_1(G_1 +_Q G_2) &= 16e_2M_1(G_1) + 10e_1M_1(G_2) - 32e_1e_2 \\ &\quad + n_1EM_1(G_2) + n_2EM_1(Q(G_1)). \end{aligned}$$

□

Let G be a graph, Then $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. For two connected graphs G_1 and G_2, $G_1 +_T G_2$ is defined with set of vertices $V(G_1 +_Q G_2) = (V(G_1) \bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_T G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(T(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_T G_2$ is shown in Figure 5.

Theorem 2.4. *Let G_1 and G_2 be two connected graphs. Then $EM_1(G_1+_TG_2) = 48e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + n_2EM_1(T(G_1)) - 64e_1e_2$.*

Proof. Let

$$d(u,v) = d_{G_1+_TG_2}(u,v) \text{ and } d((u_1,v_1)(u_2,v_2)) = d_{G_1+_TG_2}((u_1,v_1)(u_2,v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1+_TG_2$ respectively

$$\begin{aligned} EM_1(G_1 +_T G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_TG_2)} d^2((u_1,v_1)(u_2,v_2)) \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u,v_1)(u,v_2))$$

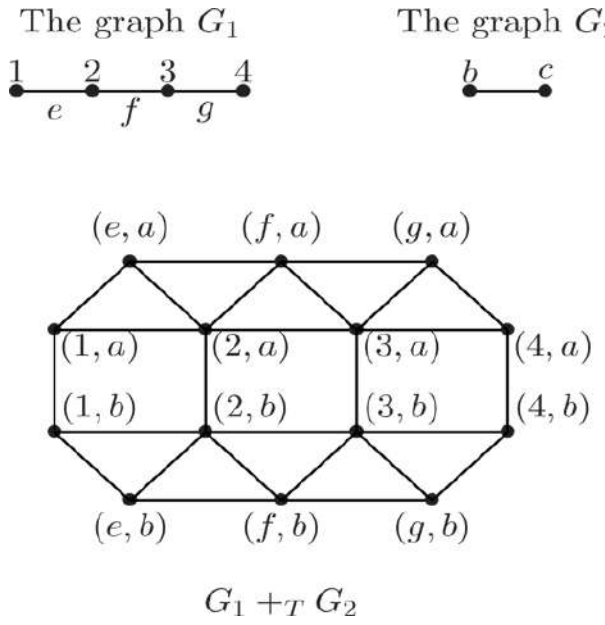


Figure 5: Graph G_1, G_2 and $G_1 +_T G_2$.

and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(T(G_1))} d^2((u_1, v)(u_2, v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u, v_1) + d(u, v_2) - 2]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [4d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\ &= 16e_2M_1(G_1) + n_1EM_1(G_2) + 16e_1M_1(G_2) - 32e_1e_2. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} d^2((u_1, v)(u_2, v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d^2((u_1, v)(u_2, v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(T(G_1))-V(G_1)}} d^2((u_1, v)(u_2, v)) \\ &= S_1' + S_2' + S_3'. \end{aligned}$$

Note that $(i)u_1, u_2 \in V(G_1)$ and $u_1u_2 \in E(T(G_1))$ if and only if $u_1u_2 \in E(G_1)$, $(ii)d_{T(G_1)}(u_1) = 2d_{G_1}(u_1)$ for $u_1 \in V(G_1)$, we have

$$\begin{aligned} S_1^{'} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(G_1),\\ u_1,u_2\in V(G_1)}} [2d_{G_2}(v)+d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2]^2 \\ &= 4e_1M_1(G_2)+16e_2M_1(G_1)-16e_1e_2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} (d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2. \end{aligned}$$

Now,

$$\begin{aligned} S_2^{'} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d_{G_2}(v)+d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d_{G_2}^2(v)+2d_{G_2}(v)(2d_{G_1}(u_1) \\ &+d_{T(G_1)}(u_2)-2)+(d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2]. \end{aligned}$$

Note that $d_{T(G_1)}(u_2) = d_{G_1}(w_i) + d_{G_1}(w_j)$ for $u_2 \in V(T(G_1)) - V(G_1)$, where u_2 is the vertex inserted into the edge w_iw_j of G_1, we have

$$\begin{aligned} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d_{T(G_1)}(u_2) &= 2\sum_{w_iw_j\in E(G_1)} [d_{G_1}(w_i)+d_{G_1}(w_j)] \\ &= 2M_1(G_1). \end{aligned}$$

So,

$$\begin{aligned} S_2^{'} &= 16e_2M_1(G_1)+2e_1M_1(G_2)-16e_1e_2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} (d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2 \end{aligned}$$

finally,

$$S_2 = 32e_2M_1(G_1)+6e_1M_1(G_2)+n_2EM_1(T(G_1))-32e_1e_2. \quad (8)$$

Therefore, from (7) and (8), we have:

$$\begin{aligned} EM_1(G_1 +_T G_2) &= 48e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) \\ &\quad +n_2EM_1(T(G_1)) - 64e_1e_2. \end{aligned}$$

□

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The isomorphic factorization of complete equipartite graphs $K_n(m)$

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Abstract. Harary, Robinson and Wormald suggest that in an international conference on combinational theory (Canberra 1977), a conjecture for a complete equipartite graph $G = K_n(m)$, the divisibility condition $t|\frac{1}{2}n(n-1)m^2$ is a sufficiency condition to $t|G$ [1][2], which were proved by Jianfang Wang [3] and S.J. Quinn [4] respectively. The first author of the paper also proved the conjecture with a new way,which is completely different from ones in [3] and [4] at the same time. Here we devote the proof. And in this paper, we give a new isomorphic factorization of K_n as well, which are usefull to the decomposition of $K_n(m)$. We proved that when t is odd or $t < \frac{n}{2}$, the t's factor of K_n is an union of a complete graph and a 3-colorable graphs; and when $t < \frac{n}{2}$ is even, the factor is an union of two complete graphs and 4-colorable graphs.

Keywords: isomorphic factorization, decomposition, complete graphs.

1. Introduction

A partition $\{E_1, E_2, \cdots, E_t\}$ of the edge set $E = E(G)$ of a graph $G = (V, E)$ is called an isomorphic factorization of the graph G, if the subgraphs $G_i = (V, E_i)$ spanned by E_i, $i = 1, 2, \cdots, t$ are isomorphic to each other. In this case, we say that t divides G, denote by $t|G$. If $G_i \cong F$, we also say that F divides G, denoted by $F|G$. An obvious necessary condition for $t|G$ is that $t||E(G)|$.

For convenience, we do not distinguish the edge set $E = E(G)$ of a graph $G = (V, E)$ from the graph itself, especially, if $E^* \subset E = E(G)$, we usually use E^* to denote the subgraph $G^* = (V, E^*)$ spanned by E^*. We denote, by (A, B), the edge set of the complete bipartite graph $K(A, B)$ with independent vertex sets A and B.Let $G = (V, E)$ be a graph, $A, B \subset V = V(G)$ are two disjoint nonempty subsets of the vertex set of the graph G, A and B are said to be completely adjacent, adjacent, and nonadjacent, if and only if, $(A, B) \subset E(G), (A, B) \cap E(G) \neq \emptyset$, and $(A, B) \cap E(G) = \emptyset$, respectively.We set $S = \bigcup_{i=1}^{t} S_i, S_i \cap S_j = \emptyset, S_i \neq \emptyset, (|S_i| = \frac{|S|}{t})$ to mean that the family $\{S_1, S_2, \cdots, S_t\}$ of nonempty sets is a partition(equipartition) of a set S.

For a graph $G = (V, E)$, a graph $G^* = (V^*, E^*)$ ia slit to be a slip of the graph G, if there is a map $\sigma : V \to \{V_1, V_2, \cdots, V_n\}$ from the vertex set

$V = V(G) = \{v_1, v_2, \cdots v_n\}$ into a partition $\{V_1, V_2, \cdots, V_n\}$ of the vertex set of G^* defined by $\sigma(v_i) = V_i$, such that V_i and V_j are completely adjacent, if and only if v_i is adjacent to v_j, else $(V_i, V_j) \cap E^* = \emptyset$. Furthermore, we say that the map σ slits G into G^*, or G^* is the graph obtained from G by slitting σ.A graph $G = (V, E)$ is said to be a quasi complete multipartite graph, if there is a partition $\{V_1, V_2, \cdots V_n\}$ of $V = V(G)$, such that

$$(V_i, V_j) \cap E(G) = \begin{cases} (V_i, V_j) \\ \emptyset \end{cases}$$

$i \neq j, i, j = 1, 2, \cdots, n$. It is easy to see that, a graph G is a quasi complete multipartite graph, if and only if, G is a slit of some graph.

Remark. An complete multipartite graph, denoted by $K(n_1, n_2, \cdots n_r)$, is a slit of K_r, where $n_i = |V_i|, i = 1, 2, \cdots, r$.

Two quasi-complete multipartite graphs G_1 and G_2 are said to be homeomorphic, if both are slits of the same graph F. Furthermore, G_1 is isomorphic to G_2, whenever, the corresponding vertex sets in G_1 and G_2, denoted by $\sigma_1(x)$ and $\sigma_2(x)$ respectively, have the same cardinality, i.e $|\sigma_1(x)| = |\sigma_2(x)|$ for every $x \in V(F)$. For integers $a, b, m > 1$, we denote $a + b(\mathrm{mod}\, m)$ to mean that $a, b \in \{1, 2, \cdots, m\}$[5] and if $a + b > m$, then , replace $a + b$ by $a + b - m$. The notation and terminology that are not defined in this paper follow [6]. For complete graphs K_n [1] and some special case of tripartite graphs $K(m, n, s)$ [2], Harary, Robinson and Wormad have proved the sufficiency of the divisibility condition and suggested two conjectures, if t is even and $t|(mn + ms + ns)$ then $t|K(m, n, s)$, which was proved by Shihui Yang [7], if $t|\frac{1}{2}n(n-1)m^2$, then $t|K_n(m)$, which was proved by Jianfang Wang [3] and S.J. Quinn [4] and Jiangdong Liao independently, may proof is different from theirs in [3] and [4], here we devote the proof to the conjecture. Shihui Yang get some sufficient conditions for complete tripartite graph $K(A, B, C)$ to have isomorphic factorization[8].

Recently, some papers have investigated the gregarious decomposition problem in a multipartite graph. [5,9,14]considered it for the path and cycle decompositions of complete equipartite graphs; and results about kite decomposition can be found in [10]; [11]proved that the equipartite graphs have a characterization; [12]considered it for the group divisible designs with two associate classes; [13]main result is to prove that there is a k-semi-perfect 1-factorization of Q_d for all k and all d, except for one possible exception when $k = 3$ and $d = 6$, the most obvious question: is it possible to find a 1-factorization M of Q_6 such that $G[M] = K_{3,3}$? In this paper, we give a new isomorphic factorization of K_n as well, which is useful to the decomposition of $K_n(m)$, we proved that when t is odd, or $t \geq \frac{n}{2}$, the t's factor of K_n is a union of a complete graph and a 3-colorable graphs, when $t < \frac{n}{2}$ is even, the factor is an union of two complete graphs and 4-colorable graphs.

2. The isomorphic factorization of K_n

Theorem 2.1. *Assume $G = K_n, t|\frac{1}{2}n(n-1)$ then $t|G$, and each $t's$ factor is a 3-colorable or a disconnected graph whose components are 3-colorable graphs or 4-colorable complete graphs.*

Proof. We will prove the proposition by four case:

Case 1. [1] $t > n$, then $\frac{1}{2}n(n-1) = tr, r < \frac{n-1}{2}$. We can label the edges by using $1, 2, \cdots, \frac{1}{2}n(n-1)$ such that the edges labeled by $i+1, i+2, \cdots, i+r(\mathrm{mod}\, \frac{1}{2}n(n-1))$ is a matching.

Case 2. [1] The $t's$ factor is a union of some K_3 and K_2 with a vertex in common, when $t = [\frac{n}{2}]$, $n = 2t$ and $[\frac{n}{2}] < t \leq n$. Denote $V(K_n) = \{1, 2, \cdots, n\}$, K_n can be decompose into t factors $F_i, i = 1, 2, \cdots, t$ as follows: $3, n, 3, n-1, \cdots, t-1, t+3, t, t+2, t+3$. F_i can obtained from F_1 by using the element plus $i-1(\mathrm{mod}\, n)$ replace the corresponding element,

$F_2 : 4, 3, 1, 4, n, \cdots, t, t+4, t+1, t+3, t+4$

$F_3 : 5, 4, 2, 5, 1, \cdots, t+1, t+5, t+2, t+4, t+5$

. . . . . .

$F_t : t+2, t+1, t-1, t+1, t-2, \cdots, n-2, n+2, n-1, n+1, n+2(\mathrm{mod}\, n)$

and F_i has the property that $V(F_i) \cap \{i, i+t\} = \emptyset$. For $n = 2t+1$, denote $V(K_n) = \{0, 1, 2, \cdots, n\}$, let $\tilde{F}_i = F_i \cup \{(0, i+1), (0, i+1+t)\}, i = 1, 2, \cdots, t$. Each F_i is the same as in the case that $n = 2t$ and $i+1, i+1+t \in \{1, 2, \cdots, n\}$. We have $V(\tilde{F}_i) \cap \{i, i+t\} = \emptyset$.

Case 3. t is odd and $t < \frac{n}{2}$. Set $n = at+b, 0 \leq b < t, V(K_n) = \{1, 2, \cdots, n\}, V^i = \{a(i-1)+1, a(i-1)+2, \cdots, ai\}, i = 1, 2, \cdots, t, V^0 = \{at+1, at+2, \cdots, at+b\}$. Denote by $K(V^i)$ the complete graph with vertex set V^i, then K_n can be represented as $K_n = K(V^1, V^2, \cdots, V^t) \cup K(V^0, V^1 \cup V^2 \cup \cdots \cup V^t) \cup (\bigcup_{i=0}^{t} K(V^i))$. Set $F_i = K(V^i) \cup K(V^{i+1}, V^{i+2} \cup V^{i+3} \cup \cdots \cup V^{i+\frac{t-1}{2}} \cup V^0) \cup M^i, i = 1, 2, \cdots, t, i+j(\mathrm{mod}\, t)$, where M^i is a t's factor of $K(V^0)$, having $r = \frac{b(b-1)}{2t}$ edges, as in case(1)(Notice that it is easy to see $t|\frac{1}{2}b(b-1)$), and the isomorphic factorization of K_a is obtained. And we have F_i is a union of complete graph K_b and a 3-color graph.

Case 4. t is even and $t < \frac{n}{2}$. Set $n = 2at+b, 0 \leq b < 2t, V(K_n) = \{1, 2, \cdots, n\}, V^i = \{a(i-1)+1, a(i-1)+2, \cdots, ai\}, i = 1, 2, \cdots, 2t$, $V^0 = \{2at+1, 2at+2, \cdots, 2at+b\}$, $K_n = K(V^1, V^2, \cdots V^{2t}) \cup K(V^0, V^1 \cup V^2 \cup \cdots \cup V^{2t}) \cup (\bigcup_{i=0}^{2t} K(V^i))$.

We describe a subgraph of $K(V^1, V^2, \cdots, V^{2t})$ by a set sequence as follows

$F_1^1 : V^3V^2V^{2t}V^3V^{2t-1} \cdots V^{t-1}V^{t+3}V^tV^{t+2}V^{t+3}$ which is similar as case 2, where two sets, which are neighbor in the sequence are completely adjacent. For any integer $i \geq 2$, F_1^i can be obtained by using the scripts plus $i-1(\mathrm{mod}\, 2t)$ to replace the corresponding scripts, so

$F_1^2 : V^4V^3V^1V^4V^{2t} \cdots V^tV^{t+4}V^{t+1}V^{t+3}V^{t+4}$, where $V(F_1^i) \cap (V^i \cup V^{i+t}) = \emptyset$. Since $b < 2t$, by the case 2 $K(V^0)$ can be decomposed into t isomorphic factors, each factor F_0^i is a union of some K_3 and K_2 with a vertex in common.

Set $F^i = F_0^i \cup F_1^i \cup K(V^0, V^{i+1} \cup V^{i+t+1}) \cup K(V^i) \cup K(V^{i+t})$, $i+1, i+t+1 (\mathrm{mod}\, 2t), i = 1, 2, \cdots, 2t$, which is an isomorphic factorization of K_n, each factor is a union of two complete graphs and a 4-colorable graph, and if F_0^i has no K_3, then $F_i = F_0^i \cup F_1^i \cup K(V^0, V^{i+1} \cup V^{i+t+1})$ is 3-colorable. □

3. The isomorphic factorization of $K_n(m)$

3.1 Preliminary results

Lemma 3.1. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a 3-colorable quasi complete multipartite graph. If $q||V^i|, i = 1, 2, \cdots, n$, then $q^2|G$.*

Proof. Let β: $V(G) \to \{1, 2, 3\}$ be a 3-coloring of the graph G, such that $|\beta(V^i)| = 1$. And denote $\beta(i) = \beta(x)$, $x \in V_i$, $i = 1, 2, \cdots, n$.

$V^i = \bigcup_{r=1}^q V_r^i$, $|V_r^i| = \frac{1}{q}|V^i|$, $i = 1, 2, \cdots, n$. Set $A_r = \bigcup_{\beta(i)=1} V_r^i$, $B_r = \bigcup_{\beta(i)=2} V_r^i$, $C_r = \bigcup_{\beta(i)=3} V_r^i$. $A = \bigcup_{r=1}^q A_r$, $B = \bigcup_{r=1}^q B_r$, $C = \bigcup_{r=1}^q C_r$. Thus we have $G \subset K(A, B, C), q||A|, q||B|, q||C|$, and so $q^2|K(A, B, C)$, and $K(A_r, B_s, C_{r+s}), r, s = 1, 2, \cdots, q$, $r + s (\mathrm{mod} q)$ are q^2 isomorphic factors of complete tripartite graph $K(A, B, C)$[7]. Let $G^{rs} = G \cap K(A_r, B_s, C_{r+s})$, then, it is clear, that each G^{rs} is a q^2's factor of G, and the proof is complete. □

Lemma 3.2. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a 3-color quasi complete multipartite graph. If $q||V^i|, d|q, i = 1, 2, \cdots, n$, then $q|G$. Furthermore, if $d|q$, then $dq|G$.*

Proof. Set $V^i = \bigcup_{r=1}^q V_r^i, |V_r^i| = \frac{1}{q}|V^i|$, and $E^r = \bigcup_{(i,j)\in(I,J)}(V^i, V_r^j)$,

$$(I, J) = \{(i, j) | 1 \le i < j \le n, (V^i, V^j) \subset E(G)\}, r = 1, 2, \cdots, q.$$

It is easy to see that $\{E^1, E^2, \cdots, E^q\}$ is an isomorphic factorization of G. For the case that $d|q$, let $q = dq_1$, then $dq = d^2q$, by Lemma 3.1, $d^2|G$, and then decompose each d^2 s-factor into q_1 factors and the proof is complete. □

3.2 Main theorems

Theorem 3.1. *Assume $G = K(V^1, V^2, \cdots, V^n)$ is a complete multipartite graph, $q||V^i|, i = 1, 2 \cdots, n$ and $q \ge 2n - 3$, then $q^2|G$.*

Theorem 3.2. *Assume $G = G(V^1, V^2, \cdots, V^n)$ is a quasi-complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \ge n$, then $q^2|G$.*

Theorem 3.3. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a quasi complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \ge 2[\frac{n+2}{3}] - 1$, then $q^2|G$.*

Theorem 3.4. *Assume $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ and $q \ge [\frac{n}{2}]$, then $q^2|G$.*

Theorem 3.5. *Assume* $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ *and* $1 < q < \frac{n}{2}$ *then* $q^2|G$.

Theorem 3.6. *Let* $G = K_n(m) = K(V^1, V^2, \cdots, V^n)$ *be a complete equipartite graph. If* $t|\frac{1}{2}n(n-1)m^2$ *then* $t|G$.

3.3 Proof of the main theorems

Theorem 3.1. *Assume* $G = K(V^1, V^2, \cdots, V^n)$ *is a complete multipartite graph,* $q||V^i|, i = 1, 2\cdots, n$ *and* $q \geq 2n-3$, *then* $q^2|G$.

Proof. Set $V^i = \bigcup_{r=1}^{q} V_r^i, q||V^i|, i=1, 2, \cdots, n$, and $E^{rs} = \bigcup_{(i,j)\in(I,J)}(V^i_{\varphi_s(j)+r-1}, V^j_{\varphi_s(j)+r+s-2}) = (\bigcup_{j=2}^{n}((\bigcup_{i=1}^{j-1}(V^i_{\varphi_s(j)+r-1}), V^j_{\varphi_s(j)+r+s-2})$, $\varphi_s(j) + r - 1(\text{mod } q)$, $\varphi_s(j) + r + s - 2(\text{mod } q)$, $r, s = 1, 2, \cdots, q$. $\varphi_s(j)$ is defined as follows:

(i) $\varphi_s(2) = 1$, and

$$(ii)\varphi_s(j) = \begin{cases} \varphi_s(j-1) + 1, & j\geq 3, \varphi(j-1)+1 \neq \varphi_s(t)+s, 2\leq t\leq j-1 \\ \varphi_s(j-1) + s(\text{mod } q), & \text{else.} \end{cases}$$

So, for instance, $E^{1s}, s = 1, 2, \cdots, q$ as follows

$E^{11} = (V_1^1, V_1^2) \cup (V_2^1 \cup V_2^2, V_2^3) \cup (V_3^1 \cup V_3^2 \cup V_3^3, V_3^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V_{n-1}^i, V_{n-1}^n)$

$E^{12} = (V_1^1, V_2^2) \cup (V_3^1 \cup V_3^2, V_4^3) \cup (V_5^1 \cup V_5^2 \cup V_5^3, V_6^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V_{2n-3}^i, V_{2n-2}^n)$

. . . . . .

$E^{1s} = (V_1^1, V_s^2) \cup (V_2^1 \cup V_2^2, V_{s+1}^3) \cup \cdots \cup (\bigcup_{i=1}^{s-1} V_{s-1}^i, V_{2s-2}^s) \cup (\bigcup_{i=1}^{s} V_{2s-1}^i, V_{3s-2}^{s+1})$
$\cup \cdots \cup (\bigcup_{i=1}^{n-1} V^i_{\varphi_s(n)}, V^n_{\varphi_s(n)+s-1})$

$E^{1q} = (V_1^1, V_q^2) \cup (V_2^1 \cup V_2^2, V_1^3) \cup (V_3^1 \cup V_3^2 \cup V_3^3, V_2^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V_{n-1}^i, V_{n-2}^n)$

And E^{rs} can be obtained from E^{1s} by using the subscripts in E^{1s} plus $r-1(\text{mod } q)$ to replace the corresponding subscripts respectively.

In order to prove that $\{E^{rs}|r, s = 1, 2, \cdots, q\}$ is an isomorphic factorization of G into q^2 factors. We first indicate that $\varphi_s(2) < \varphi_s(3) < \cdots < \varphi_s(n) \leq 2n-3$. From definition of the function, $\varphi_s(j)$ is increasing.

$s = 1, \varphi_1(2) = 1, \varphi_1(j) = \varphi_1(j-1) + 1 = j - 1, \varphi_1(n) = n - 1$, $s = 2, \varphi_2(j - 1) + 1 = \varphi_2(j-1) + s - 1, \quad j \geq 3$, hence $\varphi_2(j) = \varphi_2(j-1) + s = \varphi_2(j-1) + 2$, thus $\varphi_2(2) = 1, \varphi_2(3) = 3, \cdots, \varphi_2(j) = 2j - 3, \cdots, \varphi_2(n) = 2n - 3$.

$s \geq n$, since $\varphi_s(t) + s - 1 \geq \varphi_s(2) + s - 1 = s \geq n, 2 \leq t \leq j - 1$, hence, $\varphi_s(2) = 1, \varphi_s(3) = 2, \cdots, \varphi_s(j-1) + 1 = j - 2 + 1 = j - 1 = \varphi_s(j), \cdots, \varphi_s(n) = n-1$, $2 < s < n$, then $n = ks+h, 0 \leq h < s$, when $3 \leq j \leq s$, since $\varphi_s(t)+s-1 \geq \varphi_s(2)+s-1 = s$, hence $\varphi_s(2) = 1, \varphi_s(3) = 2, \cdots, \varphi_s(j) = j-1, \cdots, \varphi_s(s) = s-1$, $\varphi_s(2) + s - 1 = s = \varphi_s(s) + 1$, so we have $\varphi_s(s+1) = \varphi_s(s) + s = 2s - 1$.

Set $t_1 = min\{j = \varphi_s(j-1)+1 = \varphi_s(t)+s-1\}$ for some t, $2 \leq t \leq j-1, 3 \leq j \leq n$. Thus we have, by the definition of $\varphi_s(j)$, $t_1 = s+1$, and $\varphi_s(t_1) = 2s-1 = 2t_1 - 3$ recurrently, we define $t_k = min\{j|j > t_{k-1}\varphi_s(j-1) + 1 = \varphi_s(t) + s - 1$, for some t, $2 \leq t \leq j - 3$.

Thus, we have $t_1 < t_2 < \cdots < t_k \leq n$.

We will prove, by induction, that $\varphi_s(t_k) = 2t_k - 3$. For $k = 1$, we are done. For $k \geq 1$, we suppose that $\varphi_s(t_k) = 2t_k - 3$, for $k+1$, since $1 \leq i \leq s-2$, $\varphi_s(t_k)+i < \varphi_s(t_k)+s-1)$ and, for $2 \leq t \leq t_k-1$, $\varphi_s(t)+s-1 \leq \varphi_s(t_k-1)+s-1 = \varphi_s(t_k)-1 < \varphi_s(t_k)+i$, hence we have $\varphi_s(t_k+i) = \varphi_s(t_k+i-1)+1 = \varphi_s(t_k)+i-1+1 = \varphi_s(t_k)+i$, and so $\varphi_s(t_k+s-2)+1 = \varphi_s(t_k)+s-2+1 = \varphi_s(t_k)+s-1$, hence, by the definition we have $t_{k+1} = t_k + s - 1$, $\varphi_s(t_{k+1}) = \varphi_s(t_k + s - 1) = \varphi_s(t_k + s - 2) + s = \varphi_s(t_k) + s - 2 + s = 2(t_k) + 3 + 2s - 2 = 2t_{k+1} - 3$.

Denote $h = max\{t_1, t_2, \cdots, t_k, \cdots\}$, if $h = n$, we have $\varphi_s(n) = 2n - 3$. If $h < n$, we have $\varphi_s(h + i) = \varphi_s(h + i - 1) + 1 = \varphi_s(h) + i - 1 + 1 = \varphi_s(h) + i$, $\varphi_s(n) = \varphi_s(h) + n - h = 2h - 3 + n - h = n + h - 3 < 2n - 3$.

Hence, when $q \geq 2n-3$, $\varphi_s(2), \varphi_s(3), \cdots, \varphi_s(n)$ are $n-1$ different elements. Hence, there are no two elements, in the index set $\{\varphi_s(j) + r - 1 | 2 \leq j \leq n\}$ (as well as in the set $\{\varphi_s(j) + r - 1 (\mathrm{mod}\, q) | 2 \leq j \leq n\}$, and $\{\varphi_s(j) + r + s - 1 (\mathrm{mod} q) | 2 \leq j \leq n\}$ are the same, and so E^{rs} is a union of $n-1$ complete bipartite graphs which and are isomorphic to each other.

Next, we prove that for any $e = (u, v) \in E(G)$, there is one and only one set $E^{r_1 s_1}$, such that $e \in E^{r_1 s_1}$. Since, for $e = (u, v)$, there is unique i, j, r_0, s_0, such that $u \in V^i_{r_0}$ and $v \in V^j_{s_0}$, since u, v are adjacent. We have $i \neq j$, and with out loss of generality, we assume that $i < j, e \in (V^i_{r_0}, V^j_{s_0})$, set $\varphi_s(j) + r - 1 = r_0, \varphi_s(j) + r + s - 2 = s_0, r_1 = r_0 + 1 - \varphi_s(j) (\mathrm{mod}\, q), s_1 = s_0 + 1 - r_0 (\mathrm{mod}\, q)$, we have $e \in E^{r_1 s_1}$, which say that, there are no two sets in the family $\{E^{rs} | r, s = 1, 2, \cdots, q\}$ are intersect, and $E(G) = \bigcup_{r=1}^{q} \bigcup_{s=1}^{q} E^{rs}$. This completes the proof. □

Lemma 3.3. *Let $G = K(V^1, V^2, \cdots, V^n)$ be a complete multipartite graph, $q || V^i |, i = 1, 2 \cdots, n$ and $q \geq 2n - 1$. Set $V^i = \bigcup_{r=1}^{q} V^i_r, |V^i_r| = \frac{1}{q}|V^i|, r = 1, 2, \cdots, q$, then there is a function $\tau(s)$ such that*

$$E^{rs} = \bigcup_{1 \leq i < j \leq n} (V^i_{\varphi_s(j)+r-1+\tau(s)}, V^j_{\varphi_s(j)+r+s-2+\tau(s)}), r, s = 1, 2, \cdots, q$$

is an isomorphic factorization of G, where $\varphi_s(j)$ is defined as in Theorem 3.1, and satisfies that $V(E^{rs}) \cap V^j_r = \emptyset, j = 1, 2, \cdots, n$.

Proof. Let $\tilde{E^{rs}} = \bigcup_{1 \leq i < j \leq n} (V^i_{\varphi_s(j)+r-1}, V^j_{\varphi_s(j)+r+s-2})$, $r, s = 1, 2, \cdots, q$. We known from Theorem 3.1 that $\{\tilde{E^{rs}} | r, s = 1, 2, \cdots, q\}$ is a factorization of G into q^2 factors.

Set $\tilde{u}(r, s) = \{\varphi_s(j) + r - 1, \varphi_s(j) + r + s - 2 | j = 2, 3, \cdots, n\} (\mathrm{mod}\ q)$, $Z_q = \{1, 2, \cdots, q\}$. Since $q \geq 2n - 1$, $Z_q - \tilde{u}(r, s) \neq \emptyset$.

So, there is an element, denoted by $\tau_r(s)$, such that $\tau_{rs} \in Z_q - \tilde{u}(r, s)$, especially, we have $\tau_{1s} \in Z_q - \tilde{u}(1, s)$. Set $\tau(s) = q - \tau_1(s) + 1$. We will have prove that $\tau(s) = q - \tau_1(s) + 1$ is the index function that we want to find.

Set $u(r, s) = \{\varphi_s(j) + r - 1 + \tau(s), \varphi_s(j) + r + s - 2 + \tau(s) | j = 1, 2, \cdots, n\} mod q$. We will prove that $r \notin u(r, s)$. By contrary, suppose that $r \in u(r, s)$, one of the following two cases would appear.

(1) $r = \varphi_s(j) + r - 1 + \tau(s) (\mathrm{mod}\, q)$, then $\varphi_s(j) - 1 + \tau(s) = \varphi_s(j) + q - 1 - \tau_1(s) + 1 = 0 (\mathrm{mod}\, q)$, thus we have $\tau_1(s) = \varphi_s(j) = \varphi_s(j) + 1 - 1 \in \tilde{u}(1, s)$.

(2) $r = \varphi_s(j) + r + s - 2 + \tau(s) (\mathrm{mod}\, q)$, then $\varphi_s(j) + s - 2 + q - \tau_1(s) + 1 = 0 (\mathrm{mod}\, q)$, and we have $\tau_1(s) = \varphi_s(j) + s - 1 = \varphi_s(j) + 1 + s - 2 \in \tilde{u}(1, s)$.

The two cases contradict the hypotheses that $\tau_1(s) \notin \tilde{u}(1, s)$, which proves that $r \notin u(r, s)$, and $V(E^{rs}) \cap V_r^j = \emptyset$, $j = 1, 2, \cdots, n, r, s = 1, 2, \cdots, q$, E^{rs} are obviously isomorphic to each other. The rest is to prove that it is a partition of $E(G)$ as well, and for which we need only to prove that $(\varphi_{s_1}(j) + r_1 - 1 + \tau(s_1), \varphi_{s_1}(j) + r_1 + s_1 - 2 + \tau(s_1)) = (\varphi_{s_2}(j) + r_2 - 1 + \tau(s_2), \varphi_{s_2}(j) + r_2 + s_2 - 2 + \tau(s_2))$, whenever $(r_1, s_1) \neq (r_2, s_2)$.

If $\varphi_{s_1}(j) + r_1 - 1 + \tau(s_1) = \varphi_{s_2}(j) + r_2 - 1 + \tau(s_2)$, and $\varphi_{s_1}(j) + r_1 - 2 + \tau(s_1) = \varphi_{s_2}(j) + r_2 - 2 + \tau(s_2)$, then we have $s_1 = s_2$, and so $\tau(s_1) = \tau(s_2), \varphi_{s_1}(j) = \varphi_{s_2}(j)$ and, thus $r_1 = r_2$, which contradict that $(r_1, s_1) \neq (r_2, s_2)$. □

Theorem 3.2. *Assume $G = G(V^1, V^2, \cdots, V^n)$ is a quasi-complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \geq n$, then $q^2|G$.*

Proof. Set $K_n(m) = K(V^1, V^2, \cdots, V^n)$ be the complete n-partite graph with the same independent vertex sets as the graph G.

Denote $n_1 = [\frac{n+1}{2}], n_2 = [\frac{n}{2}], n = n_1 + n_2$, and $A = V^1 \cup V^2 \cup \cdots \cup V^{n_1}, B = V^{n_1+1} \cup V^{n_2+1} \cup \cdots \cup V^n$. Set $V^i = \bigcup_{r=1}^q V_r^i, |V_r^i| = \frac{m}{q}, i = 1, 2, \cdots, n$, $A_r = \bigcup_{i=1}^m V_r^i$, $B_r = \bigcup_{r=n_1+1}^n V_r^i, r = 1, 2, \cdots, q$, and we have

$$A = \bigcup_{r=1}^{q} A_r, B = \bigcup_{r=1}^{q} B_r.$$

Since $q \geq n \geq 2[\frac{n+1}{2}] - 1 \geq [\frac{n}{2}] - 1$, according to Lemma 3.3, we construct the q^2 isomorphic factors G_1^{rs} and G_2^{rs} of $G_1 = K(V^1, V^2, \cdots, V^{n_1})$ and $G_2 = K(V^{n_1}, \cdots, V^n)$, $r, s = 1, 2, \cdots, q$ respectively, such that $V(G_1^{rs}) \cap A_r = \emptyset$, $V(G_2^{rs}) \cap B_r = \emptyset$.

Set $G^{rs} = G_1^{sr} \cup G_2^{rs} \cup K(A_r, B_s)$, $r, s = 1, 2, \cdots, q$, and so $\tilde{G^{rs}} = G^{rs} \cap G$, $r, s = 1, 2, \cdots, q$ are q^2 isomorphic factors of G. □

Theorem 3.3. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a quasi complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \geq 2[\frac{n+2}{3}] - 1$, then $q^2|G$.*

Proof. Denote $n_1 = [\frac{n+2}{3}], n_2 = [\frac{n+1}{3}], n_3 = [\frac{n}{3}], n = n_1 + n_2 + n_3$. Set $V^i = \bigcup_{r=1}^q V_r^i, |V_r^i| = \frac{1}{q}|V^i|, i = 1, 2, \cdots, n$, $A = \bigcup_{i=1}^{n_1} V^i$, $B = \bigcup_{i=1}^{n_2} V^{i+n_1}$, $C = \bigcup_{i=1}^{n_3} V^{i+n_1+n_2}$. Then, $K(V^1, V^2, \cdots, V^n) = G_1 \cup G_2 \cup G_3 \cup K(A, B, C), G_1 = K(V^1, \cdots, V^{n_1}), G_2 = K(V^{n_1+1}, \cdots, V^{n_1+n_2}), G_3 = K(V^{n_1+n_2+1}, \cdots, V^n)$,

$$A_r = \bigcup_{i=1}^{n_1} V_r^i, B_r = \bigcup_{i=1}^{n_2} V_r^{i+n_1}, C_r = \bigcup_{i=1}^{n_3} V_r^{i+n_1+n_2}$$

decompose G_1, G_2, G_3 into q^2 factors, as Lemma 3.3, $G_1^{rs}, G_2^{rs}, G_3^{rs}$ respectively, such that

$$V(G_1^{rs}) \cap A_r = \emptyset, V(G_2^{rs}) \cap B_r = \emptyset, V(G_3^{rs}) \cap C_r = \emptyset$$

and decompose $K(A, B, C)$ into $K(A_r, B_s, C_{r+s}), r, s = 1, 2, \cdots, q$. Set

$$G^{rs} = G_1^{sr} \cup G_2^{rs} \cup G_3^{r+s,s} \cup K(A_r, B_s, C_{r+s})$$

$\tilde{G^{rs}} = G^{rs} \cap G, r, s = 1, 2, \cdots, q$, then $\{\tilde{G^{rs}} | r, s = 1, 2, \cdots, q\}$ is an isomorphic factorization of G into q^2 factors. □

Theorem 3.4. *Assume $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ and $q \geq [\frac{n}{2}]$, then $q^2|G$.*

Proof. When $q \geq n-1, n \geq 4$, we have $q \geq n-1 \geq 2[\frac{n+2}{3}] - 1$, the proof is obtained from Theorem 3.3, so we assume that $[\frac{n}{2}] \leq q \leq n-2$. Set $n_1 = [\frac{n-q+1}{2}], n_2 = [\frac{n-q}{2}]$, then we have $n = n_1 + n_2 + q$. Set

$$V^i = \bigcup_{r=1}^{q} V_r^i, |V_r^i| = \frac{1}{q}m, 1 = 1, 2, \cdots, n,$$
$$A = \bigcup_{i=1}^{n_1} V^i, B = \bigcup_{i=1}^{n_2} V^{i+n_1}, A_r = \bigcup_{i=1}^{n_1} V_r^i, B_r \bigcup_{i=1}^{n_2} V_r^{i+n_1}.$$

Denote $C^i = V^{i+n_1+n_2}, i = 1, 2, \cdots, q, C = \bigcup_{i=1}^{q} C^i, C_r = \bigcup_{i=1}^{q} C_r^i$.
Case 1. q is odd. Decompose $K(C^1, C^2, \cdots, C^q)$ into $G_3^{rs} = \bigcup_{i=1}^{q_1}(C_s^r, C^{r+i}), r, s = 1, 2, \cdots, q, r + i(\mathrm{mod}\, q), q_1 = \frac{q-1}{2}$. Since $q \geq [\frac{n}{2}]$, $q \geq 2n_1 - 1 \geq 2n_2 - 1$. By the same way as we do in Lemma 3.3. Set G_1^{rs} and G_2^{rs}, $r, s = 1, 2, \cdots, q$ are q^2 isomorphic factors of $K(V^1, V^2, \cdots, V^{n_1})$ and $K(V^{n_1+1}, V^{n_2} + 2, \cdots V^{n_1+n_2})$ respectively, and satisfy that $V(G_1^{rs}) \cap A_r = \emptyset, V(G_2^{rs}) \cap B_r = \emptyset$. Thus $G^{rs} = G_1^{sr} \cup G_2^{rs} \cup G_3^{r+s,s} \cup K(A_r, B_s, C_{r+s})$, $r, s = 1, 2 \cdots, q, r + s(\mathrm{mod}\, q)$ is an isomorphic factorization of $G = K_n(m)$.
Case 2. q is even, from Theorem 2.1, let $F_1, F_2, \cdots F_{q_1}$ is an isomorphic factorization of K_q with $V(F_i) \cap (i, i + q_1) = \emptyset, q_1 = \frac{q}{2}, i + q_1(\mathrm{mod}\ q)$. Each F_i is a 3-colorable graph. Let $\tilde{F_i}$ be a slit of F_i, which is obtained by using C^r to replace r in F_i. So $\tilde{F_i}, i = 1, 2, \cdots, q_1$ is an isomorphic factorization of $K(C^1, C^2, \cdots, C^q)$, and each factor is a 3-color quasi-complete graph. Since $q = 2q_1, q|m$, so from Lemma 2 we can decompose each $\tilde{F_i}$ into $2q$ isomorphic factors, denoted by G_3^{is} and $G_3^{i+q_1,s}, i, s = 1, 2, \cdots, q$.

Thus, we have that $V(G_3^{rs}) \cap (C^r \cup C^{r+q_1}) = \emptyset, r + q_1(\mathrm{mod}\, q)$. Set $G^{rs} = G_1^{r+s,s} \cup G_2^{sr} \cup G_3^{rs} \cup K(A_{r+s}, B_s, C_r)$, $r, s = 1, 2 \cdots, q, r + s(\mathrm{mod}\, q)$, then it is an isomorphic factorization of G and each factor has 4 components. □

Theorem 3.5. *Assume $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ and $1 < q < \frac{n}{2}$ then $q^2|G$.*

Proof. Set $n = 2eq+f, 0 \le f \le 2q-1, V^i = \bigcup_{r=1}^{q} V_r^i, |V_r^i| = \frac{m}{q}, i = 1, 2, \cdots, n,$

$$B^i = V^{f+e(i-1)+1} \cup V^{f+e(i-1)+2} \cup \cdots \cup V^{f+ei}, i = 1, 2 \cdots, 2q,$$
$$G^i = K(V^{f+e(i-1)+1}, \cdots, V^{f+ei}) = G[B^i]$$

$G[V'] \subset G$ as the subgraph induced by $V' \subset V(G)$, $G^0 = K(V^1, V^2, \cdots, V^f)$, $A = \bigcup_{i=1}^{f} V^i, B = \bigcup_{i=1}^{2q} B^i$. Since $q > \frac{f}{2}$, we decompose G^0 into q^2 factors $G_0^{rs}, r, s = 1, 2, \cdots, q$.

Set $G = K(B^1, B^2, \cdots, B^{2q}) \cup (\bigcup_{i=1}^{2q} G^i) \cup G^0 \cup K(A, B)$.
By Theorem 2.1, let $F^r, r = 1, 2, \cdots, q$ be q isomorphic factors of K_{2q}, where $V(K_{2q}) = \{1, 2, \cdots, 2q\}$ and $V(F^r) \cap \{r, r+q\} = \emptyset$, $\tilde{F^r}$ is a slit of F^r by using B^i replace i, F^r is a quasi-complete multipartite graph decompose each $\tilde{F^r}$ into q isomorphic factors, denote by $\tilde{F^{rs}}, r, s = 1, 2, \cdots, q$, decompose each G^i into q factors, denoted $G_s^i, i = 1, 2, \cdots, 2q, s = 1, 2, \cdots, q$, $q \ge \frac{f}{2}$, we can decompose. G^0 into q^2 factors, denote G_0^{rs}, and then set

$$G^{rs} = \tilde{F^{rs}} \cup G_s^r \cup G_s^{r+q} \cup G_0^{rs} \cup K(AB_s^r) \cup K(A, B_s^{r+q}),$$

where $B_s^i = V_s^{f+e(i-1)+1} \cup \cdots \cup V_s^{f+ei}, i = 1, 2, \cdots, 2q$. $G^{rs}, r, s = 1, 2, \cdots, q$ are q^2 isomorphic factor of G. □

Theorem 3.6. *Let $G = K_n(m) = K(V^1, V^2, \cdots, V^n)$ be a complete equipartite graph. If $t|\frac{1}{2}n(n-1)m^2$ then $t|G$.*

Proof. Let $F = K_n, V(F) = \{1, 2, \cdots, n\}$, then $G = K_n[\bar{K_m}]$ is a slit obtained from F by using V^i to replace i. By the condition, we have $t = t_1pq^2, t_1|\frac{1}{2}n(n-1), pq|m$. We first decompose $F = K_n$ into t_1 isomorphic factors F^i and then obtain t_1 isomorphic factors of G, by setting $G_i = F^i[\bar{K_m}]$, which is a quasi complete equipartite graph, and then decompose each G_i into q^2 factor G_i^{rs}, and then decompose each G_i^{rs} into p factors, and complete the decomposition. The key is to decompose G_i into G_i^{rs}. For t_1 is odd or $t_1 \ge [\frac{n}{2}]$, by Theorem 2.1 we known that F^i as well as G_i are both unions of some complete (equipartite) graphs and (quasi complete) 3-colourable (equipartite) graphs. So we can decompose G_i into q^2 isomorphic factors G_i^{rs} by using the preceeding lemmas and theorems. The rest we need only to consider the case that $t_1 < \frac{n}{2}$ is even. In this case, the t_1's factor of K_n, denoted by F_i is a union of two complete graphs and a 4-colorable graph $F_i^{'}$, as in the case 4 of Theorem 2.1. $G_i = G_i^{'} \cup G_{i2} \cup G_{i3}$, where G_{i2}, G_{i3} both are complete equipartite graphs, we can decompose G_{i2} and G_{i3} into q^2 factors as we have done in the preceeding, so we need only to discuss the decomposition of $G_i^{'}$.

If F_0^i as in the case 4 of Theorem 2.1, has no K_3, then both $F_i^{'}$ and $G_i^{'}$ are 3-colorable, so we have proved that $q^2|G_i^{'}$. If F_0^i has a K_3 then $G_i^{'}$ has a corresponding complete tripartite subgraph $K(V^{i_1}, V^{i_2}, V^{i_3})$. For convenience,

denote it by $K(A,B,C)$, then $G_i^{''}-G_i^{'}-(B,C)$ is 3-colorable. So we can decompose $G_i^{''}$ into q^2 factors, denoted by H^{rs}, such that $(A_{r+s},B_r)\cup(A_{r+s},C_r)\subset H^{rs}$, $r,s=1,2,\cdots,q$, and then construct the factor $\tilde{H^{rs}}$ of $G_i^{'}$ from H^{rs} as follows.

(1) When $q\geq 3$, set

$$\tilde{H^{rs}}=\begin{cases}H^{rs}\cup(B_{r+1},C_{r+s}), & s=1,2,\cdots,q-1 \quad r+j(\mathrm{mod}\,q)\\ H^{rs}\cup(B_{r+2},C_{r+1}), & s=q\end{cases}$$

Since $V(H^{rs})\cap(B_{r+1}\cup B_{r+s})=\emptyset, s\neq q$, and $V(H^{rq})\cap(B_{r+2}\cup C_{r+1})=\emptyset$, which guarantee the property that $\tilde{H^{rs}}$ are isomorphic to each other. The rest we need to see that each edge $e=(u,v)\in(B,C)$ is in some $\tilde{H^{rs}}$, see $u\in B_i, v\in C_j$ for some i,j. If $j-i=-1=q-1(\mathrm{mod}\,q)$ then we have $s=q$, and $r=i-2(\mathrm{mod}\,q)$, thus $e\in\tilde{H}^{i-2,q}$. If $j-i\neq q-1(\mathrm{mod}\,q)$ then $r=i-1, s=j-r=j-i+1(\mathrm{mod}\,q)$ and so $e\in\tilde{H}^{i-1,j-i+1}$.

(2) $q=2$, $(B,C)=(B_1,C_1)\cup(B_1,C_2)\cup(B_2,C_1)\cup(B_2,C_2)$, let

$$\begin{aligned}\tilde{H^{11}}&=(H^{11}-(A_2,B_1)-(A_2,C_1))\cup(A_2,B)(B_1,C_1),\\ \tilde{H^{12}}&=(H^{12}-(A_1,B_2)-(A_1,C_2))\cup(A_1,B)(B_1,C_2),\\ \tilde{H^{21}}&=(H^{21}-(A_1,B_1)-(A_1,C_1))\cup(A_1,C)(B_2,C_1),\\ \tilde{H^{22}}&=(H^{22}-(A_2,B_2)-(A_2,C_2))\cup(A_2,C)(B_2,C_2).\end{aligned}$$

It is easy to see that $H^{rs}, r,s=1,2,\cdots,q$ are q^2 isomorphic factors of $G_i^{'}$.

If F_0^i has more than one K_3, then $G_i^{'}$ has corresponding complete tripartite graphs as $K(A,B^j,C^j), j=1,2,\cdots,k$. And thus $G_i^{''}=G_i^{'}-\bigcup_{j=1}^{k}(B^j\cup C^j)$ is 3-colorable, and the vertex in the set $\bigcup_{j=1}^{k}(B^j\cup C^j)$ may have same color of a 3-coloring of $G_i^{''}$. So we can decompose $G_i^{''}$ into $H^{rs}, r,s=1,2,\cdots,q$. Such that $\bigcup_{j=1}^{k}(A_{r+s},B_r^j\cup C_r^j)\subset H^{rs}, j=1,2,\cdots,k$. And then get the factors $\tilde{H^{rs}}$ of $G_i^{'}$ from H^{rs} as we have done as above, and the proof is complete. □

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Lyapunov-type inequalities for ψ-Laplace equations

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Abstract. In this paper, we establish several Lyapunov-type inequalities for a class of ψ-Laplace equations

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0$$

with Dirichlet boundary conditions, where ψ, f are nonlinear functions defined on $\mathbb{R}$ and one of them is imposed on structural conditions of Tolksdorf type. The obtained Lyapunov-type inequalities are extensions and complements of the known results in the sense that compared with the existing literature, neither sub-multiplicative property of ψ nor convexity of $\frac{1}{\psi(t)}$ (or $\psi(t)t$) is required in this paper.

Keywords: Lyapunov inequality; ψ-Laplace, nonlinear differential equation.

1. Introduction

For the Hill's equation

$$u''(x) + r(x)u(x) = 0, \quad x \in (a, b), \tag{1a}$$

$$u(a) = u(b) = 0, \tag{1b}$$

where r is a continuous and nonnegative function defined on $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, if there exists a nontrivial solution u of (1), then the inequality

$$\int_a^b r(x)\mathrm{d}x \geq \frac{4}{b-a} \tag{2}$$

holds. This result is originally due to Lyapunov [12], and is known as "Lyapunov inequality". Lyapunov inequality and many of its generalizations have proved

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to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations, and also in time scales. In the past few years, independent works appeared generalizing Lyapunov inequality for the p-Laplacian, by using Hölder, Jensen or Cauchy-Schwarz inequalities. A thorough literature review of Lyapunov-type inequalities and their applications can be found in the survey articles by Brown and Hinton [3], Cheng [6] and Tiryaki [18]. Some other related topics can be found in the resent articles [1, 2, 4, 5, 8, 9, 10, 13, 14, 15, 17, 19, 21, 22, 23, 24, 25] and the references therein.

It should be mentioned that De Nápoli and Pinasco considered Lyapunov-type inequalities for certain nonlinear differential equations (ψ-Laplace equations) in [7]. The main result they obtained is the following theorem.

Theorem A. Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function such that $t\psi(t)$ is a convex function in t. Besides, suppose that there exists a constant $k > 0$ such that $\psi(2t) \le k\psi(t)$ for any $t \ge 0$. If r is a positive integrable function and the following equation with Dirichlet boundary conditions

$$(\psi(u'))' + r\psi(u) = 0 \ \text{ in } (a,b), \tag{3a}$$

$$u(a) = u(b) = 0, \tag{3b}$$

admits a nontrivial solution, then

$$2\left(\frac{k}{2}\right)^{[1-\log_2(b-a)]} \le \int_a^b r(x)\mathrm{d}x,$$

where $[v]$ is the largest integer less than or equal to v.

In 2011, Sáchez and Vergara extended (3) to the following equations having a general nonlinear form (see [16]):

$$(\psi(u'))' + \lambda r f(u) = 0 \ \text{ in } (a,b), \tag{4a}$$

$$u(a) = u(b) = 0, \tag{4b}$$

where $\lambda > 0$ is a constant and
($\boldsymbol{A_1}$) $f \in C(\mathbb{R};\mathbb{R})$ is odd and satisfies $tf(t) > 0$ for $t \ne 0$;
($\boldsymbol{A_2}$) $r \in C([a,b];(0,+\infty))$;
($\boldsymbol{A_3}$) ψ is odd, increasing, and sub-multiplicative on $[0,+\infty)$, and $\frac{1}{\psi(t)}$ is convex on $(0,+\infty)$.

Under the above assumptions, the authors obtained the following theorem.

Theorem B. If u is a nontrivial solution of (4), satisfying $u(x) \ne 0$ for all $x \in (a,b)$, then the following inequality holds:

$$\frac{2}{\psi\left(\frac{b-a}{2}\right)} \le \lambda \int_a^b \frac{f(u(x))}{\psi(u(x))}\mathrm{d}x,$$

whenever the integral exists.

It should be noticed that the convexity of $t\psi(t)$ proposed in [7], the convexity of $\frac{1}{\psi(t)}$ and the sub-multiplicative property of ψ proposed in [16] play essential roles in the establishment of a Lyapunov-type inequality for ψ-Laplace equations, which motivate us to consider the problem of a large class of nonlinear equations. More precisely, in this paper, we establish Lyapunov-type inequalities for ψ-Laplace equations governed by (5) (or (4)) with structural conditions of Tolksdorf type. With these conditions, the function ψ is allowed to permit much more nonlinearities than those in [7, 16], e.g., there is no a convexity assumption on $t\psi(t)$ or $\frac{1}{\psi(t)}$, and no a sub-multiplicative assumption on ψ (see Remark 2.2 in Section 1 and examples in Section 4). Moreover, in view of $(\boldsymbol{H_4})$ in Section 2, we do not impose any odd-even assumptions on f, and require less sign conditions of f than those in [16].

The rest of this paper is organized as follows. Section 2 presents the considered problem and the main results on Lyapunov-type inequalities. Some remarks on the structural conditions are also provided in this section. Detailed proofs of Lyapunov-type inequalities are given in Section 3. Additional examples satisfying our structural conditions are provided in Section 4.

2. Problem setting and main results

In this paper, we establish Lyapunov-type inequalities for the following equation

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0, \qquad x \in (a,b), \tag{5a}$$

$$u(a) = u(b) = 0, \tag{5b}$$

where ψ and f satisfy the following structural conditions:

$(\boldsymbol{H_1})$ $\psi, f \in C(-\infty,\infty) \cap C^1(0,\infty)$ with $f \not\equiv 0$ on $(-\infty,\infty)$;

$(\boldsymbol{H_2})$ ψ is odd on $(-\infty,\infty)$;

$(\boldsymbol{H_3})$ $f(t) \geq 0$ for all $t \in [0,\infty)$;

$(\boldsymbol{H_4})$ There exists $k_0 > 0$ such that $|f(t)| \leq k_0\psi(|t|)$ for all $t \in (-\infty,\infty)$.

Moreover, ψ or f satisfies the structural condition of Tolksdorf type:

$(\boldsymbol{H_\psi})$ There exist constants $\delta_0, \delta_1 \geq 0$ such that

$$\delta_0\psi(t) \leq t\psi'(t) \leq \delta_1\psi(t), \quad \forall t > 0;$$

or,

$(\boldsymbol{H_f})$ There exist constants $\theta_0, \theta_1 \geq 0$ such that

$$\theta_0 f(t) \leq tf'(t) \leq \theta_1 f(t), \quad \forall t > 0.$$

Throughout this paper, we always assume that $r \in L^1(a,b)$ with $r \not\equiv 0$ on (a,b), and conditions $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ are satisfied. Moreover, we always assume that (5) has a non-trivial solution u in the sense that $u \in C^1(a,b) \cap C([a,b])$, $\psi(u'(x))$ is absolutely continuous in x, and u satisfies (5) almost everywhere in (a,b).

The first main result is stated as follows, which can be seen as a complement of the work of [7] and [16] in the setting of functions satisfying the structural condition $(\boldsymbol{H_\psi})$ or $(\boldsymbol{H_f})$.

Theorem 2.1. *The following statements hold true:*

(*i*) *If ψ satisfies $(\boldsymbol{H_\psi})$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\delta_0}{1+\delta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$;

(*ii*) *If f satisfies $(\boldsymbol{H_f})$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\theta_0}{1+\theta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

Corollary 2.1. *The following statements hold true:*

(*i*) *If $\psi(t) = f(t) = |t|^{p-2}t$ $(p>1)$, then* $\int_a^b |r(x)|dx \geq \frac{2^p}{(b-a)^{p-1}}$, *which is the same as one of the results obtained in [10, 13];*

(*ii*) *If $\psi(t) = f(t) = |t|^{a-1}t\log_c(b|t|+d)$, $a,b>0, c,d>1$, then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{1+(1+a)\ln d} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a+\frac{1}{\ln d}}\right\}$;

(*iii*) *If $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)}$, $b>0, c,d>1, a>\frac{1}{\ln d}$, then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{(1+a)\ln d-1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a-\frac{1}{\ln d}}\right\}$.

If we make a further assumption that $\psi(t)t$ (or $f(t)t$) is convex on $[0+\infty)$, we can obtain the following stronger results, which can be seen as extensions of the work in [7].

Theorem 2.2. *The following statements hold true:*

(*i*) *If ψ satisfies $(\boldsymbol{H_\psi})$ and $\psi(t)$, t is convex in $t \in [0,+\infty)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$;

(*ii*) *If f satisfies $(\boldsymbol{H_f})$ and $f(t)$, t is convex in $t \in [0,+\infty)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

We provide some remarks on the structural conditions.

Remark 2.1. (*i*) We point out that a slight version of the structural condition $(\boldsymbol{H_\psi})$ (or $(\boldsymbol{H_f})$) was firstly introduced by Tolksdorf in [20], i.e., $0 < \delta_0 \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1, \forall t > 0$, which is a generalization of the natural conditions of Ladyzhenskaya and Uralltseva for elliptic equations (see [11]). It should be noticed that $\frac{t\psi'(t)}{\psi(t)}$ is always required to be positive in [11, 20], while in this paper $\psi'(t)$ in $(\boldsymbol{H_\psi})$ (or $f'(t)$ in $(\boldsymbol{H_f})$) can be

zero at some point $t_0 > 0$, i.e., $\delta_0 = 0$ (or $\theta_0 = 0$). Indeed, considering $\psi(t) = \begin{cases} \frac{7}{8} \cdot (2t)^{\frac{3}{7}}, \ 0 < t < \frac{1}{2}, \\ (t-1)^3 + 1, \ \frac{1}{2} \le t \le \frac{3}{2}, \\ \frac{3}{4}t, \ t > \frac{3}{2}, \end{cases}$ we have $\psi \in C^1((0,+\infty))$ and $0 \le \frac{t\psi'(t)}{\psi(t)} \le 3$ for all $t > 0$. The lower boundedness of $\frac{t\psi'(t)}{\psi(t)}$ can be achieved when $\psi'(1) = 3(t-1)^2\big|_{t=1} = 0$.

(*ii*) By $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$, it follows that $\psi(0) = f(0) = 0$ and $\psi(t) \ge 0$ for any $t \ge 0$. Furthermore, if ψ (or f) satisfies $(\boldsymbol{H_\psi})$ (or $(\boldsymbol{H_f})$), then $\psi'(t) \ge 0$ (or $f'(t) \ge 0$), which guarantees the increasing monotonicity of $\psi(t)$ (or $f(t)$) in $t \ge 0$.

Remark 2.2. The following two examples show that our assumptions on ψ (or f) are much weaker than those in [7, 16] (see Theorem A and B) in a certain sense. More examples of functions satisfying $(\boldsymbol{H_\psi})$ or $(\boldsymbol{H_f})$ are provided in Section 4.

(i) Let $\psi_0(t) = t + \frac{1}{t}$ with $t \in [\frac{6}{5}, \sqrt{3}]$. Due to the continuities of ψ_0 and ψ_0', there exist $\widehat{\delta_0}, \widehat{\delta_1} > 0$ such that $\widehat{\delta_0} \le \frac{t\psi_0'(t)}{\psi_0(t)} \le \widehat{\delta_1}$ for all $t \in [\frac{6}{5}, \sqrt{3}]$. Let $p = \frac{11}{61}, a = \left(\frac{6}{5} + \frac{5}{6}\right)\left(\frac{6}{5}\right)^{-p}$ and $q = \frac{1}{2}, b = \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)(\sqrt{3})^{-q}$. Let $\psi(t) = \begin{cases} at^p, \ 0 < t < \frac{6}{5} \\ \psi_0(t), \ \frac{6}{5} \le t \le \sqrt{3} \\ bt^q, \ t > \sqrt{3} \end{cases}$. By direct computations, we have $\left(\frac{1}{\psi(t)}\right)'' = \left(\frac{1}{\psi_0(t)}\right)'' = \frac{-2}{(t+1)^3} < 0$ for $t \in (\frac{6}{5}, \sqrt{3})$. Thus $\frac{1}{\psi}$ is not convex on $[0,+\infty)$, i.e, ψ does not satisfy the condition $(\boldsymbol{A_3})$. However, ψ defined as above satisfies $(\boldsymbol{H_\psi})$ with $\delta_0 = \min\{\widehat{\delta_0}, p, q\}$ and $\delta_1 = \max\{\widehat{\delta_1}, p, q\}$ in this paper.

(ii) Let $\psi_0(t) = \sin t$ with $t \in (0, \frac{\pi}{2})$. Then $(\psi_0(t)t)'' = -t\sin t + 2\cos t \to -\frac{\pi}{2} < 0$ as $t \to \left(\frac{\pi}{2}\right)^-$. Thus there exists $[t_0, t_1] \subset (0, \frac{\pi}{2})$ such that $(\psi_0(t)t)'' < 0$ for all $t \in [t_0, t_1]$. Let $\psi(t) = \begin{cases} at^p, \ 0 < t < t_0 \\ \psi_0(t), \ t_0 \le t \le t_1 \\ bt^q, \ t > t_1 \end{cases}$, where $p = t_0 \cot t_0 > 0, a = t_0^{-p} \sin t_0 > 0$ and $q = t_1 \cot t_1 > 0, b = t_1^{-q} \sin t_1 > 0$. Note that there exist $\widehat{\delta_0}, \widehat{\delta_1} > 0$ such that $\widehat{\delta_0} \le \frac{t\psi_0'(t)}{\psi_0(t)} \le \widehat{\delta_1}$ for all $t \in [t_0, t_1]$. It's easy to see that ψ satisfies $(\boldsymbol{H_\psi})$ with $\delta_0 = \min\{\widehat{\delta_0}, p, q\}$ and $\delta_1 = \max\{\widehat{\delta_1}, p, q\}$. However, $(\psi(t)t)'' < 0$ for $t \in [t_0, t_1]$. That is to say such a function ψ does not satisfy the convexity condition in [7] while it still satisfies $(\boldsymbol{H_\psi})$ in this paper.

3. Proof of main results

We present first some auxiliary results needed for the main proof. Let $\Psi(t) = \int_0^t \psi(s)\mathrm{d}s$ for $t \geq 0$.

Lemma 3.1. *Assume that ψ satisfies $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ and $(\boldsymbol{H_\psi})$. The following statements hold true:*

(i) $\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t),\ \forall s, t \geq 0$;

(ii) Ψ *is C^2-contiuous on $(0, +\infty)$ and convex on $[0, +\infty)$;*

(iii) $\frac{t\psi(t)}{1+\delta_1} \leq \Psi(t) \leq \frac{t\psi(t)}{1+\delta_0},\ \forall t \geq 0.$

Proof. Let $h_0(t) = \frac{\psi(t)}{t^{\delta_0}}, h_1(t) = \frac{\psi(t)}{t^{\delta_1}}$ for $t > 0$. By $(\boldsymbol{H_\psi})$, it follows that

$$h_0'(t) = \frac{\psi'(t)t^{\delta_0} - \psi(t)\delta_0 t^{\delta_0 - 1}}{t^{2\delta_0}} = \frac{t\psi'(t) - \psi(t)\delta_0}{t^{\delta_0+1}} \geq 0,$$

which implies that $h_0(t)$ is increasing in $t > 0$. Therefore $h_0(st) \leq h_0(t)$ for $0 \leq s \leq 1$. It follows that

$$\psi(st) \leq s^{\delta_0}\psi(t),\ \forall t > 0, 0 \leq s \leq 1. \tag{6}$$

Similarly, one may prove that $h_1(t)$ is decreasing in $t > 0$. Then $h_1(st) \leq h_1(t)$ for $s \geq 1$. It follows that

$$\psi(st) \leq s^{\delta_1}\psi(t),\ \forall t > 0, s \geq 1. \tag{7}$$

By (6) and (7), we have

$$\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t),\ \forall t > 0, s \geq 0,$$

which and the continuity of ψ in $t = 0$ yield (i).

(ii) is obvious since $\Psi''(t) = \psi'(t) \geq 0$ for $t > 0$ (see Remark 2.1) and $\Psi(t)$ is continuous in $t = 0$.

To conclude (iii), let $\Psi_0(t) = (1+\delta_0)\Psi(t) - t\psi(t)$ and $\Psi_1(t) = (1+\delta_1)\Psi(t) - t\psi(t)$ for $t \geq 0$. It is easy to see that $\Psi_0'(t) \leq 0$ and $\Psi_1'(t) \geq 0$ for $t > 0$. Then $\Psi_0(t) \leq \Psi_0(0) = 0$ and $\Psi_1(t) \geq \Psi_1(0) = 0$, which and the continuities of Ψ_0, Ψ_1 yield (iii). □

Remark 3.1. Let $F(t) = \int_0^t f(s)\mathrm{d}s$ for $t \geq 0$. For the function f satisfying $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ and $(\boldsymbol{H_f})$, and the function F, one may obtain similar properties as above.

Proof of Theorem 2.1. Without loss of generality, assume that $|u(c)| = \max\limits_{x\in[a,b]}|u(x)| > 0$ with $c \in (a,b)$. Note that

$$|u(c)| = \frac{1}{2}\Big(\big|\int_a^c u'(x)\mathrm{d}x\big| + \big|\int_c^b u'(x)\mathrm{d}x\big|\Big) \le \frac{1}{2}\int_a^b |u'(x)|\mathrm{d}x.$$

Firstly, we prove Theorem 2.1 (i) under the assumption that ψ satisfies the structural condition $(\boldsymbol{H_\psi})$. Indeed, by the monotonicity of ψ and Lemma 3.1 (i) (iii), we get

$$\begin{aligned}
&\psi(|u(c)|)|u(c)|\\
\le&\frac{1}{2}\cdot\psi\Big(\frac{1}{2}\int_a^b|u'(x)|\mathrm{d}x\Big)\cdot\int_a^b|u'(x)|\mathrm{d}x\\
=&\frac{b-a}{2}\cdot\psi\Big(\frac{b-a}{2}\frac{1}{b-a}\int_a^b|u'(x)|\mathrm{d}x\Big)\cdot\frac{1}{b-a}\int_a^b|u'(x)|\mathrm{d}x\\
\le&\max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0},\left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot\psi\Big(\frac{1}{b-a}\int_a^b|u'(x)|\mathrm{d}x\Big)\cdot\frac{1}{b-a}\int_a^b|u'(x)|\mathrm{d}x\\
\le&\max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0},\left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot(1+\delta_1)\cdot\Psi\Big(\frac{1}{b-a}\int_a^b|u'(x)|\mathrm{d}x\Big)\\
\le&\max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0},\left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot(1+\delta_1)\cdot\frac{1}{b-a}\cdot\int_a^b\Psi(|u'(x)|)\mathrm{d}x,
\end{aligned}$$

where in the last inequality we used the convexity of Ψ (see Lemma 3.1 (ii)). It follows that

$$\psi(|u(c)|)|u(c)| \le (1+\delta_1)\cdot\max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\cdot\int_a^b\Psi(|u'(x)|)\mathrm{d}x. \tag{8}$$

Multiplying (5) by u, integrating over (a,b), using Lemma 3.1 (iii), $(\boldsymbol{H_2})$-$(\boldsymbol{H_4})$ and (8), we get

$$\begin{aligned}
\int_a^b\Psi(|u'|)\mathrm{d}x\le&\frac{1}{1+\delta_0}\int_a^b\psi(|u'|)|u'|\mathrm{d}x\\
=&\frac{1}{1+\delta_0}\int_a^b\psi(u')u'\mathrm{d}x\\
=&\frac{1}{1+\delta_0}\int_a^b r(x)f(u)u\mathrm{d}x\\
\le&\frac{1}{1+\delta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b|r(x)|\mathrm{d}x\\
\le&\frac{k_0}{1+\delta_0}\max_{x\in[a,b]}(\psi(|u|)|u|)\int_a^b|r(x)|\mathrm{d}x\\
\le&\frac{k_0(1+\delta_1)}{1+\delta_0}\cdot\max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\cdot\int_a^b\Psi(|u'|)\mathrm{d}x\cdot\int_a^b|r(x)|\mathrm{d}x.
\end{aligned}$$

Note that $\int_a^b \Psi(|u'|)\mathrm{d}x > 0$, otherwise, $\int_a^b \Psi(|u'|)\mathrm{d}x = 0$. By (8) and Lemma 3.1 (i), we have,

$$\begin{aligned} 0 \leq & \psi(t)|u(c)| = \psi\left(\frac{t}{u(c)}u(c)\right)|u(c)| \\ & \leq \max\left\{\left(\frac{t}{u(c)}\right)^{\delta_0}, \left(\frac{t}{u(c)}\right)^{\delta_1}\right\}\psi(u(c))|u(c)| \\ & \leq 0, \quad \forall t \geq 0, \end{aligned}$$

which implies $\psi \equiv 0$ for all $t \in [0, +\infty)$. Then by the odd property of ψ, we have $\psi \equiv 0$ for all $t \in (-\infty, +\infty)$. Due to $(\boldsymbol{H_4})$, $f \equiv 0$ for all $t \in (-\infty, +\infty)$, which is a contradiction with the assumption $(\boldsymbol{H_1})$.

Then we get

$$\int_a^b |r(x)|\mathrm{d}x \geq \frac{1+\delta_0}{k_0(1+\delta_1)} \cdot \min\left\{\frac{2^{1+\delta_0}}{(b-a)^{\delta_0}}, \frac{2^{1+\delta_1}}{(b-a)^{\delta_1}}\right\}.$$

Theorem 2.1 (i) has been proved.

Now for Theorem 2.1 (ii), we proceed in a similar way as above. Indeed, if f satisfies the structural condition $(\boldsymbol{H_f})$, proceeding as in (8), we get

$$f(|u(c)|)|u(c)| \leq (1+\theta_1) \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'(x)|)\mathrm{d}x. \tag{9}$$

Multiplying (5) by u, integrating over (a, b), using Lemma 3.1 (iii) and $(\boldsymbol{H_2})$-$(\boldsymbol{H_4})$, we get

$$\begin{aligned} \int_a^b F(|u'|)\mathrm{d}x \leq & \frac{1}{1+\theta_0}\int_a^b f(|u'|)|u'|\mathrm{d}x \\ \leq & \frac{k_0}{1+\theta_0}\int_a^b \psi(|u'|)|u'|\mathrm{d}x \\ = & \frac{k_0}{1+\theta_0}\int_a^b \psi(u')u'\mathrm{d}x \\ = & \frac{k_0}{1+\theta_0}\int_a^b r(x)f(u)u\mathrm{d}x \\ \leq & \frac{k_0}{1+\theta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|\mathrm{d}x \\ \leq & \frac{k_0}{1+\theta_0}f(|u(c)|)|u(c)|\int_a^b |r(x)|\mathrm{d}x, \end{aligned}$$

which and (9) give

$$\int_a^b F(|u'|)\mathrm{d}x \leq \frac{k_0(1+\theta_1)}{1+\theta_0} \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x. \tag{10}$$

Note that $\int_a^b F(|u'|)\mathrm{d}x > 0$, otherwise, we can argue as in the proof of (i) to conclude $f(t) \equiv 0$ for any $t \in (-\infty, +\infty)$. Finally, (10) implies the desired result. □

Proof of Theorem 2.2. The proof of Theorem 2.2 is a slight modification of the proof of Theorem 2.1. Indeed, let $\Phi(t) = \psi(t)t$ for $t \geq 0$, and let $c, u(c)$ be defined as in the proof of Theorem 2.1. If ψ satisfies the structural condition $(\boldsymbol{H_\psi})$, arguing as in (8), we get

$$\begin{aligned}
&\psi(|u(c)|)|u(c)| \\
&\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \psi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right) \cdot \frac{1}{b-a}\int_a^b |u'|\mathrm{d}x \\
&= \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \Phi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right),
\end{aligned}$$

which and the convexity of Φ give

(11)
$$\psi(|u(c)|)|u(c)| \leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \frac{1}{b-a}\int_a^b \Phi(|u'|)\mathrm{d}x.$$

Concerning with (5) and (11), we have

$$\begin{aligned}
\int_a^b \Phi(|u'|)\mathrm{d}x &= \int_a^b \psi(|u'|)|u'|\mathrm{d}x \\
&= \int_a^b \psi(u')u'\mathrm{d}x \\
&= \int_a^b r(x)f(u)u\mathrm{d}x \\
&\leq k_0\psi(|u(c)|)|u(c)| \int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0}{2}\max\left\{\left(\frac{b-a}{2}\right)^{\delta_0}, \left(\frac{b-a}{2}\right)^{\delta_1}\right\} \cdot \int_a^b \Phi(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x,
\end{aligned}$$

which yields the desired result in Theorem 2.2 (i). The desired result in Theorem 2.2 (ii) can be proved in a similar way. □

Proof of Corollary 2.1. For (i), it should be noticed that $\delta_0 = \theta_0 = \delta_1 = \theta_1 = p - 1$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

For (ii), it should be noticed that for $t \geq 0$,

$$\psi(t) = f(t) = t^a \log_c(bt + d),\ a, b > 0, c, d > 1.$$

Then we have

$$\begin{aligned}
a \leq &\frac{t\psi'(t)}{\psi(t)} = a + \frac{bt}{(bt+d)\ln c \cdot \log_c(bt+d)} \\
\leq &a + \frac{bt}{(bt+d)\ln c \cdot \log_c d} \\
\leq &a + \frac{1}{\ln d},\quad \forall t > 0.
\end{aligned}$$

Thus $\delta_0 = \theta_0 = a > 0, \delta_1 = \theta_1 = a + \frac{1}{\ln d} > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

For (iii), it should be noticed that for $t \geq 0$,

$$\psi(t) = f(t) = \frac{t^a}{\log_c(bt+d)},\ b > 0, c, d > 1, a > \frac{1}{\ln d}.$$

Then we have

$$a - \frac{1}{\ln d} \leq \frac{t\psi'(t)}{\psi(t)} = a - \frac{bt}{(bt+d)\ln(bt+d)} \leq a, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a - \frac{1}{\ln d} > 0, \delta_1 = \theta_1 = a > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$. □

4. Examples

In this section, we provide more examples of $\psi(t)$ (or $f(t)$) satisfying $(\boldsymbol{H_\psi})$ (or $(\boldsymbol{H_f})$). For simplicity, we only restrict $\psi(t)$ (or $f(t)$) to the case $t \in [0, +\infty)$, since one may construct functions by odd or even extensions to $t \in (-\infty, +\infty)$.

Example 4.1.

$$\psi(t) = f(t) = \ln(1+at) + bt, \quad \forall a > 0, b > 0. \tag{12}$$

For (12), we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{at}{1+at}\frac{1}{\ln(1+at)+bt} + \frac{bt}{\ln(1+at)+bt} \leq \frac{a}{b} + 1, \quad \forall t > 0.$$

Note that $\ln(1+at) \leq at$ for all $t > 0$, it follows that

$$\frac{b}{a+b} \leq \frac{t\psi'(t)}{\psi(t)} \leq \frac{a}{b} + 1, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = \frac{b}{a+b} > 0, \delta_1 = \theta_1 = \frac{a}{b} + 1 > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.2.

$$\psi(t) = f(t) = (1+t)\ln(1+t) - t. \tag{13}$$

For (13), firstly, note that $\psi'(t) = \ln(1+t) \geq 0$ for any $t \geq 0$. Thus $\psi(t) \geq \psi(0) = 0$. By direct computations, we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{t\ln(1+t)}{(1+t)\ln(1+t) - t} = \frac{t}{(1+t) - \frac{t}{\ln(1+t)}}, \quad \forall t > 0.$$

Since $\ln(1+t) \leq t$ for all $t > 0$, it follows that

$$\frac{t}{(1+t) - \frac{t}{\ln(1+t)}} \geq \frac{t}{(1+t) - \frac{t}{t}} = 1, \quad \forall t > 0.$$

Now we prove that for any $t > 0$, there holds

$$\frac{t\ln(1+t)}{(1+t)\ln(1+t)-t} \le 2. \tag{14}$$

Indeed, let $h_1(t) = t\ln(1+t) - 2((1+t)\ln(1+t)-t) = 2t - t\ln(1+t) - 2\ln(1+t)$. Then $h_1'(t) = 1 - \ln(1+t) - \frac{1}{1+t}$. Let $h_2(t) = (1+t) - (1+t)\ln(1+t) - 1 = t - (1+t)\ln(1+t)$. It is easy to check that $h_2'(t) = -\ln(1+t) < 0$ for any $t > 0$. Thus $h_2(t) \le h_2(0) = 0$, which leads to $h_1'(t) \le 0$ for any $t > 0$. Therefore $h_1(t) \le h_1(0) = 0$. As a consequence, (14) holds true for any $t > 0$. Finally $\delta_0 = \theta_0 = 1, \delta_1 = \theta_1 = 2$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.3.

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \le t < t_0, \\ bt^q + c, t \ge t_0, \end{cases} \tag{15}$$

where $a, b, c, p, q, t_0 > 0$ satisfying $at_0^p = bt_0^q + c$ and $apt_0^{p-1} = bqt_0^{q-1}$.

For (15), we have $\psi = f \in C^1((0,+\infty))$ and $\min\{p,q\} \le \frac{t\psi'(t)}{\psi(t)} \le \max\{p,q\}$. Thus $\delta_0 = \theta_0 = \min\{p,q\} > 0, \delta_1 = \theta_1 = \max\{p,q\} > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.4. The following example is interesting since ψ or f is with a variable exponent:

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \le t < t_0, \\ bt^{g(t)-1}, t \ge t_0, \end{cases}$$

where $t_0 > 1$, $a, b, p > 0$ and $g \in C^1([t_0, +\infty)$ satisfying

$$\begin{cases} c \le g'(t)t\ln t + g(t) - 1 \le d, \forall t \ge t_0, \\ p = g'(t_0)t_0\ln t_0 + g(t_0) - 1, \\ a = bt_0^{g(t_0)-1-p}, \end{cases}$$

with some constants c, d satisfying $d \ge c > 0$. Note that $\frac{t(bt^{g(t)-1})'}{bt^{g(t)-1}} = tg'(t)\ln t + g(t) - 1$. By direct computations, one may verify that $\psi = f \in C^1((0,+\infty))$, and satisfy $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$ with $\delta_0 = \theta_0 = \min\{p,c\} = c, \delta_1 = \theta_1 = \max\{p,d\} = d$.

Example 4.5. In [16], the authors provided two examples of $\psi(t)$, i.e,

(*i*) $\psi(t) = |t|^a\varphi_p(t)$ with $a > 1 - p$, and

(*ii*) $\psi(t) = (\ln(|t| + a))^b\varphi_p(t)$ with $a \ge e, b > 0$,

showing that $\frac{1}{\psi(t)}$ is convex in $t > 0$, where $\varphi_p(t) = |t|^{p-2}t$ $(p > 1)$. We point out that $\psi(t)$ given by (i) or (ii) also satisfies the structural condition $(\boldsymbol{H_\psi})$. Indeed, for (i), it is easy to see that $\frac{t\psi'(t)}{\psi(t)} = a + p - 1 > 0$ for $t > 0$. Thus $\delta_0 = \delta_1 = a + p - 1$ in $(\boldsymbol{H_\psi})$. For (ii), by direct computations, we have $\frac{t\psi'(t)}{\psi(t)} = p + \frac{bt}{(t+a)\ln(t+a)}$ for $t > 0$. Note that $0 \le \frac{bt}{(t+a)\ln(t+a)} \le \frac{b}{a\ln a}$ for $t > 0$. Then $\delta_0 = p, \delta_1 = p + \frac{b}{a\ln a}$ in $(\boldsymbol{H_\psi})$.

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On a class of Lorentzian para-Kenmotsu manifolds admitting the Weyl-projective curvature tensor of type $(1,3)$

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Abstract. In this present paper, we consider a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds admitting the Weyl-projective curvature tensor of type $(1,3)$. We study and have shown that Lorentzian para-Kenmotsu manifolds admitting a flat curvature tensor, an irrotational curvature tensor and a conservative curvature tensor are an Einstein manifolds of constant scalar curvature. Further we study Lorentzian para-Kenmotsu manifolds satisfying the curvature condition $R(X,Y)\cdot W_2 = 0$. At the end, we construct an example of a 3-dimensional Lorentzian para-Kenmotsu manifold admitting Weyl-projective curvature tensor which verifies the results discussed in the present work.

Keywords: Lorentzian paracontact manifolds, W_2-curvature tensor, Einstein manifold, scalar curvature.

1. Introduction

In 1989, K. Matsumoto [3] introduced the notion of Lorentzian paracontact and in particular, Lorentzian para-Sasakian (LP-Sasakian) manifolds. Later, these manifolds have been widely studied by many geometers such as Matsumoto and Mihai [4], Mihai and Rosca [6], Mihai, Shaikh and De [7], Venkatesha and

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Bagewadi [13], Venkatesha, Pradeep Kumar and Bagewadi [14, 15] and obtained several results of these manifolds.

In 1970 [9], Pokhariyal and Mishra introduced new tensor fields, called W_2 and E, on a Riemannian manifold. The Weyl-projective curvature tensor W_2 of type $(1,3)$ on a Riemannian manifold M with respect to Riemannian connection is given by [9]:

$$W_2(X,Y,Z,U)=R(X,Y,Z,U)+\frac{1}{n-1}[g(X,Z)S(Y,U)-g(Y,Z)S(X,U)], \tag{1}$$

where $R(X,Y)$ is the Riemannian curvature tensor and $S(X,Y)$ is the Ricci tensor of M.

The above expression can be written as:

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{2}$$

where $QX = (n-1)X$.

Further, Pokhariyal [8] studied the properties of these tensor fields on a Sasakian manifold. In 1986, Matsumoto, Ianus and Mihai have extended these concepts to almost paracontact structures and studied para-Sasakian manifolds admitting these tensor fields [5] and these results were further generalised by De and Sarkar, in 2009 [2]. In 1995, Sinha and Sai Prasad [12] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds and in 2015 [11], Sai Prasad and Satyanarayana have studied W_2-tensor field in an SP-Kenmotsu manifold.

In the present work, we investigate a class of Lorentzian para-Kenmotsu manifolds [1, 10] admitting the Weyl-projective curvature tensor W_2 of type $(1,3)$. The present work is organised as follows: Section 2 is equipped with some prerequisites about Lorentzian para-Kenmotsu manifolds. Further on, we consider Lorentzian para-Kenmotsu manifolds admittting the Weyl-projective curvature tensor W_2 of type $(1,3)$ and studied the geometrical properties of these manifolds, in particular, when the Weyl-projective curvature tensor W_2 is either flat (or) irrotational (or) conservative. In section 6, we studied the properties of W_2-semisymmetric Lorentzian para-Kenmotsu manifolds.

Finally, in section 7, we construct an example of a 3-dimensional Lorentzian para-Kenmotsu manifold admitting Weyl-projective curvature tensor which verifies the results discussed in the sections 3, 4 and 5.

2. Preliminaries

An n-dimensional differentiable manifold M admitting a (1, 1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

and

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \ rank\phi = n-1 \tag{4}$$

is called Lorentzian almost paracontact manifold [3].

In a Lorentzian almostpara contact manifold, we have

$$\Phi(X,Y) = \Phi(Y,X), \tag{5}$$

where $\Phi(X,Y) = g(X,\phi Y)$.

A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{6}$$

for any vector fields X and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

In the Lorentzian para-Kenmotsu manifold, the following relations hold good:

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi, \tag{7}$$

and

$$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y). \tag{8}$$

Further, on a Lorentzian para-Kenmotsu manifold M, the following relations hold [1]:

$$\begin{aligned}
&(a)\ g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),\\
&(b)\ R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X,\\
&(c)\ R(X,Y)\xi = \eta(Y)X - \eta(X)Y;\ \text{when } X \text{ is orthogonal to } \xi,\\
&(d)\ S(X,\xi) = (n-1)\eta(X),\\
&(e)\ S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y).
\end{aligned} \tag{9}$$

A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor $S(X,Y)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{10}$$

where a and b are scalar functions on M.

A Lorentzian para-Kenmotsu manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor $S(X,Y)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Phi(X,Y), \tag{11}$$

where a,b,c are scalar functions on M and $\Phi(X,Y) = g(\phi X,Y)$. If $c = 0$, the manifold reduces to an η-Einstein manifold.

3. Flat W_2-curvature tensor in Lorentzian para-Kenmotsu manifolds

Let us consider a Lorentzian para-Kenmotsu manifold and suppose that it admits W_2-curvature tensor which is flat. Then, we have

$$g(W_2(X,Y)Z,\phi W)=0. \tag{12}$$

In view of (1), (12) can be written as

$$g(R(X,Y)Z,\phi W)+\frac{1}{n-1}\big\{g(X,Z)S(Y,\phi W)-g(Y,Z)S(X,\phi W)\big\}=0. \tag{13}$$

Now, by putting $Y=Z=\xi$ in (13), we get

$$g(R(X,\xi)\xi,\phi W)+\frac{1}{n-1}\big\{\eta(X)S(\xi,\phi W)+S(X,\phi W)\big\}=0; \tag{14}$$

which, by using (9)(c) and (9)(d), further reduces to:

$$-g(X,\phi W)+\frac{1}{n-1}S(X,\phi W)=0, \tag{15}$$

and this implies that

$$S(X,\phi W)=(n-1)g(X,\phi W). \tag{16}$$

By replacing W by ϕW in (16) and on using (3), we get

$$S(X,W)=(n-1)g(X,W), \tag{17}$$

which on contraction gives

$$r=n(n-1). \tag{18}$$

Hence, we can state the following:

Theorem 3.1. *The Lorentzian para-Kenmotsu manifold admitting a flat W_2-curvature tensor is an Einstein manifold and is a space of constant scalar curvature.*

4. Irrotational W_2-curvature tensor in Lorentzian para-Kenmotsu manifolds

Definition 4.1. The rotation (*curl*) of a Weyl-projective curvature tensor W_2 in a Lorentzian para-Kenmotsu manifold M is defined as:

$$\begin{aligned} RotW_2 &= (\nabla_U W_2)(X,Y)Z+(\nabla_X W_2)(U,Y)Z \\ &\quad +(\nabla_Y W_2)(X,U)Z-(\nabla_Z W_2)(X,Y)U. \end{aligned} \tag{19}$$

In consequence of Bianchi's second identity for Riemannian connection ∇, (19) becomes

$$RotW_2 = -(\nabla_Z W_2)(X,Y)U. \tag{20}$$

Let us suppose that the W_2 curvature tensor is irrotational. Then Rot $W_2 = 0$ and hence from (20), we have

$$(\nabla_Z W_2)(X,Y)U = 0; \tag{21}$$

which can be written as:

$$\nabla_Z(W_2(X,Y)U) = W_2(\nabla_Z X,Y)U + W_2(X,\nabla_Z Y)U + W_2(X,Y)\nabla_Z U. \tag{22}$$

By replacing $U = \xi$ in (22), we get

$$\nabla_Z(W_2(X,Y)\xi) = W_2(\nabla_Z X,Y)\xi + W_2(X,\nabla_Z Y)\xi + W_2(X,Y)\nabla_Z \xi. \tag{23}$$

On the other hand, by substituting $Z = \xi$ in (1) and on using (9)(c) and (9)(d), we get

$$W_2(X,Y)\xi = k\big[\eta(Y)X - \eta(X)Y\big]; \tag{24}$$

where

$$k = \Big[1 - \frac{1}{n-1}\Big\{\frac{r}{n-1} - 1\Big\}\Big]. \tag{25}$$

In view of (24) and (25), (23) becomes

$$W_2(X,Y)Z = k\big[g(Y,Z)X - g(X,Z)Y\big]. \tag{26}$$

Further, equations (1) and (26) gives

$$S(Y,Z) = (n-1)g(Y,Z), \tag{27}$$

which on contraction gives

$$r = n(n-1). \tag{28}$$

In consequence of the above equations from (25) to (28) and (1), we get

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{29}$$

Thus we have the following assertion:

Theorem 4.1. *The Lorentzian para-Kenmotsu manifold admitting an irrotational W_2-curvature tensor is a space of constant curvature.*

5. Conservative W_2-curvature tensor in Lorentzian para-Kenmotsu manifolds

On differentiating (1) with respect to U, we get

$$(\nabla_U W_2)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{n-1}\big[g(X,Z)(\nabla_U Q)Y - g(Y,Z)(\nabla_U Q)(X)\big], \tag{30}$$

which on contracting becomes

$$(divW_2)(X,Y)Z = \big[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\big] + \frac{1}{2(n-1)}\big[g(X,Z)dr(X) - g(Y,Z)dr(Y)\big]. \tag{31}$$

Let us suppose that W_2-curvature tensor is conservative, i. e., $divW_2 = 0$. Then (31) can be written as:

$$\big[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\big] = \frac{1}{2(n-1)}\big[g(Y,Z)dr(X) - g(X,Z)dr(Y)\big]. \tag{32}$$

By putting $X = \xi$ in (32), we have

$$\big[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)\big] = \frac{1}{2(n-1)}\big[g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)\big]. \tag{33}$$

On the other hand, since ξ is a killing vector and the scalar curvature r remains invariant, we have $L_\xi r = 0$, where L denotes the Lie derivative.

Then, the relation

$$\begin{aligned}(\nabla_\xi S)(Y,Z) &= \xi S(Y,Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z)\\ &= (L_\xi S)(Y,Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi)\end{aligned}$$

yields

$$(\nabla_\xi S)(Y,Z) = 0. \tag{34}$$

Then from (34) and (33), we have

$$\begin{aligned}&-\big[\nabla_Y S(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z)\big]\\ &= \frac{1}{2(n-1)}\big[g(Y,Z)dr(\xi) - \eta(Z)dr(Y)\big].\end{aligned} \tag{35}$$

By using (7), (9)(d) and $dr(\xi) = 0$ the above expression becomes

$$\begin{aligned}&[-\nabla_Y\{(n-1)\eta(Z)\} + S(-\phi^2 Y, Z) + (n-1)\eta(\nabla_Y Z)]\\ &= \frac{-1}{2(n-1)}[\eta(Z)dr(Y)].\end{aligned} \tag{36}$$

On simplifying (36), we get

$$[(n-1)g(Y,Z) - S(Y,Z)] = \frac{-1}{2(n-1)}[\eta(Z)dr(Y)]. \tag{37}$$

By putting $Z = \phi Z$ in (37), we get

$$(n-1)g(Y,\phi Z) = S(Y,\phi Z) \tag{38}$$

and this implies that

$$S(Y,Z) = (n-1)g(Y,Z), \tag{39}$$

which on contracting gives

$$r = n(n-1), \tag{40}$$

where

$$r = \sum_{i=1}^{3} \epsilon_i \; S(e_i,e_i) \text{ and } \epsilon_i = g(e_i,e_i), \text{ which is constant.} \tag{41}$$

So, one can state that:

Theorem 5.1. *A Lorentzian para-Kenmotsu manifold admitting a conservative W_2-curvature tensor is an Einstein manifold and it is of constant scalar curvature.*

6. W_2-semisymmetric para-Kenmotsu manifolds

We define a W_2-semisymmetric Lorentzian para-Kenmotsu manifold as:

Definition 6.1. An n-dimensional Lorentzian para-Kenmotsu manifold is called W_2-semisymmetric if its W_2-curvature tensor satisfies the condition

$$R(X,Y) \cdot W_2 = 0, \tag{42}$$

where $R(X,Y)$ is considered to be the derivation of tensor algebra at each point of the manifold for tangent vectors X and Y.

It can be easily shown that, on Lorentzian para-Kenmotsu manifold the W_2-curvature tensor satisfies the condition

$$W_2(X,Y,Z,\xi) = 0. \tag{43}$$

In consequence of $QX = hX$, (1) becomes

$$W_2(X,Y)Z = R(X,Y)Z + \frac{h}{(n-1)}\big\{g(X,Z)Y - g(Y,Z)X\big\}. \tag{44}$$

In view of (9)(a) and (44), we obtain

$$\eta(W_2(X,Y)Z) = \Big(1 - \frac{h}{n-1}\Big)\big\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\big\}. \tag{45}$$

Replacing Z by ξ in (45), we have

$$\eta(W_2(X,Y)\xi) = 0. \tag{46}$$

Now

$$\begin{aligned}(R(X,Y)\cdot W_2)(Z,U)V &= R(X,Y)W_2(Z,U)V - W_2(R(X,Y)Z,U)V\\ &\quad - W_2(Z,R(X,Y)U)V - W_2(Z,U)R(X,Y)V.\end{aligned} \tag{47}$$

Using $R(X,Y)\cdot W_2 = 0$ in the above equation, we obtain $R(X,Y)W_2(Z,U)V - W_2(R(X,Y)Z,U)V - W_2(Z,R(X,Y)U)V - W_2(Z,U)R(X,Y)V = 0$. By taking the inner product of the above relation with ξ, we get

$$\begin{aligned}&g(R(X,Y)W_2(Z,U)V,\xi) - g(W_2(R(X,Y)Z,U)V,\xi)\\ &- g(W_2(Z,R(X,Y)U)V,\xi) - g(W_2(Z,U)R(X,Y)V,\xi) = 0.\end{aligned} \tag{48}$$

Putting $X = \xi$ in (48) and then using (9)(b), we obtain

$$\begin{aligned}&- W_2(Z,U,V,Y) - \eta(Y)\eta(W_2(Z,U)V) + \eta(Z)\eta(W_2(Y,U)V)\\ &+ \eta(U)\eta(W_2(Z,Y)V) + \eta(V)\eta(W_2(Z,U)Y) - g(Y,Z)\eta(W_2(\xi,U)V)\\ &- g(Y,U)\eta(W_2(Z,\xi)V - g(Y,V)\eta(W_2(Z,U)\xi = 0.\end{aligned} \tag{49}$$

In consequence of (45), the above equation gives

$$\begin{aligned}-W_2(Z,U,V,Y) &+ \eta(V)\Big(1 - \frac{h}{n-1}\Big)\big[\eta(Z)g(U,Y) - \eta(U)g(Y,Z)\big]\\ &+ g(Y,Z)\Big(1 - \frac{h}{n-1}\Big)\big[g(U,V) + \eta(U)\eta(V)\big]\\ &- g(Y,U)\Big(1 - \frac{h}{n-1}\Big)\big[\eta(Z)\eta(V) + g(Z,V)\big]\\ &- g(Y,V)\Big(1 - \frac{h}{n-1}\Big)\big[\eta(Z)\eta(U) - \eta(U)\eta(Z)\big] = 0,\end{aligned} \tag{50}$$

which on simplification, we obtain

$$W_2(Z,U,V,Y) = \Big(1 - \frac{h}{n-1}\Big)\big[g(Y,Z)g(U,V) - g(Y,U)g(Z,V)\big], \tag{51}$$

which implies

$$W_2(Z,U)V = \Big(1 - \frac{h}{n-1}\Big)\big[g(U,V)Z - g(Z,V)U\big]. \tag{52}$$

Thus in view of (44) and (52), we obtain

$$R(Z,U)V = g(U,V)Z - g(Z,V)U. \tag{53}$$

Thus we have the following:

Theorem 6.1. *A Lorentzian para-Kenmotsu manifold satisfying* $R(X,Y)\cdot W_2 = 0$ *is a space of constant curvature.*

7. Example of a 3-dimensional Lorentzian para-Kenmotsu manifold

Example 7.1. We consider a 3-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M_3 given by

$$e_1 = x\frac{\partial}{\partial x} = \xi, \quad e_2 = x\frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial z}. \tag{54}$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M_3 and hence form a basis of $\chi(M_3)$.

The Lorentzian metric $g(X, Y)$ is defined by:

$$g(e_i, e_j) = \begin{cases} -1, & \text{if } i = j = 1 \\ 1, & \text{if } i = j = 2 \text{ or } 3 \\ 0, & \text{if } i \neq j;\ i,\ j = 1, 2, 3. \end{cases}$$

Let η be the 1-form defined by:

$$\eta(Z) = g(Z, e_1), \text{ for any } Z \in \chi(M_3).$$

Let ϕ be a (1, 1)-tensor field on M_3 defined by:

$$\phi(e_1) = 0, \phi(e_2) = -e_2, \phi(e_3) = -e_3 \text{ and}$$
$$\phi^2(e_1) = 0, \phi^2(e_2) = e_2, \phi^2(e_3) = e_3.$$

The linearity of ϕ and $g(X, Y)$ yields that

$$\eta(e_1) = -1,\ \phi^2(Z) = Z + \eta(Z)e_1 \text{ ; and}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any vector fields $X,\ Y,\ Z \in \chi(M_3)$.

Thus, for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines an Lorentzian almostpara-contact structure on M_3.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_2, e_3] = 0.$$

The Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{55}$$

By using the above Koszul's formula and on taking $e_1 = \xi$, we get the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0; \\ &\nabla_{e_2} e_1 = -e_2, \nabla_{e_2} e_2 = -e_1, \nabla_{e_2} e_3 = 0; \\ &\nabla_{e_3} e_1 = -e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -e_1. \end{aligned} \tag{56}$$

From the above calculations, we see that the manifold under consideration satisfies all the properties of Lorentzian para-Kenmotsu manifold i.e., $\nabla_X \xi = -\Phi^2 X = -X - \eta(X)\xi$ and $(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X$ for all $e_1 = \xi$. Thus, the manifold M_3 under consideration with the structure (ϕ, ξ, η, g) is a Lorentzian para-Kenmotsu manifold.

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{57}$$

Then by using (56) and (57), the non-vanishing components of the curvature tensor are obtained as:

$$\begin{aligned} &R(e_1,e_2)e_1 = e_2, R(e_1,e_2)e_2 = e_1, R(e_1,e_3)e_1 = e_3, \\ &R(e_1,e_3)e_3 = e_1, R(e_2,e_3)e_2 = -e_3, R(e_2,e_3)e_3 = e_2. \end{aligned} \tag{58}$$

With the help of above expressions of the curvature tensors, it follows that

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{59}$$

This proves that the manifold M_3 under consideration is a Lorentzian para-Kenmotsu manifold and it admits a Weyl-projective curvature tensor.

Let X, Y and Z be any three vector fields given by:

$$X = a_1 e_1 + a_2 e_2 + a_3 e_3, Y = b_1 e_1 + b_2 e_2 + b_3 e_3, Z = c_1 e_1 + c_2 e_2 + c_3 e_3; \tag{60}$$

where a_i, b_i, c_i are all non-zero real numbers, for all $i = 1, 2, 3$.

By putting $Z = \xi = e_1$ in (2) and on using (60), we get that

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y = a_1 b_2 e_2 + a_1 b_3 e_3 - a_2 b_1 e_2 - a_3 b_1 e_3;$$

and hence we get that $W_2(X,Y)\xi = 0$, where $QX = (n-1)X$.

Further, in view of (2) and (60), we get

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y &= (c_1 e_2 + c_2 e_1)(a_1 b_2 - a_2 b_1) \\ &+ (a_1 b_3 - a_3 b_1)(c_1 e_3 + c_3 e_1) + (a_2 b_3 - a_3 b_2)(c_3 e_2 - c_2 e_3); \end{aligned}$$

and hence we have $W_2(X,Y)Z = 0$.

This proves that the manifold M_3 under consideration is a Lorentzian para-Kenmotsu manifold and it admits a flat Weyl-projective curvature tensor.

From the equation (58), we obtain the Ricci tensors and the scalar curvatures as follows: $S(e_1,e_1) = -2$, $S(e_2,e_2) = 2$, $S(e_3,e_3) = 2$ and $r = 6$, where $S(X,Y) = \sum_{i=1}^{3} \epsilon_i \, g(R(e_i,X)Y,e_i)$, $r = \sum_{i=1}^{3} \epsilon_i \, S(e_i,e_i)$ and $\epsilon_i = g(e_i,e_i)$.

The above arguments verifies the results discussed in sections 3, 4 and 5.

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On Laplacian eigenvalues of $\mathbb{N}$-sum graphs and $\mathbb{Z}$-sum graphs and few more properties

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Abstract. The concept of sum graphs was introduced by Harary [4]. A graph G is a sum graph if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. Harary extended the concept to allow any integers and called them as integral sum graphs. To distinguish between the two types, we call graphs that use only positive integers as $\mathbb{N}$-*sum graphs* and those with any integers as $\mathbb{Z}$-*sum graphs* [9]. In this paper we investigate the Laplacian eigenvalues of $\mathbb{N}$-*Sum Graphs and* $\mathbb{Z}$-*Sum Graphs* and its *anti-sum graphs*. Also, we obtain a few more properties of $\mathbb{N}$-sum graph.

Keywords: $\mathbb{N}$-sum graph, $\mathbb{Z}$-sum graph, Laplacian matrix, operation on graphs, isomorphism of two graphs.

1. Introduction

Harary [4] introduced the concepts of sum and integral sum graphs. A graph G is a sum graph if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. The graphs that use only positive integers for labeling are called $\mathbb{N}$-*sum graphs* and those with any integers are called $\mathbb{Z}$-*sum graphs* [9]. If G is a sum or integral sum graph with respect to a label set S, then graph G can be denoted by $G^+(S)$. The complement of a sum graph G satisfies the property that $e = uv$ is an edge of G^c if and only if the sum of the labels on vertices u and v is not a vertex label in G^c [9]. Harary [4] introduced a family of sum graphs G_n over a set of positive integers, $\{1, 2, ..., n\}$. In 1994, Harary [4] also introduced a family of integral sum graphs $G_{n,n}$ over a set of integers, $\{-n, -n+1, ..., n\}$ and specified the structure of these graphs in terms of the graphs G_n as $G_{n,n} = K_1 \nabla (G_n \nabla G_m)$, where ∇ denotes join of graphs. Vilfred [8] generalized integral sum graph $G_{n,n}$ to $G_{m,n}$, $m, n \in \mathbb{N} \cup \{0\}$.

In this paper, we investigate Laplacian eigenvalues and Laplacian eigenvectors of G_n, $G_{m,n}$ and G_n^c. The terms that are not defined in this paper, one may refer [3].

We use the following known results in this paper.

Theorem 1.1 ([8]). *For $n \geq 2$, the degree of the vertex with label i in G_n is*

$$deg(i) = \begin{cases} n-i-1, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ n-i, & \lfloor \frac{n}{2} \rfloor \leq i \leq n. \end{cases}$$

Theorem 1.2 ([7]). *For all $n \in \mathbb{N}$, $G_{n+1} \cong_{wvl} G_n^c \cup \{v_{n+1}\}$, i.e G_{n+1} is isomorphic to $G_n^c \cup \{v_{n+1}\}$ without the vertex labels , where v_{n+1} is an isolated vertex with label $n+1$.*

Theorem 1.3 ([5]). *Let G and H be graphs with Laplacian eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$, respectively. Then Laplacian eigenvalues of the join $G\nabla H$ is $n+k, k+\alpha_1, k+\alpha_2, ..., k+\alpha_{n-1}, n+\beta_1, n+\beta_2, ..., n+\beta_{k-1}, 0$.*

Theorem 1.4 ([8]). *For $m, n \in \mathbb{N} \cup \{0\}$, $G_{m,n} \cong_{wvl} K_1 \nabla G_m \nabla G_n$*

Theorem 1.5 ([8]). *For $m, n > 0$, $K_1 \nabla G_m \cong_{wvl} G_{0,m} \cong_{wvl} G_{m+1}^c$.*

2. Results on sum graphs

2.1 Laplacian eigenvalues of G_n and G_n^c

One may observe that in Laplacian matrix of G_n, $L(G_n) = (a_{i,j})_{n \times n}$, the entries $a_{i,i} = a_{i+1,i+1} + 1$, except for the $a_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ and $a_{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 1}$. Also it is interesting to observe that for the matrix $L(G_n)$, $(1, 1, ..., 1, 1)$ and $(0, 0, ..., 0, 1)$ are eigenvectors each with eigenvalue 0. Thus, $L(G_n)$ has 0 as an eigenvalue with multiplicity at least 2.

Theorem 2.1. *For any $n \in \mathbb{N}$($n \geq 4$), there exist a G_n with Laplacian eigenvalues 1 and $n-1$ with the corresponding eigenvectors $(0, -\frac{1}{n-3}, -\frac{1}{n-3}, ..., -\frac{1}{n-3}, 1, 0)$ and $(-(n-2), 1, ..., 1, 0)$.*

Proof. For convenience, the odd and even cases on the order of the graph G_n are considered separately, and because the two cases are so similar, we prove only the even case here. Invoking Theorem 1.1, we have the degree sequence of G_{2k} is $\{2k-2, 2k-3, ..., k-1, k-1, ..., 1, 0\}$, while that of G_{2k+1} is $\{2k-1, 2k-2, ..., k, k, ..., 1, 0\}$. Let the graph be G_{2n}. Then for $n \geq 2$, we observe the following:

i) $d(v_1) = 2n-2$ in G_{2n}.

ii) $a_{i,i} = a_{i+1,i+1} + 1$ except for $a_{n,n}$ and $a_{n+1,n+1}$ in $L(G_{2n})$, $1 \leq i \leq 2n-1$.

iii) $d(v_n) = d(v_{n+1}) = n-1$ in G_{2n}.

Let $V = (0, -\frac{1}{2n-3}, -\frac{1}{2n-3}, ..., -\frac{1}{2n-3}, 1, 0)$ and R_i be the i^{th} row of $L(G_{2n})$, $i = 1, 2, ..., 2n$.

Claim: V is an eigenvector with eigenvalue 1 for $L(G_{2n})$.

Consider the product $R_k \times V^T$ of dimension 1×1. If the resultant element is the k^{th} element of V, then 1 is an eigenvalue. Clearly, $R_1 \times V^T = 0 = R_{2n} \times V^T$. Correspondingly we get 0 as the first and last element in V. Using (i) and

(ii), for $2 \leq k \leq n$, $d(v_k) = 2n - (k+1)$ and the row is of the form $R_k = [-1, -1, ..., -1, 2n-(k+1), -1, .., -1, 0, .., 0]$. In the sum graph G_{2n}, v_k is non-adjacent to vertices v_{2n+m-k} for $m = 1, 2, 3, ..., k$. So the last k entries of the row R_k are zero and -1 appears in this row $2n-(k+1)$ times. Thus we get, $R_k \times V^T = -\frac{1}{2n-3}$, which is the k^{th} term of V, $2 \leq k \leq n$. For $n+1 \leq k \leq 2n-2$, say $k = n+a, a = 1, 2, ..., n-2$, then using (i) and (ii) $d(v_k) = n - a$ and the row $R_k = [-1, -1, ..., -1, 0, .., 0, n-a, 0, ..., 0]$ and -1 appears in the row $(n-a)$ times. This implies, $R_k \times V^T = -\frac{1}{2n-3}$, for $n = 1 \leq k \leq 2n-2$. $R_{2n-1} = [-1, 0, ..., 0, 1, 0]$ and clearly $R_{2n-1} \times V^T = 1$.

Similarly, we get $(n-1)$ as an eigenvalue for eigenvector $(-(2n-2), 1, ..., 1, 0)$. Hence the result is true for the graph G_{2n}. □

Theorem 2.2. *Let $\lambda \neq 0$ be an eigen value of $L(G_n)$ and $(x_1, x_2, ..., x_n)$ be the corresponding eigenvector. Then $(0, x_1, x_2, ..., x_n, 0)$ is an eigenvector for $L(G_{n+2})$ with eigenvalue $(\lambda + 1)$.*

Proof. Let R_i denote the i^{th} row of $L(G_n)$, $1 \leq i \leq n$. Let $V = (x_1, x_2, ..., x_n)$ be an eigenvector of $L(G_n)$, with eigenvalue $\lambda \neq 0$. Then, $L(G_n).V^T = \lambda.V^T$, $n > 2$.

$$ie, \begin{pmatrix} n-2 & -1 & -1 & . & . & . & -1 & -1 & 0 \\ -1 & n-3 & -1 & . & . & . & -1 & 0 & 0 \\ & & & & & & & & \\ -1 & -1 & . & . & \lfloor n/2 \rfloor - 1 & . & . & 0 & 0 \\ -1 & -1 & . & . & . & \lfloor n/2 \rfloor - 1 & . & 0 & 0 \\ & & & & & & & & \\ -1 & 0 & . & . & . & . & . & 1 & 0 \\ 0 & 0 & . & . & . & . & . & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$= \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix}$$

Expanding the above we get,

$$\begin{aligned} (n-2)x_1 - x_2 - \cdots - x_{n-1} &= \lambda x_1 \quad (1) \\ -1.x_1 + (n-3)x_2 - \cdots - x_{n-2} &= \lambda x_2 \quad (2) \\ &\vdots \\ -1.x_1 - 0 - \cdots - 0 + x_{n-1} &= \lambda x_{n-1} \quad (n-1) \\ 0.x_1 + 0.x_2 \cdots + 0.x_{n-1} + 0.x_n &= \lambda x_n \quad (n) \end{aligned}$$

Let $U = (0, x_1, x_2, ..., x_n, 0)$, $n > 2$.

Claim: $L(G_{n+2}) \times U^T = (\lambda + 1)U^T$.

Consider each row R_i of $L(G_{n+2})$, $1 \leq i \leq n+2$, then $R_1 \times U^T = -x_1 - x_2 - \cdots - x_n = -1(1, 1, ..., 1).(x_1, x_2, ..., x_{n-1}, x_n) = 0$, since it is the dot product of two eigenvectors of symmetric matrix $L(G_n)$.

Expanding $L(G_{n+2}) \times U^T$ and using equations from (1) to (n), for each $R_i \times U^T$, we get $(\lambda + 1)x_{i-1}$ as common term, $2 \leq i \leq n + 1$. Also every entires in R_{n+2} is zero and so $R_{n+2} \times U^T = 0 = (\lambda + 1).0$.

Thus, whenever $\lambda \neq 0$ is an eigenvalue for $L(G_n)$, $(\lambda + 1)$ is an eigenvalue for $L(G_{n+2})$. □

We deonte the set of all Laplacian eigenvalues of a graph G by $\mu(G)$.

Theorem 2.3. *For $n \geq 2$, Laplacian eigenvalues of G_n are $\{0, 0, 1, 2, ..., n - 2, n - 1\} \setminus \{\lfloor n/2 \rfloor\}$ and of G_n^c are $\{0, 1, 2, ..., n - 1, n\} \setminus \{\lceil n/2 \rceil\}$.*

Moreover, $\mu(G_{n+1}) = \{0\} \cup \mu(G_n^c)$.

Proof. We use principle of mathematical induction on n to prove the Theorem. Since we are using Theorem 2.2 in the proof, we consider even and odd case separetly. Consider the Laplacian matrix $L(G_{2n})$. For $n = 1$, $L(G_2)$ is zero matrix and hence the Theorem holds. For $L(G_4)$, we get $(1, 1, 1, 1)$ and $(0, 0, 0, 1)$ as the eigenvectors corresponding to eigenvalue 0 and using Theorem 2.1, we get $(-2, 1, 1, 0)$ and $(0, -1, 1, 0)$ as the eigenvectors corresponding to eigenvalues 3 and 1, respectively. Thus $\mu(G_4) = \{0, 0, 1, 3\}$. Note that when $n = 4$, $\lfloor n/2 \rfloor = 2$ is not an eigenvalue for $L(G_4)$. Thus the result is true for $n = 4$. Now consider $L(G_6)$. Clearly, 0 is an eigenvalue with multiplicity two and using Theorem 2.1, 1 and 7 are eigenvalues of $L(G_6)$. Since 1 and 3 are the eigenvalues of $L(G_4)$, 2 and 4 are eigenvalues of $L(G_6)$ using Theorem 2.2. Thus $\mu(G_6) = \{0, 0, 1, 2, 4, 5\}$. Note that 3 is not an eigenvalue of $L(G_6)$. Assume that Theorem 2.3 is true for $n = k$. ie. $\mu(G_{2k}) = \{0, 0, 1, 2..., k - 1, k + 1, ..., 2k - 2, 2k - 1\}$. Consider the graph $G_{2(k+1)}$. Using Theorem 2.1 and Theorem 2.2 we get $\mu(G_{2(k+1)}) = \{0, 0, 1, 2, ..., k, k + 2, ..., 2k, 2k + 1\}$. Thus by Induction, the Theorem holds for every $n \in \mathbb{N}$, i.e $\mu(G_n) = \{0, 0, 1, 2, ..., n - 2, n - 1\} \setminus \{\lfloor n/2 \rfloor\}$. Similarly we can prove the result for $L(G_{2n+1})$. Hence the result is true for all G_n, $n \in \mathbb{N}$ and $n \geq 2$.

To prove for anti-sum graph we use Theorem 1.2. An isolated vertex give an empty row and column. Thus $\mu(G_{n+1}) = \mu(G_n^c) \cup \{0\}$. i.e, eigenvalues of G_{n+1} are the set of all eigenvalues of G_n^c and 0. Now we have $\mu(G_n) = \{0, 0, 1, 2, ..., n - 2, n - 1\} \setminus \{\lfloor n/2 \rfloor\}$. Thus we get eigenvalues of G_n^c is $\{0, 1, 2, ..., n - 1, n\} \setminus \lceil n/2 \rceil$, since $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. □

From Theorem 2.3 and Theorem 1.5 we have the following results.

Corollary 2.1. *Characteristic polynomial of G_n and G_n^c are*

$$\Phi(G_n, x) = (-1)^n x^2 (x - 1)(x - 2)...(x - (n - 1))/(x - \lfloor n/2 \rfloor),$$
$$\Phi(G_n^c, x) = (-1)^n x(x - 1)(x - 2)...(x - n))/(x - \lceil n/2 \rceil).$$

Corollary 2.2. $\mu(G_{0,m}) = \mu(G^c_{m+1})$.

Corollary 2.3. *Let $d(v_i)$ denote the degree of vertex v_i, then for a sum graph G_n and for $v_i \in G_n$, $\Sigma_i d(v_i) = \Sigma_i \mu_i(G_n) = 2\lfloor n/2 \rfloor(\lceil n/2 \rceil - 1)$, $\Sigma_i d(v_i) = \Sigma_i \mu_i(G^c_n) = 2\lfloor n/2 \rfloor \lceil n/2 \rceil$.*

2.2 Properties of sum graph

Theorem 2.4. *For $n > 0$, $n \in \mathbb{N}$, $G_n - \{v_{\lfloor n/2 \rfloor}\} \cong_{wvl} G_{n-1}$.*

Proof. For $n = 1, 2$, the case is trivial. Consider odd and even cases on the order of the graph G_n separately. We prove only even case here, because the two cases are so similar. Let graph be G_{2r}, $r \geq 2$. Our aim is to prove $G_{2r} - \{v_r\} \cong_{wvl} G_{2r-1}$, $r > 0$. Let $V(G_{2r}) = \{v_1, v_2, ..., v_{2r}\}$ and $V(G_{2r-1}) = \{w_1, w_2, ..., w_{2r-1}\}$, i be the sum label of the vertex v_i in G_{2r} and j be the sum label of the vertex w_j in G_{2r-1}, $i = 1, 2, ..., 2r$ and $j = 1, 2, ..., 2r - 1$. Clearly, the set $K = \{v_1, v_2, ..., v_r\}$ is a clique and set $L = \{v_{r+1}, v_{r+2}, ..., v_{2r}\}$ is an independent set in G_{2r}. Also it is clear that the set $K_1 = \{v_1, v_2, ..., v_{r-1}, v_{r+1}\}$ is a clique and the set $L_1 = \{v_{r+2}, v_{r+3}, ..., v_{2r}\}$ is an independent set in $G_{2r} - \{v_r\}$ and the set $K_2 = \{w_1, w_2, ..., w_r\}$ is a clique and the set $L_2 = \{w_{r+1}, w_{r+2}, ..., w_{2r-1}\}$ is an independent set in (G_{2r-1}).

Define the mapping $f : V(G_{2r} - \{v_r\}) \to V(G_{2r-1})$ by $f(v_i) = w_i$, $1 \leq i \leq r - 1$ and $f(v_j) = w_{j-1}$, $r + 1 \leq j \leq 2r$. Clearly, this mapping preserves the adjacency between clique set K_1 and K_2. We now show that this map preserves adjacency between Independent sets L_1 and L_2. Suppose $v_j \in L_1$ and $v_i \in K_1$, then v_j is adjacent to v_i if and only if $1 \leq i + j \leq 2r$, by definition of G_n. And this is same as $3 \leq i + j \leq 2r$, since $i, j \neq 0$ and $r > 1$. Clearly, $f(v_j) = w_{j-1} \in L_2$ and $f(v_i) = w_i \in K_2$. And w_j is adjacent to w_i if and only if $2 \leq i + j - 1 \leq 2r - 1$, ie. $3 \leq i + j \leq 2r$. This implies v_j is adjacent to v_i if and only if $f(v_j)$ is adjacent to $f(v_i)$. Hence the adjacency is preserved.

Similarly we can prove that $G_{2r+1} - \{v_{\lfloor (2r+1)/2 \rfloor}\} \cong_{wvl} G_{2r}$. □

Theorem 2.5. *For $n > 0$, $G_{0,n} - \{v_{\lfloor n/2 \rfloor}\} \cong_{wvl} G^c_n$.*

Proof. For convenience, the odd and even cases on the order of the graph G_n are considered separetely and because the two cases are so similar, we prove only the even case here. Let graph be $G_{0,2r}$, $r > 0$. We prove $G_{0,2r} - \{v_r\} \cong_{wvl} G^c_{2r}$. Let $V(G_{0,2r} - \{v_r\}) = \{v_0, v_1, ..., v_{r-1}, v_{r+1}, ..., v_{2r}\}$ and $V(G^c_{2r}) = \{w_1, w_2, ..., w_{2r}\}$, both subscript-labeled. Clearly, in $G_{0,2r} - \{v_r\}$, the set $K_1 = \{v_0, v_1, ..., v_{r-1}, v_{r+1}\}$ induces a clique and the set $L_1 = \{v_{r+2}, v_{r+3}, ..., v_{2r}\}$ induces an independent set. For (G^c_{2r}), clearly, the set $K_2 = \{w_{2r}, w_{2r-1}, ..., w_r\}$ induces a clique and the set $L_2 = \{w_{r-1}, w_{r-2}, ..., w_1\}$ induces an independent set.

Define the mapping $f : V(G_{0,2r} - \{v_r\}) \to V(G^c_{2r})$ by $f(v_i) = w_{2r-i}$ for $i = 0, 1, 2, ..., r - 1$ and $f(v_j) = w_{2r-j+1}$ for $j = r + 1, r + 2, ..., 2r$. Clearly, this mapping preserves the adjacency between clique set K_1 and K_2. We now show that this map preserves adjacency between Independent sets L_1 and L_2. Suppose

$v_j \in L_1$ and $v_i \in K_1$, then v_j is adjacent to v_i if and only if $1 \leq i + j \leq 2r$. Clearly, $f(v_j) = w_{2r-j+1} \in L_2$ and $f(v_i) = w_{2r-i} \in K_2$. And w_{2r-j+1} is adjacent to w_{2r-i} if and only if $2r - j + 1 + 2r - i \geq 2r + 1$, ie. $i + j \leq 2r$ and clearly $0 \leq i+j$. This implies, v_i adjacent to v_j if and only if $f(v_i)$ is adjacent to $f(v_j)$. Hence the adjacency is preserved in this mapping.

Similarly we can prove that $G_{0,2r+1} - \{v_{\lfloor (2r+1)/2 \rfloor}\} \cong_{wvl} G^c_{2r+1}$. □

3. Laplacian eigenvalues of $G_{m,n}$

From the above discusions on Laplacian eigenvalues of sum graph and invoking Theorem 1.4 and Theorem 1.5 one may conclude that

Lemma 3.1. *For $m, n \in \mathbb{N} \cup \{0\}$, $G_{m,n} \cong_{wvl} G^c_{m+1} \nabla G_n$.*

Also considering Theorem 1.3 and Theorem 2.3 we get the Laplacian eigenvalues of $G_{m,n}$ as follows

Theorem 3.1. *For $m, n \in \mathbb{N} \cup \{0\}$, $\mu(G_{m,n}) = \{n + m + 1, m + 1 + n - 1, m + 1 + n - 2, .., m + 1 + 2, m + 1 + 1, m + 1 + 0, n + m + 1, n + m, ..., n + 2, n + 1, 0\} \backslash \{n + \lceil m + 1/2 \rceil, m + 1 + \lfloor n/2 \rfloor\}$.*

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A note on $(\mathbf{m}, \mathbf{n})$-full stability Banach algebra modules relative to an ideal H of $A_{m\times n}$

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Abstract. In this paper the concept of (m, n)- fully stable Banach Algebra-module relative to ideal ($F-(m,n)-S-B-A$-module relative to ideal) is introducing, we study some properties of $F-(m,n)-S-B-A$-module relative to ideal and another characterization is given.

Keywords: fully stable Banach A-module relative to ideal, fully (m, n)-stable Banach A-module relative to ideal, multiplication (m, n)- A-module relative to ideal.

1. Introduction

"A non-empty set A is an algebra if, $(A, +, \cdot)$ is a vector space over a field F, $(A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in F$, for every $a, b \in A$" [1]. In [2] "a ring R is an algebra $< R, +, \cdot, -, 0 >$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $< R, +, -, 0 >$ is an abelian group, $< R, \cdot >$ is a semigroup and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y + z)$. "Let A be an algebra, recall that a Banach space E is a Banach left A-module (B-A- module) if E is a left A-module, and $\|a \cdot x\| \leq \|a\| \, \|x\|$ $(a \in A, x \in E)$"[1]. Following [3] "a map from a left B-A-module X into a left Banach A-module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a \cdot x) = a \cdot Tx$ for all $a \in A, x \in X$". In [4], "a submodule N of an R-module M is said to be stable, if $f(N) \subseteq N$ for each R-homomorphism $f : N \longrightarrow M$. M is called a fully stable module, each submodule of M is stable". " Let X be Banach A - module, X is called fully stable Banach A-module relative to ideal K of A, if for every submodule N of X and for each multiplier $\theta : N \longrightarrow X$ satisfy $\theta(N) \subseteq N + KX$" [5]. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of N. Let M be a right Banach Algebra-module and N be a left R-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in M^{m\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well defined element in $M^{1\times n}$

(resp. $N^{n\times k}$). "If $X \subseteq M^{1\times m}$, $S \subseteq R^{n\times n}$ and $Y \subseteq N^{n\times k}$ define

$$\begin{aligned}
\ell_{M^{l\times m}}(S) &= \Big\{u \in M^{l\times m} | us = 0; \forall s \in S\Big\} \\
r_{N^{n\times k}}(S) &= \Big\{v \in N^{n\times k} | sv = 0; \forall s \in S\Big\} \\
\ell_{R^{m\times n}}(Y) &= \big\{s \in R^{m\times n} | sy = 0; \forall y \in Y\big\} \\
r_{R^{m\times n}}(X) &= \big\{s \in R^{m\times n} | xs = 0; \forall x \in X\big\}
\end{aligned}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$" [6]. In this paper for two fixed positive integers n, m the concept of (m,n)- full stable and (m,n)-fully pseudo stable Banach Algebra modules relative to ideal have been introduced.

2. Fully (m,n)-stable Banach algebra modules relative to ideal

"A left B-A-module X is n-generated for $n \in N$ if there exists $x_1, \ldots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k \cdot x_k$ for some $a_1, \ldots, a_n \in A$. A module which is 1-generated is called a cyclic module" [7]. A right R-module M is called fully (m,n)-stable relative to ideal A of $R^{n\times m}$, if $\theta(N) \subseteq N + M^n A$ for each n-generated submodule of M^m and R-homomorphism $\theta : N \longrightarrow M$ [8].

Definition 2.1. Let K be $B-A$ - module, K is called (m,n)-fully stable $B-A$-module relative to ideal H of $A^{m\times n}$, if for every m-generated submodule L of K^n and for each multiplier $\theta : L \longrightarrow K^n$ wsatisfy $\theta(L) \subseteq L + K^m H$ for two fixed positive integers n, m.

In [5] "for a nonempty subset M in a left $B-A$-module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A : a \cdot x = 0 \text{ for all } x \in M\}$".

Notetion 2.2. Let X be a $B-A$-module

1. $L_{x_1,x_2,\ldots,x_n} = \{\oplus l_{x_i} | n \in N, x_i \in X, i = 1,2,\ldots,n\}$

   $K_{y_1,y_2,\ldots,y_n} = \{\oplus k_{y_i} | k \in K, y_i \in X, i = 1,2,,n\}$

2. $\ell_{A^{m\times n}} L_{x_1,x_2,\ldots,x_n} = \{a \in A^{m\times n}, a \cdot (\oplus l_{x_i}) = 0, \forall l_{x_i} \in L_{x_1,x_2,\ldots,x_n}\}$

   $\ell_{A^{m\times n}} K_{y_1,y_2,\ldots,y_n} = \{a \in A^{m\times n}, a \cdot (\oplus k_{y_i}) = 0, \forall k_{y_i} \in K_{y_1,y_2,\ldots,y_n}\}$

Proposition 2.3. *A $B-A$-module X is (m,n)-fully stable, if and only if any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and*

$$\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$$

of X^n, *if* $\beta_j \notin \sum_{i=1}^{n} \alpha_i A + X^m H$, *for each* $j = 1, \ldots, m$, $i = 1, \ldots, n$, $\alpha_i \in \{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ *and* $\beta_j \in \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ *implies*

$$\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \not\subseteq \mathfrak{r}_{An}\left(\{K_{x_1}, K_{x_1,x_2}, \ldots, K_{x_1,x_2,\ldots,x_m}\}\right).$$

where H *of* $A^{m\times n}$.

Proof. Assume that K is $F-(m,n)-S-B-A$-module and there exist two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n such that if $K_{y_j} \notin \sum_{i=1}^{n} A\alpha_i + X^m H$, for each $j = 1, \ldots, m$ and

$$\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}\right).$$

Define $f : \sum_{i=1}^{n} \alpha_i A \longrightarrow X^n$ by $f\left(\sum_{i=1}^{n} a_i L_{x_i}\right) = \sum_{i=1}^{n} a_i K_{y_i}$.

Let $L_{x_i} = (k_{1i}, k_{2i}, \ldots, k_{ni})$. If $\sum_{i=1}^{n} a_i L_{x_i} = 0$, then $\sum_{i=1}^{n} a_i k_{ij} = 0$, $j = 1, 2, \ldots, m$, implies that $rL_{x_j} = 0$ where $r = (r_1, \ldots, rn)$ and hence

$$r^T \in \mathfrak{r}A_n\left\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\right\}.$$

By assumption $rK_{y_j} = 0$ $j = 1, \ldots, m$ so $\sum_{i=1}^{n} r_i K_{y_i} = 0$. This shows that f is well defined. It is an easy matter to see that f is multiplier. (m,n)-fully stability of M implies that there exists $t = (t_1, \ldots, t_n) \in A^n$ such that $f\left(\sum_{i=1}^{n} r_i L_{x_i}\right) = \sum_{k=1}^{n} t_k \left(\sum_{i=1}^{n} r_i L_{x_i}\right) + b = \sum_{k=1}^{n} \sum_{i=1}^{n} \left(t_k r_i\right) L_{x_i} + b$ for each $\sum_{i=1}^{n} r_i L_{x_i} \in \sum_{i=1}^{n} L_{x_i} A$ and $b \in X^m H$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = f\left(L_{x_i}\right) = \sum_{k=1}^{n} t_k L_{x_i} + b \in \sum_{i=1}^{n} L_{x_i} A + X^m H$, which is contradiction.

Conversely assume that there exists m-generated $B-A$-submodule of K^n and multiplier $\mu : \sum_{i=1}^{n} L_{x_i} A \longrightarrow K^n$ such that $\mu\left(\sum_{i=1}^{n} L_{x_i} A\right) \not\subset \sum_{i=1}^{n} L_{x_i} A + X^m H$. Then, there exists an element $\beta \left(= \sum_{i=1}^{n} r_i L_{x_i}\right) \in \sum_{i=1}^{n} L_{x_i} A$ such that $\mu\left(K_y\right) \notin \sum_{i=1}^{n} L_{x_i} A + X^m H$. Take $K_{y_i} = K_y$, $j = 1, \ldots, m$, then we have m-element subset $\{\mu\left(K_y\right), \ldots, \mu\left(K_y\right)\}$, such that $\mu\left(K_y\right) \notin \sum_{i=1}^{n} L_{x_i} A + X^m H$, $j = 1, \ldots, m$. Let $\eta = (t_1, \ldots, t_n) \in \mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right)$, then $\eta\alpha_j = 0$, i.e $\sum_{i=1}^{n} t_i a_{ij} = 0$, for each $j = 1, \ldots, m$, $L_{x_j} = (a_{1j}, a_{2j}, \ldots, a_{nj})$ and $\{\mu\left(K_y\right), \ldots, \mu\left(K_y\right)\}$

$$\eta = \sum_{k=1}^{n} t_k \mu\left(K_y\right) = \sum_{k=1}^{n} t_k \mu\left(\sum_{i=1}^{n} r_i L_{x_i}\right) = \sum_{k=1}^{n} \mu\left(\sum_{i=1}^{n} t_k r_i L_{x_i}\right) = 0$$

hence $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{\mu\left(K_y\right), \ldots, \mu\left(K_y\right)\}\right)$, thus $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \cdot, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{\mu\left(K_{y_1}\right), \ldots, \mu\left(K_{y_1,y_2,\ldots,y_m}\right)\}\right)$ which is a contradiction. Thus X is $F-(m,n)-S-B-A$-module relative to ideal H of $A^{m\times n}$. □

Corollary 2.4. *Let* X *be an* $F-(m,n)-S-B-A$*-module, then for any two* m*-element subsets* $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ *and* $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ *of* X^n, $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{K_{x_1}, K_{x_1,x_2}, \ldots, K_{x_1,x_2,\ldots,x_m}\}\right)$ *implies that* $L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_m} + AX^m H = K_{y_1}A + K_{y_1,y_2}A + K_{y_1,y_2,\ldots,y_m}A + X^m H$, H *of* $A^{m\times n}$.

In [9], $AB-A$ - module X is said to satisfy Baer criterion if each submodule of X satisfies Baer criterion, that is for every submodule N of X and $A-$ multiplier $\theta : N \longrightarrow X$, there exists an element a in A such that $\theta(n) = an$ for all $n \in N$".

Definition 2.5. A $B - A$- module X is said to satisfy Baer (m, n)-criterion relative to an ideal H of $A^{m\times n}$ if each submodule of X satisfies Baer (m, n)-criterion relative to an ideal H, that is for every m-generated submodule L of X^n and A - multiplier $\theta : L \longrightarrow X^n$, there exists an element a in A such that $\theta(l) - al \in X^m H$ for all $l \in L$.

Proposition 2.6. *If X satisfies Baer $(m, 1)$-criterion and $\mathfrak{r}_A(L\cap M) = \mathfrak{r}_A(L)+ \mathfrak{r}_A(M)$ for each m-generated submodules of X^n, then X satisfies Baer (m, n)-criterion. ,H of $A^{m\times n}$.*

Proof. Let $P = Ax_1 + Ax_2 + \ldots + Ax_m$ be an m-generated submodule of X^n and $f : P \longrightarrow X^n$ a multiplier. We use induction on m. It is clear that M satisfies Bear (m, n)- criterion, if $m = 1$. Suppose that X satisfies Bear (m, n)-criterion for all k-generated submodule of X_n, for $k \leq n - 1$. Write $L = Ax_1$, $M = Ax_2 + \ldots + Ax_m$, then for each $w_1 \in L$ and $w_2 \in M$ $f|_L(w_1) = y_1w_1$, $f|_M(w_2) = y_2w_2$ for some $y_1, y_2 \in A$. It is clear $y_1 - y_2 \in \mathfrak{r}_A(L \cap M) = \mathfrak{r}_A(L) + \mathfrak{r}_A(M)$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in \mathfrak{r}_A(L)$, $z_2 \in \mathfrak{r}_A(M)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $w = w_1 + w_2 \in P$ with $w_1 \in L$ and $w_2 \in M$, $f(w) = f(w_1) + f(w_2) = w_1y_1 + w_2y_2 = w_1(y_1 - z_1) + w_2(y_2 + z_2) = w_1y + w_2y = (w_1 + w_2)y = wy$. □

Proposition 2.7. *Let X be a $B - A$- module. Then X satisfies Baer (m, n) criterion if and only if $\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A) \subseteq L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A + X^m H$ for n-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ of X^n, H of $A^{m\times n}$.*

Proof. Suppose that Baer (m, n)-criterion holds for m-generated submodule of X^n , let $L_{x_i} = (k_{i1}; k_{i2}, \ldots, k_{im})$, for each $i = 1, \ldots, n$ and $K_y = \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\} \in \ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A)$, $K_{y_i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$.

Define $\mu : L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A \longrightarrow X^n$ by $\mu\left(\sum_{i=1}^n L_{x_i}a_i\right) = \sum_{i=1}^n K_{y_i}a_i$. If $\sum_{i=1}^n L_{x_i}a_i$, then $\sum_{i=1}^n k_{ij}a_i = 0$. $j = 1, \ldots, m$, this implies that $L_{x_i}r = 0$ where $r = (r_1, \ldots, r_n)$ and hence

$$r \in \mathfrak{r}_{An}\left(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A\right).$$

By assumption $rL_{x_i} = 0$, $i = 1, \ldots, n$ so $\sum_{i=1}^n K_{y_i}a_i = 0$. This show that f is well defined. It is an easy matter to see that μ is an multiplier. By assumption there exists $t \in A$ such that $\mu\left(\sum_{i=1}^n L_{x_i}a_i\right) = t\left(\sum_{i=1}^n K_{y_i}a_i\right) = \sum_{i=1}^n K_{y_i}(ta_i)$ for each $\sum_{i=1}^n L_{x_i}a_i \in \sum_{i=1}^n L_{x_i}A$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = \mu\left(\sum_{i=1}^n L_{x_i}\right) = \sum_{i=1}^n L_{x_i}t \in \sum_{i=1}^n L_{x_i}A$ which is contradiction.

This implies that $\ell_{X^n}\mathfrak{r}_{An}\left(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A\right) \subseteq L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A + X^mH$.

Conversely, assume that $\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A) = L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A + X^mH$, for each $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ in X^n. Then, for each multiplier $f : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \longrightarrow X^n$ and

$$s = (s_1, \ldots, s_n) \in r_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A), \sum_{k=1}^{n} s_k(\sum_{i=1}^{n} L_{x_i}t_i) = 0,$$

for each $\sum_{i=1}^{n} L_{x_i}t_i \in \sum_{i=1}^{n} L_{x_i}A$, hence

$$\sum_{k=1}^{n} s_k f(\sum_{i=1}^{n} L_{x_i}t_i) = \sum_{k=1}^{n} f(\sum_{i=1}^{n} L_{x_i}s_kt_i) = 0,$$

thus $f(\sum_{i=1}^{n} L_{x_i}t_i) \in \ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A) \subseteq L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A + X^mH$. Then X satisfies Baer (m,n)-criterion. □

Corollary 2.8. *Let X be a $B-A$- module. Then X is $F-(m,n)-S-B-A$-module if and only if*

$$\begin{aligned}&\mathfrak{r}X_n\ell_{A^n}\left(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}\right)\\ &\subseteq AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} + X^mH\end{aligned}$$

for n-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ of X_n, H of $A^{m\times n}$.

Following [5] "Let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi -injective if, $\phi : N \longrightarrow X$ is A-module homomorphisms such that $\|\phi\| \leq 1$, there exists A-module homomorphism $\theta : X \longrightarrow X$, such that $\theta \circ i = \phi$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α".

Following [5], "Let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi-α-injective relative to ideal H of A if, $\phi : N \longrightarrow X$ is A-module homomorphisms such that $\|\phi\| \leq 1$, there exists A-module homomorphism $\theta : X \longrightarrow X$, such that $(\theta \circ i)(n)\phi(n) \in XH$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X".

The concepts quasi $(m,n) - \alpha - B - A$ - module relative to ideal for some α is introducing.

Definition 2.9. Let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi $(m,n) - \alpha$-injective relative to ideal H of $A^{m\times n}$ if, $\phi : N \longrightarrow X^n$ is A-module homomorphisms such that $\|\phi\| \leq 1$, there exists A-module homomorphism $\theta : X^n \longrightarrow X^n$, such that $(\theta \circ i)(n)\phi(n) \in X^mH$ and $\|\phi\| \leq \alpha$ where i is an isometry from m-generated submodule N of X. We shall say that X is quasi (m,n)- injective relative to ideal if it is quasi $(m,n) - \alpha$ injective relative to ideal for some α. , H of $A^{m\times n}$.

Proposition 2.10. *If X is $F-(m,n)-S-B-A$-module relative to ideal then X is quasi $(m,n)-B-A$-module relative to ideal H of $A^{m\times n}$.*

Proof. Let $N = \alpha_1 A + \ldots + \alpha_n A$ be m-generated submodule of X^n, $\alpha_i \in X^n$ let $\alpha > 1$ and f be any A-module homomorphism from N to X^n such that $\|f\| \leq 1$. Since X is $F-(m,n)-S-B-A$-module relative to ideal, then $f(\alpha 1A + \ldots + \alpha_n A) \subseteq \alpha_1 A + \ldots + \alpha_n A + X^m H$, then there exist $t = (t_1, \ldots, t_n) \in A_n$ and $wX^m H$. Let $a_i = (0, \ldots, 1, 0, \ldots, 0)$ such that $f(\sum_{i=1}^n \alpha_i) = t(\sum_{i=1}^n \alpha_i) + w$. Define $g : X^n \longrightarrow X$ by $g(\alpha_i) = tT\alpha_i$ it is clear that g is well defined A-module homomorphism. Now $f(\sum_{i=1}^n \alpha_i) - g(\sum_{i=1}^n \alpha_i) = t(\sum_{i=1}^n \alpha_i) + w - t(\sum_{i=1}^n \alpha_i) = w \in X^m H$ and since for each $y \in \alpha_1 A + \ldots + \alpha_n A$, $y = \sum_{i=1}^n \alpha_i s_i$ for some $s = \{(\} s_1, \ldots, s_n) \in A$, $f(y)g(y) = f(\sum_{i=1}^n \alpha_i s_i) - g(\sum_{i=1}^n \alpha_i s_i) = f((\sum_{i=1}^n \alpha_i) s) - g((\sum_{i=1}^n \alpha_i) s) = (f(\sum_{i=1}^n \alpha_i) g(\sum_{i=1}^n \alpha_i)) s \in X^m H$, therefore X is quasi $(m,n) - B - A$ -module relative to ideal. □

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On the p-supersolvability of finite groups

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Abstract. A subgroup H of a finite group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. Obviously, the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups. In this paper, we investigate the influence of CSS-subgroups on the p-supersolvability of finite groups. Some further results are obtained.

Keywords: CSS-subgroups, p-solvable groups, $p-$supersolvable groups.

1. Introduction

All groups considered in this paper are finite.

Recall a subgroup H of a group G is said to be $c-$normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G = Core_G(H)$(see [6]); The other fundamental concept deriving from [4] is that of $SS-$quasinormality: a subgroup H of G is said to be an SS-quasinormal subgroup of G if there is a supplement B of H in G such that H permutes with every Sylow subgroup of B. Combining these two concepts, the authors in [8] introduce the concept of the $CSS-$ subgroups: A subgroup H of a group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. In this case, K is called a normal CSS-supplement of H in G.

Obviously, a $c-$normal subgroup must be a $CSS-$subgroup, while a $SS-$quasinormal subgroup also can be regarded as a $CSS-$subgroup if we take the

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normal subgroup $K = G$ in the definition of the $CSS-$subgroups, which indicates that the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups.

It is noteworthy that many authors have studied the structure of a group G by using the $c-$normality and the $SS-$quasinormality, for example, [2, 7, 5, 1, 4, 6].

As for $c-$normal subgroups, the author in [6] asserts that G is suppersolvable if every maximal subgroup of Sylow subgroup of G is $c-$normal in G; In [5], the authors generalize the result and prove that G is supersolvable if G/N is supersolvable and every maximal subgroup of Sylow subgroup of N is $c-$normal in G, where N is a normal subgroup of G; Referring to $SS-$quasinormal subgroups, the authors show that a group G is supersolvable if some maximal subgroups of Sylow subgroups are $SS-$quasinormal in G, which are unified in [8] by using the conception of the $CSS-$subgroup and minimizing the number of maximal subgroups. In this paper, we proceed with the study about $CSS-$subgroups and obtain the $p-$supersolvability of G for a fixed prime p by considering some maximal subgroups of some Sylow $p-$subgroup:

Theorem 3.1. *Let G be a p-solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Based on this result, we also consider the case of quotient group:

Theorem 3.2. *Let G be a p-solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

It is necessary to point out that, in this article, the notation $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$ be a set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, where d be the smallest generator number of a p-group P; Apart from this, all unexplained notations and terminology are standard.

2. Preliminaries

In this section we list some known results which are needed in the sequel.

Lemma 2.1 ([4, Lemma 2.1])**.** *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

(3) *If $N \leq K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.*

(4) *If K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2 ([8, Lemma 2.3]). *Let H be a subgroup of a group G. We have:*

(1) *If H is a CSS-subgroup of G and $H \leq M \leq G$, then H is a CSS-subgroup of M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$, then H is a CSS-subgroup of G if and only if H/N is a CSS-subgroup of G/N.*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is a CSS-subgroup of G, then HN/N is a CSS-subgroup of G/N.*

Lemma 2.3 ([8, Lemma 2.5]). *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.4 ([4, Lemma 2.5]). *If a p-subgroup P of G is SS-quasinormal, where p is a prime. Then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.5 ([3, A. 1.2]). *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1). $U \cap VW = (U \cap V)(U \cap W)$.

(2). $UV \cap UW = U(V \cap W)$.

Lemma 2.6. *Let P be a Sylow $p-$subgroup of G, and N a normal subgroup of G, then $\Phi(PN/N) = \Phi(P)N/N$.*

Proof. Keeping in mind that $\Phi(P) = P'P^p$, so

$$\begin{aligned} \Phi(PN/N) &= (PN/N)'(PN/N)^p \\ &= P'N/N \cdot P^pN/N \\ &= P'P^pN/N \\ &= \Phi(P)N/N. \end{aligned}$$

Lemma 2.7. *Let H be a subgroup of G, N contained in H be a normal subgroup of G and $P \in Syl_p(H)$. Assume that some $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$, if every member in $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then:*

(1) $\mathcal{M}_d(PN/N) = \{P_iN/N \mid P_iN/N \lessdot PN/N, 1 \leq i \leq d\}$.

(2) Every member in $\mathcal{M}_d(PN/N)$ is a CSS-subgroup of G/N.

Proof. (1) It is of course by Lemma 2.6.

(2) For every $P_iN/N \in \mathcal{M}_d(PN/N)$, since $P_i \in \mathcal{M}_d(P)$, the hypotheses show that there exists a normal subgroup K such that $G = P_iK$ and $P_i \cap K$ is $SS-$quasinormal in G, so $G/N = P_iN/N \cdot KN/N$. Now, our task is to show $(P_iN \cap KN)/N$ is $SS-$quasinormal in G/N. In fact, note that $P_iN/N \lessdot PN/N$, so we can conclude that $P_i \cap N = P \cap N$; On the other hand, assume that N_q is a Sylow $q-$subgroup of N, where q is a prime with $q \neq p$, there exists a

Sylow $q-$subgroup Q with $Q \leq K$ since $G = P_1K$ and $K \trianglelefteq G$, so $N_q \leq N \cap K$. Therefore $N = (P_i \cap N) \cdot (K \cap N)$, hence $P_iK \cap N = N = (P_i \cap N) \cdot (K \cap N)$, which leads to $P_iN \cap KN = (P_i \cap K)N$ by Lemma 2.5, Lemma 2.1(2) shows that $(P_i \cap K)N/N$ is $SS-$quasinormal, so the conclusion (2) holds.

3. Main results

Now we are equipped to prove the main results.

Theorem 3.1. *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Proof. Suppose that the theorem is not true, and let G be a counter-example of minimal order with $\mathcal{M}_d(P) = \{P_1, \cdots P_d\}$ and $\bigcap_{i=1}^{d} P_i = \Phi(P)$. We proceed in the following steps:

Step 1. $O_{p'}(G) = 1$.

Otherwise, considering the quotient group $G/O_{p'}(G)$. In view of the minimal choice of G, Lemma 2.2 shows that $G/O_{p'}(G)$ is $p-$supersolvable, and so is G, a contradiction.

Step 2. *G has a unique minimal normal subgroup N, and* $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. By Lemma 2.7, the factor group $\overline{G} = G/N$ satisfies the hypotheses of the theorem. The choice of G implies that $\overline{G}$ is p-supersolvable. Notice that the class of p-supersolvable groups is a saturated formation, so we may assume that N is a unique minimal normal subgroup of G. On the other hand, if $\Phi(G) > 1$, then $N \leq \Phi(G)$ by the above, therefore the supersolvability of G/N indicates that G is supersolvable, and so is G, a contradiction.

Step 3. $N = O_p(G)$.

Keeping in mind that G is $p-$solvable, so N is a (normal) p-subgroup by Step 1. By employing $\Phi(G) = 1$ in Step 2, we have that there exists a maximal subgroup M of G such that $G = NM$. Also, $\Phi(N) \leq \Phi(G) = 1$, it is followed that N is an elementary abelian $p-$group, which yields to $N \cap M = 1$ by the minimal normality of N. On the other hand, it is clear that $G = O_p(G)M$, we are now in a position to apply Lemma 2.3 concluding that $O_p(G) \cap M \trianglelefteq G$, so $N \leq O_p(G) \cap M$ if $O_p(G) \cap M > 1$, which leads to $N \leq M$, a contradiction. Therefore $O_p(G) \cap M = 1$. By the above that $G = NM$ and $N \cap M = 1$, which forces us conclude that $N = O_p(G)$.

Step 4. *Without loss of generality, we may assume that $P = NP_1$. Set $N_1 = N \cap P_1$, then $N_1 \lessdot N$.*

If $N \leq \Phi(P)$, then $N \leq \Phi(G)$ contrary to the Step 1. Thus $N \not\leq \Phi(P) = \bigcap_{i=1}^{d} P_i$, we may assume that $N \not\leq P_1$, so $P = NP_1$, and

$$|N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p.$$

Step 5. *Let K_1 be a $CSS-$supplement of P_1 in G, then $P \cap K_1 = (P_1 \cap K_1)N$ and $P_1 \cap K_1 \lessdot P \cap K_1$.*

By the hypotheses of the theorem that P_1 is a CSS-subgroup of G, so there exists a normal subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is SS-quasinormal in G. Also $G = PK_1$, by computing the order of G, we have that $P_1 \cap K_1 \lessdot P \cap K_1$.

On the other hand, the normality of K_1 asserts that $P \cap K_1$ is a Sylow p-subgroup of K_1, and

$$P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N.$$

Step 6. *Let K_{1q} be a Sylow q-subgroup of K_1, where $q \neq p$, then $N_1 = N \cap (P_1 \cap K_1)K_{1q}$.*

Let K_{1q} be a Sylow q-subgroup of K_1, where $q \neq p$. Clearly, K_{1q} is also a Sylow q-subgroup of G. Since $P_1 \cap K_1$ is SS-quasinormal in G, by Lemma 2.4, we have $(P_1 \cap K_1)K_{1q}$ is a subgroup, and so is $(P_1 \cap K_1)K_{1q}N$. Concentrate on the group $(P_1 \cap K_1)K_{1q}N$, the following task is to show that $N_1 = N \cap (P_1 \cap K_1)K_{1q}$.

Firstly, we show $N_1 \leq N \cap (P_1 \cap K_1)K_{1q}$. Notice that $N_1 = P_1 \cap N$ and $N \leq K_1$, so $N_1 \leq P_1 \cap K_1$, and hence $N_1 \leq N \cap (P_1 \cap K_1)K_{1q}$.

Secondly, we are required to prove that $|N_1| = |N \cap (P_1 \cap K_1)K_{1q}|$. Considering $|(P_1 \cap K_1)K_{1q}N|$. On the one hand, it is clear that

$$|(P_1 \cap K_1)K_{1q}N| = |(P_1 \cap K_1)NK_{1q}| = |(P_1 \cap K_1)N||K_{1q}|,$$

according to Step 5, $(P_1 \cap K_1)N = P \cap K_1$, so

$$|(P_1 \cap K_1)K_{1q}N| = |(P \cap K_1)||K_{1q}|.$$

On the other hand,

$$|(P_1 \cap K_1)K_{1q}N| = \frac{|(P_1 \cap K_1)K_{1q}| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1q}|} = \frac{|(P_1 \cap K_1)| \cdot |K_{1q}| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1q}|}.$$

Comparing the two equalities, we are allowed to conclude that

$$|P \cap K_1| = \frac{|(P_1 \cap K_1)| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1q}|},$$

by the Step 5 that $P_1 \cap K_1 \lessdot P \cap K_1$, we are forced to deduce that $N \cap (P_1 \cap K_1)K_{1q} \lessdot N$, in view of $N_1 \lessdot N$ by Step 4, while $N_1 \leq N \cap (P_1 \cap K_1)K_{1q}$, and therefore $N_1 = N \cap (P_1 \cap K_1)K_{1q}$.

Step 7. *The final contradiction.*

It is clear that $N_1 = N \cap (P_1 \cap K_1)K_{1q} \trianglelefteq (P_1 \cap K_1)K_{1q}$. Therefore N_1 is normal in the subgroup $\langle N, (P_1 \cap K_1)K_{1q} | q \in \pi(G), q \neq p\rangle = K_1$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq P_1$, then $N_1 \trianglelefteq P_1K_1 = G$. The minimality of N yields $N_1 = 1$. Consequently, N is a cyclic subgroup of order p. Since G/N is p-supersolvable, G is p-supersolvable, a contradiction.

Theorem 3.2. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Proof. Let G be a minimal counter-example. We claim:

Step 1. *G has a unique minimal normal subgroup N contained in H, G/N is p-supersolvable and $N \not\leq \Phi(G)$.*

Let N contained in H be a minimal normal subgroup of G, Lemma 2.7 shows that G/N is $p-$supersolvable. Notice that the $p-$supersolvable class is saturated, so we may assume that $N \not\leq \Phi(G)$ and N is a unique minimal normal subgroup contained in H.

Step 2. *$O_{p'}(G) = 1$ and $\Phi(G) = 1$.*

It is clear that $O_{p'}(G) = 1$ by Lemma 2.7. Now, we show $\Phi(G) = 1$. Otherwise, concentrating on the quotient group $G/H\Phi(G)$. Obviously,

$$(G/\Phi(G))/(H\Phi(G)/\Phi(G)) \cong G/H\Phi(G) \cong (G/H)/(H\Phi(G)/H),$$

we have that $(G/\Phi(G))/(H\Phi(G)/\Phi(G))$ is $p-$supersolvable, Lemma 2.7 shows that $H\Phi(G)/\Phi(G)$ satisfies the hypotheses of the theorem, so the choice of G indicates that $G/\Phi(G)$ is $p-$supersolvable, and so is G, a contradiction.

Step 3. *$N = O_p(H)$*

As G is p-solvable, we have N is a p-subgroup by Step 2, $\Phi(N) \leq \Phi(G) = 1$ by Step 1. It follows that there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Applying Lemma 2.3, we have $O_p(H) \cap M \trianglelefteq G$, so $O_p(H) \cap M = 1$, thus $N = O_p(H)$.

Step 4. *Without loss of generality, we may assume that $P = NP_1$. Set $N_1 = N \cap P_1$, we have $N_1 \lessdot N$.*

Since $N \not\leq \Phi(P)$, then there exists a $P_1 \in \mathcal{M}_d(P)$ such $N \not\leq P_1$, so $P = NP_1$, set $N_1 = P_1 \cap N$, clearly $N_1 \lessdot N$.

Step 5. *Let K_1 be a $CSS-$supplement of P_1 in G, then $N \leq K_1$, $P \cap K_1 = (P_1 \cap K_1)N$ and $P_1 \cap K_1 \lessdot P \cap K_1$.*

Notice that P_1 is a $CSS-$subgroup of G, so there exists a normal subgroup K_1 such that $G = P_1K_1$, also, $G = PK_1$, therefore $P_1 \cap K_1 \lessdot P \cap K_1$.

Next, we shall show that $N \leq K_1$. Since $H \cap K_1 \trianglelefteq G$, then either $N \leq H \cap K_1$ or $H \cap K_1 = 1$ by Step 1. If $H \cap K_1 = 1$, then $P_1 = P$ as $G = P_1K_1 = PK_1$, a contradiction, which shows that $N \leq H \cap K_1 \leq K_1$.

Therefore, $P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N$.

Step 6. *The final conclusion.*

By using a similar argument to the proof of Theorem 3.1, we have the similar conclusions to Step 6 and Step 7, so G is $p-$supersolvable.

Notice that the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups, so we have the following:

Corollary 3.3. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a $c-$normal subgroup of G, then G is p-supersolvable.*

Corollary 3.4. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a $SS-$quasinormal subgroup of G, then G is p-supersolvable.*

In Corollary 3.3 and Corollary 3.4, if we take $H = G$, we have that:

Theorem 3.5. *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a $c-$ normal subgroup of G, then G is p-supersolvable.*

Theorem 3.6 ([4, Theorem 1.3])**.** *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a $SS-$quasinormal subgroup of G, then G is p-supersolvable.*

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A corollary that provides seat arrangements for even numbers of seats

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Abstract. Let $1, 2, 3, \ldots, n$ be n students and $s_1, s_2, s_3, \ldots, s_n$ be n row seats. A seat is arranged for each student on each day of the following n days. It is required that each student shall have different seat every day of the n days. Also, it is required that for these n day every student shall have a chance to sit next to every other student on one of his side and shall have another chance to sit next to every other student on the other of his side. In this paper, it is shown that such arrangements are possible when the number of students is even. Also, an algorithm for such arrangements is provided.

Keywords: combinatorics, seat arrangement.

1. Introduction

Let $1, 2, 3, \ldots, n$ be n students and $s_1, s_2, s_3, \ldots, s_n$ be n row seats. A seat is arranged for each student on each day of the following n days. It is required that each student shall have different seat every day of the n days. Also, it is required that for these n day every student shall have a chance to sit next to every other student on one of his side and shall have another chance to sit next to every other student on the other of his side. For example, when $n = 4$ the following is one possible way of the arrangement for 4 days.

∗. Corresponding author

	s_1	s_2	s_3	s_4
Day 1	1	2	3	4
Day 2	2	4	1	3
Day 3	3	1	4	2
Day 4	4	3	2	1

Another example of the arrangement is when $n = 6$.

	s_1	s_2	s_3	s_4	s_5	s_6
Day 1	1	2	3	4	5	6
Day 2	2	4	6	1	3	5
Day 3	3	6	2	5	1	4
Day 4	4	1	5	2	6	3
Day 5	5	3	1	6	4	2
Day 6	6	5	4	3	2	1

From the first example, we can see that the arrangements for Day 3, and Day 4 are simply the reverses of Day 2, and Day 1 respectively. Also, from the second example, the arrangements for Day 4, Day 5, Day 6 are the reverses of Day 3, Day 2, and Day 1 respectively.

In [1], it is shown that the arrangement is possible when $n = p - 1$ for every prime number $p \geq 3$ and the algorithm (Seat Arrangement Algorithm or SAA) for the arrangement is also provided.

Theorem 1.1 ([1]). *The SAA is possible when the number of students $n = p-1$ for any given prime number $p \geq 3$.*

When n is odd numbers, it is not difficult to verify that it is not possible to arrange the seats if all above conditions are required.

The above theorem shows that we can always have the required seat arrangement if $n = p - 1$ where $p \geq 3$ is prime number. We note that $n = p - 1$ is even, so there is a question that whether it is still possible to have the required seat arrangements for some other even values of n. Consider the following arrangement when $n = 8$ which satisfies the conditions of seat arrangement.

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
1	2	3	4	5	6	7	8
2	4	1	3	6	8	5	7
3	8	4	7	2	5	1	6
4	6	2	8	1	7	3	5
8	7	6	5	4	3	2	1
7	5	8	6	3	1	4	2
6	1	5	2	7	4	8	3
5	3	7	1	8	2	6	4

We note that $n = 8$ is not of the form $n = p - 1$, since 9 is not prime. In fact, in section 2, we shall show that seat arrangements can be arranged for all even values of $n \geq 2$.

2. Seat arrangements when n are even numbers

When $n = 8$, for example, one can try to arrange the seats and shall quickly find out that, without an algorithm, it is not an obvious work to do the arrangement. We have tried to find practical algorithms to serve the purpose, and have found that the simple algorithm can follow from Theorem 2.1.

Theorem 2.1 ([2])**.** *The graph K_{2n+1} is the sum of n spanning cycles.*

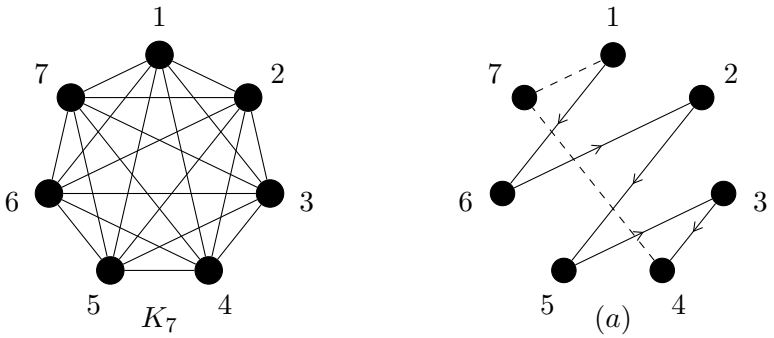
From, Theorem 2.1, we can have Corollary 2.1 which guarantees the possibility of seat arrangements when n are even numbers.

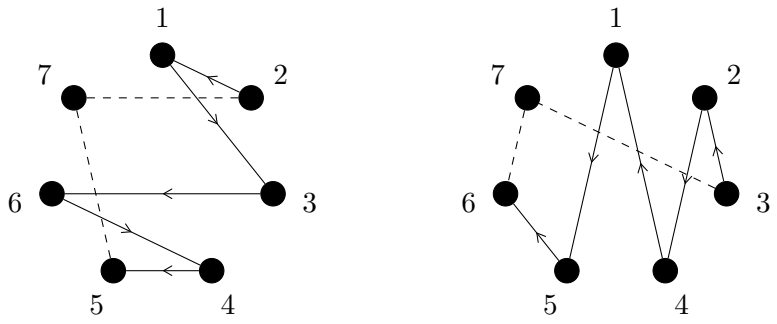
Corollary 2.1. *The seat arrangements are possible for all numbers of students that are positive even numbers.*

Proof. From the proof of Theorem 2.1, there are $2n + 1$ numbers of points : $v_1, v_2, v_3, \ldots, v_{2n+1}$. The proof shows the constructions of n paths: $P_1, P_2, \ldots, P_n$ from $2n$ points $v_1, v_2, v_3, \ldots, v_{2n}$. Each path consists of $2n - 1$ lines, and so all paths together have $n(2n - 1) = \binom{2n}{2}$ lines, i.e. all lines that join between $2n$ points. All these lines are different, and, according to the proof of Theorem 2.1, the j-th points of P_i are different. We can see that $P_1, P_2, \ldots, P_n$ can represent seat arrangement for Day 1, Day 2, ... , Day n respectively. The reverse paths of $P_1, P_2, , P_n$ can represent the arrangements for Day $n + 1$, Day $n + 2$, ... , Day $2n$. □

For example, when $2n = 6$, we use Theorem 2.1 to construct 3 paths of K_7, see Fig. 2.1 which is from [2].

1	6	2	5	3	4
2	1	3	6	4	5
3	2	4	1	5	6





(b) (c)

Fig. 2.1

These 3 paths provide seat arrangements for Day 1, Day 2, Day 3. Together with the reverses of the three paths, we have the following seat arrangements for 6 days:

	s_1	s_2	s_3	s_4	s_5	s_6
Day 1	1	6	2	5	3	4
Day 2	2	1	3	6	4	5
Day 3	3	2	4	1	5	6
Day 4	4	3	5	2	6	1
Day 5	5	4	6	3	1	2
Day 6	6	5	1	4	2	3

3. Algorithm for seat arrangements when number of students is even

We shall simplify the path construction in the proof of Theorem 2.1 in obtaining the seat arrangement algorithm. We only need to construct Day 1 arrangement, the other day arrangements shall readily follow.

For example, consider the above seat arrangement when $2n = 6$. First, consider the arrangements for s_1, s_3, and s_5.

s_1	s_3	s_5
1	2	3
2	3	4
3	4	5
4	5	6
5	6	1
6	1	2

and consider the arrangements for s_2, s_4, and s_6.

s_2	s_4	s_6
6	5	4
1	6	5
2	1	6
3	2	1
4	3	2
5	4	3

Each column of the arrangement is simply related to clockwise circular arrangement of 1, 2, 3, 4, 5, and 6, see Fig. 3.1.

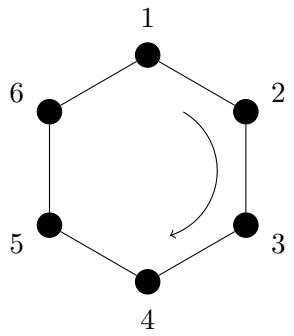


Fig. 3.1

We can generalize the idea to arrange for $2n$ students $1, 2, 3, \ldots, 2n$. Note that we only need to construct the seat arrangement for Day 1 and all the arrangements for other days shall follow by using the clockwise circular arrangements for each column, see Fig. 3.2, as mentioned in the case $2n = 6$ above.

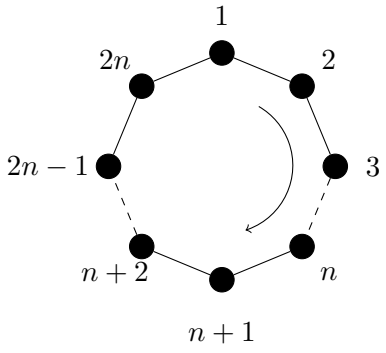


Fig. 3.2

The algorithm for general cases is as follow. For Day 1 arrangement, let $1, 2, 3, \ldots, n$ sit on seats $s_1, s_3, s_5, \ldots, s_{2n-1}$ respectively, and let $2n, 2n-1, 2n-$

$2, \ldots, n+1$ sit on seat $s_2, s_4, s_6, \ldots, s_{2n}$ respectively. From this, the arrangements for Day 2, Day 3, ..., Day $2n$ shall follow.

s_1	s_3	s_5	...	s_{2n-3}	s_{2n-1}
1	2	3	...	$n-1$	n
2	3	4	...	n	$n+1$
3	4	5	...	$n+1$	$n+2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$2n-1$	$2n$	1	...	$n-3$	$n-2$
$2n$	1	2	...	$n-2$	$n-1$

s_2	s_4	s_6	...	s_{2n-2}	s_{2n}
$2n$	$2n-1$	$2n-2$	...	$n+2$	$n+1$
1	$2n$	$2n-1$	...	$n+3$	$n+2$
2	1	$2n$	...	$n+4$	$n+3$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$2n-2$	$2n-3$	$2n-4$	...	n	$n-1$
$2n-1$	$2n-2$	$2n-3$	...	$n+1$	n

For example, consider K_9 or $K_{2(4)+1,}$ we can construct 4 paths (and another 4 reverse paths) by using the above algorithm.

For Day 1 arrangement. let 1,2,3,4 sit on seats s_1, s_3, s_5, s_7 respectively, and let 8, 7, 6, 5 sit on seats s_2, s_4, s_6, s_8 respectively. After that, we can arrange seats, using the above algorithm, as follow.

s_1	s_3	s_5	s_7
1	2	3	4
2	3	4	5
3	4	5	6
4	5	6	7
5	6	7	8
6	7	8	1
7	8	1	2
8	1	2	3

s_2	s_4	s_6	s_8
8	7	6	5
1	8	7	6
2	1	8	7
3	2	1	8
4	3	2	1
5	4	3	2
6	5	4	3
7	6	5	4

Combining the above arrangements we obtain seat arrangement for $2n = 8$ students.

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
1	8	2	7	3	6	4	5
2	1	3	8	4	7	5	6
3	2	4	1	5	8	6	7
4	3	5	2	6	1	7	8
5	4	6	3	7	2	8	1
6	5	7	4	8	3	1	2
7	6	8	5	1	4	2	3
8	7	1	6	2	5	3	4

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[1] V. Longani, H. Yingtaweesittikul, *Seat arrangement problems*, Thai J. of Mathematics, 14 (2016), 383-390.

[2] F. Harary, *Graph theory*, Addison-Wesley Publishing Company, Boston, 1969.

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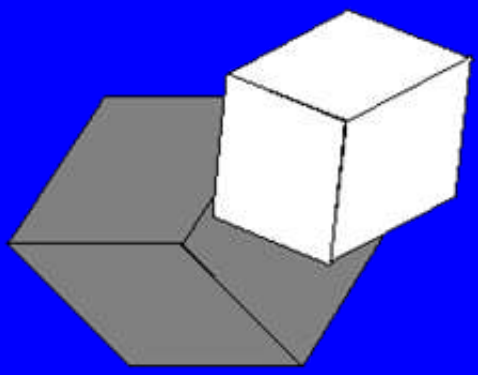
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