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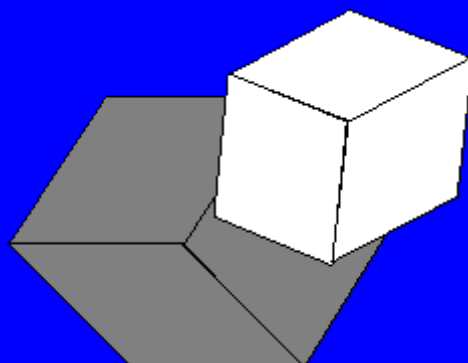
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In memoriam of Professor **Giuseppe Manuppella**

The “Italian Journal of Pure and Applied Mathematics“ cannot more take advantage of the precious collaboration of prof. Giuseppe Manuppella, who has passed away, for Mathematics event organization.

The members of Editorial Board express their deep sorrow for this loss.

The Managing Board Chief regrets the loss of Prof. Giuseppe Manuppella. He has been a great man of science and a very dear friend.

All they who knew him will remember always his scientific value and his wonderful human qualities.

Piergiulio Corsini
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Two-level secret sharing schemes based on reverse super edge magic labelings

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Abstract. In this paper, we propose two-level secret sharing scheme based on a reverse edge magic labeling of star graphs. It is a scheme which creates two types of hierarchical sets. The first set contains share that are more powerful than the share in the second set. We build banking secret sharing scheme that will share a secret among one bank manager and several sets of authorized staff members.

Keywords: two-level secret sharing schemes, edge-magic total labelling.

1. Introduction

In 1979, Blakley [3], Shamir [7], and Chaum [5] introduced the notion of secret sharing scheme. A secret sharing scheme is a method of allocating a secret S among a finite set of participants $P = \{p_1, p_2, \dots, p_n\}$ in such a way that if the participants in $A \subseteq P$ are capable to know the secret, then their partial information by pooling together, they can reconstruct the secret S ; but any $B \subseteq P$, which is not qualified to know S , cannot reconstruct the secret. The key S which is chosen by a special participant d , called a dealer, and it usually assumed that $d \notin P$. The share means that dealer gives partial information to each participant which is a tool to reveal the secret S . An access structure Γ is the family of all the subsets of participants that are able to reconstruct the secret. The sets belonging to the access structure Γ are called authorized sets and those not belonging to the access structure are termed as unauthorized sets.

A two-level secret sharing scheme is a scheme which produces two kinds of hierarchical sets. The first set (the highest rank) contains shares that are

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more powerful (important) than the shares in the second set. In our previous paper, we construct a secret sharing scheme based on reverse edge magic graph labeling. In this paper, we continue our work to plan a two-level secret sharing scheme based on reverse edge magic graph labeling. We provide two different types of such schemes, with two different possible applications.

The first scheme distributes shares of a secret among two sets. The first set (the highest rank set) contains a single person s_0 , called a supervisor, and the second set contains a number of chosen people. In the first scheme, the access structure Γ is the family of all sets of the form $\{s_0, p\}$ where p belongs to the second set. We call the first scheme the supervision al secret sharing scheme.

In the proposed second scheme, the first set (the highest rank set) contains a single person s_0 . The second set S_2 contains k departments, namely $S_2 = \{D_1, D_2, \dots, D_k\}$ where each department D_i consists of d_i authorised people. In the second scheme, the access structure Γ in this second scheme is the family of all sets of the form $\{s_0, x_1, x_2, \dots, x_k\}$ where $x_i \in D_i$. We name this scheme the departmental secret sharing scheme. These two secret sharing schemes are constructed by reverse edge-magic labeling. The definitions of reverse edge-magic labeling and its related results will be existing in Section II. The proposed schemes will be enlightened in Section III. A conclusion is located in Section IV.

2. Basic theory

In this paper, we considered finite and simple graphs and used general reference for graph-theoretic ideas in [?] . The graph G with the vertex-set $V(G)$ and the edge-set $E(G)$ is said to be a reverse edge-magic (REM) labeling then the one-to-one mapping

$$f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) + E(G)|\}$$

satisfying the property that there exists an integer k such that

$$f(xy) - \{f(x) + f(y)\} = k$$

for each edge xy in G . We call $f(xy) - \{f(x) + f(y)\} = k$ the reverse edge difference of edge xy , and k the reverse magic constant of graph G . In particular, if $f(V(G)) = \{1, 2, \dots, |V(G)|\}$ then f is called reverse super edge-magic labeling. A graph is called reverse(super) edge-magic if it admits any reverse (super) edge-magic labeling.

Venkata Ramana et al [9] introduced and studied the notion of reverse edge-magic graphs with a different name, i.e., graphs with reverse magic valuations, while the term of reverse super edge-magic graphs was firstly introduced by them. They showed that a star $S_{n+1} = K_1, n$ is the only complete bipartite graph which is reverse super edge-magic total. They also showed that any odd cycle is reverse super edge-magic, but every wheel is not. Since then, some of authors have studied reverse(super) edge-magic properties in graphs, see, for

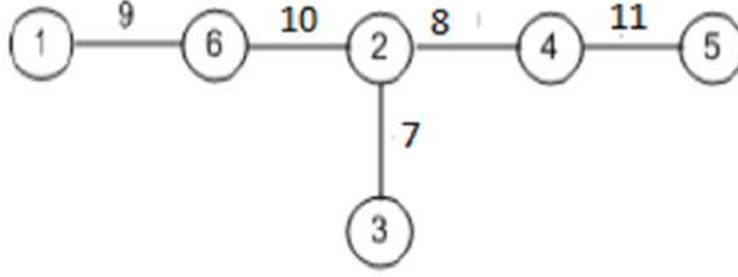


Figure 1: Reverse Super Edge-Magic Labeling

instances, [8, 9]. Figure 1 shows a reverse super edge-magic labeling on graph tree on 6 vertices with the reverse magic constant $k = 2$.

In the following section, we suggest two schemes for secret sharing based on an reverse super edge-magic labeling on a star S_n . The distribution and reconstruction algorithms in this schemes is calculated to work based on our knowledge of reverse super edge-magic labeling, in specific on stars .

3. Proposed schemes

Since the notion of secret sharing scheme introduced, various different types of secret sharing schemes have been proposed (by many authors). They used mathematical structures, such as vector spaces, polynomial, block design, room squares [5], and latin squares [7], to design secret sharing schemes. In this paper, we proposed two schemes for secret sharing based on edge-magic total labeling on graphs.

3.1 Supervisional secret sharing scheme

In supermarkets or banks, when a cashier or teller wants to change or cancel an (important) transaction, on their computer, they will need their supervisor approval to do such a thing. The supervisor will enter a password and then the cashier or teller will complete the entry by his/her own password. In this situation, we can see that the supervisor has more power password then the cashier or teller.

Based on this situation, we build supervisional secret sharing scheme that will share a secret among one supervisor and set of participants under his/her supervision (staffs). In the following we propose an algorithm for this purpose.

Algorithm 3.1

Distribution Algorithm

Input:

- The length of bank manager's share, n_1 (n_1 even).

- The length of bank employee's share, n_2 (n_2 even).
- The number of bank employee, r .

Steps:

- 1) Build a reverse edge magic labelling on star S_n .
- 2) Select the label randomly for the center from the integer set $(1, n+1, 2n+1)$, then we have a particular sum k .
- 3) Compute the labelling.
- 4) Set k as a secret

Output:

- The share for the bank manager, s_0 (s_0 even).
- The share for each his/her bank employee, p_i for $i = 1, 2, \dots, r$.

Reconstruction Algorithm

Input:

- 1) The share of the supervisor, s_0 (s_0 even)
- 2) The share of one of the staff, L .
- 3) The size of the star, n (kept by the system)

Steps

- 1) Built the reverse edge magic number as $H = f(xy) - \{f(x) + f(y)\}$.
- 2) If H is the reverse edge magic constant k , then the secret is revealed, otherwise secret is not revealed.

Output: The secret is revealed or not.

In this scheme, the secret is an edge-magic total labeling on graph S_n . The secret will be shared to a bank manager and his/her employee.

3.2 Departmental secret sharing scheme

In institutions with several departments, in some situation, to do an agreement, it will need some approval, those are from the head of the institution and faculty from each department. They give their approval by signing the agreement.

Based on this, we build departmental secret sharing scheme that will share a secret among one head of institution and several faculties. These sets represent the departments and the faculties be the authorized representatives.

Algorithm 3.2

Distribution Algorithm (with two departments)

Input:

- The length of share for the director.
- The length of share for each HOD in Department A .
- The length of share for each HOD in Department B .
- The number of Faculties in Department A .
- The number of Faculties in Department B .

Steps:

- 1) Build a reverse edge magic labelling on Tree $\langle K_{1,n_1}, K_{1,n_2} \rangle$
- 2) Select , at random, a reverse edge magic labelling k on the Tree $\langle K_{1,n_1}, K_{1,n_2} \rangle$
- 3) Set λ as a secret.

Output:

- descriptionThe share for the director.
- The share for Faculty in Department A .
- The share for Faculty in Department B .

Reconstruction Algorithm

Input:

The share of Faculty in Department A.

The share of Faculty in Department B.

The share of Director.

The size of a tree

Steps

- 1) Define $k = f(xy) - \{f(x) + f(y)\}$
- 2) If $k = \lambda$ then secret is revealed, then otherwise secret is not revealed.

4. Conclusion

This paper proposed two level schemes of secret sharing based on reverse edge-magic labeling on graph G . In these methods, the secret is a chosen reverse edge-magic labeling, in particular, on star S_n and tree $\langle K_{1,n_1}, K_{1,n_2} \rangle$. To distribute the shares, the algorithms work based on a reverse edge magic labeling. The reconstruction algorithms for these schemes are based on our knowledge on k for star/ Tree.

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Edge maximal W_7 -free graphs

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Abstract. Extremal graph theory is one of the most important subjects in graph theory. In this paper we give an upper bound to the number of edges of graphs which are W_7 -free. In fact we prove that if G is a graph on n -vertices which is W_7 -free, then

$$|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2.$$

Keywords: extremal graphs, wheels.

1. Introduction

In this paper we only consider finite, undirected simple graphs. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. A cycle on n -vertices is denoted by C_n . The degree of a vertex v in $V(G)$ is denoted by $d(v)$. Moreover, we denote the minimum degree of vertices of G by $\delta(G)$ and the maximum degree by $\Delta(G)$. Also, we let $\varepsilon(G)$ and $v(G)$ denote $|E(G)|$ and $|V(G)|$, resp. The graph formed by taking a cycle C_{n-1} on $n-1$ vertices, and a vertex $u \notin V(C_{n-1})$ by joining u to each vertex of C_{n-1} is called a wheel and is denoted by W_n . We say that a graph G is H -free if G does not contain a subgraph isomorphic to the graph H as a subgraph. If $u \in V(G)$, then we denote by $N_G(u)$ or simply by $N(u)$ the set of vertices of G adjacent to u and we denote by $N[u]$ the set $N(u) \cup u$. Two nonadjacent vertices u, v in $V(G)$ are called twins if $N(u) = N(v)$. If H_1 and H_2 are vertex disjoint subgraphs of a graph G , then we let $E(H_1 \cup H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\varepsilon(H_1, H_2) = |E(H_1 \cup H_2)|$. Let $\mathcal{F}$ be a set of graphs and n be a positive integer and let $\mathcal{G}(n, \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$ -free graphs on n -vertices, and

$$f(n, \mathcal{F}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n, \mathcal{F})\}.$$

The branch of graph theory which aims at finding the values of the function $f(n, \mathcal{F})$ is extremal graph theory which was initiated by Turan's [16] in 1941. Many researchers have studied these problems; for example, see [1]–[5], [9], [11], [13]–[17]. Next, we state some of the results found.

Theorem 1.1 (Erdős-stone-simonovite). *Let $\mathcal{F}$ be any finite set of graphs and let r be the minimum number of chromatic number of F in $\mathcal{F}$, then*

$$f(n, \mathcal{F}) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$

Theorem 1.2 ([15]). *Let G be a graph on n -vertices such that F is wheel free, then $\varepsilon(G) \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor$.*

Al-Rhayel et al. in [1] proved that $f(n, W_5) \leq \lfloor \frac{n-2}{4} \rfloor + \lfloor \frac{s}{4} \rfloor$ for $n \geq 3$ where $s = n$ if $n \neq 4k + 2$ and $s = n - 1$ if $n = 4k + 2$ and

$$f(n, W_6) \leq \lfloor \frac{n^2}{3} \rfloor$$

for $n \geq 6$.

Theorem 1.3 ([7]). *Let G be a graph on n -vertices with $\varepsilon(G) > \lfloor \frac{n^2}{4} \rfloor$, then G contains a cycle of every length L with $3 \leq L \leq \lfloor \frac{n+3}{2} \rfloor$.*

Haggkvist et al. [12] proved that: $f(n, C_r) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 1$, for all r . Jia [14] proved that $f(n, C_5) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3$ for $n \geq 9$. Bataineh et al. [5] proved that

$$f(n, \theta_7) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 3.$$

Jaradat et al. [13] proved that $f(n, \theta_{2k+1}) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3$.

Finally, we state Turan's theorem [16].

Theorem 1.4. *Let G be a graph on n -vertices without a k -clique, then*

$$|E(G)| \leq \frac{(k-2)n^2}{2(k-1)}.$$

2. Edge maximal W_7 -free graphs on $n(\leq 10)$ -vertices

Throughout this section, let G be a graph on n -vertices ($n \leq 10$); clearly, if $n \leq 6$, then G has no W_7 as a subgraph, so we will only deal with the cases $n = 7, 8, 9$, and 10. Most of this section's results can be found in [2].

Theorem 2.1. *Let G be a graph on 7-vertices if G is W_7 -free then $|E(G)| \leq 17$.*

Proof. Let $\{x_1, \dots, x_7\}$ be the vertices of G : if G contains K_6 as a subgraph, let $\{x_1, \dots, x_6\}$ be the vertices of K_6 ; hence, x_7 is the seventh vertex of G , then clearly x_7 can be adjacent to at most two vertices of K_6 , and hence $|E(G)| \leq 15 + 2 = 17$. Figure 1 below shows such a graph: notice that the addition of any new edge to this graph produces W_7 as a subgraph of G .

If G does not contain K_6 as a subgraph, but contains K_5 : let $\{x_1, \dots, x_5\}$ be the vertices of K_5 . Now $x_6 \in V(G)$, if $|N_{K_5}(x_6)| = 5$, then G contains K_6

which is a contradiction, hence $|N_{K_5}(x_6)| \leq 4$. In fact, it is easy to check that we may assume $N_{K_5}(x_6) = \{x_1, \dots, x_4\}$, $x_7 \in V(G)$. Figure 2 shows such a graph G , in which the addition of any new edge produces K_6 or W_7 as a subgraph of G , and clearly $|E(G)| = 16 \leq 17$. If G does not contain K_5 but contains K_4 as a subgraph of G , let $\{x_1, \dots, x_4\}$ be the vertices of K_4 , $x_5 \in V(G)$, if $N_{K_4}(x_5) = \{x_1, \dots, x_4\}$, then G contains K_5 , which is a contradiction. Hence, we may assume that $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$ and let G_1 be the resulting graph on $\{x_1, \dots, x_5\}$. If $|N_{G_1}(x_6)| = 5$, then $G[x_1, \dots, x_6]$ contains K_5 as a subgraph, also a contradiction. Hence $|N_{G_1}(x_6)| \leq 4$, and clearly, we may assume $|N_{G_1}(x_6)| = \{x_1, x_2, x_4, x_5\}$. Let G_2 be the resulting graph on $\{x_1, \dots, x_6\}$. Now $x_7 \in V(G)$, and we may assume $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$. Figure 3 displays such a graph G , where the addition of any new edge to G produces a K_5 or a W_7 as subgraph of G , hence $|E(G)| = 6 + 3 + 8 = 17$. $\square$

Theorem 2.2. *Let G be a graph on 8 vertices, if G is W_7 -free, then $|E(G)| \leq 21$.*

Proof. Let $\{x_1, \dots, x_8\}$ be the vertices of G . If G contains K_6 as a subgraph of G , let $\{x_1, \dots, x_6\}$ be the vertices of K_6 . $x_7, x_8 \in V(G)$ and, as above (case $n = 7$), $|N_{K_6}(x_i)| \leq 2$, $i \in \{7, 8\}$. Clearly, we may assume $N_{K_6}(x_7) = \{x_1, x_2\}$, $N_{K_6}(x_8) = \{x_3, x_4\}$, and $x_7x_8 \in E(G)$. Figure 4 shows such a graph G , where the addition of any new edge to G produces W_7 as a subgraph of G , hence $|E(G)| = 20$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \dots, x_5\}$ be the vertices K_5 . $x_6 \in V(G)$, if $|N_{K_5}(x_6)| = 5$, then we get K_6 as a subgraph of G , hence $|N_{K_5}(x_6)| \leq 4$, and as in case $n = 7$, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$ and $N_{G_1}(x_7) = \{x_1, x_6\}$, where G_1 is the resulting graph on $\{x_1, \dots, x_6\}$. Let H be the resulting graph on $x_1, \dots, x_7$. Now $x_8 \in V(G)$ and clearly we may assume $N_H(x_8) = \{x_1, x_6\}$, and $x_7x_8 \in E(G)$. Notice that Figure 5 displays such a graph G , where the addition of any new edge to this graph produces K_6 or W_7 as a subgraph of G ; hence $|E(G)| = 19$. If G does not contain K_5 but contains K_4 as a subgraph, let $\{x_1, \dots, x_4\}$ be the vertices of K_4 . Now $x_5, x_6, x_7 \in V(G)$ and by analysis similar to that used for case $n = 7$, we may assume $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$, $N_{G_1}(x_6) = \{x_1, x_2, x_4, x_5\}$ and $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$, where G_1 is the resulting graph on $\{x_1, \dots, x_5\}$, and G_2 is the resulting graph on $\{x_1, \dots, x_6\}$. Let G_3 be the resulting graph on $\{x_1, \dots, x_7\}$ (see figure 6 below). $x_8 \in V(G)$, and one can check that we may assume $N_{G_3}(x_8) = \{x_3, x_4, x_5, x_6\}$. Figure 7 shows such a graph G , where the addition of any new edge to this graph produces K_5 or W_7 as a subgraph of G , hence $|E(G)| = 21$. $\square$

Theorem 2.3. *Let G be a graph on 9-vertices if G is W_7 -free then*

$$|E(G)| \leq 25.$$

Proof. Let $\{x_1, \dots, x_9\}$ be the vertices of G . If G contains K_6 as a subgraph, let $\{x_1, \dots, x_6\}$ be the vertices of G . Notice that if $v \in \{x_7, x_8, x_9\}$, then

$|E(v, K_6)| \leq 2$, and if $B = G[x_7, x_8, x_9]$, then $|E(B)| \leq 3$, hence $|E(G)| \leq 15 + 6 + 3 = 24$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \dots, x_5\}$ be the vertices of K_5 . $x_6 \in V(G)$. Clearly, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let G_1 be the resulting graph on $\{x_1, \dots, x_6\}$. Now $\{x_7, x_8, x_9\} \subseteq V(G)$; clearly, $|N_{G_1}(x_i)| \leq 2$, for $i = 7, 8, 9$ and we may assume x_7x_8, x_7x_9 and x_8x_9 are edges in G ; hence, $|E(G)| \leq 10 + 4 + 6 + 3 = 23$. Figure 8 below shows such a graph G , where the addition of any new edge produces K_6 or W_7 as a subgraph of G .

If G does not contain K_5 but contains K_4 as a subgraph, then by analysis as in previous cases, one can easily check that the graph G given in Figure 9 ensures that G contains K_4 and the addition of any new edge to G produces K_5 or W_7 as a subgraph of G , and $|E(G)| \leq 6 + 3 + 16 = 25$. $\square$

Theorem 2.4. *Let G be a graph on 10-vertices, if G is W_7 -free then*

$$|E(G)| \leq 30.$$

Proof. Let $\{x_1, \dots, x_{10}\}$ be vertices of G . If G contains K_6 as a subgraph, then let $\{x_1, \dots, x_6\}$ be the vertices of K_6 . Notice that if $v \in \{x_7, x_8, x_9, x_{10}\}$, then $|E(v, K_6)| \leq 2$ and if $B = G[x_7, x_8, x_9, x_{10}]$, then $|E(B)| \leq 6$, hence $|E(G)| \leq 15 + 8 + 6 = 29$. If G does not contain K_6 but contains K_5 as a subgraph, let $\{x_1, \dots, x_5\}$ be the vertices of K_5 . $x_6 \in V(G)$, and clearly we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let G_1 be the resulting graph on $\{x_1, \dots, x_6\}$, and following an analysis like for $n = 9$, we notice that $\{x_7, x_8, x_9, x_{10}\} \subseteq V(G)$, and $G[x_7, x_8, x_9, x_{10}] = K_4$ and $|N_{G_1}(x_i)| \leq 2$, $i = 7, 8, 9, 10$, hence $|E(G)| \leq 10 + 4 + 8 + 6 = 28$. If G does not contain K_5 but contains K_4 as a subgraph, let $\{x_1, \dots, x_4\}$ be the vertices of K_4 , then following an analysis as in previous cases (for more details consult [2]), we get $|E(G)| \leq 30$. $\square$

Remark 2.5. The bound in Theorem 2.4 is sharp as illustrated in example 2.1 below.

Example 2.1. Let G_1 be the graph shown in Figure 10, where each vertex in H_1 is joined to each vertex of H_2 , then clearly G_1 has exactly 30 edges and is W_7 -free. and the addition of any new edge to G_1 produces W_7 as a subgraph of G_1 .

3. Main results

Let G be a graph of order $n \geq 11$, following Moon [15]. Let H_n be the class of graphs obtained by partitioning the vertices of G into two sets, P and Q , such that P has $\lfloor \frac{n+1}{2} \rfloor$ vertices and Q has $\lfloor \frac{n}{2} \rfloor$ vertices, and there are as many edges joining pairs of vertices in P and Q as consistent with the requirement that no two of these edges have a vertex in common, and each vertex in P is joined to each vertex in Q .

Remark 3.1. 1. If two edges, say in P , have a common vertex, say x_1x_2, x_2x_3 , then let y_1y_2, y_3y_4 be two edges in Q , and notice that x_2 is adjacent to every vertex of the 6-cycle $(x_1y_1y_2x_3y_4y_3x_1)$. This produces W_7 as subgraph of G .

2. Figure 11 below shows a graph G in H_n : clearly G is W_7 -free and $\varepsilon(G) \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$, for all $n \geq 7$. So we conclude that:

$$f(n, W_7) \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor.$$

Lemma 3.2. *Let G be a W_7 -free graph on n -vertices ($n \geq 7$) such that K_6 is a subgraph of G , then $|E(v, K_6)| \leq 2$ for all $v \in V(G - K_6)$.*

Proof. Let $\{x_1, x_2, \dots, x_6\}$ be the vertices of K_6 , if $|N_{K_6}(v)| = 3$. Then we may assume $N_{K_6}(v) = \{x_1, x_2, x_3\}$; clearly x_2 is adjacent to every vertex of the 6-cycle $(x_1vx_3x_4x_5x_6)$, and hence W_7 is produced as a subgraph of G , which is a contradiction. $\square$

In Remarks 3.3 and 3.4 below we construct a class of graphs F_n such that if G belongs to F_n , then G is W_7 -free and $|E(G)| \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$.

Remark 3.3. Let G be a graph of order $n \geq 11$. Let F_n be the class of graphs obtained by partitioning the vertices of G into two sets P, Q such that P has $\lfloor \frac{n+1}{2} \rfloor$ vertices and Q has $\lfloor \frac{n}{2} \rfloor$ vertices, and there are many edges joining vertices in P and Q , as consistent with the following requirements.

1. P has no vertices of degree 3.
2. If H has component of P , then H is one of the following: a single vertex, a P_2 , a C_3 or a C_4 .
3. Q has exactly one edge joining any two vertices of Q and the rest of the vertices in Q are vertices of degree 0.
4. Each vertex in P is joined to each vertex in Q ; figure 12 below shows a graph G in the class F_n .

Remark 3.4. 1. Remark 3.3 above shows that a graph G in F_n is W_7 -free.

2. If C_5 is a component in P , $\{x_1, x_2, x_3, x_4, x_5\}$ be this cycle, and let y_1, y_2 be the edge in G , then y_1 is adjacent to each vertex of the cycle $(y_2x_1x_2x_3x_4x_5)$; hence, W_7 is produced as a subgraph of G .

3. If P has a vertex x_1 of degree 3, then let $N(x_1) = \{x_2, x_3, x_4\}$, let y_1y_2 be the edge in G , let y_3 be a vertex of degree 0 in G and notice that under the above construction, x_1 is adjacent to every vertex of the cycle $(x_2y_3x_4y_2x_3y_1)$, and this produces W_7 as a subgraph of G .

4. The maximum number of edges in P is obtained if all components of P are either C_3 's, C_4 's or a combination of both, and hence this implies that each vertex of P has degree 2, thus

$$|E(G)| \geq \lfloor \frac{n+2}{2} \rfloor \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{4} \rfloor + 1 \geq \lfloor \frac{n^2 + 2n + 1}{4} \rfloor + 1 \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1.$$

Theorem 3.5. *Let G be a W_7 -free graph on n - vertices ($n \geq 10$). If G contains exactly one copy of K_6 as a subgraph, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. We use induction on n , for $n = 7$ see Theorem 2.1. So assume the result is true for all graphs having less than n -vertices, and assume G is a graph on n -vertices. Let $B = G - K_6$, then B is a graph with $|V(B)| < n$, hence

$$|E(B)| \leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2,$$

notice that if $x \in V(B)$, then $E(x, K_6) \leq 2$, since otherwise W_7 is produced as a subgraph of G ; Thus $|E(B, K_6)| \leq 2(n-6)$, and

$$\begin{aligned} |E(G)| &\leq |E(K_6)| + |E(B, K_6)| + |E(B)|, \\ &\leq 15 + 2(n-6) + \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2, \\ &\leq \frac{12 + 8n + n^2 - 12n + 36 + n - 5 + 8}{4}, \\ &= \frac{n^2 + 2n + 2 - 5n + 49}{4}, \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 10. \end{aligned}$$

□

Theorem 3.6. *Let G be a W_7 -free graph on n - vertices ($n \geq 8$). If G contains exactly one copy of K_5 as a subgraph, but it does not contain K_6 , then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. we use induction on n . The result being proved for $n = 7, 8, 9, 10$ (see Theorems 2.1, 2.2, 2.3 and 2.4). Assume the result holds for all graphs having less than n -vertices, and let G is a graph on n -vertices. Let G_1 be the graph on 6-vertices shown in Figure 13. Let $B_1 = G - G_1$, then $|V(B_1)| < n$, and hence $|E(B_1)| \leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2$.

If $x \in V(B_1)$, then clearly $E(x, K_6) \leq 2$, as otherwise either K_6 or W_7 is produced as a subgraph of G , hence $|E(B_1, K_6)| \leq 2(n-6)$; therefore

$$\begin{aligned} |E(G)| &\leq |E(G_1)| + |E(B_1)| + |E(B_1, K_6)|, \\ &\leq 14 + \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 2 + 2(n-6), \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 8. \end{aligned}$$

□

Theorem 3.7. *Let G be a W_7 -free graph on n - vertices ($n \geq 10$). If G contains K_4 but it does not contain K_5 as a subgraph, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2$.*

Proof. we use induction on n ($n \geq 10$). Let $\{x_1, \dots, x_n\}$ be the vertices of G , and $\{x_1, \dots, x_4\}$ be the vertices of K_4 . $x_5 \in V(G)$, hence $|N_{K_4}(x_5)| \leq 3$, otherwise we get K_5 as a subgraph of G . Clearly we may assume that $N_{K_4}(x_5) = \{x_1, x_2, x_3\}$. Let G_1 be resulting graph on $\{x_1, x_2, x_3, x_4, x_5\}$, note that $x_6 \in V(G)$. If $|N_{G_1}(x_6)| = 5$, then K_5 is produced as a subgraph of G , hence $|N_{G_1}(x_6)| \leq 4$, and clearly we may assume $N_{G_1}(x_6) = \{x_1, x_2, x_4, x_5\}$. Let G_2 be a resulting graph on $\{x_1, \dots, x_6\}$, now $x_7 \in V(G)$, if $|N_{G_2}(x_7)| = 5$, then K_5 is produced as a subgraph of G , hence $|N_{G_2}(x_7)| \leq 4$, and we may assume that $N_{G_2}(x_7) = \{x_3, x_4, x_5, x_6\}$. Let G_3 be the resulting graph on $\{x_1, \dots, x_7\}$. Notice that the addition of any new edge to G_3 either produces K_5 or W_7 as a subgraph of G . Now $x_8 \in V(G)$, and we may assume that $N_{G_3}(x_8) = \{x_3, x_4, x_5, x_6\}$. Let G_4 be the resulting graph on $\{x_1, \dots, x_8\}$, and notice that the addition of any new edge to G_4 either produces K_5 or W_7 as a subgraph of G . Also $x_9 \in V(G)$, and by analysis as above we conclude that $|N_{G_4}(x_9)| \leq 4$, and may assume that $N_{G_4}(x_9) = \{x_1, x_2, x_7, x_8\}$. Let G_5 be the resulting graph on $\{x_1, \dots, x_9\}$, and notice that the addition of any new edge to G_5 (see Figure 14) produces either K_5 or W_7 as a subgraph of G , hence $|E(G_5)| \leq 25$. Let $B = G - G_5$, hence $|V(B)| = n - 9 < n$, and by induction hypothesis we have:

$$|E(B)| \leq \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2.$$

Note that if $x \in V(B)$, then $|E(x, G_5)| \leq 4$, hence $|E(B, K_5)| \leq 4(n-9)$, and we conclude that:

$$\begin{aligned} |E(G)| &\leq |E(G_5)| + |E(B)| + |E(B, G_5)|, \\ &\leq 25 + \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2 + 4(n-9), \\ &\leq 25 + \lfloor \frac{(n-9)^2}{4} \rfloor + \lfloor \frac{n-8}{2} \rfloor + 2 + 4n + 36, \\ &\leq \lfloor \frac{n^2 - 18n + 81}{4} \rfloor + \lfloor \frac{2n - 16}{4} \rfloor + 4n - 9, \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{-18n + 81 + 2n - 16 + 16n - 36}{4} \rfloor, \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + \frac{27-2n}{4}, \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, \quad \text{if } n \geq 10. \end{aligned}$$

□

Theorem 3.8. *Let G be a graph of order $n \geq 11$, if G is W_7 -free, then*

$$|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2.$$

Proof. To prove this theorem we use strong mathematical induction on m , where m is the number of vertex disjoint K_5 subgraphs contained in G . If $m = 0$, then done by Theorem 3.7, hence we assume that the result holds for all values less than m . We need to prove that the result holds for m , let G be a graph on n -vertices having m vertex disjoint K_5 subgraphs. Let K_5 be a subgraph of G , $\{x_1, x_2, x_3, x_4, x_5\}$ be the vertices of K_5 , and $B = G - K_5$.

Case 1. There exist $x_6 \in V(B)$ such that x_6 is adjacent to all vertices of K_5 , hence K_6 is produced as a subgraph of G . let $B_1 = G - K_6$; then $\varepsilon(B_1, K_6) \leq 2$, for all $x \in B_1$, hence $\varepsilon(B_1, K_6) \leq 2(n - 6)$, and by induction hypothesis

$$\varepsilon(B_1) \leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 1,$$

hence

$$\begin{aligned} \varepsilon(G) &\leq \varepsilon(B_1) + \varepsilon(K_6, B_1) + \varepsilon(K_6) \\ &\leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 1 + 2(n-6) + 15 \\ &\leq \lfloor \frac{n^2 - 12n + 36}{4} \rfloor + \lfloor \frac{2n-10}{4} \rfloor + 2n + 4 \\ &\leq \lfloor \frac{n^2}{4} \rfloor - 3n + 9 + \lfloor \frac{n+1}{2} \rfloor - 3 + 2n + 4 \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 1 \quad \text{if } -n + 10 \leq 1 \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 1 \quad \text{if } n \geq 9 \\ &\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2 \quad \text{if } n \geq 9. \end{aligned}$$

Case 2. There exist $x_6 \in V(B)$ such that x_6 is adjacent to four vertices of K_5 ; clearly, we may assume $N_{K_5}(x_6) = \{x_1, x_2, x_3, x_4\}$. Let H be the resulting graph on $\{x_1, \dots, x_6\}$, and let $B_2 = G - H$. We claim that $\varepsilon(y, H) \leq 2$ for all y in $V(B_2)$. To prove this claim, assume $|N_H(y)| = 3$, since $N_H(x_5) = N_H(x_6)$ and $N_H(x_1) = N_H(x_2) = N_H(x_3) = N_H(x_4)$; then $N_H(y)$ must be one of the following sets: $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_5\}$, $\{x_1, x_5, x_6\}$, if $N_H(y) = \{x_1, x_2, x_3\}$. Then, x_2 is adjacent to every vertex of the 6-cycle $(yx_3x_6x_4x_5x_1)$. If $N_H(y) = \{x_1, x_2, x_5\}$, then x_2 is adjacent to every vertex of the 6-cycle $(yx_1x_6x_3x_4x_5)$. If $N_H(y) = \{x_1, x_5, x_6\}$, then x_1 is adjacent to every vertex of the 6-cycle $(yx_6x_2x_3x_4x_5)$. In any case, W_7 is produced as a subgraph of G and this proves our claim. Now

$$\begin{aligned} \varepsilon(G) &\leq \varepsilon(B_2) + \varepsilon(H, B_1) + \varepsilon(H) \\ &\leq \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 1 + 2(n-6) + 14 \\ &\leq \lfloor \frac{n^2}{4} \rfloor - 3n + 9 + \lfloor \frac{n+1}{2} \rfloor - 3 + 1 + 2n + 2 \end{aligned}$$

$$\begin{aligned}
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 1 && \text{if } -n+9 \leq 1 \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 1 && \text{if } n \geq 8 \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2 && \text{if } n \geq 8.
\end{aligned}$$

Case 3. There exists a vertex $x_6 \in V(B)$ such that x_6 is adjacent to 3-vertices of K_5 ; clearly, we may assume these vertices to be $\{x_1, x_2, x_3\}$. Let H_1 be the resulting graph on $\{x_1, \dots, x_6\}$, $B_3 = G - H_1$ and $y \in V(B_3)$. We claim that $\varepsilon(y, H_1) \leq 3$ for all $y \in V(B_3)$. To prove this claim, notice that if $|N_{H_1}(y)| = 4$, then $x_6 \in N_{H_1}(y)$, since otherwise y is adjacent to 4 vertices of K_5 , and we go back to case 2 above and since $N_{H_1}(x_1) = N_{H_1}(x_2) = N_{H_1}(x_3)$ and $N_{H_1}(x_4) = N_{H_1}(x_5)$ then we may assume that $N_{H_1}(y)$ to be one of the following sets: $\{x_1, x_2, x_3, x_6\}$ or $\{x_1, x_2, x_5, x_6\}$. If $N_{H_1}(y) = \{x_1, x_2, x_5, x_6\}$, then x_2 is adjacent to every vertex of the 6-cycle $(yx_1x_5x_4x_3x_6)$. If $N_{H_1}(y) = \{x_1, x_2, x_3, x_6\}$, then again x_2 is adjacent to every vertex of 6-cycle $(yx_1x_5x_4x_3x_6)$. In any case, W_7 is produced as a subgraph of G , hence $|N_{H_1}(y)| \leq 3$. Clearly, if $N_{H_1}(y) = \{x_1, x_2, x_3\}$, then this bound is achieved, and we conclude that $\varepsilon(y, H_1) \leq 3$ for all $y \in V(B_3)$, and hence

$$\begin{aligned}
\varepsilon(G) &\leq \varepsilon(H_1) + \varepsilon(B_3) + \varepsilon(B_3, H_1) \\
&\leq 13 + \lfloor \frac{(n-6)^2}{4} \rfloor + \lfloor \frac{n-5}{2} \rfloor + 1 + 3(n-6) \\
&\leq \lfloor \frac{n^2 - 12n + 36}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor - 3 + 1 + 3n - 18 \\
&\leq \lfloor \frac{n^2}{4} \rfloor - 3n + 9 + \lfloor \frac{n+1}{2} \rfloor - 3 + 14 + 3n - 18 \\
&\leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor + 2, && \text{for all } n.
\end{aligned}$$

□

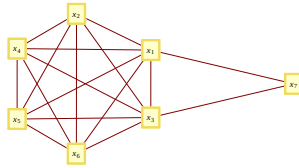


Figure 1:

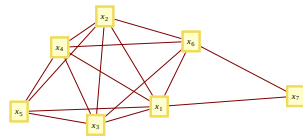


Figure 2:

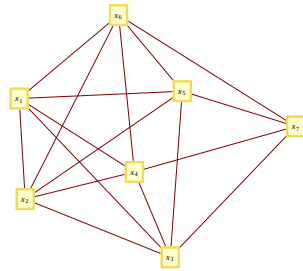


Figure 3:

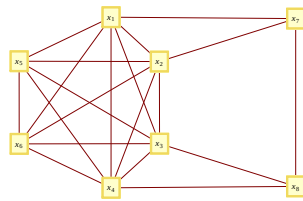


Figure 4:

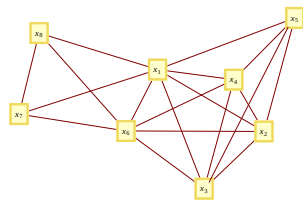


Figure 5:

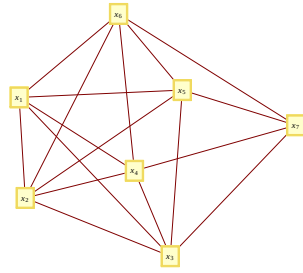


Figure 6:

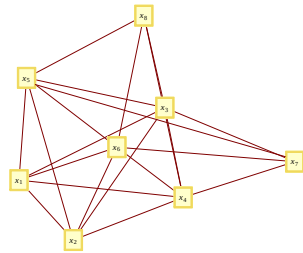


Figure 7:

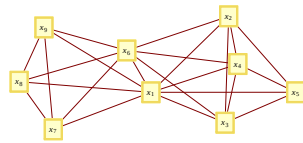


Figure 8:

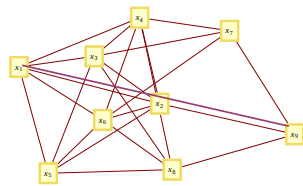


Figure 9:

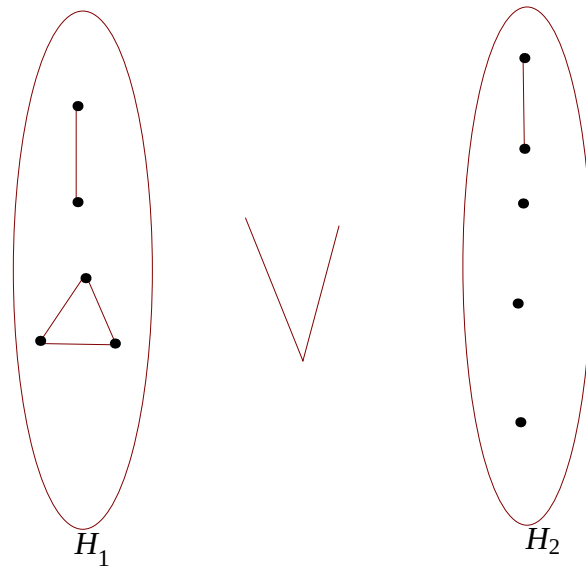


Figure 10:

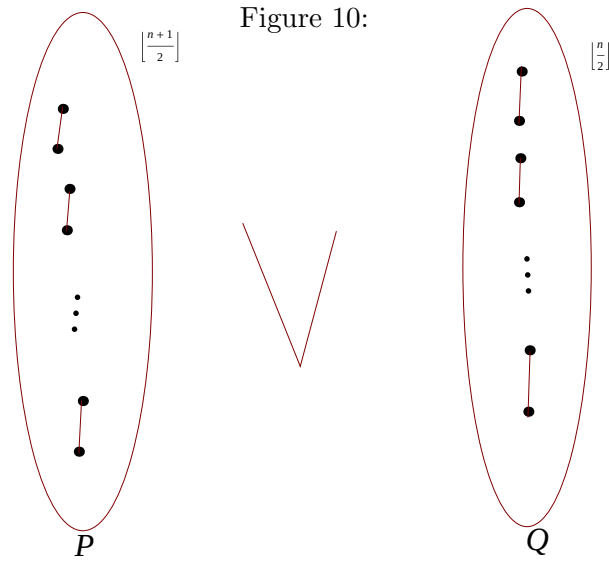


Figure 11:

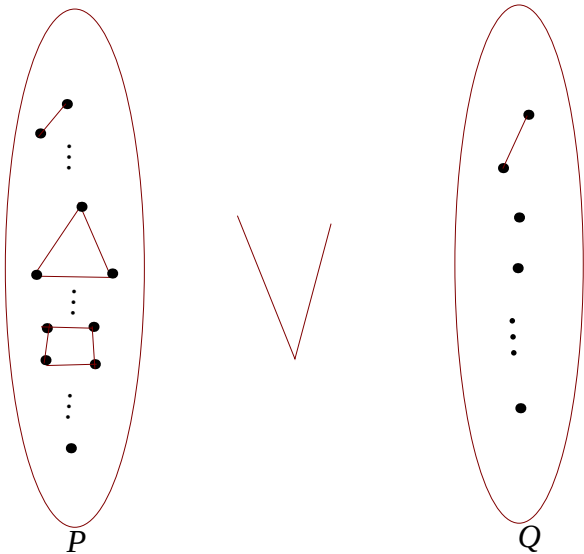


Figure 12:

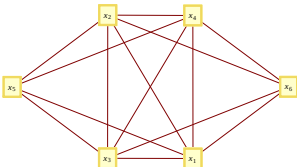


Figure 13:

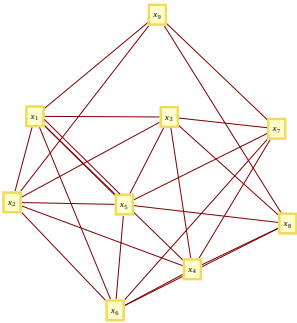


Figure 14:

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On Lie ideals of inverse semirings

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Abstract. The purpose of this paper is to study Lie ideals of inverse semirings, thereby extending a few well-known results of I.N Herstein and C.Lanski in the setting of inverse semirings.

Keywords: inverse semiring, Lie ideals, commutators, prime inverse semiring.

1. Introduction

Lie ring structure of an associative ring R was first considered by I.N Herstein in 1950's. He studied Lie ideals and their relationship with R in series of papers ([9],[10],[11]). In ensuing years, the study of Lie ideals generalized to prime and semiprime rings and rings with involution. Herstein's work on Lie ideals proved to be a source of motivation for many researchers to look into the Lie theory in various other directions and settings ([1],[3],[4],[5],[8],[11],[14],[18],[19],[20]). Recently, M.A Javed and M.Asalam [16] introduced the notion of Lie ideals in inverse semirings and investigated some commutativity conditions on inverse semirings with the help of Lie ideals and derivations.

Our objective is to explore and generalize Lie type results of rings in the setting of semirings, thereby we extend a few results on Lie ideals of [10],[14] and [18] to inverse semirings. These results will be helpful in investigating the theory of Lie derivation and higher derivation of Lie ideals of semirings.

Herstein [10] proved a result on Lie ideals which says that; A non-zero Lie ideal U of a 2-torsion free simple ring R is either contained in center of R or contains a non-zero ideal of R . Later, in dealing with some problems on von Neumann algebra, Herstein [14] extended this result to semiprime rings as follows; Let R be a semiprime 2-torsion free ring and V be subring of R . Suppose U is a Lie ideal of R such that $[V, U] \subset V$ then either $[V, U] = 0$ or V contains non-zero ideal of R . Lanski [18] proved the same result with prime rings (Theorem 12 of [18]). In this paper, we establish these two main results of Lie ideals for prime inverse semirings.

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By S , we mean a semiring with commutative addition and an absorbing zero. A semiring S is called an inverse semiring[17] if for every $a \in S$ there exists a unique element $\acute{a} \in S$ such that $a + \acute{a} + a = a$ and $\acute{a} + a + \acute{a} = \acute{a}$. Karvellas [17] proved that for all $a, b \in S$, $(a.b)^\acute{ } = \acute{a}.b = a.\acute{b}$ and $\acute{a}\acute{b} = ab$. Bandler and Petrich [2] studied inverse semirings with some conditions (A1)-(A4). Throughout this paper, S will denote inverse semiring which satisfies (A-2) condition of [2] i.e; for every $a \in S$, $a + \acute{a} \in Z(S)$, where $Z(S)$ is center of S . This class of inverse semiring is known as MA semiring[15], which is indeed useful in developing the Lie type theory of semirings and studying certain additive mappings in semirings (see for example [21], [22], [23]). Commutative inverse semirings and distributive lattices are natural examples of MA-semiring. Also, if R is a ring and T is a subset of all ideals of R then $S = \{(a, I) : a \in R, I \in T\}$ is MA-semiring(see [15]). For more examples, we refer readers to [15]. A semiring S is prime if $aSb = 0$ implies either $a = 0$ or $b = 0$. S is 2-torsion free if $2x = 0$, $x \in S$ implies $x = 0$. By [15], a commutator $[\cdot, \cdot]$ in inverse semiring is defined as $[x, y] = xy + \acute{y}x$. Unlike a commutator of a ring, it is not necessary that $[x, x] = 0$ for every $x \in S$. We will make use of fundamental identities of commutators $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = [x, z]y + x[y, z]$ and the jacobian identity $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (see [15]). By [16], an additively closed subset U of S is called Lie ideal of S if $[U, S] \subset U$. By $[U, V]$, we mean an additive subsemigroup of S generated by the elements of the form $[u, v]$, where $u \in U$ and $v \in V$. A Lie ideal U is said to be commutative Lie ideal if $[U, U] = 0$. An additive mapping d on S is called derivation if $d(xy) = d(x)y + xd(y)$, $x, y \in S$.

We shall need following two results in our arguments.

Lemma 1 (Lemma 1.2. of [21]). *If $x, y, z \in S$ then the following identities are valid:*

- (1) $[xy, x] = x[y, x]$, $[x, yx] = [x, y]x$, $[x, xy] = x[x, y]$, $[yx, x] = [y, x]x$;
- (2) $y[x, z] = [x, y]z + [x, y]\acute{z}$, $[x, y]z = \acute{y}[x, z] + [x, yz]$;
- (3) $x[y, z] = [xy, z] + [x, z]\acute{y}$, $[x, z]y = [xy, z] + \acute{x}[y, z]$.

Theorem 2 (Theorem 2.7 of [16]). *Let S be a 2-torsion free prime inverse semiring and d be a non-zero derivation of S . If $a \in S$ and $[d(S), a] = 0$. Then $a \in Z(S)$.*

We will make frequent use of the following lemma.

Lemma 3. *Let S be an inverse semiring and $a \in S$. If $[a, x] = 0$ for all $x \in S$. Then $a \in Z(S)$.*

Proof. *We have, $0 = [a, x] = ax + x\acute{a}$. Adding xa on both sides we have, $xa = ax + x(\acute{a} + a) = ax + (\acute{a} + a)x = ax$. Thus $a \in Z(S)$.*

The main results of this paper deal with the case in which S is prime and so we assume S to be prime and 2-torsion free, in the subsequent discussions.

Theorem 4. *Let U be a non-zero Lie ideal and subsemiring of S then either $U \subset Z(S)$ or U contains a non-zero ideal of S .*

Proof. Suppose that U is commutative, that is; $[U, U] = 0$. Let $a \in U$ and $s \in S$ then $[a, s] \in U$ and $[a, [a, s]] = 0$. Replacing s by $st, t \in S$ in last equation we get, $[a, s][a, t] = 0$. Replacing t by st in last expression, we have $0 = [a, s]s[a, t]$. But S is prime so $[a, s] = 0$. Hence by Lemma 3, $U \subset Z(S)$.

Suppose that U is non-commutative. Let $a, b \in U$ such that $[a, b] \neq 0$. As U is a Lie ideal so for $s \in S$ we have,

$$[a, sb] = [a, s]b + s[a, b] \in U.$$

U is also subsemiring of S so $[a, s]b \in U$. Thus we have, $[a, s](b + \acute{b}) + s[a, b] \in U$ or $(as + s\acute{a})(b + \acute{b}) + s[a, b] \in U$ or $(s + \acute{s})ab + s(b + \acute{b})\acute{a} + sab + s\acute{b}a \in U$ or $s[a, b] \in U$.

For $t \in S$ we have, $[s[a, b], t] \in U$ or $s[a, b]t + \acute{t}s[a, b] \in U$. From this, $s[a, b]t + \acute{t}s[a, b] + ts[a, b] \in U$ or $s[a, b]t \in U$. Clearly, $S[a, b]S$ is an ideal of S contained in U . If we say $S[a, b]S = (0)$ then either $S[a, b] = (0)$ or $[a, b] = 0$, which gives a contradiction. Hence, $S[a, b]S$ is non-zero ideal contained in U . This establishes the theorem.

Corollary 5. *Let $a \in S$ and $[a, [a, s]] = 0$ for all $s \in S$. Then $a \in Z(S)$.*

Lemma 6. *Let I be a non-zero ideal of S such that $[a, I] = 0, a \in S$. Then $a \in Z(S)$.*

Proof. Let $s \in S$ and $b \in I$ then $[a, sb] = 0$. Thus we have, $[a, s]I = 0$. But S is prime so $[a, s] = 0$. Hence by Lemma 3, $a \in Z(S)$.

Lemma 7. *Let U be a Lie ideal of S then either $[U, S] = 0$ or $[T(S), S] = 0$, where $T(U) = \{x \in S : [x, U] = 0\}$.*

Proof. It is easy to see that $T(U)$ is a Lie ideal and subsemiring of S so by Theorem 4 either $[T(U), S] = 0$ or $T(U)$ contains a non-zero ideal, say I . If $I \subset T(U)$ then from Lemma 6, we obtain $[U, S] = 0$.

The following corollary immediately follows from Lemma 3.

Corollary 8. *Let U be a Lie ideal of S then either $U \subset Z(S)$ or $T(U) \subset Z(S)$.*

Lemma 9. *Let U_1 and U_2 be Lie ideals of S such that $[[U_1, U_2], S] = 0$. Then either $[U_1, S] = 0$ or $[U_2, S] = 0$.*

Proof. If $[U_1, U_2] = 0$ then by Lemma 7, either $[U_1, S] = 0$ or $[U_2, S] = 0$. Let $[U_1, U_2] \neq 0$. Suppose $[U_1, S] \neq 0$ and $[U_2, S] \neq 0$. Let $u \in U_1, v \in U_2$ and $s \in S$ then

$$[[u, [v, vs]], S] = 0.$$

Using Lemma 1 we have, $[[u, v][v, s] + v[u, [v, s]], S] = 0$. In particular, $[[u, v][v, s] + v[u, [v, s]], v] = 0$. From this, we obtain

$$(1.1) \quad [u, v][v, s]v + v[u, [v, s]]v + \acute{v}[u, v][v, s] + \acute{v}v[u, [v, s]] = 0.$$

By Lemma 3 and the given hypothesis we have, $[U_1, U_2] \in Z(S)$ which implies that $v[[u, [v, s]], v] + [u, v][[v, s], v] = 0$ and hence $[u, v][[v, s], v] = 0$. Therefore, $[u, v]S[[v, s], v] = S[u, v][[v, s], v] = 0$. But S is prime so $[[v, s], v] = 0$ which by Lemma 7 implies that either $[v, S] = 0$ or $[[v, s], S] = 0$. From Corollary 5, $[v, S] = 0$ which contradicts that $[U_2, S] = 0$. This completes the proof.

Corollary 10. *Let U_1 and U_2 be Lie ideals of S and $[[U_1, U_2], S] = 0$. Then either $U_1 \subset Z(S)$ or $U_2 \subset Z(S)$.*

Lemma 11. *Let U be a Lie ideal and $V \subset S$ such that $[[U, V], S] = 0$. Then either $[U, S] = 0$ or $[V, S] = 0$.*

Proof. Consider $T = \{x \in S : [[x, U], S] = 0\}$ then using jacobian identity we can show that T is a Lie ideal of S . Hence, by Lemma 9 either $[U, S] = 0$ or $[T, S] = 0$.

Lemma 12. *If U is a Lie ideal of S , $x \in S$ and $xUx = 0$ then either $x = 0$ or $U \subset Z(S)$.*

Proof. For $s \in S$ and $u \in U$, we have $x[u, s]x = 0$. Replacing s by u_1xs , $u_1 \in U$, we get $xuu_1xsx = 0$. But S is prime so either $x = 0$ or $xuu_1x = 0$. Suppose $x \neq 0$. Replacing s by u_1u_2xs in $x[u, s]x = 0$ we have xuu_1u_2sx . Continuing this, we arrive at $x\bar{U}x = 0$ where $\bar{U}$ is subsemiring of S generated by U . If $U \not\subset Z(S)$ then $\bar{U} \not\subset Z(S)$. By Theorem 4, $\bar{U}$ contains a non-zero ideal, say I . Hence $xIx = 0$ but S is prime so $x = 0$, a contradiction. Hence lemma is proved.

The next lemma is an extension of Lemma 10 of [18] in a canonical fashion.

Lemma 13. *Let $U \not\subset Z(S)$ be a Lie ideal of S and V be an additive subsemigroup of S and $[U, V] \subset V$. If $v^2 = 0, \forall v \in V$ then $V = 0$.*

Lemma 14. *Let $U \not\subset Z(S)$ be a Lie ideal of S and V be an additive subsemigroup of S such that $[V, U] \subset V$. If $[V, V] = 0$ then $V \subset Z(S)$.*

Proof. Put $I = [V, U]$ then $[I, I] = 0$. Let $a \in I, u \in U$ and $s \in S$ then using Lemma 1 we have, $0 = [a, [a, [u, us]]] = 2[a, u][a, [a, s]]$. S is 2-torsion free so $[a, u][a, [a, s]] = 0$. Replace s by au , we obtain

$$(1.2) \quad [a, u]^3 = 0, a \in I, u \in U.$$

Also,

$$(1.3) \quad [a, [a, u]] = a^2u + 2au\acute{a} + ua^2 = 0.$$

If $a \in I$ such that $a^3 = 0$ then from (3), $a^3ua + 2a^2u\acute{a}a + au\acute{a}a^3 = 0$ which gives $a^2ua^2 = 0$. By Lemma 12, $a^2 = 0$. Again, by (3) we get $a = 0$. Hence, if $a \in I$ and $a^3 = 0$ then $a = 0$. So from (2), $[a, u] = 0, u \in U, a \in I$. By Lemma 7, $[I, S] = 0$ and therefore $[V, S] = 0$, by Lemma 9.

Theorem 15. *Let U be a Lie ideal and $V \subset U$ be an additive subsemigroup of S such that $[V, U] \subset V$. If $[[V, V], S] = 0$ then $[V, U] = 0$.*

Proof. If $[V, V] = 0$ then by above Lemma $[V, S] = 0$ and hence $[V, U] = 0$, as desired. Suppose $v_1, v_2 \in V$ such that $\alpha = [v_1, v_2] \neq 0$ then $[\alpha, S] = 0$ or $\alpha \in Z(S)$, by Lemma 3. For $s \in S$, let $d(s) = [v_1, s]$ then by given hypothesis we have

$$[d^2(u), S] = 0, \forall u \in U.$$

For $x \in S$, let $u_1 = [v_2, x] \in U$ then $v_2u_1 \in U$. Thus we have $[d^2(v_2u_1), S] = 0$. But $d^2(v_2u_1) = v_2d^2(u_1) + 2\alpha d(u_1)$. Therefore, $[v_2d^2(u_1) + 2\alpha d(u_1), S] = 0$. In particular, $0 = [v_2d^2(u_1) + 2\alpha d(u_1), v_2] = 2\alpha[d(u_1), v_2]$ or $\alpha[d(u_1), v_2] = 0$.

As S is prime so non-zero elements of $Z(S)$ cannot be zero divisors in S , thus we have $[d(u_1), v_2] = 0$. Hence $[[v_2, d(x)], v_2] = 0$. Replacing x by v_1x in last expression and using it again, we have $0 = [[v_2, d(v_1x)], v_2] = [[v_2, v_1d(x)], v_2] = \alpha[v_2, d(x)] + v_1[[v_2, d(x)], v_2] + [\acute{\alpha}, v_2]d(x) + \acute{\alpha}[d(x), v_2] = 2\alpha[v_2, d(x)]$.

Therefore, $\alpha[v_2, d(x)] = 0$. Primeness of S implies that $[v_2, d(x)] = 0, x \in S$. By Theorem 2, $v_2 \in Z(S)$. As $[[v_1, v_2], S] = 0$ so $[[v_1, v_2], v_2] = 0$ or $[v_1, v_2]v_2 = 0$. This implies that either $\alpha = 0$ or $v_2 = 0$, a contradiction. Hence theorem is proved.

Theorem 16. *Let U be a Lie ideal of S and V a subsemiring of S such that $[V, U] \subset V$. Then either $[U, S] = 0, [V, S] = 0$ or V contains a non-zero ideal of S .*

Proof. If $[V, U] = 0$ then by Lemma 7, either $[V, S] = 0$ or $[U, S] = 0$. Suppose $[V, U] \neq 0$. As U is Lie ideal we have $[V, U] \subset V \cap U$. Let $s \in S, t \in [V, U]$ and $v \in V$ then $[v, [t, ts]] \in [V, U] \subset V$ or $[v, t[t, s]] \in V$, by Lemma 1. Also, $t[v, [t, s]] \in V$. Thus by Lemma 1 we have $[v, t[t, s]] = [v, [t, ts]] + \acute{t}[v, [t, s]] \in V$, that is;

$$(1.4) \quad [v, t][t, s] \in V, \quad t \in [V, U], v \in V.$$

Again, using Lemma 1 we obtain

$$(1.5) \quad [v, t][t, w]s = [v, t][t, ws] + [v, t]w[t, s].$$

Using the fact that $[V, U] \subset V$ and V is subsemiring of S we have, $[v, t]w[t, s] = [v, t](w+\acute{w}+w)[t, s] = [v, t][t, s](w+\acute{w}) + [v, t]w[t, s] = [v, t][t, s]w + [v, t][w, [t, s]] \in V$. This together with (4) and (5) gives $[v, t][t, w]s \in V, \forall w, v \in V, t \in [V, U], s \in S$. Hence $[u, [v, t][t, w]s] \in V$ for all $u \in U, s \in S$. So we have,

$$u[v, t][t, w]s = [u, [v, t][t, w]s] + [v, t][t, w]su \in V.$$

Continuing in this manner, we arrive at $\bar{U}[v, t][t, w]S \subset V$ where $\bar{U}$ is subsemiring generated by U . Because $[V, U] \neq 0$, hence by Lemma 3 $U \not\subset Z(S)$ so $\bar{U} \not\subset Z(S)$. By Theorem 4, $\bar{U}$ contains a non-zero ideal say, I of S . Then $I[v, t][t, w]S$ is an ideal of S in V . Our theorem is proved if we show $I[v, t][t, w] \neq 0$. Suppose $I[v, t][t, w] = 0$ then primeness of S implies that

$$(1.6) \quad [v, t][t, w] = 0, \forall v, w \in V, t \in [V, U].$$

Replacing v by vv_1 in (6) we have

$$(1.7) \quad [v, t]v_1[t, w] = 0, \forall w, v, v_1 \in V, t \in [V, U].$$

Let $s \in S, a \in V$ and $t_1 \in [V, U]$ then $u = [[a, t_1], s] \in U$ which implies that $[u, [a, t_1]] \in [V, U] \subset V$. But $[u, [a, t_1]] = 2[a, t_1]s[a, t_1]$. Thus $2[a, t_1]s[a, t_1] \in V$. From this and (7) we get

$$(1.8) \quad [v, t][a, t_1]S[a, t_1][t, w] = 0.$$

Primeness of S implies that either $[v, t][a, t_1] = 0$ or $[a, t_1][t, w] = 0$. Suppose for all $v, a \in V$ and $t, t_1 \in [V, U]$

$$(1.9) \quad [v, t][a, t_1] = 0.$$

Put $V_1 = [V, [V, U]]$ then V_1 is a Lie ideal of U i.e; $[V_1, U] \subset V_1$ and from (9), we have

$$(1.10) \quad [v, t]V_1 = 0.$$

Thus $[v, t][v_1, u] = 0, u \in U, v_1 \in V_1$. From this, we have $[v, t]UV_1 = 0$. In particular, $[v, t]U[v, t] = 0$. By Lemma 12, either $[v, t] = 0$ or $U \subset Z(S)$. But $[V, U] \neq 0$ so $[v, t] = 0, v \in V, t \in [V, U]$. Hence $V_1 = [V, [V, U]] = 0$. Put $I = [V, U]$ then $[I, U] \subset I$ and $[I, I] \subset [V, [V, U]] = 0$. It follows from Lemma 14 that $[I, S] = 0$. By Corollary 10, we get $[V, S] = 0$. Thus assuming V does not contains non-zero ideal, we obtain the desired result. Similarly, the case with $[a, t_1][t, w] = 0$. This completes the proof.

Corollary 17. *Let U be a Lie ideal of S and V a subsemiring of S such that $[V, U] \subset V$. Then either $U \subset Z(S)$, $V \subset Z(S)$ or V contains a non-zero ideal of S .*

Theorem 18. *Let U be a Lie ideal and V an additive subsemigroup of S such that $[V, U] \subset V$. Then either $V \subset Z(S)$, $U \subset Z(S)$ or $0 \neq [M, S] \subset V$ for an ideal $0 \neq M$ of S .*

Proof. Put $I = [V, U]$ and $T = \{x \in S : [x, S] \subset I\}$. It is easy to see that T is subsemiring of S , since for $t_1, t_2 \in T$ and $x \in S$ we have, $[t_1 t_2, x] = t_1 t_2 x + t_1(t_2 + t_2')x + \acute{x}t_1 t_2 = t_1 t_2 x + (t_2 + t_2')\acute{x}t_1 + \acute{x}t_1 t_2 = [t_1, t_2 x] + [t_2, \acute{x}t_1] \in I$.

Moreover, by jacobian identity we have

$$[[I, I], S] \subset [I, [I, S]] \subset [I, [U, S]] \subset [V, U] = I.$$

which gives that $[I, I] \subset T$. Consider a subsemiring T_0 of T which is generated by $[I, I]$. As $[[I, I], U] \subset [I, [I, U]] \subset [I, I]$ therefore, $[T_0, U] \subset T_0$. By Theorem 16, either $[T_0, S] = 0$, $[U, S] = 0$ or T_0 contains a non zero ideal say M of S . If $[T_0, S] = 0$ then $[[I, I], S] = 0$ and hence $[I, U] = 0$, by Theorem 15. Set $T_1 = \{x \in S; [x, U] = 0\}$. Then T_1 is Lie ideal of S , because $I \subset T_1$. So by Lemma 9 and Lemma 11 we have either $[U, S] = 0$ or $[V, S] = 0$. If $M \subset T_0 \subset T$ then $[M, S] \subset I \subset V$. Hence proved the theorem.

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Modification of conformable fractional derivative with classical properties

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Abstract. In this paper, a new modification of the conformable fractional derivative that uses limit approach with classical properties, linearity, product rule, semi-chain rule, quotient rule, ets. is given. Moreover, we prove that such generalization formula does not satisfy the classical chain rule.

Keywords: Conformable fractional derivative.

1. Introduction

Fractional calculus nowadays is an interesting subject in the area of mathematical analysis. The idea comes from the question asked by L'Hospital in 1695 about the meaning of $\frac{d^n f}{dx^n}$, if $n = \frac{1}{2}$, [5]. Many researchers tried to generalize the concept of the usual derivative to a fractional one. Many definitions appears and most of them use integral form, see [1, 4, 6, 7, 8, 9]. Unfortunately all the existing fractional derivatives do not satisfy the familiar product rule, quotient rule and chain rule for the derivative of two functions and most of them except Caputo derivative don't satisfy the derivative of the constant function is zero. Moreover all of them do not have a corresponding Mean Value Theorem nor Rolle's Theorem. To find a solution for some of these difficulties an interesting definition for fractional derivative that uses limit approach is given by Khalil et. al, [3] as an extension of the usual definition of derivative.

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Definition 1. ([3]). Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The α 'th order conformable fractional derivative of f is defined by

$$(1.1) \quad T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0$ and $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{\epsilon \rightarrow 0^+} T_\alpha(f)(t)$ exists, then $T_\alpha(f)(0) = \lim_{\epsilon \rightarrow 0^+} T_\alpha(f)(t)$.

Following Khalil, Katugampola, [2] introduce a new generalization of the conformable fractional derivative that generalizes results in [3]. Properly, the authors in [2, 3], extend the usual definition of the derivative of a function f at a point a .

Definition 2. ([2]). Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $t > 0$. Then the fractional derivative of order α is defined by

$$(1.2) \quad D_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}$$

for all $t > 0$ and $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{\epsilon \rightarrow 0^+} D_\alpha(f)(t)$ exists, then $D_\alpha(f)(0) = \lim_{\epsilon \rightarrow 0^+} D_\alpha(f)(t)$.

Clearly Definition 2 generalizes Definition 1. More precisely if we truncate the series, $te^{\epsilon t^{-\alpha}} = \sum_{k=0}^{\infty} \frac{\epsilon^k t^{1-\alpha k}}{k!}$ when $k = 1$, we get the formula in Definition 1.

A function f is said to be α -differentiable if the conformable fractional derivatives of order α exists. If $\alpha = 1$, then the first derivative

$$D_1 f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon) - f(t)}{\epsilon} = f'(t).$$

The authors in [2, 3] as a consequence of their definitions showed that the α -fractional derivatives in (1.1) and (1.2) satisfy product rule, quotient rule and proved some results similar to that in classical calculus such as Mean Value Theorem and Rolle's Theorem.

In this paper we give a new definition for α -fractional derivative that generalizes both Definition 1 and 2. Also a simple example is given to prove that such generalization formulas don't obey the classical chain rule.

2. New generalization of fractional derivative

In this section we present our main new definition of α -fractional derivative and study its classical properties. Infact we prove that such derivative satisfies product rule, quotient rule, semi chain rule. Moreover we prove some corresponding results similar to Mean Value Theorem and Rolle's Theorem. We start by the following definitions that generalizes (1.1) and (1.2).

Definition 1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $t > 0$. Then the fractional derivative of f of order α is defined by,

$$(2.1) \quad M^\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon},$$

for all $t > 0$ and $\alpha \in (0, 1)$, where, g is a continuously differentiable function such that $g(0) = g'(0) = 1$. If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} M^\alpha(f)(t)$ exists, then $M^\alpha(f)(0) = \lim_{t \rightarrow 0^+} M^\alpha(f)(t)$.

One should notice that α -differentiable function need not be differentiable, for example, if $f(t) = 3t^{\frac{1}{3}}$, then $M^{\frac{1}{3}}(f)(0) = \lim_{t \rightarrow 0^+} M^{\frac{1}{3}}(f)(t) = 1$. While $M^1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

The following result is a generalization of Theorem 2.1 in [3] and Theorem 2.2 in [2].

Theorem 2. *If $f : [0, \infty) \rightarrow R$ is α -differentiable at a point $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0 .*

Proof. Since f is α -differentiable at t_0 , taking the limit as $\epsilon \rightarrow 0$ of both sides of the equation

$$f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0) = \frac{f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0)}{\epsilon} \epsilon,$$

we get

$$\lim_{\epsilon \rightarrow 0} [f(t_0 g(\epsilon t_0^{-\alpha})) - f(t_0)] = M^{(\alpha)}(f)(t_0) \lim_{\epsilon \rightarrow 0} \epsilon = 0.$$

If we put $h = \epsilon t_0^{1-\alpha} + O(\epsilon^2)$, then $\lim_{\epsilon \rightarrow 0} f(t_0 + h) = f(t_0)$, which completes the proof. $\square$

One of the main results of this paper is the following theorem.

Theorem 3. *Let $\alpha \in (0, 1]$ and f, L be α -differentiable functions at a point $t > 0$. Then,*

- (1) $M^\alpha(af + bL)(t) = aM^\alpha(f)(t) + bM^\alpha(L)(t)$, for all $a, b \in R$.
- (2) $M^\alpha(C) = 0$, for all constant functions, $f(t) = C$.
- (3) $M^\alpha(fL)(t) = gM^\alpha(f)(t) + fM^\alpha(L)(t)$.
- (4) $M^\alpha\left(\frac{f}{L}\right) = \frac{LM^\alpha(f)(t) - fM^\alpha(L)(t)}{L(t)^2}$.
- (5) If, in addition, f is differentiable, then $M^\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$.

Proof. Parts (1) and (2) can be proved directly from the definition. To prove (3), fix $\alpha \in (0, 1]$. Then for $t > 0$,

$$\begin{aligned} M^\alpha(fL)(t) &= \lim_{\epsilon \rightarrow 0} \frac{(fL)(tg(\epsilon t^{-\alpha})) - (fL)(t)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(tg(\epsilon t^{-\alpha}))L(tg(\epsilon t^{-\alpha})) - f(t)L(tg(\epsilon t^{-\alpha})) + f(t)L(tg(\epsilon t^{-\alpha})) - f(t)L(t)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} L(tg(\epsilon t^{-\alpha})) \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon} + f(t) \lim_{\epsilon \rightarrow 0} \frac{L(tg(\epsilon t^{-\alpha})) - L(t)}{\epsilon} \\ &= L(tg(0))M^\alpha(f)(t) + f(t)M^\alpha(L)(t) \\ &= L(t)M^\alpha(f)(t) + f(t)M^\alpha(L)(t). \end{aligned}$$

The proof of part (4) is similar.

Now we prove part (5). Since f is differentiable at $t > 0$, putting $tg(\epsilon t^{-\alpha}) = t + \epsilon t^{1-\alpha} + O(\epsilon^2)$ and $h = \epsilon t^{1-\alpha} (1 + O(\epsilon))$ in (2.1) taking in account $g(0) = g'(0) = 1$, we have

$$\begin{aligned} M^\alpha(f)(t) &= \lim_{\epsilon \rightarrow 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon} \\ M^\alpha(f)(t) &= \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha} + O(\epsilon^2)) - f(t)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(t + h) - f(t)}{\frac{1}{ht^{\alpha-1}/(1+O(\epsilon))}} \\ &= t^{1-\alpha} \frac{df}{dt}(t). \end{aligned}$$

This ends the proof. □

As an application to part (5) Theorem 3, we have

Theorem 4. *Let $\alpha \in (0, 1]$ and $a, n \in \mathbb{R}$ and $t > 0$. Then,*

- (1) $M^\alpha(t^n) = nt^{n-\alpha}$.
- (2) $M^\alpha(e^{at}) = at^{1-\alpha}e^{at}$.
- (3) $M^\alpha(\sin(at)) = at^{1-\alpha}\cos(at)$.
- (4) $M^\alpha(\cos(at)) = -at^{1-\alpha}\sin(at)$.
- (5) $M^\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

Also an unusual results is given in the following theorem.

Theorem 5. *Let $\alpha \in (0, 1]$ and $t \in \mathbb{R}$. Then,*

- (1) $M^\alpha(\sin(\frac{1}{\alpha}t^\alpha)) = \cos(\frac{1}{\alpha}t^\alpha)$.
- (2) $M^\alpha(\cos(\frac{1}{\alpha}t^\alpha)) = -\sin(\frac{1}{\alpha}t^\alpha)$.
- (3) $M^\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

Definition 6. *Let f and L be α -differentiable function at a point $t > 0$. We say that*

- (1) *f and g satisfy the semi-chain rule if*

$$M^\alpha(f \circ L)(t) = f'(L(t))M^\alpha(L)(t),$$

provided that f is continuously differentiable.

- (2) *f and g satisfy the classical chain rule if*

$$M^\alpha(f \circ L)(t) = M^\alpha(f)(L(t))M^\alpha(L)(t).$$

In the following theorem we will show that conformable fractional derivatives do not obey the classical chain rule.

Theorem 7. For $\alpha \in (0, 1]$,

- (1) M^α satisfy the semi-chain rule.
- (2) M^α does not satisfy the classical chain rule.

Proof. (1) Let f be a differentiable function for $t > 0$ and L be α -differentiable at the point $t_0 > 0$. Then

$$\begin{aligned} M^\alpha(f \circ L)(t_0) &= \lim_{\epsilon \rightarrow 0} \frac{f(L(t_0 g(\epsilon t_0^{-\alpha}))) - f(L(t_0))}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(L(t_0 g(\epsilon t_0^{-\alpha}))) - f(L(t_0))}{L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0)} \frac{L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0)}{\epsilon}. \end{aligned}$$

Since, f is differentiable, by the use of Mean Value Theorem there exists a point $c \in (L(t_0), L(t_0 g(\epsilon t_0^{-\alpha})))$ such that

$$f(L(t_0 g(\epsilon t_0^{-\alpha}))) - f(L(t_0)) = f'(c) (L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0)).$$

Consequently,

$$(2.2) \quad M^\alpha(f \circ L)(t_0) = \lim_{\epsilon \rightarrow 0} f'(c) \frac{L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0)}{\epsilon}.$$

But $L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0) = L(t_0(1 + \epsilon t_0^{-\alpha} + O(\epsilon^2))) - L(t_0)$ which turns to zero as ϵ turns to zero. It follows that $c \rightarrow L(t_0)$ as $\epsilon \rightarrow 0$ and hence from (2.2), we get

$$M^\alpha(f \circ L)(t_0) = \lim_{\epsilon \rightarrow 0} f'(c) \lim_{\epsilon \rightarrow 0} \frac{L(t_0 g(\epsilon t_0^{-\alpha})) - L(t_0)}{\epsilon} = f'(L(t_0)) M^\alpha(L)(t_0).$$

To prove (2), it is enough to give the following simple example: Put $f(t) = \sin(t)$ and $L(t) = t^3$. Then applying Theorem 3 part(5) we get,

$$M^\alpha(f \circ L)(t) = t^{1-\alpha} \frac{d(f \circ L)}{dt}(t) = t^{1-\alpha} (3t^2) \cos(t^3) = 3t^{3-\alpha} \cos(t^3)$$

and

$$M^\alpha(f)(L(t)) = M^\alpha(\sin t)(t^3) = (t^3)^{1-\alpha} \cos t^3,$$

$$M^\alpha(L)(t) = (t)^{1-\alpha} \frac{d}{dt} L(t) = 3t^{3-\alpha}.$$

Hence

$$M^\alpha(f \circ L)(t) = 3t^{3-\alpha} \cos t^3 \neq 3t^{6-4\alpha} \cos t^3 = M^\alpha(f)(L(t)) M^\alpha(L)(t).$$

This completes the proof of the theorem. □

3. More generalization and applications

First of all if we let the function g in Definition 2.1 to be any polynomial of degree n such that $g(0) = g'(0) = 1$, we get a generalization of Definition 2.7 in [2]. Furthermore the following definition generalizes the conformable fractional derivatives defined in [3, 10].

Definition 8. Let $k_1, k_2 : [0, \infty) \times [0, 1] \rightarrow [0, \infty)$, be continuously differentiable on a neighborhood of (α, t) , and $t > 0, \alpha \in (0, 1], f : [0, \infty) \rightarrow \mathbb{R}$. Then fractional derivative of f of order α is defined as

$$(3.1) \quad D^\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t(1 + \epsilon k_1(t, \alpha))) - f(t)}{\epsilon k_2(t, \alpha)}.$$

If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} D^\alpha(f)(t)$ exists, then $D^\alpha(f)(0) = \lim_{t \rightarrow 0^+} D^\alpha(f)(t)$.

Indeed if $k_1(t, \alpha) = t^{-\alpha}$ and $k_2(t, \alpha) = 1$, we have Khalil Definition [3], while if $k_1(t, \alpha) = t^{-\alpha-1}(t-a)$, and $k_2(t, \alpha) = 1 - at^{-\alpha}$, (3.1) turns to be precisely

$$D^\alpha(f) = \lim_{\epsilon \rightarrow 0} \frac{f(t(1 + \epsilon t^{-\alpha-1}(t-a))) - f(t)}{\epsilon(1 - at^{-\alpha})} = D_\alpha^\alpha(f)(t),$$

which coincide with the definition of Sarikaya, budak, and Usta in [10].

Theorem 9. Let $k_1, k_2 : [0, \infty) \times [0, 1] \rightarrow [0, \infty)$, be continuously differentiable on a neighborhood of (α, t) , and $t > 0, \alpha \in (0, 1], f : [0, \infty) \rightarrow \mathbb{R}$. Then fractional derivative $D^\alpha(f)$ of order α defined as

$$D^\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t(1 + \epsilon k_1(t, \alpha))) - f(t)}{\epsilon k_2(t, \alpha)}$$

doesn't satisfy the classical chain rule unless $\alpha = 1$.

Proof. Note that: for $\alpha \neq 1, tk_1(t, \alpha) \neq k_2(t, \alpha)$. Other wise

$$\begin{aligned} D^\alpha(f)(t) &= \lim_{\epsilon \rightarrow 0} \frac{f(t(1 + \epsilon k_1(t, \alpha))) - f(t)}{\epsilon k_2(t, \alpha)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(t+h) - f(t)}{h} = f'(t), \end{aligned}$$

where $h = \epsilon tk_1(t, \alpha)$. Hence $tk_1(t, \alpha) \neq k_2(t, \alpha)$.

To prove the theorem it is enough to show that if f and L are differentiable non constant functions, then for $\alpha \neq 1, D^\alpha$ doesn't satisfy the classical chain rule.

Since f and L are differentiable, then

$$\begin{aligned} D^\alpha(L)(t) &= \lim_{\epsilon \rightarrow 0} \frac{L(t(1 + \epsilon k_1(t, \alpha))) - L(t)}{\epsilon k_2(t, \alpha)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{L(t(1 + \epsilon k_1(t, \alpha))) - L(t)}{\epsilon tk_1(t, \alpha)} \cdot \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} \\ (3.2) \quad &= \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} L'(t). \end{aligned}$$

Similarly,

$$(3.3) \quad D^\alpha(f)(L(t)) = \frac{L(t)k_1(L(t), \alpha)}{k_2(L(t), \alpha)} f'(L(t))$$

and

$$(3.4) \quad D^\alpha(f \circ L)(t) = \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} (f \circ L)'(t) = \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} f'(L(t))L'(t).$$

Now suppose that D^α does satisfy the classical chain rule. Then using (3.2) and (3.3) we have:

$$(3.5) \quad \begin{aligned} D^\alpha(f \circ L)(t) &= D^\alpha(f)(L(t)) D^\alpha(L)(t) \\ &= \frac{L(t)k_1(L(t), \alpha)}{k_2(L(t), \alpha)} f'(L(t)) \frac{tk_1(t, \alpha)L'(t)}{k_2(t, \alpha)} \\ &= \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} f'(L(t))L'(t). \end{aligned}$$

Thus from (3.4) and (3.5), we get

$$f'(L(t))L'(t) \frac{tk_1(t, \alpha)}{k_2(t, \alpha)} \left(1 - \frac{L(t)k_1(L(t), \alpha)}{k_2(L(t), \alpha)} \right) = 0.$$

The case $f'(L(t)) = 0$, $L'(t) = 0$ or $tk_1(t, \alpha) = 0$, is impossible. Hence

$$\frac{L(t)k_1(L(t), \alpha)}{k_2(L(t), \alpha)} = 1$$

and this implies that $L(t)k_1(L(t), \alpha) = k_2(L(t), \alpha)$ which is impossible since in this case α turns to be 1. This completes the proof. $\square$

The following two theorems extend the Rolle's and the Mean Value Theorem for α -fractional derivative proved in [2, 3].

Theorem 10 (Roll's theorem for α -fractional differentiable functions). *Let $a, b > 0$ and f be a real valued continuous function on $[a, b]$ with $f(a) = f(b)$. If f is α -differentiable on (a, b) for some $\alpha \in (0, 1)$, then there exists $c \in (a, b)$, such that $M^\alpha(f)(c) = 0$.*

Proof. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there is a point $c \in (a, b)$, at which the function f has local extrema. But f is α -differentiable at c . So

$$M^\alpha(f)(c) = \lim_{\epsilon \rightarrow 0^-} \frac{f(cg(\epsilon c^{-\alpha})) - f(c)}{\epsilon} = \lim_{\epsilon \rightarrow 0^+} \frac{f(cg(\epsilon c^{-\alpha})) - f(c)}{\epsilon}.$$

Since the two limits have opposite signs it follows that $M^\alpha(f)(c) = 0$. $\square$

Theorem 11 (Generalized Mean Value Theorem for α -fractional differentiable functions). *Let f and h be two continuous functions on $[a, b]$ with $a > 0$. If f and h are both α -differentiable functions on (a, b) , then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))M^\alpha(h)(c) = (h(b) - h(a))M^\alpha(f)(c).$$

Proof. Let $u(x) = (f(b) - f(a))h(x) - (h(b) - h(a))f(x)$. Then u satisfies the conditions of Roll's Theorem on $[a, b]$.

Hence there exists $c \in (a, b)$ such that $M^\alpha(u)(c) = 0$. Consequently, $(f(b) - f(a))M^\alpha(h)(c) = (h(b) - h(a))M^\alpha(f)(c)$. $\square$

Corollary 12. (Mean Value Theorem for α -fractional differentiable functions).

Let f be a continuous function on $[a, b]$ with $a > 0$. If f is α -differentiable on (a, b) for some $\alpha \in (0, 1)$, then there exists $c \in (a, b)$ such that

$$M^{(\alpha)}(f)(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}.$$

Proof. Taking $h(x) = \frac{x^\alpha}{\alpha}$ in Theorem 11, we get the result. $\square$

For the case $\alpha \in (n, n+1]$, we have the following generalization:

Definition 13. *Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$ and f be an n -differentiable at $t > 0$. Then the α -fractional derivative of f defined by*

$$M^\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f^{(n)}(tg(\epsilon t^{-\alpha})) - f^{(n)}(t)}{\epsilon}$$

if the limit exists.

As a direct consequence of Definition 13 and part (5) of Theorem 3, one can show that

$$M^\alpha(f)(t) = t^{n+1-\alpha} f^{(n+1)}(t),$$

where $\alpha \in (n, n+1]$ and f is $(n+1)$ -differentiable at $t > 0$.

4. Fractional integral

According to results obtained in [2] and [3] due to our similar results in Theorem 3 and Theorem 4, it is nice to say that we can still have the same definition of fractional integral. We start by the following definition.

Definition 14. *Let $a \geq 0, t \geq a$ and let f be a function defined on $(a, t]$. Then for $\alpha \in \mathbb{R}$, the α -fractional Integral of f is defined by $I_a^\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$ provided the Riemann improper integral exists.*

The following theorem shows an interesting property of the above definition that the α -fractional integral and α -derivatives are inverse each others.

Theorem 15. Let $a \geq 0, t \geq a$ and let f be a continuous function defined on $(a, t]$. Then for $\alpha \in R$ such that $I_a^\alpha f$ exists. Then $M^\alpha(I_a^\alpha f)(t) = f(t)$, for $t \geq a$.

Proof. Since f is continuous, $I_a^\alpha f$ is clearly differentiable. Therefore, using part (5) of Theorem 3, we have

$$M^\alpha(I_a^\alpha f)(t) = t^{1-\alpha} \frac{d}{dt}(I_a^\alpha(f)(t)) = t^{1-\alpha} \frac{d}{dt} \int_a^t \frac{f(x)}{x^{1-\alpha}} dx = t^{1-\alpha} \frac{f(t)}{t^{1-\alpha}} = f(t).$$

□

Using Definition 14 one can easily prove the following Theorem.

Theorem 16. Let $\alpha \in (0, 1]$ and f, g be α -Fractional Integrable on $[0, t]$. Then,

- (1) $I_a^\alpha(bf + cg)(t) = bI_a^\alpha(f)(t) + cI_a^\alpha(g)(t)$, for all $a, b \in R$.
- (2) $I_a^\alpha(t^n) = \frac{1}{(n+\alpha)}[t^{n+\alpha} - a^{n+\alpha}]$, for all $n \in R$.
- (3) $I_a^\alpha(c) = \frac{c}{\alpha}[t^\alpha - a^\alpha]$, for all constant functions, $f(t) = c$.

The following theorem shows that the usual integration by parts can be applicable in the case of conformable fractional derivative.

Theorem 17. (Integration by parts) Let f, g be α -integrable functions. Then $I_a^\alpha(fM^\alpha(g)) = fg - I_a^\alpha(gM^\alpha(f))$.

Proof. Using Theorem 3 part 3, we have

$$M^{(\alpha)}(fg) = gM^{(\alpha)}(f) + fM^{(\alpha)}(g).$$

Integrate both sides of the above equation we get:

$$I_a^\alpha(M^{(\alpha)}(fg)) = I_a^\alpha(gM^{(\alpha)}(f)) + I_a^\alpha(fM^{(\alpha)}(g)).$$

Consequently,

$$I_a^\alpha(fM^{(\alpha)}(g)) = fg - I_a^\alpha(gM^{(\alpha)}(f)).$$

This completes the proof. □

We end this paper by the following example:

Example 18. Evaluate $I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t)$

Solution 19. Using integration by parts we have

$$\begin{aligned} I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t) &= t[e^t - 1] - I_0^{\frac{1}{2}}(t^{\frac{1}{2}}(e^t - 1)) \\ &= t[e^t - 1] - \int_0^t \frac{x^{\frac{1}{2}}(e^x - 1)}{x^{\frac{1}{2}}} dx = (t - 1)e^t + 1. \end{aligned}$$

The same alternative solution can be found as

$$I_0^{\frac{1}{2}}(t^{\frac{3}{2}}e^t) = \int_0^t \frac{x^{\frac{3}{2}}e^x}{x^{\frac{1}{2}}} dx = \int_0^t xe^x dx = \int_0^t xe^x dx = (t - 1)e^t + 1.$$

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Properties of γ - P_S - R_0 and γ - P_S - R_1 spaces

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Abstract. This paper introduces some more γ - P_S - separation axioms called γ - P_S - R_0 and γ - P_S - R_1 by using γ - P_S -open sets and τ_{γ} - P_S -closure of a set. Some properties of these spaces are constructed.

Keywords: γ - P_S -open set, γ - P_S - R_0 space, γ - P_S - R_1 space.

1. Introduction

Kasahara [12] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α -continuous functions with α -closed graphs and α -compact spaces. After this, Jankovic [11] introduced the concept of α -closure of a set in X via α -operation and investigated further characterizations of

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function with α -closed graph. Later, Ogata [15] defined and studied the concept of γ -open sets, and applied it to investigate operation-functions and operation-separation axioms. Since that, γ operation on τ has attracted the attention of many researchers. Among them are γ -preopen sets [13] and γ -semiopen sets [14].

Recently, the notion of γ - P_S -open sets had been defined by Asaad, Ahmad and Omar [4]. This set is stronger than γ -preopen set. Besides that, they also introduced γ -locally indiscrete and γ -hyperconnected spaces [6]. In addition, Asaad, Ahmad and Omar ([3], [7]) introduced and studied the notion of γ - P_S - T_i spaces for $i = 0, \frac{1}{2}, 1, 2$. Later, in [1] they studied γ - P_S -regular and γ - P_S -normal spaces and investigated some of their characterizations.

The aim of this paper is to introduce some more γ - P_S - separation axioms called γ - P_S - R_0 and γ - P_S - R_1 by using γ - P_S -open sets and τ_γ - P_S -closure of a set. Some properties of these spaces are constructed. In this paper, the pairs (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms assumed unless otherwise mentioned.

2. Preliminaries and basic definitions

An operation γ on the topology τ on X is a mapping $\gamma: \tau \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [15]. A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ -open [15] if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ -open set is called γ -closed. The class of all γ -open sets of X is denoted by τ_γ . The τ_γ -closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ -closed sets containing A and it is denoted by $\tau_\gamma\text{-Cl}(A)$ [15], and the τ_γ -interior of a subset A of X with an operation γ on τ is defined as the union of all γ -open sets containing A [14] and it is denoted by $\tau_\gamma\text{-Int}(A)$ [14]. An operation γ on τ is said to be regular if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$ [15].

A subset A of a topological space (X, τ) with an operation γ on τ is said to be γ -preopen [13] (respectively, γ -semiopen [14]) if $A \subseteq \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A))$ (respectively, $A \subseteq \tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A))$). The complement of a γ -semiopen [14] set is called γ -semiclosed. A γ -preopen subset A of a topological space (X, τ) is called γ - P_S -open [4] if for each $x \in A$, there exists a γ -semiclosed set F such that $x \in F \subseteq A$. The complement of a γ - P_S -open set is called γ - P_S -closed. The τ_γ - P_S -closure [4] of a subset A of a space (X, τ) is defined as the intersection of all γ - P_S -closed sets of X containing A and it is denoted by $\tau_\gamma\text{-}P_S\text{Cl}(A)$. The τ_γ - P_S -interior [4] of a subset A of a space (X, τ) is defined as the union of all γ - P_S -open sets of X contained in A and it is denoted by $\tau_\gamma\text{-}P_S\text{Int}(A)$. A subset A of X is γ - P_S -closed if and only if $\tau_\gamma\text{-}P_S\text{Cl}(A) = A$ [4]. A point $x \in \tau_\gamma\text{-}P_S\text{Cl}(A)$ if and only if $A \cap U \neq \emptyset$ for every γ - P_S -open set U of X containing x [4]. A

subset A of a topological space (X, τ) with an operation γ on τ is said to be γ - P_S -generalized closed (shortly γ - P_S - g -closed) [3] if τ_γ - P_S - $Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is γ - P_S -open set in X . For any topological space (X, τ) and any operation γ on τ , we denote the class of all γ - P_S -open (respectively, γ - P_S -closed) sets of X by τ_γ - P_S - $O(X)$ or τ_γ - P_S - $O(X, \tau)$ (respectively, τ_γ - P_S - $C(X)$ or τ_γ - P_S - $C(X, \tau)$).

A space (X, τ) with an operation γ on τ is said to be γ - P_S - T_0 [7] if for each pair of distinct points x, y in X , there exists a γ - P_S -open set G such that either $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$. A space (X, τ) with an operation γ on τ is said to be γ - P_S - T_1 [7] (respectively, γ -semi T_1 [14]) if for each pair of distinct points x, y in X , there exist two γ - P_S -open (respectively, γ -semiopen) sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. A space (X, τ) with an operation γ on τ is said to be γ - P_S - T_2 [7] if for each pair of distinct points x, y in X , there exist two γ - P_S -open sets G and H containing x and y respectively such that $G \cap H = \emptyset$. A space (X, τ) with an operation γ on τ is said to be γ - P_S - $T_{\frac{1}{2}}$ [3] if every γ - P_S - g -closed set in X is γ - P_S -closed.

3. Properties of γ - P_S - R_0 spaces

In this section, another type of γ - P_S -separation axioms which is γ - P_S - R_0 spaces will be presented. Some of its properties will be discussed.

The definition of γ - P_S - R_0 space in terms of γ - P_S -open and τ_γ - P_S -closure operator is defined as follows:

Definition 3.1. A topological space (X, τ) with an operation γ on τ is said to be γ - P_S - R_0 if every γ - P_S -open set contains the τ_γ - P_S -closure of each of its singletons.

In other words, a space (X, τ) is said to be γ - P_S - R_0 if G is a γ - P_S -open set and $x \in G$, then τ_γ - P_S - $Cl(\{x\}) \subseteq G$.

Some important properties of γ - P_S - R_0 spaces and relations between γ - P_S - R_0 space and other types of γ - P_S -separation axioms are established.

Theorem 3.2. For any topological space (X, τ) and any operation γ on τ . If τ_γ - P_S - $Cl(\{x\}) \subseteq F$ whenever F is γ -semiclosed set in X and $x \in F$. Then (X, τ) is γ - P_S - R_0 space.

Proof. Let G be any γ - P_S -open set in X such that $x \in G$. Then there exists a γ -semiclosed set F in X such that $x \in F \subseteq G$. So by hypothesis, τ_γ - P_S - $Cl(\{x\}) \subseteq F$ and hence τ_γ - P_S - $Cl(\{x\}) \subseteq G$. This means that (X, τ) is γ - P_S - R_0 . $\square$

Remark 3.3. Let $A \subseteq (X, \tau)$, τ_γ - $Cl(\tau_\gamma$ - $Int(A)) \subseteq \tau_\gamma$ - P_S - $Cl(A)$.

Theorem 3.4. If a topological space (X, τ) is γ - P_S - R_0 , then τ_γ - $Cl(\tau_\gamma$ - $Int(\{x\})) \subseteq G$ whenever G is γ - P_S -open set in X and $x \in G$.

Proof. Straightforward from Definition 3.1 and Remark 3.3. $\square$

Theorem 3.5. *Let (X, τ) be a topological space and γ be an operation on τ . Then the following conditions are equivalent:*

- (i) X is γ - P_S - R_0 .
- (ii) *If for each γ - P_S -closed set E in X such that $x \notin E$, then there exists a γ - P_S -open set G such that $E \subseteq G$ and $x \notin G$.*
- (iii) *If for each γ - P_S -closed set E in X such that $x \notin E$, then τ_γ - P_S - $Cl(\{x\}) \cap E = \emptyset$.*
- (iv) *If for each distinct points $x, y \in X$, then either τ_γ - P_S - $Cl(\{x\}) \cap \tau_\gamma$ - P_S - $Cl(\{y\}) = \emptyset$ or τ_γ - P_S - $Cl(\{x\}) = \tau_\gamma$ - P_S - $Cl(\{y\})$.*

Proof. (i) $\Rightarrow$ (ii) Let E be any γ - P_S -closed set in X and $x \notin E$. Since (X, τ) is γ - P_S - R_0 , then τ_γ - P_S - $Cl(\{x\}) \subseteq X \setminus E$. Let $G = X \setminus \tau_\gamma$ - P_S - $Cl(\{x\})$. Then G is γ - P_S -open set in X such that $E \subseteq G$ and $x \notin G$.

(ii) $\Rightarrow$ (iii) Let E be any γ - P_S -closed set in X does not containing x . Then by (ii), there exists $G \in \tau_\gamma$ - P_S - $O(X, \tau)$ such that $E \subseteq G$ and G does not containing x . This means that $\{x\} \cap G = \emptyset$ and hence τ_γ - P_S - $Cl(\{x\}) \cap G = \emptyset$ since G is γ - P_S -open set. That is, τ_γ - P_S - $Cl(\{x\}) \cap E = \emptyset$.

(iii) $\Rightarrow$ (iv) Let τ_γ - P_S - $Cl(\{x\}) \neq \tau_\gamma$ - P_S - $Cl(\{y\})$ for distinct elements x and y in X . Then there exists an element $z \in \tau_\gamma$ - P_S - $Cl(\{x\})$ such that $z \notin \tau_\gamma$ - P_S - $Cl(\{y\})$ (or $z \in \tau_\gamma$ - P_S - $Cl(\{y\})$ such that $z \notin \tau_\gamma$ - P_S - $Cl(\{x\})$). Then there exists γ - P_S -open set G of X such that $z \in G$ and $z \notin G$. So $\{x\} \cap G \neq \emptyset$ and $\{y\} \cap G = \emptyset$. This means that $x \in G$ and $x \notin \tau_\gamma$ - P_S - $Cl(\{y\})$. Therefore, by (iii), τ_γ - P_S - $Cl(\{x\}) \cap \tau_\gamma$ - P_S - $Cl(\{y\}) = \emptyset$.

(iv) $\Rightarrow$ (i) Let G be any γ - P_S -open set in X such that $x \in G$. For each point $y \in G$ and $x \neq y$. Then $x \notin \tau_\gamma$ - P_S - $Cl(\{y\})$ which implies that τ_γ - P_S - $Cl(\{x\}) \neq \tau_\gamma$ - P_S - $Cl(\{y\})$. So by hypothesis, τ_γ - P_S - $Cl(\{x\}) \cap \tau_\gamma$ - P_S - $Cl(\{y\}) = \emptyset$ for each $y \in X \setminus G$. Hence τ_γ - P_S - $Cl(\{x\}) \cap (\bigcup_{y \in X \setminus G} \tau_\gamma$ - P_S - $Cl(\{y\})) = \emptyset$ for each $y \in X \setminus G$. On the other hand, since G is γ - P_S -open set in X and $y \in X \setminus G$ implies τ_γ - P_S - $Cl(\{y\}) \subseteq X \setminus G$. That is, $X \setminus G = \bigcup_{y \in X \setminus G} \tau_\gamma$ - P_S - $Cl(\{y\})$. Therefore, τ_γ - P_S - $Cl(\{x\}) \cap X \setminus G = \emptyset$ and hence τ_γ - P_S - $Cl(\{x\}) \subseteq G$. Then by Definition 3.1, a space X is γ - P_S - R_0 . $\square$

Corollary 3.6. *A space (X, τ) is γ - P_S - R_0 if and only if τ_γ - P_S - $Cl(\{x\}) \neq \tau_\gamma$ - P_S - $Cl(\{y\})$ implies τ_γ - P_S - $Cl(\{x\}) \cap \tau_\gamma$ - P_S - $Cl(\{y\}) = \emptyset$ for every x and y in X .*

Proof. Directly follows from Theorem 3.5. $\square$

Definition 3.7. Let A be any subset of a topological space (X, τ) and γ be an operation on τ . Then the γ - P_S -kernel of A is denoted by γ - P_S - $ker(A)$ and is defined as follows:

$$\gamma$$
- P_S - $ker(A) = \cap \{G : A \subseteq G \text{ and } G \in \tau_\gamma$ - P_S - $O(X, \tau)\}$

In other words, $\gamma\text{-}P_Sker(A)$ is the intersection of all $\gamma\text{-}P_S$ -open sets of (X, τ) containing A .

The following remark follows directly from Definition 3.7.

Remark 3.8. Let $A \subseteq (X, \tau)$ and γ be an operation on τ . Then the following are true:

- (i) $A \subseteq \gamma\text{-}P_Sker(A)$.
- (ii) If A is $\gamma\text{-}P_S$ -open set in X , then $\gamma\text{-}P_Sker(A) = A$.

The following example shows that the converse of the above remark is not true in general.

Example 3.9. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define an operation $\gamma: \tau \rightarrow P(X)$ as follows:
For every $A \in \tau$

$$\gamma(A) = \begin{cases} A, & \text{if } a \in A \\ Cl(A), & \text{if } a \notin A. \end{cases}$$

Obviously, $\tau_\gamma = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$. So $\tau_\gamma\text{-}P_S O(X) = \{\emptyset, X, \{a\}, \{a, c\}, \{b, c\}\}$. Therefore, the $\gamma\text{-}P_Sker(\{b\}) = \{b, c\}$ and hence $\gamma\text{-}P_Sker(A) \not\subseteq A$. Again, $\gamma\text{-}P_Sker(\{c\}) = \{c\}$, but the set $\{c\}$ is not $\gamma\text{-}P_S$ -open in X .

Theorem 3.10. Let A be a subset of a topological space (X, τ) and γ be an operation on τ . Then $\gamma\text{-}P_Sker(A) = \{x \in X: \tau_\gamma\text{-}P_S Cl(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in \gamma\text{-}P_Sker(A)$. Suppose that $\tau_\gamma\text{-}P_S Cl(\{x\}) \cap A = \emptyset$. Then $x \notin X \setminus \tau_\gamma\text{-}P_S Cl(\{x\})$ and $X \setminus \tau_\gamma\text{-}P_S Cl(\{x\})$ is $\gamma\text{-}P_S$ -open set in X such that $A \subseteq X \setminus \tau_\gamma\text{-}P_S Cl(\{x\})$. Hence $x \notin \gamma\text{-}P_Sker(A)$. This is contradiction of the hypothesis. So $\tau_\gamma\text{-}P_S Cl(\{x\}) \cap A \neq \emptyset$.

Now, for the other part, let $x \in X$ such that $\tau_\gamma\text{-}P_S Cl(\{x\}) \cap A \neq \emptyset$. Suppose that $x \notin \gamma\text{-}P_Sker(A)$. Then there exists a $\gamma\text{-}P_S$ -open set G containing A and $x \notin G$ and hence $x \notin A$. So, let $y \in \tau_\gamma\text{-}P_S Cl(\{x\}) \cap A$. Then G is a $\gamma\text{-}P_S$ -open set containing y but does not contain x . This means that $\tau_\gamma\text{-}P_S Cl(\{x\}) \cap A = \emptyset$ which is contradiction of the assumption. Therefore, $x \in \gamma\text{-}P_Sker(A)$. $\square$

Theorem 3.11. Let (X, τ) be a topological space with an operation γ on τ and let $x, y \in X$. Then $y \in \gamma\text{-}P_Sker(\{x\})$ if and only if $x \in \tau_\gamma\text{-}P_S Cl(\{y\})$.

Proof. Suppose that $y \notin \gamma\text{-}P_Sker(\{x\})$. Then there exists a $\gamma\text{-}P_S$ -open set G containing x such that $y \notin G$. That is, $x \notin \tau_\gamma\text{-}P_S Cl(\{y\})$.

Conversely, the proof is similar to the above case. $\square$

Theorem 3.12. Let $A \subseteq (X, \tau)$ and γ be an operation on τ . Then A is $\gamma\text{-}P_S$ -g-closed if and only if $\tau_\gamma\text{-}P_S Cl(A) \subseteq \gamma\text{-}P_Sker(A)$.

Proof. Suppose that A is γ - P_S - g -closed. Then $\tau_\gamma\text{-}P_S\text{-}Cl(A) \subseteq G$, whenever $A \subseteq G$ and G is γ - P_S -open. Let $x \in \tau_\gamma\text{-}P_S\text{-}Cl(A)$. Suppose that $x \notin \gamma\text{-}P_S\text{-}sk(A)$. Then there exists a γ - P_S -open set G containing A and $x \notin G$. This implies that $x \notin \tau_\gamma\text{-}P_S\text{-}Cl(A)$. Which is contradiction to the hypothesis. Hence $x \in \gamma\text{-}P_S\text{-}sk(A)$ and so $\tau_\gamma\text{-}P_S\text{-}Cl(A) \subseteq \gamma\text{-}P_S\text{-}sk(A)$.

Conversely, let $\tau_\gamma\text{-}P_S\text{-}Cl(A) \subseteq \gamma\text{-}P_S\text{-}sk(A)$. If G is any γ - P_S -open set containing A . Then $\gamma\text{-}P_S\text{-}sk(A) \subseteq G$. That is $\tau_\gamma\text{-}P_S\text{-}Cl(A) \subseteq \gamma\text{-}P_S\text{-}sk(A) \subseteq G$. Therefore, A is γ - P_S - g -closed set in X . $\square$

Definition 3.13 ([7]). A topological space (X, τ) with an operation γ on τ , is said to be γ - P_S -symmetric if for each x, y in X , then $x \in \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$ implies that $y \in \tau_\gamma\text{-}P_S\text{-}Cl(\{x\})$.

Theorem 3.14 ([7]). Let (X, τ) be a topological space and γ be an operation on τ . Then X is γ - P_S -symmetric if and only if the singleton set $\{x\}$ is γ - P_S - g -closed, for each $x \in X$.

Corollary 3.15. Let (X, τ) be a topological space and γ be an operation on τ . Then X is γ - P_S -symmetric if and only if $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \subseteq \gamma\text{-}P_S\text{-}sk(\{x\})$ for every $x \in X$.

Proof. The proof is immediate consequence of Theorem 3.12 and Theorem 3.14. $\square$

Recall that a topological space (X, τ) with an operation γ on τ is γ -locally indiscrete if every γ -open subset of X is γ -closed, or every γ -closed subset of X is γ -open [6].

Lemma 3.16 ([7]). If (X, τ) is γ -locally indiscrete space, then (X, τ) is γ - P_S -symmetric.

Recall that a topological space (X, τ) with an operation γ on τ is γ -hyperconnected if $\tau_\gamma\text{-}Cl(G) = X$ for every γ -open set G of X [6].

Lemma 3.17 ([7]). If (X, τ) is γ -hyperconnected space and γ is a regular operation on τ , then (X, τ) is γ - P_S -symmetric.

Corollary 3.18. If (X, τ) is either γ -locally indiscrete or γ -hyperconnected space, then $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \subseteq \gamma\text{-}P_S\text{-}sk(\{x\})$ for every $x \in X$.

Proof. Straightforward from Lemma 3.16, Lemma 3.17 and Corollary 3.15. $\square$

Theorem 3.19. For any points x and y in a topological space (X, τ) with an operation γ on τ . Then $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$ if and only if $\gamma\text{-}P_S\text{-}sk(\{x\}) \neq \gamma\text{-}P_S\text{-}sk(\{y\})$.

Proof. Let $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. This means that there is a point p in X such that $p \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$ and $p \notin \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. Then there exists a $\gamma\text{-}P_S$ -open set containing p and x , but not containing y . Hence by Definition 3.7, $y \notin \gamma\text{-}P_S\text{ker}(\{x\})$ and so $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$.

Conversely, suppose that $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$. That is, there is a point p in X such that $p \in \gamma\text{-}P_S\text{ker}(\{x\})$ and $p \notin \gamma\text{-}P_S\text{ker}(\{y\})$. When the point $p \in \gamma\text{-}P_S\text{ker}(\{x\})$, then by Theorem 3.10, $\tau_\gamma\text{-}P_S\text{Cl}(\{p\}) \cap \{x\} \neq \emptyset$. This means that $x \in \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$. Now when the point $p \notin \gamma\text{-}P_S\text{ker}(\{y\})$. Then by Theorem 3.10, $\tau_\gamma\text{-}P_S\text{Cl}(\{p\}) \cap \{y\} = \emptyset$. Since $x \in \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$ which implies that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \subseteq \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$. So $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \{y\} = \emptyset$. Therefore, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. This completes the proof. $\square$

Theorem 3.20. *A space (X, τ) is $\gamma\text{-}P_S\text{-}R_0$ if and only if $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$ implies $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\}) = \emptyset$ for every x and y in X .*

Proof. Assume that (X, τ) is $\gamma\text{-}P_S\text{-}R_0$ space. By Theorem 3.19, if $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$, then $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ for every points x and y in X . Hence by Corollary 3.6, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \tau_\gamma\text{-}P_S\text{Cl}(\{y\}) = \emptyset$ for every points x and y in X . To prove $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\}) = \emptyset$. Suppose there is a point $p \in X$ such that $p \in \gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\})$. Then $p \in \gamma\text{-}P_S\text{ker}(\{x\})$ and $p \in \gamma\text{-}P_S\text{ker}(\{y\})$. When $p \in \gamma\text{-}P_S\text{ker}(\{x\})$, then by Theorem 3.11, $x \in \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$. Since $x \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$. This means that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \tau_\gamma\text{-}P_S\text{Cl}(\{p\}) \neq \emptyset$. Thus by Theorem 3.5, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) = \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$. When $p \in \gamma\text{-}P_S\text{ker}(\{y\})$, then by the same way, we can obtain $\tau_\gamma\text{-}P_S\text{Cl}(\{y\}) = \tau_\gamma\text{-}P_S\text{Cl}(\{p\})$. This means that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) = \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ which is contradiction. Therefore, $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\}) = \emptyset$ for every x and y in X .

Conversely, suppose that $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$ implies $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\}) = \emptyset$ for every x and y in a space (X, τ) . If $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$, then by Theorem 3.19, $\gamma\text{-}P_S\text{ker}(\{x\}) \neq \gamma\text{-}P_S\text{ker}(\{y\})$. So by hypothesis, $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{y\}) = \emptyset$. This implies that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \tau_\gamma\text{-}P_S\text{Cl}(\{y\}) = \emptyset$ since $p \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ which implies that $p \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$ and hence $x \in \gamma\text{-}P_S\text{ker}(\{p\})$. So $\gamma\text{-}P_S\text{ker}(\{x\}) \cap \gamma\text{-}P_S\text{ker}(\{p\}) \neq \emptyset$. Then by hypothesis, $\gamma\text{-}P_S\text{ker}(\{x\}) = \gamma\text{-}P_S\text{ker}(\{p\})$. Similarly, we have $\gamma\text{-}P_S\text{ker}(\{y\}) \cap \gamma\text{-}P_S\text{ker}(\{p\}) \neq \emptyset$. Then by hypothesis, $\gamma\text{-}P_S\text{ker}(\{y\}) = \gamma\text{-}P_S\text{ker}(\{p\}) = \gamma\text{-}P_S\text{ker}(\{x\})$. This is a contradiction. Therefore, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap \tau_\gamma\text{-}P_S\text{Cl}(\{y\}) = \emptyset$. Consequently, by Corollary 3.6, (X, τ) is $\gamma\text{-}P_S\text{-}R_0$ space. $\square$

Theorem 3.21. *The following properties of a topological space (X, τ) with an operation γ on τ are equivalent:*

- (i) X is $\gamma\text{-}P_S\text{-}R_0$.

- (ii) If for every nonempty subset A of X and every γ - P_S -open set U in X such that $A \cap U \neq \emptyset$, then there exists a γ - P_S -closed set E such that $A \cap E \neq \emptyset$ and $E \subseteq U$.
- (iii) $U = \cup\{E : E \subseteq U \text{ and } E \in \tau_\gamma\text{-}P_S C(X, \tau)\}$ for every $U \in \tau_\gamma\text{-}P_S O(X, \tau)$.
- (iv) $E = \cap\{U : E \subseteq U \text{ and } U \in \tau_\gamma\text{-}P_S O(X, \tau)\}$ for every $E \in \tau_\gamma\text{-}P_S C(X, \tau)$.
- (v) $\tau_\gamma\text{-}P_S Cl(\{x\}) \subseteq \gamma\text{-}P_S ker(\{x\})$ for every element x in X .

Proof. (i) $\Rightarrow$ (ii) Let A be a nonempty subset of X and U is γ - P_S -open set in X such that $A \cap U \neq \emptyset$. Then there exists a point $x \in A \cap U$ and hence $x \in U$. Since (X, τ) is γ - P_S - R_0 , then $\tau_\gamma\text{-}P_S Cl(\{x\}) \subseteq U$. Let $E = \tau_\gamma\text{-}P_S Cl(\{x\})$. Then E is γ - P_S -closed set in X such that $A \cap E \neq \emptyset$ and $E \subseteq U$.

(ii) $\Rightarrow$ (iii) Let $U \in \tau_\gamma\text{-}P_S O(X, \tau)$. Then $\cup\{E : E \subseteq U \text{ and } E \in \tau_\gamma\text{-}P_S C(X, \tau)\} \subseteq U$. Let x be any point in U . Then there exists $E \in \tau_\gamma\text{-}P_S C(X, \tau)$ such that $x \in E$ and $E \subseteq U$. So $x \in E \subseteq \cup\{E : E \subseteq U \text{ and } E \in \tau_\gamma\text{-}P_S C(X, \tau)\}$. Therefore, $U = \cup\{E : E \subseteq U \text{ and } E \in \tau_\gamma\text{-}P_S C(X, \tau)\}$.

(iii) $\Rightarrow$ (iv) It is clear.

(iv) $\Rightarrow$ (v) Let x be any element of X and let $y \in X$ such that $y \notin \gamma\text{-}P_S ker(\{x\})$. This means that there exists a γ - P_S -open set V in X containing x such that $y \notin V$ and hence $\tau_\gamma\text{-}P_S Cl(\{y\}) \cap V = \emptyset$. Then by (iv), $\cap\{U : \tau_\gamma\text{-}P_S Cl(\{y\}) \subseteq U \text{ and } U \in \tau_\gamma\text{-}P_S O(X, \tau)\} \cap V = \emptyset$. This means that there exists γ - P_S -open set U in X such that $x \notin U$ and $\tau_\gamma\text{-}P_S Cl(\{y\}) \subseteq U$. Then $\{x\} \cap U = \emptyset$ and hence $\tau_\gamma\text{-}P_S Cl(\{x\}) \cap U = \emptyset$ and $y \notin \tau_\gamma\text{-}P_S Cl(\{x\})$. Therefore, $y \notin \tau_\gamma\text{-}P_S Cl(\{x\})$. Hence $\tau_\gamma\text{-}P_S Cl(\{x\}) \subseteq \gamma\text{-}P_S ker(\{x\})$.

(v) $\Rightarrow$ (i) Let G be any γ - P_S -open set in X containing x . Let $y \in \gamma\text{-}P_S ker(\{x\})$, then by Theorem 3.11, $x \in \tau_\gamma\text{-}P_S Cl(\{y\})$ and hence $y \in U$. This follows that $\gamma\text{-}P_S ker(\{x\}) \subseteq U$. Thus $x \in \tau_\gamma\text{-}P_S Cl(\{x\}) \subseteq \gamma\text{-}P_S ker(\{x\}) \subseteq U$. Then a space (X, τ) is γ - P_S - R_0 . $\square$

Theorem 3.22. A topological space (X, τ) is γ - P_S - R_0 if and only if $\tau_\gamma\text{-}P_S Cl(\{x\}) = \gamma\text{-}P_S ker(\{x\})$ for all x in X .

Proof. Let (X, τ) be a γ - P_S - R_0 space. Then by Theorem 3.21 (v), $\tau_\gamma\text{-}P_S Cl(\{x\}) \subseteq \gamma\text{-}P_S ker(\{x\})$ for every element x in X . Let $y \in \gamma\text{-}P_S ker(\{x\})$ and by Theorem 3.11, $x \in \tau_\gamma\text{-}P_S Cl(\{y\})$. So by Corollary 3.6, $\tau_\gamma\text{-}P_S Cl(\{x\}) = \tau_\gamma\text{-}P_S Cl(\{y\})$. Thus $y \in \tau_\gamma\text{-}P_S Cl(\{x\})$. This means that $\gamma\text{-}P_S ker(\{x\}) \subseteq \tau_\gamma\text{-}P_S Cl(\{x\})$. It follows that $\tau_\gamma\text{-}P_S Cl(\{x\}) = \gamma\text{-}P_S ker(\{x\})$ for all x in X .

Conversely, the proof is follows directly from Theorem 3.21 and hence it is omitted. $\square$

Theorem 3.23. The following properties are equivalent for a topological space (X, τ) with an operation γ on τ :

- (i) X is γ - P_S - R_0 .

(ii) $x \in \tau_\gamma\text{-}P_SCl(\{y\})$ if and only if $y \in \tau_\gamma\text{-}P_SCl(\{x\})$.

Proof. (i) $\Rightarrow$ (ii) Let X be a $\gamma\text{-}P_S\text{-}R_0$ space and let $x \in \tau_\gamma\text{-}P_SCl(\{y\})$. Then $\tau_\gamma\text{-}P_SCl(\{x\}) \cap \tau_\gamma\text{-}P_SCl(\{y\}) \neq \emptyset$. Hence by Corollary 3.6, $\tau_\gamma\text{-}P_SCl(\{x\}) = \tau_\gamma\text{-}P_SCl(\{y\})$. So $y \in \tau_\gamma\text{-}P_SCl(\{x\})$. Similarly, $y \in \tau_\gamma\text{-}P_SCl(\{x\})$ implies $x \in \tau_\gamma\text{-}P_SCl(\{y\})$.

(ii) $\Rightarrow$ (i) Let G be any $\gamma\text{-}P_S$ -open set in X containing x . Let $y \notin G$, then $x \notin \tau_\gamma\text{-}P_SCl(\{y\})$ and hence $y \notin \tau_\gamma\text{-}P_SCl(\{x\})$ (by hypothesis). So $\tau_\gamma\text{-}P_SCl(\{x\}) \subseteq G$. This shows that X is $\gamma\text{-}P_S\text{-}R_0$ space. $\square$

Corollary 3.24. *Let (X, τ) be a topological space and γ be an operation on τ . Then (X, τ) is $\gamma\text{-}P_S\text{-}R_0$ if and only if (X, τ) is $\gamma\text{-}P_S$ -symmetric.*

Proof. This is an immediate consequence of Theorem 3.23 and Definition 3.13. $\square$

Corollary 3.25 can be constructed by applying Corollary 3.15, Corollary 3.6 and Corollary 3.24.

Corollary 3.25. *Let (X, τ) be a topological space and γ be an operation on τ . Then X is $\gamma\text{-}P_S$ -symmetric if and only if $\tau_\gamma\text{-}P_SCl(\{x\}) = \gamma\text{-}P_Sker(\{x\})$ for every $x \in X$.*

Proof. Straightforward. $\square$

Consequently, some results on $\gamma\text{-}P_S\text{-}R_0$ space are obtained by using Corollary 3.24.

Corollary 3.26. *Let (X, τ) be a topological space and γ be an operation on τ . Then X is $\gamma\text{-}P_S\text{-}R_0$ if and only if every singleton sets in X is $\gamma\text{-}P_S$ -g-closed.*

Proof. Directly follows from Corollary 3.24 and Theorem 3.14. $\square$

Lemma 3.27 ([7]). *For any topological space (X, τ) with an operation γ on τ . If $\tau_\gamma\text{-}P_SO(X) = \tau_\gamma\text{-}P_SC(X)$, then (X, τ) is $\gamma\text{-}P_S$ -symmetric.*

From Corollary 3.24 and Lemma 3.27, we have the following corollary.

Corollary 3.28. *If $\tau_\gamma\text{-}P_SO(X, \tau) = \tau_\gamma\text{-}P_SC(X, \tau)$, then the topological space (X, τ) is $\gamma\text{-}P_S\text{-}R_0$.*

Proof. Obvious. $\square$

The proof of the next corollary follows directly from Corollary 3.24, Lemma 3.16 and Lemma 3.17.

Corollary 3.29. *If (X, τ) is either γ -locally indiscrete or γ -hyperconnected space, then the topological space (X, τ) is $\gamma\text{-}P_S\text{-}R_0$.*

Proof. It is clear. $\square$

Theorem 3.30 ([7]). *If (X, τ) is γ - P_S - T_1 space, then it is γ - P_S -symmetric.*

Theorem 3.31 ([7]). *A space (X, τ) is γ - P_S - T_1 if and only if (X, τ) is γ - P_S - T_0 and γ - P_S -symmetric.*

The relation between the γ - P_S - R_0 and γ - P_S - T_1 spaces are shown in the following theorem.

Theorem 3.32. *If (X, τ) is γ - P_S - T_1 space, then it is γ - P_S - R_0 .*

Proof. The proof is an immediate consequence of Corollary 3.24 and Theorem 3.30. $\square$

The converse of Theorem 3.32 does not true in general as stated in the following Example 3.33.

Example 3.33. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a, b\}, \{c\}, X\}$. Then $P_S O(X) = \tau$. Define an operation γ on τ by:

For every $A \in \tau$

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{c\} \\ X, & \text{if } A \neq \{c\}. \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, X, \{c\}\}$ and $\tau_{\gamma-P_S} O(X) = \{\emptyset, X\}$. Then it is easy to show that the space (X, τ) is γ - P_S - R_0 , but it is not γ - P_S - T_1 .

However, the converse of Theorem 3.32 is true when a space (X, τ) is γ - P_S - T_0 as shown in the following Corollary 3.34.

Corollary 3.34. *Let (X, τ) be a topological space and γ be an operation on τ . Then (X, τ) is γ - P_S - T_1 if and only if (X, τ) is γ - P_S - R_0 and γ - P_S - T_0 .*

Proof. This is an immediate consequence of Corollary 3.24 and Theorem 3.31. $\square$

Theorem 3.35. *The following conditions are equivalent for a topological space (X, τ) with an operation γ on τ :*

- (i) X is γ - P_S - R_0 .
- (ii) If F is γ - P_S -closed set in X , then $F = \gamma$ - P_S ker(F).
- (iii) If F is γ - P_S -closed set in X containing x , then γ - P_S ker($\{x\}$) $\subseteq F$.
- (iv) γ - P_S ker($\{x\}$) $\subseteq \tau_{\gamma-P_S} Cl(\{x\})$ for every element $x \in X$.

Proof. (i) $\Rightarrow$ (ii) Let F be a γ - P_S -closed set in a γ - P_S - R_0 space (X, τ) and $x \notin F$. Then $X \setminus F$ is γ - P_S -open set such that $x \in X \setminus F$. By hypothesis, $\tau_{\gamma-P_S} Cl(\{x\}) \subseteq X \setminus F$. This implies that $\tau_{\gamma-P_S} Cl(\{x\}) \cap F = \emptyset$. Hence by Theorem 3.10, $x \notin \gamma$ - P_S ker(F). So γ - P_S ker(F) $\subseteq F$. But in general, $F \subseteq \gamma$ - P_S ker(F) (by Remark 3.8 (i)). This follows that $F = \gamma$ - P_S ker(F).

(ii) $\Rightarrow$ (iii) For any γ - P_S -closed set F in X containing x , $\{x\} \subseteq F$ which implies that γ - $P_Sker(\{x\}) \subseteq \gamma$ - $P_Sker(F) = F$.

(iii) $\Rightarrow$ (iv) Since τ_γ - $P_SCl(\{x\})$ is γ - P_S -closed containing x . Then by (3), γ - $P_Sker(\{x\}) \subseteq \tau_\gamma$ - $P_SCl(\{x\})$.

(iv) $\Rightarrow$ (i) Suppose $x \in \tau_\gamma$ - $P_SCl(\{y\})$, then by Theorem 3.11, $y \in \gamma$ - $P_Sker(\{x\})$. By using (iv), we get $y \in \tau_\gamma$ - $P_SCl(\{x\})$. So $x \in \tau_\gamma$ - $P_SCl(\{y\})$ implies $y \in \tau_\gamma$ - $P_SCl(\{x\})$. Similarly, we can show that $y \in \tau_\gamma$ - $P_SCl(\{x\})$ implies that $x \in \tau_\gamma$ - $P_SCl(\{y\})$. Therefore, by Theorem 3.23, a space (X, τ) is γ - P_S - R_0 . $\square$

Definition 3.36 ([5]). Let (X, τ) and (Y, σ) be two topological spaces. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called (γ, β) - P_S -irresolute at a point $x \in X$ if for each β - P_S -open set V of Y containing $f(x)$, there exists a γ - P_S -open set U of X containing x such that $f(U) \subseteq V$. If f is (γ, β) - P_S -irresolute at every point x in X , then f is said to be (γ, β) - P_S -irresolute.

Theorem 3.37 ([5]). *The following properties are equivalent for any function $f: (X, \tau) \rightarrow (Y, \sigma)$, where γ and β are operations on τ and σ respectively.*

(i) f is (γ, β) - P_S -irresolute.

(ii) The inverse image of every β - P_S -open set of Y is γ - P_S -open set in X .

(iii) The inverse image of every β - P_S -closed set of Y is γ - P_S -closed set in X .

Definition 3.38 ([5]). A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called (γ, β) - P_S -closed if for every γ - P_S -closed set V of X , then $f(V)$ is β - P_S -closed set in Y .

Theorem 3.39. *Let γ and β be operations on τ and σ respectively. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective (γ, β) - P_S -irresolute and (γ, β) - P_S -closed function and (X, τ) is γ - P_S - R_0 , then (Y, σ) is also β - P_S - R_0 .*

Proof. Let U be any β - P_S -open set in (Y, σ) and y be any point in U . Since f is (γ, β) - P_S -irresolute, then by Theorem 3.37 (ii), $f^{-1}(U)$ is γ - P_S -open set in (X, τ) . Since (X, τ) is γ - P_S - R_0 space, for a point $x \in f^{-1}(y)$, τ_γ - $P_SCl(\{x\}) \subseteq f^{-1}(U)$. Since f is (γ, β) - P_S -closed, then σ_β - $P_SCl(\{y\}) = \sigma_\beta$ - $P_SCl(\{f(x)\}) \subseteq f(\tau_\gamma$ - $P_SCl(\{x\})) \subseteq U$. Therefore, (Y, σ) is β - P_S - R_0 space. $\square$

Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation γ on τ is γ -continuous [9] if $f^{-1}(V)$ is γ -open set in X , for every open set V of Y . Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation β on σ is β -open [2] if the image of every open set in X is open set in Y .

Lemma 3.40 ([5]). *If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is both γ -continuous and β -open, then f is (γ, β) - P_S -irresolute.*

Corollary 3.41. *Let γ and β be operations on τ and σ respectively. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective (γ, β) - P_S -closed, γ -continuous and β -open function and (X, τ) is γ - P_S - R_0 , then (Y, σ) is also β - P_S - R_0 .*

Proof. Follows directly from Theorem 3.39 and Lemma 3.40. $\square$

Definition 3.42. A topological space (X, τ) with an operation γ on τ is said to be weakly γ - P_S - R_0 if $\bigcap_{x \in X} \tau_\gamma P_S Cl(\{x\}) = \emptyset$.

Theorem 3.43. Let (X, τ) be a topological space and γ be an operation on τ . Then (X, τ) is weakly γ - P_S - R_0 if and only if γ - P_S ker($\{x\}) \neq X$ for every $x \in X$.

Proof. Let (X, τ) be a weakly γ - P_S - R_0 space. Suppose that γ - P_S ker($\{y\}) = X$ for every $y \in X$. Then $\{y\} \neq G$ where G is any proper γ - P_S -open subset of X . Then $y \in \tau_\gamma P_S Cl(\{x\})$ for every $x \in X$ and hence $y \in \bigcap_{x \in X} \tau_\gamma P_S Cl(\{x\})$. This is a contradiction.

Conversely, let γ - P_S ker($\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \bigcap_{x \in X} \tau_\gamma P_S Cl(\{x\})$. Then every γ - P_S -open set containing y must contain every point of x . Hence X is the unique γ - P_S -open set containing y . So γ - P_S ker($\{y\}) = X$. This is a contradiction. Therefore, (X, τ) is weakly γ - P_S - R_0 space. $\square$

The following Remark 3.44 follows directly from Definitions 3.1 and 3.42.

Remark 3.44. Every γ - P_S - R_0 space is weakly γ - P_S - R_0 .

But in general the converse of the above Remark 3.44 does not hold as shown by the following Example 3.45.

Example 3.45. In Example 3.9, the space (X, τ) is weakly γ - P_S - R_0 , but it is not γ - P_S - R_0 since $\{a, c\}$ is γ - P_S -open set in X containing c such that $\tau_\gamma P_S Cl(\{c\}) = \{b, c\} \not\subseteq \{a, c\}$.

Remark 3.46. Let γ and β be operations on τ and σ respectively. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an injective (γ, β) - P_S -closed function and (X, τ) is weakly γ - P_S - R_0 , then (Y, σ) is also weakly β - P_S - R_0 .

4. Properties of γ - P_S - R_1 spaces

Now, γ - P_S - R_1 space in terms of γ - P_S -open and $\tau_\gamma P_S$ -closure operator is defined as follows:

Definition 4.1. A topological space (X, τ) with an operation γ on τ is said to be γ - P_S - R_1 if for x and y in X with $\tau_\gamma P_S Cl(\{x\}) \neq \tau_\gamma P_S Cl(\{y\})$, there exist disjoint γ - P_S -open sets G and H such that $\tau_\gamma P_S Cl(\{x\}) \subseteq G$ and $\tau_\gamma P_S Cl(\{y\}) \subseteq H$.

The relation between the γ - P_S - R_0 and γ - P_S - R_1 spaces are shown in the following theorem.

Theorem 4.2. If (X, τ) is γ - P_S - R_1 space, then it is γ - P_S - R_0 .

Proof. Let G be any γ - P_S -open set in γ - P_S - R_1 space (X, τ) containing x . Let $y \notin G$. If $y \notin G$, and since $x \notin \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$. Then $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$. Thus, there exists a γ - P_S -open set H such that $\tau_\gamma\text{-}P_S\text{-}Cl(\{y\}) \subseteq H$ and $x \notin H$. This implies that $y \notin \tau_\gamma\text{-}P_S\text{-}Cl(\{x\})$ and hence $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \subseteq G$. Therefore, the space (X, τ) is γ - P_S - R_0 . $\square$

But the converse of the above Theorem 4.2 does not true in general as seen from the following example.

Example 4.3. Let a space $X = \{x, y, z\}$ with the discrete topology τ on X . Define an operation γ on τ by: for every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{x, y\} \text{ or } \{x, z\} \text{ or } \{y, z\} \\ X & \text{if otherwise} \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, \{x, y\}, \{x, z\}, \{y, z\}, X\} = \tau_\gamma\text{-}P_S\text{-}O(X)$. Then the space (X, τ) is γ - P_S - R_0 , but it is not γ - P_S - R_1 since for the points $x, y \in X$ such that $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) = \{x\} \neq \{y\} = \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$, but there is no disjoint γ - P_S -open sets G and H such that $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \subseteq G$ and $\tau_\gamma\text{-}P_S\text{-}Cl(\{y\}) \subseteq H$.

Theorem 4.4 ([7]). *Let (X, τ) be a topological space and γ be an operation on τ . Then X is γ - P_S - T_1 if and only if every singleton set in X is γ - P_S -closed.*

Lemma 4.5 ([7]). *Let (X, τ) be a topological space and γ be an operation on τ . Then the following statements are holds:*

- (i) *If X is γ - P_S - T_2 , then it is γ - P_S - T_1 .*
- (ii) *If X is γ - P_S - T_1 , then it is γ - P_S - $T_{\frac{1}{2}}$.*
- (iii) *If X is γ - P_S - $T_{\frac{1}{2}}$, then it is γ - P_S - T_0 .*

Theorem 4.6. *If (X, τ) is γ - P_S - T_2 space, then it is γ - P_S - R_1 .*

Proof. Let x and y be any elements in γ - P_S - T_2 space (X, τ) such that $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}P_S\text{-}Cl(\{y\})$. Then by Lemma 4.5 (i), (X, τ) is γ - P_S - T_1 and hence by Theorem 4.4, $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) = \{x\}$ and $\tau_\gamma\text{-}P_S\text{-}Cl(\{y\}) = \{y\}$ for every $x, y \in X$. Then $x \neq y$ and hence by hypothesis, there exist disjoint γ - P_S -open sets G and H containing x and y respectively. This follows that $\tau_\gamma\text{-}P_S\text{-}Cl(\{x\}) \subseteq G$ and $\tau_\gamma\text{-}P_S\text{-}Cl(\{y\}) \subseteq H$. Consequently, (X, τ) is γ - P_S - R_1 space. $\square$

So, the converse of the Theorem 4.6 is not true in general as described in the following Example 4.7.

Example 4.7. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define an operation $\gamma: \tau \rightarrow P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$. Hence $\tau_\gamma\text{-}P_S\text{-}O(X) = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. Hence the space (X, τ) is γ - P_S - R_1 , but it is not γ - P_S - T_2 .

The converse of the Theorem 4.6 is only true when a space (X, τ) is γ - P_S - T_0 as shown in the following Theorem 4.8.

Theorem 4.8. *Let (X, τ) be a topological space and γ be an operation on τ . Then (X, τ) is γ - P_S - T_2 if and only if (X, τ) is γ - P_S - R_1 and γ - P_S - T_0 .*

Proof. Let x and y be any distinct points in X and (X, τ) be a γ - P_S - R_1 and γ - P_S - T_0 space. Then by Theorem 4.2, (X, τ) is γ - P_S - R_0 and hence by Corollary 3.34, (X, τ) is γ - P_S - T_1 . Then by Theorem 4.4, τ_γ - P_S - $Cl(\{x\}) = \{x\}$ and τ_γ - P_S - $Cl(\{y\}) = \{y\}$ for every distinct points $x, y \in X$. This means that τ_γ - P_S - $Cl(\{x\}) \neq \tau_\gamma$ - P_S - $Cl(\{y\})$ and hence by Definition 4.1, there exist disjoint γ - P_S -open sets G and H such that τ_γ - P_S - $Cl(\{x\}) \subseteq G$ and τ_γ - P_S - $Cl(\{y\}) \subseteq H$. It follows that $x \in G$ and $y \in H$. Therefore, (X, τ) is γ - P_S - T_2 .

Conversely, if (X, τ) is γ - P_S - T_2 space, then by Lemma 4.5 and Theorem 4.6, (X, τ) is γ - P_S - T_0 and γ - P_S - R_1 respectively. This completes the proof. $\square$

Therefore, from Theorem 4.6, Theorem 4.8 and Lemma 4.5, we have the following corollary.

Corollary 4.9. *Let (X, τ) be any topological space and γ be an operation on τ . Then the following statements are equivalent:*

- (i) (X, τ) is γ - P_S - R_1 and γ - P_S - T_0 .
- (ii) (X, τ) is γ - P_S - R_1 and γ - P_S - $T_{\frac{1}{2}}$.
- (iii) (X, τ) is γ - P_S - R_1 and γ - P_S - T_1 .
- (iv) (X, τ) is γ - P_S - T_2 .

Proof. It is clear and hence it is omitted. $\square$

Recall that a topological space (X, τ) with an operation γ on τ is γ - P_S -regular if for each γ - P_S -closed set F of X not containing x , there exist disjoint γ - P_S -open sets G and H such that $x \in G$ and $F \subseteq H$.

Lemma 4.10 ([1]). *If (X, τ) is γ - P_S -regular and γ - P_S - T_1 space. Then it is γ - P_S - T_2 .*

Corollary 4.11. *If (X, τ) is γ - P_S - T_1 space. Then every γ - P_S -regular space (X, τ) is γ - P_S - R_1 .*

Proof. This is an immediate consequence of Lemma 4.10 and Theorem 4.6. $\square$

Recall that a topological space (X, τ) with an operation γ on τ is γ - P_S -normal [1] if for each pair of disjoint γ - P_S -closed sets E, F of X , there exist disjoint γ - P_S -open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

Lemma 4.12 ([1]). *If (X, τ) is γ - P_S - T_1 space. Then every γ - P_S -normal space X is γ - P_S -regular.*

The proof of the following corollary is follows directly from Corollary 4.11 and Lemma 4.12.

Corollary 4.13. *If (X, τ) is γ - P_S -normal and γ - P_S - T_1 space. Then it is γ - P_S - R_1 .*

Proof. Obvious. $\square$

The following are some significant characterizations of γ - P_S - R_1 spaces.

Remark 4.14. A topological space (X, τ) is γ - P_S - R_1 if and only if for x and y in X with $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$, there exist disjoint γ - P_S -closed sets E and F such that $x \in E$, $y \notin E$, $y \in F$, $x \notin F$ and $E \cup F = X$.

Theorem 4.15. *A topological space (X, τ) is γ - P_S - R_1 if and only if for x and y in X with $\tau_\gamma\text{-}P_S\text{Sk}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Sk}(\{y\})$, there exist disjoint γ - P_S -open sets G and H such that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \subseteq G$ and $\tau_\gamma\text{-}P_S\text{Cl}(\{y\}) \subseteq H$.*

Proof. Follows directly from Theorem 3.19. $\square$

Theorem 4.16. *A topological space (X, τ) is γ - P_S - R_1 if and only if for $x \in X \setminus \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ implies that x and y have disjoint γ - P_S -open neighborhoods.*

Proof. Suppose that $x \in X \setminus \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. Then $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ and hence x and y have disjoint γ - P_S -open neighborhoods.

Conversely, first, to show that (X, τ) is γ - P_S - R_0 . Let W be a γ - P_S -open set and $x \in W$. Suppose that $y \notin W$. Then $\tau_\gamma\text{-}P_S\text{Cl}(\{y\}) \cap W = \emptyset$ and $x \notin \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. There exist γ - P_S -open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \subseteq \tau_\gamma\text{-}P_S\text{Cl}(U)$ and $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \cap V \subseteq \tau_\gamma\text{-}P_S\text{Cl}(U) \cap V = \emptyset$. Therefore, $y \notin \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$. Consequently, $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \subseteq G$ and (X, τ) is γ - P_S - R_0 . Next, to show that (X, τ) is γ - P_S - R_1 . Suppose that $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. Then, we can assume that there exists $p \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$ such that $p \notin \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. There exist γ - P_S -open sets G and H such that $p \in G$, $y \in H$ and $G \cap H = \emptyset$. Since $p \in \tau_\gamma\text{-}P_S\text{Cl}(\{x\})$, $x \in G$. Since (X, τ) is γ - P_S - R_0 , we obtain $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \subseteq G$, $\tau_\gamma\text{-}P_S\text{Cl}(\{y\}) \subseteq H$ and $G \cap H = \emptyset$. This shows that (X, τ) is γ - P_S - R_1 space. $\square$

Corollary 4.17. *A topological space (X, τ) is γ - P_S - R_1 if and only if $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ implies that x and y have disjoint γ - P_S -open neighborhoods.*

Proof. Assume that $x \in X \setminus \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$. Then $\tau_\gamma\text{-}P_S\text{Cl}(\{x\}) \neq \tau_\gamma\text{-}P_S\text{Cl}(\{y\})$ and hence by hypothesis, x and y have disjoint γ - P_S -open neighborhoods. Thus by Theorem 4.16, (X, τ) is γ - P_S - R_1 space. The proof of the converse part is obvious. $\square$

5. Conclusion

This study introduces some types of γ - P_S - separation axioms such as γ - P_S - R_0 and γ - P_S - R_1 by using γ - P_S -open sets and $\tau_\gamma\text{-}P_S$ -closure of a set. Some properties of these spaces have been constructed.

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Synchronization of two three-species food chain system with Beddington-DeAngelis functional response using active controllers based on the Lyapunov function

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Abstract. In this paper master-slave scheme is used to investigate the problem of synchronization of two three species models with Beddington-DeAngelis type of functional response. Using Lyapunov stability theory the analytical expression for nonlinear active controllers have been deduced for the synchronization of (I) two identical three-species web chain but with different growth rate and (II) two identical chaotic three-species food chain with different initial conditions. The bifurcation diagram of the food chain system has been used to find the intrinsic growth rate parameter for which the dynamics of three-species food chain exhibit either regular or chaotic behavior in phase space. Based on the chosen intrinsic growth parameters that characterizes the complex dynamics of the master and slave system, the active controller functions, have been obtained, which causes global asymptotic stability of synchronization errors, for the case I and II respectively. The efficacy of the controller functions, in the synchronization process, is demonstrated using the time series plot of species population and phase plots before and after the activation.

Keywords: Bennington-DeAngelis response, active control, Lyapunov function, Chaos control.

1. Introduction

Chaotic systems and attractors have been investigated in a variety of fields including physics [18], biology [14], chemistry [8], marketing and finance [28,2]. Chaotic systems are sensitive to initial conditions and despite being identical

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can lead to different trajectories as a result of slightly different initial conditions. As a result the problem of synchronization of chaotic system becomes very challenging. The possibility of synchronization of two identical chaotic system with different initial conditions was first demonstrated by Pecora and Carroll [30].

Soon after several methods like adaptive control method [43], sliding mode control method [17,40], nonlinear feedback method [24], active control method [1,4,12,39,41], back-stepping design [45], impulsive control [44], passive control [42] and anti-synchronization method [38] have been proposed for synchronization of chaotic system in different areas. Further, in recent years the problems of chaos control dynamics and synchronization has been investigated in specific systems i.e., Rossler and Chen system [22], Hasting Powell food chain model [23], coupled electrical oscillators [27], tritrophic food chain meta-communities [6], supply chain systems [16,47], delay coupled heterogeneous networks [32], Chua and Lure system [34]. The interdisciplinary nature of the problem therefore led to significant developments in the area of synchronization of chaotic dynamical systems in network of oscillators [31].

Several investigations in ecology and animal sciences [7,10,19,21,26,33,5] have revealed plant and animal species showing synchronous behavior over large areas as a consequence of population dispersal and meteorological driving forces. Therefore in ecological systems, study of chaotic behavior and its control through synchronization is a subject of great interest. Modeling of ecological food chains is significant to promote mutual coexistence and sustainable growth of species. In modeling of ecological systems one of the central goals is to understand the predator-prey interactions and its dynamics. The functional response of the predator is one of the most significant aspect of prey-predator relationship [5,13,15,20,46]. In recent years the complex dynamics of predator prey system has been extensively studied using various forms of function response [3,11,48]. Further, chaotic dynamics is observed in a system involving three-species over a wide range of parameter space [9,11,25,29,36,48].

The three species food chain considered involves at the top level super predators preying on the middle level predators which subsequently preys on the prey at the lowest level. The model further incorporates a growth rate term that includes the resource utilization efficiency [46]. However, the population interaction between the species is described by Beddington-DeAngelis functional response [43]. In section 2, we briefly outline the details of various features of the model including the linear stability analysis of the model [48]. The bifurcation diagram of this model exhibit complex chaotic dynamics i.e., a limit cycle, period-doubling and chaos depending on the value of the chosen control parameter, say, the intrinsic growth rate parameter of the species at the bottom level. In section 3, the problem of synchronization of two such identical three-species food chains is investigated using the Lyapunov stability theory for stabilizing the synchronizations errors. The controllers have been designed using the method of active control in the master-slave framework for synchronization involving (I)

two identical three-species food web chain but with different intrinsic growth rate parameter.

Numerical results are presented for the settings when the master three-species food chain behavior is described by a (i) limit cycle and (ii) chaotic attractor while the slave system is chaotic in the absence of any control. In both the cases, the application of the designed controller function is shown to result in synchronization of the slave system with the master. The mathematical analysis is also extended to study the synchronization of (II) two identical but chaotic three-species food chains. The efficacy of the derived nonlinear controller is shown using the time series and the phase plots before and after the onset of synchronization. At the end, in section 4, we summarize the main results of our work.

2. Mathematical model

The basic equation governing the dynamics of three species food chain model comprising of prey X , predator Y and a super predator Z with Beddington-DeAngelis type functional response can be described by the following equations[48]:

$$\begin{aligned} (1) \quad \dot{X} &= \frac{R_1(J_0 - X)X}{J_1 - X} - K_1(X, Y)Y, \\ (2) \quad \dot{Y} &= L_1 K_1(X, Y)Y - K_2(Y, Z)Z - M_1 Y, \\ (3) \quad \dot{Z} &= L_2 K_2(Y, Z)Z - M_2 Z, \end{aligned}$$

where $K_1(X, Y) = \frac{E_1 X}{F_1 Y + G_1 X + H_1}$ and $K_2(Y, Z) = \frac{E_2 Y}{F_2 Z + G_2 Y + H_2}$ are the Beddington-DeAngelis type functional response.

For the two responses E_i , F_i and G_i ($i = 1, 2$) are the saturating parameters. The maximum harvest rate of predator from prey is given by $\frac{E_i}{H_i}$ while constants $\frac{H_i}{F_i}$ and $\frac{H_i}{G_i}$ ($i = 1, 2$) represent half saturation. Intrinsic growth rate of prey species X is denoted by R_1 and $R_1 J_0$ is the carrying capacity. For the available resources J_1 is the limiting value. It becomes equivalent to carrying capacity if no resources are wasted in the system under ideal circumstances which is not possible in real systems where resources get wasted. Efficiency of nutrient utilization in species X is given by the ration $\frac{J_0}{J_1}$ and is either zero or one. For all times $X < J_0 < J_1$ and when the ratio $\frac{J_0}{J_1}$ is close to one it implies higher efficiency of nutrient utilization while if the ratio is closer to zero it implies lower efficiency indicating restricting of increment in population by limiting of resources. L_i and M_i represent the prey to predator conversion rate and species death rate of Y and Z respectively. The system parameters R_0 , J_0 , J_1 , E_i , F_i , G_i , H_i and M_i , ($i = 1, 2$) have positive values and state space of the system is given as $R_t^3 = \{(X, Y, Z) \in R^3 : X \geq 0, Y \geq 0, Z \geq 0\}$.

In terms of new variables x , y and z defined as,

$$x = \frac{X}{H_1}, \quad y = \frac{Y}{H_1}, \quad z = \frac{Z}{H_1} \quad \text{and} \quad t = M_1 T,$$

we may rewrite the system equations (1)-(3) in the following form,

$$(4) \quad \dot{x} = \frac{r_1 x(k_0 - x)}{(k_1 - x)} - \frac{\alpha_1 xy}{\alpha_2 y + \alpha_3 x + 1},$$

$$(5) \quad \dot{y} = \frac{\alpha_4 xy}{\alpha_2 y + \alpha_3 x + 1} - \frac{\alpha_5 yz}{\alpha_6 z + \alpha_7 y + \alpha_8} - y,$$

$$(6) \quad \dot{z} = \frac{\alpha_9 yz}{\alpha_6 z + \alpha_7 y + \alpha_8} - \alpha_{10} z,$$

where $r_1 = \frac{R_1}{M_1}$, $k_0 = \frac{J_0}{H_1}$, $k_1 = \frac{J_1}{H_1}$, $\alpha_1 = \frac{E_1}{M_1}$, $\alpha_2 = F_2$, $\alpha_3 = G_1$, $\alpha_4 = \frac{E_1 L_1}{M_1}$, $\alpha_5 = \frac{E_2}{M_1}$, $\alpha_6 = F_2$, $\alpha_7 = G_2$, $\alpha_8 = \frac{H_2}{H_1}$, $\alpha_9 = \frac{E_2 L_2}{M_1}$ and $\alpha_{10} = \frac{M_2}{M_1}$.

The Jacobian for system eqs.(4)-(6) may be written as,

$$(7) \quad J = \begin{bmatrix} x \frac{\partial \Gamma}{\partial x} + \Gamma & x \frac{\partial \Gamma}{\partial y} & x \frac{\partial \Gamma}{\partial z} \\ y \frac{\partial \Omega}{\partial x} & y \frac{\partial \Omega}{\partial y} + g & y \frac{\partial \Omega}{\partial z} \\ z \frac{\partial \Theta}{\partial x} & z \frac{\partial \Theta}{\partial y} & z \frac{\partial \Theta}{\partial z} + \Theta \end{bmatrix},$$

where $\Gamma(x, y, z) = \frac{r_1(k_0 - x)}{(k_1 - x)} - \frac{\alpha_1 y}{\alpha_2 y + \alpha_3 x + 1}$, $\Omega(x, y, z) = \frac{\alpha_4 x}{\alpha_2 y + \alpha_3 x + 1} - \frac{\alpha_5 z}{\alpha_6 z + \alpha_7 y + \alpha_8} - 1$ and $\Theta(x, y, z) = \frac{\alpha_9 y}{\alpha_6 z + \alpha_7 y + \alpha_8} - \alpha_{10}$.

The system eqs.(4)-(6) has four positive equilibrium points (i) $E_1(0, 0, 0)$, (ii) $E_2(k_0, 0, 0)$, (iii) $E_3(x^*, y^*, 0)$ and (iv) $E_4(\bar{x}, \bar{y}, \bar{z})$.

Following [48], we briefly consider the problem of linear stability of the equilibrium points. For linear stability analysis at any of these equilibrium points, the Jacobian matrix of the system needs to be evaluated using the following values of the various biological relevant parameters:

$$k_0 = 3, \quad k_1 = 4, \quad \alpha_1 = 4.0, \quad \alpha_2 = 0.20, \quad \alpha_3 = 0.8, \quad \alpha_4 = 3.0, \quad \alpha_5 = 2.0, \quad \alpha_6 = 0.15,$$

$$\alpha_7 = 0.80, \quad \alpha_8 = 0.50, \quad \alpha_9 = 1.5, \quad \alpha_{10} = 0.20.$$

(i) The equilibrium point $E_1 = (0, 0, 0)$ is found to be a saddle point as the eigenvalues of the Jacobian matrix are $\lambda_1 = \frac{r_0 k_0}{k_1} > 0$, $\lambda_2 = -1 < 0$ and $\lambda_3 = -\alpha_{10} < 0$.

(ii) For the equilibrium point $E_2 = (k_0, 0, 0)$, the eigenvalues of the corresponding Jacobian matrix are $\lambda_1 = -\frac{r_0 k_0}{k_1 - k_0} < 0$, $\lambda_2 = \frac{k_0(\alpha_4 - \alpha_3) - 1}{k_0 \alpha_3 + 1} > 0$ and $\lambda_3 = -\alpha_{10} < 0$. Therefore the equilibrium point is a saddle point.

(iii) The equilibrium point $E_3(x^*, y^*, 0)$ suggest the possibility of survival of the predator y in the absence of the super-predator z . In the absence of the super-predator, the subsystem comprising of eqs.(4)-(5) only results in the

following positive solutions,

$$(8) \quad y^* = \frac{(\alpha_4 - \alpha_3)x^* - 1}{\alpha_2}, \quad x^*(\alpha_4 - \alpha_3) > 1,$$

$$(9) \quad x^* = \frac{\chi_2 \pm \sqrt{\chi_2^2 - 4\chi_1\chi_3}}{2\chi_1}, \quad \chi_1 > 0 \quad \text{and} \quad (\chi_2^2 - 4\chi_1\chi_3) > 0,$$

where $\chi_1 = [\alpha_1(\alpha_4 - \alpha_3) - r_0\alpha_2\alpha_4]$, $\chi_2 = [\alpha_1(\alpha_4 - \alpha_3)k_1 - r_0k_0\alpha_2\alpha_4 + \alpha_1]$ and $\chi_3 = \alpha_1k_1$. The stability of the equilibrium point (x^*, y^*) corresponding to the subsystem may be investigated using the following Jacobian matrix,

$$(10) \quad J(x^*, y^*) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

where

$$(11) \quad \begin{aligned} J_{11} &= x^* \left[\frac{r_1(k_0 - k_1)}{(k_1 - x^*)^2} + \frac{\alpha_3\alpha_1y^*}{(\alpha_2y^* + \alpha_3x^* + 1)^2} \right], \\ J_{12} &= \left[\frac{-\alpha_1x^*(\alpha_3x^* + 1)}{(\alpha_2y^* + \alpha_3x^* + 1)^2} \right], \\ J_{21} &= \left[\frac{\alpha_4y^*(\alpha_2y^* + 1)}{(\alpha_2y^* + \alpha_3x^* + 1)^2} \right], \\ J_{22} &= \left[\frac{-\alpha_2\alpha_4x^*y^*}{(\alpha_2y^* + \alpha_3x^* + 1)^2} \right]. \end{aligned}$$

The condition for stability of the equilibrium point for the subsystem is that $\Delta_1 = -(J_{11} + J_{22}) > 0$ and $\Delta_2 = (J_{11}J_{22} - J_{12}J_{21}) > 0$.

The stability of the third equilibrium point $E_3(x^*, y^*, 0)$ can be ensured only if the equilibrium solution of the subsystem $E(x^*, y^*)$ is stable. The three eigenvalues of the Jacobian matrix for the equilibrium solution $E_3(x^*, y^*, 0)$ satisfies the following constraints:

$$(12) \quad \begin{aligned} \lambda_1 + \lambda_2 &= x^* \left[\frac{r_1(k_0 - k_1)}{(k_1 - x^*)^2} + \frac{(\alpha_3\alpha_1 - \alpha_2\alpha_4)x^*y^*}{(\alpha_2y^* + \alpha_3x^* + 1)^2} \right], \\ \lambda_1\lambda_2 &= -\frac{r_1(k_0 - k_1)\alpha_2x^*y^*}{(k_1 - x^*)^2(\alpha_2y^* + \alpha_3x^* + 1)} + \frac{\alpha_1y^*}{(\alpha_2y^* + \alpha_3x^* + 1)^2}, \\ \lambda_3 &= \frac{\alpha_9y^*}{(\alpha_7y + \alpha_8)} - \alpha_{10}. \end{aligned}$$

The stability of the equilibrium solution $E_3(x^*, y^*, 0)$ in the direction of increasing z requires $\lambda_3 < 0$ or $\alpha_{10} > \frac{\alpha_9y^*}{\alpha_7y^* + \alpha_8}$.

(iv) The equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ corresponds to the coexistence of the species. From the system eqs.(4)-(6), it is readily observed that

$$(13) \quad \begin{aligned} \bar{y} &= \frac{\alpha_{10}(\alpha_8 + \alpha_6\bar{z})}{\alpha_9 - \alpha_{10}\alpha_7}, \\ \bar{z} &= \left[\frac{u_4(k_0 - \bar{x})\bar{x}}{u_1(k_1 - \bar{x})} - \frac{u_2}{u_1} \right], \\ \bar{x} &= \frac{u_3 \pm (u_3^2 - 4u_2k_1u_4)^{1/2}}{2u_4}, \end{aligned}$$

where $u_1 = \alpha_1\alpha_6\alpha_9\alpha_{10} + \alpha_1\alpha_5\alpha_{10}(\alpha_9 - \alpha_7\alpha_{10})$, $u_2 = \alpha_1\alpha_8\alpha_9\alpha_{10}$, $u_3 = u_2 + r_1\alpha_4\alpha_9(\alpha_9 - \alpha_{10}\alpha_7)k_0$, and $u_4 = r_1\alpha_4\alpha_9(\alpha_9 - \alpha_{10}\alpha_7)$.

The linear stability analysis of the equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ of the system results in the following characteristic equation for the Jacobian $J(\bar{x}, \bar{y}, \bar{z})$ matrix,

$$(14) \quad \lambda^3 + \omega_1\lambda^2 + \omega_2\lambda + \omega_3 = 0.$$

where the coefficients $\omega_1 = -(P_{11} + P_{22} + P_{33})$, $\omega_2 = P_{11}P_{22} - P_{12}P_{21} + P_{22}P_{33} - P_{23}P_{32} + P_{11}P_{33}$, $\omega_3 = P_{11}P_{23}P_{32} - P_{11}P_{22}P_{33} + P_{12}P_{21}P_{33}$ and further the various elements P_{ij} , $i, j = 1, 2, 3$ of the Jacobian matrix are,

$$(15) \quad \begin{aligned} P_{11} &= \bar{x} \left[\frac{r_1(k_0 - k_1)}{(k_1 - \bar{x})^2} + \frac{\alpha_3\alpha_1\bar{y}}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} \right], \\ P_{12} &= -\frac{\alpha_1\bar{x}(\alpha_3\bar{x} + 1)}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} < 0, \\ P_{21} &= \frac{\alpha_4\bar{y}(\alpha_2\bar{y} + 1)}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} > 0, \\ P_{22} &= \bar{y} \left[-\frac{\alpha_2\alpha_4\bar{x}}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} + \frac{\alpha_5\alpha_7\bar{z}}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2} \right], \\ P_{23} &= -\frac{\alpha_5\bar{y}(\alpha_7\bar{y} + \alpha_8)}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2} < 0, \quad P_{13} = 0, \quad P_{31} = 0 \\ P_{32} &= \frac{\alpha_9\bar{z}(\alpha_6\bar{z} + \alpha_8)}{((\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2)} > 0, \quad P_{33} = -\frac{\alpha_6\alpha_9\bar{z}\bar{y}}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2} < 0. \end{aligned}$$

Following Routh-Hurwitz criterion, the equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ is asymptotically stable if $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1\omega_2 > \omega_3$. Violation of these conditions results in limit cycle, period doubling bifurcation leading to chaotic behavior of the system. In order to find the parameter range over which dynamical transition occur, we have computed the bifurcation diagram (Fig.1) of the system viz. eqs.(4)-(6). The bifurcation diagram reveals the dynamical complexity of the system based on simulation results on varying the intrinsic growth rate parameter in the range $1.8 \leq r_0 \leq 8.0$ while retaining the values of other parameters of the system. The transition from limit cycle behavior for $r_0 < 2.5$ to period doubling cascade is initiated in the region $2.5 \leq r_0 \leq 4.2$. The system dynamics is observed to be chaotic for $4.2 \leq r_0 \leq 6.8$. The phenomena of reverse period doubling is observed beyond $r_0 = 6.8$.

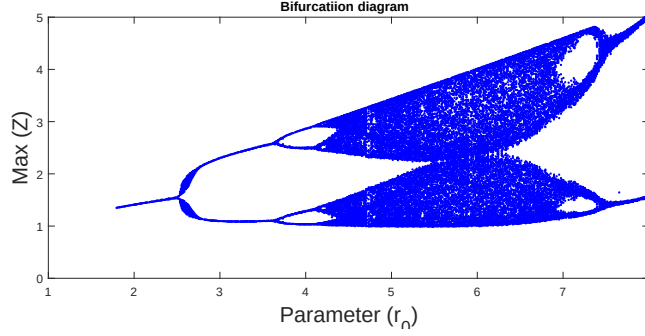


Figure 1: Bifurcation Diagram for Three species food chain model with Beddington-DeAngelis type functional response.

3. Synchronization of two identical type of three species food chains but with different growth rate

In this section, we consider the problem of synchronization of (a) two different three-species food web chains and (b) two identical three-species food web chain with different initial conditions.

3.1 Synchronization of two different three-species food chains

The three-species food web models considered here are similar to the system, eqs.(4)-(6). The master and slave system are therefore described by the following sets of coupled nonlinear differential equations,

Master system:

$$\begin{aligned}
 \dot{x}_1 &= \frac{r_1 x_1 (k_0 - x_1)}{(k_1 - x_1)} - \frac{\alpha_1 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1}, \\
 \dot{y}_1 &= \frac{\alpha_4 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1} - \frac{\alpha_5 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - y_1, \\
 \dot{z}_1 &= \frac{\alpha_9 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - \alpha_{10} z_1.
 \end{aligned}
 \tag{16}$$

Slave system:

$$\begin{aligned}
 \dot{x}_2 &= \frac{r_2 x_2 (k_0 - x_2)}{(k_1 - x_2)} - \frac{\alpha_1 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} + \nu_1, \\
 \dot{y}_2 &= \frac{\alpha_4 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} - \frac{\alpha_5 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - y_2 + \nu_2, \\
 \dot{z}_2 &= \frac{\alpha_9 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - \alpha_{10} z_2 + \nu_3.
 \end{aligned}
 \tag{17}$$

where r_1 and r_2 corresponds to intrinsic growth rate parameter of the master and slave system respectively. However, all other biologically relevant parameters are having the same meaning as in section 2 and their values are kept same

in the two food web models. The variables ν_i , $i = 1, 2, 3$ refers to the control functions to be determined for effective synchronization of the two three-species food web with different intrinsic growth parameter. Here the master and slave system for the three species food web chains differs only with respect to different intrinsic growth parameters in the absence of controller functions. However, mathematical analysis remain valid if we change the master or slave system by any other model of three species food chain [21].

The master system, eq.(16), when deducted from slave system, eq.(17), allow us to determine the dynamical system involving the state errors e_1 , e_2 and e_3 for designing the controllers. The state errors e_1 , e_2 and e_3 are thus defined as,

$$(18) \quad \begin{aligned} e_1 &= x_2 - x_1, & \dot{e}_1 &= \dot{x}_2 - \dot{x}_1 \\ e_2 &= y_2 - y_1, & \dot{e}_2 &= \dot{y}_2 - \dot{y}_1 \\ e_3 &= z_2 - z_1, & \dot{e}_3 &= \dot{z}_2 - \dot{z}_1 \end{aligned}$$

Therefore from eqs.(16)-(18), the differential equations for the state errors e_i , $i = 1, 2, 3$ could be written as,

$$(19) \quad \begin{aligned} \dot{e}_1 &= \left[\frac{(r_1 k_0 k_1 - r_1 k_1 (x_1 + x_2) + r_1 x_1 x_2 +)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 - \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_2 \\ &\quad + \epsilon \frac{[k_0 k_1 x_2 - k_1 x_2^2 - k_0 x_1 x_2 + x_1 x_2^2]}{(k_1 - x_2)(k_1 - x_1)} - \left[\frac{\alpha_1 (x_2 y_2 - x_1 y_1)}{\beta_1} \right] + \nu_1, \\ \dot{e}_2 &= \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 + \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} - 1 \right] e_2 - \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 \\ &\quad + \left[\frac{\alpha_4 (x_2 y_2 - y_2 y_1)}{\beta_2} - \frac{\alpha_5 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] + \nu_2, \\ \dot{e}_3 &= \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 + \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} \right] e_3 + \left[\frac{\alpha_9 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] + \nu_3 \end{aligned}$$

where we have defined the parameters β_1 , β_2 and the difference between the growth rate of the two food chains, ϵ , as:

$$(20) \quad \begin{aligned} \beta_1 &= (\alpha_2 y_2 + \alpha_3 x_2 + 1)(\alpha_2 y_1 + \alpha_3 x_1 + 1) \\ \beta_2 &= (\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8)(\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8) \\ \epsilon &= r_2 - r_1 \end{aligned}$$

The system defined by eq.(19) is called error system and to ensure the asymptotic stability of error system about origin, we define the Lyapunov function ϕ as,

$$(21) \quad \phi(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2).$$

From eq.(19) and eq.(21), it is observed that

$$\begin{aligned}
 \dot{\phi} = & e_1^2 \left[\frac{(r_1 k_0 k_1 - r_1 k_1 (x_1 + x_2) + r_1 x_1 x_2)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} \right] - \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_1 e_2 \\
 (22) \quad & - \left[\frac{\alpha_1 (x_2 y_2 - x_1 y_1)}{\beta_1} \right] e_1 + \epsilon \frac{[k_0 k_1 x_2 - k_1 x_2^2 - k_0 x_1 x_2 + x_1 x_2^2]}{(k_1 - x_2)(k_1 - x_1)} \\
 & + \nu_1 e_1 + \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 e_2 - \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 e_2 \\
 & + \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} - 1 \right] e_2^2 \\
 & + \left[\frac{\alpha_4 (x_2 y_2 - x_1 y_1)}{\beta_1} - \frac{\alpha_5 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] e_2 \\
 & + \nu_2 e_2 + \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 e_3 + \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} \right] e_3^2 \\
 & + \left[\frac{\alpha_9 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] e_3 + \nu_3 e_3
 \end{aligned}$$

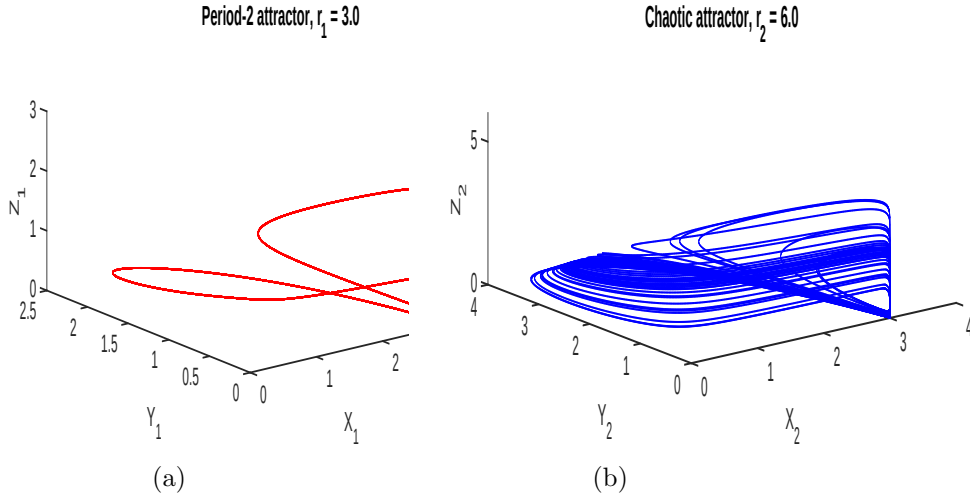


Figure 2: Phase portrait of three-species food chain (a) regular master system with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$, with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

3.1.1 Active controller functions for two three-species food chain with different intrinsic growth rate

Under active control synchronization scheme, we find the functional form of controller ν_1 , ν_2 and ν_3 such that Lyapunov function $\phi(e_1, e_2, e_3)$ satisfy the

condition $\dot{\phi} < 0$ [13,21]. There could be several control functions which possibly yield different synchronization times and satisfy the requirement that $\dot{\phi} < 0$. However, in the following, we chose the the following functional form of the active controller ν_1 , ν_2 and ν_3 such that $\dot{\phi} < 0$.

Controller function:

$$\begin{aligned}
 \nu_1 = & - \left[\frac{(r_1 k_0 k_1 - r_1 k_1 (x_1 + x_2) + r_1 x_1 x_2)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} + 1 \right] e_1 \\
 & + \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_2 + \left[\frac{\alpha_1 (x_2 y_2 - x_1 y_1)}{\beta_1} \right] \\
 & - \epsilon \frac{[k_0 k_1 x_2 - k_1 x_2^2 - k_0 x_1 x_2 + x_1 x_2^2]}{(k_1 - x_2)(k_1 - x_1)} \\
 (23) \quad \nu_2 = & - \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 + \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 - \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 \\
 & + \left[-\frac{\alpha_4 (x_2 y_2 - x_1 y_1)}{\beta_1} + \frac{\alpha_5 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] \\
 \nu_3 = & - \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 - \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} + 1 \right] e_3 - \left[\frac{\alpha_9 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right]
 \end{aligned}$$

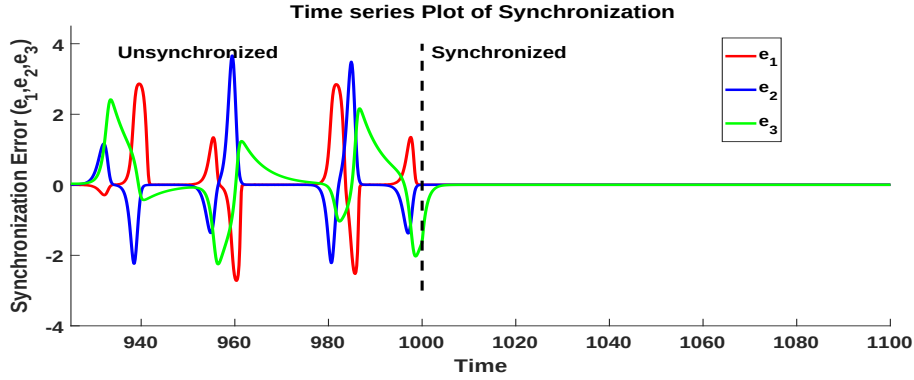


Figure 3: The evolution of state errors before and after the application of controller, eq.20, when the regular master system with $r_1 = 3.0$ drives the chaotic slave system with $r_2 = 6.0$, with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

For the problem of synchronization of two different three-species food web chain, the simulation of the master system is effected with initial condition $(0.3, 0.3, 0.22)$ and $(0.15, 0.15, 0.11)$ for the slave system in the absence of controller functions. We further chose the intrinsic growth rate $r_1 = 3.0$ for the master or drive system so that its response is that of a period-2 attractor in phase space (Fig.2a). We further take $\epsilon = 3.0$ such that the intrinsic growth

rate parameter of the driven system is $r_2 = 6.0$ which results in a chaotic response in the absence of the controller function (Fig.2b).

The effect of the application of controller functions, ν_i , $i = 1, 2, 3$ given in eq.(23), on the state errors e_i , $i = 1, 2, 3$, is illustrated in Fig.3. It is observed that the state errors for the unsynchronized three-species food chains are significant whereas the application of controller function at time $t = 1000$ and thereafter results in the synchronization of the two food chains. In Fig.4, we further display the simulation results regarding the species population before and after the synchronization of the two system take place. Fig.5a shows the phase plot of the master food chain system with $r_1 = 3.0$ while Fig.5b illustrates the behavior of chaotic slave system with $r_2 = 6.0$ before and after switching on the controller function, eq.(23). The transition of the chaotic slave system with $r_2 = 6.0$ to regular behavior as a result of the application of active controller function, eq.(23), is shown in red color.

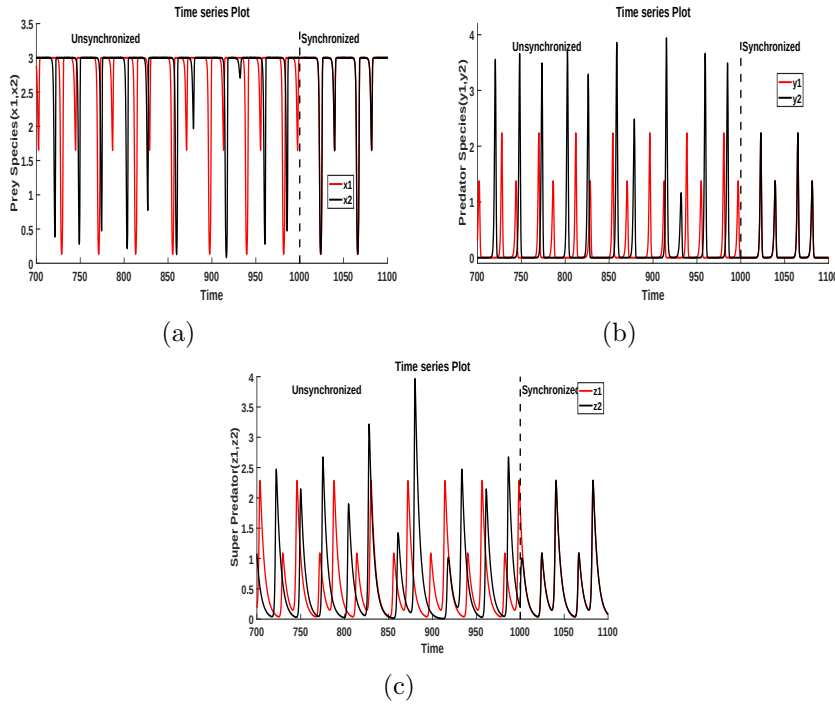


Figure 4: The evolution of species population before and after the application of active controller, eq.(20), when the regular master system with $r_1 = 3.0$ drives the chaotic slave system with $r_2 = 6.0$, with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$, with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

Simulation results are also obtained for the situation when the master three-species food chain is chaotic say with $r_1 = 5.0$ and the slave system is also chaotic with $r_2 = 6.0$. Fig.6a shows time series of the state errors before and

after switching the active controller. Fig.6b shows the time series of evolution of top-predator before and after synchronization. Fig.7a,b illustrates the phase plot of the chaotic master food chain with $r_1 = 5.0$ and the slave chaotic system with $r_2 = 6.0$ before and after synchronization.

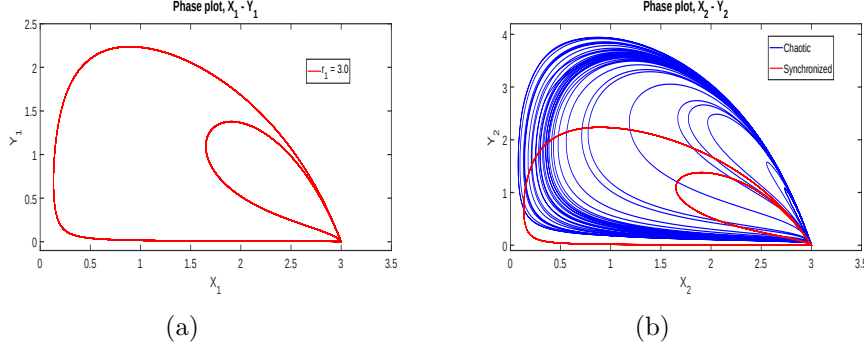


Figure 5: The phase plot of (a) regular master system with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$ (blue) and the synchronized state (red), with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

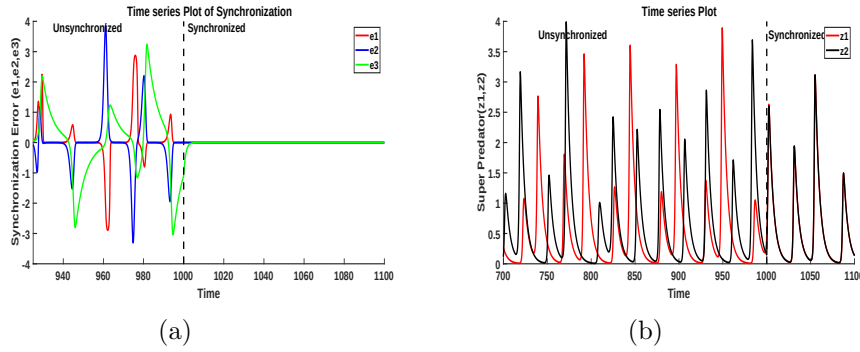


Figure 6: The evolution of (a) state errors and (b) top-predator population before and after the application of controller, eq.(20), when the chaotic master system with $r_1 = 5.0$ drives the chaotic slave system with $r_2 = 6.0$ with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

3.2 Synchronization of two identical chaotic three-species food chains with same intrinsic growth rate ($\epsilon = 0$) but different initial conditions

The time evolution of a trajectory, in the phase space of a system exhibiting chaotic dynamics, is very sensitively dependent on initial conditions. Two

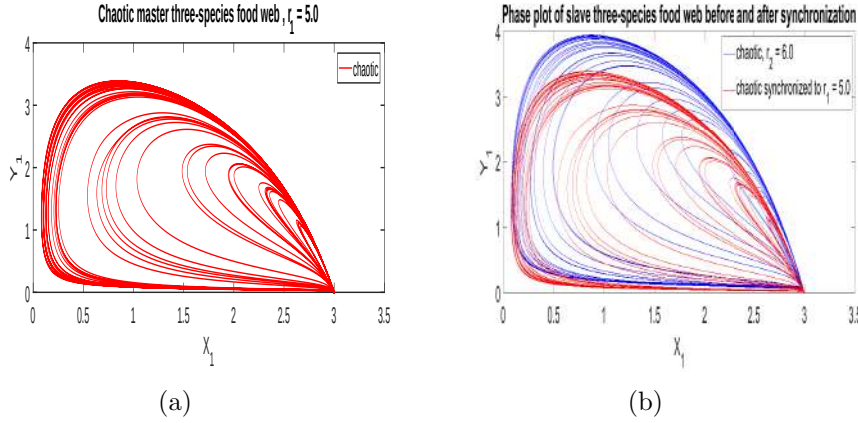


Figure 7: The phase plot of (a) chaotic master system with $r_1 = 5.0$, (b) chaotic slave system with $r_2 = 6.0$ (blue) and the synchronized state (red), with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

very close trajectories asymptotically diverges from each other depending on the magnitude of the positive Lyapunov characteristic exponent of the system. Therefore, two identical chaotic three-species food web chain evolves differently in phase space for different initial conditions that characterizes evolution of each of the systems. It is within this scenario, we investigate the problem of synchronization of the following two identical three species food chain models. We consider the following master chaotic system with initial conditions $(0.30, 0.30, 0.22)$

Master system:

$$\begin{aligned}
 \dot{x}_1 &= \frac{r_1 x_1 (k_0 - x_1)}{(k_1 - x_1)} - \frac{\alpha_1 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1}, \\
 \dot{y}_1 &= \frac{\alpha_4 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1} - \frac{\alpha_5 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - y_1, \\
 \dot{z}_1 &= \frac{\alpha_9 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - \alpha_{10} z_1.
 \end{aligned}
 \tag{24}$$

Slave system:

$$\begin{aligned}
 \dot{x}_2 &= \frac{r_1 x_2 (k_0 - x_2)}{(k_1 - x_2)} - \frac{\alpha_1 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} + \nu_1, \\
 \dot{y}_2 &= \frac{\alpha_4 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} - \frac{\alpha_5 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - y_2 + \nu_2, \\
 \dot{z}_2 &= \frac{\alpha_9 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - \alpha_{10} z_2 + \nu_3.
 \end{aligned}
 \tag{25}$$

Master drives the slave system with initial condition $(0.15, 0.15, 0.11)$ while retaining the values of all other parameters of the two systems. As earlier,

ν_1 , ν_2 and ν_3 in the slave system corresponds to unknown control functions. In view of the different initial conditions, the chaotic dynamics of the master system differs from that of the slave system in phase space in the absence of the controlling functions, for values of the intrinsic growth variable r_1 in the range $4.2 \leq r_0 \leq 6.8$.

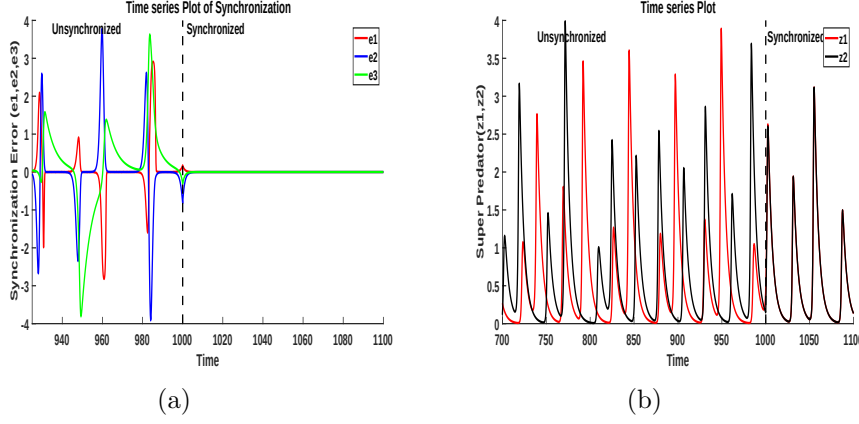


Figure 8: The evolution of (a) state errors and (b) top-predator population before and after the application of controller, eq.(20), in case of identical three-species food web with different initial conditions. Initial condition for the master system is $(0.30, 0.30, 0.22)$ whereas for the slave system initial condition is $(0.15, 0.15, 0.11)$. Other parameter values are $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

Defining the state errors e_1 , e_2 and e_3 as in eq.(18), we obtain the following nonlinear differential equations for the evolution of state errors,

$$\begin{aligned}
 \dot{e}_1 &= \left[\frac{(r_1 k_0 k_1 - r_1 k_1 (x_1 + x_2) + r_1 x_1 x_2)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 \\
 (26) \quad &- \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_2 - \left[\frac{\alpha_1 (x_2 y_2 - x_1 y_1)}{\beta_1} \right] + \nu_1 \\
 \dot{e}_2 &= \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 + \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} - 1 \right] e_2 - \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 \\
 &+ \left[\frac{\alpha_4 (x_2 y_2 - y_2 y_1)}{\beta_2} - \frac{\alpha_5 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] + \nu_2 \\
 \dot{e}_3 &= \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 + \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} \right] e_3 + \left[\frac{\alpha_9 \alpha_8 (y_2 z_2 - y_1 z_1)}{\beta_2} \right] + \nu_3
 \end{aligned}$$

where β_1 and β_2 are defined in eq.(20).

As earlier, we define the Lyapunov function ϕ as,

$$(27) \quad \phi = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2).$$

Following the method described in section 3.1, the asymptotic stability of error system about origin is ensured if

$$\begin{aligned} \dot{\phi} = & e_1^2 \left[\frac{(r_1 k_0 k_1 - r_1 k_1(x_1 + x_2) + r_1 x_1 x_2)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} \right] \\ & - \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_1 e_2 + \nu_1 e_1 + \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 e_2 \\ & - \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 e_2 + \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} - 1 \right] e_2^2 \\ & + \left[\frac{\alpha_4(x_2 y_2 - x_1 y_1)}{\beta_1} - \frac{\alpha_5 \alpha_8(y_2 z_2 - y_1 z_1)}{\beta_2} \right] e_2 \\ & - \left[\frac{\alpha_1(x_2 y_2 - x_1 y_1)}{\beta_1} \right] e_1 + \nu_2 e_2 + \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 e_3 \\ & + \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} \right] e_3^2 + \left[\frac{\alpha_9 \alpha_8(y_2 z_2 - y_1 z_1)}{\beta_2} \right] e_3 + \nu_3 e_3. \end{aligned}$$

3.2.1 Active controller functions for two identical system with different initial conditions

In the following, we deduce the following functional form of the active controller ν_1 , ν_2 and ν_3 such that $\dot{\phi} < 0$.

Controller function:

$$\begin{aligned} \nu_1 = & - \left[\frac{(r_1 k_0 k_1 - r_1 k_1(x_1 + x_2) + r_1 x_1 x_2)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1 \alpha_2 y_1 y_2}{\beta_1} + 1 \right] e_1 \\ & + \left[\frac{\alpha_1 \alpha_3 x_1 x_2}{\beta_1} \right] e_2 + \left[\frac{\alpha_1(x_2 y_2 - x_1 y_1)}{\beta_1} \right] \\ (28) \quad \nu_2 = & - \left[\frac{\alpha_4 \alpha_2 y_1 y_2}{\beta_1} \right] e_1 + \left[\frac{\alpha_5 \alpha_7 y_1 y_2}{\beta_2} \right] e_3 \\ & - \left[\frac{\alpha_4 \alpha_3 x_1 x_2}{\beta_1} - \frac{\alpha_5 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 \\ & + \left[-\frac{\alpha_4(x_2 y_2 - x_1 y_1)}{\beta_1} + \frac{\alpha_5 \alpha_8(y_2 z_2 - y_1 z_1)}{\beta_2} \right] \\ \nu_3 = & - \left[\frac{\alpha_9 \alpha_6 z_1 z_2}{\beta_2} \right] e_2 - \left[\frac{\alpha_9 \alpha_7 y_1 y_2}{\beta_2} - \alpha_{10} + 1 \right] e_3 \\ & - \left[\frac{\alpha_9 \alpha_8(y_2 z_2 - y_1 z_1)}{\beta_2} \right]. \end{aligned}$$

As a result, we find that the state errors e_i , $i = 1, 2, 3$ satisfy the following set of equations

$$(29) \quad \dot{e}_i = -e_i, \quad i = 1, \dots, 3$$

which suggest the synchronization of the two identical systems with different initial conditions.

For numerical computation regarding synchronization of two identical system, we consider both the master and slave chaotic system with $r_1 = 6.0$ but with different initial conditions i.e, $(0.30, 0.30, 0.22)$ for the master and $(0.15, 0.15, 0.11)$ for the slave system respectively. Using the fourth order Runge-Kutta scheme with the foregoing initial conditions, eq.(24) and (25) are integrated to provide the state error functions before and after the activation of controller, eq.(29). Fig.8a illustrates the behavior of error functions whereas Fig.8b describes the oscillations of top-predator population before and after switching on the controller at time $t = 1000$. Fig.9a,b describes the phase plot of the master three-species food web chain with $r_1 = 6.0$ and its synchronization with the identical chaotic slave system, however with different initial condition as above before and after the controller is activated at time $t = 1000$, respectively.

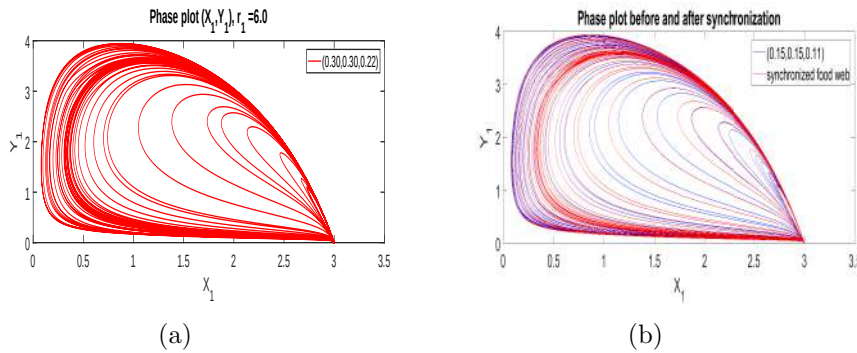


Figure 9: The phase plot of (a) chaotic master system with $r_1 = 6.0$ with initial conditions $(0.30, 0.30, 0.22)$, (b) chaotic slave system with $r_1 = 6.0$ (blue) with initial conditions $(0.15, 0.15, 0.11)$ and the synchronized state (red), with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

4. Conclusion

In this paper two three-species food chain models with Beddington-DeAngelis type functional response of prey-predator interaction has been considered for the study of synchronization under master-slave scheme. For our investigation, we consider the model of three species food web chain that incorporates the effects of resource utilization [43] and Beddington-DeAngelis type functional response. The details of analytical results concerning the linear stability of various possible equilibrium points of such a system has been provided along with appropriate conditions for their stability. The bifurcation diagram of the model has been constructed by noting the maximum of the top-predator population as the intrinsic growth parameter r_1 is varied (Fig.1). It reveals the dynamical

transition of from limit cycle behavior to period doubling bifurcation and chaos with interspersed thin regular windows in the region $2 \leq r_1 \leq 6.8$. However, for the range $6.8 \leq r_1 \leq 8.0$ the model behavior switches over to limit cycle behavior following reverse period-doubling scenario. The problem of synchronization of two such three-species food chain is investigated by establishing active controllers using the Lyapunov stability theory. The effectiveness of the analytically derived active controller function has been tested numerically for the case of (I) two three-species food web chains with different intrinsic growth rate and (II) two identical three-species food web chain with same intrinsic growth but with different initial conditions. We, however, take the same values of other biologically relevant parameters in the two models.

For case I, we first consider the master three-species food chain with intrinsic growth parameter $r_1 = 3.0$ that drives the slave three-species model with intrinsic growth $r_2 = 6.0$. Accordingly, the population dynamics of the master and slave system corresponds to that of period-2 and chaotic attractor in phase space (Fig. 2). Fig. 3 describes the behavior of state error functions before and after the controller function is switched on at time $t = 1000$. The efficacy of the non-linear active controller function, eq.(23), with regard to synchronization of master-slave system is further shown in terms of the time series of species population (Fig.4) and the phase plot (Fig.5). The same controller function is observed to be effective in synchronizing the master chaotic three-species model with $r_1 = 5.0$ and the slave system with $r_2 = 6.0$ (Fig. 6,7).

For case II, we consider investigate the synchronization of two identical three-species food chain with same value of the intrinsic growth parameter i.e. $r_1 = r_2$ or $\epsilon = 0$ but with different initial conditions. In this case the active controller function is different from the case I and as earlier is observed to be effective in the synchronization of the two systems (Fig.8,9). The process of control of response system using the nonlinear active controller function for synchronization of the two three-species food chain involves basically the state error functions representing the difference in the growth of the species population in the two model. As a result, the diffusion of the species tends to balance the growth rate of master with that of the slave system causing synchronization. It is noted that the method is suitable for synchronization of any two models of the three-species food chains treated as master-slave system. We intend to extend the technique of finding the active controller function in this paper to the setting where several parameters of the master three-species food differ from that of the slave system subsequently.

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The complexity of the graph access structures on seven participants

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Abstract. In this paper, we deal with determining the values for the complexity of the graph access structures on seven participants. As [10], it was determined exact values of the 91 of the complexities of access structures. In this paper we give simple proof by entropy and shannon inequalities for determining the lower bound of complexity which used for determining of lower bound of the 51 graph access structure. Further, we establish the exact values of the complexity of five graphs, which remained as open problems in [10]. Also we obtain the tight bounds for complexity of five access structures.

Keywords: Perfect secret sharing scheme, Complexity, Entropy method, Graph access structures.

1. Introduction

A secret sharing scheme is a method for sharing a secret among a set P of participants, by distributing the share relating to the secret among them, in such a way that only certain *qualified subsets* of participants are able to recover the secret from their shares. The secret sharing scheme is called *perfect* if any non-qualified subsets of participants cannot obtain any information about the secret.

The *access structure* Γ is the collection of all subsets of P that are qualified to reconstruct the secret. A subset $A \in \Gamma$ is called *minimal qualified subset* of P , if for every $B \in \Gamma, B \subseteq A$ implies $B = A$. The collection of all minimal qualified subsets of P is denoted by $[\Gamma^-]$, which is called the basis of Γ . The *complexity* of secret sharing scheme is the ratio between the maximum size of the shares given to any participant in P , and the size of the secret. The complexity of an access structure Γ , denoted by $\sigma(\Gamma)$, is defined as the infimum of the complexities of all secret sharing schemes with access structure Γ .

The graph access structure on a set P of participants is an access structure that contains only minimal qualified subsets of cardinality two. The complexity for the graph access structures have been considered in several papers, such as [1, 2, 3, 4, 6, 8, 11, 13, 14]. Van Dijk studied the complexity of the 112

graph access structures on six participants. He determined the exact values of complexity in 94 cases [14]. Subsequently in [13], Sun and Chen improved the known upper bound and in [6], Gharahi and Dehkordi established the exact values of complexity for five the graph access structures which remained unsolved in Van Dijk paper. Recently, Padro and Vazquez presented lower bound of complexity by resorting to linear programming approach [9]. The complexity of the graph access structures on seven participants have been studied in [10]. Yun Song, Zhihui Li and Weicong Wang in [10], compute the complexity of the graph access structures of 111 connected graphs that have six, seven and eight vertices and in 91 cases, determined the exact values of the complexity. In this paper, we give a simple and different proof of their results and obtain the exact values of the complexity of five the graph access structures. We also introduce new decompositions which improves the upper bound of complexity of five the access structures.

2. Definitions and basic properties

We give a definition of a perfect secret sharing in terms of *shannon entropy*, as in [4].

Consider two sets of random variables X and Y , the shannon entropy function H satisfied the following properties

1. *non-negativity*, i.e., $H(X) \geq 0$,
2. *monotonicity*, i.e., if $X \subseteq Y$ then $H(Y) \geq H(X)$,
3. *submodularity*, i.e., $H(x) + H(Y) \geq H(X \cup Y) + H(X \cap Y)$.

The information inequalities that can be obtained from the above inequalities are called *Shannon inequalities*. In any secret sharing scheme, there is one random variable, denoted by S , for the secret and, for each participant $p \in P$, there is one random variable for the share given to p . We denote the joint distribution of the shares given to the participants in $X \subseteq P$ by X . A *perfect secret sharing scheme* with the access structure Γ is a sharing of the secrets in S among a set P of participants in such a way that:

1. $H(S|A) = 0$, for all $A \in \Gamma$,
2. $H(S|A) = H(S)$, for all $A \notin \Gamma$.

Let Σ is a secret sharing scheme for the access structure Γ and the set of secrets S . The *complexity* of Σ is defined by

$$\sigma(\Sigma) = \frac{\max_{p \in P} H(p)}{H(s)}$$

and the *complexity* of an access structure Γ is defined as

$$\sigma(\Gamma) = \inf_{\Sigma} \sigma(\Sigma),$$

where the infimum is taken over all secret sharing schemes realizing Γ . For each important participant $p \in P$, $H(s) \leq H(p)$ [1], so $\sigma(\Gamma) \geq 1$. The access structure with complexity equal to one is called *ideal*. Throughout this paper only perfect secret sharing schemes are considered. The following theorem is obvious [1].

Theorem 2.1. *Let Γ be an access structure on P and let Γ' be an induced subset of Γ , i.e., for some subset $P' \subseteq P$*

$$\Gamma' = \{X : X \in \Gamma, X \subseteq P'\}.$$

Then Γ' is an access structure on P' . And, if there exists a perfect secret sharing scheme for Γ and secret set S with complexity σ then this scheme is also a perfect secret sharing scheme for Γ' and S having complexity at most equal to σ .

In graph access structure all minimal qualified subsets have exactly two different participants. When the access structure Γ is based on a connected graph G , then the participants and the minimal qualified subsets are corresponding to the vertices and the edges of the graphs G , respectively. From 2.1 we immediately obtain:

Corollary 2.2. *Suppose G is a graph and G' is an induced subgraph of G . Then $\sigma(G) \geq \sigma(G')$.*

Theorem 2.3 ([3]). *The access structure Γ based on a connected graph G is ideal if and only if G is a complete multipartite graph.*

From the following theorem as in [14], we can determine the complexity of graph access structures on seven participants by the complexity of graph access structure is known on six participants.

Theorem 2.4. *Let G be a graph with vertices A and B such that AD is an edge iff BD is an edge for all vertices D . Define G' by deleting edges AD , for all vertices D , and by deleting vertex A . Then $\sigma(G) = \sigma(G')$.*

Since the complexity of graph access structures is known for graphs on six participants [10, 6, 14], we can obtain the exact value of complexity 55 graph with seven participants [10].

3. The upper bound of the complexity

To drive the upper bound on the complexity of access structure Γ , we apply λ -decomposition technique from [12]. Let Γ be an access structure having basis $[\Gamma]^-$ and $\lambda \geq 1$, a λ -decomposition of $[\Gamma]^-$ consist of set $\{\Gamma^1, \dots, \Gamma^t\}$ such that the following properties are satisfied:

1. $\Gamma^h \subseteq [\Gamma]^-$ for $1 \leq h \leq t$.

2. For any $A \in [\Gamma]^-$, there exist λ indices $i_1 < \dots < i_\lambda$ such that $A \in \Gamma^{i_j}$ for $1 \leq j \leq \lambda$.

For $1 \leq h \leq t$, define $P_h = \bigcup_{B \in \Gamma^h} B$; P_h denotes the set of participants in scheme with access structure $[\Gamma]^-$.

Theorem 3.1 ([12]). *Let Γ be an access structure on n participants, having basis $[\Gamma]^-$ and suppose that $\{\Gamma^1, \dots, \Gamma^t\}$ is a λ -decomposition of $[\Gamma]^-$. Let for every Γ^h ($1 \leq h \leq t$), there exist a secret sharing scheme with complexity σ_{ih} that $p_i \in P_h$, then there exist a secret sharing scheme Σ with access structure Γ where*

$$\sigma(\Sigma) = \max \left\{ \frac{\sum_{\{h: p_i \in P_h\}} \sigma_{ih}}{\lambda} : 1 \leq i \leq n \right\}.$$

Corollary 3.2 ([12]). *Let Γ be an access structure on n participants and suppose that $\{\Gamma^1, \dots, \Gamma^t\}$ is a λ -decomposition of $[\Gamma]^-$ such that for every $1 \leq h \leq t$, Γ^h is ideal. If $R = \max \{|h : p_i \in P_h| : 1 \leq i \leq n\}$ then $\sigma(\Gamma) \leq \frac{R}{\lambda}$.*

4. The lower bound of the complexity

To derive the lower bound on the complexity of access structure Γ , we use entropy method as described in [1]. First, we state the following lemma which will be used in this section.

Lemma 4.1 ([5]). *Let Γ be an access structure on a set P of participants, then*

1. *if $A \subseteq B$, $A \notin \Gamma$, and $B \in \Gamma$, then $H(B) \geq H(A) + H(S)$,*
2. *if $A \in \Gamma$, $B \in \Gamma$ but $A \cap B \notin \Gamma$, then*

$$H(A) + H(B) \geq H(A \cup B) + H(A \cap B) + H(S).$$

Now we prove the following theorem which will be needed when we obtain the lower bound on the complexity. As in [10], they proved the following theorem by concepts of information theory, but we only use 4.1 and submodularity of H .

Theorem 4.2. *Let Γ be a graph access structure on a set of participants $P = \{a, b, c, d, e, f\}$ such that $ab, bd, de, df, fc \in \Gamma$ and $ef, ec, eb, ea, fb, fa \notin \Gamma$. Then $\sigma(\Gamma) \geq \frac{5}{3}$.*

Proof. From 4.1, we obtain

$$\begin{aligned} H(ed) + H(bd) &\geq H(ebd) + H(d) + H(S) \\ H(ed) + H(df) &\geq H(edf) + H(d) + H(S) \\ H(ebdf) &\geq H(ebf) + H(S) \\ H(efc) &\geq H(ec) + H(S) \\ H(eabf) &\geq H(eaf) + H(S) \end{aligned}$$

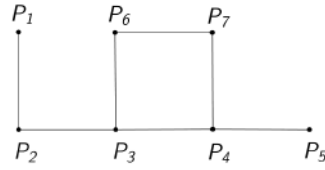
and by the submodularity of H , we have

$$\begin{aligned}
H(e f) + H(e c) &\geq H(e f c) + H(e) \\
H(e b d) + H(e d f) &\geq H(e b d f) + H(e d) \\
H(e b f) + H(e a f) &\geq H(e a b f) + H(e f) \\
H(e) + H(d) &\geq H(e d) \\
H(b) + H(d) &\geq H(b d) \\
H(d) + H(f) &\geq H(d f).
\end{aligned}$$

By adding the above inequalities, we get $H(b) + H(d) + H(f) \geq 5H(S)$. $\square$

From 4.2 and 2.2, we can obtain the lower bound of $\frac{5}{3}$ for 51 graph access structures on seven participants that have six, seven and eight edges, as given in [10].

Example 4.3. Suppose Γ_{20} is the access structure of the following graph:



By using Theorem 4.2, with $a = P_2, b = P_3, c = P_6, d = P_4, e = P_5, f = P_7$, we conclude that the complexity of Γ_{20} is at least $\frac{5}{3}$.

As in [6], we give the following theorem which will be need when we obtain the lower bounds on the complexity.

Theorem 4.4 ([6]). *Let Γ be a graph access structure on a set participants $P = \{a, b, c, d, e, f\}$ such that $ab, ac, ad, cd, cf, de \in \Gamma$ and $bc, bd, be, ce, bf, ef \notin \Gamma$ then $\sigma(\Gamma) \geq \frac{7}{4}$.*

In [10], authors calculated the complexity of 111 graph access structures and obtained the exact values of 91 connected graphs. They showed $\frac{5}{3} \leq \sigma(\Gamma_i) \leq 2$ for $i = 44, 79, 80, 83, 85$. Here we consider the problem of determining the exact values for the complexity of the graph access $\Gamma_{44}, \Gamma_{79}, \Gamma_{80}, \Gamma_{83}$ and Γ_{85} .

Example 4.5. The complexity is $\frac{7}{4}$ for each of the access structures $\Gamma_{44}, \Gamma_{79}, \Gamma_{80}, \Gamma_{83}$ and Γ_{85} .

Proof. We consider the following lables on the access structures of $\Gamma_{44}, \Gamma_{79}, \Gamma_{80}, \Gamma_{83}$ and Γ_{85} .

By using Theorem 4.4 and Corollary 2.2, we have $\sigma(\Gamma_i) \geq \frac{7}{4}$ for $i = 44, 79, 80, 83, 85$. H. Sun and B. Chen, in [13], by weighted decomposition, proved that $\sigma(\Gamma_{44}) \leq \frac{7}{4}$.

For Γ_{79} , by 2.4, we can delete edge ag and then consider the following 2-decomposition:

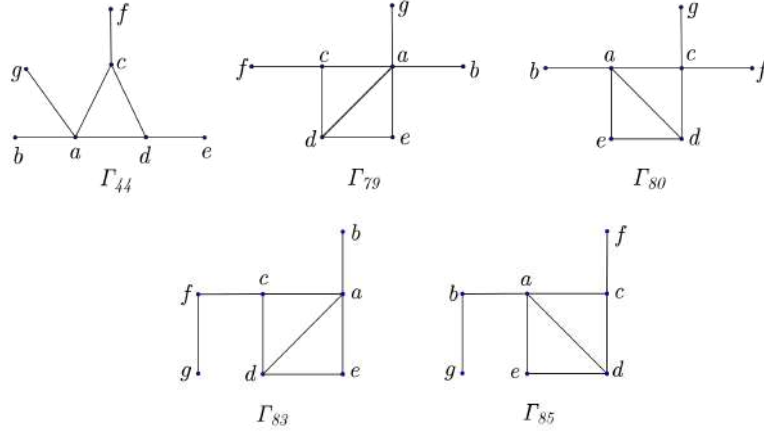
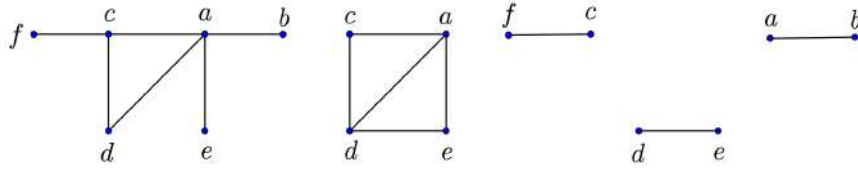
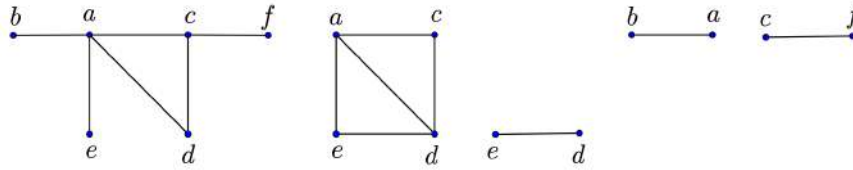


Figure 1: Lables on the access structures


 Figure 2: 2-decomposition of Γ_{79}

Now, by 3.1, we have $\sigma(\Gamma_{79}) \leq \frac{7}{4}$.

For the complexity of Γ_{80} , by 2.4, we can delete edge cg and then consider the following 2-decomposition:

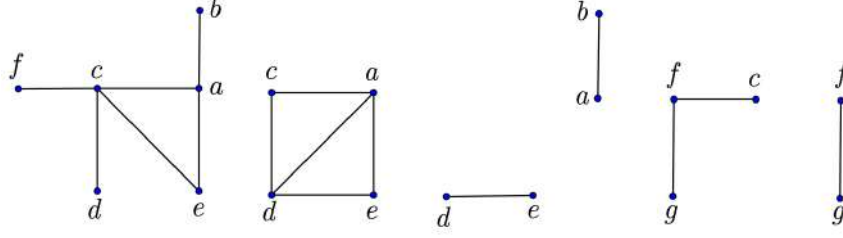
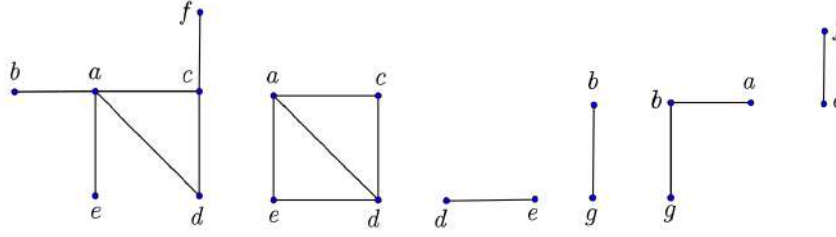

 Figure 3: 2-decomposition of Γ_{80}

And by 3.1, we have $\sigma(\Gamma_{80}) \leq \frac{7}{4}$.

For the complexity of Γ_{83} and Γ_{85} we consider the following decompositions:

Now, by 3.1, we have $\sigma(\Gamma_i) \leq \frac{7}{4}$ that $i = 83, 85$. $\square$

In [10], authors proved that $\frac{5}{3} \leq \sigma(\Gamma_i) \leq 2$ for $i = 43, 47, 60, 61$ and 93, which the lower bound was obtained from Theorem 4.2. Here, for each of these access structures, we obtain tight bounds of the complexity.

Figure 4: 2– decomposition of Γ_{83} Figure 5: 2– decomposition of Γ_{85}

Example 4.6. The complexity is between $\frac{7}{4}$ and $\frac{15}{8}$ for Γ_i that $i = 43, 47, 61$ and 93.

Proof. We consider the following lables on the access structures of $\Gamma_{43}, \Gamma_{47}, \Gamma_{61}$, and Γ_{93} .

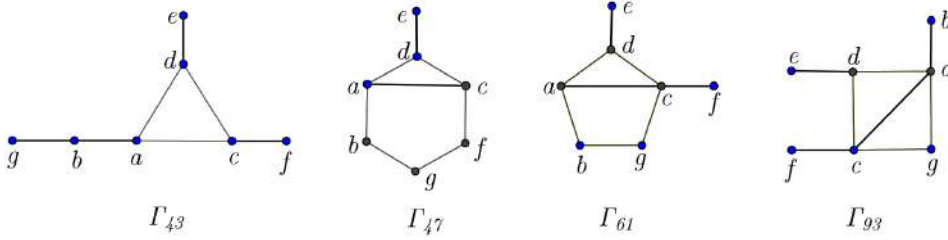


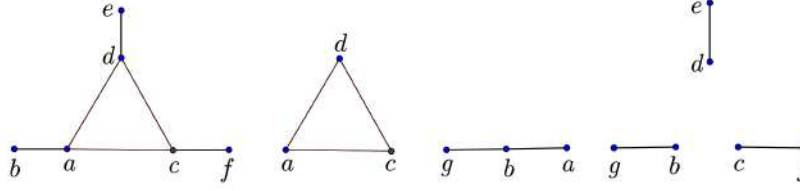
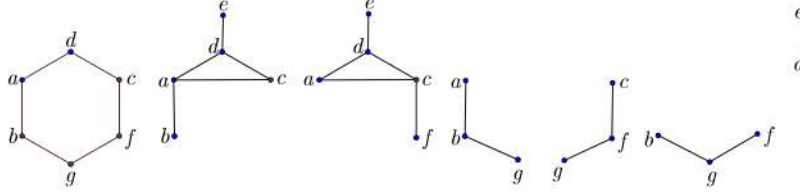
Figure 6: Lables on the access structures

By using 4.4 and 2.2, we have $\sigma(\Gamma_i) \geq \frac{7}{4}$ for $i = 43, 47, 61$ and 93.

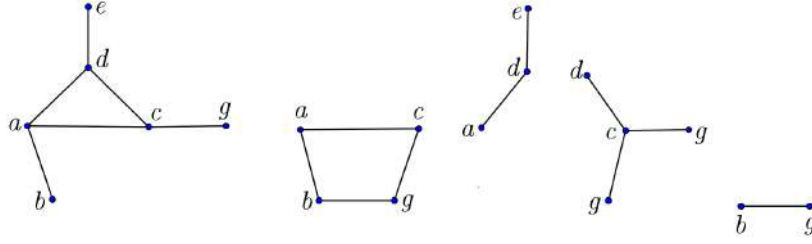
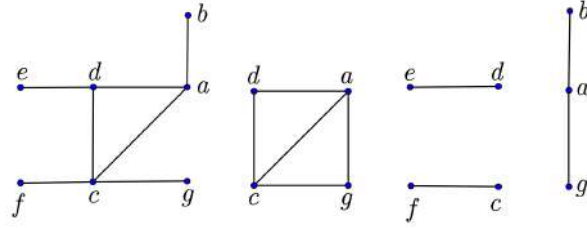
For the upper bound of $\Gamma_{43}, \Gamma_{47}, \Gamma_{61}$, and Γ_{93} we consider the following decomposition and by 3.1, we have $\sigma(\Gamma_i) \leq \frac{15}{8}$; $i = 43, 61$ and 93 and $\sigma(\Gamma_{47}) \leq \frac{11}{6}$. $\square$

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Figure 7: 2— decomposition of Γ_{43} Figure 8: 3— decomposition of Γ_{47}

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Figure 9: 2— decomposition of Γ_{61} Figure 10: 2— decomposition of Γ_{93}

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The convergence of the solutions of a system of max-type difference equations of higher order

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Abstract. In this paper, we study the convergence of the solutions of the following system of max-type difference equations

$$x_n = \max\left\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^\alpha}\right\}, \quad y_n = \max\left\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-r}^\alpha}\right\}, \quad n = 0, 1, \dots,$$

where $m, r \in \{1, 2, \dots\}$, $A \in (0, +\infty)$ and $\alpha \in (0, 1)$ and the initial values $x_{-d}, x_{-d+1}, \dots, x_{-1}, y_{-d}, y_{-d+1}, \dots, y_{-1} \in (0, +\infty)$ with $d = \max\{m, r\}$. We show that: (1) If $0 < A \leq 1$ and $\{(x_n, y_n)\}_{n \geq -d}$ is a solution of the above system, then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 1$; (2) If $A > 1$ and $\{(x_n, y_n)\}_{n \geq -d}$ is a solution of the above system, then for any $0 \leq k \leq 2m - 1$, x_{2mn+k} and y_{2mn+k} are eventually monotone.

Keywords: System of max-type difference equations, positive solution, eventual monotonicity.

1. Introduction

Recently there has been a great interest in studying the properties of solutions of many max-type difference equations and systems, such as eventual periodicity, the boundedness character and eventual monotonicity (see [1-12]).

In 2009, Gelişken and Çinar [13] investigated the asymptotic behavior and the periodicity of the positive solutions of the following max-type difference equation

$$(1.1) \quad x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^\alpha}\right\}, \quad n \in \mathbf{N}_0 \equiv \{0, 1, 2, \dots\},$$

where $A \in \mathbf{R}^+ \equiv (0, +\infty)$ and $\alpha \in (0, 1)$, and showed that every positive solution of (1.1) converges to 1 or is eventually periodic with period 2.

*. Corresponding author

In 2012, we [14] studied the convergence of the positive solutions of the following max-type difference equation

$$(1.2) \quad x_n = \max\left\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^\alpha}\right\}, \quad n \in \mathbf{N}_0,$$

where $A \in \mathbf{R}^+$, $m, k \in \mathbf{N} \equiv \{1, 2, \dots\}$ and $\alpha \in (0, 1)$.

In 2008, Sun [15] studied the asymptotic behavior of the following max-type difference equation

$$(1.3) \quad x_n = \max\left\{\frac{A}{x_{n-1}^\alpha}, \frac{B}{x_{n-2}^\beta}\right\}, \quad n \in \mathbf{N}_0,$$

where $A, B \in \mathbf{R}^+$ and $\alpha, \beta \in (0, 1)$, and showed that every positive solution of (1.3) converges to $\max\{1/A^{\frac{1}{\alpha+1}}, 1/B^{\frac{1}{\beta+1}}\}$.

In 2009, Stević [16] showed that every positive solution of the following max-type difference equation

$$(1.4) \quad x_n = \max\left\{\frac{A_i}{x_{n-i}^{\alpha_i}} : 1 \leq i \leq k\right\}, \quad n \in \mathbf{N}_0$$

converges to $\max\{1/A_i^{\frac{1}{\alpha_i+1}} : 1 \leq i \leq k\}$, where $A_i \in \mathbf{R}^+$ and $\alpha_i \in (0, 1)$ for every $1 \leq i \leq k$.

In 2011, we [17] showed that every positive solution of the following max-type difference equation

$$(1.5) \quad x_n = \max\left\{\frac{A_i}{x_{n-m_i}^{\alpha_i}} : 1 \leq i \leq k\right\}, \quad n \in \mathbf{N}_0$$

converges to $\max\{1/A_i^{\frac{1}{\alpha_i+1}} : 1 \leq i \leq k\}$, where $m_i \in \mathbf{N}$, $A_i \in \mathbf{R}^+$ and $\alpha_i \in (0, 1)$ for every $1 \leq i \leq k$.

In this paper, we investigate the convergence of the following system of max-type difference equations

$$(1.6) \quad x_n = \max\left\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^\alpha}\right\}, \quad y_n = \max\left\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-r}^\alpha}\right\}, \quad n \in \mathbf{N}_0,$$

where $m, r \in \mathbf{N}$, $A \in \mathbf{R}^+$ and $\alpha \in (0, 1)$ and the initial values $x_{-d}, x_{-d+1}, \dots, x_{-1}, y_{-d}, y_{-d+1}, \dots, y_{-1} \in \mathbf{R}^+$ with $d = \max\{m, r\}$. We show that:

(1) If $0 < A \leq 1$ and $\{(x_n, y_n)\}_{n \geq -d}$ is a solution of (1.6), then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 1$;

(2) If $A > 1$ and $\{(x_n, y_n)\}_{n \geq -d}$ is a solution of (1.6), then for any $0 \leq k \leq 2m - 1$, x_{2mn+k} and y_{2mn+k} are eventually monotone.

2. The case $0 < A \leq 1$

In this section, we study the convergence of the solutions of (1.6) when $0 < A \leq 1$.

Let $\{(x_n, y_n)\}_{n \geq -d}$ be a solution of (1.6) with the initial values $x_{-d}, x_{-d+1}, \dots, x_{-1}, y_{-d}, y_{-d+1}, \dots, y_{-1} \in \mathbf{R}^+$. Then we obtain from (1.6) that for any $n \in \mathbf{N}_0$,

$$(2.1) \quad x_n y_{n-r}^\alpha \geq 1, \quad y_n x_{n-r}^\alpha \geq 1.$$

For all $n \geq d$, write

$$(2.2) \quad X_n = \max\{x_n, x_{n-1}, \dots, x_{n-2d}, 1\}, \quad Y_n = \max\{y_n, y_{n-1}, \dots, y_{n-2d}, 1\}.$$

Lemma 2.1.

- (1) $x_{n+1} \leq X_n$ and $y_{n+1} \leq Y_n$ for all $n \geq d$ and X_n, Y_n are decreasing ($n \geq d$).
- (2) $x_n \geq A/Y_d$ and $y_n \geq A/X_d$ for any $n \geq d + m + 1$.

Proof. By (1.6) and (2.1), we obtain that for any $n \geq d$,

$$(2.3) \quad \begin{aligned} x_{n+1} &= \max\left\{\frac{Ax_{n-m+1-r}^\alpha}{y_{n-m+1}x_{n-m+1-r}^\alpha}, \frac{x_{n-r+1-r}^{\alpha^2}}{y_{n-r+1}x_{n-r+1-r}^{\alpha^2}}\right\} \\ &\leq \max\{x_{n-m+1-r}^\alpha, x_{n-r+1-r}^{\alpha^2}\} \\ &\leq \max\{x_{n-m+1-r}, x_{n-r+1-r}, 1\} \\ &\leq X_n. \end{aligned}$$

Thus

$$(2.4) \quad X_{n+1} = \max\{x_{n+1}, \dots, x_{n+1-2d}, 1\} \leq \max\{x_{n+1}, X_n\} = X_n.$$

In same fashion, we also obtain that for any $n \geq d$,

$$(2.5) \quad y_{n+1} \leq Y_n, \quad Y_{n+1} \leq Y_n.$$

Hence it follows that for all $n \geq d + m + 1$,

$$(2.6) \quad x_n \geq \frac{A}{y_{n-m}} \geq \frac{A}{Y_{n-m-1}} \geq \frac{A}{Y_d}, \quad y_n \geq \frac{A}{x_{n-m}} \geq \frac{A}{X_{n-m-1}} \geq \frac{A}{X_d}.$$

The proof is complete. □

By Lemma 2.1 we write

$$(2.7) \quad \liminf_{n \rightarrow \infty} y_n = y > 0, \quad \liminf_{n \rightarrow \infty} x_n = x > 0$$

and

$$(2.8) \quad \lim_{n \rightarrow \infty} X_n = X, \quad \lim_{n \rightarrow \infty} Y_n = Y.$$

Then

$$(2.9) \quad X \geq 1, \quad Y \geq 1.$$

Remark 2.2. Note that from (2.1) we see that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ or $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$.

Lemma 2.3.

(1) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then $\limsup_{n \rightarrow \infty} x_n = X$.*

(2) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then $\limsup_{n \rightarrow \infty} y_n = Y$.*

Proof. Assume that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Then by Remark 2.2 we see that there exists a subsequence $x_{r_1}, x_{r_2}, \dots, x_{r_n}, \dots$ with $x_{r_n} = X_{k_n} \geq x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, which implies

$$(2.10) \quad \limsup_{n \rightarrow \infty} x_n \geq X \geq 1.$$

On the other hand, by $x_{n+1} \leq X_n$ for all $n \geq d$ we obtain

$$(2.11) \quad \limsup_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} X_n = X.$$

Thus $\limsup_{n \rightarrow \infty} x_n = X$. The other case is treated similarly, so we omit the detail. The proof is complete. $\square$

Lemma 2.4. (1) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then $X = y = 1$.*

(2) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then $Y = x = 1$.*

Proof. Assume that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Then by Lemma 2.1 we may assume that there exist $0 < m_1 < m_2 < \dots < m_n < \dots$ and $0 < s_1 < s_2 < \dots < s_n < \dots$ such that

$$(2.12) \quad \begin{aligned} y_{s_n} &\rightarrow \liminf_{n \rightarrow \infty} y_n = y > 0, \\ x_{s_n-m} &\rightarrow A_1, \\ x_{s_n-r} &\rightarrow A_2, \\ x_{m_n} &\rightarrow \limsup_{n \rightarrow \infty} x_n = X \geq 1, \\ y_{m_n-m} &\rightarrow B_1, \\ y_{m_n-r} &\rightarrow B_2. \end{aligned}$$

By taking the limit in the following relationship

$$(2.13) \quad x_{m_n} = \max\left\{\frac{A}{y_{m_n-m}}, \frac{1}{y_{m_n-r}^\alpha}\right\}$$

as $n \rightarrow \infty$, it follows

$$(2.14) \quad X = \max\left\{\frac{A}{B_1}, \frac{1}{B_2^\alpha}\right\} \leq \max\left\{\frac{A}{B_1}, \frac{1}{B_2}\right\} \leq \frac{1}{y}.$$

Thus

$$(2.15) \quad Xy \leq 1.$$

We claim $X = 1$. In fact, if $X > 1$, then by (2.15) and taking the limit in the following relationship

$$(2.16) \quad y_{s_n} = \max\left\{\frac{A}{x_{s_n-m}}, \frac{1}{x_{s_n-r}^\alpha}\right\}$$

as $n \rightarrow \infty$, we obtain

$$(2.17) \quad 1 > \frac{1}{X} \geq y = \max\left\{\frac{A}{A_1}, \frac{1}{A_2^\alpha}\right\} \geq \frac{1}{A_2^\alpha} > \frac{1}{A_2} \geq \frac{1}{X},$$

which is a contradiction. This implies $X = 1$. Again by (2.15) and taking the limit in (2.16) as $n \rightarrow \infty$, we obtain

$$(2.18) \quad 1 \geq y = \max\left\{\frac{A}{A_1}, \frac{1}{A_2^\alpha}\right\} \geq \frac{1}{A_2^\alpha} \geq \frac{1}{A_2} \geq \frac{1}{X} \geq 1.$$

Thus $y = 1$. The other case is treated similarly, so we omit the detail. The proof is complete. $\square$

Theorem 2.5. $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 1$.

Proof. If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ and there exists a sequence $0 < s_1 < s_2 < \dots$ such that $y_{s_n} \geq 1$ for any $n \in \mathbf{N}$, then by Lemma 2.4 we have $X = Y = x = y = 1$, which implies $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 1$.

If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ and there exists a $N \in \mathbf{N}$ such that $y_n < 1$ for any $n \geq N$, then by Lemma 2.4 we have $X = y = 1$. Thus $1 = y \leq \limsup_{n \rightarrow \infty} y_n \leq 1$, which implies $\lim_{n \rightarrow \infty} y_n = 1$. By taking the limit in the following relationship

$$(2.19) \quad x_n = \max\left\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^\alpha}\right\}$$

as $n \rightarrow \infty$, it follows

$$(2.20) \quad \lim_{n \rightarrow \infty} x_n = \max\left\{\frac{A}{1}, \frac{1}{1}\right\} = 1.$$

In same fashion, we also show that if there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$ and there exists a $N \in \mathbf{N}$ such that $x_n < 1$ for any $n \geq N$, then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 1$. The proof is complete. $\square$

2. The case $A > 1$

In this section, we study the convergence of the solutions of (1.6) when $A > 1$. Let $x_n = \sqrt{A}x'_n$, $y_n = \sqrt{A}y'_n$ and $A' = 1/A^{\frac{1+\alpha}{2}}$. Then (1.6) reduces to the system of the following difference equations

$$x'_n = \max\left\{\frac{1}{y'_{n-m}}, \frac{A'}{y'^{\alpha}_{n-r}}\right\}, \quad y'_n = \max\left\{\frac{1}{x'_{n-m}}, \frac{A'}{x'^{\alpha}_{n-r}}\right\}, \quad n \in \mathbf{N}_0.$$

In the following, we study the system of the following difference equations

$$(3.1) \quad x_n = \max\left\{\frac{1}{y_{n-m}}, \frac{A}{y_{n-r}^{\alpha}}\right\}, \quad y_n = \max\left\{\frac{1}{x_{n-m}}, \frac{A}{x_{n-r}^{\alpha}}\right\}, \quad n \in \mathbf{N}_0,$$

where $A \in (0, 1)$. Let $\{(x_n, y_n)\}_{n \geq -d}$ be a solution of (3.1) with the initial values $x_{-d}, x_{-d+1}, \dots, x_{-1}, y_{-d}, y_{-d+1}, \dots, y_{-1} \in \mathbf{R}^+$. Then we have from (3.1) that for any $n \in \mathbf{N}_0$,

$$(3.2) \quad x_n y_{n-m} \geq 1, \quad y_n x_{n-m} \geq 1.$$

By (3.1) and (3.2) we obtain the following statements:

(S₁) For any $n \geq d$,

$$(3.3) \quad x_n \leq \max\{x_{n-2m}, Ax_{n-r-m}^{\alpha}\}, \quad y_n \leq \max\{y_{n-2m}, Ay_{n-r-m}^{\alpha}\}.$$

(S₂) If $x_n = 1/y_{n-m}$ (resp. $y_n = 1/x_{n-m}$) for some $n \geq m$, then

$$(3.4) \quad x_n = \frac{x_{n-2m}}{y_{n-m}x_{n-2m}} \leq x_{n-2m} \text{ (resp. } y_n \leq y_{n-2m}\text{)}.$$

(S₃) If $x_n = A/y_{n-r}^{\alpha} > 1/y_{n-m}$ (resp. $y_n = A/x_{n-r}^{\alpha} > 1/x_{n-m}$) for some $n \geq d$, then

$$(3.5) \quad \begin{aligned} x_{n-2m} &< x_{n-2m}x_n y_{n-m} \\ &= \max\left\{x_n, \frac{x_n x_{n-2m} y_{n-r}^{\alpha} A}{x_{n-r-m}^{\alpha} y_{n-r}^{\alpha}}\right\} \\ &\leq \max\{x_n, x_{n-2m} A^2\} \\ &= x_n \\ \text{(resp. } y_{n-2m} &\leq y_n\text{)}. \end{aligned}$$

Lemma 3.1. *If there exists $M \in \mathbf{N}$ such that $\{y_{M+2mn}\}_{n \geq 0}$ (respectively $\{x_{M+2mn}\}_{n \geq 0}$) is monotone, then $\{x_{M+2mn+r}\}_{n \geq 0}$ (respectively $\{y_{M+2mn+r}\}_{n \geq 0}$) is eventually monotone.*

Proof. If there exists $K \in \mathbf{N}$ such that $x_{M+2mn+r} = 1/y_{M+2mn-m+r}$ for all $n \geq K$, then by (3.4) we know that $x_{M+2mn+r} \leq x_{M+2m(n-1)+r}$ for all $n \geq K$. Thus $\{x_{M+2mn+r}\}_{n \geq K}$ is decreasing.

If there exists $K \in \mathbf{N}$ such that $x_{M+2mn+r} > 1/y_{M+2mn-m+r}$ for all $n \geq K$, then by (3.5) we know that $x_{M+2mn+r} > x_{M+2m(n-1)+r}$ for all $n \geq K$. Thus $\{x_{M+2mn+r}\}_{n \geq K}$ is increasing.

In the following, we assume that there exists a sequence $1 < s_1 < t_1 < s_2 < t_2 < \dots < s_n < t_n < \dots$ such that

$$(3.6) \quad x_{M+2mk+r} = \frac{A}{y_{M+2mk}^\alpha} > \frac{1}{y_{M+2mk+r-m}}, \quad \text{for every } s_n \leq k < t_n$$

and

$$(3.7) \quad x_{M+2mk+r} = \frac{1}{y_{M+2mk+r-m}}, \quad \text{for every } t_n \leq k < s_{n+1}.$$

By (3.5) we see that for any $n \in \mathbf{N}$,

$$(3.8) \quad x_{M+2ms_{n+1}+r} = \frac{A}{y_{M+2ms_{n+1}}^\alpha} > x_{M+2m(s_{n+1}-1)+r} \geq \frac{A}{y_{M+2m(s_{n+1}-1)}^\alpha}.$$

Then $y_{M+2ms_{n+1}} < y_{M+2m(s_{n+1}-1)}$ and $\{y_{M+2mn}\}_{n \geq 0}$ is decreasing.

For every $s_n \leq k < t_n$, by (3.5) we have $x_{M+2m(k-1)+r} < x_{M+2mk+r}$. For every $t_n \leq k < s_{n+1}$, it follows from $A^2 < 1 \leq y_{M+2mk}^\alpha x_{M+2mk-m}^\alpha$ and (3.4) that

$$(3.9) \quad \begin{aligned} \frac{A}{y_{M+2mk}^\alpha} &\geq \frac{A}{y_{M+2m(t_n-1)}^\alpha} = x_{M+2m(t_n-1)+r} \\ &\geq x_{M+2mk+r} = \frac{1}{y_{M+2mk+r-m}^\alpha} \\ &= \min\left\{x_{M+2m(k-1)+r}, \frac{x_{M+2mk-m}^\alpha}{A}\right\} \\ &= x_{M+2m(k-1)+r} \geq x_{M+2mk+r}, \end{aligned}$$

which implies $x_{M+2m(k-1)+r} = x_{M+2mk+r}$. Thus $\{x_{M+2mn+r}\}_{n \geq 0}$ is eventually increasing. The other case is treated similarly, so we omit the detail. The proof is complete.

For all $n \geq d$, write

$$(3.10) \quad X_n = \max\{x_n, x_{n-1}, \dots, x_{n-2d}, 1\}, \quad Y_n = \max\{y_n, y_{n-1}, \dots, y_{n-2d}, 1\}.$$

Lemma 3.2. $x_{n+1} \leq X_n$ and $y_{n+1} \leq Y_n$ for all $n \geq d$ and X_n, Y_n are decreasing ($n \geq d$).

Proof. By (3.3) we obtain that for any $n \geq d$,

$$(3.11) \quad \begin{aligned} x_{n+1} &\leq \max\{x_{n-2m+1}, x_{n-r+1-m}, 1\} \leq X_n, \\ y_{n+1} &\leq \max\{y_{n-2m+1}, y_{n-r+1-m}, 1\} \leq Y_n. \end{aligned}$$

Thus

$$(3.12) \quad \begin{aligned} X_{n+1} &= \max\{x_{n+1}, \dots, x_{n+1-2d}, 1\} \leq \max\{x_{n+1}, X_n\} = X_n, \\ Y_{n+1} &= \max\{y_{n+1}, \dots, y_{n+1-2d}, 1\} \leq \max\{y_{n+1}, Y_n\} = Y_n. \end{aligned}$$

The proof is complete.

By Lemma 3.2 we write

$$(3.13) \quad \lim_{n \rightarrow \infty} X_n = X, \quad \lim_{n \rightarrow \infty} Y_n = Y.$$

Then

$$(3.14) \quad X \geq 1, \quad Y \geq 1.$$

Remark 3.3. Note that from (3.2) we see that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ or $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$.

Lemma 3.4. (1) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then:*

$$(i) \quad \limsup_{n \rightarrow \infty} x_n = X.$$

$$(ii) \quad \text{There exists a } M \in \mathbf{N} \text{ such that } x_{M+2nm} \text{ is decreasing with } \lim_{n \rightarrow \infty} x_{M+2nm} = X \text{ and } x_{M+2nm} = 1/y_{M+2nm-m}.$$

(2) *If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then*

$$(i) \quad \limsup_{n \rightarrow \infty} y_n = Y.$$

$$(ii) \quad \text{There exists a } M \in \mathbf{N} \text{ such that } y_{M+2nm} \text{ is decreasing with } \lim_{n \rightarrow \infty} y_{M+2nm} = Y \text{ and } y_{M+2nm} = 1/x_{M+2nm-m}.$$

Proof. If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then there exists a subsequence $x_{r_1}, x_{r_2}, \dots, x_{r_n}, \dots$ with $x_{r_n} = X_{k_n} \geq x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Thus we see that $x_{r_n} = X_{k_n} \geq X$ and

$$(3.15) \quad \limsup_{n \rightarrow \infty} x_n \geq X \geq 1.$$

On the other hand, by $x_{n+1} \leq X_n$ for all $n \geq d$ we obtain

$$(3.16) \quad \limsup_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} X_n = X.$$

Thus $\limsup_{n \rightarrow \infty} x_n = X$.

By $\limsup_{n \rightarrow \infty} x_n = X$ we see that there exists a $N \in \mathbf{N}$ such that $x_n < X/A < (X/A)^{\frac{1}{\alpha}}$ for any $n \geq N$. Then $Ax_n^\alpha < X$ for any $n \geq N$. We can assume that there exist a sequence $N + 2d \leq r_1 < r_2 < \dots < r_n < \dots$ and some $0 \leq c < 2m$ such that $x_{2r_n m + c} \geq X$ and

$$(3.17) \quad \lim_{n \rightarrow \infty} x_{2k_n m + c} = X.$$

From (3.3) we have

$$\begin{aligned}
 (3.18) \quad X &\leq x_{2mr_n+c} \leq x_{2mr_n-2m+c} \leq \dots \\
 &\leq x_{2mr_{n-1}+c} \leq x_{2mr_{n-1}-2m+c} \leq \dots \\
 &\dots \\
 &\leq x_{2mr_1+c}.
 \end{aligned}$$

Let $M = 2mr_1 + c$. Then x_{M+2nm} is decreasing and $\lim_{n \rightarrow \infty} x_{M+2nm} = X$.

By (3.2) we obtain

$$(3.19) \quad \frac{A}{y_{M+2mn-r}^\alpha} \leq Ax_{M+2mn-m-r}^\alpha < X \leq x_{M+2nm}.$$

Then $x_{M+2nm} = 1/y_{M+2nm-m}$. The other case is treated similarly, so we omit the detail. The proof is complete.

If m and r are odd and (1) of Lemma 3.4 holds, then for all $n \geq d$, write

$$\begin{aligned}
 (3.20) \quad X'_n &= \max\{x_{M+2n-1}, x_{M+2n-3}, \dots, x_{M+2n-2d-1}, 1\}, \\
 Y'_n &= \max\{y_{M+2n}, y_{M+2n-2}, \dots, y_{M+2n-2d}, 1\}.
 \end{aligned}$$

If m and r are odd and (2) of Lemma 3.4 holds, then for all $n \geq d$, write

$$\begin{aligned}
 (3.21) \quad X'_n &= \max\{x_{M+2n}, x_{M+2n-2}, \dots, x_{M+2n-2d}, 1\}, \\
 Y'_n &= \max\{y_{M+2n-1}, y_{M+2n-3}, \dots, y_{M+2n-2d-1}, 1\}.
 \end{aligned}$$

where M is as in Lemma 3.4.

Lemma 3.5. *If (1) of Lemma 3.4 holds, then $x_{M+2n+1} \leq X'_n$ and $y_{M+2n+2} \leq Y'_n$ for all $n \geq d$ and X'_n, Y'_n are decreasing ($n \geq d$). If (2) of Lemma 3.4 holds, then $x_{M+2n+2} \leq X'_n$ and $y_{M+2n+1} \leq Y'_n$ for all $n \geq d$ and X'_n, Y'_n are decreasing ($n \geq d$).*

Proof. If (1) of Lemma 3.4 holds, then by (3.3) we obtain that for any $n \geq d$,

$$\begin{aligned}
 (3.22) \quad x_{M+2n+1} &\leq \max\{x_{M+2n-2m+1}, x_{M+2n-r+1-m}, 1\} \leq X'_n, \\
 y_{M+2n+2} &\leq \max\{y_{M+2n-2m+2}, y_{M+2n-r+2-m}, 1\} \leq Y'_n.
 \end{aligned}$$

Thus

$$\begin{aligned}
 (3.23) \quad X'_{n+1} &= \max\{x_{M+2n+1}, \dots, x_{M+2n+1-2d}, 1\} \leq X'_n, \\
 Y'_{n+1} &= \max\{y_{M+2n+2}, \dots, y_{M+2n+2-2d}, 1\} \leq Y'_n.
 \end{aligned}$$

The other case is treated similarly, so we omit the detail. The proof is complete.

Let

$$(3.24) \quad \lim_{n \rightarrow \infty} X'_n = X', \quad \lim_{n \rightarrow \infty} Y'_n = Y'.$$

Remark 3.6. Note that from (3.2) we see that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{M+2k_n-1} \geq 1$ for any $n \in \mathbf{N}$ or $y_{M+2k_n} \geq 1$ for any $n \in \mathbf{N}$ or $x_{M+2k_n} \geq 1$ for any $n \in \mathbf{N}$ or $y_{M+2k_n-1} \geq 1$ for any $n \in \mathbf{N}$.

Using arguments similar to ones developed in the proof of Lemma 2.4, we can show the following Lemma 3.7.

Lemma 3.7. (1) *Assume that (1) of Lemma 3.4 holds. If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{M+2k_n-1} \geq 1$ for any $n \in \mathbf{N}$, then*

- (i) $\limsup_{n \rightarrow \infty} x_{M+2n-1} = X'$.
- (ii) *There exists a $N \in \mathbf{N}$ such that $x_{M+2N+2nm-1}$ is decreasing with $\lim_{n \rightarrow \infty} x_{M+2N+2nm-1} = X'$ and $x_{M+2N+2nm-1} = 1/y_{M+2N+2nm-1-m}$.*

If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{M+2k_n} \geq 1$ for any $n \in \mathbf{N}$, then

- (iii) $\limsup_{n \rightarrow \infty} y_{M+2n} = Y'$.
- (iv) *There exists a $N \in \mathbf{N}$ such that $y_{M+2N+2nm}$ is decreasing with $\lim_{n \rightarrow \infty} y_{M+2N+2nm} = Y'$ and $y_{M+2N+2nm} = 1/x_{M+2N+2nm-m}$.*

(2) *Assume that (2) of Lemma 3.4 holds. If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{M+2k_n} \geq 1$ for any $n \in \mathbf{N}$, then*

- (i) $\limsup_{n \rightarrow \infty} x_{M+2n} = X'$.
- (ii) *There exists a $N \in \mathbf{N}$ such that $x_{M+2N+2nm}$ is decreasing with $\lim_{n \rightarrow \infty} x_{M+2N+2nm} = X'$ and $x_{M+2N+2nm} = 1/y_{M+2N+2nm-m}$.*

If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $y_{M+2k_n-1} \geq 1$ for any $n \in \mathbf{N}$, then

- (iii) $\limsup_{n \rightarrow \infty} y_{M+2n-1} = Y'$.
- (iv) *There exists a $N \in \mathbf{N}$ such that $y_{M+2N+2nm-1}$ is decreasing with $\lim_{n \rightarrow \infty} y_{M+2N+2nm-1} = Y'$ and $y_{M+2N+2nm-1} = 1/x_{M+2N+2nm-1-m}$.*

Theorem 3.8. *For any $0 \leq k \leq 2m-1$, $\{x_{2nm+k}\}_{n \geq 0}$ and $\{y_{2nm+k}\}_{n \geq 0}$ are eventually monotone.*

Proof. First we suppose that $\gcd(m, r) = 1$.

If m is even and r is odd, or m is odd and r is even, then by Lemma 3.4 there are two cases to consider.

Case 1. There exists a $M \in \mathbf{N}$ such that x_{M+2nm} is decreasing and $x_{M+2nm} = 1/y_{M+2nm-m}$. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m-1$, $\{x_{M+2nm+2kr}\}_{n \geq 0}$, $\{x_{M+2nm+(2k+1)r-m}\}_{n \geq 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{y_{M+2nm+2kr-m}\}_{n \geq 0}$ are eventually monotone.

Case 2. There exists a $M \in \mathbf{N}$ such that y_{M+2nm} is decreasing and $y_{M+2nm} = 1/x_{M+2nm-m}$. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq$

$m - 1$, $\{y_{M+2nm+2kr}\}_{n \geq 0}$, $\{y_{M+2nm+(2k+1)r-m}\}_{n \geq 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{x_{M+2nm+2kr-m}\}_{n \geq 0}$ are eventually monotone.

Since $\gcd(m, r) = 1$, it follows that

$$(3.25) \quad \{2kr : 0 \leq k \leq m - 1\} = \{0, 2, \dots, 2m - 2\} \pmod{2m},$$

which implies

$$(3.26) \quad \{2kr + r - m : 0 \leq k \leq m - 1\} = \{1, 3, \dots, 2m - 1\} \pmod{2m}$$

and

$$(3.27) \quad \begin{aligned} & \{2kr + r : 0 \leq k \leq m - 1\} \cup \{2kr - m : 0 \leq k \leq m - 1\} \\ &= \{0, 1, 2, \dots, 2m - 1\} \pmod{2m}. \end{aligned}$$

Thus $\{x_{M+2nm+k}\}_{n \geq 0}$ and $\{y_{M+2nm+k}\}_{n \geq 0}$ are eventually monotone for every $k \in \{0, 1, 2, \dots, 2m - 1\}$. In the following, we assume that m and r are odd. Then by Lemma 3.7 there are four cases to consider.

Case 1. There exist $M, N \in \mathbf{N}$ such that x_{M+2nm} and $x_{M+2N+2nm-1}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m - 1$, $\{x_{M+2nm+2kr}\}_{n \geq 0}$, $\{x_{M+2N+2nm+2kr-1}\}_{n \geq 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{y_{M+2N+2nm+(2k+1)r-1}\}_{n \geq 0}$ are eventually monotone.

Case 2. There exist $M, N \in \mathbf{N}$ such that x_{M+2nm} and $y_{M+2N+2nm}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m - 1$, $\{x_{M+2nm+2kr}\}_{n \geq 0}$, $\{x_{M+2N+2nm+(2k+1)r}\}_{n \geq 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{y_{M+2N+2nm+2kr}\}_{n \geq 0}$ are eventually monotone.

Case 3. There exist $M, N \in \mathbf{N}$ such that y_{M+2nm} and $y_{M+2N+2nm-1}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m - 1$, $\{y_{M+2nm+2kr}\}_{n \geq 0}$, $\{y_{M+2N+2nm+2kr-1}\}_{n \geq 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{x_{M+2N+2nm+(2k+1)r-1}\}_{n \geq 0}$ are eventually monotone.

Case 4. There exist $M, N \in \mathbf{N}$ such that y_{M+2nm} and $x_{M+2N+2nm}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m - 1$, $\{y_{M+2nm+2kr}\}_{n \geq 0}$, $\{y_{M+2N+2nm+(2k+1)r}\}_{n \geq 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n \geq 0}$ and $\{x_{M+2N+2nm+2kr}\}_{n \geq 0}$ are eventually monotone.

Since $\gcd(m, r) = 1$, it follows that

$$(3.28) \quad \begin{aligned} & \{2kr : 0 \leq k \leq m - 1\} \cup \{2N + 2kr - 1 : 0 \leq k \leq m - 1\} \\ &= \{(2k + 1)r : 0 \leq k \leq m - 1\} \cup \{2N + (2k + 1)r - 1 : 0 \leq k \leq m - 1\} \\ &= \{2kr : 0 \leq k \leq m - 1\} \cup \{2N + (2k + 1)r : 0 \leq k \leq m - 1\} \\ &= \{(2k + 1)r : 0 \leq k \leq m - 1\} \cup \{2N + 2kr : 0 \leq k \leq m - 1\} \\ &= \{0, 1, 2, \dots, 2m - 1\} \pmod{2m}. \end{aligned}$$

Thus $\{x_{M+2nm+k}\}_{n \geq 0}$ and $\{y_{M+2nm+k}\}_{n \geq 0}$ are eventually monotone for every $k \in \{0, 1, 2, \dots, 2m - 1\}$.

If $\gcd(m, r) = d > 1$, then we write $m = dm_1$ and $r = dr_1$ with $\gcd(m_1, r_1) = 1$. Consider the system of difference equations

$$(3.29) \quad x_n = \max\left\{\frac{1}{y_{n-dm_1}}, \frac{A}{y_{n-dr_1}^\alpha}\right\}, \quad y_n = \max\left\{\frac{1}{x_{n-dm_1}}, \frac{A}{x_{n-dr_1}^\alpha}\right\}, \quad n \in \mathbf{N}_0.$$

Write $x_{n,i} = x_{nd+i}$ and $y_{n,i} = y_{nd+i}$ for every $0 \leq i \leq d-1$ and $n \in \mathbf{N}_0$. Then (3.29) reduces to the equations

$$(3.29,i) \quad x_{n,i} = \max\left\{\frac{1}{y_{n-m_1,i}}, \frac{A}{y_{n-r_1,i}^\alpha}\right\}, \quad y_{n,i} = \max\left\{\frac{1}{x_{n-m_1,i}}, \frac{A}{x_{n-r_1,i}^\alpha}\right\},$$

$$0 \leq i \leq d-1, \quad n \in \mathbf{N}_0.$$

By an analogous way as in the above, we obtain that if $\{x_{n,i}, y_{n,i}\}_{n \geq 0}$ is a positive solution of (3.29, i) for every $0 \leq i \leq d-1$, then $\{x_{2m_1n+k,i}\}_{n \geq 0}$ and $\{y_{2m_1n+k,i}\}_{n \geq 0}$ are eventually monotone for every $0 \leq k \leq 2m_1-1$. Thus for every $0 \leq k \leq 2m-1$, $\{x_{2mn+k}\}_{n \geq 0}$ and $\{y_{2mn+k}\}_{n \geq 0}$ are eventually monotone. The proof is complete.

Remark 3.9. It follows from Theorem 3.8 that if $A > 1$ and $\{(x_n, y_n)\}_{n \geq -d}$ is a positive solution of (1.6), then for any $0 \leq k \leq 2m-1$, $\{x_{2nm+k}\}_{n \geq 0}$ and $\{y_{2nm+k}\}_{n \geq 0}$ are eventually monotone.

Remark 3.10. In [18], we showed that if $\alpha, A \in (0, 1)$ and $k \in \mathbf{N}$, then the following equation

$$(3.30) \quad x_n = \max\left\{\frac{1}{x_{n-1}}, \frac{A}{x_{n-2k-1}}\right\}$$

has a positive solution $\{z_n\}_{n \geq -2k-1}$ satisfying the following conditions:

- (1) $z_{2n+1} = Az_{2n-2k-1}^\alpha$ for any $n \in \mathbf{N}$.
- (2) $z_{2n+2} = 1/z_{2n+1}$ for any $n \in \mathbf{N}$.
- (3) $z_{2n+2} < z_{2n}$ for any $n \in \mathbf{N}$.

Let $x_n = y_n = z_n$ for any $n \geq -2k-1$. Then $\{(x_n, y_n)\}_{n \geq -2k-1}$ is a solution of the following equation

$$(3.31) \quad x_n = \max\left\{\frac{1}{y_{n-1}}, \frac{A}{y_{n-2k-1}^\alpha}\right\}, \quad y_n = \max\left\{\frac{1}{x_{n-1}}, \frac{A}{x_{n-2k-1}^\alpha}\right\}, \quad n \in \mathbf{N}_0.$$

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$\Delta^*(\mathcal{H})$ operators and the hyperinvariant subspace problem

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Abstract. This paper is devoted to study the relation between an operator T that satisfies the operator equation $TK - KT^* = C$ for some compact operator K and some rank one operator C , and the hyperinvariant subspace problem.

Keywords: Hyperinvariant subspace, rank-one operator, commutators.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Recall that a closed linear manifold (subspace) M of $\mathcal{H}$ is said to be a nontrivial invariant subspace for an operator $T \in \mathcal{B}(\mathcal{H})$ if $TM \subseteq M$. If, moreover, $0 \neq M \neq \mathcal{H}$ then M is called a nontrivial invariant subspace. M is said to be a hyperinvariant subspace for T if $SM \subseteq M$ for every operator S that commutes with T .

The motivation of this note is the remarkable result obtained by V. Lomonosov in [9], he proved that any nonscalar operator commutes with a nonzero compact operator has a nontrivial hyperinvariant subspace. H.W. Kim, C. Pearcy, and A.L. Shields had defined a class, $\Delta(\mathcal{H})$, of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that $TK - KT$ has rank one. It was shown that every operator in $\Delta(\mathcal{H})$ has a nontrivial hyperinvariant subspace [7]. This class was extensively studied in [6] and [8].

In this paper, we define the class $\Delta^*(\mathcal{H})$ of all operators $T \in \mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that $TK - KT^*$ has rank one. The main purpose is to obtain some results concerning the relation between the class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem. In section 2, we study the basic properties of $\Delta^*(\mathcal{H})$ and give some examples. However, we can not prove that the nonscalar elements in $\Delta^*(\mathcal{H})$ have nontrivial hyperinvariant subspaces. In section 3, we define a subclass $\Delta^*\mathcal{F}(\mathcal{H})$ of $\Delta^*(\mathcal{H})$, $\Delta^*\mathcal{F}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{there exists a finite rank operator } F \text{ such that } TF - FT^* \text{ has rank one}\}$. We show that every nonscalar element in $\Delta^*\mathcal{F}(\mathcal{H})$ has a nontrivial hyperinvariant subspace. Furthermore, we prove that $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue provided that $T \neq \lambda I, \lambda \in \mathbb{R}$.

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2. The class $\Delta^*(\mathcal{H})$: basic properties

We introduce a new class of operators, denoted by $\Delta^*(\mathcal{H})$. The aim of this section is to study some basic properties of $\Delta^*(\mathcal{H})$.

Definition 2.1. By $\Delta^*(\mathcal{H})$ we mean the set of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that the rank of $TK - KT^*$ is equal to one. That is, $\Delta^*(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{there exists a compact operator } K \text{ such that } TK - KT^* \text{ has rank one}\}$.

Remark 2.2. One can easily see that $\Delta^*(\mathcal{H})$ is not empty. In fact $iI \in \Delta^*(\mathcal{H})$, for let $K = x \otimes y$ be the rank one operator that sends y to x provided that x and y are nonzero unit vectors, then clearly K is compact and $T = iT, TK - KT^* = 2i(x \otimes y)$ is of rank one.

We begin with the following proposition

Proposition 2.3. 1. For any scalar operator λI , $\lambda I \in \Delta^*(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$.

2. $T \in \Delta^*(\mathcal{H})$ iff $\alpha T + \beta I \in \Delta^*(\mathcal{H})$ for each $\alpha \neq 0, \beta \in \mathbb{R}$

Proof. 1. It is clear that if $\lambda \in \mathbb{R}$, $\lambda K - K(\lambda I)^* = 0$ for any compact operator K , thus $\lambda I \notin \Delta^*(\mathcal{H})$. If $\lambda \notin \mathbb{R}$ then $\lambda = a + ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Thus, if K is an operator of rank one,

$$\lambda K - K(\lambda I)^* = aK + ibK - (aK - ibK) = 2ibK$$

which is also of rank one. Consequently, $\lambda I \in \Delta^*(\mathcal{H})$.

2. Suppose that $T \in \Delta^*(\mathcal{H})$, then there exists a compact operator K such that $TK - KT^*$ has rank one. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ then $(\alpha T + \beta I)K - K(\alpha T + \beta I)^* = \alpha(TK - KT^*)$ which has rank one. Thus, $\alpha T + \beta I \in \Delta^*(\mathcal{H})$. The converse is trivial, just assume $\alpha = 1, \beta = 0$. $\square$

Remark 2.4. The class $\Delta^*(\mathcal{H})$ is not closed under the usual algebraic operations. For example, it is not a linear manifold since clearly $0 \in \Delta^*(\mathcal{H})$. Now, if $T_1, T_2 \in \Delta^*(\mathcal{H})$, it does not always follow that $T_1 + T_2 \in \Delta^*(\mathcal{H})$ or $T_1 T_2 \in \Delta^*(\mathcal{H})$. In fact, if $T_1 = iI$ and $T_2 = iI$ then Proposition 2.3 implies that both $T_1, T_2 \in \Delta^*(\mathcal{H})$ while neither $T_1 + T_2 = 0$ nor $T_1 T_2 = I$ belongs to $\Delta^*(\mathcal{H})$.

Now, we go through some examples of operators that belong to $\Delta^*(\mathcal{H})$. We give first the following remark which will be used several times in our work.

Remark 2.5. For any nonzero vectors $f, g \in \mathcal{H}$, for any $T \in \mathcal{B}(\mathcal{H})$,

$$T(f \otimes g) - (f \otimes g)T^* = (Tf \otimes g) - (f \otimes Tg).$$

Proposition 2.6. Let T be any nonzero operator which is not 1-1, then $T \in \Delta^*(\mathcal{H})$.

Proof. By assumption, there exist nonzero vectors $x, y \in \mathcal{H}$ such that $Tx \neq 0$ and $Ty = 0$. Let $K = x \otimes y$, then K is of rank one hence compact and

$$TK - KT^* = T(x \otimes y) - (x \otimes y)T^* = Tx \otimes y - x \otimes Ty = Tx \otimes y$$

which has rank one. Thus $T \in \Delta^*(\mathcal{H})$. $\square$

Corollary 2.7. *Any nonzero nilpotent operator belongs to $\Delta^*(\mathcal{H})$.*

Proof. Let T be a nonzero nilpotent operator, then there exists $n \in \mathbb{N}$ such that $T^n = 0$ and $T^{n-1} \neq 0$. Let $f \in \mathcal{H}$ be such that $T^{n-1}f \neq 0$ then $T(T^{n-1}f) = T^n f = 0$ so T is not 1-1. Hence by proposition 6, $T \in \Delta^*(\mathcal{H})$. $\square$

Corollary 2.8. *Any nonzero finite rank operator belongs to $\Delta^*(\mathcal{H})$.*

Proposition 2.9. *Let T be a nonzero operator. Assume there exists a polynomial p such that $p(T) \neq 0$ but $Tp(T) = 0$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. By hypothesis, there exist nonzero vectors $f, g \in \mathcal{H}$ such that $p(T)f \neq 0$ and $Tg \neq 0$. Let $K = p(T)f \otimes g$ then

$$TK - KT^* = Tp(T)f \otimes g - p(T)f \otimes Tg = -p(T)f \otimes Tg,$$

is of rank one. Thus, $T \in \Delta^*(\mathcal{H})$. $\square$

Corollary 2.10. *Every nonzero idempotent operator which is not the identity belongs to $\Delta^*(\mathcal{H})$.*

Proof. Let T be an idempotent operator such that $T \neq I$. If $p(x) = x - 1$ is a polynomial then p satisfies the condition of Proposition 3.9, thus $T \in \Delta^*(\mathcal{H})$. $\square$

Recall that an operator T is called algebraic if $p(T) = 0$ for some polynomial p .

Proposition 2.11. *Let T be a nonzero algebraic operator. If T is non invertible then $T \in \Delta^*(\mathcal{H})$.*

Proof. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$; $a_i \in \mathbb{R}, i = 0, 1, \dots, n$ be a polynomial such that $p(T) = 0$. If $a_0 \neq 0$ then $a_n T^n + a_{n-1} T^{n-1} + \dots + a_1 T = -a_0 I$, So

$$T(a_n T^{n-1} + a_{n-1} T^{n-2} + \dots + a_1 I) = -a_0 I,$$

which contradicts the non invertibility of T . Let $q(T) = a_n T^{n-1} + a_{n-1} T^{n-2} + \dots + a_1 I$, then $Tq(T) = 0$. One can assume that the degree of p is the smallest one among those polynomials that annihilate T . Thus, $q(T) \neq 0$ and therefore proposition 9 implies that $T \in \Delta^*(\mathcal{H})$. $\square$

Proposition 2.12. *If $T, K, R \in \mathcal{B}(\mathcal{H})$ with K compact and if p is a polynomial with real coefficients such that $p(T)K - Kp(T)^* = R$, then there exists a compact operator K' such that $TK' - K'T^* = R$.*

Proof. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$; $a_i \in \mathbb{R}, i = 0, 1, \dots, n$ and $a_n \neq 0$. Then $p(T)K - Kp(T)^* = a_n(T^n K - K(T^*)^n) + a_{n-1}(T^{n-1} K - K(T^*)^{n-1}) + \dots + a_1(TK - KT^*)$.

Now, for any $\ell \in \mathbb{N}$,

$$\begin{aligned} T^\ell K - K(T^*)^\ell &= T(T^{\ell-1} K + T^{\ell-2} K T^* + T^{\ell-3} K(T^*)^2 + \dots + K(T^*)^{\ell-1}) \\ &\quad - (T^{\ell-1} K + T^{\ell-2} K T^* + T^{\ell-3} K(T^*)^2 + \dots + K(T^*)^{\ell-1}) T^* \\ &= T K' - K' T^*, \end{aligned}$$

where $K' = \sum_{i=0}^{\ell-1} T^{\ell-1-i} K(T^*)^i$. Clearly K' is compact as the set of compact operators form an ideal in $\mathcal{B}(\mathcal{H})$. Thus, there exist compact operators $K_1, K_2, \dots, K_n$ such that

$$\begin{aligned} p(T)K - K(p(T))^* &= a_n(TK_n - K_n T^*) + a_{n-1}(TK_{n-1} - K_{n-1} T^*) \\ &\quad + \dots + a_1(TK_1 - K_1 T^*) \\ &= T(a_n K_n + a_{n-1} K_{n-1} + \dots + a_1 K_1) - (a_n K_n + a_{n-1} K_{n-1} + \dots + a_1 K_1) T^*. \end{aligned}$$

Let $K' = a_n K_n + a_{n-1} K_{n-1} + \dots + a_1 K_1$, then K' is compact and the result follows. Clearly if $R = 0$ then choose $K' = 0$. $\square$

Corollary 2.13. *Let p be a polynomial with real coefficients and $T \in \mathcal{B}(\mathcal{H})$. If $p(T) \in \Delta^*(\mathcal{H})$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Since $p(T) \in \Delta^*(\mathcal{H})$, then there exists a compact operator K such that $p(T)K - K(p(T))^* = R$ where R is a rank one operator. The result follows directly by the previous proposition. $\square$

Combine the above corollary with Proposition 2.3, we get

Theorem 2.14. *Let p be a polynomial with real coefficients except for the constant term to be non real and let $T \in \mathcal{B}(\mathcal{H})$. If $p(T) = 0$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial with $a_i \in \mathbb{R}, i = 0, 1, \dots, n$ and $a_n \neq 0$. If $p(T) = 0$ then $p(T) - a_0 I = -a_0 I$ which by Proposition 2.3 belongs to $\Delta^*(\mathcal{H})$. But $p(T) - a_0 I$ is a polynomial with real coefficients and $p(T) - a_0 I \in \Delta^*(\mathcal{H})$. Thus, by Corollary 2.13, $T \in \Delta^*(\mathcal{H})$. $\square$

In what follows in this section, we assume that T is invertible operator.

Proposition 2.15. *Let T be an invertible operator. If $T \in \Delta^*(\mathcal{H})$ then $T^{-1} \in \Delta^*(\mathcal{H})$. follows in this section, we assume that T is invertible operator.*

Proof. Let K be a compact operator such that $TK - KT^*$ has rank one. Now,

$$TK - KT^* = (TKT^*)(T^*)^{-1} - T^{-1}(TKT^*).$$

Since TKT^* is compact, the result follows. $\square$

Theorem 2.16. *Let T be an invertible operator. If there exist nonzero vectors $f, g \in \mathcal{H}$ such that*

$$\sum_{n=0}^{\infty} \|T^n f\| \|T^{-n} g\| < \infty.$$

Then $T \in \Delta^*(\mathcal{H})$.

Proof. Let $K_n = \sum_{k=0}^n T^k f \otimes T^{-k} g$, then K_n is of finite rank for each n , if $K = \sum_{k=0}^{\infty} T^k f \otimes T^{-k} g$, then by hypothesis

$$\|K - K_n\| = \left\| \sum_{k>n} T^k f \otimes T^{-k} g \right\| \leq \sum_{k>n} \|T^k f\| \|T^{-k} g\| \rightarrow 0.$$

Consequently, $K_n \rightarrow K$, hence K is compact. Now

$$\begin{aligned} TK - KT^* &= T \left(\sum_{k=0}^{\infty} T^k f \otimes T^{-k} g \right) - \left(\sum_{k=0}^{\infty} T^k f \otimes T^{-k} g \right) T^* \\ &= \sum_{k=0}^{\infty} T^{k+1} f \otimes T^{-k} g - \sum_{k=0}^{\infty} T^k f \otimes T^{-k+1} g \\ &= Tf \otimes g + T^2 f \otimes T^{-1} g = T^3 f \otimes T^{-2} g + \dots \\ &= -f \otimes Tg. \end{aligned}$$

But $Tg \neq 0$ as T is invertible, hence $TK - KT^*$ has rank one, so $T \in \Delta^*(\mathcal{H})$. $\square$

As an illustration we have

Example 2.17. Let $\{e_n\}_{n \in \mathbf{Z}}$ be an orthonormal basis of $\mathcal{H}$. Define an operator B by $Be_n = w_{n+1}e_{n+1}$, $n \in \mathbf{Z}$, where $w_{n+1} = \frac{1}{2}$ when $n \geq 0$, and $w_{n+1} = 4$ when $n \leq 0$. This operator is called weighted bilateral shift operator with weights w_n . It is well-known that B is invertible operator and $B^{-1}e_n = \frac{1}{w_n}e_{n-1}$, $n \in \mathbf{Z}$. Note also that $B^{-k}e_n \rightarrow 0$ as $k \rightarrow \infty$ for each $n \in \mathbf{Z}$ and $B^k e_n \rightarrow 0$ as $k \rightarrow \infty$ for each $n \in \mathbf{Z}$. Easily one can see that $\sum_{k=0}^{\infty} \|B^k e_n\| \|B^{-k} e_n\| < \infty$ for any fixed $n \in \mathbf{Z}$. Thus, Theorem 2.16 implies that $B \in \Delta^*(\mathcal{H})$.

Compare the following result with Theorem 2.16,

Proposition 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$, M be a nonzero subspace which is invariant under T and $T_1 = T|_M$ is invertible. If there exist nonzero vectors $f \in \mathcal{H}$ and $g \in M$ such that $\sum_{k=0}^{\infty} \|T^k f\| \|T_1^{-k} g\| < \infty$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Let $K = \sum_{k=0}^{\infty} T^k f \otimes T_1^{-k} g$ then the hypothesis implies that the series converges in the operator norm topology, and hence K is a compact operator. Furthermore,

$$\begin{aligned} TK - KT^* &= \left(K = \sum_{k=0}^{\infty} T^{k+1} f \otimes T_1^{-k} g \right) - \left(\sum_{k=0}^{\infty} T^k f \otimes T_1^{-k+1} g \right) \\ &= -f \otimes T_1 g. \end{aligned}$$

Since T_1 is invertible then $T_1 g \neq 0$ and thus $TK - KT^*$ has rank one, i.e., $T \in \Delta^*(\mathcal{H})$. $\square$

Example 2.19. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\sigma(T) = \{0, 2\}$. Then $\sigma(T) = \{0\} \cup \{2\}$. Thus Riesz decomposition theorem implies that $\mathcal{H}$ splits into the direct sum of two nontrivial hyperinvariant subspaces M_1 and M_2 such that

$$\sigma(T|M_1) = \{0\}, \sigma(T|M_2) = \{2\}.$$

Clearly $T = T|M_2$ is invertible and the spectral radius $r(T|M_1) = 0$ while $r(T^{-1}) = \frac{1}{2} < 1$, consequently $\sum_{k=0}^{\infty} \|T^k f\| \|T^{-k} g\| < \infty$ for any nonzero vectors $f \in \mathcal{H}, g \in M_2$. Thus, $T \in \Delta^*(\mathcal{H})$.

3. The class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem

In this section we try to find a relation between the class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem. For this purpose we study a subset of $\Delta^*(\mathcal{H})$, denoted by $\Delta^*\mathcal{F}(\mathcal{H})$ which is defined as follows.

Definition 3.1. By $\Delta^*\mathcal{F}(\mathcal{H})$ we mean the set of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a finite rank operator F such that the rank of $TF - FT^*$ is equal to one. That is $\Delta^*\mathcal{F}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{there exists a finite rank operator } F \text{ such that } TF - FT^* \text{ has rank one}\}$.

Remark 3.2. Clearly $\Delta^*\mathcal{F}(\mathcal{H}) \subseteq \Delta^*(\mathcal{H})$, however the converse is not true as we see later. It is easy to verify that $\Delta^*\mathcal{F}(\mathcal{H})$ is nonempty set, for example λI belongs to $\Delta^*\mathcal{F}(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$ (see proof of Proposition 2.3).

The following proposition exhibits several cases for an operator T belongs to $\Delta^*\mathcal{F}(\mathcal{H})$. For the proof just notice that the compact operator found in the corresponding cases appeared in the last section is, in fact, of finite rank.

Proposition 3.3. *Let T be a nonzero operator, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$ if it satisfies any of the following statements.*

1. T is nilpotent.
2. T is of finite rank.
3. $T \neq I$ is idempotent.
4. T is non invertible algebraic operator.
5. T is invertible with $T^{-1} \in \Delta^*\mathcal{F}(\mathcal{H})$.

The following theorem reflects the importance of the class $\Delta^*\mathcal{F}(\mathcal{H})$. For this purpose we need some preliminaries.

Lemma 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. T has a nontrivial hyperinvariant subspace iff T^* has so.*

Proof. Suppose that M is a nontrivial hyperinvariant subspace for T , let $S \in \{T^*\}'$, the commutant of T^* , i.e., $ST^* = T^*S$, then $TS^* = S^*T$. Hence, $S^*M \subseteq M$ which implies that $SM^\perp \subset M^\perp$ [2]. Since M is nontrivial and $M \oplus M^\perp = \mathcal{H}$ then $M^\perp$ is also nontrivial subspace. Consequently $M^\perp$ is a nontrivial hyperinvariant subspace for T^* . The converse is obvious since $(T^*)^* = T$. $\square$

Lemma 3.5. *Let M be a nontrivial invariant subspace for an operator T . If M is of finite dimension then T has an eigenvalue.*

Proof. Suppose that an operator T has a finite dimensional nontrivial invariant subspace M , then $T|_M$ represents an operator on a finite dimension Hilbert space over the complex numbers, hence by the Fundamental theorem of algebra, it has an eigenvalue. $\square$

It is known that, for operators A and B in $\mathcal{B}(\mathcal{H})$, if $AB - BA$ has rank one, then either $\ker A$ or $\text{rng} A$ is invariant under B . This fact is due to Laffey [3]. As a modification to Laffey's theorem, it has been shown that for operators A, B and $T \in \mathcal{B}(\mathcal{H})$, if $AB - TA$ has rank one, then either $\ker A$ is invariant under B or $\text{rng} A$ is invariant under T (see [1]). In particular we can prove the following lemma

Lemma 3.6. *Let $A, T \in \mathcal{B}(\mathcal{H})$. If $TA - AT^*$ has rank one, then either $\ker A$ is invariant under T^* or $\text{rng} A$ is invariant under T .*

Proof. Assume $\ker A$ is not invariant under T^* , then $\ker A$ must be nontrivial and hence there exists a nonzero vector $x \in \mathcal{H}$ such that $Ax = 0$ and $AT^*x \neq 0$. Since $TA - AT^*$ has rank one, then there exists a nonzero vector $z \in \mathcal{H}$ such that for each $y \in \mathcal{H}$, $(TA - AT^*)y = c_y z$ where $c_y z$ is a complex number that may depend on y . In particular, $(TA - AT^*)x = -AT^*x \neq 0$, hence $(TA - AT^*)y = -c_y AT^*x$ for each $y \in \mathcal{H}$. Therefore $TAy = AT^*y - c_y AT^*x = AT^*(y - c_y x)$ for each $y \in \mathcal{H}$. Consequently, $TA\mathcal{H} \subseteq AT^*\mathcal{H} \subseteq A\mathcal{H}$. i.e., $\text{rng} A$ is invariant under T . $\square$

Now we are ready to give the following theorem. Recall that if F is a finite rank operator then so is F^* [4, p.49].

Theorem 3.7. *Let T be any non-real scalar operator. If $T \in \Delta^*\mathcal{F}(\mathcal{H})$ then T has a nontrivial hyperinvariant subspace.*

Proof. The assumption implies that there exists a finite rank operator F such that $TF - FT^*$ has rank one. Clearly $F \neq 0$. On the other hand, since F is of finite rank, then $\ker F \neq 0$ and $\text{rng} F$ is a finite dimensional subspace. Thus, by Lemma 3.6, either $\ker F$ is a nontrivial invariant subspace for T^* , hence $\text{rng}(F^*) = (\ker F)^\perp$ is a nontrivial invariant subspace for T or $\text{rng} F$ is a nontrivial invariant subspace for T . Since F^* is also of finite rank, i.e., $\text{rng} F^*$ is a finite dimensional subspace, then we get that either $\text{rng} F$ or $\text{rng} F^*$ is a nontrivial invariant subspace for T . In both cases T has a finite dimensional nontrivial

invariant subspace thus by Lemma 5, T has an eigenvalue, say λ . Moreover, since $T \neq \lambda I$ by assumption, $\ker(T - \lambda I)$ is a nontrivial invariant subspace for T . Consequently, $M = \text{rng}(T^* - \bar{\lambda}I)$ is a nontrivial invariant subspace for T^* . Now, for each $S \in \{T^*\}'$, the commutant of T^* ,

$$S(T^* - \bar{\lambda}I)\mathcal{H} = (T^* - \bar{\lambda}I)S\mathcal{H} \subseteq \text{rng}(T^* - \bar{\lambda}I),$$

i.e., $S(\text{rng}(T^* - \bar{\lambda}I)) \subseteq \text{rng}(T^* - \bar{\lambda}I)$ which implies that $SM \subseteq M$. Thus M is a nontrivial invariant subspace for T^* and consequently by Lemma 3.4, T has a nontrivial hyperinvariant subspace. $\square$

Remark 3.8. The proof of Theorem 7 shows that any operator T which belongs to $\Delta^*\mathcal{F}(\mathcal{H})$ has a nontrivial invariant subspace of finite dimension and thus by Lemma 3.5 has an eigenvalue. The following proposition shows that the converse is true provided that T is nonscalar operator.

Proposition 3.9. *Let T be any non-real scalar operator. If T has an eigenvalue, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $Tf = \lambda f$ for some nonzero vector $f \in \mathcal{H}$. Moreover there exists a nonzero vector $g \in \mathcal{H}$ such that $Tg \neq \bar{\lambda}g$ otherwise $T = \bar{\lambda}I$ which contradicts the assumption. Let $F = f \otimes g$ then,

$$\begin{aligned} TF - FT^* &= T(f \otimes g) - (f \otimes g)T^* \\ &= TF \otimes g - f \otimes Tg - \lambda f \otimes g - f \otimes Tg \\ &= f \otimes \bar{\lambda}g - f \otimes Tg - f \otimes (\bar{\lambda}g - Tg). \end{aligned}$$

Since $\bar{\lambda}g \neq Tg$ then $TF - FT^*$ has rank one. Hence $T \in \Delta^*\mathcal{F}(\mathcal{H})$. $\square$

The following theorem characterizes the elements in $\Delta^*\mathcal{F}(\mathcal{H})$.

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue provided that $T \neq \lambda I$ where $\lambda \in \mathbb{R}$.*

Proof. If $T = \lambda I$ then clearly λ has an eigenvalue for T . Thus $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$. Let T be any non-real scalar operator, then Remark 3.8 and Proposition 3.9 imply that $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue. $\square$

It is worthy to notice that the operator B defined in Example 2.17 does not belong to $\Delta^*\mathcal{F}(\mathcal{H})$ as $\sigma_p(B) = \phi$ ([5, p.229]). However, $B \in \Delta^*(\mathcal{H})$ Example 2.17.

Theorem 3.11. *$T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has a finite dimensional nontrivial invariant subspace provided that $T \neq \lambda I, \lambda \in \mathbb{R}$.*

Proof. If $T \in \Delta^*\mathcal{F}(\mathcal{H})$ then Remark 3.8 shows that T has a finite dimensional non invariant subspace. Suppose that T has a finite dimensional nontrivial invariant subspace and $T \neq \lambda I, \lambda \in \mathbb{R}$, then Lemma 3.5 implies that T has an eigenvalue and thus the results follows from Theorem 3.10. $\square$

Remark 3.12. If U is the unilateral shift operator then $U \notin \Delta^*\mathcal{F}(\mathcal{H})$ since $\sigma_p(U) = \phi$ ([5, p.227]). On the other hand, U has a nontrivial invariant subspace, defined as follows.

Fix a positive integer k and consider $M_k =$ the linear manifold generated by $\{e_n\}, n \geq k$, where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. Since $Ue_n = e_{n+1}$ for all $n \geq 1$ by definition, then $UM_k \subseteq M_k$ for all $k \geq 1$. Thus, $M_k, k \geq 2$ is a nontrivial invariant subspace of U . Clearly, M_k is of infinite dimension for all $k \geq 2$.

The following example shows that $\Delta^*\mathcal{F}(\mathcal{H})$ is not selfadjoint class.

Example 3.13. Let U be the unilateral shift operator then $\sigma_p(U) = \phi$ while $\sigma_p(U^*) \neq \phi$ as $U^*e_1 = 0$ ([5, p.227]). Consequently, Theorem 3.10 implies that $U^* \in \Delta^*\mathcal{F}(\mathcal{H})$ while $U \notin \Delta^*\mathcal{F}(\mathcal{H})$.

On the other hand, for any nonscalar operator T , if either T or T^* belongs to $\Delta^*\mathcal{F}(\mathcal{H})$, then Lemma 3.4 and Theorem 3.10 imply that T has a nontrivial hyperinvariant subspace.

In what follows, more results about $\Delta^*\mathcal{F}(\mathcal{H})$ are obtained.

Proposition 3.14. Let $T \in \Delta^*\mathcal{F}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ satisfies $VT = SV$ for some 1-1 operator V , then $S \in \Delta^*\mathcal{F}(\mathcal{H})$.

Proof. Let F be a finite rank operator such that $TF - FT^*$ has rank one, then there exist nonzero vectors $f, g \in \mathcal{H}$ such that $TF - FT^* = f \otimes g$. Now, $V(TF - FT^*)V^* = V(f \otimes g)V^*$, so $VTFV^* - VFT^*V^* = Vf \otimes Vg$, thus

$$S(VFV^*) - (VFV^*)S^* = Vf \otimes Vg.$$

Since V is 1-1 then $Vf \neq 0$ and $Vg \neq 0$ hence $Vf \otimes Vg$ is of rank one. Consequently, $S \in \Delta^*\mathcal{F}(\mathcal{H})$ as VFV^* is of finite rank. $\square$

Corollary 3.15. Let $T \in \Delta^*\mathcal{F}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. If T is similar to S , then $S \in \Delta^*\mathcal{F}(\mathcal{H})$.

Recall that the compression spectrum of an operator T , denoted by $\Gamma(T)$, is the set of all complex numbers λ such that $\overline{\text{rng}(T - \lambda I)} \neq \mathcal{H}$, and $\sigma(T) = \Gamma(T) \cup \sigma_{ap}(T)$ ([5, p.41]).

Proposition 3.16. Let $T \in \mathcal{B}(\mathcal{H})$. If $\Gamma(T) \neq \phi$ then $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.

Proof. Let $\lambda \in \Gamma(T)$ then $\overline{\text{rng}(T - \lambda I)} \neq \mathcal{H}$. Since $\overline{\text{rng}(T - \lambda I)} \oplus \ker(T^* - \bar{\lambda}I) = \mathcal{H}$ then $\ker(T^* - \bar{\lambda}I) \neq 0$, i.e., $\bar{\lambda} \in \sigma_p(T^*)$ and by Proposition 3.9, $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$. $\square$

Proposition 3.17. Let $T \in \mathcal{B}'(\mathcal{H})$. If there exists $\lambda \in \sigma(T)$ such that $\text{rng}(T - \lambda I)$ is closed then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.

Proof. Let $\lambda \in \sigma(T)$ satisfy $\text{rng}(T - \lambda I)$ is closed. Since $\sigma(T) = \Gamma(T) \cup \sigma_{ap}(T)$, then either $\lambda \in \Gamma(T)$ which implies by Proposition 3.16, that $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$ or $\lambda \in \sigma_{ap}(T)$, thus, $\lambda \in \sigma_p(T)$. Consequently, $T \in \Delta^*\mathcal{F}(\mathcal{H})$ Proposition 3.9. $\square$

Corollary 3.18. *Let T be a non invertible operator. If $\text{rng}T$ is closed then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Since T is not invertible, then $0 \in \sigma(T)$ and the result follows directly from Proposition 3.17. $\square$

Corollary 3.19. *Let T be a normal operator that satisfies the conditions in Proposition 3.17, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Since $\sigma(T) = \sigma_{ap}(T)$ then $\ker(T - \lambda I) \neq 0$, i.e., $\lambda \in \sigma_p(T)$. Hence $T \in \Delta^*\mathcal{F}(\mathcal{H})$ Proposition 3.9. $\square$

Recall that an operator T is isometric if $ST = I$ for some $S \in \mathcal{B}(\mathcal{H})$, equivalently, $\|Tx\| = \|x\|$ for each $x \in \mathcal{H}$ ([2, p.143]).

Corollary 3.20. *Let T be an isometric which is not unitary, then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Clearly T is 1-1 by the definition. Since any isometry operator is unitary iff it is invertible, then T is not invertible. It is easy to check that $\text{rng}T$ is closed. Thus, by Corollary 3.18, either T or T^* belongs to $\Delta^*\mathcal{F}(\mathcal{H})$. $\square$

Recall that an operator T is called a Fredholm operator if $\text{rng}T$ is closed and both $\ker T$ and $(\text{rng}T)^\perp$ are finite dimensional ([5, p.96]). For example it is easy to check that the unilateral shift operator is Fredholm operator.

Corollary 3.21. *Let T be a non invertible Fredholm operator, then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Just apply Corollary 3.18. $\square$

The following result related to the size of $\Delta^*\mathcal{F}(\mathcal{H})$.

Proposition 3.22. *The class $\Delta^*\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{B}(\mathcal{H})$ in the norm operator topology.*

Proof. Since the set of all operators with nonempty point spectrum is dense in $\mathcal{B}(\mathcal{H})$, the result is an immediate consequence of Theorem 3.10. $\square$

Corollary 3.23. *The class $\Delta^*\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{B}(\mathcal{H})$ in the norm operator topology and hence dense in both the operator strong and weak topology.*

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Reconstructing the diffusion coefficient in fractional diffusion equations

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Abstract. We consider the inverse problem of identifying the diffusion coefficient in a time-fractional diffusion equation from noisy interior measurements of the density. To overcome the instability issue of the problem, we employ a mollification based technique to smooth out the noisy data. We derive consistency and stability results, and provide numerical examples to show the feasibility of the proposed approach.

Keywords: inverse problems, coefficient identification, fractional diffusion.

1. Introduction

Fractional partial differential equations are extensions of the classical models for which the usual derivatives are replaced by fractional-order ones. In the past few decades, such equations have gained increasing currency due to a promising wide range of applications in physics, geophysics, polymer sciences, and other areas; see for example [27, 11, 15] and the references therein.

This article is devoted to the problem of identifying (reconstructing) the variable parameter q in the time-fractional diffusion equation

$$(1) \quad \frac{\partial^\alpha u}{\partial t^\alpha}(x, t) = \frac{\partial}{\partial x} \left(q(x, t) \frac{\partial u}{\partial x}(x, t) \right) + f(x, t), \quad (x, t) \in (0, 1)^2,$$

where the time-fractional derivative is taken in the Caputo sense. We shall assume $\alpha \in (0, 1)$, and we further impose the boundary condition

$$(2) \quad q(0, t) \frac{\partial u}{\partial x}(0, t) = g(t).$$

More precisely, we study the following *inverse problem*: assume that q and u are related by equations (1)–(2), estimate $q|_{\Omega}$ from a noisy (and probably discrete) measurement u^{ϵ} of u , where Ω is some suitable compact subset of $[0, 1]^2$.

In many scientific application the data u^{ϵ} can be acquired easily using non-destructive, and in many cases, noninvasive, procedures such as the Infrared Thermography. However, measuring the parameter q (characterizing the material properties of the hosting medium) would inevitably require a direct access and cannot be done remotely or nondestructively. For these reasons, this inverse problem is very important and can be used in various environmental, physical, or engineering applications.

Equation (1), and its variations, are found to be adequate models to describe anomalous diffusion, such as the sub-diffusion, which have been observed, for example, in transport processes in porous media, protein diffusion within cells, transport of ions in column experiments, movement of a material along fractals, etc.; see [30, 23, 8, 14, 12] for derivation and more concrete examples. For the analytical and numerical aspects of fractional-differential equations, the reader may consult [27, 11, 26, 1, 16, 22].

Inverse problems involving classical differential equations have been fairly examined in the past few decades. However, only a humble number publications treating fractional inverse problems have been established in spite of its significance in a variety of scientific and engineering applications. We mention some of the recent inverse problems involving fractional differential equations. Wang and Liu [32] considered the inverse problem of the determination of the initial distribution from internal measurements of $u(\cdot, T)$, their problem is in 2D and they used total variation regularization to obtain stable approximations of the backward problem. In [10], Deng and Yang proposed a numerical method based on the idea of reproducing kernel approximation to reconstruct the unknown initial heat distribution from a finite set of scattered measurements of transient temperature at a fixed final time. Zhang and Xu [34] considered the problem of identifying the time-independent source term f from the additional boundary data $u(0, \cdot)$. In [20], LI and GUO considered the identification of the diffusion coefficient and the order of the fractional derivative from the boundary data $u(0, \cdot)$. A tutorial on inverse problems for anomalous diffusion equations is given by Jin and Rundell [18]. The article summarizes some recent results and sheds the light on several open problems on the subject. See also [33, 21, 31, 17, 2, 3, 4] for other related inverse problems.

The aforementioned inverse problem is ill-posed; clearly if $\partial_x u$ vanishes on some open subset of Ω , then (1) provides no information about the parameter on that set, and hence no uniqueness. Moreover, the need to differentiate the (noisy) data makes the inverse solution instable with respect to perturbation in the data. To retain stability, we employ the regularization by mollification technique which has been introduced in [24, 25]. This amounts to computing the coefficient in (1)–(2) with u replaced by a (smoothed) mollified version of

the data u^ϵ . In section 3, we prove consistency and stability theorems and give error bounds for the proposed method.

The rest of this paper is organized as follows. In Section 2 we present some definitions and lay the theoretical background, in Section 3 we prove the main results, numerical examples are contained in Section 4.

2. Preliminaries

In this section we collect some definitions and preliminary results pertaining to this work. We use the notation $\|f\|_K$ to denote the maximum norm of f over the set K , $C^n(K)$ to denote the space of n -times continuously differentiable functions, and $L^1(K)$ for the space of Lebesgue integrable functions over K .

2.1 Fractional calculus

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. It can be traced back to the year 1695 in a letter from Marquis de L'Hôpital to Gottfried Wilhelm Leibniz asking of the meaning of $d^n y/dx^n$ with $n = 1/2$. Since then several definitions have been given trying to accommodate the meaning of non-integer order derivatives and integrals.

To date, however, there is no unified definition of fractional derivative, but there are some accepted and common definitions in the literature. These definitions include but not limited to the Riemann-Liouville, Caputo, and Grönwald Letnikov definitions [27, 19]. Recently new definitions of fractional derivatives with applications to real world problems have emerged. In [7], Atangana and Baleanu introduced a new fractional derivative definition based on the generalized Mittag-Leffler function. Caputo and Fabrizio [9] presented a new definition of fractional derivative with a smooth kernel based on the exponential function. Comparison between various types of fractional derivatives applied to the fractional Allen Cahn and Freedman models can be found in [6, 13]. Saad et al. [29] used the optimal q-HAM analysis method to solve the fractional gas dynamics equation. An extension of the standard model of cubic isothermal auto-catalytic chemical system to the fractional case along with their numerical solutions using the q-homotopy analysis transform method is considered in [28]. See also the work by Al-Jamel et al. [5] where they studied the forced damped harmonic oscillator within the context of memory-dependent fractional derivative.

Contrary to the Riemann-Liouville definition, Caputo's definition of fractional derivative appears more frequently in the contexts of mathematical physics since it allows traditional (non-fractional) initial and boundary conditions to be incorporated in the mathematical model [27]. Therefore, in our model given by equation (1), we use the Caputo fractional derivative given by the following definition.

Definition 2.1. *The Caputo fractional derivative of order $\alpha \in (0, 1)$ of a function f is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad t > 0.$$

This definition can be adopted for functions of several variables as well. For instance, we define

$$\frac{\partial^\alpha f}{\partial t^\alpha}(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial f}{\partial t}(x, s) ds,$$

and similarly for the fractional derivative of f with respect to the spatial variable x . More about the properties and applications of Caputo derivatives can be found in [27, 19].

2.2 Mollification

We shall need the following definitions and results in the sequel. For more details, readers may consult the textbook [24].

Definition 2.2. *The δ -mollifier, denoted by ρ_δ , is defined by*

$$\rho_\delta(x) = \frac{1}{A\delta} \begin{cases} \exp(-x^2/\delta^2), & |x| \leq 3\delta, \\ 0, & |x| > 3\delta. \end{cases}$$

where $A = \int_{-3}^3 \exp(-s^2) ds$.

Definition 2.3. *The δ -mollification of $f \in L^1(I)$ on K_δ , denoted by $J_\delta f$, is defined by*

$$(J_\delta f)(x) = \int_{-3\delta}^{3\delta} \rho_\delta(s) f(x-s) ds = \int_{x-3\delta}^{x+3\delta} \rho_\delta(x-s) f(s) ds, \quad x \in K_\delta,$$

where $I = [0, 1]$ and $K_\delta = [3\delta, 1-3\delta]$.

Let K be any compact subset of K_δ . Then we have the following auxiliary results [24].

Lemma 2.1. *If $\|f''\|_I \leq M$, then*

$$\begin{aligned} \|(J_\delta f)' - f'\|_K &\leq 3M\delta, \\ \|D^\alpha(J_\delta f) - D^\alpha f\|_K &\leq \frac{3M}{\Gamma(2-\alpha)}\delta. \end{aligned}$$

Lemma 2.2. *Suppose that $f, f^\epsilon \in C(I)$ with $\|f - f^\epsilon\|_I \leq \epsilon$. Then*

$$\begin{aligned} \|(J_\delta f)' - (J_\delta f^\epsilon)'\|_K &\leq \frac{4}{A} \frac{\epsilon}{\delta}, \\ \|D^\alpha(J_\delta f) - D^\alpha(J_\delta f^\epsilon)\|_K &\leq \frac{1}{\Gamma(2-\alpha)} \frac{4}{A} \frac{\epsilon}{\delta}. \end{aligned}$$

We conclude with the following corollary which we shall need in the sequel.

Corollary 2.1. *If $f \in C^1(I)$, then*

$$\min_K \{|f'(x)|\} \leq \|(J_\delta f)'\|_K \leq \|f'\|_I.$$

Proof. This follows from the fact that $(J_\delta f)'(x) = (J_\delta f')(x)$ and the mean-value theorem for integrals. $\square$

3. Main results

We now prove stability, consistency, and convergence results for the mollification method. For convenience, we use the notations $J^\delta u(x, t)$ and $J_\delta u(x, t)$ to denote the δ -mollification with respect to the variables x and t , respectively.

Assume that $\Omega \subset K_\delta \times K_\delta$, and let q_δ and $q_{\delta,\epsilon}$ be the coefficients obtained by solving equations (1)–(2) when u is replaced by $J_\delta u$ and $J_\delta u^\epsilon$, respectively, then we have the following consistency and stability results:

Lemma 3.1. *If $f, g \in C(I)$, $\max \{\|\partial_{xx} u\|_I, \|\partial_{tt} u\|_I\} \leq U < \infty$, and $\min_\Omega |\partial_x u(x, t)| \geq R > 0$, then there exists a constant C independent of δ such that*

$$\|q - q_\delta\|_\Omega \leq C\delta.$$

Proof. Set $E_\delta(x, t) = q(x, t) - q_\delta(x, t)$. Integrating both sides of (1) from 0 to x and using the boundary condition (2) yield

$$\begin{aligned} \partial_x u(x, t) \partial_x J^\delta u(x, t) E_\delta(x, t) &= \left(\int_0^x [\partial_t^\alpha u(s, t) - f(s, t)] ds + g(t) \right) \partial_x J^\delta u(x, t) \\ &\quad - \left(\int_0^x [\partial_t^\alpha J_\delta u(s, t) - f(s, t)] ds + g(t) \right) \partial_x u(x, t). \end{aligned}$$

Let $\tilde{u}_\delta = J^\delta u - u$. Using Corollary 2.1, Lemma 2.1, and simple manipulation, for all (x, t) in Ω we have

$$\begin{aligned} R^2 |E_\delta(x, t)| &\leq \left(\int_0^x [|f(s, t)| + |\partial_t^\alpha u(s, t)|] ds + |g(t)| \right) \|\partial_x \tilde{u}_\delta\|_\Omega \\ &\quad + \int_0^x |\partial_t^\alpha J_\delta u(s, t) - \partial_t^\alpha u(s, t)| ds \|\partial_x u\|_\Omega \\ &\leq 3U\delta \left(\|f\|_I + \frac{\|\partial_t u\|_I}{\Gamma(2-\alpha)} + \|g\|_\Omega \right) + \frac{3U\delta \|\partial_x u\|_\Omega}{\Gamma(2-\alpha)} \leq C\delta, \end{aligned}$$

for some constant C independent of δ , which completes the proof. $\square$

Lemma 3.2. *Suppose that f , g , and u satisfy the hypothesis of Lemma 3.1. If $\|u - u^\epsilon\|_I \leq \epsilon$ and $2\epsilon < AR\delta$, then there exists a constant C independent of ϵ and δ such that*

$$\|q_\delta - q_{\delta,\epsilon}\|_\Omega \leq C \frac{\epsilon}{RA\delta - 2\epsilon}.$$

Proof. Set $E_{\delta,\epsilon} = q_\delta - q_{\delta,\epsilon}$. Then from (1) and (2), for all $(x, t) \in \Omega$ we have

$$\begin{aligned} \left(\partial_x J^\delta u \cdot \partial_x J^\delta u^\epsilon \cdot E_{\delta,\epsilon} \right) (x, t) &= \left(\int_0^x [f(s, t) - \partial_t^\alpha J_\delta u(s, t)] ds + g(t) \right) \partial_x J^\delta u^\epsilon \\ &\quad - \left(\int_0^x (f(s, t) - \partial_t^\alpha J_\delta u^\epsilon(s, t)) ds + g(t) \right) \partial_x J^\delta u. \end{aligned}$$

It follows from Lemma 2.2 and the reverse triangle inequality that

$$0 < R - \frac{2\epsilon}{A\delta} \leq \left| \partial_x J^\delta u^\epsilon(x, t) \right|$$

for all $(x, t) \in \Omega$, and so, with a little algebra, we have

$$\begin{aligned} \left(R^2 - \frac{2R\epsilon}{A\delta} \right) |E_{\delta,\epsilon}| &\leq \int_0^x |\partial_t^\alpha J_\delta u^\epsilon(s, t) - \partial_t^\alpha J_\delta u(s, t)| ds \left| \partial_x J^\delta u(x, t) \right| \\ &\quad + \left(\int_0^x [|f(s, t)| + |\partial_t^\alpha J_\delta u(s, t)|] ds + |g(t)| \right) \|\partial_x \tilde{u}\|_\Omega, \end{aligned}$$

where $\tilde{u} = J^\delta(u^\epsilon - u)$. Thus, using Lemma 2.2 and Corollary 2.1 we obtain

$$\left(R^2 - \frac{2R\epsilon}{A\delta} \right) \|E_{\delta,\epsilon}\|_\Omega \leq \frac{2\epsilon}{A\delta} \left(\|f\|_I + \|g\|_\Omega + \frac{\|\partial_t u\|_I}{\Gamma(2-\alpha)} \right) + \frac{2\epsilon \|\partial_x u\|_I}{A\Gamma(2-\alpha)\delta} \leq C \frac{\epsilon}{\delta},$$

which ends the proof. $\square$

From the triangle inequality, Lemma 3.1 and Lemma 3.2, we obtain the main convergence result.

Theorem 3.1. *Under the assumptions of Lemma 3.1 and Lemma 3.2, we have*

$$\|q - q_{\delta,\epsilon}\|_\Omega \leq C \left(\delta + \frac{\epsilon}{RA\delta - 2\epsilon} \right)$$

for some constant C independent of ϵ and δ .

Remark 3.1. If δ is chosen so that $\delta = O(\epsilon^\mu)$ for some $0 < \mu < 1$, then we obtain the convergence result

$$\|q - q_{\delta,\epsilon}\|_\Omega \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Moreover, taking $\mu = 1/2$, we have the optimal convergence rate

$$\|q - q_{\delta,\epsilon}\|_\Omega = O(\epsilon^{1/2}).$$

4. Numerical examples

In this section we demonstrate how to implement the proposed method in a practical algorithm and we present several numerical examples.

Upon integrating (1) and using the boundary condition (2), the approximate inverse solution can be written as

$$(3) \quad q_{\delta,\epsilon}(x, t) = \frac{\int_0^x [\partial_t^\alpha J_\delta u^\epsilon(s, t) - f(s, t)] ds + g(t)}{\partial_x J_\delta u^\epsilon(x, t)}.$$

In practice the measured data is discrete, and so, the right-hand side of (3) must be discretized. To this end, we approximate the integral using a composite trapezoidal rule, and the fractional derivative is approximated using the algorithm proposed in [25]. For the partial derivative in the denominator, we use the centered difference approximation.

For simplicity, the noisy data u^ϵ is computed according to the formula

$$u^\epsilon(x_i, t_j) = u(x_i, t_j) + \epsilon \gamma_{i,j}, \quad i = 1, \dots, N_x, \quad j = 1, \dots, N_t,$$

where ϵ represents the noise level, and $\gamma_{i,j}$ is a uniformly distributed random number in $[-1, 1]$. In the experiments below, we take $\Omega = [0, 1]^2$ and we use uniform grid points with $N_x = N_t = 150$. The mollification parameter δ , representing the degree of smoothing, is determined by the Principle of Generalized Cross Validation as described in the article [24].

The error in the approximation is measured using the relative root-mean-square error which is defined as

$$\text{RMS} = \frac{\sqrt{\sum_{i=1}^{N_x} \sum_{j=1}^{N_t} [q(x_i, t_j) - q_{\delta,\epsilon}(x_i, t_j)]^2}}{\sqrt{\sum_{i=1}^{N_x} \sum_{j=1}^{N_t} [q(x_i, t_j)]^2}}.$$

We note that the RMS is just the discrete version of the relative L^2 -error.

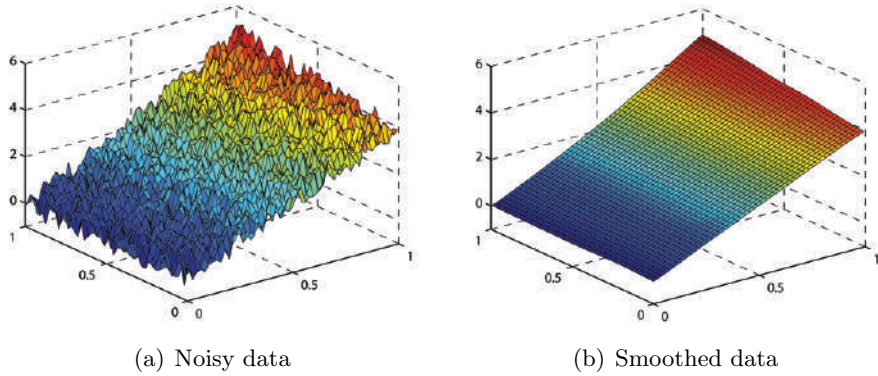


Figure 1: Smoothing the data of Example 4.1.

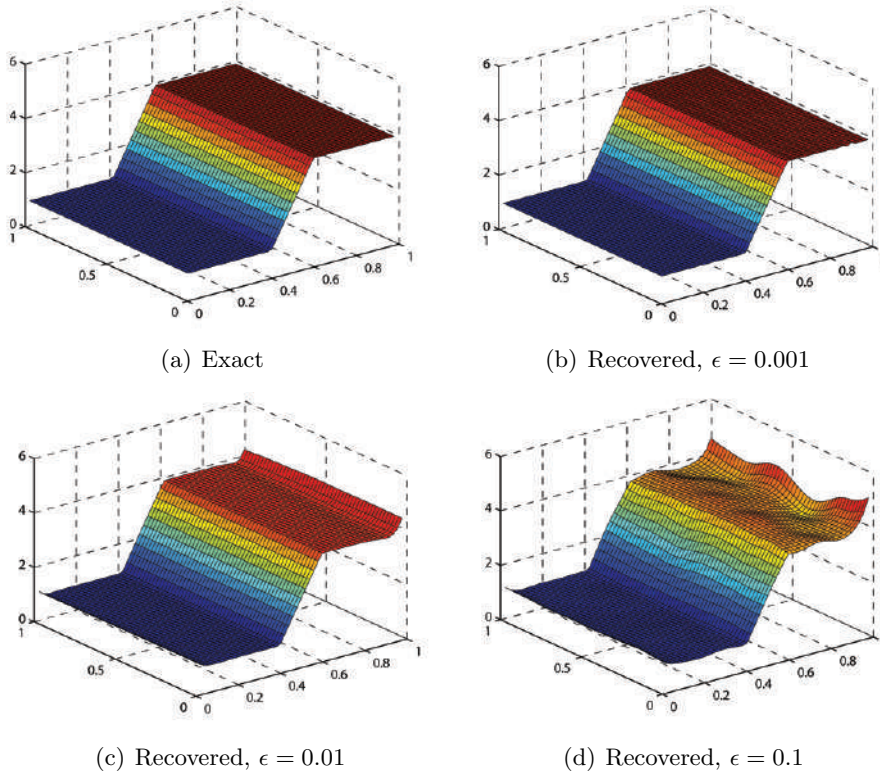
Table 1: Error results for Example 4.1 at several noise levels ϵ .

Noise Level	0.1%	1%	10%
(i) Raw	1.39E-2	1.41E-1	7.39E+1
(ii) Mollified	4.80E-3	3.56E-2	4.10E-2

Example 4.1. We consider equations (1)–(2) with $u(x, t) = tx^2 + 4x$ and $\alpha = 0.75$. The function f is chosen so that the exact diffusion coefficient is

$$q(x, t) = \begin{cases} 1, & 0 \leq x \leq 0.4, \\ 15x - 5, & 0.4 < x \leq 0.6, \\ 4, & 0.6 < x \leq 1. \end{cases}$$

The error results using the noisy (Raw) and smoothed (Mollified) data are presented in Table 1. Figure 1 shows the plots for the exact data, noisy data, and the denoised (smoothed) data when the noise level in the data is $\epsilon = 10\%$. The recovered parameter q at different noise levels are shown in Figure 2.

Figure 2: Exact and recovered parameter q in Example 4.1.

Example 4.2. In this experiment, we consider equations (1)–(2) with $\alpha = 0.5$ and $u(x, t) = 8x \exp(t) \operatorname{erfc}(\sqrt{t})$, where $\operatorname{erfc}(\cdot)$ is the complementary error

function. The source term f is chosen so that the exact diffusion coefficient is

$$q(x, t) = (1 - 0.4 \sin(10x + 2t))^{-1}.$$

The error results using the noisy and smoothed data are summarized in Table 2. Plots for the recovered parameter q at different noise levels are shown in Figure 3.

Table 2: Error results for Example 4.2 at several noise levels ϵ .

Noise Level	0.1%	1%	10%
(i) Raw	1.50E-2	1.57E-1	9.07E+2
(ii) Mollified	5.60E-3	3.67E-2	4.75E-2

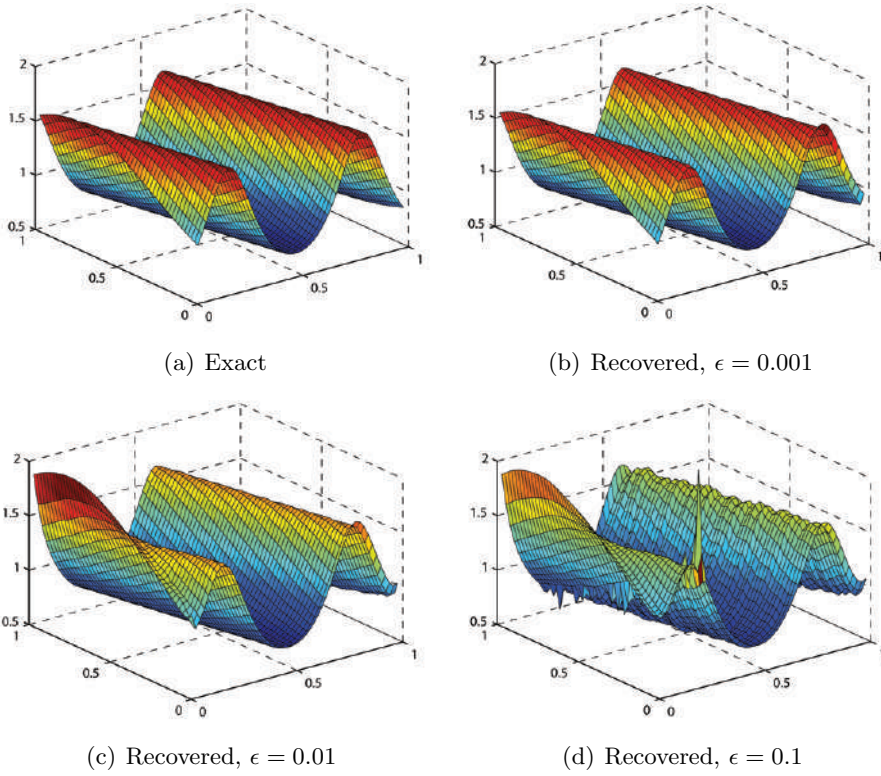


Figure 3: Exact and recovered parameter q in Example 4.2.

5. Conclusion

We considered mollification based technique to solve the inverse problem of identifying the diffusion coefficient in a one-dimensional time-fractional diffusion equation from noisy interior data. We proved consistency and convergence

theorems for the proposed approach. Numerical examples are also given to elucidate and validate the proposed method. Numerical results show that this approach is powerful and indicate noteworthy results.

We observe from the numerical tests above that for small noise levels the mollification method gives slight improvement in the error. However for larger noise levels, the method shows noteworthy improvements over using directly the raw (non-smoothed) data. In the future, we hope to carry the analysis for the discretized problem, and generalize this approach to problems in higher spatial dimensions which is more relevant in scientific applications.

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Structures in a differentiable manifold and their applications to the tangent bundle

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Abstract. The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair, E. T. Davies, P. Dombrowski, S. Ianus, A. J. Ledger and K. Yano, V. Oproiu, C. Udriste, Yano and Davies, Yano and Ishihara and among others [15, 23, 24, 25, 26]. It is well known that different structures defined on a manifold M can be lifted to the same type of structures on its tangent bundle. However, when we consider the generalized almost r-contact structure not the same type of structure is obtained on the tangent bundle. The aim of this study is to investigate the prolongations of G -structures immersed in the generalized almost r-contact structure on a manifold M to its tangent bundle $T(M)$. Moreover, An almost contact structure, Lorentzian almost paracontact structure etc. on the tangent bundle $T(M)$ are discussed.

Keywords: complete lift, vertical lift, tangent bundle, G -structure, Hsu-structure, Lorentzian almost paracontact structure, integrability.

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1. Introduction

The study was made based on the general theory of prolongations, the geometric properties of the prolongations of pseudo-group structures and G -structures to tangent bundles [3]. The previous study investigated the prolongation of G -structures to tangent bundles of first and higher orders and showed that the integrability of G -structures is equivalent to the integrability of its prolongations [1, 2]. In 2006 Das and Nivas [7] defined certain structures on tangent bundle and studied prolongation of F -structures to the tangent bundles of order r . The prolongations of different structures like F -structure, golden structure, G -structure has been studied [5, 6, 11, 12].

In the previous papers[10, 13, 22], we have studied the complete and vertical lifts in Quarter-symmetric semi-metric connection, Quarter-symmetric metric connection to tangent bundles and semi-symmetric non-metric connection on a Kähler manifold. Earlier investigators studied the complete and horizontal lifts of golden structure in the tangent bundle [11].

The paper is structured as follows: In Section 2, let us recall the definitions of vertical lift, complete lift, Hsu-structure and generalized almost r -contact structure. Section 3 deals with the study of prolongation of tensor fields and G -structure to the tangent bundle and the integrability of the prolongation of a G -structure. In Section 4, we present results on some classical G -structures defined by tensor fields immersed in the generalized almost r -contact structure to tangent bundle. Moreover, our goal is to see as to what kind of structure is defined on the tangent bundle $T(M)$ when we consider an almost contact structure, Lorentzian almost paracontact structure etc. in the last Section.

2. Preliminaries

Let M be an n -dimensional differentiable manifold of class C^∞ and $T_p(M)$ be the tangent space of M at a point of M . Then the set

$$T(M) = \bigcup_{p \in M} T_p(M)$$

is called the tangent bundle over the manifold M ([4]).

2.1 Vertical lifts

If f is a function in M , we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \rightarrow M$ and $f : M \rightarrow R$, so that

$$f^V = f \circ \pi$$

thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h) then

$$f^V(\tilde{p}) = f^V(x, y) = f \circ \pi(\tilde{p}) = f(p) = f(x),$$

thus, the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f ([4]).

2.2 Complete lifts

If f is a function in M , we write f^C for the function in $T(M)$ defined by

$$f^C = i(df)$$

and call f^C the complete lift of the function f . The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$. Suppose that $X \in \mathfrak{S}_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $X \in \mathfrak{S}_0^1(M)$. Then a 1-form ω^C in $T(M)$ defined by

$$\omega^C(X^C) = (\omega(X))^C$$

X being an arbitrary vector field in M . We call ω^C the complete lift of ω .

We use vertical and complete lifts of functions, vector fields, forms and tensor fields in the sense of ([4]).

2.3 Hsu-structure

The base space M is said to possess a Hsu-structure if there exists on M a tensor field F of type $(1, 1)$ such that

$$(1) \quad F^2 = a^r I,$$

where I is the unit tensor field and a is a real or imaginary number ([8, 9]).

2.4 Generalized almost r -contact structure

The manifold M will be said to possess the generalized almost r -contact structure if there exists on M a tensor field F of type $(1, 1)$, $r(C^\infty)$ contravariant vector fields U_p and $r(C^\infty)$ 1-forms ω_p satisfying ([7])

$$(2) \quad F^2 = a^r I + \epsilon \sum_{p=1}^r \omega_p \otimes U_p,$$

where

$$(3) \quad \begin{aligned} (i) \quad & FU_p = 0, \\ (ii) \quad & \omega_p \circ F = 0, \\ (iii) \quad & \omega_p(U_q) = -\frac{a^r}{\epsilon} \delta_q^p, \end{aligned}$$

where $p, q = 1, 2, \dots, r$ and δ_q^p denote the Kronecker delta while ϵ is non-zero real numbers. The manifold M is called a generalized almost r -contact manifold and manifold with a generalized almost r -contact structure or in short with an $(F, U_p, \omega_p, a, \epsilon)$ -structure. The structure is said to be normal if the tensor $S = [F, F] + \epsilon \sum_{p=1}^r \omega_p \otimes U_p$ vanishes.

3. Prolongations of G -structure immeresed in generalized almost r -contact structure to tangent bundle

Let there be given a Lie subgroup G of $GL(n, R)$ and a tensor field $\overset{o}{F}$ of type (1,1) in R^n , which is invariant by G . An n -dimensional manifold M is assumed to admit a G -structure P . We take a coordinate neighborhood $\{U, X^h\}$ of M and an n -frame $\{X_{(i)}\}$ in U , which is adapted to the G -structure P . Thus, if we put

$$(4) \quad \overset{o}{F} = \overset{o}{F}_i^{h} X_{(h)} \theta^{(i)}$$

in U , $\{\theta^{(i)}\}$ being the coframe dual to $\{X_{(i)}\}$ in U and $\overset{o}{F}_i^{h}$ components of $\overset{o}{F}$ in R^n . The local tensor field F , defined by equation (4) in each coordinate neighborhood U , is determined independently of the choice of the adapted frame $\{X_{(i)}\}$ and hence defines globally in M a tensor field denoted by F , which is called the tensor field induced in M from $(\overset{o}{F}, P)$ ([4]).

Some classical G -structures defined by tensor fields immersed in the generalized almost r -contact structure to tangent bundle.

(I) $G = GL(n, C)$. Let $\overset{o}{F}$ be a tensor of type (1,1) in R^n such that $F^2 = a^r I$ and denoted by $GL(n, C)$ the group of all the linear transformations which leave $\overset{o}{F}$ invariant. Then the complete lift $\overset{o}{F}^{\overset{o}{C}}$ of $\overset{o}{F}$ to $T(R^n)$ is a tensor of type (1,1) satisfying $F^2 = a^r I$ and the tangent group $T(G)$ leaves $\overset{o}{F}^{\overset{o}{C}}$ invariant. Thus we obtain

$$T(G) = GL(n, C).$$

Therefore, we have the following theorem:

Theorem 3.1. *If P is Hsu-structure (as a G -structure) on manifold M , then the prolongation $\tilde{P}$ of P to the tangent bundle $T(M)$ is the Hsu-structure determined*

by the complete lift $\overset{\circ}{F}^C$ of F to $T(M)$. The prolongation $\tilde{P}$ of P to $T(M)$ is the Hsu-structure if and only if P is integrable.

(II) $G = GL(s, C) \times GL(m, R)$. Let $\overset{\circ}{F}$ be a tensor of type (1,1) and of rank $2s$ in $(2s + m)$ such that $F^3 - a^r F = 0$. If we denote by G the group of all the elements of $GL(n, R)$, which leave $\overset{\circ}{F}$ invariant, then we easily obtain $T(G) \subset GL(2s, C) \times GL(2m, R) \subset GL(n, R)$. Then the complete lift $\overset{\circ}{F}^C$ of $\overset{\circ}{F}$ to satisfies $F^3 - a^r F = 0$ and is of $2s$ rank. Thus, we obtain

$$T(G) \subset GL(2s, C) \times GL(2m, R) \subset GL(2n, R).$$

Therefore, we have the following theorem:

Theorem 3.2. *If a manifold M admits Hsu-structure P (as a G -structure) determined by a tensor field F of type (1,1) and of rank s everywhere such that $F^3 - a^r F = 0$, then on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P to $T(M)$ is the Hsu-structure determined by the complete lift $\overset{\circ}{F}^C$ of $\overset{\circ}{F}$ to $T(M)$ where $\overset{\circ}{F}^C$ is of rank $2s$. The Hsu-structure P is integrable in M if and only if the prolongation $\tilde{P}$ of P to $T(M)$ is integrable.*

(III) $G = GL(n, C) \times I$. Let $\overset{\circ}{F}$ be a tensor of type (1, 1) and of rank $2s$, contravariant vector fields $\overset{\circ}{U}_p$ and 1-forms $\overset{\circ}{\omega}_p, p = 1, 2, \dots, r$ in R^{2n+r} such that

$$(5) \quad \overset{\circ}{F}^2 = a^r I + \epsilon \sum_{p=1}^r \overset{\circ}{\omega}_p \otimes \overset{\circ}{U}_p,$$

where

$$(6) \quad \begin{aligned} (i) \quad & \overset{\circ}{F} \overset{\circ}{U}_p = 0, \\ (ii) \quad & \overset{\circ}{\omega}_p \circ \overset{\circ}{F} = 0, \\ (iii) \quad & \overset{\circ}{\omega}_p(\overset{\circ}{U}_q) = -\frac{a^r}{\epsilon} \delta_q^p, \end{aligned}$$

then if we denote by G the group of all the elements of $GL(2n + r, R)$, which leave $\overset{\circ}{F}, \overset{\circ}{U}_p, \overset{\circ}{\omega}_p, p = 1, 2, \dots, r$ invariant, then we easily obtain

$$G = GL(n, C) \times I \subset GL(2n + r, R),$$

where I denotes the trivial group. The complete lifts $\overset{\circ}{F}^C, \overset{\circ}{U}_p^C, \overset{\circ}{\omega}_p^C$, the vertical lifts $\overset{\circ}{F}^V, \overset{\circ}{U}_p^V, \overset{\circ}{\omega}_p^V, p = 1, 2, \dots, r$ satisfy

$$(\overset{\circ}{F}^C)^2 = a^r I + \epsilon \sum_{p=1}^r \left\{ \overset{\circ}{\omega}_p^V \otimes \overset{\circ}{U}_p^C + \overset{\circ}{\omega}_p^C \otimes \overset{\circ}{U}_p^V \right\},$$

$$\begin{aligned}
& \overset{o}{F} \overset{o}{U}_p = 0, \overset{o}{F} \overset{o}{U}_p^V = 0, \overset{o}{\omega}_p \overset{o}{F} = 0, \overset{o}{\omega}_p^V \overset{o}{F} = 0, \\
(7) \quad & (\overset{o}{\omega}_p^C)(\overset{o}{U}_q^C) = 0, (\overset{o}{\omega}_p^V)(\overset{o}{U}_q^V) = 0, (\overset{o}{\omega}_p^C)(\overset{o}{U}_q^V) = -\frac{a^r}{\epsilon} \delta_q^p, (\overset{o}{\omega}_p^V)(\overset{o}{U}_q^C) = -\frac{a^r}{\epsilon} \delta_q^p
\end{aligned}$$

and the tangent group $T(G)$ of G leaves $\overset{o}{F}, \overset{o}{U}_p, \overset{o}{\omega}_p, p = 1, 2, \dots, r$ invariant. Thus we obtain

$$T(G) \subset GL(2n, C) \times I \subset GL(4n + 2r, R).$$

Therefore, we have the following Theorem:

Theorem 3.3. *If a manifold M of $2n + r$ dimensions admits the generalized almost r -contact structure P (as a G -structure) determined by a tensor field F of type $(1, 1)$, contravariant vector fields U_p and 1-forms $\omega_p, p = 1, 2, \dots, r$ such that*

$$(8) \quad F^2 = a^r I + \epsilon \sum_{p=1}^r \omega_p \otimes U_p$$

where

$$\begin{aligned}
(9) \quad & (i) \quad FU_p = 0, \\
& (ii) \quad \omega_p \circ F = 0, \\
& (iii) \quad \omega_p(U_q) = -\frac{a^r}{\epsilon} \delta_q^p,
\end{aligned}$$

then, on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is a framed the generalized almost r -contact structure ([18, 14]).

If we put

$$(10) \quad \tilde{J} = \overset{o}{F} + \frac{\epsilon}{a^{\frac{r}{2}}} \sum_{p=1}^r \left\{ \overset{o}{\omega}_p^V \otimes \overset{o}{U}_p^C + \overset{o}{\omega}_p^C \otimes \overset{o}{U}_p^V \right\},$$

then in view of the equations (7)-(9) it is easily shown that $\tilde{J}^2 = a^r I_{T(M)}$ and that $\tilde{J}$ is left-invariant by the tangent group $T(G)$. Thus we obtain

$$T(G) \subset GL(n, C) \times I \subset GL(n + r, R) \subset GL(2n + 2r, R).$$

Therefore, we have the following theorem:

Theorem 3.4. *If a manifold M of $2n + r$ dimensions admits the generalized almost r -contact structure P (as a G -structure) determined by $(F, U_p, \omega_p, a, \epsilon)$ given in theorem (3.3), then, on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is the generalized almost r -contact structure determined by the tensor field*

$$(11) \quad \tilde{J} = \overset{o}{F} + \frac{\epsilon}{a^{\frac{r}{2}}} \sum_{p=1}^r \left\{ \overset{o}{\omega}_p^V \otimes \overset{o}{U}_p^C + \overset{o}{\omega}_p^C \otimes \overset{o}{U}_p^V \right\}.$$

4. Conclusion

In this section, we will discuss to what kind of structure is defined on tangent bundle when we consider an almost contact structure, Lorentzian almost paracontact structure etc.

Putting $r = 1, a = -1$ in equation (1), we have

$$F^2 = -I$$

then, the tensor field F of type (1,1) is called almost complex structure.

Putting $r = 1, a = -1, \epsilon = 1$ in equation (2), we have

$$F^2 = -I + \omega \otimes U, \quad \omega(U) = 1$$

then, the tensor field F of type (1,1) is called an almost contact structure on M ([21, 19]).

The G -structures defined by tensor fields immersed in an almost contact structure to tangent bundle then Theorem (3.1), Theorem (3.3) and Theorem (3.4) change as Theorem (4.1), Theorem (4.2) and Theorem (4.3) respectively. Thus, we have

Theorem 4.1. *If P is almost complex structure (as a G -structure) on manifold M , then the prolongation $\tilde{P}$ of P to the tangent bundle $T(M)$ is almost complex structure determined by the complete lift $\overset{o}{F}$ of F to $T(M)$. The prolongation $\tilde{P}$ of P to $T(M)$ is a complex structure if and only if P is integrable.*

Theorem 4.2. *If a manifold M admits almost contact structure P (as a G -structure) determined by a tensor field F of type (1, 1), contravariant vector field U and 1-form ω such that*

$$(12) \quad F^2 = -I + \omega \otimes U$$

where

$$(13) \quad \begin{aligned} (i) \quad & FU = 0, \\ (ii) \quad & \omega \circ F = 0, \\ (14) \quad & \omega(U) = 1, \end{aligned}$$

then on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is a framed almost contact structure.

Theorem 4.3. *If a manifold M admits almost contact structure P (as a G -structure) determined by (F, U, ω) given in theorem 4.2, then, on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is almost contact structure determined by the tensor field*

$$(15) \quad \tilde{J} = \overset{o}{F} - \left\{ \overset{o}{\omega} \otimes \overset{o}{U} + \overset{o}{\omega} \otimes \overset{o}{U} \right\}.$$

Putting $r = 0, \epsilon = 1$ in equation (2), we have $F^2 = I$. The tensor field F of type (1,1) is called an almost product structure.

$$F^2 = I + \omega \otimes U, \quad \omega(U) = -1.$$

Then, the tensor field F of type (1,1) is called Lorentzian almost paracontact structure on M ([20]).

The G -structures defined by tensor fields immersed in Lorentzian almost paracontact structure to tangent bundle then Theorem (3.1), Theorem (3.3) and Theorem (3.4) change as Theorem (4.4), Theorem (4.5) and Theorem (4.6) respectively. Thus, we have

Theorem 4.4. *If P is almost product structure (as a G -structure) on manifold M , then the prolongation $\tilde{P}$ of P to the tangent bundle $T(M)$ is almost product structure determined by the complete lift $\overset{o}{F}$ of F to $T(M)$. The prolongation $\tilde{P}$ of P to $T(M)$ is almost product structure if and only if P is integrable.*

Theorem 4.5. *If a manifold M admits Lorentzian almost paracontact structure P (as a G -structure) determined by a tensor field F of type (1, 1), contravariant vector field U and 1-form ω such that*

$$(16) \quad F^2 = I + \omega \otimes U$$

where

$$(17) \quad \begin{aligned} (i) \quad & FU = 0, \\ (ii) \quad & \omega \circ F = 0, \\ (18) \quad & \omega(U) = -1, \end{aligned}$$

then on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is a framed Lorentzian almost paracontact structure.

Theorem 4.6. *If a manifold M admits Lorentzian almost paracontact structure P (as a G -structure) determined by (F, U, ω) given in theorem 4.5, then, on the tangent bundle $T(M)$, the prolongation $\tilde{P}$ of P is Lorentzian almost paracontact structure determined by the tensor field*

$$(19) \quad \tilde{J} = \overset{o}{F} + \overset{o}{\omega} \otimes \overset{o}{U} + \overset{o}{\omega} \otimes \overset{o}{U}.$$

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Warped product submanifolds of a generalized Sasakian space form admitting nearly cosymplectic structure

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Abstract. In this paper we consider semi-invariant warped product submanifolds of the type $M = N_T \times_{\psi} N_{\perp}$ of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure and obtained a characterizing inequality ($\|\sigma_{\mu}(D, D^{\perp})\|^2 \geq f_2 \cdot \alpha \cdot \beta + 2\|\mathcal{P}_{D^{\perp}} D\|^2 + 2\|\mathcal{Q}_{D^{\perp}} D\|^2$) for existence of warped product submanifolds. Moreover, some special cases are also discussed. The results in the present paper generalize the existing results available in the literature.

Keywords: warped product, semi-invariant submanifolds, nearly cosymplectic manifold.

1. Introduction

The conception of warped product of manifolds was instigated by R. L. Bishop and B. O. Neill in order to set up a massive velocity of manifolds of negative curvatures [9]. The formulation of warped product on submanifolds was given by B. Y. Chen (c.f., [10], [11]). Basically, B. Y. Chen [10] studied warped product of the types $N_{\perp} \times_{\psi} N_T$ and $N_T \times_{\psi} N_{\perp}$ and produced a sharp relationship between the warping function ψ and squared norm of second fundamental form for CR-warped product submanifolds in the setting of Kaehler manifolds. Later, I. Hesigawa and I. Mihai [12] investigated contact version of CR-warped product submanifolds in the setting of Sasakian manifolds. A step forward, I. Mihai [15] developed the analogous inequality for contact CR-warped products in the Sasakian space forms. Furthermore, contact CR-warped product submanifolds in Kenmotsu space form were also studied by K. Arslan et al. [2] and they achieved a sharp inequality for the squared norm of the second fundamental form and the warping function.

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M. Atceken deliberate the contact CR-warped product submanifolds in the setting of cosymplectic and Kenmotsu space forms (see [3], [4]) and compute a necessary and sufficient condition for contact CR-product. In this sequence, S. Sular and C. Özgür in [17] investigated contact CR-warped product submanifolds of trans-Sasakian generalized Sasakian space forms and obtained a more general inequality for squared norm of second fundamental form and warping function. Sufficient work has been done for contact CR-warped product submanifolds of almost contact metric manifolds (see [2], [5], [6], [18]).

Stimulated by the analysis of above researchers, in the present paper, we consider semi invariant warped product submanifolds of a generalized Sasakian space form admitting nearly cosymplectic structure and obtained some interesting results and some special cases are also discussed and correlated them with the previous works.

The paper is categorized as: section 2 is introductory which holds a brief information about almost contact metric manifolds and in particular nearly cosymplectic manifolds. Moreover, in this section the definition of a generalized Sasakian space form is given. In section 3, warped product of manifolds are discussed and we also collect some useful results for later use. Finally, we establish a characterizing inequality for squared norm of second fundamental form of contact CR-warped product submanifolds in the background of generalized Sasakian space forms admitting nearly cosymplectic structures.

2. Preliminaries

The tensorial equations of an almost contact metric manifold $\bar{M}$ with dimension $(2n + 1)$ are given by

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$

where ϕ, ξ and η are the $(1, 1)$ tensor field, a characteristic vector field and 1-form respectively.

The Riemannian metric g exists on an almost contact metric manifold $\bar{M}$ satisfies the following conditions

$$(2.2) \quad \eta(U) = g(U, \xi), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for all $U, V \in T\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J(U, f \frac{d}{dt}) = (\phi U - f\xi, \eta(U) \frac{d}{dt}),$$

where f is C^∞ -function on $\bar{M} \times R$, has no torsion, that is J is integrable and the condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ . Finally, the *fundamental 2-form* Φ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and η are closed [8]. The structure is said to be nearly cosymplectic if ϕ is killing i.e.,

$$(2.3) \quad (\bar{\nabla}_U \phi)U = 0,$$

for each $U \in T\bar{M}$.

Proposition 2.1 ([7]). *On a nearly cosymplectic manifold, the vector field ξ is killing.*

From the above proposition we have $\bar{\nabla}_U \xi = 0$, for any $U \in T\bar{M}$, where $\bar{M}$ is a nearly cosymplectic manifold.

Given an almost contact metric manifold $\bar{M}$, it is said to be a generalized Sasakian space form [1] if there exist three functions f_1 , f_2 and f_3 on $\bar{M}$ such that

$$(2.4) \quad \begin{aligned} \bar{R}(U, V)W &= f_1\{g(V, W)U - g(U, W)V\} \\ &+ f_2\{g(U, \phi W)\phi V - g(V, \phi W)\phi U \\ &+ 2g(U, \phi V)\phi W\} + f_3\{\eta(U)\eta(W)V \\ &- \eta(V)\eta(W)U + g(U, W)\eta(V)\xi - g(V, W)\eta(U)\xi\}, \end{aligned}$$

for any vector fields U, V, W on $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ then $\bar{M}$ is Sasakian space form [8], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}$ is a Kenmotsu space form [13], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}$ is a cosymplectic space form [14].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g , and if ∇ and $\nabla^\perp$ are the induced connection on the tangent bundle TM and the normal bundle $T^\perp M$ of M , respectively, then the Gauss and the Weingarten formulae are given by

$$(2.5) \quad \bar{\nabla}_U V = \nabla_U V + \sigma(U, V),$$

$$(2.6) \quad \bar{\nabla}_U N = -A_N U + \nabla_U^\perp N,$$

for each $U, V \in TM$ and $N \in T^\perp M$, where σ and A_N are the second fundamental form and the shape operator respectively, for the immersion of M in $\bar{M}$, they are related as

$$(2.7) \quad g(\sigma(U, V), N) = g(A_N U, V),$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M .

The mean curvature vector H of M is given by

$$H = \frac{1}{n} \sum_{i=1}^n \sigma(u_i, u_i),$$

where n is the dimension of M and $\{u_1, u_2, \dots, u_n\}$ is a local orthonormal frame of vector fields on M . The squared norm of the second fundamental form is defined as

$$(2.8) \quad \|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(u_i, u_j), \sigma(u_i, u_j)).$$

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold if $\sigma(U, V) = 0$, for each $U, V \in TM$, and totally umbilical submanifold if $\sigma(U, V) = g(U, V)H$.

For any $U \in TM$, we write

$$(2.9) \quad \phi U = PU + FU,$$

where PU is the tangential component and FU is the normal component of ϕU .

Similarly, for $N \in T^\perp M$, we can write

$$(2.10) \quad \phi N = tN + fN,$$

where tN and fN are the tangential and normal components of ϕN respectively.

The covariant differentiation of the tensors ϕ , P , F , t and f are defined as respectively

$$(2.11) \quad (\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi \bar{\nabla}_U V,$$

$$(2.12) \quad (\bar{\nabla}_U P)V = \nabla_U P V - P \nabla_U V,$$

$$(2.13) \quad (\bar{\nabla}_U F)V = \nabla_U^\perp F V - F \nabla_U V,$$

$$(2.14) \quad (\bar{\nabla}_U t)N = \nabla_U t N - t \nabla_U^\perp N,$$

$$(2.15) \quad (\bar{\nabla}_U f)N = \nabla_U^\perp f N - f \nabla_U^\perp N.$$

Furthermore, for any $U, V \in TM$, the tangential and normal parts of $(\bar{\nabla}_U \phi)V$ are denoted by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ i.e.,

$$(2.16) \quad (\bar{\nabla}_U \phi)V = \mathcal{P}_U V + \mathcal{Q}_U V.$$

Since ξ is the killing vector field, then it is easy to verify the following property

$$(2.17) \quad (\bar{\nabla}_U \phi)\phi V = -\phi(\bar{\nabla}_U \phi V).$$

On using equations (2.6)-(2.14) and (2.17), we may obtain that

$$(2.18) \quad \mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V),$$

$$(2.19) \quad \mathcal{Q}_U V = (\bar{\nabla}_U F)V + \sigma(U, PV) - fh(U, V).$$

On a submanifold M of a nearly cosymplectic manifold by (2.3) and (2.16)

$$(2.20) \quad (a) \quad \mathcal{P}_U V = -\mathcal{P}_V U, \quad (b) \quad \mathcal{Q}_U V = -\mathcal{Q}_V U$$

for any $U, V \in TM$.

Now we have the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which can be verified easily

- (P₁) (i) $\mathcal{P}_{U+V}W = \mathcal{P}_UW + \mathcal{P}_VW$ (ii) $\mathcal{Q}_{U+V}W = \mathcal{Q}_UW + \mathcal{Q}_VW$,
(P₂) (i) $\mathcal{P}_U(V+W) = \mathcal{P}_UV + \mathcal{P}_UW$ (ii) $\mathcal{Q}_U(V+W) = \mathcal{Q}_UV + \mathcal{Q}_UW$,
(P₃) (i) $g(\mathcal{P}_UV, W) = -g(W, \mathcal{P}_UW)$ (ii) $g(\mathcal{Q}_UV, N) = -g(W, \mathcal{P}_UN)$,

for all $U, V, W \in TM$ and $N \in T^\perp M$.

An m -dimensional Riemannian submanifold M of an almost contact metric manifold $\bar{M}$, where ξ is tangent to M , is called *semi-invariant submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^\perp$ is anti invariant, that is

$$TM = D \oplus D^\perp \oplus \langle \xi \rangle,$$

where $\phi D \subseteq D$ and $\phi D^\perp \subseteq T^\perp M$ and $\langle \xi \rangle$ denotes 1-dimensional distribution which is spanned by ξ .

If μ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of semi-invariant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$(2.21) \quad T^\perp M = \mu \oplus \phi D^\perp.$$

A semi-invariant submanifold M is called semi-invariant product if the distribution D and $D^\perp$ are parallel on M . In this case M is foliated by the leaves of these distributions. In general, if N_1 and N_2 are Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric g defined as

$$(2.22) \quad g(U, V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V),$$

where π_1 and π_2 are the projection maps of M onto N_1 and N_2 , respectively, and $d\pi_1$ and $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of semi-invariant product submanifold, one can consider warped product of manifolds which are defined as follows

Definition 2.2. Let (R, g_R) and (S, g_S) be two Riemannian manifolds with Riemannian metric g_R and g_S respectively and ψ be a positive differentiable function on R . The warped product of R and S is the Riemannian manifold $(R \times S, g)$, where

$$g = g_R + \psi^2 g_S.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively.

The warped product manifold $(R \times S, g)$ is denoted by $R \times_\psi S$. If U is the tangent vector field to $M = R \times_\psi S$ at (α, β) then

$$(2.23) \quad \|U\|^2 = \|d\pi_1 U\|^2 + \psi^2(\alpha) \|d\pi_2 U\|^2.$$

The following Lemma was proved by R. L. Bishop and B. O'Neill [9].

Theorem 2.1. *Let $M = R \times_\psi S$ be a warped product manifolds. If $X, Y \in TR$ and $V, W \in TS$ then*

$$(i) \quad \nabla_X Y \in TR$$

$$(ii) \quad \nabla_X V = \nabla_V X = \left(\frac{X\psi}{\psi}\right)V,$$

$$(iii) \quad \nabla_V W = \frac{-g(V, W)}{\psi} \nabla \psi.$$

From the above theorem, for the warped product $M = R \times_f S$ it is easy to conclude that

$$(2.24) \quad \nabla_X V = \nabla_V X = (X \ln \psi) V,$$

for any $X \in TR$ and $V \in TS$.

$\text{grad } \psi$ is the gradient of ψ and is defined as

$$(2.25) \quad g(\text{grad } \psi, U) = U\psi,$$

for all $U \in TM$.

Corollary 2.1. *On a warped product manifold $M = R \times_\psi S$, the following statements hold*

$$(i) \quad N_1 \text{ is totally geodesic in } M,$$

$$(ii) \quad N_2 \text{ is totally umbilical in } M.$$

In what follows, $N_\perp$ and N_T will denote a totally real and holomorphic submanifold respectively of an almost Hermitian manifold $\bar{M}$.

A warped product manifold is said to be trivial if its warping function ψ is constant.

Let M be a m -dimensional Riemannian manifold with Riemannian metric g and let $\{u_1, \dots, u_m\}$ be an orthogonal basis of TM . For a smooth function ψ on M the Hessian of ψ are defined as

$$(2.26) \quad H^\psi(U, V) = UV\psi - (\nabla_U V)\psi = g(\nabla_U \text{grad } \psi, V),$$

for any $U, V \in TM$. The Laplacian of ψ is defined by

$$(2.27) \quad \Delta\psi = \sum_{i=1}^m \{(\nabla_{u_i} u_i)\psi - u_i u_i \psi\} = - \sum_{i=1}^m g(\nabla_{u_i} \text{grad } \psi, u_i).$$

It is evident from the above two equations that Laplacian is the negative of the Hessian. Moreover from the integration theory on manifolds, for a compact orientable Riemannian manifold M without boundary, we have

$$(2.28) \quad \int_M \Delta \psi dV = 0,$$

where dV is the volume element of M ([16]).

3. Semi-invariant warped product submanifolds

Recently, Siraj Uddin et al. [18] studied warped product semi-invariant submanifolds of nearly cosymplectic manifolds and prove the following theorem

Theorem 3.1. *A warped product submanifold $M = M_1 \times_f M_2$ of a nearly cosymplectic manifold $\bar{M}$ is a usual Riemannian product if the structure vector field ξ is tangent to M_2 , where M_1 and M_2 are the Riemannian submanifolds of $\bar{M}$.*

In this section we consider contact CR-warped product of the type $N_T \times_\psi N_\perp$ of the nearly cosymplectic manifold $\bar{M}$, where N_T and $N_\perp$ are the invariant and anti-invariant submanifolds respectively of $\bar{M}$. Now we obtain some basic results in the following lemma.

Lemma 3.1. *Let $M = N_T \times_\psi N_\perp$ be a semi-invariant warped product submanifold of a nearly cosymplectic manifold $\bar{M}$. Then*

- (i) $g(\sigma(U, W), \phi W) = -\phi U \ln \psi \|W\|^2,$
- (ii) $g(\sigma(\phi U, W), \phi W) = U \ln \psi \|W\|^2,$
- (iii) $g(\phi \sigma(U, W), \mathcal{P}_W U) = -\|\mathcal{P}_W U\|^2,$
- (iv) $g(\sigma(\phi U, W) - \phi \sigma(U, W), \mathcal{Q}_W U) = \|\mathcal{Q}_W U\|^2,$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. From the Gauss formula, we have

$$g(\sigma(U, W), \phi W) = -g(\bar{\nabla}_W \phi W, U).$$

By use of (2.3), we get

$$g(\sigma(U, W), \phi W) = g(\bar{\nabla}_W W, \phi U),$$

or

$$g(\sigma(U, W), \phi W) = -g(\bar{\nabla}_W \phi U, W) = -g(\nabla_W \phi U, W).$$

In view of (2.24), we get

$$g(\sigma(U, W), \phi W) = -\phi U \ln \psi g(W, W),$$

which is the part (i) of Lemma.

Part (ii) is the part (iii) of Lemma 3.4 in [18]. To prove (iii), consider

$$g(\phi\sigma(U, W), \mathcal{P}_W U) = g(\phi\bar{\nabla}_W U - \phi\nabla_W U, \mathcal{P}_W U).$$

By use of (2.5) and (2.11) the last equation gives

$$g(\phi\sigma(U, W), \mathcal{P}_W U) = g(\bar{\nabla}_W \phi U - (\bar{\nabla}_W \phi)U, \mathcal{P}_W U).$$

From the Gauss formula and (2.16), we get

$$g(\phi\sigma(U, W), \mathcal{P}_W U) = \phi U \ln \psi g(W, \mathcal{P}_W U) - \|\mathcal{P}_W U\|^2,$$

in view of part (i) of P_3 and (2.3), we have

$$g(\phi\sigma(U, W), \mathcal{P}_W U) = -\|\mathcal{P}_W U\|^2.$$

Finally, applying the Gauss formula in the following expression

$$g(\sigma(\phi U, W), \mathcal{Q}_W U) = g(\bar{\nabla}_W \phi U - \nabla_W \phi U, \mathcal{Q}_W U),$$

or equivalently,

$$g(\sigma(\phi U, W), \mathcal{Q}_W U) = g(\mathcal{Q}_W U, \mathcal{Q}_W U) + g(\phi\bar{\nabla}_W U, \mathcal{Q}_W U),$$

again applying (2.5) and (2.11) in above equation, we get

$$g(\sigma(\phi U, W), \mathcal{Q}_W U) = \|\mathcal{Q}_W U\|^2 + g(\phi\sigma(U, W), \mathcal{Q}_W U) + U \ln \psi g(\phi W, \mathcal{Q}_W U).$$

In view of part (ii) of P_3 and (2.3) the last term in above equation becomes zero and finally, we get part (iv) of Lemma.

Now we have the following theorem

Theorem 3.2. *Let $M = N_T \times_\psi N_\perp$ be a semi-invariant warped product submanifold of a nearly cosymplectic manifold $\bar{M}$, then*

$$\begin{aligned} \|\sigma(U, W)\|^2 + \|\sigma(\phi U, W)\|^2 &= 2g(\sigma(\phi U, W), \phi\sigma(U, W)) + (U \ln \psi)^2 \|W\|^2 \\ (3.1) \qquad \qquad \qquad &+ (\phi U \ln \psi)^2 \|W\|^2 + \|\mathcal{P}_W U\|^2 + \|\mathcal{Q}_W U\|^2. \end{aligned}$$

Proof. From the Gauss formula, we have

$$g(\sigma(\phi U, W), \phi\sigma(U, W)) = g(\phi\sigma(U, W), \bar{\nabla}_W \phi U - \nabla_W \phi U).$$

In view of (2.11), (2.23) and part (i) of Lemma 3.1, the above equation takes the form

$$g(\sigma(\phi U, W), \phi\sigma(U, W)) = g(\phi\sigma(U, W), (\bar{\nabla}_W \phi)U - \phi\bar{\nabla}_W U) - (\phi U \ln \psi)^2 \|W\|^2,$$

by (2.5) and (2.16), the above equation yields

$$(3.2) \quad \begin{aligned} g(\sigma(\phi U, W), \phi\sigma(U, W)) &= g(\phi\sigma(U, W), \mathcal{P}_W U) + g(\phi\sigma(U, W), \mathcal{Q}_W U) \\ &\quad + \|\sigma(U, W)\|^2 - (\phi U \ln \psi)^2 \|W\|^2. \end{aligned}$$

The second term on the right hand side can be solved as follows

$$g(\phi\sigma(U, W), \mathcal{Q}_W U) = g(\phi\bar{\nabla}_W U - \phi\nabla_W U, \mathcal{Q}_W U).$$

In view of (2.11) and (2.24) the above equation takes the form

$$g(\phi\sigma(U, W), \mathcal{Q}_W U) = -\|\mathcal{Q}_W U\|^2 + g(\sigma(\phi U, W), \mathcal{Q}_W U) - U \ln \psi g(\phi W, \mathcal{Q}_W U),$$

on using (2.17), part (ii) of P_3 , the last term of above equation becomes zero and finally, we have

$$(3.3) \quad g(\phi\sigma(U, W), \mathcal{Q}_W U) = -\|\mathcal{Q}_W U\|^2 + g(\sigma(\phi U, W), \mathcal{Q}_W U).$$

Applying part (ii) of P_3 in (3.3), we get

$$g(\phi\sigma(U, W), \mathcal{Q}_W U) = -\|\mathcal{Q}_W U\|^2 - g((\bar{\nabla}_W \phi)\sigma(\phi U, W), U).$$

By use of (2.11), above equation gives

$$g(\phi\sigma(U, W), \mathcal{Q}_W U) = -\|\mathcal{Q}_W U\|^2 - g(\bar{\nabla}_W \phi \sigma(\phi U, W) - \phi \bar{\nabla}_W \sigma(\phi U, W), U),$$

or

$$\begin{aligned} g(\phi\sigma(U, W), \mathcal{Q}_W U) &= -\|\mathcal{Q}_W U\|^2 - g(\phi\sigma(\phi U, W), \nabla_W U) \\ &\quad - g(\sigma(\phi U, W), \phi\sigma(U, W)) + g(\sigma(\phi U, W), \bar{\nabla}_W \phi U). \end{aligned}$$

On using (2.5), (2.24) and part (i) of Lemma 3.1 above equation reduced to

$$(3.4) \quad \begin{aligned} g(\phi\sigma(U, W), \mathcal{Q}_W U) &= -\|\mathcal{Q}_W U\|^2 - (U \ln \psi)^2 \|W\|^2 - g(\sigma(\phi U, W), \phi\sigma(U, W)) \\ &\quad + \|\sigma(\phi U, W)\|^2. \end{aligned}$$

Using (3.4) and part (iii) of Lemma 3.1, in (3.2), we get the required result.

Now we will prove the following theorem for semi-invariant warped product submanifolds of a nearly cosymplectic generalized Sasakain space form.

Theorem 3.3. *Let $M = N_T \times_f N_\perp$ be a semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly Cosymplectic structure, then*

$$(3.4) \quad \begin{aligned} \|\sigma(U, W)\|^2 + \|\sigma(\phi U, W)\|^2 &= H^{\ln \psi}(U, U) + H^{\ln \psi}(\phi U, \phi U) + f_2 \cdot \|U\|^2 \cdot \|W\|^2 \\ &\quad + 2\|\mathcal{P}_W U\|^2 + 2\|\mathcal{Q}_W U\|^2 + (U \ln \psi)^2 g(W, W) + (\phi U \ln \psi)^2 g(W, W), \end{aligned}$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. Suppose M be a semi-invariant warped product submanifolds of a nearly Cosymplectic generalized Sasakian space form then by (2.4), we have

$$(3.5) \quad \bar{R}(U, \phi U, W, \phi W) = -f_2 g(U, U)g(W, W).$$

On the other hand by Coddazi equation

$$\begin{aligned} \bar{R}(U, \phi U, W, \phi W) &= U g(\sigma(\phi U, W), \phi W) \\ &- g(\bar{\nabla}_U \phi W, \sigma(\phi U, W)) - g(\sigma(\nabla_U \phi U, W), \phi W) \\ &- g(\sigma(\nabla_U W, \phi U), \phi W) - \phi U g(\sigma(U, W), \phi W) + g(\sigma(U, W), \bar{\nabla}_{\phi U} \phi W) \\ &+ g(\sigma(\nabla_{\phi U} U, W), \phi W) + g(\sigma(\nabla_{\phi U} W, U), \phi W), \end{aligned}$$

on using (2.5), (2.23), decomposition (2.16), parts (i) and (ii) of Lemma 3.1, above equation takes the form

$$\begin{aligned} \bar{R}(U, \phi U, W, \phi W) &= U(U \ln \psi)g(W, W) \\ &+ 2(U \ln \psi)^2 g(W, W) - g(\sigma(\phi U, W), \mathcal{Q}_U W) \\ (3.6) \quad &- g(\sigma(\phi U, W), \phi \sigma(U, W)) - U \ln \psi g(\sigma(\phi U, W), \phi W) \\ &- g(\sigma(\nabla_U \phi U, W), \phi W) - (U \ln \psi)^2 g(W, W) \\ &+ \phi U (\phi U \ln \psi) g(W, W) + 2\phi U \ln \psi g(\nabla_{\phi U} W, W) \\ &+ g(\sigma(U, W), \mathcal{Q}_{\phi U} W) + g(\sigma(U, W), \phi \sigma(\phi U, W)) \\ &+ \phi U \ln \psi g(\sigma(U, W), \phi W) + g(\sigma(\nabla_{\phi U} U, W), \phi W) \\ &- (\phi U \ln \psi)^2 g(W, W). \end{aligned}$$

From part (ii) of Lemma 3.1 and (2.7)

$$g(A_{\phi W} W, \phi U) = U \ln \psi g(W, W).$$

Since $\nabla_U U \in TN_T$, then we can replace U by $\nabla_U U$ as

$$g(A_{\phi W} W, \phi \nabla_U U) = \nabla_U U \ln \psi g(W, W).$$

Applying the Gauss formula, we find

$$(3.7) \quad g(A_{\phi W} W, \phi \bar{\nabla}_U U - \phi \sigma(U, U)) = \nabla_U U \ln \psi g(W, W).$$

On using (2.11), (2.3) and (2.24) in the following equation

$$g(\sigma(U, U), \phi W) = -g(\phi \bar{\nabla}_U U, W),$$

one can conclude that $\sigma(X, X) \in \mu$, using this fact in (3.7), we get

$$g(A_{\phi W} W, \phi \bar{\nabla}_U U) = \nabla_U U \ln \psi g(W, W).$$

By use of (2.5) and (2.3), the above expression reduced to

$$g(A_{\phi W}W, \nabla_U \phi U) = \nabla_U U \ln \psi g(W, W),$$

or

$$(3.8) \quad g(\sigma(\nabla_U \phi U, W), \phi W) = \nabla_U U \ln \psi g(W, W).$$

Similarly, we can get

$$(3.9) \quad g(\sigma(\nabla_{\phi U} U, W), \phi W) = \nabla_{\phi U} \phi U \ln \psi g(W, W).$$

Moreover, using (2.20)(b) as follows

$$\begin{aligned} & -g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W) = g(\sigma(\phi U, W), \mathcal{Q}_W U) \\ & -g(\sigma(U, W), \mathcal{Q}_W \phi U), \end{aligned}$$

or

$$\begin{aligned} & -g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W) = g(\sigma(\phi U, W), \mathcal{Q}_W U) \\ & -g(\sigma(U, W), (\bar{\nabla}_W \phi) \phi U), \end{aligned}$$

using (2.17), it is easy to see that

$$(3.10) \quad \begin{aligned} & -g(\sigma(\phi U, W), \mathcal{Q}_U W) + g(\sigma(U, W), \mathcal{Q}_{\phi U} W) \\ & = g(\sigma(\phi U, W) - \phi \sigma(U, W), \mathcal{Q}_W U) - g(\phi \sigma(U, W), \mathcal{P}_W U). \end{aligned}$$

By making use of (3.8), (3.9) and (3.10) in (3.6), we get

$$\begin{aligned} \bar{R}(U, \phi U, W, \phi W) &= \{U(U \ln \psi) + \phi U(\phi U \ln \psi) \\ & - (\nabla_U U \ln \psi + \nabla_{\phi U} \phi U \ln \psi)\} \|W\|^2 \\ & - 2g(\phi \sigma(U, W), \sigma(\phi U, W)) - g(\phi \sigma(U, W), \mathcal{P}_W U) \\ & + g(\sigma(\phi U, W) - \phi \sigma(U, W), \mathcal{Q}_W U). \end{aligned}$$

By parts (iii) and (iv) of Lemma 3.1 and (2.26), we have

$$(3.11) \quad \begin{aligned} \bar{R}(U, \phi U, W, \phi W) &= H^{\ln \psi}(U, U) + H^{\ln \psi}(\phi U, \phi U) + \|\mathcal{P}_W U\|^2 \\ & + \|\mathcal{Q}_W U\|^2 - 2g(\phi \sigma(U, W), \sigma(\phi U, W)). \end{aligned}$$

Applying (3.1) and (3.5) in (3.11), we get

$$\begin{aligned} & -f_2 \cdot \|U\|^2 \cdot \|W\|^2 = H^{\ln \psi}(U, U) + H^{\ln \psi}(\phi U, \phi U) + 2\|\mathcal{P}_W U\|^2 + 2\|\mathcal{Q}_W U\|^2 \\ & - \|\sigma(U, W)\|^2 - \|\sigma(\phi U, W)\|^2 + (U \ln \psi)^2 \|W\|^2 + (\phi U \ln \psi)^2 \|W\|^2, \end{aligned}$$

which is the required result.

Note 3.1. If for semi-invariant warped product submanifolds $M = N_T \times_\psi N_\perp$ the ambient manifold is cosymplectic space form $\bar{M}(c)$, then $f_2 = \frac{c}{4}$ and $\mathcal{P} = \mathcal{Q} = 0$.

In view of Note 3.1 we have the following corollary

Corollary 3.1. *Let $M = N_T \times_f N_\perp$ be a semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$, then we have*

$$\begin{aligned} \|\sigma(U, W)\|^2 + \|\sigma(\phi U, W)\|^2 &= H^{\ln \psi}(U, U) + H^{\ln \psi}(\phi U, \phi U) + \frac{c}{4}\|U\|^2 \cdot \|W\|^2 \\ &+ \{(U \ln \psi)^2 + (\phi U \ln \psi)^2\}g(W, W), \end{aligned}$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Let $\{u_0 = \xi, u_1, u_2, \dots, u_\alpha, \phi u_1, \phi u_2, \dots, \phi u_\alpha, u^1, u^2, \dots, u^\beta\}$ be an orthonormal frame of TM such that $\{u_1, u_2, \dots, u_\alpha, \phi u_1, \phi u_2, \dots, \phi u_\alpha\}$ are tangential to TN_T and $\{u^1, u^2, \dots, u^\beta\}$ are tangential to $TN_\perp$. Moreover assume that $\{\phi u^1, \phi u^2, \dots, \phi u^\beta\}$ and $\{n_1, n_2, \dots, n_{2r}\}$ are tangential to $\phi N_\perp$ and μ respectively.

Finally, we will prove the main theorem

Theorem 3.4. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a nearly cosymplectic generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then M is a semi-invariant warped product if*

$$\|\sigma_\mu(D, D^\perp)\|^2 \geq f_2 \cdot \alpha \cdot \beta + 2\|\mathcal{P}_{D^\perp} D\|^2 + 2\|\mathcal{Q}_{D^\perp} D\|^2,$$

where σ_μ denotes the component of σ in μ , $2\alpha + 1$ and β are the dimensions of N_T and $N_\perp$.

Proof. By the definition of Laplacian of $\ln \psi$

$$\begin{aligned} -\Delta \ln \psi &= \sum_{i=1}^{\alpha} g(\nabla_{u_i} \text{grad } \ln \psi, u_i) + \sum_{i=1}^{\alpha} g(\nabla_{\phi u_i} \text{grad } \ln \psi, \phi u_i) \\ &+ \sum_{j=1}^{\beta} g(\nabla_{u^j} \text{grad } \ln \psi, u^j) + g(\nabla_\xi \text{grad } \ln \psi, \xi). \end{aligned}$$

Since, the ambient manifold $\bar{M}$ is nearly cosymplectic manifold and moreover, the structure vector field ξ is killing on $\bar{M}$, the induced connection is Levi-Civita and $\text{grad } \psi \in TN_T$, then by some straight forward calculations it is easy to see that $g(\nabla_\xi \text{grad } \ln \psi, \xi) = 0$.

Hence, by use of (2.29), the last equation can be written as

$$-\Delta \ln \psi = \sum_{i=1}^{\alpha} \{H^{\ln \psi}(u_i, u_i) + H^{\ln \psi}(\phi u_i, \phi u_i)\} + \sum_{j=1}^{\beta} g(\nabla_{u^j} \text{grad } \ln \psi, u^j).$$

Further by a simple calculations, we can get the following

$$(3.12) \quad -\Delta \ln \psi = \sum_{i=1}^{\alpha} \{H^{\ln \psi}(u_i, u_i) + H^{\ln \psi}(\phi u_i, \phi u_i)\} + \beta \|\text{grad } \ln \psi\|^2.$$

In (3.4) using local frame of vector fields on N_T and $N_{\perp}$ and summing both sides over $i = 1, 2, \dots, \alpha$ and $j = 1, 2, \dots, \beta$, we have

$$(3.13) \quad \begin{aligned} & \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\sigma(u_i, u^j)\|^2 + \|\sigma(\phi u_i, u^j)\|^2) \\ &= \sum_{i=1}^{\alpha} \{H^{\ln \psi}(u_i, u_i) + H^{\ln \psi}(\phi u_i, \phi u_i)\} \beta \\ &+ f_2 \cdot \alpha \cdot \beta + \beta \sum_{i=1}^{\alpha} \{(u_i \ln \psi)^2 + (\phi u_i \ln \psi)^2\} \|Z\|^2 \\ &+ 2 \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\mathcal{P}_{u^j} u_i\|^2 + \|\mathcal{Q}_{u^j} u_i\|^2). \end{aligned}$$

Furthermore, we can write the second fundamental form σ as follows

$$\sigma(u_i, u^j) = \sum_{k=1}^{\beta} g(\sigma(u_i, u^j), \phi u^k) \phi u^k + \sum_{t=1}^{2r} g(\sigma(u_i, u^j), n_t) n_t,$$

for each $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. Taking the inner product of the above equation with $\sigma(u_i, u^j)$ we get

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \|\sigma(u_i, u^j)\|^2 = \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(\sigma(u_i, u^j), \phi u^k)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{t=1}^{2r} g(\sigma(u_i, u^j), n_t)^2,$$

then making use of part (i) of Lemma 3.1, the last equation takes the form

$$(3.14) \quad \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\sigma(u_i, u^j)\|^2) = \beta \sum_{i=1}^{\alpha} (\phi u_i \ln \psi)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \|\sigma_{\mu}(u_i, u^j)\|^2.$$

Similarly, we can get

$$(3.15) \quad \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \|\sigma(\phi u_i, u^j)\|^2 = \beta \sum_{i=1}^{\alpha} (u_i \ln \psi)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \|\sigma_{\mu}(\phi u_i, u^j)\|^2.$$

By use of (3.12), (3.14) and (3.15) in (3.13), we have

$$\begin{aligned}
 (3.16) \quad & \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\sigma_{\mu}(u_i, u^j)\|^2 + \|\sigma_{\mu}(\phi u_i, u^j)\|^2) \\
 & = (-\Delta \ln \psi - \beta \|\operatorname{grad} \ln \psi\|^2) \beta + f_2 \cdot \alpha \cdot \beta \\
 & + 2 \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\mathcal{P}_{u^j} u_i\|^2 + \|\mathcal{Q}_{u^j} u_i\|^2).
 \end{aligned}$$

Now, we use the following notations

$$\begin{aligned}
 \|\sigma_{\mu}(D, D^{\perp})\|^2 &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \|\sigma_{\mu}(u_i, u^j)\|^2 + \|\sigma_{\mu}(\phi u_i, u^j)\|^2, \\
 \|\mathcal{P}_{D^{\perp}} D\|^2 + \|\mathcal{Q}_{D^{\perp}} D\|^2 &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (\|\mathcal{P}_{u^j} u_i\|^2 + \|\mathcal{Q}_{u^j} u_i\|^2).
 \end{aligned}$$

In view of above notations, (3.16) can be written as

$$\begin{aligned}
 (3.17) \quad & -\beta \Delta \ln \psi = \|\sigma_{\mu}(D, D^{\perp})\|^2 + \beta^2 \|\operatorname{grad} \ln \psi\|^2 - f_2 \cdot \alpha \cdot \beta \\
 & - 2\|\mathcal{P}_{D^{\perp}} D\|^2 - 2\|\mathcal{Q}_{D^{\perp}} D\|^2.
 \end{aligned}$$

From (2.28), we can conclude that

$$\begin{aligned}
 (3.18) \quad & \int_M \{ \|\sigma_{\mu}(D, D^{\perp})\|^2 + \beta^2 \|\operatorname{grad} \ln \psi\|^2 - f_2 \cdot \alpha \cdot \beta \\
 & - 2\|\mathcal{P}_{D^{\perp}} D\|^2 - 2\|\mathcal{Q}_{D^{\perp}} D\|^2 \} dV = 0.
 \end{aligned}$$

Here, if

$$\|\sigma_{\mu}(D, D^{\perp})\|^2 \geq f_2 \cdot \alpha \cdot \beta + 2\|\mathcal{P}_{D^{\perp}} D\|^2 + 2\|\mathcal{Q}_{D^{\perp}} D\|^2,$$

(3.17) and the above inequality implies that $\|\operatorname{grad} \ln \psi\| = 0$ i.e., ψ is constant, since $\beta \neq 0$, which proves the Theorem completely.

Now, we have some consequences of above findings as follows

Corollary 3.2. *Let $M = N_T \times_{\psi} N_{\perp}$ be a compact orientable semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$, if*

$$\|\sigma_{\mu}(D, D^{\perp})\|^2 \geq \frac{c \cdot \alpha \cdot \beta}{4},$$

then M is contact semi-invariant product

Note 3.2. Almost similar result was also proved by M. Atceken in [3] in the form of the Theorem 3.4.

Corollary 3.3. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure. Then M is semi invariant product if and only if*

$$(3.19) \quad \|\sigma_\mu(D, D^\perp)\|^2 = f_2 \cdot \alpha \cdot \beta + 2\|\mathcal{P}_{D^\perp} D\|^2 + 2\|\mathcal{Q}_{D^\perp} D\|^2.$$

Proof. Assume that M is compact semi-invariant warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting nearly cosymplectic structure satisfying (3.19), then from (3.18), ψ is constant i.e., M is contact semi-invariant product.

Conversely, if M is semi-invariant product, i.e., ψ is constant and from part (i) of Lemma 3.1 $\sigma(X, Z) \in \mu$, for all $X \in TN_T$ and $Z \in TN_\perp$ then it is easy to see that (3.19) holds.

Corollary 3.4. *Let $M = N_T \times_\psi N_\perp$ be a compact orientable semi-invariant warped product submanifold of a cosymplectic space form $\bar{M}(c)$. Then M is semi-invariant product if and only if*

$$\|\sigma_\mu(D, D^\perp)\|^2 = \frac{c \cdot \alpha \cdot \beta}{4}.$$

Note 3.3. Proposition 3.5 in [3] is approximate similar to Corollary 3.4.

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Structures on Galois connections

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Abstract. In this paper it is proved that $G(Q, P)$ (the set of all Galois connections of P into Q) is a meet semilattice if and only if both P and Q are meet semilattices and in this case, the meet operation in $G(Q, P)$ is point-wise and P and Q are isomorphic to meet subsemilattices of $G(Q, P)$.

Keywords: residuated mappings, Galois connections, p.o.set, lattice and sublattice.

1. Introduction

U.M. Swamy and A. V. S. N. Murty [5] studied an important properties of Residuated mappings and it is observed that, for any p.o.sets P and Q , f is a Galois connection of P into Q if and only if it is a residuated mapping of P into Q^d , the dual of Q (that is, $a \leq b$ in Q^d if $b \leq a$ in Q). Some observations necessitate in depth study of the abstract Galois connections [3] between general partially ordered sets and in particular, from a Heyting algebra into an algebraic lattice, in view of the well-known Birkhoff's theorem [1] and [2] which states that the lattice of subalgebras of any algebra is an algebraic lattice and that any algebraic lattice is isomorphic to the lattice of subalgebras of a suitable algebra. Here we first discuss certain properties of Galois connections and then the order structure on the set of Galois connections between certain given partially ordered sets.

2. Preliminaries

Let P and Q be any partially ordered sets (p.o.sets). A mapping $f : P \rightarrow Q$ is said to be an isotone (antitone) if, for any $a, b \in P$, $a \leq b$ implies $f(a) \leq f(b)$ ($f(a) \geq f(b)$). An antitone $f : P \rightarrow Q$ is said to be a Galois connection ([3]) of P into Q if there exists an antitone $g : Q \rightarrow P$ such that $a \leq g(f(a))$ and $x \leq f(g(x))$ for all $a \in P$ and $x \in Q$; in other words, $g \circ f$ and $f \circ g$ are extensive maps. If $f : P \rightarrow Q$ is a Galois connection, then the antitone $g : Q \rightarrow P$ satisfying the above property is unique and is called the dual of f and is denoted by f^* . If f is a Galois connection, then so is f^* and $(f^*)^* = f$. The following is a straight forward verification.

Theorem 2.1. Let P and Q be p.o. sets and $f : P \rightarrow Q$ an antitone. Then the following are equivalent to each other.

1. f is a Galois connection
2. There exists an antitone $g : Q \rightarrow P$ such that $a \leq g(x) \Leftrightarrow x \leq f(a)$, for any $a \in P$ and $x \in Q$.
3. For each $x \in Q$, the set $\{a \in P \mid x \leq f(a)\}$ has the largest element.

If $f : P \rightarrow Q$ is a Galois connection, then f can be expressed in terms of its dual and vice versa as follows. For any $a \in P$ and $x \in Q$, $f^*(x) =$ The greatest in $\{b \in P \mid x \leq f(b)\}$ and $f(a) =$ The greatest in $\{y \in Q \mid a \leq f^*(y)\}$.

Definition 2.2. A mapping $f : P \rightarrow Q$ of p.o.sets is said to be

1. A complete join- meet homomorphism if, for any $A \subseteq P$ for which $\text{Sup}_P A$ exists in P . $\text{Inf}_Q f(A)$ exists in Q and is equal to $f(\text{Sup}_P A)$
2. A complete meet- join homomorphism if, for any $A \subseteq P$ for which $\text{Inf}_P A$ exists in P . $\text{Sup}_Q f(A)$ exists in Q and equal to $f(\text{Inf}_P A)$.

The above two concepts of homomorphism are independent of each other and each of them is an antitone. The following can be easily verified.

Theorem 2.3. Any Galois connection $f : P \rightarrow Q$ is a complete join – meet homomorphism.

The converse of the above theorem is not true, in the sense that a complete join-meet homomorphism need not be a Galois connection; for, if P is with the trivial ordering (that is, $a \leq b$ only when $a = b$), then every mapping of P into P is a complete join-meet homomorphism, while the Galois connections of P into P are bijections of P onto P . However, if P is a complete lattice (a p.o.set in which every subset has supremum and infimum), then every complete join-meet homomorphism of P into any p.o.set Q is a Galois connection. In fact, we have the following [5].

Theorem 2.4. The following are equivalent to each other for any p.o.set P .

1. P is a complete lattice;
2. Every complete join-meet homomorphism of P into P is a Galois connection;
3. Every complete join-meet homomorphism of P into any p.o.set Q is a Galois connection.

If P and Q are p.o.sets and $R \subseteq Q$, then a map $f : P \rightarrow R$ may be a Galois connection without being a Galois connection of P into Q . In this connection, we have the following which is a consequence of theorem 3.8 of [5].

Theorem 2.5. Let R be a nonempty subset of complete lattice Q . Suppose R is complete with respect to the ordering induced by that of Q . Then the following are equivalent.

1. For any complete P , every Galois connection of P into R is a Galois connection of P into Q .
2. For any subset A of R , $\text{Inf}_R A = \text{Inf}_Q A$.

3. Structures on Galois connections

For any p.o.sets P and Q , let $G(P, Q)$ denote the set of all Galois connections of P into Q . Naturally, we are interested only in the case when $G(P, Q)$ is non-empty. In this context, it can be seen that, a p.o.set Q has largest element (denoted by 1) if and only if $G(P, Q)$ is non-empty for any p.o.set P with largest element; for, the constant map f , defined by $f(a) = 1$ for all $a \in P$, is a Galois connection. For this reason, we shall consider only non-trivial bounded p.o.sets; that is, p.o.sets with the largest element and smallest element denoted by 1 and 0 respectively and $1 \neq 0$. For any $f, g \in G(P, Q)$ we shall define $f \leq g$ if $f(a) \leq g(a)$ for all $a \in P$. It can be easily proved that $f \leq g$ if and only if $f^* \leq g^*$ and hence $f \mapsto f^*$ is an order isomorphism of $G(P, Q)$ onto $G(Q, P)$. In the following, we shall embed P and Q in $G(P, Q)$.

Theorem 3.1. Let P and Q be bounded p.o.sets. For any $a \in P$ and $x \in Q$, define $f_a, g_x : P \rightarrow Q$ by

$$f_a(b) = \begin{cases} 1, & \text{if } b \leq a \\ 0, & \text{if } b \not\leq a \end{cases} \quad \text{and} \quad g_x(b) = \begin{cases} x, & \text{if } b \neq 0 \\ 1, & \text{if } b = 0 \end{cases}$$

Then f_a and g_x are Galois connections and $a \mapsto f_a$ and $x \mapsto g_x$ are embeddings of P and Q respectively in $G(P, Q)$.

Proof. It can be easily verified that f_a and g_x are antitones. For any $a \in P$ and $x \in Q$, define $f_a^*, g_x^* : Q \rightarrow P$ by

$$f_a^*(y) = \begin{cases} a, & \text{if } y \neq 0 \\ 1, & \text{if } y = 0 \end{cases} \quad \text{and} \quad g_x^*(y) = \begin{cases} 1, & \text{if } y \leq x \\ 0, & \text{if } y \not\leq x \end{cases}$$

Then f_a^* and g_x^* are antitones, $b \leq f_a^*(f_a(b))$ and $y \leq f_a(f_a^*(y))$ for all $b \in P$ and $y \in Q$. Therefore f_a is a Galois connection and its dual is f_a^* . Similarly g_x is a Galois connection and g_x^* is its dual. It can be easily verified

that $a \leq b \Leftrightarrow f_a \leq f_b$ and $x \leq y \Leftrightarrow g_x \leq g_y$ for any $a, b \in P$ and $x, y \in Q$. Thus $a \mapsto f_a$ is an embedding of P into $G(P, Q)$ and $x \mapsto g_x$ is an embedding of Q into $G(P, Q)$.

A p.o.set is called a meet (join) semilattice if any two of its elements have the infimum (supremum, respectively) in it. The infimum and supremum of two elements a and b will be denoted by $a \wedge b$ and $a \vee b$ respectively when they exist.

Theorem 3.2. $G(P, Q)$ is a meet semilattice if and only if both P and Q are meet semilattices and, in this case, the meet operation in $G(P, Q)$ is point-wise and P and Q are isomorphic to meet subsemilattices of $G(P, Q)$.

Proof. Suppose that P and Q are meet semilattices. For any $f, g \in G(P, Q)$, define $f \wedge g$ and $f^* \wedge g^* : Q \rightarrow P$ point-wise; that is, $(f \wedge g)(a) = f(a) \wedge g(a)$ and $(f^* \wedge g^*)(x) = f^*(x) \wedge g^*(x)$ for all $a \in P$ and $x \in Q$. Then, clearly $f \wedge g$ and $f^* \wedge g^*$ are antitones. Also for any $a \in P$ and $x \in Q$,

$$\begin{aligned} (f \wedge g)(f^* \wedge g^*)(x) &= f(f^*(x) \wedge g^*(x)) \wedge g(f^*(x) \wedge g^*(x)) \\ &\geq f(f^*(x)) \wedge g(g^*(x)) \\ &\geq x \end{aligned}$$

and, similarly, $(f^* \wedge g^*)((f \wedge g)(x)) \geq x$.

Therefore, $(f \wedge g)$ is a Galois connection and its dual is $f^* \wedge g^*$. Thus $G(P, Q)$ is a meet sublattice and the meet operation $\wedge$ is point wise.

And, similarly, $(f^* \wedge g^*)((f \wedge g)(x)) \geq x$. Therefore, $f \wedge g$ is a Galois correction and its dual is $f^* \wedge g^*$. Thus $G(P, Q)$ is a meet sublattice and the meet operation $\wedge$ is point-wise.

Conversely, suppose that $G(P, Q)$, is a meet semilattice. Since $G(Q, P) \cong G(P, Q)$, it follows that $G(Q, P)$ is also a meet semilattice. Now, let $x, y \in Q$ and g_x and g_y be the corresponding (as in theorem 2.1) Galois connections from P into Q . Let $g = g_x \wedge g_y$ in $G(P, Q)$ and choose $0 \neq a \in P$. Then $g \leq g_x$ and $g \leq g_y$ and hence $g(a) \leq g_x(a) = x$ and $g(a) \leq g_y(a) = y$ so that $g(a)$ is a lower bound of x and y . If z is any lower bound of x and y , then $g_z \leq g_x$ and $g_z \leq g_y$ and hence $g_z \leq g_x \wedge g_y = g$ so that $z = g_z(a) \leq g(a)$. Thus $g(a)$ is the infimum of x and y and, in fact, $g = g_{g(a)}$. Therefore, Q is a meet semilattice and $x \mapsto g_x$ is an isomorphism of Q onto a meet subsemilattice of $G(P, Q)$. Similarly it can be verified that P is a meet semilattice which is isomorphic to a meet subsemilattice of $G(Q, P) (\cong G(P, Q))$. Note that, even if P and Q are complete lattices, the supremum $f \vee g$ in $G(P, Q)$ may not be the point-wise; for, consider the following.

Example 3.3. Let P be the interval $[0, 1]$ of real numbers with usual ordering and Q the set of all subgroups of the group $(\mathbb{Z}, +)$ of integers with the inclusion

ordering. Define $f, g : P \rightarrow Q$ by

$$f(a) = \begin{cases} 0, & \text{if } a = 0 \\ \mathbb{Z}, & \text{if } a = 1 \\ 2^n\mathbb{Z}, & \text{if } \frac{n-1}{n} < a \leq \frac{n}{n+1} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$

and

$$g(a) = \begin{cases} 0 & \text{if } a = 1 \\ \mathbb{Z}, & \text{if } a = 0 \\ 3^n\mathbb{Z}, & \text{if } \frac{n-1}{n} < a \leq \frac{n}{n+1} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$

Then f and g are Galois connections and the dual f^* of f is given by

$$f^*(H) = \begin{cases} 1, & \text{if } H = \{0\} \\ 0, & \text{if } H \not\subseteq 2^n\mathbb{Z} \text{ for any } n \in \mathbb{Z}^+ \\ \frac{n}{n+1}, & \text{if } H \subseteq 2^n\mathbb{Z} \text{ and } H \not\subseteq 2^{n+1}\mathbb{Z} \end{cases}$$

and analogously g^* . If h is the point wise supremum of f and g , then

$$h(a) = f(a) \vee g(a) \begin{cases} \{0\}, & \text{if } a = 1 \\ \mathbb{Z}, & \text{if } a < 1 \end{cases}$$

for any $a \in P$.

Note that h is not a Galois connection; for, if so, and if h^* is the dual of h , then

$$\frac{n}{n+1} \leq h^*(h(\frac{n}{n+1})) = h^*(\mathbb{Z}),$$

for all $n \in \mathbb{Z}^+$ and hence $h^*(\mathbb{Z}) = 1$ so that $\mathbb{Z} \subseteq h(h^*(\mathbb{Z})) = h(1) = \{0\}$, which is an absurd. Thus the point-wise supremum of f and g is not a Galois connection.

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On exact sequences of the rigid fibrations

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Abstract. In 2002, Biss investigated on a kind of fibration which is called rigid covering fibration (we rename it by rigid fibration) with properties similar to covering spaces. In this paper, we obtain a relation between arbitrary topological spaces and its rigid fibrations. Using this relation we obtain a commutative diagram of homotopy groups and quasi-topological homotopy groups and deduce some results in this field.

Keywords: fibration, rigid covering fibration, topological homotopy group, exact sequence.

1. Introduction and motivation

The quasi-topological n th homotopy group of the pointed space (X, x) , denoted by $\pi_n^{qtop}(X, x)$, is a quasi-topological group of the familiar homotopy group $\pi_n(X, x)$ which is endowed with the quotient topology induced by the natural surjective map $q : \Omega^n(X, x) \rightarrow \pi_n(X, x)$, where $\Omega^n(X, x)$ is the n th loop space of (X, x) with the compact-open topology (see [1, 2, 4]).

In 2002, Biss [1] investigated on a kind of fibration which is called rigid covering fibration (we rename it by rigid fibration) with properties similar to

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covering spaces. He proved that there is a universal rigid fibration for a space if and only if its topological fundamental group is totally path disconnected [1, Theorem 4.3]. The rigid fibrations generalize classical covering spaces, the hypothesis being that the topological fundamental group is totally disconnected rather than discrete as in the classical theory. Since rigid fibrations have unique path lifting, one can apply standard arguments to extend a number of classical theorems about covering spaces to the rigid fibration setting. For instance, every morphism in the category of rigid fibrations over X is itself a rigid fibration [1, Lemma 4.5]. Also, Biss proved that the coset space of any subgroup $\pi \leq \pi_1^{qtop}(HE)$ has no nonconstant paths, where HE is the Hawaiian earring space. Therefore, for any subgroup π , there is a rigid fibration $p : E \rightarrow HE$ with $p_*\pi_1(E) = \pi$. On the other hand Brazas [3] introduced the $\mathcal{C}$ -covering map for a space X which has a unique lifting property with respect to maps on the objects of $\mathcal{C}$, where $\mathcal{C}$ is the category of path-connected spaces having the unit disk as an object, and then he defined the category of $\mathcal{C}$ -covering maps for a space X denoted by $Cov_{\mathcal{C}}(X)$. The $\mathcal{C}$ -covering maps generalize classical covering maps. In this paper, we consider the category of rigid fibration maps of a space X , $RF(X)$, and show that it is a subcategory of $wCov_{\mathcal{C}}(X)$. Then we obtain some results in rigid fibrations. Also, we consider rigid fibrations of two topological spaces X and Y and obtain a relation between these spaces and its rigid fibrations. More precisely, if $p : E_H \rightarrow X$ and $q : E_G \rightarrow Y$ are two rigid fibrations of X and Y , respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$ and $g : X \rightarrow Y$ is a continuous map such that $g_*(H) \subseteq G$, then there is a map $\tilde{f} : E_H \rightarrow E_G$ such that $q \circ \tilde{f} = g \circ p$ and vice versa. Then with the above conditions, we obtain the following commutative diagram in Set_* with exact rows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_n(F_1) & \xrightarrow{i_*} & \pi_n(E_H) & \xrightarrow{p_*} & \pi_n(X) & \xrightarrow{d} & \pi_{n-1}(F_1) & \longrightarrow & \cdots \\
& & \downarrow (f|_{F_1})_* & & \downarrow f_* & & \downarrow g_* & & \downarrow (f|_{F_1})_* & & \\
\cdots & \longrightarrow & \pi_n(F_2) & \xrightarrow{j_*} & \pi_n(E_G) & \xrightarrow{q_*} & \pi_n(Y) & \xrightarrow{d'} & \pi_{n-1}(F_2) & \longrightarrow & \cdots
\end{array}$$

In follow, by using this diagram, we deduce some results about homotopy groups and quasi-topological homotopy groups.

2. Preliminaries

In this section, we recall some of the main notion and results of rigid fibrations.

Definition 2.1 ([1, Definition 2.1]). *Let X be a topological space. A fibration $p : E \rightarrow X$ is said to be a covering fibration if $p_* : \pi_i(E) \rightarrow \pi_i(X)$ is an isomorphism, for $i \geq 2$, and an injection for $i = 1$.*

We recall that a fibration $p : E \rightarrow X$ has unique path lifting property if for any path $\gamma : I \rightarrow X$ in X and $e \in E$ with $p(e) = \gamma(0)$, there is a unique

lift $\tilde{\gamma} : I \longrightarrow E$ with $\gamma = p \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = e$. One can show that a fibration has unique path lifting property if and only if all of the fibers have no nonconstant paths [6, Theorem 2.2.5].

Definition 2.2 ([1, Definition 4.1]). *Let X be a topological space. A fibration $p : E \longrightarrow X$ is called a rigid covering fibration if it is a covering fibration and if, in addition, each fiber has no nonconstant paths.*

If we use the homotopy sequence of fibrations for rigid covering fibrations, we see that the condition “covering fibration” in Definition 2.2 can be replaced with “fibration”. Therefore we can say that A fibration $p : E \longrightarrow X$ is called a rigid covering fibration (rename it to rigid fibration) if each fiber has no nonconstant paths.

Theorem 2.3 ([1, Theorem 4.3]). *Let X be a space, and let $\pi < \pi_1(X)$ be a subgroup of the fundamental group of X . If the left coset space $\pi_1^{qtop}(X)/\pi$ has no nonconstant paths, then there is a rigid covering fibration $p : E \longrightarrow X$ with $p_*\pi_1(E) = \pi$.*

For a pointed map $f : (X, x_0) \longrightarrow (Y, y_0)$, the mapping fiber is the pointed space $Mf = \{(x, \omega) \in X \times Y^I : \omega(0) = y_0 \text{ and } \omega(1) = f(x)\}$, the base point of this space is (x_0, ω_0) , where ω_0 is the constant path at y_0 . Also, there are an injection $k : \Omega(Y, y_0) \longrightarrow Mf$ given by $k(\omega) = (x_0, \omega)$ and an obvious map $\lambda : X \longrightarrow Mf$ by $\lambda(x) = (x, \omega_0)$ (see [5]).

3. Main results

In this section, we obtain some results in rigid fibrations and intend then to obtain a relation between arbitrary topological spaces and its rigid fibrations. Then we obtain a commutative diagram of homotopy groups and quasi-topological homotopy groups and deduce some results in this field.

Theorem 3.1. *Let $p : E \longrightarrow X$ be a rigid fibration. If A is any path component of E , then $p|_A : A \longrightarrow p(A)$ is a rigid fibration.*

Proof. It follows from [6, Lemma 2.3.1] and Definition 2.2. □

Theorem 3.2. *Let $p : E \longrightarrow X$ be a map. If E is locally path connected, then p is a rigid fibration if and only if for each path component A of E , $p(A)$ is a path component of X and $p|_A : A \longrightarrow p(A)$ is a rigid fibration.*

Proof. It follows from [6, Theorem 2.3.2] and Definition 2.2. □

The following theorems imply that if $\mathcal{C}$ is the category of connected locally path connected spaces, then every rigid fibration is a weak $\mathcal{C}$ -covering map in the sense of [3].

Theorem 3.3 ([6, Lemma 2.2.4]). *A map with unique path lifting has the unique lifting property for path connected spaces.*

Theorem 3.4 ([6, Theorem 2.4.5]). *Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a fibration with unique path lifting property. Let Y be a connected locally path connected space. A necessary and sufficient condition that a map $f : (Y, y_0) \rightarrow (X, x_0)$ has a lifting $(Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ is that, in $\pi_1(X, x_0)$,*

$$f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Remark 3.5. It follows easily from homotopy lifting property that for a fibration with unique path lifting property $p : E \rightarrow X$, if E is a nonempty path connected space and X is connected and locally path connected, then p is a homeomorphism if and only if $p_* : \pi_1(E) \rightarrow \pi_1(X)$ is an isomorphism [6, Corollary 2.4.6].

We know that every fibration with unique path lifting property whose base space is locally path connected and semilocally simply connected and whose total space is locally path connected is a covering map [6, Theorem 2.4.10]. In follow, we exhibit two examples of rigid fibration which are not covering map.

Example 3.6. It is known that $\pi_1^{qtop}(HE)$ has no nonconstant paths, where HE is the Hawaiian earring space [1]. Therefore there is a universal rigid fibration $p : E \rightarrow HE$. p is not a covering map, because HE is not semilocally simply connected and so it has not any universal covering map, by [5, Corollary 10.37].

Example 3.7. Let $p : \mathbb{R} \rightarrow S^1$ be a map with $p(t) = \exp(it)$. Since p is a covering map, so it is a rigid fibration, therefore $h = \prod_{n \in \mathbb{N}} p : \prod_{n \in \mathbb{N}} \mathbb{R} \rightarrow \prod_{n \in \mathbb{N}} S^1$ is a rigid fibration by [6, Theorem 2.2.7]. Also the map $q : \mathbb{R} \times (\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}) \rightarrow S^1$ by $q(t, s) = p(t)$ is a rigid fibration. Note that every fiber of the maps h and q is totally path disconnected which is not discrete. Hence the rigid fibrations h and q are not covering.

We are inspired by the results of [3] and obtain the following results.

Definition 3.8. *A subgroup $H \leq \pi_1(X, x_0)$ is a rigid fibration subgroup (for simplicity, RF subgroup) if there is a rigid fibration $p : (E, e_0) \rightarrow (X, x_0)$ with $p_*\pi_1(E, e_0) = H$.*

Remark 3.9. Let H be a subgroup of $\pi_1(X)$. If the left coset space $\pi_1^{qtop}(X)/H$ has no nonconstant paths, then H is a RF subgroup of $\pi_1(X)$, by Theorem 2.3. As an example, every subgroup of fundamental group of Hawaiian earring, $\pi_1(HE)$, is a RF subgroup. Because the left coset space $\pi_1^{qtop}(HE)/H$ has no nonconstant paths for every subgroup H of $\pi_1(HE)$ [1].

Example 3.10. Let HA be the harmonic archipelago space. Any proper subgroup of $\pi_1(HA)$ is not a RF subgroup. Indeed, $\pi_1^{qtop}(HA)$ is indiscrete and therefore, for any subgroup $H \leq \pi_1(HA)$, the left coset space $\pi_1^{qtop}(HA)/H$ is not totally path disconnected.

Theorem 3.11. *If $\{H_i\}_{i \in I}$ is any set of RF subgroup of $\pi_1(X, x_0)$, then:*

- (i) $\prod_i H_i$ is a rigid fibration subgroup.
- (ii) $\bigcap_i H_i$ is a rigid fibration subgroup.

Proof. (i) If H_i is a RF subgroup of $\pi_1(X, x_0)$, then there is a rigid fibration $p_i : (E_i, e_i) \rightarrow (X, x_0)$ with $p_{i*}\pi_1(E_i, e_i) = H_i$ for all $i \in I$. By putting $(E, e) = (\prod_i E_i, (e_i))$, the product $p = \prod p_i : (E, e) \rightarrow (\prod_i X, x_0)$ is a rigid fibration by [6, Theorem 2.2.7] and

$$p_*\pi_1(E, e) = p_*\pi_1(\prod_i E_i, (e_i)) = \prod_i p_{i*}\pi_1(E_i, e_i) = \prod_i H_i.$$

Thus $\prod_i H_i$ is a RF subgroup.

(ii) It follows from a similar argument of [3, Theorem 2.36] by applying [6, Theorem 2.2.7]. $\square$

For a given space X , let $RF(X)$ denote the category of rigid fibrations over X which is the category whose objects are rigid fibrations $p : E \rightarrow X$ and morphisms are commutative triangles of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array} .$$

Let $DCov(X)$ denote the category of disk-coverings over X and $GSet$ denote the category of G -Sets and G -equivariant functions. One can see that for a topological space X , $RF(X) \subseteq DCov(X)$. Recall that the functor $\mu : DCov(X) \rightarrow \pi_1(X, x_0)Set$ was defined as follows: On objects, μ is defined as the fiber $\mu(p) = p^{-1}(x_0)$. If $q : E_0 \rightarrow X$ is a disk-covering and $f : E \rightarrow E_0$ is a map such that $q \circ f = p$, then $\mu(f)$ is the restriction of f to a $\pi_1(X, x_0)$ -equivariant function $p^{-1}(x_0) \rightarrow q^{-1}(x_0)$. μ is a faithful functor [3, Lemma 2.5], and so the restriction on $RF(X)$ is also faithful. With the same argument of [3, Theorem 2.11], we have the following theorem.

Theorem 3.12. *The functor $\mu : RF(X) \rightarrow \pi_1(X, x_0)Set$ is fully faithful.*

The functor μ in Theorem 3.12 is not necessarily an equivalence of categories. Because subgroups $H \leq \pi_1(X, x_0)$ exist which are not RF subgroups (see Example 3.10).

Every morphism in $RF(X)$ is itself a rigid fibration [1, lemma 4.5]. By this fact and [6, Theorem 2.2.6] we have the following result.

Proposition 3.13. *Suppose that $p : E \rightarrow X$ and $q : X \rightarrow Y$ are maps:*

- (i) *If p and q are rigid fibrations, then so is $q \circ p$,*
- (ii) *If q and $q \circ p$ are rigid fibrations, then so is p .*

Remark 3.14. Let $g : X \rightarrow Y$ be a rigid fibrations of Y ; then there is a functor $\mathcal{G} : RF(X) \rightarrow RF(Y)$ that is defined identity on morphisms and on objects as follows. If $p : E \rightarrow X$ is a rigid fibration of X , we define $\mathcal{G}(p) = g \circ p$.

The following theorem is one of the main results of this paper.

Theorem 3.15. *Let $p : E_H \rightarrow X$ and $q : E_G \rightarrow Y$ be two rigid fibrations of X and Y , respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$.*

(i) *If $g : X \rightarrow Y$ is a continuous map such that $g_*(H) \leq G$, then there is a map $f : E_H \rightarrow E_G$ such that $q \circ f = g \circ p$.*

(ii) *If $g : X \rightarrow Y$ is a rigid fibration such that $g_*(H) \leq G$, then $f : E_H \rightarrow E_G$ defined in part (i) is a rigid fibration.*

(iii) *If $g : X \rightarrow Y$ is a continuous map and there is a map $f : E_H \rightarrow E_G$ such that $q \circ f = g \circ p$, then $g_*(H) \leq G$.*

Proof. (i) Consider the map $g \circ p : E_H \rightarrow Y$. Since q is a rigid fibration, it has lifting property for path connected spaces. Since

$$(f \circ p)_*(\pi_1(H)) = g_*(H) \leq G = q_*\pi_1(E_G),$$

there exist a map $f : E_H \rightarrow E_G$ such that $q \circ f = g \circ p$.

(ii) Since $g \circ p$ and q are two rigid fibrations of Y , f is a rigid fibration of E_H by Proposition 3.13.

(iii) Since $q \circ f = g \circ p$, by applying the functor π_1 , $q_* \circ f_* = g_* \circ p_*$ and then

$$g_*(H) = g_* \circ p_*(\pi_1(E_H)) = q_* \circ f_*(\pi_1(E_H)) \leq q_*(\pi_1(E_G)) = G.$$

□

Theorem 3.16. *Let H and G be two RF subgroups of $\pi_1(X)$ and $\pi_1(Y)$, respectively. If $g : X \rightarrow Y$ is a rigid fibration such that $g_*(H) \leq G$, then $g_*(H)$ is a RF subgroup of $\pi_1(Y)$.*

Proof. By hypotheses, there exist two rigid fibrations $p : E_H \rightarrow X$ and $q : E_G \rightarrow Y$ of X and Y , respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$. Using Theorem 3.15 part (ii), there is a rigid fibration $f : E_H \rightarrow E_G$ with $q \circ f = g \circ p$. The composition $q \circ f : E_H \rightarrow Y$ is a rigid fibration of Y such that

$$(q \circ f)_*(\pi_1(E_H)) = (g \circ p)_*(\pi_1(E_H)) = g_*(H).$$

□

The following result can be concluded from part (i) of Theorem 3.15.

Theorem 3.17. *Let $p : E_H \rightarrow X$ and $q : E_G \rightarrow Y$ be two rigid fibrations of X and Y with fibers F_1 and F_2 , respectively, such that $p_*\pi_1(E_H) = H$ and $q_*\pi_1(E_G) = G$. Let $g : X \rightarrow Y$ be a continuous map such that $g_*(H) \leq G$; then*

(i) there is a commutative diagram in Set_* with exact rows:

$$(1) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(F_1) & \xrightarrow{i_*} & \pi_n(E_H) & \xrightarrow{p_*} & \pi_n(X) & \xrightarrow{d} & \pi_{n-1}(F_1) & \longrightarrow & \cdots \\ & & \downarrow (f|_{F_1})_* & & f_* \downarrow & & g_* \downarrow & & (f|_{F_1})_* \downarrow & & \\ \cdots & \longrightarrow & \pi_n(F_2) & \xrightarrow{j_*} & \pi_n(E_G) & \xrightarrow{q_*} & \pi_n(Y) & \xrightarrow{d'} & \pi_{n-1}(F_2) & \longrightarrow & \cdots, \end{array}$$

where the existence of the map $f : E_H \longrightarrow E_G$ follows from Theorem 3.15.

(ii) there is a commutative diagram in Top_* with exact rows:

$$(2) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n^{qtop}(F_1) & \xrightarrow{i_*} & \pi_n^{qtop}(E_H) & \xrightarrow{p_*} & \pi_n^{qtop}(X) & \xrightarrow{d} & \pi_{n-1}^{qtop}(F_1) & \longrightarrow & \cdots \\ & & \downarrow (f|_{F_1})_* & & f_* \downarrow & & g_* \downarrow & & (f|_{F_1})_* \downarrow & & \\ \cdots & \longrightarrow & \pi_n^{qtop}(F_2) & \xrightarrow{j_*} & \pi_n^{qtop}(E_G) & \xrightarrow{q_*} & \pi_n^{qtop}(Y) & \xrightarrow{d'} & \pi_{n-1}^{qtop}(F_2) & \longrightarrow & \cdots. \end{array}$$

(iii) there is a long exact sequence of homotopy groups.

$$(3) \quad \cdots \longrightarrow \pi_n(E_H) \longrightarrow \pi_n(E_G) \oplus \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_{n-1}(E_H) \longrightarrow \cdots.$$

(iv) there is a long exact sequence of topological homotopy groups.

$$(4) \quad \cdots \longrightarrow \pi_n^{qtop}(E_H) \longrightarrow \pi_n^{qtop}(E_G) \oplus \pi_n^{qtop}(X) \longrightarrow \pi_n^{qtop}(Y) \longrightarrow \pi_{n-1}^{qtop}(E_H) \longrightarrow \cdots.$$

Proof. (i) Since $g_*(H) \leq G$, there is a map $f : E_H \longrightarrow E_G$ such that $q \circ f = g \circ p$, by Theorem 3.15. Since $f \circ i = j \circ f|_{F_1}$, $q \circ f = g \circ p$ and π_n is a functor, the first two squares commute. To see commutativity of the last squares, consider the following diagram:

$$(5) \quad \begin{array}{ccccc} \Omega X & \xrightarrow{k} & Mp & \xleftarrow{\lambda} & F_1 \\ \downarrow g_\# & & \downarrow l & & \downarrow f|_{F_1} \\ \Omega Y & \xrightarrow{k'} & Mq & \xleftarrow{\lambda'} & F_2, \end{array}$$

where the maps $g_\#$ and l are induced maps. It is easy to see that Diagram (5) is commutative. Therefore the induced diagram by the functor π_n , that is the last square in Diagram (2), is commutative.

(ii) It follows from a similar argument of part (i) by applying the functor π_n^{qtop} .

(iii) The result holds from [5, Lemma 6.2] and part (i).

(iv) The result holds from [5, Lemma 6.2] and part (ii). $\square$

Remark 3.18. Let $H < \pi_1(X)$, let $G < \pi_1(Y)$, and let the left coset spaces $\pi_1^{qtop}(X)/H$ and $\pi_1^{qtop}(Y)/G$ have no nonconstant paths. By Theorem 2.3, Diagrams (1) and (2) are commutative in Set_* and Top_* , respectively, where $g : X \longrightarrow Y$ is a continuous map with $g_*(H) \leq G$.

The following result follows from using Diagrams (3), (4) and Theorem 2.3.

Corollary 3.19. *Let $\pi_1^{qtop}(X)$ and $\pi_1^{qtop}(Y)/G$ have no nonconstant paths, where $G < \pi_1(Y)$. If there is a continuous map $g : X \rightarrow Y$, then*

$$\pi_n(E_G) \oplus \pi_n(X) \cong \pi_n(Y)$$

and

$$\pi_n^{qtop}(E_G) \oplus \pi_n^{qtop}(X) \cong \pi_n^{qtop}(Y).$$

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On norm inequalities and orthogonality of commutators of derivations

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Abstract. Let H be a complex separable Hilbert space and $B(H)$ the algebra of all bounded linear operators on H . In this paper, we give considerable generalizations of the inequalities for norms of commutators of normal operators. Let $S, T \in B(H)$ be positive normal operators with the cartesian decomposition $S = A + iC$ and $T = B + iD$ such that $a_1 \leq A \leq a_2$, $b_1 \leq B \leq b_2$, $c_1 \leq C \leq c_2$ and $d_1 \leq D \leq d_2$ for some real numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 we have shown that $\|ST - TS\| \leq \frac{1}{2} \sqrt{(a_2 - a_1)^2 + (c_2 - c_1)^2} \sqrt{(b_2 - b_1)^2 + (d_2 - d_1)^2}$. Moreover, orthogonality and norm inequalities for commutators of derivation are also established. We have shown that if the pair of operators (S, T) satisfies Fuglede-Putnam's property and $C \in \ker(\delta_{S,T})$ where $C \in B(H)$ then $\|\delta_{S,T}X + C\| \geq \|C\|$.

Keywords: commutator, norm inequality, orthogonality and derivation.

1. Introduction

Studies on commutators and their norm inequalities have been considered by several mathematicians [1], [2] and [3]. Very interesting results have been obtained in special cases however, a generalization in infinite dimensional complex separable Hilbert space remain interesting. At this point we start by defining some key terms that are useful in the sequel.

Definition 1.1. An element $S \in B(H)$ is a commutator if there exist $X, T \in B(H)$ such that $S = XT - TX$; is positive if $\|S\| \geq 0$; normal if $SS^* = S^*S$; and self-adjoint if $S = S^*$.

Definition 1.2. For $S, T \in B(H)$, let $\delta_{S,T}$ denote the operator on $B(H)$ defined by $\delta_{S,T}X = SX - XT$ is called the generalized derivation. If $S = T$, $\delta_S X = SX - XS$ is called the inner derivation induced by $S \in B(H)$.

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Definition 1.3. Let $S, T \in B(H)$. We say that the pair S, T satisfies $(FP)_{B(H)}$ the Fuglede-Putnam's property, if $SC = CT$ where $C \in B(H)$ implies $S^*C = CT^*$.

2. Inequalities for norms of commutators

Our aim in this section is to establish some inequalities of norms of commutators of normal operators that can be obtained naturally from cartesian decomposition and various vector inequalities in inner product spaces for example, reverse of quadratic Schwarz inequality. In the following result, we obtain a norm inequality for commutators of normal operators.

Theorem 2.1. Let $S, T \in B(H)$ be positive normal operators with the cartesian decomposition $S = A + iC$ and $T = B + iD$ such that $a_1 \leq A \leq a_2$, $b_1 \leq B \leq b_2$, $c_1 \leq C \leq c_2$ and $d_1 \leq D \leq d_2$ for some real numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 then,

$$(1) \quad \|ST - TS\| \leq \frac{1}{2} \sqrt{(a_2 - a_1)^2 + (c_2 - c_1)^2} \sqrt{(b_2 - b_1)^2 + (d_2 - d_1)^2}.$$

Proof. Since $S, T \in B(H)$ are normal such that $S = A + iC$ and $T = B + iD$ are the cartesian decomposition of S and T . Then $S - z$ and $T - w$ are normal for all complex numbers z and w such that $a = \frac{a_1 + a_2}{2}$, $b = \frac{b_1 + b_2}{2}$, $c = \frac{c_1 + c_2}{2}$, $d = \frac{d_1 + d_2}{2}$, $z = a + ic$ and $w = b + id$. Then

$$\begin{aligned} \|ST - TS\| &= \|(S - z)(T - w) - (T - w)(S - z)\| \\ &\leq \|S - z\| \|T - w\| + \|T - w\| \|S - z\| \\ (2) \quad &\leq 2\|S - z\| \|B - w\|. \end{aligned}$$

Following an analogous argument of [4] we have

$$(3) \quad \|S - z\|^2 \leq \|A - a\|^2 + \|C - c\|^2.$$

Similarly

$$(4) \quad \|T - w\|^2 \leq \|B - b\|^2 + \|D - d\|^2.$$

Suppose $A, B, C, D \in B(H)$ are self-adjoint with $a_1 \leq A \leq a_2$, $b_1 \leq B \leq b_2$, $c_1 \leq C \leq c_2$ and $d_1 \leq D \leq d_2$ for some real numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 and $a = \frac{a_1 + a_2}{2}$, then $-(\frac{a_1 + a_2}{2}) \leq A - a \leq \frac{a_1 + a_2}{2}$ and so $\|A - a\| \leq \frac{a_1 + a_2}{2}$. Similarly, $\|B - b\| \leq \frac{b_1 + b_2}{2}$, $\|C - c\| \leq \frac{c_1 + c_2}{2}$ and $\|D - d\| \leq \frac{d_1 + d_2}{2}$. Which upon substituting in Inequality 3 and Inequality 4 we obtain

$$(5) \quad \|S - z\| \leq \sqrt{\frac{(a_2 - a_1)^2}{2} + \frac{(c_2 - c_1)^2}{2}}$$

and

$$(6) \quad \|T - w\| \leq \sqrt{\frac{(b_2 - b_1)^2}{2} + \frac{(d_2 - d_1)^2}{2}}.$$

Substituting Inequality 5 and Inequality 6 into Inequality 2 we obtain

$$\|ST - TS\| \leq 2\sqrt{\frac{(a_2 - a_1)^2}{2} + \frac{(c_2 - c_1)^2}{2}} \sqrt{\frac{(b_2 - b_1)^2}{2} + \frac{(d_2 - d_1)^2}{2}}$$

which upon simplification yields

$$\|ST - TS\| \leq \frac{1}{2}\sqrt{(a_2 - a_1)^2 + (c_2 - c_1)^2} \sqrt{(b_2 - b_1)^2 + (d_2 - d_1)^2}.$$

□

Corollary 2.2. *Let $S, T \in B(H)$ be normal operators with the cartesian decomposition $S = A + iC$ and $T = B + iD$ such that C and D are positive, then*

$$(7) \quad \|ST - TS\| \leq \frac{1}{2}\sqrt{4\|A\|^2 + \|C\|^2} \sqrt{4\|B\|^2 + \|D\|^2}.$$

Proof. Consider Inequality 2 and let $a_1 = -\|A\|$, $a_2 = \|A\|$, $c_1 = 0$, $c_2 = \|C\|$, $b_1 = -\|B\|$, $b_2 = \|B\|$, $d_1 = 0$ and $d_2 = \|D\|$. Substituting in Inequality 8 we obtain

$$\|ST - TS\| \leq \frac{1}{2}\sqrt{4\|A\|^2 + \|C\|^2} \sqrt{4\|B\|^2 + \|D\|^2}.$$

□

Remark 2.3. In Corollary 2.2 if we instead of the assumption that C and D are positive, we can assume that S and T are positive, then we obtain the inequality

$$\|ST - TS\| \leq \frac{1}{2}\sqrt{\|A\|^2 + 4\|C\|^2} \sqrt{\|B\|^2 + 4\|D\|^2}.$$

Corollary 2.4. *Let $S \in B(H)$ with the cartesian decomposition $S = A + iC$ such that A and C are positive. Then $\|S^*S - SS^*\| \leq \frac{1}{2}(\|A\|^2 + \|C\|^2)$*

Proof. Let $S \in B(H)$ has the cartesian decomposition $S = A + iC$. Also let A and C be self-adjoint and $S^*S - SS^* = 2i(AC - CA)$. Using [[1], Inequality 36] and the arithmetic geometric mean inequality, we have $\|S^*S - SS^*\| = 2\|AC - CA\| \leq \|A\|\|C\| \leq \frac{1}{2}(\|A\|^2 + \|C\|^2)$. □

Theorem 2.5. *Let $S, T \in B(H)$ be normal operators with the cartesian decomposition $S = A + iC$ and $T = B + iD$ such that $a_1 \leq A \leq a_2$, $b_1 \leq B \leq b_2$, $c_1 \leq C \leq c_2$ and $d_1 \leq D \leq d_2$ for some real numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 and X and Y are compact then, $s_j(SX - YT) \leq \max(\|A\|, \|B\|)s_j(X \oplus Y)$ for $j = 1, 2, \dots$*

Proof. Let $a = \frac{a_1+a_2}{2}$, $b = \frac{b_1+b_2}{2}$, $c = \frac{c_1+c_2}{2}$, $d = \frac{d_1+d_2}{2}$, $z = a + ic$ and $w = b + id$. We have, $(SX - YT) = (S - \frac{z+w}{2})X - Y(T - \frac{z+w}{2})$. Taking the norms

we obtain $s_j\|SX - YT\| \leq \|(S - \frac{z+w}{2})\| + \|T - (\frac{z+w}{2})\|s_j(X \oplus Y)$. Since S and T are normal then $S - z$ and $T - w$ is normal. It follows by analogy that

$$\begin{aligned}
 s_j\|SX - YT\| &\leq \|(S - z) - \frac{z+w}{2}\| + \|(T - w) - \frac{z+w}{2}\|s_j(X \oplus Y) \\
 &\leq (\|S - z\| + \|T - w\| + |z - w|)s_j(X \oplus Y) \\
 &\leq (\sqrt{\|A - a\|^2 + \|C - c\|^2} + \sqrt{\|B - b\|^2 + \|D - d\|^2} \\
 &\quad + \sqrt{(a - b)^2 + (c - d)^2}) \quad \text{by (Equation 5)} \\
 (8) \quad &\leq (\|A - a\| + \|B - b\| + \|C - c\| + \|D - d\| + |a - b| + |c - d|)s_j(X \oplus Y) \\
 &\leq \left(\frac{a_2 - a_1 + b_2 - b_1 + c_2 - c_1 + d_2 - d_1}{2} \right. \\
 &\quad \left. + \left| \frac{a_2 - a_1 + b_2 - b_1 + c_2}{2} - \frac{c_1 + d_2 - d_1}{2} \right| \right) s_j(X \oplus Y) \\
 &= \frac{(b_2 - a_1) + (a_2 - b_1) + |(b_2 - a_1) - (a_2 - b_1)|}{2} \\
 &\quad + \frac{(d_2 - c_1) + (c_2 - d_1) + |(d_2 - c_1) - (c_2 - d_1)|}{2} s_j(X \oplus Y) \\
 &= (\max(b_2 - a_1, a_2 - b_1) + \max(d_2 - c_1, c_2 - d_1))s_j(X \oplus Y).
 \end{aligned}$$

Letting $a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = 0$, $b_2 = \|A\|$ and $d_2 = \|B\|$ then we have $s_j(SX - YT) \leq \max(\|A\|, \|B\|)s_j(X \oplus Y)$ for $j = 1, 2, \dots$ $\square$

Corollary 2.6. *Let $S \in B(H)$ be normal with the cartesian decomposition $S = A + iC$ if $a_1 \leq A \leq a_2$ and $c_1 \leq C \leq c_2$ for some real numbers a_1, a_2, c_1 and c_2 and if X, Y are compact, then $s_j\|SX - YT\| \leq \|A\|s_j(X \oplus Y)$ for $j = 1, 2, \dots$*

Proof. From Inequality 8 replacing T, b, d and w by S, a, c and z we obtain

$$\begin{aligned}
 s_j(SX - YS) &\leq \|S - z\| + \|S - z\| \leq 2\|S - z\| \\
 &\leq 2\sqrt{\frac{(a_2 - a_1)^2}{2} + \frac{(c_2 - c_1)^2}{2}}s_j(X \oplus Y) \quad \text{by (Equation 5)} \\
 &\leq \sqrt{(a_2 - a_1)^2 + (c_2 - c_1)^2}s_j(X \oplus Y).
 \end{aligned}$$

Let $a_1 = c_1 = c_2 = 0$ and $\|a_2\| = \|A\|$ we obtain $s_j(SX - YS) \leq \|A\|s_j(X \oplus Y)$ for $j = 1, 2, \dots$ $\square$

Lemma 2.7. *Let $S, T \in B(H)$ be normal operators belonging to the norm ideal associated with the Hilbert Schmidt norm $\|\cdot\|_2$ such that there product ST is normal. Then*

$$(9) \quad \|ST\|_2 \leq \|TS\|_2.$$

Proof. Let $w(A)$ denote the numerical radius of S . Then from $w(S) \leq \|S\|$ and if S is normal, $w(S) = \|S\|$. Moreover, for any two normal operators S and T we have

$$(10) \quad w(ST) = \|TS\|.$$

Suppose ST is normal, then we have $\|ST\|_2 \leq w(ST) = w(TS) \leq \|TS\|_2$. $\square$

Lemma 2.8. *Let S and T be as Lemma 2.7 above. If ST is a normal operator then $\| |ST|^{\frac{1}{2}} \| \leq \| |TS|^{\frac{1}{2}} \|$.*

Proof. Invoking Equation 10 we have $\| |ST|^{\frac{1}{2}} \| = w(|ST|^{\frac{1}{2}}) = w(|TS|^{\frac{1}{2}}) \leq \| |TS|^{\frac{1}{2}} \|$. $\square$

Theorem 2.9. *Let X be a positive definite operator and let S and T be normal operators belonging to the norm ideal associated with the Hilbert Schmidt norm $\|\cdot\|_2$. Then $\|S - T\|_2^2 \leq \|SX - XT\|_2^2 \|X^{-1}S - TX^{-1}\|_2^2$.*

Proof. Suppose S and T are self-adjoint then we can write $\|S - T\|_2 = \| |(S - T)^2|^{\frac{1}{2}} \|_2 = \| |(S - T)X^{-\frac{1}{2}}X^{\frac{1}{2}}(S - T)|^{\frac{1}{2}} \|_2$. Using Lemma 2.7 and Lemma 2.8 we get

$$\begin{aligned} \|S - T\|_2 &\leq \| |X^{\frac{1}{2}}(S - T)^2X^{-\frac{1}{2}}|^{\frac{1}{2}} \|_2 \\ &\leq (\|X^{\frac{1}{2}}(S - T)X^{\frac{1}{2}}\|_2 \|X^{-\frac{1}{2}}(S - T)X^{-\frac{1}{2}}\|_2)^{\frac{1}{2}} \\ &\leq (\|Re[(S - T)X]\|_2 \|Re[X^{-1}(S - T)]\|_2)^{\frac{1}{2}}. \end{aligned}$$

Since $TX - XT$ and $TX^{-1} - X^{-1}T$ are skew-Hermitian [5] then $Re[(S - T)X] = Re[(S - T)X + (TX - XT)] = Re(SX - XT)$, and $Re[X^{-1}(S - T)] = Re[X^{-1}(S - T) + (TX^{-1} - X^{-1}T)] = Re(X^{-1}S - TX^{-1})$. This implies that $\|S - T\|_2 \leq (\|Re(SX - XT)\|_2 \|Re(X^{-1}S - TX^{-1})\|_2)^{\frac{1}{2}}$. But $\|ReS\| \leq \|S\|$ for any operator. So we have $\|S - T\|_2 \leq (\|SX - XT\|_2 \|X^{-1}S - TX^{-1}\|_2)^{\frac{1}{2}}$ which upon squaring we obtain the required result. $\square$

Corollary 2.10. *Let $S, T, X \in B(H)$ such that S and T are positive, then $\|SX - XT\|_2 \leq \|X\|_2 (\|S\|_2^2 + \|T\|_2^2)^{\frac{1}{2}}$.*

Proof. $\|SX - XT\|_2 = \|(S - T)X\|_2 \leq \|S - T\|_2 \|X\|_2$. Since S and T are positive i.e $\|S - T\|_2 = (\|S\|_2^2 + \|T\|_2^2)^{\frac{1}{2}}$, then we have by [1] $\|SX - XT\|_2 \leq \|X\|_2 (\|S\|_2^2 + \|T\|_2^2)^{\frac{1}{2}}$. $\square$

Theorem 2.11. *Let $S, T \in B(H)$ be positive and self-adjoint operators and $ST - TS$ be also positive. If $n > 0$ is defined such that $\|ST - TS\| \leq n$, then*

$$(11) \quad \|STx\|^2 \geq \frac{1}{n^2} (\|STx\|^4 - |\langle (ST)^2 x, x \rangle|^2) \quad \forall x \in H, \|x\| = 1.$$

Proof. We employ the reverse of quadratic Schwarz inequality in i.e.

$$0 \leq \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \leq \frac{1}{|\alpha|^2} \|a\|^2 \|a - \alpha b\|^2.$$

For every $a, b \in H$, let $\alpha = 1$, $a = ST$, $b = TS$, we have

$$\|STx\|^2 \|TSx\|^2 - |\langle STx, TSx \rangle|^2 \leq \|STx\|^2 \|STx - TSx\|^2.$$

Since $ST = TS$, we have $\|STx\|^4 - |\langle (ST)^2x, x \rangle| \leq n^2\|STx\|^2$ which upon simplification yield, $\|STx\|^2 \geq \frac{1}{n^2}(\|STx\|^4 - |\langle (ST)^2x, x \rangle|)$. $\frac{1}{2}$ is the best constant possible in Inequality 11 in the sense that it cannot be replaced by a smaller quantity in general. The equality case is realized in Inequality 11 if, for instance one takes $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and a unit vector $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore Inequality 11 becomes $\|STx\|^2 \geq \frac{1}{2}(\|STx\|^4 - |\langle (ST)^2x, x \rangle|) \forall x \in H, \|x\| = 1$. $\square$

3. Orthogonality of commutators of derivations

In this section, we give some new results on orthogonality of commutators of normal derivation with respect to Fuglede-Putnam's property and norm-attainable operators.

Lemma 3.1. *Let $S, T, C \in B(H)$. Then the following are equivalent*

- i. *The pair (S, T) has the property $(FP)_{B(H)}$.*
- ii. *If $SC = CT$, then $R(C)$ reduces S , $\ker(C)^\perp$ reduces T and $S|_{\overline{R(C)}}$ and $T|_{\ker(C)^\perp}$ are normal operators where R and the $\ker$ denote the range and the kernel.*

Proof. (1) $\Rightarrow$ (2) Analogously by the proof of [3] Since $SC = CT$ and the pair (S, T) has the property $(FP)_{B(H)}$, $S^*C = CT^*$ this implies that $\overline{R(C)}$ and $\ker(C)^\perp$ are the reducing subspaces for S and T . If $S(SC) = (SC)T$, by $(FP)_{B(H)}$ we obtain $S^*(SC) = (SC)T^*$ and the identity $S^*C = CT^*$ implies that $S^*SC = SS^*C$. This shows that $S|_{\overline{R(C)}}$ is normal. Indeed, (T^*, S^*) satisfies $(FP)_{B(H)}$ and $(T^*C^* = C^*S^*)$. Similarly $T^*|_{\overline{R(C)}} = (T|_{\ker(C)})^*$.

(2) $\Rightarrow$ (1) If $C \in B(H)$ such that $SC = CT$. Let $S = S_1 \oplus S_2$ with respect to the orthogonal decomposition $H = \overline{R(C)} \oplus \overline{R(C)}^\perp$, $T = T_1 \oplus T_2$ with respect to $H = \ker(C) \oplus \ker(C)^\perp$ and $X : \overline{R(C)} \oplus \overline{R(C)}^\perp \rightarrow \ker(C)^\perp \oplus \ker(C)$ have the matrix representation $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$. From $SC = CT$, it follows that $S_1C_1 = C_1T_1$. Since S_1 and T_1 are normal operators, then applying the Fuglede-Putnam's property, we obtain $S_1^*C_1 = C_1T_1^*$ which implies that $S^*C = CT^*$. $\square$

Theorem 3.2. *Let $S, T, X \in B(H)$. If the pair of operators (S, T) satisfies Fuglede-Putnam's property and $C \in \ker(\delta_{S,T})$ where $C \in B(H)$ then $\|\delta_{S,T}X + C\| \geq \|C\|$.*

Proof. Since the pair (S, T) satisfies the $(FP)_{B(H)}$ property it follows that from Lemma 3.1 that $\overline{R(C)}$ reduces S , $\ker^\perp(C)$ reduces T and $S|_{\overline{R(C)}}$, $T|_{\ker^\perp(C)}$ are normal operators. Letting $C_o : \ker^\perp(C) \rightarrow \overline{R(C)}$ be the quasi-affinity defined by setting $C_1x = Cx$ for each $x \in \ker^\perp(C)$, then it results that $\delta_{S,T}(C_o) = \delta_{S_1^*, T_1^*}(C_o) = 0$. By Lemma 3.1, we have the matrix representation $S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$, $T =$

$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$, $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ Since S_1 and T_1 are two normal operators, then $\|\delta_{S,T}(X) + C\| = \left\| \begin{pmatrix} \delta_{S_1,T_1}X + C_1 & * \\ * & * \end{pmatrix} \right\| \geq \|\delta_{S_1,T_1}(X) + C_1\| = \|C\|$. $\square$

Corollary 3.3. *Let $S, T, X \in B(H)$ and $C \in \ker(\delta_{S,T})$ then $\|\delta_{S,T}X + C\| \geq \|C\|$.*

Proof. On $H \oplus H$ consider the operator M, N and Y defined as:

$$N = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}, \quad M = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then N is normal, $M \in N'$ and $\delta_N(Y) + M = \begin{bmatrix} 0 & \delta_{S,T}X + C \\ 0 & 0 \end{bmatrix}$.

Applying Theorem 3.2 to the operators N, M and Y and $M \in \delta_{S,T}$ and $\|\delta_N Y + M\| \geq \|M\|$. Therefore, $C \in \delta_{S,T}$ and $\|\delta_{S,T}X + C\| \geq \|C\|$. $\square$

4. Conclusion

In this paper, we have given results on norm inequality for commutators and also orthogonality of these commutators in Banach algebras.

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A fully fuzzy DEA approach for network cost efficiency measurement based on ranking functions

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Abstract. The purpose of this paper is to evaluate the cost-efficiency of the fully fuzzy network data envelopment analysis. Since actual measurement of real-world data is practically impossible, assuming that data is accurate in solving problems is not a valid hypothesis. One of the ways to deal with inaccurate data is fuzzy data. In this paper, linear ranking functions are used to transform the fully fuzzy cost efficiency model into a precise linear programming problem. By assuming triangular fuzzy numbers, the fuzzy cost efficiency of the decision makers is measured. Finally, a numerical example is shown for the proposed method.

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Keywords: data envelopment analysis, uncertain data, STOM algorithm, Multi objective linear, programming.

1. Introduction

Data Envelopment Analysis (DEA) is one of the most effective tools for measuring the efficiency and performance of organizations. On the other hand, in many DEA models, intra-organizational relationships have been ignored, and decision makers are thought as a black box, which only considers the inputs and outputs of the system. Unlike the black box model, the Network Data Envelopment Analysis (NDEA) model regards all internal processes in performance evaluation. For example, many companies are composed of several divisions with linked activities as in Diagram 1. In this case, the company is comprised of three divisions, each using its input resources to generate the outputs. However, there are linked activities or intermediate products that are shown by the links $1 \rightarrow 2$ and $1 \rightarrow 3$, and the link $2 \rightarrow 3$. The link $1 \rightarrow 2$ shows that some outputs of division 1 are used as inputs in division 2. In the current DEA models, each activity must belong to an input or output, not both, so these models cannot be formulated with intermediate products. For the first time, in

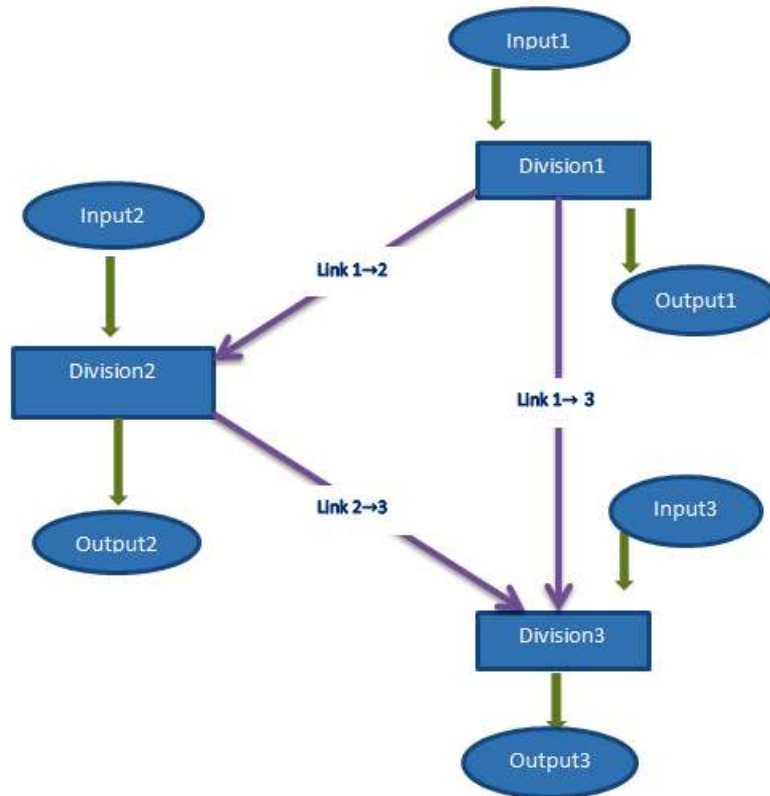


Figure 1: A company with three linked activities

2000, Fare and Grosskopf provided Data Envelopment Analysis models. Later, their models were expanded by several authors. Lewis and Sexton [9] presented a multi-stage NDEA model in 2004 as an extension of the Lewis and Sexton's two-step DEA model [4]. This article solves a DEA model independently for each node. Based on the Slack Based Measure (SBM) model, Tone and Tsutsui [11] further presented a NDEA model in 2009.

As a DEA model, Cost Efficiency (CE), evaluates the ability of a Decision Making Unit (DMU) to generate current outputs at the lowest cost and with assumed input prices. In fact, cost efficiency is a useful choice of inputs in terms of their price used to minimize production costs. Given the fact that in the real world we are dealing with network data envelopment analysis, it is important for managers to evaluate the revenue efficiency in NDEA. In 2013, Bani-Hashemi and Tohidi [1] proposed a model for assessing the cost efficiency of network data envelopment analysis models.

In the classical models, it is assumed that all data is accurate. However, accurate data is not always available, because the nature of data can be vague and unclear. Hence, one of the important ways to deal with inaccurate data is to consider fuzzy data. Only in [7] and [8], the fuzzy CEs (FCEs) are discussed with the fuzzy input- outputs and fuzzy input prices. But in none of these studies, fuzzy cost efficiency measurements in fully fuzzy NDEA (FFNDEA) have not been cited. In this article, the fully fuzzy models of network data envelopment analysis (fuzzy input-outputs and fuzzy input prices) were used in evaluating the fuzzy cost efficiency. Here, the ranking function method is used. Therefore, the ranking functions transform the full fuzzy model of network cost efficiency into a precise linear programming problem for measuring the fuzzy network cost efficiency.

The rest of the article will be as follows. In Section 2, the fuzzy clauses are discussed. In the next section, the problem of fuzzy linear programming and its transformation into a precise problem are studied. In Section 4 will measure the cost-effectiveness in NDEA. In Section 5, the proposed method for measuring fuzzy cost efficiency in FFNDEA is presented and, based on the proposed method, a numerical example is solved in the last section.

2. Fuzzy premises

2.1 Basic definitions of fuzzy

In this section, the basic definitions and the symbols of the fuzzy sets [12, 13], fuzzy Numbers, ranking function [5], and the concept of Fully Fuzzy Linear Programming (FFLP) used in this article are introduced.

Definition 2.1 ([12]). A fuzzy set $\tilde{A}$ is defined in the reference set X with $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ is the membership function and $\mu_{\tilde{A}}(x)$ is the degree of x in A .

Definition 2.2 ([13]). Assuming that X is a reference set, a fuzzy set $\tilde{A}$ is said to be convex X if and only if for every $x_1, x_2 \in X$:

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \quad \forall \lambda \in [0, 1]$$

Definition 2.3. Assuming that X is the reference set, then the fuzzy set A is called normal provided that there exist $x \in X$ so that $\mu_{\tilde{A}}(x) = 1$.

Definition 2.4. A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line R such that

1. it exists exactly one $x_0 \in R$ $\mu_{\tilde{A}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{A}$).
2. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 2.5 ([13]). A triangular fuzzy number (TFN), $\tilde{A} = (a^l, a^m, a^u)$ is a fuzzy number with the given membership function $\mu_{\tilde{A}}$

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a^l)/(a^m - a^l), & a^l < x \leq a^m \\ (x - a^u)/(a^u - a^m), & a^m \leq x < a^u \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. A triangular fuzzy number $\tilde{A} = (a^l, a^m, a^u)$ is called a non-negative number if and only if $a^l \geq 0$, $a^m - a^l \geq 0$, $a^u - a^m \geq 0$ and it is a positive number if and only if $a^l > 0$, $a^m - a^l \geq 0$, $a^u - a^m \geq 0$.

Definition 2.7. The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$ is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$. The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α -cut set: $A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$

Definition 2.8 ([5]). Suppose $\mathfrak{F}$ a set of all triangular fuzzy numbers. If $\tilde{A} \in \mathfrak{F}$ and $[A_\alpha^l, A_\alpha^u]$, $\alpha \in [0, 1]$ the α -cut is $\tilde{A}$. Then, the ranking function of a function $\mathfrak{R} : \mathfrak{F} \rightarrow \mathbb{R}$ is defined as:

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \int_0^1 (A_\alpha^l + A_\alpha^u) d\alpha.$$

If $\tilde{A} = (a^l, a^m, a^u)$ is a triangular fuzzy number, then

$$\mathfrak{R}(\tilde{A}) = \frac{1}{4}(a^l + 2a^m + a^u).$$

Definition 2.9 ([5]). If $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ are two triangular fuzzy numbers, then order of $\tilde{A}$ and $\tilde{B}$ based on the ranking function $\mathfrak{R}$ will be as follows:

- (i) $\tilde{A} \preceq \tilde{B} \iff \mathfrak{R}(\tilde{A}) \leq \mathfrak{R}(\tilde{B})$
- (ii) $\tilde{A} \succcurlyeq \tilde{B} \iff \mathfrak{R}(\tilde{A}) \geq \mathfrak{R}(\tilde{B})$
- (iii) $\tilde{A} \approx \tilde{B} \iff \mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$

And the features of Linearity will be:

$$\mathfrak{R}(k\tilde{A} + \tilde{B}) = k\mathfrak{R}(\tilde{A}) + \mathfrak{R}(\tilde{B}), \quad k \in \mathbb{R}.$$

2.2 Math operations on triangular fuzzy numbers

If $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ are two triangular fuzzy numbers, then the mathematical operations on triangular fuzzy numbers will be as follows:

- (i) $\tilde{A} \oplus \tilde{B} \approx (a^l + b^l, a^m + b^m, a^u + b^u)$ Addition
- (ii) $\tilde{A} \ominus \tilde{B} \approx (a^l - b^u, a^m - b^m, a^u - b^l)$ Subtraction
- (iii) $\tilde{A} \otimes \tilde{B} \approx (a^l b^l, a^m b^m, a^u b^u), \quad \tilde{A}, \tilde{B} \succ \tilde{0}$ Multiplication
- (iv) $\frac{\tilde{A}}{\tilde{B}} \approx \frac{(a^l, a^m, a^u)}{(b^l, b^m, b^u)} \approx \left(\frac{a^l}{b^u}, \frac{a^m}{b^m}, \frac{a^u}{b^l} \right), \quad \tilde{A}, \tilde{B} \succ \tilde{0}$ Division
- (v) $\forall k \in \mathbb{R}, k\tilde{A} \approx \begin{cases} (ka^l, ka^m, ka^u), & k > 0 \\ (ka^u, ka^m, ka^l), & k < 0 \end{cases}$ Scalar multiplication

3. Fuzzy linear programming problem

A linear programming problem with fuzzy coefficients and variables is called a full fuzzy linear programming problem. A full fuzzy linear programming problem with m constraints and n fuzzy variables are defined by the following model:

$$\begin{aligned} \tilde{Z} &= \max(\text{or min})(\tilde{C}^T \otimes \tilde{X}) \\ \text{subject to } \tilde{A} \otimes \tilde{X} &\preceq, \approx, \succeq \tilde{b}; \tilde{X} \succ \tilde{0} \end{aligned} \quad (P1)$$

where $\tilde{C} = [\tilde{c}_j]_{n \times 1}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_j]_{m \times 1}$, and $\tilde{a}_{ij}, \tilde{c}_j, \tilde{b}_i \in \mathfrak{F}$, $\tilde{x}_j$ are non-negative fuzzy numbers and $\tilde{0} = (0, 0, 0)$.

Definition 3.1 ([6]). The fuzzy optimal solution to the full fuzzy linear programming problem (P1) will be $\tilde{X} = [\tilde{x}_j]_{n \times 1}$. if it fits with the following conditions:

- 1) $\tilde{x}_j$ is a non-negative fuzzy number,
- 2) $\tilde{A} \otimes \tilde{X} \preceq, \approx, \succeq \tilde{b}$,

and 3) If there exist any non-negative fuzzy number such as $\tilde{Y} = [\tilde{y}_j]_{n \times 1}$, in case $\tilde{A} \otimes \tilde{X} \preceq, \approx, \succeq \tilde{b}$, then $\Re(\tilde{C}^T \otimes \tilde{X}) \geq \Re(\tilde{C}^T \otimes \tilde{Y})$ for the maximization problem and $\Re(\tilde{C}^T \otimes \tilde{X}) \leq \Re(\tilde{C}^T \otimes \tilde{Y})$ for the minimization problem.

Definition 3.2 ([6]). Assume that $\tilde{X} = [\tilde{x}_j]_{n \times 1}$ is the fuzzy optimal solution for full fuzzy linear problem (P1). If there exist any non-negative fuzzy number such as $\tilde{Y} = [\tilde{y}_j]_{n \times 1}$, so that $\tilde{A} \otimes \tilde{Y} \preceq, \approx, \succeq \tilde{b}$, and $\Re(\tilde{C}^T \otimes \tilde{X}) = \Re(\tilde{C}^T \otimes \tilde{Y})$, then $\tilde{Y} = [\tilde{y}_j]_{n \times 1}$ is called a fuzzy optimal solution of (P1). Suppose $\tilde{c}_j = (c_j^l, c_j^m, c_j^u)$, $\tilde{x}_j = (x_j^l, x_j^m, x_j^u)$, $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^m, a_{ij}^u)$ and $\tilde{b}_i = (b_i^l, b_i^m, b_i^u)$ represents triangular

fuzzy numbers. Then, the fuzzy decision parameters and variables in the model (P1) are converted as follows:

$$\begin{aligned} \tilde{Z} &= \max(\text{or } \min) \left(\sum_{j=1}^n (c_j^l, c_j^m, c_j^u) \otimes (x_j^l, x_j^m, x_j^u) \right) \\ \text{subject to } & \sum_{j=1}^n (a_{ij}^l, a_{ij}^m, a_{ij}^u) \otimes (x_j^l, x_j^m, x_j^u) \preccurlyeq, \approx, \succcurlyeq (b_i^l, b_i^m, b_i^u) \quad \forall i; (x_j^l, x_j^m, x_j^u) \\ & \succcurlyeq \tilde{0} \quad \forall j \quad (P2) \end{aligned}$$

After performing the mathematical operations discussed in Section 2-2, the model (P2) is converted to the following form:

$$\begin{aligned} \tilde{Z} &= \max(\text{or } \min) \left(\sum_{j=1}^n c_j^l x_j^l, \sum_{j=1}^n c_j^m x_j^m, \sum_{j=1}^n c_j^u x_j^u \right) \\ \text{subject to } & \left(\sum_{j=1}^n a_{ij}^l x_j^l, \sum_{j=1}^n a_{ij}^m x_j^m, \sum_{j=1}^n a_{ij}^u x_j^u \right) \preccurlyeq, \approx, \succcurlyeq (b_i^l, b_i^m, b_i^u) \quad \forall i; (x_j^l, x_j^m, x_j^u) \\ & \succcurlyeq \tilde{0} \quad \forall j \quad (P3) \end{aligned}$$

Now, using Nasseri et al.'s algorithm [6] and the ranking method, the FFLP (P2) turns into a precise linear programming problem. The steps in the algorithm are briefly summarized below:

Step 1. Transform full fuzzy objective function using its ranking function $\left(\Re \left(\sum_{j=1}^n c_j^l x_j^l, \sum_{j=1}^n c_j^m x_j^m, \sum_{j=1}^n c_j^u x_j^u \right) \right)$ into the exact form.

Step 2. Full fuzzy constraints of the model (P2) using the following ranking functions are as follows:

$$\begin{aligned} \sum_{j=1}^n a_{ij}^l x_j^l &\leq, =, \geq b_i^l \quad \forall i \\ \sum_{j=1}^n a_{ij}^m x_j^m &\leq, =, \geq b_i^m \quad \forall i \\ \sum_{j=1}^n a_{ij}^u x_j^u &\leq, =, \geq b_i^u \quad \forall i \end{aligned}$$

Step 3. The non-negative Fuzzy constraints, i.e., $(x_j^l, x_j^m, x_j^u) \succcurlyeq \tilde{0} \quad \forall j$ in the model (P2), which guarantees the decision variables assessment as non-triangular fuzzy numbers, will be as follows:

$$x_j^l \geq 0, x_j^m - x_j^l \geq 0, x_j^u - x_j^m \geq 0, \quad \forall j.$$

Thus, using the above steps, the model (P2) turns into the exact linear programming problem:

$$\begin{aligned}
Z = \max (\text{or } \min) & \Re \left(\sum_{j=1}^n c_j^l x_j^l, \sum_{j=1}^n c_j^m x_j^m, \sum_{j=1}^n c_j^u x_j^u \right) \\
\text{subject to} & \sum_{j=1}^n a_{ij}^l x_j^l \leq, =, \geq b_i^l \quad \forall i \\
& \sum_{j=1}^n a_{ij}^m x_j^m \leq, =, \geq b_i^m \quad \forall i \\
& \sum_{j=1}^n a_{ij}^u x_j^u \leq, =, \geq b_i^u \quad \forall i \\
& x_j^l \geq 0, x_j^m - x_j^l \geq 0, x_j^u - x_j^m \geq 0, \quad \forall j
\end{aligned} \tag{P4}$$

Theorem 3.3. *Each feasible solution in the model (P4) is also a feasible solution in the model (P3). The proof is provided in [8].*

Theorem 3.4. *The optimal solution of the model (P4) is the optimal solution for the model (P3) as well. The proof is provided in [8].*

4. Cost efficiency evaluation in network data envelopment analysis

There are drawbacks in the traditional methods of evaluating cost efficiency. These shortcomings are due to the set of defined production possibility P .

$$(1) \quad P = \{(x, y) | x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0\}$$

The production possibility set P is defined only on the basis of the technical factors $X = (x_1, \dots, x_n) \in \mathbb{R}^{m \times n}$ and $Y = (y_1, \dots, y_n) \in \mathbb{R}^{s \times n}$ and has no relationship with the input price, $C = (c_1, \dots, c_n)$. Tone [11] defines a set of production possibility set based on cost as follows:

$$(2) \quad P_c = \{(\bar{x}, y) | \bar{x} \geq \bar{X}\lambda, y \leq Y\lambda, \lambda \geq 0\}$$

where $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$ and $\bar{x}_j = (c_{1j}x_{1j}, \dots, c_{mj}x_{mj})^T$. Here, it is assumed that the matrices C and X are non-negative, and all inputs are cost-related. It is also assumed that the elements $\bar{x}_{ij} = (c_{ij}x_{ij})$ ($\forall (i, j)$) are inhomogeneous units such as dollar, in the sense that the multiplication of these elements is significant.

Based on the definition of the set of new production possibility set P_c , the new technical efficiency $\bar{\theta}^*$ is given as the optimal solution to the linear

programming problem:

$$\begin{aligned} \bar{\theta}^* &= \min \bar{\theta} \\ \text{s.t. } \quad &\bar{\theta} \bar{x}_0 \geq \bar{X} \lambda \\ &y_0 \leq Y \lambda \\ &\lambda \geq 0. \end{aligned}$$

The new cost efficiency $\bar{\gamma}^*$ is as follows:

$$(3) \quad \bar{\gamma}^* = e \bar{x}_0^* / e \bar{x}_0,$$

where $e \in \mathbb{R}^m$, is a row vector with the elements 1 and $\bar{x}_0^*$ is the optimal solution to the linear programming problem below:

$$\begin{aligned} [Ncost] \quad &\min e \bar{x} \\ \text{s.t. } \quad &\bar{x} \geq \bar{X} \lambda \\ &y_0 \leq Y \lambda \\ &\lambda \geq 0. \end{aligned}$$

5. Network data envelopment analysis based on SBM model

The commonly used DEA models measure the relative efficiency of multiple input/output decision maker units. One of the drawbacks of such models is the neglect of intermediate products or linked activities. In this section, the network data envelopment and the characteristics of the production possibility set is analyzed.

Suppose n is the decision maker available in Section K . m_k and r_k are the numbers of inputs and outputs in the k^{th} section. The link from division k to division h is represented by (h, k) and the set of all links is shown by L . The observed data is $\{x_j^k \in R_+^{m_k}\} (j = 1, \dots, n, k = 1, \dots, K)$, $\{y_j^k \in R_+^{r_k}\} (j = 1, \dots, n, k = 1, \dots, K)$ and $\{z_j^{(k,h)} \in R_+^{t_{(k,h)}}\} (j = 1, \dots, n, (k, h) \in L)$.

The set of production possibility set in network data envelopment analysis is defined as follows:

$$\begin{aligned} P &= \left\{ (x^k, y^k, z^{(h,k)}) \mid x^k \geq X^k \lambda^k, y^k \leq Y^k \lambda^k, z^{(k,h)} = z^{(k,h)} \lambda^k \text{ (as outputs } k), z^{(k,h)} \right. \\ &\quad \left. = z^{(k,h)} \lambda^h \text{ (as inputs } h), \lambda \geq 0 \right\}. \end{aligned}$$

Assume that the following model (with input-oriented) has a return to the variable scale and DMU_o , ($o = 1, \dots, n$) unit under evaluation. Since the SBM

model needs positive data, this paper assumes that all data are positive.

$$\begin{aligned}
 [NSBM] \quad \theta_0 = \min \quad & \sum_{k=1}^K w^k \left[1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_i^{k-}}{x_{io}^k} \right) \right] \\
 \text{s.t} \quad & x_o^k = X^k \lambda^k + s^{k-} \\
 & y_o^k = Y^k \lambda^k - s^{k+} \\
 & \lambda^k, \lambda^h, s^{k-}, s^{k+} \geq 0 \\
 & z_o^{(k,h)} = z^{(k,h)} \lambda^k \quad (\forall (k, h)), \quad (a) \\
 & z_o^{(k,h)} = z^{(k,h)} \lambda^h \quad (\forall (k, h)), \\
 & \text{or} \\
 & z_o^{(k,h)} \lambda^k = z^{(k,h)} \lambda^k \quad (\forall (k, h)), \quad (b),
 \end{aligned}$$

where $y^k = (y_1^k, \dots, y_n^k) \in R^{r_k \times n}$, $X^k = (x_1^k, \dots, x_n^k) \in R^{m_k \times n}$, $z^{(k,h)} = (z_1^{(k,h)}, \dots, z_n^{(k,h)}) \in R^{t_{(k,h)} \times n}$, s^{k-} (s^{k+}) are the auxiliary vectors of the input (output). Given the linked constraints, there are several choices that can be made in two possible ways:

(a) In the first case, the values of fixed intermediate current are taken into account.

$$\begin{aligned}
 z_o^{(k,h)} &= z^{(k,h)} \lambda^k \quad (\forall (k, h)), \quad (a) \\
 z_o^{(k,h)} &= z^{(k,h)} \lambda^h \quad (\forall (k, h)).
 \end{aligned}$$

(b) In the latter case, the average flow rates in the link can be freely reduced or increased.

$$z^{(k,h)} = z^{(k,h)} \lambda^h \quad (\forall (k, h)).$$

6. Cost efficiency in network data envelopment analysis

The production possibility set based on cost for NDEA will be as follows [1]:

$$\begin{aligned}
 P_c = \left\{ (\bar{x}^k, y^k, \bar{z}^{(k,h)}) \mid \bar{x}^k \geq \bar{X}^k \lambda^k, y^k \leq Y^k \lambda^k, z^{(k,h)} = z^{(k,h)} \lambda^k \text{ (as outputs } k), \bar{z}^{(k,h)} \right. \\
 \left. = \bar{z}^{(k,h)} \lambda^h \text{ (as inputs } h), \lambda \geq 0 \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 \bar{X}^k &= (\bar{x}_1^k, \dots, \bar{x}_n^k), & \bar{x}_j^k &= (c_{1j}^k x_{1j}^k, \dots, c_{mj}^k x_{mj}^k) \\
 \bar{z}^{(k,h)} &= (\bar{z}_1^{(k,h)}, \dots, \bar{z}_n^{(k,h)}), & \bar{z}_j^{(k,h)} &= (c_{1j}^k z_{1j}^{(k,h)}, \dots, c_{mj}^k z_{mj}^{(k,h)}).
 \end{aligned}$$

Based on this set, a new production possibility, $\bar{\gamma}^{*k}$, is obtained from the following linear programming problem:

$$\begin{aligned}
 [NNCost] \quad & \min \sum_{k=1}^K \bar{x}^k + \sum_h \bar{z}^{(k,h)} \\
 \text{s.t.} \quad & \bar{x}^k \geq \bar{X}^k \lambda^k, \quad k = 1, \dots, K \\
 & y_o \leq Y^k \lambda^k, \quad k = 1, \dots, K \\
 & z_o^{(k,h)} = z^{(k,h)} \lambda^k \quad (\forall (k, h)), \quad (a) \\
 & \bar{z}_o^{(k,h)} = \bar{z}^{(k,h)} \lambda^h \quad (\forall (k, h)) \\
 & \text{or} \\
 & z_o^{(k,h)} = \bar{z}^{(k,h)} \lambda^h \quad (\forall (k, h)) \quad (b) \quad (P5) \\
 & \lambda^k, \lambda^h \geq 0.
 \end{aligned}$$

The new network cost efficiency for each division k is obtained from the following equation:

$$\bar{\gamma}^{*k} = \sum_{k=1}^K \bar{x}_o^{*k} + \sum_h \bar{z}_o^{*(k,h)} \bigg/ \sum_{k=1}^K \bar{x}_o^k + \sum_h \bar{z}_o^{(k,h)},$$

where $e \in R^m$ is a row vector with elements is 1.

7. Proposed fuzzy cost efficiency method in fully fuzzy network data envelopment analysis

In the real world, input-output data and their corresponding costs are not accurately observed and may be available in inappropriate forms such as fuzzy numbers, in particular triangular fuzzy numbers. Many researchers have investigated the cost efficient with fuzzy and intermediate data. Yet, in these studies, only the decision parameters are considered as fuzzy and the decision variables are precise quantifiers. However, in this paper, the fully-fuzzy models of NDEA are used to measure the cost-effectiveness of a fully fuzzy environment in which all the decision making parameters and variables are represented by triangular fuzzy numbers.

To measure Fuzzy cost efficiency in NDEA, the model (P4) was expanded to a Fully-Fuzzy environment. Suppose n decision maker unit is available in division K . m_k and r_k are the number of Fuzzy inputs and outputs in the k^{th} division. The link from division k to division h is represented by (k, h) and the set of all links with L . The observed Fuzzy data $\tilde{x}_j^k$, $\tilde{y}_j^k$, $\tilde{z}_j^{(k,h)}$ and $\tilde{c}_j^k$ (where $j = 1, \dots, n$, $k = 1, \dots, K$) show fuzzy inputs, fuzzy outputs in each division, fuzzy linked activities from division k to division h , and fuzzy input units in each division, respectively. If these data are triangular fuzzy numbers, we will

have:

$$\begin{aligned}
 \tilde{x}_j^k &= (x_j^{l,k}, x_j^{m,k}, x_j^{u,k}), & j &= 1, \dots, n, \quad k = 1, \dots, K \\
 \tilde{y}_j^k &= (y_j^{l,k}, y_j^{m,k}, y_j^{u,k}), & j &= 1, \dots, n, \quad k = 1, \dots, K \\
 \tilde{c}_j^k &= (c_j^{l,k}, c_j^{m,k}, c_j^{u,k}), & j &= 1, \dots, n, \quad k = 1, \dots, K \\
 \tilde{z}_j^{(k,h)} &= (z_j^{l,(k,h)}, z_j^{m,(k,h)}, z_j^{u,(k,h)}), & j &= 1, \dots, n, \quad (k, h) \in L.
 \end{aligned}$$

As mentioned earlier, the model (P5) will become a fully fuzzy model as follows:

$$\begin{aligned}
 [FFNNCost] \quad \min \quad & \sum_{k=1}^K \tilde{x}^k \oplus \sum_h \tilde{z}^{(k,h)} \\
 & \tilde{x}^k \succcurlyeq \sum_{j=1}^n \tilde{X}_j^k \otimes \tilde{\lambda}_j^k, \quad k = 1, \dots, K \\
 & \tilde{y}_o^k \preccurlyeq \sum_{j=1}^n \tilde{y}_j^k \otimes \tilde{\lambda}_j^k, \quad k = 1, \dots, K
 \end{aligned} \tag{P6}$$

$$\begin{aligned}
 \tilde{z}_o^{(k,h)} &\approx \sum_{j=1}^n \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^k, \quad \forall (k, h) \\
 \tilde{z}_o^{(k,h)} &\approx \sum_{j=1}^n \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^h, \quad \forall (k, h)
 \end{aligned} \tag{a}$$

or

$$\begin{aligned}
 \sum_{j=1}^n \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^k &\approx \sum_{j=1}^n \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^h, \quad \forall (k, h) \\
 \tilde{\lambda}_j^k, \tilde{\lambda}_j^h &\succcurlyeq \tilde{0} \quad j, k,
 \end{aligned} \tag{b}$$

where $\tilde{\lambda}_j^k = (\lambda_j^{l,k}, \lambda_j^{m,k}, \lambda_j^{u,k})$ are non-negative triangular Fuzzy variables and

$$\begin{aligned}
 \tilde{X}^k &= (\tilde{x}_1^k, \dots, \tilde{x}_n^k), & \tilde{x}_j^k &= (\tilde{c}_{1j}^k \tilde{x}_{1j}^k, \dots, \tilde{c}_{mj}^k \tilde{x}_{mj}^k) \\
 \tilde{z}^{(k,h)} &= (\tilde{z}_1^{(k,h)}, \dots, \tilde{z}_n^{(k,h)}), & \tilde{z}_j^{(k,h)} &= (\tilde{c}_{1j}^k \tilde{z}_{1j}^{(k,h)}, \dots, \tilde{c}_{mj}^k \tilde{z}_{mj}^{(k,h)}).
 \end{aligned}$$

Model (P6) is a fully fuzzy cost efficiency model in fuzzy NDEA. After replacing the triangular fuzzy variables and parameters in model (P6) and using mathematical operations on triangular fuzzy numbers and the steps of the Nasseri algorithm, the fully fuzzy linear programming model (P6) becomes the exact linear programming:

$$\min \quad \frac{1}{4} \left[\sum_{k=1}^K \bar{x}^{l,k} + \sum_h \bar{z}^{l,(k,h)} + 2 \left(\sum_{k=1}^K \bar{x}^{m,k} + \sum_h \bar{z}^{m,(k,h)} \right) + \sum_{k=1}^K \bar{x}^{u,k} + \sum_h \bar{z}^{u,(k,h)} \right]$$

$$\begin{aligned}
\text{s.t.} \quad & \bar{x}^{l,k} \geq \sum_{j=1}^n \bar{X}_j^{l,k} \lambda_j^{l,k}, & k = 1, \dots, K \\
& \bar{x}^{m,k} \geq \sum_{j=1}^n \bar{X}_j^{m,k} \lambda_j^{l,k}, & k = 1, \dots, K \\
& \bar{x}^{u,k} \geq \sum_{j=1}^n \bar{X}_j^{u,k} \lambda_j^{l,k}, & k = 1, \dots, K \\
& y_o^{l,k} \leq \sum_{j=1}^n y_j^{l,k} \lambda_j^{l,k}, & k = 1, \dots, K \quad (P7) \\
& y_o^{m,k} \leq \sum_{j=1}^n y_j^{m,k} \lambda_j^{m,k}, & k = 1, \dots, K \\
& y_o^{u,k} \leq \sum_{j=1}^n y_j^{u,k} \lambda_j^{u,k}, & k = 1, \dots, K \\
& z_o^{l,(k,h)} = \sum_{j=1}^n z_j^{l,(k,h)} \lambda_j^{l,k}, & \forall (k, h) \\
& z_o^{m,(k,h)} = \sum_{j=1}^n z_j^{m,(k,h)} \lambda_j^{m,k}, & \forall (k, h) \\
& z_o^{u,(k,h)} = \sum_{j=1}^n z_j^{u,(k,h)} \lambda_j^{u,k}, & \forall (k, h) \\
& \bar{z}_o^{l,(k,h)} = \sum_{j=1}^n \bar{z}_j^{l,(k,h)} \lambda_j^{l,h}, & \forall (k, h) \quad (a) \\
& \bar{z}_o^{m,(k,h)} = \sum_{j=1}^n \bar{z}_j^{m,(k,h)} \lambda_j^{m,h}, & \forall (k, h) \\
& \bar{z}_o^{u,(k,h)} = \sum_{j=1}^n \bar{z}_j^{u,(k,h)} \lambda_j^{u,h}, & \forall (k, h)
\end{aligned}$$

OR

$$\begin{aligned}
& \sum_{j=1}^n z_j^{l,(k,h)} \lambda_j^{l,k} = \sum_{j=1}^n \bar{z}_j^{l,(k,h)} \lambda_j^{l,h}, & \forall (k, h) \\
& \sum_{j=1}^n z_j^{m,(k,h)} \lambda_j^{m,k} = \sum_{j=1}^n \bar{z}_j^{m,(k,h)} \lambda_j^{m,h}, & \forall (k, h) \quad (b) \\
& \sum_{j=1}^n z_j^{u,(k,h)} \lambda_j^{u,k} = \sum_{j=1}^n \bar{z}_j^{u,(k,h)} \lambda_j^{u,h}, & \forall (k, h)
\end{aligned}$$

$$\begin{aligned} \lambda_j^{l,k} &\geq 0, \lambda_j^{m,k} - \lambda_j^{l,k} \geq 0, \lambda_j^{u,k} - \lambda_j^{m,k} \geq 0 \quad \forall j, k \\ \lambda_j^{l,k} &\geq 0, x_j^{m,k} - x_j^{l,k} \geq 0, x_j^{u,k} - x_j^{m,k} \geq 0 \quad \forall j, k. \end{aligned}$$

Theorem 7.1. *The optimal solution for the model (P7) will be a model optimization solution (P6). The proof of this is similar to the proof of Theorem 3.3.*

Definition 7.2. The fuzzy cost efficiency of the i^{th} DMU in the FFDEA is defined as the ratio of the minimum fuzzy cost to the observed fuzzy cost of DMU_i :

$$\begin{aligned} \tilde{\gamma}_i^{*k} &= \frac{\sum_{k=1}^K \tilde{x}_i^{*k} \oplus \sum_h \tilde{z}_i^{*(k,h)}}{\sum_{k=1}^K \tilde{x}_i^k \oplus \sum_h \tilde{z}_i^{*(k,h)}} \\ &= \frac{\left(\sum_{k=1}^K \tilde{x}_i^{l,k*} + \sum_h \tilde{z}_i^{l,(k,h)*}, \sum_{k=1}^K \tilde{x}_i^{m,k*} + \sum_h \tilde{z}_i^{m,(k,h)*}, \sum_{k=1}^K \tilde{x}_i^{u,k*} + \sum_h \tilde{z}_i^{u,(k,h)*} \right)}{\left(\sum_{k=1}^K \tilde{x}_i^{l,k} + \sum_h \tilde{z}_i^{l,(k,h)}, \sum_{k=1}^K \tilde{x}_i^{m,k} + \sum_h \tilde{z}_i^{m,(k,h)}, \sum_{k=1}^K \tilde{x}_i^{u,k} + \sum_h \tilde{z}_i^{u,(k,h)} \right)} \\ &= \left(\frac{\sum_{k=1}^K \tilde{x}_i^{l,k*} + \sum_h \tilde{z}_i^{l,(k,h)*}}{\sum_{k=1}^K \tilde{x}_i^{l,k} + \sum_h \tilde{z}_i^{l,(k,h)}}, \frac{\sum_{k=1}^K \tilde{x}_i^{m,k*} + \sum_h \tilde{z}_i^{m,(k,h)*}}{\sum_{k=1}^K \tilde{x}_i^{m,k} + \sum_h \tilde{z}_i^{m,(k,h)}}, \frac{\sum_{k=1}^K \tilde{x}_i^{u,k*} + \sum_h \tilde{z}_i^{u,(k,h)*}}{\sum_{k=1}^K \tilde{x}_i^{u,k} + \sum_h \tilde{z}_i^{u,(k,h)}} \right). \end{aligned}$$

In other words, using the definition 2.8

$$\Re \left(\sum_{k=1}^k \tilde{x}_i^{*k} \oplus \sum_h \tilde{z}_i^{*(k,h)} \right) \approx \Re \left(\sum_{k=1}^k \tilde{x}_i^k \oplus \sum_h \tilde{z}_i^{(k,h)} \right).$$

Definition 7.3. i^{th} DMU in the network data envelopment analysis is called Fuzzy Cost Efficiency if the observed Fuzzy Cost and the minimum Fuzzy Cost equal DMU_i , i.e.,

$$\sum_{k=1}^k \tilde{x}_i^{*k} \oplus \sum_h \tilde{z}_i^{*(k,h)} \approx \sum_{k=1}^k \tilde{x}_i^k \oplus \sum_h \tilde{z}_i^{(k,h)}.$$

In other words, using definition 2.8

$$\Re \left(\sum_{k=1}^k \tilde{x}_i^{*k} \oplus \sum_h \tilde{z}_i^{*(k,h)} \right) \approx \Re \left(\sum_{k=1}^k \tilde{x}_i^k \oplus \sum_h \tilde{z}_i^{(k,h)} \right).$$

8. Numerical example

In this section, an illustrative example of electric power companies are presented for describing network DEA. As we know, the vertically integrated electric power companies consist of several divisions such as generation, transmission and distribution. For illustrative purpose, ten vertically integrated electric power companies in the U.S in 1994 [16]. The inputs, outputs and links are as follows:

Generation (Div1):

Input1 = Labor input (number of employees)

Transmission (Div2):

Input2 = Labor input (number of employees)

Output2 = Electric power sold to large customers

Distribution (Div3):

Input3 = Labor input (number of employees)

Output3 = Electric power sold to small customers

Link (1-2) = Electric power generated (output from Generation Division and input to Transmission Division)

Link (2-3) = Electric power sent (output from Transmission Division and input to Distribution Division)

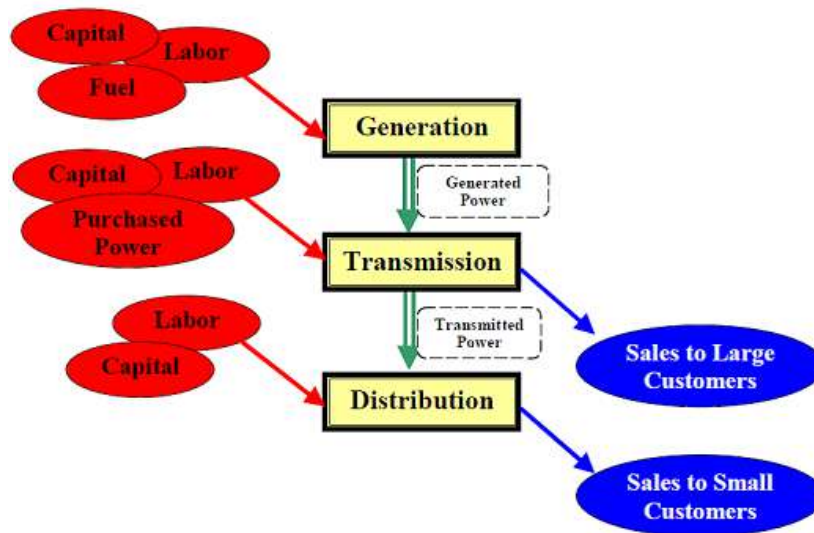


Figure 2: Vertically integrated electric power companies

Table 1: Fuzzy inputs, fuzzy outputs, fuzzy input cost in three divisions

DMU	Div1		Div2			Div3		
	Input1	C1	Input2	C2	Output2	Input3	C3	Output3
A	(0.836,0.838,0.840)	(248,250,253)	(0.275,0.277,0.279)	(118,120,123)	(0.876,0.879,0.881)	(0.960,0.962,0.965)	(218,220,223)	(0.335,0.337,0.340)
B	(1.231,1.233,1.235)	(497,500,502)	(0.130,0.132,0.133)	(178,180,183)	(0.535,0.538,0.540)	(0.440,0.443,0.445)	(208,210,213)	(0.15,0.18,0.20)
C	(0.318,0.321,0.323)	(123,125,128)	(0.042,0.045,0.048)	(289,290,293)	(0.909,0.911,0.914)	(0.482,0.485,0.487)	(137,140,142)	(0.195,0.198,0.200)
D	(1.480,1.483,1.485)	(110,113,115)	(0.110,0.111,0.113)	(57,60,62)	(0.55,0.57,0.59)	(0.465,0.467,0.470)	(147,150,153)	(0.488,0.491,0.495)
E	(1.590,1.592,1.595)	(48,50,53)	(0.205,0.208,0.211)	(83,85,88)	(1.085,1.086,1.089)	(1.070,1.073,1.075)	(197,200,202)	(0.370,0.372,0.375)
F	(0.76,0.79,0.81)	(42,45,48)	(0.136,0.139,0.141)	(93,95,98)	(0.720,0.722,0.724)	(0.543,0.545,0.548)	(80,85,89)	(0.250,0.253,0.255)
G	(0.449,0.451,0.454)	(92,95,96)	(0.073,0.075,0.077)	(98,100,102)	(0.507,0.509,0.511)	(0.365,0.366,0.368)	(147,150,153)	(0.238,0.241,0.244)
H	(0.405,0.408,0.410)	(447,450,452)	(0.072,0.074,0.076)	(138,140,143)	(0.617,0.619,0.621)	(0.226,0.229,0.231)	(228,230,235)	(0.095,0.097,0.099)
I	(1.860,1.864,1.865)	(198,200,203)	(0.059,0.061,0.063)	(129,130,135)	(1.021,1.023,1.025)	(0.689,0.691,0.693)	(158,160,165)	(0.35,0.38,0.40)
J	(1.220,1.222,1.225)	(8,10,14)	(0.147,0.149,0.151)	(43,45,47)	(0.765,0.769,0.771)	(0.336,0.337,0.339)	(448,450,453)	(0.175,0.178,0.180)

Table 2: Fuzzy unit input link cost

Link			
Link12	Lc1	Link23	Lc2
(0.891,0.894,0.897)	(340,345,348)	(0.360,0.362,0.365)	(160,162,165)
(0.675,0.678,0.780)	(210,212,215)	(0.185,0.188,0.190)	(120,123,125)
(0.835,0.836,0.838)	(170,173,175)	(0.205,0.207,0.210)	(340,345,347)
(0.865,0.869,0.872)	(398,400,405)	(0.514,0.516,0.520)	(452,456,460)
(0.690,0.693,0.695)	(25,27,30)	(0.405,0.407,0.410)	(65,67,70)
(0.961,0.966,0.970)	(94,96,100)	(0.265,0.269,0.273)	(75,78,80)
(0.645,0.647,0.650)	(80,83,85)	(0.255,0.257,0.259)	(185,189,192)
(0.752,0.756,0.760)	(285,289,290)	(0.101,0.103,0.105)	(85,90,92)
(1.189,1.191,1.194)	(100,104,107)	(0.400,0.402,0.405)	(17,19,23)
(0.790,0.792,0.795)	(45,47,50)	(0.185,0.187,0.190)	(325,327,330)

Here, it is assumed that the intermediate flow values in the link can be freely reduced or increased; thus, the proposed model for evaluating fuzzy cost efficiency is as follows:

$$\begin{aligned}
\min \quad & \frac{1}{4} \left[\sum_{k=1}^K \bar{x}^{l,k} + \sum_h \bar{z}^{l,(k,h)} + 2 \left(\sum_{k=1}^K \bar{x}^{m,k} + \sum_h \bar{z}^{m,(k,h)} \right) + \sum_{k=1}^K \bar{x}^{u,k} + \sum_h \bar{z}^{u,(k,h)} \right] \\
\text{s.t.} \quad & \bar{x}^{l,k} \geq \sum_{j=1}^n \bar{X}_j^{l,k} \lambda_j^{l,k}, & k = 1, \dots, K \\
& \bar{x}^{m,k} \geq \sum_{j=1}^n \bar{X}_j^{m,k} \lambda_j^{m,k}, & k = 1, \dots, K \\
& \bar{x}^{u,k} \geq \sum_{j=1}^n \bar{X}_j^{u,k} \lambda_j^{u,k}, & k = 1, \dots, K \\
& y_o^{l,k} \leq \sum_{j=1}^n y_j^{l,k} \lambda_j^{l,k}, & k = 1, \dots, K \\
& y_o^{m,k} \leq \sum_{j=1}^n y_j^{m,k} \lambda_j^{m,k}, & k = 1, \dots, K \\
& y_o^{u,k} \leq \sum_{j=1}^n y_j^{u,k} \lambda_j^{u,k}, & k = 1, \dots, K \\
& \sum_{j=1}^n \bar{z}_j^{l,(k,h)} \lambda_j^{l,k} = \sum_{j=1}^n \bar{z}_j^{l,(k,h)} \lambda_j^{l,h}, & \forall (k, h) \\
& \sum_{j=1}^n \bar{z}_j^{m,(k,h)} \lambda_j^{m,k} = \sum_{j=1}^n \bar{z}_j^{m,(k,h)} \lambda_j^{m,h}, & \forall (k, h)
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^n z_j^{u,(k,h)} \lambda_j^{u,k} &= \sum_{j=1}^n z_j^{u,(k,h)} \lambda_j^{u,h}, & \forall (k, h) \\
\lambda_j^{l,k} &\geq 0, \quad \lambda_j^{m,k} - \lambda_j^{l,k} \geq 0, \quad \lambda_j^{u,k} - \lambda_j^{m,k} \geq 0 & \forall j, k \\
\bar{x}_j^{l,k} &\geq 0, \quad \bar{x}_j^{m,k} - \bar{x}_j^{l,k} \geq 0, \quad \bar{x}_j^{u,k} - \bar{x}_j^{m,k} \geq 0, & \forall j, k \\
\bar{z}_j^{l,k} &\geq 0, \quad \bar{z}_j^{m,k} - \bar{z}_j^{l,k} \geq 0, \quad \bar{z}_j^{u,k} - \bar{z}_j^{m,k} \geq 0, & \forall j, k
\end{aligned}$$

The above model is solved using GAMS software and the results are shown in Table 3 As shown in the table and according to Definition 7.3, DMU_D is cost

Table 3: Evaluating and ranking revenue efficiency

DMU_s	$\tilde{\gamma}^{*k}$	$R(\tilde{\gamma}^{*k})$	Rank
A	(0.326,0.375,0.391)	0.375	8
B	(0.150,0.178,0.195)	0.178	10
C	(0.565,0.607,0.781)	0.607	3
D	(0.972,1,1.0002)	1	1
E	(0.370,0.457,0.463)	0.457	6
F	(0.542,0.666,0.730)	0.666	2
G	(0.479,0.556,0.630)	0.556	4
H	(0.345,0.418,0.504)	0.418	7
I	(0.206,0.239,0.274)	0.239	9
J	(0.300,0.447,0.486)	0.447	5

efficient. In the last column of Table 3, the cost efficiency of the decision making units is arranged based on the ranking function of Definition 2.8.

Conclusion

In this paper, a new idea was developed for expanding the classic cost efficiency model into a fully fuzzy environment in a network data envelopment analysis. In the fuzzy cost efficiency model, it is assumed that the inputs and outputs and their corresponding costs are fuzzy triangular numbers. Then, using the ranking function method, the fully fuzzy cost efficiency model is transformed into a precise model, and ultimately, the final fuzzy cost function is obtained as a triangular fuzzy number.

Since cost efficiency sensitivity analysis helps the manager or decision maker to modify the amount of inputs under evaluation to minimize cost. Therefore, future work can include sensitivity analysis of performance, as well as finding the appropriate stability area to maintain cost efficiency in precise and imprecise network data envelopment analysis.

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Numerical solution of nonlinear oscillatory differential equations using shifted second kind Chebyshev wavelet method

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Abstract. A numerical method for solving Oscillatory differential equations using shifted second kind Chebyshev wavelets is presented. An operational matrix of derivative is introduced and utilized to reduce the oscillatory differential equations and its initial conditions to a set of algebraic equations in the unknown expansion coefficients. Illustrative examples are included to demonstrate the validity and applicability of the method. Numerical solution of Van der Pol oscillator problem and Duffing Equations are obtained by using this method.

Keywords: Chebyshev wavelets, operational matrix of derivative, oscillatory differential equations.

1. Introduction

The study of non linear oscillatory problems is of crucial importance in solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. Several techniques such as variational iteration method [1], Adomian decomposition method [2], Homotopy Perturbation Method [3], differential transform method [4] have been used for solving these problems.

Orthogonal functions and polynomial series have received considerable attention in dealing with numerical solution of these problems. The approach is based on reducing the given problem to a system of algebraic equations by approximating the unknown function in terms of truncated orthogonal series. Walsh functions, block pulse functions, Laguerre polynomials, Legendre poly-

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nomials, Chebyshev polynomials and Fourier series are widely used orthogonal functions.

In this paper we will apply wavelet based numerical method for solution of nonlinear oscillatory differential equations. Wavelets have properties like orthogonality and compact support. These properties are quite useful for approximating any function in terms of truncated orthogonal series.

Operational matrix of derivative and integration of wavelets are widely used for numerical solution of differential equations [5] [8] [10], fractional differential equations [6], problems of calculus of variations [9]. The method consists of reducing the oscillatory differential equation into a set of algebraic equations by expanding the candidate function as wavelet basis with unknown coefficients.

The paper is organized as follow In section 2 we describe the second kind Chebyshev Polynomials and their shifted form, Shifted second kind Chebyshev wavelet and its operational matrix of derivative. In Section 3 Mathematical formulation of proposed method is given. In Section 4 some examples are solved using the proposed numerical method. Conclusion is drawn in Section 5.

2. Second kind Chebyshev polynomials and their shifted form

The second kind Chebyshev Polynomials are defined on $[-1, 1]$ by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad t = \cos\theta.$$

These polynomials are orthogonal w.r.t to the weight function $\sqrt{1-t^2}$ on the interval $[-1, 1]$ i.e.

$$\int_{-1}^1 \sqrt{1-t^2} U_m(t) U_n(t) dt = \begin{cases} 0, & n = m \\ \frac{\pi}{2}, & n \neq m \end{cases}$$

and also satisfy the following recursive formula[7]

$$\begin{aligned} U_0(t) &= 1, \\ U_1(t) &= 2t, \\ U_{m+1}(t) &= 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \dots \end{aligned}$$

The Shifted second kind Chebyshev polynomials are defined on $[0, 1]$ by $U_n^*(t) = U_n(2t-1)$. These polynomials are orthogonal on $[0, 1]$ w.r.t the weight function $\sqrt{t-t^2}$ i.e.

$$\int_0^1 \sqrt{t-t^2} U_n^*(t) U_m^*(t) dt = \begin{cases} 0, & n = m \\ \frac{\pi}{8}, & n \neq m. \end{cases}$$

2.1 Wavelets and shifted second kind Chebyshev wavelets

Wavelet methods have been used extensively for the solution of various problems of science and engineering. Wavelets constitute a family of functions constructed

from dilation and translation of a single function called the mother wavelet[11], [12],[14], [15]. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in R, \quad a \neq 0.$$

The second kind Chebyshev wavelet $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments $n = 0, 1, 2, \dots, 2^k - 1$, k can assume any positive integer, m is the degree of the second kind Chebyshev polynomials and t denotes the time. They are defined on the interval $[0,1]$ as

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n), & \text{if } \frac{n}{2^k} \leq t \leq \frac{n+1}{2^k} \\ 0, & \text{otherwise,} \end{cases}$$

where $m = 0, 1, 2, \dots, M$.

A function $y(t)$ defined over $L^2[0,1]$ can be expanded in terms of shifted second kind Chebyshev wavelets ψ_{nm} as:

$$(2.1) \quad y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t),$$

where $c_{nm} = \langle y(t), \psi_{nm}(t) \rangle$ denotes the inner product and is given by $c_{nm} = \int_0^1 \sqrt{1-t^2} y(t) \psi_{nm}(t) dt$. If the infinite series in (2.1) is truncated, then (2.1) can be written as

$$y(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(t) = C^T \Psi(t),$$

where C and $\Psi(t)$ are $2^k(M+1) \times 1$ matrices given by

$$C = \{c_{0,0}, c_{0,1}, \dots, c_{0,M}, \dots, c_{2^k-1,0}, c_{2^k-1,1}, \dots, c_{2^k-1,M}\}^T \quad \text{and} \\ \Psi(t) = \{\psi_{0,0}, \psi_{0,1}, \dots, \psi_{0,M}, \dots, \psi_{2^k-1,0}, \psi_{2^k-1,1}, \dots, \psi_{2^k-1,M}\}^T.$$

The first derivative of second kind Chebyshev wavelet vector $\psi(t)$ can be defined as

$$\frac{d\psi}{dt} = D\psi(t),$$

where D is $2^k(M+1)$ square operational matrix of derivative of shifted second kind Chebyshev wavelet and is given by

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{pmatrix}$$

where F is $(M + 1)$ square matrix whose $(r, s)th$ element is given by

$$F_{r,s} = \begin{cases} 2^{k+2}s, & r \geq 2, r > s, (r+s)\text{odd} \\ 0, & \text{otherwise.} \end{cases}$$

Generally, the operational matrix of n th order derivative is given by

$$\frac{d^n \psi(t)}{dt} = D^n \psi(t), \quad n = 1, 2, \dots$$

3. Mathematical formulation

In this paper we will consider the following general Oscillatory differential equation

$$(3.1) \quad u'' + f(t, u, u') = 0$$

with the initial conditions $u(0) = a$, $u'(0) = b$ where u is displacement, t is time and $'$ denotes differentiation w.r.t t .

The method consist of reducing the given problem to a set of algebraic equations by expanding the unknown function $u(t)$ as shifted second kind Chebyshev wavelets with unknown coefficients c_{nm} :

$$(3.2) \quad u(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(t) = C^T \Psi(t)$$

u' and u'' can also be approximated as

$$(3.3) \quad u'(t) = C^T D \Psi(t), \quad u''(t) = C^T D^2 \Psi(t).$$

Substitution of (3.2) and (3.3) in (3.1) reduces the equation (3.1) to the following form

$$(3.4) \quad C^T D^2 \psi(t) + f(t, C^T \psi(t), C^T D \psi(t)) = 0.$$

Now, we will collocate (3.4) at $2^k(M + 1) - 2$ Chebyshev points which yields $2^k(M + 1) - 2$ algebraic equations in unknown coefficients c_{nm} .

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$(3.5) \quad C^T \psi(0) = a, \quad C^T D \psi(0) = b.$$

Equations (3.4) and (3.5) give $2^k(M + 1)$ algebraic equations which can be solved for the unknown components of the vector C . So the underlying Oscillatory differential equation together with initial conditions is converted into system of algebraic equations via operational matrix of derivatives. This system can be solved for the unknown C and an approximate wavelet solution to $u(t)$ given in (3.2) can be obtained.

4. Numerical examples

All numerical computations have been done by taking $M = 2, k = 0$. The matrices D, D^2, C and ψ which are used in calculation are given by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}, \quad \psi(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 2 \\ 8t - 4 \\ 32t^2 - 32t + 6 \end{pmatrix},$$

$$C = (c_{0,0} \quad c_{0,1} \quad c_{0,2})^T = \sqrt{\frac{\pi}{2}} (c_0 \quad c_1 \quad c_2)^T.$$

Example 1 (Van Der Pol Oscillator problem). Consider the Van Der Pol Oscillator problem $u'' + u' + u + u^2 u' = 2 \cos t - \cos^3 t$ with initial conditions $u(0) = 0, u'(0) = 1$.

The analytic solution of this problem is given by $u(t) = \sin t$.

For numerical solution we approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$(4.1) \quad u(t) = C^T \Psi(t).$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$(4.2) \quad \begin{aligned} u'(t) &= C^T D \Psi(t), \\ u''(t) &= C^T D^2 \Psi(t). \end{aligned}$$

Substituting (4.1) and (4.2) into given equation, we have

$$C^T D^2 \Psi(t) + C^T D \Psi(t) + C^T \Psi(t) + (C^T \Psi(t))^2 (C^T D \Psi(t)) = 2 \cos t - \cos^3 t$$

which is equivalent to

$$(4.3) \quad \begin{aligned} &64c_2 + 8c_1 + (64t - 32)c_2 + 2c_0 + (8t - 4)c_1 + (32t^2 - 32t + 6)c_2 + (2c_0 \\ &+ (8t - 4)c_1 + (32t^2 - 32t + 6)c_2)^2 (8c_1 + (64t - 32)c_2) = 2 \cos t - \cos^3 t. \end{aligned}$$

Collocating eq.(4.3) in $2^k(M+1) - 2$ Chebyshev points (i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$(4.4) \quad \begin{aligned} &64c_2 + 8c_1 - 22.6304c_2 + (2c_0 - 2.8288c_1 + 2.0011c_2) \\ &+ (4c_0^2 + 8.0021c_1^2 + 4.0044c_2^2 \\ &- 11.3152c_0c_1 - 11.3215c_1c_2 + 8.0044c_0c_2)(8c_1 - 22.6304c_2) = 1.0104. \end{aligned}$$

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$(4.5) \quad \begin{aligned} c_0 - 2c_1 + 3c_2 &= 0, \\ 8c_1 - 32c_2 &= 1. \end{aligned}$$

Table 1: Absolute error of exact and approximate solution

t	<i>Exact Solution</i>	<i>Wavelet method</i>	<i>Variational iteration method</i>
0	0	0.000000106	0
.1	0.09983341665	0.09932344712	.09983338930
.2	0.1986693308	0.1972934769	0.1986676148
.3	0.2955202067	0.2939101953	0.2955011070
.4	0.3894183423	0.3891736023	0.3893138342
.5	0.4794255386	0.483083698	0.4790386507
.6	0.5646424734	0.5756404823	0.5635252927
.7	0.6442176872	0.6668439553	0.6415028932
.8	0.7173560909	0.7566941169	0.7115469778
.9	0.7833269096	0.8451909671	0.7720563532
1	0.8414709848	0.932334506	0.8212443348

Equation (4.4) together with (4.5) gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2 . By using Newton's iterative formula we get $c_0 = 0.2394273$, $c_1 = 0.1165418$, $c_2 = -0.002114549$. So the unknown function u can be found using (4.1). A comparison between the wavelet method and Variational iteration method [16] is shown in Table 1.

Example 2. $u'' - u + u^2 + u'^2 = 1$ with initial conditions $u(0) = 2, u'(0) = 0$ The analytic solution of this problem is $u(t) = 1 + \cos t$. For numerical solution we approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$(4.6) \quad u(t) = C^T \Psi(t).$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$(4.7) \quad \begin{aligned} u'(t) &= C^T D \Psi(t), \\ u''(t) &= C^T D^2 \Psi(t) \end{aligned}$$

by substituting (4.6) and (4.7) into given equation, one can have

$$C^T D^2 \psi(t) - C^T \psi(t) + (C^T \psi(t))^2 + (C^T D \psi(t))^2 = 1$$

which is equivalent to

$$(4.8) \quad \begin{aligned} &64c_2 - (2c_0 + (8t - 4)c_1 + (32t^2 - 32t + 6)c_2) + (2c_0 + (8t - 4)c_1 \\ &+ (32t^2 - 32t + 6)c_2)^2 + (8c_1 + (64t - 32)c_2)^2 = 1. \end{aligned}$$

Collocating eq.(4.8) in $2^k(M+1) - 2$ Chebyshev points (i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$(4.9) \quad \begin{aligned} &64c_2 - (2c_0 - 2.8288c_1 + 2.0011c_2) + 4c_0^2 + 8.0021c_3^2 \\ &+ 4.0044c_2^2 - 11.3152c_0c_1 - 11.3215c_1c_2 + 8.0044c_0c_2 \\ &+ 64c_1^2 + 512.1350c_2^2 - 362.0864c_2c_1 = 1. \end{aligned}$$

Table 2: Absolute error of exact and approximate solution

t	<i>Exact Solution</i>	<i>Wavelet method</i>	<i>Variational iteration method</i>
0	2.0	2.0	2.000000000
.1	1.995004165	1.995053594	1.994995832
.2	1.980066578	1.980214374	1.979933244
.3	1.955336489	1.955482342	1.954661486
.4	1.921060994	1.920857498	1.918927628
.5	1.877582562	1.87633984	1.872374035
.6	1.825335615	1.82192937	1.814534782
.7	1.764842187	1.757626086	1.744830994
.8	1.696706709	1.68343	1.662565054
.9	1.621609968	1.599341082	1.566913614
1	1.540302306	1.50535936	1.456919365

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$(4.10) \quad \begin{aligned} c_0 - 2c_1 + 3c_2 &= 1, \\ c_1 - 4c_2 &= 0. \end{aligned}$$

Equation (4.9) together with (4.10) gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2 . By using Newton's iterative formula we get $c_0 = 0.9227124$, $c_1 = -0.06183008$, $c_2 = -0.01545752$. So the unknown function u can be found using (4.6). A comparison between the wavelet method and variational iteration method [16] is shown in Table 2.

Example 3 (Duffing Equation). Consider one dimensional unforced Duffing oscillator $u'' + \alpha u + \epsilon u^3 = 0$ with initial conditions $u(0) = 1, u'(0) = 0$. Here α controls the size of the stiffness and ϵ controls the amount of nonlinearity in the restoring force.

For $\alpha = 1$ analytic solution of this problem is given by

$$u(t) = \cos t + \epsilon \left[\frac{1}{32} \cos 3t - \frac{1}{32} \cos t - \frac{3}{8} t \sin t \right], \epsilon \rightarrow 0^+.$$

Now, we will solve this problem numerically on the interval $[0,1]$ for $\epsilon = 10^{-4}$. To find numerical solution approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$(4.11) \quad u(t) = C^T \Psi(t).$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$(4.12) \quad \begin{aligned} u'(t) &= C^T D \Psi(t), \\ u''(t) &= C^T D^2 \Psi(t), \end{aligned}$$

Table 3: Absolute error of exact and approximate solution

t	<i>Exact Solution</i>	<i>Numerical Solution</i>	<i>Absolute error</i>
0	1.0	1.0	4.0e-8
.1	0.9950037	0.9950525	0.00004883866
.2	0.9800646	0.9802101	0.0001455221
.3	0.9553321	0.9554728	0.0001407008
.4	0.9210534	0.9208406	0.0002128132
.5	0.8775711	0.8763134	0.001257611
.6	0.8253196	0.8218914	0.00342826
.7	0.7648213	0.7575744	0.00724695
.8	0.6966807	0.6833624	0.01331828
.9	0.6215788	0.5992556	0.02232319
1	0.540266	0.5052538	0.03501217

by substituting (4.11) and (4.12) into given equation, one can have

$$C^T D^2 \psi(t) + C^T \psi(t) + 10^{-4} (C^T \psi(T))^3 = 0$$

which is equivalent to

$$(4.13) \quad \begin{aligned} &64c_2 + 2c_0 + (8t - 4)c_1 + (32t^2 - 32t + 6)c_2 \\ &+ 10^{-4}(2c_0 + (8t - 4)c_1 + (32t^2 - 32t + 6)c_2)^3 = 0. \end{aligned}$$

Collocating eq.(4.13) in $2^k(M+1) - 2$ Chebyshev points (i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$(4.14) \quad 64c_2 + (2c_0 - 2.8288c_1 + 2.0011c_2) + (10^{-4})(2c_0 - 2.8288c_1 + 2.0011c_2)^3 = 0.$$

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$(4.15) \quad \begin{aligned} 2c_0 - 4c_1 + 6c_2 &= 1, \\ c_1 - 4c_2 &= 0. \end{aligned}$$

Equation (4.14) together with (4.15) gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2 . By using Newton's iterative formula we get $c_0 = 0.4226959$, $c_1 = -0.06184327$, $c_2 = -0.01546082$. So the unknown function u can be found using (4.11). A comparison between the wavelet method and the method discussed in [17] shows that wavelet method can be adapted in a simple way to solve this kind of problems numerically. Table 3 shows the absolute error of wavelet method.

5. Conclusion

A computational method based on Chebyshev wavelets and its operational matrix of derivative is proposed for solving oscillatory differential equations. The

main advantage of the proposed method is that it transforms the oscillatory differential equations into non-linear systems of algebraic equations which can be easily solved by Newton's iterative method. Some examples are given to verify and demonstrate the effectiveness of the proposed method. Comparison of the exact solution and wavelet solution shows that the method is highly accurate.

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Compactness of topological spaces with grills

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Abstract. This work aimed at creation of expansion of topological structures using the concept of grill for the first time. In this respect, we obtained new findings such as αg -compact, αg -compact sets and countably αg -compact spaces. On the other hand, we studied the properties of these concepts and their relationships to each others and their previous counterparts.

Keywords: g -compact, αg -compact, αg -compact sets and countably αg -compact spaces.

1. Introduction and preliminaries

Topological constructions are currently used in applied topical areas such as artificial intelligence, information systems , economics, and data analysis, cf

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[14], [15], [16]. The concept of a grill has been introduced by Choquet [4], with the aim of generalizing the concept of a topological space. It generates finer topology which helps in investigating properties that were difficult to be deduced using the original topology. There are convenient relations between the concept of a grill and some well known concepts such as ideals, nets and filters. In [13], Roy and Mukherjee introduced and investigated the notion of a topology τ_g associated to a grill g on a topological space (X, τ) . During, the past ten years, the study of continuity and compactness, nano closed sets and irresolute functions has been generalized. Using the concept of a grill and many interesting constructions, properties and characterization have been deduced, cf [1, 5, 6, 7, 8, 9]. Nassef and Azzam [10, 11] reported a new topological operators via grill. Azzam [2, 3] demonstrated generalized irresolute and quasi-irresolute functions and nano generalized closed sets via grill. Recently, Roy and Mukherjee [13] introduced g -compactness and investigated its relation with compactness. In this paper our purpose is to present some new types of compactness with grill such as αg -compactness, sets αg -compactness and countably αg -compactness. Also, some of its properties and characterizations are obtained.

Definition 1.1 ([4]). A nonempty subcollection g of a space X which carries topology τ is named a grill on this space if the following conditions are true:

- (1) $\emptyset \notin g$,
- (2) $A \in g$ and $A \subseteq B \subseteq X \Rightarrow B \in g$,
- (3) If $A \cup B \in g$ for $A, B \subseteq X$, then $A \in g$ or $B \in g$.

Definition 1.2 ([12]). Let (X, τ) be a topological space and g be a grill on X . We define a mapping $\Phi : P(X) \rightarrow P(X)$, denoted by $\Phi_g(A, \tau)$ (for $A \in P(X)$ which stands the power set of X) or $\Phi_g(A)$ or simply by $\Phi(A)$ (when it is known which topology and grill on X we are talking about), called the operator associated with the grill g and the topology τ , and is defined by $\Phi_g(A) = \{x \in X : A \cap U \in g, \forall U \in \tau(x)\}$, where $\tau(x)$ stands for the collection of all open neighborhoods of x . We also define a mapping $\psi : P(X) \rightarrow P(X)$ by $\psi(A) = A \cup \Phi(A)$, for all $A \in P(X)$, which is a Kuratowski closure operator and hence induces a topology τ_g on X .

Definition 1.3 ([12]). Corresponding to a grill g on a topological space (X, τ) , there exists a unique topology τ_g on X given by $\tau_g = \{U \subseteq X : \psi(X \setminus U) = X \setminus U\}$, where for any $A \subseteq X$, $\psi(A) = A \cup \Phi(A)$.

Theorem 1.4 ([12]). (a) If g_1 and g_2 are two grills on a space (X, τ) with $g_1 \subseteq g_2$, then $\tau_{g_2} \subseteq \tau_{g_1}$.

(b) If g is a grill on a space (X, τ) and $B \notin g$, then B is closed in (X, τ_g) .

(c) For any subset A of a space (X, τ) and any grill g on X , $\Phi(A)$ is τ_g -closed.

Remark 1.5 ([12]). [12] Let (X, τ) be a topological space and g be a grill on X . Then $B(g, \tau) = \{V \setminus A : V \in \tau \text{ and } A \notin g\}$ is obviously an open base for τ_g .

Corollary 1.6 ([12]). *For any grill g on a topological (X, τ) , $\tau \subseteq B(g, \tau) \subseteq \tau_g$.*

Definition 1.7 ([13]). Let g be a grill on a topological space (X, τ) . A cover $\{U_\gamma : \gamma \in \Lambda\}$ of X is said to be a g -cover if there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\gamma \in \Lambda_0} U_\gamma \notin g$. A cover which is not a g -cover of X is named a $\bar{g}$ -cover.

Definition 1.8. A subset A of a space (X, τ) is called α -open if

$$A \subseteq \text{Int}(\text{cl}(\text{Int}(A))).$$

Definition 1.9 ([6]). Let (X, τ) be a topological space. A cover $U = \{U_\gamma : \gamma \in \Lambda\}$ of X is said to be α -open cover if each member of U is an α -open set of X .

2. αg -compact spaces

We give a brief expansion of compactness using the grill concept.

Definition 2.1. Let g be a grill on a topological space (X, τ) . Then (X, τ) is said to be αg -compact space if every α -open cover of X is a g -cover.

Example 2.2. Let $X = [0, \infty[$, $\tau = \{\emptyset, X\} \cup \{]r, \infty[: r \geq 0\}$. (X, τ) is g -compact space. Then it is αg -compact space because if $\{[r_\lambda, \infty[: r_\lambda \geq 0, \lambda \in \Lambda\}$ is an α -open cover of X , then there exist $\lambda_0 \in \Lambda$ such that $r_{\lambda_0} = 0$ and $X \setminus [r_{\lambda_0}, \infty[= \emptyset \notin g$.

Remark 2.3. Any open set in (X, τ) is an α -open. But the converse need not be true.

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$. Clearly $\{a, b\}$ is an α -open set which is not open set.

Remark 2.5. Every α -compact space (X, τ) is clearly αg -compact for any grill g on X .

Proof. Indeed, if $\{U_\beta : \beta \in \Lambda\}$ is any α -open cover of X of an α -compact space (X, τ) , then there exists a finite subcover $\{U_\beta : \beta \in \Lambda_0\}$ of X . Since $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta = \emptyset \notin g$, then (X, τ) is αg -compact. $\square$

Proposition 2.6. *Let $g = P(X) \setminus \emptyset$, then αg -compactness of a space (X, τ) reduces to the compactness and α -compactness of (X, τ) .*

Proof. First, we want to show that every α -compact (X, τ) is compact space. Let $\{U_\beta : \beta \in \Lambda\}$ be any open cover of X . Since every open set is α -open, then $\{U_\beta : \beta \in \Lambda\}$ is α -open cover of X . Since (X, τ) is α -compact space, then there exists a finite subcover $\{U_\beta : \beta \in \Lambda_0\}$ of X and (X, τ) is compact space. Now let (X, τ) be αg -compact, where $g = P(X) \setminus \emptyset$. Let $\{U_\beta : \beta \in \Lambda\}$ be any α -open cover of X . Since (X, τ) is αg -compact space, then there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \notin g$. Since $g = P(X) \setminus \emptyset$, then $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta = \emptyset$. Then $\{U_\beta : \beta \in \Lambda_0\}$ is a finite subcover of X . So (X, τ) is α -compact space and compact space. $\square$

Proposition 2.7. *Let $g = P(X) \setminus \emptyset$ be a grill on a space (X, τ) and the space (X, τ_g) is αg -compact. Then (X, τ) is α -compact and compact (as $\tau \subseteq \tau_g$) and hence is αg -compact.*

Proof. Let (X, τ_g) be αg -compact space, where $g = P(X) \setminus \emptyset$. Now we want to show that (X, τ) is α -compact space. Let $\{U_\beta : \beta \in \Lambda\}$ be any $\tau\alpha$ -open cover of X . Since $\tau \subseteq \tau_g$, then $\{U_\beta : \beta \in \Lambda\}$ be $\tau_g\alpha$ -open cover of X . Since (X, τ_g) be αg -compact space, then there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \notin g$. Since $g = P(X) \setminus \emptyset$, then $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta = \emptyset$. Then $\{U_\beta : \beta \in \Lambda_0\}$ is a finite subcover of X . So (X, τ) is α -compact space and hence compact space. $\square$

Remark 2.8. Every αg -compact space (X, τ) is clearly g -compact for any grill g on X .

Proof. Trivial. $\square$

Theorem 2.9. *Let g be a grill on a topological space (X, τ) . Then (X, τ_g) is αg -compact if (X, τ) is αg -compact (where τ_g is discrete topology on X).*

Proof. Let $\{U_\beta : \beta \in \Lambda\}$ be a basic $\tau_g\alpha$ -open cover of X . Then by definition of a base for τ_g , we get for each $\beta \in \Lambda$, $U_\beta = V_\beta \setminus H_\beta$, where $V_\beta \in \tau$ and $H_\beta \notin g$. Then $\{V_\beta : \beta \in \Lambda\}$ is a τ -open cover of X . since every open set is α -open, then $\{V_\beta : \beta \in \Lambda\}$ is a $\tau\alpha$ -open cover of X . Since (X, τ) is αg -compact, then there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\beta \in \Lambda_0} V_\beta \notin g$. Then $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta = X \setminus \bigcup_{\beta \in \Lambda_0} (V_\beta \setminus H_\beta) \subseteq X \setminus (\bigcup_{\beta \in \Lambda_0} V_\beta \setminus \bigcup_{\beta \in \Lambda_0} H_\beta) \subseteq (X \setminus \bigcup_{\beta \in \Lambda_0} V_\beta) \cup (\bigcup_{\beta \in \Lambda_0} H_\beta) \notin g$. So (X, τ_g) is αg -compact. $\square$

Remark 2.10. Let g be a grill on a topological space (X, τ) . Then (X, τ_g) is g -compact if (X, τ) is αg -compact.

Proof. Trivial. $\square$

3. Grills and compactness of Hausdorff spaces

We will establish the relation between grill and α -quasi H-closed.

Definition 3.1. A topological space (X, τ) is said to be α -quasi H-closed (α -QHC, in short) if for every α -open cover U of X , there is a finite sub-collection U_0 of U such that $X = \bigcup \{Cl(u) : u \in U_0\}$. A Hausdorff α -quasi H-closed space is called an α H-closed space.

Proposition 3.2. *Let g be a grill on a topological space (X, τ) , such that $\tau \setminus \emptyset \subseteq g$. If (X, τ) is αg -compact then (X, τ) is α -QHC*

Proof. Let $\{U_\beta : \beta \in \Lambda\}$ be any α -open cover of X . Since (X, τ) is αg -compact, then there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \notin g$. Then $Int(X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta) = \emptyset$. For otherwise, $Int(X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta) \in \tau \setminus \emptyset \subseteq g$ and hence

$X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \in g$, a contradiction. It follows that, $X \setminus Cl(X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta)^c = \emptyset$. Hence, $X = \bigcup_{\beta \in \Lambda_0} Cl(U_\beta)$ and (X, τ) is α -QHC space. $\square$

Proposition 3.3. *Let (X, τ) be an α -QHC space, where τ is a discrete topology on X . Then (X, τ) is αg_δ -compact, where g_δ is the grill given by $g_\delta = \{A \subseteq X : IntCl(A) \neq \emptyset\}$.*

Proof. Let (X, τ) be an α -QHC space. Let $U = \{U_\beta : \beta \in \Lambda\}$ be any α -open cover of X . Then there is a finite sub-collection $U_0 = \{U_\beta : \beta \in \Lambda_0\}$ of U such that $X = \bigcup \{Cl(U_\beta) : U_\beta \in U_0\}$. Then $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \notin g_\delta$. Otherwise, $X \setminus \bigcup_{\beta \in \Lambda_0} U_\beta \in g_\delta$ and hence $X \setminus ClInt(\bigcup_{\beta \in \Lambda_0} U_\beta) \neq \emptyset$. Thus $X \setminus \bigcup_{\beta \in \Lambda_0} Cl(U_\beta) \neq \emptyset$, a contradiction. Therefore (X, τ) is αg_δ -compact. $\square$

Definition 3.4. Let (X, τ) be a topological space and g be a grill on X . Then the space X is said to be αg -regular if for any α -closed set F in X with $x \notin F$, there exists disjoint α -open sets U and V such that $x \in U$ and $F \setminus V \notin g$.

Proposition 3.5. *Let g be a grill on a Hausdorff space (X, τ) . If (X, τ) is αg -compact then it is αg -regular.*

Proof. Let F be any α -closed subset of X and $x \notin F$. Since X is a Hausdorff space, then for each $y \in F$ there exists two disjoint open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$. Thus, $\{V_y : y \in F\} \cup \{X \setminus F\}$ is an α -open cover of X . Since (X, τ) is αg -compact, then there exists finitely many points $y_1, y_2, y_3, \dots, y_n \in F$ such that $X \setminus [(\bigcup_{i=1}^n V_{y_i}) \cup (X \setminus F)] \notin g$. Let $G = X \setminus \bigcup_{i=1}^n ClV_{y_i}$ and $H = \bigcup_{i=1}^n V_{y_i}$. Then G and H are disjoint non-empty α -open sets in X such that $x \in G$ (Since $x \notin ClV_{y_i}$ for all $i = 1, 2, 3, \dots, n$) and $F \setminus H = F \cap [X \setminus \bigcup_{i=1}^n V_{y_i}] = X \setminus [(\bigcup_{i=1}^n V_{y_i}) \cup (X \setminus F)] \notin g$. So (X, τ) is αg -regular. $\square$

Corollary 3.6. *Let g be a grill on a Hausdorff space (X, τ) such that $\tau \setminus \emptyset \subseteq g$. If (X, τ) is αg -compact then it is αH -closed and αg -regular.*

Proof. Direct consequence of Propositions 3.2. and 3.5. $\square$

Theorem 3.7. *Let U be an α -open subbase for a topological space (X, τ) . If X has an open $\bar{g}$ -cover then there is a $\bar{g}$ -cover of X , which consists of elements of U .*

Proof. Let C be the collection of all open $\bar{g}$ -covers of X . Clear that C is partially ordered set. Then by hypothesis C is non-empty. Let $\{P_\beta\}$ be a linearly ordered (chain) subset of C . Then $\bigcup_\beta P_\beta$ is a covering of X . We claim that it is open $\bar{g}$ -cover of X . For, if not, then there exist $G_1, G_2, G_3, \dots, G_n \in \bigcup_\beta P_\beta$ such that $X \setminus \bigcup_{i=1}^n G_i \notin g$. Now, there exists a $P_{\beta_0} \in C$ such that $G_1, G_2, G_3, \dots, G_n \in P_{\beta_0}$. Thus $P_{\beta_0} \notin C$, a contradiction. Consequently by Zorn's lemma, C contains a maximal element P . Thus if H is an open set and $H \notin P$, then there exist finitely many $G_1, G_2, G_3, \dots, G_n \in P$ such that $X \setminus (H \cup G_1 \cup G_2 \cup \dots \cup G_n) \notin g$. It is clear that the family of open sets which do not belong to P form a filter.

To complete the proof it is sufficient to show that $U \cap P$ is a $\bar{g}$ -cover of X . Let $x \in X$. Since P is an open cover of X , there exists an open set $G \in P$ such that $x \in G$. Since U is α -open subbase for (X, τ) , there exist $H_1, H_2, \dots, H_n \in U$ such that $x \in H_1 \cap H_2 \cap \dots \cap H_n \subseteq G$. It follows that there exists an H_i (for some $i = 1, 2, \dots, n$) such that $H_i \in P$. For otherwise, if $H_i \notin P$ for all $i = 1, 2, \dots, n$, then $\bigcap_{i=1}^n H_i \in \tau \setminus P$ (i.e., $\bigcap_{i=1}^n H_i \notin P$). Thus $G \in \tau \setminus P$ and $G \notin P$, a contradiction. Therefore $x \in H_i \in U \cap P$ and consequently, $U \cap P$ is a $\bar{g}$ -cover of X . $\square$

4. αg -compact sets relative to a space

Definition 4.1. Let g be a grill on a topological space (X, τ) , a subset A of a space (X, τ) is said to be αg -compact relative to X if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by α -open sets of X , there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda \notin g$.

Theorem 4.2. The following are equivalent for a subset A of X :

- 1- A is αg -compact relative to X , where the grill $g = P(X) \setminus \emptyset$;
- 2- A is α -compact relative to X .

Proof. 1. $\implies$ 2. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of A by α -open sets of X . Since A is αg -compact relative to X , then there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda \notin g$. Since $g = P(X) \setminus \emptyset$, then $A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = \emptyset$. Then $A = \bigcup_{\lambda \in \Lambda_0} U_\lambda$, thus A is α -compact relative to X .

2. $\implies$ 1. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of A by α -open sets of X . Since A is α -compact relative to X , then there exists a finite subcover $\{U_\lambda : \lambda \in \Lambda_0\}$ of A (i.e., $A \subseteq \bigcup_{\lambda \in \Lambda_0} U_\lambda$). Then $A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = \emptyset \notin g$, thus A is αg -compact relative to X . $\square$

Proposition 4.3. Let g be a grill on a topological space (X, τ) , if $A_i, i = 1, 2$ are αg -compact subsets relative to a space (X, τ) , then $A_1 \cup A_2$ is αg -compact relative to X .

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of $A_1 \cup A_2$ by α -open sets of X . Then it is an α -open cover of A_i for $i = 1, 2$. Since A_i is αg -compact relative to X , then there exists a finite subset Λ_1 of Λ such that $A_1 \setminus \bigcup_{\lambda \in \Lambda_1} U_\lambda \notin g$ and there exists a finite subset Λ_2 of Λ such that $A_2 \setminus \bigcup_{\lambda \in \Lambda_2} U_\lambda \notin g$. Since $(A_1 \setminus \bigcup_{\lambda \in \Lambda_1} U_\lambda) \cup (A_2 \setminus \bigcup_{\lambda \in \Lambda_2} U_\lambda) \supseteq (A_1 \cup A_2) \setminus \bigcup_{\lambda \in \Lambda_1 \cup \Lambda_2} U_\lambda \notin g$. Then there exists a finite subset $\Lambda_1 \cup \Lambda_2$ of Λ such that $(A_1 \cup A_2) \setminus \bigcup_{\lambda \in \Lambda_1 \cup \Lambda_2} U_\lambda \notin g$, thus $A_1 \cup A_2$ is αg -compact relative to X . $\square$

Theorem 4.4. Let (X, τ) be a space with a grill g on X . A is αg -compact relative to X , then $(A, \tau/A)$ is g/A -compact.

Proof. Let $\{(U_\lambda \cap A) : \lambda \in \Lambda\}$ be τ/A -open cover of A , where $U_\lambda \in \tau$ for each $\lambda \in \Lambda$. Now $\{U_\lambda : \lambda \in \Lambda\}$ is a cover of A by α -open subsets of X . Since A is

αg -compact relative to X , then there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda \notin g$. Thus $A \cap [A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda] \notin A \cap g$. Since $A \setminus \bigcup_{\lambda \in \Lambda_0} (U_\lambda \cap A) = A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = A \cap [A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda] \notin A \cap g$, then there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\lambda \in \Lambda_0} (U_\lambda \cap A) \notin g/A$. Thus $(A, \tau/A)$ is g/A -compact. $\square$

Example 4.5. The converse of Theorem 4.4 is not true. Indeed, let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. The set $U = \{a, b\}$ is an α -open set in τ . Let $A = \{b, c\}$ and $\tau/A = \{A, \emptyset\}$. Obviously, $U \cap A = \{b\}$ not open in τ/A .

5. Countably αg -compact spaces

Definition 5.1. Let g be a grill on a topological space (X, τ) , a space (X, τ) is said to be countably αg -compact if for every countable α -open cover $\{U_n : n \in N\}$ of X there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} U_n \notin g$, where N denotes the set of positive integers.

Proposition 5.2. Let g be a grill on a topological space (X, τ) , if the space (X, τ) is countably αg -compact, then for any countable family $\{f_n : n \in N\}$ of α -closed sets of X such that

$\bigcap \{f_n : n \in N\} = \emptyset$, there exists a finite subset N_0 of N such that $\bigcap \{f_n : n \in N_0\} \notin g$.

Proof. Let $\{f_n : n \in N\}$ be a countable family of α -closed sets of X such that $\bigcap \{f_n : n \in N\} = \emptyset$. Then $\{X \setminus f_n : n \in N\}$ is a countable α -open cover of X . Then, there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} (X \setminus f_n) \notin g$. This leads to $\bigcap_{n \in N_0} [X \setminus (X \setminus f_n)] \notin g$ (i.e., $\bigcap_{n \in N_0} f_n \notin g$). $\square$

Proposition 5.3. If (X, τ) is countably αg -compact, g and $\acute{g}$ are two grills on X such that $g \supseteq \acute{g}$ then (X, τ) is countably $\alpha \acute{g}$ -compact.

Proof. Let $\{U_n : n \in N\}$ be a countable α -open cover of X . Since (X, τ) is a countably αg -compact, then there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} U_n \notin g$. Since $g \supseteq \acute{g}$, thus $X \setminus \bigcup_{n \in N_0} U_n \notin \acute{g}$. Thus (X, τ) is countably $\alpha \acute{g}$ -compact. $\square$

Theorem 5.4. If $g = P(X) \setminus \emptyset$ is the grill on the space (X, τ) , then the following are equivalent

- 1- The space (X, τ) is countably α -compact;
- 2- The space (X, τ) is countably αg -compact.

Proof. 1. $\implies$ 2.

Let $\{U_n : n \in N\}$ be a countable α -open cover of X . By 1., there exists a finite subcover $\{U_n : n \in N_0\}$ of X . Then $X \setminus \bigcup_{n \in N_0} U_n = \emptyset \notin g$. Thus (X, τ) is countably αg -compact.

2. $\implies$ 1.

Let $\{U_n : n \in N\}$ be a countable α -open cover of X . By 2., there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} U_n \notin g$. Since $g = P(X) \setminus \emptyset$, then $X \setminus \bigcup_{n \in N_0} U_n = \emptyset$. This means that $\{U_n : n \in N_0\}$ is a finite subcover of X . Thus (X, τ) is countably α -compact. $\square$

Proposition 5.5. *If (X, τ) is countably α -compact, then (X, τ) is countably αg -compact, where g is a grill on X .*

Proof. It is clear. $\square$

Definition 5.6. A space (X, τ) is called α -Lindelöf if and only if every α -open cover of X has a countable subcover.

Theorem 5.7. *If (X, τ) is a countably αg -compact and α -Lindelöf space, then (X, τ) is αg -compact, where g is a grill on (X, τ) .*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an α -open cover of X . Since (X, τ) is α -Lindelöf, there exists a countable subset Λ_1 of Λ such that $X = \bigcup_{\lambda \in \Lambda_1} U_\lambda$. But (X, τ, g) is countably αg -compact and hence there exists a finite subset Λ_0 of Λ_1 such that $X \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda \notin g$. Thus (X, τ) is αg -compact. $\square$

Theorem 5.8. *Let $f : (X, \tau, g) \rightarrow (Y, \sigma)$ be an α -irresolute surjection. If (X, τ, g) is countably αg -compact, then $(Y, \sigma, f(g))$ is countably $\alpha f(g)$ -compact.*

Proof. Let $\{V_n : n \in N\}$ be a countable α -open cover of Y . Since f is α -irresolute, then $\{f^{-1}(V_n) : n \in N\}$ is a countable α -open cover of X and hence there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} f^{-1}(V_n) \notin g$. Since f is surjective, we have $Y \setminus \bigcup_{n \in N_0} V_n = f[X \setminus \bigcup_{n \in N_0} f^{-1}(V_n)] \notin f(g)$. Thus $(Y, \sigma, f(g))$ is countably $\alpha f(g)$ -compact. $\square$

Theorem 5.9. *Let $f : (X, \tau, g) \rightarrow (Y, \sigma)$ be an α -continuous surjection. If (X, τ, g) is countably αg -compact, then $(Y, \sigma, f(g))$ is countable $f(g)$ -compact.*

Proof. Let $\{V_n : n \in N\}$ be a countable open cover of Y . Then $\{Y \setminus V_n : n \in N\}$ is a countable closed subsets of Y . since f is α -continuous, then $\{f^{-1}(Y \setminus V_n) : n \in N\}$ is a countable α -closed subsets of X . Then $\{X \setminus f^{-1}(Y \setminus V_n) : n \in N\}$ is a countable α -open subsets of X . Since $X \setminus f^{-1}(Y \setminus V_n) = f^{-1}(V_n)$, then $\{f^{-1}(V_n) : n \in N\}$ be a countable α -open cover of X . Since (X, τ, g) is countably αg -compact, there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} f^{-1}(V_n) \notin g$. Thus $f(X \setminus \bigcup_{n \in N_0} f^{-1}(V_n)) \notin f(g)$, this leads to $Y \setminus \bigcup_{n \in N_0} V_n \notin f(g)$ and hence $(Y, \sigma, f(g))$ is $f(g)$ -compact. $\square$

Theorem 5.10. *If $f : (X, \tau) \rightarrow (Y, \sigma, g)$ is pre- α -open bijection and (Y, σ, g) is countably αg -compact, then (X, τ) is countably $\alpha f^{-1}(g)$ -compact.*

Proof. Since f is pre- α -open, then $f(F)$ is α -open in (Y, σ, g) for every α -open set F in (X, τ) . Since f is bijection, then $f^{-1} : (Y, \sigma, g) \rightarrow (X, \tau)$ exists and an α -irresolute surjection. Therefore, the proof follows from the last theorem. $\square$

Theorem 5.11. *If (X, τ, g) is countably αg -compact and g is superset of τ , then (X, τ) is feebly compact.*

Proof. Let $\{U_n : n \in N\}$ be a countable open cover of X , so it is α -open cover of X . Since (X, τ, g) is countably αg -compact, there exists a finite subset N_0 of N such that $X \setminus \bigcup_{n \in N_0} U_n \notin g$. Since $\tau \subseteq g$, then $X \setminus \bigcup_{n \in N_0} U_n \notin \tau$. Thus $\text{Int}(X \setminus \bigcup_{n \in N_0} U_n) = \emptyset$, then $X \setminus \text{Cl}(\bigcup_{n \in N_0} U_n) = \emptyset$. So $X = \text{Cl}\{\bigcup_{n \in N_0} U_n\}$ and hence (X, τ) is feebly compact. $\square$

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Free commutative B -algebras

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Abstract. In this paper, the notion free commutative B -algebras was introduced. In order to characterize free commutative B -algebras and obtain its properties, the concept of external direct product of B -algebras was also introduced.

Keywords: B -algebras, direct product, external direct product, internal direct product, free commutative B -algebras, B -homomorphism, basis and combination.

1. Introduction

In [8], Neggers, J. and Kim, H. S. introduced the notion of B -algebras. A B -algebra is a nonempty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms for all $x, y, z \in X$: (B1) $x * x = 0$, (B2) $x * 0 = x$, (B3) $(x * y) * z = x * (z * (0 * y))$. B -algebra can be derived from a given group and a group can be derived from a B -algebra. From this relationship, parallel concepts and results were formulated and established. A B -algebra $(X; *, 0)$ is said to be *commutative* if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$. They also introduced the concepts of subalgebras and normality in [9]. A nonempty subset A of X is said to be a *subalgebra* of X if $a * b \in A$ for all $a, b \in A$. It is said to be *normal* in X if for any $x, y, a, b \in X$ with $x * y, a * b \in A$ implies $(x * a) * (y * b) \in A$.

In [10], Walendziak, A. characterized normality in B -algebras. The concept of B -homomorphism was introduced by Neggers, J., and Kim, H. S. [9].

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A map $\phi : X \rightarrow Y$, where $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ are B-algebras, is called a *B-homomorphism* if $\phi(x *_X y) = \phi(x) *_Y \phi(y)$ for any $x, y \in X$. If ϕ is onto (respectively, one-to-one), then ϕ is called a *B-epimorphism* (respectively, *B-monomorphism*). Moreover, if ϕ is a bijection, then ϕ is called a *B-isomorphism*. The *kernel* of ϕ , denoted by $\text{Ker } \phi$, is defined to be the set $\{x \in X : \phi(x) = 0\}$. The following properties for a B-algebra are used in this paper : for any $x, y, z \in X$, (P1) $x*y = 0$ implies $x = y$, (P2) $0*(0*x) = x$, (P3) $x*(y*z) = (x*(0*z))*y$, and (P4) $x*y = 0*(y*x)$. If X is commutative, then (P5) $(x*y)*z = (x*z)*y$ and (P6) $x*(x*y) = y$ [8, 11, 6]. In [4], Gonzaga, N. C. and Vilela, J. P. introduced cyclic B-algebras as well as established the laws of exponents for B-algebras. In [1], Endam, J. C. and Teves, R. C. established some properties of cyclic B-algebras. In [7], Lincong, J. A. and Endam, J. C. introduced the concept of direct product of B-algebras and established its structure. From these parallel concepts and results, one may wonder if every concept in groups has its counterpart for B-algebras and if every result is applicable to B-algebras. In this paper, we introduced the concept of free commutative B-algebra. In order to investigate and characterize this type of B-algebra, we also introduced the external direct product of B-algebras. Throughout this paper, X means a B-algebra $(X; *, 0)$.

2. Preliminaries

This section presents some concepts and results needed in this paper. We start with some examples of a B-algebra.

Example 2.1. [[1]] Let $X = \{e, a, b, c\}$ be a set whose binary operation $*$ is shown in the following Cayley Table.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	a	e	a
c	c	b	a	e

By routine calculations, X is a commutative B-algebra. X is called the *Klein B-algebra* K_4 .

Example 2.2. [[7]] Consider the set of integers $\mathbb{Z}$ and the usual subtraction of integers “ $-$ ”. Then $(\mathbb{Z}; -, 0)$ is a B-algebra. We denote $(\mathbb{Z}; -, 0)$ by $*\mathbb{Z}$. Since for every $x, y \in \mathbb{Z}$, $x - (0 - y) = x + y = y + x = y - (0 - x)$, $*\mathbb{Z}$ is commutative.

In [2], the intersection of a family of subalgebras of a B-algebra X is a subalgebra of X .

Definition 2.3 ([4]). Let $A \subseteq X$ be nonempty and $\{N_\alpha : \alpha \in I\}$ be a collection of subalgebras of X with $A \subseteq N_\alpha$ for all $\alpha \in I$. Then $\bigcap_{\alpha \in I} N_\alpha$ is called the subalgebra X generated by A , denoted by $\langle A \rangle_B$. If A is finite, then we say

that $\langle A \rangle_B$ is finitely generated. If $A = \{a\}$, then $\langle A \rangle_B = \langle a \rangle_B$ is called the cyclic subalgebra of X generated by a . If $X = \langle S \rangle_B$, then S is called a set of generators for X .

Remark 2.4 ([4]). Let S be a subset of X . Then $\langle S \rangle_B$ is the smallest subalgebra of X containing S . If either $S = \emptyset$ or $S = \{0\}$, then $\langle S \rangle_B = \{0\}$. If S is a subalgebra of X , then $\langle S \rangle_B = S$. In particular, $\langle X \rangle_B = X$.

Let $x \in X$. From [8], Neggers, J. and Kim, H. S. defined $x^n = x^{n-1} * (0 * x)$ ($n \geq 1$) and $x^0 = 0$. Note that $x^1 = x^0 * (0 * x) = 0 * (0 * x) = x$ by (P2). Gonzaga, N. C. and Vilela, J. P. [4] defined $-x = 0 * x$ and $x^{-n} = (-x)^n$ for each $n \geq 1$. In [1], Endam, J. C. and Teves, R. C. defined $x^m = 0 * x^{-m}$ for $m \leq -1$.

By (B3), $(x * y) * z = x * (z * (0 * y))$. For convenience, we directly write $(x * y) * z = x * (z * y^{-1})$ at times. Similarly, by (P3) $x * (y * z) = (x * z^{-1}) * y$.

Definition 2.5 ([3]). The order of $a \in X$, denoted by $|a|_B$, is the smallest positive integer k such that $a^k = 0$.

$a \in X$ is said to be of finite order if $|a|_B < +\infty$; otherwise, it is of infinite order.

Theorem 2.6 ([3]). Let $a \in X$. Then $|a|_B = |\langle a \rangle_B|$. In particular, if $|a|_B = n < +\infty$, then $\langle a \rangle_B = \{0, a, a^2, \dots, a^{n-1}\}$.

Theorem 2.7 ([4]). If $\emptyset \neq A \subseteq X$, then $\langle A \rangle_B$ consists of all finite products $(\dots((a_1^{n_1} * a_2^{n_2}) * a_3^{n_3}) * \dots * a_{t-1}^{n_{t-1}}) * a_t^{n_t}$ where $a_i \in A$ and $n_j \in \mathbb{Z}$ for all $i, j = 1, 2, \dots, t$, with $t < \infty$.

Corollary 2.8 ([4]). Let $x \in X$ and $m, n \in \mathbb{Z}$. Then:

$$(i) \quad x^{-n} = (-x)^n = -(x^n) = 0 * x^n;$$

$$(ii) \quad x^m * x^n = x^{m-n}.$$

Theorem 2.9 ([8]). Let $x \in X$. Then $(x^m)^n = x^{mn}$ for any $m, n \in \mathbb{Z}$.

Corollary 2.10 ([4]). If $a_i, b_j \in X$ and $m_i, n_j \in \mathbb{Z}$ for $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$, $r, s < +\infty$, then

$$\begin{aligned} & [\dots((a_1^{m_1} * a_2^{m_2}) * a_3^{m_3}) * \dots * a_r^{m_r}] * [\dots((b_1^{n_1} * b_2^{n_2}) * b_3^{n_3}) * \dots * b_s^{n_s}] \\ &= (\dots(((\dots((a_1^{m_1} * a_2^{m_2}) * a_3^{m_3}) * \dots * a_r^{m_r}) * b_s^{-n_s}) * b_{s-1}^{-n_{s-1}}) * \dots * b_2^{-n_2}) * b_1^{n_1}. \end{aligned}$$

Definition 2.11 ([2]). Let H and K be subalgebras of X . Define the subset HK of X to be the set $HK = \{x \in X : x = h * (0 * k), h \in H, k \in K\}$.

Theorem 2.12. Let H and K be subalgebras of X . Then

- (i) [1] HK is a subalgebra of X if and only if $HK = \langle H \cup K \rangle_B$;

- (ii) [2] HK is a subalgebra of X if and only if $HK = KH$;
- (iii) [2] $HK = KH$ whenever H and K are normal in X .

Theorem 2.13 ([10]). *A subalgebra N of X is normal if and only if $x*(x*y) \in N$ for any $x \in X$, $y \in N$.*

Remark 2.14 ([5]). Let N_1 and N_2 be normal subalgebras of X such that $N_1 \cap N_2 = \{0\}$. Then $x*(0*y) = y*(0*x)$ for all $x \in N_1$ and $y \in N_2$.

Theorem 2.15 ([3]). *Every infinite cyclic B-algebra is isomorphic to the B-algebra ${}^*\mathbb{Z}$.*

Proposition 2.16 ([9]). *Let $\varphi : X \rightarrow Y$ be a B-homomorphism. Then φ is one-to-one if and only if $\text{Ker } \varphi = \{0_X\}$.*

Definition 2.17 ([5]). *Let $\{X_i : i \in I\}$ be a family of B-algebras. The direct product of this family, denoted by $\prod_{i \in I} X_i$, is the set of all functions $f : I \rightarrow \bigcup_{i \in I} X_i$ given by $f(i) \in X_i$ for each $i \in I$ and whose operation is component-wise.*

Theorem 2.18 ([5]). *If $\{X_i : i \in I\}$ is a family of B-algebras, then*

- (i) $\prod_{i \in I} X_i$ is a B-algebra;
- (ii) X_i is commutative if and only if $\prod_{i \in I} X_i$ is commutative.

Theorem 2.19 ([5]). *Let $\{f_i : X_i \rightarrow Y_i : i \in I\}$ be a family of B-homomorphisms and let $f = \prod f_i$ be the map $\prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$, given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a B-homomorphism such that $f(\prod_{i \in I} X_i) = \prod_{i \in I} Y_i$ and $\text{Ker } f = \prod_{i \in I} \text{Ker } f_i$. Consequently, f is a B-monomorphism [respectively B-epimorphism] if and only if each f_i is.*

Definition 2.20 ([5]). *Let J_1 and J_2 be normal subalgebras of X . Then X is called the internal product of J_1 and J_2 if every $a \in A$ can be uniquely expressed as $a = a_1 * (0 * a_2)$, where $a_1 \in J_1$ and $a_2 \in J_2$.*

Theorem 2.21 ([5]). *Let J_1 and J_2 be normal subalgebras of X . Then X is an internal direct product of J_1 and J_2 if and only if $A = J_1 J_2$ and $J_1 \cap J_2 = \{0\}$.*

3. Some properties of B-algebras

If N is a subalgebra of a commutative B-algebra X , then by (P6), $x*(x*y) = y \in N$ for any $x \in X$, $y \in N$. By Theorem 2.13, it follows that any subalgebra is normal in a commutative B-algebra. The following Lemma follows from (P4).

Lemma 3.1. *For all $x, y \in X$, $(x*y)^{-1} = y*x$.*

Proposition 3.2. *Let $x, y \in X$. Then for all $n, m \in \mathbb{Z}$, $(x*y^n)*y^m = x*y^{n+m}$.*

Proof. Let $x, y \in X$. Then by (B3) and Corollary 2.8(i) and (ii), $(x * y^n) * y^m = x * (y^m * (0 * y^n)) = x * (y^m * y^{-n}) = x * y^{m-(-n)} = x * y^{m+n}$. $\square$

Proposition 3.3. *Let X be a commutative B -algebra, $x_1, x_2, \dots, x_k \in X$ and $n_1, n_2, \dots, n_k \in \mathbb{Z}$. Then*

$$(i) \quad \begin{aligned} & [\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_k^{n_k}] * x_1^m \\ & \quad = (\dots((x_1^{n_1-m} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k}; \end{aligned}$$

$$(ii) \quad \begin{aligned} & [\dots(\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i}) * \dots * x_k^{n_k}] * x_i^m \\ & \quad = (\dots(\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i+m}) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k}. \end{aligned}$$

for all $i = 2, \dots, k$.

Proof. (i) By (P5) and Corollary 2.8(ii),

$$\begin{aligned} & [\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_k^{n_k}] * x_1^m \\ & \quad = [\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_1^m] * x_k^{n_k} \\ & \quad \vdots \\ & \quad = (\dots((x_1^{n_1} * x_1^m) * x_2^{n_2}) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k} \\ & \quad = (\dots((x_1^{n_1-m} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k}. \end{aligned}$$

(ii) If $k = 2$ (implying that $i = 2$), then the conclusion follows from Proposition 3.2. We now consider $k > 2$. By Proposition 3.2 and (P5),

$$\begin{aligned} & [\dots(\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i}) * \dots * x_k^{n_k}] * x_i^m \\ & \quad = [\dots(\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i}) * \dots * x_i^m] * x_k^{n_k} \\ & \quad \vdots \\ & \quad = (\dots((\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i}) * x_i^m) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k} \\ & \quad = (\dots(\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_i^{n_i+m}) * \dots * x_{k-1}^{n_{k-1}}) * x_k^{n_k}. \end{aligned}$$

This completes the proof. $\square$

Proposition 3.4. *Let X be a commutative B -algebra and $x_1, x_2, \dots, x_k \in X$. Then for all $n_i, m_i \in \mathbb{Z}$,*

$$\begin{aligned} & [\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_k^{n_k}] * [\dots((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * \dots * x_k^{m_k}] \\ & \quad = (\dots((x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}) * \dots * x_{k-1}^{n_{k-1}-m_{k-1}}) * x_k^{n_k-m_k}. \end{aligned}$$

Proof. Let X be a commutative B -algebra, $x_1, x_2, \dots, x_k \in X$ and $n_i, m_i \in \mathbb{Z}$. By Proposition 3.3 and Corollary 2.10,

$$\begin{aligned} & [\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_k^{n_k}] * [\dots((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * \dots * x_k^{m_k}] \\ & \quad = (\dots(((\dots((x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}) * \dots * x_k^{n_k}) * x_k^{-m_k}) * x_{k-1}^{-m_{k-1}}) * x_{k-2}^{-m_{k-2}}) * \\ & \quad \quad \dots * x_2^{-m_2}) * x_1^{m_1} \\ & \quad = (\dots((x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}) * \dots * x_{k-1}^{n_{k-1}-m_{k-1}}) * x_k^{n_k-m_k}. \end{aligned}$$

□

4. Direct product and free commutative B-algebra

Proposition 4.1. *Let $\{X_i : i \in I\}$ be a family of B-algebras and $\{u_i\} \in \prod_{i \in I} X_i$. Then $\{u_i\}^n = \{u_i^n\}$ for all $n \in \mathbb{Z}$.*

Proof. Let $\{X_i : i \in I\}$ be a family of B-algebras. By Theorem 2.18(i), $(\prod_{i \in I} X_i; \otimes, \{0\})$ is a B-algebra. Let $\{u_i\} \in \prod_{i \in I} X_i$ and $n \in \mathbb{Z}$.

Case 1. $n = 0$. Then $\{u_i\}^0 = \{0\} = \{0^n\}$.

Case 2. $n > 0$. Now,

$$\begin{aligned}
 \{u_i\}^n &= \{u_i\}^{n-1} \otimes (\{0\} \otimes \{u_i\}) \\
 &= [\{u_i\}^{n-2} \otimes (\{0\} \otimes \{u_i\})] \otimes \{0 * u_i\} \\
 &= (\{u_i\}^{n-2} \otimes \{0 * u_i\}) \otimes \{0 * u_i\} \\
 &\vdots \\
 &= \underbrace{[\dots [(\{u_i\} \otimes \{0 * u_i\}) \otimes \{0 * u_i\}] \otimes \dots \otimes \{0 * u_i\}] \otimes \{0 * u_i\}}_{n \text{ terms}} \\
 &= [\dots [(\{u_i * (0 * u_i)\}) \otimes \{0 * u_i\}] \otimes \dots \otimes \{0 * u_i\}] \otimes \{0 * u_i\} \\
 &= [\dots [(\{u_i^2\}) \otimes \{0 * u_i\}] \otimes \dots \otimes \{0 * u_i\}] \otimes \{0 * u_i\} \\
 &\vdots \\
 &= \{u_i^{n-1}\} \otimes (\{0 * u_i\}) \\
 &= \{u_i^{n-1} * (0 * u_i)\} \\
 &= \{u_i^n\}.
 \end{aligned}$$

Case 3. $n < 0$. Then $-n > 0$. By Case 2 and Corollary 2.8(i), $\{u_i\}^n = \{0\} \otimes \{u_i\}^{-n} = \{0\} \otimes \{u_i^{-n}\} = \{0 * u_i^{-n}\} = \{u_i^n\}$. □

Definition 4.2. *The external direct product of a family of B-algebras $\{(X_i; *_i, 0_i) : i \in I\}$, denoted by $\prod_{i \in I}^w X_i$, is the set of all $f \in \prod_{i \in I} X_i$ such that $f(i) = 0_i$ for all but a finite number of $i \in I$. If all of the B-algebras X_i are commutative, $\prod_{i \in I}^w X_i$ is usually called the external direct sum and is denoted by $\sum_{i \in I} X_i$.*

If I is finite, then the external direct product coincides with direct product. The following remark follows from Definition 4.2 and definition of subalgebra.

Remark 4.3. If $\{X_i : i \in I\}$ is a family of B-algebras, then $\prod_{i \in I}^w X_i$ is a subalgebra of $\prod_{i \in I} X_i$.

Consider the B-algebra ${}^*\mathbb{Z} = (\mathbb{Z}; -, 0)$. For clarity, we emphasize that ${}^*\mathbb{Z}$ denotes the B-algebra and $\mathbb{Z}$ the set of integers. To avoid confusion, we also emphasize that $\prod^w {}^*\mathbb{Z}$ (respectively $\sum {}^*\mathbb{Z}$) would denote the B-algebra and $\prod^w \mathbb{Z}$ (respectively $\sum \mathbb{Z}$) the set of elements. The following results are proved similarly as in Theorems 2.18(ii) and 2.19.

Remark 4.4. Let $\{(A_i; *, 0) : i \in I\}$ be a family of B-algebras. Then A_i is commutative if and only if $\prod_{i \in I} {}^w A_i$ is commutative.

Remark 4.5. Let $\{f_i : X_i \rightarrow Y_i : i \in I\}$ be a family of B-homomorphisms and let $f = \prod f_i$ be the map $\prod_{i \in I} {}^w X_i \rightarrow \prod_{i \in I} {}^w Y_i$, given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a B-homomorphism such that $f(\prod_{i \in I} {}^w X_i) = \prod_{i \in I} {}^w Y_i$ and $\text{Ker } f = \prod_{i \in I} {}^w \text{Ker } f_i$. Consequently, f is a B-monomorphism [respectively B-epimorphism] if and only if each f_i is.

The following remark follows from Theorems 2.12 and 2.21.

Remark 4.6. Let H and K be normal subalgebras of X . Then X is the internal direct product of H and K if and only if $X = \langle H \cup K \rangle_B$ and $H \cap K = \{0\}$.

Theorem 4.7. Let $\{N_i : i \in I\}$ be a family of normal subalgebras of X such that $X = \langle \bigcup_{i \in I} N_i \rangle_B$ and $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle_B = \{0\}$ for each $k \in I$. Then $X \cong \prod_{i \in I} {}^w N_i$.

Proof. Let $x \in N_{i_s}$ and $y \in N_{i_t}$ with $s \neq t$. By Remark 2.14, $x * y^{-1} = x * (0 * y) = y * (0 * x) = y * x^{-1}$. By (B3), Remark 2.14 and (P4) we also have $x^{-1} * y = (0 * x) * y = 0 * (y * (0 * x)) = 0 * (x * (0 * y)) = (0 * y) * x = y^{-1} * x$.

Let $\{u_i\} \in \prod_{i \in I} {}^w N_i$. Then there exists $I' = \{i_1, i_2, \dots, i_n\} \subseteq I$ such that $u_i = 0$ for all $i \in I \setminus I'$ and $u_{i_1}, u_{i_2}, \dots, u_{i_n} \neq 0$. For each $j = 1, 2, \dots, n$, there is a normal subalgebra N_{i_j} such that $u_{i_j} \in N_{i_j}$. By hypothesis, it follows that $u_{i_j} \notin N_{i_k}$ for all $j \neq k$. Define $f : \prod_{i \in I} {}^w N_i \rightarrow X$ by $f(\{u_i\}) = (\dots((u_{i_1} * u_{i_2}^{-1})u_{i_3}^{-1}) * \dots * u_{i_{n-1}}^{-1}) * u_{i_n}^{-1}$. By (B3), (P3), and Corollary 2.8(i), notice that for $a \in N_r, b \in N_s$ and $c \in N_t$, where $r, s, t \in I$ are distinct, we have $(a * b^{-1}) * c^{-1} = (b * a^{-1}) * c^{-1} = b * (c^{-1} * (0 * a^{-1})) = b * (c^{-1} * a) = b * (a^{-1} * c) = (b * c^{-1}) * a^{-1}$. Similarly, $(a * c^{-1}) * b^{-1} = (c * b^{-1}) * a^{-1} = (c * a^{-1}) * b^{-1}$. This could be extended to any finite number of elements. This implies that the arrangement of the elements does not matter. It follows that f is well-defined.

Let $\{a_i\}, \{b_i\} \in \prod_{i \in I} {}^w N_i$. Then there exist $I', I'' \subseteq I$ with $|I'|, |I''| < \infty$ such that $a_i, b_j = 0$ for all $i \in I \setminus I'$ and $j \in I \setminus I''$. Since $|I'|, |I''| < \infty$, $|I' \cup I''| = n < \infty$. Note that the operation $*$ is the same in every N_i . Thus,

$$\begin{aligned} f(\{a_i\} \otimes \{b_i\}) &= f(\{a_i * b_i\}) \\ &= [\dots[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1}] * \\ &\quad \dots * (a_{i_{n-1}} * b_{i_{n-1}})^{-1}] * (a_{i_n} * b_{i_n})^{-1}, \end{aligned}$$

where $i_j \in I' \cup I''$ for all $j = 1, 2, \dots, n$. Consider $(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}$. By Lemma 3.1, (B3) and (P3),

$$\begin{aligned} (a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1} &= (a_{i_1} * b_{i_1}) * (b_{i_2} * a_{i_2}) \\ &= a_{i_1} * ((b_{i_2} * a_{i_2}) * b_{i_1}^{-1}) \end{aligned}$$

$$\begin{aligned}
&= a_{i_1} * (b_{i_1} * (b_{i_2} * a_{i_2})^{-1}) \\
&= a_{i_1} * (b_{i_1} * (a_{i_2} * b_{i_2})) \\
&= a_{i_1} * ((b_{i_1} * b_{i_2}^{-1}) * a_{i_2}) \\
&= (a_{i_1} * a_{i_2}^{-1}) * (b_{i_1} * b_{i_2}^{-1}).
\end{aligned}$$

Similarly, $[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1} = ((a_{i_1} * a_{i_2}^{-1}) * a_{i_3}^{-1}) * ((b_{i_1} * b_{i_2}^{-1}) * b_{i_3}^{-1})$. Continuing in this manner, we obtain $[...[(a_{i_1} * b_{i_1}) * (a_{i_2} * b_{i_2})^{-1}] * (a_{i_3} * b_{i_3})^{-1}] * ... * (a_{i_{n-1}} * b_{i_{n-1}}) * (a_{i_n} * b_{i_n})^{-1} = [...((a_{i_1} * a_{i_2}^{-1}) * a_{i_3}^{-1}) * ... * a_{i_n}^{-1}] * [...((b_{i_1} * b_{i_2}^{-1}) * b_{i_3}^{-1}) * ... * b_{i_n}^{-1}] = f(\{a_i\}) * f(\{b_i\})$.

Hence, $f(\{a_i\} \otimes \{b_i\}) = f(\{a_i\}) * f(\{b_i\})$. Accordingly, f is a B -homomorphism.

Let $y \in X = \langle \bigcup_{i \in I} N_i \rangle_B$. Then $y = (...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_{n-1}^{m_{n-1}}) * x_n^{m_n}$ for some $x_j \in \bigcup_{i \in I} N_i$, $n_j \in \mathbb{Z}$. For each $j = 1, 2, \dots, n$, there exists N_{i_j} such that $x_j \in N_{i_j}$. Since N_{i_j} is a subalgebra, $a_j = x_j^{m_j} \in N_{i_j}$ for each j . Hence, $a_j^{-1} = x_j^{-m_j} \in N_{i_j}$ for each j . By hypothesis, it follows that $a_j \notin N_{i_k}$ for all $j \neq k$. Let $\{u_i\} \in \prod_{i \in I} N_i$ be such that $u_i = 0$ for all $i \neq i_j$, $j = 1, 2, \dots, n$, $u_{i_1} = a_1$ and $u_{i_k} = a_k^{-1}$, $k = 2, \dots, n$. Since $m < \infty$, $\{u_i\} \in \prod_{i \in I} {}^w N_i$ and

$$\begin{aligned}
f(\{u_i\}) &= (...((a_1 * (a_2^{-1})^{-1}) * (a_3^{-1})^{-1}) * ... * (a_{n-1}^{-1})^{-1}) * (a_n^{-1})^{-1} \\
&= (...((a_1 * a_2) * a_3) * ... * a_{n-1}) * a_n \\
&= (...((x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}) * ... * x_{n-1}^{m_{n-1}}) * x_n^{m_n} \\
&= y.
\end{aligned}$$

Hence, f is onto.

Let $\{u_i\} \in \prod_{i \in I} {}^w N_i$. Then there exists $I' = \{i_1, i_2, \dots, i_n\} \subseteq I$ such that $u_i = 0$ for all $i \in I \setminus I'$ and $u_{i_1}, u_{i_2}, \dots, u_{i_n} \neq 0$. Suppose $\{u_i\} \in \text{Ker } f$. Then $f(\{u_i\}) = 0$. Thus, $(...((u_{i_1} * u_{i_2}^{-1})u_{i_3}^{-1}) * ... * u_{i_{n-1}}^{-1}) * u_{i_n}^{-1} = 0$. By (P1), it follows that $(...((u_{i_1} * u_{i_2}^{-1})u_{i_3}^{-1}) * ... * u_{i_{n-2}}^{-1}) * u_{i_{n-1}}^{-1} = u_{i_n}^{-1}$. Since $u_{i_n} \in N_{i_n}$, $u_{i_n}^{-1} = 0 * u_{i_n} \in N_{i_n}$. Hence, $u_{i_n}^{-1} \in N_{i_n} \cap \langle \bigcup_{i \neq i_n} N_i \rangle_B$. By hypothesis, it follows that $u_{i_n}^{-1} = 0$. Accordingly, $u_{i_n} = (u_{i_n}^{-1})^{-1} = 0^{-1} = 0$. Repeating the process, we have $u_{i_j} = 0$ for all j . Thus, $u_i = 0$ for all $i \in I$. Hence, $\{u_i\} = \{0\}$. By Proposition 2.16, f is one-to-one. Therefore, f is a B -isomorphism. $\square$

In view of Remark 4.6 and Theorem 4.7, we shall extend the notion of internal direct product to an arbitrary family of normal subalgebras.

Definition 4.8. *The B -algebra X satisfying the conditions of Theorem 4.7 is called the internal direct product of the family $\{N_i : i \in I\}$ (or the internal direct sum if X is commutative).*

Example 4.9. *Consider the Klein B -algebra K_4 described in Example 2.1 and its subalgebras $\langle a \rangle_B = \{e, a\}$ and $\langle b \rangle_B = \{e, b\}$. Since K_4 is commutative, $\langle a \rangle_B$ and $\langle b \rangle_B$ are normal in K_4 . Notice that $\langle a \rangle_B \cap \langle b \rangle_B = \{e\}$. Since $e, a, b, c =$*

$a * b \in \langle \langle a \rangle_B \cup \langle b \rangle_B \rangle_B$, $K_4 = \langle \langle a \rangle_B \cup \langle b \rangle_B \rangle_B$. Therefore, K_4 is the internal direct product of $\langle a \rangle_B$ and $\langle b \rangle_B$.

Example 4.10. Consider the B -algebra ${}^*\mathbb{Z} = (\mathbb{Z}; -, 0)$ and $X = \sum {}^*\mathbb{Z}$ (external direct sum) where the copies of ${}^*\mathbb{Z}$ is indexed by an arbitrary set I . For each $i \in I$, let $N_i = \{\{u_j\} \in \sum \mathbb{Z} : u_i \in \mathbb{Z}, u_j = 0 \text{ for all } j \neq i\}$. Clearly, N_i is a subalgebra of X for all $i \in I$. Since ${}^*\mathbb{Z}$ is commutative, X is also commutative, by Remark 4.4. Hence, N_i is normal in X for all i .

Let $\{u_i\} \in \sum \mathbb{Z}$. Then $u_i \in \mathbb{Z}$ and $u_i = 0$ for all but a finite number of $i \in I$. Thus, there exists $t \in \mathbb{Z}^+$ and $k_1, k_2, \dots, k_t \in I$ such that $u_{k_j} \neq 0$ and $u_i = 0$ for all $i \neq k_j$ for all $j = 1, 2, \dots, t$.

Consider $k_1 \in I$. Then for each $i \in I$, we can write $u_i = w'_{1,i} - w_{1,i}$ where $w'_{1,i} = u_i$ and $w_{1,i} = 0$ for all $i \neq k_1$, $w'_{1,k_1} = 0$ and $w_{1,k_1} = -u_{k_1}$. Thus, $\{u_i\} = \{w'_{1,i} - w_{1,i}\} = \{w'_{1,i}\} * \{w_{1,i}\}$. Consider $k_2 \in I$. Then for each $i \in I$, we can write $w'_{1,i} = w'_{2,i} - w_{2,i}$ where $w'_{2,i} = w'_{1,i} = u_i$ and $w_{2,i} = 0$ for all $i \neq k_2$, $w'_{2,k_2} = 0$ and $w_{2,k_2} = -w'_{1,k_2} = -u_{k_2}$. Thus,

$$\{u_i\} = \{w'_{1,i}\} * \{w_{1,i}\} = \{w'_{2,i} - w_{2,i}\} * \{w_{1,i}\} = (\{w'_{2,i}\} * \{w_{2,i}\}) * \{w_{1,i}\}.$$

Continuing this process, we have

$$\begin{aligned} \{u_i\} &= (\{w'_{2,i}\} * \{w_{2,i}\}) * \{w_{1,i}\} \\ &= ((\{w'_{3,i}\} * \{w_{3,i}\}) * \{w_{2,i}\}) * \{w_{1,i}\} \\ &\vdots \\ &= [\dots((\{w'_{t-1,i}\} * \{w_{t-1,i}\}) * \{w_{t-2,i}\}) * \dots * \{w_{2,i}\}] * \{w_{1,i}\} \end{aligned}$$

where $w_{j,i} = 0$ for all $i \neq k_j$ and $w_{j,k_j} = -u_{k_j}$ for all $j = 1, 2, \dots, t-1$. Since there are t k_j 's, it follows that $w'_{t-1,i} = 0$ for all $i \neq k_t$ and $w'_{t-1,t} = u_{k_t}$. Let $w'_{t-1,i} = w_{t,i}$. Thus, $\{u_i\} = [\dots((\{w_{t,i}\} * \{w_{t-1,i}\}) * \{w_{t-2,i}\}) * \dots * \{w_{2,i}\}] * \{w_{1,i}\}$ where $w_{j,i} = 0$ for all $i \neq k_j$, $w_{j,k_j} = -u_{k_j}$, $j = 1, 2, \dots, t-1$, and $w_{t,i} = 0$ for all $i \neq k_t$ and $w_{t,k_t} = u_{k_t}$. Hence, $\{w_{j,i}\} \in N_{j,k_j}$ for $j = 1, 2, \dots, t$. Accordingly, $\{u_i\} \in \langle \bigcup_{i \in I} N_i \rangle_B$. Thus, $\sum \mathbb{Z} \subseteq \langle \bigcup_{i \in I} N_i \rangle_B$. Hence, $\sum \mathbb{Z} = \langle \bigcup_{i \in I} N_i \rangle_B$.

Suppose there exists $\{u_i\} \in N_k \cap \langle \bigcup_{q \neq k} N_q \rangle_B$ such that $\{u_i\} \neq \{0\}$. Then $\{u_i\} \in N_k$, $\{u_i\} \in \langle \bigcup_{q \neq k} N_q \rangle_B$ and there exists $r \in I$ such that $u_r \neq 0$. Thus, $u_i = 0$ for all $i \neq k$ and $u_k \in \mathbb{Z}$, $u_k \neq 0$. Also,

$$\{u_i\} = (\dots((\{w_{1,i}\}^{n_1} \otimes \{w_{2,i}\}^{n_2}) \otimes \{w_{3,i}\}^{n_3}) \otimes \dots \otimes \{w_{t-1,i}\}^{n_{t-1}}) \otimes \{w_{t,i}\}^{n_t}$$

for some $\{w_{s,i}\} \in \bigcup_{q \neq k} N_q$, $s = 1, 2, \dots, t$. Hence, for each s , there exists N_{q_s} such that $\{w_{s,i}\} \in N_{q_s}$, $q_s \neq k$. It follows that $w_{s,q_k} = 0$. By Proposition 4.1, $\{w_{s,i}\}^{n_s} = \{w_{s,i}^{n_s}\}$. Thus, $w_{s,q_k}^{n_s} = 0^{n_s} = 0$ for all $s = 1, 2, \dots, t$.

Consider $\{w_{1,i}\}^{n_1} \otimes \{w_{2,i}\}^{n_2}$. We have $w_{1,q_k}^{n_1} = 0$ and $w_{2,q_k}^{n_2} = 0$. It follows that by Proposition 4.1 and definition of $\otimes$, $\{w_{1,i}\}^{n_1} \otimes \{w_{2,i}\}^{n_2} = \{w_{1,i}^{n_1}\} \otimes$

$\{w_{2,i}^{n_2}\}$ where $w_{1,q_k}^{n_1} * w_{2,q_k}^{n_2} = 0 * 0 = 0$, by (B1). Continuing in this manner, it follows that

$$(\dots((w_{1,q_k}^{n_1} * w_{2,q_k}^{n_2}) * w_{3,q_k}^{n_3}) * \dots * w_{t-1,q_k}^{n_{t-1}}) * w_{t,q_k}^{n_t} = 0$$

But $u_k \neq 0$ and $u_k = (\dots((w_{1,q_k}^{n_1} * w_{2,q_k}^{n_2}) * w_{3,q_k}^{n_3}) * \dots * w_{t-1,q_k}^{n_{t-1}}) * w_{t,q_k}^{n_t}$, a contradiction. Hence, $\{u_i\} = \{0\}$. Therefore, X is the internal direct sum of the family $N_i = \{\{u_j\} \in \sum \mathbb{Z} : u_i \in \mathbb{Z}, u_j = 0 \text{ for all } j \neq i\}$.

Definition 4.11. Let X be a B-algebra and $S \subseteq X$. A combination of elements of S is a finite product $[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k}$ where $x_i \in S$ and $n_i \in \mathbb{Z}$ for all $i = 1, 2, \dots, k$, with $k < \infty$.

Definition 4.12. Let $(X; *, 0)$ be a B-algebra. A subset $S \subseteq X$ is said to be a basis for X if $X = \langle S \rangle_B$ and for distinct $x_1, x_2, \dots, x_k \in S$ and $n_i \in \mathbb{Z}$, $[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k} = 0$ implies $n_i = 0$ for all $i = 1, 2, \dots, k$.

Definition 4.13. Let X be a commutative B-algebra. X is said to be a free commutative B-algebra if it has a nonempty basis.

Example 4.14. Consider the B-algebra ${}^*\mathbb{Z} = (\mathbb{Z}, -, 0)$ and $\{1\} \subseteq \mathbb{Z}$. Note that ${}^*\mathbb{Z} = \langle 1 \rangle_B$. Also, for all $n \in \mathbb{Z}$, $1^n = 0$ if and only if $n = 0$. Thus, $\{1\}$ is a basis for $\mathbb{Z}$. Therefore, ${}^*\mathbb{Z} = (\mathbb{Z}; -, 0)$ is a free commutative B-algebra.

Example 4.15. By Remark 2.4, $\langle \emptyset \rangle_B = \{0\}$. Thus, $\{0\}$ is not a free commutative B-algebra.

Lemma 4.16. Let $(F; *, 0)$ be a free commutative B-algebra with basis S . Then $0 \notin S$.

Proof. Suppose $0 \in S$. Note that $0^1 = 0$. Since $1 \neq 0$, we get a contradiction to the definition of a basis. Hence, $0 \notin S$. $\square$

Theorem 4.17. Let $(F; *, 0)$ be a free commutative B-algebra with basis S . Then the representation of every element of F as a combination of elements of S is unique.

Proof. Let $(F; *, 0)$ be a free commutative B-algebra with basis S and $u \in F$, $u \neq 0$. Then $u = [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k}$ for some $x_i \in S$, $n_i \in \mathbb{Z}$ and $x_i \neq 0$ for all i , by Lemma 4.16. Suppose $x_i = x_j$ for some $i \neq j$. Without loss of generality, we assume that $i < j$. By Proposition 3.3, we may assume that $x_i \neq x_j$ for all $i \neq j$. Suppose

$$\begin{aligned} u &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ u &= [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_{t-1}^{m_{t-1}}] * y_t^{m_t} \end{aligned}$$

for some $x_i, y_j \in S$, $n_i, m_j \in \mathbb{Z}$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, t$. Again, we may assume that $x_p \neq x_q$ and $y_p \neq y_q$ for all $p \neq q$. If $k > t$, then we may write

$$\begin{aligned} u &= [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_{t-1}^{m_{t-1}}] * y_t^{m_t} \\ &= [\underbrace{\dots[[\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_t^{m_t}] * 0] * 0] * \dots * 0]}_{k \text{ terms}}] * 0 \\ &= [\dots[[\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_t^{m_t}] * y_{t+1}^0] * \dots * y_{k-1}^0] * y_k^0 \end{aligned}$$

where $y_i \in S$ for each $i = t+1, t+2, \dots, k$. Thus, we may assume that $k = t$.

Case 1: $x_i = y_j$ for all $i, j = 1, 2, \dots, k$. By Proposition 3.4,

$$\begin{aligned} 0 &= u * u \\ &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_k^{n_k}] * [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_k^{m_k}] \\ &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_k^{n_k}] * [\dots[(x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * \dots * x_k^{m_k}] \\ &= [\dots[(x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * \dots * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Since S is a basis, it follows that $n_i - m_i = 0$, that is, $n_i = m_i$ for all i .

Case 2. $x_i \neq y_i$ for some $i = 1, 2, \dots, k$. By Corollary 2.10, Proposition 3.3 and (P5) for $i = 1$,

$$\begin{aligned} 0 &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_k^{n_k}] * [\dots[(y_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * \dots * x_k^{m_k}] \\ &= [\dots[[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_k^{n_k}] * x_k^{-m_k}] * x_{k-1}^{-m_{k-1}}] * \dots * x_2^{-m_2}] * y_1^{m_1} \\ &= [\dots[(x_1^{n_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * \dots * x_k^{n_k-m_k}] * y_1^{m_1} \\ &= [\dots[[\dots[(x_1^{n_1} * y_1^{m_1}) * x_2^{n_2-m_2}] * x_3^{n_3-m_3}] * \dots * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Thus, $n_j - m_j = 0$, that is, $n_j = m_j$ for all $j = 2, 3, \dots, k$ and $n_1, m_1 = 0$. By Corollary 2.10, Proposition 3.3 and (P5) for $i = 2, 3, \dots, k$,

$$\begin{aligned} 0 &= u * u \\ &= [\dots[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_i^{n_i}] * \dots * x_k^{n_k}] * \\ &\quad [\dots[\dots[(x_1^{m_1} * x_2^{m_2}) * x_3^{m_3}] * \dots * y_i^{m_i}] * \dots * x_k^{m_k}] \\ &= [\dots[[\dots[[\dots[[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_i^{n_i}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k}] * \\ &\quad x_k^{-m_k}] * x_{k-1}^{-m_{k-1}}] * \dots] * x_{i+1}^{-m_{i+1}}] * y_i^{-m_i}] * x_{i-1}^{-m_{i-1}}] * \dots * x_2^{-m_2}] * x_1^{m_1} \\ &= [\dots[[\dots[[\dots[(x_1^{n_1-m_1} * x_2^{n_2-m_2}) * x_3^{n_3-m_3}] * \dots * x_{i-1}^{n_{i-1}-m_{i-1}}] * x_i^{n_i}] * y_i^{-m_i}] * \\ &\quad x_{i+1}^{n_{i+1}-m_{i+1}}] * \dots * x_{k-1}^{n_{k-1}-m_{k-1}}] * x_k^{n_k-m_k}. \end{aligned}$$

Thus, $n_j - m_j = 0$, that is, $n_j = m_j$ for all $j \neq i$ and $n_i, -m_i = 0$, that is, $n_i, m_i = 0$. Hence, $n_i, m_i = 0$, for all $i = 1, 2, \dots, k$. Consequently, $x_i^{n_i} = x_i^0 = 0 = y_i^0 = x_i^{n_i}$. Hence,

$$\begin{aligned} u &= [\dots[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * 0] * x_{i+1}^{n_{i+1}}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ u &= [\dots[\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * 0] * y_{i+1}^{m_{i+1}}] * \dots * y_{k-1}^{m_{k-1}}] * y_k^{m_k} \end{aligned}$$

with $x_j = y_j$ and $n_j = m_j$ for all $j \neq i$. By (B2), 0 plays no role in each equality. Accordingly, we may disregard each i^{th} term. It follows that if

$$\begin{aligned} u &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \quad \text{and} \\ u &= [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots * y_{k-1}^{m_{k-1}}] * y_k^{m_k}, \end{aligned}$$

then $x_j = y_j$ and $n_j = m_j$ for all j . This proves the uniqueness of the representation. $\square$

The following theorem is a characterization of a free commutative B-algebra.

Theorem 4.18. *Let $(F; *, 0)$ be a commutative B-algebra. Then the following are equivalent:*

- (i) F has a nonempty basis;
- (ii) F is the internal direct sum of a family of infinite cyclic subalgebras;
- (iii) F is (isomorphic to) a direct sum of copies of the B-algebra ${}^*\mathbb{Z}$.

Proof. (i) $\Rightarrow$ (ii) Suppose F has a nonempty basis S . Let $x \in S$ and $n \in \mathbb{Z}$. By the definition of a basis, it follows that if $x^n = 0$, then $n = 0$. Thus, $|x|_B = +\infty$. Hence, by Theorem 2.6, it follows that $\langle x \rangle_B$ is an infinite cyclic subalgebra of F . Since F is commutative, $\langle x \rangle_B$ is normal in F . Since for each $x \in S$, $\langle x \rangle_B \subseteq \langle S \rangle_B = F$, it is clear that $F = \left\langle \bigcup_{x \in S} \langle x \rangle_B \right\rangle_B$. Suppose there exists $z \in S$ such that $\langle z \rangle_B \cap \left\langle \bigcup_{x \in S, x \neq z} \langle x \rangle_B \right\rangle_B \neq \langle 0 \rangle_B = \{0\}$. Then there exists $w \in \langle z \rangle_B \cap \left\langle \bigcup_{x \in S, x \neq z} \langle x \rangle_B \right\rangle_B$ such that $w \neq 0$. Thus, $w \in \langle z \rangle_B$ and $w \in \left\langle \bigcup_{x \in S, x \neq z} \langle x \rangle_B \right\rangle_B$, $w \neq 0$. Hence, $w = z^n$ for some $n \in \mathbb{Z}$ and $w = [\dots[(y_1^{n_1} * y_2^{n_2}) * y_3^{n_3}] * \dots * y_{k-1}^{n_{k-1}}] * y_k^{n_k}$ for some distinct $y_1, y_2, \dots, y_k \in \bigcup_{x \in S, x \neq z} \langle x \rangle_B$, $n_i \in \mathbb{Z}$. Note that if $y_i = 0$ for some $i = 2, 3, \dots, k$, then y_i plays no role in the combination. If $y_1 = 0$, then $y_1^{n_1} * y_2^{n_2} = 0 * y_2^{n_2} = y_2^{-n_2}$. In these cases, we just change the variables as well as the indexing. Hence, we may assume that $y_i \neq 0$ for all i . Now, for each i , there exists $x_i \in S \setminus \{z\}$ such that $y_i \in \langle x_i \rangle$. Hence, $y_i = x_i^{m_i}$ for some $m_i \in \mathbb{Z}$. By Theorem 2.9,

$$\begin{aligned} z^n &= w \\ &= [\dots[(y_1^{n_1} * y_2^{n_2}) * y_3^{n_3}] * \dots * y_{k-1}^{n_{k-1}}] * y_k^{n_k} \\ &= [\dots[((x_1^{m_1})^{n_1} * (x_2^{m_2})^{n_2}) * (x_3^{m_3})^{n_3}] * \dots * (x_{k-1}^{m_{k-1}})^{n_{k-1}}] * (x_k^{m_k})^{n_k} \\ &= [\dots[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * \dots * x_{k-1}^{m_{k-1} n_{k-1}}] * x_k^{m_k n_k}, \end{aligned}$$

$x_i \in S \setminus \{z\}$, $m_i n_i \in \mathbb{Z}$. Thus,

$$0 = z^n * z^n = [\dots[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * \dots * x_k^{m_k n_k}] * z^n$$

with $z \neq x_1, x_2, \dots, x_k$. If the x'_i 's are distinct, it follows that $m_i n_i, n = 0$ for all i since S is a basis. If $x_p = x_t$ for some $p \neq t$, then by Proposition 3.3, we have

$$\begin{aligned} 0 &= [\dots[\dots[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * \dots * x_p^{m_p n_p}] * \dots * x_t^{m_t n_t}] * \\ &\quad \dots * x_k^{m_k n_k}] * z^n \\ &= [\dots[\dots[(x_1^{m_1 n_1} * x_2^{m_2 n_2}) * x_3^{m_3 n_3}] * \dots * x_i^{m_p n_p \pm m_t n_t}] * \dots * x_{t-1}^{m_{t-1} n_{t-1}}] * \\ &\quad x_{t+1}^{m_{t+1} n_{t+1}}] * \dots * x_k^{m_k n_k}] * z^n. \end{aligned}$$

Thus, $m_i n_i = 0$ for all $i \neq p, t$ and $n, m_p n_p \pm m_t n_t = 0$. In both cases, $n = 0$. Thus, $w = z^n = z^0 = 0$, a contradiction. Hence, $\langle z \rangle_B \cap \left\langle \bigcup_{x \neq z} \langle x \rangle_B \right\rangle_B = \{0\}$. It follows that $F = \sum_{x \in S} \langle x \rangle_B$.

(ii) $\Rightarrow$ (iii) Let $S \subseteq F$ such that $F = \sum_{x \in S} \langle x \rangle_B$, where $\langle x \rangle_B$ is infinite for each $x \in S$. By Theorem 2.15, it follows that $\langle x \rangle_B \cong {}^*\mathbb{Z}$. Thus, for each $x \in S$, there exists a B-isomorphism $f_x : \langle x \rangle_B \rightarrow {}^*\mathbb{Z}$. Let $f = \prod_{x \in S} f_x$ be the map $\sum_{x \in S} \langle x \rangle_B \rightarrow \sum {}^*\mathbb{Z}$ given by $f(\{u_x\}) = \{f_x(u_x)\}$ and where the copies of ${}^*\mathbb{Z}$ are indexed by S . By Remark 4.5, f is a one-to-one and onto B-homomorphism, since each f_x is. Hence, $F = \sum_{x \in S} \langle x \rangle_B \cong \sum {}^*\mathbb{Z}$.

(iii) $\Rightarrow$ (i) Suppose $F \cong \sum {}^*\mathbb{Z}$ where the copies of ${}^*\mathbb{Z}$ are indexed by a nonempty set S . Then for all $\{u_x\} \in \sum \mathbb{Z}$, $x \in S$, $u_x = 0$ for all but a finite number of $x \in S$. Note that $\sum {}^*\mathbb{Z} = (\sum \mathbb{Z}; *, \{0\})$ is a B-algebra where $\{u_x\} * \{v_x\} = \{u_x - v_x\}$ for all $\{u_x\}, \{v_x\} \in \sum \mathbb{Z}$.

For each $y \in S$, let $\alpha_y = \{u_x\} \in \sum \mathbb{Z}$ such that $u_x = 0$ for all $x \neq y$ and $u_y = 1$. Let $A = \{\alpha_y : y \in S\}$. Since $S \neq \emptyset$, it follows that $\emptyset \neq A \subseteq \sum \mathbb{Z}$. We show that A is a basis for $\sum {}^*\mathbb{Z}$.

Let $a \in \sum \mathbb{Z}$. Then $a = \{u_x\}$ where $x \in S$, $u_x \in \mathbb{Z}$ and $u_x = 0$ for all but a finite $x \in S$. Thus, there exists $k \in \mathbb{Z}^+$ and $y_1, y_2, \dots, y_k \in S$ such that $u_{y_i} \neq 0$ and $u_x = 0$ for all $x \neq y_i$ for all $i = 1, 2, \dots, k$.

Consider $y_1 \in S$. Then for each $x \in S$, we can write $u_x = w'_{1,x} - w_{1,x}$ where $w'_{1,x} = u_x$ and $w_{1,x} = 0$ for all $x \neq y_1$, $w'_{1,y_1} = 0$ and $w_{1,y_1} = -u_{y_1}$. Thus, $a = \{u_x\} = \{w'_{1,x} - w_{1,x}\} = \{w'_{1,x}\} * \{w_{1,x}\}$. Consider $y_2 \in S$. Then for each $x \in S$, we can write $w'_{1,x} = w'_{2,x} - w_{2,x}$ where $w'_{2,x} = w'_{1,x} = u_x$ and $w_{2,x} = 0$ for all $x \neq y_2$, $w'_{2,y_2} = 0$ and $w_{2,y_2} = -w'_{1,y_2} = -u_{y_2}$. Thus,

$$a = \{w'_{1,x}\} * \{w_{1,x}\} = \{w'_{2,x} - w_{2,x}\} * \{w_{1,x}\} = (\{w'_{2,x}\} * \{w_{2,x}\}) * \{w_{1,x}\}.$$

Continuing this process, we have

$$\begin{aligned} a &= (\{w'_{2,x}\} * \{w_{2,x}\}) * \{w_{1,x}\} \\ &= ((\{w'_{3,x}\} * \{w_{3,x}\}) * \{w_{2,x}\}) * \{w_{1,x}\} \\ &\quad \vdots \\ &= [\dots((\{w'_{k-1,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}) * \dots * \{w_{2,x}\}] * \{w_{1,x}\} \end{aligned}$$

where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = -u_{y_i}$ for all $i = 1, 2, \dots, k-1$. Since there are k y'_i 's, it follows that $w'_{k-1,x} = 0$ for all $x \neq y_k$ and $w'_{k-1,k} = u_{y_k}$. Let

$w'_{k-1,x} = w_{k,x}$. Thus, $a = [\dots((\{w_{k,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}) * \dots * \{w_{2,x}\}] * \{w_{1,x}\}$ where $w_{i,x} = 0$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i}$, $i = 1, 2, \dots, k-1$, and $w_{k,x} = 0$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k}$.

Since $1^n = n$ and $0^n = 0$ for all $n \in \mathbb{Z}$, it follows that $w_{i,x} = 0 = 0^{u_{y_i}}$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i} = 1^{-u_{y_i}}$, $i = 1, 2, \dots, k-1$ and $w_{k,x} = 0 = 0^{u_{y_k}}$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k} = 1^{u_{y_k}}$.

For each i , let $\alpha_{y_i} = \{v_{i,x}\}$. Then $v_{i,x} = 0$ for all $x \neq y_i$ and $v_{i,y_i} = 1$, $i = 1, 2, \dots, k$. Thus, $(v_{i,x})^{-u_{y_i}} = 0^{-u_{y_i}} = 0$ for all $x \neq y_i$ and $(v_{i,y_i})^{-u_{y_i}} = 1^{-u_{y_i}} = -u_{y_i}$, $i = 1, 2, \dots, k-1$ and $(v_{k,x})^{u_{y_k}} = 0^{u_{y_k}} = 0$ for all $x \neq y_k$ and $(v_{k,y_k})^{u_{y_k}} = 1^{u_{y_k}} = u_{y_k}$. Hence, $w_{i,x} = (v_{i,x})^{-u_{y_i}}$ for all $i = 1, 2, \dots, k-1$ and $w_{k,x} = (v_{k,x})^{u_{y_k}}$. Thus, by Proposition 4.1, $\{w_{i,x}\} = \{(v_{i,x})^{-u_{y_i}}\} = \{v_{i,x}\}^{-u_{y_i}} = (\alpha_{y_i})^{-u_{y_i}}$ for all $i = 1, 2, \dots, k-1$ and similarly, $\{w_{k,x}\} = (\alpha_{y_k})^{u_{y_k}}$. Thus,

$$\begin{aligned} a &= [\dots((\{w_{k,x}\} * \{w_{k-1,x}\}) * \{w_{k-2,x}\}) * \dots * \{w_{2,x}\}] * \{w_{1,x}\} \\ &= [\dots((\alpha_{y_k})^{u_{y_k}} * (\alpha_{y_{k-1}})^{-u_{y_{k-1}}}) * \dots * (\alpha_{y_2})^{-u_{y_2}}] * (\alpha_{y_1})^{-u_{y_1}} \end{aligned}$$

where $u_{y_i} \in \mathbb{Z}$. By Theorem 2.7, it follows that $a \in \langle A \rangle_B$. Accordingly, $\sum \mathbb{Z} \subseteq \langle A \rangle_B$. Since $A \subseteq \sum \mathbb{Z}$, $\langle A \rangle_B \subseteq \sum \mathbb{Z}$. Thus, $\sum \mathbb{Z} = \langle A \rangle_B$.

Let $\alpha_{y_1}, \alpha_{y_2}, \dots, \alpha_{y_k}$, be distinct elements of A with

$$[\dots[(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}] * (\alpha_{y_3})^{n_3}] * \dots * (\alpha_{y_{k-1}})^{n_{k-1}}] * (\alpha_{y_k})^{n_k} = \{0\},$$

$y_i \in S$, $n_i \in \mathbb{Z}$, $i = 1, 2, \dots, k$. By definition of elements of A , it follows that $y_1, y_2, \dots, y_k$ are also distinct elements of A .

For each i , $\alpha_{y_i} = \{w_{i,x}\}$ where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = 1$. Now, by Proposition 4.1, $(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2} = \{w_{1,x}\}^{n_1} * \{w_{2,x}\}^{n_2} = \{(w_{1,x})^{n_1}\} * \{(w_{2,x})^{n_2}\} = \{(w_{1,x})^{n_1} - (w_{2,x})^{n_2}\}$. For all $x \neq y_1, y_2$, $(w_{1,x})^{n_1} - (w_{2,x})^{n_2} = 0^{n_1} - 0^{n_2} = 0$. For $x = y_1$, $(w_{1,y_1})^{n_1} - (w_{2,y_1})^{n_2} = 1^{n_1} - 0^{n_2} = n_1 - 0 = n_1$. For $x = y_2$, $(w_{1,y_2})^{n_1} - (w_{2,y_2})^{n_2} = 0^{n_1} - 1^{n_2} = 0 - n_2 = -n_2$.

Let $(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2} = \{v_x\}$. Then for all $x \neq y_1, y_2$, $v_x = 0$, $v_{y_1} = n_1$ and $v_{y_2} = -n_2$. By Proposition 4.1, $((\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}) * (\alpha_{y_3})^{n_3} = \{v_x\} * \{w_{3,x}\}^{n_3} = \{v_x\} * \{(w_{3,x})^{n_3}\} = \{v_x - (w_{3,x})^{n_3}\}$. For all $x \neq y_1, y_2, y_3$, $v_x - (w_{3,x})^{n_3} = 0 - 0^{n_3} = 0$. For $x = y_1$, $v_{y_1} - (w_{3,y_1})^{n_3} = n_1 - 0^{n_3} = n_1 - 0 = n_1$. For $x = y_2$, $v_{y_2} - (w_{3,y_2})^{n_3} = -n_2 - 0^{n_3} = -n_2 - 0 = -n_2$. For $x = y_3$, $v_{y_3} - (w_{3,y_3})^{n_3} = 0 - 1^{n_3} = 0 - n_3 = -n_3$.

Continuing this process, we obtain

$$[\dots[(\alpha_{y_1})^{n_1} * (\alpha_{y_2})^{n_2}] * (\alpha_{y_3})^{n_3}] * \dots * (\alpha_{y_{k-1}})^{n_{k-1}}] * (\alpha_{y_k})^{n_k} = \{z_x\},$$

where $z_x = 0$ for all $x \neq y_1, y_2, \dots, y_k$, $z_{y_1} = n_1$ and $z_{y_i} = -n_i$ for all $i = 2, 3, \dots, k$. By hypothesis, $\{z_x\} = \{0\}$. Hence, $0 = z_{y_1} = n_1$, $0 = z_{y_i} = -n_i$ for all $i = 2, 3, \dots, k$. Accordingly, $n_i = 0$ for all $i = 1, 2, \dots, k$.

Therefore, A is a basis for $\sum^* \mathbb{Z}$. Since $F \cong \sum^* \mathbb{Z}$, it follows that F also has a nonempty basis. $\square$

Theorem 4.19. *Let $(F; *, 0)$ be a free commutative B-algebra. Then there exists a nonempty set S and a function $\iota : S \rightarrow F$ with the following property: given a commutative B-algebra Y and a function $f : S \rightarrow Y$, there exists a unique B-homomorphism $g : F \rightarrow Y$ such that $g\iota = f$.*

Proof. Let $S \neq \emptyset$ be basis for F and consider the inclusion map $\iota : S \rightarrow F$ given by $\iota(x) = x$ for all $x \in S$. Let $(Y; *', 0')$ be a commutative B-algebra and $f : S \rightarrow Y$. Let $u \in F$. Then $u = [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k}$ for some $x_i \in S$, $n_i \in \mathbb{Z}$. Define $g : F \rightarrow Y$ by

$$\begin{aligned} g(u) &= g([\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k}) \\ &= [\dots[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' \dots *' f(x_{k-1})^{n_{k-1}}] *' f(x_k)^{n_k} \end{aligned}$$

for all $u = [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k} \in F$.

Let $u, v \in F$. Then

$$\begin{aligned} u &= [\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k} \text{ and} \\ v &= [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots] * y_t^{m_t} \end{aligned}$$

for some $x_i, y_j \in S$, $n_i, m_j \in \mathbb{Z}$. Suppose $u = v$. By Theorem 4.17, $k = t$, $x_i = y_i$, and $n_i = m_i$ for all i . By the well-definedness of f , $f(x_i) = f(y_i)$ for all i . Accordingly,

$$\begin{aligned} g(u) &= [\dots[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' \dots *' f(x_{k-1})^{n_{k-1}}] *' f(x_k)^{n_k} \\ &= [\dots[(f(y_1)^{m_1} *' f(y_2)^{m_2}) *' f(y_3)^{m_3}] *' \dots *' f(y_{k-1})^{n_{k-1}}] *' f(y_k)^{m_k} \\ &= g(v). \end{aligned}$$

Thus, g is well-defined and for all $x \in S$, $g\iota(x) = g(\iota(x)) = g(x) = f(x)$. By Corollary 2.10,

$$\begin{aligned} g(u * v) &= g\left([\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k} * \right. \\ &\quad \left. [\dots[(y_1^{m_1} * y_2^{m_2}) * y_3^{m_3}] * \dots] * y_t^{m_t}\right) \\ &= g\left([\dots[[\dots[(x_1^{n_1} * x_2^{n_2}) * x_3^{n_3}] * \dots] * x_k^{n_k}] * y_s^{-m_t}] * y_{t-1}^{-m_{t-1}}] * \dots * y_2^{-m_2}] * y_1^{m_1}\right) \\ &= [\dots[[\dots[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' \dots *' f(x_k)^{n_k}] *' f(y_t)^{-m_t}] *' f(y_{t-1})^{-m_{t-1}}] *' \dots *' f(y_2)^{-m_2}] *' f(y_1)^{m_1}] \\ &= (\dots[(f(x_1)^{n_1} *' f(x_2)^{n_2}) *' f(x_3)^{n_3}] *' \dots *' f(x_k)^{n_k}) *' \\ &\quad (\dots[(f(y_1)^{m_1} *' f(y_2)^{m_2}) *' f(y_3)^{m_3}] *' \dots *' f(y_t)^{m_t}) \\ &= g(u) *' g(v). \end{aligned}$$

Hence, g is a B-homomorphism.

Suppose there exists $g' : F \rightarrow Y$ such that $g'\iota = f$. Since S generates F , g' is completely determined by its action on S . Let $x \in S$. Then

$$g'(x) = g'(\iota(x)) = g'\iota(x) = f(x) = g\iota(x) = g(\iota(x)) = g(x).$$

Hence, $g = g'$. Accordingly, g is unique. $\square$

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Associativity of max-min composition of three fuzzy relations

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Abstract. In this paper, we introduce some concepts and definitions related to the Max-Min composition of fuzzy relations. We also prove the associativity of Max-Min composition of three fuzzy relations.

Keywords: fuzzy sets, fuzzy relations, composition of fuzzy relations.

1. Introduction

Fuzzy relations play an essential role in fuzzy diagnosis, fuzzy control, and fuzzy modeling [1]. They have applications in fields such as Sociology, Economics, Psychology, and Medicine. They are also an appropriate tool for characterizing correspondences between objects. The utilization of fuzzy relations started from the fact that real-life objects can be related to each other to a certain degree [2]. However, in many cases, we can consider that a certain object X is in relation R with another object Y to a certain degree, but it is possible that we are not so sure about it [3]. In other words, it may be an uncertainty about the degree that is allocated to the relationship between X and Y . In fuzzy set theory, there is no means to incorporate that uncertainty in the membership degrees. Atanassov in 1983 suggested to use intuitionistic fuzzy sets to solve this.

The authors in [4] studied the utilization of fuzzy composition to solve the problem using matrix which is an efficient solution when the particular data base is given. As a case study, they have explained the technique to find the good resistor for long life with respect to resistor size, among the given set of resistors. The definition of rules is based on sampled data for each resistor. The sampled data for each resistor is expressed in sq.mm. The number of rules is two, the first rule is used for the relation between size and power dissipation, and the

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second rule is used for the relationship between power dissipation and lifetime of a particular resistor. The better resistor can be suggested using composition of fuzzy relation in accord with his size for longer life time.

Shakhatreh M.A and Hayajenh M.A. They gave new definition for the fuzzy partition which is called SH fuzzy partition [5] using fuzzy equivalence classes [6, 7], these concepts will help us to use the associativity of Max-Min composition of three fuzzy relations and in general n fuzzy relation to build new concepts related to the title of this paper.

In this paper, we study the associativity of Max-Min composition of three fuzzy relations. To prove the *Associativity of Max-Min composition of three fuzzy relations*, we should know in advance the definitions of the fuzzy set, the equality of two fuzzy sets, the fuzzy relations and the Max-Min composition of two fuzzy relations. At the end of the paper, we present the proof of Associativity of max-min composition of three fuzzy relations.

2. Definitions and preliminaries

First, we present some definitions and preliminaries related to fuzzy sets and compositions of fuzzy relations.

Definition 2.1 (Fuzzy Sets, see [8]). *Let X be a set of objects characterized by a membership (characteristic) function $f_{\mathbf{A}}(x)$ which associates with each object in X a real number in the interval $[0, 1]$, with the value of $f_{\mathbf{A}}(x)$ at x representing the grade of membership of x in $\mathbf{A}$.*

Definition 2.2 (Equality of Two Fuzzy Sets, see [9]). *Let $\mathbf{A}$ and $\mathbf{B}$ be two fuzzy sets in $X \neq \emptyset$. Then $\mathbf{A} = \mathbf{B}$ iff $\mu_{\mathbf{A}}(x) = \mu_{\mathbf{B}}(x) \forall x \in X$.*

Definition 2.3 (Fuzzy Relations, see [10]). *Let X and Y be nonempty sets. Then $\mathbf{R} = \{(x, y), \mu_{\mathbf{R}}(x, y)\} : (x, y) \in X \times Y\}$ is called a fuzzy relation in $X \times Y$.*

Definition 2.4 (Max-Min Composition of Two Fuzzy Relations, see [8]). *Let X, Y and Z be nonempty sets, $\mathbf{R}_1$ be a fuzzy relation in $X \times Y$, and $\mathbf{R}_2$ be a fuzzy relation in $Y \times Z$. Then*

$$\mathbf{R}_2 \circ \mathbf{R}_1 = \left\{ \left((x, z), \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \mu_{\mathbf{R}_2}(y, z) \right\} \right\} \right) : x \in X, y \in Y, z \in Z \right\}$$

is called Max-Min Composition of $\mathbf{R}_1$ and $\mathbf{R}_2$.

3. Associativity of max-min composition of three fuzzy relations

In Theorem 3.1, we prove the associativity of Max-Min composition of three fuzzy relations.

Theorem 3.1 (Associativity of Max-Min Composition of Three Fuzzy Relations). *Let $\mathbf{R}_1$ be a fuzzy relation in $X \times Y$, $\mathbf{R}_2$ be a fuzzy relation in $Y \times Z$, and $\mathbf{R}_3$ be a fuzzy relation in $Z \times W$, then:*

$$\mathbf{R}_3 \circ (\mathbf{R}_2 \circ \mathbf{R}_1) = (\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1.$$

Proof. It is sufficient to prove that:

$$\mu_{\mathbf{R}_3 \circ (\mathbf{R}_2 \circ \mathbf{R}_1)}(x, w) = \mu_{(\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1}(x, w) \quad \forall (x, w) \in X \times W.$$

We have:

$$\begin{aligned} \mu_{\mathbf{R}_2 \circ \mathbf{R}_1}(x, z) &= \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \mu_{\mathbf{R}_2}(y, z) \right\} \right\}, \\ \mu_{\mathbf{R}_3 \circ \mathbf{R}_2}(y, w) &= \max_z \left\{ \min \left\{ \mu_{\mathbf{R}_2}(y, z), \mu_{\mathbf{R}_3}(z, w) \right\} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} \mu_{\mathbf{R}_3 \circ (\mathbf{R}_2 \circ \mathbf{R}_1)}(x, w) &= \max_z \left\{ \min \left\{ \mu_{\mathbf{R}_2 \circ \mathbf{R}_1}(x, z), \mu_{\mathbf{R}_3}(z, w) \right\} \right\} \\ &= \max_z \left\{ \min \left\{ \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \mu_{\mathbf{R}_2}(y, z) \right\} \right\}, \mu_{\mathbf{R}_3}(z, w) \right\} \right\} \\ &= \max_z \left\{ \min \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y_k), \mu_{\mathbf{R}_2}(y_k, z) \right\}, \mu_{\mathbf{R}_3}(z, w) \right\} \right\}, \end{aligned}$$

for some $y_k \in Y$

$$= \min \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y_k), \mu_{\mathbf{R}_2}(y_k, z_m) \right\}, \mu_{\mathbf{R}_3}(z_m, w) \right\},$$

for some $z_m \in Z$.

Now, suppose that:

$$\mu_{\mathbf{R}_1}(x, y_k) = a, \quad \mu_{\mathbf{R}_2}(y_k, z_m) = b, \quad \text{and} \quad \mu_{\mathbf{R}_3}(z_m, w) = c.$$

Then, we have:

$$(1) \quad \mu_{\mathbf{R}_3 \circ (\mathbf{R}_2 \circ \mathbf{R}_1)}(x, w) = \min \{ \min \{ a, b \}, c \} = \min \{ a, b, c \}.$$

Also,

$$\begin{aligned} \mu_{(\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1}(x, w) &= \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \mu_{\mathbf{R}_3 \circ \mathbf{R}_2}(y, w) \right\} \right\} \\ &= \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \max_z \left\{ \min \{ \mu_{\mathbf{R}_2}(y, z), \mu_{\mathbf{R}_3}(z, w) \} \right\} \right\} \right\} \\ &= \max_y \left\{ \min \left\{ \mu_{\mathbf{R}_1}(x, y), \min \left\{ \mu_{\mathbf{R}_2}(y, z_m), \mu_{\mathbf{R}_3}(z_m, w) \right\} \right\} \right\}, \end{aligned}$$

for some $z_m \in Z$ and

$$= \min \left\{ \mu_{\mathbf{R}_1}(x, y_k), \min \left\{ \mu_{\mathbf{R}_2}(y_k, z_m), \mu_{\mathbf{R}_3}(z_m, w) \right\} \right\},$$

for some $y_k \in Y$.

Now, suppose that:

$$\mu_{\mathbf{R}_1}(x, y_k) = a, \quad \mu_{\mathbf{R}_2}(y_k, z_m) = b, \quad \text{and} \quad \mu_{\mathbf{R}_3}(z_m, w) = c.$$

Then, we have

$$\begin{aligned} (2) \quad \mu_{(\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1}(x, w) &= \min \{a, \min \{b, c\}\} \\ &= \min \{a, b, c\} \end{aligned}$$

From (1) and (2), we have

$$\mu_{(\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1}(x, w) = \mu_{(\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1}(x, w) \quad \forall (x, w) \in X \times W,$$

and therefore:

$$\mathbf{R}_3 \circ (\mathbf{R}_2 \circ \mathbf{R}_1) = (\mathbf{R}_3 \circ \mathbf{R}_2) \circ \mathbf{R}_1,$$

which completes the proof. $\square$

Conclusion

In this paper, we study the associativity of Max-Min composition of three fuzzy relations. First, we present some concepts and definitions related to the Max-Min composition of fuzzy relations. The definitions cover fuzzy sets, equality of two fuzzy sets, fuzzy relations, and Max-Min composition of two fuzzy relations. After that, we prove the associativity of Max-Min composition of three fuzzy relations.

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On centralizers and multiplicative generalized derivations of semiprime ring

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Abstract. In this note we gave the description of commutativity of prime and semiprime rings with the help of some identities involving multiplicative generalized derivation and multiplicative left centralizer.

Keywords: semiprime (prime) ring, multiplicative generalized derivation, multiplicative left centralizer.

1. Introduction

Throughout the paper R will be an associative ring. We shall denote by $Z(R)$ the centre of ring R . A ring R is said to be a prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall denote for any pair of elements $x, y \in R$ the commutator $xy - yx$ and anticommutator $x \circ y = xy + yx$. We extensively use the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$, (ii) $[xy, z] = [x, z]y + x[y, z]$ and anticommutator identities $x \circ yz = (x \circ y)z - y[x, z]$, $xy \circ z = (x \circ z)y - x[y, z]$. By a derivation, we mean an additive mapping $d : R \rightarrow R$ satisfying $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such

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that $d_a(x) = [a, x]$, for all $x \in R$. Following Hvala [7] generalized derivation is an additive mapping $F : R \rightarrow R$ satisfying $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ associated with a derivation d . Also, he calls the maps of the form $x \mapsto ax + xb$ where a, b are fixed elements in R by the inner generalized derivations. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map g satisfying $g(xy) = g(x)y$ for all $x, y \in R$).

In [4] Daif introduced the notion of multiplicative derivation as follows: Let $d : R \rightarrow R$ be a map. If $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$; then d is said to be multiplicative derivation. Thus the concept of multiplicative derivation involves the concept of derivation. Daif and El-Sayiad [5] introduced the multiplicative generalized derivation as follows: Let $F : R \rightarrow R$ be an arbitrary map and $d : R \rightarrow R$ be a derivation such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$, F is called a multiplicative generalized derivation associated with d . Hence the idea of multiplicative generalized derivation involves the concept of generalized derivation. A map H is said to be a multiplicative left centralizer if it satisfies $H(xy) = H(x)y$ for all $x, y \in R$, more details of such mappings can be found in [9]. In [6], the authors extend the previous definition by defining a mapping $F : R \rightarrow R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is any mapping (neither a derivation nor an additive map). Moreover, multiplicative (generalized)-derivation with $d = 0$ covers the requirement of multiplicative centralizers (not necessarily additive). Therefore, every generalized derivation is a multiplicative (generalized)-derivation on R . But converse need not be true.

Example 1. Let K be any arbitrary ring and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in K \right\}$,

be a ring. Define the mappings $F, d : R \rightarrow R$ such that $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$

$\begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that F is a multiplicative generalized derivation associated with d but F is not a generalized derivation.

A number of authors in literature has investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations generalized derivations, multiplicative generalized derivations for references [1, 2, 4, 6]. In [1], authors studied the following situations: (i) $F(xy) \in Z(R)$, (ii) $F([x, y]) = 0$, (iii) $(F(xy) \pm yx) \in Z(R)$ and (iv) $(F(xy) \mp [x, y]) \in Z(R)$; for all x, y in some nonzero left ideal of semiprime ring R , where F is a generalized derivation of R . Recently, Dhara et.al. [6] studied the commutativ-

ity in semiprime rings with left ideal and multiplicative (generalized)-derivation. Very recently authors in [3] study the following conditions: Let R be a semiprime ring, $F : R \rightarrow R$ be a multiplicative (generalized)- derivation associated with the map d and the map $H : R \rightarrow R$ be a multiplicative left centralizer such that i) $F(xy) \pm H(xy) = 0$, ii) $F(xy) \pm H(yx) = 0$, iii) $F(x)F(y) \pm H(xy) = 0$, iv) $F(xy) \pm H(xy) \in Z(R)$, v) $F(xy) \pm H(yx) \in Z(R)$, vi) $F(x)F(y) \pm H(xy) \in Z(R)$ for all $x, y \in R$. In this paper we investigate the commutativity of left ideal of semiprime rings by considering some identities involving multiplicative generalized derivation and multiplicative left centralizer.

2. Preliminaries

Firstly we would prefer to fix some results which will be useful in our proofs.

Lemma 2.1 ([8]). *Let R be a semiprime ring, then:*

- (1) *The center of R contains no nonzero nilpotent elements.*
- (2) *R does not contain any nonzero nilpotent left ideal.*

Lemma 2.2. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated map d such that $F(xy) = 0$ for all $x, y \in I$, then either $F(I)I = IF(I) = Id(I) = 0$.*

Proof. Consider $F(xy) = 0$ for all $x, y \in I$. Replace y by yz to get $0 = F(xyz) = F(xy)z + xyd(z) = xyd(z)$ for all $x, y \in I$. This implies that $xyd(z) = 0$ for all $x, y, z \in I$. Replace y by ry to get $xryd(z) = 0$ for all $x, y, z \in I, r \in R$. Therefore we have $IRId(I) = 0$, that is, $Id(I)RId(I) = 0$. Semiprimeness of R yields that $Id(I) = 0$. By a simple calculation we can get $F(I)I = IF(I) = 0$.

Lemma 2.3. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated map d such that $F([x, y]) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. Consider $F([x, y]) = 0$ for all $x, y \in I$. Replace y by yx to get $0 = F([x, y]x) = F([x, y])x + [x, y]d(x) = [x, y]d(x)$ for all $x, y \in I$. This implies that

$$(2.1) \quad [x, y]d(x) = 0 \quad \text{for all } x, y \in I.$$

substitute $d(x)y$ for y in (2.1) to find

$$(2.2) \quad [x, d(x)]yd(x) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yx in (2.2) and multiplying (2.2) by x from right and subtracting the last two resulting equation we arrive at

$$(2.3) \quad [x, d(x)]y[x, d(x)] = 0 \quad \text{for all } x, y \in I.$$

This implies that $(y[d(x), x])^2 = 0$ for all $x, y \in I$. Since R is semiprime, we have $I[x, d(x)] = 0$ for all $x \in I$ by application of Lemma 2.1. $\square$

3. Main results

Theorem 3.1. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated map d and H is multiplicative left centralizer such that $F([x, y]) \pm H(xy) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. If $H = 0$, then we have $F([x, y]) = 0$ for all $x, y \in I$. Hence conclusion follows from Lemma 2.3. Let us consider $H \neq 0$

$$(3.1) \quad F([x, y]) \pm H(xy) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yx we get

$$(3.2) \quad F([x, y])x + [x, y]d(x) + \pm H(xy)x = 0 \quad \text{for all } x, y \in I.$$

In view of (3.1), (3.2) reduces to

$$(3.3) \quad [x, y]d(x) = 0 \quad \text{for all } x, y \in I.$$

substitute $d(x)y$ for y in (3.4) to find

$$(3.4) \quad [x, d(x)]yd(x) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yx in (3.4) and multiplying (3.4) by x from right and subtracting the last two resulting equations we arrive at

$$(3.5) \quad [x, d(x)]y[x, d(x)] = 0 \quad \text{for all } x, y \in I.$$

Last equation yields that $(y[d(x), x])^2 = 0$ for all $x, y \in I$. Since R is semiprime, we have $I[x, d(x)] = 0$ for all $x \in I$ by application of Lemma 2.1. $\square$

Theorem 3.2. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated map d and H is a multiplicative left centralizer such that $F([x, y]) \pm H([x, y]) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. If $H = 0$, then we have $F([x, y]) = 0$ for all $x, y \in I$. Hence conclusion follows from Lemma 2.3. Now suppose that $H \neq 0$. Consider

$$(3.6) \quad F([x, y]) \pm H([x, y]) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yx in (3.6) we have

$$(3.7) \quad F([x, y])x + [x, y]d(x) \pm H([x, y])x = 0 \quad \text{for all } x, y \in I.$$

In view of (3.6), (3.7) yields that

$$(3.8) \quad [x, y]d(x) \quad \text{for all } x, y \in I.$$

Repeating the same arguments as in the above theorem, we get the desired result. $\square$

Theorem 3.3. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated map d and H is a multiplicative left centralizer such that $F(x)F(y) \pm H(xy) \in Z(R)$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. Consider the hypothesis

$$(3.9) \quad F(x)F(y) \pm H(xy) \in Z(R) \quad \text{for all } x, y \in I.$$

Replace y by yz in (3.9) to obtain

$$(3.10) \quad F(x)F(y)z + F(x)yd(z) \pm H(xy)z \in Z(R) \quad \text{for all } x, y, z \in I.$$

Commuting (3.10) with z , we find

$$(3.11) \quad [F(x)yd(z), z] = 0 \quad \text{for all } x, y, z \in I.$$

Substitute xu for x in (3.11) and use (3.11) to get

$$(3.12) \quad [xd(u)yd(z), z] = 0 \quad \text{for all } u, x, y, z \in I.$$

Replacing x by rx in (3.12), we have

$$(3.13) \quad [r, z]xd(u)yd(z) = 0 \quad \text{for all } u, x, y, z \in I, r \in R.$$

After simple computation we arrive at $[v, z]d(z)R[v, z]d(z) = 0$ for all $v, z \in I$. Since R is semiprime we have $[v, z]d(z) = 0$ for all $v, z \in I$. Repeating the same arguments as in the Theorem 3.1, we reach the conclusion. $\square$

Theorem 3.4. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x)F(y) \pm H([x, y]) = 0$ for all $x, y \in I$, then $xd(x^2) = 0$ for all $x \in I$.*

Proof. Consider the hypothesis

$$(3.14) \quad F(x)F(y) \pm H([x, y]) = 0 \quad \text{for all } x, y \in I.$$

Substitute yx for y in (3.14) to get

$$(3.15) \quad F(x)F(y)x + F(x)yd(x) \pm H([x, y])x = 0 \quad \text{for all } x, y \in I.$$

In view of (3.14), (3.15) reduces to

$$(3.16) \quad F(x)yd(x) = 0 \quad \text{for all } x, y \in I.$$

Replace x by xz in (3.16) and use (3.16) to find

$$(3.17) \quad F(x)zyxd(z) + xd(z)yd(x)z + xd(z)yxd(z) = 0 \quad \text{for all } x, y, z \in I.$$

In particular, take $x = z$ in above equation to obtain

$$(3.18) \quad xd(x)yd(x)x + xd(x)yxd(x) = 0 \quad \text{for all } x, y \in I.$$

Which gives that

$$(3.19) \quad xd(x)yd(x^2) = 0 \quad \text{for all } x, y \in I.$$

A simple calculation yields that $xd(x^2)Rxd(x^2) = 0$ for all $x \in I$. Since R is semiprime, we have $xd(x^2) = 0$ for all $x \in I$. $\square$

Theorem 3.5. *Let R be a semiprime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x)H(y) \pm x \circ y \in Z(R)$ for all $x, y \in I$, then $I[I, I] = 0$ for all $x \in I$.*

Proof. Let us consider

$$(3.20) \quad F(x)H(y) \pm x \circ y \in Z(R) \quad \text{for all } x, y \in I.$$

Replacing y by yz in (3.20) to find

$$(3.21) \quad F(x)H(y)z \pm (x \circ y)z - y[x, z] \in Z(R) \quad \text{for all } x, y, z \in I.$$

Commuting (3.21) with z , we have

$$(3.22) \quad [(F(x)H(y) \pm (x \circ y))z, z] - [y[x, z], z] = 0 \quad \text{for all } x, y, z \in I.$$

In view of (3.20), (3.22) reduces to

$$(3.23) \quad [y[x, z], z] = 0 \quad \text{for all } x, y, z \in I.$$

This implies that

$$(3.24) \quad y[[x, z], z] + [y, z][x, z] = 0 \quad \text{for all } x, y, z \in I.$$

Substitute ry for y in (3.24) and use (3.24) to get $[r, z]y[x, z] = 0$ for all $x, y, z \in I$, $r \in R$. This implies that $y[x, z]Ry[x, z] = 0$ for all $x, y, z \in I$, $r \in R$. Semiprimeness of R force that $y[x, z] = 0$ for all $x, y, z \in I$, i.e. $I[I, I] = 0$. $\square$

Theorem 3.6. *Let R be a Prime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $[F(x), H(y)] = 0$ for all $x, y \in I$, then one of the conditions hold:*

- (1) $H = 0$;
- (2) $I[I, I] = 0$ for all $x \in I$;
- (3) F acts as multiplicative left centralizer.

Proof. We have

$$(3.25) \quad [F(x), H(y)] = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yz in (3.25) and using (3.25) we obtain

$$(3.26) \quad H(y)[F(x), z] = 0 \quad \text{for all } x, y, z \in I.$$

Again replace y by yu in (3.26) to get $H(y)Ru[F(x), z] = 0$ for all $x, y, z \in I$, $r \in R$. Primeness of R yield that either $H(y) = 0$ or $u[F(x), z] = 0$ for all $x, y, z \in I$, $r \in R$. If $H(y) = 0$, then $0 = H(ry) = H(r)y = H(r)Ry$ for all $y \in I$, $r \in R$. Hence $H = 0$ because I is a nonzero ideal and R is prime. Consider the later case

$$(3.27) \quad y[F(x), z] = 0 \quad \text{for all } x, y, z \in I.$$

Substitute xz for x in (3.27) and use (3.27) to find

$$(3.28) \quad yx[d(z), z] + y[x, z]d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Replacing x by vx in (3.28), we have

$$(3.29) \quad yvx[d(z), z] + yv[x, z]d(z) + y[v, z]xd(z) = 0 \quad \text{for all } v, x, y, z \in I.$$

In view of (3.28), (3.29) reduces to $y[v, z]xd(z) = 0$ for all $v, x, y, z \in I$. This implies that $y[v, z]Rxd(z) = 0$ for all $v, x, y, z \in I$. Primeness of R forces that either $y[v, z] = 0$, i.e. $I[I, I] = 0$ or $xd(z) = 0$ for all $v, x, y, z \in I$. In later case we have $xd(z) = 0$, which gives that $F(xz) = F(x)z + xd(z) = F(x)z$ for all $x, z \in I$. Therefore F acts as multiplicative left centralizer. This complete the proof. $\square$

Theorem 3.7. *Let R be a Prime ring and I be a nonzero left ideal of R . If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x) \circ H(y) = 0$ for all $x, y \in I$, then one of the conditions hold:*

- (1) $H = 0$;
- (2) $I[I, I] = 0$ for all $x \in I$;
- (3) F acts as multiplicative left centralizer.

Proof. We have

$$(3.30) \quad F(x) \circ H(y) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by yz in (3.30) we have

$$(3.31) \quad (F(x) \circ H(y))z - H(y)[F(x), z] = 0 \quad \text{for all } x, y, z \in I.$$

By application of (3.30), (3.31) yields that $H(y)[F(x), z] = 0$ for all $x, y, z \in I$. Repeating the same steps as in the proof of Theorem 3.6, we arrive at the conclusion. $\square$

Example 2. Let $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2 \right\}$, be a ring and $I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} : b, c, d \in \mathbb{Z}_2 \right\}$ be an ideal of R . Define $F \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b+c & 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$ and $H \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ba & 0 & 0 \end{pmatrix}$. F is a multiplicative generalized derivation associated with

the map d and H is a multiplicative left centralizer. R satisfies the hypothesis of Theorem 3.6 and Theorems 3.7 but $H \neq 0$. Hence Primeness is essential condition. $\square$

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Some fixed points theorems in intuitionistic Menger spaces

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Abstract. In this paper, we established some common fixed point theorems for two pairs of weakly subsequential continuous (wsc) maps with compatibility of type (E) in an Intuitionistic Menger space (briefly IM space). We deduce important results in this line by restricting the number of mappings involved. An example is given to support the main result.

Keywords: fixed point, intuitionistic Menger space, compatible maps of type (E), weak subsequential continuous maps (wsc).

1. Introduction

Menger [5] on his famous article titled statistical metric states that large parts of metric geometry can be developed, in particular, a theory of betweenness on the basis of postulates given herein, especially, where the probabilistic situation arises. Thereafter, it became an attractive area of research for the researchers involved in the field of fixed point theory and its applications. Schweizer and Sklar [8, 9] stimulated the study further with their pioneering work on statistical metric spaces. Working on the same line, Sehgal and Bharucha-Reid [10] established some fixed points theorems involving contraction mappings on probabilistic metric spaces. Stojakovic [12, 13, 14] brought forward the legacy with his work on probabilistic metric spaces and its applications. Singh and Pant [11] gave some fixed point results for commuting maps in probabilistic metric spaces. Stepping one milestone ahead, Kutucku *et al.* [6] developed probabilistic metric spaces due to Menger [5] to intuitionistic Menger spaces and established common fixed point theorems with the help of continuous t -norm and continuous t -conorm. Rashwan and Heder [7] established some new fixed point results for

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compatible mappings in Menger spaces. Pant *et al.* [3] studied fixed points and its uniqueness for weakly compatible mappings in intuitionistic Menger spaces without any appeal to the continuity of mappings. Leaving aside the condition of continuity, Jain *et al.* [4] came out with some fixed point results for absorbing type of maps in intuitionistic Menger spaces. Leila and Aliouche [18] established a common fixed point theorem for mappings under ϕ -contractive conditions on intuitionistic Menger metric spaces. They also studied the existence and uniqueness of the solution to a nonlinear Fredholm integral equation as an application of their result.

Bouhadjera and Thobie [1] proved common fixed point theorems for pairs of subcompatible maps. Singh *et al.* [15] introduced the notion of compatibility of type (E) and proved some common fixed point theorems for it. Recently, Beloul [2] established some fixed point theorems for two pairs of self mappings satisfying contractive conditions by using the weak subsequential mappings with compatibility of type (E). The purpose of this paper is to obtain common fixed point theorems for weak subsequential continuous mappings with compatibility of type (E) in an intuitionistic Menger space.

2. Preliminaries

Definition 2.1 ([3]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be t -norm if it satisfies the following conditions:

- (i) $*$ is commutative and associative,
- (ii) $*$ is continuous
- (iii) $a * 1 = a$ for all $a \in [0, 1]$ and
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Definition 2.2 ([3]). A binary operation $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be t -conorm if it satisfies the following conditions:

- (i) $\diamond$ is commutative and associative,
- (ii) $\diamond$ is continuous,
- (iii) $a \diamond 0 = a$, for all $a \in [0, 1]$ and
- (iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Remark 2.1 ([3]). The concept of triangular norms (t -norms) and triangular conorms (t -conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [5] in his study of statistical metric spaces.

Definition 2.3 ([3]). A distance distribution function is function $F : R \rightarrow R^+$ which is left continuous on R , non-decreasing and $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) = 1$.

We shall denote by D the family of all distance distribution functions and by H a special distance distribution function in D given by

$$H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}.$$

Definition 2.4 ([3]). A non-distance distribution function is function $L : R \rightarrow R^+$ which is left continuous on R , non-increasing and $\inf_{t \in R} L(t) = 1$, $\sup_{t \in R} L(t) = 0$.

We shall denote by E the family of all non distance distribution functions and by G a special non distance distribution function in E given by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}.$$

Definition 2.5 ([3]). Given an arbitrary set X , a continuous t -norm $*$, a continuous t -conorm $\diamond$, a probabilistic distance F and a probabilistic non-distance L on X , the 5-tuple $(X, F, L, *, \diamond)$ is said to be an intuitionistic Menger space if the following conditions are satisfied for all $x, y, z \in X$ and $s, t \geq 0$.

$$(IM1) \quad F(x, y, t) + L(x, y, t) \leq 1,$$

$$(IM2) \quad F(x, y, 0) = 0,$$

$$(IM3) \quad F(x, y, t) = H(t) \text{ iff } x = y,$$

$$(IM4) \quad F(x, y, t) = F(y, x, t),$$

$$(IM5) \quad \text{If } F(x, y, t) = 1 \text{ and } F(y, z, s) = 1, \text{ then } F(x, z, t + s) = 1,$$

$$(IM6) \quad F(x, z, t + s) \geq F(x, y, t) * F(y, z, s),$$

$$(IM7) \quad L(x, y, 0) = 1,$$

$$(IM8) \quad L(x, y, t) = G(t) \text{ iff } x = y,$$

$$(IM9) \quad L(x, y, t) = L(y, x, t),$$

$$(IM10) \quad \text{If } L(x, y, t) = 0 \text{ and } L(y, z, s) = 0, \text{ then } L(x, z, t + s) = 0,$$

$$(IM11) \quad L(x, z, t + s) \leq L(x, y, t) \diamond L(y, z, s).$$

The functions $F(x, y, t)$ and $L(x, y, t)$ denote the degree of nearness and degree of non- nearness between x and y with respect to t , respectively.

Definition 2.6 ([6]). Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space with $t * t \geq t$ and $(1 - t)\Diamond(1 - t) \leq (1 - t)$ for all $0 < t < 1$. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if, for any $\epsilon > 0$ and $k \in (0, 1)$, there exists a positive integer N such that $F(x_n, x, \epsilon) > 1 - k$ and $L(x_n, x, \epsilon) < k$ whenever $n \geq N$.

Lemma 2.1 ([3]). Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space. If there exists a constant $k \in (0, 1)$, and two elements $x, y \in X$ such that for all $t > 0$,

$$F(x, y, kt) \geq F(x, y, t) \text{ and } L(x, y, kt) \leq L(x, y, t).$$

Then $x = y$.

Lemma 2.2 ([6]). Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space with $t * t \geq t$ and $(1 - t)\Diamond(1 - t) \leq (1 - t)$ for all $0 < t < 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X converging to x and y respectively. If $t \geq 0$ is a point of continuity of $F(x, y, \cdot)$ and $L(x, y, \cdot)$, then $\lim_{n \rightarrow \infty} F(x_n, y_n, t) = F(x, y, t)$ and $\lim_{n \rightarrow \infty} L(x_n, y_n, t) = L(x, y, t)$.

Singh *et al.* [15, 16] introduced the notion of compatibility of type (E), A -compatibility of type (E) and S -compatibility of type (E), we redefine all these notions further in the setting of intuitionistic Menger space as follows:

Definition 2.7. Self maps A and S on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be compatible of type (E), if $\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = Az$ and $\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$.

Definition 2.8. Self maps A and S on an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to be A -compatible of type (E), if $\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$. Pair A and S are said to be S -compatible of type (E), if $\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = Az$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$.

Remark 2.2. It is also interesting to see that if A and S are compatible of type (E), then they are A -Compatible and S -Compatible of type (E), but the converse need not be true (see Example 1 [2]). Bouhadjera and Thobie [1] introduced the concept of subsequential continuity as follows:

Definition 2.9. Self maps A and S of a metric space (X, d) are said to be subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$ and $\lim_{n \rightarrow \infty} ASx_n = At$ and $\lim_{n \rightarrow \infty} SAx_n = St$.

Motivated by the Definition 2.9 and [2], we define the following in the setting of intuitionistic Menger space.

Definition 2.10. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be weakly subsequentially continuous (in short wsc), if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$ or $\lim_{n \rightarrow \infty} SAsx_n = Sz$

Definition 2.11. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be S subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$ and $\lim_{n \rightarrow \infty} SAsx_n = Sz$.

Definition 2.12. Self maps A and S defined on an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be A subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$.

Remark 2.3. If the pair of mappings $\{A, S\}$ is A -subsequentially continuous (or S -subsequentially continuous) then it is wsc but never be subsequentially continuous (see Example 3[2]).

Definition 2.13 ([17]). Let Ψ be the class of all non decreasing mappings $\psi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $\eta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that

- (i) $\lim_{n \rightarrow \infty} \psi^n(s) = 1, \forall s \in (0, 1]$,
- (ii) $\psi(s) > s, \forall s \in (0, 1)$,
- (iii) $\psi(1) = 1$,
- (iv) $\lim_{n \rightarrow \infty} \eta^n(r) = 0, \forall r \in [0, 1)$,
- (v) $\eta(r) < r, \forall r \in (0, 1)$,
- (vi) $\eta(0) = 0$. For examples we refer to [17].

3. Main results

Theorem 3.1. Let A, B, S and T be four self mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ with continuous t -norm $*$ and continuous t -conorm $\diamond$ satisfying $t * t \geq t$ and $(1 - t)\diamond(1 - t) \leq (1 - t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:

$$(3.1) \quad F(Ax, By, kt) \geq \psi[\min\{F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t), F(Sx, By, t), F(Ty, Ax, t)\}],$$

$$(3.2) \quad L(Ax, By, kt) \leq \eta[\max\{L(Sx, Ty, t), L(Ax, Sx, t), L(By, Ty, t), L(Sx, By, t), L(Ty, Ax, t)\}].$$

If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequential continuous and compatible of type (E), then A, B, S and T have a unique common fixed point in X .

Proof. Since the pair $\{A, S\}$ is weakly subsequential continuous, we can assume that it is A -subsequentially continuous and compatible of type (E). There exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$. The compatibility of type (E) implies that $\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = Sz$ and $\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAsx_n = Az$. Therefore $Az = Sz$, whereas in respect of the pair $\{B, T\}$, suppose that it is B -subsequentially continuous. Then, there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = w$, for some $w \in X$ and $\lim_{n \rightarrow \infty} BTy_n = Bw$. The pair $\{B, T\}$ is compatible of type (E), so $\lim_{n \rightarrow \infty} B^2y_n = \lim_{n \rightarrow \infty} BTy_n = Tw$ and $\lim_{n \rightarrow \infty} T^2y_n = \lim_{n \rightarrow \infty} TBy_n = Bw$, for some $w \in X$. This gives $Bw = Tw$. Hence z is a coincidence point of the pair $\{A, S\}$ whereas w is a coincidence point of the pair $\{B, T\}$. Now we prove that $z = w$. Choose a $t > 0$ satisfying (3.1) and (3.2). Without loss of generality, we can assume that t and kt are points of continuity of $F(z, w, \cdot)$, $F(z, z, \cdot)$, $F(w, w, \cdot)$, $L(z, w, \cdot)$, $L(z, z, \cdot)$, $L(w, w, \cdot)$, $F(Az, w, \cdot)$, $F(Sz, w, \cdot)$, $F(z, Bz, \cdot)$, $F(z, Tz, \cdot)$, $L(Az, w, \cdot)$, $L(Sz, w, \cdot)$, $L(z, Bz, \cdot)$ and $L(z, Tz, \cdot)$. This is so because these functions are monotonic on $\mathbb{R}$ and hence have at most countable number of discontinuities in $(0, b)$ for any $b > 0$. So we may choose t sufficiently small that $0 < kt < t < b$ and both kt and t are points of continuity of all the functions mentioned above. By putting $x = x_n$ and $y = y_n$ in inequality (3.1), we have

$$F(Ax_n, By_n, kt) \geq \psi[\min\{F(Sx_n, Ty_n, t), F(Ax_n, Sx_n, t), F(By_n, Ty_n, t), F(Sx_n, By_n, t), F(Ty_n, Ax_n, t)\}].$$

Taking the limit as $n \rightarrow \infty$ and using Lemma 2.2, we get

$$F(z, w, kt) \geq \psi[\min\{F(z, w, t), F(z, z, t), F(w, w, t), F(z, w, t), F(w, z, t)\}].$$

So, $F(z, w, kt) \geq \psi[\min\{F(z, w, t), 1, 1, F(z, w, t), F(w, z, t)\}]$. This gives, for all $t > 0$,

$$(3.3) \quad F(z, w, kt) \geq \psi[F(z, w, t)] \geq F(z, w, t).$$

Again, by putting $x = x_n$ and $y = y_n$ in inequality (3.2), we have

$$L(Ax_n, By_n, kt) \leq \eta[\max\{L(Sx_n, Ty_n, t), L(Ax_n, Sx_n, t), L(By_n, Ty_n, t), L(Sx_n, By_n, t), L(Ty_n, Ax_n, t)\}]$$

Taking the limit as $n \rightarrow \infty$, and using Lemma 2.2 we get

$$L(z, w, kt) \leq \eta[\max\{L(z, w, t), L(z, z, t), L(w, w, t), L(z, w, t), L(w, z, t)\}].$$

So, we have for all $t > 0$,

$$L(z, w, kt) \leq \eta[\max\{L(z, w, t), 0, 0, L(z, w, t), L(w, z, t)\}].$$

$$(3.4) \quad L(z, w, kt) \leq \eta[L(z, w, t)] \leq L(z, w, t),$$

for all $t > 0$.

By Lemma 2.1, (3.3) and (3.4), we have $z = w$. Now we prove that $Az = z$. By putting $x = z$ and $y = y_n$ in the inequality (3.1), we get $F(Az, By_n, kt) \geq \psi[\min\{F(Sz, Ty_n, t), F(Az, Sz, t), F(By_n, Ty_n, t), F(Sz, By_n, t), F(Ty_n, Az, t)\}]$.

Letting $n \rightarrow \infty$ and using Lemma 2.2, we obtain

$$F(Az, w, kt) \geq \psi[\min\{F(Sz, w, t), F(Az, Sz, t), F(w, w, t), F(Sz, w, t), F(w, Az, t)\}].$$

This gives $F(Az, w, kt) \geq \psi[\min\{F(Sz, w, t), 1, 1, F(Sz, w, t), F(w, Az, t)\}]$. But $Az = Sz$. Thus, for all $t > 0$,

$$(3.5) \quad F(Az, w, kt) \geq \psi[F(Az, w, t)] \geq F(Az, w, t).$$

Again, by putting $x = z$ and $y = y_n$ in the inequality (3.2), we get $L(Az, By_n, kt) \leq \eta[\max\{L(Sz, Ty_n, t), L(Az, Sz, t), L(By_n, Ty_n, t), L(Sz, By_n, t), L(Ty_n, Az, t)\}]$.

Taking the limit as $n \rightarrow \infty$ and using Lemma 2.2, we obtain

$$L(Az, w, kt) \leq \eta[\max\{L(Sz, w, t), L(Az, Sz, t), L(w, w, t), L(Sz, w, t), L(w, Az, t)\}].$$

This gives $L(Az, w, kt) \leq \eta[\max\{L(Sz, w, t), 0, 0, L(Sz, w, t), L(w, Az, t)\}]$ and so for all $t > 0$,

$$(3.6) \quad L(Az, w, kt) \leq \eta[L(Az, w, t)] \leq L(Az, w, t)$$

By Lemma 2.1, (3.5) and (3.6) we get, $Az = w$. Since $Az = Sz$, we have $Az = Sz = w = z$. Now we prove that $Bz = z$.

By putting $x = x_n$ and $y = z$ in the inequality (3.1), we get $F(Ax_n, Bz, kt) \geq \psi[\min\{F(Sx_n, Tz, t), F(Ax_n, Sx_n, t), F(Bz, Tz, t), F(Sx_n, Bz, t), F(Tz, Ax_n, t)\}]$.

Taking the limit as $n \rightarrow \infty$ and using Lemma 2.2, we obtain

$$F(z, Bz, kt) \geq \psi[\min\{F(z, Tz, t), F(z, z, t), F(Bz, Tz, t), F(z, Bz, t), F(Tz, z, t)\}],$$

$$F(z, Bz, kt) \geq \psi[\min\{F(z, Tz, t), 1, 1, F(z, Bz, t), F(Tz, z, t)\}].$$

Since $z = w$ and $Bw = Tw$, then $Bz = Tz$. Thus, for all $t > 0$,

$$(3.7) \quad F(z, Bz, kt) \geq \psi[F(z, Bz, t)] \geq F(z, Bz, t)$$

Again, by putting $x = x_n$ and $y = z$ in the inequality (3.2), we get $L(Ax_n, Bz, kt) \leq \eta[\max\{L(Sx_n, Tz, t), L(Ax_n, Sx_n, t), L(Bz, Tz, t), L(Sx_n, Bz, t), L(Tz, Ax_n, t)\}]$.

Taking the limit as $n \rightarrow \infty$ and using Lemma 2.2, we get

$$L(z, Bz, kt) \leq \eta[\max\{L(z, Tz, t), L(z, z, t), L(Bz, Tz, t), L(z, Bz, t), L(Tz, z, t)\}],$$

which gives $L(z, Bz, kt) \leq \eta[\max\{L(z, Tz, t), 0, 0, L(z, Bz, t), L(Tz, z, t)\}]$ and so for all $t > 0$,

$$(3.8) \quad L(z, Bz, kt) \leq \eta[L(z, Bz, t)] \leq L(z, Bz, t)$$

By Lemma 2.1, (3.7) and (3.8), we obtain, $Bz = z$. Since $Bz = Tz$, we have $Bz = Tz = z$. So, in all $z = Az = Bz = Sz = Tz$, that is, z is a common fixed point of A , B , S and T .

To prove uniqueness, let $z^* \in X$ be such that $z^* = Az^* = Bz^* = Sz^* = Tz^*$. By putting $x = z$ and $y = z^*$ in the inequality (3.1), we get $F(Az, Bz^*, kt) \geq \psi[\min\{F(Sz, Tz^*, t), F(Az, Sz, t), F(Bz^*, Tz^*, t), F(Sz, Bz^*, t), F(Tz^*, Az, t)\}]$.

That is, $F(z, z^*, kt) \geq \psi[\min\{F(z, z^*, t), F(z, z, t), F(z^*, z^*, t), F(z, z^*, t), F(z^*, z, t)\}]$. So, $F(z, z^*, kt) \geq \psi[\min\{F(z, z^*, t), 1, 1, F(z, z^*, t), F(z^*, z, t)\}]$. Thus, for all $t > 0$,

$$(3.9) \quad F(z, z^*, kt) \geq \psi[F(z, z^*, t)] \geq F(z, z^*, t).$$

By putting $x = z$ and $y = z^*$ in the inequality (3.2), we get $L(Az, Bz^*, kt) \leq \eta[\max\{L(Sz, Tz^*, t), L(Az, Sz, t), L(Bz^*, Tz^*, t), L(Sz, Bz^*, t), L(Tz^*, Az, t)\}]$.

That is, $L(z, z^*, kt) \leq \eta[\max\{L(z, z^*, t), L(z, z, t), L(z^*, z^*, t), L(z, z^*, t), L(z^*, z, t)\}]$. So, $L(z, z^*, kt) \leq \eta[\max\{L(z, z^*, t), 0, 0, L(z, z^*, t), L(z^*, z, t)\}]$. Thus, for all $t > 0$,

$$(3.10) \quad L(z, z^*, kt) \leq \eta[L(z, z^*, t)] \leq L(z, z^*, t).$$

By Lemma 2.1, (3.9) and (3.10) we get, $z = z^*$. □

If we put $A = B$ in Theorem 3.1 we have the following corollary for three mappings:

Corollary 3.1. *Let A , S and T be three self mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ with continuous t -norm $*$ and continuous t -conorm $\diamond$ satisfying $t * t \geq t$ and $(1 - t) \diamond (1 - t) \leq (1 - t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$\begin{aligned} F(Ax, Ay, kt) &\geq \psi[\min\{F(Sx, Ty, t), F(Ax, Sx, t), \\ &\quad F(Ay, Ty, t), F(Sx, Ay, t), F(Ty, Ax, t)\}], \\ L(Ax, Ay, kt) &\leq \eta[\max\{L(Sx, Ty, t), L(Ax, Sx, t), \\ &\quad L(Ay, Ty, t), L(Sx, Ay, t), L(Ty, Ax, t)\}]. \end{aligned}$$

If the pairs $\{A, S\}$ and $\{A, T\}$ are weakly subsequential continuous and compatible of type (E), then A , S and T have a unique common fixed point in X .

Alternatively, if we set $S = T$ in Theorem 3.1, we'll have the following corollary for three self mappings:

Corollary 3.2. *Let A , B and S be four self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ with continuous t -norm $*$ and continuous t -conorm $\Diamond$ satisfying $t * t \geq t$ and $(1 - t)\Diamond(1 - t) \leq (1 - t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$\begin{aligned} F(Ax, By, kt) &\geq \psi[\min\{F(Sx, Sy, t), F(Ax, Sx, t), \\ &\quad F(By, Sy, t), F(Sx, By, t), F(Sy, Ax, t)\}], \\ L(Ax, By, kt) &\leq \eta[\max\{L(Sx, Sy, t), L(Ax, Sx, t), \\ &\quad L(By, Sy, t), L(Sx, By, t), L(Sy, Ax, t)\}]. \end{aligned}$$

If the pairs $\{A, S\}$ and $\{B, S\}$ are weakly subsequential continuous and compatible of type (E), then A , B and S have a unique common fixed point in X .

If we put $S = T$ in Corollary 3.1, we have the following result for two self mappings:

Corollary 3.3. *Let A , S be two self mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ with continuous t -norm $*$ and continuous t -conorm $\Diamond$ satisfying $t * t \geq t$ and $(1 - t)\Diamond(1 - t) \leq (1 - t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ the following conditions are satisfied:*

$$\begin{aligned} F(Ax, Ay, kt) &\geq \psi[\min\{F(Sx, Sy, t), F(Ax, Sx, t), \\ &\quad F(Ay, Sy, t), F(Sx, Ay, t), F(Sy, Ax, t)\}], \\ L(Ax, Ay, kt) &\leq \eta[\max\{L(Sx, Sy, t), L(Ax, Sx, t), \\ &\quad L(Ay, Sy, t), L(Sx, Ay, t), L(Sy, Ax, t)\}]. \end{aligned}$$

If the pair $\{A, S\}$ is weakly subsequential continuous and compatible of type (E), then A and S have a unique common fixed point in X .

Example 3.1. Let $X = [0, 2]$ with metric $d(x, y) = |x - y|$ and for each $t \in (0, 1)$, define

$$\begin{aligned} F(x, y, t) &= \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases} \\ L(x, y, t) &= \begin{cases} \frac{|x - y|}{t + |x - y|}, & \text{if } t > 0; \\ 1, & \text{if } t = 0, \end{cases} \end{aligned}$$

for all $x, y \in X$. Clearly $(X, F, L, *, \Diamond)$ is an intuitionistic Menger space where $*$ is defined by $t * t \geq t$ and $\Diamond$ defined by $(1 - t)\Diamond(1 - t) \leq (1 - t)$. Let us define mappings A, B, S and T as follows:

$$A(x) = B(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{3}{4}, & 1 < x \leq 2 \end{cases}.$$

$$S(x) = T(x) = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ 2, & 1 < x \leq 2 \end{cases}.$$

Let us consider a sequence $\{x_n\}$ in X defined by $x_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} Ax_n = 1 = \lim_{n \rightarrow \infty} Sx_n$ and $\lim_{n \rightarrow \infty} ASx_n = 1 = A(1)$; $\lim_{n \rightarrow \infty} A^2x_n = 1 = S(1)$, $\lim_{n \rightarrow \infty} S^2x_n = 1 = A(1)$. Hence $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E). Proceeding in the same way, we can easily show that $\{B, T\}$ is weakly subsequentially continuous and compatible of type (E). For each $r, s \in [0, 1]$ and $\psi, \eta \in \Psi$, define the maps $\psi, \eta : [0, 1] \rightarrow [0, 1]$ as $\psi(s) = \frac{2s}{s+1}$ and $\eta(r) = \frac{r}{2r-1}$.

Next, for $k = \frac{3}{4}$, we'll consider the different cases for which the following inequalities holds:

$$F(Ax, By, kt) \geq \psi[\min\{F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t), F(Sx, By, t), F(Ty, Ax, t)\}]$$

and

$$L(Ax, By, kt) \leq \eta[\max\{L(Sx, Ty, t), L(Ax, Sx, t), L(By, Ty, t), L(Sx, By, t), L(Ty, Ax, t)\}].$$

1. If $x, y \in [0, 1]$, we have $F(Ax, By, kt) = 1 \geq \psi(\frac{t}{t + \frac{|x-y|}{2}}) \geq \frac{t}{t + \frac{|x-y|}{2}} = F(Sx, Ty, t)$ and $L(Ax, By, kt) = 0 \leq \eta(\frac{\frac{|x-y|}{2}}{t + \frac{|x-y|}{2}}) \leq \frac{\frac{|x-y|}{2}}{t + \frac{|x-y|}{2}} = L(Sx, Ty, t)$.

$$\text{Where } \psi(\frac{t}{t + \frac{|x-y|}{2}}) = \frac{2t}{2t + \frac{|x-y|}{2}} \text{ and } \eta(\frac{\frac{|x-y|}{2}}{t + \frac{|x-y|}{2}}) = \frac{\frac{|x-y|}{2}}{\frac{|x-y|}{2} - t}.$$

2. If $x \in [0, 1]$ and $y \in (1, 2]$ we have $F(Ax, By, kt) = \frac{kt}{kt+0.25} \geq \psi(\frac{t}{t+1.25}) \geq \frac{t}{t+1.25} = F(By, Ty, t)$ and $L(Ax, By, kt) = \frac{1}{kt+0.25} \leq \eta(\frac{1.25}{t+1.25}) \leq \frac{1.25}{t+1.25} = L(By, Ty, t)$.

$$\text{Where } \psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25} \text{ and } \eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}.$$

3. If $x \in (1, 2]$ and $y \in [0, 1]$ we have $F(Ax, By, kt) = \frac{kt}{kt+0.25} \geq \psi(\frac{t}{t+1.25}) \geq \frac{t}{t+1.25} = F(Ax, Sx, t)$ and $L(Ax, By, kt) = \frac{0.25}{kt+0.25} \leq \eta(\frac{1.25}{t+1.25}) \leq \frac{1.25}{t+1.25} = L(Ax, Sx, t)$.

$$\text{Where } \psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25} \text{ and } \eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}.$$

4. If $x, y \in (1, 2]$, we have $F(Ax, By, kt) = 1 \geq \psi(\frac{t}{t+1.25}) \geq \frac{t}{t+1.25} = F(Sx, By, t)$ and $L(Ax, By, kt) = 0 \leq \eta(\frac{1.25}{t+1.25}) \leq \frac{1.25}{t+1.25} = L(Sx, By, t)$.

$$\text{Where } \psi(\frac{t}{t+1.25}) = \frac{2t}{2t+1.25} \text{ and } \eta(\frac{1.25}{t+1.25}) = \frac{1.25}{t-1.25}.$$

Thus, all the conditions of Theorem (3.1) are satisfied and A, B, S, T have a unique common fixed point $x = 1$.

Theorem 3.2. Let A, B, S and T be four self mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ with continuous t -norm $*$ and continuous t -conorm $\diamond$ satisfying $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$ for all $0 < t < 1$. Suppose $\psi, \eta \in \Psi$ and there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$ satisfying (3.1) and (3.2). Assume that

- (i) the pair $\{A, S\}$ is A -compatible of type (E) and A -subsequentially continuous.
- (ii) the pair $\{B, T\}$ is B -compatible of type (E) and B -subsequentially continuous.

Then A , B , S and T have a unique common fixed point in X .

Proof. The proof is obvious as on the lines of Theorem 3.1. □

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Nowhere-zero 3-flows in 4-connected simple graphs with independence number 3

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Abstract. It is mainly proved in this paper that every 4-connected simple graph with $\alpha(G) = 3$ admits a nowhere-zero 3-flow, which is a partial result to Tutte's 3-flow conjecture and generalizes the result by Luo *et al.* [Graphs and Combin., (29)(2013)1899–1907] that characterized all graphs with independence number at most 2.

Keywords: nowhere zero 3-flow, independence number.

1. Introduction

Graphs considered in this paper are simple, finite without loops. Terminology and notations not defined here can be seen in the book [1] by Bondy and Murty.

For a graph G , a set S of vertices in G is *independent* if no two vertices of S are adjacent in G . The maximum cardinality of an independent set of vertices of G is called the *independence number* of G and denoted by $\alpha(G)$. If V is a vertex subset of G , we denote by $G - V$ the graph obtained from G by deleting all vertices in V and all incident edges. For two subsets $A, B \subseteq V(G)$, $e(A, B)$ denotes the number of edges with one endpoint in A and the other endpoint in B . For simplicity, if $A = \{v\}$, we write $e(v, B)$ to mean $e(\{v\}, B)$ and if H_1 and H_2 are two subgraphs of G , we write $e(H_1, H_2)$ to mean $e(V(H_1), V(H_2))$. The set of neighbors of x in G is denoted by $N_G(x)$, or simply $N(x)$. In addition, $N[x] = N(x) \cup \{x\}$.

The complete graph on n vertices is denoted by K_n . Let K_n^- denote the graph obtained from K_n by deleting an edge. The *wheel* W_k ($k \geq 2$) is the graph obtained from a k -cycle by adding a new vertex, called the *center* of the

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wheel, which is adjacent to every vertex of the k -cycle. We define W_k to be *odd* (even) if k is odd (or even, respectively).

Let D be an orientation of a graph G , and f a function from $E(G)$ to Z with $-k < f(e) < k$ for every $e \in E(G)$. Then the pair (D, f) is a k -flow if it satisfies the Kirchhoff condition $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every vertex $v \in V(G)$, where $E^+(v)$ and $E^-(v)$ denote the sets of outgoing and incoming edges of v with respect to D . A k -flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$.

Nowhere-zero k -flow was first introduced by Tutte [15]. We shall restrict our attention to the case that $k = 3$ and this paper is mainly motivated by the well-known 3-flow conjecture of Tutte.

Conjecture 1.1 (Tutte [15]). *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Let Z_k denote the cyclic group of order k . A graph G is Z_k -connected if G has an orientation D such that for any function $b : V(G) \rightarrow Z_k$ with $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \rightarrow Z_k - \{0\}$ such that $b(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$ for any $v \in V(G)$, where “ $\sum$ ” refers to the addition in Z_k .

Group connectivity was first introduced by Jaeger *et al.* [9] as a generalization of nowhere-zero flows. Clearly, if G is Z_3 -connected, then G admits a nowhere-zero 3-flow.

Many authors are involved in the study of nowhere-zero 3-flows. In particular, Fan and Zhou [6, 7] used degree sum condition and Ore-condition to guarantee the existence of nowhere-zero 3-flows. Recently, Thomassen [14] first made a breakthrough in the 3-flow conjecture. He verified the following theorem.

Theorem 1.2. *Every 8-edge-connected graph admits a nowhere-zero 3-flow.*

This result was soon further improved by Lovasz *et al.* [13].

Theorem 1.3. *Every 6-edge-connected graph admits a nowhere-zero 3-flow.*

So far, Conjecture 1.1 is still open. Chvátal and Erdős [2] proved a classical result: for a 2-connected simple graph G , if $k(G) \geq \alpha(G)$, then G is hamiltonian. Recently, Han, Lai, Xiong and Yan [8] studied whether a graph with a slightly weaker Chvátal and Erdős condition is supereulerian. It is well-known that every hamiltonian graph is supereulerian which has a nowhere-zero 4-flow. In view of this, we hope to explore that which hamiltonian graphs admit a nowhere-zero 3-flow. Luo *et al.* [10] characterized all the graphs with independence number at most 2 that admit a nowhere-zero 3-flow. However, it is difficult to have a complete characterization for the case of independence number 3. In this paper, we study a slightly stronger Chvátal-Erdős condition (that is $k(G) \geq \alpha(G) + 1$) and nowhere zero 3-flows and show that Conjecture 1.1 holds for a family of graphs, as follows.

Theorem 1.4. *Every 4-connected simple graph with independence number 3 admits a nowhere-zero 3-flow.*

2. Preliminaries

In this section, we establish several lemmas. Some results in [3, 4, 5, 9, 11, 12] on group connectivity are summarized as follows.

Lemma 2.1. *Let A be an abelian group with $|A| \geq 3$. The following results are known.*

- (1) K_n and K_n^- are A -connected if $n \geq 5$.
- (2) C_n is A -connected if and only if $|A| \geq n + 1$.
- (3) W_{2k} is Z_3 -connected and W_{2k+1} is not Z_3 -connected, where k is a positive integer.
- (4) L_3^+ in Fig. 1 is Z_3 -connected.

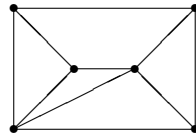


Fig. 1 The graph L_3^+

Let H be a subgraph of G . The *contraction* G/H denotes the graph obtained from G by identifying all vertices in H and then deleting all loops generating in this process.

Lemma 2.2 ([3, 4]). *Let H be a Z_3 -connected subgraph of G . The following results hold.*

- (1) *If G/H admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow.*
- (2) *If G/H is Z_3 -connected, then G is Z_3 -connected.*

Lemma 2.3 ([16]). *Let G be a 2-connected simple graph with $\delta(G) \geq 4$. If $\alpha(G) \leq 2$, then G is Z_3 -connected.*

Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \dots, v_m\}$. The graph $G_{[vv_1, vv_2]}$ is obtained from G by deleting two edges vv_1 and vv_2 and adding a new edge v_1v_2 . If $m = 2k$ is even, let

$$M = \langle \{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \dots, \{v_k, v_{2k}\} \rangle$$

be a way to pair the vertices in $N(v)$. $G_{(v, M)}$ denotes the graph obtained from $G - v$ by adding k new edges v_iv_{k+i} , where $1 \leq i \leq k$.

Lemma 2.4. *Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \dots, v_m\}$. Each of the following statements holds.*

- (1) (Lai, [11]) *If $G_{[vv_1, vv_2]}$ is Z_3 -connected, then G is Z_3 -connected.*
- (2) *If $m = 2k$ is even and $G_{(v, M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow.*

Proof. (2) Let f^* be a nowhere-zero 3-flow of $G_{(v, M)}$. If the newly added edge $v_i v_{k+i}$ is oriented from v_i to v_{k+i} , then the two edges vv_i and vv_{k+i} are oriented from v_i to v and from v to v_{k+i} , respectively.

Define $f : E(G) \rightarrow \{1, 2\}$ as

$$f(e) = \begin{cases} f^*(e), & \text{if } e \in E(G) - \{v_1 v_{k+1}, v_2 v_{k+2}, \dots, v_k v_{2k}\} \\ f^*(v_i v_{k+i}), & \text{if } e \in \{v_i v_{k+i}\} (1 \leq i \leq k). \end{cases}$$

Then f is a nowhere-zero 3-flow of G . □

Lemma 2.5. *Let G be a simple connected graph with $\delta(G) = 3$ and $\alpha(G) = 2$. If G is not Z_3 -connected, then G is one of 19 graphs shown in Fig. 2 or G can be Z_3 -contracted to one graph in $\{K'_4, K''_4\}$, where K'_4 and K''_4 denote the graphs obtained by connecting one edge and two edges between K_1 and K_4 , respectively.*

Proof. Let G be a simple graph with $\delta(G) = 3$. Then $\kappa'(G) \leq \delta(G) = 3$. If $\kappa'(G) = 3$, by [Theorem 1.3, 16], $G = H_i$, where $1 \leq i \leq 19$ but $i \notin \{15, 16\}$. If $\kappa'(G) = 1$, let e be a cut edge of G , then $G - e$ has two connected components which are complete graphs since $\alpha(G) = 2$. So $G = H_{15}$ or G can be Z_3 -contracted to K'_4 . Similarly we can prove that if $\kappa'(G) = 2$, then $G = H_{16}$ or G can be Z_3 -contracted to K''_4 . □

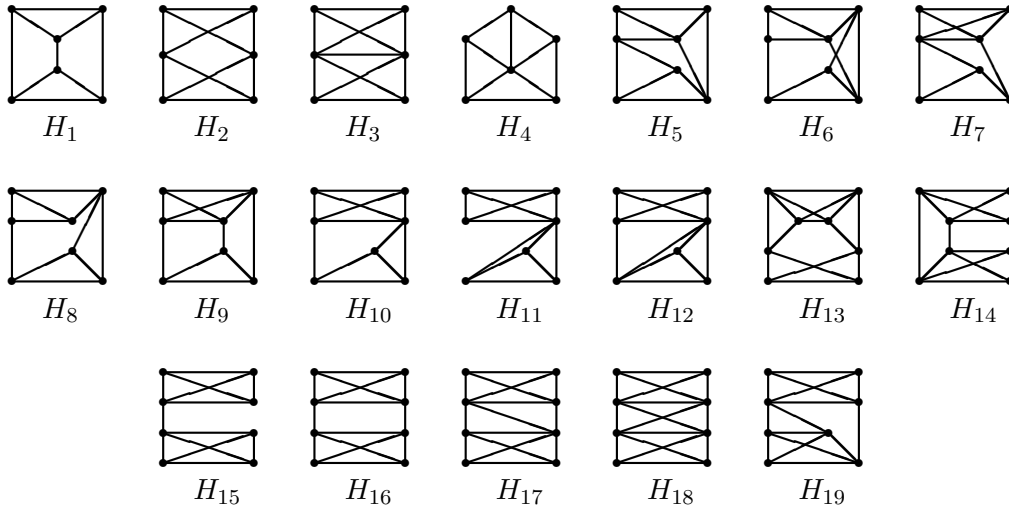


Fig. 2 19 graphs for Lemma 2.5

Lemma 2.6. *Let G be a 4-edge-connected simple graph with $\alpha(G) \leq 3$. If G contains a maximal Z_3 -connected subgraph H and $\alpha(G - H) = 2$, then G admits a nowhere-zero 3-flow.*

Proof. Let $G^* = G/H$. Then G^* is also a 4-edge-connected simple graph. If G^* is Z_3 -connected, then G admits a nowhere-zero 3-flow follows from Lemma 2.2. Next we assume that G^* is not Z_3 -connected. By Lemma 2.3, $\alpha(G^*) = 3$. Let v^* be a vertex into which H is contracted. Then $d(v^*) \geq 4$ and $\alpha(G^* - v^*) = \alpha(G - H) = 2$. If $e(v^*, G^* - v^*) = 4$, we consider the graph $G_{(v^*, M)}^*$, which may contains 2-cycles, where M is any way to pair the vertices in $N(v^*)$. We contract all generating 2-cycles, the resulting graph is denoted by $G_{(v^*, M)}^*$. It is clear that $G_{(v^*, M)}^*$ is a 4-edge connected simple graph and $\alpha(G_{(v^*, M)}^*) \leq 2$. By Lemma 2.3, $G_{(v^*, M)}^*$ is Z_3 -connected. It follows from Lemmas 2.1 and 2.2 that $G_{(v^*, M)}^*$ is also Z_3 -connected. Hence G admits a nowhere-zero 3-flow follows from Lemma 2.4. Next we suppose that $e(v^*, G^* - v^*) \geq 5$. Note that $\alpha(G^*) = 3$. There must be two vertices in $G^* - v^*$, say v_1 and v_2 , such that $\{v^*, v_1, v_2\}$ is an independent set of G^* . This implies that $|V(G^* - v^*)| \geq 7$ and $d(v_i) \geq 4$, where $i \in \{1, 2\}$. If $G^* - v^*$ is not connected, then $G^* - v^*$ has just two connected components which are complete graphs with order more than 4 since $\alpha(G^* - v^*) = 2$. Let G_1 and G_2 be two connected components of $G^* - v^*$. Then $e(v^*, G_1) \geq 2$ or $e(v^*, G_2) \geq 2$. It is clear that $G_1 + v^*$ or $G_2 + v^*$ is Z_3 -connected which is contrary to the maximum of H . Hence $G^* - v^*$ is connected and $\delta(G^* - v^*) \geq 3$. It follows from Lemma 2.5 that $G^* - v^* \in \{H_5, H_7, H_{13}, H_{14}, H_{16}, H_{17}, H_{19}\}$ or can be Z_3 -contracted to one graph in $\{K_4', K_4''\}$. It is a routine work to prove that G^* is Z_3 -connected, a contradiction. Therefore this lemma holds. $\square$

3. Proof of the main theorem

Proof of Theorem 1.4. Theorem 1.3 tells us that if $\kappa'(G) \geq 6$, then G admits a nowhere-zero 3-flow. So we only need to show that this theorem holds for $4 = \kappa(G) \leq \kappa'(G) \leq 5$. Let A be a maximal independent set of G and $B = V(G) - A$. Then $|A| = \alpha(G) = 3$ and $|B| \geq 4$ since G is a 4-connected simple graph. It follows that $|V(G)| \geq (3 + 4) = 7$. Next we proceed our proof by induction on $|V(G)|$.

Let $|V(G)| = 7$. Then $|B| = 4$ and $N(x) = B$ for any vertex $x \in A$. Hence G contains $K_{3,4}$ as a spanning subgraph. Furthermore, we get that $G[B]$ is connected. Otherwise A is a cut of G , contrary to that $\kappa(G) = 4$. So there exist two adjacent edges uv and vw in $G[B]$. Let $x, y \in A$. Then $G[\{x, y, u, v, w\}]$ contains W_4 (center at v) as a subgraph. Contracting W_4 and all 2-cycles generating in this process, we can get a K_1 . It follows from Lemma 2.1 that G is Z_3 -connected, of course, admits a nowhere-zero 3-flow.

Suppose that the theorem is true for any graph G' with $|V(G')| < |V(G)|$. Then we consider the graph G .

Case 1. There exist one vertex $x \in V(G)$ such that $d(x) = 4$.

In this case, we concentrate our attention on the graph $G_{(x,M)}$, where M is a way to pair the vertices in $N(x)$. By Lemma 2.4, if $G_{(x,M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow. Therefore we suppose that there is no such M that $G_{(x,M)}$ admits a nowhere-zero 3-flow.

Claim 1. $\alpha(G_{(x,M)}) = 3$.

Proof of Claim 1. Note that $\delta(G_{(x,M)}) \geq \delta(G) = 4$ for any way to pair the vertices in $N(x)$. It follows from Lemma 2.3 that if $\alpha(G_{(x,M)}) \leq 2$, then $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $3 \leq \alpha(G_{(x,M)}) \leq \alpha(G) = 3$, that is $\alpha(G_{(x,M)}) = 3$.

Note that $4 = \kappa(G) \leq \kappa'(G) \leq \delta(G) = 4$. We have that $\kappa'(G) = 4$. Let S be a minimal vertex cut of G which contains x . Then $|S| \geq 4$. Let G_1 and G_2 be two components (some component maybe not connected) of $G - S$ and $N(x) = \{x_1, x_2, x_3, x_4\}$. Denote by $N_1 = N(x) \cap V(G_1)$, $N_2 = N(x) \cap V(G_2)$ and $N_s = N(x) \cap S$. We assume, without loss of generality, that $|N_1| = 0$. Then $|N_2| \neq 0$, $|N_s| \neq 0$ and $|S| \geq 5$. If $|N_s| \leq 2$, there must be a way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}$ (or $G_{(x,M)}^*$) is simple and $\kappa(G_{(x,M)}) = 4$ (or $\kappa(G_{(x,M)}^*) = 4$), where $G_{(x,M)}^*$ denotes the graph obtained from $G_{(x,M)}$ by contracting some 2-cycles. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ (or $G_{(x,M)}^*$) admits a nowhere-zero 3-flow, a contradiction. Hence $|N_s| = 3$ and $|N_2| = 1$. If $G[N_s] \neq K_3$, as the above, we are done. Next we suppose that $G[N_s] = K_3$. If $e(N_2, N_s) \geq 1$, then $G_{(x,M)}[N(x)]$ is Z_3 -connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $e(N_2, N_s) = 0$. Let $N_s = \{x_1, x_2, x_3\}$ and $N_2 = \{x_4\}$. Choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}[N_s]$ is Z_3 -connected. Let H be the maximal Z_3 -connected subgraph which contains $G_{(x,M)}[N_s]$. Next we shall prove that $\alpha(G_{(x,M)} - H) \leq 2$. Note that $\alpha(G - N[x]) = 2$. Let $v_1, v_2 \in V(G - N[x])$ and $v_1v_2 \notin E(G)$. If $\{v_1, v_2, x_4\}$ is an independent set of three vertices, then $N_s \subseteq N(v_1) \cup N(v_2)$. This implies that $v_1 \in H$ or $v_2 \in H$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $|N_1| \geq 1$ and $|N_2| \geq 1$. It follows that $|N_s| \leq 2$. We discuss it in three cases.

Subcase 1.1. $|N_s| = 0$.

If $|N_1| = 1$, $|N_2| = 3$ or $|N_1| = 3$, $|N_2| = 1$, as the above, we are done. So we suppose that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2, x_4\}$. Choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

Subcase 1.2. $|N_s| = 1$.

We assume, without loss of generality, that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2\}$ and $N_s = \{x_4\}$. If $x_1x_4 \notin E(G)$ or $x_3x_4 \notin E(G)$, choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$ or $M = \langle \{x_3, x_4\}, \{x_1, x_2\} \rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By

Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $x_1x_4 \in E(G)$ and $x_3x_4 \in E(G)$.

If $x_1x_3 \in E(G)$, we consider the graph $G_{(x,M)}$, where M is any way to pair the vertices in $N(x)$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3 -connected. Let H be the maximal Z_3 -connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $\alpha(G_{(x,M)} - H) = 3$. At first, we claim that $x_2x_4 \notin E(G)$. For otherwise, $x_2 \in V(H)$. Thus $\alpha(G_{(x,M)} - H) \leq \alpha(G_{(x,M)} - N(x)) = \alpha(G - N[x]) \leq 2$, a contradiction. If G_1 is a complete graph, then $G_{(x,M)}[V(G_1) \cup \{x_4\}] \subset H$. If $y \in S - V(H)$, then $e(y, G_1) \leq 1$. Note that $\alpha(G_2) \leq 2$. Then $\alpha(G_2 + y) \leq 2$ since $\alpha(G) = 3$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Next we assume that G_1 is not a complete graph. Then $\alpha(G_1) = 2$ and $\alpha(G_1 - x_1x_3) = \alpha(G_2) = 1$. Furthermore, there must be $u \in V(G_1 - x_1x_3)$ and $v \in S - \{x, x_4\}$ such that $\{u, v, x_2\}$ is an independent set of three vertices. Note that $e(x_2, \{x_1, x_3, x_4\}) = 0$. Then $\{x_1, x_3, x_4\} \subset N(u) \cup N(v)$. So $u \in H$ or $v \in H$. This yields that $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction.

Next we suppose that $x_1x_3 \notin E(G)$. Then G_2 is a complete graph. If $x_2x_4 \in E(G)$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, we consider the graph $G_{(x,M)}$. If there exist one vertex $y' \in S - \{x, x_4\}$ such that $y' \in N(x_2) \cap N(x_4)$ and $y'x_3 \in E(G)$ or $y'x_1 \in E(G)$, then $G[\{x, x_1, x_2, x_4, y'\}] = W_4$ or $G[\{x, x_2, x_3, x_4, y'\}] = W_4$ which is Z_3 -connected. So $G[N[x] \cup \{y'\}]$ is Z_3 -connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, G is Z_3 -connected. Next we assume that $y'x_3 \notin E(G)$ and $y'x_1 \notin E(G)$. Then $G_2 + y$ is also a complete graph since $\alpha(G) = 3$. Choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$ is Z_3 -connected. Let H' be the maximal Z_3 -connected subgraph which contains $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$. If $|V(G_2)| \geq 2$, then $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. Hence $|V(G_2)| = 1$. Denote by $G_{(x,M)}^* = G_{(x,M)} / H'$. We find that $G_{(x,M)}^*$ is a 4-connected simple graph with $\alpha(G_{(x,M)}^*) \leq 3$. By Lemma 2.3 or the induction hypothesis, $G_{(x,M)}^*$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$ also admits a nowhere-zero 3-flow, a contradiction. Hence we suppose that $N(x_2) \cap N(x_4) \cap S = \{x\}$. Similarly if $N(x_2) \cap N(x_4) \cap V(G_2) \neq \emptyset$, we can also get that $G_{(x,M)}$ is Z_3 -connected, a contradiction. So $N(x_2) \cap N(x_4) \cap V(G_2) = \emptyset$. Note that $\kappa(G) = 4$. Then $|S| \geq 5$. Thus $G_{(x,M)}$ is also a 4-connected simple graph. By induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Therefore $x_2x_4 \notin E(G)$. If $|S| \geq 5$, as the above, we are done. Hence we assume that $|S| = 4$. If $N(x_1) \cap N(x_4) = \{x\}$ or $N(x_3) \cap N(x_4) = \{x\}$, choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$ or $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}$ contains only one 2-cycle. Contracting the 2-cycle, the resulting graph is denoted by $G_{(x,M)}^*$. It is easy to prove that $G_{(x,M)}^*$ is also a 4-connected simple with $\alpha(G) = 3$. By induction hypothesis, $G_{(x,M)}^*$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$, a contradiction. Hence $|N(x_1) \cap N(x_4)| \geq 2$ and $|N(x_3) \cap N(x_4)| \geq 2$. Choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$

or $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$. It is a routine work to prove that there exist a vertex subset $x_4 \in A \subseteq S$ such that $G_{(x,M)}[V(G_1) \cup A]$ is Z_3 -connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup A]) \leq 2$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3 -connected, a contradiction.

Subcase 1.3. $|N_s| = 2$.

If $|N_s| = 2$, then $|N_1| = 1$ and $|N_2| = 1$. Let $N_1 = \{x_1\}$, $N_2 = \{x_2\}$ and $N_s = \{x_3, x_4\}$.

Claim 2. $x_3x_4 \in E(G)$.

Proof of Claim 2. If $x_3x_4 \notin E(G)$, choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = \kappa(G) = 4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

It is clear that $\alpha(G_1) \leq 1$ or $\alpha(G_2) \leq 1$ since $\alpha(G_1) + \alpha(G_2) \leq 3$. We assume, without loss of generality that $G_1 = K_m$.

Claim 3. $m \geq 2$.

Proof of Claim 4. If $m = 1$, then $N(x_1) = S$. Choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$ or $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3 -connected. Let H be the maximal Z_3 -connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. So $\alpha(G_{(x,M)} - H) = 3$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, then $x_2 \in H$. Note that $\alpha(G_{(x,M)} - H) \leq \alpha(G - G[N[x]]) = 2$, a contradiction. Hence $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $y \in S - H - \{x\}$. Then $yx_3 \notin E(G)$ and $yx_4 \notin E(G)$. If there exist $z \in V(G_2 - x_2)$ such that $\{y, z, x_2\}$ is an independent set of three vertices, then $x_3z \in E(G)$ and $x_4z \in E(G)$. If not, $\{y, z, x_2, x_3\}$ or $\{y, z, x_2, x_4\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Thus $z \in H$. This implies $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Therefore $m \geq 2$.

Claim 4. $|S| = 4$.

Proof of Claim 3. If $|S| \geq 5$, by the minimum property of S , then $d(x_i) \geq 5$, where $i \in \{1, 2, 3, 4\}$. Firstly, we assume that $x_1x_3 \in E(G)$ and $x_1x_4 \in E(G)$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, then $G_{(x,M)}[N(x)]$ is Z_3 -connected. Note that $\alpha(G - N[x]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. So $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $V_1 = \{v | v \in S - \{x, x_3, x_4\} \text{ and } G[V(G_1) \cup \{v\}] \text{ is a complete graph}\}$. If there exist one vertex $u \in V(G_1 - x_1) \cup V_1$ such that $ux_3 \in E(G)$ or $ux_4 \in E(G)$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, then $G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]$ is Z_3 -connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. Hence $vx_3 \notin E(G)$ and $vx_4 \notin E(G)$ for each $v \in V(G_1 - x_1) \cup V_1$. Let $V'_1 = G_2 + S - V_1 - \{x, x_2, x_3, x_4\}$. If there exist one vertex $w \in V'_1$ such that $\{v, w, x_2\}$ is an independent set, then $wx_3 \in E(G)$ and $wx_4 \in E(G)$. For otherwise, $\{v, x_3, x_2, w\}$ or $\{v, x_4, x_2, w\}$ is an

independent set of four vertices, contrary to that $\alpha(G) = 3$. Let $V_2 = \{w | w \in V_1' \text{ and } \{v, w, x_2\} \text{ is an independent set}\}$. Then $G_{(x,M)}[V_2 \cup \{x_1, x_3, x_4\}]$ is Z_3 -connected. Note that $\alpha(G_{(x,M)} - G_{(x,M)}[V_2 \cup \{x_1, x_3, x_4\}]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. Secondly, we assume, without loss of generality, that $x_1x_3 \in E(G)$, but $x_1x_4 \notin E(G)$. If $x_2x_3 \notin E(G)$, choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$, then $\kappa(G_{(x,M)}) = \kappa(G) = 4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $x_2x_3 \in E(G)$. If $N(x_1) \cap N(x_3) \cap S = \{x\}$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, we can prove that $G_{(x,M)}[V(G_1) \cup \{x_3, x_4\}]$ is Z_3 -connected or $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)}) = 3$ since $|S| \geq 5$. By Lemma 2.6 or the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $|N(x_1) \cap N(x_3) \cap S| \geq 2$. Similarly $|N(x_3) \cap N(x_4) \cap S| \geq 2$ and $|N(x_2) \cap N(x_3) \cap S| \geq 2$. Let $y_1 \in N(x_1) \cap N(x_3)$, $y_2 \in N(x_3) \cap N(x_4)$ and $y_3 \in N(x_2) \cap N(x_3)$. If $y_1 = y_2$, then $G[\{x, x_1, x_3, x_4, y_1\}] = K_4$ (center at x_4) which is Z_3 -connected. It follows that $G[N[x] \cup \{y_1\}]$ is Z_3 -connected. Denote by $G^* = G/G[N[x] \cup \{y_1\}]$. Then $\alpha(G^*) \leq 2$ and $\delta(G^*) \geq 4$. By Lemmas 2.3 and 2.2, G is Z_3 -connected. Similarly if $y_1 = y_3$ or $y_2 = y_3$, G is also Z_3 -connected. So we suppose that $y_1 \neq y_2 \neq y_3$. Note that $\alpha(G) = 3$. Then $G[\{y_1, y_2, y_3\}]$ contains at least one edge. We assume, without loss of generality, that $y_1y_2 \in E(G)$. Then $G[N[x] \cup \{y_1, y_2, y_3\}]$ is a triangularly connected graph. It is easy to prove that $G_{(x,M)}[N[x] \cup \{y_1, y_2, y_3\}]$ is Z_3 -connected, where M is any way to pair the vertices of $N(x)$. As the above, we can prove that $G_{(x,M)}$ is Z_3 -connected, a contradiction. Thirdly, we assume that $x_1x_3 \notin E(G)$ and $x_1x_4 \notin E(G)$. If $x_2x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$, choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$ or $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, then $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)}) = 3$ since $|S| \geq 5$. As the above, we are done. Next we suppose that $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$. Choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$. It is clear that $G_{(x,M)}[\{x_2, x_3, x_4\}]$ is Z_3 -connected. Let H be the maximal Z_3 -connected subgraph which contains $G_{(x,M)}[\{x_2, x_3, x_4\}]$. Next we shall prove that $\alpha(G_{(x,M)} - H) \leq 2$. On the one hand, $\alpha(G - N[x]) = 2$. On the other hand, if there exist two vertices $u_1, u_2 \in G_{(x,M)} - H$ such that $\{x_1, u_1, u_2\}$ is an independent set, then there must be one vertex in $\{x_2, x_3, x_4\}$, say x_3 such that $\{x_1, x_3, u_1, u_2\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3 -connected, a contradiction. Therefore $|S| = 4$.

Note that $\kappa(G) = 4$. Then $e(x_i, G_1 - x_1) \geq 1$, where $i \in \{3, 4\}$. If $x_1x_3 \in E(G)$ or $x_1x_4 \in E(G)$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$ or $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$, respectively. It is easy to prove that $G_{(x,M)}[V(G_1) \cup \{x_1, x_3, x_4\}]$ is Z_3 -connected. If there exist one vertex, say $v \in V(G_1)$ such that $vy \notin E(G)$, then $\alpha(G_2 + y) \leq 2$. If $vy \in E(G)$ for each $v \in V(G_1)$, then $G_{(x,M)}[V(G_1) \cup \{x_1, x_3, x_4, y\}]$ is Z_3 -connected. By Lemma 2.6, in either case, $G_{(x,M)}$ is Z_3 -connected, a contradiction. So $x_1x_3 \notin E(G)$ and $x_1x_4 \notin E(G)$. Thus $m \geq 3$. If $e(\{x_3, x_4\}, G_1) \geq 3$, choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$. It is clear that $G[V(G_1) \cup \{x_3, x_4\}]$ is Z_3 -connected. By Lemma 2.6, $G_{(x,M)}$ is Z_3 -connected, a contradiction. So $e(\{x_3, x_4\}, G_1) = 2$. Thus $\alpha(G[V(G_2) \cup \{x, x_3, x_4\}]) \leq 2$ and $\alpha(G[V(G_2) \cup$

$\{x, y, x_3, x_4\}\} \leq 2$ if $e(y, G_1) < |V(G_1)|$. When $m \geq 5$, G_1 is Z_3 -connected follows from Lemma 2.1. Choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G[V(G_1) \cup \{x_3, x_4\}]$ is Z_3 -connected, we are done. Therefore we suppose that $3 \leq m \leq 4$. If $G_2 = K_n$, by the symmetry, $3 \leq n \leq 4$, $x_2x_i \notin E(G)$ and $e(\{x_3, x_4\}, G_2) = 2$, where $i \in \{3, 4\}$. If $x_3y \in E(G)$ or $x_4y \in E(G)$, choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G[V(G_1) \cup \{y, x_3, x_4\}]$ is Z_3 -connected. It follows from Lemmas 2.6 that $G_{(x,M)}$ is Z_3 -connected, a contradiction. Hence we assume that $x_3y \notin E(G)$ and $x_4y \notin E(G)$. Let $G_1 = G_2 = K_3$. Then $G[V(G_1 \cup \{y\})] = K_4$ and $G[V(G_2 \cup \{y\})] = K_4$. Thus G is the graph (a) showed in Figure 3 which is even. So G admits a nowhere-zero 3-flow. Let $G_1 = G_2 = K_4$. If there exist $w_1 \in (V(G_1) - x_1)$ and $w_2 \in (V(G_2) - x_2)$ such that $w_1y \notin E(G)$ and $w_2y \notin E(G)$, then $\{x, y, w_1, w_2\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence, $e(y, G_1) = 4$ or $e(y, G_2) = 4$. Thus, G contains Z_3 -connected subgraph W_4 . By Lemma 2.6, G admits a nowhere-zero 3-flow. At last, we consider that $G_1 = K_3$ and $G_2 = K_4$. Then G is even or the graph (b) showed in Figure 3, which has a $\{C_3, C_4, C_3 \oplus C_4\}$ -composition. So G also admits a nowhere-zero 3-flow. Next we suppose that $G_2 \neq K_n$. It is easy to check that $G_2 - x_2 = K_l$ since $\alpha(G) = 3$, where $l \geq 2$. Note that $\kappa(G) = 4$ and $\alpha(G_2) = 2$. We have that $1 \leq e(x_2, K_l) \leq l - 1$. Note that $\alpha(G_1 + y) \leq 2$. If there exist some way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3 -connected, then $G_{(x,M)}$ is Z_3 -connected follows from Lemma 2.6, a contradiction. Next we suppose that there is no such M that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3 -connected. If $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$, choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}[\{x_2, x_3, x_4\}]$ is Z_3 -connected. Note that $\kappa(G) = 4$. Then $e(\{x_2, x_3, x_4\}, K_l) \geq 3$. It is clear that if $l \geq 4$ or $l = 2$, then $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3 -connected, a contradiction. Hence $l = 3$. If $e(\{x_2, x_3, x_4\}, K_l) \geq 4$, as the above, we are done. Next we assume that $e(\{x_2, x_3, x_4\}, K_l) = 3$. In this case, $G[V(K_l) \cup \{y\}] = K_4$. Thus $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3 -connected, a contradiction. Therefore $x_2x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$. As the above, if $e(\{x_3, x_4\}, K_l) \geq 3$, we need only consider the case of $l = 3$ and $e(\{x_3, x_4\}, K_l) = 3$. Choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, it is easy to check that $G_{(x,M)}[V(K_l) \cup \{x_3, x_4\}]$ is Z_3 -connected or can be contracted to K_4 . For the former, we are done; for the latter, $e(x_2, V(K_l) \cup \{x_3, x_4\}) \geq 3$ if $x_2y \notin E(G)$ and $e(x_2y, V(K_l) \cup \{x_3, x_4\}) \geq 4$ if $x_2y \in E(G)$. We get that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3 -connected follows from Lemmas 2.1 and 2.2, a contradiction. Hence $e(\{x_3, x_4\}, K_l) = 2$. If $m = 2$, then $G_2 = K_3$, a contradiction. If $m \geq 5$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3 -connected, a contradiction. Therefore $3 \leq m \leq 4$. Note that $\kappa(G) = 4$. Then x_3 and x_4 have no common neighbor vertex in K_l . For $l = 3$, let $G_2 - x_2 = G[\{u_1, u_2, u_3\}]$, then $1 \leq e(x_2, \{u_1, u_2, u_3\}) \leq 2$. We assume, without loss of generality, that $x_3u_2 \in E(G)$, $x_4u_3 \in E(G)$ and $x_2u_1 \in E(G)$. If $x_2x_3 \notin E(G)$, then $x_2u_3 \in E(G)$. If not, $\{x_1, x_2, x_3, u_3\}$ constructs an independent set of four vertices, a contradiction. Similarly, if $x_2x_4 \notin E(G)$, then $x_2u_2 \in E(G)$. Note that $\alpha(G_2) = 2$. Then $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$. Choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$ or $M =$

$\langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$, we also get that $G_{(x,M)}$ is Z_3 -connected, a contradiction. For $l = 4$, let $G_2 - x_2 = G[\{u_1, u_2, u_3, u_4\}]$, then $1 \leq e(x_2, \{u_1, u_2, u_3, u_4\}) \leq 3$. If $e(x_2, \{u_1, u_2, u_3, u_4\}) = 3$, then $G[\{x_2, u_1, u_2, u_3, u_4\}]$ contains W_4 as a spanning subgraph. By Lemma 2.1, $G[\{x_2, u_1, u_2, u_3, u_4\}]$ is Z_3 -connected. Choose $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3 -connected, a contradiction. Hence $1 \leq e(x_2, \{u_1, u_2, u_3, u_4\}) \leq 2$. Thus there must be $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$ since $\alpha(G) = 3$, a contradiction.

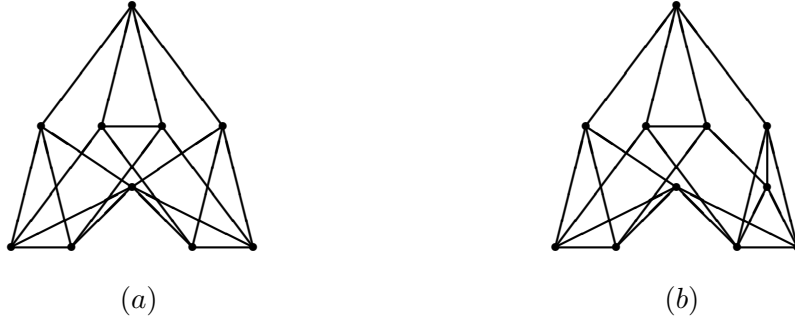


Fig. 3 Two graphs which admits a nowhere-zero 3-flow.

Case 2. $d(v) \geq 5$ for any vertex $v \in V(G)$.

We first consider that $\kappa'(G) = 4$. There must be a nontrivial minimal edge cut E_c such that $|E_c| = 4$. Let G_1 and G_2 be two components of $G - E_c$.

Subcase 2.1. Either G_1 or G_2 is Z_3 -connected.

We assume, without loss of generality, that G_1 is Z_3 -connected. Denote by $G_1^* = G/G_1$. By Lemma 2.2, it is sufficient to show that G_1^* admits a nowhere-zero 3-flow. Clearly $\alpha(G_1^*) \leq \alpha(G)$. Note that $\delta(G_1^*) \geq \kappa'(G_1^*) \geq 4$. If $\alpha(G_1^*) \leq 2$, by Lemma 2.3, G_1^* admits a nowhere-zero 3-flow. Next we assume that $\alpha(G_1^*) = 3$. If $\kappa(G_1^*) \geq \kappa(G) = 4$, by the induction hypothesis, G_1^* also admits a nowhere-zero 3-flow. Therefore it is crucial to consider that $\kappa(G_1^*) < \kappa(G)$.

Let v^* be the new vertex into which G_1 is contracted and let V^* be a nontrivial minimal vertex cut of G_1^* . Then $v^* \in V^*$. For otherwise, V^* is also a nontrivial minimum vertex cut of G , contrary to that $\kappa(G_1^*) < \kappa(G)$. Let $V_1^* = V^* - v^*$. Then V_1^* is a nontrivial minimal vertex cut of G_2 . If $|V_1^*| = 1$, we can get $\kappa(G) \leq 3$, a contradiction. If $|V_1^*| \geq 3$, then $|V^*| \geq 4$. It follows that $\kappa(G_1^*) \geq \kappa(G) = 4$, contrary to that $\kappa(G_1^*) < \kappa(G)$. Hence $|V_1^*| = 2$. Let G_{21} and G_{22} be two connected components of $G_2 - V_1^*$. If $\alpha(G_{21}) = 2$ or $\alpha(G_{22}) = 2$, we can get that $\alpha(G) = 4$, contrary to that $\alpha(G) = 3$. Therefore both G_{21}

and G_{22} are complete graphs. We assume, without loss of generality, that $E_c = \{x_1y_1, x_2y_2, x_3y_3, x_4y_4\}$. Then $x_i \neq x_j$ and $y_i \neq y_j$, where $1 \leq i < j \leq 4$. Let $X = \{x_1, x_2, x_3, x_4\} \subset V(G_1)$ and $Y = \{y_1, y_2, y_3, y_4\} \subset V(G_2)$. Then $e(X, G_{21}) = 2$ and $e(X, G_{22}) = 2$ since $\kappa(G) = 4$. Let $G_{21} = K_m$ and $G_{22} = K_n$. Then $m \geq 4$ and $n \geq 4$ since $d(v) \geq 5$ for any vertex $v \in V(G)$. If $m \geq 5$, by Lemma 2.1(1), G_{21} is Z_3 -connected. If $m = 4$, then $e(V_1^*, G_{21}) \geq 6$. So there must be one vertex, say $u^* \in V_1^*$ such that $e(u^*, G_{21}) \geq 3$. It is clear that $G_{21} + u^*$ contains W_4 as a spanning subgraph. It follows from Lemma 2.1 that $G_{21} + u^*$ is Z_3 -connected. Therefore it is easy to prove that G_1^* admits a nowhere-zero 3-flow.

Subcase 2.2. Neither G_1 nor G_2 are Z_3 -connected.

If neither G_1 nor G_2 are Z_3 -connected, we can immediately get the following claim.

Claim 5. G_i contains no Z_3 -connected subgraph H_i with $|V(G_i - H_i)| \leq 3$, where $i \in \{1, 2\}$.

Proof of Claim 5. We prove by contradiction and suppose that G_i contains Z_3 -connected subgraph H_i with $|V(G_i - H_i)| \leq 3$. Let $G_i^* = G_i/H_i$ and w_i^* be the new vertex into which H_i is contracted. Then $d(w_i^*) \geq 4$ since $\kappa'(G) \geq 4$. Note that $|V(G_i - H_i)| \leq 3$. So we have that $|V(G_i^*)| \leq 4$ and G_i^* contains 2-cycles. It is easy to prove that G_i^* is Z_3 -connected. By Lemma 2.2, G_i is Z_3 -connected, a contradiction. Hence, this claim holds.

We have that $|V(G_1)| \geq 6$ and $|V(G_2)| \geq 6$ since $\delta(G) \geq 5$. It follows that $\alpha(G_1) = \alpha(G_2) = 2$. Let A be a maximal independent set of G_1 and $B = \{y | yx \notin E(G), y \in V(G_2), x \in A\}$. Then there are at most two vertices in B whose neighbors in A . Hence $|B| \geq (6 - 2) = 4$ and $G[B]$ is a complete graph since $\alpha(G) = 3$. If $|B| \geq 5$, by Lemma 2.1, $G[B]$ is Z_3 -connected, contrary to Claim 5. So $|B| = 4$. It is just that $|A \cap \{x_1, x_2, x_3, x_4\}| = 2$ and $|V(G_2)| = 6$. We assume, without loss of generality, that $A = \{x_1, x_2\}$. Then for $1 \leq i \leq 2$, $e(y_i, B) \geq 2$ since $\delta(G) \geq 4$. Note that $G[B] = K_4$. If there exist some y_i such that $e(y_i, B) \geq 3$, then $G[B \cup \{y_i\}]$ contains Z_3 -connected spanning subgraph W_4 , contrary to Claim 5. Hence $e(y_1, B) = 2$ and $e(y_2, B) = 2$ while $y_1y_2 \in E(G)$. Let $B = \{y_3, y_4, y_5, y_6\}$. If $|N(y_1) \cap N(y_2)| \leq 1$, then G_2 contains L_3^+ or W_4 as a subgraph which is Z_3 -connected follows from Lemma 2.1(3)(4), contrary to Claim 5. Therefore we assume that $|N(y_1) \cap N(y_2)| = 2$. Then $N(y_1) \cap N(y_2) = \{y_5, y_6\}$. If $|V(G_1)| \geq 7$, then G_1 contains K_m as a subgraph, contrary to Claim 5, where $m \geq |V(G_1)| - 2 \geq 5$. So $|V(G_1)| = 6$. Let $V(G_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then $G[\{x_2, x_3, x_5, x_6\}] = K_4$ since $y_1y_4 \notin E(G)$ and $G[\{x_2, x_4, x_5, x_6\}] = K_4$ since $y_1y_3 \notin E(G)$. It is clear that G_1 contains K_5^- as a subgraph, contrary to Claim 5.

Next we consider that $\kappa'(G) = 5$. Let $S = \{v_1, v_2, v_3, v_4\}$ be a minimal cut of G . Then $G[S]$ contains at least one edge, say v_1v_2 . Let G_1 and G_2 be two components of $G - S$. Note that $\alpha(G) = 3$. Then one of G_1 and G_2 must be a

complete graph. We assume, without loss of generality, that $G_1 = K_m$. Thus $\alpha(G_2) \leq 2$. Let $A \subseteq S$ and $A \neq \emptyset$. We have the following claim.

Claim 6. If $G[V(G_1) \cup A]$ is Z_3 -connected, then G is also Z_3 -connected.

Proof of Claim 6. Let $G^* = G/G[V(G_1) \cup A]$. Note that $\alpha(G_2) \leq 2$. If $A = S$, by Lemma 2.6, G^* is Z_3 -connected. If $A \subset S$, then $e(v, G_1) = 1$, for any vertex $v \in S - A$. This yields that $\alpha(G_2 + v) \leq 2$ since $\alpha(G) = 3$. Hence $\alpha(G_2 + S - A) \leq 2$. By Lemma 2.6, G^* is Z_3 -connected. Therefore if $G[V(G_1) \cup A]$ is Z_3 -connected, G is also Z_3 -connected follows from Lemma 2.2.

Note that $\delta(G) \geq \kappa'(G) = 5$. So $m \geq 2$. If $m \geq 5$, by Lemma 2.1 G_1 is Z_3 -connected. Note that $e(V(G_1), S) \geq 5$. Then there must be one vertex $v \in S$ such that $e(v, V(G_1)) \geq 2$. Thus $G[V(G_1) \cup \{v\}]$ is Z_3 -connected. By Claim 6, G is Z_3 -connected. If $m = 4$, then $e(V(G_1), S) \geq 8$. If there exist one vertex $v \in S$ such that $e(v, V(G_1)) \geq 3$, then $G[V(G_1) \cup \{v\}] = K_5$ or K_5^- which is Z_3 -connected follows from Lemma 2.1. By Claim 6, G is Z_3 -connected. Next we suppose that $e(v, V(G_1)) = 2$ for each vertex $v \in S$. It follows that $e(\{v_1, v_2\}, V(G_1)) = 4$. If $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$, then $G[V(G_1) \cup \{v_1, v_2\}]$ contains L_3^+ or W_4 as a subgraph. It follows from Lemma 2.1 that $G[V(G_1) \cup \{v_1, v_2\}]$ is Z_3 -connected. By Claim 6, G is Z_3 -connected. Next we assume that $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| = 2$. Then $|N_{G_1}(v_3) \cap N_{G_1}(v_4)| = 2$. If $v_i v_j \in E(G)$, then $G[V(G_1) \cup S]$ is Z_3 -connected, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G is Z_3 -connected. Next we assume that $v_i v_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Let $N_{G_1}(v_3) = \{x, y\}$. We consider the graph $G_{[v_3x, v_3y]}$ which contains a 2-cycle xyx . Denote by $H_1 = G_1 + S - v_3 + xy$. It is clear that H_1 is a Z_3 -connected subgraph of $G_{[v_3x, v_3y]}$. Note that $e(S - v_3, G_2) \geq 7$ and $\kappa'(G) = 5$. Then $G_{[v_3x, v_3y]} - v_3$ is a 4-edge-connected graph. By Lemma 2.6, $G_{[v_3x, v_3y]} - v_3$ is Z_3 -connected. So is $G_{[v_3x, v_3y]}$ since $e(v_3, G_2) \geq 3$. It follows from Lemma 2.4 that G is Z_3 -connected. If $m = 3$, then $e(V(G_1), S) \geq 9$. Let $D_3 = \{v | v \in S \text{ and } e(v, G_1) = 3\}$. Then $|D_3| \geq 1$. If $|D_3| \geq 2$, then G contains Z_3 -connected subgraph K_5^- . By Lemma 2.6, G is Z_3 -connected. Hence we suppose that $|D_3| = 1$. Then $e(v, G_1) \geq 2$ for each vertex $v \in S$. So $4 \leq e(\{v_1, v_2\}, V(G_1)) \leq 5$. Let $A = \{v_1, v_2\}$. We shall show that $G[V(G_1) \cup A]$ is Z_3 -connected. If $e(\{v_1, v_2\}, V(G_1)) = 5$, then $G[V(G_1) \cup A]$ contains K_5^- as a spanning subgraph. If $e(\{v_1, v_2\}, V(G_1)) = 4$, then $e(v_i, G_1) = 2$, where $i \in \{1, 2\}$. Note that $e(u, S) \geq 3$ for each vertex $u \in V(G_1)$. Then $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$. Thus $G[V(G_1) \cup A]$ contains W_4 as a spanning subgraph. Therefore $G[V(G_1) \cup A]$ is Z_3 -connected follows from Lemma 2.1. By Claim 6, G is Z_3 -connected. At last, we assume that $m = 2$. Let $G_1 = u_1 u_2$. Then $N(u_1) = \{u_2, v_1, v_2, v_3, v_4\}$ and $N(u_2) = \{u_1, v_1, v_2, v_3, v_4\}$. We consider the graph $G_{[u_1 v_1, u_1 v_2]}$ which contains a 2-cycle $v_1 v_2 v_1$. Let H_2 be the maximal Z_3 -connected subgraph of $G_{[u_1 v_1, u_1 v_2]}[V(G_1) \cup S]$. If $v_i v_j \in E(G)$, then $H_2 = G_{[u_1 v_1, u_1 v_2]}[V(G_1) \cup S]$, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G^* is Z_3 -connected. It follows from Lemmas 2.1 and 2.4 that G is Z_3 -connected. Next we assume that $v_i v_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Thus $H_2 = v_1 u_2 v_2 + v_1 v_2$ and $v^* v_j \in E(G^*)$, where

$j \in \{3, 4\}$. Note that $d_{G^*}(v_3) \geq 5$. Next we consider the graph $G_{[v_3u_1, v_3v^*]}^*$ which contains a 2-cycle u_1v^* . Let H^* be the maximal Z_3 -connected subgraph which contains a 2-cycle u_1v^* of $G_{[v_3u_1, v_3v^*]}^*$ and v^{**} be the vertex into which H^* is contracted. Then $\alpha(G_2 + v^{**}) \leq 2$. For otherwise, there are two vertices in G_2 , say x, y such that $\{v^{**}, x, y\}$ is an independent set of three vertices. Note that $v_1v_4 \notin E(G)$. Thus $\{v_1, v_4, x, y\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence $\alpha(G_2 + v^{**} - v_3) \leq 2$. Denote by $G^{**} = G_{[v_3u_1, v_3v^*]}^*/H^*$. Then $G^{**} = G_2 + v^{**}$. Note that $\delta(G^{**} - v_3) \geq 4$. By Lemma 2.3, $G^{**} - v_3$ is Z_3 -connected. So is G^{**} since $d_{G^{**}}(v_3) \geq 3$. It follows from Lemmas 2.1 and 2.4 that both G^* and G are Z_3 -connected, a contradiction.

Therefore we conclude that if $\alpha(G) = 3$, then G admits a nowhere-zero 3-flow, this completes the proof.

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Modal operators on equality algebras

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Abstract. The main goal of this paper is to investigate modal operators on equality algebras. To begin with, we introduce the notion of modal operators on equality algebras and investigate some important properties of this operator. As applications, we give a characterization of prelinear equality algebras. In the following, we give the concepts of modal filters and modal congruences of equality algebras and obtain some related results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. Finally, using strong modal filters, we establish the uniform structures on modal equality algebras and we prove that modal equality algebras with uniform topologies are topological modal equality algebras.

Keywords: equality algebra, modal operator, modal filter, modal congruence, uniform structure.

1. Introduction

Fuzzy type theory(FTT) [12, 13, 14] was developed as a fuzzy counterpart of the classical higher-order logic. Since the truth values for algebra is no longer a residuated lattice, a specific algebra which called an EQ-algebra was proposed by Novák and De Baets in [11] which generalizes residuated lattice. In [3], it was mentioned if the product in EQ-algebras is replaced by any other smaller

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binary operation, we still obtains an EQ-algebra. Based on the above reasons, a new algebraic structure was introduced by Jenei in [4], called equality algebra, which consisting of two binary operations meet and equivalence, and constant 1.

In 1981, modal operators on Heyting algebras were introduced and studied as algebraic counter-part of the intuitionistic propositional logic by Macnab [10]. Since then, properties of modal operators were considered on other algebraic structures such as MV-algebra [2], bounded commutative residuated $R\ell$ -monoids (simply called $R\ell$ -monoids) [15], commutative reesiduated lattices [9] and so on. The essence of modal operator is closure operator, and closure operator is an important part of the theoretical study of partial order sets.

In this paper, we define modal operators for equality algebras. This paper is structured in five sections. In section 2, we recall the definition of equality algebras and their basic properties that will be used in this paper. In section 3, we introduce the notion of modal operators in equality algebras and investigate some important related properties. And we give some characterizations of prelinear equality algebras. In section 4, we introduce modal filters and modal congruences of modal equality algebras and obtain some important results. And we show that there is a one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. In section 5, we establish uniform structures by the special family of strong modal filters on equality algebras, and then induce uniform topologies. Moreover, we show that modal equality algebras with uniform topologies are topological equality algebras.

2. Preliminaries

In the section, we summarize some definitions and results about equality algebras, which will be used in the following sections of the paper.

Definition 2.1 ([3, 4]). *An algebra structure $(E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called an equality algebra, if it satisfies the following conditions: for all $x, y, z \in E$,*

(E1) *$(E, \wedge, 1)$ is a commutative idempotent integral monoid (i.e. meet semi-lattice with top element 1).*

(E2) $x \sim y = y \sim x$.

(E3) $x \sim x = 1$.

(E4) $x \sim 1 = x$.

(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$.

(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$.

(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

Where $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is an equality operation. Also, other two operations are defined, called implication and equivalence oper-

ation, respectively:

$$x \rightarrow y = x \sim (x \wedge y)x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

In what follows, we let $E = (E, \wedge, \sim, 1)$ be an equality algebra.

Proposition 2.2 ([3, 4]). *Let E be an equality algebra. Then the following properties hold: for all $x, y, z \in E$:*

- (1) $x \sim y \leq x \leftrightarrow y \leq x \rightarrow y$,
- (2) $x \rightarrow y = 1$ if and only if $x \leq y$,
- (3) $1 \rightarrow x = x, x \rightarrow 1 = 1, x \rightarrow x = 1$,
- (4) $x \leq y \rightarrow x$,
- (5) $x \leq (x \rightarrow y) \rightarrow y$,
- (6) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (7) $x \leq y \rightarrow z$ if and only if $y \leq x \rightarrow z$,
- (8) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (9) $y \leq x$ implies $x \leftrightarrow y = x \rightarrow y = x \sim y$,
- (10) $x \sim y = 1$ iff $x = y$.

Proposition 2.3 ([18]). *Let E be an equality algebra. Then the following statements hold: for all $x, y, z \in E$,*

- (i) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y$,
- (ii) $x \rightarrow y = x \rightarrow (x \wedge y)$,
- (iii) $x \sim y \leq (z \rightarrow x) \sim (z \rightarrow y)$,
- (iv) $x \sim y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
- (v) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
- (vi) $x \rightarrow y \leq (x \wedge z) \rightarrow (y \wedge z)$,
- (vii) $x \rightarrow y = ((x \rightarrow y) \rightarrow y) \rightarrow y$.

Definition 2.4 ([18]). *Let E be an equality algebra. Then, E is called prelinear, if 1 is the unique upper bound of the set $\{x \rightarrow y, y \rightarrow x\}$, for all $x, y \in E$.*

Proposition 2.5 ([18]). *If E is a prelinear equality algebra, then $(E, \leq)$ is a lattice, where the join operation is given by $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$, for all $x, y \in E$.*

Proposition 2.6 ([19, 20]). *An equality algebra E is prelinear iff $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$, for all $x, y, z \in E$.*

Definition 2.7 ([18]). *A lattice equality algebra is an equality algebra which is lattice.*

The following proposition provides some properties of lattice equality algebras.

Proposition 2.8 ([18]). *Let E be a lattice equality algebra. Then the following properties hold, for all $x, y \in E$:*

- (i) *for all indexed families $\{x_i\}_{i \in I}$ in E , we have $(\bigvee_{x \in I}) \rightarrow y = \bigwedge_{i \in I} (x_i \rightarrow y)$, provided that the infimum and supremum of $\{x_i\}_{i \in I}$ exist in E ;*
- (ii) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
- (iii) $x \rightarrow y = (x \vee y) \rightarrow y$.

Theorem 2.9 ([18]). *Any prelinear equality algebra is a distributive lattice.*

The following proposition provides some properties of prelinear equality algebras.

Proposition 2.10 ([18]). *Let E be a prelinear equality algebra. Then the following statements holds: for all $x, y, z \in E$,*

- (i) $x \leftrightarrow y = x \sim y$;
- (ii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

An equality algebra $(E, \wedge, \sim, 1)$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E , we define the negation “ $\prime$ ” on E by, $x' = x \rightarrow 0 = x \sim 0$, for all $x \in E$.

Proposition 2.11 ([18]). *Let $(E, \wedge, \sim, 0, 1)$ be a bounded lattice equality algebra. Then the following properties hold: for all $x, y \in E$,*

- (i) $(x \vee y)' = x' \wedge y'$;
- (ii) $x \leq (x')'$;
- (iii) $x \rightarrow y \leq y' \rightarrow x'$, and if it is involutive, then $x \rightarrow y = y' \rightarrow x'$.

Definition 2.12 ([4]). *Let E be an equality algebra, $F \subseteq E$. F is called a filter of E if for all $a, b \in E$,*

- (i) $1 \in F$;
- (ii) $a \in F, a \leq b \Rightarrow b \in F$;
- (iii) $a, a \sim b \in F \Rightarrow b \in F$.

We will denote the set of all filters of E by $F(E)$. Clearly, $\{1\}, E \subseteq F(E)$, and $F(E)$ is closed under arbitrary intersections. And hence we have $(F(E), \subseteq)$ is a complete lattice. A filter F of an equality algebra E is proper if $F \neq E$. A proper filter F is called maximal if $F \subseteq G \subseteq E$ implies $F = G$ for all G proper filter of E . An equality algebra $(E, \wedge, \sim, 1)$ is called simple, if $F(E) = \{\{1\}, E\}$. (See [1, 5])

Definition 2.13 ([4]). *A subset θ of $E \times E$ is called congruence of E , if it is an equivalence relation on E and for all $a, a', b, b' \in E$ such that $(a, b), (a', b') \in \theta$ the following hold:*

- (i) $(a \wedge a', b \wedge b') \in \theta$;
- (ii) $(a \sim a', b \sim b') \in \theta$.

We will denote the set of all congruences of E by $C(E)$.

Let F be a filter of E . Define the congruence relation $\equiv_F$ on E by $x \equiv_F y$ iff $x \sim y \in F$. The set of all congruence class is denote by E/F , i.e. $E/F = \{[x] \mid x \in E\}$, where $[x] = \{x \in E \mid x \equiv_F y\}$. Define $\bullet, \rightarrow$ on E/F as follows: $[x] \bullet [y] = [x \wedge y]$, $[x] \rightarrow [y] = [x \sim y]$. Therefore, $(E/F, \bullet, \rightarrow, [1])$ is an equality algebra which is called a quotient equality algebra of E with respect to F . (See [1])

Proposition 2.14 ([4]). *Let E be an equality algebra. $F \in F(E)$ iff for all $a, b \in E$,*

- (i) $1 \in F$;
- (ii) $a, a \rightarrow b \in F \Rightarrow b \in F$ holds, where $\rightarrow$ denotes the implication of E .

The next theorem makes connection between $F(E)$ and $C(E)$.

Theorem 2.15 ([4]). *Let E be an equality algebra, $\theta, \varphi \in C(E)$, $F \in F(E)$. Then:*

- (a) $[1]_\theta \in F(E)$, where $[1]_\theta = \{a \mid (a, 1) \in \theta\}$;
- (b) $[1]_{\theta_F} = F$;
- (c) $\theta_{[1]_\theta} = \theta$;
- (d) if $[1]_\theta = [1]_\varphi$, then $\theta = \varphi$.

Theorem 2.16 ([4]). *For $\theta \in C(E)$, we have $(a, b) \in \theta$ iff $(a \sim b, 1) \in \theta$.*

Next, we review some notions about uniformity which will be necessary in the following section.

Let X be a nonempty set and A, B be any subset of $X \times X$. We have the following notation:

- (1) $A \circ B = \{(x, y) \in X \times X : (x, z) \in A, (z, y) \in B, \text{ for some } z \in X\}$;
- (2) $A^{-1} = \{(x, y) \in X \times X : (y, x) \in A\}$;
- (3) $\Delta = \{(x, x) \in X \times X : x \in X\}$.

Definition 2.17 ([16, 7, 8]). *A nonempty collection K of subsets of $X \times X$ is called an uniformity on X , which satisfies the following conditions:*

- (A1) $\Delta \subseteq A$ for any $A \in K$;
- (A2) if $A \in K$, then $A^{-1} \in K$;
- (A3) if $A \in K$, then there exists $B \in K$ such that $B \circ B \subseteq A$;
- (A4) if $A, B \in K$, then $A \cap B \in K$;
- (A5) if $A \in K$ and $A \subseteq B \subseteq X \times X$, then $B \in K$. Then the pair (X, K) is called an uniform structure (uniform space).

3. Modal operators on equality algebras

In this section, we introduce the notion of modal operators on equality algebras and we investigate some important related properties of modal operators. And in the end, we give some characterizations of prelinear equality algebras.

Definition 3.1. Let E be an equality algebra. The mapping $f : E \rightarrow E$ is called a modal operator if it satisfies the following conditions: for all $x, y \in E$,

- (f1) $x \leq f(x)$;
- (f2) $f(f(x)) = f(x)$;
- (f3) $f(x \wedge y) = f(x) \wedge f(y)$;
- (f4) $f(x \sim y) \leq f(x) \sim f(y)$.

And the pair (E, f) is called an equality algebra with a modal operator (simplify, modal equality algebra).

Let (E, f) be a modal equality algebra. The kernel of f is the set $Ker(f) = \{x \in E \mid f(x) = 1\}$. A modal operator f is said to be faithful if $Ker(f) = \{1\}$.

Example 3.2. Let $(E, \wedge, \sim, 1)$ be an equality algebra. We define mappings $f_1, f_2 : E \rightarrow E$ such that $f_1(x) = x$, $f_2(x) = 1, \forall x \in E$. We can see that both f_1 and f_2 are modal operators on E . Also, f_1 is faithful since $Ker(f) = \{1\}$. Moreover, modal operators f_1 and f_2 are called trivial.

Example 3.3. Let $E = \{0, a, b, c, 1\}$ with $0 < c < a, b < 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

Then we can see that $(E, \wedge, \sim, 1)$ is an equality algebra. Now, we define a mapping f on E as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ a, & x = a, c \\ 1, & x = b, 1 \end{cases}$$

It is easy to check that f is a modal operator on E .

Example 3.4. Let $E = \{0, a, b, 1\}$ with $0 < a < b < 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Then we can see that $(E, \wedge, \sim, 1)$ is an equality algebra. Now, we define a mapping f on E as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ b, & x = a, b \\ 1, & x = 1 \end{cases}$$

It is also easily to check that f is a modal operator on E .

In what follows, we obtain some important properties of modal operators on equality algebras.

Proposition 3.5. *Let $(E, \wedge, \sim, 0, 1)$ be a bounded equality algebra and $f : E \rightarrow E$ be a modal operator. Then for $x, y \in E$ we have:*

- (1) $f(1) = 1$;
- (2) If $x \leq y$, then $f(x) \leq f(y)$;
- (3) $f(x \rightarrow y) \leq f(x) \rightarrow f(y) = f(f(x) \rightarrow f(y)) = x \rightarrow f(y) = f(x \rightarrow f(y))$;
- (4) $f(x) \leq (x \rightarrow f(0)) \rightarrow f(0)$;
- (5) $f(x) \wedge f(y) = f(f(x) \wedge f(y))$;
- (6) $f(x)' \leq f(x')$;
- (7) If $y \leq x$, then $f(x) \rightarrow f(y) = f(x) \sim f(y)$;
- (8) $f(x \sim y) \leq f(x) \rightarrow f(y)$.

Proof. (1) By (f1) of Definition 3.1, we have $1 \leq f(1)$. Thus $f(1) = 1$.

(2) For all $x, y \in E$, we have that $x \leq y$ if and only if $x \wedge y = x$. By (f3) of Definition 3.1, if $x \leq y$, then $f(x) = f(x \wedge y) = f(x) \wedge f(y)$. Thus $f(x) \leq f(y)$.

(3) For all $x, y \in E$, by (f4) of Definition 3.1, we have $f(x \rightarrow y) = f(x \sim (x \wedge y)) \leq f(x) \sim f(x \wedge y) = f(x) \sim (f(x) \wedge f(y)) = f(x) \rightarrow f(y)$. Further by the condition (f1), (f2) of Definition 3.1, we have:

$f(f(x) \rightarrow f(y)) \leq f(f(x)) \rightarrow f(f(y)) = f(x) \rightarrow f(y) \leq x \rightarrow f(y) \leq f(x \rightarrow f(y)) \leq f(x) \rightarrow f(f(y)) = f(x) \rightarrow f(y) \leq f(f(x) \rightarrow f(y))$. Hence $f(x \rightarrow f(y)) = f(f(x) \rightarrow f(y)) = f(x) \rightarrow f(y) = x \rightarrow f(y)$.

(4) By $x \leq (x \rightarrow y) \rightarrow y$ and (3), we have that $f(x) \leq (f(x) \rightarrow f(0)) \rightarrow f(0) = (x \rightarrow f(0)) \rightarrow f(0)$.

(5) $f(x) \wedge f(y) \leq f(f(x) \wedge f(y))$ is valid. Now we show that $f(f(x) \wedge f(y)) \leq f(x) \wedge f(y)$: as $f(x) \wedge f(y) \leq f(x)$ and $f(x) \wedge f(y) \leq f(y)$, by (2), we have $f(f(x) \wedge f(y)) \leq f(f(x))$ and $f(f(x) \wedge f(y)) \leq f(f(y))$, hence $f(f(x) \wedge f(y)) \leq (f(f(x)) \wedge f(f(y)))$. By (f2) of Definition 3.1, we have $f(f(x) \wedge f(y)) \leq f(x) \wedge f(y)$. Therefore $f(x) \wedge f(y) = f(f(x) \wedge f(y))$.

(6) By (f1) and (1) of Proposition 2.3, we have $f(x)' \leq x' \leq f(x')$.

(7) Since $y \leq x$, we have $f(y) \leq f(x)$ and $f(x) \rightarrow f(y) = f(x) \sim (f(x) \wedge f(y)) = f(x) \sim f(y)$.

(8) It follows from (1) of Proposition 2.2 and items (2), (3). $\square$

We know that let E be a non-empty set, an isotone mapping $f : E \rightarrow E$ is called a closure operator on E if it is such that $f = f^2 \geq id_E$. Then we have,

Remark 3.6. By the definition of a modal operator and Theorem 3.5 (2), every modal operator on an equality algebra E is a closure operator. In general, the converse is not true.

The following example will show that a closure operator on E is not a modal operator.

Example 3.7. Consider the equality algebra E of Example 3.3. Now we define a mapping f on E as follows:

$$f(x) = \begin{cases} c, & x = 0, c \\ 1, & x = a, b, 1 \end{cases}$$

It is easily to check that f is a closure operator on E . However, f is not a modal operator on E since $f(a \wedge b) = f(c) = c$ and $f(a) \wedge f(b) = 1$. Hence $f(a \wedge b) \neq f(a) \wedge f(b)$.

In what follows, we will define residuated equality algebras. And we will give a characterization of modal operator on a residuated equality algebra.

Definition 3.8. Let E be an equality algebra. Then E is called *residuated*, if (R) $(x \wedge y) \wedge z = (x \wedge y)$ iff $x \wedge (y \sim (y \wedge z)) = x$, for all $x, y, z \in E$.

Clearly, (R) can be written in a classical way as $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

Example 3.9. Let $E = \{0, a, b, 1\}$ be a chain such that $x \wedge y = \min\{x, y\}$, for all $x, y \in E$. Define the operation $\sim$ on E as follows:

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

Routine calculation shows that $(E, \wedge, \sim, 1)$ is a residuated equality algebra.

Theorem 3.10. Let E be a residuated equality algebra and $f : E \rightarrow E$ be a mapping. Then f is a modal operator on E if and only if, for each $x, y \in E$, it holds:

- (1) $x \rightarrow f(y) = f(x) \rightarrow f(y)$;
- (2) $f(x \sim y) \leq f(x) \sim f(y)$.

Proof. Firstly, assume a mapping f fulfil conditions (1) and (2) of above Theorem, now we will show that f is a modal operator on E , that is, f satisfies the conditions (f1) – (f4) of Definition 3.1: Indeed,

For any $x \in E$, by (1) we have $x \rightarrow f(x) = f(x) \rightarrow f(x) = 1$. Therefore $x \leq f(x)$. Thus (f1) holds.

For any $x \in E$, it holds $1 = f(x) \rightarrow f(x) = f(f(x)) \rightarrow f(x)$. This implies $f(f(x)) \leq f(x)$. Then by (f1), we have $f(f(x)) = f(x)$. Thus (f2) holds.

To show that f establish (f3), we first verity the following result:

$$\text{if } x \leq y, \text{ then } f(x) \leq f(y).$$

Indeed, if $x \leq y$, by (1) of Proposition 2.3 and (1) of above Theorem, we have $f(x) \rightarrow f(y) = x \rightarrow f(y) \geq y \rightarrow f(y) = 1$, hence $f(x) \rightarrow f(y) = 1$. Therefore $f(x) \leq f(y)$. Now, as $x \wedge y \leq x$ and $x \wedge y \leq y$, applying above result we can obtain that $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$, and then $f(x \wedge y) \leq f(x) \wedge f(y)$. Now we show that $f(x) \wedge f(y) \leq f(x \wedge y)$: By (f1), we have $x \wedge y \leq f(x \wedge y)$. As E be a residuated equality algebra, $x \wedge y \leq z$ iff $x \leq y \rightarrow z$ holds. Then we get $y \leq x \rightarrow f(x \wedge y) = f(x) \rightarrow f(x \wedge y)$, so $y \wedge f(x) \leq f(x \wedge y)$. Similarly, we have $f(x) \leq y \rightarrow f(x \wedge y) = f(y) \rightarrow f(x \wedge y)$, so that $f(x) \wedge f(y) \leq f(x \wedge y)$. Therefore $f(x \wedge y) = f(x) \wedge f(y)$. Thus (f3) holds.

By (2), (f4) is straightforward. Therefore, f is a modal operator on E .

Now if f be a modal operator on E , (1) and (2) is obvious. $\square$

Theorem 3.11. *Let (E, f) be a modal equality algebra. Then the following properties hold:*

- (1) $f(E) = \text{Fix}(f)$, where $\text{Fix}(f) = \{x \in E \mid x = f(x)\}$.
- (2) The image $f(E)$ is a subalgebra of E .

Proof. (1) Assume that $y \in f(E)$, then there exists $x \in E$ such that $f(x) = y$. By (f2) of Definition 3.1, we can obtain that $f(y) = f(f(x)) = f(x) = y$, this is, $y \in \text{Fix}(f)$. Conversely, assume that $y \in \text{Fix}(f)$. By $\text{Fix}(f) = \{x \in E \mid x = f(x)\}$, we have $y \in f(E)$. Therefore $f(E) = \text{Fix}(f)$ holds.

(2) For all $x, y \in f(E)$, by (f3) and (1), we have $f(x \wedge y) = f(x) \wedge f(y) = x \wedge y$. So $x \wedge y \in f(E)$. By (f4) and (1), we have $f(x \sim y) \leq f(x) \sim f(y) = x \sim y$. Then by (f1), $x \sim y \leq f(x \sim y)$. Hence, $f(x \sim y) = x \sim y$. Therefore we have $x \sim y \in f(E)$. Hence $f(E)$ is closed under the operation $\wedge$ and $\sim$. And we have $f(1) = 1$, so $1 \in f(E)$. Therefore, $f(E)$ is a subalgebra of E . $\square$

Corollary 3.12. *Let (E, f) be a modal equality algebra. If $f(E) = E$, then f is the identity map on E .*

Proof. If $f(E) = E$, for any $x \in E$, then there exists $x_0 \in E$ such that $f(x_0) = x$. Form (f2), we have $f(x) = f(f(x_0)) = f(x_0) = x$. Therefore, f is the identity map on E . $\square$

Theorem 3.13. *If (E, f) be a modal equality algebra, then $\text{Ker}(f) = \{x \in E \mid f(x) = 1\}$ is a filter.*

Proof. By (f1) of Definition 3.1, we have $1 \leq f(1)$. Then $f(1) = 1$, hence $1 \in \text{Ker}(f)$.

Let x and $x \rightarrow y \in \text{Ker}(f)$, then we get $f(x) = 1$ and $f(x \rightarrow y) = 1$. By (3) of Proposition 3.5, we can obtain that $1 = f(x \rightarrow y) \leq f(x) \rightarrow f(y) = 1 \rightarrow$

$f(y) = f(y)$, thus $f(y) = 1$. So we get $y \in \text{Ker}(f)$. Therefore, by Proposition 2.14, we obtain $\text{Ker}(f) = \{x \in E \mid f(x) = 1\}$ is a filter. $\square$

In what follows, the conditions for an equality algebra to be a prelinear equality algebra is given by a modal operator on an equality algebra.

Theorem 3.14. *Let (E, f) be a modal equality algebra. Then the following conditions are equivalent:*

- (1) E is prelinear;
- (2) $f(x \vee y) = f(x) \vee f(y)$;
- (3) $f(x \rightarrow y) \vee f(y \rightarrow x) = 1$;
- (4) $f((x \rightarrow y) \rightarrow z) \leq f((y \rightarrow x) \rightarrow z) \rightarrow z$.

Proof. (1) $\Rightarrow$ (2) Assume that E is a prelinear equality algebra. $f(x) \vee f(y) \leq f(x \vee y)$ can be proved easily by Proposition 2.3 (1). Now we will prove that $f(x \vee y) \leq f(x) \vee f(y)$. Indeed, by Proposition 2.5 and (f3) of Definition 3.1, we have for all $x, y \in E$, $f(x \vee y) = f(((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)) = f((x \rightarrow y) \rightarrow y) \wedge f((y \rightarrow x) \rightarrow x)$. By Proposition 3.5 (3), Proposition 2.3 (1), we have $f(x \vee y) \leq (f(x \rightarrow y) \rightarrow f(y)) \wedge (f(y \rightarrow x) \rightarrow f(x)) \leq (f(x \rightarrow y) \rightarrow (f(x) \vee f(y))) \wedge (f(y \rightarrow x) \rightarrow (f(x) \vee f(y)))$. Then by Proposition 2.9 and Proposition 2.8 (1), we have $f(x \vee y) \leq (f(x \rightarrow y) \vee f(y \rightarrow x)) \rightarrow (f(x) \vee f(y))$. Now, we will prove that $f(x \rightarrow y) \vee f(y \rightarrow x) = 1$. Indeed, by (f1) of Definition 3.1 and Proposition 2.5, we have $(x \rightarrow y) \vee (y \rightarrow x) \leq f(x \rightarrow y) \vee f(y \rightarrow x)$. Hence by Definition 2.4, we obtain $(x \rightarrow y) \vee (y \rightarrow x) = 1$, and hence $f(x \rightarrow y) \vee f(y \rightarrow x) = 1$. Thus, by Proposition 2.2 (3), we have $f(x \vee y) \leq 1 \rightarrow (f(x) \vee f(y)) = f(x) \vee f(y)$.

(2) $\Rightarrow$ (3) Obviously.

(3) $\Rightarrow$ (4) This follows similar to the proof of the equivalence between prelinearity and Proposition 2.6.

(4) $\Rightarrow$ (1) Taking modal operator $f = id_E$ and by Proposition 2.6, we can easily prove it. $\square$

4. Modal filters on modal equality algebras

In this section, we will introduce the modal filters and modal congruences of modal equality algebras and obtain some important results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra.

Definition 4.1. *Let (E, f) be a modal equality algebra. A non-empty subset $F \subseteq E$ is called a modal filter of (E, f) , if F is a filter of E such that $x \in F$ implies $f(x) \in F$, for all $x \in E$.*

We denote the set of all modal filters of (E, f) by $MF(E, f)$.

Example 4.2. Let (E, f) be a modal equality algebra. Then $\text{Ker}(f)$ is a modal filter of (E, f) .

Example 4.3. Consider the Example 3.3, we can easily check that the modal filter of (E, f) are $\{1\}, \{a, 1\}, \{b, 1\}, \{a, b, c, 1\}$ and E .

Example 4.4. Consider the Example 3.4, we can also easily check that the modal filter of (E, f) are $\{1\}, \{b, 1\}, \{a, b, 1\}$ and E .

Proposition 4.5. *Let (E, f) be a modal equality algebra.*

- (1) *If F is a filter of $f(E)$, then $f^{-1}(F)$ is a modal filter of (E, f) ;*
- (2) *If F is a modal filter of (E, f) , then $f(F)$ is a filter of $f(E)$.*

Proof. (1) Suppose that F is a filter of $f(E)$. Obviously, $1 \in f^{-1}(F)$. Let $x, y \in E$ such that $x \in f^{-1}(F)$ and $x \leq y$. Then $f(x) \leq f(y)$. Since $f(x) \in F$ and $f(y) \in f(E)$, we can obtain that $f(y) \in F$, that is, $y \in f^{-1}(F)$. If $x, x \sim y \in f^{-1}(F)$. Then $f(x), f(x \sim y) \in F$. Since $f(x \sim y) \leq f(x) \sim f(y)$, then $f(x) \sim f(y) \in F$, thus $f(y) \in F$, that is $y \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a filter of E .

If $x \in f^{-1}(F)$, then $f(x) \in F$, so $f(f(x)) = f(x) \in F$, that is, $f(x) \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a modal filter of (E, f) .

(2) First, we have $f(F) = F \cap f(E)$. Indeed, if $x \in F \cap f(E)$, then we have that $x \in F$ and $x \in f(E)$. Hence $f(x) \in f(F)$ and $f(x) = x$. Thus, we have $x \in f(F)$. It follows that $F \cap f(E) \subseteq f(F)$. Conversely, if $y \in f(F)$, then there exists $x \in F$ such that $y = f(x)$. Since F is a modal filter of (E, f) , we have $y = f(x) \in F$. Hence $y \in F \cap f(E)$, $f(F) \subseteq F \cap f(E)$. Therefore, $f(F) = F \cap f(E)$.

Obviously, $1 \in f(F) = F \cap f(E)$. If $x \in f(F) = F \cap f(E), y \in f(E)$ such that $x \leq y$, then $y \in f(F) = F \cap f(E)$. If $x, x \sim y \in f(F) = F \cap f(E)$, then $y \in F$. Since F is a modal filter of (E, f) , we have $f(y) \in F \cap f(E)$ and then $y \in f(F) = F \cap f(E)$. Therefore, $f(F)$ is a filter of $f(E)$. $\square$

For any modal filter F of modal equality algebra (E, f) . Defined by $f_F : E/F \rightarrow E/F$ as a mapping $f_F([x]) = [f(x)]$.

Proposition 4.6. *Let (E, f) be a modal equality algebra and F be a modal filter of modal equality algebra (E, f) . Then f_F is a modal operator on E/F .*

Proof. First, we will show that f_F is well defined. Indeed, assume that $[x] = [y]$ for $x, y \in E$. Then $(x, y) \in \theta_F$, i.e., $x \sim y \in F$. Since F is a modal filter, then we have $f(x \sim y) \in F$. Now, by (f4) of Definition 3.1, $f(x \sim y) \leq f(x) \sim f(y)$, we have $f(x) \sim f(y) \in F$. Thus, $(f(x), f(y)) \in \theta_F$. And then $[f(x)] = [f(y)]$. Therefore, f_F is well defined. Next, we will prove f_F fulfill Definition 3.1:

- (f1) $[x] \leq [f(x)] = f_F([x]);$
- (f2) $f_F([x]) = [f(x)] = [f(f(x))] = f_F f_F([x]);$
- (f3) $f_F([x] \wedge [y]) = [f(x \wedge y)] = [f(x) \wedge f(y)] = f_F([x]) \wedge f_F([y]);$
- (f4) $f_F([x] \sim [y]) = [f(x \sim y)] \leq [f(x) \sim f(y)] = f_F([x]) \sim f_F([y]).$

Therefore, f_F is a modal operator on E/F . $\square$

Above them, we can obtain that $(E/F, f_F)$ is a modal equality algebra.

Proposition 4.7. *In the modal equality algebra $(E/Ker(f), f_{Ker(f)})$, we obtain:*

- (1) $[x] \leq [y]$ iff $f(x \sim (x \wedge y)) = 1$ iff $f(x \rightarrow y) = 1$;
- (2) $[x] = [y]$ iff $f(x \sim y) = 1$.

Proof. (1) We have $[x] \leq [y]$ iff $[x] \bullet [y] = [x]$ iff $[x] = [x \wedge y]$ iff $[x] \rightarrow [x \wedge y] = [1]$ iff $[x \sim x \wedge y] = [1]$ iff $x \sim x \wedge y \in Ker(f)$ iff $f(x \sim x \wedge y) = 1$ iff $f(x \rightarrow y) = 1$.

(2) $[x] = [y]$ iff $[x] \rightarrow [y] = [1]$ iff $[x \sim y] = [1]$ iff $x \sim y \in Ker(f)$ iff $f(x \sim y) = 1$. $\square$

Definition 4.8. *Let (E, f) be a modal equality algebra and θ be a congruence on E . Then θ is called a modal congruence on (E, f) , if $(x, y) \in \theta$ implies $(f(x), f(y)) \in \theta$ for each $x, y \in E$.*

We denote the set of all modal congruences of (E, f) by $MC(E, f)$.

Next, we will show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. For obtaining this important result, we need the following proposition.

Proposition 4.9. *Let (E, f) be a modal equality algebra and θ be a modal congruence on (E, f) . Then the following statements hold:*

- (1) $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ is a modal filter of (E, f) ;
- (2) $(x, y) \in \theta$ iff $(x \sim y, 1) \in \theta$.

Proof. (1) Clearly, $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ is a filter of E . Now for each $x \in [1]_\theta$, we have $(x, 1) \in \theta$. Hence $(f(x), f(1)) \in \theta$ and then $(f(x), 1) \in \theta$. Therefore, $f(x) \in [1]_\theta$. This proves that $[1]_\theta$ is a modal filter on (E, f) .

(2) If $(x, y) \in \theta$, then $(x \sim x, x \sim y) \in \theta$, that is, $(x \sim y, 1) \in \theta$. Conversely, let $(x \sim y, 1) \in \theta$, then $[x \sim y]_\theta = [1]_\theta$ and then $[x]_\theta \sim [y]_\theta = [1]_\theta$, $[x]_\theta = [y]_\theta$. Hence, $(x, y) \in \theta$. $\square$

Theorem 4.10. *Let (E, f) be a modal equality algebra. Then there exists one to one correspondence between $MF(E, f)$ and $MC(E, f)$.*

Proof. Define two mappings as follows, $f : MF(E, f) \mapsto MC(E, f)$ by $F \mapsto \theta_F$ and $g : MC(E, f) \mapsto MF(E, f)$ by $\theta \mapsto [1]_\theta$. Let $\theta \in MC(E, f)$. Clearly, $[1]_\theta = \{x \in E \mid (x, 1) \in \theta\}$ is a filter of E . Now for each $x \in [1]_\theta$, we have $(x, 1) \in \theta$. Hence $(f(x), f(1)) \in \theta$ and then $(f(x), 1) \in \theta$. Therefore, $f(x) \in [1]_\theta$. This proves that $[1]_\theta$ is a modal filter on (E, f) . Conversely, assume that $F \in MF(E, f)$. Obviously, θ_F is a congruence on E , where $(x, y) \in \theta_F$ iff $x \sim y \in F$. Then for each $(x, y) \in \theta_F$, we have $x \sim y \in F$. Since F is a modal filter, then $f(x \sim y) \in F$. By (f4) of Definition 3.1, we obtain $f(x) \sim f(y) \in F$, thus $(f(x), f(y)) \in \theta_F$. Therefore, θ_F is a modal congruence of (E, f) . By the above arguments, we have that f and g are well defined.

For any congruence θ on (E, f) , we have $(f \circ g)(\theta) = f([1]_\theta) = \theta_{[1]_\theta}$. By Proposition 4.9, we have $(x, y) \in \theta_{[1]_\theta}$ iff $x \sim y \in [1]_\theta$ iff $(x \sim y, 1) \in \theta$ iff $(x, y) \in \theta$. Hence, $\theta_{[1]_\theta} = \theta$. That is, $f \circ g = id_{MC(E, f)}$. Similarly, we have $g \circ f = id_{MF(E, f)}$. Therefore, there exists one to one correspondence between $MF(E, f)$ and $MC(E, f)$. $\square$

5. Uniformity structures on modal equality algebras

In this section, we will consider the uniformity structures on the modal equality algebra as an application of the strong modal filters of modal equality algebras. For this, we will introduce the concept of strong modal filters of modal equality algebras firstly.

Definition 5.1. Let (E, f) be a modal equality algebra. A non-empty subset $F \subseteq E$ is called a strong modal filter of (E, f) , if F is a filter of E such that $x \in F$ iff $f(x) \in F$, for all $x \in E$.

We denote the set of all strong modal filters of (E, f) by $SMF(E, f)$.

Note that for each modal equality algebra (E, f) , we can easily check that each strong modal filter of (E, f) is a modal filter, not vice versa. Next, we will give an example to proof this result.

Example 5.2. Let E be an equality algebra such that $|E| > 1$ and a mapping f be defined by $f(x) = 1, \forall x \in E$. Then we can check that (E, f) is a modal equality algebra. Now, we can see that $F = \{1\}$ is a modal filter but is not a strong modal filter. Since for $x \neq 1, x \in E$, we have $f(x) = 1 \in F$, but $x \notin F$.

In the next, we will consider the uniformity structure on the modal equality algebra as an application of the strong modal filters of modal equality algebras.

Theorem 5.3. Let (E, f) be a modal equality algebra and F be a strong modal filter of modal equality algebra (E, f) . Define $U_F = \{(x, y) \in E \times E \mid f(x \sim y) \in F\}$ and $K^* = \{U_F \mid F \in SMF(E, f)\}$. Then K^* satisfies the conditions (A1) – (A4) of Definition 2.17.

Proof. Now, we will prove that K^* satisfies the conditions (A1) – (A4):

(A1) Let $U_F \in K^*$, $(x, x) \in \Delta$. As $x \sim x = 1$, then $f(x \sim x) = f(1) = 1 \in F$. Thus we have $(x, x) \in U_F$. Thus, $\Delta \subseteq U_F$. Therefore, (A1) holds.

(A2) If $U_F \in K^*$, We know that $(x, y) \in U_F$ iff $f(x \sim y) \in F$ iff $f(y \sim x) \in F$ iff $(y, x) \in U_F$ iff $(x, y) \in U_F^{-1}$. Thus, $U_F^{-1} = U_F$. Thus, $U_F^{-1} \in K^*$. Therefore, (A2) holds.

(A3) Let $\Lambda(F) = \{F_a \mid F_a \subseteq F\}$ be the collection of strong modal filters contained in F . Clearly, $\Lambda(F)$ is not empty. Let G be strong modal filter generated by $\cup F_a$. Then $U_G \in K^*$. Next, we will show that $U_G \circ U_G \subseteq U_F$:

If $(x, y) \in U_G \circ U_G$, then there exists $z \in E$ such that $(x, z) \in U_G$ and $(z, y) \in U_G$. And we know $f(x \sim z) \in G$ and $f(z \sim y) \in G$. Then by G is a strong modal filter, we have $x \sim z \in G$ and $z \sim y \in G$. By (E7) of Definition 2.1, we have $(x, y) \in G$. That is, $f(x \sim y) \in G$. Since G be strong modal filter generated by $\cup F_a$ and $\cup F_a \subseteq F$, then $G \subseteq F$. Thus $f(x \sim y) \in F$ and then $(x, y) \in U_F$. Therefore, $U_G \circ U_G \subseteq U_F$. Therefore, (A3) holds.

(A4) We should proof that for all $U_F, U_G \in K^*$, $U_F \cap U_G = U_{F \cap G}$. Indeed, if $(x, y) \in U_F \cap U_G$, then we have that $(x, y) \in U_F$ and $(x, y) \in U_G$. Hence $f(x \sim y) \in F$ and $f(x \sim y) \in G$. Thus, $f(x \sim y) \in F \cap G$ and then $(x, y) \in U_{F \cap G}$.

Therefore, $U_F \cap U_G \subseteq U_{F \cap G}$. Similarly, we obtain $U_F \cap U_G \supseteq U_{F \cap G}$. That is, $U_F \cap U_G = U_{F \cap G}$ holds. Therefore, (A4) holds. $\square$

Theorem 5.4. *Let (E, f) be a modal equality algebra. Define $\mathcal{K} = \{U \in E \times E \mid U_F \subseteq U, \text{ for some } U_F \in K^*\}$. Then $((E, f), \mathcal{K})$ is a uniform structure.*

Proof. By above Theorem 5.3, we can know that $\mathcal{K}$ fulfill the conditions (A1) – (A4) of Definition 2.17. Now, we prove that $\mathcal{K}$ fulfill (A5). If $U \in \mathcal{K}, U \subseteq V \subseteq E \times E$. Then there exists $U_F \in K^*$ such that $U_F \subseteq U \subseteq V$. Thus, $V \in \mathcal{K}$. Therefore, $\mathcal{K}$ is a uniformity on (E, f) and hence $((E, f), \mathcal{K})$ is a uniform structure. $\square$

Now, we define $U[x] = \{y \in E \mid (x, y) \in U\}$, for all $x \in E, U \in \mathcal{K}$. Clearly, if $V \subseteq U$, then $V[x] \subseteq U[x]$. And then the next theorem shows that we can obtain a topology on the modal equality algebra (E, f) .

Theorem 5.5. *Let (E, f) be a modal equality algebra. Define $T = \{Y \subseteq E \mid \forall x \in Y, \exists U \in \mathcal{K}, U[x] \subseteq Y\}$. Then T is a topology on the modal equality algebra (E, f) .*

Proof. Obviously, $\emptyset, E \in T$. And it clear that T is closed under arbitrary union. Next, we will show that T is closed under finite intersection. Let $Y, W \in T$ such that $x \in Y \cap W$. Then there exist $U, V \in \mathcal{K}$ such that $U[x] \subseteq Y$ and $V[x] \subseteq W$. Now, let $N = U \cap V$, then $N \in \mathcal{K}$. And then $N[x] \subseteq U[x] \cap V[x] \subseteq Y \cap W$. Thus $N[x] \subseteq Y \cap W$, and then $Y \cap W \in T$. Therefore, T is a topology on (E, f) . $\square$

Note that for any $x \in E$, $U[x]$ is a neighborhood of x .

Let Σ be an arbitrary family of strong modal filters of an equality algebra E which is closed under intersection. Then the topology T_Σ obtained from Theorem 5.5 is called a uniform topology on (E, f) induced by Σ . And if $\Sigma = \{F\}$, we denote it by T_F .

In what following, we show that modal equality algebras with uniform topologies are topological modal equality algebras. Firstly, we give the definition as follows.

Definition 5.6. *Let (E, f) be a modal equality algebra and T be a topology on (E, f) . Then $((E, f), T)$ is called a topological modal equality algebra if the operators $\wedge, \sim$ are continuous.*

Note that the operator $\diamond \in \{\wedge, \sim\}$ is continuous iff for any $x, y \in E$ and any neighborhood C of $x \diamond y$ there exist neighborhoods A and B of x and y , respectively, such that $A \diamond B \subseteq C$.

Theorem 5.7. *Let (E, f) be a modal equality algebra and Σ be an arbitrary family of strong modal filters of E which is closed under intersection. Then the structure $((E, f), T_\Sigma)$ is a topological modal equality algebra.*

Proof. By the definition of a topological modal equality algebra, we will show that $\diamond$ is continuous, where $\diamond = \{\wedge, \sim\}$. Assume $x \diamond y \in Y$, where $x, y \in E$ and Y is an open subset of E . Then there exists $U \in \mathcal{K}, U[x \diamond y] \subseteq Y$ and there exists a strong modal filter F such that $U_F \in K^*, U_F \subseteq U$. Next, we will prove that the following relation holds:

$$U_F[x] \diamond U_F[y] \subseteq U_F[x \diamond y] \subseteq U[x \diamond y].$$

Let $h \diamond k \in U_F[x] \diamond U_F[y]$ such that $h \in U_F[x]$ and $k \in U_F[y]$. Hence, we have $(x, h) \in U_F$ and $(y, k) \in U_F$, and then $f(x \sim h) \in F, f(y \sim k) \in F$. Since F is a strong modal filter, we can obtain that $x \sim h \in F$ and $y \sim k \in F$. Therefore, $(x \diamond y) \sim (h \diamond k) \in F$. And then $f((x \diamond y) \sim (h \diamond k)) \in F$. Therefore $(x \diamond y, h \diamond k) \in U_F$ and then $(x \diamond y, h \diamond k) \in U$. Hence, $h \diamond k \in U_F[x \diamond y] \subseteq U[x \diamond y]$. That is, the above relation holds. Clearly, $U_F[x]$ and $U_F[y]$ are neighborhoods of x and y , respectively. Therefore, the operator $\diamond$ is continuous. Thus the structure $((E, f), T_\Sigma)$ is a topological modal equality algebra. $\square$

6. Conclusion

The study of equality algebras is motivated by the goal to develop appropriate algebraic semantics for fuzzy type theory as we mentioned in the introduction, so a concept of fuzzy type theory should be introduced. In this paper, we applied these ideas to propose the notion of modal operators on equality algebras. Also, we obtained some properties of modal equality algebras. As applications, we give a characterization of prelinear equality algebras. And we give the concepts of modal filters and modal congruences of equality algebras and obtain some related results. Moreover, we show that there is one to one correspondence relation between modal filters and modal congruences of a modal equality algebra. Finally, using strong modal filters, we establish the uniform structures on modal equality algebras. In particular, we prove that modal equality algebras with uniform topologies are topological modal equality algebras.

Future research will focus on characterizing modal filter generated by a subset of a modal equality algebra in terms of fuzzy equality operation and researching their related properties.

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Metric projection in countably seminormed spaces

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Abstract. We introduce a definition of metric projection in countably seminormed space. In order to do this, we have to introduce a definition of uniformly convex seminormed space, projection theorem in seminormed space, a new vision of the completion of countably seminormed space and a definition of uniformly convex countably seminormed space.

Keywords: countably seminormed space, Fréchet space, completion of countably seminormed space, uniformly convex countably seminormed space, projection theorem in countably seminormed space, metric projection.

1. Introduction

Ya. Alber introduced the notion of *metric projection* which solved some fixed point problems for a non-self contraction mapping which maps a nonempty, closed and convex subset of a Banach space E into E ([1, 2, 3]).

In his definition, he depended on the existence of the projection theorem which is well known in Hilbert space and it used, for example, in Optimization theory.

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Theorem 1.1 (Projection theorem in uniformly convex Banach space, [4, 5]). *Let K be a nonempty, closed and convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$. For each $x \in E \setminus K$ there exists a unique $\bar{x} \in K$ such that*

$$\|x - \bar{x}\| = \inf_{y \in K} \|x - y\|.$$

The operator $P_K : E \longrightarrow K$ is called a *metric projection operator* if it assigns to each $x \in E$ its *nearest point* $\bar{x} \in K$. i.e. $P_K(x) = \bar{x}$ ([1, 5]).

Nashat Faried and Hany A. El-Sharkawy introduced definitions of uniformly convex, uniformly smooth countably normed spaces and the existence of the metric projection in a countably normed space ([6]).

2. Projection theorem in uniformly convex complete seminormed space

Definition 2.1 (Seminormed space and normed space, [7]). *Let E be a real vector space. A real valued function p defined on E is called a seminorm if for all $x, x_1, x_2 \in E$ and all $a \in \mathcal{R}$: $p(ax) = |a|p(x)$, $p(x_1 + x_2) \leq p(x_1) + p(x_2)$. (E, p) is the seminormed space equipped with the topology defined by seminorm p .*

If p is a seminorm and $p(x) = 0 \Rightarrow x = 0$, then p is a norm and (E, p) is a normed space.

Notation. Let L be a subspace of a topological linear space E . For a point x and a set K in E , we may write $x + L$ and $K/L = \{x + L : x \in K\}$ to denote the equivalence class $\hat{x}$ and the set of equivalence classes $\hat{K}$ belonging to the factor space E/L , respectively. So we can write $\hat{x} = x + L$ and $\hat{K} = K/L$ in $\hat{E} = E/L$.

Definition 2.2 (Normed space associated with seminormed space, [7]). For a seminormed space (E, p) there is a normed space $E/\ker p$ with the norm $\|x + \ker p\|_p = p(x)$ called the associated normed space with the seminormed space (E, p) .

Since the norm in the associated normed space is defined by the seminorm, so the following propositions and the definition are easy to introduce and verified.

Proposition 2.3 ([8]). *A seminormed space is complete if and only if its associated normed space is complete.*

Proposition 2.4. *If (E, p) is seminormed space, K is non empty closed set in E , then for $x \in E \setminus K$: $d(x, K) = \inf_{y \in K} p(x - y) > 0$.*

Proof. Since $x \notin K$ and K is closed, then $\hat{x} \notin \hat{K}$ in the associated normed space. Hence $d(\hat{x}, \hat{K}) = \inf_{\hat{y} \in \hat{K}} \|\hat{x} - \hat{y}\|_p = \inf_{y \in K} p(x - y) > 0$. $\square$

Definition 2.5 (Uniformly convex seminormed space, [8]). We call a seminormed linear space (E, p) *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $p(x) = 1$, $p(y) = 1$ and $p(x - y) \geq \varepsilon$ then $p(\frac{1}{2}(x + y)) \leq 1 - \delta$.

Proposition 2.6 ([8]). *A seminormed space (E, p) is uniformly convex if and only if its associated normed space is uniformly convex.*

This gives an equivalent definition of uniformly convex seminormed space.

In the following we give an example of uniformly convex seminormed space which are not normed space.

Example 2.7 ([8]). Let $\mathbb{Q}$ be the set of all rational numbers.

$\mathbb{Q}^2$ with the following norm $\|z = (x, y)\| = \sqrt{x^2 + y^2}$ is uniformly convex.

Defining $\bar{\mathbb{Q}}^2 = \{\{z_n\} : \{z_n\} \text{ is a Cauchy sequence in } \mathbb{Q}^2\}$ and a seminorm on $\bar{\mathbb{Q}}^2$ by $p(\{z_n\}) = \lim_{n \rightarrow \infty} \|z_n\|$. Since the associated normed space of $(\bar{\mathbb{Q}}^2, p)$ is $\mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$ and it is uniformly convex, then $(\bar{\mathbb{Q}}^2, p)$ is a uniformly convex seminormed space.

For the following theorem which we called Projection theorem in uniformly convex complete seminormed space, we give two proofs, one depending on the following lemma and the other depending on the associated normed space.

Lemma 2.8. *Let (E, p) be uniformly convex seminormed space. If $\{x_i\}$ is a sequence in E : $\lim_{i \rightarrow \infty} p(x_i) = 1$ and $\lim_{i,j \rightarrow \infty} p(\frac{x_i + x_j}{2}) = 1$. Then $\{x_i\}$ is a Cauchy sequence in (E, p) .*

Proof. Suppose contrarily that the sequence $\{x_i\}$ is not a Cauchy sequence in E . In this case there exist two subsequences of natural numbers i_k, j_k such that $p(x_{i_k} - x_{j_k}) \geq \varepsilon$ for some $\varepsilon > 0$. Let $1 \leq M < 2$ be fixed, then $p(\frac{x_i}{M}) \rightarrow \frac{1}{M} < 1$ as $i \rightarrow \infty$. Then $\exists n_0 \in \mathbb{N} : i_k, j_k \geq n_0 \implies p(\frac{x_{i_k}}{M}) \leq 1 ; p(\frac{x_{j_k}}{M}) \leq 1$, hence $p(\frac{x_{i_k} - x_{j_k}}{M}) \geq \frac{\varepsilon}{M} > \frac{\varepsilon}{2}, \forall i_k, j_k \geq n_0$. Since (E, p) is uniformly convex, then $\exists \delta > 0$ such that $p(\frac{x_{i_k} + x_{j_k}}{2M}) \leq 1 - \delta$. Taking limit as $i_k, j_k \rightarrow \infty$, we get $\frac{1}{M} \leq 1 - \delta$. Now take $M \rightarrow 1$, we get $1 \leq 1 - \delta$ which is a contradiction. $\square$

Theorem 2.9. Let (E, p) be a uniformly convex complete seminormed space. If $K \subset E$ is a nonempty closed convex set and $x \in E \setminus K$, then there exists $\bar{x} \in K$ such that $p(x - \bar{x}) = \inf_{y \in K} p(x - y)$.

Proof 1. Since $x \notin K$ and K is closed in (E, p) , then $\inf_{y \in K} p(x - y) := d > 0$. There exists a sequence $\{y_i\}$ in K : $\lim_{i \rightarrow \infty} p(x - y_i) := d > 0$, and hence $\lim_{i \rightarrow \infty} p(\frac{x - y_i}{d}) = 1$,

(1) Taking $u_i := \frac{x - y_i}{d} \in E$, then $p(u_i) \rightarrow 1$ as $i \rightarrow \infty$.

Since K is convex, then $\frac{y_i + y_j}{2} \in K, \forall i, j \in \mathbb{N}$. Then,

$$\begin{aligned} d &\leq p(x - \frac{y_i + y_j}{2}) = p(\frac{x - y_i}{2} + \frac{x - y_j}{2}) \leq p(\frac{x - y_i}{2}) + p(\frac{x - y_j}{2}) \\ (2) \quad &\implies 2 \leq p(\frac{x - y_i}{d} + \frac{x - y_j}{d}) \leq p(\frac{x - y_i}{d}) + p(\frac{x - y_j}{d}) \\ &\implies 2 \leq p(u_i + u_j) \leq p(u_i) + p(u_j) \\ &p(u_i + u_j) \rightarrow 2 \text{ as } i, j \rightarrow \infty. \end{aligned}$$

From equations (1) and (2), using Lemma (2.8), then $\{u_n\}$ is a Cauchy sequence in (E, p) . i.e., $p(u_i - u_j) \rightarrow 0$ that gives $p(\frac{y_i - y_j}{d}) \rightarrow 0$.

Therefore, $\{y_i\}$ is a Cauchy sequence in $K \subset (E, p)$.

Since K is closed in (E, p) and $y_i \in K, \forall i$, then $\bar{x} \in K$, hence $p(x - y_i) \rightarrow p(x - \bar{x})$. Since $p(x - y_i) \rightarrow d$, so from the uniqueness of the limit, we get $p(x - \bar{x}) = d : \bar{x} \in K$.

Proof 2. Since (E, p) is uniformly convex complete seminormed space, then $(E/\ker p, \|\cdot\|_p)$ is uniformly convex Banach space.

Since $K \subset E$ is a nonempty closed convex set, then $K/\ker p \subset E/\ker p$ is a nonempty closed convex set.

For $\hat{x} = x + \ker p \in (E/\ker p) \setminus (K/\ker p)$ there exist unique $\hat{\bar{x}} = \bar{x} + \ker p \in K/\ker p$ such that $\|\hat{x} - \hat{\bar{x}}\|_p = \inf_{\hat{y} \in K/\ker p} \|\hat{x} - \hat{y}\|_p$. Then

$$p(x - \bar{x}) = \inf_{y \in K} p(x - y).$$

Remark 2.10. $\bar{x} \in K$ is unique up to the points $x^* \in K$ such that $\bar{x} - x^* \in \ker p$. This was expected because the seminormed space is not Hausdorff (means it has no uniqueness of limit).

The following remarks and lemma are useful in the projection theorem in countably seminormed space.

Remark 2.11. Let K_x be the set of all points in K which has a minimum distance from $x \notin K$. i.e. $K_x = \{x^* \in K : p(x - x^*) = \inf_{y \in K} p(x - y)\}$.

Hence $p(\bar{x} - x^*) = 0, \forall \bar{x}, x^* \in K_x$.

Lemma 2.12. $K_x \subset K$ is closed in (E, p) .

Proof. Let $\{x_i\}$ be a sequence in K_x such that $\{x_i\}$ converges to $x_0 \in E$. i.e. $p(x_i - x_0) \rightarrow 0$ as $i \rightarrow \infty$. Since K is closed, then $x_0 \in K$. And $\inf_{y \in K} p(x - y) \leq p(x - x_0) \leq p(x - x_i) + p(x_i - x_0), \forall i \in \mathbb{N}$. Since $p(x - x_i) = \inf_{y \in K} p(x - y), \forall i \in \mathbb{N}$ and $p(x_i - x_0) \rightarrow 0$ as $i \rightarrow \infty$ then $x_0 \in K_x$. $\square$

3. Projection theorem in uniformly convex complete countably seminormed space

Definition 3.1 (Countably seminormed space, [7, 8, 9]). A linear space E equipped with a total countable family of seminorms $\{p_n, n \in \mathbb{N}\}$ is said to be countably seminormed space. Totality ($p_n(x) = 0, \forall n \implies x = \theta$) guarantees that E is Hausdorff. A complete countably seminormed space is called Fréchet space.

Remark 3.2 ([7, 10]). 1-Without loss of generality (by taking the equivalent system of seminorms $\dot{p}_n(x) = \max_{m=1}^n p_m(x)$), one can assume that the sequence of seminorms $\{p_n; n = 1, 2, \dots\}$ is increasing, i.e., $p_1(x) \leq p_2(x) \leq \dots \leq p_n(x) \leq \dots, \forall x \in E$.

2-Any countably seminormed space is metrizable and its metric d can be defined by $d(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)}$.

Definition 3.3 (Compatible norms, [10]). Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a linear space E are said to be *compatible* if, whenever a sequence $\{x_i\}$ in E is Cauchy with respect to both norms and converges to a limit $x \in E$ with respect to one of them, it also converges to the same limit x with respect to the other norm.

Remark 3.4 ([9, 10]). If only one of the seminorms say p_{n_0} is a norm, then by adding this norm to each of the seminorms, we will get an equivalent system of increasing norms and if these norms are pair wise compatible, then E is, in fact, a countably normed space.

Since the definition of uniform convexity is always useful on complete spaces, we quoted Merkle's work ([11]) on completion of countably seminormed space while changed some notations to be suitable for our work as will as in ([8]).

For fixed seminorm p_n one can define a seminorm $\bar{p}_n$ on the space

$$E_{p_n} = \{\{x_i\} : \{x_i\} \text{ is } p_n \text{ Cauchy sequence}\}$$

as follows

$$\bar{p}_n(\{x_i\}_{i \in \mathbb{N}}) = \lim_{i \rightarrow \infty} p_n(x_i).$$

This limit exists, because $\{p_n(x_i)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

By standard arguments one can prove that $(E_{p_n}, \bar{p}_n)$ is the completion of the countably seminormed space E if equipped with only the seminorm p_n .

In fact, defining a map $\pi : E \rightarrow E_{p_n}$ by $\pi(x) = (x, x, x, \dots)$ gives a 1-1 isometrical and isomorphical mapping of E (as a seminormed space with p_n) onto a dense linear subspace of the space $(E_{p_n}, \bar{p}_n)$.

Since for any $x \in E$ $p_n(x) \leq p_{n+1}(x), \forall n \in \mathbb{N}$, then every p_{n+1} Cauchy sequence is p_n Cauchy sequence.

So,

$$E \subset \dots \subset E_{p_{(n+1)}} \subset E_{p_n} \subset \dots \subset E_{p_1}.$$

Therefore, $\bigcap_{n \in \mathbb{N}} E_{p_n}$ is a countably seminormed space with

$$\bar{p}_1(x) \leq \bar{p}_2(x) \leq \dots \leq \bar{p}_n(x) \leq \dots, \forall x \in \bigcap_{n \in \mathbb{N}} E_{p_n},$$

but it is not, in general, a Hausdorff space because $\bar{p}_n(\{x_i\}_{i \in \mathbb{N}}) = 0$ for all $n \in \mathbb{N}$ does not necessarily imply that $\{x_i\}_{i \in \mathbb{N}}$ be the zero sequence. In fact $\bigcap_{n \in \mathbb{N}} \ker \bar{p}_n$ may not be the zero sequence.

On the factor space $\bigcap_{n \in \mathbb{N}} E_{p_n} / \bigcap_{i \in \mathbb{N}} \ker \bar{p}_i$ we define

$$\hat{p}_n(\hat{x}) = \hat{p}_n(\{x_i\}_{i \in \mathbb{N}} + \bigcap_{i \in \mathbb{N}} \ker \bar{p}_i) = \bar{p}_n(\{x_i\}).$$

Assume $\hat{E}_{p_n} = E_{p_n} / \bigcap_{i \in \mathbb{N}} \ker \bar{p}_i$. In this case $\bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}$ equipped with the seminorms $\hat{p}_1(\hat{x}) \leq \hat{p}_2(\hat{x}) \leq \dots \leq \hat{p}_n(\hat{x}) \leq \dots$, $\forall \hat{x} \in \bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}$ is countably seminormed Hausdorff space.

Defining $\hat{\pi} : E \rightarrow \hat{E}_{p_n}$ such that $\hat{\pi}(x) = (x, x, x, \dots) + \bigcap_{i \in \mathbb{N}} \ker \bar{p}_i$, we see that E is isomorphically isometric to a linear dense subset of $\hat{E}_{p_n}$, i.e.,

$$E \subset \dots \subset \hat{E}_{p_{n+1}} \subset \hat{E}_{p_n} \subset \dots \subset \hat{E}_{p_1}.$$

Proposition 3.5 ([8, 11]). *Let E be a countably seminormed space, then $\bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}$ is a complete space. Moreover, there is an isometric and isomorphic, 1-1 mapping $\hat{\pi}$ of E onto a dense subspace of $\bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}$. E is complete if and only if*

$$\hat{\pi}(E) = \bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}.$$

We may write $E = \bigcap_{n \in \mathbb{N}} \hat{E}_{p_n}$.

Definition 3.6 (Uniformly convex countably seminormed space, [8]). A countably seminormed space E is said to be *uniformly convex* if $\hat{E}_{p_n}$ is uniformly convex for all n , i.e., it is associated normed space $(\bar{E}_{p_n}, \|\cdot\|_{\bar{p}_n})$ is uniformly convex where $\bar{E}_{p_n} = \frac{E_{p_n}}{\ker \bar{p}_n}$.

In the following we give an example of uniformly convex countably seminormed space which is not countably normed space.

Example 3.7 ([8]). Let $X = \mathbb{Q}^2$. Defining $X^\infty = \mathbb{Q}^2 \times \mathbb{Q}^2 \times \mathbb{Q}^2 \times \dots$ and seminorms on X^∞ by $p_n(z) = \|z_n\| = \sqrt{x_n^2 + y_n^2}$ where $z = (z_1, z_2, z_3, \dots) \in X^\infty$ and $z_n = (x_n, y_n) \in \mathbb{Q}^2$. $(X^\infty, \{p_n, n \in \mathbb{N}\})$ is a countably seminormed space.

Defining $\bar{X}_{p_n}^\infty = \{\{z^i\} : \{z^i\} \text{ is } p_n \text{ Cauchy sequence}\}$ where $z^i = (z_1^i, z_2^i, z_3^i, \dots) \in X^\infty$ and a seminorm on $\bar{X}_{p_n}^\infty$ by $\bar{p}_n(\{z^i\}) = \lim_{i \rightarrow \infty} p_n(z^i)$.

To show that $(X^\infty, \{p_n, n \in \mathbb{N}\})$ is a uniformly convex seminormed space, we must prove that $\bar{X}_{p_n}^\infty / \bigcap_{n \in \mathbb{N}} \ker \bar{p}_n$ with the seminorm $\hat{p}_n(\{z^i\} + \bigcap_{n \in \mathbb{N}} \ker \bar{p}_n) = \bar{p}_n(\{z^i\})$ is uniformly convex or its associated normed space $\bar{X}_{p_n}^\infty / \ker \bar{p}_n$ with $\|\{z^i\} + \ker \bar{p}_n\|_{p_n} = \bar{p}_n(\{z^i\})$ is uniformly convex. In the associated normed space two elements $\{z^i\}$ and $\{\dot{z}^i\}$ where $z^i = (z_1^i, z_2^i, z_3^i, \dots)$ and $\dot{z}^i = (\dot{z}_1^i, \dot{z}_2^i, \dot{z}_3^i, \dots)$ belong to the same equivalence class if both $\{z_n^i\}$ and $\{\dot{z}_n^i\}$ are p_n Cauchy sequence (i.e. it is Cauchy sequence in the n th coordinate) and the p_n limit of the difference between $\{z_n^i\}$ and $\{\dot{z}_n^i\}$ is convergent to zero. Thus the associated normed space is $\mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$ which is uniformly convex.

In the following we give an example of countably normed space which is uniformly convex with respect to one seminorm but not uniformly convex countably seminormed space.

Example 3.8 ([8]). For $\mathbb{R}^2$ with two norms $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$ and $\|(x, y)\|_\infty = \max\{|x|, |y|\}$, $(\mathbb{R}^2, \|\cdot\|_2)$ is uniformly convex but $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not uniformly convex.

Defining $\ell^1(\mathbb{R}^2) = \{z_i : z_i \in \mathbb{R}^2, \forall i \in \mathbb{N}, \sum_{i=1}^\infty \|z_i\|_2 < \infty\}$ and $p_0(\{z_i\}) = \sum_{i=1}^\infty \|z_i\|_2$, $p_1(\{z_i\}) = \|z_1\|_2$ and $p_n(\{z_i\}) = \|z_n\|_\infty$, $\forall n = 2, 3, \dots$

$(\ell^1(\mathbb{R}^2), p_0)$ is not uniformly convex normed space. In fact for $\epsilon = 1$ and $z_1 = ((1, 0), (0, 0), (0, 0), \dots)$, $z_2 = ((0, 0), (-1, 0), (0, 0), \dots)$.

Clearly $p_0(z_1) = 1 = p_0(z_2)$, $p_0(z_1 - z_2) = 2 > \epsilon$. However $p_0(\frac{1}{2}(z_1 + z_2)) = 1$.

So, there is no $\delta > 0$ satisfying $p_0(\frac{1}{2}(z_1 + z_2)) = 1 - \delta$. $(\ell^1(\mathbb{R}^2), \{p_n, n = 0, 1, 2, \dots\})$ is countably normed space (see Remark(3.3)).

$(\ell^1(\mathbb{R}^2), p_1)$ is uniformly convex seminormed space. $(\ell^1(\mathbb{R}^2), p_n)$ is not uniformly convex seminormed space $\forall n = 2, 3, \dots$. Therefore, $(\ell^1(\mathbb{R}^2), \{p_n, n = 0, 1, 2, \dots\})$ is not uniformly convex seminormed space.

Notation. For a complete countably seminormed space $(E, \{p_n, n \in \mathbb{N}\})$, let the completion of E with respect to the seminormed p_n be E_n and for every $x \in E \subset E_n$ the seminorm which has been defined on E_n will be p_n it self i.e. $\hat{E}_{p_n} = E_n$, $\hat{p}_n(\hat{x}) = p_n(x)$, $\forall x \in E$.

In the following theorem, we establish one of the most important and interesting geometric properties in countably seminormed spaces.

Theorem 3.9. Let $(E, \{p_n, n \in \mathbb{N}\})$ be a real uniformly convex complete countably seminormed space, K be a nonempty convex proper subset of E such that K is closed in each seminormed space (E_n, p_n) . Then

$$\forall x \in E \setminus K \quad \exists! \bar{x} \in K : p_n(x - \bar{x}) = \inf_{y \in K} p_n(x - y) \quad \forall n \in \mathbb{N}.$$

In order to proof it, we introduce the following lemmas and proposition.

Lemma 3.10. Let (E, p) be uniformly convex seminormed space. If $\bar{x}$ and x^* in $S(x_0, d) := \{x : p(x_0 - x) = d\}$ and $\bar{x} \neq x^*$, then $p(x_0 - \frac{\bar{x} + x^*}{2}) < d$.

Proof. Since $\bar{x}$ and $x^* \in S(x_0, d)$, then $p(x_0 - \bar{x}) = p(x_0 - x^*) = d$.

Hence $p(\frac{x_0 - \bar{x}}{d}) = p(\frac{x_0 - x^*}{d}) = 1$.

Since (E, p) is uniformly convex, then $p(\frac{\frac{x_0 - \bar{x}}{d} + \frac{x_0 - x^*}{d}}{2}) < 1$.

Hence $p(x_0 - \frac{\bar{x} + x^*}{2}) < d$. □

Proposition 3.11 (Principle of nested sequence of closed sets). Let $(E, \{p_n, n \in \mathbb{N}\})$ be a complete countably seminormed space, K_n be a nonempty closed subset in each seminormed space (E_n, p_n) , $K_{n+1} \subset K_n$ and $p_n(x - y) = 0, \forall x, y \in K_n, \forall n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} K_n$ is non empty. Moreover the intersection is only one point in E .

Proof. Let $x_i \in K_i, \forall i \in \mathbb{N}$, we have to prove the sequence $\{x_i\}_{i=n}^\infty$ is p_n Cauchy sequence $\forall n \in \mathbb{N}$. Since $x_j - x_{j+p} \in E_j, \forall j, p \in \mathbb{N}$, then $p_n(x_j - x_{j+p}) = 0, \forall j, p \in \mathbb{N}$ and $n = 1, 2, \dots, j$. Take the limit as $j \rightarrow \infty$, then $p_n(x_j - x_{j+p}) \rightarrow 0$

as $j \rightarrow \infty, \forall n \in \mathbb{N}$. Since E is complete, then $\{x_i\}$ has a limit $\bar{x} \in E$. Since K_i is closed $\forall i \in \mathbb{N}$, then $\bar{x} \in K_i, \forall i \in \mathbb{N}$.

$$\begin{array}{ccccccccccc} E = \bigcap_{i \in \mathbb{N}} E_i & \subset & \dots & \subset & E_{i+1} & \subset & E_i & \subset & \dots & \subset & E_2 & \subset & E_1 \\ p_1 \leq p_2 \leq \dots & & p_1 \leq \dots \leq p_{i+1} & & p_1 \leq \dots \leq p_i & & p_1 \leq p_2 & & p_1 & & & & \\ \bigcap_{i \in \mathbb{N}} K_i & \subset & \dots & \subset & K_{i+1} & \subset & K_i & \subset & \dots & \subset & K_2 & \subset & K_1 \\ \bar{x} & \leftarrow & \dots & , & x_{i+1} & , & x_i & , & \dots & , & x_2 & , & x_1 \end{array}$$

Now, assume $\bar{x}, x^* \in \bigcap_{i \in \mathbb{N}} K_i$, then $p_i(\bar{x} - x^*) = 0, \forall i \in \mathbb{N}$. Hence $\bar{x} = x^*$. $\square$

Lemma 3.12. *Let $(E, \{p_n, n \in \mathbb{N}\})$ be a countably seminormed space. If K is a nonempty closed subset in each seminormed space (E, p_n) , then K is closed in $(E, \{p_n, n \in \mathbb{N}\})$.*

Let $\{x_i\}$ be a sequence in $K \subset (E, \{p_n, n \in \mathbb{N}\})$ such that $\lim_{i \rightarrow \infty} p_n(x_i - x_0) = 0, \forall n \in \mathbb{N}$. Since K is closed in each seminormed space (E, p_n) , then $x_0 \in K \subset (E, p_n), \forall n \in \mathbb{N}$. Hence $x_0 \in K \subset (E, \{p_n, n \in \mathbb{N}\})$.

Proof of Theorem 3.8. Since $x \notin K$ and K is closed in (E_n, p_n) , then there exists a set $K_{x,n}$ of points in K which has a minimum distance with respect to $p_n, \forall n \in \mathbb{N}$.

Since $p_n(x) \leq p_{n+1}(x), E_{n+1} \subset E_n, \forall n \in \mathbb{N}$ and each E_n has the seminorms $p_1, p_2, \dots, p_n$, then $K_{x,n}$ are nested decreasing sequence of closed sets. i.e. $K_{x,n+1} \subset K_{x,n} \quad n \in \mathbb{N}$. Using Proposition 3.10, $\bigcap_{n \in \mathbb{N}} K_{x,n}$ is nonempty and it has a unique point denote it by $\bar{x} \in K$.

Now we prove the uniqueness:

Assume that $x^* \in K : p_n(x - x^*) = \inf_{y \in K} p_n(x - y) := d_n, \forall n \in \mathbb{N}$ and $x^* \neq \bar{x}$. Since $\frac{\bar{x} + x^*}{2} \in K$ because of the convexity of K , then

$$d_n \leq p_n\left(x - \frac{\bar{x} + x^*}{2}\right) \leq p_n\left(\frac{x - \bar{x}}{2}\right) + p_n\left(\frac{x - x^*}{2}\right) = \frac{d_n}{2} + \frac{d_n}{2} = d_n,$$

i.e., $p_n\left(x - \frac{\bar{x} + x^*}{2}\right) = d_n, \forall n \in \mathbb{N}$ where $\bar{x} \neq \frac{\bar{x} + x^*}{2} \neq x^*$. This contradicts Lemma 3.9.

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Investigation of simulated annealing components to solve the university course timetabling problem

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Abstract. Simulated Annealing (SA) is a common meta-heuristic algorithm that has been widely used to solve complex optimization problems. This is due to its ease of implementation and capability to escape from local optimum. This research conducts an investigation on three of SA components: the initial temperature, cooling schedule and neighborhood structure. We observed that the high initial temperature leads SA to accept any solution (wasting more computational time), whilst the lower value leads SA to quickly trap in local optimum. Based on research findings from this phase, for each component we suggested a technique to overcome the limitations. The limitations are: (i) a dynamic initial temperature mechanism that dynamically chose the suitable initial temperature for each instance problem; (ii) adaptive cooling schedule that will adjust the temperature value during the search; and (iii) a new neighborhood structure that will improve the search ability by minimizing the random selection. In the second phase. The experimental results show that the proposed techniques and approaches in all phases have outperformed the SA and comparable to other approaches in the literature (tested on university course timetabling benchmark dataset ITC2007-Track3). **Keywords:** simulated annealing, meta-heuristic, university course timetabling problem, optimization, ITC2007-Track3.

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1. Introduction

Timetabling is considered as a unique case of scheduling problems [23]. Wren [23] defined the timetabling as follows:

“The allocation, subject to constraints, of given resources to objects being placed in spacetime, in such a way as to satisfy as nearly as possible a set of desirable objectives”.

Schaerf [18] categorized the educational field into three common timetabling problems: university course timetabling problem, examination timetabling problem and school timetabling problem. All of them may different but they still have the same fundamental characteristics. For example, the sizes of class for all school lectures are normally similar and have the same group of students in common that are associated with the same courses, meanwhile the university courses have different number of students’ enrolments. Table (1) summarizes some of the common meta-heuristic algorithms related to different categories which are presented by Birattari et al. [4], where $\checkmark$ means that the algorithm is totally categorized under this category and x means it does not, whilst $\neg$ means it is a partly related to this category.

Table 1: Summary of the meta-heuristic algorithms characteristics.

Category	SA	TS	GA	ACO	ILS	grasp	GLS
Trajectory	$\checkmark$	$\checkmark$	x	x	x	x	$\checkmark$
Population	x	x	$\checkmark$	$\checkmark$	x	x	
Memory	x	$\checkmark$	$\neg$	$\checkmark$	$\neg$	x	$\checkmark$
Various neighborhood	x	x	$\neg$	x	$\checkmark$	x	x
Dynamic	x	$\neg$	x	x	x	x	$\checkmark$
Nature inspired	$\checkmark$	x	$\checkmark$	$\checkmark$	x	x	x

Note: ACO: ant colony optimization; TS: Tabu search; ILS: iterated local search; GRASP: greedy randomized adaptive search procedure; SA: simulated annealing; GLS: guided local search, GA: genetic algorithm.

SA is a trajectory-based algorithm or a single point algorithm [4]. The single point algorithms consider a single element of the solution space at each iteration by jumping from position to another in the solution space.

SA uses a special strategy to escape from local optimum. At first, SA will explore the search space (diversification) by easily accepting worse solution based on probability acceptance criterion. Later on, the probability is decreased during the search space process that leads to focus more on promising regions (intensification) [1]. In fact, the acceptance probability depends on the temperature which is reduced during the search process to give balance between the diversification and intensification. However, SA could be trapped into local optimum and may consume a longer time to find good solutions [24]. Hence, many researchers attempted to improve the SA performance by using different approaches, such as adaptive SA (ASA) suggested by Ingber [10], or by

hybridize SA with other meta-heuristics approaches, such as genetic algorithm [21]. In fact, hybrid techniques can achieve different advantages; such as the best performance; by combining the advantages of each individual technique into one hybrid algorithm. Therefore, this research is motivated to investigate the process of SA performance improvement by escaping from local optimum and reducing the computational time.

This research focuses on improving a SA performance by investigating the effect of SA components: temperature, cooling schedule and neighborhood structure on the SA performance. The good initial temperature will balance the diversification and intensification properly by avoiding the waste of computational time when the initial temperature is high or when trap in local optimum which is a result of a very low temperature [22].

Elmohamed et al. [8] claimed that adjusting the decrement amount of the given temperature during the search process is an important issue, in order to avoid the wasting of computational time. Furthermore, The high temperature with slow cooling schedule leads SA to accept many bad solutions (a zigzag walk), while the low temperature with fast cooling schedule leads SA to trapped quickly in the local optimum by rejecting most of the unimproved solutions and accepting the improved solutions only (descent method). Meanwhile, an effective neighborhood structures may easily improve the solution quality [14]. Moreover, choosing the most suitable neighborhood is an important issue to avoid the discounted neighborhood and save the computational time [15].

AlHadid et al. [2] proposed a hybrid SA with EMC technique to divert the search effectively to another promising region by escaping the search space from local optimum to another promising region space. AlHadid et al. [2] stated that the proposed technique results has outperformed the standard SA and gave comparable results to other approaches when tested on ITC2007-Track3 university course timetabling datasets.

According to the previous dissection, we have identified two main research questions (in regard to SA algorithm) to be answered:

- a) How can we enhance the SA performance by escaping from local optimum?
- b) How can we reduce the SA computational time?

In order to answer the main research questions, we have identified four research questions which need to be answered, the questions are:

Q1) How to initialize the initial temperature dynamically based on the problem instance individually?

Q2) What is the effective cooling schedule that can adaptively adjust the amounts of temperature decrement during the SA search process?

Q3) How to design a good neighborhood structure that can help SA to effectively search for good quality solution and minimize the random swap and move?

Q4) How to choose a good neighborhood structure among different neighborhood structures during the search process that is able to avoid the disconnected neighborhood?

2. Related work

2.1 Dynamic initial temperature

This section presents Dynamic Initial Temperature (D-IT) which proposed by Tarawneh et al. [20] to improve performance of SA by initializing the initial temperature dynamically. Normally, when the temperatures are very high the search space of SA could be very wide by accepting many worse solutions. Meanwhile, the range of accepting worse solution(s) becomes very small when the initial temperature is low.

The proposed D-IT mechanism by Tarawneh et al. [20] is used to estimate the suitable range of accepting worse solution at early stage, by performing a preliminary experiment to clarify the acceptance ratio related to different worse solutions for each temperature and deviation average γ , where γ is calculated using the following equation:

$$(1) \quad \gamma = \frac{\sum_{i=1}^n |\Delta f|}{n}$$

Where $\Delta f = f(s^*) - f(s)$ and n is the total current iterations

Table 2: The acceptance ratio for different averages deviation under different temperatures ranges

		Temperatures									
		10000000000	1000000	10000	5000	2000	1000	500	100	10	1
γ	150	100%	100%	99.0%	95.8%	94.3%	85.5%	71.5%	27.4%	0.0%	0.0%
	50	100%	100%	99.0%	98.9%	97.6%	95.5%	88.3%	28.9%	0.0%	0.0%
	5	100%	100%	99.0%	99.0%	99.9%	99.9%	97.5%	94.9%	70.5%	0.0%

From Table (2), the high temperatures (i.e. $T = 10000000000$) lead the SA to accept every worse solution. Such as, when the γ is equal 150, 50 or 5. Namely, high temperature makes the SA to waste more computational time by accepting every worse bad solution. In this study, we select the initial temperature value based on the average deviation of the penalty value γ (see Eq.1). However, according to the preliminary experiment, we suggest several initial temperatures ranges to select the suitable one among them according to the average deviation of the penalties value that will be calculated for several iterations during the SA process. The initial temperatures range in Table 3.

Table 3: The initial temperature selection

γ	T_0
50 and more	Random (1000, 2000)
5 to 49	Random (500, 1000)
1 to 4	Random (100, 500)

Note: γ is the deviation average of the penalties

Table 4: The differences between the dynamic initial temperatures with other from the literature

Technology	First Initial Temperature	Average(Y)	Comments	Disadvantages
Dynamic	Dynamic	$ \gamma $	For several consecutive iterations, inside the SA and for every Δ (increase or decrease)	Need a good chosen cooling schedule.
SA with Estimated Temperature [16]	Infinite	$\overleftrightarrow{ \gamma }$	For every iteration (for each current solution neighbors), when is increased	The temperature is not control parameter anymore, but the acceptance probability.
Solving the Course Scheduling problem Using SA [3]	10000	$\overleftrightarrow{ \gamma }$	Before the SA starts, for several iteration, when is increased	The initial temperature still high.

Note $\overleftrightarrow{|\gamma|}$ means that the deviation average for (positive and negative values).

From Table 4, we found that the other researchers' techniques have some limitations such as initializing high initial temperature values even when they used a small neighbors size, also the temperature becomes uncontrolled SA parameter anymore when it is estimated for each iteration, which means that the problem of setting the initial temperature value is still exists. Therefore, the dynamic Initial Temperature overcomes these problems by choosing the initial temperature value properly. Where the initial temperature T_0 is selected from the Table 2 suggested according to preliminary experiment from Table 1.

2.2 Adaptive Cooling Schedule (ACS)

The performance of SA depends heavily on temperature cooling schedule [5], where the good cooling schedule is substantial feature to get the optimal solution and to reduce the consuming time [17].

In this research, two cooling schedule proposed in the literature review are investigated in order to identify the good and suitable one as follow:

1. The first one is called Static or Geometric cooling schedules, which proposed by Kirkpatrick et al. [11] as the following equation:

$T_{k+1} = \alpha T_k$; where $(0 < \alpha < 1)$ denotes the cooling rate or factor; T_k is the current initial temperature.

2. The second one is called Adaptive cooling schedules presented by Lewis et al. [12] where the temperature is decremented to be adjusted during the process of the algorithm according to the objective function values which is used to decide amount of temperature decreasing as following equations:

$$(2) \quad \lambda_0 = 1 - \beta$$

$$(3) \quad \lambda_{i+1} = \lambda_i + \frac{\beta + \beta}{M}$$

$$(4) \quad T_{i+1} = T_i - \lambda_{i+1} \left(\frac{T_0}{M} \right)$$

Where β represents a parameter to determine the value of λ which influence the concavity amount that presented in cooling schedule and M represents the iteration number used to decrease the initial temperature T_0 to a zero value or close to it. However, this work investigates both geometric and adaptive cooling schedules, which were presented by Kirkpatrick et al. [11] and Lewis et al. [12] In order to identify the suitable cooling schedule for our study.

2.3 Neighborhood structure

Lü and Hao [13] claimed that neighbourhood structure is one of the most important features of the local search algorithms, where the effective and good neighbourhoods structure influence the SA performance positively [14]. Furthermore, Fleischer and Jacobso [9] stated that to improve the solution quality and the SA performance we must chose a good neighbourhood structure and size. Also, [7] mentioned that:

A smooth topology with shallow local minima which imposed by a neighbourhood structure is preferred, rather than a bumpy topology with many deeply local minima

Several researchers claimed that SA performance is better when the neighbourhood structure size is relatively small [6]. Meanwhile, other researchers demonstrated that SA performance improves when a large neighbourhood structure is used [15].

In this work, we use three neighbourhood structures in order to improve the solution quality, two neighbourhood structures are common (simple move as NS1 and simple swap as NS2), and a new proposed neighbourhood structure NS_3 which is presented by Tarawneh et al. [20].

NS_3 : Generally, the selection mechanism in neighbourhood structure happens randomly (e.g.: - select two lectures randomly which belong to two different

rooms and slots). Thus, the search may need more time to reach a good solution. Tarawneh et al. [20] presented a new neighbourhood structure to reduce the random selection. NS_3 Neighbourhood structure mechanism calculates the total soft constraints penalties for each timeslot and sums them up for the whole week (Table 4 as example). Then, the lectures with the highest penalty swap with other lectures randomly.

Table (4) shows that the presented neighbourhood structure mechanism selects the timeslot with highest penalties (i.e. timeslot= three), and randomly select another timeslot. Then it interchanges both of them with any conflict. Therefore, the soft constraints violation weight (penalties) is calculated for each timeslot every time when a new neighbourhood structure is applied. However, this neighbourhood is used to minimize the random selection in order to avoid the disconnected neighbourhood structure that leads the search to escape from local optimum solution. it works as a flipping procedure for the SA search space.

Table 5: The soft constraints penalties for each time period and slot

Timeslots (1-6)	Room1					Room2					Room3					Penalties
	Days(1-5)					Days(1-5)					Days(1-5)					
	0	5	6	30	0	100	0	1	4	2	10	20	0	1	160	
	6	15	10	0	0	2	4	2	30	100	1	0	60	1	3	
	10	0	80	10	0	1	20	1	1	1	3	0	0	0		
	10	1	4	5	2	0	10	0	10	20	6	4	150	1	10	
	1	2	8	10	0	20	30	15	0	1	2	2	6	0	9	
2	3	6	1	8	120	30	40	0	4	0	0	9	0	1		

Figure 5 shows the SA pseudo code, which contain D-IT, A-CS and the new neighbourhood structure.

Hint: for the parameters settings, we setup them for each part of our experiments separately.

3. Experiment and result

3.1 Experimental results of the Adaptive Cooling Schedule (ACS), D-IT with adaptive neighborhood structure

In the next experiment we compare the geometric cooling schedule (GCS) presented by Kirkpatrick et al. [11] with static initial temperature and standard neighbourhoods structure N1 and N2 against the adaptive cooling seclude (ACS) presented by Lewis et al. [12] with D-IT presented by Tarawneh et al. [20] and A-CS presented by Tarawneh and Ayob [19]. The parameters setting that we used in this experiment are presented in Table 6.

NS1: Move one lecture period from the current period to another free position period.

NS2: Randomly swaps two different lectures from different time slots and rooms.

NS3: Adaptive neighborhood structure presented by Tarawneh et al. [20].

Enhanced SA pseudo code

```

Given an initial solution, Sol;
Set Best solution Sol*=Sol;
Set the iteration counter, i = 0;
Set the initial deviation average, = zero;
Set the initial temperature counter iterations, k;
Set the minimum temperature,  $T_{min}$ ;
while termination criteria does not met do
    i=i+1 // update the iterations counter.
    Generate k neighbors from N neighborhood structures;
    Assign the best neighbor to Sol";
    Calculate  $\Delta = f(Sol'') - f(Sol)$ 
    //Calculate the quality deference between the new solution and the
    current solution.
    if  $i < k$  then
         $\gamma = \frac{\gamma + abc(\Delta)}{i}$ 
        // Calculate the deviation average.
        Select  $T_0$  according to Table 3.
    end
    if  $\Delta \leq 0$  then
        // New solution.
         $Sol \leftarrow Sol''$  // Update the current solution.
        if  $f(Sol'') < f(Sol^*)$  then
            // Sol'' has better quality than Sol*.
             $Sol'' \leftarrow Sol^*$  // Update the best solution so far.
        end
    else
        end
        Generate a random number  $\eta$  between  $[0, 1]$ ;
        if  $(e^{-\Delta/T_i}) > \eta$  then
            // The solution Sol is accepted.
             $Sol \leftarrow Sol''$ 
        end
        Update the temperature; //A-CS end

```

Table 6: SA parameters setting (standard and enhanced comparison)

Parameters	Standard SA(S-SA)	Enhanced SA(E-SA)
Initial Temperature	1000000	D-IT
Neighborhoods structure	NS1 and NS2	NS3
Cooling rate	0.99	Adaptive
Termination condition	460 seconds	460 seconds

Table (6) illustrates the results of S-SA and E-SA. The first column indicates the instances, columns 2-5, 6-9 reports the best and worst solutions, Mean, and standard deviation over 31 runs for each instance and Column 10 reports the P-value.

Table 7. Computational results of the SA performance using standard components (S-SA) and Enhanced components(E-SA).

Table (7) shows that E-SA improves the SA performance compared to S-SA. SA permanence is statistically significant performance depending on enhanced simulated annealing components (in all instances except the small instances comp1 and comp11). Thus, we conclude that adaptive cooling schedule and

dynamic initial temperature with adaptive neighborhood structure will leads the SA to improve the solution quality compared to SA with standard and static components. *Note: Best results in italic bold.*

Table 7 :Enhanced components on ITC2007track³ datasets

Instances	E-SA				S-SA				P-Value
	Best	Max.	Mean	STD	Best	Max.	Mean	STD	
Comp01	5	6	5.09	0.288	5	9	5.26	0.773	0.071
Comp02	86	105	97.35	5.684	88	108	99.32	4.962	0.001
Comp03	93	121	104.81	8.023	105	134	121.06	8.095	0
Comp04	65	83	73.27	5.806	67	90	78.23	6.893	0.01
Comp05	330	350	340.48	5.372	338	356	344.78	5.789	0.001
Comp06	84	120	102	9.778	89	115	102.77	6.869	0.001
Comp07	54	71	63.65	5.251	57	82	68.84	6.187	0
Comp08	54	76	66.94	6.303	59	85	74.87	6.994	0
Comp09	143	169	157.94	7.550	149	176	161.10	8.010	0.002
Comp10	36	53	44.45	5.824	43	63	54.65	5.851	0
Comp11	0	1	0.15	0.359	0	2	0.45	0.624	0.102
Comp12	341	369	356.42	8.955	355	385	372.16	8.214	0
Comp13	95	119	105.09	7.296	96	127	111.00	10.237	0.025
Comp14	77	100	88.61	7.680	81	119	100.48	10.875	0
Comp15	83	107	94.74	6.831	87	123	103.10	9.724	0.001
Comp16	60	84	72.15	7.652	68	90	78.55	5.579	0.001
Comp17	84	103	94.58	6.071	88	119	102.06	9.536	0.002
Comp18	94	120	104.84	7.453	100	123	111.71	6.953	0.002
Comp19	91	121	105.87	8.385	85	125	111.29	8.814	0.009
Comp20	46	68	56.26	6.309	50	74	62.55	6.985	0.003
Comp21	103	122	112.42	6.656	111	135	118.58	5.993	0

4. Conclusion

In this paper, we employed the standard SA components to solve the university course timetabling (UCT) problem using the benchmark dataset ITC2007-Track3. We investigated the SA components as follows: initial temperature, cooling schedule and neighborhood's structure, by initializing the initial temperature dynamically and identifying a good cooling schedule to decrease the temperature properly. Moreover, we worked to improve the SA solution quality using adaptive neighborhood structure that swaps the selected time slot randomly with the timeslot that has the highest cost (penalty).

However, SA still could get stuck or trapped in local optimum, in the future work we suggest to hybridizing SA with other approaches or Artificial intelligence (AI) algorithms, in addition to test the solutions using a real world data set for further investigations.

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Solvability for continuous classical optimal control associated with triple hyperbolic boundary value problem

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Abstract. This article deals with the solvability for continuous classical optimal control associated with triple boundary value problem of linear hyperbolic type; for given continuous classical control vector the Galerkin method is used to prove the existence theorem of a unique state vector solution for the hyperbolic boundary value problem as well as for its adjoint equations. The existence theorem of a continuous classical optimal control vector dominating with the considered triple hyperbolic equations is proved. The directional derivative for the cost functional is derived. Finally the necessity theorem(conditions) for optimality of the problem is stated and proved.

Keywords: continuous classical optimal control, triple boundary value problem of linear hyperbolic type, Galerkin Method, the necessary conditions for optimality.

1. Introduction

Various applications in the real life are classified as an optimal control problems (OCPs) in a different field of sciences, for example in medicine [1], robots [2], engineering [3], economic [4], chemistry[5], electromagnetic [6], pharmacy [7] and many others fields. Usually, in the field of applied mathematics OCPs are dominated by ordinary or partial differential equations (ODES or PDES) which are studying by many researchers, for examples and precisely [8],[9], and [10] are studied OCP dominated by PDES of elliptic, parabolic and hyperbolic type respectively, whilst [11],[12] and [13] studied OCP which are dominating by couple of PDES (CPDES) of elliptic, parabolic and hyperbolic type, whilst [14],[15]and [16] investigated boundary OCP dominated by these three types of PDES respectively, beside these studies, [17] and [18] investigated the OCP dominated by triple linear PDEs (TLPDES) of elliptic and parabolic type. All these investigations encourage us to aim at the study for OCP dominating by TLPDES of hyperbolic type (TLHPDES). This article is concerned with the investigation for the continuous classical optimal control problem (CCOCP), it includes in the state and proof the existence theorem of a unique solution (state

vector solution SVS) for the TLHPDES (as well as the solution vector of the triple adjoint equations (TAES) associated the (TLHPDES)) by employing the Galerkin method (GAM) when the continuous classical control vector (CCCV) is fixed, also the existence theorem for a continuous classical optimal control vector (CCOCV) dominating by TLHPDES is stated and proved. The derivation of the Directional derivative (DDV) for the cost functional is obtained; finally, the necessity theorem (necessity conditions) for optimality (NTHO) of this OCP is sated and proved.

2. Problem description

Let $\Omega \subset R^2, x = (x_1, x_2), Q = [0, T] \times \Omega, \tilde{I} = [0, T], \Gamma = \partial\Omega, \Sigma = \Gamma \times \tilde{I}$, the CCOCV include in the TSE are given by the following TLHPDES:

$$\begin{aligned} (1) \quad & \varphi_{1tt} - \Delta\varphi_1 + \varphi_1 - \varphi_2 - \varphi_3 = p_1 + \alpha_1 \text{ in } Q, \\ (2) \quad & \varphi_{2tt} - \Delta\varphi_2 + \varphi_2 + \varphi_3 + \varphi_1 = p_2 + \alpha_2 \text{ in } Q, \\ (3) \quad & \varphi_{3tt} - \Delta\varphi_3 + \varphi_3 + \varphi_1 - \varphi_2 = p_3 + \alpha_3 \text{ in } Q, \end{aligned}$$

with the following boundary conditions (BCs) and the initial conditions (ICs)

$$\begin{aligned} (4) \quad & \varphi_1(x, t) = 0, \text{ on } \Sigma, \\ (5) \quad & \varphi_2(x, t) = 0, \text{ on } \Sigma, \\ (6) \quad & \varphi_3(x, t) = 0, \text{ on } \Sigma, \\ (7) \quad & \varphi_1(x, 0) = \varphi_1^0(x), \text{ and } \varphi_{1t}(x, 0) = \varphi_1^1(x), \text{ on } \Omega, \\ (8) \quad & \varphi_2(x, 0) = \varphi_2^0(x), \text{ and } \varphi_{2t}(x, 0) = \varphi_2^1(x), \text{ on } \Omega, \\ (9) \quad & \varphi_3(x, 0) = \varphi_3^0(x), \text{ and } \varphi_{3t}(x, 0) = \varphi_3^1(x) \text{ on } \Omega, \end{aligned}$$

where $(p_1, p_2, p_3) \in L^2(Q)$ is a given vector function for each $(x_1, x_2) \in \Omega$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in L^2(Q)$ is a CCCV and the corresponding SVS is $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in H^2(\Omega)$. The set of admissible CCCV is: $\vec{W}_A = \{\vec{\alpha} \in L^2(Q) | \vec{\alpha} \in \vec{A} = A_1 \times A_2 \times A_3 \subset R^3 \text{ a.e. in } Q\}$, $\vec{A}$ is convex and compact.

The cost function for $\beta > 0$ is:

$$(10) \quad G_0(\vec{\alpha}) = \|\varphi_1 - \varphi_{1d}\|_Q^2 + \|\varphi_2 - \varphi_{2d}\|_Q^2 + \|\varphi_3 - \varphi_{3d}\|_Q^2 + (\beta/2)(\|\alpha_1\|_Q^2 + \|\alpha_2\|_Q^2 + \|\alpha_3\|_Q^2).$$

Let $\vec{W} = W \times W \times W; W = H^1(\Omega)$, and $\vec{W} = \{\vec{w} : \vec{w} = (w_1, w_2, w_3) \in H^1(\Omega), w_1 = w_2 = w_3 = 0 \text{ on } \partial\Omega\}$.

The weak form(wf) of (1)- (9), is given almost everywhere on I by

$$(11a) \quad \begin{aligned} & \langle \varphi_{1tt}, w_1 \rangle + (\nabla \varphi_1, \nabla w_1) + (\varphi_1, w_1) - (\varphi_2, w_1) - (\varphi_3, w_1) \\ & = (p_1 + \alpha_1, w_1), \forall w_1 \in W_1 \end{aligned}$$

$$(11b) \quad \begin{aligned} & (\varphi_1^0, w_1) = (\varphi_1(0), w_1), \text{ and } (\varphi_1^1, w_1) \\ & = (\varphi_{1t}(0), w_1), \forall w_1 \in W_1 \end{aligned}$$

$$(12a) \quad \begin{aligned} & \langle \varphi_{2tt}, w_2 \rangle + (\nabla \varphi_2, \nabla w_2) + (\varphi_2, w_2) + (\varphi_3, w_2) + (\varphi_1, w_2) \\ & = (p_2 + \alpha_2, w_2), \forall w_2 \in W_2 \end{aligned}$$

$$(12b) \quad \begin{aligned} & (\varphi_2^0, w_2) = (\varphi_2(0), w_2), \text{ and } (\varphi_2^1, w_2) \\ & = (\varphi_{2t}(0), w_2), \forall w_2 \in W_2 \end{aligned}$$

$$(13a) \quad \begin{aligned} & \langle \varphi_{3tt}, w_3 \rangle + (\nabla \varphi_3, \nabla w_3) + (\varphi_3, w_3) + (\varphi_1, w_3) - (\varphi_2, w_3) \\ & = (p_3 + \beta_3, w_3), \forall w_3 \in W_3 \end{aligned}$$

$$(13b) \quad \begin{aligned} & (\varphi_3^0, w_3) = (\varphi_3(0), w_3), \text{ and } (\varphi_3^1, w_3) \\ & = (\varphi_{3t}(0), w_3) \forall w_3 \in W_3. \end{aligned}$$

The following assumption is important to study the CCOCV problem (CCOCVP).

2.1 Assum. (A):

The function $p_i (\forall i = 1, 2, 3)$ is satisfied the following condition w.r.t. x and t , i.e.

$$|p_i| \leq \gamma_i(x, t), \text{ where } (x, t) \in Q, \gamma_i \in L^2(Q, R).$$

3. The solution for the wf:

Theorem 3.1. *Existence of a Unique Solution for the wf: In addition to assume (A), for each given CCCV $\vec{\alpha} \in L^2(Q)$, the wf (11-13) has a unique solution $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ with $\vec{\varphi} \in L^2(I, W)$ and $\vec{\varphi}_t = (\varphi_{1t}, \varphi_{2t}, \varphi_{3t}) \in L^2(I, W^*)$.*

Proof. Let $\vec{W}_n = W_n \times W_n \times W_n \subset \vec{W}$ (for each n) be the set of continuous and piecewise affine function (CPWAFF) in Ω . $\{\vec{W}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{W}$, such that $\forall \vec{w} = (w_1, w_2) \in \vec{W}$, there exists a sequence $\{\vec{w}_n\}$ with $\vec{w}_n = (w_{1n}, w_{2n}) \in \vec{W}_n, \forall n$, and $\vec{w}_n \rightarrow \vec{w}$ strongly in $\vec{W} \Rightarrow \vec{w}_n \rightarrow \vec{w}$ strongly in $L^2(\Omega)$. $\{\vec{w}_j = (w_{1j}, w_{2j}, w_{3j}) : j = 1, 2, \dots, M(n)\}$ be a finite basis of $\vec{W}_n$ (where $\vec{w}_j$ is CPWAFF in Ω , with $\vec{w}_j(x) = 0$ on the boundary Γ) and let $\vec{\varphi}_n = (\varphi_{1n}, \varphi_{2n}, \varphi_{3n})$ be the Galerkin approximate solution (method (GAM)) to the exact solution $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ such that

$$(14) \quad \varphi_{in} = \sum_{j=1}^n c_{ij}(t) v_{ij}(x),$$

where $c_{ij}(t)$ is unknown function of t , $\forall i = 1, 2, 3, j = 1, 2, \dots, n$. Now, the GAM is used to approximate the wf (10)-(13) w.r.t. x , substituting $\vec{\varphi}_{int} = z_{in}$, (for $i = 1, 2, 3$) in the obtained approximation wf, yield to

$$(15a) \quad \begin{aligned} & \langle \varphi_{1nt}, w_1 \rangle + (\nabla \varphi_{1n}, \nabla w_1) + (\varphi_{1n}, w_1) - (\varphi_{2n}, w_1) - (\varphi_{3n}, w_1) \\ & = (p_1 + \alpha_1, w_1), \forall w_1 \in W_n, \end{aligned}$$

$$(15b) \quad \begin{aligned} & (\varphi_{1n}^0, w_1) = (\varphi_{1n}^0, w_1) \text{ and } (\varphi_{1n}^1, w_1) \\ & = (\varphi_{1n}^1, w_1), \forall w_1 \in W_n, \end{aligned}$$

$$(16a) \quad \begin{aligned} & \langle \varphi_{2nt}, w_2 \rangle + (\nabla \varphi_{2n}, \nabla w_2) + (\varphi_{2n}, w_2) + (\varphi_{3n}, w_2) + (\varphi_{1n}, w_2) \\ & = (p_2 + \alpha_2, w_2), w_2 \in W_n, \end{aligned}$$

$$(16b) \quad \begin{aligned} & (\varphi_{2n}^0, w_2) = (\varphi_2^0, w_2), \text{ and } (\varphi_{2n}^1, w_2) \\ & = (\varphi_2^1, w_2), \forall w_2 \in W_n, \end{aligned}$$

$$(17a) \quad \begin{aligned} & \langle y_{3nt}, w_3 \rangle + (\nabla \varphi_{3n}, \nabla w_3) + (\varphi_{3n}, w_3) + (\varphi_{1n}, w_3) - (\varphi_{2n}, w_3) \\ & = (p_3 + \alpha_3, w_3), \forall w_3 \in W_n, \end{aligned}$$

$$(17b) \quad \begin{aligned} & (\varphi_{3n}^0, w_3) = (\varphi_3^0, w_3), \text{ and } (\varphi_{3n}^1, w_3) \\ & = (\varphi_3^1, w_3) \forall w_3 \in W_n, \end{aligned}$$

where $\varphi_{in}^0 = \varphi_{in}^0(x) = \varphi_{in}(x, 0) \in W_n$ (respectively $z_{in}^0 = \varphi_{in}^1 = \varphi_{in}^1(x) = \varphi_{int}(x, 0) \in L^2(\Omega)$ be the projection of φ_i^0 onto W (be the projection of $\varphi_i^1 = \varphi_{it}$ onto $L^2(\Omega)$, $\forall i = 1, 2, 3$, i.e.

$$(18) \quad \varphi_{in}^0 \longrightarrow \varphi_i^0 \text{ strongly in } W, \text{ with } \|\tilde{\varphi}_n^0\|_1 \leq b_0 \text{ and } \|\varphi_n^0\|_0 \leq b_0$$

$$(19) \quad \varphi_{in}^1 \longrightarrow \varphi_i^1 \text{ strongly in } L^2(\Omega) \text{ and } \|\tilde{\varphi}_n^1\|_0 \leq b_1.$$

Substituting (14) for $i = 1, 2, 3$ in (17)-(19) respectively and then setting $w_i = w_{il}, \forall l = 1, 2, \dots, n$, the obtained equations are equivalent to the following linear system (LS) of 1st order ODEs with ICs (which has a unique solution), i.e.

$$(20a) \quad A_1 \dot{C}_1(t) + B_1 C_1(t) - EC_2(t) - FC_3(t) = b_1,$$

$$(20b) \quad A_1 C_1(0) = b_1^0, \text{ and } A_1 \bar{C}_1(0) = b_1^1,$$

$$(21a) \quad A_2 \dot{C}_2(t) + B_2 C_2(t) + GC_3(t) + HC_1(t) = b_2,$$

$$(21b) \quad A_2 C_2(0) = b_2^0, \text{ and } A_2 \bar{C}_2(0) = b_2^1,$$

$$(22a) \quad A_3 \dot{C}_3(t) + B_3 C_3(t) + RC_1(t) - WC_2(t) = b_3,$$

$$(22b) \quad A_3 C_3(0) = b_3^0, \text{ and } A_3 \bar{C}_3(0) = b_3^1.$$

Where $A_i = (a_{ilj})_{(n \times n)}, a_{ilj} = (w_{ij}, w_{il}), B_i = (b_{ilj})_{(n \times n)}, b_{ilj} = (\nabla w_{ij}, \nabla w_{il}) + (w_{ij}, w_{il}), E = (e_{lj})_{(n \times n)}, e_{lj} = (w_{2j}, w_{1l}), F = (f_{lj})_{(n \times n)}, f_{lj} = (w_{3j}, w_{1l}), G = (g_{lj})_{(n \times n)}, g_{lj} = (w_{3j}, w_{2l}), H = (h_{lj})_{(n \times n)}, h_{lj} = (w_{1j}, w_{2l}), R = (r_{lj})_{(n \times n)}, r_{lj} = (w_{1j}, w_{3l}), W = (w_{lj})_{(n \times n)}, w_{lj} = (w_{2j}, w_{3l}), b_{il}^0 = (y_i^0, w_{il}), b_i^0 = (b_{il}^0), b_i = (b_{il})_{(n \times 1)}, b_{il} = (p_i + \alpha_i, v_{il}), \dot{C}_i(t) = (\dot{c}_{ij}(t))_{(n \times 1)}, C_i(t) = (c_{ij}(t))_{(n \times 1)}, \bar{C}_i(0) = (\bar{c}_{ij}(0))_{(n \times 1)}, C_i(0) = (c_{ij}(0))_{(n \times 1)}, \forall l = 1, 2, 3, \dots, n, i = 1, 2, 3$

Then corresponding to the sequence $\{\vec{W}_n\}$, the following approximation problems are hold, i.e. for each $\vec{w}_n = (w_{1n}, w_{2n}, w_{3n}) \subset W_n$, and $n = 1, 2, \dots$

$$(23a) \quad \begin{aligned} & \langle \varphi_{1ntt}, w_{1n} \rangle + (\nabla \varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n}) - (\varphi_{2n}, w_{1n}) \\ & - (\varphi_{3n}, w_{1n}) = (p_1 + \alpha_1, w_{1n}), \end{aligned}$$

$$(23b) \quad (\varphi_{1n}^0, w_{1n}) = (\varphi_1^0, w_{1n}), (\varphi_{1n}^1, w_{1n}) = (\varphi_1^1, w_{1n})$$

$$(24a) \quad \langle \varphi_{2nt}, w_{2n} \rangle + (\nabla \varphi_{2n}, \nabla w_{2n}) + (\varphi_{2n}, w_{2n}) + (\varphi_{1n}, w_{2n}) + (\varphi_{3n}, w_{2n}) = (p_2 + \alpha_2, w_{2n}),$$

$$(24b) \quad (\varphi_{2n}^0, w_{2n}) = (\varphi_2^0, w_{2n}), (\varphi_{2n}^1, w_2) = (\varphi_2^1, w_{2n})$$

$$(25a) \quad \langle \varphi_{3nt}, w_3 \rangle + (\nabla \varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n}) + (\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n}) = (p_3 + \alpha_3, w_{3n}),$$

$$(25b) \quad (\varphi_{3n}^0, w_{3n}) = (\varphi_3^0, w_{3n}), (\varphi_{3n}^1, w_{3n}) = (\varphi_3^1, w_{3n}),$$

which has a sequence of unique solution $\{\vec{\varphi}_n\}$.

Substituting $w_{in} = \varphi_{int}$, for $i = 1, 2, 3$ in ((23a),(24a),(25a)) respectively, then adding the three obtained equations, using Lemma 1.2 in [20] for the 1st term of the L.H.S., once get

$$(26) \quad \begin{aligned} 28(d/dt)[\|\vec{\varphi}_{nt}\|_0^2 + \|\varphi_n\|_1^2] &= 2((\varphi_{2n}, \varphi_{1nt}) + (\varphi_{3n}, \varphi_{1nt}) - (\varphi_{1n}, \varphi_{2nt}) \\ &\quad - (\varphi_{3n}, \varphi_{2nt}) - (\varphi_{1n}, \varphi_{3nt}) + (\varphi_{2n}, \varphi_{3nt}) + (p_1 + \alpha_1, \varphi_{1nt}) \\ &\quad + (p_2 + \alpha_2, \varphi_{2nt}) + (p_3 + \alpha_3, \varphi_{3nt})), \\ (d/dt)[\|\vec{\varphi}_{nt}\|_0^2 + \|\vec{\varphi}_n\|_1^2] &\leq 2(|(\varphi_{2n}, \varphi_{1nt})| \\ &\quad + |(\varphi_{3n}, \varphi_{1nt})| + |(\varphi_{1n}, \varphi_{2nt})| + |(\varphi_{3n}, \varphi_{2nt})| \\ &\quad + |(\varphi_{1n}, \varphi_{3nt})| + |(\varphi_{1n}, \varphi_{3nt})| + |(\varphi_{2n}, \varphi_{3nt})| + |(p_1 + \alpha_1, \varphi_{1nt})| \\ (27) \quad &\quad + |(p_2 + \alpha_2, \varphi_{2nt})| + |(p_3 + \alpha_3, \varphi_{3nt})|). \end{aligned}$$

Using the C-S inequality for the R.H.S. of (27), integrating both sides on $[0, t]$, using

$$\|\varphi_{in}\|_0 \leq \|\varphi_{in}\|_1 \leq \|\vec{\varphi}_n\|_1, \|\varphi_{int}\|_0 \leq \|\vec{\varphi}_{nt}\|_0,$$

and assum (A), to get

$$(28) \quad \begin{aligned} &\int_0^t (d/dt)[\|\vec{\varphi}_{nt}(t)\|_0^2 + \|\vec{\varphi}_n\|_1^2] dt \leq \\ &2 \int_0^t (\|\vec{\varphi}_{nt}\|_0^2 + \|\vec{\varphi}_n\|_1^2) dt + \int_0^t \sum_{i=1}^3 (\|\gamma_i\|_0^2 + \|u_{ii}\|_0^2) dt + \int_0^t \|\vec{\varphi}_{nt}\|_0^2 dt + \\ &\sum_{i=1}^3 (\|\gamma_i\|_Q^2 + \|\check{b}_i\|_Q^2) + \lambda_1 \int_0^t (\|\vec{\varphi}_{nt}\|_0^2 + \|\vec{\varphi}_n\|_1^2) dt \leq \\ &\lambda_2 + \lambda_1 \int_0^t (\|\vec{\varphi}_{nt}\|_0^2 + \|\vec{\varphi}_n\|_1^2) dt, \end{aligned}$$

where, $\lambda_2 = \sum_{i=1}^3 (\bar{b}_1 + \check{b}_i)$, $\lambda_1 \geq 3$, with $\|\gamma_i\|_Q^2 \leq \bar{b}_i$, $\|\alpha_i\|_Q \leq \check{b}_i$, for each $i = 1, 2, 3$. Since $\|\varphi_n^0\|_1 \leq b_1$, and $\|\varphi_n^1\|_0 \leq b_0$, with $\lambda_3 = b_0 + b_1 + \lambda_2$, inequality (28) becomes $\|\vec{\varphi}_{nt}(t)\|_0^2 + \|\vec{\varphi}_n(t)\|_1^2 \leq \lambda_3 + \lambda_1 \int_0^t (\|\vec{\varphi}_n\|_0^2 + \|\vec{\varphi}_{nt}\|_1^2) dt$.

Using the Belman-Gronwall (B-G) inequality, to get $\forall t \in [0, T]$, $\|\vec{\varphi}_{nt}(t)\|_0^2 + \|\vec{\varphi}_n(t)\|_1^2 \leq \lambda_3 e^{\lambda_1} = b^2(c) \|\vec{\varphi}_{nt}(t)\|_0^2 \leq b^2(c)$ and $\|\vec{\varphi}_n(t)\|_1^2 \leq b^2(c)$, $\forall t \in [0, T]$ Easily once can obtained that $\|\vec{\varphi}_{nt}(t)\|_Q \leq b_1(c)$ and $\|\vec{\varphi}_n(t)\|_{L^2(I,V)} \leq b(c)$.

Then the Alaoglu's theorem (ATh) can be applied here, thus there exists a subsequence of $\{\vec{\varphi}_n\}_{n \in \mathbb{N}}$, for simplicity let be $\{\vec{\varphi}_n\}_{n \in \mathbb{N}}$, s.t. $\vec{\varphi}_{nt} \rightarrow \vec{\varphi}$ weakly in $L^2(Q)$ and $\vec{\varphi}_n \rightarrow \vec{\varphi}$ weakly in $L^2(I, V)$, and since Now, multiplying both sides of (23a),(24a),(25a) by

$$(29) \quad y_i(t) \in C^2[0, T], \text{ s.t. } y_i(T) = y_i(T) = 0, y_i(0) \neq 0, y_i'(0) \neq 0, \forall i = 1, 2, 3$$

integrating on $[0, T]$, finally integrating by parts twice the 1st term of each one of the obtained three equations, yield to

$$(30) \quad \begin{aligned} & - \int_0^T d/dt(\varphi_{1n}, w_{1n})(w_1')(t)dt + \int_0^T [(\nabla \varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n}) \\ & - (\varphi_{2n}, w_{1n}) - (\varphi_{3n}, w_{1n})]y_1(t)dt \\ & = \int_0^T (p_1 + \alpha_1, w_{1n})y_1(t)dt + (\varphi_{1n}^1, w_{1n})y_1(0), \end{aligned}$$

$$(31) \quad \begin{aligned} & \int_0^T (\varphi_{1n}, w_{1n})(y_1'')(t)dt + \int_0^T [(\nabla \varphi_{1n}, \nabla w_{1n}) + (\varphi_{1n}, w_{1n}) \\ & - (\varphi_{2n}, w_{1n}) - (\varphi_{3n}, w_{1n})]y_1(t)dt \\ & = \int_0^T (p_1 + \alpha_1, w_{1n})y_1(t)dt + (\varphi_{1n}^1, w_{1n})y_1(0) + (\varphi_{1n}^0, w_{1n})(y_1')(0), \end{aligned}$$

$$(32) \quad \begin{aligned} & - \int_0^T d/dt(\varphi_{2n}, w_{2n})(y_2')(t)dt + \int_0^T [(\nabla \varphi_{2n}, \nabla w_{2n}) \\ & + (\varphi_{2n}, w_{2n}) + (\varphi_{1n}, w_{2n}) + (\varphi_{3n}, w_{2n})]y_2(t)dt \\ & = \int_0^T (p_2 + \alpha_2, w_{2n})y_2(t)dt + (\varphi_{2n}^0, w_{2n})y_2(0), \\ & \int_0^T (\varphi_{2n}, w_{2n})(y_2'')(t)dt + \int_0^T [(\nabla \varphi_{2n}, \nabla w_{2n}) + (\varphi_{2n}, w_{2n}) \\ & + (\varphi_{1n}, w_{2n}) + (\varphi_{3n}, w_{2n})]y_2(t)dt \end{aligned}$$

$$(33) \quad \begin{aligned} & = \int_0^T (p_2 + \alpha_2, w_{2n})\varphi_2(t)dt + (\varphi_{2n}^0, w_{2n})\varphi_2(0) + (\varphi_{2n}^0, w_{2n})(\varphi_2')(0), \\ & - \int_0^T d/dt(\varphi_{3n}, w_{3n})(y_3')(t)dt + \int_0^T [(\nabla \varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n}) \\ & + (\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n})]y_3(t)dt \end{aligned}$$

$$(34) \quad \begin{aligned} & = \int_0^T (p_3 + \alpha_3, w_{3n})y_3(t)dt + (\varphi_{3n}^1, w_{3n})y_3(0), \\ & \int_0^T (\varphi_{3n}, w_{3n})(y_3'')(t)dt + \int_0^T [(\nabla \varphi_{3n}, \nabla w_{3n}) + (\varphi_{3n}, w_{3n}) \\ & + (\varphi_{1n}, w_{3n}) - (\varphi_{2n}, w_{3n})]y_3(t)dt \end{aligned}$$

$$(35) \quad = \int_0^T (p_3 + \alpha_3, w_{3n})\varphi_1(t)dt + (\varphi_{3n}^1, w_{3n})y_3(0) + (\varphi_{3n}^0, w_{3n})(y_3')(0).$$

First, since

$$\begin{aligned} w_{in} \longrightarrow w_i \quad \text{strongly in } W &\implies \left\{ \begin{array}{l} w_{in}y_i(t) \longrightarrow w_iy_i(t) \\ w_{in}y_i'(t) \longrightarrow w_iy_i'(t) \end{array} \right\} \quad \text{strongly in } L^2(I, W) \\ w_{in} \longrightarrow w_i \quad \text{strongly in } L^2(\Omega) &\implies \left\{ \begin{array}{l} w_{in}y_i(t) \longrightarrow w_iy_i(0) \\ w_{in}y_i'(t) \longrightarrow y_i'(t) \\ w_{in}y_i''(t) \longrightarrow w_iy_i''(t) \end{array} \right\} \quad \begin{array}{l} \text{strongly in } L^2(\Omega) \\ \text{strongly in } L^2(I, Q) \\ \text{strongly in } L^2(\Omega) \end{array} \end{aligned}$$

Second, $\varphi_{int} \longrightarrow \varphi_{it}$ weakly in $L^2(Q)$ and $\varphi_{in} \longrightarrow \varphi_i$ weakly in $L^2(I, W)$ and strongly in $L^2(Q)$.

Third, since $w_{in}y_i \longrightarrow w_iy_i$ weakly in $L^2(I, W)$, then

$$\int_0^T (p_i + \alpha_i, w_{in})y_i(t)dt \rightarrow \int_0^T (p_i + \alpha_i, w_i)y_i(t)dt, \forall i = 1, 2, 3.$$

From these convergences, (18) and (19), we can passaged the limits in (30)-(34), to get

$$\begin{aligned} & - \int_0^T (\varphi_{1t}, w_1)(y_1)'(t)dt + \int_0^T [(\nabla\varphi_1, \nabla w_1) + (\varphi_1, w_1) \\ & - (\varphi_2, w_1) - (\varphi_3, w_1)]y_1(t)dt \end{aligned}$$

$$(36) \quad = \int_0^T (p_1 + \alpha_1, w_1)y_1(t)dt + (\varphi_1^1, w_1)y_1(0)$$

$$\begin{aligned} & \int_0^T (\varphi_1, w_1)(y_1)''(t)dt + \int_0^T [(\nabla\varphi_1, \nabla w_1) + (\varphi_1, w_1) \\ & - (\varphi_2, w_1) - (\varphi_3, w_1)]y_1(t)dt \\ (37) \quad & = \int_0^T (p_1 + \alpha_1, w_1)y_1(t)dt + (\varphi_1^1, w_1)y_1(0) + (\varphi_1^0, w_1)y_1'(0) \end{aligned}$$

$$\begin{aligned} & - \int_0^T (\varphi_{2t}, w_2)(\varphi_2)'(t)dt + \int_0^T [(\nabla\varphi_2, \nabla w_2) + (\varphi_2, w_2) \\ & + (\varphi_1, w_2) + (\varphi_3, w_2)]y_2(t)dt \end{aligned}$$

$$(38) \quad = \int_0^T (p_2 + \alpha_2, w_2)y_2(t)dt + (\varphi_2^0, w_2)y_2(0)$$

$$\begin{aligned} & \int_0^T (y_2, w_2)(y_2)''(t)dt + \int_0^T [(\nabla y_2, \nabla w_2) + (y_2, w_2) \\ & + (y_1, w_2) + (y_3, w_2)]y_2(t)dt \end{aligned}$$

$$(39) \quad = \int_0^T (p_2 + \alpha_2, w_2)y_2(t)dt + (y_2^0, w_2)y_2(0) + (y_2^0, w_2)y_2'(0)$$

$$\begin{aligned} & - \int_0^T (\varphi_{3t}, w_3)(y_3)'(t)dt + \int_0^T [(\nabla\varphi_3, \nabla w_3) + (\varphi_3, w_3) \\ & + (\varphi_1, w_3) - (\varphi_2, w_3)]y_3(t)dt \end{aligned}$$

$$\begin{aligned}
(40) \quad &= \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt + (\varphi_3^1, w_3) y_3(0) \\
&\quad \int_0^T (\varphi_3, w_3) (y_3)''(t) dt + \int_0^T [(\nabla \varphi_3, \nabla w_3) + (\varphi_3, w_3) \\
&\quad + (\varphi_1, w_3) - (\varphi_2, w_3)] y_3(t) dt
\end{aligned}$$

$$(41) \quad = \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt + (\varphi_3^1, w_3) y_3(0) + (\varphi_3^0, w_3) y_3'(0)$$

Case 1. Choose $y_i \in C^2[0, T]$, s.t. $y_i(0) = (y_i)'(0) = y_i(T) = (y_i)'(T) = 0, \forall i = 1, 2, 3$. in (35), (37), (39), integration by parts twice the 1st terms in the L.H.S. of each one of the obtained equation, yield to

$$\begin{aligned}
&\int_0^T \langle \varphi_{1tt}, w_1 \rangle y_1(t) dt + \int_0^T [(\nabla \varphi_1, \nabla w_1) \\
(42) \quad &+ (\varphi_1, w_1) - (\varphi_2, w_1) - (\varphi_3, w_1)] y_1(t) dt = \int_0^T (p_1 + \alpha_1, w_1) y_1(t) dt
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \langle \varphi_{2tt}, w_2 \rangle y_2(t) dt + \int_0^T [(\nabla \varphi_2, \nabla w_2) \\
(43) \quad &+ (\varphi_2, w_2) + (\varphi_1, w_2) + (\varphi_3, w_2)] y_2(t) dt = \int_0^T (p_2 + \alpha_2, w_2) y_2(t) dt
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \langle \varphi_{3tt}, w_3 \rangle y_3(t) dt + \int_0^T [(\nabla \varphi_3, \nabla w_3) + (\varphi_3, w_3) \\
(44) \quad &+ (\varphi_1, w_3) - (\varphi_2, w_3)] y_3(t) dt = \int_0^T (p_3 + \alpha_3, w_3) y_3(t) dt
\end{aligned}$$

Which give that $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of ((11a) , (12a) , (13a)) a.e. on I.

Case 2. Choose $y_i \in C^2[0, T]$, s.t. $y_i(T) \neq 0 \& y_i(0) \neq 0, \forall i = 1, 2, 3$. Multiplying both sides of (11a), (12a), and (13a) by $y_1(t), y_2(t)$ and $y_3(t)$ respectively, integrating for $0 \leq t \leq T$, integrating by parts the 1st term in the L.H.S. of each one of the three equations, then subtracting each one of these obtained equations from those in (34) , (36) & (38) respectively, one has $(y_i^1, w_i) y_i(0) = (y_i t(0), w_i) y_i(0)$.

From the last two cases easily once get the initial conditions (11b), (12b) and (13b). To prove that $\vec{\varphi}_n \longrightarrow \vec{\varphi}$ strongly in $L^2(I, W)$, we start by integrating (26) for $0 \leq t \leq T$, to get

$$(45) \quad \|\vec{\varphi}_{nt}(T)\|_Q^2 - \|\varphi_{nt}(0)\|_Q^2 + 2 \int_0^T \|\vec{\varphi}_n(t)\|_1^2 dt = (43a) + (34b)$$

$$\begin{aligned}
(43a) = &2((\varphi_{2n}, \varphi_{1nt}) + (\varphi_{3n}, \varphi_{1nt}) - (\varphi_{1n}, \varphi_{2nt}) \\
&- (\varphi_{3n}, \varphi_{2nt}) - (\varphi_{1n}, \varphi_{3nt}) + (\varphi_{2n}, \varphi_{3nt}))
\end{aligned}$$

$$(43b) = 2((p_1 + \alpha_1, \varphi_{1nt}) + (p_2 + \alpha_2, \varphi_{2nt}) + (p_3 + \alpha_3, \varphi_{3nt})).$$

Same steps which is applied to get (26)&(45), can be also applied here when we have $\vec{\varphi}$ and $\vec{\varphi}_t$, i.e.

$$(46) \quad \|\vec{\varphi}_t(T)\|_Q^2 - \|\vec{\varphi}_t(0)\|_Q^2 + 2 \int_0^T \|\vec{\varphi}(t)\|_1^2 dt = (44a) + (44b)$$

$$(44a) = 2((\varphi_2, \varphi_{1t}) + (\varphi_3, \varphi_{1t}) - (\varphi_1, \varphi_{2t}) - (\varphi_3, \varphi_{2t}) - (\varphi_1, \varphi_{3t}) + (\varphi_2, \varphi_{3t}))$$

$$(44b) = 2((p_1 + \alpha_1, \varphi_{1t}) + (p_2 + \alpha_2, \varphi_{2t}) + (p_3 + \alpha_3, \varphi_{3t})).$$

Since

$$(47) \quad \begin{aligned} & \|\vec{\varphi}_{nt}(T) - \vec{\varphi}_t(T)\|_Q^2 - \|\vec{\varphi}_{nt}(0) - \vec{\varphi}_t(0)\|_Q^2 + 2 \int_0^T \|\vec{\varphi}_n(t) - \vec{\varphi}(t)\|_1^2 dt \\ &= (45a) - (45b) - (45c) \\ (45a) &= \|\vec{\varphi}_{nt}(T)\|_Q^2 - \|\vec{\varphi}_{nt}(0)\|_Q^2 + 2 \int_0^T \|\vec{\varphi}_n(t)\|_1^2 dt \\ (45b) &= (\vec{\varphi}_{nt}(T), \vec{\varphi}_t(T)) - (\vec{\varphi}_{nt}(0), \vec{\varphi}_t(0)) + 2 \int_0^T (\vec{\varphi}_n(t), \vec{\varphi}(t))_1 dt \\ (45c) &= (\vec{\varphi}_t(T), \vec{\varphi}_{nt}(T) - \vec{\varphi}_t(T)) - (\vec{\varphi}_t(0), \vec{\varphi}_{nt}(0) - \vec{\varphi}_t(0)) \\ & \quad + 2 \int_0^T (\vec{\varphi}(t), \vec{\varphi}_n(t) - \vec{\varphi}(t))_1 dt \end{aligned}$$

Since $\vec{\varphi}_n \rightarrow \vec{\varphi}$ strongly in $L^2(Q)$, and $\vec{\varphi}_{nt} \rightarrow \vec{\varphi}$ weakly in $L^2(Q)$, then from (45) and the assum.(A), we obtain

$$(45a) = (43a) + (43b)(44a) + (44b)$$

by the same way that it used to get (19), we can get also that

$$(48) \quad \vec{\varphi}_{nt}(T) \rightarrow \vec{\varphi}_t(T) \text{ strongly in } L(\Omega)^2.$$

On the other hand, since $\vec{\varphi}_n \rightarrow \vec{\varphi}$ weakly in $L^2(I, W)$, then using (19 & 46), we get

$$(45b) \rightarrow (44a) + (44b).$$

All the terms in (45c) imply to zero, so as the 1st two terms in the L.H.S. of (45), hence (45) gives $\int_0^T \|\vec{\varphi}_n(t) - \vec{\varphi}(t)\|_1^2 dt \rightarrow 0$ as $n \rightarrow \infty$, so we get that $\vec{\varphi}_n \rightarrow \vec{\varphi}$ strongly in $L^2(I, W)$. Uniqueness of the solution: Let $\vec{\varphi} = (\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3)$ and $\vec{\bar{\varphi}} = (\vec{\bar{\varphi}}_1, \vec{\bar{\varphi}}_2, \vec{\bar{\varphi}}_3)$ be two solutions of the wf (11-13), i.e. $\vec{\varphi}$ and $\vec{\bar{\varphi}}$ are satisfied the wf (11-13), subtracting each equation from the other and then setting $w_i = \varphi_i - \bar{\varphi}_i$, for each $i = 1, 2, 3$, yields to $\langle (\varphi_i - \bar{\varphi}_i)_{tt}, \varphi_i - \bar{\varphi}_i \rangle + \|\varphi_i - \bar{\varphi}_i\|_1^2 = 0$, $((\varphi_i - \bar{\varphi}_i)(0), (\varphi_i - \bar{\varphi}_i)(0)) = 0$ for $w_i = (\varphi_i - \bar{\varphi}_i)_t$ we have $((\varphi_i - \bar{\varphi}_i)_t(0), (\varphi_i - \bar{\varphi}_i)_t(0)) = 0$.

Adding the above equalities (for $i = 1, 2, 3$), using Lemma 1.2 in ref. [20] for the 1st in L.H.S. of the obtained equation which will be positive, integrating both sides from 0 to t , using the initial conditions, and finally applying the B

-G inequality, to get $\int_0^t \left[(d/dt) \|(\vec{\varphi} - \vec{\bar{\varphi}})_t(t)\|_Q^2 + 2 \|(\vec{\varphi} - \vec{\bar{\varphi}})\|_1^2 \right] dt \leq 2L \int_0^t \|(\vec{\varphi} - \vec{\bar{\varphi}})\|_1^2 dt \Rightarrow \|(\vec{\varphi} - \vec{\bar{\varphi}})(t)\|_1^2 = 0, \forall t \in I. \Rightarrow \|(\vec{\varphi} - \vec{\bar{\varphi}})(t)\|_{L^2(I,W)} = 0 \Rightarrow$ the solution is unique. $\square$

4. Existence of a CCOC:

Lemma 4.1. *In addition to assum. (A), assume that $\vec{\varphi}$ and $\vec{\varphi} + \delta\vec{\varphi}$ are the SVS corresponding to the CVS $\vec{\alpha}$ and $\vec{\alpha} + \delta\vec{\alpha}$ respectively with $\vec{\alpha}$ and $\delta\vec{\alpha}$ are bounded in $L^2(Q)$ then $\|\delta\vec{\varphi}_\varepsilon\|_{L^\infty(I, L^2(\Omega))} \leq a\|\delta\vec{\alpha}\|_Q, \|\delta\vec{\varphi}_\varepsilon\|_{L^2(I,V)} \leq a\|\delta\vec{\alpha}\|_Q$ and $\|\delta\vec{\varphi}_\varepsilon\|_Q \leq a\|(\delta\vec{\alpha})\|_Q$, with $a \in \mathbb{R}^+$.*

Proof. Let $\vec{\delta\alpha} = \vec{\alpha} - \vec{\bar{\alpha}}$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \vec{\bar{\alpha}} = (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \in L^2(Q)$, then $\vec{\alpha}_\varepsilon = \vec{\alpha} + \varepsilon(\vec{\delta\alpha}) \in L^2(Q)$ for $\varepsilon > 0$, and then by Theorem 3.1, $\vec{\varphi} = \vec{\varphi}_{\vec{\alpha}} = (\varphi_1, \varphi_2, \varphi_3)$ and $\vec{\varphi}_\varepsilon = \vec{\varphi}_{\vec{\alpha}_\varepsilon} = (\varphi_{1\varepsilon}, \varphi_{2\varepsilon}, \varphi_{3\varepsilon})$ are their corresponding states solutions which are satisfied the wf (11-13), by setting $(\vec{\delta\varphi})_\varepsilon = (\delta\varphi_{1\varepsilon}, \delta\varphi_{2\varepsilon}, \delta\varphi_{3\varepsilon}) = \vec{\varphi}_\varepsilon - \vec{\varphi}$, once get

$$(47a) \quad \begin{aligned} & \langle \delta\varphi_{1\varepsilon t}, w_1 \rangle + (\nabla \delta\varphi_{1\varepsilon}, \nabla w_1) + (\delta\varphi_{1\varepsilon}, w_1) - (\delta\varphi_{2\varepsilon}, w_1) - (\delta\varphi_{3\varepsilon}, w_1) \\ & = (\varepsilon \delta\alpha_1, w_1) \end{aligned}$$

$$(47b) \quad \delta\varphi_{1\varepsilon}(x, 0) = 0 \text{ and } \delta\varphi_{1\varepsilon t}(x, 0) = 0$$

$$(48a) \quad \begin{aligned} & \langle \delta\varphi_{2\varepsilon t}, w_2 \rangle + (\nabla \delta\varphi_{2\varepsilon}, \nabla w_2) + (\delta\varphi_{2\varepsilon}, w_2) + (\delta\varphi_{1\varepsilon}, w_2) + (\delta\varphi_{3\varepsilon}, w_2) \\ & = (\varepsilon \delta\alpha_2, w_2) \end{aligned}$$

$$(48b) \quad \delta\varphi_{2\varepsilon}(x, 0) = 0 \text{ and } \delta\varphi_{2\varepsilon t}(x, 0) = 0$$

$$(49a) \quad \begin{aligned} & \langle \delta\varphi_{3\varepsilon t}, w_3 \rangle + \delta\varphi_{3\varepsilon}, \nabla w_3 + (\delta\varphi_{3\varepsilon}, w_3) + (\delta\varphi_{1\varepsilon}, w_3) - (\delta\varphi_{2\varepsilon}, w_3) \\ & = (\varepsilon \delta\alpha_3, w_3) \end{aligned}$$

$$(49b) \quad \delta\varphi_{3\varepsilon}(x, 0) = 0 \text{ and } \delta\varphi_{3\varepsilon t}(x, 0) = 0.$$

Substituting $w_i = \delta\varphi_{i\varepsilon t}$, for $i = 1, 2, 3$ in (47a), (48a) & (49a) respectively, adding the three obtained equations, using the same way that it used to get (27), a similar equation will be obtained but with $(\vec{\delta\varphi})_\varepsilon$ in position of $\vec{\varphi}_n$, and then integration both sides on $[0, t]$, yield to

$$\begin{aligned} & \int_0^t (d/dt) \left[\|(\vec{\delta\varphi})_{\varepsilon t}(t)\|_Q^2 + \|(\vec{\delta\varphi})_\varepsilon\|_1^2 \right] dt \\ & \leq 2 \int_0^t (|\delta\varphi_{2\varepsilon}| + |\delta\varphi_{3\varepsilon}| + \varepsilon|\delta\alpha_1|) |\delta\varphi_{1\varepsilon} t| dt + m \\ & + 2 \int_0^t (|\delta\varphi_{1\varepsilon}| + |\delta\varphi_{2\varepsilon}| + \varepsilon|\delta\alpha_2|) |\delta\varphi_{2\varepsilon} t| dt \\ & + 2 \int_0^t (|\delta\varphi_{1\varepsilon}| + |\delta\varphi_{3\varepsilon}| + \varepsilon|\delta\alpha_3|) |\delta\varphi_{3\varepsilon} t| dt. \end{aligned}$$

Using the definitions of the norms and the relations between them, to get

$$\|(\overrightarrow{\delta\varphi})_{\varepsilon t}(t)\|_Q^2 + \|(\overrightarrow{\delta\varphi})_{\varepsilon}(t)\|_1^2 \leq \varepsilon \|(\overrightarrow{\delta\alpha})(t)\|_Q^2 + 3 \int_0^t (\|(\overrightarrow{\delta\varphi})_{\varepsilon}\|_0^2 + \|(\overrightarrow{\delta\varphi})_{\varepsilon t}\|_1^2) dt.$$

Applying the B-G inequality, with $a^2 = \varepsilon e^3$, to get $\|(\overrightarrow{\delta\varphi})_{\varepsilon t}(t)\|_0^2 + \|(\overrightarrow{\delta\varphi})_{\varepsilon}(t)\|_1^2 \leq a^2 \|(\overrightarrow{\delta\alpha})(t)\|_Q^2, \forall t \in I \Rightarrow \|(\overrightarrow{\delta\varphi})_{\varepsilon}(t)\|_1^2 \leq a^2 \|(\overrightarrow{\delta\alpha})(t)\|_Q^2, \forall t \in \bar{I}$. $\square$

From this result and from the definitions of norms the requirement norms are obtained.

Lemma 4.2. *With assuming (A), the operator $\vec{\alpha} \mapsto \vec{\varphi}_{\vec{\alpha}}$ is continuous from $L^2(Q)$ in to $L^\infty(I, L^2(\Omega))$ or into*

Proof. Let $\delta\vec{\alpha} = \vec{\vec{\alpha}} - \vec{\alpha}$ and $\delta\vec{\varphi} = \vec{\vec{\varphi}} - \vec{\varphi}$, where $\vec{\vec{\varphi}}$ and $\vec{\varphi}$ are the correspond SVS to the CVS $\vec{\vec{\alpha}}$ and $\vec{\alpha}$ using the first result in Lemma 4.1, one has $\|\vec{\vec{\varphi}} - \vec{\varphi}\|_{L^\infty(\bar{I}, L^2(\Omega))} \leq M \|\vec{\vec{\alpha}} - \vec{\alpha}\|_Q$,

If $\vec{\alpha} \xrightarrow{L^2(Q)} \vec{\alpha}$ then $\vec{\vec{\varphi}} \xrightarrow{L^\infty(\bar{I}, L^2(\Omega))} \vec{\varphi}$. Thus the operator $\vec{\alpha} \mapsto \vec{\varphi}_{\vec{\alpha}}$ is Lipschitz continues (LC) from $L^2(\Omega)$ in to $L^\infty(\bar{I}, L^2(\Omega))$.

Similar way is used to prove this operator is also LC from $L^2(Q)$ into $L^2(Q)$ and into $L^2(\bar{I}, W)$. $\square$

Lemma 4.3 ([10]). *The norm $\|\cdot\|_0$ is weakly lower semi continuous (wlsc).*

Lemma 4.4. *The cost function which is given by (10) is wlsc.*

Proof. From Lemma 4.3, $\|\vec{\alpha}\|_{L^2(Q)}$ is wlsc, now when $\vec{\alpha}_k \rightarrow \vec{\alpha}$ weakly in $L^2(Q)$, then $\vec{\varphi}_k \rightarrow \vec{\varphi}$ weakly in $L^2(Q)$ (by Lemma 4.1), and then by Lemma 4.3

$$\|\vec{\varphi} - \vec{\varphi}_d\| \leq \lim_{k \rightarrow \infty} \inf_{y_k \in V_k} \|\vec{\varphi}_k - \vec{\varphi}_d\|.$$

Thus $G_0(\vec{\alpha})$ is wlsc. $\square$

Lemma 4.5 ([13]). *The norm $\|\cdot\|_0^2$ is strictly convex.*

Theorem 4.1. *Consider the cost function (10), if $G_0(\vec{\alpha})$ is coercive, the set $\vec{A}$ is convex. Then there exists a CCOC.*

Proof. Since $G_0(\vec{\alpha}) \geq 0$ and $G_0(\vec{\alpha})$ is coercive, then there exists a minimizing sequence $\{\vec{\alpha}_k\} = (\alpha_{1k}, \alpha_{2k}, \alpha_{3k}) \in \vec{W}_A, \forall k$, s.t. $\lim_{n \rightarrow \infty} G_0(\vec{\alpha}_k) = \inf_{\vec{\alpha}_k \in \vec{W}_A} G_0(\vec{\alpha})$, and $\|\vec{\alpha}_k\| \leq c$, then by AT there exists a subsequence of $\{\vec{u}_k\}$, for simplicity let be $\{\vec{\alpha}_k\}$, s.t. $\vec{\alpha}_k \rightarrow \vec{\alpha}$ weakly in $L^2(Q)$, as $k \rightarrow \infty$.

From Theorem 3.1, corresponding to the sequence of control $\{\vec{\alpha}_k\}$ there exists a sequence of unique solutions $\{\vec{\varphi}_k\} = \vec{\varphi}_{\vec{\alpha}_k}$ such that the norms $\|\vec{\varphi}_k\|_{L^2(\bar{I}, V)}$, $\|\vec{\varphi}_{kt}\|_{L^2(Q)}$ are bounded, then by AT there exist a subsequences of $\{\vec{\varphi}_k\}$, $\{\vec{\varphi}_{kt}\}$ for simplicity let be $\{\vec{\varphi}_k\}$, $\{\vec{\varphi}_{kt}\}$ s.t. $\vec{\varphi}_k \rightarrow \vec{\varphi}$ weakly in $L^2(\bar{I}, V)$, $\vec{\varphi}_{kt} \rightarrow \vec{\varphi}$

weakly in $L^2(Q)$. Now, for each k , the solution $(\varphi_{1k}, \varphi_{2k}, \varphi_{3k})$ satisfies the wf (15-17), multiplying both sides of each equation by $y_i(t), \forall i = 1, 2, 3$ respectively (with $y_i \in C^2[0, T]$, such that $y_i(T) = y'_i(T) = 0, y_i(0) \neq 0, y'_i(0) \neq 0$). Rewriting the 1st terms in the L.H.S. of each one of them, integrating both sides from 0 to T , finally integrating by parts twice for their 1st terms, one gets the same equations like (30)-(34) with different that each $w_{in} = w_i$ and the term $\int_0^T (p_i + \alpha_i, w_{in}) y_i(t) dt$ (for all $i = 1, 2, 3$) in the R.H.S in each one of the obtained equations will be in the form

$$(50) \quad \int_0^T (p_i + \alpha_{ik}, w_i) y_i(t) dt, \forall i = 1, 2, 3, \text{ and } \forall k.$$

Hence, same technique which is used in the proof of Theorem 3.1, will be used to passage the limit (as $k \rightarrow \infty$), in both sides of the above indicated equations, except the new term (50) which also converge to the following term (since $\alpha_{ik} \rightarrow \alpha_i$, weakly in $L^2(Q)$).

$$(51) \quad \int_0^T (p_i + \alpha_i, w_i) \varphi_i(t) dt, \forall i = 1, 2, 3, \text{ and } \forall k.$$

From these convergences we get the wf like ((11a),(12a),(13a)).

Also, using the same steps which used in the proof of Theorem (3.1) can be also used here to passage the limits in the initial conditions and to get ((22b),(12b),(13b)), which give us the limit point $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of the SE. At the end since $G_0(\vec{\alpha})$ is W.L.S.C. from Lemma (4.1) and since $\vec{\alpha}_k \rightarrow \vec{\alpha}$ weakly in $(L^2(\Omega))^3$, then

$$\begin{aligned} G_0(\vec{\alpha}) &\leq \lim_{k \rightarrow \infty} \inf_{\vec{\alpha}_k \in \tilde{W}_A} G_0(\vec{\alpha}_k) = \lim_{k \rightarrow \infty} G_0(\vec{\alpha}_k) = \inf G_0(\vec{\alpha}) \\ &\implies G_0(\vec{\alpha}) \leq \inf_{\vec{\alpha} \in \tilde{W}_A} G_0(\vec{\alpha}) \\ &\implies G_0(\vec{\alpha}) = \min_{\vec{\alpha} \in \tilde{W}_A} G_0(\vec{\alpha}) \end{aligned}$$

Thus $\vec{\alpha}$ is a CCOC. □

5. The NTHO

Theorem 5.1. Consider $G_0(\vec{\alpha})$ which is given by (10) and the TAEs of the STE (1-9) are given by

$$(52a) \quad \eta_{1tt} - \nabla \eta_1 + \eta_1 + \eta_2 + \eta_3 = (\varphi_1 - \varphi_{1d}), \text{ on } Q,$$

$$(52b) \quad \eta_1 = 0 \text{ on } \sigma, \eta_1(x, T) = \eta_{1t}(x, T) = 0, \text{ on } \Omega$$

$$(53a) \quad \eta_{2tt} - \nabla \eta_2 + \eta_2 - \eta_1 - \eta_3 = (\varphi_2 - \varphi_{2d}), \text{ on } Q,$$

$$(53b) \quad \eta_2 = 0 \text{ on } \sigma, \eta_2(x, T) = \eta_{2t}(x, T) = 0 \text{ on } \Omega$$

$$(54a) \quad \eta_{3tt} - \nabla \eta_3 + \eta_3 - \eta_1 + \eta_2 = (\varphi_3 - \varphi_{3d}), \text{ on } Q,$$

$$(54b) \quad \eta_3 = 0 \text{ on } \sigma, \eta_3(x, T) = \eta_{3t}(x, T) = 0 \text{ on } \Omega.$$

And the Hamiltonian is defined:

$$H(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}) = \sum_{i=1}^3 (\eta_i(p_i(x, t) + \alpha_i) + (1/2) \sum_{i=1}^3 (\|\varphi_i - \varphi_{id}\|_Q^2 + \beta/2) \|\alpha_i\|_Q^2).$$

Then for $\vec{\alpha}' \in \vec{W}$, the directional derivative of G is given by

$$DG(\vec{\alpha}, \vec{\alpha}' - \vec{\alpha}) = \lim_{\epsilon \rightarrow 0} \frac{G(\vec{\alpha} + \epsilon(\delta\alpha)) - G(\vec{\alpha})}{\epsilon} = \int_Q H_{\vec{\alpha}}(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}) \cdot (\vec{\alpha}' - \vec{\alpha}) dx dt.$$

Proof. At first let, the wf of the adjoint equations is given, $\forall w_1, w_2, w_3 \in W$, (a.e. on I) by

$$(55a) \quad \langle \eta_{1tt}, w_1 \rangle + (\nabla \eta_1, \nabla w_1) + (\eta_1, w_1) + (\eta_2, w_1) + (\eta_3, w_1) = (\varphi_1 - \varphi_{1d}, w_1),$$

$$(55b) \quad (\eta_1(T), w_1) = (\eta_{1t}(T), w_1) = 0,$$

$$(56a) \quad \langle \eta_{2tt}, w_2 \rangle + (\nabla \eta_2, \nabla w_2) + (\eta_2, w_2) - (\eta_1, w_2) - (\eta_3, w_1) = (\varphi_2 - \varphi_{2d}, w_2),$$

$$(56b) \quad (\eta_2(T), w_2) = (\eta_{2t}(T), w_2) = 0,$$

$$(57a) \quad \langle \eta_{3tt}, w_3 \rangle + (\nabla \eta_3, \nabla w_3) + (\eta_3, w_3) - (\eta_1, w_3) + (\eta_2, w_3) = (\varphi_3 - \varphi_{3d}, w_3),$$

$$(57b) \quad (\eta_3(T), w_3) = (\eta_{3t}(T), w_3) = 0.$$

The proof of existence of a unique solution $\vec{\eta} = (\eta_1, \eta_2, \eta_3) \in L^2(Q)$ for the wf (55-57) can be done by using the same manner which is used in the proof of Theorem 3.1,

Substituting $w_i = \delta\varphi_{i\epsilon}$ for $i = 1, 2, 3$ in (55a), (56a), (57a) respectively, integrating both sides for $0 \leq t \leq T$, to get

$$(58) \quad \int_0^T \langle \delta\varphi_{1\epsilon}, \eta_{1tt} \rangle dt + \int_0^T [(\nabla \eta_1, \nabla \delta\varphi_{1\epsilon}) + (\eta_1, \delta\varphi_{1\epsilon}) + (\eta_2, \delta\varphi_{1\epsilon}) + (\eta_3, \delta\varphi_{1\epsilon})] dt = \int_0^T (\varphi_1 - \varphi_{1d}, \delta\varphi_{1\epsilon}) dt,$$

$$(59) \quad \int_0^T \langle \delta\varphi_{2\epsilon}, \eta_{2tt} \rangle dt + \int_0^T [(\nabla \eta_2, \nabla \delta\varphi_{2\epsilon}) + (\eta_2, \delta\varphi_{2\epsilon}) - (\eta_1, \delta\varphi_{2\epsilon}) - (\eta_3, \delta\varphi_{2\epsilon})] dt = \int_0^T (\varphi_2 - \varphi_{2d}, \delta\varphi_{2\epsilon}) dt,$$

$$(60) \quad \int_0^T \langle \delta\varphi_{3\epsilon}, \eta_{3tt} \rangle dt + \int_0^T [(\nabla \eta_3, \nabla \delta\varphi_{3\epsilon}) + (\eta_3, \delta\varphi_{3\epsilon}) - (\eta_1, \delta\varphi_{3\epsilon}) + (\eta_2, \delta\varphi_{3\epsilon})] dt = \int_0^T (\varphi_3 - \varphi_{3d}, \delta\varphi_{3\epsilon}) dt.$$

Now, let $\vec{\alpha}, \vec{\alpha}' \in L^2(Q)$, $\vec{\delta\alpha} = \vec{\alpha}' - \vec{\alpha}$, for $\epsilon > 0$, $\vec{\alpha}_\epsilon = \vec{\alpha} + \epsilon\vec{\delta\alpha} \in L^2(Q)$, then by Theorem 3.1, $\vec{\varphi} = \vec{\varphi}_{\vec{\alpha}}$ & $\vec{\varphi}_\epsilon = \vec{\varphi}_{\vec{\alpha}_\epsilon}$ are their corresponding solutions. Setting $\vec{\delta\varphi}_\epsilon = (\delta\varphi_{1\epsilon}, \delta\varphi_{2\epsilon}, \delta\varphi_{3\epsilon}) = \vec{\varphi}_\epsilon - \vec{\varphi}$, substituting $w_i = \eta_i$, for $i = 1, 2, 3$

in (47a),(48a),(49a) respectively, integrating both sides for $0 \leq t \leq T$, then integrating by parts twice the 1st term in the L.H.S. of each one of the obtained equation, once get

$$(61) \quad \int_0^T \langle \delta\varphi_1\epsilon, \eta_{1tt} \rangle dt + \int_0^T [(\nabla\delta\varphi_1\epsilon, \nabla\eta_1) + (\delta\varphi_1\epsilon, \eta_1) - (\delta\varphi_2\epsilon, \eta_1) - (\delta\varphi_3\epsilon, \eta_1)] dt = \int_0^T (\epsilon\delta\delta_1, \eta_1) dt,$$

$$(62) \quad \int_0^T \langle \delta\varphi_2\epsilon, \eta_{2tt} \rangle dt + \int_0^T [(\nabla\delta\varphi_2\epsilon, \nabla\eta_2) + (\delta\varphi_2\epsilon, \eta_2) + (\delta\varphi_1\epsilon, \eta_2) + (\delta\varphi_3\epsilon, \eta_2)] dt = \int_0^T (\epsilon\delta\alpha_2, \eta_2) dt,$$

$$(63) \quad \int_0^T \langle \delta\varphi_3\epsilon, \eta_{3tt} \rangle dt + \int_0^T [(\nabla\delta\varphi_3\epsilon, \nabla\eta_3) + (\delta\varphi_3\epsilon, \eta_3) + (\delta\varphi_1\epsilon, \eta_3) - (\delta\varphi_2\epsilon, \eta_3)] dt = \int_0^T (\epsilon\delta\alpha_3, \eta_3) dt,$$

$$(64) \quad \epsilon \int_0^T [(\delta\alpha_1, \eta_1) + (\delta\alpha_2, \eta_2) + (\delta\alpha_3, \eta_3)] dt = \int_0^T [(\varphi_1 - \varphi_{1d}, \delta\varphi_1\epsilon) + (\varphi_2 - \varphi_{2d}, \delta\varphi_2\epsilon) + (\varphi_3 - \varphi_{3d}, \delta\varphi_3\epsilon)] dt,$$

where $O_1(\epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$, with $O_1(\epsilon) = \|(\delta\varphi)_\epsilon\|_Q + \epsilon\|(\delta\alpha)\|_Q$

On the other hand, we have

$$(65) \quad \begin{aligned} G_0(\vec{\alpha}_\eta) - G_0(\vec{\alpha}) &= \int_Q ((\varphi_1 - \varphi_{1d})\delta\varphi_1\epsilon + \epsilon\beta\alpha_1\beta\alpha_1) dxdt \\ &+ \int_Q ((\varphi_2 - \varphi_{2d})\delta\varphi_2\epsilon + \epsilon\beta\alpha_2\delta\alpha_2) dxdt \\ &+ \int_Q ((\varphi_3 - \varphi_{3d})\delta\varphi_3\epsilon + \epsilon\beta\alpha_3\delta\alpha_3) dxdt + O_1(\epsilon), \end{aligned}$$

where $O_1(\epsilon) = \|\vec{\delta\varphi}_\epsilon\|_Q + \epsilon\|\vec{\delta\alpha}\|_Q$, with $O_1(\epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$ Now, setting (64) in (65), one have that

$$G_0(\vec{\alpha}_\epsilon) - G_0(\vec{\alpha}) = \epsilon \int_Q [(\eta_1 + \beta\alpha_1)\delta\alpha_1 + (\eta_2 + \beta\alpha_2)\delta\alpha_2 + (\eta_3 + \beta\alpha_3)\delta\alpha_3] dxdt + O_1(\epsilon).$$

Finally, dividing both sides of the above equality by ϵ , then taking the limit $\epsilon \rightarrow 0$, once get

$$DG(\vec{\alpha}, \vec{\alpha} - \vec{\alpha}) = \int_Q H_{\vec{u}}(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}) \cdot (\vec{\alpha} - \vec{\alpha}) dxdt = \int_Q (\vec{\eta} + \beta\vec{\alpha}) \cdot \vec{\delta\alpha} dxdt,$$

where $H_{\vec{\alpha}}(x, t, \vec{\varphi}, \vec{\alpha}, \vec{\eta}) = (\eta_1 + \beta\alpha_1\eta_2 + \beta\alpha_2\eta_3 + \beta\alpha_3)^T$.

□

Theorem 5.2. *The CCOC of the above problem is $DG(\vec{\alpha}, \vec{\alpha} - \vec{\alpha}) = \vec{\eta} + \beta\vec{\alpha} = 0$*

Proof. If $\vec{\alpha}$ is an CCOC of the problem, then $G_0(\vec{\alpha}) = \min_{\vec{\alpha} \in \vec{W}_A} G_0(\vec{\alpha}), \forall \vec{\alpha} \in L^2(Q)$, i.e. $DG(\vec{\alpha}, \vec{\alpha} - \vec{\alpha}) = 0 \implies \vec{\eta} + \beta\vec{\alpha} = 0, \delta\vec{\alpha} = \vec{w} - \vec{\alpha} \implies$ The NCO is $(\vec{\eta} + \beta\vec{\alpha}, \delta\vec{\alpha}) \geq 0 \implies (\vec{\eta} + \vec{\alpha}, \vec{w}) \geq (\vec{\eta} + \beta\vec{\alpha}, \vec{\alpha}), \forall \vec{w} \in L^2(Q)$. $\square$

6. Conclusions

The GM is employed to prove the existence and unique theorem for a SVS of the TSPDEs of parabolic type for fixed CCCV as well as The existence and uniqueness solution of the TAEs associated with the TSPDEs. The existence of a CCOCV associated with the considered TLPDEs of parabolic type is proved. The derivation of the Directional derivative (DDV) for the cost functional is obtained; finally, the necessity theorem (necessity conditions) for optimality (NTHO) of this OCP is sated and proved.

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Residual power series technique for solving Fokker-Planck equation

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Abstract. In this paper we used a numerical method to find the solutions of Fokker-Planck equation. The method used the residual power series method to solve the initial value Fokker-Planck problems. Numerous algorithms for linear and nonlinear Fokker-Planck equations are presented. A comparison between residual power series method, Adomian decomposition method (ADM) and the variational iteration method (VIM) is concluded that RPSM is easy to perform and yields very accurate results.

Keywords: RESIDUAL power series method, Fokker-Planck equations, Kolmogorov equation, adomian decomposition method, variational iteration method.

1. Introduction

The Fokker-Planck equation was first introduced by Anthony Fokker [1] and Max Planck [2] to describe the Brownian motion of particles. This equation plays a very important role in a wide range of natural sciences such as statistical physics, solid-state physics, quantum optics, chemical physics, and theoretical biology [3, 4]. The experiment of immersing small particles in a fluid can be characterized by the equation of motion for the distribution function $u(x, t)$ that is given by the linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \frac{1}{\gamma} \frac{\partial(x, u)}{\partial x} + \frac{\beta}{\gamma} \frac{\partial^2 u}{\partial x^2}.$$

Where γ is the particle relaxation time, x is the velocity of the particle, $\beta = \frac{KT}{m}$ is the thermal velocity, K is the Boltzman's constant, T is the temperature of the fluid, and m is the mass of the particle [3]. The general form of the linear Fokker-Planck equation is given by $\frac{\partial u}{\partial t} = -A(s) \frac{\partial u}{\partial s} + B(s) \frac{\partial^2 u}{\partial s^2}$ subject to the

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initial condition $u(s, 0) = f(s)$, $s \in \mathcal{R}$. Where $A(s)$ and $B(s)$ are positive and represented the drift and diffusion coefficients respectively. In some applications, the drift and diffusion coefficients depend on time and space. Thus, the linear Fokker-Planck equation becomes

$$(1) \quad \frac{\partial u}{\partial t} = -A(s, t) \frac{\partial u}{\partial s} + B(s, t) \frac{\partial^2 u}{\partial s^2}$$

subject to $u(s, 0) = f(s)$.

which is also called the backward Kolmogorov's equation which is a second order partial differential equation of parabolic type (for more details see [3]).

The nonlinear Fokker-Planck equation arises in several applications including plasma physics, surface physics, population dynamics, biophysics, laser physics, pattern formation, psychology and marketing [4]. A one dimensional nonlinear Fokker-Planck equation is given by

$$\frac{\partial u}{\partial t} = - \frac{\partial A(s, t, u) u}{\partial s} + \frac{\partial^2 B(s, t, u) u}{\partial s^2}.$$

The n th dimensional nonlinear Fokker-Planck equation has the following form:

$$\frac{\partial u}{\partial t} = - \sum_{i=1}^n \frac{\partial A_i(s, t, u) u}{\partial s_i} + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 B_{ij}(s, t, u) u}{\partial s_i \partial s_j},$$

where $S = (s_1, s_2, \dots, s_n)$.

Recently, vast of studies have been made to solve the Fokker-Planck equation numerically and analytically. Numerous methods have been applied to find approximated solutions of the Fokker-Planck equation [5]-[12]. Tatari et. al. [5] applied the Adomian decomposition method to solve Fokker-Planck equation without discretization. Hesam et. al. [6], applied the differential transform method to obtain an analytical solution of the Fokker-Planck equation. Aminikhah and Jamalian [7], used a combination of Laplace transform and the homotopy perturbation method to solve the nonlinear Fokker-Planck equation. Torvattanabum [8], employed the variational iterative method to obtain approximated analytical solution of the Fokker-Planck equation. Al-Jawary et. al. [12] solved the Fokker-Planck equation by the semi-analytical iterative technique (TAM).

In this work, we applied the residual power series method [13] to obtain analytical solutions for the linear and nonlinear Fokker-Planck equations. The residual power series method (henceforth referred to as RPSM) can be used to obtain a power series solutions for the Fokker-Planck equations without perturbations or discretization. The solutions induced by both the traditional Taylor series method and the RPSM are in polynomials form but the Taylor series method is computationally expensive for higher orders (see Theorem 1 below) [13]. The idea behind the RPSM is to express the solutions of the Fokker-Planck

equation as a power series about the initial point $t = t_0$. The approximated solutions can be written as $u(s, t) = \sum_{n=0}^{\infty} f_n(s) (t - t_0)^n$. The solution $u(s, t)$ can be approximated by the k th- truncated solution $u_k(s, t)$ for $k = 1, 2, 3, \dots$ and hence,

$$(2) \quad u_k(s, t) = f_0(s) + \sum_{n=1}^k f_n(s) (t - t_0)^n$$

where $f_0(s) = u_0(s, t) = u(s, 0)$. Substituting equation 1 into the Fokker-Planck equation yields what is called the k th- residual function which is denoted by $Res_k(s, t)$. The latter is infinitely many times differentiable at $t = 0$ because for each t we have

$$Res_{\infty}(s, t) = \lim_{k \rightarrow \infty} Res_k(s, t) = 0.$$

Moreover, one of a fundamental rules in the construction of the RPS method is

$$\frac{\partial^m}{\partial t^m} Res_{\infty}(s, 0) = \frac{\partial^m}{\partial t^m} Res_k(s, 0) = 0,$$

for $m = 1, 2, 3, \dots, k$. For more details about the method please see [13]-[21]. The following theorem shows the convergence of residual power series method.

Theorem 1 ([13]). *Suppose that $u(s, t)$ is the exact solution of the initial value problem 1, then the approximated solutions obtained by the residual power series method is just the Taylor expansion of $u(s, t)$.*

Corollary 1.1 ([13]). *If $u(s, t)$ is a polynomial, then the exact solution will be obtained by the residual power series method.*

This paper is organized as follows: In section 2, we introduced the algorithms that have been applied on the linear and nonlinear Fokker-Planck equations using the residual power series method. The residual power series method has been illustrated by some examples and we introduced numerical simulations for several examples of Fokker-Planck equations in section 3. In section 4, the conclusion and discussion of the efficiency of the method is presented.

2. Algorithms of the RPSM to solve Fokker-Planck Equation

In this section we will introduce some numerical algorithms to solve several types of the Fokker-Planck equations. These algorithms are based on the RPSM [13]. The backward Kolmogorov equation which is a linear Fokker-Planck equation is introduced by Risken [3]

$$(3) \quad \frac{\partial u}{\partial t} = -A(s, t) \frac{\partial u}{\partial s} + B(s, t) \frac{\partial^2 u}{\partial s^2}$$

subject to the initial condition $u(s, 0) = f(s)$.

To apply the RPS method on equation 3, we assume that $u(s, t) = \sum_{i=0}^{\infty} f_i(s)t^i$ is a solution of equation 3 and hence the n^{th} -truncated series $u_n(s, t) = \sum_{i=0}^n f_i(s)t^i$ is the approximated solutions of 3. To determine the coefficient functions $\{f_i\}_{i=0}^n$ we define $Res_n = \frac{\partial u_n}{\partial t} + A \frac{\partial u_n}{\partial s} - B \frac{\partial^2 u_n}{\partial s^2}$, and $Res_{\infty} = \lim_{k \rightarrow \infty} Res_k$. Moreover, $Res_{\infty}|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0, 1, 2, 3, \dots, n$.

Assume $u_0(s, 0) = f(s) = f_0(s)$ and hence $u_1(s, t) = f_0(s) + f_1(s) \times t$, to find $f_1(s)$ we use the fact that $Res_1|_{t=0} = 0$ which implies that

$$Res_1 = \frac{\partial u_1}{\partial t} + A \frac{\partial u_1}{\partial s} - B \frac{\partial^2 u_1}{\partial s^2} = f_1(s) + A[f'(s) + t f_1'(s)] - B[f''(s) + t f_1''(s)]$$

Upon substituting $t = 0$ we can solve for $f_1(s)$ and then we can find $u_1(s, t)$. Repeating this procedure to find the n^{th} -truncated solution $u_n(s, t) = f_0(s) + t f_1(s) + \dots + t^n f_n(s)$. The above procedure is illustrated in the following algorithm 1

Algorithm 1 RPSM algorithm for solving Kolmogorov equation

```

procedure RPS( $A, B, t_0, f_0, n$ )    ▷ The Kolmogorov equation parameters
     $u_0 = f_0$ ;
    for  $k = 1, \dots, n$  do
         $u_k = u_{k-1} + f_k t^{k-1}$ 
         $Res_k = \frac{\partial u_k}{\partial t} + A \frac{\partial u_k}{\partial s} - B \frac{\partial^2 u_k}{\partial s^2}$                                 ▷ Compute the residue
         $equ_k = \left( \frac{\partial^k}{\partial t^k} Res_k \right)_{t=0}$                                 ▷ Use the residue condition
        Solve( $equ_k = 0, f_k$ )
    end for
end for
return  $u_n$     ▷ The  $n^{th}$  approximated solution of Kolmogorov equation
end procedure

```

The general form of the linear Fokker-Planck Equation is given by

$$(4) \quad \frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s} A(s) + \frac{\partial^2}{\partial s^2} B(s) \right] u$$

subject to the initial condition $u(s, 0) = f(s)$.

To apply the RPSM on equation 4, we assume that $u(s, t) = \sum_{i=0}^{\infty} f_i(s)t^i$ is a solution of equation 4 and hence the n^{th} -truncated series $u_n(s, t) = \sum_{i=0}^n f_i(s)t^i$ is the approximated solutions of 4. To determine the coefficient functions $\{f_i\}_{i=0}^n$ we define $Res_n = \frac{\partial u_n}{\partial t} + \frac{\partial A u_n}{\partial s} - \frac{\partial^2 B u_n}{\partial s^2}$, and $Res_{\infty} = \lim_{k \rightarrow \infty} Res_k$. Moreover, $Res_{\infty}|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0, 1, 2, 3, \dots, n$.

Assume $u_0(s, 0) = f(s) = f_0(s)$ and hence $u_1(s, t) = f_0(s) + f_1(s) \times t$, to find $f_1(s)$ we use the fact that $Res_1|_{t=0} = 0$ which implies that

$$\begin{aligned} Res_1|_{t=0} &= \frac{\partial u_1}{\partial t} + \frac{\partial}{\partial s} A u_1|_{t=0} - \frac{\partial^2}{\partial s^2} B u_1|_{t=0} \\ &= f_1 + A f'(s) + f(s) \frac{\partial A}{\partial s} - B f''(s) - f'(s) \frac{\partial B}{\partial s} - f(s) \frac{\partial^2 B}{\partial s^2} - f''(s) \frac{\partial B}{\partial s}. \end{aligned}$$

Upon substituting $t = 0$ we can solve for $f_1(s)$ and then we can obtain $u_1(s, t)$ as follows

$$u_1 = f(s) + \left(-A f'(s) - f(s) \frac{\partial A}{\partial s} + B f''(s) + f'(s) \frac{\partial B}{\partial s} + f(s) \frac{\partial^2 B}{\partial s^2} + f''(s) \frac{\partial B}{\partial s} \right) \times t.$$

Repeating this procedure to find the n^{th} -truncated solution $u_n(s, t) = f_0(s) + t f_1(s) + \dots + t^n f_n(s)$. The above procedure is illustrated in the following algorithm 2

Algorithm 2 RPSM algorithm for solving Fokker-Planck equation

procedure RPS(A, B, t_0, f_0, n) $\triangleright$ The Fokker-Planck Equation parameters

$u_0 = f_0$;

for $k = 1, \dots, n$ **do**

$u_k = u_{k-1} + f_k t^{k-1}$

$Res_k = \frac{\partial u_k}{\partial t} + \frac{\partial}{\partial s} A u_k - \frac{\partial^2}{\partial s^2} B u_k$ $\triangleright$ Compute the residue

$equ_k = \left(\frac{\partial^k}{\partial t^k} Res_k \right)_{t=0}$ $\triangleright$ Use the residue condition

Solve($equ_k = 0, f_k$)

end for

end for

return u_n $\triangleright$ The n^{th} approximate solution of Fokker-Planck Equation

end procedure

The N -dimensional linear Fokker-Planck equation is given by

$$(5) \quad \frac{\partial u}{\partial t} = \left[-\sum_{i=1}^N \frac{\partial}{\partial s_i} A_i(s) + \sum_{i,j=1}^N \frac{\partial^2}{\partial s_i \partial s_j} B_{i,j}(s) \right] u$$

subject to the the initial condition $u(s, 0) = f(s)$, where $s = (s_1, \dots, s_N)$. The RPSM procedure to find the approximated solution of equation 5 is described in algorithm 3

Algorithm 3 RPSM algorithm for solving general Fokker-Planck Equation

```

procedure RPS( $A_i, B_{i,j}, t_0, f_0, n$ )  ▷ The general Fokker-Planck equation
parameters
   $u_0 = f_0$ ;
  for  $k = 1, \dots, n$  do
     $u_k = u_{k-1} + f_k t^{k-1}$ 
     $Res_k = \frac{\partial u}{\partial t} = -\sum_{i=1}^N \frac{\partial}{\partial s_i} A_i(s) u_k + \sum_{i,j=1}^N \frac{\partial^2}{\partial s_i \partial s_j} B_{i,j}(s) u_k$   ▷ Compute
the residue
     $equ_k = \left( \frac{\partial^k}{\partial t^k} Res_k \right)_{t=0}$   ▷ Use the residue condition
    Solve( $equ_k = 0, f_k$ )
  end for
end for
return  $u_n$   ▷ The  $n^{th}$  approximate solution of general Fokker-Planck
Equation
end procedure

```

The one-dimensional nonlinear Fokker-Planck equation is given by

$$(6) \quad \frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s} A(s, t, u) + \frac{\partial^2}{\partial s^2} B(s, t, u) \right] u.$$

The RPSM procedure to find the approximated solutions of equation 6 is described in a similar way as in algorithm 2.

3. Numerical discussion

In this section, the residue power series method is applied to solve several cases of the Fokker-Planck equations such as in 3, 4, 5 and 6 for different initial conditions.

Example 3.1. Consider the IVP

$$(7) \quad \begin{cases} \frac{\partial u}{\partial t} = -A(s, t) \frac{\partial u}{\partial s} + B(s, t) \frac{\partial^2 u}{\partial s^2}, \\ u(s, 0) = 1 + s \end{cases}$$

where $A(s, t) = -(s+1)$, $B(s, t) = e^t s^2$.

Assume that the solution $u(s, t)$ has the form $u(s, t) = \sum_{i=0}^{\infty} f_i(s) t^i$ and the approximated solution is $u_n(s, t) = \sum_{i=0}^n f_i(s) t^i$, then the coefficient functions $\{f_i(s)\}_{i=0}^n$ will be determined By RPSM as follows:

Let $Res_n = \frac{\partial u_n}{\partial t} - (s+1) \frac{\partial u_n}{\partial s} - e^t s^2 \frac{\partial^2 u_n}{\partial s^2}$, and $Res_{\infty} = \lim_{k \rightarrow \infty} Res_k$.

Since, $Res_{\infty}|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0, 1, 2, 3, \dots, n$, then $f_0(s) = s+1$ and hence, $u_0 = s+1$, $f_1 = s+1$ which implies that $u_1 = (s+1)(1+t)$. For

$n = 2$, $u_2 = f_0 + f_1 \times t + f_2 \times t^2$. Upon applying the condition $\frac{\partial}{\partial t} Res_2|_{t=0} = 0$, we obtained that $f_2 = \frac{s+1}{2}$ and the second term of the RPSM approximated solution is $u_2 = (s+1)(1+t+\frac{t^2}{2})$.

Similarly the third term of the RPSM approximated solution is given by $u_3 = f_0 + f_1 \times t + f_2 \times t^2 + f_3 \times t^3$ where $f_3 = \frac{s+1}{6}$. Therefore the n^{th} approximated solution can be computed as follows:

$$u_n = (s+1) \left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \dots + \frac{1}{n!}t^n \right).$$

Since $u(s, t) = \lim_{n \rightarrow \infty} u_n(s, t)$, then we conclude that $u(s, t) = (s+1)e^t$ which is the exact solution of equation 7. Figure 1 describes the exact solution $u(s, t)$ and the approximated solutions $u_5(s, t)$ and $u_3(s, t)$.

Example 3.2. Consider the IVP

$$(8) \quad \begin{cases} \frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s} A(s) + \frac{\partial^2}{\partial s^2} B(s) \right] u, \\ u(s, 0) = s \end{cases}$$

where $A(s) = -1$, $B(s) = 1$.

Consider equation 4. Let $A(s) = -1$ and $B(s) = 1$ with $u(s, 0) = s$, then by applying algorithm 2 we have:

For $n = 0$, obviously $f_0(s) = f(s) = s$ hence $u_0 = s$. For $n = 1$, $u_1 = f_0 + f_1 \times t$, then $Res_1|_{t=0} = 0$ which implies that $f_1 = 1$ and $u_1 = s + t$.

Repeating the process k -times yields $u_k = s + t$ for all k , which is the exact solution of the problem. Figure 2 describes the approximated solutions $u_0(s, t)$ and $u_1(s, t)$.

Example 3.3. Consider equation 6. Let $A(s, t, u) = 4\frac{u}{s} - \frac{s}{3}$ and $B(s, t, u) = u$ with $u(s, 0) = s^2$

$$(9) \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial s} \left(\frac{s}{3}u - 4\frac{u^2}{s} \right) + \frac{\partial^2}{\partial s^2} u^2$$

According to RPSM algorithm 2, we have $u_0 = s^2$, $u_1 = t s^2 + s^2$, $u_2 = (t^2 s^2)/2 + t s^2 + s^2$, and $u_3 = s^2(1 + t + \frac{t^2}{2} + \frac{t^3}{6})$.

Repeating this procedure k -times we get $u_k = s^2(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \dots + \frac{1}{k!}t^k)$, therefore as $k \rightarrow \infty$, $u = s^2 e^t$ which is the exact solution. Figure 3 describes the exact solution $u(s, t)$ and the approximated solutions $u_5(s, t)$ and $u_3(s, t)$.

Example 3.4. Consider equation 5. Let $A_1(s_1, s_2) = s_1$, $A_2(s_1, s_2) = 5s_1$, $B_{1,1}(s_1, s_2) = s_1^2$, $B_{1,2}(s_1, s_2) = 1$, $B_{2,1}(s_1, s_2) = 1$, $B_{2,2}(s_1, s_2) = s_2^2$ with $u(s_1, s_2, 0) = s_1$, where $s_1, s_2 \in \mathbb{R}$.

According to RPSM algorithm 3, we have $u_0(s_1, s_2, t) = s_1$, $u_1 = t s_1 + s_1$, $u_2 = (t^2 s_1)/2 + t s_1 + s_1$ and $u_3 = s_1(1 + t + \frac{t^2}{2} + \frac{t^3}{6})$.

Repeating this procedure k -times we get $u_k = s_1 (1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\dots+\frac{1}{k!}t^k)$, therefore as $k \rightarrow \infty$, $u = s_1 e^t$ which is the exact solution.

The level curves of the variable s_i at c will be defined by:

$$l_{c(s_i)} [u(s_1, \dots, s_N)] \\ = \{(u, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N) : u = u(s_1, \dots, s_{i-1}, c, s_{i+1}, \dots, s_N)\}.$$

These level curves will be used to analyze the high dimensional systems solutions obtained by RPSM. Figures 4, 5, and 6 illustrated the level curves of the solutions of the problem in example (3.4).

Example 3.5. For the sake of comparison, We will study the forward Kolmogorov equation that was considered by Odibat and Momani [10], the authors solved the problem by ADM and VIM methods to obtain numerical solutions. In this example we will solve the same problem by the RPSM method.

Consider the following forward Kolmogorov equation

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s} A(s, t) + \frac{\partial^2}{\partial s^2} B(s, t) \right] u(s, t),$$

where $A(s, t) = \frac{s}{6}$, $B(s, t) = \frac{s^2}{12}$ and $u(s, 0) = s^2$. Upon applying the RPSM method we obtain that $u_1(s, t) = s^2 + \frac{ts^2}{2}$, $u_2(s, t) = s^2 + \frac{ts^2}{2} + \frac{t^2s^2}{8}$, $u_3(s, t) = s^2 + \frac{ts^2}{2} + \frac{t^2s^2}{8} + \frac{t^3s^2}{48}$ and hence, $u(s, t) = s^2 e^{t/2}$ which is the exact solution. Figure 7 depicts the plot of the approximated solution $u_3(s, t)$ with several values of time $t = 0.2$ (blue), $t = 0.4$ (red) and $t = 0.6$ (black) and space s (Left) and the plot of the exact solution $u(s, t)$ and the approximated solutions $u_1(s, t)$, $u_2(s, t)$ and $u_3(s, t)$ with time $t = 2$ and space s (Right).

4. Conclusion

The residual power series method is introduced and applied to find the approximated solutions of Fokker-Planck equation. Numerous algorithms are presented to solve the Fokker-Planck linear and nonlinear equations. Several numerical examples are provided to illustrate the solutions which concluded that the RPSM is efficient in finding numerical solutions for such problems using very simple calculations. A comparison between RPSM and Adomian decomposition method (ADM) and the variational iteration method (VIM) [10] shows the simplicity and the powerful of the RPSM. The following table 1 illustrates the efficiency of the RPSM by introducing a comparison between the RPSM, ADM, VIM and the exact solution of the problem in Example 3.5.

All calculations in the presented examples in this work have been performed using MAPLE.

Table 1: A comparison of the relative error of the approximated solutions in example 3.5 using RPSM with the relative error of ADM and VIM. The values of ADM and VIM are taken from [10]

t	s	$E_{\text{relative RPSM}}$	$E_{\text{relative ADM}}$	$E_{\text{relative VIM}}$
0.2	0.25	0	1.5×10^{-4}	1.5×10^{-4}
	0.5	3.6×10^{-6}	1.2×10^{-4}	1.5×10^{-4}
	0.75	4.8×10^{-6}	1.5×10^{-4}	1.5×10^{-4}
	1	3.6×10^{-6}	1.5×10^{-4}	1.5×10^{-4}
0.4	0.25	6.5×10^{-5}	1.2×10^{-3}	1.2×10^{-3}
	0.5	5.9×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
	0.75	5.7×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
	1	5.7×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
0.6	0.25	3.8×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	0.5	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	0.75	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	1	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}

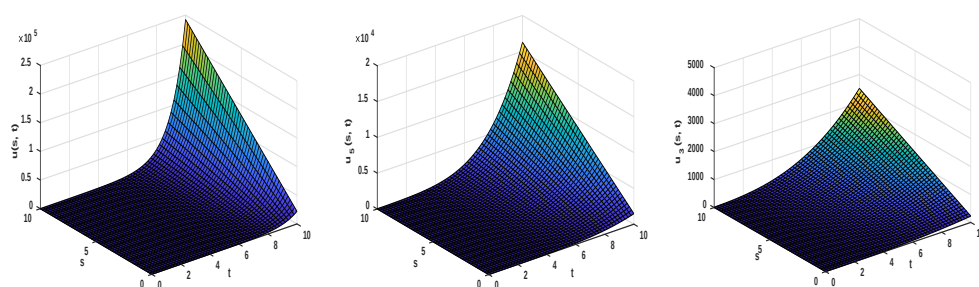


Figure 1: The 3D plot of the exact solution (Left), $u_5(s, t)$ (Middle), and $u_3(s, t)$ (Right) with time t and space s .

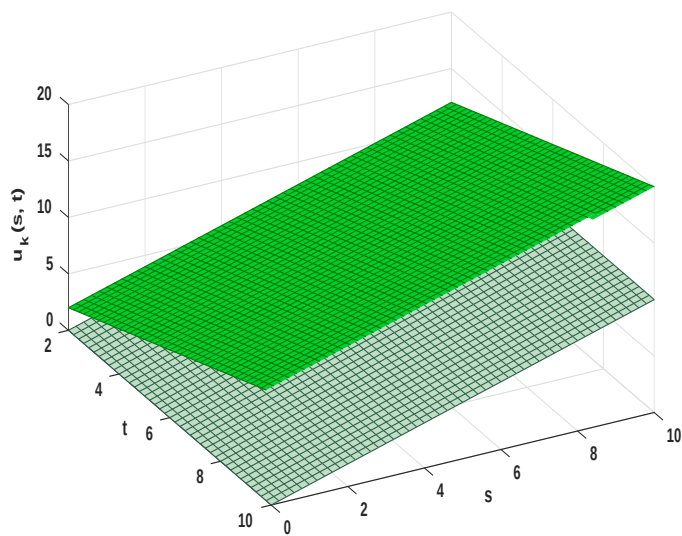


Figure 2: The approximated solutions $u_0(s, t)$ and $u_1(s, t)$ of problem 8 in Example 3.2

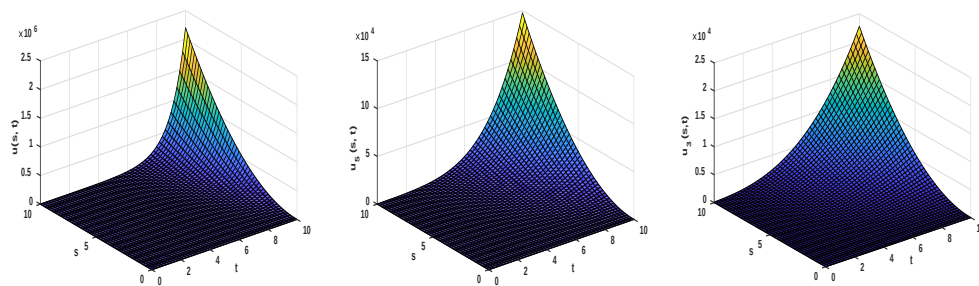


Figure 3: The 3D plot of the exact solution (Left), $u_5(s, t)$ (Middle), and $u_3(s, t)$ (Right) with time t and space s .

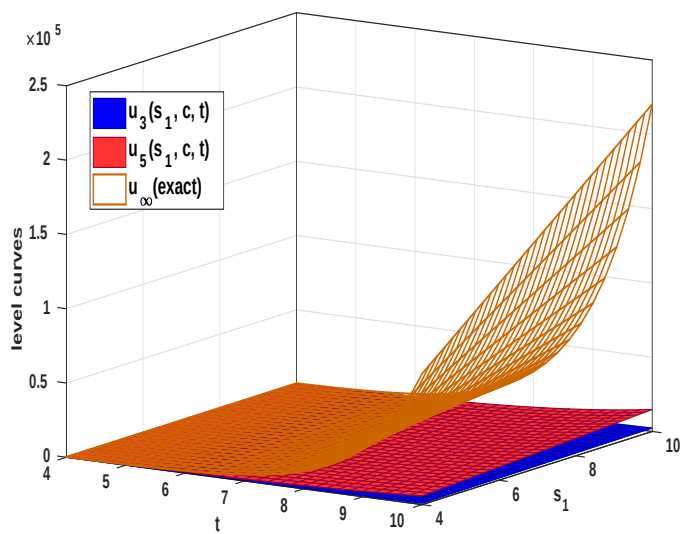


Figure 4: The plot of the level curves $l_{c(s_2)}[u(s_1, s_2, t)]$ for arbitrary value c .

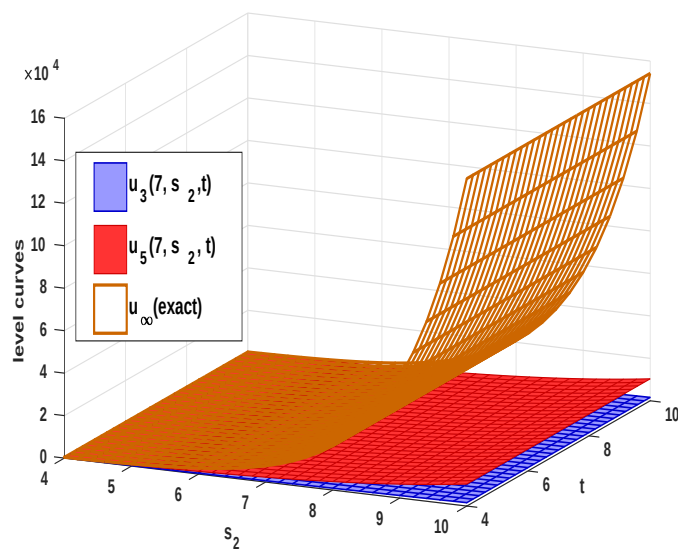


Figure 5: The plot of the level curves $l_{c(s_1)}[u(s_1, s_2, t)]$ for $c = 7$.

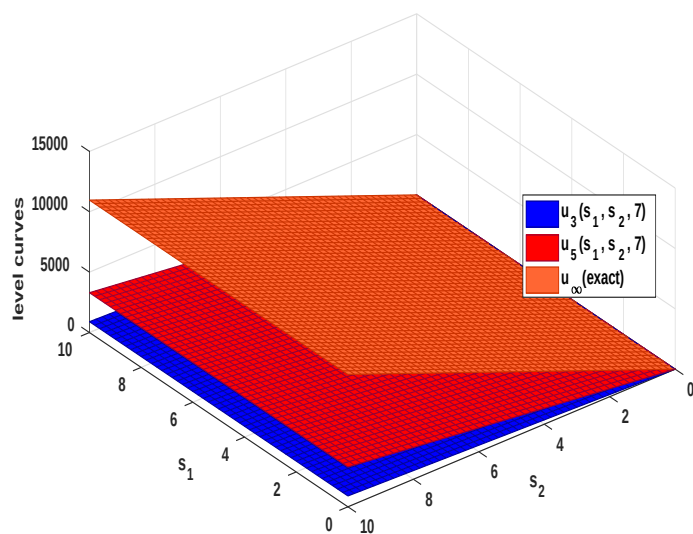


Figure 6: The plot of the level curves $l_{c(t)}[u(s_1, s_2, t)]$ for $c = 7$.

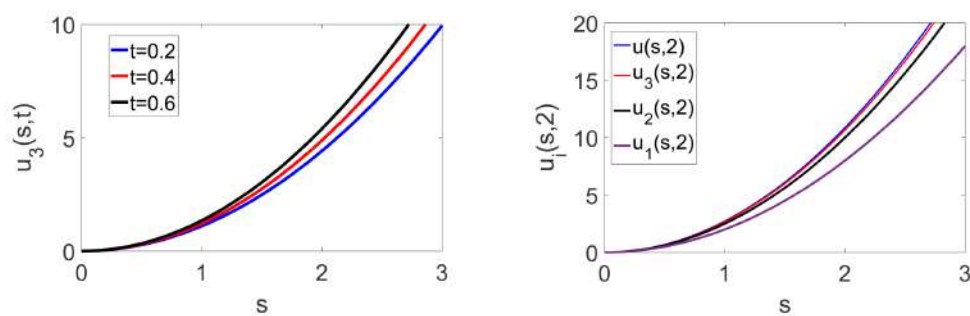


Figure 7: The plot of the approximated solution $u_3(s, t)$ with several values of time t and space s (Left) and the plot of the exact solution $u(s, t)$ and the approximated solutions $u_1(s, t)$, $u_2(s, t)$ and $u_3(s, t)$ with time $t = 2$ and space s (Right).

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Fixed point results in $\alpha - \eta$ -complete metric spaces via w -distances

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Abstract. We introduce the concepts of $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$ -rational contractions via w -distances. For such type contractions, we ensure the existence and uniqueness of a fixed point in the setting of $\alpha - \eta$ -complete metric spaces. An example is also given to support our given theorems. Our paper generalizes the work of Lakzian et al. [11] and several related articles in the literature.

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Keywords: $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$ -rational contraction mapping, $\alpha - \eta$ -complete metric space, (c) -comparison function, $\alpha - \eta$ -continuous function.

1. Introduction and preliminaries

Fixed point strategies have been connected in numerous fields in sciences, In particular, applied and natural sciences. Over the years, Several mathematicians have been generalized and amplified the famous contractive Banach contraction principle in many directions ([1, 2, 9, 10, 16, 3, 22, 5, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36]). In 2012, Samet et al. [6] established several fixed point results for $\alpha - \psi$ -contractive mappings in complete metric spaces. The concept of a w -distance metric space was presented by Kada et al [7] to generalize some important results in nonconvex minimizations. In 2012, Imdad and Rouzkard [8] used the concept of a w -distance to proved fixed point results in ordered metric spaces. Recently, Lakzian et al. [11] proved some fixed point theorems for $\alpha - \psi$ -contractions involving w -distances.

Throughout this article, we denote by $\mathbb{N}$ the set of all positive integers. Also, we denote by Ψ the family of all functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (ψ_1) ψ is nondecreasing;
- (ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, where ψ^n is the n th iterate of ψ .

These functions are known in the literature as (c) -comparison functions or Bianchini-Grandolfi gauge functions (see [31]).

Remark 1.1. For each $\psi \in \Psi$, we have:

- (ψ_3) $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for all $t > 0$;
- (ψ_4) $\psi(t) < t$ for all $t > 0$;
- (ψ_5) $\psi(0) = 0$.

Denote by Φ the set of all continuous functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Definition 1.1 ([6]). Let $f : Y \rightarrow Y$ and $\alpha : Y \times Y \rightarrow [0, \infty)$ be two given mappings. The mapping f is called α -admissible if

$$y, z \in Y, \alpha(y, z) \geq 1 \Rightarrow \alpha(fy, fz) \geq 1.$$

Definition 1.2 ([6]). Let (Y, d) be a metric space and let $f : Y \rightarrow Y$ be a given mapping. f is said an $\alpha - \psi$ -contraction if there exist two functions $\alpha : Y \times Y \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(y, z)d(fy, fz) \leq \psi(d(y, z))$$

for all $y, z \in Y$.

Afterward, many articles appeared as extensions and generalizations of Definition 1.2 (see also [13, 14, 15, 17]).

Salimi et al. [20] presented the concept of α -admissible mappings with respect to η .

Definition 1.3. Let $f : Y \rightarrow Y$ and $\alpha, \eta : Y \times Y \rightarrow [0, \infty)$. The mapping f is α -admissible with respect to η if

$$(1.1) \quad y, z \in Y, \quad \alpha(y, z) \geq \eta(y, z) \Rightarrow \alpha(fy, fz) \geq \eta(fy, fz).$$

If we take $\eta(y, z) = 1$, Definition 1.3 reduces to Definition 1.1. Also, if we take $\alpha(y, z) = 1$, then we say that f is an η -subadmissible mapping.

Hussain et al. [19] presented the concept of $\alpha - \eta$ -complete metric spaces and $\alpha - \eta$ -continuous functions.

Definition 1.4. Let (Y, d) be a metric space and $\alpha, \eta : Y \times Y \rightarrow [0, \infty)$. The metric space Y is said to be $\alpha - \eta$ -complete if every Cauchy sequence $\{y_n\}$ in Y with $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ converges in Y . Y is said an α -complete metric space if $\eta(y, z) = 1$ for all $y, z \in Y$ and one says that (Y, d) is an η -complete metric space if $\alpha(y, z) = 1$ for all $y, z \in Y$.

Example 1.1 ([19]). Let $Y = (0, +\infty)$ and $d(y, z) = |y - z|$ for all $y, z \in Y$. Note that Y is not complete. Let Y be a closed subset of Y . Define $\alpha, \eta : Y \times Y \rightarrow [0, +\infty)$ by

$$\alpha(y, z) = \begin{cases} (y + z)^2, & \text{if } y, z \in Y, \\ 5, & \text{otherwise,} \end{cases}$$

and

$$\eta(y, z) = 2yz.$$

Then (Y, d) is an $\alpha - \eta$ -complete metric space.

Definition 1.5 ([19]). Let (Y, d) be a metric space and $\alpha, \eta : Y \times Y \rightarrow [0, \infty)$. A mapping $f : Y \rightarrow Y$ is said to be $\alpha - \eta$ -continuous on (Y, d) , if for each sequence $\{y_n\}$ in Y with $y_n \rightarrow y$ as $n \rightarrow \infty$ and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ implies $f(y_n) \rightarrow f(y)$ as $n \rightarrow \infty$.

Example 1.2 ([19]). Let $Y = [0, +\infty)$ be equipped by the metric $d(y, z) = |y - z|$ for all $y, z \in Y$. Define $f : Y \rightarrow Y$ and $\alpha, \eta : Y \times Y \rightarrow [0, +\infty)$ by

$$fy = \begin{cases} y^5, & \text{if } y \in [0, 1], \\ \sin \pi y + 2, & \text{if } y \in (0, +\infty), \end{cases}$$

$$\alpha(y, z) = \begin{cases} y^2 + z^2 + 1, & \text{if } y, z \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(y, z) = y^2.$$

Clearly, f is not continuous, but f is $\alpha - \eta$ -continuous on (Y, d) . Indeed, if $y_n \rightarrow y$ as $n \rightarrow \infty$ and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$, then $y_n \in [0, 1]$ and so $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} y_n^5 = y^5 = fy$.

In 2014, Ansari [18] introduced the concept of $\mathcal{C}$ -class functions as follows:

Definition 1.6. A continuous function $\mathfrak{F} : [0, \infty)^2 \rightarrow \mathbb{R}$ is called a $\mathcal{C}$ -class function if for any $s, t \in [0, \infty)$, the following conditions hold:

- (i) $\mathfrak{F}(s, t) \leq s$;
- (ii) $\mathfrak{F}(s, t) = s$ implies that either $s = 0$ or $t = 0$.

As examples of $\mathcal{C}$ -class functions, we state:

1. $\mathfrak{F}(r, t) = r - t$ for all $r, t \in [0, \infty)$;
2. $\mathfrak{F}(r, t) = mr$ for all $r, t \in [0, \infty)$ where $0 < m < 1$;
3. $\mathfrak{F}(r, t) = \frac{r}{(1+t)^\rho}$ for all $r, t \in [0, \infty)$ where $\rho \in (0, \infty)$;
4. $\mathfrak{F}(r, t) = (r + l)^{\frac{1}{(1+t)^\rho}} - l$ for all $r, t \in [0, \infty)$ where $l > 1$ and $\rho \in (0, \infty)$;
5. $\mathfrak{F}(r, t) = r \log_{t+a} a$ for all $r, t \in [0, \infty)$ where $a > 1$;
6. $\mathfrak{F}(r, t) = r - (\frac{1+r}{2+r})(\frac{t}{1+t})$ for all $r, t \in [0, \infty)$;
7. $\mathfrak{F}(r, t) = r\beta(r)$ for all $r, t \in [0, \infty)$ where $\beta : [0, \infty) \rightarrow [0, 1)$ and is continuous;
8. $\mathfrak{F}(r, t) = r - \varphi(r)$ where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
9. $\mathfrak{F}(r, t) = sh(r, t)$ where $h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $h(t, r) < 1$ for all $t, r > 0$;
10. $\mathfrak{F}(r, t) = r - (\frac{2+t}{1+t})t$ for all $r, t \in [0, \infty)$;
11. $\mathfrak{F}(r, t) = \sqrt[n]{\ln(1 + r^n)}$ for all $r, t \in [0, \infty)$.

Definition 1.7 ([7]). Let (Y, d) be a metric space. A function $p : Y \times Y \rightarrow [0, \infty)$ is called a w -distance on Y if it satisfies the following properties:

- (p_1) $p(y, v) \leq p(y, z) + p(z, v)$ for any $y, v, z \in Y$;
- (p_2) p is lower semicontinuous in its second variable; i.e, if $y \in Y$ and $z_n \rightarrow z \in Y$, then $p(y, z) \leq \liminf_{n \rightarrow \infty} p(y, z_n)$;
- (p_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(v, y) \leq \delta$ and $p(v, z) \leq \delta$ imply $d(y, z) \leq \epsilon$.

Lakzian et al. [11] proved some fixed point theorems for $\alpha - \psi$ -contractions involving w -distances. For other results dealing with w -distances, see [4, 12].

Definition 1.8 ([11]). Let (Y, d) be a metric space with a w -distance p . The mapping $f : Y \rightarrow Y$ is said an (α, ψ, p) -contraction if there exist two functions $\alpha : Y \times Y \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$(1.2) \quad \alpha(y, z)p(fy, fz) \leq \psi(p(y, z))$$

for all $y, z \in Y$.

Theorem 1.1 ([11]). Let (Y, d) be a complete metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be an (α, ψ, p) -contractive mapping. Assume that the following conditions hold:

1. f is an α -admissible mapping;
2. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;
3. Either f is continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y \in Y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

We need the following Lemma to prove our main results.

Lemma 1.1 ([7]). Let (Y, d) be a metric space and let p be a w -distance on Y . Assume that $\{y_n\}$ and $\{z_n\}$ are sequences in Y , $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $[0, \infty)$ converging to 0, and let $y, t, z \in Y$. Then the following assertions hold:

1. If $p(y_n, z) \leq \gamma_n$ and $p(y_n, t) \leq \lambda_n$ for all $n \in \mathbb{N}$, then $z = t$. In particular, if $p(y, z) = p(y, t) = 0$, then $z = t$.
2. If $p(y_n, z_n) \leq \gamma_n$ and $p(y_n, z) \leq \lambda_n$ for all $n \in \mathbb{N}$, then $\{z_n\}$ converges to z ;
3. If $p(y_n, y_m) \leq \gamma_n$ for all $n, m \in \mathbb{N}$, then $\{y_n\}$ is a Cauchy sequence.

In this article, motivated by the idea of $\mathcal{C}$ -class function and w -distance due to Kada et al [7], we present the class of $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$ -rational contractive mappings via w -distances and prove some fixed point results for this class of mappings. We also present an example to support our work.

2. Main results

Definition 2.1. Let (Y, d) be a metric space with a w -distance p and $f : Y \rightarrow Y$ be a given mapping. We say that f is an $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$ -rational contractive mapping if there exist $\alpha, \eta : Y \times Y \rightarrow [0, \infty)$, $\psi \in \Psi$, $\mathfrak{F} \in \mathcal{C}$ and $\phi \in \Phi$ such that

$$(2.1) \quad \alpha(y, z) \geq \eta(y, z) \Rightarrow p(fy, fz) \leq \mathfrak{F}(\psi(M(y, z)), \phi(M(y, z))),$$

where

$$M(y, z) = \max\left\{p(y, z), \frac{p(y, fy)}{1 + p(y, fy)}, \frac{p(z, fz)}{1 + p(z, fz)}\right\}$$

for all $y, z \in Y$.

Theorem 2.1. *Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be an $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$ -rational contractive mapping. Assume that the following conditions hold:*

1. (Y, d) is an $\alpha - \eta$ -complete metric space;
2. f is α -admissible with respect to η ;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;
4. Either f is $\alpha - \eta$ -continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y \in Y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq \eta(y_n, y)$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Proof. From condition(3), there exists $y_0 \in Y$ such that

$$\alpha(y_0, fy_0) \geq \eta(y_0, fy_0).$$

Define the sequence $\{y_n\}$ in Y by

$$y_{n+1} = fy_n$$

for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then f has a fixed point. From now on, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Since f is α -admissible with respect to η and $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$, we obtain that

$$(2.2) \quad \alpha(y_1, y_2) = \alpha(fy_0, fy_1) \geq \eta(fy_0, fy_1) = \eta(y_1, y_2).$$

By continuing this process, we have

$$(2.3) \quad \alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$$

for all $n \in \mathbb{N}$. Since f is an $(\alpha, \eta, \mathfrak{F}, \psi, \phi)$ -rational contractive mapping, we get

$$(2.4) \quad \begin{aligned} p(y_n, y_{n+1}) &= p(fy_{n-1}, fy_n) \\ &\leq \mathfrak{F}(\psi(M(y_{n-1}, y_n)), \phi(M(y_{n-1}, y_n))), \end{aligned}$$

where

$$(2.5) \quad \begin{aligned} M(y_{n-1}, y_n) &= \max\{p(y_{n-1}, y_n), \frac{p(y_{n-1}, fy_{n-1})}{1 + p(y_{n-1}, fy_{n-1})}, \frac{p(y_n, fy_n)}{1 + p(y_n, fy_n)}\} \\ &= \max\{p(y_{n-1}, y_n), \frac{p(y_{n-1}, y_n)}{1 + p(y_{n-1}, y_n)}, \frac{p(y_n, y_{n+1})}{1 + p(y_n, y_{n+1})}, \} \\ &\leq \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}. \end{aligned}$$

Now, if $M(y_{n-1}, y_n) = p(y_n, y_{n+1})$ for some n , then from (2.4)

$$(2.6) \quad \begin{aligned} p(y_n, y_{n+1}) &\leq \mathfrak{F}(\psi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))) \\ &< \psi(p(y_n, y_{n+1})) < p(y_n, y_{n+1}), \end{aligned}$$

which is a contradiction. Thus $M(y_{n-1}, y_n) = p(y_{n-1}, y_n)$ all $n \in \mathbb{N}$. Hence

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \mathfrak{F}(\psi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))) \\ &< \psi(p(y_{n-1}, y_n)). \end{aligned}$$

Iteratively, we find that

$$(2.7) \quad p(y_n, y_{n+1}) < \psi^n(p(y_0, y_1)).$$

From (ψ_3) , we can conclude that $\lim_{n \rightarrow \infty} \psi^n(p(y_0, y_1)) = 0$. This implies that

$$(2.8) \quad \lim_{n \rightarrow \infty} p(y_n, y_{n+1}) = 0.$$

We want to show that $\lim_{n \rightarrow \infty} p(y_n, y_m) = 0$. Assume that with $n, m \in \mathbb{N}$ with $m > n$. Then

$$(2.9) \quad p(y_n, y_m) \leq \sum_{l=n}^{m-1} p(y_l, y_{l+1}) \leq \sum_{l=n}^{\infty} \psi^l(p(y_0, y_1)).$$

Since the properties of ψ , we have $\lim_{n \rightarrow \infty} \sum_{l=n}^{\infty} \psi^l(p(y_0, y_1)) = 0$. Letting $n, m \rightarrow \infty$ in 2.9, we get

$$(2.10) \quad \lim_{n \rightarrow \infty} p(y_n, y_m) = 0,$$

that is, $\{y_n\}$ is a Cauchy sequence in the $\alpha - \eta$ -complete metric space, so there exists $u \in Y$ such that $\lim_{n \rightarrow \infty} y_n = u$. Using the fact that f is $\alpha - \eta$ -continuous, we obtain

$$u = \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} f y_n = f \lim_{n \rightarrow \infty} y_n = f u.$$

That is, f has a fixed point.

Otherwise, by condition (4), we have

$$(2.11) \quad \alpha(y_n, u) \geq \eta(y_n, u)$$

for each $n \in \mathbb{N}$. From (p_2) , we get

$$p(y_n, u) \leq \liminf_{n \rightarrow 0} p(y_n, y_m) := \delta_n \text{ (say)}$$

for all $n \in \mathbb{N}$. By using (2.10), we have $\lim_{n \rightarrow \infty} \delta_n = 0$. So

$$(2.12) \quad \lim_{n \rightarrow \infty} p(y_n, u) = 0.$$

Now, from (2.1) and (2.11), we get

$$(2.13) \quad \begin{aligned} p(y_{n+1}, fu) &= p(fy_n, fu) \\ &\leq \mathfrak{F}(\psi(M(y_n, u)), \phi(M(y_n, u))), \end{aligned}$$

where

$$(2.14) \quad \begin{aligned} M(y_n, u) &= \max\{p(y_n, u), \frac{p(y_n, fy_n)}{1 + p(y_n, fy_n)}, \frac{p(u, fu)}{1 + p(u, fu)}\} \\ &= \max\{p(y_n, u), \frac{p(y_n, y_{n+1})}{1 + p(y_n, y_{n+1})}, \frac{p(u, u)}{1 + p(u, u)}\}. \end{aligned}$$

By using (2.12) and (2.8), we have

$$(2.15) \quad M(y_n, u) = \max\{0, \frac{0}{1+0}, \frac{0}{1+0}\} = 0.$$

Then (2.13) becomes

$$\lim_{n \rightarrow 0} p(y_{n+1}, fu) \leq \mathfrak{F}(\psi(0), \phi(0)),$$

which implies that $\lim_{n \rightarrow 0} p(y_{n+1}, fu) = 0$. By the triangular inequality, we obtain

$$(2.16) \quad p(y_n, fu) \leq p(y_n, y_{n+1}) + p(y_{n+1}, fu).$$

Thus,

$$(2.17) \quad \lim_{n \rightarrow 0} p(y_n, fu) = 0.$$

From Lemma 1.1, (2.12) and (2.17), we deduce that $fu = u$. Now, we suppose that $\alpha(u, u) \geq \eta(u, u)$. If $p(u, u) > 0$, from (2.4), we have

$$(2.18) \quad \begin{aligned} p(u, u) &= p(fu, fu) \\ &\leq \mathfrak{F}(\psi(M(u, u)), \phi(M(u, u))), \end{aligned}$$

where

$$(2.19) \quad \begin{aligned} M(u, u) &= \max\{p(u, u), \frac{p(u, fu)}{1 + p(u, fu)}, \frac{p(u, fu)}{1 + p(u, fu)}\} \\ &= \max\{p(u, u), \frac{p(u, u)}{1 + p(u, u)}, \frac{p(u, u)}{1 + p(u, u)}\} \\ &< \max\{p(u, u), p(u, u)\} = p(u, u). \end{aligned}$$

Therefore,

$$\begin{aligned} p(u, u) &= p(fu, fu) \\ &\leq \mathfrak{F}(\psi(p(u, u)), \phi(p(u, u))) \\ &< \psi(p(u, u)) < p(u, u), \end{aligned}$$

which is impossible. Thus $p(u, u) = 0$. □

Theorem 2.2. *Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be an $(\alpha, \eta, \mathfrak{F}, \psi, \varphi, p)$ -rational contractive mapping. Assume that the following conditions hold:*

1. (Y, d) is an $\alpha - \eta$ -complete metric space;
2. f is α -admissible with respect to η ;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;
4. if for $z \in Y$ with $z \neq fz$, $\inf\{p(y, z) + p(y, fy) : y \in Y\} > 0$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Proof. Following the proof of Theorem 2.1, $\{y_n\}$ is a Cauchy sequence in the $\alpha - \eta$ -complete metric space (Y, d) . Then there exists $u \in Y$ such that $y_n \rightarrow u$ as $n \rightarrow \infty$. Assume that $u \neq fu$. Then

$$\begin{aligned} 0 &< \inf\{p(y, u) + p(y, fy) : y \in Y\} \\ &\leq \inf\{p(y_n, u) + p(y_n, y_{n+1}) : y \in Y\} \\ &= 0, \end{aligned}$$

which is a contradiction. Thus $u = fu$. The last step of the proof follows as in Theorem 2.1 and in order to avoid duplication, the details are skipped. $\square$

If we take $\eta(y, z) = 1$, we obtain the following two corollaries.

Corollary 2.1. *Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be an $(\alpha, \mathfrak{F}, \psi, \phi)$ -rational contractive mapping. Assume that the following conditions hold:*

1. (Y, d) is an α -complete metric space;
2. f is an α -admissible mapping;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;
4. Either f is α -continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y \in Y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

Corollary 2.2. *Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be an $(\alpha, \mathfrak{F}, \psi, \phi, p)$ -rational contractive mapping. Assume that the following conditions hold:*

1. (Y, d) is an α -complete metric space;

2. f is an α -admissible mapping;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;
4. if for $z \in Y$ with $z \neq fz$, $\inf\{p(y, z) + p(y, fy) : y \in Y\} > 0$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

If we take $M(y, z) = p(y, z)$, we get the following results.

Corollary 2.3. Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be such that

$$(2.20) \quad \alpha(y, z) \geq \eta(y, z) \Rightarrow p(fy, fz) \leq \mathfrak{F}(\psi(p(y, z)), \phi(p(y, z))),$$

for all $y, z \in Y$, where $\mathfrak{F} \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$. Assume that

1. (Y, d) is an $\alpha - \eta$ -complete metric space;
2. f is an α -admissible with respect to η ;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;
4. Either f is $\alpha - \eta$ -continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y \in Y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq \eta(y_n, y)$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Corollary 2.4. Let (Y, d) be a metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be defined as in (2.20) and satisfy the following conditions:

1. (Y, d) is an α -complete metric space;
2. f is α -admissible with respect to η ;
3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;
4. if for $z \in Y$ with $z \neq fz$, $\inf\{p(y, z) + p(y, fy) : y \in Y\} > 0$.

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Corollary 2.5. Let (Y, d) be a complete metric space equipped with a w -distance p and let $f : Y \rightarrow Y$ be a mapping. Assume that there exists $\tau \in [0, 1)$ such that, for all $y, z \in Y$,

$$p(fy, fz) \leq \tau \mathfrak{F}(p(y, z), p(y, z)),$$

where $\mathfrak{F} \in \mathcal{C}$. Then there exists $u \in Y$ such that $fu = u$. Moreover, $p(u, u) = 0$.

Proof. Take in Corollary 2.3, $\alpha(y, z) = \eta(y, z) = 1$, $\psi(k) = \tau k$ and $\phi(k) = k$ where $\tau \in [0, 1)$. $\square$

Example 2.1. Let $Y = [0, \infty)$ and $d(y, z) = |y - z|$ for all $y, z \in Y$. The function p defined by $p(y, z) = y$ is a w -distance on (Y, d) . Take $\psi(k) = k$, $\phi(k) = k^2$ and $\mathfrak{F}(k, r) = \frac{k}{2+k}$ for all $k, r \in [0, \infty)$. Consider $f : Y \rightarrow Y$ as

$$f(y) = \begin{cases} \frac{1}{4}y & \text{if } 0 \leq y \leq 1, \\ 2y, & \text{if } y > 1. \end{cases}$$

Given $\alpha, \eta : Y \times Y \rightarrow [0, \infty)$ by

$$\alpha(y, z) = \begin{cases} 1 & \text{if } 0 \leq y, z \leq 1, \\ 3, & \text{otherwise,} \end{cases}$$

$$\eta(y, z) = \begin{cases} \frac{1}{5} & \text{if } 0 \leq y, z \leq 1, \\ 5, & \text{otherwise.} \end{cases}.$$

The following statements hold:

1. (Y, d) is an $\alpha - \eta$ -complete metric space;
 2. f is α -admissible with respect to η ;
 3. there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;
 4. f is $\alpha - \eta$ -continuous;
 5. f satisfies the condition (2.20).
1. We will show that (Y, d) is an $\alpha - \eta$ -complete metric space. Let $\{y_n\}$ be a Cauchy sequence in Y such that $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$, then $\{y_n\} \subseteq [0, 1]$. Since $([0, 1], d)$ is a complete metric space, then $\{y_n\}$ converges in $[0, 1]$.
 2. Let $y, z \in Y$ be such that $\alpha(y, z) \geq \eta(y, z)$. Then $y, z \in [0, 1]$. Since $fb \in [0, 1]$ for each $b \in [0, 1]$, one writes $\alpha(fy, fz) \geq \eta(fy, fz)$. Thus f is α -admissible with respect to η .
 3. Take $y_0 = 1$. Here,

$$\alpha(1, f1) = \alpha(1, \frac{1}{4}) = 1 \geq \frac{1}{5} = \eta(1, \frac{1}{4}) = \alpha(1, f1).$$

4. Let $\{y_n\}$ be in Y such that $y_n \rightarrow y$ as $n \rightarrow \infty$ and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Recall that $\{y_n\} \subseteq [0, 1]$ for all $n \in \mathbb{N}$. Also, $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} \frac{1}{4}y_n = \frac{1}{4}y = fy$. That is, f is $\alpha - \eta$ -continuous.

5. We will show that f satisfies (2.20). Let $y, z \in Y$ be such that $\alpha(y, z) \geq \eta(y, z)$. Then $y, z \in [0, 1]$. In this case, we have

$$\begin{aligned} \mathfrak{F}(\psi(p(y, z)), \phi(p(y, z))) - p(fy, fz) &= \mathfrak{F}(y, y^2) - \frac{1}{4}y \\ &= \frac{y}{2+y} - \frac{1}{4}y \\ &= \frac{2y - y^2}{8 + 4y} \\ &= \frac{y(2 - y)}{8 + 4y} \\ &\geq 0. \end{aligned}$$

Then $p(fy, fz) \leq \mathfrak{F}(\psi(p(y, z)), \phi(p(y, z)))$. Thus all assumptions of Corollary 2.3 are satisfied, and so f has a fixed point, which is, $u = 0$.

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New independent paracompact spaces

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Abstract. The purpose of the present paper is to introduce a new type of paracompactness which is called ω_δ - paracompact and to obtain some results of paracompact spaces, one of which is the image of ω_δ - paracompact is paracompact under ω_δ - continuous surjection which maps ω_δ - open sets onto open set. We give an example shows that this type of paracompact is independent with standard paracompact space.

Keywords: ω_δ -open set, ω_δ -paracompact Space, ω_δ -paracompact subset.

1. Introduction

As defined by J. Dieudonne [1], a space X is said to be paracompact if each open covering has locally finite open refinement. C. H. Dowker [2] generalize this concept and introduced the class of countably paracompact spaces. A space X is said to be countably paracompact if each countable open cover of X has a locally finite open refinement. Generalizing the concept of paracompact spaces, K. Y. Al Zoubi [7] and M. K. Singal and Shashi Prabha Arya [8]. A space X is said to be S - Paracompact if each open cover has a locally finite semi open refinement and a space X is said to be R - Paracompact if each open covering of X of cardinality R has a locally finite open refinement. Since, then a lot of work has been done of S - Paracompact spaces and many interesting results have been obtained [10, 11]. This type of paracompact space is on the set that is different of open set and other types of sets like as ω_p - open sets [4]. The

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object of the present paper is to present some results of new independent type of paracompact spaces.

2. Preliminaries

Throughout this paper a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. A subset G of a space X is called δ -open [8], if for each $x \in G$, there exists an open set U containing x such that $\text{Int}ClU \subseteq G$. For a subset A of a space X , the $\text{Int}_\delta(A)$, $Cl_\delta(A)$ will be denoted the δ -interior and δ -closure of A respectively. A space X is said to be locally-countable [5] if each point of X has a countable open neighborhood. Let (X, τ) be a space, a subset A of X is said to be ω_δ -open set [3] if for each $x \in U$, there exists an open set G containing x such that $G - \text{Int}_\delta U$ is countable. The complement of ω_δ -open set is called ω_δ -closed set. If A is a subset of a space X , then the ω_δ -Interior ($\omega_\delta \text{Int}(A)$) of A is a union of all ω_δ -open sets of X which contained in A and the ω_δ -Closure ($\omega_\delta Cl(A)$) of A is the intersection of all ω_δ -closed sets which containing A .

3. ω_δ -paracompact spaces

The main purpose of this section is to define ω_δ -Paracompact spaces, and obtain some characterizations, properties, and relationships.

A family $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of a space (X, τ) is called ω_δ -locally finite [4], if for each $x \in X$, there exist an ω_δ -open set G containing x such that $\{\lambda \in \Lambda : G \cap A_\lambda \neq \emptyset\}$ is finite.

Definition 3.1. A space X is called an ω_δ -Paracompact space, if each ω_δ -open covering of X has an ω_δ -locally finite ω_δ -open refinement.

Proposition 3.2. A topological space (X, τ) is ω_δ -Paracompact if and only if the topological space $(X, \tau_{\omega_\delta})$ is paracompact.

From [Proposition 3.8, 3], we get the following result:

Proposition 3.3. If a topological space (X, τ) is locally countable, then $(X, \tau_{\omega_\delta})$ is paracompact.

Lemma 3.4. If a covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of a space X has ω_δ -locally finite ω_δ -open refinement, then there exist an ω_δ -locally finite ω_δ -open covering $\{G_\lambda\}_{\lambda \in \Lambda}$ of X such that $G_\lambda \subseteq U_\lambda$, for all $\lambda \in \Lambda$.

Proof. Let $\{V_\gamma\}_{\gamma \in \Gamma}$ be the ω_δ -locally finite ω_δ -open refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$. Therefore, there exists a function $\beta \rightarrow \Lambda$ such that $V_\gamma \subseteq U_{\beta\gamma=\lambda}$, for each $\gamma \in \Gamma$. Let $G_\lambda = \bigcup_{\gamma \in \Gamma, \beta(\gamma)=\lambda} V_\gamma$, then the family $\{G_\lambda\}_{\lambda \in \Lambda}$ is an ω_δ -open covering of X with the property that $G_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Also, $\{U_\lambda\}_{\lambda \in \Lambda}$ is ω_δ -locally finite. If $x \in X$, then there is an ω_δ -open set W containing x such that the set $\Gamma_0 = \{\gamma \in \Gamma : W \cap V_\lambda \neq \emptyset\}$ is finite, since $W \cap G_\lambda \neq \emptyset$ if and only if $\lambda = \beta(\gamma)$,

for some $\gamma \in \Gamma_0$, so the set $\{\lambda \in \Lambda : W \cap G_\lambda \neq \emptyset\}$ is finite. Hence, the proof is complete. $\square$

Corollary 3.5. *A space X is ω_δ -paracompact if and only if for every ω_δ -open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X , there exists an ω_δ -locally finite ω_δ -open covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of X such that $V_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$.*

The ω_δ -boundary of a subset A of a space X ($\omega_\delta b(A)$) is the difference between $\omega_\delta Cl(A)$ and $\omega_\delta Int(A)$.

Corollary 3.6. *If $\{A_\lambda : \lambda \in \Lambda\}$ is an ω_δ -locally finite family of subsets of X , then:*

- 1) $\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is also ω_δ -locally finite and $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$.
- 2) $\omega_\delta b(\cup_{\lambda \in \Lambda} A_\lambda) \subseteq \cup_{\lambda \in \Lambda} \omega_\delta b(A_\lambda)$.

Proof. 1) Let $x \in X$. Since, $\{A_\lambda : \lambda \in \Lambda\}$ is ω_δ -locally finite, so there exists an ω_δ -open set G containing x such that the set $\{\lambda : \lambda \in \Lambda : G \cap A_\lambda \neq \emptyset\}$ is finite. Since, $G \cap A_\lambda = \emptyset$ if and only if $G \cap \omega_\delta Cl(A_\lambda) = \emptyset$, so $\{\lambda : \lambda \in \Lambda : G \cap \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Hence, $\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is ω_δ -locally finite. Since, $\cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\} \subseteq \omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\})$. To prove $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) \subseteq \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Let $x \notin \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Since by what we have proved above $\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$ is ω_δ -locally finite, so there exist an ω_δ -open set U containing x such that $\Lambda_0 = \{\lambda \in \Lambda : U \cap \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Set $V = U \cup (\cap\{X - \omega_\delta Cl(A_\lambda) : \lambda \in \Lambda_0\})$ is ω_δ -open subsets of X containing x such that $V \cap (\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{V \cap A_\lambda : \lambda \in \Lambda\} = \emptyset$. Thus $x \notin \omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\})$, hence $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) \subseteq \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$. Therefore, $\omega_\delta Cl(\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{\omega_\delta Cl(A_\lambda) : \lambda \in \Lambda\}$.

- 2) Since $\omega_\delta b(\cup_{\lambda \in \Lambda} A_\lambda) = \omega_\delta Cl(\cup_{\lambda \in \Lambda} A_\lambda) \cap \omega_\delta Cl(X - \cup_{\lambda \in \Lambda} A_\lambda)$
 $= \cup_{\lambda \in \Lambda} \omega_\delta Cl(A_\lambda) \cap \omega_\delta Cl(X - \cup_{\lambda \in \Lambda} A_\lambda) \subseteq \cup_{\lambda \in \Lambda} (\omega_\delta Cl(A_\lambda) \cap \omega_\delta Cl(X - A_\lambda)) = \cup_{\lambda \in \Lambda} \omega_\delta b(A_\lambda)$.

$\square$

Corollary 3.7. *Let X be an ω_δ -paracompact space, and let H and F be two subsets in which F is an ω_δ -closed subset of X which is disjoint from H . If for every $x \in F$, there exist disjoint ω_δ -open sets U_x and V_x containing x and H , respectively. Then, there are disjoint ω_δ -open sets U and V containing F and H , respectively.*

Proof. Consider the ω_δ -open covering $\{U_x\}_{x \in F} \cup \{X - F\}$ of an ω_δ -paracompact space X . Then by Corollary 3.5, there exists an ω_δ -locally finite ω_δ -open covering $\{G_x\}_{x \in F} \cup \{G\}$ of X such that $G \subseteq X - F$ and $G_x \subseteq U_x$, for each $x \in F$. Since $U_x \cap V_x = \emptyset$, then $G_x \cap V_x = \emptyset$, so $\omega_\delta Cl(G_x) \cap V_x = \emptyset$, for each $x \in F$.

Then by Proposition 3.6, the sets $U = \bigcup_{x \in F} G_x$ and $V = X - \bigcup_{x \in F} \omega_\delta Cl(G_x)$ are the required ω_δ - open sets of X . Thus, completes the proof. $\square$

Definition 3.8 ([4]). *A space X is said to be:*

- 1) $\omega_\delta - T_2$ space, if for each distinct points x and y of X , there exists disjoint ω_δ - open sets U and V containing x and y , respectively.
- 2) $\omega_\delta - T_1$ space, if for each pair of distinct points of X , there exist ω_δ - open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
- 3) ω_δ - regular space, if each ω_δ - closed subset H of X and a point x in X such that $x \notin H$, there exist disjoint ω_δ - open sets U and V containing x and H , respectively.
- 4) ω_δ - normal space, if for each pair of disjoint ω_δ - closed sets A and B in X , there exist disjoint ω_δ - open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Proposition 3.9. *Each ω_δ - paracompact $\omega_\delta - T_2$ (ω_δ - regular) space is an ω_δ - normal space.*

Proof. Let X be an ω_δ - paracompact $\omega_\delta - T_2$ space and let x_0 be any point in X , which is not in arbitrary ω_δ - closed subset F of X . Therefore, for each $x \in F$, there are disjoint ω_δ - open sets U_x and V_x containing x and $\{x_0\}$, so by Proposition 3.7, there exist disjoint ω_δ - open sets U and V containing F and x_0 . This shows that X is ω_δ - regular. Thus, X is ω_δ - paracompact ω_δ - regular space. Let F and H be any two disjoint ω_δ - closed subsets of X . Since, H is ω_δ - closed, so by ω_δ - regularity of X , for each $x \in F$, there exist disjoint ω_δ - open sets U_x and V_x containing x and H . Therefore, by Proposition 3.7, there exist disjoint ω_δ - open sets U and V containing F and H . Thus, X is an ω_δ - normal space. $\square$

Example 3.10. Since the usual space on $\mathbb{R}$ is an $\omega_\delta - T_2$ but not ω_δ - normal space, so by **Proposition 3.9**, it is not ω_δ - paracompact.

Proposition 3.11. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of subsets of a space X and $\{B_\gamma : \gamma \in \Gamma\}$ be an ω_δ - locally finite ω_δ - closed cover of X such that for each $\gamma \in \Gamma$, the set $\{\lambda \in \Lambda : B_\gamma \cap A_\lambda \neq \emptyset\}$ is finite. Then there exists an ω_δ - locally finite family $\{G_\lambda : \lambda \in \Lambda\}$ of ω_δ - open sets of X such that $A_\lambda \subseteq G_\lambda$, for each $\lambda \in \Lambda$.*

Proof. For each λ , let $G_\lambda = X - (\cup\{B_\gamma : B_\gamma \cap A_\lambda \neq \emptyset\})$. Then $A_\lambda \subseteq G_\lambda$, and since $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ - closed sets, so $\omega_\delta Cl(B_\gamma) = B_\gamma$, for all $\gamma \in \Gamma$. Since, $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ - locally finite, so by Proposition 3.6, $\omega_\delta Cl(\bigcup_{\gamma \in \Gamma} B_\gamma) = \bigcup_{\gamma \in \Gamma} \omega_\delta Cl(B_\gamma) = \bigcup_{\gamma \in \Gamma} B_\gamma$, this implies that G_λ is an ω_δ - open sets, for each $\lambda \in \Lambda$. Let $x \in X$. Since, $\{B_\gamma : \gamma \in \Gamma\}$ is ω_δ - locally finite, so there is an ω_δ - open set U containing x such that the set $\gamma_0 = \{\gamma \in \Gamma : U \cap B_\gamma \neq \emptyset\}$ is finite.

Thus, $U \cap B_\gamma = \emptyset$, for each $\gamma \notin \Gamma_0$. Therefore, $U \subseteq \cup\{B_\gamma : \gamma \in \Gamma_0\}$. Also, since for each $\gamma \in \Gamma_0$, $G_\lambda \cap B_\gamma = \emptyset$ if and only if $A_\lambda \cap B_\gamma = \emptyset$. The finiteness of $\{\lambda \in \Lambda : B_\gamma \cap A_\lambda \neq \emptyset\}$ implies the finiteness of $\{\lambda \in \Lambda : U \cap G_\lambda \neq \emptyset\}$. Thus, $\{G_\lambda : \lambda \in \Lambda\}$ is ω_δ -locally finite of ω_δ -open subsets of X . $\square$

Theorem 3.12. *A space X is ω_δ -paracompact ω_δ -normal if and only if every ω_δ -open covering of X has an ω_δ -locally finite ω_δ -closed refinement.*

Proof. Necessity. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an ω_δ -open covering of an ω_δ -paracompact ω_δ -normal space X . So by Corollary 3.5, there exists an ω_δ -locally finite ω_δ -open covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of X such that $V_\lambda \subseteq U_\lambda$, for all $\lambda \in \Lambda$. Since, X is an ω_δ -normal space, then by [Theorem 5.27, 4], there exists an ω_δ -locally finite ω_δ -closed refinement of $\{V_\lambda\}_{\lambda \in \Lambda}$ which also covers X .

Sufficiency. Let X be a space with the property that every ω_δ -open covering of X it has an ω_δ -locally finite ω_δ -closed refinement. Thus, by [Theorem 5.27, 4], X is ω_δ -normal space. It remains only to show that X is ω_δ -paracompact. Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be an ω_δ -open covering of X and $\{F_\gamma\}_{\gamma \in \Gamma}$ be an ω_δ -locally finite ω_δ -closed refinement of $\{W_\lambda\}_{\lambda \in \Lambda}$. Therefore, for each $x \in X$, there exists an ω_δ -open set U_x containing x such that the set $\{\gamma \in \Gamma : U_x \cap F_\gamma \neq \emptyset\}$ is finite. Consider $\{E_\nu\}_{\nu \in \vartheta}$ is an ω_δ -locally finite ω_δ -closed refinement of the ω_δ -open covering $\{U_x\}_{x \in X}$ of X , then for each $\nu \in \vartheta$, the set $\{\gamma \in \Gamma : F_\gamma \cap E_\nu \neq \emptyset\}$ is finite. So by Proposition 3.11, there exists an ω_δ -locally finite family $\{G_\gamma\}_{\gamma \in \Gamma}$ of ω_δ -open sets of X such that $F_\gamma \subseteq G_\gamma$, for each $\gamma \in \Gamma$, which is also covers X . Since, $\{F_\gamma\}_{\gamma \in \Gamma}$ is refinement of $\{W_\lambda\}_{\lambda \in \Lambda}$, so for each $\gamma \in \Gamma$, there is $\lambda(\gamma) \in \Lambda$ such that $F_\gamma \subseteq W_{\lambda(\gamma)}$. Therefore, $\{G_\gamma \cap W_{\lambda(\gamma)}\}_{\gamma \in \Gamma}$ is an ω_δ -locally finite ω_δ -open refinement of $\{W_\lambda\}_{\lambda \in \Lambda}$. Hence, X is ω_δ -paracompact space. $\square$

4. ω_δ -paracompact subset

In this section, we study some results of paracompactness on a subspace.

Proposition 4.1. *Every ω_δ -paracompact, δ -open subset of a space X is an ω_δ -paracompact subspace.*

Proof. Let A be an ω_δ -paracompact, δ -open subset of a space X and let $\{U_\lambda\}_{\lambda \in \Lambda}$ is a covering of A by ω_δ -open subsets of A . Then by [Theorem 3.4, 4], $\{U_\lambda\}_{\lambda \in \Lambda}$ is a covering of A by ω_δ -open subsets of X . By hypothesis, there exists an ω_δ -locally finite ω_δ -open refinement $\{W_\gamma\}_{\gamma \in \Gamma}$ of the family $\{U_\lambda\}_{\lambda \in \Lambda}$ which covers A also. Therefore, by [Proposition 3.6, 4], $\{W_\gamma \cap A\}_{\gamma \in \Gamma}$ is an ω_δ -locally finite ω_δ -open refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$ in A . Thus, A is an ω_δ -paracompact subspace of X . $\square$

Proposition 4.2. *An ω_δ -closed subset of an ω_δ -paracompact space is an ω_δ -paracompact subset.*

Proof. Let F be an ω_δ -closed subset of an ω_δ -paracompact space X and let $\{U_\lambda\}_{\lambda \in \Lambda}$ is a covering of F by ω_δ -open sets of X . Then $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X - F\}$ is an ω_δ -open covering of X . By hypothesis and by Corollary 3.5, there exists an ω_δ -locally finite ω_δ -open covering $\{W_\lambda\}_{\lambda \in \Lambda} \cup \{W\}$ of X such that $W \subseteq X - F$ and $W_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Therefore, $\{W_\lambda\}_{\lambda \in \Lambda}$ is an ω_δ -locally finite ω_δ -open refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$ which covers F . This shows that F is an ω_δ -paracompact relative to X . $\square$

Proposition 4.3. *If a space X is $\omega_\delta - T_2$ space and has a subset F which is ω_δ -paracompact relative to X , then for each $x \in X - F$, there exist two disjoint ω_δ -open sets of X containing x and F .*

Proof. Let F be an ω_δ -paracompact subset of an $\omega_\delta - T_2$ space X and let x be any point of $X - F$. Then for each $y \in F$, there exist ω_δ -open sets U_y and V_y such that $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \emptyset$. This implies that $\omega_\delta Cl(U_y) \cap V_y = \emptyset$. Hence, $x \notin \omega_\delta Cl(U_y)$, for each $y \in F$. Now, $\{U_y\}_{y \in F}$ is cover of F by ω_δ -open subsets of X . Thus, by hypothesis and by Corollary 3.5, there exists an ω_δ -locally finite covering $\{W_y\}_{y \in F}$ of F by ω_δ -open subsets of X such that for each $y \in F$, $W_y \subseteq U_y$. Therefore, $x \notin \omega_\delta Cl(W_y)$, for each $y \in F$. Hence, by Proposition 3.6, $U = \bigcup_{y \in F} W_y$ and $V = X - \bigcup_{y \in F} \omega_\delta Cl(W_y)$, which are ω_δ -open sets which containing F and x respectively. $\square$

From Proposition 4.3, we get the following result:

Corollary 4.4. *Every ω_δ -paracompact subset of an $\omega_\delta - T_2$ space is an ω_δ -closed.*

Corollary 4.5. *Every ω_δ -regular, $\omega_\delta - T_1$ space is an $\omega_\delta - T_2$ space.*

Proposition 4.6. *If X is an ω_δ -regular $\omega_\delta - T_1$ space and F is a subset of X which is ω_δ -paracompact relative to X , then for each ω_δ -open set U in X containing F in X , there exists an ω_δ -closed set H in X containing F and it is contained in U .*

Proof. Since X is ω_δ -regular $\omega_\delta - T_1$ space, so by Corollary 4.5, and Corollary 4.4, F is ω_δ -closed subset of X . Therefore, by [Theorem 5.4, 4], for each $x \in F$, there is ω_δ -open set U_x such that $x \in U_x \subseteq \omega_\delta Cl(U_x) \subseteq U$. Since F is ω_δ -paracompact relative to X , so there exists an ω_δ -locally finite family $\{V_\gamma\}_{\gamma \in \Gamma}$ of F by ω_δ -open sets of X which refines $\{U_x\}_{x \in F}$ and covers F . Therefore, by Proposition 3.6, $H = \bigcup_{\gamma \in \Gamma} \omega_\delta Cl(V_\gamma)$ is required ω_δ -closed set. $\square$

Definition 4.7. *A topological space (X, τ) is called ω_δ -connected space, if X is not a union of two nonempty disjoint ω_δ -open sets, otherwise it is ω_δ -disconnected. Obviously from Definition 4.7, we get the following result:*

Theorem 4.8. *A space X is ω_δ -disconnected if and only if there exists a nonempty proper subset of X which is both ω_δ -open and ω_δ -closed in X .*

Corollary 4.9. *A space X is ω_δ - connected if and only if the only nonempty subset of X which is both ω_δ - open and ω_δ - closed is X itself.*

Theorem 4.10. *Let X be ω_δ - disconnected space, then the following statements are equivalent:*

- 1) X is an ω_δ - paracompact space.
- 2) Every proper ω_δ - closed subset of X is ω_δ - paracompact relative to X .
- 3) Every proper δ - open, ω_δ - closed subset of X is ω_δ - paracompact subspace.
- 4) Every proper δ - open, ω_δ - clopen subset of X is ω_δ - paracompact subspace.
- 5) There exists a proper δ - open, ω_δ - clopen subset F of X such that both F and $X - F$ are ω_δ - paracompact subspace of X .

Proof. $(1 \rightarrow 2)$ and $(2 \rightarrow 3)$, Follows from Proposition 4.2 and Proposition 4.1, respectively.

$(3 \rightarrow 4)$ and $(4 \rightarrow 5)$ are obvious. We prove if (5), we get (1). Let X be a space that contains a proper δ - open, ω_δ - clopen subset F in which both F and $X - F$ are ω_δ - paracompact, and let $\{G_\lambda\}_{\lambda \in \Lambda}$ be any ω_δ - open cover of X . Then $\{F \cap G_\lambda\}_{\lambda \in \Lambda}$ and $\{(X - F) \cap G_\lambda\}_{\lambda \in \Lambda}$ are covers F and $X - F$ by ω_δ - open subset of F and $X - F$, respectively. Therefore, there exist ω_δ - locally finite refinements $\{V_\gamma\}_{\gamma \in \Gamma}$ and $\{V_\nu\}_{\nu \in \vartheta}$ of $\{F \cap G_\lambda\}_{\lambda \in \Lambda}$ and $\{(X - F) \cap G_\lambda\}_{\lambda \in \Lambda}$ covering F and $X - F$, respectively such that V_γ is an ω_δ - open in F , for each $\gamma \in \Gamma$ and V_ν is an ω_δ - open in $X - F$, for each $\nu \in \vartheta$. By [Theorem 3.4, 4], both V_γ and V_ν are ω_δ - open sets in X , for each $\gamma \in \Gamma$ and $\nu \in \vartheta$. Therefore, $\{V_\mu\}_{\mu \in \gamma \cup \vartheta}$ is an ω_δ - locally finite ω_δ - open refinement of $\{G_\lambda\}_{\lambda \in \Lambda}$ which covers X . Hence, X is ω_δ - paracompact space. $\square$

Remark 4.11. From Theorem 4.10, we notice that, if X is ω_δ - connected, then by Corollary 4.9, the only ω_δ - clopen subsets of X are empty set and X itself. So the condition that X is ω_δ - disconnected is essential.

A function $f : X \rightarrow Y$ is said to be an ω_δ - continuous [3], if the inverse image of each open subset of Y is an ω_δ - open subset in X .

Proposition 4.12. *Let $f : X \rightarrow Y$ be an ω_δ - continuous surjection which maps ω_δ - open sets onto open sets. If K is ω_δ - paracompact relative to X , then $f(K)$ is paracompact relative to Y .*

Proof. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be any covering of $f(K)$ by open sets of Y . Since, f is ω_δ - continuous surjection function, then $\{f^{-1}(G_\lambda)\}$ is a covering of $f(K)$ by ω_δ - open subsets of X . But, K is ω_δ - paracompact relative to X , thus, there exists an ω_δ - locally finite ω_δ - open family $\{V_\gamma\}_{\gamma \in \Gamma}$ of subsets of X which refines $\{f^{-1}(G_\lambda)\}_{\lambda \in \Lambda}$ and covers K , so by hypothesis, $\{f(V_\gamma)\}_{\gamma \in \Gamma}$ is a locally finite family of open subsets of Y which refines $\{G_\lambda\}_{\lambda \in \Lambda}$ and covers $f(K)$. Therefore, $f(K)$ is paracompact relative to Y . $\square$

The ω_δ -paracompactness and paracompactness are independent, as shown in the following examples:

Example 4.13. Consider the co-countable topology $(\mathbb{R}, \tau_{COC})$, $(\tau_{COC})_{\omega_\delta} = \{\emptyset, \mathbb{R}\}$, so $\mathbb{R}$ is ω_δ -paracompact, but it is not paracompact.

Example 4.14. Consider the closed unit interval I of the usual topology $(\mathbb{R}, \mathfrak{A})$. Since, I is compact subset of $\mathbb{R}$, so $(I, \mathfrak{A}_I)$ is compact space and hence it is paracompact. Since $(I, \mathfrak{A}_I)$ is $\omega_\delta - T_2$ but not ω_δ -normal, so by Proposition 3.9, it is not ω_δ -paracompact.

Theorem 4.15. *The union of ω_δ -locally finite family of ω_δ -open ω_δ -paracompact subsets of a space X is ω_δ -paracompact subset.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any ω_δ -locally finite family of ω_δ -open, ω_δ -paracompact sets and take $U = \cup\{U_\alpha : \alpha \in I\}$. Let $\{V_\beta : \beta \in J\}$ be any ω_δ -open covering of U , by ω_δ -open subsets of X . Then, for each α , $\{V_\beta \cap U_\alpha : \beta \in J\}$ is a covering of U_α by ω_δ -open sets. Since U_α is ω_δ -paracompact relative to X , then there exist ω_δ -locally finite family of ω_δ -open sets, $\{D_\lambda : \lambda \in K^\alpha\}$ of X which refines $\{V_\beta \cap U_\alpha : \beta \in J\}$ and covers U_α , where K is infinite cardinal. Consider the family $F = \{D_\lambda : \lambda \in K^\alpha, \alpha \in I\}$. Then F is ω_δ -locally finite ω_δ -open refinement of $\{V_\beta : \beta \in J\}$ and hence, U is ω_δ -paracompact relative to X . $\square$

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On a maximal subgroup of the Conway group Co_3

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Abstract. This paper is dealing with a split extension group of the form $3^5:(2 \times M_{11})$, which is a maximal subgroup of the Conway simple group Co_3 . We refer to this extension by $\bar{G}$. We firstly determine the conjugacy classes of $\bar{G}$ using the coset analysis technique. The structures of inertia factor groups were determined through deep investigation on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$. We found the inertia factors to be the groups $2 \times M_{11}$, $A_6 2$ (non-split) and $(S_3 \times S_3):2$. We then determine the Fischer matrices of $\bar{G}$ and apply the Clifford-Fischer theory to compute the ordinary character table of this group. The Fischer matrices of $\bar{G}$ are all integer valued, with sizes ranging from 1 to 4. The full character table of $\bar{G}$ is 37×37 complex valued matrix and is given at the end of this paper.

Keywords: group extensions, Matheiu group, inertia groups, Fischer matrices, character table.

1. Introduction

the Conway group Co_3 is of order 495 766 656 000. From the Atlas [15] we can see that Co_3 has 14 conjugacy classes of maximal subgroups. The fifth largest maximal subgroup is a group of the form $3^5:(2 \times M_{11})$. We refer to this group by $\bar{G}$ and clearly it has order $243 \times 2 \times |M_{11}| = 3\,849\,120$ and index 128 800 in

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Co_3 . In fact the normalizer of an elementary abelian group 3^5 in Co_3 will have the form of $\overline{G}$; that is $N_{Co_3}(3^5) = 3^5:(2 \times M_{11})$. Using GAP [16] we were able to construct this split extension group in terms of permutations of 33 points. We used the coset analysis technique to construct the conjugacy classes of $\overline{G}$, where correspond to the 20 classes of $2 \times M_{11}$, we obtained 37 classes of $\overline{G}$. Then by looking on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$, we were able to determine the structures of the inertia factor groups. Then we computed the Fischer matrices of the extension and we found to be integer valued matrices with sizes ranging from 1 to 4. Finally we were able to compute the ordinary character table of $\overline{G}$ using Clifford-Fischer theory and we supplied it at the end of this paper.

The character table of any finite group extension $\overline{G} = N \cdot G$ (here N is the kernel of the extension and G is isomorphic to $\overline{G}/N$) produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma [14]. Also there is an interesting interplay between the coset analysis and Clifford-Fischer Theory. Indeed the size of each Fischer matrix is $c(g_i)$, the number of $\overline{G}$ -classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. That is computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

By the Atlas of Wilson [25], we see that Co_3 has an absolutely irreducible module of dimension 22 over $\mathbb{F}_2$. Using the two 22×22 matrices over $\mathbb{F}_2$ that generate Co_3 together with the program for obtaining maximal subgroups of Co_3 given in [25] we were able to locate our group $\overline{G}$ inside Co_3 in terms of 22×22 matrices over $\mathbb{F}_2$. The following two elements a and b generate $\overline{G}$.

$$a = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

where $o(a) = 4$, $o(b) = 3$ and $o(ab) = 22$.

For the sake of computations, we used few GAP commands to convert the representation of our group $\overline{G}$ from matrix into permutation representation, where we were able to represent $\overline{G}$ in terms of the set $\{1, 2, \dots, 33\}$. Thus in terms of permutations and using $(c_1, c_2, \dots, c_k)$ to denote the k -cycle $(c_1 \ c_2 \ \dots \ c_k)$, the group $\overline{G}$ is generated by $\overline{g}_1$ and $\overline{g}_2$, where

$$\begin{aligned} \overline{g}_1 &= (1, 4, 23, 29)(2, 15, 19, 10)(3, 16, 24, 33)(6, 25, 26, 31)(7, 12, 17, 21) \\ &\quad (8, 9, 30, 32)(13, 20), \\ \overline{g}_2 &= (1, 3, 12)(2, 22, 18)(4, 25, 32)(5, 11, 21)(6, 23, 13)(7, 9, 28)(8, 19, 33) \\ &\quad (14, 27, 31)(15, 16, 20). \end{aligned}$$

Since $\overline{G}$ can be constructed in GAP, it is easy to obtain all its maximal subgroups. In fact $\overline{G}$ has 7 conjugacy classes of maximal subgroups. The second largest maximal subgroup of $\overline{G}$ is a group of the form $(3^4:(A_6 2)):S_3 := T$, with order 349920 and index 11 in $\overline{G}$. The group T itself has also 7 conjugacy classes of maximal subgroups and the second largest maximal subgroup of T is a group of the form $3^4:(2 \times (A_6 2)) \cong 3^4:(2 \times M_{10}) := M$. The order of M is 116640 and its index in T is 3 (and hence has index 33 in $\overline{G}$). In fact M gives rise to the above permutation representation of $\overline{G}$ on 33 points.

Remark 1. From the Atlas of Wilson [25] we see that Co_3 can be generated in terms of permutations of 276 points and thus any maximal subgroup (including our group $\overline{G}$) will be generated in terms of permutations acting on 276 points. However starting with the 22-dimensional matrix representation over $\mathbb{F}_2$ for Co_3 (available in [25]), then generating $\overline{G}$ in terms of 22×22 matrices over $\mathbb{F}_2$ (for example the above generators a and b) and then using few GAP commands, we were able to generate $\overline{G}$ in terms of permutations on only 33 points. Indeed this will work faster for the required computations.

Having $\overline{G}$ being constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\overline{G}$ possesses three proper normal subgroups of orders 243, 486 and 1924560. The normal subgroup of order 243 is an elementary abelian group isomorphic to N .

Table 1 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$[[g_{ij}]_{\overline{G}}]$	$ C_{\overline{G}}(g_{ij}) $
$g_5 = 3A$	$k_5 = 9$	$f_{51} = 1$	$m_{51} = 27$	g_{51}	3	11880	324
		$f_{52} = 2$	$m_{52} = 54$	g_{52}	3	23760	162
		$f_{53} = 2$	$m_{53} = 54$	g_{53}	9	23760	162
		$f_{54} = 4$	$m_{54} = 108$	g_{54}	9	47520	81
$g_6 = 4A$	$k_6 = 3$	$f_{61} = 1$	$m_{61} = 81$	g_{61}	4	80190	48
		$f_{62} = 2$	$m_{62} = 162$	g_{62}	12	160380	24
$g_7 = 4B$	$k_7 = 9$	$f_{71} = 1$	$m_{71} = 27$	g_{71}	4	26730	144
		$f_{72} = 4$	$m_{72} = 108$	g_{72}	12	106920	36
		$f_{73} = 4$	$m_{73} = 108$	g_{73}	12	106920	36
$g_8 = 5A$	$k_8 = 3$	$f_{81} = 1$	$m_{81} = 81$	g_{81}	5	128304	30
		$f_{82} = 2$	$m_{82} = 162$	g_{82}	15	256608	15
$g_9 = 6A$	$k_9 = 1$	$f_{91} = 1$	$m_{91} = 243$	g_{91}	6	106920	36
$g_{10} = 6B$	$k_{10} = 3$	$f_{10,1} = 1$	$m_{10,1} = 81$	$g_{10,1}$	6	106920	36
		$f_{10,2} = 2$	$m_{10,2} = 162$	$g_{10,2}$	18	213840	18
$g_{11} = 6C$	$k_{11} = 3$	$f_{11,1} = 1$	$m_{11,1} = 81$	$g_{11,1}$	6	106920	36
		$f_{11,2} = 2$	$m_{11,2} = 162$	$g_{11,2}$	6	213840	18
$g_{12} = 8A$	$k_{12} = 3$	$f_{12,1} = 1$	$m_{12,1} = 81$	$g_{12,1}$	8	80190	48
		$f_{12,2} = 2$	$m_{12,2} = 162$	$g_{12,2}$	24	160380	24
$g_{13} = 8B$	$k_{13} = 3$	$f_{13,1} = 1$	$m_{13,1} = 81$	$g_{13,1}$	8	80190	48
		$f_{13,2} = 2$	$m_{13,2} = 162$	$g_{13,2}$	24	160380	24
$g_{14} = 8C$	$k_{14} = 1$	$f_{14,1} = 1$	$m_{14,1} = 243$	$g_{14,1}$	8	240570	16
$g_{15} = 8D$	$k_{15} = 1$	$f_{15,1} = 1$	$m_{15,1} = 243$	$g_{15,1}$	8	240570	16
$g_{16} = 10A$	$k_{16} = 1$	$f_{16,1} = 1$	$m_{16,1} = 243$	$g_{16,1}$	10	384912	10
$g_{17} = 11A$	$k_{17} = 1$	$f_{17,1} = 1$	$m_{17,1} = 243$	$g_{17,1}$	11	349920	22
$g_{18} = 11B$	$k_{18} = 1$	$f_{18,1} = 1$	$m_{18,1} = 243$	$g_{18,1}$	11	349920	22
$g_{19} = 22A$	$k_{19} = 1$	$f_{19,1} = 1$	$m_{19,1} = 243$	$g_{19,1}$	22	349920	22
$g_{20} = 22B$	$k_{20} = 1$	$f_{20,1} = 1$	$m_{20,1} = 243$	$g_{20,1}$	22	349920	22

3. The inertia factor groups of $\overline{G}$

We recall that knowledge of the appropriate character tables of inertia factor groups is crucial in calculating the full character table of any group extension. Since in our extension $\overline{G}$, the normal subgroup 3^5 is abelian and the extension splits, it follows by applications of Mackey's Theorem (see for example Theorem 3.3.4 of Whitley [24]), that every character of 3^5 is extendible to an ordinary character of its respective inertia group $\overline{H}_k$. Thus all the character tables of the inertia factor groups that we will use to construct the character tables of $\overline{G}$ are the ordinary ones. Next we determine the structures of the inertia factor groups.

We have seen from Section 2 that the action of $\overline{G} = 3^5:(2 \times M_{11})$ (which can be reduced to the action of $2 \times M_{11}$) on the classes of $N = 3^5$ yielded three orbits of lengths 1, 110 and 132 (and the corresponding point stabilizers were $2 \times M_{11}$, $3^2:QD_{16}$ and S_5). By a theorem of Brauer (see for example Theorem 5.1.5 of

Mpono [22]), it follows that the action of $\overline{G}$ on $\text{Irr}(N)$ will also produce three orbits. We used Programme C of [23], which can also be found in [20, 21], to determine the lengths of the orbits of $\overline{G}$ or just $2 \times M_{11}$ on $\text{Irr}(N)$. We found that the action of $2 \times M_{11}$ on $\text{Irr}(N)$ produces three orbit of lengths 1, 22 and 220. Let H_1 , H_2 and H_3 be the respective inertia factor groups of the representatives of characters from the orbits with previous lengths. We notice that these inertia factors have indices 1, 22 and 220 respectively in $2 \times M_{11}$. Now since M_{11} has 5 maximal subgroups, the group $2 \times M_{11}$ will have 6 maximal subgroups, namely M_{11} itself together with the direct product of each maximal subgroup of M_{11} by $\mathbb{Z}_2$. Let $T_1, T_2, \dots, T_6$ be representatives of the conjugacy classes of the maximal subgroups of $2 \times M_{11}$. In Table 2, we give few information on these maximal subgroups.

Table 2: The maximal subgroups of $2 \times M_{11}$

T_i	$ T_i $	$[2 \times M_{11} : T_i]$
M_{11}	7920	2
$2 \times M_{10} \cong 2 \times (A_6 2)$	1440	11
$2 \times PSL(2, 11)$	1320	12
$2 \times (3^2:QD_{16})$	288	55
$2 \times S_5$	240	66
$2 \times GL(2, 3)$	96	165

Now the first inertia factor group H_1 of $2 \times M_{11}$ has an index 1 and thus $H_1 = 2 \times M_{11}$ itself. Since we have the character table of M_{11} (see the Atlas) we can easily construct the character table of $2 \times M_{11}$, which we supply below as Table 3.

The second inertia factor group H_2 has index 22 in $2 \times M_{11}$. From Table 2 we can see that the only index of a maximal subgroup that divides 22 is 11. It follows that H_2 is an index 2 subgroup of $2 \times M_{10} \cong 2 \times (A_6 2)$. Now the group $2 \times (A_6 2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6 2$ (twice), $2 \times A_6$, $2 \times (3^2:Q_8)$, $2 \times (5:4)$ and $2 \times QD_{16}$ with respective orders 720 (3 times), 144, 40 and 32, where Q_8 and QD_{16} are the quaternion and quasi-dihedral groups of orders 8 and 16 respectively. Therefore we can see that

$$(2) \quad H_2 \in \{A_6 2, 2 \times A_6\}.$$

Using GAP we were able to construct the character tables of $A_6 2$ and $2 \times A_6$, where below in Table 4 we only supply the character table of $A_6 2$ as the character table of $2 \times A_6$ can be constructed easily from the character tables of $\mathbb{Z}_2$ and of A_6 (available in the Atlas).

Table 3: The character table of $H_1 = G = 2 \times M_{11}$

	1A	2A	2B	2C	3A	4A	4B	5A	6A	6B	6C	8A	8B	8C	8D	10A	11A	11B	22A	22B
$ C_{H_1}(h) $	15840	15840	96	96	36	16	16	10	36	12	12	16	16	16	16	10	22	22	22	22
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	-1
χ_3	10	10	2	2	1	2	2	0	1	-1	-1	0	0	0	0	0	-1	-1	-1	-1
χ_4	10	-10	2	-2	1	2	-2	0	-1	-1	1	0	0	0	0	0	-1	-1	1	1
χ_5	10	-10	-2	2	1	0	0	0	-1	1	-1	A	-A	A	-A	0	-1	-1	1	1
χ_6	10	-10	-2	2	1	0	0	0	-1	1	-1	-A	A	-A	A	0	-1	-1	1	1
χ_7	10	10	-2	-2	1	0	0	0	1	1	1	A	A	-A	-A	0	-1	-1	-1	-1
χ_8	10	10	-2	-2	1	0	0	0	1	1	1	-A	-A	A	A	0	-1	-1	-1	-1
χ_9	11	11	3	3	2	-1	-1	1	2	0	0	-1	-1	-1	-1	1	0	0	0	0
χ_{10}	11	-11	3	-3	2	-1	1	1	-2	0	0	1	-1	-1	1	-1	0	0	0	0
χ_{11}	16	-16	0	0	-2	0	0	1	2	0	0	0	0	0	0	0	-1	-B	-B	B
χ_{12}	16	-16	0	0	-2	0	0	1	2	0	0	0	0	0	0	0	-1	-B	-B	B
χ_{13}	16	16	0	0	-2	0	0	1	-2	0	0	0	0	0	0	0	1	-B	-B	-B
χ_{14}	16	16	0	0	-2	0	0	1	-2	0	0	0	0	0	0	0	1	-B	-B	-B
χ_{15}	44	44	4	4	-1	0	0	-1	-1	1	1	0	0	0	0	-1	0	0	0	0
χ_{16}	44	-44	4	-4	-1	0	0	-1	1	1	-1	0	0	0	0	1	0	0	0	0
χ_{17}	45	45	-3	-3	0	1	1	0	0	0	0	-1	-1	-1	-1	0	1	1	1	1
χ_{18}	45	-45	-3	3	0	1	-1	0	0	0	0	1	-1	-1	1	0	1	1	-1	-1
χ_{19}	55	55	-1	-1	1	-1	-1	0	1	-1	-1	1	1	1	1	0	0	0	0	0
χ_{20}	55	-55	-1	1	1	-1	1	0	-1	-1	1	-1	1	1	-1	0	0	0	0	0

where in Table 3, $A = -i\sqrt{2}$ and $B = \frac{1}{2} - i\frac{\sqrt{11}}{2}$.

Table 4: The character table of $A_6 2$

	1a	2a	3a	4a	4b	5a	8a	8b
$ C_{H_2}(h) $	720	16	9	8	4	5	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	1	-1	-1
χ_3	9	1	0	1	-1	-1	1	1
χ_4	9	1	0	1	1	-1	-1	-1
χ_5	10	2	1	-2	0	0	0	0
χ_6	10	-2	1	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$
χ_7	10	-2	1	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$
χ_8	16	0	-2	0	0	1	0	0

We keep the information that $|\text{Irr}(A_6 2)| = 8$ and $|\text{Irr}(2 \times A_6)| = 14$ in mind and we continue to look at the third inertia factor group H_3 , where later we will determine both H_2 and H_3 simultaneously.

Now the third inertia factor group H_3 has index 220 in $2 \times M_{11}$. Again from Table 2 we can see that the only indices of maximal subgroups that divides 220 are 2, 11 and 55. Therefore H_3 is either

- index 110 subgroup of M_{11} ,
- index 20 subgroup of $2 \times (A_6 2)$ or
- index 4 subgroup of $2 \times (3^2:QD_{16})$.

Now this requires more analysis on the structures of the subgroups of M_{11} , $2 \times (A_6 2)$ and $2 \times (3^2:QD_{16})$. We consider each case of the above:

- From the Atlas we can see that an index 110 subgroup of M_{11} must be either an index 10 subgroup of $M_{10} \cong A_6 2$, or an index 2 subgroup of $3^2:QD_{16}$. Now $A_6 2$ has four maximal subgroups, namely A_6 , $3^2:Q_8$, $5:4$ and QD_{16} of respective orders 360, 72, 20 and 16. Thus H_3 is either a subgroup of A_6 with index 5 or is

isomorphic to $3^2:Q_8$. However the group A_6 does not contain a subgroup of order 72 (the largest maximal subgroup of A_6 is A_5 , which is of order 60). Therefore a subgroup of A_6 of index 10 must necessarily be isomorphic to $3^2:Q_8$.

From another side the group $3^2:QD_{16}$ has four maximal subgroups, namely $3^2:Q_8$, $3^2:8$, $(S_3 \times S_3):2$ and QD_{16} with respective orders 72, 72, 72 and 16. Therefore an index 2 subgroup of $3^2:QD_{16}$ must necessarily be isomorphic to one of the first three maximal subgroups; that is either $3^2:Q_8$, $3^2:8$ or $(S_3 \times S_3):2$. Now we conclude this case by saying that if H_3 is an index 110 subgroup of M_{11} , then $H_3 \in \{3^2:Q_8, 3^2:8, (S_3 \times S_3):2\}$.

- We now consider the case that H_3 is an index 20 subgroup of $2 \times (A_6:2)$. The $2 \times (A_6:2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6:2$ (2 isomorphic non-conjugate copies), $2 \times A_6$, $2 \times (3^2:Q_8)$, $2 \times (5:4)$ and $2 \times QD_{16}$. Because of the order, only the first four maximal subgroups ($A_6:2$, $A_6:2$, $2 \times A_6$ and $2 \times (3^2:Q_8)$) can contain H_3 with respective indices 10, 10, 10 and 2. As of the above case, if H_3 is an index 10 in $A_6:2$ (any of the non-conjugate copies), then H_3 will be isomorphic to a subgroup of the form $3^2:Q_8$.

The group $2 \times A_6$ has 6 conjugacy classes of maximal subgroups, each class represented by A_6 , $2 \times A_5$ (2 isomorphic non-conjugate copies), $2 \times (3^2:4)$ and $2 \times S_4$ (2 isomorphic non-conjugate copies) with respective orders 360, 120 (twice), 72, 48 (twice). We know that A_6 has no index 5 subgroup (see the Atlas). Therefore if H_3 is a subgroup of $2 \times A_6$, then it must be isomorphic to $2 \times (3^2:4)$.

The group $2 \times (3^2:Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$ (3 isomorphic non-conjugate copies), $3^2:Q_8$ (4 isomorphic non-conjugate copies) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Therefore $H_3 \in \{2 \times (3^2:4), 3^2:Q_8\}$.

Now gathering all the possibilities of H_3 in this case, we conclude this case by saying that if H_3 is an index 20 subgroup of $2 \times (A_6:2)$, then $H_3 \in \{3^2:Q_8, 2 \times (3^2:4)\}$.

- Here we consider the case that H_3 is an index 4 of $2 \times (3^2:QD_{16})$. Firstly $2 \times (3^2:QD_{16})$ has 8 conjugacy classes of maximal subgroups, each class represented by $(3^2:8):2$ (4 isomorphic non-conjugate copies), $2 \times (3^2:Q_8)$, $2 \times ((S_3 \times S_3):2)$, $2 \times (3^2:8)$ and $2 \times QD_{16}$ with respective orders 144 (7 times) and 32. We look at the cases $(3^2:8):2$, $2 \times (3^2:Q_8)$, $2 \times ((S_3 \times S_3):2)$ and $2 \times (3^2:8)$.

The group $(3^2:8):2$ has 4 conjugacy classes of maximal subgroups, each class represented by $(S_3 \times S_3):2$, $3^2:Q_8$, $3^2:8$ and QD_6 with respective orders 72 (3 times) and 16. Thus possibilities for H_3 are $(S_3 \times S_3):2$, $3^2:Q_8$ or $3^2:8$.

The group $2 \times (3^2:Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$, $3^2:Q_8$ (4 times), $2 \times (3^2:4)$ (twice) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Thus possibilities for H_3 are $2 \times (3^2:4)$ or $3^2:Q_8$.

The group $2 \times ((S_3 \times S_3):2)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$, $(S_3 \times S_3):2$ (4 times), $2 \times S_3 \times S_3$ (twice) and $2 \times D_8$ with respective orders 72 (7 times) and 16. Thus possibilities for H_3 are $2 \times (3^2:4)$, $(S_3 \times S_3):2$ or $2 \times S_3 \times S_3$.

The group $2 \times (3^2:8)$ has 4 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$, $3^2:8$ (twice) and 8×2 with respective orders 72 (3 times) and 16. Thus the only possibility for H_3 is to be $3^2:8$.

Now we conclude this case by saying that if H_3 is an index 4 subgroup of $2 \times (3^2:QD_{16})$, then $H_3 \in \{3^2:Q_8, 2 \times (3^2:4), (S_3 \times S_3):2, 3^2:8, 2 \times S_3 \times S_3, 3^2:8\}$.

Now combining the possibilities of H_3 in the above three cases, we deduce that

$$(3) \quad H_3 \in \{3^2:Q_8, 3^2:8, (S_3 \times S_3):2, 2 \times (3^2:4), 2 \times S_3 \times S_3\}.$$

Remark 2. Using GAP we were able to obtain that $|\text{Irr}(3^2:Q_8)| = 6$, $|\text{Irr}(3^2:8)| = |\text{Irr}((S_3 \times S_3):2)| = 9$, $|\text{Irr}(2 \times (3^2:4))| = 12$ and $|\text{Irr}(2 \times S_3 \times S_3)| = 18$. Also we recall that $|\text{Irr}(A_6:2)| = 8$ and $|\text{Irr}(2 \times A_6)| = 14$. These information will help us in determining H_2 and eliminating many possibilities for H_3 as we will see in Proposition 3.1 below.

Proposition 3.1. *The group H_2 is $A_6:2$ while H_3 is either $3^2:8$ or $(S_3 \times S_3):2$.*

Proof. By Table 1 we can see that the number of conjugacy classes of $\overline{G}$ is 37 and thus $|\text{Irr}(\overline{G})| = 37$. We have all also indicated before that since the extension $\overline{G}$ splits and the kernel 3^5 is abelian, then all the character tables of the inertia factor groups that we will use to construct the character table of $\overline{G}$ are the ordinary ones. Now by Equations (2) and (3) we have

$$(H_2, H_3) \in \{(A_6:2, 3^2:Q_8), (A_6:2, 3^2:8), (A_6:2, (S_3 \times S_3):2), (A_6:2, 2 \times (3^2:4)), (A_6:2, 2 \times S_3 \times S_3), (2 \times A_6, 3^2:Q_8), (2 \times A_6, 3^2:8), (2 \times A_6, (S_3 \times S_3):2), (2 \times A_6, 2 \times (3^2:4)), (2 \times A_6, 2 \times S_3 \times S_3)\}$$

and respectively we have

$$(4) \quad (|\text{Irr}(H_2)|, |\text{Irr}(H_3)|) \in \{(8, 6), (8, 9), (8, 9), (8, 12), (8, 18), (14, 6), (14, 9), (14, 9), (14, 12), (14, 18)\}.$$

We know that $|\text{Irr}(\overline{G})| = \sum_{i=1}^3 |\text{Irr}(H_i)|$. From Table 3 we have that $|\text{Irr}(H_1)| = 20$. It follows that $|\text{Irr}(H_2)| + |\text{Irr}(H_3)| = |\text{Irr}(\overline{G})| - |\text{Irr}(H_1)| = 37 - 20 = 17$. Now the only possible pairs that satisfying the relation $|\text{Irr}(H_2)| + |\text{Irr}(H_3)| = 17$ are (8, 9) and (8, 9); that is $(H_2, H_3) \in \{(A_6:2, 3^2:8), (A_6:2, (S_3 \times S_3):2)\}$. In either case we can see that $H_2 = A_6:2$ and H_3 is either $3^2:8$ or $(S_3 \times S_3):2$ as claimed. $\square$

The character table supplied in Table 4 is the character table of the second inertia factor group H_2 . We now proceed to determine H_3 , which by Proposition 3.1 is either $3^2:8$ or $(S_3 \times S_3):2$. Using GAP we constructed the character tables of both $3^2:8$ and $(S_3 \times S_3):2$, which we show in Tables 5 and 6 respectively.

Proposition 3.2. *The third inertia factor group H_3 is $(S_3 \times S_3):2$.*

Table 5: The character table of $3^2:8$

	1a	2a	3a	4a	4b	8a	8b	8c	8d
$ C_{3^2:8}(h) $	72	8	9	8	8	8	8	8	8
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	-1	1	A	-A	B	-B	$\overline{B}$	$-\overline{B}$
χ_4	1	-1	1	-A	A	$\overline{B}$	- $\overline{B}$	B	-B
χ_5	1	-1	1	-A	A	$\overline{B}$	- $\overline{B}$	B	-B
χ_6	1	-1	1	A	-A	-B	B	$-\overline{B}$	$\overline{B}$
χ_7	1	1	1	-1	-1	A	A	-A	-A
χ_8	1	1	1	-1	-1	-A	-A	A	A
χ_9	8	0	-1	0	0	0	0	0	0

where in Table 5, $A = -i$ and $B = -E(8)$.

Table 6: The character table of $(S_3 \times S_3):2$

	1a	2a	2b	2c	3a	3b	4a	6a	6b
$ C_{(S_3 \times S_3):2}(h) $	72	12	12	8	18	18	4	6	6
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	-1	-1	1
χ_3	1	1	-1	1	1	1	-1	1	-1
χ_4	1	-1	-1	1	1	1	1	-1	-1
χ_5	2	0	0	-2	2	2	0	0	0
χ_6	4	2	0	0	-2	1	0	-1	0
χ_7	4	-2	0	0	-2	1	0	1	0
χ_8	4	0	-2	0	1	-2	0	0	1
χ_9	4	0	2	0	1	-2	0	0	-1

Proof. By Table 1 we can see that the class $g_{10} \in [6B]_{2 \times M_{11}}$ affords 2 conjugacy classes in $\overline{G}$. This implies that $c(g_{10}) = 2$ and hence the corresponding Fischer matrix $\mathcal{F}_{10}$ to this class will be of size 2. By description of Fischer matrices (for example see [2]), the first row of $\mathcal{F}_{10}$ will correspond to the first inertia factor group $H_1 = 2 \times M_{11}$. Thus the second row of $\mathcal{F}_{10}$ must correspond to either H_2 or H_3 depending on the fusions. From Tables 4 and 5 we can see that neither H_2 nor $3^2:8$ has an element of order 6 so that it fuses into $[6B]_{2 \times M_{11}}$. This argument shows that $H_3 \neq 3^2:8$. Since by Proposition 3.1, $H_3 \in \{3^2:8, (S_3 \times S_3):2\}$ and we just established that $H_3 \neq 3^2:8$, it follows that $H_3 = (S_3 \times S_3):2$. Hence the result. $\square$

We summarize that the inertia factor groups are $H_1 = G = 2 \times M_{11}$, $H_2 = A_6 2$ (non-split) and $H_3 = (S_3 \times S_3):2$.

Next we turn to determine the fusions of classes of the inertia factor groups of the extension into the classes of $G = 2 \times M_{11}$. We have used the permutation characters of G on the inertia factor groups and the centralizer sizes to determine the fusions of these inertia factors into G . We list these fusions in Tables 7 and 8.

Table 7: The fusion of H_2 into $2 \times M_{11}$

$[g]_{H_2}$	$\longrightarrow$	$[x]_{2 \times M_{11}}$	$[g]_{H_2}$	$\longrightarrow$	$[x]_{2 \times M_{11}}$
1a		1A	4b		4B
2a		2B	5a		5A
3a		3A	8a		8A
4a		4A	8b		8B

Table 8: The fusion of H_3 into $2 \times M_{11}$

$[g]_{H_3}$	$\longrightarrow$	$[x]_{2 \times M_{11}}$	$[g]_{H_3}$	$\longrightarrow$	$[x]_{2 \times M_{11}}$
$1a$		$1A$	$3b$		$3A$
$2a$		$2B$	$4a$		$4B$
$2b$		$2B$	$6a$		$6B$
$2c$		$2C$	$6b$		$6C$
$3a$		$3A$			

4. Fischer matrices of $\overline{G}$

In this section, we use the arithmetical properties of Fischer matrices, given in Proposition 3.6 of [3], to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. To build these systems of equations, we firstly recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i , by the sizes of the centralizers of g_{ij} , $1 \leq j \leq c(g_i)$ in $\overline{G}$ and m_{ij} respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and m_{ij} , $1 \leq i \leq 20$, $1 \leq j \leq c(g_i)$. Also having obtained the fusions of the inertia factor groups into $2 \times M_{11}$, we are able to label the rows of the Fischer matrices as described in [2, 3].

Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 4 for all $i \in \{1, 2, \dots, 20\}$. Now with the help of the symbolic mathematical package Maxima [17], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, where we found that all these matrices are integer valued. Below we list these matrices.

$\mathcal{F}_1$				
g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	3	3
$ C_{\overline{G}}(g_{1j}) $		3849120	34992	29160
(k, m)	$ C_{H_k}(g_{1km}) $			
$(1, 1)$	15840	1	1	1
$(2, 1)$	720	22	-5	4
$(3, 1)$	72	220	4	-5
m_{1j}		1	110	132

$\mathcal{F}_2$	
g_2	g_{21}
$o(g_{2j})$	2
$ C_{\overline{G}}(g_{2j}) $	15840
(k, m)	$ C_{H_k}(g_{2km}) $
$(1, 1)$	15840
m_{2j}	243

$\mathcal{F}_3$					
g_3		g_{31}	g_{32}	g_{33}	g_{34}
$o(g_{3j})$		2	6	6	6
$ C_{\overline{G}}(g_{3j}) $		2592	432	324	216
(k, m)	$ C_{H_k}(g_{3km}) $				
$(1, 1)$	96	1	1	1	1
$(2, 1)$	16	6	3	-3	0
$(3, 1)$	12	8	-4	-1	2
$(3, 2)$	8	12	0	3	-3
m_{3j}		9	54	72	108

$\mathcal{F}_4$		g_{41}	g_{42}
g_4		2	6
$o(g_{4j})$		864	108
$ C_{\overline{G}}(g_{4j}) $			
(k, m)	$ C_{H_k}(g_{4km}) $		
$(1, 1)$	96	1	1
$(3, 1)$	12	8	-1
m_{4j}		27	216

$\mathcal{F}_5$	
g_5	$g_{51} \ g_{52} \ g_{53} \ g_{54}$
$o(g_{5j})$	3 3 9 9
$ C_{\overline{G}}(g_{5j}) $	324 162 162 81
$(k, m) \quad C_{H_k}(g_{5km}) $	
(1, 1) 36	1 1 1 1
(2, 1) 9	4 -2 -2 1
(3, 1) 18	2 0 -1 -1
(3, 2) 18	2 1 2 -1
m_{5j}	27 54 54 108

$\mathcal{F}_7$	
g_7	$g_{71} \ g_{72} \ g_{73}$
$o(g_{7j})$	4 12 12
$ C_{\overline{G}}(g_{7j}) $	144 36 36
$(k, m) \quad C_{H_k}(g_{7km}) $	
(1, 1) 16	1 1 1
(2, 1) 4	4 -2 1
(3, 1) 4	4 1 -2
m_{7j}	27 108 108

$\mathcal{F}_9$	
g_9	g_{91}
$o(g_{9j})$	6
$ C_{\overline{G}}(g_{9j}) $	36
$(k, m) \quad C_{H_k}(g_{9km}) $	
(1, 1) 36	1
m_{9j}	243

$\mathcal{F}_{11}$	
g_{11}	$g_{11,1} \ g_{11,2}$
$o(g_{11j})$	6 6
$ C_{\overline{G}}(g_{11j}) $	36 18
$(k, m) \quad C_{H_k}(g_{11km}) $	
(1, 1) 12	1 1
(3, 1) 6	2 -1
m_{11j}	81 162

$\mathcal{F}_{13}$	
g_{13}	$g_{13,1} \ g_{13,2}$
$o(g_{13j})$	8 24
$ C_{\overline{G}}(g_{13j}) $	48 24
$(k, m) \quad C_{H_k}(g_{13km}) $	
(1, 1) 16	1 1
(2, 1) 8	2 -1
m_{13j}	81 162

$\mathcal{F}_{15}$	
g_{15}	$g_{15,1}$
$o(g_{15j})$	8
$ C_{\overline{G}}(g_{15j}) $	16
$(k, m) \quad C_{H_k}(g_{15km}) $	
(1, 1) 16	1
m_{15j}	243

$\mathcal{F}_{17}$	
g_{17}	$g_{17,1}$
$o(g_{17j})$	11
$ C_{\overline{G}}(g_{17j}) $	22
$(k, m) \quad C_{H_k}(g_{17km}) $	
(1, 1) 22	1
m_{17j}	243

$\mathcal{F}_{19}$	
g_{19}	$g_{19,1}$
$o(g_{19j})$	22
$ C_{\overline{G}}(g_{19j}) $	22
$(k, m) \quad C_{H_k}(g_{19km}) $	
(1, 1) 22	1
m_{19j}	243

$\mathcal{F}_6$	
g_6	$g_{61} \ g_{62}$
$o(g_{6j})$	4 12
$ C_{\overline{G}}(g_{6j}) $	864 108
$(k, m) \quad C_{H_k}(g_{6km}) $	
(1, 1) 16	1 1
(2, 1) 8	2 -1
m_{6j}	81 162

$\mathcal{F}_8$	
g_8	$g_{81} \ g_{82}$
$o(g_{8j})$	5 15
$ C_{\overline{G}}(g_{8j}) $	30 15
$(k, m) \quad C_{H_k}(g_{8km}) $	
(1, 1) 10	1 1
(2, 1) 5	2 -1
m_{8j}	81 162

$\mathcal{F}_{10}$	
g_{10}	$g_{10,1} \ g_{10,2}$
$o(g_{10j})$	6 18
$ C_{\overline{G}}(g_{10j}) $	36 18
$(k, m) \quad C_{H_k}(g_{10km}) $	
(1, 1) 12	1 1
(3, 1) 6	2 -1
m_{10j}	81 162

$\mathcal{F}_{12}$	
g_{12}	$g_{12,1} \ g_{12,2}$
$o(g_{12j})$	8 24
$ C_{\overline{G}}(g_{12j}) $	48 24
$(k, m) \quad C_{H_k}(g_{12km}) $	
(1, 1) 16	1 1
(2, 1) 8	2 -1
m_{12j}	81 162

$\mathcal{F}_{14}$	
g_{14}	$g_{14,1}$
$o(g_{14j})$	8
$ C_{\overline{G}}(g_{14j}) $	16
$(k, m) \quad C_{H_k}(g_{14km}) $	
(1, 1) 16	1
m_{14j}	243

$\mathcal{F}_{16}$	
g_{16}	$g_{16,1}$
$o(g_{16j})$	10
$ C_{\overline{G}}(g_{16j}) $	10
$(k, m) \quad C_{H_k}(g_{16km}) $	
(1, 1) 10	1
m_{16j}	243

$\mathcal{F}_{18}$	
g_{18}	$g_{18,1}$
$o(g_{18j})$	11
$ C_{\overline{G}}(g_{18j}) $	22
$(k, m) \quad C_{H_k}(g_{18km}) $	
(1, 1) 22	1
m_{18j}	243

$\mathcal{F}_{20}$	
g_{20}	$g_{20,1}$
$o(g_{20j})$	22
$ C_{\overline{G}}(g_{20j}) $	22
$(k, m) \quad C_{H_k}(g_{20km}) $	
(1, 1) 22	1
m_{20j}	243

5. The character tables of $\overline{G}$

Throughout Sections 2, 3 and 4 we have found:

- the conjugacy classes of $\overline{G}$ (Table 1),
- the inertia factor groups H_1 , H_2 and H_3 of $\overline{G}$ and their character tables (Tables 3, 4 and 6). Also we obtained the fusions of classes of the inertia factors H_2 and H_3 of $\overline{G}$ into $2 \times M_{11}$ (Tables 7 and 8),
- the Fischer matrices of $\overline{G}$ (Section 4).

It follows by [2, 3] that the full character table of $\overline{G}$ can be constructed easily in the format of Clifford-Fischer theory. This table will be partitioned into 60 parts corresponding to the 20 cosets and the three inertia factor groups. The full character table of $\overline{G}$ is 37×37 $\mathbb{C}$ -valued matrix. In Table 9, we supply the character table of $\overline{G}$ in the format of Clifford-Fischer Theory. In this table we have also included the fusions of the conjugacy classes of $\overline{G}$ into the conjugacy classes of the Conway group Co_3 , where the classes of Co_3 as in the Atlas. Finally we would like to remark that the accuracy of this character table has been tested using GAP.

Table 9: The character table of $\overline{G}$

$[g_i]_{\overline{G}}$	1A			2A	2B				2C	3A				4A	4B				
$[g_{ij}]_{\overline{G}}$	1a	3a	3b	2a	2b	6a	6b	6c	2c	6d	3c	3d	9a	9b	4a	12a	4b	12b	12c
$ C_{\overline{G}}(g_{ij}) $	3849120	34992	29160	15840	2592	432	324	216	1184	108	324	162	162	81	48	24	144	36	36
$\hookrightarrow Co_3$	1A	3A	3B	2B	2A	6A	6B	6C	2B	6D	3B	3C	9A	9B	4B	12B	4A	12A	12C
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	-1	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1	-1
X3	10	10	10	-10	2	2	2	2	-2	-2	1	1	1	1	2	2	-2	-2	-2
X4	10	10	10	10	2	2	2	2	2	2	1	1	1	1	2	2	2	2	2
X5	10	10	10	-10	-2	-2	-2	-2	2	2	1	1	1	1	0	0	0	0	0
X6	10	10	10	-10	-2	-2	-2	-2	2	2	1	1	1	1	0	0	0	0	0
X7	10	10	10	10	-2	-2	-2	-2	-2	-2	1	1	1	1	0	0	0	0	0
X8	10	10	10	10	-2	-2	-2	-2	-2	-2	1	1	1	1	0	0	0	0	0
X9	11	11	11	-11	3	3	3	3	-3	-3	2	2	2	2	-1	-1	1	1	1
X10	11	11	11	11	3	3	3	3	3	3	2	2	2	2	-1	-1	-1	-1	-1
X11	16	16	16	-16	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
X12	16	16	16	-16	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
X13	16	16	16	16	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
X14	16	16	16	16	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
X15	44	44	44	-44	4	4	4	4	-4	-4	-1	-1	-1	-1	0	0	0	0	0
X16	44	44	44	44	4	4	4	4	4	4	-1	-1	-1	-1	0	0	0	0	0
X17	45	45	45	-45	-3	-3	-3	-3	3	3	0	0	0	0	1	1	-1	-1	-1
X18	45	45	45	45	-3	-3	-3	-3	-3	-3	0	0	0	0	1	1	1	1	1
X19	55	55	55	-55	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	1	1	1
X20	55	55	55	55	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1
X21	22	-5	4	0	6	3	-3	0	0	0	4	-2	-2	1	2	-1	4	-2	1
X22	22	-5	4	0	6	3	-3	0	0	0	4	-2	-2	1	2	-1	-4	2	-1
X23	198	-45	36	0	6	3	-3	0	0	0	0	0	0	0	2	-1	4	-2	1
X24	198	-45	36	0	6	3	-3	0	0	0	0	0	0	0	2	-1	-4	2	-1
X25	220	-50	40	0	-12	-6	6	0	0	0	4	-2	-2	1	0	0	0	0	0
X26	220	-50	40	0	-12	-6	6	0	0	0	4	-2	-2	1	0	0	0	0	0
X27	220	-50	40	0	12	6	-6	0	0	0	4	-2	-2	1	-4	2	0	0	0
X28	352	-80	64	0	0	0	0	0	0	0	-8	4	4	-2	0	0	0	0	0
X29	220	4	-5	0	20	-4	2	-1	-8	1	4	1	1	-2	0	0	-4	-1	2
X30	220	4	-5	0	20	-4	2	-1	8	-1	4	1	1	-2	0	0	4	1	-2
X31	220	4	-5	0	4	4	4	-5	8	-1	4	1	1	-2	0	0	-4	-1	2
X32	220	4	-5	0	4	4	4	-5	-8	1	4	1	1	-2	0	0	4	1	-2
X33	440	8	-10	0	-24	0	-6	6	0	0	8	2	2	-4	0	0	0	0	0
X34	880	16	-20	0	-16	8	2	-4	0	0	-2	-2	-5	1	0	0	0	0	0
X35	880	16	-20	0	0	0	0	0	-16	2	-2	-5	4	1	0	0	0	0	0
X36	880	16	-20	0	0	0	0	0	16	-2	-2	-5	4	1	0	0	0	0	0
X37	880	16	-20	0	16	-8	-2	4	0	0	-2	4	-5	1	0	0	0	0	0

$[g_i]_G$	5A	6A	6B	6C	8A	8B	8C	8D	10A	11A	11B	22A	22B
$[g_{ij}]_{\overline{G}}$	5a 15a	6e	6f 18a	6g 6h	8a 24a	8b 24b	8c	8d	10a	11a	11b	22a	22b
$ C_{\overline{G}}(g_{ij}) $	30 15	36	36 18	18 36	48 24	48 24	16	16	10	22	22	22	22
$\hookrightarrow C_{O_3}$	5B 15B	6D	6C 18A	6E 6D	8A 24A	8A 24A	8C	8C	10B	11A	11B	22B	22B
X1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1
X3	0	0	-1	-1	-1	1	1	0	0	0	0	-1	-1
X4	0	0	1	-1	-1	-1	-1	0	0	0	0	-1	-1
X5	0	0	-1	1	1	-1	-1	A	A	-A	-A	-A	A
X6	0	0	-1	1	1	-1	-1	-A	-A	A	A	A	-A
X7	0	0	1	1	1	1	1	A	A	-A	-A	A	-A
X8	0	0	1	1	1	1	1	-A	-A	A	A	-A	A
X9	1	1	-2	0	0	0	0	1	1	1	1	-1	-1
X10	1	1	2	0	0	0	0	-1	-1	-1	-1	-1	-1
X11	1	1	2	0	0	0	0	0	0	0	0	-1	C
X12	1	1	2	0	0	0	0	0	0	0	0	-1	C
X13	1	1	-2	0	0	0	0	0	0	0	0	1	C
X14	1	1	-2	0	0	0	0	0	0	0	0	1	C
X15	-1	-1	1	1	1	-1	-1	0	0	0	0	0	1
X16	-1	-1	-1	1	1	1	1	0	0	0	0	0	-1
X17	0	0	0	0	0	0	0	1	1	1	1	-1	-1
X18	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1
X19	0	0	-1	-1	-1	1	1	-1	-1	1	1	0	0
X20	0	0	1	-1	-1	-1	-1	1	1	1	1	0	0
X21	2	-1	0	0	0	0	0	2	-1	2	-1	0	0
X22	2	-1	0	0	0	0	0	-2	1	-2	1	0	0
X23	-2	1	0	0	0	0	0	-2	1	-2	1	0	0
X24	-2	1	0	0	0	0	0	2	-1	2	-1	0	0
X25	0	0	0	0	0	0	0	B	A	-B	-A	0	0
X26	0	0	0	0	0	0	0	-B	-A	B	A	0	0
X27	0	0	0	0	0	0	0	0	0	0	0	0	0
X28	2	-1	0	0	0	0	0	0	0	0	0	0	0
X29	0	0	0	2	-1	1	-2	0	0	0	0	0	0
X30	0	0	0	2	-1	-1	2	0	0	0	0	0	0
X31	0	0	0	-2	1	-1	2	0	0	0	0	0	0
X32	0	0	0	-2	1	1	-2	0	0	0	0	0	0
X33	0	0	0	0	0	0	0	0	0	0	0	0	0
X34	0	0	0	2	-1	0	0	0	0	0	0	0	0
X35	0	0	0	0	0	-1	2	0	0	0	0	0	0
X36	0	0	0	0	0	1	-2	0	0	0	0	0	0
X37	0	0	0	-2	1	0	0	0	0	0	0	0	0

where in Table 9, $A = -i\sqrt{2}$, $B = 2i\sqrt{2}$ and $C = \frac{1}{2} - i\frac{\sqrt{11}}{2} = b11$.

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Minimal left ideals in semirings

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Abstract. Minimal left ideals in semirings are investigated.

Keywords: semiring, semimodule, semilattice, ideal, endomorphism.

Simple semirings appear in many fields of mathematics. In order to characterize simple semirings with bi-absorbing element, it is necessary to have some information on minimal left ideals. In this note, basic properties of minimal left ideals in semirings are investigated.

1. Prerequisites

1.1 Semigroups. Let $S = S(*)$ be a semigroup. If A, B are subsets of S then $A * B = \{a * b \mid a \in A, b \in B\}$. An element $a \in S$ is called *left/right neutral (absorbing)* if $a * b = b$ / $b * b = b$ ($a * b = a$ / $b * a = a$) for every $b \in S$. We denote by $\underline{L}(S(*))$ ($\underline{R}(S(*))$) the set of left (right) absorbing elements and we put $\underline{L}^+(S(*)) = \{a \in S \mid |a * S| = 1\}$ ($\underline{R}^+(S(*)) = \{a \in S \mid |S * a| = 1\}$).

The semigroup S is called

- *left (right) cancellative* if $a * b \neq a * c$ ($b * a \neq c * a$) for all $a, b, c \in S$, $b \neq c$,
- *idempotent* if $a * a = a$ for every $a \in S$,

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- *constant* if $|S * S| = 1$.

If the semigroup S is commutative then, for $a, b \in S$, we write $a \leq b$ iff $b \in (a * S) \cup \{a\}$. We write $a < b$ iff $a \leq b$ and $a \neq b$.

1.2 Semirings. A *semiring* is an algebraic structure with two associative binary operations, usually denoted as addition and multiplication, where the addition is commutative and the multiplication distributes over the addition.

Let S be a semiring. Then $0_S \in S$ ($o_S \in S$) means that S has the uniquely determined additively neutral (absorbing) element 0_S (o_S). On the other hand, $0_S \notin S$ ($o_S \notin S$) means that no such element exists in S . Similarly, $1_S \in S$ means that S contains the uniquely determined multiplicatively neutral element 1_S . There is no specific way how to denote multiplicatively absorbing elements, but if the additively neutral element 0_S is multiplicatively absorbing then 0_S is called the *zero* element. The set of left (right) multiplicatively absorbing elements will be denoted by $\underline{L}(S)$ ($\underline{R}(S)$) in the sequel.

The semiring S is called *congruence-simple* if it has just two congruence relations. The following basic classification of congruence-simple semirings can be found in [1], Theorem 2.1 and Lemma 1.4.

Proposition 1.2.1. *Let S be a congruence-simple semiring. Then just one of the following four cases takes place:*

1. S is additively cancellative;
2. S is additively idempotent;
3. S is additively constant;
4. $o_S \in S$, $2S = \{o_S\}$ and $S + S = S$ (and S is infinite).

Every two-element semiring is congruence-simple and there are just ten two-element semirings up to isomorphism:

$S_1(+)$	$\begin{array}{c cc} & 0 & a \\ \hline 0 & 0 & a \\ a & a & 0 \end{array}$	$S_1(\cdot)$	$\begin{array}{c cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & a \end{array}$	$S_2(+)$	$\begin{array}{c cc} & 0 & a \\ \hline 0 & 0 & a \\ a & a & 0 \end{array}$	$S_2(\cdot)$	$\begin{array}{c cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & 0 \end{array}$

$S_3(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_3(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & 0 \\ o & 0 & 0 \end{array}$	$S_4(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_4(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & o & o \\ o & o & o \end{array}$

$S_5(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_5(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & 0 \\ o & 0 & o \end{array}$	$S_6(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_6(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$

$S_7(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_7(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & 0 \\ o & o & o \end{array}$	$S_8(+)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & o & o \end{array}$	$S_8(\cdot)$	$\begin{array}{c cc} & 0 & o \\ \hline 0 & 0 & o \\ o & 0 & o \end{array}$
$S_9(+)$	$\begin{array}{c cc} & a & o \\ \hline a & o & o \\ o & o & o \end{array}$	$S_9(\cdot)$	$\begin{array}{c cc} & a & o \\ \hline a & o & o \\ o & o & o \end{array}$	$S_{10}(+)$	$\begin{array}{c cc} & a & o \\ \hline a & o & o \\ o & o & o \end{array}$	$S_{10}(\cdot)$	$\begin{array}{c cc} & a & o \\ \hline a & a & o \\ o & o & o \end{array}$

1.3 Semimodules. Let S be a semiring. A (*left S -*) *semimodule* is a commutative semigroup M ($= M(+)$) together with a scalar multiplication $S \times M \rightarrow M$. The semimodule M is called

- *minimal* if $|M| \geq 2$ and $N = M$ whenever N is a non-trivial subsemimodule of M ;
- *simple* if it has just two congruence relations;
- *faithful* if for all $a, b \in S$, $a \neq b$, there is at least one $x \in M$ with $ax \neq bx$.

2. Ideals-basic facts

Let S be a non-trivial semiring. A non-empty subset I of S is called

- a *left (right) ideal* if $(I + I) \cup SI \subseteq I$ ($(I + I) \cup IS \subseteq I$);
- an *ideal* if $(I + I) \cup SI \cup IS \subseteq I$;
- a *bi-ideal* if $(S + I) \cup SI \cup IS \subseteq I$.

A (left, right, bi-) ideal is said to be

- *trivial* if $|I| = 1$;
- *proper* if $I \neq S$;
- *minimal* if it is non-trivial and $K = I$ whenever K is a non-trivial (left, right, bi-) ideal such that $K \subseteq I$.

The semiring S is called

- (*left, right, bi-*) *ideal-simple* if S has no proper non-trivial (left, right, bi-) ideal;
- (*left, right, bi-*) *ideal-free* if S has no proper (left, right, bi-) ideal.

The following four assertions are straightforward.

Lemma 2.1. *Let $a \in S$. Then:*

- (i) *The one-element set $\{a\}$ is a left (right) ideal of S iff the element a is right (left) multiplicatively absorbing.*
- (ii) *The set $\{a\}$ is a bi-ideal iff a is bi-absorbing.* □

Lemma 2.2. *Assume that $\underline{L}(S) \neq \emptyset$ ($\underline{R}(S) \neq \emptyset$). Then:*

- (i) *$\underline{L}(S)$ ($\underline{R}(S)$) is an ideal and it is the smallest left (right) ideal of S .*
- (ii) *$|\underline{L}(S)| = 1$ iff $|\underline{R}(S)| = 1$ and then $\underline{L}(S) = \{a\} = \underline{R}(S)$, where a is multiplicatively absorbing.*
- (iii) *If $|\underline{L}(S)| \geq 2$ ($|\underline{R}(S)| \geq 2$) then $\underline{R}(S) = \emptyset$ ($\underline{L}(S) = \emptyset$).*
- (iv) *$\underline{L}(S)$ ($\underline{R}(S)$) is bi-idempotent.*
- (v) *If $a, b \in \underline{L}(S)$ ($a, b \in \underline{R}(S)$) are such that $a < b$ then the two-element set $\{a, b\}$ is a minimal right (left) ideal.* □

Lemma 2.3. (i) *$\underline{L}(S) \subseteq \underline{L}^+(S)$ ($\underline{R}(S) \subseteq \underline{R}^+(S)$) and $\underline{L}^+(S)$ ($\underline{R}^+(S)$) is an ideal of S , provided that $\underline{L}^+(S) \neq \emptyset$ ($\underline{R}^+(S) \neq \emptyset$).*

- (ii) *There is a mapping $\alpha : \underline{L}^+(S) \rightarrow \underline{L}(S)$ ($\alpha : \underline{R}^+(S) \rightarrow \underline{R}(S)$) such that $aS = \{\alpha(a)\}$ ($Sa = \{\alpha(a)\}$) for every $a \in \underline{L}^+(S)$ ($a \in \underline{R}^+(S)$).*
- (iii) *$\alpha|_{\underline{L}(S)} = \text{id}$ ($\alpha|_{\underline{R}(S)} = \text{id}$).*
- (iv) *$\alpha^2 = \alpha$, $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(ca) = c\alpha(a)$ ($\alpha(ac) = \alpha(a)c$) for all $a, b \in \underline{L}^+(S)$ ($a, b \in \underline{R}^+(S)$) and $c \in S$.* □

Lemma 2.4. *If I is a bi-ideal of S then the relation $(I \times I) \cup \text{id}_S$ is a congruence of the semiring S . In particular, the semiring S is bi-ideal-simple, provided that it is congruence-simple.* □

3. Minimal left ideals-basic observations (a)

In this part, let S be a semiring and K be a minimal left ideal of S . We put $A = \{a \in K \mid Sa = K\}$, $B = K \cap \underline{R}^+(S) = \{a \in K \mid |Sa| = 1\}$, $C = K \cap \underline{R}(S) = \{a \in K \mid Sa = \{a\}\}$, $D = \{a \in K \mid Ka = K\}$, $E = \{a \in K \mid |Ka| = 1\}$ and $F = \{a \in K \mid Ka = \{a\}\}$.

The following seven assertions are easy.

Lemma 3.1. (i) *$A \cup B = K$ and $A \cap B = \emptyset$.*

- (ii) *B is a left ideal of S , provided that $B \neq \emptyset$.*
- (iii) *Either $B = \emptyset$ or $|B| = 1$ or $B = K$.*
- (iv) *$C \subseteq B$, $\underline{R}(S)K \subseteq C$ and $C \neq \emptyset$ iff $\underline{R}(S) \neq \emptyset$.*
- (v) *$D \cup E = K$ and $D \cap E = \emptyset$.*
- (vi) *$D \subseteq A$, $B \subseteq E$ and $C \subseteq F \subseteq E$.*
- (vii) *If $E \neq \emptyset$ then E is an ideal of K .* □

Lemma 3.2. *Let $B = K$ (see 3.1(iii)). Then just one of the following five cases holds:*

1. $K \subseteq \underline{R}(S)$, $|K| = 2$ and $K = \{0_K, o_K\}$;
2. $SK = \{0_K\}$ and $K = \{0_K, o_K\}$;
3. $SK = \{o_K\}$ and $K = \{0_K, o_K\}$;
4. $SK = \{o_K\} = K + K$ and $|K| = 2$;
5. $SK = \{0_K\}$ and $K(+)$ is a finite cyclic group of prime order. $\square$

Lemma 3.3. *Let $B = K$ and $|K| \geq 3$. Then $SK = \{0_K\}$ and $K(+)$ is a finite cyclic group of prime odd order.* $\square$

Lemma 3.4. *Let $|B| = 1$ (see 3.1(iii)). Then just one of the following two cases holds:*

1. $B = \{0_K\}$, $0_K K = \{0_K\} = S0_K$ and $Sa = K$ for every $a \in K \setminus \{0_K\}$. Besides, either $0_K S \subseteq \underline{R}(S)$ and $|0_K S| \geq 2$ or 0_K is multiplicatively absorbing in S , $K \subseteq S + 0_K$ and $S + 0_K$ is a bi-ideal of S ;
2. $B = \{o_K\}$, $o_K K = \{o_K\} = So_K$ and $Sa = K$ for every $a \in K \setminus \{o_K\}$. Besides, either $o_K S \subseteq \underline{R}(S)$ and $|o_K S| \geq 2$, or o_K is multiplicatively absorbing in S , $K \cap (S + o_K) = \{o_K\}$ and $S + o_K$ is a bi-ideal of S . $\square$

Lemma 3.5. *Let $B = \{0_K\}$ (see 3.4). Then:*

- (i) $E = \{a \in K \mid Ka = \{0_K\}\}$.
- (ii) $0_K \in L = \{a \in S \mid aK = \{0_K\}\}$ and L is an ideal of S .
- (iii) If $K \subseteq L$ then $KK = \{0_K\}$, $E = K$ and $K \times K = \varrho$, where ϱ is the congruence of S defined by $(a, b) \in \varrho$ iff $ac = bc$ for every $c \in K$.
- (iv) If $K \not\subseteq L$ then $LK = K \cap L = \{0_K\}$. $\square$

Lemma 3.6. *Let $B = \{o_K\}$ (see 3.4). Then:*

- (i) $E = \{a \in K \mid Ka = \{o_K\}\}$.
- (ii) $o_K \in L = \{a \in S \mid aK = \{o_K\}\}$ and L is an ideal of S .
- (iii) If $K \subseteq L$ then $KK = \{o_K\}$, $E = K$ and $K \times K \subseteq \varrho$ (see 3.5(iii)).
- (iv) If $K \not\subseteq L$ then $LK = K \cap L = \{o_K\}$. $\square$

Lemma 3.7. *Let $B = \{0_K\}$ (see 3.4) and $J = \{a \in K \mid 0_K \in K + a\}$. Then:*

- (i) J is a left ideal of S .
- (ii) Either $J = \{0_K\}$ or $J = K$.
- (iii) If $J = \{0_K\}$ then $K + (K \setminus \{0_K\}) \subseteq K \setminus \{0_K\}$.
- (iv) If $J = K$ then $K(+)$ is a group. $\square$

Lemma 3.8. *Let $B = \{0_K\} = J$ (see 3.7). Then either $E = K$ or D is a subsemiring of K (in fact, $K + D \subseteq D$).*

Proof. Assume that $D \neq \emptyset$. Clearly, $DD \subseteq D$. If $a, b \in K$ are such that $a + b \notin D$ then $a + b \in E$, $K(a + b) = \{0_K\}$ by 3.5(i), and hence $ca + ab = 0_K$ for every $c \in K$. Now, $ca = 0_K = cb$ follows from 3.7(iii), so that $a, b \in E$. $\square$

Lemma 3.9. $Da = D$ for every $a \in D$.

Proof. Clearly, $DD \subseteq D$. If $a, b \in D$ then $b = ca$ for some $c \in K$ and $K = Kb = Kca$, $|Kc| \geq 2$ and $c \in D$. $\square$

Lemma 3.10. Assume that K is multiplicatively idempotent. Then just one of the following three cases is true:

1. K is additively idempotent and $ab = a$ for all $a, b \in K$ (then $E = \emptyset$);
2. $|K| = 2$ and $K \cong S_8$ (then $E = K$);
3. $|E| = 1$.

Proof. We have $Ka = \{a^2\} = \{a\}$ for every $a \in E$, and hence $E \subseteq \underline{R}(S)$ and $E = \underline{R}(S) \cap K$. If $|E| \geq 2$ then $K \subseteq \underline{R}(S)$ and it is clear that $K \cong S_8$. On the other hand, if $E = \emptyset$ then $Ka = K$ and $baa = ba$ for all $a, b \in K$. Thus $ab = a$ for all $a, b \in K$. $\square$

4. Minimal left ideals-basic observations (b)

Let S be a semiring and K be a minimal left ideal such that $|B| = 1$ (see 3.4). Then $B = \{q\}$, where either $q = 0_K$ or $q = o_K$. Besides, $q \in \underline{R}(S)$, $K\underline{R}(S) = K \cap \underline{R}(S) = \{q\}$, $qS \subseteq \underline{R}(S)$ and $qK = \{q\}$, so that q is multiplicatively absorbing in K . We have $Sa = K$ for every $a \in K \setminus \{q\}$, $KE = \{q\}$, E is an ideal of K and $Ka = K$ for every $a \in D = K \setminus E$ (see 3.5, 3.6).

Lemma 4.1. (i) $q \notin (K \setminus \{q\})a$ for every $a \in D$.

(ii) If $a \in K$ then either $Ba \neq b$ for every $b \in K$, $b \neq q$, or a is right multiplicatively neutral in K .

Proof. (i) The set $\{b \in S \mid ba = q\}$ is a left ideal of S .

(ii) The set $\{b \in K \mid ba = b\}$ is a left ideal of S . $\square$

The following six assertions are straightforward.

Lemma 4.2. Let $a \in K$ be such that $a^2 = a$. Then:

- (i) If $a \neq q$ then a is right multiplicatively neutral in K .
- (ii) If $a \neq q$ then $(a + q)^2 = a + b + ab + b^2$ for every $b \in K$. If, moreover, $b^2 = b \neq q$ then $(a + b)^2 = 2(a + b)$.
- (iii) $(a + q)^2 = a + q$.
- (iv) If $a \neq q$ then $(ba)^2 = b^2$ for every $b \in K$.

- (v) If $a \neq q$ then $(ab)^2 = ab^2$ for every $b \in K$.
 (vi) If $b \in K$ is such that $b^2 = b \neq q$ then $2c = c(a+b)$ for every $c \in K$. If, moreover, $2c \neq q$ then $a+b \neq q$. If $2c = c \neq q$ then $(a+b)^2 = a+b$. $\square$
 Put $I_a(K) = \{a \in K \mid 2a = a\}$, $I_m(K) = \{a \in K \mid a^2 = a\}$ and $I_z(K) = \{a \in K \mid 2a = q\}$.

Lemma 4.3. (i) $q \in I_a(K)$ and $I_a(K)$ is a left ideal of S .
 (ii) Either $I_a(K) = \{q\}$ or $I_a(K) = K$ (so that K is additively idempotent). $\square$

Lemma 4.4. (i) $q \in I_z(K)$ and $I_z(K)$ is a left ideal of S .
 (ii) Either $I_z(K) = \{q\}$ or $I_z(K) = K$.
 (iii) If $I_z(K) = K$ then $I_a(K) = \{q\}$.
 (iv) If $I_z(K) = \{q\} = I_a(K)$ and either $q = o_K$, or $q = 0_K$ and $J = \{0_K\}$ (see 3.7(iii)) then, for every $a \in K \setminus \{q\}$, the additive subsemigroup $\mathbb{N}a = \{a, 2a, 3a, \dots\}$ is a copy of the additive semigroup of positive integers. $\square$

Lemma 4.5. Let $I_a(K) = K$. Then:
 (i) $I_m(K)$ is a subsemiring of K .
 (ii) $q \in I_m(K)$ and $ab = a$ for all $a, b \in I_m(K)$, $b \neq q$.
 (iii) If $a \in I_m(K) \setminus \{q\}$ then a is right multiplicatively neutral in K .
 (iv) $a + b \neq q$ for all $a, b \in I_m(K) \setminus \{q\}$. $\square$

Lemma 4.6. Let $a \in K$ be such that $a^m = a$ for some $m \geq 2$. Put $b = a + a^2 + \dots + a^{m-1}$. Then:
 (i) $ab = b = ba$.
 (ii) If $b = q = 0_K$ and $a \neq 0_K$ then $K(+)$ is a p -elementary group for a prime p (see 3.7).
 (iii) If $b \neq q$ then a is right multiplicatively neutral in K , $a \in I_m(K)$ and $b = ma$. $\square$

Lemma 4.7. Let $a \in K$ be such that $q \neq a^m \in I_m(K)$. Then $a^{m+1} = a$. $\square$

Lemma 4.8. Let $a \in K$ and $1 \leq m < n$ be such that $a^m = a^n$. Then $a^{m(n-m)} \in I_m(K)$.

Proof. As $a^n = a^m \cdot a^{n-m} = a^m$, we have $a^m = a^m \cdot a^{n-m} \cdot a^{n-m} = \dots = a^m \cdot a^{m(n-m)}$, and hence $a^{m(n-m)} \cdot a^{m(n-m)} = a^{m(n-m)}$. $\square$

Lemma 4.9. Let $a \in K \setminus I_m(K)$ be such $q \neq a + a^2 + \dots + a^k$ for every $k \geq 1$. Then either $q = 0_K = a^l$ for some $l \geq 2$ or the multiplicative subsemigroup $a^{\mathbb{N}} = \{a, a^2, a^3, \dots\}$ is a copy of the additive semigroup of positive integers.

Proof. Assume, on the contrary, that $a^m = a^n$ for some $1 \leq m < n$. By 4.8, $a_1 = a^{m(n-m)} \in I_m(K)$. By 4.7, either $a^{m(n-m)+1} = a$ or $a^{m(n-m)} = q$. If $a^{m(n-m)} = a$ then $ab = b = ba$ for $b = a + a^2 + \dots + a^{m(n-m)}$, $b \neq q$ and $a \in I_m(K)$ by 4.1(ii), a contradiction. Thus $a^{m(n-m)} = q$, and hence $q = 0_K$ (if $q = o_K$ then $a + a^2 + \dots + a^{m(n-m)} = o_K = q$). $\square$

Lemma 4.10. *Assume that $q = 0_K$.*

(i) *If $a \in E$ then $a^{\mathbb{N}} = \{0_K, a\}$ and $\mathbb{N}a \cup \{0_K\}$ is the subsemiring of K generated by a .*

(ii) *If $a \in I_m(K)$ then $a^{\mathbb{N}} = \{a\}$ and $\mathbb{N}a$ is the subsemiring of K generated by a .*

(iii) *If K is not a ring and $a \in D \setminus I_m(K)$ then $a^{\mathbb{N}}$ is a copy of the additive semigroup of positive integers, $\langle a \rangle = \{ \sum_{i=1}^m n_i a_i \mid m \geq 1, 0 \leq n_i, \sum n_i \neq 0 \}$ is the subsemiring of K generated by the element a and $\langle a \rangle \subseteq D$.*

Proof. (i) We have $a^2 = 0_K$.

(ii) We have $a^2 = a$.

It follows from 4.1(i) that $0_K \notin a^{\mathbb{N}}$. We have $a \in D$ and D is a subsemiring of K by 3.8. Thus $\langle a \rangle \subseteq D$. The rest follows from 4.9. $\square$

For $e \in I_m(K)$, let $G_e^+ = \{a \in K \mid ea = a\}$ and $G_e = G_e^+ \setminus \{q\}$.

Lemma 4.11. *Let $e \in I_m(K)$. Then $e, q \in G_e^+$ and G_e^+ is a right ideal of the semiring K . If $e = q$ then $G_e^+ = \{q\}$. If $e \neq q$ then $|G_e^+| \geq 2$ and e is multiplicatively neutral in G_e^+ .* $\square$

Lemma 4.12. *Let $e \in I_m(K) \setminus \{q\}$. Then G_e is a subgroup of the multiplicative semigroup $K(\cdot)$ and e is the multiplicatively neutral element of G_e . Moreover, $G_e \subseteq D$.*

Proof. Clearly, $e \in G_e$. If $a, b \in G_e$ then $a \in Ka$, $b \in Kb$, and hence $a, b \in D$, $ab \in D$ and $ab \in G_e$. By 4.5(iii), e is right multiplicatively neutral in K , so that e is the multiplicatively neutral element of G_e .

Let $a \in G_e$. Then $ba = e$ for some $b \in K$, $eb = bab$ and $a = ae = aba$. We have $ab \neq q$, and therefore $b \in D$ and $cb = e$ for some $c \in K$. Then $a = ea = cba = cb = c$, $ab = e$, $eb = be = b$ and we see that G_e is a multiplicative group. $\square$

Lemma 4.13. *Let $e, f \in I_m(K) \setminus \{q\}$, $e \neq f$. Then $G_e \cap G_f = \emptyset$.*

Proof. If $a \in G_e \cap G_f$ then $ea = a = fa$. But $e = ab$ for some $b \in G_e$ and we have $f = fe = fab = ab = e$. $\square$

Lemma 4.14. *Assume that no subsemigroup of the multiplicative semigroup of D is a copy of the additive semigroup of positive integers. Then $D = \bigcup G_e$, $e \in I_m(K) \setminus \{q\}$.*

Proof. Let $a \in D$. Then $ba = a$ for some $b \in D$ (see 3.9), and hence $b^m a = a$ for every $m \geq 1$. According to our assumption, $b^n \in I_m(K) \setminus \{q\}$ for some $n \geq 1$. $\square$

Lemma 4.15. *Let $e, f \in I_m(K) \setminus \{q\}$. The mapping $a \mapsto fa$ is an isomorphism of the multiplicative group G_e onto the multiplicative group G_f .*

Proof. First, $ffa = fa$, and so $fa \in G_f$. Furthermore, by 4.5(iii), $fab = fafbf$ for all $a, b \in G_e$, and therefore $a \mapsto fa$ is a multiplicative homomorphism of G_e into G_f . If $a \in G_e$ is such that $fa = f$ then a is right multiplicatively neutral in K by 4.1(ii), and so $a = ea = e$. It means that the homomorphism is injective. Finally, if $c \in G_f$ then $ec \in G_e$ and $fewc = fc = c$. $\square$

Lemma 4.16. *Assume that K is multiplicatively idempotent. Then $ab = a$ for all $a, b \in K$, $b \neq q$, and just one of the following three cases holds:*

1. $|K| = 2$ and $K \cong S_1$ (then $q = 0_K$);
2. K is additively idempotent;
3. $q = o_K$ and $2a = o_K$ for every $a \in K$.

Proof. It follows immediately from 4.2(i) that $ab = a$ for all $a, b \in K$, $b \neq q$. Now, $c(a + b) = ca + cb = 2c$ for all $c \in K$ and $a, b \in K \setminus \{q\}$. If $a + b = q$ then $2c = q$ and $I_z(K) = K$. If, moreover $q = 0_K$ then $K(+)$ is a 2-elementary group and $a = a(a + b) = aa + ab = 2a = 0_K$ for $0_K \neq a \neq b \neq 0_K$. Now, it follows easily that $|K| = 2$ and $K \cong S_1$. Finally, if $a + b \neq q$ for all $a, b \in K \setminus \{q\}$ then $K(+)$ is idempotent. $\square$

5. Partial summary (a)

Let S be a semiring and K be a minimal left ideal of S such that $B = \{0_K\}$ (see 3.4). We put $K^* = K \setminus \{0_K\}$ and $I_m(K)^* = I_m(K) \setminus \{0_K\}$.

Proposition 5.1. (i) $0_K K = \{0_K\} = S0_K$.

(ii) $0_K S \subseteq \underline{R}(S)$.

(iii) $Sa = K$ for every $a \in K^*$.

(iv) $0_K \in L = \{a \in S \mid aK = \{0_K\}\}$ and L is an ideal of S .

(v) If $K \subseteq L$ then $KK = \{0_K\}$ and $K \times K \subseteq \varrho$, where ϱ is the congruence of S defined by $(a, b) \in \varrho$ iff $ac = bc$ for every $c \in K$.

(vi) If $K \subseteq L$ and $\varrho = S \times S$ then $SK = \{0_K\}$.

- (vii) If $SK = \{0_K\}$ (or $L = S$) then either $|K| = 2$ or $K(+)$ is a finite (cyclic) group of prime order.
- (viii) If $K \not\subseteq L$ then $LK = L \cap K = \{0_K\}$.
- (ix) If $|\underline{R}(S)| = 1$ then 0_K is multiplicatively absorbing in S and $S + 0_K$ is a bi-ideal of S .
- (x) If $S + 0_K = S$ then $0_K = 0_S$.
- (xi) $E = \{a \in K \mid |Ka| = 1\} = \{a \in K \mid Ka = \{0_K\}\}$.
- (xii) $E = K$ iff $K \subseteq L$ (see (v) and (vi)).
- (xiii) $DD = D = \{a \in K \mid Ka = K\}$ and $Da = D$ for every $a \in D$.
- (xiv) $K = E \cup D$ and $E \cap D = \emptyset$.
- (xv) $0K \notin K^*D$.
- (xvi) Either K is additively idempotent or $2a \neq a$ for every $a \in K^*$.
- (xvii) Either $2a = 0_K$ for every $a \in K$ (so that $K(+)$ is a 2-elementary group), or $2b \neq 0_K$ for every $b \in K^*$ and $K = \{2a \mid a \in K\}$.
- (xviii) Either $K(+)$ is a group or $K + K^* \subseteq K^*$.
- (xix) If $a \in I_m(K)^*$ then a is right multiplicatively neutral in K .
- (xx) $I_m(K)^* \subseteq D$.

Proof. See 3.4, 3.5, 3.7, 3.8, 3.9 and 4.2, 4.3, 4.4. □

Proposition 5.2. Assume that $K(+)$ is not a group and that $KK \neq \{0_K\}$. Then:

- (i) $K + K^* \subseteq K^*$.
- (ii) D is a subsemiring of K and $K + D \subseteq D$.
- (iii) For every $e \in I_m(K)^*$, $G_e = eK^* = \{a \in K \mid ea = a\}$ is a subsemiring of K , e is the multiplicatively neutral element of G_e and the multiplicative semigroup of G_e is a group.
- (iv) $G_e^* = G_e \cup \{0_K\} = eK$ is a right ideal of K .
- (v) Either $G_e = \{e\}$ or the multiplicative group of G_e is infinite and torsion-free.
- (vi) If $e, f \in I_m(K)^*$, $e \neq f$, then $G_e \cap G_f = \emptyset$ and the mapping $G_e \rightarrow fG_e = G_f$ is an isomorphism of the semiring G_e onto the semiring G_f .

Proof. See 3.7(iii), 4.10, 4.11, 4.12, 4.13 and 4.15. □

Proposition 5.3. Assume that $K(+)$ is not a group, $KK \neq \{0_K\}$ and no multiplicative subsemigroup of D is a copy of the additive semigroup of positive integers. Then:

- (i) $D = I_m(K)^*$ is a subsemiring of K , $K + D \subseteq D$ and D is just the set of elements from K that are right multiplicatively neutral in K .
- (ii) $D \cup \{0_K\}$ is a right ideal of K .
- (iii) E is an ideal of K .
- (iv) K is additively idempotent.
- (v) The relation $(D \times D) \cup (E \times E)$ is the greatest proper congruence of the (left K -) semimodule ${}_K K$.

(vi) For every $a \in K$, $aE = \{0_K\}$, $aD = \{a\}$ and $aK = \{0_K, a\}$.

Proof. (i) It follows from 5.2(v) that $G_e = \{e\}$ for every $e \in I_m(K)^*$. Now, 4.14 implies $D = I_m(K)^*$. The rest follows from 5.2(ii).

(ii) This follows immediately from (i).

(iii) Easy to see.

(iv) If $a \in K$ and $b \in D$ then $a + a = ab + ab = a(b + b) = a$ (we have $2b \in D$).

(v) and (vi) These assertions are obvious. $\square$

5.4 Assume that $K(+)$ is not a group, $KK \neq \{0_K\}$ and $D = I_m(K)^*$ (see 5.3). Then the assertions 5.3(i), ..., (vi) are true and we define a relation ϑ on K by $(a, b) \in \vartheta$ iff $\{c \in S \mid ca \in E\} \subseteq \{c \in S \mid cb \in E\}$.

Lemma 5.4.1. ϑ is reflexive and transitive. Moreover, $(a, b) \in \vartheta$, provided that $b \leq a$.

Proof. If $b \leq a$ and $ca \in E$ then $cb \in E$, since $cb \leq ca$. $\square$

Lemma 5.4.2. If $(a, b) \in \vartheta$ then $(a + c, b + c) \in \vartheta$ and $(da, db) \in \vartheta$ for all $c \in K$ and $d \in S$.

Proof. Use 5.3(i), (iii). $\square$

Lemma 5.4.3. $(0_K, a) \notin \vartheta$ for every $a \in K^*$. $\square$

Lemma 5.4.4. (i) $\sigma = \ker(\vartheta)$ is a congruence of the (left S -) semimodule ${}_S K$.
(ii) $(0_K, a) \notin \sigma$ for every $a \in K^*$.

Proof. Use 5.4.1, 5.4.2, and 5.4.3. $\square$

Lemma 5.4.5. σ is the (unique) greatest proper congruence of the (left S -) semimodule ${}_S K$ and $\sigma \subseteq (D \times D) \cup (E \times E)$.

Proof. Let τ be a congruence of ${}_S K$ such that $\tau \not\subseteq \sigma$. Take $(a, b) \in \tau \setminus \sigma$ and assume that $ca \notin E$ and $cb \in E$ for some $c \in S$ (the other case being symmetric). Now, $ca \in D$, $dca = d$ and $dc b = 0_K$ for every $d \in K$. Consequently, $(d, 0_K) \in \tau$ and $\tau = K \times K$. $\square$

Corollary 5.4.6. *The (left S -) semimodule ${}_SK$ is simple iff $\sigma = \text{id}_K$.* $\square$

Corollary 5.4.7. *The factorsemimodule ${}_SK/\sigma$ is both minimal and simple.* $\square$

Lemma 5.4.8. *Let $a \in K$ and $F_a = \{f \in S \mid fa \in E\}$. Then:*

- (i) $F_a + F_a \subseteq F_a$ and $KF_a \subseteq E_a$.
- (ii) $S + (S \setminus F_a) \subseteq S \setminus F_a$.
- (iii) $E \subseteq F_a$.
- (iv) $F_a = S$ iff $a = 0_K$.
- (v) $K \subseteq F_a$ iff $a \in E$.
- (vi) If $a \in D$ then $K \cap F_a = \{0_K\}$.

Proof. It is easy (use 5.3). $\square$

Lemma 5.4.9. *Let $a \in K$ be such that $b = o_F \in F = F_a$. Then, for every $c \in K$, we have:*

- (i) $cba_1 = 0_K$ for every $a_1 \in K$ such that $(a, a + a_1) \in \sigma$.
- (ii) $cba_2 = c$ for every $a_2 \in K$ such that $(a, a + a_2) \notin \sigma$.

Proof. (i) We have $b \in F_a = F_{a_1+a}$, and so $b \in F_{a_1}$, $ba_1 \in E$ and $cba_1 = 0_K$.

(ii) We have $F_{a+a_2} \subset F_a$, and hence $b_1a \in E$ and $b_1a_2 \notin E$ for some $b_1 \in S$. Now, $b_1 + b = b$, $ba \in E$, $ba_2 \in D$ and $cba_2 = c$. $\square$

6. Partial summary (b)

Let S be a semiring and K be a minimal ideal of S such that $B = \{o_K\}$ (see 3.4). We put $K^\circ = K \setminus \{o_K\}$ and $I_m(K)^\circ = I_m(K) \setminus \{o_K\}$.

Proposition 6.1. (i) $o_K K = \{o_K\} = So_K$.

(ii) $o_K S \subseteq \underline{R}(S)$.

(iii) $Sa = K$ for every $a \in K^\circ$.

(iv) $o_K \in L_1 = \{a \in S \mid aK = \{o_K\}\}$ and L is an ideal of S .

(v) If $K \subseteq L_1$ then $KK = \{o_K\}$ and $K \times K \subseteq \varrho$, where ϱ is the congruence of S defined by $(a, b) \in \varrho$ iff $ac = bc$ for every $c \in K$.

(vi) If $K \subseteq L_1$ and $\varrho = S \times S$ then $SK = \{o_K\}$.

(vii) If $SK = \{o_K\}$ (or $L_1 = S$) then either $|K| = 2$ or $K(+)$ is a finite (cyclic) group of prime order.

(viii) If $K \not\subseteq L_1$ then $L_1 K = L_1 \cap K = \{o\}$.

(ix) If $|\underline{R}(S)| = 1$ then o_K is multiplicatively absorbing in S , $S + o_K$ is a bi-ideal of S and $K \not\subseteq S + o_K$.

(x) If $S = o_K = \{o_K\}$ then $o_K = o_S$.

(xi) $E = \{a \in K \mid |Ka| = 1\} = \{a \in K \mid Ka = \{o_K\}\}$.

(xii) $E = K$ iff $\subseteq L_1$ (see (v) and (vi)).

(xiii) $DD = D = \{a \in K \mid Ka = K\}$ and $Da = D$ for every $a \in D$.

(xiv) $K = E \cup D$ and $E \cap D = \emptyset$.

- (xv) $o_K \notin K^\circ D$.
- (xvi) Either K is multiplicatively idempotent or $2a \neq a$ for every $a \in K^\circ$.
- (xvii) Either $2a = o_K$ for every $a \in K$, or $2b \neq o_K$ for every $b \in K^\circ$ and $K = \{2a \mid a \in K\}$.
- (xviii) If $a \in I_m(K)^\circ$ then a is right multiplicatively neutral in K .
- (xix) $I_m(K)^\circ \subseteq D$.

Proof. See 3.4, 3.6, 3.9 and 4.2, 4.3, 4.4. □

Proposition 6.2. (i) For every $e \in I_m(K)^\circ$, the set $H_e = \{a \in K^\circ \mid ea = a\}$ is a subgroup of the multiplicative semigroup $K(\cdot)$ and e is the multiplicatively neutral element of H_e . Moreover, $H_e \subseteq D$.

- (ii) $H_e^+ = eK = H_e \cup \{o_K\}$ is a right ideal of K .
- (iii) If $a \in H_e$ is such that $a^m = a^n$ for some $1 \leq m < n$ then either $a = e$ or $o_K = a + a^2 + \dots + a^{(n-m)m}$.
- (iv) If $e, f \in I_m(K)^\circ$, $e \neq f$, then $H_e \cap H_f = \emptyset$ and the mapping $H_e \rightarrow fH_e = H_f$ is an isomorphism of the multiplicative groups.
- (v) If $e, f \in I_m(K)^\circ$ then the mapping $H_e^+ \rightarrow fH_e^+ = H_f^+$ is an isomorphism of the semirings.

Proof. See 4.9, 4.11, 4.12, 4.13 and 4.15. □

Lemma 6.3. Let $e \in I_m(K)^\circ$ be such that $2e = o_K$. Then $2K = \{o_K\} = H_e + H_e$.

Proof. First, $2K = \{o_K\}$ by 6.1(xvii). Next, $(a + e)^2 = a^2 + 2a + e = o_K$ for $a \in H_e$, and hence $a + e = o_K$ and $ab + b = o_K$ for all $a, b \in H_e$. Thus $H_e + H_e = \{o_K\}$ □

Lemma 6.4. Let $e \in I_M(K)^\circ$ be such that $e \in F_e = \{a \in H_e \mid a + e \neq o_K\}$. Then:

- (i) F_e is a subsemiring of K and the multiplicative semigroup of F_e is a group.
- (ii) If $F_e \neq \{e\}$ then the multiplicative group of F_e is infinite and torsionfree.
- (iii) If $F_e = \{e\}$ then $2a = a$ and $a + b = o_K$ for all $a, b \in H_e$, $a \neq b$.

Proof. (i) We have $a + e \in H_e$ for every $a \in H_e$. Now, $e + a + b + ab = (a + e)(b + e) \in H_e$ for $a, b \in H_e$, and hence $a + b \in F_e$ and $ab \in F_e$. If $c \in H_e$ is such that $ac = e$ then $o_K \neq (a + e)c = e + c$, so that $c \in F_e$.

(ii) Use 6.2(iii).

(iii) If $c \in H_e$ is such that $ac = e$ then $(a + b)c = e + bc$ and $bc \neq e$. Since $F_e = \{e\}$, we have $e + bc = o_K$, and hence $a + b = o_K$ as well. Of course, $2e = e$ by (i). □

Proposition 6.5. *Assume that $KK \neq \{o_K\}$ and no multiplicative subsemigroup of D is a copy of the additive semigroup of positive integers. Then:*

- (i) $D \neq \emptyset$ and $Da = D$ for every $a \in D$.
- (ii) $D = \bigcup H_e$, $e \in I_m(K)^\circ$.
- (iii) For every $e \in I_m(K)^\circ$, the multiplicative group of H_e is periodic.
- (iv) $a + b = o_K$ for all $a, b \in H_e$, $a \neq b$.
- (v) If $2e \neq o_K$ then $2a = a$ for every $a \in H_e$.
- (vi) If $2e = o_K$ then $a + b = o_K$ for all $a, b \in H_e$.

Proof. (i) See 6.1(xii), (xiii).

(ii) This is 4.14.

(iii) This follows immediately from our assumption.

(iv), (v) and (vi). If $2e = o_K$ then 6.3 applies. If $2e \neq o_K$ then $F_e = \{e\}$ by 6.4(ii) and 6.4(iii) applies. $\square$

Proposition 6.6. *Assume that K is multiplicatively idempotent and $\emptyset \neq D = I_m(K)^\circ$. Then:*

- (i) D is a subsemiring of K .
- (ii) If $a, b \in K$ are such that $a \leq b$ and $b \in D$ then $a \in D$.
- (iii) E is a bi-ideal of K .
- (iv) $aD = \{a\}$ and $aE = \{o_K\}$ for every $a \in K$.

Proof. Easy (use 4.5). $\square$

6.7 Assume that $KK \neq \{o_K\}$ and no multiplicative subsemigroup of D is a copy of the additive semigroup of positive integers (see 6.5).

Lemma 6.7.1. (i) $DD = D$.

(ii) $KE = \{o_K\}$.

(iii) $Da = D$ and $Ea = E$ for every $a \in D$.

(iv) If $a, b \in K$ are such that $ab \in E$ then either $a \in E$, or $b \in E$ and $ab = o_K$.

Proof. (i) See 6.1(xiii).

(ii) See 6.1(xi).

(iii) We have $K = Ka = Da \cup Ea$, where $Da \subseteq D$ and $Ea \subseteq E$. Thus $Da = D$ and $Ea = E$.

(iv) This follows immediately from (i). $\square$

Lemma 6.7.2. *Let $a \in D$. Then:*

- (i) $a \in H_e$ for a uniquely determined $e \in IU_m(K)^\circ$.
- (ii) $H_fa = H_f$ for every $f \in I_m(K)^\circ$.
- (iii) $aH_f = H_f$.
- (iv) $aD = H_e$ and $aK = H_e^+ = H_e \cup \{o_K\}$.

Proof. (i) By 6.5(ii), $a \in G_e$ and, by 6.2(iv), the idempotent e is determined uniquely.

(ii) We have $H_fa \subseteq DD \subseteq D$ by 6.7.1(i) and $fb a = ba$ for every $b \in H_f$. Thus $H_fa \subseteq H_f$. Since $Da = D$, we get $H_fa = H_f$.

(iii) Clearly, $aH_f \subseteq H_e$. On the other hand, if $b \in H_e$ then $b = ac$ for some $c \in H_e$ (see 6.2(i)), and hence $b = afc$, while $fc \in H_f$ (use 6.1(xviii)).

(iv) This follows from (iii). $\square$

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Minimal left ideals in some endomorphism semirings of semilattices

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Abstract. Minimal left ideals in some endomorphism semirings of semilattices are investigated.

Keywords: semiring, semimodule, semilattice, ideal, endomorphism.

1. Preliminaries

1.1 Semirings. A semiring is an algebraic structure possessing two associative binary operations (addition and multiplication), where the addition is commutative and the multiplication distributes over the addition.

Let S be a semiring. A non-empty subset I of S is called

- a *left (right) ideal* if $(I + I) \cup SI \subseteq I$ ($(I + I) \cup IS \subseteq I$);
- an *ideal* if $(I + I) \cup SI \cup IS \subseteq I$;
- a *bi-ideal* if $(S + I) \cup SI \cup IS \subseteq I$.

A (left, right, bi-) ideal I is called *minimal* if $|I| \geq 2$ and $K = I$ whenever K is a (left, right, bi-) ideal such that $K \subseteq I$ and $|K| \geq 2$. The semiring S is called

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(left-, right-, bi-) *ideal-simple* if it is non-trivial and has no proper non-trivial (left, right, bi-) ideal.

1.2 Semilattices. A *semilattice* $M (= M(+))$ is a commutative idempotent semigroup. A non-empty subset A of M is an *ideal* if $M + A = A$. Such an ideal is called *prime* if the set $M \setminus A$ is a subsemilattice of M . We have the basic order on M defined by $a \leq b$ iff $a + b = b$.

An element $w \in M$ is *neutral* (*absorbing*) if $x + w = x$ ($x + w = w$) for every $x \in M$. Such an element is unique (provided that it exists - this fact is denoted by $0_M \in M$ ($o_M \in M$)) and is denoted by $0 = 0_M$ ($o = o_M$) in the sequel. The fact that no such element exists is denoted by $0_M \notin M$ ($o_M \notin M$).

1.3 Semimodules. Let S be a semiring. A (*left* S -) *semimodule* is a commutative semigroup $M (= M(+))$ together with a scalar multiplication $S \times M \rightarrow M$. The semimodule M is called

- *minimal* if $|M| \geq 2$ and $N = M$ whenever N is a non-trivial subsemimodule of M ;
- *simple* if it has just two congruence relations;
- *faithful* if for all $a, b \in S$, $a \neq b$, there is at least one $x \in M$ with $ax \neq bx$.

1.4 Endomorphism semirings. Let $M (= M(+))$ be a commutative semigroup. The set $\underline{E} = \underline{E}(M) = \text{End}(M)$ of endomorphisms of M is a semiring via $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$. The semiring $\underline{E}$ is unitary and the multiplicatively neutral element is the identity automorphism id_M of M .

2. 0-preserving endomorphisms

Let M be a non-trivial semilattice such that $0_M \in M$. The set $\underline{E}_0 = \{f \in \underline{E} \mid f(0_M) = 0_M\}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism ξ , $\xi(M) = \{0_M\}$, is the zero element of the semiring $\underline{E}_0$. For all $u, v \in M$, we have $q_{u,v} \in \underline{E}_0$, where $q_{u,v}(x) = 0_M$ for $x \leq u$ and $q_{u,v}(y) = v$ for $y \not\leq u$. Thus $q_{u,v}(M) = \{0_M, v\}$ for $u \neq o_M$, $q_{o_M,v} = \xi$ (if $o_M \in M$) and $q_{u,0_M} = \xi$.

Now, let E be a subsemiring of $\underline{E}_0$ such that $q_{u,v} \in E$ for all $u, v \in M$. For every $u \in N = M \setminus \{o_M\}$, put $T_u = \{q_{u,v} \mid v \in M\}$. The following observations are quite easy.

Lemma 2.1. (i) $q_{u,v_1} + q_{u,v_2} = q_{u,v_1+v_2}$.

(ii) $f q_{u,v} = q_{u,f(v)}$ for every $f \in \underline{E}_0$.

(iii) $q_{u,v_1} q_{P u, v_2} = \xi$ for $v_2 \leq u$.

$q_{u,v_1} q_{u,v_2} = q_{u,v_1}$ for $v_2 \not\leq u$.

(v) $q_{u,v} q_{u,v} = \xi$ for $v \leq u$.

(vi) $q_{u,v} q_{u,v} = q_{u,v}$ for $v \not\leq u$. □

Proposition 2.2. *The set T_u is a minimal left ideal of the semiring E for every $u \in N$. The mapping $v \mapsto q_{u,v}$ is an isomorphism of the semimodule ${}_E M$ onto the semimodule ${}_E T_u$.*

Proof. It follows immediately from 2.1 that T_u is a left ideal of E and that the map $v \mapsto q_{u,v}$ is an isomorphism of the semimodules. Furthermore, if $v \neq 0_M$ then $q_{0_M, v_1} q_{u,v} = q_{u, v_1}$ for every $v_1 \in M$. Now, it is clear that T_u is a minimal left ideal of E . $\square$

Proposition 2.3. (i) $T_u q_{u,v} = \{\xi\}$ for $v \leq u$.
(ii) $T_u q_{u,v} = T_u$ for $v \not\leq u$.

Proof. Use 2.1(iii),(iv). $\square$

Proposition 2.4. (i) *The set $\{q_{u,v} \mid v \leq u\}$ is an ideal of the semiring T_u and it is the greatest proper ideal of that semiring.*

(ii) *The $\{q_{u,v} \mid v \not\leq u\} \cup \{\xi\}$ is just the set of the multiplicative idempotents of T_u ($\xi = q_{u, 0_M}$).*

(iii) *The semiring T_u is multiplicatively idempotent iff $u = 0_M$.*

(iv) *The semiring T_u is (left-) ideal-simple iff $u = 0_M$.*

Proof. It is easy. $\square$

Proposition 2.5. *The semimodule ${}_E T_u$ is simple.*

Proof. Let $\alpha \neq \text{id}$ be a congruence of T_u . Then $(q_{u, v_1}, q_{u, v_2}) \in \alpha$ for some $v_1 < v_2$, and hence $(\xi, q_{u, v_2}) = (q_{v_1, v_2} q_{u, v_1}, q_{v_1, v_2} q_{u, v_2}) \in \alpha$. Consequently, $(\xi, q_{u, v_2}) = (q_{0_M, v_3} \xi, q_{0_M, v_3} q_{u, v_2}) \in \alpha$ for every $v_3 \in M$. Thus $\alpha = T_u \times T_u$. $\square$

Proposition 2.6. *The semimodule T_u is faithful.*

Proof. Use 2.1(ii). $\square$

Lemma 2.7. *If I is a non-trivial ideal of E then $\bigcup_{u \in N} T_u \subseteq I$.*

Proof. We have $IT_u \subseteq I \cap T_u$ and the latter set is a left ideal of E . If $T_u \not\subseteq I$ then $IT_u = I \cap T_u = \{\xi\}$ and it means that $q_{u, f(v)} = f q_{u, v} = \xi$ for every $f \in I$ and $v \in M$. Since $u \neq 0_M$, we get $f(v) = 0_M$ and $f = \xi$. Thus $I = \{\xi\}$, a contradiction. $\square$

Corollary 2.8. *The set $\{q_{u_1, v_1} f_1 + \dots + q_{u_n, v_n} f_n \mid n \geq 1, u_i \in N, v_i \in M, f_i \in E\}$ is just the smallest non-trivial ideal of the semiring E .* $\square$

2.9 Let K be a minimal left ideal of the semiring E such that $K \neq T_u$ for every $u \in N$.

Lemma 2.9.1. $K \cap T_u = \{\xi\}$ for every $u \in N$.

Proof. It is obvious. $\square$

Lemma 2.9.2. $Ef = K$ for every $f \in K \setminus \{\xi\}$.

Proof. The set $K_1 = \{f \in K \mid Ef = \{\xi\}\}$ is a left ideal of E . If $K_1 = K$ then $EK = \{\xi\}$ and $K = \{\xi, f_0\}$ is two-element. Now, $q_{0_M, v}f_0 = \xi$, and hence $f_0 = \xi$, a contradiction. It follows that $K_1 = \{\xi\}$ and $Ef = K$, since Ef is a left ideal of E . $\square$

Lemma 2.9.3. *There are subsemilattices A and B of M such that:*

- (i) $A \cup B = M$ and $A \cap B = \emptyset$.
- (ii) $A = \{x \in M \mid f(x) \neq 0_M\}$ and $B = \{x \in M \mid f(x) = 0_M\}$ for every $f \in K \setminus \{\xi\}$.
- (iii) $A + M = A$ (i.e., A is a prime ideal of the semilattice M).
- (iv) $o_B \notin B$ (i.e., the prime ideal A is not principal).

Proof. If $f, g \in K \setminus \{\xi\}$ then $g = hf$ for some $h \in E$ by 2.9.2 and we see that $g(x) \neq 0_M$ implies $f(x) \neq 0_M$. Clearly, $A + M = A$ and $B + B = B$. We have $0_M \in B$ and $A \neq \emptyset$, since $f \neq \xi$. If $u = o_B \in B$ then $u \neq o_M$ and $q_{0_M, v}f = q_{u, v} \in K$ for every $v \in M$, a contradiction. $\square$

Lemma 2.9.4. *If $f \in K \setminus \{\xi\}$ and $v \in M$ then $q_{0_M, v}f = q_{A, v} \in K$, where $q_{A, v}(B) = \{0_M\}$ and $q_{A, v}(A) = \{v\}$.*

Proof. It is obvious. $\square$

Proposition 2.9.5. (i) $K = \{q_{A, v} \mid v \in M\}$.

- (ii) *The mapping $v \mapsto q_{A, v}$ is an isomorphism of the semimodule ${}_E M$ onto the semimodule ${}_E K$.*
- (iii) $Kq_{A, v} = \{\xi\}$ for every $v \in B$.
- (iv) $Kq_{A, v} = K$ for every $v \in A$.
- (v) *The set $\{q_{A, v} \mid v \in B\}$ is an ideal of the semiring K and it is the greatest proper left ideal of that semiring.*
- (vi) *The set $\{q_{A, v} \mid v \in A\} \cup \{\xi\}$ is just the set of the multiplicative idempotents of K ($\xi = q_{A, 0_M}$).*
- (vii) *The semiring K is neither multiplicatively idempotent nor ideal-simple.*

Proof. According to 2.9.4, $T_A = \{q_{A, v} \mid v \in M\} \subseteq K$. The rest is clear (see 2.2, 2.4 and 2.4). $\square$

Theorem 2.10. (i) *The set $T_u = \{q_{u, v} \mid v \in M\}$ is a minimal left ideal of the semiring E for every $u \in N$.*

- (ii) *If K is a minimal left ideal of E such that $K \neq T_u$ for every $u \in N$ then there is a non-principal prime ideal A of the semilattice M such that $K = T_A = \{q_{A, v} \mid v \in M\}$.*

(iii) *If A is a non-principal prime ideal of M such that $Q_{A, v} \in E$ for at least one $v \in M \setminus \{0_M\}$ then $T_A \subseteq E$ and T_A is a minimal left ideal of E such that $T_A \neq T_u$ for every $u \in N$.*

- (iv) If K is a minimal left ideal of E then ${}_E K$ is a faithful, simple and minimal (left E -) semimodule and the semimodules ${}_E K$ and ${}_E M$ are isomorphic.
- (v) If K is a minimal left ideal of E then the semiring K is multiplicatively idempotent if and only if $K = T_{0_M}$.

Proof. Combine the foregoing results. $\square$

Corollary 2.11. Denote by $\mathbf{A}$ the set of prime ideals A of M such that $T_A \subseteq E$ (see 2.10(ii),(iii)). Then:

- (i) $\{T_A \mid A \in \mathbf{A}\}$ is just the set of (pairwise distinct) minimal left ideals of the semiring E .
- (ii) $A_u = \{x \in M \mid x \not\leq u\} \in \mathbf{A}$ for every $u \in N$.
- (iii) The set $\sum T_A$, $A \in \mathbf{A}$, is just the smallest non-trivial ideal of the semiring E . $\square$

3. 1-preserving endomorphisms (a)

Let M be a non-trivial semilattice such that $o_M \in M$. The set $\underline{E}_1 = \underline{E}_1(M) = \{f \in \underline{E} \mid f(o_M) = o_M\}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism $\zeta = \zeta_M$, $\zeta(M) = \{o_M\}$, is the bi-absorbing element of the semiring $\underline{E}_1$. For all $u, v \in M$, $u \neq o_M$, we have $p_{u,v} \in \underline{E}_1$, where $p_{u,v}(x) = v$ for $x \leq u$ and $p_{u,v}(y) = o_M$ for $y \not\leq u$. Thus $p_{u,v}(M) = \{v, o_M\}$ and $p_{u,o_M} = \zeta$.

Now, let F be a subsemiring of $\underline{E}_1$ such that $p_{u,v} \in F$ for all $u, v \in M$, $u \neq o_M$. For every $u \in N = M \setminus \{o_M\}$, put $V_u = \{p_{u,v} \mid v \in M\}$.

The following observations are quite easy.

Lemma 3.1. (i) $p_{u,v_1} + p_{u,v_2} = p_{u,v_1+v_2}$.

(ii) $f p_{u,v} = p_{u,f(v)}$ for every $f \in \underline{E}_1$.

(iii) $p_{u,v_1} p_{u,v_2} = \zeta$ for $v_2 \not\leq u$.

(iv) $p_{u,v_1} p_{u,v_2} = p_{u,v_1}$ for $v_2 \leq u$.

(v) $p_{u,v} p_{u,v} = \zeta$ for $v \not\leq u$.

(vi) $p_{u,v} p_{uv} = p_{u,v}$ for $v \leq u$. $\square$

Proposition 3.2. The set V_u is a minimal left ideal of the semiring F for every $u \in N$. The mapping $v \mapsto p_{u,v}$ is an isomorphism of the semimodule ${}_F M$ onto the semimodule ${}_F V_u$.

Proof. It follows immediately from 3.1 that V_u is a left ideal of F and the map $v \mapsto p_{u,v}$ is an isomorphism of the semimodules. Furthermore, if $v \neq o_M$ then $p_{v,v_1} p_{u,v} = p_{u,v_1}$ for every $v_1 \in M$. Now, it is clear that V_u is a minimal left ideal of F . $\square$

Proposition 3.3. (i) $V_u p_{u,v} = \{\zeta\}$ for $v \not\leq u$.

(ii) $V_u p_{u,v} = V_u$ for $v \leq u$.

Proof. Use 3.1 (iii), (iv). $\square$

Proposition 3.4. (i) *The set $\{p_{u,v} \mid v \not\leq u\}$ is a bi-ideal of the semiring V_u and it is the greatest proper left ideal of that semiring.*

(ii) *The set $\{p_{u,v} \mid v \leq u\} \cup \{\zeta\}$ is just the set of multiplive idempotents of V_u ($\zeta = p_{u,o_M}$).*

(iii) *The semiring V_u is multiplicatively idempotent iff $u = o_N \in N$.*

(iv) *The semiring V_u is (left-) ideal-simple iff $u = o_N \in N$.*

Proof. It is easy. □

Proposition 3.5. *The semimodule ${}_F V_u$ is simple.*

Proof. Let $\alpha \neq \text{id}$ be a congruence of ${}_F V_u$. Then $(p_{u,v_1}, p_{u,v_2}) \in \alpha$ for some $v_1 < v_2$, and hence $(p_{u,v_2}, \zeta) = (p_{v_1,v_2} p_{u,v_1}, p_{v_1,v_2} p_{u,v_2}) \in \alpha$. Consequently, $(p_{u,v_1}, \zeta) \in \alpha$ and, finally, $(p_{u,v_3}, \zeta) = (p_{v_1,v_3} p_{u,v_1}, p_{v_1,v_3} \zeta) \in \alpha$ for every $v_3 \in M$. Thus $\alpha = V_u \times V_u$. □

Proposition 3.6. *The semimodule ${}_F V_u$ is faithful.*

Proof. Use 3.1(ii). □

Lemma 3.7. *If I is a non-trivial ideal of F then $\bigcup_{u \in N} V_u \subseteq I$.*

Proof. We can proceed similarly as in the proof of 2.7. □

Corollary 3.8. *The set $\{p_{u_1,v_1} f_1 + \dots p_{u_n,v_n} f_n \mid n \geq 1, u_i \in N, v_i \in M, f_i \in F\}$ is just the smallest non-trivial ideal of the semiring F .* □

3.9 Let K be a minimal left ideal of the semiring F such that $K \neq V_u$ for every $u \in N$.

Lemma 3.9.1. *$K \cap V_u = \{\zeta\}$ for every $u \in N$.*

Proof. It is obvious. □

Lemma 3.9.2. *$Ff = K$ for every $f \in K \setminus \{\zeta\}$.*

Proof. If $f \in F \setminus \{\zeta\}$ then $f(v) \neq o_M$ for at least one $v \in N$ and $p_{f(v),v} \neq \zeta$. Now, we can proceed similarly as in the proof of 2.9.2. □

Lemma 3.9.3. *There is a proper ideal A of the semilattice M such that $A = \{x \in M \mid f(x) = o_M\}$ for every $f \in K \setminus \{\zeta\}$ and we put $B = M \setminus A$.*

Proof. Use 3.9.2. □

Lemma 3.9.4. *Let $f \in K \setminus \zeta$, $v \in M$, and let $o_m \neq w \in f(M)$. Then $p_{w,v} f = p_{A,v} \in K$, where $p_{A,v}(B) = \{v\}$ and $p_{A,v}(A) = \{o_M\}$.*

Proof. We have $g = p_{w,v} f \in K$ and, by 3.9.3, $A = \{x \in M \mid g(x) = o_M\} = \{x \in M \mid f(x) \not\leq w\}$. Thus $B = \{y \in M \mid g(y) \neq o_M\} = \{y \in M \mid f(y) \leq w\}$ and $g(B) = \{v\}$. □

Proposition 3.9.5. (i) A is a prime ideal of M , $o_B \notin B$ and $K = \{p_{A,v} \mid v \in M\} = V_A$.

(ii) The mapping $v \mapsto p_{A,v}$ is an isomorphism of the semimodule ${}_F M$ onto the semimodule ${}_F K$.

(iii) $Kp_{A,v} = \{\zeta\}$ for $v \in A$.

(iv) $Kp_{A,v} = K$ for $v \in B$.

(v) The set $\{p_{A,v} \mid v \in A\}$ is a bi-ideal of the semiring K and it is the greatest proper left ideal of that semiring.

(vi) The set $\{p_{A,v} \mid v \in B\} \cup \{\zeta\}$ is just the set of multiplicative idempotents of K ($\zeta = p_{A,o_M}$).

(vii) The semiring K is neither multiplicatively idempotent nor bi-ideal-simple.

Proof. According to 3.9.4, $V_A = \{p_{A,v} \mid v \in M\} \subseteq K$. The rest is clear (see 3.2, 3.3 and 3.4) $\square$

Theorem 3.10. (i) The set $V_u = \{p_{u,v} \mid v \in M\}$ is a minimal left ideal of the semiring F for every $u \in N$.

(ii) If K is a minimal left ideal of F such that $K \neq V_u$ for every $u \in N$ then there is a non-principal prime ideal A of the semilattice M such that $K = V_A = \{p_{A,v} \mid v \in M\}$.

(iii) If A is a non-principal prime ideal of M such that $p_{A,v} \in F$ for at least one $v \in M \setminus \{o_M\}$ then $V_A \subseteq F$ and V_A is a minimal left ideal of F such that $V_A \neq V_u$ for every $u \in N$.

(iv) If K is a minimal left ideal of F then ${}_F K$ is a faithful, simple and minimal F -semimodule and the semimodules ${}_F K$ and ${}_F M$ are isomorphic.

(v) If K is a minimal left ideal of F then the semiring K is multiplicatively idempotent if and only if $o_N \in N$ and $K = V_{o_N}$.

Proof. See the foregoing results. $\square$

Corollary 3.11. Denote by $\mathbf{B}$ the set of prime ideals A of M such that $V_A \subseteq F$ (see 3.10(ii),(iii)). Then:

(i) $\{V_A \mid A \in \mathbf{B}\}$ is just the set of (pair-wise distinct) minimal left ideals of the semiring F .

(ii) $A_u = \{x \in M \mid x \not\leq u\} \in \mathbf{B}$ for every $u \in N$.

(iii) The set $\bigcup V_A, A \in \mathbf{B}$, is just the smallest non-trivial ideal of the semiring F . $\square$

4. 1-preserving endomorphisms (b)

Let M be an *antichain*, i.e., a semilattice containing at least three elements such that $o_M \in M$ and $x + y = o_M$ for all $x, y \in M$, $x \neq y$, and put $N = M \setminus \{o_M\}$. Now, a transformation f of M belongs to $\underline{E}_1$ iff $f(o_M) = o_M$ and $f(x) \neq f(y)$ for all $x, y \in M$ such that $x \neq y$ and $f(x) \neq o_M$.

Proposition 4.1. *Let E be a subsemiring of $\underline{E}_1$ such that $\zeta \in E$. The following conditions are equivalent:*

- (i) *The semimodule ${}_E M$ is minimal.*
- (ii) *$N \subseteq E(x) = \{f(x) \mid f \in E\}$ for every $x \in N$.*
- (iii) *$E(x) = M$ for every $x \in N$.*
- (iv) *The semimodule ${}_E M$ is simple.*

Proof. Clearly, the first three conditions are equivalent. Now, let α be a congruence of ${}_E M$ and $P = \{x \in M \mid (x, o_M) \in \alpha\}$. Then P is a subsemimodule of M and if M is minimal then $P = \{o_M\}$. On the other hand, if $(x, y) \in \alpha$, $x \neq y$, then $(x, o_M) = (x + x, x + y) \in \alpha$ and $x \in P$. Thus (i) implies (iv). Conversely, for every $x \in M$, the relation $\alpha_x = (E(x) \times E(x)) \cup \text{id}_M$ is a congruence of ${}_E M$. If $\alpha = M \times M$ then $E(x) = M$. If $\alpha_x = \text{id}_M$ then $E(x) = \{o_M\}$ and $x \in Q = \{y \in M \mid F(y) = \{o_M\}\}$. Of course, Q is a subsemimodule of ${}_E M$ and $\beta = (Q \times Q) \cup \text{id}_M$ is a congruence of ${}_E M$. If $\beta = \text{id}_M$ then $x = o_M$. If $\beta = M \times M$ then $Q = M$, $E(M) = \{o_M\}$ and ${}_E M$ is not simple. Thus (iv) implies (iii). $\square$

Lemma 4.2. *Let $f, g \in \underline{E}_1$. Then $f + g = \zeta$ iff $f(x) = o_M$ whenever $x \in M$ is such that $f(x) = g(x)$.*

Proof. It is obvious. $\square$

In the remaining part of this section, let E be a non-trivial subsemiring of $\underline{E}_1$ satisfying the equivalent conditions of 4.1 and let K be a left ideal of E .

Lemma 4.3. *Let $w \in M$. The mapping $\tau_w : f \mapsto f(w)$ is a homomorphism of the semimodule ${}_E K$ into the semimodule ${}_E M$. If $K(w) \neq \{o_M\}$ then $\tau(K) = M$.*

Proof. It is easy. $\square$

Proposition 4.4. *If the semimodule ${}_E K$ is simple then K is a minimal left ideal of E and the semimodules ${}_E K$ and ${}_E M$ are isomorphic.*

Proof. Since ${}_E K$ is simple, the left ideal K is non-trivial and, taking $f \in K \setminus \{o_M\}$, we find $w \in M$ with $f(w) \neq o_M$. Then $K(w) = M$ and τ_w is an isomorphism of the semimodules (use 4.3). $\square$

Lemma 4.5. *For every $w \in M$, the set $L_w = \{f \in E \mid f(w) = o_M\}$ is a left ideal of the semiring E .*

Proof. It is obvious. $\square$

Lemma 4.6. *Let $w \in M$ be such that $K \cap L_w = \{\zeta\}$ (e.g., if K is minimal and $K \not\subseteq L_w$). If $f, g \in K$ are such that $f(w) \neq g(w)$ then $f + g = \zeta$.*

Proof. We have $f + g \in K \cap L_w$. □

Lemma 4.7. *If K is minimal then $f + g = \zeta$ for all $f, g \in K$, $f \neq g$.*

Proof. Since $f \neq g$, there is $w \in M$ with $f(w) \neq g(w)$. Clearly, $K \not\subseteq L_w$ and 4.6 applies. □

Theorem 4.8. *If K is a minimal left ideal of the semiring E then the semi-modules ${}_E K$ and ${}_E M$ are isomorphic.*

Proof. Since K is non-trivial, there is $w \in M$ with $K(w) \neq \{o_M\}$. Then $K \cap L_w = \{\zeta\}$ and, in view of 4.3, we have to show that $\ker \tau_w = \text{id}_K$. If $f, g \in K$ are such that $f(w) = g(w)$ then $f(w) = (f + g)(w) = g(w)$ and either $f(w) = o_M = g(w)$ and $f = \zeta = g$, or $(f + g)(w) \neq o_M$, $f + g \neq \zeta$ and $f = g$ by 4.7. □

Remark 4.9. Assume that K is a minimal left ideal and let $w \in M$ be such that $K(w) = M$ (equivalently, $K \not\subseteq L_w$). If $f \in K$ then $Kf = \{\zeta\}$ iff $K \subseteq L_{f(w)}$. On the other hand, if $Kf \neq \{\zeta\}$ then $Kf = K$, $K(f(w)) = M$ and $K \cap L_{f(w)} = \{\zeta\}$.

Lemma 4.10. *If $f \in \underline{E}_1$ is such that $f^2 = f$ then $f(x) \in \{x, o_M\}$ for every $x \in M$.*

Proof. It is obvious. □

Lemma 4.11. *Let K be a minimal left ideal, $A = \{w \in N \mid K \subseteq L_w\}$ and let $f \in K$ be such that $f^2 = f \neq \zeta$. Then:*

- (i) $f(x) = x$ for every $x \in M \setminus A$.
- (ii) f is right multiplicatively neutral in K .
- (iii) If $A = \emptyset$ then $f = \text{id}_M$ and $K = E$.
- (iv) If $A \neq \emptyset$ then $K = \bigcap L_w$, $w \in A$.

Proof. (i) This follows from 4.10.

(ii) If $g \in K$ then $gf(x) = g(x)$ for every $x \in M \setminus A$ and $gf(y) = o_M = g(y)$ for every $y \in A$. Thus $gf = g$.

(iii) By (i), $f = \text{id}_M$.

(iv) If $g \in \bigcap L_w$ then $gf = g$. □

Corollary 4.12. *Let K be a minimal left ideal of E . Then:*

- (i) K contains at most two multiplicatively idempotent elements.
- (ii) K is not multiplicatively idempotent. □

5. 1-preserving endomorphisms (c)

Let K and L be semilattices containing at least three elements and such that $o_K \in K$ and $o_L \in L$. The set $Q = (\{o_K\} \times L) \cup (K \times \{o_L\})$ is an ideal of the cartesian product $K \times L$ and we put $M = (K \times L)/Q = \{o_M\} \cup (K' \times L')$, $K' = K \setminus \{o_K\}$, $L' = L \setminus \{o_L\}$. Clearly, $|M| \geq 5$.

If $f \in \underline{E}_1(K)$ and $g \in \underline{E}_1(L)$ then $(f \times g)(Q) \subseteq Q$ and we put $f * g = (f \times g)/Q \in \underline{E}_1(M)$. Thus $(f_1 * g_1) + (f_2 * g_2) = (f_1 + f_2) * (g_1 + g_2)$, $(f_1 * g_1)(f_2 * g_2) = f_1 f_2 * g_1 g_2$, and hence $\underline{E}_1(K) * \underline{E}_1(L)$ is a subsemiring of $\underline{E}_1(M)$. In fact, if E is a subsemiring of $\underline{E}_1(K)$ and F is a subsemiring of $\underline{E}_1(L)$ then $E * F$ is a subsemiring of $\underline{E}_1(M)$.

In the remaining part of this section, assume that L is an antichain and take a non-trivial subgroup G of $\text{Aut}(L)$ such that $f(x) \neq g(x)$ for all $x \in L'$, $f, g \in G$, $f \neq g$. Let S be a subsemiring of $\underline{E}_1(K) * (G \cup \{\zeta_L\})$ such that $p_{u,v} * g \in S$ for all $u \in K'$, $v \in K$ and $g \in G$ (see the preceding two sections). One checks easily that $o_S = \zeta_M = p_{u,o_K} * g \in S$ is the bi-absorbing element of S and $AA \subseteq A$, where $A = \{f \in \underline{E}_1(K) \mid f * g \in S, g \in G\}$. Furthermore, $S \subseteq \underline{E}_1(K) * G$ and $B = \{f \in \underline{E}_1(K) \mid f * \text{id}_M \in S\}$ is a subsemiring of $\underline{E}_1(K)$ such that $p_{u,v} \in B$ for all $u \in K'$ and $v \in K$.

Lemma 5.1. *Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then:*

- (i) *If $g_1 = g_2$ then $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = p_{u,v_1+v_2} * g_1$.*
- (ii) *If $g_1 \neq g_2$ then $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = \zeta_M$.*

Proof. We have $(p_{u,v_1} * g_1) + (p_{u,v_2} * g_2) = (p_{u,v_1+v_2} * (g_1 + g_2))$. □

Lemma 5.2. *Let $u \in K'$, $v \in K$, $g \in G$ and let $f \in \underline{E}_1(K)$ and $h \in G$ be such that $f * h \in S$. Then $(f * h)(p_{u,v} * g) = p_{u,f(v)} * (hg)$.*

Proof. It is obvious. □

Proposition 5.3. *For every $u \in K'$, the set $W_u = \{p_{u,v} * g \mid v \in K, g \in G\}$ is a minimal left ideal of the semiring S .*

Proof. According to 5.1 and 5.2, the set W_u is a left ideal of S . If $v \in K'$, $w \in K$ and $g, h \in S$ then $p_{v,w} * hg^{-1}(p_{u,v} * g) = p_{u,w} * h$. Thus $S(p_{u,v} * g) = W_u$. □

Lemma 5.4. *Let $f_1, f_2 \in \underline{E}_1(K)$ and $g_1, g_2 \in G$ be such that $f_1 * g_1 = f_2 * g_2$. Then $f_1 = f_2$ and if $f_1 \neq \zeta_K$ then $g_1 = g_2$.*

Proof. Let $x_0 \in K$ be such that $f_1(x_0) \neq f_2(x_0)$. Then $x_0 \in K'$, $(f_1(x_0), g_1(y)) \in Q$ and $(f_2(x_0), g_2(y)) \in Q$ for every $y \in L'$. If $f_1(x_0) \neq o_K$ then $g_1(y) = o_L$, a contradiction. Thus $f_1(x_0) = o_K$ and, similarly, $f_2(x_0) = o_K$. It follows that $f_1 = f_2$. The rest is clear. □

Lemma 5.5. *Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then $p_{u,v_1} * g_1 = p_{u,v_2} * g_2$ iff $v_1 = v_2$ and either $g_1 = g_2$ or $v_1 = o_K$.*

Proof. This follows from 5.4. $\square$

Proposition 5.6. *Let $u \in K'$ and $z \in L'$. Define a mapping $\nu_{u,z} : W_u \rightarrow M$ by $\nu_{u,z}(p_{u,v} * g) = (v, g(z))$ for $v \in K'$ and $\nu_{u,z}(o_S) = o_M$ (see 5.5). Then $\nu_{u,z}$ is an injective homomorphism of the semimodule ${}_S W_u$ into the semimodule ${}_S M$. The homomorphism $\nu_{u,z}$ is an isomorphism of the semimodules iff $G(z) = L'$ (i.e., the group G operates transitively on L').*

Proof. Use 5.1, ..., 5.5. $\square$

Proposition 5.7. *Let $u \in K'$. The semimodule ${}_S W_u$ is simple, minimal and faithful.*

Proof. The semimodule is minimal due to 5.3. By 5.2, $(f * h)(p_{u,v} * \text{id}_L) = p_{u,f(v)} * h$ and, using 5.5, we conclude easily the our semimodule is faithful as well.

It remains to show that ${}_S W_u$ is simple. For, let $\alpha \neq \text{id}$ be a congruence of the semimodule. Then $(p_{u,v_1} * g_1, p_{u,v_2} * g_2) \in \alpha$, where $p_{u,v_1} * g_1 \neq p_{u,v_2} * g_2$ and we can assume that $v_1 \in K'$. If $g_1 \neq g_2$ then $(p_{u,v_1} * g_1, \zeta) \in \alpha$ follows from 5.1(ii). But $S(p_{u,v_1} * g_1) = W_u$ by 5.3, and hence $\alpha = W_u \times W_u$. Assume, therefore, that $g_1 = g_2 = g$ and $v_2 \not\leq v_1$. Then $(p_{u,v_1} * g, p_{u,v_3} * g) \in \alpha$, where $v_1 < v_3 = v_1 + v_2$. From this, $(p_{u,v_4} * g, \zeta) = ((p_{v_1,v_4} * g)(p_{u,v_1} * g), (p_{v_1,v_4} * g)(p_{u,v_3} * g)) \in \alpha$ for every $v_4 \in K$. Thus $\alpha = W_u \times W_u$. $\square$

Remark 5.8. S is a subsemiring of $\underline{E}_1(M)$, and so the (left S -) semimodule ${}_S M$ is faithful. In view of 5.6, the semimodule ${}_S M$ is minimal iff G operates transitively on L' , and then the semimodules ${}_S M$ and ${}_S W_u$ are isomorphic, so that ${}_S M$ is simple by 5.7. If $y \in L'$ then $R = (K' \times G(y)) \cup \{o_M\}$ is an ideal of the semimodule ${}_S M$ and $(R \times R) \cup \text{id}_M$ is a congruence of ${}_S M$. If $\tau = M \times M$ (e.g., if ${}_S M$ is simple) then $R = M$ and G operates transitively on L' .

Proposition 5.9. (i) $W_u(p_{u,v} * g) = \{\zeta\}$ for $v \not\leq u$.
(ii) $W_u(p_{u,v} * g) = W_u$ for $v \leq u$.

Proof. It is easy. $\square$

Proposition 5.10. (i) *The set $\{p_{u,v} * g \mid v \not\leq u\}$ is a bi-ideal of the semiring W_u and it is the greatest proper left ideal of that semiring.*

(ii) *The set $\{p_{u,v} * \text{id}_L \mid v \leq u\} \cup \{\zeta\}$ is just the set of multiplicative idempotents of W_u .*

(iii) *The semiring W_u is not multiplicatively idempotent.*

(iv) *The semiring W_u is not bi-ideal-simple.*

Proof. Easy. $\square$

5.11 Let R be a minimal left ideal of the semiring S such that $R \neq W_u$ for every $u \in N$.

Lemma 5.11.1. $R \cap W_u = \{\zeta\}$.

Proof. It is obvious. $\square$

Lemma 5.11.2. $S(f * g) = R$ for every $f * g \in R \setminus \{\zeta\}$.

Proof. We have $f \in E_1(K) \setminus \{\zeta_K\}$, and hence $f(v) \neq o_K$ for at least one $v \in N$ and $(p_{f(v),v} * \text{id}_M)(f * g) = (p_{f(v),v} f * g) \neq \zeta$. Now, we can proceed similarly as in the proof of 2.9.2. $\square$

Lemma 5.11.3. *There is a proper ideal A of the semilattice K such that $A = \{x \in K \mid f(x) \neq o_K\}$ for every $f * g \in R \setminus \{\zeta\}$.*

Proof. Use 5.11.2. $\square$

Lemma 5.11.4. *Let $f * g \in R \setminus \{\zeta\}$, $v \in M$ and let $o_K \neq w \in f(K)$. Then $p_{w,v} f * g = p_{A,v} * g \in K$, where $B = K \setminus A$, $p_{A,v}(B) = \{v\}$ and $p_{A,v}(A) = \{o_K\}$.*

Proof. We have $p_{w,v} f * g = (p_{w,v} * \text{id}_L)(f * g) \in R$ and, by 5.11.3, $A = \{x \in K \mid p_{w,v} f(x) = o_K\} = \{x \in K \mid f(x) \not\leq w\}$. Thus $B = \{y \in K \mid f(y) \neq o_K\} = \{y \in K \mid f(y) \leq w\}$ and $p_{w,v} f(B) = \{v\}$. $\square$

Proposition 5.11.5. (i) A is a prime ideal of the semilattice K , $o_B \notin B$ and $W_A = \{p_{A,v} * g \mid v \in K, g \in G\} = R$.

(ii) The mapping $\nu_{A,z} : R \rightarrow M$, where $z \in L'$ and $\nu_{A,z}(p_{A,v} * g) = (v, g(z))/Q$ is an injective homomorphism of the semimodule ${}_S R$ into the semimodule ${}_S M$. This homomorphism is an isomorphism of the semimodules iff G operates transitively on L' .

(iii) $R(p_{A,v} * g) = \{\zeta\}$ for $v \in A$.

(iv) $R(p_{A,v} * g) = R$ for $v \in B$.

(v) The set $\{p_{A,v} * g \mid v \in A, g \in G\}$ is a bi-ideal of the semiring R and it is the greatest proper left ideal of that semiring.

(vi) The set $\{p_{A,v} * \text{id}_L \mid v \in B\} \cup \{\zeta\}$ is just the set of multiplicative idempotents of R .

(vii) The semiring R is neither multiplicatively idempotent nor bi-ideal-simple.

Proof. (i) First, take $f * g \in R \setminus \{\zeta\}$. By 5.11, we get $p_{A,v} * g \in R$ for every $v \in K$. If $v \in K'$ and $h \in G$ then $p_{A,v} * h = (p_{A,v} * hg^{-1})(p_{A,v} * g) \in R$. Thus $W_A \subseteq R$. On the other hand, W_A is a non-trivial left ideal of S and, R being minimal, we find that $R = W_A$.

(ii) We can proceed similarly as in the proof of 5.6.

The remaining assertions are easy to check. $\square$

Theorem 5.12. (i) *The set $W_u = \{p_{u,v} * g \mid v \in K, g \in G\}$ is a minimal left ideal of the semiring S for every $u \in K'$.*

(ii) *if R is a minimal left ideal of S such that $R \neq W_u$ for every $u \in K'$ then there is a non-principal prime ideal A of the semilattice K such that $R = W_A = \{p_{A < v} * g \mid v \in K, g \in G\}$, where $p_{A,v}(A) = \{o_K\}$ and $p_{A,v}(K \setminus A) = \{v\}$.*

(iii) *If A is a non-principal prime ideal of K such that $p_{A,v} * g \in S$ for at least one $v \in K'$ and at least one $g \in G$ then $W_A \subseteq S$ and W_A is a minimal left ideal of S such that $W_A \neq W_u$ for every $u \in K'$.*

(iv) *If P is a minimal left ideal of S then ${}_S P$ is a faithful, simple and minimal (left S -) semimodule. Besides, if $z \in L'$ then ${}_S P$ is isomorphic to the subsemimodule ${}_S(K \times G(z))/Q$ of ${}_S M$.*

(v) *If P_1 and P_2 are minimal left ideals of S then the semimodules ${}_S P_1$ and ${}_S P_2$ are isomorphic.*

(vi) *If P is a minimal left ideal of S then the semiring P is neither multiplicatively idempotent nor bi-ideal-simple.*

Proof. Combine the foregoing results. □

Corollary 5.13. *Denote by $\mathbf{C}$ the set of prime ideals A of K such that $W_A \subseteq S$. Then:*

(i) *The set $\{W_A \mid A \in \mathbf{C}\}$ is just the set of (pair-wise distinct) minimal left ideals of the semiring S .*

(ii) *$A_u = \{x \in K \mid x \not\leq u\} \in \mathbf{C}$ for every $u \in K'$.*

(iii) *The set $\bigcup W_A$, $A \in \mathbf{C}$, is just the smallest non-trivial ideal of the semiring S .* □

Proposition 5.14. *The following conditions are equivalent:*

(i) *There is a minimal left ideal R of S such that $R(+)$ is an antichain.*

(ii) *For every minimal left ideal P of the semiring S , the semilattice $P(+)$ is an antichain.*

(iii) *The semilattice K is an antichain.*

(iv) *The semilattice M is an antichain.*

Proof. It is easy. □

Lemma 5.15. *The following conditions are equivalent for a minimal left ideal P of S :*

(i) *The set $P \setminus \{\zeta\}$ is a subsemigroup of the multiplicative semigroup of P .*

(ii) *$K' + K' = K'$ and $P = W_A$, where $A = \{o_K\}$.*

Proof. It is easy. □

6. 0,1-preserving endomorphisms

Let M be a non-trivial semilattice such that $0_M, o_M \in M$. The set $\underline{E}_{0,1} = \{f \in \underline{E} \mid f(0_M) = 0_M, f(o_M) = o_M\}$ is a unitary subsemiring of $\underline{E}$. For every $u \in N \setminus \{o_M\}$, we have $r_u \in \underline{E}_{0,1}$, where $r_u(x) = 0_M$ for $x \leq u$ and $r_u(y) = o_M$ for $y \not\leq u$. Clearly, r_{0_M} is additively absorbing.

Now, let E be a subsemiring of $\underline{E}_{0,1}$ such that $r_u \in E$ for every $u \in N$. For every pair $(u, v) \in N \times N$, $u < v$, the set $X_{(u,v)} = \{r_u, r_v\}$ is a minimal left ideal of E .

6.1 Let K be a minimal left ideal of E such that $K \neq X_{(u,v)}$ for all $u, v \in N$, $u < v$.

Lemma 6.1.1. *Let $u \in N$, $f \in K$ and $A = A_{f,u} = \{x \in M \mid f(x) \not\leq u\}$. Then:*
 (i) *A is a prime ideal of M .*
 (ii) *$r_u f = r_A \in K$, where $r_A(A) = \{o_M\}$ and $r_A(B) = \{o_M\}$, $B = M \setminus A$.*
 (iii) *If $u = o_B \in B$ then $r_A = r_U$.*

Proof. It is easy. □

Lemma 6.1.2. *Let A_1, A_2 be prime ideals of M such that $r_{A_1}, r_{A_2} \in K$. Then either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.*

Proof. Let, on the contrary, $A_1 \not\subseteq A_2 \not\subseteq A_1$. The set $A_3 = A_1 \cup A_2$ is a prime ideal and $r_{A_3} = r_{A_1} + r_{A_2} \in K$. Now, the sets $K_1 = \{r_{A_1}, r_{A_3}\}$ and $K_2 = \{r_{A_2}, r_{A_3}\}$ are two-element left ideals contained in K . Consequently, $K_1 = K_2 = K$, $r_{A_1} = r_{A_2}$ and $A_1 = A_2$, a contradiction. □

Corollary 6.1.3. *Let A_1, A_2 be two different prime ideals such that $r_{A_1}, r_{A_2} \in K$. Then $K = \{r_{A_1}, r_{A_2}\}$ and either $A_1 \subset A_2$ or $A_2 \subset A_1$.* □

Now, assume that $r_A \in K$ for exactly one prime ideal A and put $B = M \setminus A$ (see 6.1.1).

Lemma 6.1.4. *$f(A) = \{o_M\}$ for every $f \in K$.*

Proof. We have $A = \{x \in M \mid f(x) \not\leq u\}$ for every $u \in N$ (use 6.1.1). Thus $f(A) = \{o_M\}$. □

Lemma 6.1.5. *$f(B) = \{0_M\}$ for every $f \in K$.*

Proof. We have $B = M \setminus A = \{y \in M \mid f(y) \leq 0_M\}$. □

Corollary 6.1.6. *$K = \{r_A\}$.* □

Theorem 6.2. (i) *For all $u, v \in N$, $u < v$, the set $\{r_u, r_v\}$ is a minimal left ideal of E .*

(ii) *If K is a minimal left ideal of E then there are prime ideals A_1, A_2 of M such that $A_1 \subset A_2$ and $K = \{r_{A_1}, r_{A_2}\}$.*

Proof. See 6.1. □

Theorem 6.3. Denote by $\mathbf{D}$ the set of prime ideals of M such that $r_A \in E$ (see 6.1). Then:

- (i) Minimal left ideals of the semiring E are just the two-element sets $\{r_{A_1}, r_{A_2}\}$, $A_1, A_2 \in \mathbf{D}$, $A_1 \subset A_2$.
- (ii) $A_u = \{x \in M \mid x \not\leq u\} \in \mathbf{D}$ for every $u \in N$.
- (iii) $P = M \setminus \{0_M\} \in \mathbf{D}$ and $r_P = r_{0_M}$.
- (iv) For every $A \in \mathbf{D}$, $A \neq P$, the two-element set $\{r_A, r_{0_M}\}$ is a minimal left ideal of E .
- (v) $\{r_A \mid A \in \mathbf{D}\}$ is an ideal and it is the smallest right ideal of E .

Proof. It is easy (use 6.2). □

7. Non-preserving endomorphisms

Let M be a non-trivial semilattice and let E be a subsemiring of $\underline{E}(M)$ containing all constant endomorphisms μ_u , $\mu_u(M) = \{u\}$, $u \in M$. Now, the set $Q = \{\mu_u \mid u \in M\}$ is an ideal and, in fact, it is the smallest left ideal of the semiring E . The two-element sets $\{\mu_u, \mu_v\}$, where $u < v$, are minimal right ideals and there are no more. The semimodules ${}_E M$ and ${}_E Q$ are isomorphic via $u \mapsto \mu_u$.

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On some (p, q) - φ relative Gol'dberg type and (p, q) - φ relative Gol'dberg weak type based growth properties of entire functions of several complex variables

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Abstract. The primary concern of this paper is to introduce the notion of (p, q) - φ relative Gol'dberg type, (p, q) - φ relative Gol'dberg weak type of entire functions of several complex variables and study some growth properties based upon them.

Keywords: Entire functions of several complex variables, (p, q) - φ relative Gol'dberg order, (p, q) - φ relative Gol'dberg lower order, (p, q) - φ relative Gol'dberg type, (p, q) - φ relative Gol'dberg weak type.

1. Introduction, definitions and notations

The symbols $\mathbb{C}^n$ and R^n will stand for complex and real n -spaces respectively. In addition, let us assume that the points $(z_1, z_2, \dots, z_n)$, $(m_1, m_2, \dots, m_n)$ of $\mathbb{C}^n$ or I^n be represented by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. Then the modulus of z , denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \dots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then the expression $z_1^{m_1} \dots z_n^{m_n}$ will be denoted by z^m where $\|m\| = m_1 + \dots + m_n$.

Consider $D \subseteq \mathbb{C}^n$ to be an arbitrary bounded complex n -circular domain with center at the origin of coordinates. Then for any entire function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists such that $\lim_{R \rightarrow \infty} M_{f,D}^{-1}(R) = \infty$.

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For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p , q and m always denote positive integers. Also throughout the paper an entire function $f(z)$ of n -complex variables will stand for an entire function $f(z)$ for any bounded complete n -circular domain D with center at origin in $\mathbb{C}^n$. Taking this into account, we recall that Datta et al. [6] defined the concept of (p, q) -th Gol'dberg order and (p, q) -th Gol'dberg lower order of an entire function $f(z)$ of n -complex variables where $p \geq q$ in the following way:

$$\rho_D^{(p,q)}(f) = \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}$$

$$\lambda_D^{(p,q)}(f) = \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}.$$

For $p = 2$ and $q = 1$, the symbols $\rho_D^{(2,1)}(f)$ and $\lambda_D^{(2,1)}(f)$ are respectively denoted by $\rho_D(f)$ and $\lambda_D(f)$ which are actually classical growth indicators (see e.g. [9, 10]). However in the line of Gol'dberg (see e.g. [9, 10]), it may be easily established that $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are independent of the choice of the domain D , and therefore one can write $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ respectively.

In [13], Shen et al. introduced the definition of (p, q) - φ order of an entire function. For details about (p, q) - φ order, one may see [13]. Consequently the definition of (p, q) - φ Gol'dberg order and (p, q) - φ Gol'dberg lower order of an entire function $f(z)$ of n -complex variables are established in [4] which are as follows:

Definition 1 ([4]). *Let $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function. Then the (p, q) - φ Gol'dberg order $\rho_D^{(p,q)}(f, \varphi)$ and (p, q) - φ Gol'dberg lower order $\lambda_D^{(p,q)}(f, \varphi)$ of an entire function $f(z)$ of n -complex variables are defined as*

$$\rho_D^{(p,q)}(f, \varphi) = \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}$$

$$\lambda_D^{(p,q)}(f, \varphi) = \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}.$$

The above definition avoids the restriction $p \geq q$. However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f, \varphi)$ and $\lambda_D^{(p,q)}(f, \varphi)$ are the same is called a function of regular (p, q) - φ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p, q) - φ Gol'dberg growth. For any non-decreasing unbounded function $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$, if it is assumed that $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$, then one can easily verify that $\rho_D^{(p,q)}(f, \varphi)$ and $\lambda_D^{(p,q)}(f, \varphi)$ are independent of

the choice of the domain D , and therefore one can use the symbols $\rho^{(p,q)}(f, \varphi)$ and $\lambda^{(p,q)}(f, \varphi)$ instead of $\rho_D^{(p,q)}(f, \varphi)$ and $\lambda_D^{(p,q)}(f, \varphi)$ respectively.

If $\varphi(R) = R$ and $p \geq q$, then Definition 1 coincides with the definition of (p, q) -th Gol'dberg order and (p, q) -th Gol'dberg lower order introduced by Datta et al. [6].

Concerning this we just state the following definition:

Definition 2. *An entire function $f(z)$ of n -complex variables is said to have index-pair (p, q) - φ if $b < \rho^{(p,q)}(f, \varphi) < \infty$ and $\rho^{(p-1, q-1)}(f, \varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f, \varphi) < \infty$, then*

$$\begin{cases} \rho^{(p-n, q)}(f, \varphi) = \infty, & \text{for } n < p, \\ \rho^{(p, q-n)}(f, \varphi) = 0, & \text{for } n < q, \\ \rho^{(p+n, q+n)}(f, \varphi) = 1, & \text{for } n = 1, 2, \dots \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f, \varphi) < \infty$,

$$\begin{cases} \lambda^{(p-n, q)}(f, \varphi) = \infty, & \text{for } n < p, \\ \lambda^{(p, q-n)}(f, \varphi) = 0, & \text{for } n < q, \\ \lambda^{(p+n, q+n)}(f, \varphi) = 1, & \text{for } n = 1, 2, \dots \end{cases}$$

Consequently for $\varphi(R) = R$, Definition 2 reduces to the the definition of index-pair (p, q) of an entire function $f(z)$ of n -complex variables. For detail about index-pair (p, q) of an entire function $f(z)$ of n -complex variables, one may see [3].

However for any two entire functions $f(z)$ and $g(z)$ of n -complex variables, Mondal et al. [11] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$. In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p, q) -th relative Gol'dberg order. With this in view one can introduce the following definition in the light of index-pair.

Definition 3 ([3]). *Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively. Then the (p, q) -th relative Gol'dberg order $\rho_{g,D}^{(p,q)}(f)$ and (p, q) -th relative Gol'dberg lower order $\lambda_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as*

$$\frac{\rho_{g,D}^{(p,q)}(f)}{\lambda_{g,D}^{(p,q)}(f)} = \lim_{R \rightarrow +\infty} \sup \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R}.$$

Definition 3 avoids the restriction $p \geq q$ of Definition 1.3 of [1]. In view of Theorem 2.1 of [1] one can easily prove that $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain D , and therefore one can write $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$.

Further an entire function $f(z)$ of n -complex variables for which $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ are the same is called a function of regular relative (p, q) Gol'dberg growth with respect to an entire function $g(z)$ of n -complex variables. Otherwise, $f(z)$ is said to be irregular relative (p, q) Gol'dberg growth with respect to $g(z)$.

In [4], Biswas has further introduced the notion of relative (p, q) -th Gol'dberg type $\sigma_{g,D}^{(p,q)}(f)$, relative (p, q) -th Gol'dberg lower type $\bar{\sigma}_{g,D}^{(p,q)}(f)$, relative (p, q) -th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$ and another growth indicator $\bar{\tau}_{g,D}^{(p,q)}(f)$ in the light of index-pair which are as follows:

Definition 4 ([3]). *Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively, such that $0 < \rho_g^{(p,q)}(f) < +\infty$. Then the relative (p, q) -th Gol'dberg type $\sigma_{g,D}^{(p,q)}(f)$, relative (p, q) -th Gol'dberg lower type $\bar{\sigma}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\frac{\sigma_{g,D}^{(p,q)}(f)}{\bar{\sigma}_{g,D}^{(p,q)}(f)} = \lim_{R \rightarrow +\infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} R) \rho_g^{(p,q)}(f)}$$

Definition 5 ([3]). *Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively, such that $0 < \lambda_g^{(p,q)}(f) < +\infty$. Then the relative (p, q) -th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$, and the growth indicator $\bar{\tau}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\frac{\bar{\tau}_{g,D}^{(p,q)}(f)}{\tau_{g,D}^{(p,q)}(f)} = \lim_{R \rightarrow +\infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} R) \lambda_g^{(p,q)}(f)}$$

Since Gol'dberg has shown that (see [9, 10]) Gol'dberg type depends on the domain D , all the growth indicators defined in Definition 4 and Definition 5 also depend on D (cf. [3]).

Now in order to make some progress in the study of relative Gol'dberg order, in [4], the definition of (p, q) - φ relative Gol'dberg order and the (p, q) - φ relative Gol'dberg lower order in view of index-pair are given which are as follows:

Definition 6 ([4]). *Let $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function. Also let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively. The (p, q) - φ relative Gol'dberg order and the (p, q) - φ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f, \varphi) = \limsup_{R \rightarrow +\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}$$

and

$$\lambda_{g,D}^{(p,q)}(f, \varphi) = \liminf_{R \rightarrow +\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

Further an entire function $f(z)$ of n -complex variables for which $\rho_{g,D}^{(p,q)}(f, \varphi)$ and $\lambda_{g,D}^{(p,q)}(f, \varphi)$ are the same is called a function of regular (p, q) - φ relative Gol'dberg growth with respect to an entire function $g(z)$ of n -complex variables. Otherwise, $f(z)$ is said to be irregular (p, q) - φ relative Gol'dberg growth with respect to $g(z)$.

In this paper, we assume that the nondecreasing unbounded function $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ always satisfies $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [4] have already shown that $\rho_{g,D}^{(p,q)}(f, \varphi)$ and $\lambda_{g,D}^{(p,q)}(f, \varphi)$ are independent of the choice of the domain D when $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ is a nondecreasing unbounded function and satisfies $\lim_{R \rightarrow +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$, so here we shall always use the notations $\rho_g^{(p,q)}(f, \varphi)$ and $\lambda_g^{(p,q)}(f, \varphi)$ instead of $\rho_{g,D}^{(p,q)}(f, \varphi)$ and $\lambda_{g,D}^{(p,q)}(f, \varphi)$ respectively.

Now, for the development of such growth indicators, one may introduce (p, q) - φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f, \varphi)$, (p, q) - φ relative Gol'dberg lower type $\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)$, (p, q) - φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f, \varphi)$ and another growth indicator $\bar{\tau}_{g,D}^{(p,q)}(f, \varphi)$ with the help of index-pair in the following way:

Definition 7. Let $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively, such that $0 < \rho_g^{(p,q)}(f, \varphi) < +\infty$. Then the (p, q) - φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f, \varphi)$ and the (p, q) - φ relative Gol'dberg lower type $\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\frac{\sigma_{g,D}^{(p,q)}(f, \varphi)}{\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)} = \lim_{R \rightarrow +\infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} \varphi(R)) \rho_g^{(p,q)}(f, \varphi)}.$$

Definition 8. Let $\varphi(R) : [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n -complex variables with index-pairs (m, q) and (m, p) respectively, such that $0 < \lambda_g^{(p,q)}(f, \varphi) < +\infty$. Then the (p, q) - φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f, \varphi)$ and the growth indicator $\bar{\tau}_{g,D}^{(p,q)}(f, \varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\frac{\bar{\tau}_{g,D}^{(p,q)}(f, \varphi)}{\tau_{g,D}^{(p,q)}(f, \varphi)} = \lim_{R \rightarrow +\infty} \sup \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} \varphi(R)) \lambda_g^{(p,q)}(f, \varphi)}.$$

As earlier it was discussed that the relative (p, q) -th Gol'dberg type, relative (p, q) -th Gol'dberg lower type and all the other growth indicators mentioned in Definition 4 and Definition 5 depend on the particular choice of the domain D , naturally the growth indicators introduced in Definition 7 and Definition 8, in general, also depend upon the domain D .

In this paper we wish to study some growth properties based upon the notion of (p, q) - φ relative Gol'dberg type, (p, q) - φ relative Gol'dberg weak type and the other growth indicators of entire functions of several complex variables.

2. Lemmas

In this section we state some lemmas which will be frequently used to prove the main results of the paper.

Lemma 1 ([5]). *Let $g(z)$ be any entire function of n complex variables with regular (m, p) - φ growth and with non zero finite (m, p) - φ Gol'dberg order. Also let $f(z)$ be another entire function of n complex variables with index-pair (m, q) . Then*

$$\rho_g^{(p,q)}(f, \varphi) = \frac{\rho^{(m,q)}(f, \varphi)}{\rho^{(m,p)}(g)}.$$

Further when $0 < \lambda^{(m,q)}(f, \varphi) < \infty$, then

$$\lambda_g^{(p,q)}(f, \varphi) = \frac{\lambda^{(m,q)}(f, \varphi)}{\lambda^{(m,p)}(g)}.$$

The above lemma can easily be derived from the conclusion of Theorem 1 already mentioned in [5]. In the similar fashion, from Theorem 1 in [5] the following lemma may be proved:

Lemma 2 ([5]). *let $f(z)$ be any entire function of regular (m, q) - φ Gol'dberg growth with non zero finite (m, q) - φ Gol'dberg order. Also let $g(z)$ be another entire function with $0 < \lambda^{(m,p)}(g) < \infty$. Then*

$$\rho_g^{(p,q)}(f, \varphi) = \frac{\lambda^{(m,q)}(f, \varphi)}{\lambda^{(m,p)}(g)}.$$

Further if $0 < \rho^{(m,p)}(g) < \infty$, then

$$\lambda_g^{(p,q)}(f, \varphi) = \frac{\rho^{(m,q)}(f, \varphi)}{\rho^{(m,p)}(g)}.$$

3. Main results

In this section we present the main results of the paper.

Theorem 1. *Suppose that $g(z)$ and $f(z)$ are any two entire functions of n complex variables. Also let $g(z)$ be of regular (m, p) -Gol'dberg growth. Then*

$$\begin{aligned} & \left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \leq \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \\ & \leq \min \left\{ \left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\} \end{aligned}$$

$$\leq \max \left\{ \left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\}$$

$$\leq \sigma_{g,D}^{(p,q)}(f, \varphi) \leq \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

Proof. In view of the first part of Lemma 1, we obtain from the definitions of $\sigma_{g,D}^{(p,q)}(f, \varphi)$ and $\bar{\sigma}_{g,D}^{(p,q)}(f, \varphi)$ that

$$\begin{aligned} \log \sigma_{g,D}^{(p,q)}(f, \varphi) &= \limsup_{R \rightarrow +\infty} [\log^{[p]} M_{g,D}^{-1}(R) - \rho_g^{(p,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))] \\ &\quad i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \\ &= \limsup_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R) \\ &\quad - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))] \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) &= \liminf_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R) \\ &\quad - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))]. \end{aligned} \quad (3.2)$$

Now from the definitions of $\sigma_D^{(m,q)}(f, \varphi)$ and $\bar{\sigma}_D^{(m,q)}(f, \varphi)$, it follows that

$$\log \sigma_D^{(m,q)}(f, \varphi) = \limsup_{R \rightarrow +\infty} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))] \quad (3.3)$$

and

$$\log \bar{\sigma}_D^{(m,q)}(f, \varphi) = \liminf_{R \rightarrow +\infty} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))] . \quad (3.4)$$

Similarly, from the definitions of $\sigma_D^{(m,p)}(g)$ and $\bar{\sigma}_D^{(m,p)}(g)$, we get that

$$\log \sigma_D^{(m,p)}(g) = \limsup_{R \rightarrow +\infty} [\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)] \text{ and} \quad (3.5)$$

$$\log \bar{\sigma}_D^{(m,p)}(g) = \liminf_{R \rightarrow +\infty} [\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)] . \quad (3.6)$$

Therefore, from (3.2), (3.4) and (3.5), it follows that

$$\begin{aligned} \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) &= \liminf_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)) \\ &\quad - (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \end{aligned}$$

$$\begin{aligned}
& i.e., \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \geq \frac{1}{\rho^{(m,p)}(g)} [\liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \\
& \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) - \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
(3.7) \quad & i.e., \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \geq \frac{1}{\rho^{(m,p)}(g)} (\log \bar{\sigma}_D^{(m,q)}(f, \varphi) - \log \sigma_D^{(m,p)}(g)) .
\end{aligned}$$

Similarly, from (3.1), (3.3) and (3.6), we have

$$\begin{aligned}
& \log \sigma_{g,D}^{(p,q)}(f, \varphi) = \limsup_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \\
& \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)) - (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
& i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \leq \frac{1}{\rho^{(m,p)}(g)} [\limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \\
& \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) - \liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
(3.8) \quad & i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \leq \frac{1}{\rho^{(m,p)}(g)} (\log \sigma_D^{(m,q)}(f, \varphi) - \log \bar{\sigma}_D^{(m,p)}(g)) .
\end{aligned}$$

Again, in view of (3.2), (3.3), (3.4), (3.5) and (3.6), we obtain that

$$\begin{aligned}
& \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) = \liminf_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)) \\
& - (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
& i.e., \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \leq \frac{1}{\rho^{(m,p)}(g)} \times \\
& \min[\liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& + \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)), \\
& \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& + \liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
& i.e., \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \leq \frac{1}{\rho^{(m,p)}(g)} \times \\
& \min[\liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))
\end{aligned}$$

$$\begin{aligned}
& - \liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)), \\
& \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& - \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)) \\
(3.9) \quad & i.e., \log \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \leq \frac{1}{\rho^{(m,p)}(g)} \min[\log \bar{\sigma}_D^{(m,q)}(f, \varphi) \\
& - \log \bar{\sigma}_D^{(m,p)}(g), \log \sigma_D^{(m,q)}(f, \varphi) - \log \sigma_D^{(m,p)}(g)] .
\end{aligned}$$

Further from (3.1), (3.3), (3.4), (3.5) and (3.6), we obtain that

$$\begin{aligned}
& \log \sigma_{g,D}^{(p,q)}(f, \varphi) = \limsup_{R \rightarrow +\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)) \\
& - (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
& i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \geq \frac{1}{\rho^{(m,p)}(g)} \times \\
& \max[\liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& + \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)), \\
& \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& + \liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
& i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \geq \frac{1}{\rho^{(m,p)}(g)} \times \\
& \max[\liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& - \liminf_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)), \\
& \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) \\
& - \limsup_{R \rightarrow +\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))] \\
(3.10) \quad & i.e., \log \sigma_{g,D}^{(p,q)}(f, \varphi) \geq \frac{1}{\rho^{(m,p)}(g)} \max[\log \bar{\sigma}_D^{(m,q)}(f, \varphi) \\
& - \log \bar{\sigma}_D^{(m,p)}(g), \log \sigma_D^{(m,q)}(f, \varphi) - \log \sigma_D^{(m,p)}(g)] .
\end{aligned}$$

Thus the theorem follows from (3.7), (3.8), (3.9) and (3.10) . $\square$

Theorem 2. Let $g(z)$ and $f(z)$ be any two entire functions of n complex variables such that $0 < \lambda^{(m,p)}(g) < \infty$ and $f(z)$ be of regular (m, q) - φ Gol'dberg

growth with non zero finite (m, q) - φ Gol'dberg order. Then

$$\begin{aligned}
 & \left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \bar{\sigma}_{g,D}^{(p,q)}(f, \varphi) \\
 & \leq \min \left\{ \left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\} \\
 & \leq \max \left\{ \left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\} \\
 & \leq \sigma_{g,D}^{(p,q)}(f, \varphi) \leq \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}.
 \end{aligned}$$

Proof. From the definitions of $\bar{\tau}_D^{(m,q)}(f, \varphi)$ and $\tau_D^{(m,q)}(f, \varphi)$, we obtain that

$$\begin{aligned}
 \log \bar{\tau}_D^{(m,q)}(f, \varphi) &= \limsup_{R \rightarrow +\infty} [\log^{[m]} R - \lambda^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_f^{-1}(R))] \text{ and} \\
 \log \tau_D^{(m,q)}(f, \varphi) &= \liminf_{R \rightarrow +\infty} [\log^{[m]} R - \lambda^{(m,q)}(f, \varphi) \cdot \log^{[q]} \varphi(M_f^{-1}(R))].
 \end{aligned}$$

Similarly, from the definitions of $\bar{\tau}_D^{(m,p)}(g)$ and $\tau_D^{(m,p)}(g)$, it follows that

$$\begin{aligned}
 \log \bar{\tau}_D^{(m,p)}(g) &= \limsup_{R \rightarrow +\infty} [\log^{[m]} R - \lambda^{(m,p)}(g) \cdot \log^{[p]} M_g^{-1}(R)] \text{ and} \\
 \log \tau_D^{(m,p)}(g) &= \liminf_{R \rightarrow +\infty} [\log^{[m]} R - \lambda^{(m,p)}(g) \cdot \log^{[p]} M_g^{-1}(R)].
 \end{aligned}$$

Now, using the same method of Theorem 1, one may easily prove the conclusion of the present theorem with the help of Lemma 2, (3.1), (3.2) and the above inequalities. $\square$

Similarly in the line of Theorem 1 and Theorem 2 and using the second part of Lemma 1 and Lemma 2, the following two theorems may easily be proved and so their proofs are omitted:

Theorem 3. Suppose that $g(z)$ and $f(z)$ are any two entire functions of n complex variables such that $g(z)$ be of regular (m, p) -Gol'dberg growth with non zero finite (m, p) -th Gol'dberg order and $0 < \lambda^{(m,q)}(f, \varphi) < \infty$. Then

$$\left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f, \varphi)$$

$$\begin{aligned}
&\leq \min \left\{ \left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\} \\
&\leq \max \left\{ \left[\frac{\tau_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\bar{\tau}_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\} \\
&\leq \bar{\tau}_{g,D}^{(p,q)}(f, \varphi) \leq \left[\frac{\bar{\tau}_D^{(m,q)}(f, \varphi)}{\tau_D^{(m,p)}(g)} \right]^{\frac{1}{\lambda^{(m,p)}(g)}}.
\end{aligned}$$

Theorem 4. Let $g(z)$ and $f(z)$ be any two entire functions of n complex variables such that $0 < \rho^{(m,p)}(g) < \infty$ and $f(z)$ be of regular (m, q) - φ Gol'dberg growth with non zero finite (m, q) - φ Gol'dberg order. Then

$$\begin{aligned}
&\left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f, \varphi) \\
&\leq \min \left\{ \left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\} \\
&\leq \max \left\{ \left[\frac{\bar{\sigma}_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\sigma_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\} \\
&\leq \bar{\tau}_{g,D}^{(p,q)}(f, \varphi) \leq \left[\frac{\sigma_D^{(m,q)}(f, \varphi)}{\bar{\sigma}_D^{(m,p)}(g)} \right]^{\frac{1}{\rho^{(m,p)}(g)}}.
\end{aligned}$$

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A note on relative φ -order and relative φ -lower order of entire functions of several complex variables

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Abstract. In this paper we introduce the notions of relative φ -order and relative φ -lower order of entire functions of several complex variables. We study then some growth properties in connection with relative φ -order and relative φ -lower order of entire functions of several complex variables.

Keywords: Entire functions, relative φ -order, relative φ - lower order, several complex variables, Property (R), Property (P), growth.

1. Introduction, definitions and notations

Let f be a non-constant entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max \{|f(z_1, z_2)| : |z_i| \leq r_i, i = 1, 2\}$. Then in view of maximum principle and Hartogs theorem [8, p. 2, p. 51], $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2 . In this connection the following definition is well known:

Definition 1 ([1, 8]). *The order $v_2\rho(f)$ and the lower order $v_2\lambda(f)$ of an entire function f of two complex variables are defined as*

$$\frac{v_2\rho(f)}{v_2\lambda(f)} = \lim_{r_1, r_2 \rightarrow \infty} \sup \frac{\log \log M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

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The equivalent formula for $v_2\rho(f)$ [8, p. 338] is

$$v_2\rho(f) = \inf \mu > 0 : M_f(r_1, r_2) < \exp[(r_1 r_2)^\mu] \text{ for all } r_1 \geq R(\mu), r_2 \geq R(\mu) .$$

Similarly, one can define $v_2\lambda(f)$ as

$$v_2\lambda(f) = \sup \mu > 0 : M_f(r_1, r_2) > \exp[(r_1 r_2)^\mu] \text{ for all } r_1 \geq R(\mu), r_2 \geq R(\mu) .$$

In [6], Chyzhykov et al. introduced the definition of φ -order of a meromorphic function on single variable in the unit disc. For details about φ -order, one may see [6]. Extending this notion, it is natural for us to give the φ -order of entire function holomorphic in the closed polydisc $\{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2\}$ for all $r_1 \geq 0, r_2 \geq 0\}$ which is as follows:

Definition 2. Let $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function of two variables r_1 and r_2 . The φ -order of an entire function f of two complex variables denoted by $v_2\rho(f, \varphi)$ is defined as:

$$v_2\rho(f, \varphi) = \inf \{\mu > 0 : M_f(r_1, r_2) < \exp[(\varphi_1(r_1, r_2) \varphi_2(r_1, r_2))^\mu], \\ r_1 \geq R(\mu), r_2 \geq R(\mu)\} .$$

If we consider $\varphi(r_1, r_2) = \varphi_1(r_1, r_2) \varphi_2(r_1, r_2) = r_1 r_2$ or $\varphi_1(r_1, r_2) = \varphi_2(r_1, r_2) = (r_1 r_2)^{\frac{1}{2}}$, then Definition 2 coincides with Definition 1.

Analogously, one can define the φ -lower order of f of two complex variables denoted by $v_2\lambda(f, \varphi)$ as follows :

$$v_2\lambda(f, \varphi) = \sup \{\mu > 0 : M_f(r_1, r_2) > \exp[(\varphi_1(r_1, r_2) \varphi_2(r_1, r_2))^\mu]; \\ r_1 \geq R(\mu), r_2 \geq R(\mu)\} ,$$

where $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function of two variables r_1 and r_2 .

Now if we consider Definition 1 for single variable, then the definition coincides with the classical definition of order (see [14]) which is as follows:

Definition 3 ([14]). The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function f are defined in the following way:

$$\frac{\rho(f)}{\lambda(f)} = \lim_{r \rightarrow \infty} \sup \frac{\log \log M_f(r)}{\log r} ,$$

where $M_f(r) = \max \{|f(z)| : |z| = r\}$.

Further if f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists and is such that $\lim_{s \rightarrow \infty} M_f^{-1}(s) = \infty$. Bernal [2], [3] introduced the definition of relative order of f with respect to g , denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf \{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ = \limsup_{r \rightarrow \infty} \frac{\log M_g^{-1}(M_f(r))}{\log r} .$$

The definition coincides with the classical one [14] if $g(z) = \exp z$.

During the past decades, several authors (see [5], [9], [10], [11], [12], [13]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 4 ([4]). *The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:*

$${}_{v_2}\rho_g(f) = \inf \{ \mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu); r_1 \geq R(\mu), r_2 \geq R(\mu) \}$$

where f and g are entire functions holomorphic in the closed polydisc

$$U = \{ (z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0 \}$$

and the definition coincides with Definition 1 {see [4]} if $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [7] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 5 ([7]). *Let $f(z_1, z_2, \dots, z_n)$ and $g(z_1, z_2, \dots, z_n)$ be any two entire functions of n variables $z_1, z_2, \dots, z_n$ with maximum modulus functions $M_f(r_1, r_2, \dots, r_n)$ and $M_g(r_1, r_2, \dots, r_n)$ respectively then the relative order of f with respect to g , denoted by ${}_{v_n}\rho_g(f)$ is defined by*

$$\begin{aligned} {}_{v_n}\rho_g(f) &= \inf \{ \mu > 0 : M_f(r_1, r_2, \dots, r_n) < M_g(r_1^\mu, r_2^\mu, \dots, r_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, \dots, n \} . \end{aligned}$$

Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$$\begin{aligned} {}_{v_n}\lambda_g(f) &= \sup \{ \mu > 0 : M_f(r_1, r_2, \dots, r_n) > M_g(r_1^\mu, r_2^\mu, \dots, r_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, \dots, n \} . \end{aligned}$$

Now in order to make some progress in the study of relative order of entire functions of several complex variables, one may introduce the definition of relative φ -order between two entire functions of several complex variables in the following way:

Definition 6. *Let $\varphi_i(r_1, r_2, \dots, r_n) \mid i = 1, 2, \dots, n : [0, +\infty) \times [0, +\infty) \times \dots \times [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function of n variables $r_1, r_2, \dots, r_n$. Also let f and g be any two entire functions of n complex variables with maximum modulus functions $M_f(r_1, r_2, \dots, r_n)$ and $M_g(r_1, r_2, \dots, r_n)$ respectively then the relative φ -order of f with respect to g , denoted by ${}_{v_n}\rho_g(f, \varphi)$ is defined by*

$$\begin{aligned} {}_{v_n}\rho_g(f, \varphi) &= \inf \{ \mu > 0 : M_f(r_1, r_2, \dots, r_n) < M_g(\varphi_1^\mu, \varphi_2^\mu, \dots, \varphi_n^\mu); \\ &\text{for } r_i \geq R(\mu), i = 1, 2, \dots, n \} , \end{aligned}$$

where $\varphi_i \mid i = 1, 2, \dots, n$ stand for $\varphi_i(r_1, r_2, \dots, r_n) \mid i = 1, 2, \dots, n$.

If we consider $\varphi_1(r_1, r_2, \dots, r_n) = r_1$, $\varphi_2(r_1, r_2, \dots, r_n) = r_2$, ..., $\varphi_n(r_1, r_2, \dots, r_n) = r_n$, then Definition 6 coincides with Definition 5. Also if we consider $n = 2$ and $g(z_1 z_2) = \exp(z_1 z_2)$, then Definition 6 coincides with Definition 2.

Likewise, one can define the relative φ -lower order of f with respect to g denoted by $_{v_n}\lambda_g(f, \varphi)$ as follows:

$$_{v_n}\lambda_g(f, \varphi) = \sup \{ \mu > 0 : M_f(r_1, r_2, \dots, r_n) > M_g(\varphi_1^\mu, \varphi_2^\mu, \dots, \varphi_n^\mu); \\ \text{for } r_i \geq R(\mu), i = 1, 2, \dots, n \},$$

where $\varphi_i \mid i = 1, 2, \dots, n : [0, +\infty) \times [0, +\infty) \times \dots \times [0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing unbounded function of n variables $r_1, r_2, \dots, r_n$.

Further an entire function f of several complex variables for which relative φ -order and relative φ -lower order with respect to another entire function g of several complex variables are the same is called a function of regular relative φ -growth with respect to g . Otherwise, f is said to be irregular relative φ -growth with respect to g .

In this connection just we state the following two definitions which will be needed in the sequel:

Definition 7 ([7]). *The function $f(z_1, z_2, \dots, z_n)$ is said to have Property (R) if for any $\sigma > 1$ and for all large $r_1, r_2, \dots, r_n$,*

$$[M_f(r_1, r_2, \dots, r_n)]^2 < M_f(r_1^\sigma, r_2^\sigma, \dots, r_n^\sigma) \quad .$$

For examples of functions with or without the Property (R), one may see [7].

Definition 8. *A pair of functions $f(z_1, z_2, \dots, z_n)$ and $g(z_1, z_2, \dots, z_n)$ of n complex variables are mutually said to have Property (P) if for all sufficiently large values of $r_1, r_2, \dots, r_n$, both*

$$M_{f.g}(r_1, r_2, \dots, r_n) > M_f(r_1, r_2, \dots, r_n)$$

and

$$M_{f.g}(r_1, r_2, \dots, r_n) > M_g(r_1, r_2, \dots, r_n)$$

hold simultaneously.

One can easily verify that the functions $f(z_1, z_2, \dots, z_n) = \exp(z_1 z_2 \dots z_n)$ and $g(z_1, z_2, \dots, z_n) = \exp(z_1 z_2 \dots z_n)^2$ have the Property (P).

Here, in this paper, we aim at investigating some basic properties of relative φ -order and relative φ -lower order of entire functions of several complex variables with respect to another one under somewhat different conditions. We do not explain the standard definitions and notations in the theory of entire function of several complex variables as those are available in [8].

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 1 ([7]). *Suppose that f be a non constant entire function of several complex variables, $\alpha > 1$ and $0 < \beta < \alpha$. Then, $M_f(\alpha r_1, \alpha r_2, \dots, \alpha r_n) > \beta M_f(r_1, r_2, \dots, r_n)$ for all sufficiently large $r_1, r_2, \dots, r_n$.*

3. Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let us consider f_1, f_2 and g_1 be any three entire functions of several complex variables. Also let at least f_1 or f_2 be of regular relative φ -growth with respect to g_1 . Then*

$${}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) \leq \max \{ {}_{v_n}\lambda_{g_1}(f_1, \varphi), {}_{v_n}\lambda_{g_1}(f_2, \varphi) \}.$$

The equality holds when any one of ${}_{v_n}\lambda_{g_1}(f_i, \varphi) > {}_{v_n}\lambda_{g_1}(f_j, \varphi)$ with at least f_j is of regular relative φ -growth with respect to g_1 where $i, j = 1, 2$ and $i \neq j$.

Proof. If ${}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) = 0$ then the result is obvious. So we suppose that ${}_{v_n}\lambda_{g_1}(f_1 \pm f_2, \varphi) > 0$. We can clearly assume that ${}_{v_n}\lambda_{g_1}(f_k, \varphi)$ is finite for $k = 1, 2$. Further let $\max \{ {}_{v_n}\lambda_{g_1}(f_1, \varphi), {}_{v_n}\lambda_{g_1}(f_2, \varphi) \} = \Delta$ and f_2 be of regular relative φ -growth with respect to g_1 .

Now for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\lambda_{g_1}(f_1, \varphi)$, we have for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_{f_1}(r_1, r_2, \dots, r_n) < M_{g_1}(\varphi_1^{({}_{v_n}\lambda_{g_1}(f_1, \varphi) + \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_{g_1}(f_1, \varphi) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_{g_1}(f_1, \varphi) + \varepsilon)})$$

$$(3.1) \quad \text{i.e., } M_{f_1}(r_1, r_2, \dots, r_n) < M_{g_1}(\varphi_1^{(\Delta + \varepsilon)}, \varphi_2^{(\Delta + \varepsilon)}, \dots, \varphi_n^{(\Delta + \varepsilon)}).$$

Also for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\rho_{g_1}(f_2, \varphi) (= {}_{v_n}\lambda_{g_1}(f_2, \varphi))$, we obtain for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_{f_2}(r_1, r_2, \dots, r_n) <$$

$$M_{g_1}(\varphi_1^{({}_{v_n}\lambda_{g_1}(f_2, \varphi) + \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_{g_1}(f_2, \varphi) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_{g_1}(f_2, \varphi) + \varepsilon)})$$

$$(3.2) \quad \text{i.e., } M_{f_2}(r_1, r_2, \dots, r_n) < M_{g_1}(\varphi_1^{(\Delta + \varepsilon)}, \varphi_2^{(\Delta + \varepsilon)}, \dots, \varphi_n^{(\Delta + \varepsilon)}).$$

Since $M_{f_1 \pm f_2}(r_1, r_2, \dots, r_n) \leq M_{f_1}(r_1, r_2, \dots, r_n) + M_{f_2}(r_1, r_2, \dots, r_n)$ for sufficiently large $r_1, r_2, \dots, r_n$, we obtain from (3.1) and (3.2) for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$(3.3) \quad M_{f_1 \pm f_2}(r_1, r_2, \dots, r_n) < 2M_{g_1}(\varphi_1^{(\Delta + \varepsilon)}, \varphi_2^{(\Delta + \varepsilon)}, \dots, \varphi_n^{(\Delta + \varepsilon)}).$$

Therefore in view of Lemma 1, we obtain from (3.3) for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$\begin{aligned} M_{f_1 \pm f_2}(r_1, r_2, \dots, r_n) &< M_{g_1} \left(3\varphi_1^{(\Delta+\varepsilon)}, 3\varphi_2^{(\Delta+\varepsilon)}, \dots, 3\varphi_n^{(\Delta+\varepsilon)} \right) \\ \text{i.e., } M_{f_1 \pm f_2}(r_1, r_2, \dots, r_n) &< M_{g_1} \left(\varphi_1^{(\Delta+3\varepsilon)}, \varphi_2^{(\Delta+3\varepsilon)}, \dots, \varphi_n^{(\Delta+3\varepsilon)} \right). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$v_n \lambda_{g_1}(f_1 \pm f_2, \varphi) \leq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Similarly, if we consider that f_1 is of regular relative φ -growth with respect to g_1 or both f_1 and f_2 are of regular relative φ -growth with respect to g_1 , then one can easily verify that

$$(3.4) \quad v_n \lambda_{g_1}(f_1 \pm f_2, \varphi) \leq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Now let $v_n \lambda_{g_1}(f_1, \varphi) > v_n \lambda_{g_1}(f_2, \varphi)$ and at least f_2 be of regular relative φ -growth with respect to g_1 . Also let $f = f_1 \pm f_2$. Then in view of (3.4) we get that $v_n \lambda_{g_1}(f, \varphi) \leq v_n \lambda_{g_1}(f_1, \varphi)$. As, $f_1 = (f \pm f_2)$ and in this case we obtain that $v_n \lambda_{g_1}(f_1, \varphi) \leq \max \{ v_n \lambda_{g_1}(f, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}$. As we assume that $v_n \lambda_{g_1}(f_2, \varphi) < v_n \lambda_{g_1}(f_1, \varphi)$, therefore we have $v_n \lambda_{g_1}(f_1, \varphi) \leq v_n \lambda_{g_1}(f, \varphi)$ and hence

$$v_n \lambda_{g_1}(f_1 \pm f_2, \varphi) \geq v_n \lambda_{g_1}(f_1, \varphi) = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Further if we consider $v_n \lambda_{g_1}(f_1, \varphi) < v_n \lambda_{g_1}(f_2, \varphi)$ and at least f_1 be of regular relative φ -growth with respect to g_1 , then one can also verify that

$$(3.5) \quad v_n \lambda_{g_1}(f_1 \pm f_2, \varphi) \geq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

So the conclusion of the second part of the theorem follows from (3.4) and (3.5). $\square$

Theorem 2. *Let us consider f_1, f_2 be any two entire functions of several complex variables with relative φ -order $v_n \rho_{g_1}(f_1, \varphi)$ and $v_n \rho_{g_1}(f_2, \varphi)$ with respect to another entire function g_1 of several complex variables. Then*

$$v_n \rho_{g_1}(f_1 \pm f_2, \varphi) \leq \max \{ v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_1}(f_2, \varphi) \}.$$

The equality holds when $v_n \rho_{g_1}(f_1, \varphi) \neq v_n \rho_{g_1}(f_2, \varphi)$.

We omit the proof of Theorem 2 as it can easily be carried out in the line of Theorem 1.

Theorem 3. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables such that $v_n \lambda_{g_1}(f_1, \varphi)$ and $v_n \lambda_{g_2}(f_1, \varphi)$ exit. Then*

$$v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) \geq \min \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi) \}.$$

The equality holds when $v_n \lambda_{g_1}(f_1, \varphi) \neq v_n \lambda_{g_2}(f_1, \varphi)$.

Proof. If $v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) = \infty$ then the result is obvious. So we suppose that $v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) < \infty$. We can clearly assume that $v_n \lambda_{g_k}(f_1, \varphi)$ is finite for $k = 1, 2$. Further let $\Psi = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $v_n \lambda_{g_k}(f_1, \varphi)$, we have for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_{g_k}(\varphi_1^{(v_n \lambda_{g_k}(f_1, \varphi) - \varepsilon)}, \varphi_2^{(v_n \lambda_{g_k}(f_1, \varphi) - \varepsilon)}, \dots, \varphi_n^{(v_n \lambda_{g_k}(f_1, \varphi) - \varepsilon)}) < M_{f_1}(r_1, r_2, \dots, r_n)$$

where $k = 1, 2$.

Therefore from above we get for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$(3.6) \quad M_{g_k}(\varphi_1^{(\Psi - \varepsilon)}, \varphi_2^{(\Psi - \varepsilon)}, \dots, \varphi_n^{(\Psi - \varepsilon)}) < M_{f_1}(r_1, r_2, \dots, r_n) \quad \text{where } k = 1, 2.$$

Since $M_{g_1 \pm g_2}(r_1, r_2, \dots, r_n) \leq M_{g_1}(r_1, r_2, \dots, r_n) + M_{g_2}(r_1, r_2, \dots, r_n)$ for sufficiently large $r_1, r_2, \dots, r_n$, we obtain from above and Lemma 1 for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$\begin{aligned} M_{g_1 \pm g_2}(\varphi_1^{(\Psi - \varepsilon)}, \varphi_2^{(\Psi - \varepsilon)}, \dots, \varphi_n^{(\Psi - \varepsilon)}) &< M_{g_1}[\varphi_1^{(\Psi - \varepsilon)}, \varphi_2^{(\Psi - \varepsilon)}, \dots, \varphi_n^{(\Psi - \varepsilon)}] \\ &+ M_{g_2}[\varphi_1^{(\Psi - \varepsilon)}, \varphi_2^{(\Psi - \varepsilon)}, \dots, \varphi_n^{(\Psi - \varepsilon)}] \\ \text{i.e., } M_{g_1 \pm g_2}(\varphi_1^{(\Psi - \varepsilon)}, \varphi_2^{(\Psi - \varepsilon)}, \dots, \varphi_n^{(\Psi - \varepsilon)}) &< 2M_{f_1}(r_1, r_2, \dots, r_n) \\ \text{i.e., } M_{g_1 \pm g_2}\left(\left(\frac{1}{3}\right)\varphi_1^{(\Psi - \varepsilon)}, \left(\frac{1}{3}\right)\varphi_2^{(\Psi - \varepsilon)}, \dots, \left(\frac{1}{3}\right)\varphi_n^{(\Psi - \varepsilon)}\right) &< M_{f_1}(r_1, r_2, \dots, r_n) \\ \text{i.e., } M_{g_1 \pm g_2}(\varphi_1^{(\Psi - 3\varepsilon)}, \varphi_2^{(\Psi - 3\varepsilon)}, \dots, \varphi_n^{(\Psi - 3\varepsilon)}) &< M_{f_1}(r_1, r_2, \dots, r_n). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$(3.7) \quad v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) \geq \Psi = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}.$$

Now let $v_n \lambda_{g_1}(f_1, \varphi) < v_n \lambda_{g_2}(f_1, \varphi)$ and $g = g_1 \pm g_2$. Then in view of (3.7) we get that $v_n \lambda_g(f_1, \varphi) \geq v_n \lambda_{g_1}(f_1, \varphi)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $v_n \lambda_{g_1}(f_1, \varphi) \geq \min \{v_n \lambda_g(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}$. As we assume that $v_n \lambda_{g_1}(f_1, \varphi) < v_n \lambda_{g_2}(f_1, \varphi)$, therefore we have $v_n \lambda_{g_1}(f_1, \varphi) \geq v_n \lambda_g(f_1, \varphi)$ and hence

$$v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) \leq v_n \lambda_{g_1}(f_1, \varphi) = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}.$$

Similarly, if we consider $v_n \lambda_{g_1}(f_1, \varphi) > v_n \lambda_{g_2}(f_1, \varphi)$, then one can also derive that

$$(3.8) \quad v_n \lambda_{g_1 \pm g_2}(f_1, \varphi) \leq \Psi = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}.$$

So the conclusion of the second part of the theorem follows from (3.7) and (3.8). $\square$

Theorem 4. Let f_1, g_1 and g_2 be any three entire functions of several complex variables such that $v_n \rho_{g_1}(f_1, \varphi)$ and $v_n \rho_{g_2}(f_1, \varphi)$ exist. Also let f_1 be of regular relative φ -growth with respect to at least any one of g_1 or g_2 . Then

$$v_n \rho_{g_1 \pm g_2}(f_1, \varphi) \geq \min\{v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi)\}.$$

The equality holds when any one of $v_n \rho_{g_i}(f_1, \varphi) < v_n \rho_{g_j}(f_1, \varphi)$ with at least f_1 is of regular relative φ -growth with respect to g_j where $i, j = 1, 2$ and $i \neq j$.

We omit the proof of Theorem 4 as it can easily be carried out in the line of Theorem 3.

Theorem 5. Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Then

$$v_n \rho_{g_1 \pm g_2}(f_1 \pm f_2, \varphi) \leq \max[\min\{v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi)\}, \min\{v_n \rho_{g_1}(f_2, \varphi), v_n \rho_{g_2}(f_2, \varphi)\}]$$

when the following two conditions hold:

- (i) $v_n \rho_{g_i}(f_1, \varphi) < v_n \rho_{g_j}(f_1, \varphi)$ with at least f_1 is of regular relative φ -growth with respect to g_j for $i = 1, 2, j = 1, 2$ and $i \neq j$; and
- (ii) $v_n \rho_{g_i}(f_2, \varphi) < v_n \rho_{g_j}(f_2, \varphi)$ with at least f_2 is of regular relative φ -growth with respect to g_j for $i = 1, 2, j = 1, 2$ and $i \neq j$.

The equality holds when

$$v_n \rho_{g_1}(f_i, \varphi) < v_n \rho_{g_1}(f_j, \varphi) \text{ and } v_n \rho_{g_2}(f_i, \varphi) < v_n \rho_{g_2}(f_j, \varphi)$$

hold simultaneously for $i = 1, 2; j = 1, 2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get that

$$\begin{aligned} & \max[\min\{v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi)\}, \min\{v_n \rho_{g_1}(f_2, \varphi), v_n \rho_{g_2}(f_2, \varphi)\}] \\ &= \max[v_n \rho_{g_1 \pm g_2}(f_1, \varphi), v_n \rho_{g_1 \pm g_2}(f_2, \varphi)] \\ (3.9) \quad & \geq v_n \rho_{g_1 \pm g_2}(f_1 \pm f_2, \varphi). \end{aligned}$$

Since $v_n \rho_{g_1}(f_i, \varphi) < v_n \rho_{g_1}(f_j, \varphi)$ and $v_n \rho_{g_2}(f_i, \varphi) < v_n \rho_{g_2}(f_j, \varphi)$ hold simultaneously for $i = 1, 2; j = 1, 2$ and $i \neq j$, we obtain that

$$\begin{aligned} & \text{either } \min\{v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi)\} > \min\{v_n \rho_{g_1}(f_2, \varphi), v_n \rho_{g_2}(f_2, \varphi)\} \text{ or} \\ & \min\{v_n \rho_{g_1}(f_2, \varphi), v_n \rho_{g_2}(f_2, \varphi)\} > \min\{v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi)\} \text{ holds.} \end{aligned}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } v_n \rho_{g_1 \pm g_2}(f_1, \varphi) > v_n \rho_{g_1 \pm g_2}(f_2, \varphi) \text{ or } v_n \rho_{g_1 \pm g_2}(f_2, \varphi) > v_n \rho_{g_1 \pm g_2}(f_1, \varphi)$$

which is the condition for holding equality in (3.9).

Hence the theorem follows. $\square$

Theorem 6. Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Then

$$v_n \lambda_{g_1 \pm g_2}(f_1 \pm f_2, \varphi) \geq \min[\max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\}, \\ \max\{v_n \lambda_{g_2}(f_1, \varphi), v_n \lambda_{g_2}(f_2, \varphi)\}]$$

when the following two conditions hold:

- (i) $v_n \lambda_{g_1}(f_i, \varphi) > v_n \lambda_{g_1}(f_j, \varphi)$ with at least f_j is of regular relative φ -growth with respect to g_1 for $i = 1, 2, j = 1, 2$ and $i \neq j$; and
(ii) $v_n \lambda_{g_2}(f_i, \varphi) > v_n \lambda_{g_2}(f_j, \varphi)$ with at least f_j is of regular relative φ -growth with respect to g_2 for $i = 1, 2, j = 1, 2$ and $i \neq j$.

The equality holds when $v_n \lambda_{g_i}(f_1, \varphi) < v_n \lambda_{g_j}(f_1, \varphi)$ and $v_n \lambda_{g_i}(f_2, \varphi) < v_n \lambda_{g_j}(f_2, \varphi)$ hold simultaneously for $i = 1, 2; j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 1 and Theorem 3, we obtain that

$$(3.10) \quad \min[\max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\}, \\ \max\{v_n \lambda_{g_2}(f_1, \varphi), v_n \lambda_{g_2}(f_2, \varphi)\}] \\ = \min[v_n \lambda_{g_1}(f_1 \pm f_2, \varphi), v_n \lambda_{g_2}(f_1 \pm f_2, \varphi)] \geq v_n \lambda_{g_1 \pm g_2}(f_1 \pm f_2, \varphi).$$

Since $v_n \lambda_{g_i}(f_1, \varphi) < v_n \lambda_{g_j}(f_1, \varphi)$ and $v_n \lambda_{g_i}(f_2, \varphi) < v_n \lambda_{g_j}(f_2, \varphi)$ hold simultaneously for $i = 1, 2; j = 1, 2$ and $i \neq j$, we get that

$$\text{either } \max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\} < \max\{v_n \lambda_{g_2}(f_1, \varphi), v_n \lambda_{g_2}(f_2, \varphi)\} \text{ or} \\ \max\{v_n \lambda_{g_2}(f_1, \varphi), v_n \lambda_{g_2}(f_2, \varphi)\} < \max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem hold, it follows from above that

$$\text{either } v_n \lambda_{g_1}(f_1 \pm f_2, \varphi) < v_n \lambda_{g_2}(f_1 \pm f_2, \varphi) \text{ or } v_n \lambda_{g_2}(f_1 \pm f_2, \varphi) < v_n \lambda_{g_1}(f_1 \pm f_2, \varphi)$$

which is the condition for holding equality in (3.10).

Hence the theorem follows. $\square$

Theorem 7. Let f_1, f_2 and g_1 be any three entire functions of several complex variables. Also let at least f_1 or f_2 be of regular relative φ -growth with respect to g_1 . Then

$$v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \leq \max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\}$$

provided g_1 has the Property (R). The equality holds when f_1 and f_2 satisfy Property (P).

Proof. Suppose that $v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) > 0$. Otherwise if $v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) = 0$ then the result is obvious. Let us consider that f_2 is of regular relative φ -growth with respect to g_1 . Also suppose that $\max\{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi)\} = \Delta$. We can clearly assume that $v_n \lambda_{g_1}(f_k, \varphi)$ is finite for $k = 1, 2$.

Since $M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) < M_{f_1}(r_1, r_2, \dots, r_n) \cdot M_{f_2}(r_1, r_2, \dots, r_n)$ for all large $r_1, r_2, \dots, r_n$, we have from (3.1), (3.2) for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) < \left[M_{g_1} \left(\varphi_1^{(\Delta+\varepsilon)}, \varphi_2^{(\Delta+\varepsilon)}, \dots, \varphi_n^{(\Delta+\varepsilon)} \right) \right]^2.$$

Also, in view of Definition 7, we obtain from above for any $\delta > 1$ and for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) < M_{g_1} \left(\varphi_1^{\delta(\Delta+\varepsilon)}, \varphi_2^{\delta(\Delta+\varepsilon)}, \dots, \varphi_n^{\delta(\Delta+\varepsilon)} \right),$$

since g_1 has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \rightarrow 1^+$, we get from above that

$$v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \leq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Similarly, if we consider that f_1 is of regular relative φ -growth with respect to g_1 or both f_1 and f_2 are of regular relative φ -growth with respect to g_1 , then also one can easily verify that

$$(3.11) \quad v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \leq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Now let f_1 and f_2 satisfy Property (P), then of course we have $M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) > M_{f_1}(r_1, r_2, \dots, r_n)$ and $M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) > M_{f_2}(r_1, r_2, \dots, r_n)$ for all sufficiently large values of $r_1, r_2, \dots, r_n$. Therefore from the definition of relative φ -lower order, we get for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_{f_1}(r_1, r_2, \dots, r_n) < M_{f_1 \cdot f_2}(r_1, r_2, \dots, r_n) \\ \leq M_{g_1} \left(\varphi_1^{(v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) + \varepsilon)}, \varphi_2^{(v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) + \varepsilon)}, \dots, \varphi_n^{(v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) + \varepsilon)} \right).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \geq v_n \lambda_{g_1}(f_1, \varphi)$. Similarly $v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \geq v_n \lambda_{g_1}(f_2, \varphi)$ and therefore

$$(3.12) \quad v_n \lambda_{g_1}(f_1 \cdot f_2, \varphi) \geq \Delta = \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}.$$

Hence the theorem follows from (3.11) and (3.12). $\square$

Now we state the following theorem which can easily be carried out in the line of Theorem 7 and therefore its proof is omitted.

Theorem 8. *Let us consider f_1, f_2 be any two entire functions of several complex variables with relative φ -order $v_n \rho_{g_1}(f_1, \varphi)$ and $v_n \rho_{g_1}(f_2, \varphi)$ with respect to another entire function g_1 of several complex variables. Then*

$$v_n \rho_{g_1}(f_1 \cdot f_2, \varphi) \leq \max \{ v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_1}(f_2, \varphi) \}$$

provided g_1 has the Property (R). The equality holds when f_1 and f_2 satisfy Property (P).

Theorem 9. Let f_1 , g_1 and g_2 be any three entire functions of several complex variables. Also let $\lambda_{g_1}(f_1)$ and $\lambda_{g_2}(f_1)$ exist. Then

$$v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) \geq \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (P).

Proof. Suppose that $v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) < \infty$. Otherwise if $v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) = \infty$ then the result is obvious. Also suppose that $\min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\} = \Psi$. We can clearly assume that $v_n \lambda_{g_k}(f_1, \varphi)$ is finite for $k = 1, 2$.

As $M_{g_1 \cdot g_2}(r_1, r_2, \dots, r_n) < M_{g_1}(r_1, r_2, \dots, r_n) \cdot M_{g_2}(r_1, r_2, \dots, r_n)$ for all large $r_1, r_2, \dots, r_n$, we get in view of (3.6) for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_{g_1 \cdot g_2}(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, \dots, \varphi_n^{(\Psi-\varepsilon)}) < [M_{f_1}(r_1, r_2, \dots, r_n)]^2$$

$$i.e., \left[M_{g_1 \cdot g_2}(\varphi_1^{(\Psi-\varepsilon)}, \varphi_2^{(\Psi-\varepsilon)}, \dots, \varphi_n^{(\Psi-\varepsilon)}) \right]^{\frac{1}{2}} < M_{f_1}(r_1, r_2, \dots, r_n).$$

Now in view of Definition 7 we obtain from above for any $\delta > 1$ and for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_{g_1 \cdot g_2}\left(\varphi_1^{\frac{(\Psi-\varepsilon)}{\delta}}, \varphi_2^{\frac{(\Psi-\varepsilon)}{\delta}}, \dots, \varphi_n^{\frac{(\Psi-\varepsilon)}{\delta}}\right) < M_{f_1}(r_1, r_2, \dots, r_n)$$

since $g_1 \cdot g_2$ has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \rightarrow 1^+$, we obtain from above that

$$(3.13) \quad v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) \geq \Psi = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}.$$

Now let g_1 and g_2 satisfy Property (P), then of course we have $M_{g_1 \cdot g_2}(r_1, r_2, \dots, r_n) > M_{g_1}(r_1, r_2, \dots, r_n)$ and $M_{g_1 \cdot g_2}(r_1, r_2, \dots, r_n) > M_{g_2}(r_1, r_2, \dots, r_n)$ for all sufficiently large values of $r_1, r_2, \dots, r_n$. Therefore from the definition of relative φ -lower order, we get for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$\begin{aligned} & M_{g_1}(\varphi_1^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}, \varphi_2^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}, \dots, \varphi_n^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}) \\ & < M_{g_1 \cdot g_2}(\varphi_1^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}, \varphi_2^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}, \dots, \varphi_n^{(v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) - \varepsilon)}) \\ & \leq M_{f_1}(r_1, r_2, \dots, r_n). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $v_n \lambda_{g_1}(f_1, \varphi) \geq v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi)$. Similarly $v_n \lambda_{g_2}(f_1, \varphi) \geq v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi)$ and therefore

$$(3.14) \quad v_n \lambda_{g_1 \cdot g_2}(f_1, \varphi) \leq \Psi = \min \{v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_2}(f_1, \varphi)\}.$$

Hence the theorem follows from (3.13) and (3.14). $\square$

Theorem 10. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables. Also let f_1 be of regular relative φ -growth with respect to at least any one of g_1 or g_2 . Then*

$$v_n \rho_{g_1 \cdot g_2}(f_1, \varphi) \geq \min \{ v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi) \}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (P).

We omit the proof of Theorem 10 as it can easily be carried out in the line of Theorem 9.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of Theorem 8, Theorem 7, Theorem 9 and Theorem 10 and

in the line of Theorem 5 and Theorem 6 respectively.

Theorem 11. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Also let $g_1 \cdot g_2$ satisfy the Property (R). Then,*

$$v_n \rho_{g_1 \cdot g_2}(f_1 \cdot f_2, \varphi) = \max \{ \min \{ v_n \rho_{g_1}(f_1, \varphi), v_n \rho_{g_2}(f_1, \varphi) \}, \min \{ v_n \rho_{g_1}(f_2, \varphi), v_n \rho_{g_2}(f_2, \varphi) \} \},$$

when the following four conditions hold:

- (i) f_1 is of regular relative φ -growth with respect to at least any one of g_1 or g_2 ;
- (ii) f_2 is of regular relative φ -growth with respect to at least any one of g_1 or g_2 ;
- (iii) f_1 and f_2 satisfy Property (P); and
- (iv) g_1 and g_2 satisfy Property (P).

Theorem 12. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Also let $g_1 \cdot g_2, g_1$ and g_2 satisfy the Property (R). Then,*

$$v_n \lambda_{g_1 \cdot g_2}(f_1 \cdot f_2, \varphi) = \min \{ \max \{ v_n \lambda_{g_1}(f_1, \varphi), v_n \lambda_{g_1}(f_2, \varphi) \}, \max \{ v_n \lambda_{g_2}(f_1, \varphi), v_n \lambda_{g_2}(f_2, \varphi) \} \},$$

when the following four conditions hold:

- (i) At least f_1 or f_2 is of regular relative φ -growth with respect to g_1 ;
- (ii) At least f_1 or f_2 is of regular relative φ -growth with respect to g_2 ;
- (iii) f_1 and f_2 satisfy Property (P);
- (iv) g_1 and g_2 satisfy Property (P).

Theorem 13. *Let f, g and h be any three entire functions of several complex variables such that $0 < v_n \lambda_h(f, \varphi) \leq v_n \rho_h(f, \varphi) < \infty$ and $0 < v_n \lambda_h(g) \leq v_n \rho_h(g) < \infty$. Then*

$$\begin{aligned} \frac{v_n \lambda_h(f, \varphi)}{v_n \rho_h(g)} &\leq v_n \lambda_g(f, \varphi) \leq \min \left\{ \frac{v_n \lambda_h(f, \varphi)}{v_n \lambda_h(g)}, \frac{v_n \rho_h(f, \varphi)}{v_n \rho_h(g)} \right\} \\ &\leq \max \left\{ \frac{v_n \lambda_h(f, \varphi)}{v_n \lambda_h(g)}, \frac{v_n \rho_h(f, \varphi)}{v_n \rho_h(g)} \right\} \leq v_n \rho_g(f, \varphi) \leq \frac{v_n \rho_h(f, \varphi)}{v_n \lambda_h(g)}. \end{aligned}$$

Proof. From the definitions of ${}_{v_n}\rho_g(f, \varphi)$ it follows for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$(3.15) \quad M_f(r_1, r_2, \dots, r_n) < M_g(\varphi_1^{({}_{v_n}\rho_g(f, \varphi) + \varepsilon)}, \varphi_2^{({}_{v_n}\rho_g(f, \varphi) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\rho_g(f, \varphi) + \varepsilon)})$$

and for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity we obtain that

$$(3.16) \quad M_f(r_1, r_2, \dots, r_n) > M_g(\varphi_1^{({}_{v_n}\rho_g(f, \varphi) - \varepsilon)}, \varphi_2^{({}_{v_n}\rho_g(f, \varphi) - \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\rho_g(f, \varphi) - \varepsilon)}).$$

Similarly from the definition of ${}_{v_n}\lambda_g(f, \varphi)$, we have for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$(3.17) \quad M_f(r_1, r_2, \dots, r_n) > M_g(\varphi_1^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)})$$

and also for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$(3.18) \quad M_f(r_1, r_2, \dots, r_n) < M_g(\varphi_1^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)}).$$

Further from the definitions of ${}_{v_n}\rho_h(g)$ it follows for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$(3.19) \quad M_g(\varphi_1, \varphi_2, \dots, \varphi_n) < M_h(\varphi_1^{({}_{v_n}\rho_h(g) + \varepsilon)}, \varphi_2^{({}_{v_n}\rho_h(g) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\rho_h(g) + \varepsilon)})$$

and for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity we obtain that

$$(3.20) \quad M_g(\varphi_1, \varphi_2, \dots, \varphi_n) > M_h(\varphi_1^{({}_{v_n}\rho_h(g) - \varepsilon)}, \varphi_2^{({}_{v_n}\rho_h(g) - \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\rho_h(g) - \varepsilon)}).$$

Likewise from the definitions of ${}_{v_n}\lambda_h(g)$ it follows for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$(3.21) \quad M_g(\varphi_1, \varphi_2, \dots, \varphi_n) > M_h(\varphi_1^{({}_{v_n}\lambda_h(g) - \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_h(g) - \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_h(g) - \varepsilon)})$$

and for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity we obtain that

$$(3.22) \quad M_g(\varphi_1, \varphi_2, \dots, \varphi_n) < M_h(\varphi_1^{({}_{v_n}\lambda_h(g) + \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_h(g) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_h(g) + \varepsilon)}).$$

Now, from (3.18) and in view of (3.19), for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity, we get that

$$M_f(r_1, r_2, \dots, r_n) < M_h(\varphi_1^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)({}_{v_n}\rho_h(g) + \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)({}_{v_n}\rho_h(g) + \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_g(f, \varphi) + \varepsilon)({}_{v_n}\rho_h(g) + \varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$(3.23) \quad \begin{aligned} {}_{v_n}\lambda_h(f, \varphi) &\leq {}_{v_n}\lambda_g(f, \varphi) \cdot {}_{v_n}\rho_h(g) \\ \text{i.e., } \frac{{}_{v_n}\lambda_h(f, \varphi)}{{}_{v_n}\rho_h(g)} &\leq {}_{v_n}\lambda_g(f, \varphi). \end{aligned}$$

Analogously from (3.17) and in view of (3.20) it follows for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_f(r_1, r_2, \dots, r_n) > M_h(\varphi_1^{(v_n \lambda_g(f, \varphi) - \varepsilon)(v_n \rho_h(g) - \varepsilon)}, \varphi_2^{(v_n \lambda_g(f, \varphi) - \varepsilon)(v_n \rho_h(g) - \varepsilon)}, \dots, \varphi_n^{(v_n \lambda_g(f, \varphi) - \varepsilon)(v_n \rho_h(g) - \varepsilon)}).$$

As $\varepsilon (> 0)$ is arbitrary, so we obtain from above that

$$(3.24) \quad \begin{aligned} v_n \rho_h(f, \varphi) &\geq v_n \lambda_g(f, \varphi) \cdot v_n \rho_h(g), \\ i.e., v_n \lambda_g(f, \varphi) &\leq \frac{v_n \rho_h(f, \varphi)}{v_n \rho_h(g)}. \end{aligned}$$

Similarly from (3.16) and in view of (3.21) we have for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_f(r_1, r_2, \dots, r_n) > M_h(\varphi_1^{(v_n \rho_g(f, \varphi) - \varepsilon)(v_n \lambda_h(g) - \varepsilon)}, \varphi_2^{(v_n \rho_g(f, \varphi) - \varepsilon)(v_n \lambda_h(g) - \varepsilon)}, \dots, \varphi_n^{(v_n \rho_g(f, \varphi) - \varepsilon)(v_n \lambda_h(g) - \varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$(3.25) \quad \begin{aligned} v_n \rho_h(f, \varphi) &\geq v_n \rho_g(f, \varphi) \cdot v_n \lambda_h(g), \\ i.e., v_n \rho_g(f, \varphi) &\leq \frac{v_n \rho_h(f, \varphi)}{v_n \lambda_h(g)}. \end{aligned}$$

Further from (3.15) and in view of (3.19) we get for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_f(r_1, r_2, \dots, r_n) < M_h(\varphi_1^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \rho_h(g) + \varepsilon)}, \varphi_2^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \rho_h(g) + \varepsilon)}, \dots, \varphi_n^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \rho_h(g) + \varepsilon)}).$$

As $\varepsilon > 0$ is arbitrary, we obtain from above that

$$(3.26) \quad \begin{aligned} v_n \rho_h(f, \varphi) &\leq v_n \rho_g(f, \varphi) \cdot v_n \rho_h(g), \\ i.e., v_n \rho_g(f, \varphi) &\geq \frac{v_n \rho_h(f, \varphi)}{v_n \rho_h(g)}. \end{aligned}$$

Moreover, from (3.15) and in view of (3.22) we have for a sequence values of $r_1, r_2, \dots, r_n$ tending to infinity that

$$M_f(r_1, r_2, \dots, r_n) < M_h(\varphi_1^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \lambda_h(g) + \varepsilon)}, \varphi_2^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \lambda_h(g) + \varepsilon)}, \dots, \varphi_n^{(v_n \rho_g(f, \varphi) + \varepsilon)(v_n \lambda_h(g) + \varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$v_n \lambda_h(f, \varphi) \leq v_n \rho_g(f, \varphi) \cdot v_n \lambda_h(g)$$

$$(3.27) \quad i.e., \quad {}_{v_n}\rho_g(f, \varphi) \geq \frac{{}_{v_n}\lambda_h(f, \varphi)}{{}_{v_n}\lambda_h(g)}.$$

Likewise from (3.17) and (3.21), it follows for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_f(r_1, r_2, \dots, r_n) > M_h(\varphi_1^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)({}_{v_n}\lambda_h(g) - \varepsilon)}, \varphi_2^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)({}_{v_n}\lambda_h(g) - \varepsilon)}, \dots, \varphi_n^{({}_{v_n}\lambda_g(f, \varphi) - \varepsilon)({}_{v_n}\lambda_h(g) - \varepsilon)}).$$

As $\varepsilon > 0$ is arbitrary, we obtain from above that

$$(3.28) \quad \begin{aligned} {}_{v_n}\lambda_h(f, \varphi) &\geq {}_{v_n}\lambda_g(f, \varphi) \cdot {}_{v_n}\lambda_h(g), \\ i.e., \quad {}_{v_n}\lambda_g(f, \varphi) &\leq \frac{{}_{v_n}\lambda_h(f, \varphi)}{{}_{v_n}\lambda_h(g)}. \end{aligned}$$

Hence the theorem follows from (3.23), (3.24), (3.25), (3.26), (3.27) and (3.28). $\square$

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On drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations

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Abstract. This paper is concerned with the drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations. Firstly, the Girsanov transformation is used to simplify the equation because of the expression of the drift coefficient. Secondly, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. Then, we prove that the maximum likelihood estimator exists in the interval when the sample size is large enough. Finally, the strong consistency of the estimator and the asymptotic normality of the error of estimation are proved.

Keywords: drift parameter estimation, mean-reversion, nonlinear nonhomogeneous, Girsanov transformation, consistency, asymptotic normality.

1. Introduction

Stochastic differential equations are very important for model building in physics, engineering, medical science and life sciences. Recently, stochastic differential equations, especially mean-reversion type stochastic differential equations, have been widely used to describe the price dynamics of a financial asset such as interest rate, discount bonds and futures, ([2, 6, 8, 13, 16]). For example, the Black-Scholes option pricing model described by a geometric Brownian motion ([3]), the Vasicek and Cox-Ingersoll-Ross models developed based on two specific mean-reversion diffusion processes ([17, 4, 5]). Therefore, stochastic differential equations are the basic stochastic modeling instruments in the modern financial theory. However, in practice, part or all of the parameters in stochastic models are always unknown, but the observed values are known. Hence, the unknown parameters are needed to be estimated based on observations. The parameters in the model may be finite-dimensional or infinite-dimensional. They have to be estimated either from one or many continuous realizations of the process if a continuous realization is possible, or from a discrete sampled data set for the process if the process is not continuously observable.

In the past decades, some methods have been used to estimate the parameters in stochastic differential equations based on continuous observations. For

example, Kutoyants [10] and Wei [19] discussed the consistency and asymptotic normality of the maximum likelihood estimator based on likelihood ratio function and maximum probability estimation respectively. Barczy [1] analyzed the consistency of the maximum likelihood estimator and given the sufficient condition for the estimation error to satisfy the normal distribute. Kan [9] used filter theory to prove the consistency of the Bayes estimator. Mendy [14] applied least squares method to estimate the parameter in Ornstein-Uhlenbeck process driven by fractional Brownian motion. Wen [20] studied the local asymptotic normality of the maximum likelihood estimator for McKean-Vlasov stochastic differential equation. Other methods such as M-estimation method ([21, 22]) and sequential estimation method ([7]) were used to estimate the parameters as well.

Although the parameter estimation for mean-reversion type stochastic differential equations has been studied by some authors such as Li [12] and Wei [18], the drift and diffusion coefficients do not depend on the time t , so the ergodic theorem could be used. However, the nonlinear nonhomogeneous stochastic differential equation could describe the stochastic phenomenon much better and the ergodic theorem could not be applied. Therefore, it is of great importance to study this topic. In this paper, parameter estimation is investigated for nonlinear nonhomogeneous stochastic differential equation. Since the stochastic differential equation is nonhomogeneous, the ergodic theorem used in Li [12] and Wei [18] can not be applied in this paper. Hence, the Girsanov transformation is used firstly to simplify the equation. Then, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. The existence and strong consistency of the maximum likelihood estimator are proved and the asymptotic normality of the error of estimation is discussed by using the central limit theorem, Lepingle's law of large numbers and the law of large numbers for square-integrable martingales.

This paper is organized as follows. In Section 2 the stochastic differential equation is simplified and a new family of probability measures has been indexed, the likelihood function is given as well. The main results are given in Section 3. In this section, the existence and strong consistency of the maximum likelihood estimator are proved and the asymptotic normality of the error of estimation is analyzed. The conclusion is given in Section 4.

2. Problem formulation and preliminaries

We study the parameter estimation problem for the following stochastic differential equation:

$$(1) \quad \begin{cases} dX_t = (\gamma + \mu(t, X_t))a(t, X_t, \theta)dt + b(t, X_t)dB_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases}$$

where $\theta \in \Theta$ is an unknown parameter, $\gamma \neq 0$ is a constant, $\mu(t, x)$ is twice differentiable with respect to x and differentiable with respect to t , both $a(t, x, \theta)$

and $b(t, x)$ are continuous with respect to t and x , $(B_t, t \geq 0)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Suppose Equation (1) satisfies the conditions that ensure the existence and uniqueness of the solution.

Because of the complexity of Equation 1, the Girsanov transformation will be used to simplify the drift coefficient. From now on, we will work under the assumptions as follows:

Assumption 1. $|\mu(t, x)a(t, x, \theta) - \mu(t, y)a(t, y, \theta)| \leq K|x - y|$, K is a positive constant.

Assumption 2. $Q_\theta(\int_0^T a^2(t, X_t, \theta)dt < \infty) = 1$ where Q is the probability measure defined on $(\Omega, \mathcal{F}_T)$.

Assumption 3. $\mathbb{E}_\theta[\int_0^T (a'(t, X_t, \theta)dt)^2] < \infty$ and $\mathbb{E}_\theta[\int_0^T (a''(t, X_t, \theta)dt)^2] < \infty$.

Assumption 4. for any θ , there exists a neighborhood V_θ of θ such that $Q_\theta(\int_0^\infty \frac{(a(t, X_t, \theta') - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt = \infty) = 1$, for all $\theta' \in V_\theta - \theta$.

Firstly, we introduce the Girsanov theorem below.

Lemma 1 ([11]). *Let $Y(t)$ be an Itô process of the form*

$$dY(t) = a(t, \omega)dt + dB(t); t \leq T,$$

where $T \leq \infty$ is a given constant and $B(t)$ is Brownian motion. Put

$$M_t = \exp(-\int_0^t a(s, \omega)dB_s - \frac{1}{2}\int_0^t a^2(s, \omega)ds); t \leq T.$$

Assume that $a(s, \omega)$ satisfies Novikov's condition

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T a^2(s, \omega)ds)] < \infty,$$

where $\mathbb{E}$ is the expectation with respect to P . Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T dP(\omega).$$

Then $Y(t)$ is a Brownian motion with respect to the probability law Q , for $t \leq T$.

According to the Girsanov theorem, Equation 1 could be written as

$$(2) \quad dX_t = ((\gamma + \mu(t, X_t))a(t, X_t, \theta) + b(t, X_t)\alpha(t, X_t))dt + b(t, X_t)d\tilde{B}_t,$$

where $\alpha(t, X_t)$ satisfies the Novikov condition, $\tilde{B}_t = B_t - \int_0^t \alpha(s, X_s)ds$ is an $\{\mathcal{F}_t\}_{t \in [0,1]}$ -Brownian motion under the probability Q_t .

Let

$$\mu(t, x)a(t, x, \theta) + b(t, x)\alpha(t, x) = 0,$$

hence

$$\mu(t, X_t)a(t, X_t, \theta) + b(t, X_t)\alpha(t, X_t) = 0.$$

Therefore, Equation 2 becomes

$$(3) \quad dX_t = \gamma a(t, X_t, \theta)dt + b(t, X_t)d\tilde{B}_t.$$

According to the Girsanov theorem, the continuous-time log-likelihood function has the following expression

$$(4) \quad \ell_T(\theta) = \gamma \int_0^T \frac{a(t, X_t, \theta)}{b^2(t, X_t)} dX_t - \frac{1}{2} \gamma^2 \int_0^T \frac{a^2(t, X_t, \theta)}{b^2(t, X_t)} dt.$$

3. Main results and proofs

In the following theorem, the strongly consistency of the maximum likelihood estimator is proved.

Theorem 1. *Under Assumptions 1-4, when $T \rightarrow \infty$,*

$$\widehat{\theta}_T \xrightarrow{a.s.} \theta.$$

Proof. It is easy to check that, for any $\delta > 0$ such that $\theta \pm \delta \in \Theta$,

$$\begin{aligned} \ell_T(\theta \pm \delta) - \ell_T(\theta) &= \gamma \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))}{b^2(t, X_t)} dX_t \\ &\quad - \frac{1}{2} \gamma^2 \int_0^T \frac{(a^2(t, X_t, \theta \pm \delta) - a^2(t, X_t, \theta))}{b^2(t, X_t)} dt \\ &= \gamma \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))}{b(t, X_t)} d\tilde{B}_t \\ &\quad - \frac{1}{2} \gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt. \end{aligned}$$

Then we can obtain that

$$(5) \quad \frac{\ell_T(\theta \pm \delta) - \ell_T(\theta)}{\gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt} = \frac{\gamma \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))}{b(t, X_t)} d\tilde{B}_t}{\gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt} - \frac{1}{2}.$$

Under Assumption 4, by using Lepingle's law of large numbers, when $T \rightarrow \infty$, it follows that

$$(6) \quad \frac{\gamma \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))}{b(t, X_t)} d\tilde{B}_t}{\gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt} \xrightarrow{a.s.} 0.$$

Hence, when $T \rightarrow \infty$, it can be check that

$$(7) \quad \frac{\ell_T(\theta \pm \delta) - \ell_T(\theta)}{\gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt} \xrightarrow{a.s.} -\frac{1}{2}.$$

Note that $\gamma^2 \int_0^T \frac{(a(t, X_t, \theta \pm \delta) - a(t, X_t, \theta))^2}{b^2(t, X_t)} dt > 0$ a.s. for large T by Assumption 4. Therefore, for almost every δ and θ , there exists T_0 such that for $T \geq T_0$

$$(8) \quad \ell_T(\theta \pm \delta) < \ell_T(\theta).$$

Since $\ell_T(\theta)$ is continuous on the compact set $[\theta - \delta, \theta + \delta]$, it has a local maximum and the maximum is attained at an element $\widehat{\theta}_T \in (\theta - \delta, \theta + \delta)$. As $\ell_T(\theta)$ is differentiable with respect to θ , it follows that $\ell'_T(\widehat{\theta}_T) = 0$ and $|\widehat{\theta}_T - \theta| < \delta$.

Therefore, when $T \rightarrow \infty$, $\widehat{\theta}_T \xrightarrow{a.s.} \theta$. The proof is complete. $\square$

Let $N_T(\theta) = \int_0^T \frac{(a'(t, X_t, \theta))^2}{b^2(t, X_t)} dt$. In the following theorem, the asymptotic normality of the error of estimation is proved .

Theorem 2. *Under Assumptions 1-4, when $T \rightarrow \infty$,*

$$\sqrt{N_T(\theta)}(\widehat{\theta}_T - \theta) \xrightarrow{d} N(0, 1).$$

Proof. Expanding $\ell'_T(\theta)$ around $\widehat{\theta}_T$, it follows that

$$(9) \quad \ell'_T(\theta) = \ell'_T(\widehat{\theta}_T) + \ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta}_T),$$

where $\widetilde{\theta}_T$ is between $\widehat{\theta}_T$ and θ .

In view of Theorem 1, it is known that $\ell'_T(\widehat{\theta}_T) = 0$, then

$$(10) \quad \ell'_T(\theta) = \ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta}_T).$$

Since $\widehat{\theta}_T \xrightarrow{a.s.} \theta$ when $T \rightarrow \infty$ and $\ell''_T(\theta)$ is continuous in θ , it follows that when $T \rightarrow \infty$,

$$(11) \quad \ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta}_T) \xrightarrow{P} \ell''_T(\theta).$$

By using the central limit theorem and the law of large numbers for square-integrable martingales, it is easy to check that when $T \rightarrow \infty$, $N_T(\theta) \rightarrow \infty$ and

$$\frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} \xrightarrow{d} N(0, 1).$$

Since

$$(12) \quad \frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} = \frac{\ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta}_T)}{\sqrt{N_T(\theta)}},$$

when $T \rightarrow \infty$, we can obtain that

$$(13) \quad \frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} \xrightarrow{P} \frac{\ell''_T(\theta)(\theta - \widehat{\theta}_T)}{\sqrt{N_T(\theta)}}.$$

Hence, when $T \rightarrow \infty$, it follows that

$$(14) \quad \frac{\ell''_T(\theta)(\theta - \widehat{\theta}_T)}{\sqrt{N_T(\theta)}} \xrightarrow{d} N(0, 1).$$

As

$$(15) \quad \frac{\ell''_T(\theta)}{N_T(\theta)} \xrightarrow{P} -1,$$

it can be check that

$$\sqrt{N_T(\theta)}(\widehat{\theta}_T - \theta) \xrightarrow{d} N(0, 1).$$

The proof is complete. $\square$

4. Conclusion

The current work concerns the drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations. The Girsanov theorem has been used to simplify the equations and a new family of probability measures has been indexed. The existence and strong consistency of the estimator and asymptotic normality of the error of estimation have been proved by applying the central limit theorem, Lepingle's law of large numbers and the law of large numbers for square-integrable martingales. Further topics will include the parameter estimation for stochastic differential equations driven by lévy process.

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A hybrid DESA-MOLP method for finding most preferred solution

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Abstract. Data Envelopment Scenario Analysis (DESA) commonly discussed past performance and Multiple Objective Linear Programming (MOLP) well known to plan for future performance goals. The aim of our study is showing an equivalence relation between MOLP and DESA model with using a direction distance function designed to account for desirable and undesirable inputs and outputs together with uncontrollable variables. In addition, we will use the DESA model because in this model we can see decrease all inputs and increases all outputs and estimate one problem instead of n problems. This equivalence model can be effectively used to support interactive processes and performance measures designed to establish future performance goals while taking into account the preferences of decision makers (DMs). In particular, it allows DMs to consider different efficiency improvement strategies when subject to budgetary restrictions. In this context, we tried to solve IDESA models with interactive STOM procedure. The STOM method is the responsible method, because it can estimate any efficient solution, and it indicates Most Preferred Solution (MPS).

Keywords: Data envelopment analysis, Uncertain data, STOM algorithm, Multi objective linear, programming.

1. Introduction

The first investigations into Data Envelopment Analysis (DEA) was conducted in 1978 by Charnes et al. also in the last few years, much more information on Multi Objective Linear Programming (MOLP) has become available. DEA and MOLP are useful for management to draw a scheme for future. DEA is used to evaluate the past preferences while MOLP is used for future preferences.

A hybrid DEA-MOLP model has been identified as being Tavana et al (2017). Recent developments in this model have led to using interactive MOLP to solve DEA model and finding Most Preferred Solution (MPS).

Data Envelopment Scenario Analysis (DESA) model was chosen because it is one of the most practical ways to this study. With this in mind we tried to show a hybrid Imprecise DESA-MOLP and use the interactive MOLP method with the name of STOM method (M) to solve the IDESA.

Imprecise DEA(IDEA) is defined by Cooper et al. (1999) to mean that data were vague, which results in a non-linear DEA model and we can transform it into a linear problem. Also, we obtain a double of interval DEA models that will be utilized for interval data rather than for crisp data. In addition, we would not use the MPS in the IDESA models by the interactive algorithm like STOM method.

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2. Data Envelopment Scenario Analysis (DESA)

Thanassoulis and Dyson (1992) developed on a new method for DEA and concluded that discuss the uselessness of radial projection in target setting. In this model, we have $I = \{1, \dots, m\}$ showing that I is a set of inputs and $O = \{1, \dots, s\}$ showing that O is a set of outputs. By substituting $I \cong I_g \cup \bar{I}_g$ and $O \cong O_g \cup \bar{O}_g$, where O_g and I_g are used to show outputs and inputs where borders of success are used in the target model. The DESA problem can be expressed as follows:

$$\begin{aligned}
 (1) \quad & \min_{\lambda_j, Z_k, \theta_i} \quad \sum_{k \in O} P_k^+ Z_k - \sum_{i \in I} P_i^- \theta_i \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} = \theta_i x_{io} \quad i \in I \\
 & \quad \quad \sum_{j=1}^n \lambda_j y_{kj} = Z_k y_{ko} \quad k \in O \\
 & \quad \quad \theta_i x_{io} \geq G_i \quad i \in I_g \\
 & \quad \quad Z_k y_{ko} \leq G'_k \quad k \in O_g \\
 & \quad \quad A_i \leq \theta_i \leq 1/B_i, \quad A_i, B_i \in [0, 1], \quad \forall i \in I \\
 & \quad \quad \Gamma_k \leq 1/Z_k \leq 1/\Delta_k, \quad \Gamma_k, \Delta_k \in [0, 1], \quad \forall k \in O
 \end{aligned}$$

P_i^- and P_k^+ are the decision makers' preferences for the recovery of inputs and outputs. θ_i is the contraction rate of input i and Z_k is the development rate of output k . G_i and G'_k are borders for i th input and k th output, respectively. (A_i, B_i) and (Γ_k, Δ_k) are remarked as the width and height borders for θ_i , Z_k respectively.

3. DESA method with imprecise data

DESA model with imprecise data has many valuable applications, the key advantage is to calculate one model instead of n models, this method can be

formulated as follows:

$$\begin{aligned}
 (2) \quad & \max_{\lambda_j, Z_k, \theta_i} \sum_{k \in O} P_k^+ Z_k - \sum_{i \in I} P_i^- \theta_i - M \sum_{i \in I_f} s'_i - M \sum_{k \in O_f} s''_k \\
 & \text{s.t.} \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij} = \theta_i \sum_{j=1}^n x_{ij} \quad i \in I \\
 & \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj} = Z_k \sum_{j=1}^n y_{kr} \quad k \in O \\
 & \quad \sum_{j=1}^n \theta_i x_{ij} \geq G_i^L - s'_i \quad i \in I_f \\
 & \quad \sum_{r=1}^n Z_k y_{kr} \leq G_k'^U - s''_k \quad k \in O_f \\
 & \quad \sum_{j=1}^n \lambda_{jr} = 1 \quad \forall r \\
 & \quad \lambda_j \geq 0 \quad \forall j, \quad \theta_i \text{ free}, \quad Z_k \text{ free}, \quad k \in O \\
 & \quad s'_i \geq 0 \quad \forall i \in I_f, \quad s''_k \geq 0 \quad \forall k \in O_f
 \end{aligned}$$

$|G_i^L, G_i^U|$ and $|G_k'^L, G_k'^U|$ are the interval resources for all input consumption and all output generation, respectively. s'_i, s''_k are intended for the legalization of all diminution and all generation, respectively. M shows a penalty factor that has to be intended by the decision maker.

4. Imprecise DESA model based on interval arithmetic

The first systematic report on change DEA model to two models was carried out in 2005 by Wang, so that we could be able to use this method for changing the DESA model to two new models with the names of undesirable model and desirable model. The most remarkable result to emerge from data is that best lower bound and best upper bound for each DMU, where

$$(3) \quad x_{ij} \in [x_{ij}^L, x_{ij}^U], \quad y_{kj} \in [y_{kj}^L, y_{kj}^U], \quad \theta_i \in [\theta_i^L, \theta_i^U], \quad Z_k \in [Z_k^L, Z_k^U].$$

G_i^U is the limit to all use of input x_{ij}^U , and $G_k'^L$ is the limit to total production of output y_{kr}^L

$$\begin{aligned}
 & \max_{\lambda_j, Z_k, \theta_i} \sum_{k \in O} P_k^+ Z_k^L - \sum_{i \in I} P_i^- \theta_i^L - M \sum_{i \in I_f} s'_i - M \sum_{k \in O_f} s''_k \\
 & \text{s.t.} \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij}^L = \theta_i^L \sum_{j=1}^n x_{ij}^U \quad i \in I
 \end{aligned}$$

$$\begin{aligned}
(4) \quad & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj}^U = Z_k^L \sum_{j=1}^n y_{kr}^L & k \in O \\
& \sum_{j=1}^n \theta_i^L x_{ij}^U \geq G_i^U - s'_i & i \in I_f \\
& \sum_{r=1}^n Z_k^L y_{kr}^L \leq G_k^L - s''_k & k \in O_f \\
& \sum_{j=1}^n \lambda_{jr} = 1 & \forall r \\
& \lambda_{jr} \geq 0 \quad \forall j, \quad \theta_i^L \text{ free}, \quad Z_k^L \text{ free}, \quad k \in O \\
& s'_i \geq 0 \quad \forall i \in I_f, \quad s''_k \geq 0 \quad \forall k \in O_f.
\end{aligned}$$

Where G_i^L is the limit to total consumption of input x_{ij}^L and G_k^U is the limit to total production of output y_{kr}^U . According to the above definition, model (3) is the lower bound of the best possible relative efficiency and model (4) is the upper bound of the best possible relative efficiency.

5. A hybrid imprecise DESA- MOLP model

This paper focuses on relation between multi-objective linear program (MOLP) and DESA model with imprecise data, suppose that:

$$(5) \quad \bar{f}_k(\lambda) = \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj}^U \quad k = 1, \dots, s,$$

$$(6) \quad \tilde{f}_i(\lambda) = \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{ij}^L \quad i = 1, \dots, s.$$

So that

$$\lambda = (\lambda_{11}, \lambda_{jr}, \dots, \lambda_{nn})^T.$$

It is assumed that $f'_k = \bar{f}_k(\lambda^*)$ is the maximum possible rate for all the k th outputs and $f''(i) = \tilde{f}_i(\lambda^*)$ is the minimum possible rate for all the i th inputs. where λ^* can be computed by finding the lower part of the models:

$$\begin{aligned}
(7) \quad & \max \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj}^U \\
& \text{s.t.} \quad \lambda \in \Omega
\end{aligned}$$

$$\begin{aligned}
(8) \quad & \min \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{ij}^L \\
& \text{s.t.} \quad \lambda \in \Omega.
\end{aligned}$$

Such that $f' = [f'_1, \dots, f'_s]$ and $f'' = [f''_1, \dots, f''_m]$ indicate the desirable points of model (5) and model (6). Suppose that:

$$(9) \quad \begin{aligned} \sum_{j=1}^n x_{ij} &> 0 \quad i = 1, \dots, m, \\ \sum_{r=1}^n y_{kr} &> 0 \quad k = 1, \dots, s, \\ w_k &= \frac{1}{\sum_{r=1}^n y_{kr}^U} \quad \forall k, \\ v_i &= \frac{1}{\sum_{j=1}^n x_{ij}^L} \quad \forall i. \end{aligned}$$

And

$$(10) \quad \begin{aligned} \bar{f}_k^* &= \frac{F^{\max}}{w_k} = F^{\max} \sum_{r=1}^n y_{kr}^U, \quad \bar{f}_i^* = \frac{H^{\min}}{v_i} = H^{\min} \sum_{j=1}^n x_{ij}^L \\ \tau_i &= \theta_i^U - H^{\min}, \quad \gamma_k = F^{\max} - Z_k^U. \end{aligned}$$

Such that

$$(11) \quad F^{\max} = \max_{1 \leq k \leq s} \{w_k f'_k\} = \max_{1 \leq k \leq s} \left\{ \frac{f'_k}{\sum_{r=1}^n y_{kr}^U} \right\},$$

$$(12) \quad H^{\min} = \min_{1 \leq i \leq m} \{v_i f''_i\} = \min_{1 \leq i \leq m} \left\{ \frac{f''_i}{\sum_{j=1}^n x_{ij}^L} \right\}.$$

We can write the equivalence model as the following model:

$$(13) \quad \begin{aligned} \max_{\lambda_j, Z_k, \theta_i} \quad & \sum_{k \in O} P_k^+ Z_k^U - \sum_{i \in I} P_i^- \theta_i^U \\ \text{s.t.} \quad & \bar{f}_i(\lambda) - \theta_i^U \frac{1}{v_i} = 0 \quad i = 1, \dots, m \\ & Z_k^U \frac{1}{w_k} - \bar{f}_k(\lambda) = 0 \quad k = 1, \dots, s \\ & g_i v_i \leq \theta_i^U \quad i \in I_f \\ & g'_k w_k \geq Z_k^U \quad k \in O_f, \lambda \in \Omega. \end{aligned}$$

Also, we can write

$$(14) \quad \begin{aligned} \tilde{f}_i(\lambda) - \theta_i^U \frac{1}{v_i} = 0 &\Leftrightarrow v_i \tilde{f}_i(\lambda) = \theta_i^U \\ &\Leftrightarrow v_i \tilde{f}_i(\lambda) - H^{\min} = \theta_i^U - H^{\min} \Leftrightarrow v_i(\tilde{f}_i(\lambda) - \tilde{f}_i^*) = \theta_i^U - H^{\min}. \end{aligned}$$

Therefore,

$$(15) \quad \begin{aligned} Z_k^U \frac{1}{w_k} \bar{f}_k(\lambda) = 0 &\Leftrightarrow -w_k \bar{f}_k(\lambda) = -Z_k^U \\ &\Leftrightarrow F^{\max} - w_k \bar{f}_k(\lambda) = F^{\max} - Z_k^U \Leftrightarrow w_k(\bar{f}_k^* - \bar{f}_k(\lambda)) = F^{\max} - Z_k^U. \end{aligned}$$

Such that $P_k^+ = P_i^- = 1$, $k \in O$, $i \in I$. Accordingly, above changes lead to the following model:

$$\begin{aligned} \max_{\lambda_j, Z_k, \theta_i} \left(\sum_{k \in O} Z_k^U - \sum_{i \in I} \theta_i^U \right) &= \min_{\lambda_j, Z_k, \theta_i} \left(\sum_{i \in I} \theta_i^U + \sum_{k \in O} (-Z_k^U) \right) \\ \Rightarrow \min \left(\sum_{i \in I} (\theta_i^U - H^{\min}) + \sum_{k \in O} (F^{\max} - Z_k^U) \right) &= \min \left(\sum_{i \in I} \tau_i + \sum_{k \in O} \gamma_k \right). \end{aligned}$$

Also, for any $\lambda \in \Omega$, we have:

$$\begin{aligned} F^{\max} - Z_k^U &\geq w_k \bar{f}_k^* - Z_k^U \geq w_k \bar{f}_k(\lambda) - Z_k^U = 0 & k = 1, \dots, s \\ \theta_i^U - H^{\min} &\geq \theta_i^U - v_i \tilde{f}_i^* \geq \theta_i^U - v_i \tilde{f}_i(\lambda) = 0 & i = 1, \dots, m \\ \bar{f}_k^* = \frac{F^{\max}}{w_k} &\geq \frac{w_k f'_k}{w_k} = f'_k = \max_{\lambda \in \Omega} \bar{f}_k(\lambda) & k = 1, \dots, s \\ f_i^* = \frac{H^{\min}}{v_i} &\leq \frac{v_i f''_i}{v_i} = f''_i = \min_{\lambda \in \Omega} \tilde{f}_i(\lambda) & i = 1, \dots, m. \end{aligned}$$

Now imprecise DESA model can be write as a minimax formulation as follows:

$$(16) \quad \begin{aligned} \min \quad & \sum_{i \in I} \tau_i \sum_{k \in O} \gamma_k \\ \text{s.t.} \quad & v_i(\tilde{f}_i(\lambda) - \tilde{f}_i^*) = \tau_i & i \in I \\ & w_k(\bar{f}_k^* - \bar{f}_k(\lambda)) = \gamma_k & k \in k = O \\ & g_i v_i - H^{\min} \leq \tau_i & i \in I_f \\ & g'_k w_k - M \leq \gamma_k & k \in O_f \\ & \sum_{j=1}^n \lambda_{jr} = 1, \forall r \lambda_{jr} \geq 0, M \geq 0 \quad j, r = 1, \dots, n. \end{aligned}$$

Now imprecise DESA model and minimax MOLP formulation are equivalent, at this point an interactive method can be used for solve the imprecise DESA models and find the most preferred solution.

6. STOM algorithm for solving DESA model with imprecise data

STOM method is an interactive method, this method helps to DM for search the MPS on the efficient frontier by systematically changing the weighting parameters. STOM algorithm is defined as follows:

Step 1. During the first phase, solve the combined-oriented DESA model (7), (8) for all DMUs, so that we would be able to obtain the inefficient and efficient DMUs

Step 2. In the second step, we could be to check MPS with collect the target unit based on formulation (16), if the DM admit the target unit obtained as MPS, then stop. Elsewhere, go to step 3.

Step 3. Set the initial weighting parameters for all DMUs, and reach the initial solution of the decision variables.

Suppose for given positive weight vectors $w^t = \{w_1^t, \dots, w_k^t, \dots, w_s^t\}$ and $v^t = \{v_1^t, \dots, v_i^t, \dots, v_m^t\}$.

The optimal solution of the minimax model is given by $\lambda^t = \{\lambda_{11}^t, \dots, \lambda_{jr}^t, \dots, \lambda_{nn}^t\}$, which must be an efficient solution. The optimal values of the dual variable of the k th and i th objective constraint $w_k(\bar{f}_k^* - \bar{f}_k(\lambda)) = \gamma_k$, $v_i(\bar{f}_i(\lambda) - \bar{f}_i^*) = \tau_i$ are given by β_k^t and α_i^t . Iteration $t = 0$. Select an initial point $\lambda^0 = [\lambda_{11}^0, \dots, \lambda_{nn}^0]$. An initial solution can be obtained by solving minimax model by setting

$$v_i^t = \left(\frac{1}{X_i^* - f_i''} \right) \quad \forall i = 1, \dots, m$$

and

$$w_k^t = \left(\frac{1}{f_k' - Y_k'^*} \right) \quad \forall k = 1, \dots, s.$$

Also, the initial dual variable values $\beta^t = [\beta_1^t, \dots, \beta_s^t]^T$, $\alpha^t = [\alpha_1^t, \dots, \alpha_m^t]^T$ for the first s and i constraint on the upper bounds of outputs and lower bounds of inputs.

Step 4. we are now ready to find the new reference point by $\bar{q}^t = (\bar{q}_1^t, \dots, \bar{q}_s^t)$ and $\tilde{q}^t = (\tilde{q}_1^t, \dots, \tilde{q}_m^t)$ Where $\tilde{q}_i^t = f_i^t + \tilde{\tau}^t \cdot \Delta f_i''' \quad \forall i = 1, \dots, m$,
 $\bar{q}_k^t = f_k^t + \bar{\gamma}^t \cdot \Delta f_k'' \quad \forall k = 1, \dots, s$

$$\begin{aligned} \bar{\mu}_k^t &= \frac{1}{|\bar{q}_k^t - \bar{f}_k^t|} & \forall k = 1, \dots, s, \\ \tilde{\mu}_i^t &= \frac{1}{|\tilde{q}_i^t - \tilde{f}_i^t|} & \forall i = 1, \dots, m. \end{aligned}$$

Let (λ^*, f^*) be a solution.

Step 5. given the new point of reference obtained in step 4, the new weights are computed as follows:

$$\begin{aligned} w_k^t &= \frac{1}{f'_k - Y_k'^*} & \forall k = 1, \dots, s_1, \\ w_k^t &= \frac{1}{f'_k - Y_k'^*} & \forall k = 1, \dots, s_2, \\ V_i^t &= \frac{1}{f'_i - x_i'^*} & \forall i = 1, \dots, m_1, \\ V_i^t &= \frac{1}{f'_i - Y_i'^*} & \forall i = 1, \dots, m_2. \end{aligned}$$

Step 6. after determined the new weights in step 5, we could solve the super ideal point method (16) and obtain the optimal value.

Step 7. select the most preferred solutions by the decision maker.

with the completion of these steps and solve formulation (16), we are now ready to obtain the new target unit of DMU_p . Decision maker select the best solutions, otherwise we go to step 4.

7. Numerical example

The following table summarizes the data pertaining to seven banks. These data were prepared as described by Yang (2004), the sample was selected on the basis of four inputs and two outputs. This sample was solved by STOM algorithm, the method has several benefits, for instance, it searches for the MPS on the efficient frontier. We noted from Table 1 that four inputs are branches, number of ATMs, staff and asset size. The two outputs are customer satisfaction and total revenue. On the other hand, for this study, it is assumed that the data have been changed from certain to imprecise, as shown in Table 2.

Table 1: Original Data Set

DMU	Bank	Branches	ATMs	Staff	Asset size	Customer	Total
1	Abby	0.77	2.18	2.35	2.96	6.79	10.57
2	Barclay	1.95	3.19	8.43	3.53	2.55	13.35
3	Halifax	0.80	2.10	3.21	2.41	9.17	8.14
4	HSBC	1.75	4.00	13.30	4.85	5.82	23.67
5	Lloyds TSB	2.50	4.30	9.27	2.40	6.57	14.01
6	Nat West	1.73	3.30	7.70	3.09	4.86	12.04
7	RBS	0.65	1.73	2.65	1.34	7.28	7.36

8. Conclusions

The evidence from this study intimates that a hybrid DESA-MOLP model developed to consider strategic environments in which the value of inputs and outputs are uncertain and the basis on which to apply the STOM interactive

Table 2: Interval Data Set

DMU	Bank	Branch	ATMs	Staff	Asset size	Customer	Total
1	Abby	0.77	[2.18,2.2]	[2.35,2.50]	[2.96,3]	6.79	10.57
2	Barclay	1.95	[3.19,3.20]	[8.43,8.50]	[3.53,4]	2.55	13.35
3	Halifax	0.80	[2.10,2.2]	[3.21,3.50]	[2.41,2.5]	9.17	8.14
4	HSBC	1.75	[4.00,4.1]	[13.3,13.5]	[4.85,5]	5.82	23.67
5	Lloyds TSB	2.50	[4.30,4.7]	[9.27,9.3]	[2.40,2.43]	6.57	14.01
6	Nat West	1.73	[3.30,3.40]	[7.7,7.8]	[3.09,3.2]	4.86	12.04
7	RBS	0.65	[1.73,1.80]	[2.65,2.7]	[1.34,1.4]	7.28	7.36

method so as to achieve the MPS along the efficient frontier for each DMU. We are confident that our procedure might be useful for finding any efficient solution. Also, this approach results in decreasing total input and increasing total output at the same time. Also, it handles one problem instead of solving n independent linear programming models.

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Prime-valent one-regular graphs of order $8p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify one-regular graphs of prime valency and order $8p$ for each prime p . By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there are only two infinite families of such graphs: one is the cycle C_{8p} with valency 2, the other is the $\mathbb{Z}_p$ -cover CQ_p of hypercube Q_3 with valency 3 and $3|(p-1)$.

Keywords: one-regular graph, arc-transitive graph, covering graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [14, 16] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G , that is, the subgroup of G fixing the point v . We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X , denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G -vertex-transitive* if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\text{Aut}(X)$ -vertex-transitive. An *s -arc* in a graph is an ordered $(s+1)$ -tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph X is said to be *(G, s) -arc-transitive* or *(G, s) -regular* if G is transitive or regular on the set of s -arcs in X , respectively. A (G, s) -arc-transitive graph is said to be *(G, s) -transitive* if it is not $(G, s+1)$ -arc-transitive. In particular, a $(G, 1)$ -arc-transitive graph is called *G -symmetric*. A graph X is simply called *s -arc-transitive*, *s -regular* or *s -transitive* if it is $(\text{Aut}(X), s)$ -arc-transitive, $(\text{Aut}(X), s)$ -regular or $(\text{Aut}(X), s)$ -transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n , respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n , that is, the cycle

C_n , which is 1-regular with full automorphism group D_{2n} . Let p be a prime. Classifying s -transitive and s -regular graphs has received considerable attention. The classification of s -transitive graphs of order p and $2p$ was given in [3] and [4], respectively. Wang [15] characterized the prime-valent s -transitive graphs of order $4p$. The classification of cubic, pentavalent and heptavalent s -transitive graphs of order $8p$ was given in [8], [13] and [12], respectively.

For 2-valent case, s -transitivity always means 1-regularity, and for cubic case, s -transitivity always means s -regularity by Miller [7]. However, for the other prime-valent case, this is not true, see for example [9] for pentavalent case and [10] for heptavalent case. Thus, characterization and classification of prime-valent s -regular graphs is very interesting and also reveals the s -regular global and local actions of the permutation groups on s -arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $8p$ for each prime p .

2. Preliminary results

Let X be a connected G -symmetric-transitive graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G . The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [11, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G -symmetric graph with $G \leq \text{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G . Then one of the following holds:*

- (1) N is transitive on $V(X)$;
- (2) X is bipartite and N is transitive on each part of the bipartition;
- (3) N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q -valent G/N -symmetric graph.

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [4], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \dots, p-1\}$ and $V' = \{0', 1', \dots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of $\mathbb{Z}_p^*$ of order q . Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q -valent symmetric graph of order $2p$ with p, q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p, q)$ with $q|(p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then $\text{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p, q) = (11, 5)$ then $\text{Aut}(G(2p, q)) = \text{PGL}(2, 11)$.*

From [6, pp.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.3. *Let G be a non-abelian simple group. Suppose that the order $|G|$ has at most three different prime divisors. Then G is called K_3 simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$ -groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\text{PSL}(2, 17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\text{PSL}(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$\text{PSL}(2, 7)$	$2^3 \cdot 3 \cdot 7$	$\text{PSU}(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$\text{PSL}(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$\text{PSU}(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $8p$ for each prime p . Let q be a prime. In what follows, we always denote by X a connected q -valent one-regular graph of order $8p$. Set $A = \text{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 8pq$. Clearly, if $q = 2$, then $X \cong C_{8p}$ with $A \cong D_{16p}$.

Let $q = 3$. Then by [8, Theorem 5.1], we can have the classification of cubic one-regular graphs of order $8p$. For convenience, we use the same notation CQ_p as in [8] to denote the cyclic $\mathbb{Z}_p$ -covering graph of the three-dimensional hypercube Q_3 .

Lemma 3.1. *If $q = 3$, then $X \cong CQ_p$ and $A \cong \mathbb{Z}_p \rtimes (A_4 \times \mathbb{Z}_2)$ with $3|(p-1)$.*

For $q = 5$ or 7 , by [13, Theorem 3.1] and [12, Theorem 1.1], it is easy to see that there is no new graph. For $p = 2$ or 3 , by [5], there is no prime-valent one-regular graph of order 16 or 24 . Thus, we treat with the case $p \geq 5$ and $q > 7$ by proving the following lemma.

Lemma 3.2. *Let $p \geq 5$ and $q > 7$. Then there is no new graph.*

Proof. Recall that $|A| = 8pq$ and $A_v \cong \mathbb{Z}_q$. By Proposition 2.3, the order of each K_3 simple group has a divisor 3 . It forces that A is solvable. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Case 1: Suppose that $p = q$. Then $|A| = 8p^2$ and $A_v \cong \mathbb{Z}_p$.

Let P be a Sylow p -subgroup of A . Then $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, we have that P is normal in A . This means that P is the only Sylow p -subgroup of A . Since $A_v \cong \mathbb{Z}_p$, we have that $A_v \leq P$, that is, $A_v = P_v \neq 1$. By Proposition 2.1, P is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P| = p^2$ and $|V(X)| = 8p$.

Case 2: Suppose that $p \neq q$. Then $|A| = 8pq$ and $A_v \cong \mathbb{Z}_q$.

Since $|A| = 8pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q -subgroup of A . It forces that the Sylow q -subgroups of A cannot be normal in A . Recall that A is solvable. Thus, all normal subgroups of A are solvable. It follows that A has a maximal normal r -subgroup with $r = 2$ or p .

Assume that A has a maximal normal p -subgroup M . Then $M \cong \mathbb{Z}_p$ because $|A| = 8pq$. Clearly, M acting on $V(X)$ has 8 orbits. By Proposition 2.1, X_M is a q -valent symmetric graph of order 8. By [5], $X_M \cong Q_3$ with $q = 3$ or K_8 with $q = 7$. This contradicts our hypothesis. Thus, the Sylow p -subgroup of A cannot be normal in A .

Assume that A has a maximal normal 2-subgroup N . Since N is a 2-subgroup, we have that N acting on $V(X)$ has at least p orbits. By Proposition 2.1, X_N is a q -valent symmetric graph of order $8p/|N|$. Recall that $q > 7$ is an odd prime and there is no graph of odd order and odd valency. Thus, $|N| \neq 8$ and $|N| = 2$ or 4 .

Let $|N| = 2$. Then X_N is a q -valent symmetric graph of order $4p$ and $|A/N| = 4pq$. Note that A is solvable. Thus, A/N is also solvable. Since $A_v \cong \mathbb{Z}_q$, we have that A/N has no normal q -subgroup. If A/N has a normal p -subgroup $K/N \cong \mathbb{Z}_p$. Then $|K| = 2p$. Since $p \geq 5$, we have that K has a normal Sylow p -subgroup M by Sylow Theorem. It forces that M is characteristic in K and hence normal in A . By the above argument, this is impossible. This implies that A/N has a non-trivial normal 2-subgroup, this contradicts the maximality of N .

Let $|N| = 4$. Then X_N is a q -valent symmetric graph of order $2p$ and $|A/N| = 2pq$. Recall that $q \neq p$ and $q \neq 5$. By Proposition 2.2, $X_N \cong K_{2p}$ with $q = 2p - 1$ or $G(2p, q)$ with $A \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. For the former, $q = 2p - 1$ is a prime. Thus, $A/N \lesssim S_{2p}$ and A/N is 2-transitive on $V(X)$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since A/N is solvable, we have that A/N is affine. It forces that A/N must have a normal subgroup isomorphic to $\mathbb{Z}_p$. By the above argument, this is impossible. For the later, $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that A/N has a normal Sylow p -subgroup. Similarly, since $p \geq 5$, we can easily deduce that A has a normal Sylow p -subgroup, a contradiction. $\square$

Combining the above arguments with the cases $q = 2, 5, 7$ and $p = 2, 3$, and by Lemmas 3.1-3.2, we have the following result.

Theorem 3.3. *Let p, q be two primes and X a connected q -valent one-regular graph of order $8p$. Then $X \cong C_{8p}$ with $\text{Aut}(X) \cong D_{16p}$ or $X \cong CQ_p$ with $\text{Aut}(X) \cong \mathbb{Z}_p \rtimes (A_4 \times \mathbb{Z}_2)$ and $3|(p - 1)$.*

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Prime-valent one-regular graphs of order $16p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify one-regular graphs of prime valency and order $16p$ for each prime p . By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there are only two infinite families of such graphs: one is the cycle C_{16p} with valency 2, the other is the $\mathbb{Z}_p$ -cover CF_{16p} of Möbius-Kantor graph F_{16} with valency 3 and $6|(p-1)$.

Keywords: one-regular graph, arc-transitive graph, covering graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [15, 17] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G , that is, the subgroup of G fixing the point v . We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X , denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G -vertex-transitive* if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\text{Aut}(X)$ -vertex-transitive. An *s -arc* in a graph is an ordered $(s+1)$ -tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph X is said to be *(G, s) -arc-transitive* or *(G, s) -regular* if G is transitive or regular on the set of s -arcs in X , respectively. A (G, s) -arc-transitive graph is said to be *(G, s) -transitive* if it is not $(G, s+1)$ -arc-transitive. In particular, a $(G, 1)$ -arc-transitive graph is called *G -symmetric*. A graph X is simply called *s -arc-transitive*, *s -regular* or *s -transitive* if it is $(\text{Aut}(X), s)$ -arc-transitive, $(\text{Aut}(X), s)$ -regular or $(\text{Aut}(X), s)$ -transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n , respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n , that is, the cycle

C_n , which is 1-regular with full automorphism group D_{2n} . Let p be a prime. Classifying s -transitive and s -regular graphs has received considerable attention. The classification of s -transitive graphs of order p and $2p$ was given in [4] and [5], respectively. Wang [16] characterized the prime-valent s -transitive graphs of order $4p$. The classification of cubic, pentavalent and heptavalent s -transitive graphs of order $16p$ was given in [14], [10] and [11], respectively.

For 2-valent case, s -transitivity always means 1-regularity, and for cubic case, s -transitivity always means s -regularity by Miller [8]. However, for the other prime-valent case, this is not true, see for example [9] for pentavalent case and [12] for heptavalent case. Thus, characterization and classification of prime-valent s -regular graphs is very interesting and also reveals the s -regular global and local actions of the permutation groups on s -arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $16p$ for each prime p .

2. Preliminary results

Let X be a connected G -symmetric-transitive graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G . The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [13, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G -symmetric graph with $G \leq \text{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G . Then one of the following holds:*

- (1) N is transitive on $V(X)$;
- (2) X is bipartite and N is transitive on each part of the bipartition;
- (3) N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q -valent G/N -symmetric graph.

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \dots, p-1\}$ and $V' = \{0', 1', \dots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of $\mathbb{Z}_p^*$ of order q . Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q -valent symmetric graph of order $2p$ with p, q primes. Then X is isomorphic to K_{2p} with $q = 2p - 1$, $K_{p,p}$ or $G(2p, q)$ with $q \mid (p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then $\text{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p, q) = (11, 5)$ then $\text{Aut}(G(2p, q)) = \text{PGL}(2, 11)$.*

From [7, pp.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.3. *Let G be a non-abelian simple group. Suppose that the order $|G|$ has at most three different prime divisors. Then G is called K_3 simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$ -groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\text{PSL}(2, 17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\text{PSL}(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$\text{PSL}(2, 7)$	$2^3 \cdot 3 \cdot 7$	$\text{PSU}(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$\text{PSL}(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$\text{PSU}(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $16p$ for each prime p . Let q be a prime. In what follows, we always denote by X the q -valent one-regular graph of order $16p$. Set $A = \text{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 16pq$. Clearly, if $q = 2$, then $X \cong C_{16p}$ with $A \cong D_{32p}$.

Let $q = 3$. Then by [14, Theorem 2.7], we can have the classification of cubic one-regular graphs of order $16p$. For convenience, we use the same notation CF_{16p} as in [14] to denote the cyclic $\mathbb{Z}_p$ -covering graph of Möbius-Kantor graph F_{16} . Denote by $D_8 \circ \mathbb{Z}_4$ the central product of D_8 and $\mathbb{Z}_4$.

Lemma 3.1. *If $q = 3$, then $X \cong CF_{16p}$ and $A \cong \mathbb{Z}_p \rtimes ((D_8 \circ \mathbb{Z}_4) \rtimes \mathbb{Z}_3)$ with $6|(p-1)$.*

For $q = 5$ or 7 , by [10, Theorem 1.1] and [11, Theorem 1.1], it is easy to see that there is no new graph. Thus, we treat with the case $p \geq 3$ and $q > 7$ by proving the following lemma.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then there is no new graph.*

Proof. Recall that $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$. If A is non-solvable, then A must contain a non-solvable composition factor H , which is isomorphic to a non-abelian simple group. It forces that $|H| \mid 16pq$ and H is a K_3 simple group. However, since $q > 7$ and $p \geq 3$, this is impossible by Proposition 2.3. Thus, A is solvable. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Case 1: Suppose that $p = q$. Then $|A| = 16p^2$ and $A_v \cong \mathbb{Z}_p$.

Let P be a Sylow p -subgroup of A . Then $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p -subgroups of A is $kp+1 = |A|$:

$|N_A(P)|$ for some integer k . Since $|A| = 16p^2$, we have that $(kp+1)|16$. Suppose that P is not normal in A . Then $kp+1 > 1$ and hence $k \geq 1$. However, $p > 7$ implies that $kp+1$ cannot be a divisor of 16 for each $k \geq 1$, a contradiction. Thus, P is normal in A . This means that P is the only Sylow p -subgroup of A . Since $A_v \cong \mathbb{Z}_p$, we have that $A_v \leq P$, that is, $A_v = P_v \neq 1$. By Proposition 2.1, P is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P| = p^2$ and $|V(X)| = 16p$.

Case 2: Suppose that $p \neq q$. Then $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$.

Since $|A| = 16pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q -subgroup of A . It forces that the Sylow q -subgroups of A cannot be normal in A . Recall that A is solvable. Thus, all normal subgroups of A are solvable. It follows that A has a maximal normal r -subgroup with $r = 2$ or p .

Assume that A has a maximal normal p -subgroup M . Then $M \cong \mathbb{Z}_p$ because $|A| = 16pq$. Clearly, M acting on $V(X)$ has 16 orbits. By Proposition 2.1, X_M is a q -valent symmetric graph of order 16. By [6], X_M is isomorphic to Möbius-Kantor graph F_{16} with $q = 3$, Q_4^d with $q = 5$, or $K_{8,8} - 8K_2$ with $q = 7$, where Q_4^d denotes the graph obtained by connecting all long diagonal of 4-cube Q_4 . This contradicts our hypothesis that $q > 7$. Thus, the Sylow p -subgroup of A cannot be normal in A .

Assume that A has a maximal normal 2-subgroup N . Since N is a 2-subgroup, we have that N acting on $V(X)$ has at least p orbits. By Proposition 2.1, X_N is a q -valent symmetric graph of order $16p/|N|$. Recall that $q > 7$ is an odd prime and there is no graph of odd order and odd valency. Thus, $|N| \neq 16$ and $|N| = 2, 4$ or 8 .

Let $|N| = 2$. Then X_N is a q -valent symmetric graph of order $8p$ and $|A/N| = 8pq$. Note that A is solvable. Thus, A/N is also solvable. Since $A_v \cong \mathbb{Z}_q$, we have that A/N has no normal q -subgroup. If A/N has a normal p -subgroup $K/N \cong \mathbb{Z}_p$. Then $|K| = 2p$. Since $p \geq 3$, we have that K has a normal Sylow p -subgroup M by Sylow Theorem. It forces that M is characteristic in K and hence normal in A . By the above argument, this is impossible. This implies that A/N has a non-trivial normal 2-subgroup, this contradicts the maximality of N .

Let $|N| = 4$. Then X_N is a q -valent symmetric graph of order $4p$ and $|A/N| = 4pq$. If A/N has a normal Sylow q -subgroup $H/N \cong \mathbb{Z}_q$, then H is normal in A . Clearly, $|H| = 4q$. Note that $q > 7$. Thus, Sylow Theorem implies that H has a normal Sylow q -subgroup $Q \cong \mathbb{Z}_q$. It forces that Q is characteristic in H and hence normal in A . This is impossible because a Sylow q -subgroup of A cannot be normal in A . If A/N has a normal Sylow p -subgroup $K/N \cong \mathbb{Z}_p$, then $|K| = 4p$ and K is normal in A . Recall $p \geq 3$. Suppose that $p = 3$. Then X_N is a q -valent symmetric graph of order 12. Since $q > 7$, by [6], the only possible is $X_N \cong K_{12}$. In this case, $|A/N| = 12 \cdot 11$ and $A/N \lesssim S_{12}$. However, by Magma [3], S_{12} has no arc-transitive subgroup of order $12 \cdot 11$, a contradiction. Suppose that $p \geq 5$. Then by Sylow Theorem K must have a

normal Sylow p -subgroup P . It forces that P is characteristic in K and hence normal in A . This is also impossible because by the above argument A has no normal Sylow p -subgroup.

Let $|N| = 8$. Then X_N is a q -valent symmetric graph of order $2p$ and $|A/N| = 2pq$. Recall that $q \neq p$ and $q \neq 3$. By Proposition 2.2, $X_N \cong K_{2p}$ with $q = 2p - 1$ or $G(2p, q)$ with $q|(p - 1)$ with $A \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. For the former, $q = 2p - 1$ is a prime. Thus, $A/N \lesssim S_{2p}$ and A/N is 2-transitive on $V(X)$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since A/N is solvable, we have that A/N is affine. It forces that A/N must have a normal subgroup isomorphic to $\mathbb{Z}_p$. By the above argument, this is impossible. For the later, $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that A/N has a normal Sylow p -subgroup. Since $q > 7$ and $q|(p - 1)$, we have that $p \geq 23$. With an easy calculation by using Sylow Theorem, we can deduce that A has a normal Sylow p -subgroup, a contradiction. $\square$

Combining the above arguments with the cases $q = 2, 5, 7$, and by Lemmas 3.1-3.2, we have the following result.

Theorem 3.3. *Let p, q be two primes and X a connected q -valent one-regular graph of order $16p$. Then $X \cong C_{16p}$ with valency 2 and $\text{Aut}(X) \cong D_{32p}$ or $X \cong CF_{16p}$ with valency 3, $\text{Aut}(X) \cong \mathbb{Z}_p \rtimes ((D_8 \circ \mathbb{Z}_4) \times \mathbb{Z}_3)$ and $6|(p - 1)$.*

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Visualization of cordial graph in human excretion track

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Abstract. The quantitative research study of this paper is to show the relationship between cordial graphic topological space and its uses in the blood circulation of the human kidney.

Keywords: Alexandroff space, sum cordial graphs, subbasis.

1. Introduction

Linear graphs are used in mathematics field to model many real life phenomena. A linear graph (finite graph) G is Hausdroff space (T_2 -Space) that is written as the uniform of finitely many edges (arcs), each pair of which intersect in at most a common end vertex (point). Any graph G is determined completely (up to homeomorphism) by listing its vertices (points) and specifying which pairs of vertices (points) have an edge (arc) joining them; however we shall look at them as interesting spaces that in some sense are generalizations of simple curves.

The concept of the topological space grew out of the study of the real line and metric space. The definition of a topological space that's currently standard was a long time on being developed. Several mathematicians like Frechet, Hausdroff, et al proposed many alternative definitions over a amount of years throughout the primary decades of the 20th century. Properly speaking, a topological space

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is a set X along with a collection of subsets of X , known as open sets, such $\emptyset$ set and X are each open and such, arbitrary unions and finite intersections of open sets are open. Exploiting this terminology, one can say that a topological space as being one thing sort of a truckload full of gravel, the pebbles and every one unions of collections of pebbles being open sets. If now, we have a tendency to smash the pebbles into smaller ones the collection of open sets has been enlarged and also the topology, just like the gravel is said to have been created finer by the mathematical operation. We were able to specify the topology by describing the whole collection τ of open sets. If the topology generated by the basis then the topology is called standard topology on the real line. The topology generated by the subbasis S is defined to be the collection τ of all unions of finite intersections of elements of S . The subbasis were defined as, If X is a set, a subbasis S for a topology on X is a collection of subsets of X whose union equals X .

Topological space is called by Alexandroff space if every point has a minimal neighborhood or equivalently, has unique minimal base. This is also equivalent to the fact that the intrersection of every family of open set is open. Alexandroff space was first introduced by P. Alexandroff in 1937 (citer1).

Shiama was introduced the concepts of sum cordial labeling of graph [9]. If the vertices or edges or both of the graph $G = (V(G), E(G))$ are assigned values subject to certain conditions it is known as graph labeling ([3]).

Blood circulation is the very important for a healthy body, because blood is a living fluid. It transports, oxygen and other essential substances throughout the body, fights sickness, and performs other vital functions.

The kidneys are two bean shaped organs in the renal systems. They will help the body pass waste as urine and filter blood before sending it back to the human heart. They perform vital role like maintaining overall fluid balance, regulating and filtering minerals from food, filtering waste materials from food, medications and toxic substances and creating hormones that will help to produce red blood cells, promote bone health and regulate blood pressure. In this paper we analyzed, interior of the path and closure of the path are ever used in human kidney.

2. Preliminaries

Definition 2.1 ([1]). *Let X be a topological space, then X is an Alexandroff space if arbitrary intersection of open sets are open.*

Definition 2.2 ([3]). *A mapping $f : V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f .*

The induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0), e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^* .

Definition 2.3 ([3]). A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits labeling.

Definition 2.4 ([3]). A binary vertex labeling of a graph G with induce edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(uv) = |f(u) + f(v)| \pmod{2}$ is called sum cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called sum cordial if it admits sum cordial labeling.

Definition 2.5 ([9]). p_n is path of length $n - 1$.

Definition 2.6 ([10]). The path p is called Topological open subgraph if the subgraph not contained its end point. The path p is called Topological closed subgraph if the subgraph contained its initial and its end points.

Definition 2.7 ([10]). The path p is called Topological open path if the path not contained its end point. The path p is called Topological closed path if the path contained its initial and its end points.

Definition 2.8 ([9]). Let $G = (V(G), E(G))$ be a simple graph with sum cordial labeling and with out isolated vertex. Define S_{0G} and S_{1G} as follows. $S_{0G} = \{A_{v(0)} | v \in V\}$ and $S_{1G} = \{A_{v(1)} | v \in V\}$ such that $A_{v(0)}$ and $A_{v(1)}$ is the set of all vertices adjacent to v of G having label 0 and 1, respectively. Since G has no isolated vertex, $S_{0G} \cup S_{1G}$ forms a subbasis for a topology τ_{CG} on V is called Cordial graphic topology of G and it is denoted by (V, τ_{CG}) .

Definition 2.9 ([9]). Cordial graphic topology of G is called Cordial Alexandroff topological space if and only if arbitrary intersection of members of $S_{0G} \cup S_{1G}$ is open in τ_{CG} .

Proposition 2.10 ([9]). Suppose that $G = (V, E)$ is a sum cordial graph then the graph G admits the cordial alexandroff topological space.

3. New approach of closure and interior on cordial graphic topological space

Definition 3.1. Let $G = (V, E)$ be a sum cordial graph which admits cordial graphic topology τ_{CG} and p be the path of graph G , then the interior of $V(p)$ has the form $Int[V(p)] = \{v \in V(p) | A_v \cap V(p) \neq \emptyset\}$ where A_v is the set of all vertices adjacent to v having the label 0 or 1.

Definition 3.2. Let $G = (V, E)$ be a sum cordial graph which admits cordial graphic topology τ_{CG} and p be the path of graph G , then the closure of $V(p)$ has the form $Cl[V(p)] = V(p) \cup \{v \in V | A_v \cap V(p) \neq \emptyset\}$ where A_v is the set of all vertices adjacent to v having the label 0 or 1.

Example 3.3. Let us consider the sum cordial graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$. From the following Figure 1, we have $A_{v_1}(0) = \{v_2, v_3\}$,

$A_{v_2}(1) = \{v_1, v_3, v_4\}$, $A_{v_3}(0) = \{v_1, v_2\}$, $A_{v_4}(1) = \{v_2\}$, $S_{0G} = \{\{v_2, v_3\}, \{v_1, v_2\}\}$ and $S_{1G} = \{\{v_1, v_3, v_4\}, \{v_2\}\}$. Thus, $S_{0G} \cup S_{1G} = \{\{v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3, v_4\}, \{v_2\}\}$, $\tau_{CG} = \{V, \emptyset, \{v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3, v_4\}, \{v_2\}, \{v_2\}, \{v_1, v_2, v_3\}, \{v_3\}, \{v_1, v_3\}, \{v_1\}\}$ the graph admits cordial graphic topology.

Let us take path $p = v_1 e_1 v_2 e_3 v_3$ so $V(p) = \{v_1, v_2, v_3\}$. Thus, we have $Cl(V(p)) = \{v_1, v_2, v_3, v_4\}$ and $Int(V(p)) = \{v_1, v_2, v_3\}$

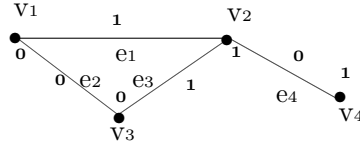


Figure 1:

Proposition 3.4. Let $G = (V, E)$ be a sum cordial graph which admits cordial graphic topology. If p_1 and p_2 are two paths of G then:

- (i) $V(p_1) \subseteq Cl(V(p_1))$;
- (ii) If $p_1 \subseteq p_2$ then $Cl(V(p_1)) \subseteq Cl(V(p_2))$.

Proof. (i) Let us assume that the vertex $v \in p_1$ having the label 0 or 1, so we have $v \in V(p_1)$, then from the definition of closure $Cl(V(p_1)) = V(p_1) \cup \{v \in V | A_v \cap V(p_1) \neq \emptyset\}$ so $v \in Cl(V(p_1))$, therefore $V(p_1) \subseteq Cl(V(p_1))$.

(ii) from (i) $V(p_1) \subseteq Cl(V(p_1))$ so $V(p_2) \subseteq Cl(V(p_2))$ and since $p_1 \subseteq p_2$ which implies that $V(p_1) \subseteq V(p_2)$ then $Cl(V(p_1)) \subseteq Cl(V(p_2))$. $\square$

Example 3.5. Let G be a simple graph with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. From the following Figure 2, we have

Here $v_f(0) = 3, v_f(1) = 2, e_f(0) = 3, e_f(1) = 3$. Therefore, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, the graph admits sum cordial labeling.

Now, $A_{v_1}(0) = \{v_2, v_5\}$, $A_{v_2}(0) = \{v_1, v_3, v_4\}$, $A_{v_3}(0) = \{v_2, v_4\}$, $A_{v_4}(1) = \{v_2, v_3, v_5\}$, $A_{v_5}(1) = \{v_1, v_4\}$, $S_{0G} = \{\{v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_2, v_4\}\}$ and $S_{1G} = \{\{v_2, v_3, v_5\}, \{v_1, v_4\}\}$.

Thus, $S_{0G} \cup S_{1G} = \{\{v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3, v_5\}, \{v_1, v_4\}\}$ and $\tau_{CG} = \{V, \emptyset, \{v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3, v_5\}, \{v_1, v_4\}, \{v_2, v_5, v_4\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_4\}, \{v_4\}, \{v_2\}, \{v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_3, v_4\}\}$. Here the graph admits cordial graphic topology.

Consider, $\tau_{CG}^c = \{\emptyset, V, \{v_1, v_3, v_4\}, \{v_2, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_4\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_3\}, \{v_5\}, \{v_1\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_5\}, \{v_3\}, \{v_1, v_5\}, \{v_3, v_5\}\}$. Let us consider the paths $p_1 = v_1 e_5 v_5$ and $p_2 = v_1 e_1 v_2 e_6 v_4 e_4 v_5$, so $V(p_1) = \{v_1, v_5\}$, $V(p_2) = \{v_1, v_2, v_4, v_5\}$. Thus, $Cl[V(p_1)] = \{v_1, v_2, v_4, v_5\}$ and $Cl[V(p_2)] = \{v_1, v_2, v_3, v_4, v_5\}$. Here, we proved (i) $V(p_1) \subseteq Cl(V(p_1))$ and (ii) If $p_1 \subseteq p_2$ then $Cl(V(p_1)) \subseteq Cl(V(p_2))$.

Proposition 3.6. Let $G = (V, E)$ be a sum cordial graph which admits cordial graphic topology. If p_1 and p_2 are two paths of G then $Cl[V(p_1) \cup V(p_2)] = Cl[V(p_1)] \cup Cl[V(p_2)]$.

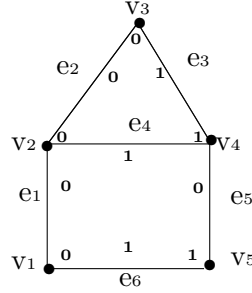


Figure 2:

Proof. Let us assume that $p_1 \cup p_2 = p_3$, where p_1 and p_2 are the open paths then $v \in p_3$, $v \in V$ having the label 0 or 1. Thus we have $V(p_3) = V(p_1) \cup V(p_2)$, then $v \in V(p_3)$, from above proposition we have, $v \in Cl[V(p_3)]$ which implies that $v \in Cl[V(p_1) \cup V(p_2)]$.

If $v \in Cl[V(p_1)] \cup Cl[V(p_2)]$, then $v \in Cl[V(p_1)]$ or $v \in Cl[V(p_2)]$ by using definition of closure we have $v \in V(p_1)$ or $v \in V(p_2)$. So $v \in V(p_1) \cup V(p_2)$ again using closure definition we have $v \in Cl[V(p_1) \cup V(p_2)]$. On the other hand, let us assume that $v \in V(p_3)$ which implies that $v \in V(p_1) \cup V(p_2)$ then $V(p_1) \cup V(p_2) = V[p_1 \cup p_2] = V(p_3)$. Now let $v \in V(p_3) \Rightarrow V(p_1 \cup p_2)$ then $v \in V(p_1)$ or $v \in V(p_2)$. If $v \in V(p_1)$ then from above proposition 3.4 we have $v \in Cl[V(p_1)]$ or if $v \in V(p_2)$ then from above proposition, we have $v \in Cl[V(p_2)]$, thus we have $v \in Cl[V(p_1)] \cup Cl[V(p_2)]$. $\square$

Example 3.7. Let us consider the sum cordial graph with $V = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4\}$.

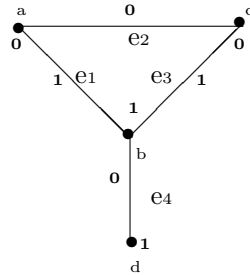


Figure 3:

From Figure 3, we have $\tau_{CG} = \{V, \emptyset, \{b, c\}, \{a, c, d\}, \{a, b\}, \{b\}, \{a, b, c\}, \{c\}, \{a, c\}, \{a\}\}$. Let us consider the paths, $p_1 = ae_1be_3$, $p_2 = ae_2ce_3$, so $V(p_1) =$

$\{a, b\}$ and $V(p_2) = \{a, c\}$. Here $Cl[V(p_1)] = \{a, b, c\}$, $Cl[V(p_2)] = \{a, b, c\}$ and $Cl[V(p_3)] = \{a, b, c\}$ (i.e) $Cl[V(p_1 \cup p_2)] = \{a, b, c\}$. Thus, we get $Cl[V(p_1) \cup V(p_2)] = Cl[V(p_1)] \cup Cl[V(p_2)]$.

Proposition 3.8. *Let $G = (V, E)$ be a sum cordial graph which admits cordial graphic topology. If p_1 and p_2 are two paths of G then $Cl[V(p_1) \cap V(p_2)] \subseteq Cl[V(p_1)] \cap Cl[V(p_2)]$.*

Proof. Let us assume that $p_1 \cap p_2 = p_3$, where p_1 and p_2 are the open paths then $v \in p_3$, $v \in V$ having the label 0 or 1.

Let us take $v \in Cl[V(p_1) \cap V(p_2)]$, then by using proposition 3.4 we have $v \in V(p_1) \cap V(p_2)$. This implies that $v \in V(p_1)$ and $v \in V(p_2)$, again using above proposition 3.4 we get $v \in Cl[V(p_1)]$ and $v \in Cl[V(p_2)]$, using this which implies that $v \in Cl[V(p_1)] \cap Cl[V(p_2)]$. $\square$

Example 3.9. Let us consider the following paths from Figure 2 $p_1 = v_1 e_1 v_2 e_2 v_3 e_4$ and $p_2 = v_2 e_4 v_4 e_5$ so, $V(p_1) = \{v_1, v_2, v_3\}$ and $V(p_2) = \{v_2, v_4\}$.

Here, $Cl[V(p_1)] = \{v_1, v_2, v_3, v_4, v_5\}$, $Cl[V(p_2)] = \{v_1, v_2, v_3, v_4, v_5\}$, $Cl[V(p_1)] \cap Cl[V(p_2)] = \{v_1, v_2, v_3, v_4, v_5\}$. Now, consider $V(p_3) = V(p_1) \cap V(p_2) = \{v_2\} \Rightarrow Cl[V(p_3)] = \{v_1, v_2, v_3, v_4\}$. That is $Cl[V(p_1) \cap V(p_2)] = \{v_1, v_2, v_3, v_4\}$. Thus, we have $Cl[V(p_1) \cap V(p_2)] \subset Cl[V(p_1)] \cap Cl[V(p_2)]$.

Example 3.10. Let us consider the following paths from Figure 3 $p_1 = a e_1 b e_3$ and $p_2 = a e_2 v_4 c e_3$ so, $V(p_1) = \{a, b\}$ and $V(p_2) = \{a, c\}$.

Here $Cl[V(p_1)] = \{a, b, c, d\}$, $Cl[V(p_2)] = \{a, b, c\}$, $Cl[V(p_1)] \cap Cl[V(p_2)] = \{a, b, c\}$. Now, consider $V(p_3) = V(p_1) \cap V(p_2) = \{a\} \Rightarrow Cl[V(p_3)] = \{a, b, c\}$. That is $Cl[V(p_1) \cap V(p_2)] = \{a, b, c\}$. Thus, we have $Cl[V(p_1) \cap V(p_2)] = Cl[V(p_1)] \cap Cl[V(p_2)]$.

4. Relation between cordial graphic topology and Kidney

In this section, we shall study the interior and closure of topological graph theory which plays a vital role in human kidney.

Firstly, we divide the human heart and kidney into a set of vertices and set of edges in order to find cordial graphic topology on it.

The vertex x_1 represent Pulmonary veins.

The vertex x_2 represent Left atrium.

The vertex x_3 represent Left ventricle.

The vertex x_4 represent aorta.

The vertex x_5 represent abdominal aorta.

The vertex x_6 represent left kidney.

The vertex x_7 represent right kidney.

The vertex x_8 represent inferior vena cave.

The vertex x_9 represent superior vena cava.

The vertex x_{10} represent Pulmonary artery.

The vertex x_{11} represent Left lung.

The vertex x_{12} represent Right lung.

The edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$ represent the blood circulate way through the human heart to kidney.

The following Figure 5 represents the blood circulation in human heart to kidney via graphs.

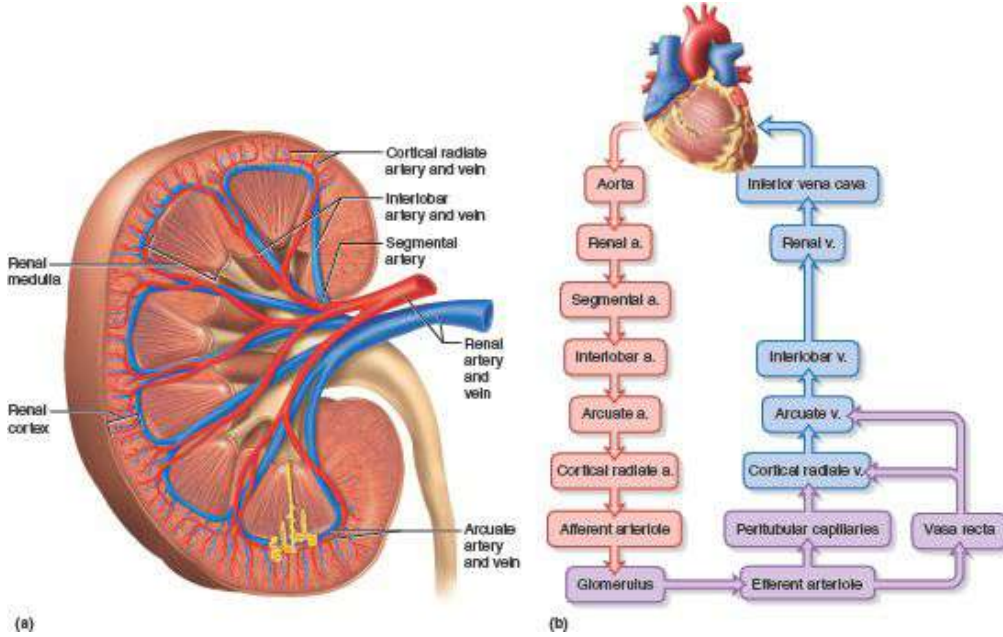


Figure 4: Blood circulation in human heart and Kidney
[colour figure can be viewed at <https://www.pinterest.com/pin/305470787201652538/>]

From the following Figure 6 we have, $v_f(0) = 6; v_f(1) = 6$ and $e_f(0) = 6; e_f(1) = 6$. Thus, we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, the graph admits sum cordial labeling. From graph we have $A_{x_1}(1) = \{x_2\}$, $A_{x_2}(1) = \{x_3\}$, $A_{x_3}(0) = \{x_4\}$, $A_{x_4}(0) = \{x_5\}$, $A_{x_5}(0) = \{x_6, x_7\}$, $A_{x_6}(0) = \{x_8\}$, $A_{x_7}(1) = \{x_8\}$, $A_{x_8}(1) = \{x_9\}$, $A_{x_9}(0) = \{x_{10}\}$, $A_{x_{10}}(1) = \{x_{11}, x_{12}\}$.

$S_{0G} = \{A_{x_3}(0), A_{x_4}(0), A_{x_5}(0), A_{x_6}(0), A_{x_9}(0)\}$ and $S_{1G} = \{A_{x_1}(1), A_{x_2}(1), A_{x_7}(1), A_{x_8}(1), A_{x_{10}}(1)\}$, $S_{0G} \cup S_{1G} = \{\{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6, x_7\}, \{x_8\}, \{x_9\}, \{x_{10}\}, \{x_{11}, x_{12}\}\}$ and $\tau_{CG} = \{V, \emptyset, \{x_2\}, \{x_3\}, \{x_4\}, \{x_4\}, \{x_6, x_7\}, \{x_8\}, \{x_9\}, \{x_{10}\}, \{x_{11}, x_{12}\}\{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}\{x_2, x_6, x_7\}, \{x_2, x_8\}, \{x_2, x_9\}, \{x_2, x_{10}\}, \{x_2, x_{11}, x_{12}\}, \{x_3, x_4\}, \{x_3, x_6, x_7\}, \{x_3, x_5\}, \{x_3, x_8\}, \{x_3, x_9\}, \{x_3, x_{11}, x_{12}\}, \{x_4, x_5\}, \{x_4, x_8\}, \{x_4, x_6, x_7\}, \{x_4, x_6\}, \{x_4, x_{10}\}, \{x_4, x_{11}, x_{12}\}, \{x_5, x_6, x_7\}, \{x_5, x_8\}, \{x_5, x_9\}, \{x_5, x_{10}\}, \{x_5, x_{11}, x_{12}\}, \{x_6, x_7, x_8\}, \{x_6, x_9\}, \{x_6, x_{10}\}, \{x_6, x_{11}, x_{12}\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_8, x_{11}, x_{12}\}, \{x_9, x_{10}\}, \{x_9, x_{11}, x_{12}\}, \{x_{10}, x_{11}, x_{12}\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_3, x_6, x_7\}, \{x_2, x_3, x_8\}, \{x_2, x_3, x_9\},$

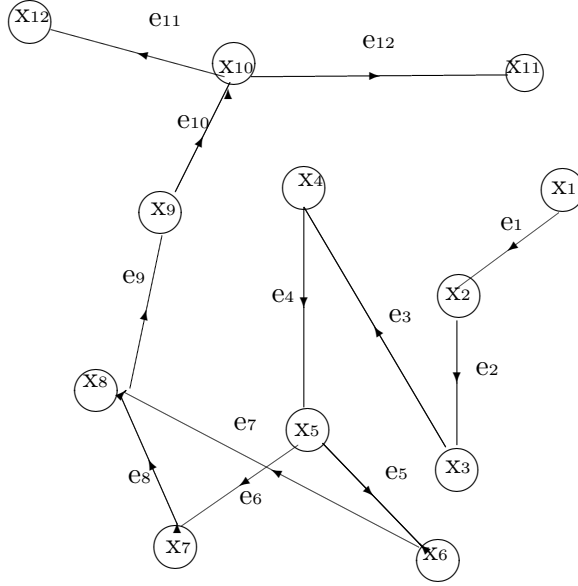


Figure 5:

$\{x_2, x_3, x_{10}\}, \{x_2, x_3, x_{11}, x_{12}\}, \{x_4, x_5, x_6, x_7\}, \{x_4, x_5, x_8\}, \{x_4, x_5, x_9\},$
 $\{x_4, x_5, x_{10}\}, \{x_4, x_5, x_{11}, x_{12}\}, \{x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9\}, \{x_6, x_7, x_8, x_{10}\},$
 $\{x_6, x_7, x_8, x_{11}, x_{12}\}, \{x_2, x_6, x_7, x_8\}, \{x_3, x_6, x_7, x_8\}, \{x_4, x_6, x_7, x_8\},$
 $\{x_5, x_6, x_7, x_8\}, \{x_9, x_{10}, x_{11}, x_{12}\}, \{x_2, x_9, x_{10}\}, \{x_3, x_9, x_{10}\}, \{x_4, x_9, x_{10}\},$
 $\{x_5, x_9, x_{10}\}, \{x_6, x_7, x_9, x_{10}\}, \{x_8, x_9, x_{10}\}, \{x_{10}, x_{11}, x_6, x_{12}, x_2\},$
 $\{x_3, x_{10}, x_{11}, x_{12}\}, \{x_4, x_{10}, x_{11}, x_{12}\}, \{x_5, x_{10}, x_{11}, x_{12}\}, \{x_6, x_7, x_{10}, x_{11}, x_{12}\},$
 $\{x_8, x_{10}, x_{11}, x_{12}\}, \{x_2, x_3, x_4, x_6, x_7\}, \{x_2, x_3, x_4, x_8\}, \{x_2, x_3, x_4, x_9\},$
 $\{x_2, x_3, x_4, x_{10}\}, \{x_2, x_3, x_4, x_{11}, x_{12}\}, \{x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_2, x_3, x_4, x_5, x_8\},$
 $\{x_2, x_3, x_4, x_5, x_9\}, \{x_2, x_3, x_4, x_5, x_{10}\}, \{x_2, x_3, x_4, x_5, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}, \{x_2, x_3, x_4, x_5, x_6, x_7, x_{10}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{11}, x_{12}\}, \{x_2, x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\},$
 $\{x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}, \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_5, x_8, x_9, x_{10}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_6, x_7, x_{10}, x_{11}, x_{12}\},$
 $\{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \dots\}.$

Let us consider the path without it's end point $p = x_1e_1x_2e_2e_3$ so we have $V(p) = \{x_1, x_2\}$, by using the resultant closure of the path is $Cl[V(p)] = \{x_1, x_2\}$. Medically, once we apply this instance within human body, we can realize it's true. As a result of the blood flow in an exceedingly heart in an

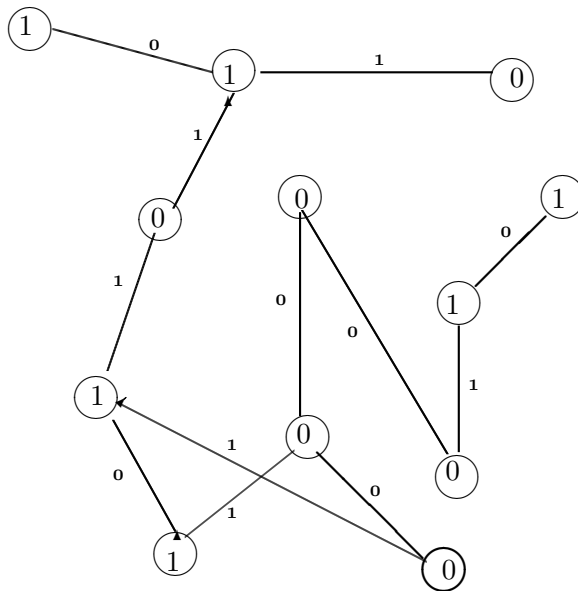


Figure 6: Sum cordial labeling on blood circulation in human heart to kidney

exceedingly directed path that means that the blood should be pass through every successive point till it completes its cycle.

Let us consider the path $p = x_4e_4x_5e_5v_6$ so, we have $V(p) = \{x_4, x_5, x_6\}$, by using the interior definition we get the interior of the path is $Int[V(p)] = \{x_4, x_5\}$, here the interior of the path does not consist the end point. But we can apply this resultant interior of the path in to the kidney.

Conclusion

Abdominal Aorta x_5 supplies blood supply to Left and Right Kidney via Left renal artery x_6 and Right renal artery x_7 .

Problem occurring in x_6 Left renal artery affects Left Kidney alone and Problem occurring in x_7 Right renal artery affects Right Kidney alone.

Human body maintains excretory/function with one Kidney alone, even if there is problem in other Kidney. It is god's gift for us that we can live with one Kidney alone.

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Although, any errors are our own and should not tarnish the reputations of these esteemed person.

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Stochastic optimal control model practices in development, finance and the industrial production

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Abstract. The systems mainly study in this work are dynamic to know, It have unfold all the time, besides are ask by the EDS of ITO and sometimes called models of diffusions. The basic source is the white noise, and since the systems are dynamic, the proposed decision (controls) are based on available information from controllers, it must also change over time.

Keywords: deterministic c.o.s, strong and weak formulations, dynamic programming, singular controls.

1. Introduction

The controllers choose an optimal decision from among all that are possible to achieve the best result.

Any optimization problem is called the optimal stochastic control problem, and the domain of these problems covers a variety of systems (physics, biology, economics, and management ...), just to mention a few.

In our work we will establish rigorous mathematical frameworks for the problems of the S.O.C.

Our work is organized as follows:

- firstly the formulation of deterministic S.O.C problems, and the reason for doing so is not only that the deterministic situation itself contains important results but also so that readers can see the essential deference between the deterministic case and the case of stochastic problem.
- second we will give several practical examples arise in development, finance, and industrial production, it can formulate as a model of optimal stochastic control given.
- third we present the strong and weak formulation of problems of (S.O.C) it mainly to treat during this project.

Despite not being studied in this work, the two approaches, namely the Pontryagin maximum principle, and the dynamic programming of Bellman are also main for these problems.

2. Formulation of deterministic S.O.C problem

starting with the production of the problem plan; a machine produce only one type of product, the raw materials produced by the machine, and the final products are stored.

Suppose that in time t the rate of production is $u(t)$ and the inventory level is $x(t)$, if the rate request for this product is a known function $Z(t)$ and the inventory is x_0 at the time $t = 0$, therefore the relationship between the quantities is given by:

$$(2.1) \quad \begin{cases} \dot{x}(t) = u(t) - Z(t) \\ x(0) = x_0 \end{cases}.$$

Here $x(t)$ can take positive or negative values, the product is surplus when $x(t) > 0$ and descends in the opposite direction suppose the cost of finding inventory x and the production rate u by unit time is $h(x, u)$.

A typical example of h is:

$$(2.2) \quad h(x, u) = c^+ x^+ + c^- x^- + pu$$

Or $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, $c^+, c^- \geq 0$ are:

- the marginal cost / penalty for the excess / goes down respectively and p is the unit cost of production
- the care of the chosen production $u(\cdot)$ to minimize the total discount cost on the horizon plan $[0, T]$.

The following functional will minimize it:

$$(2.3) \quad J(u(\cdot)) = \int_0^T e^{-\lambda t} h(x(t), u(t)) dt.$$

Or λ is the discount rate, note here that $u(\cdot)$ is a function on $[0, T]$, which is called production plan if the machine has a maximum production rate k (called production capacity), therefore for any production plan we have:

$$(2.4) \quad 0 \leq u(t) \leq k, \forall t \in [0, T].$$

If the storage size $b > 0$, then the inventory level $x(t)$ it must satisfy the constraint:

$$(2.5) \quad x(t) \leq b$$

Any production plan that verified (2.4), (2.5), (2.1) is called an eligible plan.

The problem is to minimize the cost (2.3) on all the admissible plans.

The above problem is perhaps one of the simplest examples for a (S.O.C) problem, it is deterministic because there is no doubt in the dynamic system (2.1), and (2.4), (2.5).

Now we will present the general formulation of the problem of deterministic optimal control.

Let: $T, 0 < T \leq +\infty, x_0 \in \mathbb{R}^n$, and a metric space Γ .

The dynamics of the system under consideration are given by the following **O.D.E**:

$$(2.6) \quad \begin{cases} \dot{x}(t) = b(t, x(t), u(t)) \\ x(0) = x_0 \end{cases} \quad t \in [0, T]$$

with $b : [0, T] \times \mathbb{R}^n \times \Gamma \rightarrow \mathbb{R}^n$ a given application.

When $T = +\infty$, $[0, T]$ is replaced by $[0, \infty)$.

A measurable application $u(.) : [0, T] \rightarrow \Gamma$ is called control and x_0 it is called the initial state and x . The solution of (2.6) is called the state trajectory corresponds to $u(.)$.

In the example of production plan the dynamic is: (2.1), the control is the rate of production, and the state is the levels of inventory.

We assume that for all $x_0 \in \mathbb{R}^n$ and all control $u(.)$ There is a unique solution: $x(.) \equiv x(., u(.))$ of (2.6).

On such case one obtains an input-output relation with input $u(.)$ and output $x(.)$.

Also call (2.6) a control system.

A special case of (2.6) is the linear case where the system to control is in the form:

$$(2.7) \quad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0 \end{cases} \quad t \in [0, T]$$

with $A : [0, T] \rightarrow \mathbb{R}^{n \times n}$ et $B : [0, T] \rightarrow \mathbb{R}^{n \times k}$ are measurable application, sometimes we will note (2.7) by linear system with varied time.

Since $A(.)$ and $B(.)$ Depend, moreover the following system is called linear system of time invariant:

$$(2.8) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases} \quad t \in [0, T]$$

with: $A \in \mathbb{R}^{n \times n}$ et $B \in \mathbb{R}^{n \times k}$.

As we saw in the example of production plan, the constrained state is given by:

$$(2.9) \quad x(t) \in S(t), \forall t \in [0, T].$$

And the control constraint is given by:

$$(2.10) \quad u(t) \in U(t), t \in [0, T],$$

where $S(t) : [0, T] \rightarrow 2^{\mathbb{R}^n}$ and $U(t) : [0, T] \rightarrow 2^{\Gamma}$ are multifunctional (i.e: for all $t \in [0, T]$, $S(t) \subseteq \mathbb{R}^n$ and $U(t) \subseteq \Gamma$).

We will only deal with cases where the control $u(t)$ is time invariant ; $u(t) \equiv u$.

We recall that U itself can look like a metric space, so from now we replace Γ by U .

Let : $V[0, T] = \{u : [0, T] \rightarrow U : u(\cdot) \text{ is measurable} \}$.

Any $u(\cdot) \in V[0, T]$ is called a feasible control, by following either the functional cost to measure the execution of the control:

$$(2.11) \quad J(u(\cdot)) = \int_0^T f(t, x(t), u(t)) dt + h(x(T)).$$

For an application: $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$.

The first and second right-hand term (2.11) are called the initial cost and the terminal cost, respectively.

Definition 2.1. A control $u(\cdot)$ is called admissible and $(x(\cdot), u(\cdot))$ an admissible pair if:

- 1) $u(\cdot) \in V[0, T]$
- 2) $x(\cdot)$ is the only solution to equation (2.6) under $u(\cdot)$.
- 3) the state (2.9) is satisfied.
- 4) $t \rightarrow f(t, x(t), u(t)) \in L^1[0, T]$.

The set of all admissible controls is denoted by $U_{ad}[0, T]$.

Our problem of the c-o-determinist is as follows:

2.0.1 Problème (D.O.C)

Minimize (2.11) on $U_{ad}[0, T]$.

Our problem is finite if (2.11) admits a lower bound and it is unique if there exists $\bar{u}(\cdot) \in U_{ad}[0, T]$ tell that:

$$(2.12) \quad J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}[0, T]} J(u(\cdot)).$$

Any $\bar{u}(\cdot) \in U_{ad}[0, T]$ verified (2.12) is called optimal control, which corresponds to the state trajectory $\bar{x}(u) \equiv x(\cdot; \bar{u}(\cdot))$ et $(\bar{x}(\cdot), \bar{u}(\cdot))$ re called the optimal state path and pair, respectively. An important and special case is such that the control system given by (2.7).

(2.9) is absent "there is no constrained state ($S(t) \equiv \mathbb{R}^n$)"

The constraint of the control u is $\mathbb{R}^k$, and the functional cost is of the form:

$$(2.13) \quad J(u(\cdot)) = \frac{1}{2} \int [\langle Q(t)x(t), x(t) \rangle + \langle R(t)u(t), u(t) \rangle] dt + \frac{1}{2} \langle G x(T), x(T) \rangle.$$

For some suitable symmetric matrix G and matrix values $Q(\cdot)$ and $R(\cdot)$ The problem (D.O.C) is called problem of the quadratic linear optimal control (LQ problem).

And finally, we will notice in the example of the production plan the terminal cost h equals 0.

In general, a problem of optimal control with:

- 1) $h = 0$ is called the Lagrange problem.
- 2) $f = 0$ is a Mayer's problem.
- 3) si: $f \neq 0, h \neq 0$ is a Bolza problem.

It is very clear that these three problems are mathematically equivalent.

2.1 Optimal stochastic control model given

This section is an overview of several practical problems.

2.2 Production planning

Consider the problem of optimal production planning, and suppose a filtered probability space: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ check the usual conditions on which a standard Brownian Motion $W(t)$ is defined.

The requested process $z(t)$ is not deterministic, rather it is given by the following:

$$(3.1) \quad z(t) = z_0 + \int_0^t \xi(s) ds + \int_0^t \sigma(s) dW(s), \quad t \in [0, T].$$

$\xi(t)$: represents the rate requested at time t , and the term $\int_0^t \sigma(s) dW(s)$ represents the fluctuation.

In addition, it covers these processes to avoid damage to inventories, which they have entirely variable, we assume that $\xi(t)$ and $\sigma(t)$ are $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted. The integrals in (3.1) are well defined. To meet demand the factory is to adjust the rate of production all the time to accommodate the change in the situation.

Suppose the machine is trustworthy (seriously: it never arrived).

And as in the deterministic case; at time t the production rate is $u(t)$ and the inventory level is $x(t)$.

If at time $t = 0$ the inventory is x_0 then the system is modeled as follows:

$$(3.2) \quad \begin{cases} dx(t) = (u(t) - z(t)) dt, & x(0) = x_0 \\ dz(t) = \xi(t) dt + \sigma(t) dW(t), & z(0) = z_0 \end{cases}.$$

Another time, control or the rate of production $u(t)$ is an object of production capacity

$$(3.3) \quad 0 \leq u(t) \leq k, \quad t \in [0, T] \quad \text{A.s.}$$

There is another implicit constraint on the rate of production $u(t)$ due to the stochastic environment of the problem, namely for all time, production

management to make a decision based only on past information, rather any information in the future, moreover the control or the decision; it is wrong that must be non anticipatory.

The precise mathematical translation for this constraint will study it.

Here to avoid infinite accumulation, the maximum production capacity is completely large enough to meet the demand.

Then the following minimum condition must be imposed for the problem to have a meaning:

$$(3.4) \quad E \int_0^T z(t) dt \equiv z_0 T + E \int_0^T \int_0^t \xi(s) ds dt < kT.$$

On the other side the inventory status is not necessarily exceeded the size

$$(3.5) \quad b \cdot X(t) \leq b, \quad \forall t \in [0, T], \quad \mathbf{A.s.}$$

The total expected cost is as follows:

$$(3.6) \quad J(u(\cdot)) = E \left\{ \int_0^T e^{-\gamma t} f(x(t), u(t)) dt + e^{-\gamma T} h(x(T)) \right\}.$$

Where the first term represents the total cost of inventory and production, and the second term is the inventory penalty at the end of production (example: disposition of the cost), and finally γ is the discount rate.

The objective of production management is to choose a suitable production plan $u(\cdot)$ tell that: (3.2), (3.3) et (3.5) are satisfied and $J(u(\cdot))$ is minimize.

2.3 Investment and consumption

Suppose that there is a market in which $(n+1)$ possession are in trade, one of the possession is called engagement which in the price process $P_0(t)$, and we have the following **O.D.E** (déterminitise):

$$(3.7) \quad \begin{cases} dP_0(t) = r(t)P_0(t) dt, & t \in [0, T] \\ P_0(0) = P_0 > 0 \end{cases}.$$

With $r(t) > 0$ is called the interest rate (of the commitment), it is clear that $P_0(t)$ increase steadily over time and the commitment is more of that called a possession with risk.

The other n holdings are called stocks, with a price process: $P_1(t), \dots, P_n(t)$, satisfies the **SDE**:

$$(3.8) \quad \begin{cases} dP_i(t) = P_i(t) \{b_i(t) dt + \langle \sigma_i(t), dW(t) \rangle\} \\ P_i(0) = P_i > 0 \end{cases}, \quad t \in [0, T].$$

With $b_i : [0, T] \times \Omega \rightarrow \mathbb{R}$ and $b_i(t) > 0$ (**p.s**) is called the rate of appreciation, and $\sigma_i : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ is the volatile (inconstant) or else the stock dispersion, all these processes are assumed so that they are $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted.

Here $W(t)$ is a standard Brownian motion defined on some "fixed" complete filtered probability spaces ; $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

The diffusion term in (3.8) return the stock price fluctuation, the latter is called risk possession and we have:

$$(3.9) \quad Eb_i(t) > r(t) > 0, \forall t \in [0, T], \quad 1 \leq i \leq n.$$

It is a very natural assumption, whereas in the opposite case none should invest in risky stocks, however, the following discussion is not dependent on these assumption.

Now we consider an investor with total opulence at time $t \geq 0$ is denoted by $x(t)$.

Suppose that it decides to contain $N_i(t)$ portions of the i -th possession ($i = 0, \dots, n$) at time t , then:

$$(3.10) \quad x(t) = \sum_{i=0}^n N_i(t) P_i(t), \quad t \geq 0.$$

Let $c(t)$ the consumption rate, therefore:

$$(3.11) \quad \begin{aligned} x(t + \Delta t) - x(t) &= \sum_{i=0}^n N_i(t) [P_i(t + \Delta t) - P_i(t)] \\ &+ \sum_{i=0}^n \mu_i(t) N_i(t) P_i(t) \Delta t - c(t) \Delta t. \end{aligned}$$

When $\Delta t \rightarrow 0$ we obtain::

$$(3.12) \quad \begin{cases} dx(t) = \sum_{i=0}^n N_i(t) dP_i(t) + \sum_{i=0}^n \mu_i(t) N_i(t) P_i(t) dt - c(t) dt \\ = \left\{ r(t) N_0(t) P_0(t) dt + \sum_{i=0}^n [b_i(t) + \mu_i(t)] N_i(t) P_i(t) dt - c(t) dt \right\} dt \\ + \left\langle \sum_{i=0}^n \sigma_i(t) u_i(t), dW(t) \right\rangle \end{cases}$$

or

$$(3.13) \quad u_i(t) \triangleq N_i(t) P_i(t), \quad i = 0, 1, \dots, n$$

Note that the value of the investor opulence market is in the i -ème commitment / stock.

When $u_i(t) < 0, (i = 0, 1, \dots, n)$ the investor is completely selling the i -th stock.

When $u_0(t) < 0$, the investor is borrowing the quantity $|u_0(t)|$ with rate $r(t)$, it is clear that by changing $u_i(t)$, the investor changes the allocation of his wealth in the $(n+1)$ possession.

Note $u(t) = (u_1(t), \dots, u_n(t))$ the investor's portfolio.

We cannot include the allocation to the commitment in the portfolio, as it will only be determined by the allocation to stocks, which gives total opulence.

As in the case of stochastic production planning any eligible portfolio is required so that must be non-anticipatory.

Now for $x(0) = x_0 > 0$, the investor will choose the investment portfolio $u(\cdot)$ and the consumption plan $c(\cdot)$ so that:

$$x(t) > 0, \forall t \in [0, T] \quad \mathbf{A.s}$$

and that the utility of delivery:

$$(3.15) \quad J(u(\cdot), c(\cdot)) = E \left\{ \int_0^T e^{-\gamma t} \varphi(c(t)) dt + e^{-\gamma T} h(x(T)) \right\}.$$

Is to maximize, with $\gamma > 0$ is the discount rate, $\varphi(c)$ the instantaneous utility for consumption c and $e^{-\gamma T} h(x(T))$ is the discount utility which is derived from what we're looking for.

We may also impose some constraints for the problem above.

For example a constraint can be bounded with a cost vene and then it has the constraints:

$$u_i(t) \geq -L_i, \forall t \in [0, T] \quad \mathbf{A.s}, \quad i = 0, 1, \dots, n.$$

For some $L_i > 0$ and if $L_i = 0$ sa implies that the short sale is very low.

2.4 The strong and weak formulation of (S.O.C) problems

The examples presented in the previous section have certain common characteristics: there is a diffusion system, which is described by the stochastic differential equation of Itô, there are many other decisions that can influence the dynamics of the system. The objective is to optimize (maximize or minimize) the criteria for selecting a non-anticipatory decision among those meeting all the constraints. These problems are called stochastic optimal control problems. We will now present two mathematical formulations (strong and weak formulations) of the optimal stochastic control problems in the two subsections, respectively.

2.5 Strong formulation

Let be the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, on which a standard Brownian motion $W(\cdot)$ is defined.

Consider the following controlled SDE:

$$(4.1) \quad \begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

With:

$$b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}.$$

U : separable metric space. and $t \in [0, \infty[$, fixed.

The function $u(\cdot)$ is called control representing the action, decision, or policy of the decision makers (the controllers).

At any time, the controller is well informed about certain information (as specified by the filtration of information $\{\mathcal{F}_t\}_{t \geq 0}$ of what happened at that time, but not able to predict what will happen next due to the uncertainty of the system (consequently, for all t the controller cannot exercise his decision $u(t)$ before time t actually).

This nonanticipative restriction in mathematical terms can be represented as " $u(\cdot)$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted" (we recall that any process $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted is progressively measurable). Know, control $u(\cdot)$ Is taken from the set:

$$\mathcal{U}[0, T] \stackrel{D}{=} \left\{ u : [0, T] \times \Omega \rightarrow U, \quad u(\cdot) \text{ est } \{\mathcal{F}_t\}_{t \geq 0} - \text{adapté} \right\}.$$

Any $u(\cdot) \in \mathcal{U}[0, T]$ is called feasible control, moreover there are some constrained states. For all $u(\cdot) \in \mathcal{U}[0, T]$ the equation (4.1) is with random coefficients.

Let $S(t) : [0, T] \rightarrow 2^{\mathbb{R}^n}$ be a multifunction.

The forced state may be granted by:

$$(4.2) \quad x(t) \in S(t), \quad \forall t \in [0, T] \quad \mathbf{A.s.}$$

Note that other types of constrained state are also possible, the following one introduces the functional cost:

$$(4.3) \quad J(u(\cdot)) = E \int_0^T f(t, x(t), u(t)) dt + h(x(T)).$$

Definition 2.2. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ a filtered probability space, $W(t)$ a $\mathcal{F}_t$ -Brownien motion

A control $u(\cdot)$ Is called an admissible control (in the strong sense) and

$(x(\cdot), u(\cdot))$ an eligible peer (in the strong sense) if:

- 1) $u(\cdot) \in \mathcal{U}[0, T]$.
- 2) $x(\cdot)$ Is the unique solution of (4.1) .
- 3) some imposed constraint states (for example (4.1)) are satisfied.
- 4) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T, \mathbb{R})$ et $h(x(T)) \in L^1_{\mathcal{F}}(\Omega, \mathbb{R})$.

The set of all admissible controls is noted by: $\mathcal{U}_{ad}^F[0, T]$.

Our problem of stochastic optimal control under the strong formulation can be indicated as follows:

Problem(SF): Minimize (4.3) on $\mathcal{U}_{ad}^F[0, T]$.

Our objective is to find: $\bar{u}(\cdot) \in \mathcal{U}_{ad}^F[0, T]$, "if there is" tell that:

$$(4.4) \quad J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}^F[0, T]} J(u(\cdot)).$$

Our problem is S-finite if the right side of (4.4) is finite, and it is said to (unique) S-solved if there is a (unique) $\bar{u}(\cdot) \in \mathcal{U}_{ad}^F[0, T]$ such as (4.4) holds.

Any $\bar{u}(\cdot) \in \mathcal{U}_{ad}^F[0, T]$ satisfactory (4.4) is called F-optimal control. The corresponding state process of $\bar{x}(\cdot)$ And the state control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ are called an F-optimal state process and F-optimal pair, respectively.

2.6 Weak formulation

We note that in the strong formulation, the underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ as well as the Brownian motion $W(t)$ are all fixed.

In certain situations, it will be practical or necessary to vary $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ as well as $W(t)$ and to consider them as parts of the control. this is the case, for example, when we apply the dynamic programming method to solve a stochastic optimal control problem initially under the strong formulation. Therefore, we need another formulation of the problem.

Definition 2.3. *The six-tuple $q = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W(\cdot), u(\cdot))$ is called an admissible control (in the weak sense) and $(x(\cdot), u(\cdot))$ an admissible peer (in the weak sense) if*

A1) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a filtered probability space satisfying the usual conditions ;

A2) $W(t)$ a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$;

A3) $u(\cdot)$ Is a process $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted on $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{U}$

A4) $x(\cdot)$ Is the only solution of equation (4.1) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ under $u(\cdot)$;

A5) certain prescribed state constraints (example (4.2)) are satisfactory ;

A6) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$. Here, the spaces $L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$ are defined on the given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ associated with the six-tuple q .

The set of all admissible controls is designated by $\mathcal{U}_{ad}^f[0, T]$. Sometimes we can write $u(\cdot) \in \mathcal{U}_{ad}^f[0, T]$ instead of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W(\cdot), u(\cdot)) \in \mathcal{U}_{ad}^f[0, T]$ if it is clear from the context that the weak formulation is under review. our problem of stochastic optimal control in the framework of the weak formulation can be indicated as follows:

Problem (Wf): Minimize (4.3) on $\mathcal{U}_{ad}^f[0, T]$.

Our objective is to find: $\bar{q} \in \mathcal{U}_{ad}^f[0, T]$, "if there is" tell that:

$$(4.5) \quad J(\bar{q}) = \inf_{q \in \mathcal{U}_{ad}^f[0, T]} J(q) .$$

As in the strong formulation our problem is finite if the right side of (4.5) is finite.

We can point out here that the strong formulation is indeed in the practical world, but the weak formulation is sometimes auxiliary, mathematically for the objective model, and generally we have problems with the strong formulation.

We emphasize here that the strong formulation is that which follows from Practical World, while the weak formulation sometimes serves as an auxiliary but also an effective mathematical model aiming at solving problems in the end with the strong formulation. One of the main reasons for which this work is perhaps that the objective of a stochastic control problem is to minimize the mathematical expectation of a random variable which depends only on the distribution of the processes. So, if the solutions of (4.1) in the different probability spaces have the same probability distribution, we have more freedom in choosing a suitable probability space to work with. the weak formulation fails if none of the coefficients given are also random (ie that they depend on ω explicitly), because in this case, the probability space must be specified and fixed a priori.

And as in the deterministic case, the following types of equations of state are of particular interest:

$$(4.6) \quad \begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum [C_j(t)x(t) + D_j(t)u(t)]dW^j(.), \\ x(0) = x_0, \end{cases}.$$

Or $A(\cdot), B(\cdot), C_j(\cdot), D_j(\cdot)$ are the appropriate size values matrix (suitable), the functional cost is quadratic:

$$(4.7) \quad J(u(\cdot)) = E \left\{ \frac{1}{2} \int_0^T [\langle Q(t)x(t), x(t) \rangle + \langle R(t)u(t), u(t) \rangle] dt + \frac{1}{2} \langle Gx(T), x(T) \rangle \right\}.$$

The problems **(SF)** or **(WF)** with the equation of state (4.6) and the functional cost (4.7) is called quadratic linear stochastic optimal control problem **(LQSOCP)**.

We note that in this formulation $T > 0$ is a constant value, therefore the problems **(SF)** or **(WF)** are also mentioned as a problem with fixed duration (fixed).

Remark 2.1. everywhere in this work when the type of formulation (strong or weak) under study is clear from the context, we will simply refer to a control as "admissible control" and the set of controls admissible by $\mathcal{U}_{ad}[0, T]$, omitting the prefixes **W** or **S**.

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Orbit-maximal green sequences and general-maximal green sequences

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Abstract. In this paper we introduce the orbit-maximal green sequences for a skew symmetrizable matrix and set the relation between orbit-maximal green sequences and the maximal green sequences of its folding matrix. We also generalize the concept of maximal green sequences, that is, the so-called general-maximal green sequence and define the red size of this generalization.

Keywords: cluster algebra, maximal green sequence, orbit-maximal green sequence, general-maximal green sequence.

1. Introduction

Cluster algebra is a very important mathematical innovation, which was introduced by S. Fomin and A. Zelevinsky in [5]. Its importance and effectiveness come from its connection with different branches of mathematics and mathematical physics such as Teichmüller theory, canonical basis, total positivity, Poisson-Lie groups, Calabi-Yau algebras and representation of finite dimensional algebras. One of the most interesting things in studying cluster algebra is that this study can be associated with combinatorial tools or geometrical description.

Maximal green sequences are a very important concept related to cluster algebras established by Keller [8] for the skew symmetric case, then and after the sign coherence property was proved for the skew-symmetrizable case [7], the definition was expanded to include the skew-symmetrizable matrices.

In this paper we approach this concept from different angles presenting new related definitions with some new results. We also give a generalization in a

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way that makes this generalized case a property owned by any given skew-symmetrizable matrix of finite size.

Firstly, we recall the definition of a group of automorphisms for a skew-symmetrizable matrix as mentioned in [4], [11]. Under the action of the group, mutations in directions of indices of the same orbit commute [4] which make it possible to introduce the so-called *orbit-mutation* in [4]. We firstly note that after mutating in direction of any orbit, all the columns in the C -matrix whose indices are of the same orbit have the same sign (Lemma 3.11). This allows us to define the *orbit-maximal green sequence*. We also prove that every orbit-maximal green sequence for a skew-symmetrizable matrix in a stable admissible pair corresponds to a maximal green sequence for its folding matrix (Theorem 3.19). Using this result we prove that the minimum length of any orbit-maximal green sequence for a skew-symmetrizable matrix is the number of the orbits (Corollary 3.22). Then, we study the relation between the orbit-maximal green sequences of a skew-symmetrizable matrix and the maximal green sequences of this matrix and prove that every orbit-maximal green sequence corresponds to a maximal green sequence (Theorem 3.25). The raised question here is when and how to create a corresponding orbit-maximal green sequence built from a given maximal green sequence? We discuss this question for maximal green sequences obtained from an admissible numbering by source for acyclic skew-symmetrizable matrices.

On the other hand, we introduce the *general-maximal green sequence* which produces a maximum number of red indices. This generalization makes every skew-symmetrizable matrix own a general-maximal green sequence. The interesting question here is that: how many red indices can be obtained maximally? We call this maximal number the *red size*. We prove that the red size of any decomposable skew-symmetrizable matrix is greater than or equals the sum of the red sizes of its skew-symmetrizable components (Theorem 4.8). Finally, we define *general orbit-maximal green sequences* and study the connection with general-maximal green sequences of a skew-symmetrizable matrix under the action of an automorphism group.

2. Preliminaries

Recall that $B = (b_{ij}) \in M_n(\mathbb{Z})$ is said to be a *skew-symmetrizable* matrix if there is a diagonal matrix D with positive integer entries such that DB is a skew-symmetric matrix. We call D the symmetrizer matrix.

Let $\tilde{B} = (b_{ij}) \in M_{n+m,n}(\mathbb{Z})$ then $\tilde{B}$ is said to be skew-symmetrizable if its principal part which is the square sub-matrix $B = (b_{i,j})$ for $1 \leq i, j \leq n$ is skew-symmetrizable. We call $\tilde{B}$ the *extended matrix* of B .

The mutation of a matrix $\widetilde{B}$ in direction k where $1 \leq k \leq n$ is the matrix $\mu_k(B) = B' = (b'_{ij})$ where:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}.$$

This is called the *exchange relation of the mutation*. Given a skew symmetrizable matrix $\widetilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$, let $\widetilde{B}_\sigma = \begin{pmatrix} B_\sigma \\ C_\sigma \end{pmatrix}$ be the matrix obtained from $\widetilde{B}$ by a composition of mutations $\mu_\sigma = \mu_{k_s} \mu_{k_{s-1}} \dots \mu_{k_1}$, $1 \leq k_j \leq n$ for every $1 \leq j \leq s$. The lower part of $\widetilde{B}_\sigma$ is called the *C-matrix* of $\widetilde{B}_\sigma$. By the sign coherence property proved in [7], all the entries of a given column in C_σ are of the same sign.

Definition 2.1. An index k for $1 \leq k \leq n$ is called *green* (respectively, *red*) if the entries of the column indexed by k in C_σ are non-negative (respectively, non-positive).

Definition 2.2. A sequence of indices $(k_1, k_2, \dots, k_s)$, where $1 \leq k_i \leq n$ for all $i \in \{1, 2, \dots, s\}$, is called a *green sequence* if k_i is green in $\mu_{k_{i-1}} \dots \mu_{k_1}(\widetilde{B})$ for $1 \leq i \leq s$. Such sequence is called *maximal* if $\mu_{k_s} \dots \mu_{k_1}(\widetilde{B})$ does not have any green indices.

By convention, we denote $\mu_{k_0}(\widetilde{B}) = \widetilde{B}$.

3. Orbit-maximal green sequences

Suppose we have a skew-symmetrizable matrix equipped with a group of automorphisms, we will expand the sign-coherence idea to the orbit-sign coherence idea and introduce the orbit-maximal green sequences.

Definition 3.1. Let $B \in M_n(\mathbb{Z})$ be a skew-symmetrizable matrix and S_n be the group of permutations on the set $\{1, 2, \dots, n\}$. Let $g \in S_n$, then g is said to be an *automorphism* of B if $b_{gi, gj} = b_{ij}$ for every $1 \leq i, j \leq n$.

Definition 3.2. Let G be a subgroup of S_n , G is said to be an *automorphism group* of B if for every $g \in G$, g is an automorphism of B .

We denote the G -orbits by $\bar{i}$ for $1 \leq i \leq n$.

Definition 3.3. An automorphism g of a skew-symmetrizable matrix B is said to be *admissible* if:

$$b_{i_1 j} b_{i_2 j} \geq 0, \quad b_{i_1 i_2} = 0 \text{ for any } i_1, i_2 \text{ falling in the same } G\text{-orbit and } 1 \leq j \leq n.$$

Definition 3.4. Let G be a group of automorphisms of a skew-symmetrizable matrix B , the group G is said to be an *admissible automorphism group* and the pair (B, G) is said to be *admissible* if g is an admissible automorphism for every $g \in G$.

By [4, Lemma 2.12], we note that mutations in directions falling in the same orbit commute whenever we have an admissible pair (B, G) , $\mu_{i_2}\mu_{i_1}(B) = \mu_{i_1}\mu_{i_2}(B)$, for every $i_1, i_2 \in \bar{i}$. Thus, the so-called orbit-mutation can be defined as follows:

Definition 3.5. Let (B, G) be an admissible pair, the *orbit-mutation* in direction $\bar{i}$ is the composition $\mu_{\bar{i}}(B) = \prod_{k \in \bar{i}} \mu_k(B)$.

Remark 3.6. It is clear that $c_{ij} = c_{gi, gj}$ for any $g \in S_n$ and any $c_{ij} \in I_n$. Thus any subgroup of S_n can be considered as an automorphism group for the identity matrix, even though I_n is not skew-symmetrizable.

Remark 3.7. The sign-coherence property for the C -matrices obtained from the extended matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ by orbit-mutations, can be regarded as a suitable substitution of the admissibility condition to be used in the results of this paper.

An admissible automorphism group of an extended skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ is defined to be any admissible automorphism group of its principle part B .

We denote by $\tilde{B}^{\bar{k}}$ the matrix obtained from $\tilde{B}$ by the orbit-mutation in direction $\bar{k}$, that is,

$$\tilde{B}^{\bar{k}} = \mu_{\bar{k}}(\tilde{B}) = \begin{pmatrix} B^{\bar{k}} \\ C^{\bar{k}} \end{pmatrix}$$

Assume $(\tilde{B}, G)$ is an admissible pair. Then, by induction, it follows from [4], [11] that the orbit-mutation of a skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ in direction $\bar{k}$ for $1 \leq k \leq n$ is given by the following form:

$$(1) \quad b_{ij}^{\bar{k}} = \begin{cases} -b_{ij}, & \text{if } i \text{ or } j \in \bar{k} \\ b_{ij} + \frac{1}{2} \sum_{t \in \bar{k}} (|b_{it} \mid b_{tj} + b_{it} \mid b_{tj}|), & \text{otherwise} \end{cases}$$

$$(2) \quad c_{ij}^{\bar{k}} = \begin{cases} -c_{ij}, & \text{if } j \in \bar{k} \\ c_{ij} + \frac{1}{2} \sum_{t \in \bar{k}} (|c_{it} \mid b_{tj} + c_{it} \mid b_{tj}|), & \text{otherwise} \end{cases}$$

Remark 3.8. Let $(\tilde{B}, G)$ be an admissible pair, it is clear that $i \in \bar{k}$ if and only if $g(i) \in \bar{k}$ for every $g \in G$ and $1 \leq i, k \leq n$. If i or $j \in \bar{k}$ then $b_{ij}^{\bar{k}} = -b_{ij}$, $c_{ij}^{\bar{k}} = -c_{ij}$, and $b_{g(i)g(j)}^{\bar{k}} = -b_{g(i)g(j)}$, $c_{g(i)g(j)}^{\bar{k}} = -c_{g(i)g(j)}$. Hence $b_{ij}^{\bar{k}} = b_{g(i)g(j)}^{\bar{k}}$ and $c_{ij}^{\bar{k}} = c_{g(i)g(j)}^{\bar{k}}$ in this case.

Clearly if t runs over $\bar{k}$, then $g(t)$ runs over $\bar{k}$ for every $g \in G$, then: $\frac{1}{2} \sum_{t \in \bar{k}} (|b_{it} \mid b_{tj} + b_{it} \mid b_{tj}|) = \frac{1}{2} \sum_{f=g(t) \in \bar{k}} (|b_{g(i)f} \mid b_{fg(j)} + b_{g(i)f} \mid b_{fg(j)}|)$

$\frac{1}{2} \sum_{t \in \bar{k}} (|c_{it}| |b_{tj} + c_{it}| |b_{tj}|) = \frac{1}{2} \sum_{f=g(t) \in \bar{k}} (|c_{g(i)f}| |b_{fg(j)} + c_{g(i)f}| |b_{fg(j)}|)$ thus $b_{ij}^{\bar{k}} = b_{g(i)g(j)}^{\bar{k}}$ and $c_{ij}^{\bar{k}} = c_{g(i)g(j)}^{\bar{k}}$ for every $k \in \{1, 2, \dots, n\}$ and $g \in G$, in case we mutate in the direction $\bar{k}$ where i and j are not in $\bar{k}$.

From the previous discussion, we note that if G is an automorphism group for $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$, then it is also an automorphism group of $\tilde{B}^{\bar{k}} = \mu_{\bar{k}}(\tilde{B}) = \begin{pmatrix} B^{\bar{k}} \\ C^{\bar{k}} \end{pmatrix}$ for $1 \leq k \leq n$.

In the following discussion we need to deal with a composition of orbit-mutations, but first a condition needs to be established to guarantee the admissibility after such composition.

Definition 3.9. An admissible pair $(\tilde{B}, G)$ is said to be *stable* if for any sequence of orbit-mutations $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$, the pair $(\mu_{\bar{k}_s} \mu_{\bar{k}_{s-1}} \dots \mu_{\bar{k}_1}(\tilde{B}), G)$ is an admissible pair for every $1 \leq j \leq s$.

Remark 3.10. By [4], [11], the relations (1) and (2) are used to obtain an orbit-mutation for any skew-symmetrizable matrix in a stable admissible pair.

Let $\mu_{\sigma_s}(\tilde{B})$ be the matrix obtained by a composition of orbit-mutations where $\sigma_s := (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$. Because the orbit-mutation is nothing but a composition of mutations in directions of all indices falling in this specific orbit, it is clear that the property of sign-coherence proved in [7, Theorem 3] is well preserved for the C -matrix C^{σ_s} of the extended skew-symmetrizable matrix $\tilde{B}^{\sigma_s} = \begin{pmatrix} B^{\sigma_s} \\ C^{\sigma_s} \end{pmatrix}$.

Lemma 3.11. Let $(\tilde{B}, G)$ be a stable admissible pair and let $\tilde{B}^{\bar{k}} = \mu_{\bar{k}}(\tilde{B}) = \begin{pmatrix} B^{\bar{k}} \\ C^{\bar{k}} \end{pmatrix}$ be the orbit-mutation in direction $\bar{k}$ for $1 \leq k \leq n$. If i_1, i_2 are of the same orbit, the columns indexed by i_1, i_2 in the C -matrix $C^{\bar{k}}$ have the same sign.

Proof. For any i_1, i_2 falling in the same orbit, there is $g \in G$ such that $g(i_1) = i_2$. Note that g is an automorphism for $\mu_{\bar{k}}(\tilde{B})$, thus $c_{li_1}^{\bar{k}} = c_{g(l)g(i_1)}^{\bar{k}} = c_{g(l)i_2}^{\bar{k}}$ for any $1 \leq l \leq n$. By the sign coherence property the two columns i_1, i_2 in the C -matrix $C^{\bar{k}}$ have the same sign since they have two equal entries. $\square$

In the light of the previous lemma we denote the sign of an orbit $\bar{i}$ by $sgn(\bar{i})$ for any $i \in \{1, 2, \dots, n\}$. Now we can establish the definitions of the orbit-maximal green sequence.

Definition 3.12. Let $(\tilde{B}, G)$ be a stable admissible pair and let C^{σ_s} be the C -matrix of $\tilde{B}^{\sigma_s}$ obtained from $\tilde{B}$ by a sequence of orbit-mutations, $\tilde{B}^{\sigma_s} = \mu_{\sigma_s}(\tilde{B})$ where $\sigma_s := (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$ such that $1 \leq k_j \leq n$ and $1 \leq j \leq s$, then the orbit $\bar{i}$ is said to be *green* (respectively, *red*) if $sgn(\bar{i})$ is nonnegative (respectively, nonpositive) in C^{σ_s} .

Definition 3.13. Let $(\tilde{B}, G)$ be a stable admissible pair, $\sigma_s := (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$ be a sequence of orbit-mutations and let C^{σ_j} be the C -matrix of $\tilde{B}^{\sigma_j}$ obtained from $\tilde{B}$ by the sequence of orbit-mutations $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_j)$ for $1 \leq j \leq s$, then $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$ is said to be an *orbit-green sequence* if for every $1 \leq j \leq s$, $\bar{k}_j$ is a green orbit in $C^{\sigma_{j-1}}$. The sequence $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$ is said to be *orbit-maximal green sequence* if $\tilde{B}^{\sigma_s}$ doesn't have green orbits.

Now we define the folding of a matrix as mentioned in [4] and [11].

Definition 3.14. Let $(\tilde{B}, G)$ be a stable admissible pair, the folding matrix of $\tilde{B}$ is given by $\pi(\tilde{B}) = \begin{pmatrix} \pi(B) \\ \pi(C) \end{pmatrix} \in M_{2m \times m}(\mathbb{Z})$, where :

$$(3) \quad \pi(B) = (b_{i\bar{j}}) \quad \text{such that, } b_{i,\bar{j}} = \sum_{f \in \bar{i}} b_{fj}$$

$$(4) \quad \pi(C) = (c_{i\bar{j}}) \quad \text{such that, } c_{i,\bar{j}} = \sum_{f \in \bar{i}} c_{fj}$$

and m is the number of the G -orbits.

By [4, Lemma 2.5], the folding matrix of a skew-symmetrizable matrix is skew-symmetrizable.

Lemma 3.15. Let $(\tilde{B}, G)$ be a stable admissible pair, such that $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$, then $\pi(\tilde{B}) = \begin{pmatrix} \pi(B) \\ I_m \end{pmatrix}$, where m is the number of the G -orbits.

Proof. Let $c_{i\bar{j}} \in \pi(I_n)$. By the definition of the folding matrix and the nature of the the entries of I_n , we have:

$$c_{i\bar{j}} = \begin{cases} 1, & \text{if } \bar{i} = \bar{j} \\ 0, & \text{if } \bar{i} \neq \bar{j} \end{cases}$$

□

Since the folding of an extended skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ in a stable admissible pair $(\tilde{B}, G)$ is of the form $\pi(\tilde{B}) = \begin{pmatrix} \pi(B) \\ I_m \end{pmatrix}$, where $\pi(B)$ is a skew-symmetrizable matrix, the natural question here is: what is the relation between orbit-maximal green sequences for the matrix $\tilde{B}$ and maximal green sequences for its folding matrix $\pi(\tilde{B})$.

The following Proposition shows the entries of the folding C -matrix after mutation in direction $\bar{k}$ as an index.

Proposition 3.16 ([11, Proposition 5.8]). Let B be any skew-symmetrizable integer matrix and let σ be a stable admissible automorphism of B . The matrix $\pi(C)$ which is the C -matrix of the matrix $\pi(\tilde{B})$ satisfies the recursion relation:

$$(5) \quad c'_{ij} = \begin{cases} -c_{i\bar{j}}, & \text{if } \bar{j} = \bar{k} \\ c_{i\bar{j}} + \frac{1}{2}(|c_{i\bar{k}}| b_{\bar{k}\bar{j}} + c_{i\bar{k}} |b_{\bar{k}\bar{j}}|), & \text{otherwise} \end{cases}$$

The following Lemma is a generalization of Theorem 2.24 in [4].

Lemma 3.17. Let $(\tilde{B}, G)$ be a stable admissible pair and B is of the size $n \times n$, then $\pi(\mu_{\bar{k}}(\tilde{B})) = \mu_{\bar{k}}(\pi(\tilde{B}))$ for every $1 \leq k \leq n$.

Proof. As for the principal part of the extended matrix $\tilde{B}$, it was proved in [4, Theorem 2.24].

For the C -matrix, we have the following points:

- the orbit-mutation of this matrix which is given in (2)
- the folding of this matrix which is given in (4)
- the mutation of the folding C -matrix which is given in (5).
- the sign coherence property.

By the previous points, we get the desired result following the same steps of the proof in Theorem 2.24 [4]. $\square$

Remark 3.18. According to the familiar method of indexing a matrix and to avoid ambiguity, the matrix $\pi(B) = (b_{i\bar{j}})$ for $1 \leq i, j \leq n$ will be referred to as $F = (F_{i^f j^f})$ for $1 \leq i^f, j^f \leq m$ such that i^f represents the order of the row or column indexed by $\bar{i}$ in $\pi(B)$ and m is the number of the orbits. Here, i^f is called the *natural corresponding index* of $\bar{i}$.

Theorem 3.19. Let $(\tilde{B}, G)$ be a stable admissible pair. Then $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s)$ is an orbit-maximal green sequence for B if and only if the natural corresponding sequence $(k_1^f, k_2^f, \dots, k_s^f)$ is a maximal green sequence for its folding matrix $\pi(B) = F$.

Proof. Let $\mu_{k_j^f} \mu_{k_{j-1}^f} \dots \mu_{k_1^f}(\pi(\tilde{B})) = \begin{pmatrix} (\pi(B))^{\sigma_j} \\ (\pi(C))^{\sigma_j} \end{pmatrix}$, and $\pi(\mu_{\bar{k}_j} \mu_{\bar{k}_{j-1}} \dots \mu_{\bar{k}_1}(\tilde{B})) = \pi \begin{pmatrix} B^{\sigma_j} \\ C^{\sigma_j} \end{pmatrix} = \begin{pmatrix} \pi(B^{\sigma_j}) \\ \pi(C^{\sigma_j}) \end{pmatrix}$ for every $1 \leq j \leq s$.

By Lemma 3.17 we have:

$$\begin{pmatrix} (\pi(B))^{\sigma_j} \\ (\pi(C))^{\sigma_j} \end{pmatrix} = \begin{pmatrix} \pi(B^{\sigma_j}) \\ \pi(C^{\sigma_j}) \end{pmatrix}.$$

That is, $(\pi(C))^{\sigma_j}$ can be considered as the folding of the matrix C^{σ_j} , by the definition of the folding matrix:

$$c_{i\bar{k}_{j+1}}^{\sigma_j} = \sum_{t \in \bar{i}} c_{tk_{j+1}}^{\sigma_j} \quad \text{such that} \quad c_{tk_{j+1}}^{\sigma_j} \in C^{\sigma_j}.$$

By the sign coherence property, we have that $c_{i\bar{k}_{j+1}}^{\sigma_j}$ is non-negative (non-positive) if and only if $c_{tk_{j+1}}^{\sigma_j}$ is non-negative (non-positive) for every $t \in \bar{i}$ and $1 \leq j \leq s-1$. That is, the column indexed by k_{j+1}^f in $(\pi(\tilde{B}))^{\sigma_j}$ is green (respectively, red) if and only if the column indexed by k_{j+1} is green (respectively, red) in $\tilde{B}^{\sigma_j}$ and as a result the orbit $\bar{k}_{j+1}$ is also green (respectively, red) for every $1 \leq j \leq s-1$ by Lemma 3.11. Moreover, $(\pi(\tilde{B}))^{\sigma_s}$ has no green indices if and only if $\tilde{B}^{\sigma_s}$ has no green orbits, and the result follows. $\square$

The result which determines the minimal length of any maximal green sequence for a skew-symmetric matrix [2, Corollary 2.18] is clearly true in the skew-symmetrizable case.

Proposition 3.20 ([2, Corollary 2.18]). If B is a skew-symmetrizable matrix of the size $n \times n$ which admits a maximal green sequence, then the minimal length of any maximal green sequence for B is n .

Definition 3.21. The length of an orbit-maximal green sequence is the number of the orbits in this sequence.

Corollary 3.22. Let $(\tilde{B}, G)$ be a stable admissible pair, the minimum length of any orbit-maximal green sequence is the number of G -orbits.

Proof. By Theorem 3.19 every orbit-maximal sequence for $\tilde{B}$ corresponds to a maximal green sequence for its folding matrix $\pi(\tilde{B}) \in M_{m \times m}(\mathbb{Z})$ where m is the number of the orbits. By Proposition 3.20 the minimum length of any such maximal green sequence is m , and this is the required. $\square$

Example 3.23. $\tilde{B} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 0 & -2 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an acyclic skew symmetric

matrix, the pair $(\tilde{B}, G)$ is an admissible pair where $G = \langle (2, 3) \rangle$, by [4, Theorem 2.23] this pair is stable admissible.

$$\mu_{\bar{1}}\mu_{\bar{2}}\mu_{\bar{4}}(\tilde{B}) = \begin{pmatrix} 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 0 & -2 & -2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ thus } (\bar{4}, \bar{2}, \bar{1}) \text{ is an orbit-maximal}$$

green sequence. The folding matrix $\pi(\tilde{B}) = \begin{pmatrix} 0 & 2 & 0 \\ -4 & 0 & 4 \\ 0 & -2 & 0 \\ - & - & - \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The natural

corresponding sequence for the sequence $(\bar{4}, \bar{2}, \bar{1})$ is $(3, 2, 1)$. It is easy to check that $(3, 2, 1)$ is a maximal green sequence of the folding matrix.

Clearly, the existence of orbit-maximal green sequences for a given skew-symmetrizable matrix B is related to two factors; the matrix entries and the group G itself for which the pair (B, G) is a stable admissible pair. We will prove that every orbit-maximal green sequence is corresponding to a maximal green sequence.

Lemma 3.24. Let $(\tilde{B}, G)$ be an admissible pair, the entries of the column i_j in the C -matrix are invariant under any mutation in direction $i_l \neq i_j$, where i_l, i_j are of the same orbit $\bar{i}$.

Proof. Let c'_{ki_j} be one of these entries then $c'_{ki_j} = c_{ki_j} + \frac{1}{2}(|c_{ki_l}b_{li_j} + c_{ki_l}b_{li_j}|)$ for every $1 \leq k \leq n$. Since the pair is admissible, we have $b_{li_j} = 0$ and thus $c'_{ki_j} = c_{ki_j}$. $\square$

In the following Theorem, we denote the indices of the orbit $\bar{i}_s$ by $i_{s1}, i_{s2}, \dots, i_{sl_s}$ where $1 \leq s \leq m$ and m is the number of the orbits, that is l_s denotes the number of the indices in the orbit $\bar{i}_s$.

Theorem 3.25. Let $(\tilde{B}, G)$ be a stable admissible pair, then every orbit-green sequence (respectively, orbit-maximal green sequence) of $(\tilde{B}, G)$ corresponds to a green sequence (respectively, maximal green sequence) of $\tilde{B}$.

Proof. (i) Suppose $(\bar{i}_1, \bar{i}_2, \dots, \bar{i}_k)$ is an orbit-green sequence, then $\bar{i}_s$ is a green orbit in the C -matrix of the extended matrix $\mu_{\bar{i}_{s-1}} \dots \mu_{\bar{i}_2} \mu_{\bar{i}_1}(\tilde{B})$ for every $1 \leq s \leq k$. By the definition of orbit mutation, the stability of the admissible condition and Lemma 3.24 the column indexed by i_{fg} is green in the C -matrix of the

extended matrix $\mu_{i_{fg-1}} \cdots \mu_{i_{f1}} \cdots \mu_{i_{1l_1}} \cdots \mu_{i_{11}}(\tilde{B})$ for every $1 \leq f \leq k$ and every $1 \leq g \leq l_f$. Hence the sequence $(i_{11}, \dots, i_{1l_1}, \dots, i_{k1}, \dots, i_{kl_k})$ is a green sequence.

(ii) Suppose $(\bar{i}_1, \bar{i}_2, \dots, \bar{i}_k)$ is an orbit-maximal green sequence, then by the proof of the part (i), $(i_{11}, \dots, i_{1l_1}, \dots, i_{k1}, \dots, i_{kl_k})$ is a green sequence which changes the matrix $\begin{pmatrix} B \\ I_n \end{pmatrix}$ into the matrix $\mu_{i_{kl_k}} \cdots \mu_{i_{k1}} \cdots \mu_{i_{1l_1}} \cdots \mu_{i_{11}} \begin{pmatrix} B \\ I_n \end{pmatrix} \simeq \begin{pmatrix} B \\ -I_n \end{pmatrix}$ and thus this green sequence is maximal. $\square$

We do not think that it is always possible to construct an orbit-maximal green sequence from a given maximal green sequence for a skew-symmetrizable matrix $\tilde{B}$ since a stable admissible pair $(\tilde{B}, G)$ may not exist. Let us suppose that such pair does exist and $\tilde{B}$ admits a maximal green sequence, still we do not think that when changing index mutation into orbit-mutation, the green indices are guaranteed to remain green. This raises the following question:

Question 3.26. When can a maximal green sequence for a skew-symmetrizable matrix $\tilde{B}$ in a stable admissible pair $(\tilde{B}, G)$ be reconstructed as an orbit-maximal green sequence?

In an attempt to answer this question, we will discuss a special case of both skew-symmetrizable matrices and maximal green sequences.

Definition 3.27. A skew-symmetrizable matrix B is said to be *acyclic* if there are no entries $i_1, \dots, i_k$ such that $b_{i_{j+1}i_j} > 0$ for every $1 \leq j \leq k-1$ and $i_k = i_1$.

Definition 3.28. A *source* in a skew-symmetrizable matrix B of the size $n \times n$ is an index i where $1 \leq i \leq n$ and $b_{ki} \geq 0$ for all $1 \leq k \leq n$.

Definition 3.29. An *admissible numbering by sources* of a skew-symmetrizable matrix $\tilde{B}$ is an n -tuple $(i_1, i_2, \dots, i_n)$ such that the indices of B are $\{i_1, \dots, i_n\}$ with i_1 a source in $\tilde{B}$, and for any $2 \leq k \leq n$, the vertex i_k is a source in $\mu_{i_{k-1}} \cdots \mu_{i_1}(\tilde{B})$.

In [2, Lemma 2.20] the following Proposition was considered for quivers (in the skew-symmetric case). It can be easily checked that it is true for the skew-symmetrizable case following the same steps.

Proposition 3.30 ([2, Lemma 2.20]). Let $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ be a skew-symmetrizable matrix. Then any admissible numbering by sources of $\tilde{B}$ is a maximal green sequence.

We will call such maximal green sequence an *admissible numbering maximal green sequence*.

Lemma 3.31. If i is a source in a skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ and suppose j is another green (respectively, red) source in this matrix, then j will remain a green (respectively, red) source in the matrix $\mu_i(B)$.

Proof. The result follows by substituting $b_{ij} = 0$ in the exchange relations. $\square$

Theorem 3.32. Let $(\tilde{B}, G)$ be a stable admissible pair such that $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ is an acyclic skew-symmetrizable matrix and let $s = (s_1, s_2, \dots, s_n)$ be an admissible numbering maximal green sequence of $\tilde{B}$.

(i) We can always re-arrange this sequence in a way such that the indices of the orbit represented by s_1 are successive and are followed by the indices of the orbit represented by s_2 (in case they are not of the same orbit) and so on, to produce a new admissible numbering maximal green sequence of $\tilde{B}$.

(ii) This new admissible numbering maximal green sequence corresponds to an orbit-maximal green sequence.

Proof. (i) If the indices of s are arranged in the way described above, there is nothing to do. Suppose that there are two consecutive indices s_k, s_{k+1} which do not meet the above description and they are the first indices to do that. We denote the orbits of G by $\bar{i}_1, \bar{i}_2, \dots, \bar{i}_m$ and the indices of each orbit by $\bar{i}_g = \{i_{g1}, \dots, i_{gl_g}\}$ for $1 \leq g \leq m$. Without losing of generality we suppose the first index of the orbit $\bar{i}_t$ in s is $i_{t1}, s_k = i_{tw}, s_{k+1} = i_{d1}$, with $t \neq d$ and $1 \leq w < l_t$. Mutations before starting mutating in directions of indices of the orbit $\bar{i}_t$ can be regarded as orbit-mutations. We denote the matrix obtained by this orbit-mutations B' . i_{t1} is a source in the matrix B' . Since G is a group of automorphisms, $b_{qi_{t1}} \geq 0$ if and only if $b_{qi_{tf}} \geq 0$ for every $1 \leq q \leq n$ and every $1 \leq f \leq l_t$. Hence i_{tf} is also a source in the matrix B' for every $1 \leq f \leq l_t$. Since B' is obtained by a composition of orbit-mutations and i_{t1} is green, then by Lemma 3.11 all the indices of the orbit $\bar{i}_t$ are green. By Lemma 3.31 i_{tf} will remain a green source in the matrix $\mu_{i_{tf-1}} \dots \mu_{i_{t1}}(B')$ for every $1 \leq f \leq l_t$. $s_{k+1} = i_{d1}$ is a green source in $\mu_{i_{tw}} \dots \mu_{i_{t1}}(B')$ again by Lemma 3.31 it will remain a green source in $\mu_{\bar{i}_t}(B')$. By iterating the same previous discussion we can put the indices of the orbit $\bar{i}_d$ together such that every i_{dr} is a green source in the matrix $\mu_{i_{dr-1}} \dots \mu_{i_{d1}} \mu_{\bar{i}_t}(B')$ for every $1 \leq r \leq l_d$. We do the same for all orbits whose indices are not successive. Finally we get a new admissible numbering maximal green sequence arranged as required.

(ii) By the first part of the proof, we can transform s into a new admissible numbering maximal green sequence s' where the indices of each orbit are successive $s' = (i_{11}, \dots, i_{1l_1}, \dots, i_{m1}, \dots, i_{ml_m}) = (\bar{i}_1, \dots, \bar{i}_m)$, where $\bar{i}_g$ is a green orbit in $\mu_{\bar{i}_{g-1}} \dots \mu_{\bar{i}_1}(B)$ for every $1 \leq g \leq m$ and $\mu_{\bar{i}_m} \dots \mu_{\bar{i}_1}(\tilde{B})$ has no green orbits. By the definition of the orbit-maximal green sequence, the result follows. $\square$

4. General-maximal green sequences

It is not necessary for a skew symmetrizable matrix to have a maximal green sequence, this fact is a motivation to set a more general concept.

Definition 4.1. Let $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ be a skew-symmetrizable matrix and let $\tau = (k_1, \dots, k_{s-1}, k_s)$ be a green sequence, such sequence is called a *general-maximal green sequence* if $\mu_{k_s} \mu_{k_{s-1}} \dots \mu_{k_1}(\tilde{B})$ has the maximum number of red indices among the red indices obtained by any other green sequences.

Definition 4.2. Let $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ be a skew-symmetrizable matrix and let $\tau = (k_1, \dots, k_{s-1}, k_s)$ be a general-maximal green sequence for this matrix, the maximum number of red indices obtained by τ is called the *red size* of $\tilde{B}$. A general-maximal green sequence becomes a maximal green sequence when the red size of $\tilde{B}$ is n .

A general-maximal green sequence which is not a maximal green sequence is called a *proper general-maximal green sequence*.

Proposition 4.3 ([10, Theorem 12]). Let $Q_{a,b,c}$ a cyclic quiver with three vertices 1, 2, 3, such that there are a -many arrows from 1 to 2 and b -many arrows from 2 to 3 and c -many arrows from 3 to 1. If $a, b, c \geq 2$, then the quiver $Q_{a,b,c}$ does not admit a maximal green sequence

Now we can find a general-maximal green sequence for 3×3 quivers which do not admit a maximal green sequence.

Example 4.4. $B = \begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & -3 \\ -4 & 3 & 0 \end{pmatrix}$. By Proposition 4.3 this matrix does not admit a maximal green sequence thus its red size is at most 2.

$$\begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & -3 \\ -4 & 3 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_2} \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & 3 \\ 5 & -3 & 0 \\ \hline 1 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_1} \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & -12 \\ -5 & 12 & 0 \\ \hline -1 & 3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_3} \begin{pmatrix} 0 & 57 & -5 \\ -57 & 0 & 12 \\ 5 & -12 & 0 \\ \hline -1 & 3 & 0 \\ -3 & 8 & 0 \\ 0 & 12 & -1 \end{pmatrix}.$$

Thus the green sequence $(2, 1, 3)$ is a general-maximal green sequence since it gives two red indices.

Since we deal with matrices of finite size, a general-maximal green sequences always exists, yet determining the red size of a given skew-symmetrizable matrix is not easy, especially in the infinite-mutation case. In the following we will discuss the red size of a reducible skew symmetrizable matrix. For more detailed information and results about reducibility we refer to [3]. We recall that an integer matrix is said to be a *column (row) sign coherent* if the nonzero elements which are of the same column (row) have the same sign. An integer matrix $B_2 \in M_{m \times n}(\mathbb{Z})$ is said to be *uniformly column (row) sign coherent* with respect to a skew symmetrizable matrix $B_1 \in M_{n \times n}(\mathbb{Z})$, if for any composition of mutations $\mu_{k_s} \dots \mu_{k_2} \mu_{k_1}$ where $1 \leq k_i \leq n$ for $1 \leq i \leq s$, the lower part of the matrix $\mu_{k_s} \dots \mu_{k_1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is still column (row) sign coherent.

Proposition 4.5 ([3, Corollary 3.3]). Any integer matrix with non-negative entries is uniformly column sign coherent with respect to any skew symmetrizable matrix which shares the same number of columns.

Definition 4.6. Let B be a skew symmetrizable matrix. We say that B is *reducible*, if B can be written as a block matrix as follows

$$B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$$

such that $B_1 \in M_{n_1 \times n_1}(\mathbb{Z})$, $B_4 \in M_{n_2 \times n_2}(\mathbb{Z})$, and the proper submatrix $B_2 \in M_{n_2 \times n_1}(\mathbb{Z})$ has non-negative entries.

Proposition 4.7 ([3, Proposition 3.7]). Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a skew-symmetrizable matrix, with $B_1 \in M_{n \times n}(\mathbb{Z})$ and $B_4 \in M_{m \times m}(\mathbb{Z})$. Then B_2 is uniformly sign coherent with respect to B_1 if and only if B_4 is invariant under any composition of mutations $\mu_{k_s} \mu_{k_{s-1}} \dots \mu_{k_1} := \mu_{\sigma_s}$, where $1 \leq k_i \leq n$, $1 \leq i \leq s$.

In [3], it was proved that for a reducible matrix $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$, k (respectively, j) is a maximal green sequence for B_1 (respectively, B_4) if and only if $\tilde{k} = (k^j)$ is a maximal green sequence for B . In the following theorem, we find the relation between the red size of a reducible skew-symmetrizable matrix and the red sizes of the skew symmetrizable matrices which appear in its decomposition.

Theorem 4.8. Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a reducible skew-symmetrizable matrix with $B_1 \in M_{n_1 \times n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2 \times n_2}(\mathbb{Z})$, $\sigma_s := (k_1, \dots, k_s)$ is a general-maximal green sequence for B_1 with red size k , and $\tau_r := (l_1, \dots, l_r)$ is a general-maximal green sequence for B_4 with red size l . Then m the red size of B is greater than or equal to $k + l$.

Proof. When applying the sequence of mutations σ_s , we will get k red indices. Because of the non negative entries of the submatrix $\begin{pmatrix} B_2 \\ I_{n_1} \\ 0 \end{pmatrix}$ it is uniformly sign coherent with respect to B_1 by Proposition 4.5. By Proposition 4.7, the submatrix $\begin{pmatrix} B_4 \\ 0 \\ I_{n_2} \end{pmatrix}$ is mutation invariant in directions $1 \leq k_s, \dots, k_1 \leq n$. As a result when applying the sequence of mutations τ_r we will get l new red indices while the C -matrix $\mu_{k_s} \dots \mu_{k_1} \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} = \begin{pmatrix} C^{\sigma_s} \\ 0 \end{pmatrix}$ remains invariant since $\begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix}$ is uniformly column sign coherent with respect to the skew symmetrizable B_4 because of its non-negative entries. Then the total number of the red indices that we get after applying the composition $(\tau_r \sigma_s)$ is $k + l$ and the result follows. $\square$

And here we raise the following question:

Question 4.9. Does the red size of a reducible skew-symmetrizable matrix equal the sum of the red sizes of the skew-symmetrizable matrices which appear in its decomposition ?

Corollary 4.10. Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ be a reducible skew-symmetric matrix of the size $n \times n$ with $B_1 \in M_{n_1 \times n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2 \times n_2}(\mathbb{Z})$, $\sigma_s := (k_1, \dots, k_s)$ is a general-maximal green sequence for B_1 with red size k , and $\tau_r := (l_1, \dots, l_r)$ is a general-maximal green sequence for B_4 with red size l . If σ_s or τ_r is proper, then the red size m of B is determined by $l + k \leq m < n$.

Proof. The left part of the inequality is obtained from Theorem 4.8. If $m = n$, then B admits a maximal green sequence. By [10, Theorem 1.4.1] any induced sub-matrix also admits a maximal green sequence. Hence, $k = n_1$ and $l = n_2$ which contradicts with the hypothesis that one the general-maximal sequences σ_s, τ_r is proper. $\square$

Since every orbit-maximal green sequence in a stable admissible pair corresponds to a maximal green sequence, see Theorem 3.25, some skew-symmetrizable matrices admit orbit-maximal green sequences while others do not. By following the same method of maximal green sequence generalization, we define the general orbit-maximal green sequence.

Definition 4.11. For a stable admissible pair $(\tilde{B}, G)$ and an orbit-green sequence $\sigma = (\bar{k}_1, \dots, \bar{k}_{s-1}, \bar{k}_s)$, this sequence is said to be a *general orbit-maximal green sequence* if $\mu_\sigma(\tilde{B})$ has the maximum number of red orbits among the red orbits obtained by any other orbit-green sequences. Similarly the number of these red orbits (which are obtained maximally) is called the *orbit-red size* of $\tilde{B}$.

It is clear that every orbit-maximal green sequence for a skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z})$ in a stable admissible pair $(\tilde{B}, G)$ is a general orbit-maximal green sequence whose orbit red size is the number of orbits obtained under the action of G .

Lemma 4.12. Let B be a skew-symmetrizable matrix of the size $n = 3 \times 3$ and (B, G) be a stable admissible pair with G not trivial (G is not the identity or S_3), then every general orbit-maximal green sequence has an orbit red size $m = 2$ (that is, it is an orbit-maximal green sequence).

Proof. Since G is not trivial, G has two orbits $j = \{i_1, i_2\}$ and $m = \{i_3\}$. By the admissibility $b_{i_1 i_2} = 0$ and $b_{i_1 i_3}, b_{i_2 i_3} \geq 0$. Hence B is acyclic with i_3 is the only source and i_1, i_2 are two sinks or i_3 is the only sink and i_1, i_2 are two sources. Thus $(m, j), (j, m)$ are two orbit-maximal green sequence and $m = 2$ is the maximum number of red orbits to be obtained. $\square$

In analogy to Theorem 3.25, we get the following result:

Corollary 4.13. For a stable admissible pair $(\tilde{B}, G)$, every general orbit-maximal green sequence corresponds to a general-maximal green sequence.

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Integral inequalities of Hermite-Hadamard type for (α, s) -convex and (α, s, m) -convex functions

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Abstract. In the paper, the authors introduce the concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and establish some new integral inequalities of the Hermite-Hadamard type for (α, s) -convex and (α, s, m) -convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

Keywords: integral inequality, (α, s) -convex function, (α, s, m) -convex function, Hermite-Hadamard type, beta function, gamma function, polygamma function.

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1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([24]). For $f : [0, b] \rightarrow \mathbb{R}_0 = [0, \infty)$ with $b > 0$ and some $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y),$$

for $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

Definition 1.3 ([12]). For $f : [0, b] \rightarrow \mathbb{R}_0$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y),$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

Definition 1.4 ([4, 9]). Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be s -convex (in the second sense) if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y),$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5 ([28]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.6 ([33]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}_0$ is said to be extended (s, m) -convex if

$$f(tx + m(1 - t)y) \leq t^s f(x) + m(1 - t)^s f(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

The famous Hermite-Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

Theorem 1.1. If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 1.2 ([5]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

Theorem 1.3 ([10, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|)$$

and

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} \right. \\ &\quad \left. + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}. \end{aligned}$$

For more information on integral inequalities of the Hermite-Hadamard type for various kinds of convex functions, please refer to the monograph [6, 8, 13, 14], to recently published papers [2, 3, 5, 7, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32], and to the closely related references therein.

In this paper, we introduce two new concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and present inequalities of Hermite-Hadamard type for functions whose twice differentiation are of (α, s, m) -convexity.

2. Two definitions and two lemmas

We introduce the notions of “ (α, s) -convex function” and “ (α, s, m) -convex function”.

Definition 2.1. *For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be (α, s) -convex if*

$$f(tx + (1-t)y) \leq t^{\alpha s} f(x) + (1-t^{\alpha})^s f(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2. *For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, s, m) -convex if*

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^{\alpha})^s f(y),$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2, we see that

1. if $s = 1$, then $f(x)$ is an (α, m) -convex function on $(0, b]$;
2. if $\alpha = 1$, then $f(x)$ is an extended (s, m) -convex function on $(0, b]$;
3. if $\alpha = m = 1$, then $f(x)$ is an extended s -convex function on $(0, b]$;
4. if $\alpha = s = m = 1$, then $f(x)$ is a convex function on $(0, b]$.

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite-Hadamard type for (α, s) -convex and (α, s, m) -convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I$ with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\begin{aligned} \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx &= \frac{(b-a)^2}{16} \\ &\times \int_0^1 t(1-t) \left[f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right) \right] \, dt. \end{aligned}$$

Proof. Integrating by parts and changing variable of integral give

$$\begin{aligned} &\int_0^1 t(1-t) f''\left(at + (1-t)\frac{a+b}{2}\right) \, dt \\ &= \frac{2}{b-a} \int_0^1 (1-2t) f'\left(at + (1-t)\frac{a+b}{2}\right) \, dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{8}{(b-a)^2} \int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right) \, dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{16}{(b-a)^3} \int_a^{(a+b)/2} f(x) \, dx \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 t(1-t) f''\left(t\frac{a+b}{2} + (1-t)b\right) \, dt \\ &= \frac{4}{(b-a)^2} \left[f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{16}{(b-a)^3} \int_{(a+b)/2}^b f(x) \, dx. \end{aligned}$$

The proof of Lemma 2.1 is complete. $\square$

Lemma 2.2. *Let $\alpha \in (0, 1]$ and $s \in [-1, 1]$. Then*

1. when $s \in (-1, 1]$, we have

$$\begin{aligned} (2.1) \quad M(\alpha, s) &\triangleq \int_0^1 t(1-t)(1-t^\alpha)^s \, dt \\ &= \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]; \end{aligned}$$

2. when $s = -1$, we have

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right],$$

where $\Gamma(x)$, $B(x, y)$, and $\psi(x)$ are the classical Euler gamma, beta, and psi functions defined respectively by

$$(2.2) \quad \Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

and

$$(2.3) \quad \psi(x) = \frac{d \ln \Gamma(x)}{dx} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right] dt$$

for $\Re(x), \Re(y) > 0$.

Proof. Let $u = t^\alpha$ for $t \in [0, 1]$. If $s \in (-1, 1]$, we have

$$\begin{aligned} M(\alpha, s) &= \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{\alpha} \int_0^1 (u^{2/\alpha-1} - u^{3/\alpha-1})(1-u)^s du = \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]. \end{aligned}$$

When $s = -1$, from the formulas

$$\psi(z) + \gamma = \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \quad \text{and} \quad \gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

in [1, p. 259, 6.3.22], it is easily to deduce

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \int_0^1 \frac{u^{2/\alpha-1} - u^{3/\alpha-1}}{1-u} du = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right].$$

The proof of Lemma 2.2 is complete. $\square$

3. Integral inequalities of Hermite-Hadamard type

Now we start out to establish some new integral inequalities of the Hermite-Hadamard type for (α, s, m) -convex functions.

Theorem 3.1. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \\ (3.1) \quad & \times \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + m M(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + m M(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. By Lemma 2.1 and Hölder's integral inequality, we obtain

$$\begin{aligned}
 & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\
 (3.2) \quad & \left. + \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \leq \frac{(b-a)^2}{16} \\
 & \times \left\{ \left[\int_0^1 t(1-t) dt \right]^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\
 & \left. + \left(\int_0^1 t(1-t) dt \right)^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\}.
 \end{aligned}$$

From the (α, s, m) -convexity of $|f''|^q$ and Lemma 2.2, we arrive at

$$\begin{aligned}
 & \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \\
 & \leq \int_0^1 t(1-t) \left(t^{\alpha s} |f''(a)|^q + m(1-t)^{\alpha s} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right) dt \\
 & = \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + m M(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q
 \end{aligned}$$

and

$$\begin{aligned}
 (3.3) \quad & \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \\
 & \leq \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + m M(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q.
 \end{aligned}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete. $\square$

Corollary 3.1. For $\alpha \in (0, 1]$ and $s \in (-1, 1]$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$, then

$$\begin{aligned}
 & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + M(\alpha, s) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\
 & \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + M(\alpha, s) |f''(b)|^q \right]^{1/q} \right\},
 \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. This is a special case of Theorem 3.1 for $m = 1$. $\square$

Theorem 3.2. For $(\alpha, m) \in (0, 1]^2$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is $(\alpha, -1, m)$ -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This follows from similar argument to Theorem 3.1. $\square$

Corollary 3.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an $(\alpha, -1)$ -convex function on I for $\alpha \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(2-\alpha)(3-\alpha)} |f''(a)|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This is a special case of Theorem 3.2 for $m = 1$. $\square$

Theorem 3.3. For $(\alpha, m) \in (0, 1]^2$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q > 1$ and $q \geq \ell$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By Lemma 2.1, Hölder's integral inequality, and the (α, s, m) -convexity of $|f''|^q$, one has

$$\begin{aligned}
& \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left\{ \left[\int_0^1 t^\ell \left| f'' \left(at + (1-t)\frac{a+b}{2} \right) \right|^q dt \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 t^\ell \left| f'' \left(t\frac{a+b}{2} + (1-t)b \right) \right|^q dt \right]^{1/q} \right\} \\
& \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\
& \quad \times \left\{ \left[\int_0^1 t^\ell \left(t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right) dt \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 t^\ell \left(t^{\alpha s} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} \\
& = \frac{(b-a)^2}{16} \left[B \left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\
& \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} \right. \\
& \quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}.
\end{aligned}$$

The proof of Theorem 3.3 is thus complete. $\square$

Corollary 3.3. Under assumptions of Theorem 3.3, if $\ell = 0$, then

$$\begin{aligned}
& \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16} \\
& \quad \times \left[B \left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \right. \right. \\
& \quad \left. \left. \times \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[\frac{|f''(\frac{a+b}{2})|^q}{\alpha s + 1} + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\},
\end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Corollary 3.4. For $\alpha \in (0, 1]$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$

is an (α, s) -convex function on I for $q \geq 1$ and $q \geq \ell$, then

$$\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) |f''(b)|^q \right]^{1/q} \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. This is a special case of Theorem 3.3 for $m = 1$. $\square$

Theorem 3.4. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $b^* > 0$. If $f \in L_1([a, b])$ for $a, b \in (0, b^*]$ and $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}} dx$$

and

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max \left\{ \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right), \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{a}{m}\right) + \frac{1}{\alpha s + 1} f(b) \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By the (α, s, m) -convexity of f , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}} \int_0^1 \left[f(ta + (1-t)b) + m(2^\alpha - 1)^s f\left(\frac{(1-t)a + tb}{m}\right) \right] dt.$$

Letting $x = ta + (1-t)b$ or $x = (1-t)a + tb$ for all $t \in [0, 1]$ leads to

$$\int_0^1 f(c[ta + (1-t)b]) dt = \int_0^1 f(c[(1-t)a + tb]) dt = \frac{1}{b-a} \int_a^b f(cx) dx$$

for $c \in \mathbb{R}$.

On the other hand, when putting $x = ta + (1-t)b$ for all $t \in [0, 1]$, by the (α, s, m) -convexity of f , we can gain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \int_0^1 \left[t^{\alpha s} f(a) + m(1-t^\alpha)^s f\left(\frac{b}{m}\right) \right] dt \\ &= \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right). \end{aligned}$$

The proof of Theorem 3.4 is complete. $\square$

Corollary 3.5. *Under conditions of Theorem 3.4, if $m = 1$, then*

$$\begin{aligned} \frac{2^{\alpha s}}{(2^{\alpha} - 1)^s + 1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \max\left\{\frac{f(a)}{\alpha s + 1} + \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(b), \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(a) + \frac{f(b)}{\alpha s + 1}\right\}. \end{aligned}$$

Remark 3.1. This paper is a corrected and revised version of the preprint [27].

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Robust non-negative matrix factorization for subspace learning

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Abstract. Conventional non-negative matrix factorization methods fail to clustrering due to many noisy data. Therefore, this paper proposes a novel non-negative matrix factorization method to learn a robust subspace from the noisy space. Considering matrix completion and non-negative matrix factorization, the proposed method can recover the contaminated data and learn a low-dimensional effective subspace. Experiments on the ORL face dataset with Salt & Pepper noise show that our proposed method is effective and robust.

Keywords: non-negative matrix factorization, matrix completion, robustness, clustering.

1. Introduction

Non-negative matrix factorization (NMF) [1] can discover a non-negative properties from a high-dimensional data space. Therefore, NMF has received considerable attention in data mining [2, 3], signal processing [4] and computer vision [5]. NMF aims to decompose the original data matrix into two low-dimensional

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matrices such that their product approximates the data matrix. However, NMF cannot learn an effective low-dimensional subspace due to the noisy data.

Many variant NMF models have shown the robustness and effectiveness for subspace learning. L_1 -NMF [6] was proposed to handle the Laplace noise by minimizing the non-smooth L_1 norm function, however its related optimization has the defect of slow convergence. To solve this problem, Manhattan NMF [7] is proposed to optimize L_1 -NMF by transforming the non-smooth function into a smooth one and solving it by Nesterov's optimization method [8]. Zhang et al. [9] proposed robust non-negative matrix factorization to recover the undamaged matrix by subtracting one error matrix from the damaged data matrix. Although previous variants have made contribution to handle noises, they fail to learn a robust subspace with considerable contaminated data.

Recently, with the popularity of compressed sensing (CS) theory [10, 11], matrix completion (MC) [12] has been paid considerable attention. When a few entries of the original data matrix are missing, corrupted or contaminated, MC can recover the damaged data from the other undamaged data. At present, MC has applied many areas such as image inpainting [13], data denosing [14] and recommended system [15].

In this paper, a novel method called non-negative matrix completion factorization (NMCF) is proposed to learn a robust and effective subspace from a contaminated face dataset. The advantages of NMCF are to recover the noisy data from MC and learn a robust subspace from NMF. The optimization algorithm of NMCF is designed according to the block-coordinate-descent-method [16] and Nesterov's optimization method [17]. Finally, the image inpainting and clustering on the contaminated ORL dataset demonstrate the effectiveness and robustness of NMCF.

2. Robust non-negative matrix factorization

2.1 Non-negative matrix factorization

Supposed that $\{v_i\}_{i=1}^n \in R^m$ is the feature description of i -th image and $V = [v_1, \dots, v_n] \in R^{m \times n}$ is an image matrix. By NMF, V is decomposed into two non-negative matrices $W \in R^{m \times r}$ and $H \in R^{r \times n}$ such that their product approximates V . Commonly, there are many loss functions to minimize the difference, however it is not the main topic in this paper. We propose the Euclidean distance to measure the reconstruction error. Hence, the optimization function of NMF is as follows:

$$(1) \quad \min_{W, H} \quad F(W, H) = \frac{1}{2} \|V - WH\|_F^2$$

$$s.t. \quad W \geq 0, H \geq 0.$$

It is obvious that (1) are not convex in optimizing W and H simultaneously. The common optimization structure for (1) is to optimize W and H iteratively until convergence. Based on this structure, Lee and Seung [18] provided a classical

algorithm for (1). The Multipartite Update algorithm minimizing (1) can be summarized as

$$(2) \quad W_{ik} \leftarrow W_{ik} \frac{(VH^T)_{ik}}{(WHH^T)_{ik}},$$

$$(3) \quad H_{kj} \leftarrow H_{kj} \frac{(W^TV)_{kj}}{(W^TW H^T)_{kj}}.$$

2.2 Matrix completion

Given a damaged image matrix $M \in R^{m \times n}$, each column of M is an image vector and most of pixels in each image may be damaged. MC aims to recover the damaged pixels from other undamaged pixels. MC can be viewed as an optimization problem in the following form:

$$(4) \quad \min_X \quad \text{rank}(X)$$

$$(5) \quad s.t. \quad P_\Omega(M) = P_\Omega(X),$$

where $X \in R^{m \times n}$ is the image inpainting matrix, Ω is the damaged area, and $P_\Omega(M_{ij})$ is defined by

$$(6) \quad P_\Omega(M_{ij}) = \begin{cases} M_{ij}, & (i, j) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

2.3 Non-negative matrix completion factorization

NMF is a dimensionality reduction method which is useful for clustering, signal processing and image recognition. However, it performs unsatisfactorily in dimensionality reduction when the original images are damaged. In this section, a novel matrix factorization method, called nonnegative matrix completion factorization (NMCF), which is used for dimensionality reduction. The proposed method can learn a good parts-based representation from the damaged images.

Consider a damaged grayscale image matrix $M \in R^{m \times n}$ consisting of n damaged images, among which the pixel values 0 or 255 are the damaged area. Hence, a weight matrix $S \in R^{m \times n}$ corresponding to the damaged or undamaged area can be described as

$$(7) \quad S_{ij} = \begin{cases} 0, & M(i, j) \text{ is the damaged pixel} \\ 1, & \text{otherwise.} \end{cases}$$

Supposed that each entry of an estimate recovered matrix $V \in R^{m \times n}$ can be recovered by M_{ij} . According to the definition of S in (7), the following term

can be used to measure the approximation between M_{ij} and V_{ij} :

$$\begin{aligned}
 (8) \quad R &= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n (M_{ij} - V_{ij})^2 \times S_{ij} \\
 &= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n ((M_{ij} - V_{ij}) \times S_{ij})^2 \\
 &= \frac{1}{2} \| (V - M) \otimes S \|_F^2.
 \end{aligned}$$

By minimizing (8), we expect if any point M_{ij} is damaged, M_{ij} can be recovered to be V_{ij} . Given a damaged image matrix $M \in R^{m \times n}$, our proposed method aims to find its corresponding recovered matrix V , and to decompose V into two non-negative matrices W and H . Non-negative matrix completion factorization (NMCF) can be minimized by Euclidean distance as follows:

$$\begin{aligned}
 (9) \quad \min_{V, W, H} \quad & F(V, W, H) = \frac{1}{2} (\| V - WH \|_F^2 + \| (V - M) \otimes S \|_F^2) \\
 \text{s.t.} \quad & V \geq 0, W \geq 0, H \geq 0.
 \end{aligned}$$

3. Algorithm minimizing non-negative matrix completion factorization

In this section, we propose Nesterov's method to optimize (9) instead of the Multipartite Update method. Firstly, we propose the block-coordinate-descent method to divide (9) into three optimization problems. Secondly, the Nesterov's optimization theory is applied to optimize each sub-problem. Thirdly, we optimize each sub-problem alternatively until convergence.

Because (9) is not a convex problem, it is difficult to obtain the global solution. Fortunately, the block-coordinate-descent method can search a local solution. This approach alternatively optimizes one block of variables with the remaining blocks fixed. For NMCF, there are three block variables V , W and H . Hence, the block-coordinate-descent method alternatively minimizes the following three optimization problems:

$$\begin{aligned}
 (10) \quad \min_V \quad & F(V) = \frac{1}{2} (\| V - WH \|_F^2 + \| (V - M) \otimes S \|_F^2) \\
 \text{s.t.} \quad & V \geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 (11) \quad \min_W \quad & F(W) = \frac{1}{2} \| V - WH \|_F^2 \\
 \text{s.t.} \quad & W \geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 (12) \quad \min_H \quad & F(H) = \frac{1}{2} \| V - WH \|_F^2 \\
 \text{s.t.} \quad & H \geq 0.
 \end{aligned}$$

According to Lemma 1 and Lemma 2, (10), (11) and (12) can be easily solved by the optimal gradient method [17].

Lemma 1 ([17]). The objective functions $F(V)$, $F(W)$ and $F(H)$ are convex.

Proof. Given any $V_1, V_2 \in R^{m \times n}$ and $\lambda \in (0, 1)$, we have

$$\begin{aligned} & F(V_1 + (1 - \lambda)V_2) - (\lambda F(V_1) + (1 - \lambda)F(V_2)) \\ = & -\frac{\lambda(1 - \lambda)}{2}(\|V_1 - V_2\|_F^2 + \|(V_1 - V_2) \otimes S\|_F^2) \leq 0. \end{aligned}$$

Based on the definition of convex function, it is clear that $F(V)$ is convex. According to Lemma 1 in [17], we know that $F(W)$ and $F(H)$ are convex. This completes the proof.

Lemma 2 ([17]). The gradients of $F(V)$, $F(W)$ and $F(H)$ are differentiable and Lipschitz continuous. Their Lipschitz constants are $\|I + S\|_F$, $\|HH^T\|$ and $\|WW^T\|$, separately.

Proof. According to (10), we can get the gradient of $F(V)$ by

$$\nabla_V F(V) = V - WH + (V - M) \otimes S.$$

For any V_1 and V_2 , we have

$$\begin{aligned} & \|\nabla_V F(V_1) - \nabla_V F(V_2)\|_F \\ = & \|(V_1 - V_2) \otimes (I + S)\|_F \\ = & \sum_{i=1}^m \sum_{j=1}^n \{(I + S)_{ij}(V_1 - V_2)_{ij}\} \\ \leq & \sum_{i=1}^m \sum_{j=1}^n (I + S)_{ij} \sum_{i=1}^m \sum_{j=1}^n (V_1 - V_2)_{ij} \\ \leq & \|I + S\|_F \|V_1 - V_2\|_F, \end{aligned}$$

where $I \in R^{m \times n}$ is a matrix whose entries are all one. Hence, $F(V)$ is differentiable and Lipschitz continuous, and its Lipschitz constant is $\|I + S\|_F$. According to (11) and (12), we can get the gradients of $F(W)$ and $F(H)$ by

$$\begin{aligned} \nabla_W F(W) &= WHH^T - VH^T, \\ \nabla_H F(H) &= W^TWH - W^TV. \end{aligned}$$

According to [17], $F(W)$ and $F(H)$ have been proved to be differentiable and Lipschitz continuous. This completes the proof.

Consider the following optimization problem:

$$(13) \quad \min_{X \geq 0} F(X),$$

where $X \in R^{m \times n}$ and $F : R^{m \times n} \rightarrow R$. Recent researches [8, 19] show that Nesterov's optimal gradient method optimizing problems can achieve the convergence rate $O(\frac{1}{k^2})$. Since $F(X)$ is convex and Lipschitz continuous, Nesterov's method can easily optimize (13). Generally, two sequences Y_k and X_k are alternatively updated in each iteration. For $k \geq 1$, we have

$$\begin{aligned} X_k &= \arg \min_{X \geq 0} \{ \phi_k(Y_k, X) \\ (14) \quad &= F(X) + \langle \nabla F(X), X - Y_k \rangle + \frac{L}{2} \|X - Y_k\|_F^2 \}, \end{aligned}$$

$$(15) \quad Y_{k+1} = X_k + \frac{\alpha_k - 1}{\alpha_{k+1}} (X_k - X_{k-1}),$$

where $\phi_k(Y_k, X)$ is the proximal function, L is the Lipschitz constant of $F(X)$, $\langle \cdot, \cdot \rangle$ denotes the sum of the element-wise multiplication of two matrices and $\alpha_{k+1} = \frac{1 + \sqrt{4\alpha_k^2 + 1}}{2}$. Using the Lagrange multiplier method and K.K.T. conditions, (14) can be solved by

$$(16) \quad X_k = \max(Y_k - \frac{1}{L} \nabla F(X), 0).$$

By updating X_k , α_{k+1} and Y_{k+1} iteratively until convergence, the optimal solution of (13) can be obtained. Based on above analysis, we summarize Nesterov's method to optimize (9) in Algorithm 1.

4. Experiments

In this section, two experiments are implemented on the ORL face image database [20] to demonstrate the effectiveness and robustness of our proposed method. Firstly, Salt & Pepper noise is considered to corrupt $\%p$ pixels in all face images ($p = 5, 20, 35, 50$). Secondly, we apply NMCF and NMF to recover the corruptions. Thirdly, we evaluate the effectiveness of NMCF and NMF by clustering the low-dimensional subspace of the damaged images.

The first experiment is to evaluate the robustness of NMCF and NMF in recovering the damaged images. We verify whether NMCF and NMF have the capacity to deal with Salt & Pepper noise. Let $m = 1024, n = 400, r = 50, n = 20$ and *number* = 100. Figure 1 mainly shows contaminated face images and recovered images by NMCF and NMF. In each sub-figure, the first two groups of images are the example images and contaminated images, and the last two groups of images are the recovered images by NMCF and NMF, separately. By contrast, NMCF can recover images when different percentages of pixels are contaminated. However, NMF fails to recover contaminated images when face images have more contaminated pixels.

The second experiment is to evaluate effectiveness of parts-based representation. After dimensional reduction by NMCF and NMF, we can achieve low-dimensional features for clustering. We compare the clustering effect by Kmeans

Algorithm 1 NMCF**Input:** $V, W, H, n, number$ **Output:** V^*, W^*, H^*

```

for  $i = 0$  to  $number$  do
  1.  $Y_0 = V, \alpha_0 = 1, L = \|I + S\|_F$ 
  for  $k = 0$  to  $n$  do
    2.  $V_k = \max(Y_k - \frac{1}{L} \nabla_V F(V), 0)$ 
    3.  $\alpha_{k+1} = \frac{1 + \sqrt{4\alpha_k^2 + 1}}{2}$ 
    4.  $Y_{k+1} = V_k + \frac{\alpha_k - 1}{\alpha_{k+1}} (V_k - V_{k-1})$ 
  end for
  5.  $V = V_k$ 
  6.  $Y_0 = W, \alpha_0 = 1, L = \|HH^T\|_2$ 
  for  $k = 0$  to  $n$  do
    7.  $W_k = \max(Y_k - \frac{1}{L} \nabla_W F(W), 0)$ 
    8.  $\alpha_{k+1} = \frac{1 + \sqrt{4\alpha_k^2 + 1}}{2}$ 
    9.  $Y_{k+1} = W_k + \frac{\alpha_k - 1}{\alpha_{k+1}} (W_k - W_{k-1})$ 
  end for
  10.  $W = W_k$ 
  11.  $Y_0 = H, \alpha_0 = 1, L = \|WW^T\|_2$ 
  for  $k = 0$  to  $n$  do
    12.  $H_k = \max(Y_k - \frac{1}{L} \nabla_H F(H), 0)$ 
    13.  $\alpha_{k+1} = \frac{1 + \sqrt{4\alpha_k^2 + 1}}{2}$ 
    14.  $Y_{k+1} = H_k + \frac{\alpha_k - 1}{\alpha_{k+1}} (H_k - H_{k-1})$ 
  end for
  15.  $H = H_k$ 
end for
16.  $V^* \leftarrow V, W^* \leftarrow W, H^* \leftarrow H$ 

```

,NMCF and NMF in terms of two metrics including Normalized Mutual Information (NMI) and Accuracy (AC) [21]. The final clustering results on the contaminated dataset are presented in Table 1. It is clear that our proposed NMCF performs more robustly and effectively when the percentage of the contaminated pixels is increasing.

5. Conclusion

This paper proposes a non-negative matrix completion factorization method to learn a robust and effective subspace from the contaminated data. The loss function based on NMF and MC is optimized by Nesterov's methods, hence the convergence speed of the proposed optimization problem is very fast. According to experiments on the ORL face dataset with Salt & Pepper noise, there are two

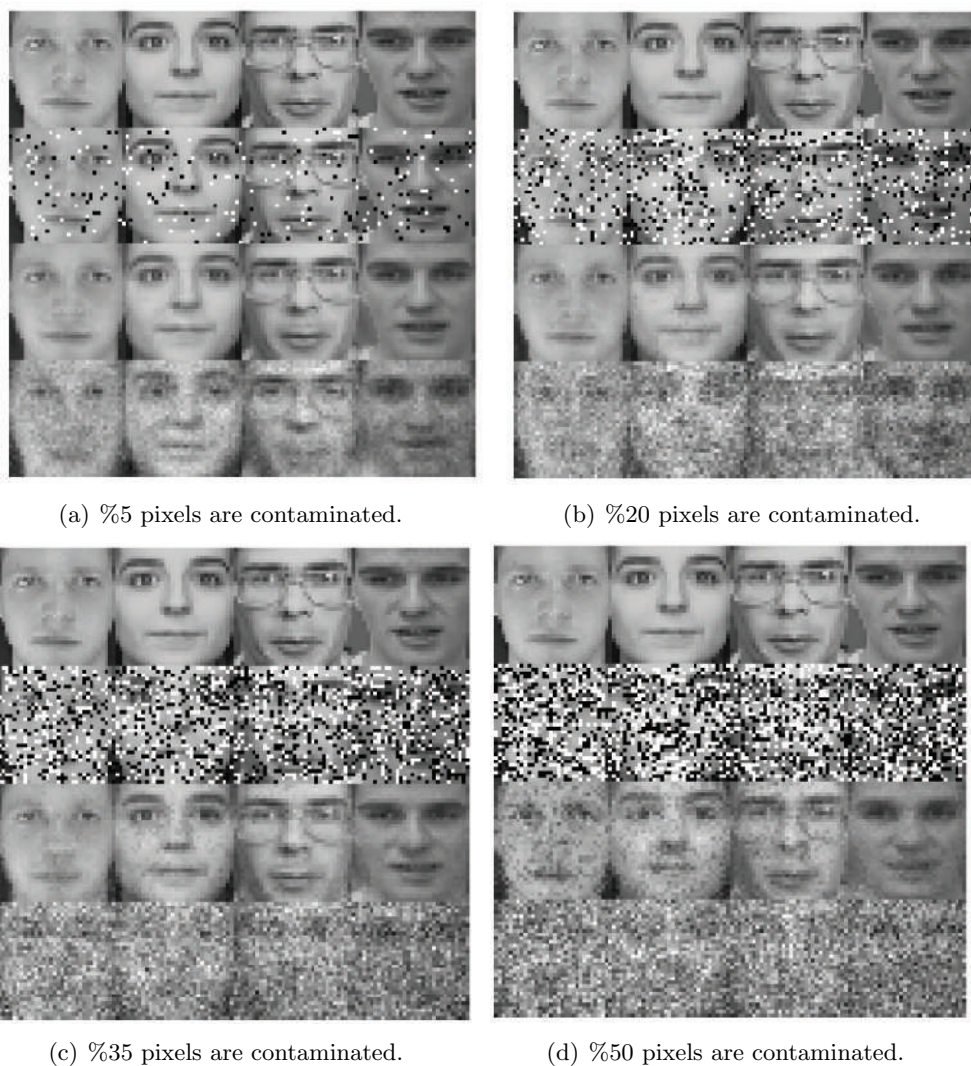


Figure 1. Contaminated and recovered samples.

Table 1. Clustering performance on contaminated images by Kmeans, NMCF and NMF

	NMI				Accuracy			
	p=5	p=20	p=35	p=50	p=5	p=20	p=35	p=50
Kmeans	76.28	70.95	56.96	46.02	57.00	51.50	35.00	22.00
NMCF	77.42	78.10	75.44	68.49	60.50	63.50	58.00	48.25
NMF	76.90	51.46	40.19	39.60	62.00	63.50	21.00	18.75

significant advantages in our proposed method. Firstly, the contaminated pixels can be repaired from other uncontaminated pixels. Secondly, a low-dimensional

effective subspace from the contaminated face dataset is learned and the clustering effect is better than NMF and Kmeans.

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Convolution conditions for q -Sakaguchi-Janowski type functions

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Abstract. In this paper, we derived convolution conditions for Sakaguchi-Janowski type functions. Those results contains some interesting corollaries as special cases.

Keywords: Janowski functions, Sakaguchi functions, convolution, q -derivative.

1. Introduction and definitions

Bieberbach conjecture, Milin conjecture, Robertson conjecture, Sheil-small conjecture, Rogosinski conjecture, Littlewood conjecture, etc, attracted eminent mathematicians to work in the theory of univalent functions. Attempts to prove or disprove these conjectures inspired research not only to develop elegant and useful techniques in complex analysis but also led to the introduce and study of various subclasses of univalent functions. Functions with positive real part play a crucial role in Geometric Function Theory as its significance can be seen from that all the simple subclasses of the class of univalent functions have been

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defined by using this concept. Motivated by this class Janowski [6] defined the class $P(A, B)$.

Let $\mathcal{A}$ denote the class of functions of form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and normalized by $f(0) = f'(0) - 1 = 0$. Let Ω denote the class of analytic functions ω in $\mathcal{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$. We denote by $P(A, B)$ the Janowski class containing functions p of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad -1 \leq B < A \leq 1, \quad \omega \in \Omega.$$

Definition 1.1. The convolution or Hadamard product, of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n (|z| < R_1) \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n (|z| < R_2),$$

is defined as the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad |z| < R_1 R_2.$$

The fractional q -calculus is the extension of the ordinary fractional calculus in the q -theory. The theory of q -calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q -difference and q -integral equations, and in q -transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [1, 2, 3, 9], and [15].

In early twentieth century, Jackson [4] initiated q -calculus and developed the concept of the q -derivative and q -integral.

Definition 1.2. For a function $f \in \mathcal{S}$ given by (1) and $0 < q < 1$, the q -derivative of f is defined by

$$(2) \quad D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}, \quad \text{where } (0 < q < 1)$$

Equivalently (2), may be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Note that as $q \rightarrow 1^-$, $[n]_q \rightarrow n$. For a function $f(z) = z^n$, we can observe that

$$D_q f(z) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}.$$

Then

$$\lim_{q \rightarrow 1^-} D_q f(z) = \lim_{q \rightarrow 1^-} [n]_q z^{n-1} = n z^{n-1} = f'(z),$$

where $f'(z)$ is the ordinary derivative. As a right inverse, Jackson [5] presented the q -integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(z q^n),$$

provided that the series converges.

Under the hypothesis of the definition of q -difference operator, we have the following rules:

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$, where a and b any real (or complex) constants;

(ii) $D_q(f(z)g(z)) = g(qz)D_q f(z) + f(z)D_q g(z) = f(z)D_q g(z) + D_q f(z)g(qz)$;

(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}$.

Now, we introduce the following classes of analytic functions

(i) $f \in S_q^*(A, B, t)$ if $\frac{(1-t)zD_q f(z)}{f(z)-f(tz)} \in P(A, B)$;

(ii) $f \in K_q(A, B, t)$ if and only if $zD_q f \in S_q^*(A, B, t)$;

(iii) $f \in S_q^{\lambda}(A, B, t)$ if $\frac{e^{i\lambda} \frac{(1-t)zD_q f(z)}{f(z)-f(tz)} - i \sin \lambda}{\cos \lambda} \in P(A, B)$;

(iv) $f \in K_q^{\lambda}(A, B, t)$ if and only if $zD_q f \in S_q^{\lambda}(A, B, t)$;

where in (iii) and (iv) λ is real and $|\lambda| < \frac{\pi}{2}$.

Remark 1.3. (i) As $q \rightarrow 1^-$, $t = 0$, $A = 1 - 2\alpha$ and $B = -1$, we get the class $S^*(\alpha, 1, t)$ introduced by Owa [8].

(ii) As $q \rightarrow 1^-$, $t = -1$, $A = 1 - 2\alpha$ and $B = -1$, we get the class $S^*(\alpha, 1, -1)$ introduced by Sakaguchi [12].

2. Main results

Theorem 2.1. The function $f \in K_q(A, B, t)$ in $|z| < R \leq 1$ if and only if

$$\frac{1}{z} \left[f * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0,$$

$|z| < R$, $|x| = 1$, where $u_n(t) = \sum_{j=1}^n t^{j-1}$ and $|x| = 1$.

Proof. The function $f \in K_q(A, B, t)$ if and only if

$$\frac{(1-t)D_q(zD_qf(z))}{D_qf(z) - D_qf(tz)} \neq \frac{1+Ax}{1+Bx}, \quad (|z| < R, |x| = 1, Bx \neq -1)$$

which implies

$$(3) \quad (1-t)D_q(zD_qf(z))(1+Bx) - (1+Ax)(D_qf(z) - D_qf(tz)) \neq 0.$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_qf = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

$$D_q(zD_qf) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (3) is equivalent to

$$(1+Bx) \left[D_qf * \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \right] - D_qf * \sum_{n=1}^{\infty} u_n(t)(1+Ax)z^{n-1}$$

$$= D_qf * \sum_{n=1}^{\infty} [(1+Bx)[n]_q - u_n(t)(1+Ax)] z^{n-1}$$

$$= D_qf * \left(\frac{-u_n(t)(1+Ax)}{1-z} + \frac{1+Bx}{(1-z)(1-qz)} \right)$$

$$= D_qf * \left(\frac{(1+Bx) - (1-qz)u_n(t)(1+Ax)}{(1-z)(1-qz)} \right).$$

Thus

$$(4) \quad \frac{1}{z} \left[zD_qf * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2}{(1-z)(1-qz)} \right] \neq 0.$$

Since $zD_qf * g = f * zD_qg$, we can write (4) as

$$\frac{1}{z} \left[f * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0,$$

$|z| < R$, $|\rho| = 1$, which completes the proof. $\square$

As $q \rightarrow 1^-$ and $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.2. *The function $f \in K(A, B)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z} \left[f * \frac{xz + \frac{(Ax+Bx+2)}{B-A} z^2}{(1-z)^3} \right] \neq 0.$$

Remark 2.3. As $q \rightarrow 1^-$, $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing convex functions as in Silverman et al. in [13] with a suitable modification.

Theorem 2.4. The function $f \in S_q^*(A, B, t)$ in $|z| < R \leq 1$ if and only if

$$\frac{1}{z} \left[f * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2}{(1-z)(1-qz)} \right] \neq 0, \quad (|z| < R, |x| = 1),$$

where $u_n(t) = \sum_{j=1}^n t^{j-1}$.

Proof. Since $f \in S_q^*(A, B, t)$ if and only if $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d_q \zeta \in K_q(A, B, t)$, we have

$$\begin{aligned} & \frac{1}{z} \left[g * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \\ &= \frac{1}{z} \left[f * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2}{(1-z)(1-qz)} \right]. \end{aligned}$$

Thus the result follows from Theorem 2.1. $\square$

As $q \rightarrow 1^-$, $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.5. The function $f \in S^*(A, B)$ in $|z| < R \leq 1$ if and only if

$$\frac{1}{z} \left[f * \frac{xz + \frac{1+Ax}{B-A} z^2}{(1-z)^2} \right] \neq 0, \quad (|z| < R, |x| = 1).$$

As a corollary we can derive coefficient inequalities for the class $S_q^*(A, B, t)$.

Corollary 2.6. A function $f \in \mathcal{A}$ is in the class $S_q^*(A, B, t)$ if and only if

$$f(z) = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

where $A_n = \frac{([n]_q - u_n(t)) + ([n]_q B - A u_n(t)) \rho}{\rho(B-A)} a_n$ and $u_n(t) = \sum_{j=1}^n t^{j-1}$.

Proof. A function $f \in S_q^*(A, B, t)$ if and only if

$$\frac{(1-t)zD_q f(z)}{f(z) - f(tz)} \neq \frac{1+A\rho}{1+B\rho}.$$

That is

$$(1+B\rho)(1-t)(zD_q f(z)) - (1+A\rho)[f(z) - f(tz)] \neq 0$$

which implies

$$(B - A)\rho z \left[1 + \sum_{n=2}^{\infty} [n]_q (1 + B\rho) - u_n(t)(1 + A\rho) \right] a_n z^n \neq 0.$$

This simplifies into

$$1 + \sum_{n=2}^{\infty} \frac{([n]_q - u_n(t))([n]_q B - A u_n(t))\rho}{\rho(B - A)} a_n z^{n-1} \neq 0,$$

which completes the proof. $\square$

Remark 2.7. As $q \rightarrow 1^-$, $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing starlike functions as in Silverman et al. in [13] with a suitable modification.

Theorem 2.8. For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have $f \in K_q^\lambda(A, B, t)$ if and only if

$$\frac{1}{z} \left[f * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)} qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)} qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0,$$

$|z| < R$, $|x| = 1$, where $\phi = (A \cos \lambda + iB \sin \lambda) e^{-i\lambda}$ and $u_n(t) = \sum_{j=1}^n t^{j-1}$.

Proof. We have, $f \in K_q^\lambda(A, B, t)$ if and only if

$$\frac{e^{i\lambda} \frac{(1-t)D_q(zD_q f(z))}{D_q f(z) - D_q f(tz)} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + Ax}{1 + Bx}, \quad (|z| < R, |x| = 1, x \neq -1)$$

which implies

$$(5) \quad (1-t)(D_q(zD_q f(z))(1+Bx) - (1+\phi x)(D_q f(z) - D_q f(tz))) \neq 0.$$

Proceeding exactly as in Theorem 2.1 we arrive at the result by replacing A by ϕ . $\square$

As $q \rightarrow 1^-$, $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.9. For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have

$$e^{i\lambda} \frac{(zf'(z))'}{f'(z)} \in P(A, B)$$

if and only if

$$\frac{1}{z} \left[f * \frac{xz + \left(x + \frac{2(1+\rho e^{i\lambda}(A \cos \lambda + iB \sin \lambda))}{B - e^{-i\lambda}(A \cos \lambda + iB \sin \lambda)} \right) z^2}{(1-z)^3} \right] \neq 0.$$

Theorem 2.10. For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have $f \in S_q^{*\lambda}(A, B, t)$ if and only if

$$\frac{1}{z} \left[f * \frac{z + \frac{([2]_q - 1)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)}qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)}qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0, |z| < R, |x| = 1,$$

where $\phi = (A \cos \lambda + iB \sin \lambda) e^{-i\lambda}$ and $u_n(t) = \sum_{j=1}^n t^{j-1}$.

Proof. The result follows from Theorem 2.8 in the same manner that Theorem 2.4 followed from Theorem 2.1. $\square$

For $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.11. For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have

$$e^{i\lambda} \frac{zf'(z)}{f(z)} \in P(A, B)$$

if and only if

$$\frac{1}{z} \left[f * \left(\frac{[B - e^{i\lambda}(A \cos \lambda + iB \sin \lambda)]xz + [1 + xe^{i\lambda}(A \cos \lambda + iB \sin \lambda)]z^2}{(1-z)^2} \right) \right] \neq 0.$$

As a corollary we can derive coefficient inequalities for the class $S_q^{*\lambda}(A, B, t)$.

Corollary 2.12. A function $f \in \mathcal{A}$ is in the class $S_q^{*\lambda}(A, B, t)$ if and only if

$$f(z) = 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0,$$

where $d_n = \frac{([n]_q - u_n(t)) + ([n]_q B - \gamma u_n(t))\rho}{\rho(B-A)} a_n$ and $\gamma = (A \cos \lambda + iB \sin \lambda) e^{-i\lambda}$.

Proof. A function $f \in S_{\lambda}^*(A, B, t)$ if and only if

$$\frac{e^{i\lambda} \frac{(1-t)zD_q f(z)}{f(z)-f(tz)} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + A\rho}{1 + B\rho}.$$

That is

$$(1 + B\rho)(1 - t)(zD_q f(z)) - (1 + \gamma\rho)[f(z) - f(tz)] \neq 0.$$

The rest of the proof follows as in Corollary 2.6. $\square$

Remark 2.13. As $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing spirallikeness of functions as in Silverman et al. in [13].

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Characterization of 1-uniform dcsl graphs and learning graphs

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Abstract. A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \rightarrow 2^X$, X being a non empty ground set, such that the corresponding induced function $f^\oplus : E(G) \rightarrow 2^X \setminus \{\emptyset\}$ given by $f^\oplus(u, v) = f(u) \oplus f(v)$ satisfies $|f^\oplus(u, v)| = k_{(u,v)}^f d_G(u, v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ denotes the path distance between u and v and $k_{(u,v)}^f$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen. A dcsl f of G is k -uniform if all the constants of proportionality with respect to f are equal to k , and if G admits such a dcsl then G is called a k -uniform dcsl graph. Let $\mathcal{F}$ be a family of subsets of a set X . A graph G is defined to be a learning graph, if it is a $\mathcal{F}$ -induced graph of some learning space $\mathcal{F}$. In this paper, we characterize 1-uniform dcsl learning graphs and discuss the embedding problems.

Keywords: dcsl graphs, 1-uniform dcsl graphs, wg-family of sets, learning graphs.

1. Introduction

Throughout this paper by a graph we mean a connected, finite, simple graph. Unless otherwise mentioned, for all terminology in graph theory the reader is referred to [4]. Acharya [1] introduced the notion of vertex set valuation as a set analogue of number valuation. For a graph $G = (V, E)$ and a non empty

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set X , Acharya defined a set valuation of G as an injective set valued function $f : V(G) \rightarrow 2^X$, and he defined a set-indexer as a set valuation such that the function $f^\oplus : E(G) \rightarrow 2^X \setminus \{\emptyset\}$ given by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X .

Acharya and Germina,[2] introduced the particular kind of set valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them [2]. In other words, they discussed the question whether one can determine those graphs $G = (V, E)$ that admit an injective function $f : V \rightarrow 2^X$, X being a non empty ground set such that the cardinality of the symmetric difference $f^\oplus(uv)$ is proportional to the usual path distance $d_G(u, v)$ between u and v in G , for each pair of distinct vertices u and v in G . They called f a *distance compatible set labeling* (dcsl) of G , and the ordered pair (G, f) , a distance compatible set labeled (dcsl) graph. Thus

Definition 1.1 ([2]). Let $G = (V, E)$ be any connected graph. A distance compatible set labeling (dcsl) of a graph G is an injective set assignment $f : V(G) \rightarrow 2^X$, X being a non empty ground set, such that the corresponding induced function $f^\oplus : E(G) \rightarrow 2^X \setminus \{\emptyset\}$ given by $f^\oplus(u, v) = f(u) \oplus f(v)$ satisfies $|f^\oplus(u, v)| = k_{(u,v)}^f d_G(u, v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ denotes the path distance between u and v and $k_{(u,v)}^f$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen.

A distance compatible set labeling f of G is k -uniform if all the constants of proportionality with respect to f in Definition 1.1 are equal to k , and if G admits such a distance compatible set labeling then G is called a k -uniform distance compatible set labeled graph.

Listed below are the definitions and known results which are used in this paper.

Definition 1.2 ([7]). A family of sets $\mathcal{F}$ is well-graded if any two sets in $\mathcal{F}$ can be connected by a sequence of sets formed by single element insertion and deletion, without redundant operations, such that all intermediate sets in the sequence belong to $\mathcal{F}$.

Definition 1.3 ([7]). Let $\mathcal{F}$ be a family of subsets of a set X . A tight path between two distinct sets P and Q (or from P to Q) in $\mathcal{F}$ is a sequence $P_0 = P, P_1, P_2 \dots P_n = Q$ in $\mathcal{F}$ such that $d(P, Q) = |P \oplus Q| = n$ and $d(P_i, P_{i+1}) = 1$ for $0 \leq i \leq n - 1$. The family $\mathcal{F}$ is well-graded family (or wg-family), if there is a tight path between any two of its distinct sets.

Definition 1.4 ([7]). Any family $\mathcal{F}$ of subsets of X defines a graph $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$, where $E_{\mathcal{F}} = \{\{P, Q\} \subseteq \mathcal{F} : |P \oplus Q| = 1\}$ and we call $G_{\mathcal{F}}$, an $\mathcal{F}$ -induced graph.

Definition 1.5 ([7]). A family of sets $\mathcal{F}$ is closed under union or $\bigcup$ -closed if for any nonempty $\mathcal{H} \subseteq \mathcal{F}$, we have $U\mathcal{H} \in \mathcal{F}$. The span of a family of sets $\mathcal{F}$ is the family $\mathcal{F}'$ containing any set which is the union of some nonempty subfamily of $\mathcal{F}$.

Definition 1.6 ([6]). The pair $(Q, \mathcal{K})$ in which Q is a nonempty set, and $\mathcal{K}$ is a family of subsets of Q , containing atleast Q and the empty set is called a learning space, if it satisfies the following conditions:

(1) For any two sets K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \dots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$, for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

(2) If K, L are two sets satisfying $K \subset L$ and q is an item such that $K \cup \{q\} \in \mathcal{K}$, and $q \notin L$, then $L \cup \{q\} \in \mathcal{K}$.

One may note that a well-graded family which is closed under union and containing the empty set, is a learning space. Germina et al. [10] defined a learning graph as

Definition 1.7. A graph G is said to be a learning graph, if it is an $\mathcal{F}$ -induced graph of some learning space $\mathcal{F}$.

In this paper, we discuss the structure of a 1-uniform dcsl graph whose collection of vertex labeling induces a learning space. That is, we analyse those classes of 1-uniform dcsl graphs that are also learning graphs. Following result has been established by Germina and Jinto [9].

Theorem 1.8 ([9]). *The path P_n on n vertices is a learning graph.*

One may note that the 1- uniform dcsl f of a graph G need not necessarily be unique. For example, the 1- uniform dcsl f of a path P_n is not unique. It is also to be noted that not all 1-uniform labeling of path is a learning space. Consider the 1- uniform dcsl labeling of a path $P_5 = u_1, u_2, u_3, u_4, u_5$ with the assignments of the vertices respectively as $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$ which is a 1 -uniform dcsl. However, since empty set is not being assigned to any vertex, with this labeling P_5 is not a learning graph. Again, consider the labeling of the path P_5 with the assignments of the vertices respectively as $\{1, 2\}$, $\{1\}$, $\{\emptyset\}$, $\{3\}$, $\{3, 4\}$ which is a 1 - uniform dcsl, but, is not a learning graph with respect to this labeling as collection of vertex labeling is not $\bigcup$ -closed. However, consider the 1-uniform dcsl for a path $P_n = u_1, u_2, u_3, \dots, u_n$ given as, $\{\emptyset\}$ $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\dots$, $\{1, 2, 3, 4, \dots, n-1\}$. With this assignments, path P_n is a learning graph. One may note that, there are many other classes of 1- uniform graphs that are not learning graphs. For example, even cycles C_{2n} , are 1-uniform but, C_{2n} , ($n \geq 3$) is not a learning graph.

Germina et.al. proved the following results [10]:

Theorem 1.9 ([10]). *The star graph $K_{1,n}, n \geq 3$ is not a learning graph.*

Theorem 1.10 ([10]). *If G is a 1-uniform dcsl graph with the collection of vertex labeling $\mathcal{F}$, contains empty set, then the graph $G_{Span\mathcal{F}}$ is a learning graph.*

Since all learning graphs are trivially 1-uniform, the classes of 1-uniform graphs which are learning graphs seems to be an interesting problem.

2. Main results

Towards discussing this problem, we first study the structure of $\mathcal{B}$ -induced 1-uniform distance compatible set labeled graph with $\mathcal{B}$ as the basis and, the structure of the respective $\mathcal{F}$ -induced graph.

David Eppstein et. al. [5] defined the base of a $\bigcup$ -closed family of sets as follows.

Definition 2.1 ([5]). A base of a $\bigcup$ -closed family of sets $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ of $\mathcal{F}$, spanning $\mathcal{F}$, where minimal is meant with respect to set inclusion. That is, if $Span(\mathcal{H}) = \mathcal{F}$ for some $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$.

Correspondingly, Germina and Jinto [9] defined the span of a 1- uniform dcsl graph G as

Definition 2.2 ([9]). The span of a 1- uniform dcsl graph G is the graph induced by the span of the vertex labelings of G , denoted by $SpanG$.

One may note that a family $\mathcal{B}$, spanning a family $\mathcal{F}$, is a base of $\mathcal{F}$, if and only if, none of the sets in $\mathcal{B}$ is the union of some other sets in $\mathcal{B}$. Let $\mathcal{F}$ be a $\bigcup$ -closed wg-family, and $\mathcal{B}$ be a base of $\mathcal{F}$. Then $\mathcal{B}$ need not necessarily be a wg-family and hence $G_{\mathcal{B}}$, the $\mathcal{B}$ -induced graph in general, need not necessarily be a 1- uniform dcsl graph. In particular, the $\mathcal{B}$ -induced graph may even be disconnected so that it is not 1-uniform dcsl graph.

Theorem 2.3. A tree T is a learning graph if and only if the tree is isomorphic to path P_n .

Proof. Let T be a learning tree. Let $\mathcal{B}$, be the basis so that none of the sets in $\mathcal{B}$ is the union of some other sets in $\mathcal{B}$ and let $\mathcal{F}$ be a $\bigcup$ -closed wg-family of sets with $\mathcal{B}$ as basis. Let $G_{\mathcal{F}}$, be the $\mathcal{F}$ -induced graph, which is a well graded graph and is 1-uniform dcsl, say with 1 - uniform dcsl f . $G_{\mathcal{F}}$, being a well graded graph, there exists a tight path between every pair of vertices u and v of T . Consider any such tight path say $P_k = u_1, u_2, \dots, u_k$ of length k in a tree T . Hence, the labeling of P_k with respect to which P_k , is a tight path is either of the following:

(i) $f(u_1) = \{\emptyset\}$, $f(u_2)$ is any singleton set, $f(u_3)$ is any two element set containing $f(u_2)$. Proceeding in this order with $f(u_k)$ is assigned with a set of order k in such a way that, $f(u_k)$ contains $f(u_{k-1})$ and the assignment is minimal with respect to set inclusion.

Or (ii) $f(u_1)$ is assigned any singleton set, $f(u_2)$ any two element set containing $f(u_1)$; $f(u_3)$ is any three element set containing $f(u_2) \dots$, in the order with $f(u_k)$ is assigned with a set of order k in such a way that, $f(u_k)$ contains $f(u_{k-1})$ and the assignment is minimal with respect to set inclusion.

Since T is a learning tree, $\emptyset$ should necessarily be assigned to one of the vertices of T .

Case (i): One possibility is that $\emptyset$ is assigned to a pendent vertex say, u_1 of T . That is $f(u_1) = \{\emptyset\}$. Now consider the vertices in the 1-neighbourhood of vertex u_1 . T being a learning graph, all these vertices in the 1-neighbourhood of u_1 should necessarily be assigned with singleton sets, say $\{1\}, \{2\}, \dots, \{k\}$, where k is the degree of u_1 . Then, all the vertices in the 2- neighbourhood of u_1 should necessarily be assigned with sets of cardinality two containing the singleton set to which they are adjacent. Denote these vertices that are adjacent to vertices with assignment as singleton sets, by $\{u_{11}, u_{12}, \dots, u_{1x_i}\}$. Among these vertices that are adjacent to the vertex with labeling $\{1\}$ shall necessarily receive the assignments as $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \dots, \{1, x_j\}$, where $x_j = \deg(u_{11}) - 1$;

Those vertices that are adjacent to the vertex with labeling $\{2\}$ should necessarily receive the assignments as $\{2, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \dots, \{2, x_l\}$, where $x_l = \deg(u_{12}) - 1$;

Those vertices that are adjacent to the vertex with labeling $\{3\}$ shall receive the assignments as $\{3, 4\}, \{3, 5\}, \{3, 6\}, \dots, \{3, x_k\}$, where $x_k = \deg(u_{13}) - 1$.

Proceeding this way of assignment to the vertices in the immediate neighbourhoods shall necessarily receive the labeling with respect to the pre-assigned vertex. Also note that the assignment followed is minimal with respect to set inclusion. With this assignment that occurred in a natural way, we reach a contradiction either to the fact that the set assignment is not U -closed or the symmetric difference between the set assignments of pair of vertices in T is not equal to the distance between the vertices. Hence, in this case, for T to be a learning graph, $T \cong P_n$.

Case (ii): $\emptyset$ is assigned to a non pendent vertex say, v of T . As in Case(i), we should necessarily give an assignment to the vertices in the first neighbourhood of v , with singleton sets; to the vertices in the second neighbourhood of v with sets of cardinality two containing the singleton set, the assignment of the vertex to which it is adjacent. And then proceed to the next immediate neighbourhood vertices and give the assignments following the rule that the assignment followed is minimal with respect to set inclusion. In this case also, we reach a contradiction either to the fact that the set assignment is not $\cup$ -closed or the symmetric difference between the set assignments of pair of vertices in T is not equal to the distance between the vertices. Hence, for T to be a learning graph, then, $T \cong P_n$.

Conversely, let $T \cong P_n$, a path of length n . Let $P_n = u_1, u_2, \dots, u_n$. Then assign the vertices $\{u_1, u_2, \dots, u_n\}$ respectively as $\{\emptyset\}, \{1\}, \{1, 2\}, \dots, \{1, 2, 3, \dots, n\}$, with respect to which P_n is a learning graph. $\square$

Theorem 2.4. A learning tree T is isometrically embeddable in a partial cube of dimension equal to the length of the eccentric path of the tree T .

Proof. Let T be a learning tree. Let $\mathcal{B}$ be a basis and $\mathcal{G}_{\mathcal{B}}$ be the $\mathcal{B}$ -induced graph. Hence, $\mathcal{B}$ contains all unions and is minimal with respect to inclusion. Invoking Theorem 2.3, $\mathcal{G}_{\mathcal{B}}$ is a 1-uniform dcsl path. Being a 1-uniform dcsl graph, $\mathcal{G}_{\mathcal{B}}$ is bipartite and so is the span $\mathcal{F}$. The span $\mathcal{F}$ is always 1-uniform and there exists a unique vertex u such that $f(u) = \{\emptyset\}$ and such that all $u - v$ paths are tight. Now, if T is a learning graph, there exists exactly one such tight path and hence is embeddable in a partial cube. Now consider the path of maximum length in T , which is nothing but the eccentric path of tree T . By choosing the initial vertex of the eccentric path and assigning it with $\emptyset$ and proceeding as in Theorem 2.3, we get an assignment with respect to which the tree T is a learning graph. Hence, a learning tree T can be isometrically embedded in a partial cube of dimension equal to the length of the eccentric path of the tree. $\square$

As we have already discussed, not all 1-uniform dcsl graphs are learning graphs. The 1-uniform dcsl f of a graph G need not necessarily be unique. Hence, we expect 1-uniform dcsl for a given graph, with respect to which the assignment may or may not induce a learning graph. Hence, a 1-uniform graph with a 1-uniform dcsl f is a learning graph only when one of the vertices receives emptyset as an assignment and the given assignment is union closed. That is, learning graph not only depend on the structure of a graph, but also, depend on the specific assignment chosen. Hence, given a 1-uniform dcsl f for a given graph G , characterizing the 1-uniform dcsl f with respect to which G is a learning graph seems to be an interesting problem.

Germina et. al. [10] proved the following theorems

Theorem 2.5. The base $\mathcal{B}$ -induced graph $G_{\mathcal{B}}$ is 1-uniform distance compatible set labeled graph if and only if for every pair of vertices $u, v \in V(G_{\mathcal{B}})$ with $f(u) = \{\emptyset\}$, there exist a tight path between u and v .

Theorem 2.6. If G is a 1-uniform dcsl graph. Then, G is a learning graph if and only if there exists a collection $\{\mathcal{F}\}$, of vertex labeling of G , with $\emptyset \in \{\mathcal{F}\}$ which is well graded.

Proof. Setting, $\{f(V(G))\} = \{\mathcal{F}\}$, where f is the 1-uniform dcsl of G , a learning graph. Then G being a learning graph, $\{\mathcal{F}\}$ is well graded and necessarily, $\emptyset \in \{\mathcal{F}\}$.

Conversely, let if $\emptyset \in \{\mathcal{F}\}$ and $\{\mathcal{F}\}$, is a well graded family of sets. Being well graded family of sets, for every pair of vertices (u, v) in G , there exists a tight $u - v$ path, and $\{\mathcal{F}\}$, is $\cup$ -closed. Hence, setting $\{f(V(G))\} = \{\mathcal{F}\}$, G is a learning graph. $\square$

Theorem 2.7. Let $\mathcal{B}$ be a basis with $\emptyset \in \mathcal{B}$ and let $\mathcal{G}_{\mathcal{B}}$ be the induced graph induced by the basis $\mathcal{B}$. Then, classes of $\mathcal{F}$ -induced graph $G_{\mathcal{F}}$ in which the base

$\mathcal{B}$ -induced graph is a learning graph and is isomorphic to a bipartite graph that are embedable in a partial cube.

Proof. Let $\mathcal{B}$ be a basis with $\emptyset \in \mathcal{B}$ and $\mathcal{G}_{\mathcal{B}}$ be the $\mathcal{B}$ -induced graph. Hence, $\mathcal{B}$ contains all unions and is minimal with respect to inclusion. Assume $\mathcal{G}_{\mathcal{B}}$ is 1-uniform dcsl. Being a 1-uniform dcsl, $\mathcal{G}_{\mathcal{B}}$ is bipartite and so is the span $G_{\mathcal{F}}$ with $\{\emptyset\}$ as an assignment of one of the vertex. The span $\mathcal{F}$ always induces a 1-uniform dcsl and there exists a unique vertex u such that $f(u) = \{\emptyset\}$ and that all $u - v$ paths are tight. Also, the $\mathcal{F}$ induced graph $G_{\mathcal{F}}$ which is 1-uniform is a bipartite graph. Also, if $G_{\mathcal{F}}$ is a learning graph, invoking Theorem 2.6, and Theorem 2.7, G is isometrically embedable in a partial cube. $\square$

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Solutions of nonlinear equations to describe physical models in plasma

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Abstract. In this article, the solutions of the models that describe the behavior of one of the physical generators in plasma were obtained using soliton solutions. The behavior of ions and electrons in the plasma, and their relationship to the magnetic field, was demonstrated using the Korteweg–de Vries (KDV) equation and analyzed for its dispersion of the solitons obtained from the KDV equation solutions.

Keywords: degenerate pressure, degenerate ions, electrons fluid, reductive perturbation method, soliton solution.

1. Introduction

Plasma physics is the key to modern-day technology in applications such as electronics, plasma ions, and plasma lasers. The returns of plasma science have a great direct effect on industry and on the economy, with ample studies and attention dedicated to the field [1]-[12].

It is also very clear from the existing research that the acquisition of energy from thermal fusion reactions in plasma is not far-fetched. This will lead to increased interest and active participation in this area. From the recent results, it has been proven that there is an urgent need to build nuclear fusion reactors because their capabilities far exceed traditional sources. They even surpass the capacity of nuclear fission, all while maintaining a pure, safe, and clean source of energy [7, 8, 9, 10, 11, 12].

There are also many applications and uses, such as access to ozone gas, separation of isotopes, printing techniques, manufacture of generators, manufacture of thin films, development of communication devices and controls, and television screens, among others [1, 2, 3, 4, 5, 6].

Crookes released a fourth meaning of the term “plasma” to describe the confirmed part of gas permeability. Solids are transformed into plasma by heating through the liquid and gas states. When the atoms forming the gas ionize at temperatures greater than 100,000 Galvin, the material becomes uniform/homogenous; this is plasma. Langmuir and Tonks described plasma as a

semi-electrolytic fluid composed of charged particles and other moderates that behave collectively. If the force produced by a loudspeaker generates sound waves, the charged particles in their motion form local clusters of positive or negative charges, which in turn generate electric fields and thus generate magnetic fields from the movement of molecules. Nuclear fission does not produce any radioactive products that pollute the environment of nuclear fusion.

Interactions through the magnetic and electric fields generate waves that represent waves of sound in a low-pressure ionized liquid. To avoid the dispersion of waves caused by the presence of ions and the fluctuation of electrons, we the behavior of the movements of ions and electrons in the absence of a magnetic field was investigated through equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

2. Physical models in plasma

Conservation mass and momentum to ion fluid within a one-dimensional space system [1, 2, 3, 4, 5], which is given by

$$(1) \quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_i) = 0,$$

$$(2) \quad m_i n_i \left(\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} \right) + \frac{\partial p_i}{\partial x} = e n_i E$$

where n_i and n_e are the density number of ions and electrons, p_i and p_e are their pressures, v_i and v_e are their velocities, m_i is the mass of an ions, E is the electric field, and e is its charge. If we neglect its pressure, i.e. $p_i = 0$, then one has [4, 6]

$$(3) \quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_i) = 0,$$

$$(4) \quad m_i \left(\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} \right) = e E.$$

While conservation of mass and momentum to the electron fluid system is given by

$$(5) \quad \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} (n_e v_e) = 0,$$

$$(6) \quad m_e n_e \left(\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} \right) + \frac{\partial p_e}{\partial x} = -e n_e E$$

where $-e$ is the charge of an electron and we denote its mass by m_e . Noting that we can neglect the inertia of electrons (i.e. $m_e = 0$) and the reason for this

is that the electrons are much lighter than ions. Hence, Eq. (6) can be reduced to

$$(7) \quad \frac{\partial p_e}{\partial x} = -en_e E.$$

Assuming an isothermal state of the electron fluid, the integration of Eq. (7) yields to following equation

$$(8) \quad p_e = kTn_e$$

where k and T represent the Boltzmann's constant and the temperature, respectively. Inserting Eq. (8) into Eq. (7) we get

$$(9) \quad kT \frac{\partial n_e}{\partial x} = en_e \frac{\partial \phi}{\partial x}$$

where $E = -\frac{\partial \phi}{\partial x}$. Using Eq. (9), one observe that n_e can be represent in terms of ϕ as

$$(10) \quad n_e = n_0 \exp\left(\frac{e\phi}{kT}\right)$$

in which the constant n_0 denoted to the density number of electrons at $\phi = 0$.

The Maxwell model to electrostatic field E can be generated by a charge density σ , which can be given by

$$\epsilon_0 \nabla \cdot E = \sigma,$$

where ϵ_0 is a dielectric constant so that

$$(11) \quad \epsilon_0 \frac{\partial E}{\partial x} = e(n_i - n_e).$$

Eq. (11) can be represent in terms of ϕ as

$$(12) \quad -\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\epsilon_0} (n_i - n_e).$$

Note that, the number density of the electrons n_e can be eliminate from Eq. (12) using Eq. (10). By dropping the ion-superscript from the ion-variables (n_i, n_e) , it follows the improved system of equations in term of (n, v, ϕ) such that

$$(13) \quad \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0,$$

$$(14) \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{e}{m} \frac{\partial \phi}{\partial x} = 0,$$

$$(15) \quad -\frac{\partial^2 \emptyset}{\partial x^2} + \frac{e n_0}{\epsilon_0} \exp\left(\frac{e \emptyset}{k T}\right) = \frac{e}{\epsilon_0} n.$$

By defining the Debye length, λ_0 , and the ion-acoustic sound speed, c_0 , respectively as follows:

$$(16) \quad \lambda_0^2 = \frac{\epsilon_0 k T}{n_0 e^2}; \quad c_0^2 = \frac{k T}{m}.$$

Then, it can be non-dimensionalize the Eqs. (13)-(15). According to the conditions, such parameters λ_0 and c_0 will be different in plasma. For instance, in a dense laboratory plasma, one can get that $n_0 \approx 10^{20} m^{-3}$, $T \approx 60.000 K$, and $\lambda_0 \approx 10^{-6} m$, while in the solar wind near the earth, one can get that $n_0 \approx 10^7 m^{-3}$, $T \approx 120.000 K$, and $\lambda_0 \approx 10 m$.

Define the dimensionless variables

$$(17) \quad \bar{x} = \lambda_0^{-1} x, \quad \bar{t} = \lambda_0^{-1} c_0 t, \quad \bar{n} = \frac{n}{n_0}, \quad \bar{v} = \frac{v}{c_0}, \quad \bar{\emptyset} = \frac{e \emptyset}{k T}, \quad \rho = \exp(\emptyset).$$

then drop the "bars", we have the following non-dimensional equations

$$(18) \quad \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0,$$

$$(19) \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial \emptyset}{\partial x} = 0,$$

$$(20) \quad \frac{\partial^2 \emptyset}{\partial x^2} + n = p.$$

3. Solution

We studied the properties of soliton ion acoustic waves in plasma. According to Washimi and Taniuti [9], the reductive perturbation approach can be applied to the main equations. Consequently, the coordinates of space and time are stretched in the sense of the formed relations as follows:

$$(21) \quad \xi = \sqrt{\epsilon} (x - \lambda t), \quad \tau = \sqrt{\epsilon^3} t,$$

where ϵ is a small positive parameter and λ is the wave velocity.

Using Eq. (21), Eqs. (18)–(20) become

$$(22) \quad -\lambda \frac{\partial n}{\partial \xi} + \frac{\partial}{\partial \xi} (nv) + \epsilon \frac{\partial n}{\partial \tau} = 0,$$

$$(23) \quad -\lambda \frac{\partial v}{\partial \xi} + \frac{\partial \emptyset}{\partial \xi} + v \frac{\partial v}{\partial \xi} + \epsilon \frac{\partial v}{\partial \tau} = 0,$$

$$(24) \quad p - \epsilon \frac{\partial^2 \emptyset}{\partial \xi^2} = n.$$

Thus, we need the asymptotic solution of Eqs. (22)–(24) of the form

$$(25) \quad n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \dots,$$

$$(26) \quad v = \epsilon v_1 + \epsilon^2 v_2 + \dots,$$

$$(27) \quad \emptyset = 1 + \epsilon \emptyset_1 + \epsilon^2 \emptyset_2 + \dots,$$

$$(28) \quad \emptyset = 1 + \epsilon \emptyset_1 + \epsilon^2 \left(\emptyset_2 + \frac{\emptyset_1^2}{2} \right) + \dots$$

Equating the coefficients of ϵ and ϵ^2 of the Taylor expansions in Eqs.(25)–(28) we find that

$$(29) \quad \frac{\partial v_1}{\partial \xi} - \lambda \frac{\partial n_1}{\partial \xi} = 0,$$

$$(30) \quad -\lambda \frac{\partial v_1}{\partial \xi} + \frac{\partial \emptyset_1}{\partial \xi} = 0,$$

$$(31) \quad \emptyset_1 - n_1 = 0.$$

Equating the coefficients of ϵ^2 , we find that

$$(32) \quad \frac{\partial v_2}{\partial \xi} - \lambda \frac{\partial n_2}{\partial \xi} + \frac{\partial}{\partial \xi} (n_1 v_1) + \frac{\partial n_1}{\partial \tau} = 0,$$

$$(33) \quad -\lambda \frac{\partial v_2}{\partial \xi} + \frac{\partial \emptyset_2}{\partial \xi} + v_1 \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial \tau} = 0,$$

$$(34) \quad \emptyset_2 - n_2 + \frac{1}{2} \emptyset_1^2 + \frac{\partial^2 \emptyset_1}{\partial \xi^2} = 0.$$

Taking Φ_1 from Eqs.(29)–(31), we have the following homogeneous linear system for (n_1, v_1) :

$$(35) \quad \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} \frac{\partial n_1}{\partial \xi} \\ \frac{\partial v_1}{\partial \xi} \end{pmatrix} = 0.$$

The system (35) has a nontrivial solution when $\lambda^2 = 1$. From definiteness and accordingly to a right-moving wave, we suppose that $\lambda = 1$. Then, one get

$$(36) \quad \begin{pmatrix} n_1 \\ v_1 \end{pmatrix} = \varphi(\xi, \tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \emptyset_1 = \varphi(\xi, \tau),$$

where $\varphi(\xi, \tau)$ is an arbitrary scalar-valued function. Setting $\lambda = 1$ and eliminating Φ_2 from Eqs. (32)–(34), we get the following non-homogeneous linear system for (n_2, v_2) :

$$(37) \quad \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} \frac{\partial n_2}{\partial \xi} \\ \frac{\partial v_2}{\partial \xi} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \xi} (n_1 v_1) + \frac{\partial n_1}{\partial \tau} \\ -\emptyset_1 \frac{\partial \emptyset_1}{\partial \xi} + \frac{\partial^3 \emptyset_1}{\partial \xi^3} + v_1 \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial \tau} \end{pmatrix} = 0.$$

The system (37) can be solved in term of (n_2, v_2) if and only if the non-homogeneous terms are orthogonal to the null vector $(1, 1)$. It can be deduced from this condition using Eq. (6) that $\varphi(\xi, \tau)$ satisfies a KDV equation:

$$(38) \quad \varphi_\tau + \varphi \varphi_\xi + \frac{1}{2} \varphi_{\xi\xi\xi} = 0.$$

It's worth to mention that the linearized dispersion relation of Eq. (38) together with original system agrees with a long-wave expansion. If it satisfies Eq. (38), then we can solve Eq. (37) in term of (n_2, v_2) . However, the solution will be the sum of non-homogeneous solution with arbitrary multiple $\varphi_2(\xi, \tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of the homogeneous. Hence, the asymptotic solution can be found in the same manner. Further, for the order ϵ^k , the non-homogeneous solution of a linear equation can be obtained in term of (n_k, v_k) as the following form

$$(39) \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial n_k}{\partial \xi} \\ \frac{\partial v_k}{\partial \xi} \end{pmatrix} + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix} = 0,$$

where f_{k-1} and g_{k-1} are depending on $(n_1, v_1), \dots, (n_{k-1}, v_{k-1})$, and Φ_k can be expressed explicitly in terms of $n_1, \dots, n_k$. The condition $f_{k-1} + g_{k-1} = 0$ ensures that this equation can be solved in term of (n_k, v_k) that satisfied whenever φ_{k-1} satisfies a suitable equation. Therefore, the solution in term of (n_k, v_k) involves an arbitrary function of integration φ_k . An equation for φ_k follows formula of solvability condition at the $(k+1)$ -order equations. Using the sine-cosine method, it can obtain a traveling wave solution for the KDV equation (38) by setting the wave variable $\theta = \xi - b\tau$, where b is a constant, which is reduced Eq. (38) into the following ordinary differential equation:

$$(40) \quad -b\varphi' + \varphi\varphi' + \frac{1}{2}\varphi''' = 0,$$

where $e' = \frac{d}{d\theta}$. Integrating Eq. (40) once, and considering the integration constant as zero, we can find

$$(41) \quad -b\varphi + \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi'' = 0.$$

We take [10, 11, 12]

$$(42) \quad \varphi(\theta) = a \cos^m(\mu\theta),$$

$$(43) \quad \varphi'(\theta) = -a\mu m \cos^{m-1}(\mu\theta) \sin(\mu\theta),$$

$$(44) \quad \varphi''(\theta) = -a\mu^2 m^2 \cos^m(\mu\theta) + a\mu^2 m(m-1) \cos^{m-2}(\mu\theta).$$

Substituting Eqs. (42)–(44) into Eq. (41) yields

$$(45) \quad -\left(ab + \frac{1}{2}a\mu^2 m^2\right) \cos^m(\mu\theta) + \frac{1}{2}a\mu^2 m(m-1) \cos^{m-2}(\mu\theta) + \frac{1}{2}a^2 \cos^{2m}(\mu\theta) = 0.$$

The following algebraic system can be obtained by equating the exponents and coefficients of each pair of cosine functions:

$$(46) \quad m \neq 0, 2m = m-2, ab + \frac{1}{2}a\mu^2 m^2 = 0, \frac{1}{2}a^2 + \frac{1}{2}a\mu^2 m(m-1) = 0.$$

From system (46), we obtain

$$(47) \quad m = -2, \quad \mu = \sqrt{\frac{-b}{2}}, \quad a = 3b.$$

Then, we obtain the formal solitary wave solutions for Eq. (38)

$$(48) \quad \varphi(\xi, \tau) = 3b \operatorname{sech}^2 \frac{1}{2} \left(\sqrt{2b}(\xi - b\tau) \right), \quad b > 0$$

or

$$(49) \quad \varphi(\xi, \tau) = 3b \operatorname{sec}^2 \frac{1}{2} \left(\sqrt{-2b}(\xi - b\tau) \right), \quad b < 0.$$

Then, we can write the solution of the system (18)–(20) as

$$(50) \quad \begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3b \operatorname{sech}^2 \frac{1}{2} \left(\sqrt{2b}(\xi - b\tau) \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\epsilon^2)$$

or

$$(51) \quad \begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3b \operatorname{sec}^2 \frac{1}{2} \left(\sqrt{-2b}(\xi - b\tau) \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\epsilon^2).$$

In summary, the leading-order asymptotic solution of the system (18)–(20) as $\epsilon \rightarrow 0^+$ is

$$(52) \quad \begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \epsilon \varphi \left(\sqrt{\epsilon}(x-t), \sqrt{\epsilon^3}t \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\epsilon^2).$$

We expect this asymptotic solution to be valid for long times, in the order

$$\tau = O(1) \quad \text{or} \quad t = O(\epsilon^{-\frac{3}{2}}).$$

4. Conclusion

Recently, the new modes of waves associated with plasma have become an attractive topic for many researchers. In this effort, an analytical plasma model was applied showing that the nonlinear evolution of charge fluctuations has a significant effect on the instability of charged ions. Moreover, a new simulation model that can be used for studying the unique physical properties of waves in plasmas, caused by charging effects, was established.

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Integral inequalities of Hermite-Hadamard type for extended (s, m) -GA- ε -convex functions

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Abstract. In the paper, by introducing the concept of (s, m) -GA- ε -functions and using Hölder's integral inequality, the authors develop several new integral inequalities of the Hermite-Hadamard type for extended (s, m) - ε -GA-convex functions.

Keywords: Hermite-Hadamard's integral inequality, convex function, extended (s, m) - ε -GA-convex function.

1. Introduction

Throughout this paper, we use the following notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

Let us recall and review some concepts of convex functions.

Definition 1.1 ([13]). *A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called to be convex on an interval I if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

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Definition 1.2 ([2]). For some $s \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called to be s -convex in the second sense on an interval I if

$$(1.1) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.3 ([23]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called to be extended s -convex on a interval I if the inequality (1.1) holds for all $x, y \in I$ and $\lambda \in (0, 1)$.

Definition 1.4 ([19]). For some $m \in (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}$ is called to be m -convex on $[0, b]$ if

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda f(x) + m(1 - \lambda)f(y)$$

for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

Definition 1.5 ([5, 15]). For some $(s, m) \in (0, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}$ is called to be (s, m) -convex on $[0, b]$ if

$$(1.2) \quad f(\lambda x + m(1 - \lambda)y) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y)$$

for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

Definition 1.6 ([27]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}$ is called to be extended (s, m) -convex on $[0, b]$ if the inequality (1.2) is valid for all $x, y \in [0, b]$ and $\lambda \in (0, 1)$.

Definition 1.7 ([14]). A function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is called to be geometrically-arithmetically-convex (or say, GA-convex) on I if

$$f(x^\lambda y^{1-\lambda}) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.8 ([16]). For $\varepsilon \geq 0$, a function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is called to be ε -GA-convex on I if

$$f(x^\lambda y^{1-\lambda}) \leq \lambda f(x) + (1 - \lambda)f(y) + \varepsilon$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.9 ([11, 12]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : (0, b] \rightarrow \mathbb{R}$ is called to be extended (s, m) -GA-convex on $(0, b]$ if

$$f(x^\lambda y^{m(1-\lambda)}) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y)$$

for all $x, y \in (0, b]$ and $\lambda \in (0, 1)$.

The well-known integral Hermite-Hadamard inequality for convex functions reads that, if f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$

In recent decades, plenty of integral inequalities of the Hermite-Hadamard type for various kinds of convex functions have been established in, for example, [1, 3, 4, 6, 7, 8, 17, 20, 22, 25, 26]. In particular, some integral inequalities of the Hermite-Hadamard type for geometrically convex functions and for s -GA-convex functions were studied in [9, 10, 18, 21, 24, 28].

2. A new definition and two lemmas

Now we extend Definition 1.9 by introducing the definition of extended (s, m) - ε -GA-convex functions.

Definition 2.1. For some $(s, m) \in [-1, 1] \times (0, 1]$ and $\varepsilon \geq 0$, a function $f : (0, b] \rightarrow \mathbb{R}_0$ is called to be extended (s, m) - ε -GA-convex on $(0, b]$ if

$$f(x^\lambda y^{m(1-\lambda)}) \leq \lambda^s f(x) + m(1-\lambda)^s f(y) + \varepsilon$$

for all $x, y \in (0, b]$ and $\lambda \in (0, 1)$.

It is clear that, when $\varepsilon = 0$ in Definition 2.1, one will derive Definition 1.9.

In order to prove our main results, we need the following lemmas.

Lemma 1. Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be differentiable on the interior I° and $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $\lambda \in (0, 1)$, then

$$\begin{aligned} & f(a^{1-\lambda}b^\lambda) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \\ &= (\ln b - \ln a) \left[\int_0^\lambda t a^{1-t} b^t f'(a^{1-t}b^t) \, dt - \int_\lambda^1 (1-t) a^{1-t} b^t f'(a^{1-t}b^t) \, dt \right]. \end{aligned}$$

Proof. Integrating by part and changing variables of integration yield

$$\begin{aligned} & (\ln b - \ln a) \left[\int_0^\lambda t a^{1-t} b^t f'(a^{1-t}b^t) \, dt - \int_\lambda^1 (1-t) a^{1-t} b^t f'(a^{1-t}b^t) \, dt \right] \\ &= \int_0^\lambda t f'(a^{1-t}b^t) \, d(a^{1-t}b^t) - \int_\lambda^1 (1-t) f'(a^{1-t}b^t) \, d(a^{1-t}b^t) \\ &= \lambda f(a^{1-\lambda}b^\lambda) + (1-\lambda) f(a^{1-\lambda}b^\lambda) - \int_0^1 f(a^{1-t}b^t) \, dt \\ &= f(a^{1-\lambda}b^\lambda) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx. \end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. Let $x, y > 0$, $\lambda \in (0, 1)$, and $s \in [-1, 1]$. Then

$$F(x, y; \lambda) = \int_0^\lambda tx^ty^{1-t} dt = \begin{cases} \frac{\lambda y^{1-\lambda}[L(x^\lambda, y^\lambda) - x^\lambda]}{\ln y - \ln x}, & x \neq y; \\ \frac{\lambda^2 x}{2}, & x = y, \end{cases}$$

$$\begin{aligned} G(x, y; \lambda, s) &= \int_0^\lambda t[tx + (1-t)y]t^s dt \\ &= \frac{(s+2)\lambda x + (s+3 - (s+2)\lambda)y}{(s+2)(s+3)} \lambda^{s+2}, \end{aligned}$$

and

$$\begin{aligned} H(x, y; \lambda, s) &= \int_0^\lambda t[tx + (1-t)y](1-t)^s dt \\ &= \begin{cases} \frac{2x + (s+1)y - [(1-\lambda)(s+1)(1+2\lambda+\lambda s)y + (2+\lambda(s+1)(2+2\lambda+\lambda s)x)](1-\lambda)^{s+1}}{(s+1)(s+2)(s+3)}, & -1 < s \leq 1; \\ \frac{\lambda^2(y-x) - 2(\ln(1-\lambda) + \lambda)x}{2}, & s = -1, \end{cases} \end{aligned}$$

where the logarithmic mean $L(x, y)$ is defined as

$$L(x, y) = \begin{cases} \frac{y-x}{\ln y - \ln x}, & x \neq y; \\ x, & x = y. \end{cases}$$

3. Main results

We are now in a position to establish some integral inequalities of the Hermite-Hadamard type for functions of the extended (s, m) - ε -GA-convexity.

Theorem 1. Let $(s, m) \in [-1, 1] \times (0, 1]$ and $\lambda \in (0, 1)$ and let $f : (0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on $(0, b^*]$, where $a, b \in (0, b^*]$, $a < b$, $b^{1/m} \leq b^*$, and $f' \in L_1([a, b])$. If $|f'|^q$ is extended (s, m) - ε -GA-convex on $(0, \max\{b^{1/m}, b\}]$ for $q \geq 1$, then

$$\begin{aligned} &\left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ &\leq \frac{\ln b - \ln a}{6^{1/q}} [F^{1-1/q}(a, b; \lambda) (6G(a, b; \lambda, s) |f'(a)|^q \\ (3.1) \quad &+ 6mH(a, b; \lambda, s) |f'(b^{1/m})|^q + \varepsilon \lambda^2 [2\lambda a + (3-2\lambda)b]^{1/q} \\ &+ F^{1-1/q}(b, a; 1-\lambda, s) (6H(b, a; 1-\lambda, s) |f'(a)|^q \\ &+ 6mG(b, a; 1-\lambda, s) |f'(b^{1/m})|^q + \varepsilon (1-\lambda)^2 [(1+2\lambda)a + 2(1-\lambda)b]^{1/q}], \end{aligned}$$

where $F(a, b; \lambda)$, $G(a, b; \lambda, s)$, and $H(a, b; \lambda, s)$ are defined as in Lemma 2.

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned}
 & \left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\
 & \leq (\ln b - \ln a) \left[\int_0^\lambda t a^t b^{1-t} |f'(a^t b^{1-t})| dt + \int_\lambda^1 (1-t) a^t b^{1-t} |f'(a^t b^{1-t})| dt \right] \\
 (3.2) \quad & \leq (\ln b - \ln a) \left[\left(\int_0^\lambda t a^t b^{1-t} dt \right)^{1-1/q} \left(\int_0^\lambda t a^t b^{1-t} |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right. \\
 & \quad \left. + \left(\int_\lambda^1 (1-t) a^t b^{1-t} dt \right)^{1-1/q} \left(\int_\lambda^1 (1-t) a^t b^{1-t} |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right].
 \end{aligned}$$

By Lemma 2, we can obtain

$$(3.3) \quad \int_0^\lambda (1-t) a^t b^{1-t} dt = F(a, b; \lambda, s)$$

and

$$(3.4) \quad \int_\lambda^1 (1-t) a^t b^{1-t} dt = F(a, b; 1-\lambda, s).$$

Using the extended (s, m) - ε -GA-convexity of $|f'|^q$, the geometric-arithmetic mean inequality, and Lemma 2 gives

$$\begin{aligned}
 & \int_0^\lambda t a^t b^{1-t} |f'(a^t b^{1-t})|^q dt \\
 (3.5) \quad & \leq \int_0^\lambda t [ta + (1-t)b] [t^s |f'(a)|^q + m(1-t)^s |f'(b^{1/m})|^q + \varepsilon] dt \\
 & = G(a, b; \lambda, s) |f'(a)|^q + mH(a, b; \lambda, s) |f'(b^{1/m})|^q + \frac{1}{6} \varepsilon \lambda^2 [2\lambda a + (3-2\lambda)b]
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_\lambda^1 (1-t) a^t b^{1-t} |f'(a^t b^{1-t})|^q dt \leq \int_\lambda^1 (1-t) [ta + (1-t)b] [t^s |f'(a)|^q \\
 (3.6) \quad & + m(1-t)^s |f'(b^{1/m})|^q + \varepsilon] dt = H(b, a; 1-\lambda, s) |f'(a)|^q \\
 & + mG(b, a; 1-\lambda, s) |f'(b^{1/m})|^q + \frac{\varepsilon(1-\lambda)^2 [(1+2\lambda)a + 2(1-\lambda)b]}{6}.
 \end{aligned}$$

Substituting inequalities (3.3), (3.4), (3.5), and (3.6) into the inequality (3.2) results in the inequality (3.1). The proof of Theorem 1 is complete. $\square$

Corollary 1. Under conditions of Theorem 1, if $s = -1$, we have

$$\begin{aligned} & \left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \frac{\ln b - \ln a}{6^{1/q}} [F^{1-1/q}(a, b; \lambda, -1) (3(\lambda^2(a-b) + 2\lambda b) |f'(a)|^q \\ & \quad + 3m[\lambda^2(b-a) - 2a(\ln(1-\lambda) + \lambda)] |f'(b^{1/m})|^q + \varepsilon \lambda^2 [2\lambda a + (3-2\lambda)b])^{1/q} \\ & \quad + F^{1-1/q}(b, a; 1-\lambda, -1) (3[(1-\lambda)^2(a-b) - 2b(1-\lambda + \ln \lambda)] |f'(a)|^q \\ & \quad + 3m[(b-a)(1-\lambda)^2 + 2a(1-\lambda)] |f'(b^{1/m})|^q \\ & \quad + \varepsilon(1-\lambda)^2 [(1+2\lambda)a + 2(1-\lambda)b])^{1/q}]. \end{aligned}$$

Theorem 2. Let $(s, m) \in [-1, 1] \times (0, 1]$ and $\lambda \in (0, 1)$ and let $f : (0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on $(0, b^*]$, where $a, b \in (0, b^*]$, $a < b$, $b^{1/m} \leq b^*$, and $f' \in L_1([a, b])$. If $|f'|^q$ is extended (s, m) - ε -GA-convex on $(0, \max\{b^{1/m}, b\}]$ for $q > 1$, then

$$\begin{aligned} & \left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \leq \frac{\ln b - \ln a}{6^{1/q}} \left[F^{1-1/q}(a, b; \lambda) \right. \\ (3.7) \quad & \times \left(\frac{\lambda^{s+2}}{s+2} |f'(a)|^q + mT(\lambda, s) |f'(b^{1/m})|^q + \frac{1}{2} \varepsilon \lambda^2 \right)^{1/q} + F^{1-1/q}(b, a; \\ & \left. 1-\lambda, s) T(1-\lambda, s) |f'(a)|^q + \frac{m(1-\lambda)^{s+2}}{s+2} |f'(b^{1/m})|^q + \frac{\varepsilon(1-\lambda)^2}{2} \right], \end{aligned}$$

where

$$T(\lambda, s) = \int_0^\lambda t(1-t)^s dt = \begin{cases} \frac{1 - (1-\lambda)^{s+1}(1+\lambda+\lambda s)}{(s+1)(s+2)}, & -1 < s \leq 1; \\ -\lambda - \ln(1-\lambda), & s = -1. \end{cases}$$

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned} & \left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq (\ln b - \ln a) \left[\int_0^\lambda t a^t b^{1-t} |f'(a^t b^{1-t})| dt + \int_\lambda^1 (1-t) a^t b^{1-t} |f'(a^t b^{1-t})| dt \right] \\ (3.8) \quad & \leq (\ln b - \ln a) \left[\left(\int_0^\lambda t (a^t b^{1-t})^{q/(q-1)} dt \right)^{1-1/q} \left(\int_0^\lambda t |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_\lambda^1 (1-t) (a^t b^{1-t})^{q/(q-1)} dt \right)^{1-1/q} \left(\int_\lambda^1 (1-t) |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right]. \end{aligned}$$

By Lemma 2, we obtain

$$(3.9) \quad \int_0^\lambda t (a^t b^{1-t})^{q/(q-1)} dt = F(a^{q/(q-1)}, b^{q/(q-1)}; \lambda, s)$$

and

$$(3.10) \quad \int_{\lambda}^1 (1-t) (a^t b^{1-t})^{q/(q-1)} dt = F(b^{q/(q-1)}, a^{q/(q-1)}; 1-\lambda, s).$$

From the extended (s, m) - ε -GA-convexity of $|f'|^q$, we have

$$(3.11) \quad \begin{aligned} \int_0^{\lambda} t a^t b^{1-t} |f'(a^t b^{1-t})|^q dt \\ \leq \int_0^{\lambda} t [t^s |f'(a)|^q + m(1-t)^s |f'(b^{1/m})|^q + \varepsilon] dt \\ = \frac{\lambda^{s+2}}{s+2} |f'(a)|^q + mT(\lambda, s) |f'(b^{1/m})|^q + \frac{1}{2} \varepsilon \lambda^2 \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} \int_{\lambda}^1 (1-t) |f'(a^t b^{1-t})|^q dt \\ \leq \int_{\lambda}^1 (1-t) [t^s |f'(a)|^q + m(1-t)^s |f'(b^{1/m})|^q + \varepsilon] dt \\ = T(1-\lambda) |f'(a)|^q + \frac{m(1-\lambda)^{s+2}}{s+2} |f'(b^{1/m})|^q + \frac{1}{2} \varepsilon (1-\lambda)^2. \end{aligned}$$

Substituting (3.9), (3.10), (3.11), and (3.12) into the inequality (3.8) leads to (3.7). The proof of Theorem 2 is complete. $\square$

Theorem 3. Let $(s, m) \in [-1, 1] \times (0, 1]$ and $\lambda \in (0, 1)$ and let $f : (0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on $(0, b^*]$, where $a, b \in (0, b^*]$, $a < b$, $b^{1/m} \leq b^*$, and $f' \in L_1([a, b])$. If $|f'|^q$ is extended (s, m) - ε -GA-convex on $(0, \max\{b^{1/m}, b\}]$ for $q > 1$, then

$$(3.13) \quad \begin{aligned} & \left| f(a^{\lambda} b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \frac{\ln b - \ln a}{6^{1/q}} \left[(\lambda L(a^{\lambda q/(q-1)}, b^{\lambda q/(q-1)}) b^{(1-\lambda)q/(q-1)})^{1-1/q} \right. \\ & \times \left(\frac{\lambda^{q+s+1}}{q+s+1} |f'(a)|^q + mB(\lambda, q+1, s+1) |f'(b^{1/m})|^q + \frac{1}{q+1} \varepsilon \lambda^{q+1} \right)^{1/q} \\ & + ((1-\lambda) a^{\lambda q/(q-1)} L(a^{(1-\lambda)q/(q-1)}, b^{(1-\lambda)q/(q-1)}))^{1-1/q} \left(B(1-\lambda, \right. \\ & \times q+1, s+1) |f'(a)|^q + \frac{m(1-\lambda)^{q+s+1}}{q+s+1} |f'(b^{1/m})|^q + \frac{\varepsilon(1-\lambda)^{q+1}}{q+1} \left. \right)^{1/q} \Big], \end{aligned}$$

where $B(\lambda, \alpha, \beta)$ is incomplete Beta function defined by

$$B(\lambda, \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\lambda} t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha, \beta > 0.$$

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned}
 & \left| f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\
 & \leq (\ln b - \ln a) \left[\int_0^\lambda t a^t b^{1-t} |f'(a^t b^{1-t})| dt + \int_\lambda^1 (1-t) a^t b^{1-t} |f'(a^t b^{1-t})| dt \right] \\
 (3.14) \quad & \leq (\ln b - \ln a) \left[\left(\int_0^\lambda (a^t b^{1-t})^{q/(q-1)} dt \right)^{1-1/q} \left(\int_0^\lambda t^q |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right. \\
 & \quad \left. + \left(\int_\lambda^1 (a^t b^{1-t})^{q/(q-1)} dt \right)^{1-1/q} \left(\int_\lambda^1 (1-t)^q |f'(a^t b^{1-t})|^q dt \right)^{1/q} \right],
 \end{aligned}$$

where

$$(3.15) \quad \int_0^\lambda (a^t b^{1-t})^{q/(q-1)} dt = \lambda b^{(1-\lambda)q/(q-1)} L(a^{\lambda q/(q-1)}, b^{\lambda q/(q-1)})$$

and

$$(3.16) \quad \int_\lambda^1 (a^t b^{1-t})^{q/(q-1)} dt = (1-\lambda) a^{\lambda q/(q-1)} L(a^{(1-\lambda)q/(q-1)}, b^{(1-\lambda)q/(q-1)}).$$

From the extended (s, m) - ε -GA-convexity of $|f'|^q$, we have

$$\begin{aligned}
 (3.17) \quad & \int_0^\lambda t^q |f'(a^t b^{1-t})|^q dt \leq \int_0^\lambda t^q \left[t^s |f'(a)|^{q+m} (1-t)^s |f'(b^{1/m})|^q + \varepsilon \right] dt \\
 & = \frac{\lambda^{q+s+1}}{q+s+1} |f'(a)|^q + m B(\lambda, q+1, s+1) |f'(b^{1/m})|^q + \frac{1}{q+1} \varepsilon \lambda^{q+1}.
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_\lambda^1 (1-t)^q |f'(a^t b^{1-t})|^q dt \leq \int_\lambda^1 (1-t)^q \left[t^s |f'(a)|^q \right. \\
 & \quad \left. + m(1-t)^s |f'(b^{1/m})|^q + \varepsilon \right] dt \\
 (3.18) \quad & = B(1-\lambda, q+1, s+1) |f'(a)|^q \\
 & \quad + \frac{m(1-\lambda)^{q+s+1}}{q+s+1} |f'(b^{1/m})|^q + \frac{1}{q+1} \varepsilon (1-\lambda)^{q+1}.
 \end{aligned}$$

Substituting (3.15), (3.16), (3.17), and (3.18) into the inequality (3.14) derives the inequality (3.13). The proof of Theorem 3 is complete. $\square$

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On Nc -continuous functions

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Abstract. In this research we introduce new types of c -continuous functions by using N -open sets, also we submit some other kinds of functions by using the same sets such as N^* -open and N^{**} -open functions, we discuss the relation between them. And we show the relation between some types of normal spaces. We support our work by some facts and examples.

Keywords: Nc -continuous function, N^* c -continuous function, N^{**} c -continuous function, N^* -open function, N^{**} -open function.

1. Introduction

Fashionable, the variant forms of functions have a considerable influence in the topological spaces and many researchers have written about plentiful sorts of it, such as the researchers Gentry et al. [1], they introduced the concept of c -continuous functions. We will use N -open sets [2] to define new types of c -continuous function called Nc -continuous, N^*c -continuous and $N^{**}c$ -continuous function. Also, we will introduce these types of Nc -continuous functions by using N -closed sets, as well we will submit the relation between those kinds of functions. By the same way we will introduce some kinds of N -open function and the relation between them. And we will highlight on new types of N -normal spaces. We will support our work with some examples, propositions and theorems.

2. On Nc -continuous functions

Now we introduce the function of c -continuous and some types of Nc -continuous. Both of ∂ and ρ are top. spaces.

Definition 2.1 ([1],[3]). The function $m : (H, \partial) \longrightarrow (K, \rho)$ is c -continuous in an element h of the space (H, ∂) if to any open set J in the space (K, ρ) where $m(h) \in J$ and its complement is compact, we delimit an open set C in (H, ∂) with $h \in C, m(C) \subseteq J$. If m is c -continuous at every element in (H, ∂) , hence m is c -continuous on (H, ∂) .

Example 2.2. $m : (H, \partial_D) \longrightarrow (K, \rho)$ is c-continuous function.

Definition 2.3. $m : (H, \partial) \longrightarrow (K, \rho)$ is Nc -continuous function if to any element $h \in (H, \partial)$ and to any open set J in (K, ρ) , $m(h) \in J$ and J^C is compact, we could delimit N -open set $C \in (H, \partial)$ such that $h \in C$ and $m(C) \subseteq J$.

Example 2.4. $m : (H, \partial_{ind}) \longrightarrow (K, \rho)$, where H is finite set, is Nc -continuous function. We found the following proposition in [4], [5] without proof, we will submit its proof here.

Proposition 2.5. If J is open set, it is an N -open set, because if J is open set in the space (H, ∂) , then to any $h \in J$, we could determine an open set J in (H, ∂) where $h \in J$ and $J - J$ is finite set, so J is N -open set.

Example 2.6. Every set $\emptyset \neq J \neq H$ in the indiscrete space (H, ∂_{ind}) with H is finite set, is N -open set but not open.

Definition 2.7. Give m as a function where $m : (H, \partial) \longrightarrow (K, \rho)$, f is N^*c -continuous function if to any element $h \in (H, \partial)$ and every N -open set J in (K, ρ) , $m(h)$ is a subset J and J^C is compact, we select an open set C in (H, ∂) with $m(C) \subseteq J$.

Example 2.8. $m : (H, \partial_D) \longrightarrow (K, \rho_{cof})$ where ρ is finite set, m is N^* -continuous function.

Definition 2.9. Suppose $m : (H, \partial) \longrightarrow (K, \rho)$, hence m is N^{**} c-continuous function if to any point h in (H, ∂) and every N -open set J in (K, ρ) , $m(h) \subseteq J$ and J^C is compact, we could select an N -open set $C \in (H, \partial)$ with $m(C) \subseteq J$.

Example 2.10. $I_H : (H, \partial_{EX}) \longrightarrow (H, \partial_{EX})$ is N^{**} c-continuous function, X is finite set, $\mathfrak{S}_{EX} = \{C \subseteq H, h_0 \in C \text{ for some } h_0 \in H\} \cup \{\emptyset\}$.

Definition 2.11 ([6]). Give H as a space, $J \subseteq H$, h is any element in J , then h is an N -interior point for J if we select an N -open set C in H and $h \subseteq C \subseteq J$.

Remark 2.12. If $\{J_\alpha | \alpha \in \Lambda\}$ is a collection of N -open sets, then $U_{\alpha \in \Lambda} J_\alpha$ is also N -open.

Proof. Since $\{J_\alpha | \alpha \in \Lambda\}$ is a combination of N -open sets in H , let $h \in U_{\alpha_i \in \Lambda} J_{\alpha_i}$, so $h \in J_{\alpha_i}$ for some $\alpha_i \in \Lambda$, therefore we can find an open set C_{α_j} in the space H and $h \in C_{\alpha_j}$, where $C_{\alpha_j} - J_{\alpha_j}$ is finite, but $C_{\alpha_j} - U_{\alpha_i \in \Lambda} J_{\alpha_i} \subseteq C_{\alpha_j} - J_{\alpha_j}$, then $C_{\alpha_j} - U_{\alpha_i \in \Lambda} J_{\alpha_i}$ is finite, hence $U_{\alpha_i \in \Lambda} J_{\alpha_i}$ is N -open set. $\square$

We found the following proposition in [7] without proof, we will introduce its proof.

Proposition 2.13. Suppose (H, ∂) be a space, J be a set in H , so J is N -open set iff $J = J^{0N}$.

Proof. Let J be an N -open set in H , to any element h in J , $h \in J \subseteq H$, so h is an N -interior point to J , and since h is arbitrary point in J , so any point in J is an N -interior point to J . Conversely, since J is a union of whole its points which are N -interior points, so $J = \bigcup_{i \in I} C_i$, C_i are N -open sets, $\forall i \in I$, where every N -open set from them containing at least one point of J (since every point in J is an N -interior point), so by remark (3.12), J is an N -open set. $\square$

Theorem 2.14. Give $m : H \longrightarrow K$, then m is Nc -continuous iff to all open set J in K with compact complement, $m^{-1}(J)$ is N -open set in H .

Proof. If m is Nc -continuous function, J is an open set in the space K and has a compact complement, let h inside $m^{-1}(J)$, so $m(h)$ is in J , therefore we can select an N -open set C in H with $h \in C$ and $m(C) \subseteq J$, then $G \subseteq m^{-1}(m(C)) \subseteq m^{-1}(J)$, hence h is N -interior point to $m^{-1}(J)$ which implies $m^{-1}(J)$ is N -open set (by proposition (3-13)). Conversely, if P is an open set in K with $m(h)$ in P where h is inside H , and has a compact complement, hence $h \in m^{-1}(P)$, but $m^{-1}(P)$ is N -open subset of H (hypothesis), and $m(m^{-1}(P)) \subseteq P$, Now let V equal to $m^{-1}(P)$, then m is Nc -continuous. $\square$

Example 2.15. $m : (H, \partial_{ind}) \longrightarrow (K, \rho)$ with H is finite, is Nc -continuous function. In the same way of proofing the previous theorem, we can proof the next proposition.

Proposition 2.16. If $m : H \longrightarrow K$, so m is $N^*(N^{**})$ c -continuous function iff to every N -open set J of K and J^C is compact, $m^{-1}(J)$ is open (N -open) set in H .

Example 2.17. A function $I_H : (H, \partial_D) \longrightarrow (H, \partial_{ind})$ where H is finite set, is $N^*(N^{**})$ c -continuous function.

Definition 2.18 ([8]). An element h in a space H is a condensation point to a subset P of H , when any open set C of H having the point h , $C \cap P$ is infinite. When the set P includes all its condensation points, then it is N -closed set. And P^C is N -open set.

Example 2.19. Every set in the indiscrete space (H, ∂_{ind}) where H is finite, is N -closed set. We found the next Remark in [9] without proof, we introduce its proof for completeness.

Remark 2.20. Suppose (H, ∂) is a space, hence any closed set in H is an N -closed, because when P is a closed set inside H , and $h \in H$ and h does not belong to P , so $h \in P^C$, since P is closed set then P^C is open. But $P \cap P^C = \emptyset$ which is finite, so h is not a condensation point for P , hence every condensation point for P is inside it, therefore P is N -closed.

Theorem 2.21. $m : H \longrightarrow K$ is Nc -continuous function iff for any closed and compact set P of K , its inverse image is N -closed set in H .

Proof. Give m as Nc-continuous function and P is closed and compact set in K , so P^C is an open set of K and its complement is compact (given), since m is Nc-continuous function so $m^{-1}(P^C)$ is N-open set in the space H , but $m^{-1}(P^C) = (m^{-1}(P))^c$, then $((m^{-1}(P))^c)^c = m^{-1}(P)$ is N-closed set of H . Conversely, suppose J is an open set in K and J^C is compact, $m^{-1}(J^C) = (m^{-1}(J))^c$ is N-closed subset of the space H , and the complement of the last one is equal to $m^{-1}(J)$ is N-open set of H , therefore m is Nc-continuous function. $\square$

Example 2.22. $I_H : (H, \partial_{ind}) \longrightarrow (H, \partial)$ where H is finite, is Nc-continuous function. We can proof the following two propositions by the same way of proof the last theorem.

Proposition 2.23. If $m : H \longrightarrow K$ is N^* c-continuous function iff for every N-closed and compact subset P of K , its inverse image is closed set of H .

Example 2.24. $I_H : (H, \partial_D) \longrightarrow (H, \partial_D)$ where H is finite set, is N^* c-continuous function.

Proposition 2.25. If $m : H \longrightarrow K$ is N^{**} c-continuous function iff to any N-closed and compact set P of K , its inverse image is N-closed set of H .

Example 2.26. The function in example (3-24) satisfies the previous proposition.

Remark 2.27. 1. If m is c-continuous function, then it is Nc-continuous.

2. If m is N^* -continuous function, so m is c-continuous.

3. When m is N^* -continuous function, hence it is N^{**} c-continuous.

4. When m is c-continuous function, it is not necessarily N^{**} c-continuous.

5. If m is N^{**} c-continuous function, it is not needs to be c-continuous.

6. If m is N^* -continuous function, hence it is Nc-continuous.

7. When m is N^{**} c-continuous function, so it is Nc-continuous.

The converses of (1, 2, 3, 6, 7) is untrue, the next example shows that.

Example 2.28. 1. If H is a finite set, then $m : (H, \partial_{ind}) \longrightarrow (H, \partial_D)$ is Nc-continuous function but not c-continuous.

2. Give H as a finite set, then $I_H : (H, \partial_{ind}) \longrightarrow (H, \partial_{ind})$ is c-continuous function but not N^* -c-continuous.

3. Suppose H is a finite set, $m : (H, \partial_{ind}) \longrightarrow (H, \partial_D)$ is N^{**} c-continuous function but not N^* -c-continuous.

4. $m : (\mathbb{R}, \partial_D) \longrightarrow (\mathbb{R}, \partial_{ind})$ is c-continuous function but not N^{**} c-continuous.

5. Give H as a finite set, then $m : (H, \partial_{ind}) \longrightarrow (H, \partial_D)$ is N^{**} c-continuous function but not c-continuous.

6. $m : (H, \partial_{ind}) \longrightarrow (H, \partial_{ind})$ where H is finite, it is Nc-continuous function but not N^* c-continuous.

7. $m : (H, \partial_{ind}) \longrightarrow (H, \partial_{ind})$ where H is finite, it is Nc-continuous function but not N^{**} c-continuous.

Definition 2.29 ([9]). (H, ∂) is NT_1 -space if to all distinct points q, r in it, we could select two N -open sets J, C with $q \in J, r$ is not in J , and $r \in C, q$ is not in C .

Example 2.30. The discrete space $(\mathfrak{R}, \partial_D)$ is NT_1 -space. We found the following proposition in [9] without proof, we introduce its proof for completeness.

Proposition 2.31. The space (H, ∂) is NT_1 -space iff each singleton point in it is N -closed set.

Proof. If H is NT_1 -space, r is not in H , to any $q \in H$ then $q \neq r$, but H is NT_1 -space, so choose N -open sets J, C in H and q inside J, r does not belong to J and r belong to C, q does not belong to C , so $C \cap \{q\} = \emptyset$ which is finite set, so r is not condensation point for $\{q\}$, since r is arbitrary, so $\{q\}$ contains each of its condensation points and then it is N -closed set. Contrariwise, if $\{q\}$ is an N -closed set to any $q \in H, r$ is any element in H with q not equal to r , so $\{q\}, \{r\}$ are N -closed subsets in H (by hypothesis), hence $\{q\}^c$ is N -open subset from H where r is in $\{q\}^c, q$ is not in $\{q\}^c$ and $\{r\}^c$ is N -open subset from H such that q in $\{r\}^c, r$ is not in $\{r\}^c$ and therefore H is NT_1 -space. $\square$

Theorem 2.32. If (K, ρ) is T_1 -space and $m : H \rightarrow K$ is Nc -continuous and one to one function, hence (H, ∂) is NT_1 -space.

Proof. If h is in H , so $\{m(h)\}$ is a subset of K but K is a NT_1 -space, so $\{m(h)\}$ is closed and compact set in $K, \{m(h)\}^C$ is open in K with closed and compact complement, $m^{-1}(\{m(h)\}^C) = m^{-1}(K - \{m(h)\}) = H - m^{-1}(\{m(h)\}) = H - \{h\}$ which is N -open subset from H (because m is Nc -continuous function), hence h is N -closed subset from H , therefore H is NT_1 -space (proposition (3.31)). The following proposition can be proof by the same way of the proof for previous theorem. $\square$

Proposition 2.33. When K is NT_1 -space and $m : H \rightarrow K$ is N^*c -continuous and one to one function, then H is T_1 -space.

Proposition 2.34. When K is NT_1 -space and $m : H \rightarrow K$ is $N^{**}c$ -continuous and one to one function, then H is NT_1 -space.

Example 2.35. $m : (H, \partial_D) \rightarrow (K, \rho_D)$, satisfying (3-32), (3-33) and (3-34).

Definition 2.36 ([9], [10], [11]). (H, ∂) is N -normal space if to any closed sets O and $P, O \cap P = \emptyset$, we could delimit two N -open sets J, C such that $J \cap C = \emptyset$, with $O \subseteq J$ and $P \subseteq C$.

Example 2.37. The space (H, ∂_{ind}) with H is finite set, is N -normal space.

Definition 2.38 ([12]). (H, ∂) is N^* -normal space if to all N -closed sets O and P where $O \cap P = \emptyset$, there are two open sets J, C and $J \cap C = \emptyset$, with $O \subseteq J$ and $P \subseteq C$.

Example 2.39. The discrete space (H, ∂_D) where H is finite set, is N^* -normal space.

Definition 2.40 ([12]). A space X is N^{**} -normal space if for each N -closed sets F and H with $F \cap H = \emptyset$, we can find N -open sets U and G where $U \cap G = \emptyset$, with $F \subseteq U$ and $H \subseteq G$.

Example 2.41. The discrete space $(\mathbb{X}, \mathfrak{L}_D)$ where $\mathbb{X}$ is finite set, is N^{**} -normal space.

Remark 2.42. 1. If H is normal space, so it is N -normal ([12]).

2. When H is N^* -normal space, then it is normal ([12]).

3. If H is normal space, it is not necessarily $N^{(**)}$ -normal.

4. If H is $N^{(**)}$ -normal space, it is not needs to be normal.

5. When H is N^* -normal space, hence it is $N^{(**)}$ -normal.

6. When HN^* -normal space, so it is N -normal.

7. If H is $N^{(**)}$ -normal space, then it is N -normal.

The converse of (1, 2, 5, 6, 7) is not true.

Example 2.43. 1. Let $H = \{2, 3, 4\}$ and $\partial = \{\emptyset, H, \{2\}, \{2, 3\}, \{2, 4\}\}$, hence (H, ∂) is N -normal space but not normal.

2. Let $H = \{1, 2, 3\}$ and $\partial = \{\emptyset, H, \{1\}, \{2\}, \{1, 2\}\}$, so (H, ∂) is normal space while not N^* -normal.

3. $(\mathcal{R}, \partial_u)$ is normal space while not N^{**} -normal.

4. Let $H = \{2, 3, 4\}$ and $\partial = \{\emptyset, H, \{2\}, \{2, 3\}, \{2, 4\}\}$, then (H, ∂) is N^{**} -normal space but not normal.

5. The indiscrete space (H, ∂_{ind}) where $H = \{1, 2, 3\}$ is N^{**} -normal space but not N^* -normal.

6. Let $H = \{2, 3, 4\}$ and $\partial = \{\emptyset, H, \{2\}, \{2, 3\}, \{2, 4\}\}$, then (H, ∂) is N -normal space but not N^* -normal.

Definition 2.44 ([13]). $m : H \rightarrow K$ is open function when $m(J)$ is open set in K to all open set J in H .

Example 2.45. $m : (H, \partial_D) \rightarrow (K, \partial_D)$ is open function.

Definition 2.46 ([14]). $m : H \rightarrow K$ is N -open function if to all open set J in H , $m(J)$ is N -open subset from K .

Example 2.47. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{ind})$, where H is finite, is N -open function.

Definition 2.48. $m : H \rightarrow K$ is $N^*(N^{**})$ -open function when to any N -open subset J in H , $m(J)$ is open (N -open) set from K .

Example 2.49. 1. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{ind})$, where H is finite, is N^{**} -open function.

2. $m : (\mathcal{R}, \partial_D) \rightarrow (\mathcal{R}, \partial_D)$ is N^* -open function.

Remark 2.50. 1. Every open function is N -open.

2. Every N^* -open function is open.
3. Not every open function is N^{**} -open.
4. Not every N^{**} -open function is open.
5. Every N^* -open function is N -open.
6. Every N^{**} -open function is N -open.

The converse of (1, 2, 5, 6) is untrue, we give the next example to that.

Example 2.51. 1. $I_H : (H, \partial_D) \rightarrow (H, \partial_D)$, where H is finite, is N -open function but not open.

2. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{ind})$, where H is finite, is open function but not N^* -open.

3. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{cof})$ is open function but not N^{**} -open.

4. $I_H : (H, \partial_D) \rightarrow (H, \partial_{ind})$, where H is finite, is N^{**} -open function but not open.

5. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{cof})$, where H is finite set, is N -open function but not N^* -open.

6. $I_H : (H, \partial_{ind}) \rightarrow (H, \partial_{cog})$ where $m(h) = h$ to any $h \in H$, H is finite set, is N -open function but not N^{**} -open.

Definition 2.52 ([15]). (H, ∂_D) is T_2 -space if to every not equal points q, r in H , we could find two open sets J, C with $q \in J, r \in C$ and $J \cap C = \emptyset$.

Example 2.53. (H, ∂) is T_2 -space.

Definition 2.54. (H, ∂_D) is NT_2 -space if to not equal element $q, r \in H$, there are two N -open sets J, C in the space H , where $q \in J, r \in C$ and $J \cap C = \emptyset$.

Theorem 2.55. If $m : H \rightarrow K$ is Nc -continuous, open and surjective where H is N^* -normal space and K is T_1 -space, hence K is T_2 -space.

Proof. Suppose $q \neq r$ are elements in K , because K is T_1 -space, $\{q\}, \{r\}$ are closed sets in K , and they are compact since they are finite sets, then the inverse image of these sets are N -closed sets in H (Theorem (3.21)), H is N^* -normal space, then $\exists$ two open sets J, C in H , where $(m^{-1}(\{q\})) \subseteq J$ and $(m^{-1}(\{r\})) \subseteq C$, hence $m(m^{-1}(\{q\})) \subseteq m(J)$, $m(m^{-1}(\{r\})) \subseteq m(C)$, where $m(m^{-1}(\{q\}))$ and $m(m^{-1}(\{r\}))$ are open subsets from K (because m is open), $m(J) \cap m(C) = m(J \cap C) = m(\emptyset) = \emptyset$, so K is T_2 -space. $\square$

Definition 2.56 ([16]). If $m : H \rightarrow K$ is N -continuous if to any open subset J in H , $m(J)$ is N -open set in H .

Example 2.57. When H is finite set, $m : (H, \partial_{ind}) \rightarrow (H, \partial_D)$, is N -continuous function.

Definition 2.58. If $m : H \rightarrow K$ is N^* -continuous if to any N -open J subset in K , $m^{-1}(J)$ is open set in H .

Example 2.59. $m : (H, \partial_D) \rightarrow (H, \partial_{ind})$ with H is finite set, is N^* -continuous function.

Definition 2.60. If $m : H \rightarrow K$ is N^{**} -continuous if to all N -open subset J from K , $m^{-1}(J)$ is N -open subset from H .

Example 2.61. $m : (H, \partial_D) \rightarrow (H, \partial_{ind})$ with H is finite set, is N^{**} -continuous function.

Theorem 2.62. Let $m : (H, \partial) \rightarrow (K, \rho)$, $m_* : (H, \partial) \rightarrow (K, \rho^*)$, and $i : (K, \rho') \rightarrow (K, \rho^*)$, and suppose that ρ^* contains a base consist from open sets having compact complement in ρ' . Whenever i is continuous function, i^{-1} is Nc -continuous. m is Nc -continuous if m_* is Nc -continuous, where $m_*(h) = m(h)$, for any h inside H .

Proof. Suppose m is Nc -continuous function and J is open set in (K, ρ^*) , since ρ^* contains a base consist from open and compact sets in ρ' , so J^C is compact, and $\rho^* \subset \rho'$, so $J \subset \rho'$ and because m is Nc -continuous, so $m^{-1}(J)$ is N -open set in H , but $m_*(h) = m(h)$, for any h in H , so $m_*^{-1}(J)$ is N -open set in H , therefore m_* is Nc -continuous function. Now to prove that i is continuous function, suppose J is open set in (K, ρ^*) , so it is an open set (K, ρ') , hence i is continuous function (since $i^{-1}(J) = J$). To manifestation that i^{-1} is Nc -continuous, suppose J is an open subset from (K, ρ') and its complement is compact, so $J \subset \rho^*$, then J is N -open subset from (K, ρ^*) (by proposition (3-5)), so i^{-1} is Nc -continuous function (since $(i^{-1})^{-1}(J) = i(J) = J$). $\square$

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More accurate Young, Heinz-Heron mean and Heinz inequalities for scalar and matrix

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Abstract. In this paper, we mainly give some refinements of Heinz mean-Heron mean inequality $F_{\alpha(v)}(a, b) \geq H_v(a, b)$ where $\alpha(v) = 1 - 4(v - v^2)$. More precisely, interpolation between them are established motivated by refinements of Young inequality. The matrix versions of these inequalities are also obtained in the last part.

Keywords: Young inequality, Heinz mean, Heron mean, matrix inequality.

1. Introduction

Nowadays, refinements, generalizations and extension to a multidimensional case of classical Young inequality, which states that if $a, b \geq 0$ and $v \in [0, 1]$, then

$$(1) \quad a \nabla_v b \geq a \sharp_v b$$

with equality if and only if $a = b$, have arisen researchers' attention. The weighted arithmetic and geometric means are defined, respectively, as follows $a \nabla_v b = (1 - v)a + vb$ and $a \sharp_v b = a^{1-v}b^v$. After discovering, the Young inequality was studied by numerous researchers, who either reproved it using various techniques, or applied and generalized it in many different directions.

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Replacing (v, b) by $(2v, \sqrt{ab})$ when $v \in [0, \frac{1}{2}]$ and (v, b) by $(2-2v, \sqrt{ab})$ when $v \in [\frac{1}{2}, 1]$ in (1) we got [1]

$$(2) \quad a\nabla_v b \geq a\sharp_v b + 2r_0(a\nabla b - a\sharp b),$$

where $r_0 = \min\{v, 1-v\}$.

Replacing (v, b) , (v, a) , (v, a) and (v, b) by $(2v, \sqrt{ab})$, $(1-2v, \sqrt{ab})$, $(2v-1, \sqrt{ab})$ and $(2-2v, \sqrt{ab})$ respectively when $v \in [0, \frac{1}{4}]$, $v \in (\frac{1}{4}, \frac{1}{2}]$, $v \in (\frac{1}{2}, \frac{3}{4}]$ and $v \in (\frac{3}{4}, 1]$ in (2) we got [2]

$$\begin{cases} a\nabla_v b \geq a\sharp_v b + 2v(a\nabla b - a\sharp b) + r_1(a + a\sharp b - 2a\sharp_{\frac{1}{4}} b), & 0 \leq v \leq \frac{1}{2} \\ a\nabla_v b \geq a\sharp_v b + 2(1-v)(a\nabla b - a\sharp b) + r_1(b + a\sharp b - 2a\sharp_{\frac{3}{4}} b), & \frac{1}{2} < v \leq 1 \end{cases}$$

where $r_1 = \min\{r_0, 1-r_0\}$.

Inequality (1) in multidimensional case has been obtained by M. Sababheh and D. Choi [3]

$$a\nabla_v b \geq a\sharp_v b + S_N(v; a, b),$$

where $S_N(v; a, b) = \sum_{j=1}^N s_j(v) \left(\sqrt[2^j]{a^{2^{j-1}-k_j(v)} b^{k_j(v)}} - \sqrt[2^j]{a^{2^{j-1}-k_j(v)-1} b^{k_j(v)+1}} \right)^2$, $s_j(v) = (-1)^{r_j(v)} 2^{j-1} v + (-1)^{r_j(v)+1} [\frac{r_j(v)+1}{2}]$, $k_j(v) = [2^{j-1}v]$ and $r_j(v) = [2^j v]$. Here $[x]$ is the greatest integer less than or equal to x .

Although classical, the refinements of Young inequality is still of great interest to numerous authors.

M. Krnić [4] obtained that

$$(3) \quad a\nabla_v b \geq a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b)$$

when $(2v-1)(b-a) \geq 0$ and reverse inequality when $(2v-1)(b-a) \leq 0$. Recently, M. Sababheh refined M. Krnić's result in [5] using the method of inductive quadratic interpolation.

Refining Young inequality or Young type inequalities can go forever, in the sense that one can find further refining terms for all refinements in the literature and extend it to matrix, norm and determinant inequality. We encourage the reader to figure different methods to obtain refinements of these inequalities.

So far the mentioned inequalities were involving a combination of a, b , i.e. $a\nabla_v b = (1-v)a + vb$. There exist some inequalities involving a combination of $a\nabla b, a\sharp b$. In the following some of them are listed.

Both means, Heinz and Heron means, interpolate between the geometric and arithmetic means, defined respectively for $a, b > 0$ as follows

$$H_v(a, b) = \frac{a\sharp_v b + a\sharp_{1-v} b}{2} \quad \text{and} \quad F_v(a, b) = (1-v)a\sharp b + va\nabla b.$$

It follows from inequality (1) when $v = \frac{1}{2}$ that the Heinz and Heron means interpolate between geometric mean and arithmetic mean

$$a\sharp b \leq H_v(a, b) \leq a\nabla b \quad \text{and} \quad a\sharp b \leq F_v(a, b) \leq a\nabla b.$$

R. Bhatia [6] proved that the Heinz and Heron means satisfy the following inequality

$$(4) \quad F_{\alpha(v)}(a, b) \geq H_v(a, b),$$

where $\alpha(v) = 1 - 4(v - v^2)$.

One of the questions that arises from this work is the following. Does this inequality between means of positive numbers lead to further refinements?

F. Kittaneh and Y. Manasrah [7] proved that

$$(5) \quad a\nabla b \geq H_v(a, b) + r_0(a\nabla b - a\sharp b).$$

W. Liao and P. Long [8] refined the above result as follows

$$a\nabla b \geq H_v(a, b) + 2r_0(a\nabla b - a\sharp b) + r_1(a\nabla b + a\sharp b - a\sharp_{\frac{1}{4}}b - a\sharp_{\frac{3}{4}}b)$$

and

$$(a\nabla b)^2 \geq H_v(a, b)^2 + 2r_0((a\nabla b)^2 - (a\sharp b)^2) + 4r_1[(a\nabla b)^2 - (a\nabla b)(a\sharp b)].$$

Further refinements of Heinz inequality have been obtained by numerous researchers. In what follows, we mainly present some refinements of general Heinz inequality and Heinz-Heron mean inequality motivated by the Young-type inequalities presented as above.

The organization of the paper is as follows. We firstly present several refinements of Young inequality and Heinz-Heron mean inequality with a series of similar inequalities are then presented. In the last part of this paper, these refinements are applied to matrix case.

2. Main results

In this section we mainly present some inequalities for scalars and then we present several matrix inequalities.

2.1 Scalar results

In this part of the paper, we mainly show some scalar results. The main approach we used in proving the scalar results is some delicate and tricky computations.

Theorem 2.1. *If $a, b > 0$ and $v \in [0, 1]$, then*

$$(6) \quad a\nabla_v b \geq a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b)$$

holds for $(\frac{b}{a})^{v-\frac{1}{2}} \geq 1$.

Proof. If $a, b > 0$ and $v \in [0, 1]$, the inequality (6) for $a = 1, b = t > 0$, becomes

$$1\nabla_v t \geq 1\sharp_v t + 4v(1-v)(1\nabla t - 1\sharp t).$$

Let

$$f(t) = 1 - v + vt - t^v - 4v(1-v)\left(\frac{1+t}{2} - \sqrt{t}\right).$$

Then

$$f'(t) = v - vt^{v-1} - 4v(1-v)\left(\frac{1}{2} - \frac{1}{2\sqrt{t}}\right)$$

and

$$f''(t) = v(1-v)t^{v-2} - v(1-v)t^{-\frac{3}{2}} = v(1-v)t^{-\frac{3}{2}}(t^{v-\frac{1}{2}} - 1).$$

In view of $t^{v-\frac{1}{2}} \geq 1$, then $t^{v-\frac{1}{2}} - 1 \geq 0$ which means $f''(t) \geq 0$. Hence, $f'(t)$ is increasing on $(0, \infty)$, $f'(1) = 0$ and $f(1) = 0$. So, $f(t)$ is decreasing on $(0, 1]$ and increasing on $(1, \infty)$. Consequently, $f(t) \geq 0$ for $v \in [0, 1]$ and $t^{v-\frac{1}{2}} \geq 1$.

This proved that

$$a\nabla_v b \geq a\sharp_v b + 4v(1-v)(a\nabla b - a\sharp b).$$

hold for $a, b > 0$, $v \in [0, 1]$ and $(\frac{b}{a})^{v-\frac{1}{2}} \geq 1$. □

Remark 2.2. In view of $4v(1-v) - 8v(1-2v) \leq 0$ when $v \in [0, \frac{1}{3}]$, it is natural to consider refining inequality (6). Using the same method of Theorem 2.1, we have

$$a\nabla_v b \geq a\sharp_v b + 8r_0(1-2r_0)(a\nabla b - a\sharp b)$$

for $a, b > 0$, $v \in [0, 1]$ and $(\frac{b}{a})^{v-\frac{1}{2}} \geq 2$.

In the following, we refine the Heinz and Heron means inequality using the same method as above. As $a\sharp b \leq H_v(a, b) \leq F_\alpha(a, b) \leq a\nabla b$, we are interested in finding an upper bound or a lower bound of $F_\alpha(a, b) - H_v(a, b)$.

Theorem 2.3. *The Heinz and Heron means satisfy*

$$(7) \quad F_\alpha(a, b) \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b)$$

for $a, b > 0$, $v \in [0, 1]$, $\alpha = 1 - 4(v - v^2)$ and $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$.

Proof. Inequality (7), in expanded form, says

$$4(v-v^2)\sqrt{ab} + (1-4(v-v^2))\frac{a+b}{2} \geq \frac{a^{1-v}b^v + a^vb^{1-v}}{2} + 4v(1-v)\left(\frac{a+b}{2} - \sqrt{ab}\right).$$

Put $a = 1, b = t$,

$$4(v-v^2)\sqrt{t} + (1-4(v-v^2))\frac{1+t}{2} \geq \frac{t^v + t^{1-v}}{2} + 4v(1-v)\left(\frac{1+t}{2} - \sqrt{t}\right).$$

Let

$$f(t) = 4(v - v^2)\sqrt{t} + (1 - 4(v - v^2))\frac{1+t}{2} - \frac{t^v + t^{1-v}}{2} - 4v(1-v)\left(\frac{1+t}{2} - \sqrt{t}\right).$$

Then

$$f'(t) = 4(v - v^2)\frac{1}{2\sqrt{t}} + (1 - 4(v - v^2))\frac{1}{2} - \frac{vt^{v-1} + (1-v)t^{-v}}{2} - 4v(1-v)\left(\frac{1}{2} - \frac{1}{2\sqrt{t}}\right)$$

and

$$\begin{aligned} f''(t) &= 4(v - v^2)\left(-\frac{1}{4}\right)t^{-\frac{3}{2}} - \frac{v(v-1)t^{v-2} + v(v-1)t^{-v-1}}{2} - v(1-v)t^{-\frac{3}{2}} \\ &= v(1-v)t^{-\frac{3}{2}}\left[\frac{t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v}}{2} - 2\right]. \end{aligned}$$

In view of $t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v} \geq 4$, we have $f''(t) \geq 0$. Then $f(1) = f'(1) = 0$ which means that $f(t)$ is decreasing on $(0, 1]$ and increasing on $(1, \infty)$, respectively. Consequently, $f(t) \geq 0$ for $v \in [0, 1]$ and $t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v} \geq 4$. This proved that

$$F_\alpha(a, b) \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b)$$

holds for $a, b > 0$, $v \in [0, 1]$, $\alpha = 1 - 4(v - v^2)$ and $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$. $\square$

We emphasize that the significance of this result is to lead to as many refining inequalities as we wish. Provided that $f(v) = 4v(1-v)(a\nabla b - a\sharp b)$, we can refine $f(v)$ to find an upper bound or a lower bound. Follow the same logic of Theorem 2.3, if we choose $f(v) = 8v(1-2v)$, then we can have the following corollary.

Corollary 2.4. *If $a, b > 0$ and $v \in [0, 1]$, then we can have*

$$F_\alpha(a, b) \geq H_v(a, b) + 8v(1-2v)(a\nabla b - a\sharp b)$$

for $\alpha = 1 - 4(v - v^2)$ and $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 6$.

Put

$$f(t) = 4(v - v^2)\sqrt{t} + (1 - 4(v - v^2))\frac{1+t}{2} - \frac{t^v + t^{1-v}}{2} - 8v(1-2v)\left(\frac{1+t}{2} - \sqrt{t}\right),$$

the detail proof of Corollary 2.4 is similar to the above theorems. Here we leave it to the reader.

Remark 2.5. It is obvious that inequality (7) is a refinement of inequality (4). But it is not a general refinement since inequality (7) constraints $(\frac{b}{a})^{v-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-v} \geq 4$. One way to refine inequality (4) without constraints of $\frac{b}{a}$ is to weaken α which we present as follows.

Theorem 2.6. *The Heinz and Heron means satisfy*

$$(8) \quad F_{\beta}(a, b) \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b)$$

for $a, b > 0$, $v \in [0, 1]$ and $\beta = 1 - 8 \max\{v, 1-v\} \min\{1-2v, 2v-1\}$.

Proof. If $a, b > 0$ and $v \in [\frac{1}{2}, 1]$, inequality (8) for $a = 1, b = t > 0$, becomes

$$8v(1-2v)\sqrt{t} + (1-8v(1-2v))\frac{1+t}{2} \geq \frac{t^v + t^{1-v}}{2} + 4v(1-v)\left(\frac{1+t}{2} - \sqrt{t}\right).$$

Let

$$f(t) = 8v(1-2v)\sqrt{t} + (1-8v(1-2v))\frac{1+t}{2} - \frac{t^v + t^{1-v}}{2} - 4v(1-v)\left(\frac{1+t}{2} - \sqrt{t}\right).$$

Then

$$f'(t) = 8v(1-2v)\frac{1}{2\sqrt{t}} + (1-8v(1-2v))\frac{1}{2} - \frac{vt^{v-1} + (1-v)t^{-v}}{2} - 4v(1-v)\left(\frac{1}{2} - \frac{1}{2\sqrt{t}}\right)$$

and

$$\begin{aligned} f''(t) &= -2v(1-2v)t^{-\frac{3}{2}} + \frac{v(1-v)t^{v-2} + v(1-v)t^{-v-1}}{2} - v(1-v)t^{-\frac{3}{2}} \\ &= v(1-v)t^{-\frac{3}{2}}\left(\frac{t^{v-\frac{1}{2}} + t^{\frac{1}{2}-v}}{2} + 2\frac{v}{1-v} - 3\right). \end{aligned}$$

Since $\frac{2v}{1-v} - 3 \geq -1$ for $v \in [\frac{1}{2}, 1]$, then $f''(t) \geq 0$, $f'(1) = f(1) = 0$ which means that $f(t)$ is decreasing on $(0, 1]$ and increasing on $(1, \infty)$, respectively. Hence, we have $f(t) \geq 0$ for $v \in [\frac{1}{2}, 1]$. Consequently,

$$F_{\beta}(a, b) \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b) \quad (*)$$

holds for $v \in [\frac{1}{2}, 1]$ and $\beta = 1 - 8 \max\{v, 1-v\} \min\{1-2v, 2v-1\}$.

Replacing v by $1-v$ and exchanging a and b in $(*)$, then we have inequality (8) holds for $v \in [0, \frac{1}{2}]$ and $\beta = 1 - 8 \max\{v, 1-v\} \min\{1-2v, 2v-1\}$. $\square$

In view of $a\nabla b \geq F_{\alpha}(a, b)$, we take inequality (8) as a refinement of inequality (5). Next, we present a squared version refinement of inequality (5).

Theorem 2.7. *If $a, b > 0$ and $v \in [0, 1]$, then we can have*

$$(9) \quad (a+b)^2 \geq 4(H_v(a, b))^2 + 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2)$$

for $(2v-1)(b^2 - a^2) \geq 0$.

Proof. From inequality (8), we have

$$\frac{a+b}{2} \geq H_v(a, b) + 4v(1-v)(a\nabla b - a\sharp b).$$

By virtue of replacing a by a^2 and b by b^2 , respectively, then

$$\begin{aligned} (a+b)^2 - 4H_v(a, b)^2 &= a^2 + b^2 - (a^{1-v}b^v)^2 - (a^vb^{1-v})^2 \\ &= (1-v)a^2 + vb^2 - (a^{1-v}b^v)^2 + va^2 + (1-v)b^2 - (a^vb^{1-v})^2 \\ &\geq 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2) \quad \text{by (1.3)} \end{aligned}$$

holds for $(2v-1)(b^2 - a^2) \geq 0$. Hence,

$$(a+b)^2 \geq 4(H_v(a, b))^2 + 8v(1-v)(a^2\nabla b^2 - a^2\sharp b^2).$$

□

2.2 Matrix inequalities

To state our results in this part, we introduce the following matrix theories.

Let $M_n(C)$ be the algebra of $n \times n$ complex matrices. For Hermitian matrices $A, B \in M_n(C)$, we write that $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite and $A \geq B$ if $A - B \geq 0$.

In this part, the weighted arithmetic and geometric matrix means are defined for $A, B \geq 0$ as follows

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

while we use the standard notations $A\nabla B$ and $A\sharp B$ for $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively. The scalar Young inequality $a\nabla_v b \geq a\sharp_v b$ can be used to prove the matrix Young inequality

$$A\nabla_v B \geq A\sharp_v B$$

holds for $A, B \geq 0$ and $v \in [0, 1]$. The matrix version of the Heinz means is defined by

$$H_v(A, B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2},$$

where $A, B \geq 0$ and $v \in [0, 1]$. The matrix version of the Heron means is defined by

$$F_v(A, B) = (1-v)A\sharp B + vA\nabla B$$

for $0 \leq v \leq 1$.

To reach inequality for bounded self-adjoint matrix on Hilbert space, we shall use the following monotonicity property for matrix functions: If $X \in M_n(C)$ with a spectrum $S_p(X)$ and f, g are continuous real-valued functions on an interval containing $S_p(X)$, then

$$f(t) \geq g(t), t \in S_p(X) \Rightarrow f(X) \geq g(X).$$

For more details about this property, the reader is referred to [9].

Now we are in a position to give our first matrix inequality.

Theorem 2.8. Let $A, B > 0$ and $v \in [0, 1]$, then we can have

$$(10) \quad F_\beta(A, B) \geq H_v(A, B) + 4v(1-v)(A\nabla B - A\sharp B)$$

for $\beta = 1 - 8 \max\{v, 1-v\} \min\{1-2v, 2v-1\}$.

For $A = (a_{ij}) \in M_n(C)$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$. $\|\cdot\|_2$ has the unitarily invariant property: $\|UAV\|_2 = \|A\|_2$ for all $A \in M_n(C)$ and all unitary matrices $U, V \in M_n(C)$.

Theorem 2.9. Suppose that two matrices A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:

- (i) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;
- (ii) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$.

Then

$$(11) \quad \|AX + XB\|_2^2 \geq \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2.$$

for $A, B, X > 0$, $v \in [0, 1]$ and $(2v-1)((M')^2 - (m')^2) \geq 0$.

Proof. Since $A, B > 0$, then by spectral theorem there are unitary matrices U, V and diagonal matrices $D_1 = \text{diag}(\lambda_i)$ and $D_2 = \text{diag}(\mu_i)$ for $i = 1, \dots, n$, such that

$$A = UD_1U^* \quad \text{and} \quad B = VD_2V^*.$$

For our computations, let $Y = U^*XV = [y_{ij}]$. Then we have

$$AX + XB = U[(\lambda_i + \mu_j)y_{ij}]V^*.$$

If $v \in [0, 1]$, utilizing (9) and the unitarily invariant property of $\|\cdot\|_2$, we have

$$\begin{aligned} \|AX + XB\|_2^2 &= \sum_{i,j=1}^n (\lambda_i + \mu_j)^2 |y_{ij}^2| \\ &\geq \sum_{i,j=1}^n [(\lambda_i^{1-v}\mu_j^v + \lambda_i^v\mu_j^{1-v})^2 + 8v(1-v)(\frac{\lambda_i^2 + \mu_j^2}{2} - \sqrt{\lambda_i^2\mu_j^2})] |y_{ij}^2| \\ &= \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2 \end{aligned}$$

for $A, B > 0$, $v \in [0, 1]$ and $(2v-1)((M')^2 - (m')^2) \geq 0$. That is

$$\|AX + XB\|_2^2 \geq \|A^{1-v}XB^v + A^vXB^{1-v}\|_2^2 + 4v(1-v)\|AX - XB\|_2^2.$$

□

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Polynomial form fuzzy numbers and their application in linear programming with fuzzy variables

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Abstract. In this paper, a special class of fuzzy numbers, whose members are called polynomial form fuzzy numbers, is considered. Some properties of its members are presented. Furthermore, linear programming problems with 2-degree polynomial form fuzzy variables are studied. A method using a linear ranking function is proposed to solve the problem. The method generalizes the known methods based on linear ranking functions to solve linear programming problems with trapezoidal variables. To illustrate the method, a transportation problem with fuzzy values of the demands and the supplies is solved and the obtained results are discussed.

Keywords: polynomial form fuzzy numbers, fuzzy linear programming, fuzzy variables, ranking function, transportation problem.

1. Introduction

Fuzzy set theory has been applied to many disciplines such as control theory [7, 13], management sciences [14], and mathematical modeling [14]. In many applications, it is convenient to work with the same type of fuzzy numbers. Many authors have restricted attention to some specific class of fuzzy numbers, such as the class of triangular or trapezoidal fuzzy numbers, to use its many properties [6, 8, 10, 12]. There are many works which authors have tried to approximate a fuzzy number by a simpler one [1, 4, 5]. Obviously, if we use a defuzzification rule which replaces a fuzzy number by a crisp one, we generally lose too much important information. Interval approximations are also considered for fuzzy numbers. For instance, in [5], a fuzzy computation problem is converted into an interval arithmetic problem. But, in this case, we also lose the concept of fuzzy centrality.

This paper contains two main parts. In the first part, we consider a class of fuzzy numbers, called polynomial form fuzzy numbers (PFNs), which is also studied by some authors [1, 2, 3, 16, 20]. We present some new algebraic and analytical properties of the class. For instance, we prove that the class of polynomial form fuzzy numbers is a vector space on the real field and also, it is dense in the set of fuzzy numbers. The class also enjoys many other properties. So it can be a convenient candidate to approximate fuzzy numbers to its members.

The second part is about a method for solving a specific class of fuzzy programming problems. Let us first review the literature in this field. The concept of fuzzy mathematical programming was first proposed by Tanaka et al. [15]. The first formulation of fuzzy linear programming (FLP) is proposed by Zimmermann [22]. Afterwards, many authors have considered various kinds of FLP problems and have proposed several approaches for solving these problems [6, 12]. In fact, one of most convenient methods is based on the concept of comparison of fuzzy numbers by using linear ranking functions. At first, Maleki et al. in [9] proposed use of linear ranking function for solving fuzzy programming. Subsequently, many authors applied the concept of ranking functions in linear programming [10, 11].

In the second part, we consider a linear programming problem with 2-degree polynomial form fuzzy variables. Using a linear ranking function, we present a method to convert the problem into one with crisp variables. Since any trapezoidal fuzzy number is a PFN of degree 1, the proposed method is a generalization of the known methods to solve linear programming problems with trapezoidal variables by using ranking functions [8, 10].

The structure of the present paper is as follows: In Section 2, we give some preliminaries and notions. Definition of a PFN and its properties are presented in Section 3. In Section 4, we consider a linear programming problem with 2-degree polynomial form fuzzy variables and propose a method to solve the problem. In this section, we also solve a transportation problem with fuzzy values of the demands and the supplies to illustrate the proposed method. Finally, we conclude in Section 5.

2. Preliminaries

In this section, we present a review of the basic terminology used throughout this paper.

Let X denote a universal set. A fuzzy subset $\tilde{a}$ of X is defined by its membership function as follows:

$$\tilde{a} : X \rightarrow [0, 1],$$

which assigns to each element $x \in X$ a real number $\tilde{a}(x)$ where the value of $\tilde{a}(x)$ represents the degree of membership of x in $\tilde{a}$. The fuzzy set $\tilde{a}$ on X is said to be normal if there exists $x_0 \in X$ such that $\tilde{a}(x_0) = 1$. For $\alpha \in (0, 1]$, the α -cut

of a fuzzy set $\tilde{a}$ is the crisp set

$$\tilde{a}_\alpha = \{x \in X : \tilde{a}(x) \geq \alpha\}.$$

The following theorem states fuzzy sets can be obtained directly from its α -cuts.

Theorem 2.1 (Theorem 2.7 in [17]). *For every fuzzy set $\tilde{a}$,*

$$\tilde{a} = \bigcup_{\alpha \in (0,1]} \alpha \tilde{a}_\alpha.$$

is called a fuzzy number. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$. A fuzzy vector $\tilde{\mathbf{a}}$ is a member of $\mathcal{F}^n(\mathbb{R})$.

A fuzzy number $\tilde{a}$ is said to be an LR fuzzy number whenever its core is a singleton. We say that a fuzzy number $\tilde{a}$ is nonnegative, denoted by $\tilde{a} \geq 0$, if its support contains no negative number. A special type of fuzzy numbers is the linear trapezoidal fuzzy number (TFN). A TFN $\tilde{a}$ is denoted by (a^1, a^2, a^3, a^4) where $a^1 \leq a^2 \leq a^3 \leq a^4$ and its membership function is as follows:

$$\tilde{a}(x) = \begin{cases} \frac{x-a^1}{a^2-a^1} & x \in [a^1, a^2], \\ 1 & x \in (a^2, a^3), \\ \frac{x-a^4}{a^3-a^4} & x \in [a^3, a^4], \\ 0 & \text{otherwise.} \end{cases}$$

Since any real number c and any interval $[a, b]$ can be written respectively as $c = (c, c, c, c)$ and $[a, b] = (a, a, b, b)$, it is obvious that TFNs are an extension of real numbers and intervals. The LR case of a trapezoidal fuzzy number, in which $a_2 = a_3$, is called a triangular fuzzy number.

Let $\tilde{a} = (a^1, a^2, a^3, a^4)$ and $\tilde{b} = (b^1, b^2, b^3, b^4)$ be TFNs and $c \in \mathbb{R}$. The following formulas for addition of two TFNs and multiplication of a TFN by a scalar are drawn from the extension principle of Zadeh [21].

- $c\tilde{a} = (ca^1, ca^2, ca^3, ca^4)$, if $c \geq 0$;
- $c\tilde{a} = (ca^4, ca^3, ca^2, ca^1)$, if $c < 0$;
- $\tilde{a} + \tilde{b} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4)$;
- $\tilde{a} - \tilde{b} = (a^1 - b^4, a^2 - b^3, a^3 - b^2, a^4 - b^1)$.

When we use fuzzy numbers for practical applications, it is sometimes needed to compare them. There exist several methods to compare fuzzy numbers. Most convenient one of these methods is based on the concept of comparing fuzzy numbers by ranking functions [9]. In fact, an efficient approach for ordering the elements of $\mathcal{F}(\mathbb{R})$ is to define a ranking function $\mathcal{R} : \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$ which maps any fuzzy number into the real line. Let us define an order on fuzzy numbers by a ranking function $\mathcal{R}$. Suppose $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers. $\tilde{a}$ is less than or equal to $\tilde{b}$, denoted by $\tilde{a} \preceq \tilde{b}$, if the value of its ranking function is less than or equal to that of $\tilde{b}$, i.e. $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$. Similarly, $\tilde{a}$ and $\tilde{b}$ in $\mathcal{F}(\mathbb{R})$ are equal, denoted by $\tilde{a} \approx \tilde{b}$, whenever the values of their ranking functions are the same, i.e. $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$.

3. Polynomial form fuzzy numbers

In this section, we introduce polynomial form fuzzy numbers and present some properties of them.

Definition 3.1 ([1, 2]). A fuzzy number $\tilde{a}$ with α -cut $\tilde{a}_\alpha = [p_a^-(\alpha), p_a^+(\alpha)]$, $\alpha \in (0, 1]$, is k -degree polynomial form if p_a^- and p_a^+ are polynomials of degree at most k .

It is obvious that any trapezoidal fuzzy number is a polynomial form fuzzy number of degree 1. Let $\mathcal{PF}^k(\mathbb{R})$ be the set of all k -degree polynomial form fuzzy numbers and $\mathcal{PF}(\mathbb{R})$ be the set of all polynomial form fuzzy numbers. Obviously, we have $\mathcal{PF}^k(\mathbb{R}) \subseteq \mathcal{PF}^{k+1}(\mathbb{R})$ and $\mathcal{PF}(\mathbb{R}) = \cup_{k=0}^{\infty} \mathcal{PF}^k(\mathbb{R})$. The following lemma is about the arithmetic of polynomial form fuzzy numbers.

Lemma 3.1. Let $\tilde{a} \in \mathcal{PF}^k(\mathbb{R})$, $\tilde{b} \in \mathcal{PF}^l(\mathbb{R})$ and $c \in \mathbb{R} \setminus \{0\}$. The following statements hold.

- (a) $\tilde{a} + \tilde{b}$ is a polynomial form fuzzy number of degree $\max\{k, l\}$.
- (b) $c \cdot \tilde{a}$ is a polynomial form fuzzy number of degree k .
- (c) If $\tilde{a} \geq 0$, then $\tilde{a} \times \tilde{b}$ is a polynomial form fuzzy number of degree $k + l$.
- (d) The fuzzy number $\tilde{a}$ is determined uniquely by its $(k + 1)$ α -cuts.

Proof. By the interval arithmetic, for every $\alpha \in [0, 1]$, we know that

$$\begin{aligned} (\tilde{a} \odot \tilde{b})_\alpha &= [p_{\tilde{a} \odot \tilde{b}}^-(\alpha), p_{\tilde{a} \odot \tilde{b}}^+(\alpha)] \\ &= [\min\{p_a^-(\alpha) \odot p_b^-(\alpha), p_a^+(\alpha) \odot p_b^-(\alpha), p_a^-(\alpha) \odot p_b^+(\alpha), p_a^+(\alpha) \odot p_b^+(\alpha)\}, \\ &\quad \max\{p_a^-(\alpha) \odot p_b^-(\alpha), p_a^+(\alpha) \odot p_b^-(\alpha), p_a^-(\alpha) \odot p_b^+(\alpha), p_a^+(\alpha) \odot p_b^+(\alpha)\}], \end{aligned}$$

where $\odot \in \{+, -, \times\}$. Therefore,

$$\begin{aligned} [p_{\tilde{a}+\tilde{b}}^-(\alpha), p_{\tilde{a}+\tilde{b}}^+(\alpha)] &= [p_a^-(\alpha) + p_b^-(\alpha), p_a^+(\alpha) + p_b^+(\alpha)], \\ [p_{c\tilde{a}}^-(\alpha), p_{c\tilde{a}}^+(\alpha)] &= [cp_a^-(\alpha), cp_a^+(\alpha)] \quad \forall c \geq 0, \\ [p_{c\tilde{a}}^-(\alpha), p_{c\tilde{a}}^+(\alpha)] &= [cp_a^+(\alpha), cp_a^-(\alpha)] \quad \forall c < 0, \\ [p_{\tilde{a} \times \tilde{b}}^-(\alpha), p_{\tilde{a} \times \tilde{b}}^+(\alpha)] &= [p_a^-(\alpha) \times p_b^-(\alpha), p_a^+(\alpha) \times p_b^+(\alpha)] \quad \forall \tilde{a} \geq 0. \end{aligned}$$

These together with Theorem 2.1 complete the proofs of (a), (b) and (c). For proving (d), note that the α -cut of any $\tilde{a} \in \mathcal{PF}^k(\mathbb{R})$ is $[p_a^-(\alpha), p_a^+(\alpha)]$ in which $p_a^-(\alpha)$ and $p_a^+(\alpha)$ are polynomials of degree k . Hence, if $k + 1$ number of α -cuts of $\tilde{a}$ are given, then we can obtain uniquely $p_a^-(\alpha)$ and $p_a^+(\alpha)$ by interpolating endpoints of α -cuts. This together with Theorem 2.1 imply the result. $\square$

From Lemma 3.1, we imply that $\mathcal{PF}^k(\mathbb{R})$ and $\mathcal{PF}(\mathbb{R})$ are vector spaces over the field $\mathbb{R}$. Using this fact, any $\tilde{a} \in \mathcal{PF}^k(\mathbb{R})$ with α -cut $\tilde{a}_\alpha = [p_a^-(\alpha), p_a^+(\alpha)]$ can be expressed in the coordinate representation as follows:

$$\tilde{a} = (a_0^-, a_1^-, \dots, a_k^-, a_0^+, a_1^+, \dots, a_k^+) \in \mathbb{R}^{2k+2}$$

in which

$$\begin{aligned} p_{\tilde{a}}^{-}(\alpha) &= a_0^{-} + a_1^{-}\alpha + a_2^{-}\alpha^2 + \dots + a_k^{-}\alpha^k, \\ p_{\tilde{a}}^{+}(\alpha) &= a_0^{+} + a_1^{+}\alpha + a_2^{+}\alpha^2 + \dots + a_k^{+}\alpha^k. \end{aligned}$$

Therefore, we can imagine that $\mathcal{PF}^k(\mathbb{R})$ is a vector subspace of $\mathbb{R}^{2k+2}$.

There exist various methods based on α -cuts to rank fuzzy numbers (for details see [17]). Yager in [19] proposed a ranking function which is based on α -cuts. For any fuzzy number $\tilde{a}$, this function is the integral of the mean of its α -cuts. In order to compare polynomial form fuzzy numbers, we used the ranking function introduced by Yager [19].

Definition 3.2. Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$ with α -cut $\tilde{a}_{\alpha} = [p_{\tilde{a}}^{-}(\alpha), p_{\tilde{a}}^{+}(\alpha)]$. Yager's ranking function $\mathcal{R} : \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$ is defined as follows:

$$(1) \quad \mathcal{R}(\tilde{a}) = \frac{1}{2} \int_0^1 (p_{\tilde{a}}^{-}(\alpha) + p_{\tilde{a}}^{+}(\alpha)) d\alpha.$$

We say that the fuzzy number $\tilde{a}$ is less than or equal to the fuzzy number $\tilde{b}$, denoted by $\tilde{a} \leq_{\mathcal{R}} \tilde{b}$, if and only if $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$. Similarly, $\geq_{\mathcal{R}}$ and $=_{\mathcal{R}}$ are defined. It is obvious that the ranking function $\mathcal{R}$ has the property of linearity, i.e.

$$(2) \quad \mathcal{R}(c\tilde{a} + \tilde{b}) = c\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b}), \quad \forall \tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R}), \quad \forall c \in \mathbb{R}.$$

Lemma 3.2. Suppose that $\tilde{a} = (a_0^{-}, a_1^{-}, \dots, a_k^{-}, a_0^{+}, a_1^{+}, \dots, a_k^{+}) \in \mathcal{PF}^k(\mathbb{R})$. Then

$$\mathcal{R}(\tilde{a}) = \sum_{i=0}^k \frac{a_i^{-} + a_i^{+}}{2i + 2}.$$

Proof.

$$\mathcal{R}(\tilde{a}) = \frac{1}{2} \int_0^1 (p_{\tilde{a}}^{-}(\alpha) + p_{\tilde{a}}^{+}(\alpha)) d\alpha = \frac{1}{2} \int_0^1 \sum_{i=0}^k (a_i^{-} \alpha^i + a_i^{+} \alpha^i) d\alpha = \sum_{i=0}^k \frac{a_i^{-} + a_i^{+}}{2i + 2}.$$

□

3.1 Approximating fuzzy numbers

In many applications of fuzzy mathematics it is more convenient to work with the fuzzy numbers belonging to one specific class due to use of its properties. If a problem has not this property, then one can approximate its fuzzy parameters with members of a class. The class $\mathcal{PF}(\mathbb{R})$ is suitable because it provides us many properties (see [2, 3]). In [1], the authors approximate fuzzy numbers with polynomial form fuzzy numbers by converting them into some linear programming problem. Here, using the well-known Weierstrass Theorem [18], we also present a theoretical result. The Weierstrass Theorem [18] shows that the continuous real-valued functions on a compact interval can be uniformly approximated by polynomials. At first, it is needed to introduce the notion of convergence for a sequence of fuzzy numbers.

Definition 3.3. A sequence of fuzzy numbers $\{\tilde{p}_n\}$ with $(\tilde{p}_n)_\alpha = [p_{\tilde{p}_n}^-(\alpha), p_{\tilde{p}_n}^+(\alpha)]$ is convergent to $\tilde{a}$ with $\tilde{a}_\alpha = [p_{\tilde{a}}^-(\alpha), p_{\tilde{a}}^+(\alpha)]$, if the sequences of functions $\{p_{\tilde{p}_n}^-\}$ and $\{p_{\tilde{p}_n}^+\}$ are respectively convergent to $p_{\tilde{a}}^-$ and $p_{\tilde{a}}^+$.

Theorem 3.1. Let $\tilde{a}$ be a fuzzy number with α -cut $\tilde{a}_\alpha = [p_{\tilde{a}}^-(\alpha), p_{\tilde{a}}^+(\alpha)]$ where the functions $p_{\tilde{a}}^-$ and $p_{\tilde{a}}^+$ are continues on $[0, 1]$. Then there is a sequence $\{\tilde{p}_n\} \subseteq \mathcal{PF}(\mathbb{R})$ that converges to $\tilde{a}$.

Proof. Since the functions $p_{\tilde{a}}^-$ and $p_{\tilde{a}}^+$ are continues, according to the Weierstrass Theorem, it follows that there exist sequences $\{p_n^-\}$ and $\{p_n^+\}$ such that p_n^- and p_n^+ are convergent to $p_{\tilde{a}}^-$ and $p_{\tilde{a}}^+$, respectively. We know that the functions $p_{\tilde{a}}^-$ and $p_{\tilde{a}}^+$ are increasing and decreasing, respectively. Then we can assume that the members of $\{p_n^-\}$ and $\{p_n^+\}$ are also increasing and decreasing, respectively. Define the fuzzy number $\tilde{p}_n$, $n \in \mathbb{N}$, with α -cut $(\tilde{p}_n)_\alpha = [p_n^-(\alpha), p_n^+(\alpha)]$. Obviously, $\tilde{p}_n \in \mathcal{PF}(\mathbb{R})$ and the sequence $\{\tilde{p}_n\}$ converges to $\tilde{a}$. $\square$

The following immediate result from Theorem 3.1 is so important to motivate use of polynomial form fuzzy numbers.

Corollary 3.1. Polynomial form fuzzy numbers are dense in the set of continuous fuzzy numbers.

4. Linear programming with fuzzy 2-degree polynomial form variables

In this section we consider a linear programming problem with polynomial form fuzzy variables formulated as follows:

$$\begin{aligned} (3a) \quad & \min \quad \mathbf{c}^T \tilde{\mathbf{x}}, \\ (3b) \quad & \text{s.t.} \quad A\tilde{\mathbf{x}} =_{\mathcal{R}} \tilde{\mathbf{b}}, \\ (3c) \quad & \mathbf{l} \leq \tilde{\mathbf{x}} \leq \mathbf{u}, \end{aligned}$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $\tilde{\mathbf{b}} = [\tilde{b}_1, \dots, \tilde{b}_m]^T \in (\mathcal{PF}^2(\mathbb{R}))^m$, $\mathbf{c} = [c_1, \dots, c_n]^T$, $\mathbf{l} = [l_1, \dots, l_n]^T \in \mathbb{R}^n$, $\mathbf{u} = [u_1, \dots, u_n]^T \in \mathbb{R}^n$ are given and the fuzzy vector $\tilde{\mathbf{x}} = [\tilde{x}_1, \dots, \tilde{x}_n]^T \in (\mathcal{PF}^2(\mathbb{R}))^n$ is to be determined.

The constraints (3b) are in the form of the soft equalities, namely, their left side fuzzy number is approximately equal to the right side one. Depending on the problem type, one can replace them with constraints in the form of the inequalities $\leq_{\mathcal{R}}$ and $\geq_{\mathcal{R}}$. The constraints (3c) containing the hard inequality $\leq$ are called the bound constraints. These constraints guarantee that the support of $\tilde{x}_j$, $j = 1, \dots, n$, belongs to $[l_j, u_j]$. To define the soft equalities and inequalities, we use a ranking function $\mathcal{R} : \mathcal{PF}^2(\mathbb{R}) \rightarrow \mathbb{R}$ defined as

$$(4) \quad \mathcal{R}(\tilde{x}) = r_0^- x_0^- + r_1^- x_1^- + r_2^- x_2^- + r_0^+ x_0^+ + r_1^+ x_1^+ + r_2^+ x_2^+,$$

for every $\tilde{x} = (x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+) \in \mathcal{PF}^2(\mathbb{R})$ in which $r_1^-, r_2^-, r_3^-, r_1^+, r_2^+, r_3^+$ are nonnegative real numbers. It is easy to see that $\mathcal{R}$ has the linearity property,

that is, $\mathcal{R}(\tilde{x} + c\tilde{y}) = \mathcal{R}(\tilde{x}) + c\mathcal{R}(\tilde{y})$ for every $\tilde{x}, \tilde{y} \in \mathcal{PF}^2(\mathbb{R})$ and every $c \in \mathbb{R}$. A specific case of the function $\mathcal{R}$ is Yager's ranking function defined as

$$(5) \quad \mathcal{R}(\tilde{x}) = \frac{1}{2}x_0^- + \frac{1}{4}x_1^- + \frac{1}{6}x_2^- + \frac{1}{2}x_0^+ + \frac{1}{4}x_1^+ + \frac{1}{6}x_2^+,$$

for every $\tilde{x} = (x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+) \in \mathcal{PF}^2(\mathbb{R})$ (see Lemma 3.1). From the viewpoint of the linear algebra, the ranking function $\mathcal{R}$ is a linear functional on the vector space $\mathcal{PF}^2(\mathbb{R})$.

To obtain an optimal solution of the problem (3), we will reduce the problem to a linear programming problem with crisp variables. Before stating details, we need some initial notions of the linear programming.

Definition 4.1. A fuzzy vector $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$ is called a fuzzy feasible solution to the problem (3), if $\tilde{\mathbf{x}}$ satisfies the constraints (3b)-(3c).

Definition 4.2. A fuzzy feasible solution $\tilde{\mathbf{x}}^* \in (\mathcal{PF}^2(\mathbb{R}))^n$ is an optimal solution of the problem (3) if

$$\mathcal{R}(c\tilde{\mathbf{x}}^*) \leq \mathcal{R}(c\tilde{\mathbf{x}})$$

for any fuzzy feasible solution $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$.

Since any decision variable $\tilde{x}_j$, $j = 1, 2, \dots, n$, is a fuzzy 2-degree polynomial form number, it can be denoted by

$$(6) \quad \tilde{x}_j = ((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+),$$

whose α -cut is $(\tilde{x}_j)_\alpha = [(x_j)_0^- + (x_j)_1^- \alpha + (x_j)_2^- \alpha^2, (x_j)_0^+ + (x_j)_1^+ \alpha + (x_j)_2^+ \alpha^2]$ for every $\alpha \in [0, 1]$. Based on the linearity property of the ranking function $\mathcal{R}$, notice that a feasible solution $\tilde{\mathbf{x}}^* \in (\mathcal{PF}^2(\mathbb{R}))^n$ is optimal if

$$\sum_{j=1}^n c_j \mathcal{R}(\tilde{x}_j^*) \leq \sum_{j=1}^n c_j \mathcal{R}(\tilde{x}_j)$$

or equivalently,

$$(7) \quad \begin{aligned} & \sum_{j=1}^n c_j (r_0^-(x_j^*)_0^- + r_1^-(x_j^*)_1^- + r_2^-(x_j^*)_2^- + r_0^+(x_j^*)_0^+ + r_1^+(x_j^*)_1^+ + r_2^+(x_j^*)_2^+) \\ & \leq \sum_{j=1}^n c_j (r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+), \end{aligned}$$

for every feasible solution $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$. Similarly, the linearity of the ranking function $\mathcal{R}$ implies that a solution $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$ satisfies the constraints (3b) if

$$\sum_{j=1}^n a_i \mathcal{R}(\tilde{x}_j) = \mathcal{R}(\tilde{b}_i), \quad i = 1, \dots, m,$$

where a_i is the i th row of the matrix A . On the other hand, it is notable that the variable $\tilde{x}_j$ satisfies its bound constraint if $l_j \leq (x_j)_0^-$ and $(x_j)_0^+ \leq u_j$. Hence, $\tilde{\mathbf{x}} \in (\mathcal{PF}^2(\mathbb{R}))^n$ is a feasible solution of the problem (3) if

$$\begin{aligned} & \sum_{j=1}^n a_{ij}(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+) \\ &= r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \dots, m, \\ & (x_j)_0^- \geq l_j \quad j = 1, 2, \dots, n, \\ & (x_j)_0^+ \leq u_j \quad j = 1, 2, \dots, n. \end{aligned}$$

Therefore, solving the problem (3) is equivalent to solving the following optimization problem.

$$\begin{aligned} (8a) \quad & \min \sum_{j=1}^n c_j(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+), \\ (8b) \quad & \text{s.t.} \sum_{j=1}^n a_{ij}(r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+) \\ (8c) \quad & = r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \dots, m, \\ (8d) \quad & (x_j)_0^- \geq l_j \quad j = 1, 2, \dots, n, \\ & (x_j)_0^+ \leq u_j \quad j = 1, 2, \dots, n, \\ (8e) \quad & \tilde{x}_j = ((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+) \in \mathcal{PF}^2(\mathbb{R}) \quad j = 1, 2, \dots, n. \end{aligned}$$

The problem (8) is a linear programming problem with the additional conditions (8e). These conditions guarantee that a fuzzy 2-degree polynomial form number $\tilde{x}_j$ can be constructed from any $((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+) \in \mathbb{R}^6$. Note that if we eliminate the conditions (8e), then the obtained optimal solution may not be a member of $\mathcal{PF}^2(\mathbb{R})$ and consequently not be an optimal solution of the problem (3). Now we try that replace the conditions (8e) with some linear constraints.

Lemma 4.1. $(x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+)$ is a member of $\mathcal{PF}^2(\mathbb{R})$ if and only if the following constraints hold.

$$\begin{aligned} (9) \quad & x_1^- \geq 0, \\ (10) \quad & x_1^+ \leq 0, \\ (11) \quad & x_1^- + 2x_2^- \geq 0, \\ (12) \quad & x_1^+ + 2x_2^+ \leq 0, \\ (13) \quad & x_0^- + x_1^- + x_2^- \leq x_0^+ + x_1^+ + x_2^+. \end{aligned}$$

Proof. We know that $(x_0^-, x_1^-, x_2^-, x_0^+, x_1^+, x_2^+)$ is a member of $\mathcal{PF}^2(\mathbb{R})$ iff the polynomials $p^-(\alpha) = x_0^- + x_1^-\alpha + x_2^-\alpha^2$ and $p^+(\alpha) = x_0^+ + x_1^+\alpha + x_2^+\alpha^2$ are respectively nondecreasing and nonincreasing on $[0, 1]$, and equivalently, the values of the first derivatives of the functions $p^-(.)$ and $p^+(.)$ are respectively nonnegative and nonpositive on the unit interval $[0, 1]$. Since the first derivatives

of the functions are linear, it is sufficient that the conditions (9)-(12) are satisfied. Furthermore, the condition (13) guarantees that the left spread always appear before the right spread. $\square$

Theorem 4.1. *Solving the problem (3) is equivalent to solving the problem*

$$(14a) \quad \min \sum_{j=1}^n c_j (r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+),$$

$$(14b) \quad s.t. \sum_{j=1}^n a_{ij} (r_0^-(x_j)_0^- + r_1^-(x_j)_1^- + r_2^-(x_j)_2^- + r_0^+(x_j)_0^+ + r_1^+(x_j)_1^+ + r_2^+(x_j)_2^+)$$

$$(14c) \quad = r_0^-(b_i)_0^- + r_1^-(b_i)_1^- + r_2^-(b_i)_2^- + r_0^+(b_i)_0^+ + r_1^+(b_i)_1^+ + r_2^+(b_i)_2^+ \quad i = 1, \dots, m,$$

$$(14d) \quad (x_j)_0^- \geq l_j \quad j = 1, 2, \dots, n,$$

$$(14e) \quad (x_j)_0^+ \leq u_j \quad j = 1, 2, \dots, n,$$

$$(14f) \quad (x_j)_1^- \geq 0 \quad j = 1, \dots, n,$$

$$(14g) \quad (x_j)_1^+ \leq 0 \quad j = 1, \dots, n,$$

$$(14h) \quad (x_j)_1^- + 2(x_j)_2^- \geq 0 \quad j = 1, \dots, n,$$

$$(14i) \quad (x_j)_1^+ + 2(x_j)_2^+ \leq 0 \quad j = 1, \dots, n,$$

$$(14j) \quad (x_j)_0^- + (x_j)_1^- + (x_j)_2^- \leq (x_j)_0^+ + (x_j)_1^+ + (x_j)_2^+ \quad j = 1, \dots, n,$$

where $\tilde{x}_j = ((x_j)_0^-, (x_j)_1^-, (x_j)_2^-, (x_j)_0^+, (x_j)_1^+, (x_j)_2^+)$ for $i = 1, \dots, n$.

Proof. Based on Lemma 4.1 and the above argument, the proof is trivial. $\square$

Remark 4.1. It is notable that any optimal solution $\tilde{\mathbf{x}}^* \in (\mathcal{PF}^2(\mathbb{R}))^n$ of the problem is an LR fuzzy number whenever the constraints (14j) are in the equality form.

Remark 4.2. If some constraints (3c) are in the form of $\geq_{\mathcal{R}}$ or $\leq_{\mathcal{R}}$, then the corresponding constraints (14b) are respectively in the form of $\geq$ and $\leq$.

A defect of the methods based on ranking functions for solving the linear programming problem with fuzzy variables is that the corresponding crisp linear programming problem has usually many optimal solutions because the number of its decision variables is several times greater than that of the corresponding fuzzy linear programming problem. As an instance, the problem (3) contains n fuzzy variables while the problem (14) has $6 \times n$ crisp variables as well as some additional constraints. Therefore, we have to choose the optimal solutions of the crisp problem (14) more carefully to be able to construct the fuzzy solutions with more quality. A remedy is to we restrict the support of fuzzy variables. Let s_{max} be the maximum length of the supports of the fuzzy right side numbers $\tilde{b}_i$, i.e.,

$$s_{max} = \max\{(b_i)_0^+ - (b_i)_0^- : i = 1, \dots, m\}.$$

To obtain the fuzzy variables $\tilde{x}_j$ which are rather like right side numbers, we can add the constraints

$$(15) \quad (x_j)_0^+ - (x_j)_0^- \leq s_{max} \quad j = 1, 2, \dots, n,$$

to the problem (14) to restrict the length of the support of $\tilde{x}_j$'s.

	Market 1	Market 2	Market 3
Source 1	22	20	25
Source 2	18	21	24

Table 1: The unit cost of shipment from sources to markets.

4.1 An applicable example

Here, we consider an instance of the well-known transportation problem in which the values of demands and supplies are polynomial form fuzzy numbers. Since the technological coefficients of the problem are 0 or 1, it follows that the decision variables have to take fuzzy values. Hence, this problem is an instance of the problems which can be formulated only with fuzzy variables.

Let us state the problem. A lumber company has two sources of wood and 3 markets where wood is demanded. The approximate quantity of wood available in the next month in the two sources of supply is $\tilde{78} = (76, 0, 2, 80, -1, -1)$ and $\tilde{69} = (67, 1.5, 0.5, 71, -1, -1)$ million board feet, respectively. The amount that can be sold at the 3 markets is approximately equal to $\tilde{28} = (26, 0, 2, 30, -1, -1)$, $\tilde{46} = (44.5, 1, 0.5, 48.5, -1.5, -1)$, and $\tilde{37} = (35, 1.5, 0.5, 39, -0.5, -1.5)$ million board feet, respectively. Figure 2 shows the fuzzy values of the demands and the supplies. The company currently transports all of the wood by trains whose capacity is 35 million board feet. The unit cost of shipment is described in Table 1. The management needs to decide to how ship the wood from the sources to the markets with minimum cost. The problem can be formulated as an instance of the problem (3) in the following form.

$$\begin{aligned}
 \min \quad & z = 22\tilde{x}_{11} + 20\tilde{x}_{12} + 25\tilde{x}_{13} + 18\tilde{x}_{21} + 21\tilde{x}_{22} + 24\tilde{x}_{23} \\
 s.t. \quad & \tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} \leq_{\mathcal{R}} \tilde{78}, \\
 & \tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} \leq_{\mathcal{R}} \tilde{69}, \\
 & \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{28}, \\
 & \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{46}, \\
 & \tilde{x}_{12} + \tilde{x}_{22} =_{\mathcal{R}} \tilde{37}, \\
 & 0 \leq \tilde{x}_{ij} \leq 35 \quad i = 1, 2, \quad j = 1, 2, 3.
 \end{aligned}$$

Since the right side numbers are LR fuzzy numbers, it is natural that variables are also LR fuzzy number. So we converted the problem into the corresponding crisp problem with the constraints (14j) in the equality form. We applied Yager's ranking function defined as (5) to solve the problem. We added the constraints (15) to the problem for $s_{\max} = 4$. We used the software Matlab, version 2016a, to solve the crisp linear programming problem by 5 different algorithms named "interior point legacy", "interior point", "active set", "simplex", and "dual-simplex". The optimal solutions obtained are depicted in Figure 2. It is notable that the optimal solution obtained by the simplex algorithm has crisp values

since the simplex algorithm, looks for an optimal basic solution, that is, an optimal solution with the most number of zero variables.

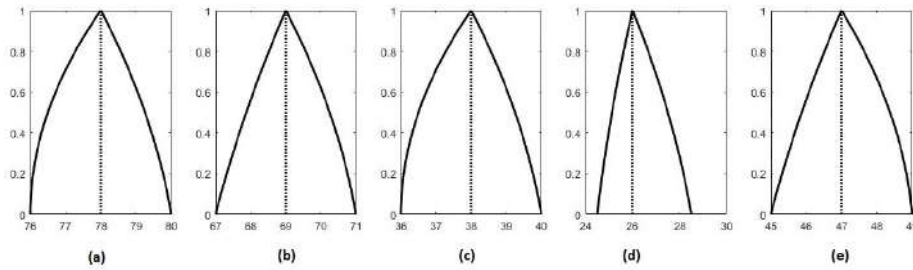


Figure 1: The fuzzy values of demands and supplies: (a) the supply of the first source (b) the supply of the second source; (c) the demand of the first market; (d) the demand of the second market; (e) the demand of the third market.

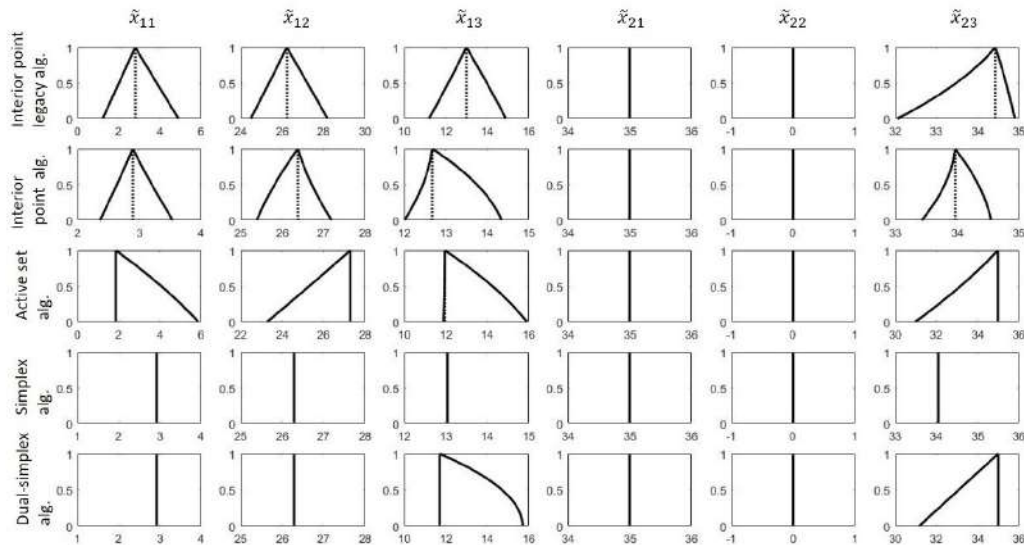


Figure 2: Optimal solutions obtained by Matlab.

5. Conclusion

In this paper, a special class of fuzzy numbers, whose members are called polynomial form fuzzy numbers, is investigated. Some properties of the class are presented. Specially, it is shown that its members are dense in the set of continuous fuzzy numbers. Furthermore, a linear programming problem with fuzzy

2-degree polynomial form variables is studied. To solve the problem, a method based on Yager's ranking function is proposed. Since any fuzzy trapezoidal number is also a fuzzy 2-degree polynomial form number, the proposed method generalizes the known methods using this ranking function to solve linear programming problems with trapezoidal variables [8].

Even though, the class of polynomial form fuzzy numbers contains many algebraic properties, it is less interested by authors (specially in the field of fuzzy optimization). As future works, we propose use of this class in different fields of fuzzy optimization such as fuzzy game theory and fuzzy regression.

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Certain types of functions by using supra $\hat{\omega}$ -open sets

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Abstract. The target of this research is to present and discuss some kinds of functions in the supra spaces and some of their strongest and weakest forms, and we will introduce the relation between these forms for each type. The results that we reached will be supported by proofs and examples.

Keywords: supra* $\hat{\omega}$ -continuous function, supra* $\hat{\omega}$ -irresolute function, supra* $\hat{\omega}^*$ -compact function, supra* $\hat{\omega}^{**}$ -compact function, strongly supra* $\hat{\omega}$ -closed function.

1. Introduction

The functions are important connectors between different spaces, and they are one of the common concepts in the world of topology, we will deal with some types of functions connect between supra spaces. The first researcher who introduced the supra spaces was Mashhour, A. S [1] (1983). A good number of researchers followed him dealt with many issues in this spaces such as the researchers Vidyarani, L. and Vigneshwaran, M. where they presented supra N -compact also supra N -connected in the supra spaces [2]. Also, we will introduce some functions in this spaces like supra* $\hat{\omega}$ -continuous, strongly supra* $\hat{\omega}$ -continuous, supra* $\hat{\omega}$ -irresolute, supra* $\hat{\omega}$ -compact, supra* $\hat{\omega}^*$ -compact, supra* $\hat{\omega}^{**}$ -compact and supra* $\hat{\omega}$ -closed, totally supra* $\hat{\omega}$ -closed, strongly supra* $\hat{\omega}$ -closed functions, we will give some theorems and propositions in this paper.

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2. On supra topology

At the beginning we will present the definition of supra space and some of its properties, also its relationship with the topological space. As well as the concept of supra $\hat{\omega}$ -open, supra $\hat{\eta}$ -open sets and the relationship between them and with the supra open sets. We will use the abbreviation "su." to express the "supra".

Definition 2.1 ([2], [3], [4]). Let X be a non-empty set, the su. topology μ is a sub collection of $P(X)$ in which ϕ , X and $\bigcup_{\alpha \in \Lambda} W_\alpha$ are belong to μ , where $W_\alpha \in \mu$. The su. space is denoted by (X, μ) , any set W in μ is su. open set, W^c is su. closed set.

Definition 2.2 ([5], [6]). The su. closure for a set W in a su. space X , (symbolizes it $cl^\mu(W)$) defined as: $cl^\mu(W) = \bigcap \{M | M^c \in \mu \text{ and } W \subseteq M\}$. While the su. interior for W , (symbolizes it $int^\mu(W)$) is defined by: $int^\mu(W) = \bigcup \{B | B \in \mu \text{ and } B \subseteq W\}$.

Remark 2.3 ([7]). Any topology is su. topology, since every topology includes ϕ, X and it is closed under the infinite union. This remark is irreversible.

Example 2.4. In the su. space (X, μ) , where $X = \{1, 2, 3\}$, $\mu = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$, μ is su. topology on X but not topology since $\{1, 3\} \cap \{2, 3\} = \{3\} \notin \mu$.

Definition 2.5. Any set W in (X, μ) is called su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set, if for any element $a \in W$ there is a set $B \in \mu$ containing a , with $B - W$ is countable (resp. finite). W^c is called su. $\hat{\omega}$ -closed (resp. su. $\hat{\eta}$ -closed) set.

Example 2.6. The set $R - \{0\}$ in the co-finite su. space (R, μ_{cof}) is su. $\hat{\omega}$ -open and su. $\hat{\eta}$ -open set.

Remark 2.7. Any su. open set is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open), since if (X, μ) is a su. space, $W \in \mu$ and $a \in W$, take $B \in \mu$, put $B = W$, thus $a \in B$ and $B - W = W - W = \phi$ is countable (resp. finite), so W is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set.

Remark 2.8. Any su. $\hat{\eta}$ -open set is su. $\hat{\omega}$ -open, because, if W is a su. $\hat{\eta}$ -open subset of (X, μ) , so for any $a \in W$ there is $B \in \mu$, with $a \in B$ such that $B - W$ is a finite set, accordingly it is a countable set (since every finite set is countable), therefore W is a su. $\hat{\omega}$ -open set. This remark is irreversible.

Example 2.9. The set $\{x\}$ in the indiscrete su. space (Z, μ_{ind}) , where Z is the set of all integers, is su. $\hat{\omega}$ -open set but neither su. open set nor su. $\hat{\eta}$ -open.

Definition 2.10 ([9], [10]). If $f^{-1}(W)$ is su. open (resp. su. closed) set in the su. space (X, μ_X) for any su. open (resp. su. closed) set W in the su. space $(Y,$

μ_Y). Then, the function f from the su. space (X, μ_X) into the space (Y, μ_Y) is called su*. continuous function.

Example 2.11. $f : (X, \mu_D) \longrightarrow (Y, \mu_Y)$ is su*. continuous function.

Definition 2.12. Let V be any su. open subset of the su. space (Y, μ_Y) , and a be any point in the su. space (X, μ_X) with $f(a) \in V$, if there is a su. $\hat{\omega}$ -open subset W of X containing a , and $f(W) \subseteq V$, then $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is called a su*. $\hat{\omega}$ -continuous function at the point a . When f is su*. $\hat{\omega}$ -continuous at any point in its domain, then it is called a su*. $\hat{\omega}$ -continuous function.

Example 2.13. The function f from the excluded point su. space (X, μ_{EX}) where X is a countable set into any su. space (Y, μ_Y) is su*. $\hat{\omega}$ -continuous function, where $\mu_{EX} = \{W \subseteq X, a_o \notin W \text{ for some } a_o \in X\} \cup \{X\}$.

Definition 2.14. Let V be any su. open subset of the su. space (Y, μ_Y) , and a be any point in the su. space (X, μ_X) with $f(a) \in V$, if there is a su. $\hat{\eta}$ -open subset W of X containing a , and $f(W) \subseteq V$, then $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is called a su*. $\hat{\eta}$ -continuous function at the point a . When f is su*. $\hat{\eta}$ -continuous at any point in its domain, then it is called a su*. $\hat{\eta}$ -continuous function.

Example 2.15. The function $f : (X, \mu_{EX}) \longrightarrow (Y, \mu_Y)$ is su*. $\hat{\omega}$ -continuous function.

Proposition 2.16 *The infinite union of su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) sets is also su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open).*

Proof. Let $\{U_\alpha | \alpha \in \Lambda\}$ be a collection of su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) sets in a su. space X , and let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$, so there is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set U_{α_j} for some $\alpha_j \in \Lambda$ with $x \in U_{\alpha_j}$, from definition of su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set, there is a su. open set G_j in X containing x and $G_j - U_{\alpha_j}$ is countable (resp. finite), since $G_j - \bigcup_{\alpha \in \Lambda} U_\alpha \subseteq G_j - U_{\alpha_j}$, then $G_j - \bigcup_{\alpha \in \Lambda} U_\alpha$ is countable (resp. finite), therefore $\bigcup_{\alpha \in \Lambda} U_\alpha$ is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set.

Lemma 2.17. *The finite union of su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) sets is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set.*

Proof. Let W_1 and W_2 be su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) sets in a su. space (X, μ_X) , and $a \in W_1 \cup W_2 \implies a \in W_1$ or $a \in W_2$, if $a \in W_1 \implies$ there is $B \in \mu$ with $a \in B$ and $B - W_1$ is countable (resp. finite), but $B - (W_1 \cup W_2) \subseteq B - W_1 \implies B - (W_1 \cup W_2)$ is countable (resp. finite) $\implies W_1 \cup W_2$ is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open). By the same way if $a \in W_2$.

Definition 2.18. A point a is a su. $\hat{\omega}$ -interior (resp. su. $\hat{\eta}$ -interior) point for a subset W of the su. space (X, μ) , if there is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set B in X , in which $a \in B \subseteq W$.

Example 2.19. Any point in any subset of the co-finite su. space (Z, μ_{cof}) is su. $\hat{\omega}$ -interior point.

Proposition 2.20. A subset W of (X, μ) is su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set iff all the points in W are su. $\hat{\omega}$ -interior (resp. su. $\hat{\omega}$ -interior) points to it.

Proof. Suggest $a \in W$, where W is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) subset of X , since each set is a subset of itself, so a is su. $\hat{\omega}$ -interior (resp. su. $\hat{\eta}$ -interior) point to W . conversely, since W is a union of its points, and every point in W is su. $\hat{\omega}$ -interior (resp. su. $\hat{\eta}$ -interior) point to it, that follows for each $a \in W$ there is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set B_a in X , in which $a \in B_a \subseteq W$, in order that W is a union of the su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) sets B_a for any a in W . Hence W is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) set (by proposition (2.16)).

Example 2.21. Any set in the co-finite su. space (Z, μ_{cof}) is su. $\hat{\omega}$ -open set, so every point in this set is su. $\hat{\omega}$ -interior point.

Theorem 2.22. The function $f : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is su^* . $\hat{\omega}$ -continuous function iff $f^{-1}(V)$ is su. $\hat{\omega}$ -open set in X , for any $V \in \mu_Y$.

Proof. Suppose V is a su. open set in Y with $f(a) \in V$, where $a \in X \implies f^{-1}(f(a)) \in f^{-1}(V) \implies a \in f^{-1}(V) \implies f(a) \in f(f^{-1}(V)) \subseteq V$, therefore just by taking W (required set in definition (2.12)) equal to $f^{-1}(V)$ we get the required. Conversely, if f is su^* . $\hat{\omega}$ -continuous function and $V \in \mu_Y$ with $f(a) \in V$, in which a is any point in X , since f is su^* . $\hat{\omega}$ -continuous, thus there is a su. $\hat{\omega}$ -open set W in X where $a \in W$ and $f(W) \subseteq V \implies f(a) \in f(W) \subseteq V \implies a \in W \subseteq f^{-1}(f(W)) \subseteq f^{-1}(V)$, then a is a su. $\hat{\omega}$ -interior point to $f^{-1}(V)$, then $f^{-1}(V)$ is a su. $\hat{\omega}$ -open set in X (by proposition (2.20)).

Example 2.23. The function $f : (X, \mu_{EX}) \rightarrow (X, \mu_{EX})$ is su^* . $\hat{\omega}$ -continuous function.

Corollary 2.24. Any function $f : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is su^* . $\hat{\omega}$ -continuous if and only if $f^{-1}(V)$ is su. $\hat{\omega}$ -closed set in X for each $V^c \in \mu_Y$.

Proof. Let f be a su^* . $\hat{\omega}$ -continuous function and $V^c \in \mu_Y$, thus $f^{-1}(V^c)$ is su. $\hat{\omega}$ -open set in X , since $(f^{-1}(V))^c = f^{-1}(V^c)$, hence $(f^{-1}(V))^c$ is su. $\hat{\omega}$ -open set, thus $((f^{-1}(V))^c)^c = f^{-1}(V)$ is su. $\hat{\omega}$ -closed set in X . Conversely, take V as a su. open set in Y so V^c is su. closed set and then $f^{-1}(V)^c$ is su. $\hat{\omega}$ -closed (Given) which is equal to $(f^{-1}(V))^c$, and $((f^{-1}(V))^c)^c = f^{-1}(V)$ is su. open set of X , therefore f is su^* . $\hat{\omega}$ -continuous function.

Example 2.25. The function $f : (X, \mu_{ind}) \rightarrow (Y, \mu_Y)$ is su^* . $\hat{\omega}$ -continuous function, where X is a finite set.

Definition 2.26. Whenever the inverse image of any $V \in \mu_Y$, is su. $\hat{\eta}$ -open set in a su. space (X, μ_X) , so f from (X, μ_X) into (Y, μ_Y) is called a su^* . $\hat{\eta}$ -continuous function.

Example 2.27. The function f from the included point su. space (X, μ_I) into the discrete su. space (X, μ_D) where X is a finite set, is su^* . $\hat{\eta}$ -continuous function, where $\mu_I = \{W \subseteq X | a_o \in W, \text{ for some } a_o \in X\} \cup \{\phi\}$.

Proposition 2.28. *If the inverse image of all su. closed set in a su. space (Y, μ_Y) is su. $\hat{\eta}$ -closed in a su. space (X, μ_X) , then f from the space X into the space Y is su^* . $\hat{\eta}$ -continuous function.*

Example 2.29. The function f from the discrete su. space (X, μ_D) into any su. space (Y, μ_Y) is su^* . $\hat{\eta}$ -continuous function.

Remark 2.30. Any su^* . $\hat{\eta}$ -continuous function f is su^* . $\hat{\omega}$ -continuous, because if we take $f : (X, \mu_X) \rightarrow (Y, \mu_Y)$ as a su^* . $\hat{\eta}$ -continuous function and $W \in \mu_Y$, then $f^{-1}(W)$ is a su. $\hat{\eta}$ -open set in (X, μ_X) , hence it is su. $\hat{\omega}$ -open (by Remark 2.8), therefore f is su^* . $\hat{\omega}$ -continuous.

Example 2.31. The function f from the indiscrete su. space (Z, μ_{ind}) into any su. space (Y, μ_Y) is su^* . $\hat{\omega}$ -continuous function but not su^* . $\hat{\eta}$ -continuous.

Definition 2.32. Whenever $f^{-1}(M) \in \mu_X$ for any su. $\hat{\omega}$ -open set M in (Y, μ_Y) , hence $f : (X, \mu_X) \rightarrow (Y, \mu_Y)$ is strongly su^* . $\hat{\omega}$ -continuous function.

Example 2.33. The function $f : (X, \mu_X) \rightarrow (X, \mu_{ind})$ where $f(a) = d$ for any $a \in X$, satisfying the preceding definition.

Proposition 2.34. *Give f as a function from the su. space (X, μ_X) into the su. space (Y, μ_Y) , when $f^{-1}(V)$ is su. closed set in the su. space (X, μ_X) for any su. $\hat{\omega}$ -closed set V in the su. space (Y, μ_Y) , then f is strongly su^* . $\hat{\omega}$ -continuous function.*

Example 2.35. The function $f : (R, \mu_{ind}) \rightarrow (R, \mu_D)$ is strongly su^* . $\hat{\omega}$ -continuous function, where f is a constant function.

Definition 2.36. If $f^{-1}(M) \in \mu_X$ of each su. $\hat{\eta}$ -open set M in the su. space (Y, μ_Y) , then the function f from X into Y is called strongly su^* . $\hat{\eta}$ -continuous function.

Example 2.37. The identity function from the co-countable su. space (X, μ_{coc}) into the same su. space where X is uncountable set, is strongly su^* . $\hat{\eta}$ -continuous function.

Proposition 2.38. *The function f from the su. space (X, μ_X) into the su. space (Y, μ_Y) , is strongly su^* . $\hat{\eta}$ -continuous function, whenever the inverse image of any su. $\hat{\eta}$ -closed set in Y is su. closed set in X .*

Example 2.39. The function $f : (X, \mu_D) \longrightarrow (Y, \mu_Y)$ is also strongly su^* . $\hat{\eta}$ -continuous function.

Remark 2.40. Every strongly su^* . $\hat{\omega}$ -continuous function is strongly su^* . $\hat{\eta}$ -continuous function, because if $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is strongly su^* . $\hat{\omega}$ -continuous function and W is su. $\hat{\eta}$ -open set in Y , then it is su. $\hat{\omega}$ -open set (by remark (2.8)), but f is strongly su^* . $\hat{\omega}$ -continuous function, so $f^{-1}(W) \in \mu_X$, which implies f is strongly su^* . $\hat{\eta}$ -continuous function. The converse is incorrect.

Example 2.41. The identity function from the co-finite su. topology defined on a countable set into the indiscrete su. topology defined on the same set, is strongly su^* . $\hat{\eta}$ -continuous function but not strongly su^* . $\hat{\omega}$ -continuous function.

Definition 2.42. Whenever the inverse image for any su. $\hat{\omega}$ -open set in (Y, μ_Y) is su. $\hat{\omega}$ -open set in (X, μ_X) , so that $f : X \longrightarrow Y$ is called su^* . $\hat{\omega}$ -irresolute function.

Example 2.43. A function f from the included point su. space (Z, μ_I) into the co-finite su. space (Z, μ_{cof}) is su^* . $\hat{\omega}$ -irresolute function.

Proposition 2.44. *If $f^{-1}(V)$ is su. $\hat{\omega}$ -closed set in the su. space (X, μ_X) for any su. $\hat{\omega}$ -closed set V in the su. space (Y, μ_Y) , then the function f from the su. space (X, μ_X) into the su. space (Y, μ_Y) is su^* . $\hat{\omega}$ -irresolute function.*

Example 2.45. The identity function I_X from the excluded point su. space (X, μ_{EX}) into the same su. space is su^* . $\hat{\omega}$ -irresolute function.

Definition 2.46. When the inverse image of any su. $\hat{\eta}$ -open set in (Y, μ_Y) is su. $\hat{\eta}$ -open set in (X, μ_X) , hence the function f from X into Y is called su^* . $\hat{\eta}$ -irresolute function.

Example 2.47. The function f from elective su. topology on an infinite set to the indiscrete su. topology on the same set in which $f(a) = d$ for any a in the domain, is su^* . $\hat{\eta}$ -irresolute function.

Proposition 2.48. *If $f^{-1}(V)$ is su. $\hat{\eta}$ -closed set of the su. space (X, μ_X) for any su. $\hat{\eta}$ -closed set V in the su. space (Y, μ_Y) , hence the function f from the space X into the space Y is su^* . $\hat{\eta}$ -irresolute function.*

Example 2.49. The function f from a su. space (X, μ_X) into a su. space (Y, μ_Y) in which X is finite, is su^* . $\hat{\eta}$ -irresolute function.

Remark 2.50. There is no relation between su^* . $\hat{\omega}$ -irresolute and su^* . $\hat{\eta}$ -irresolute function.

Example 2.51. The identity function I_R from the co-finite su . space (R, μ_{cof}) into the co-countable su . space (R, μ_{coc}) is su^* . $\hat{\omega}$ -irresolute but not su^* . $\hat{\eta}$ -irresolute function.

Definition 2.52 ([8]). A point a in (X, μ) is a condensation point to a subset M of X if $W \cap M$ is uncountable set for all $W \in \mu$ containing a .

Lemma 2.53. *Whenever a subset M of a su . space (X, μ) contains all its condensation points, then it is su . $\hat{\omega}$ -closed subset of X .*

Proof. Suppose M contains all its condensation points, to prove M is su . $\hat{\omega}$ -closed set, that means to prove M^c is su . $\hat{\omega}$ -open set. Let $x \in M^c$, so it is not a condensation point to M (from hypothesis), then there is $U \in \mu$ with $x \in U$ and either $U \cap M = \emptyset$ or $U \cap M$ is countable, if $U \cap M = \emptyset$, then $x \in U \subseteq M^c$, and since U is su . open set then it is su . $\hat{\omega}$ -open (by remark (2.7)), therefore x is a su . $\hat{\omega}$ -interior point to M^c , that means M^c is a su . $\hat{\omega}$ -open set, and then M is su . $\hat{\omega}$ -closed set in X . Now if $U \cap M$ is countable, let $G = U - M$, so $x \in G \subseteq M^c$, to prove G is su . $\hat{\omega}$ -open set, let $y \in G$, so there is a su . open set U in X containing y and $U - G$ is countable, thus G is su . $\hat{\omega}$ -open set, and then M^c is $\hat{\omega}$ -open set, therefore M is su . $\hat{\omega}$ -closed.

Example 2.54. The set of all irrational numbers Q^c in the co-finite su . space (R, μ_{cof}) is not su . $\hat{\omega}$ -closed set, but the set of all rational numbers Q is su . $\hat{\omega}$ -closed.

Lemma 2.55. 1- Any su . closed set M in (X, μ) is su . $\hat{\omega}$ -closed.

Proof. Consider M is a su . closed set in X and $a \in X$ with $a \notin M$, so $a \in M^c \in \mu$, but $M \cap M^c = \emptyset$ is countable set, then a is not condensation point to M , since a is arbitrary point in X , hence M includes all its condensation points, therefore M is su . $\hat{\omega}$ -closed in X (by lemma (2.53)).

2- Any su . closed set M in (X, μ) is su . $\hat{\eta}$ -closed.

Proof. Consider M is a su . closed set in X , so M^c is su . open set in X , and by (remark (2.7)) it is su . $\hat{\eta}$ -open set, therefore $(M^c)^c$ is su . $\hat{\eta}$ -closed set.

Example 2.56. In the excluded point su . space (X, μ_{EX}) where X is a finite set, every set is su . $\hat{\omega}$ -closed and su . $\hat{\eta}$ -closed set but not all these sets are su . closed, for instance $\{x\}$ is su . $\hat{\omega}$ -closed and su . $\hat{\eta}$ -closed set but not su . closed.

Remark 2.57. Every su . $\hat{\eta}$ -closed set is su . $\hat{\omega}$ -closed, since if M is su . $\hat{\eta}$ -closed set in a su . space (X, μ_X) then M^c is su . $\hat{\eta}$ -open set and the by remark (2.8) it is su . $\hat{\omega}$ -open, hence $(M^c)^c = M$ is su . $\hat{\omega}$ -closed.

Example 2.58. The set $Z - \{x\}$ in the indiscrete su. space (Z, μ_{ind}) is su. $\hat{\omega}$ -closed but not su. $\hat{\eta}$ -closed set.

Remark 2.59. Any su*. continuous function is su*. $\hat{\omega}$ -continuous and su*. $\hat{\eta}$ -continuous function.

Definition 2.60. A class $\{W_\alpha | \alpha \in \Lambda\}$ of su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) subsets of (X, μ) is called a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) cover to a subset S of X whenever $S \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha$, when $S = X$, then $\{W_\alpha | \alpha \in \Lambda\}$ is a su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) cover to X .

Definition 2.61. A set S in (X, μ) is a su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) if any su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) cover for S possesses a finite sub cover, when $X = S$, then X is su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) space.

Example 2.62. Let (R, μ_{ind}) be the indiscrete su. space where $\mu_{ind} = \{\phi, R\}$, it is su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) space. Since any su. $\hat{\omega}$ -open (resp. su. $\hat{\eta}$ -open) cover can be reduced to a finite sub cover.

Remark 2.63. Every su. $\hat{\omega}$ -compact set is su. $\hat{\eta}$ -compact, because if $G = \{W_\alpha | \alpha \in \Lambda\}$ is an $\hat{\eta}$ -open cover for a set M in a su. space (X, μ) where each $W_\alpha, \alpha \in \Lambda$ is su. $\hat{\eta}$ -open set in X and hence it is su. $\hat{\omega}$ -open set (by remark (2.8)), but M is su. $\hat{\omega}$ -compact, then G owns a finite sub cover, so M is su. $\hat{\eta}$ -compact set. This remark is irreversible.

Example 2.64. The set Z with the indiscrete su. space is su. $\hat{\eta}$ -compact space, while it is not su. $\hat{\omega}$ -compact space.

Definition 2.65. Whenever the inverse image of each su. compact subset of the su. space (Y, μ_Y) is su. compact subset of the su. space (X, μ_X) , then the function f from (X, μ_X) into (Y, μ_Y) is called a su*. compact function.

Definition 2.66. Whenever the inverse image of each su. compact subset of the su. space (Y, μ_Y) is su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) subset of the su. space (X, μ_X) , then the function f from (X, μ_X) into (Y, μ_Y) is called a su*. $\hat{\omega}^*$ -compact (resp. su*. $\hat{\eta}^*$ -compact) function.

Example 2.67. 1- The identity function I_R from the co-finite su. space (R, μ_{cof}) into the discrete su. space (R, μ_D) is su*. compact.

2- The identity function I_R from (R, μ_{ind}) into the same space is su*. $\hat{\omega}^*$ -compact function.

3- The function $f : (R, \mu_{ind}) \longrightarrow (R, \mu_R)$ is su*. $\hat{\eta}^*$ -compact function.

Remark 2.68. Any su*. $\hat{\omega}^*$ -compact function is su*. $\hat{\eta}^*$ -compact, since if $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is a su*. $\hat{\omega}$ -compact function, and K is a su. compact

set in the su. space (Y, μ_Y) , so $f^{-1}(K)$ is su. $\hat{\omega}$ -compact set in the supra space (X, μ_X) (because f is su*. $\hat{\omega}^*$ -compact function) and by remark (2.63), $f^{-1}(K)$ is su. $\hat{\eta}$ -compact set, therefore f is su*. $\hat{\eta}^*$ -compact function.

Example 2.69. The identity function $I_Z : (Z, \mu_{ind}) \longrightarrow (Z, \mu_{ind})$ is su*. $\hat{\eta}^*$ -compact but not su*. $\hat{\omega}^*$ -compact function.

Definition 2.70. Whenever $f^{-1}(K)$ is su. compact set in (X, μ_X) for any su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) subset K in (Y, μ_Y) , then $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is called su*. $\hat{\omega}$ -compact (resp. su*. $\hat{\eta}$ -compact) function.

Example 2.71. 1- The function $f : (R, \mu_{ind}) \longrightarrow (R, \mu_{cof})$ is su*. $\hat{\omega}$ -compact function.

2- The function $f : (X, \mu_{cof}) \longrightarrow (Y, \mu_Y)$ is su*. $\hat{\eta}$ -compact function.

Remark 2.72. Every su*. $\hat{\eta}$ -compact function is su*. $\hat{\omega}$ -compact. Since if $f : X \longrightarrow Y$ is su*. $\hat{\eta}$ -compact function and K is su. $\hat{\omega}$ -compact set in Y , consequently, K is su. $\hat{\eta}$ -compact set (by remark (2.63)), but f is su*. $\hat{\eta}$ -compact function, hence $f^{-1}(K)$ is su. compact set in X , so f is su*. $\hat{\omega}$ -compact.

Definition 2.73. Whenever $f^{-1}(K)$ is su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) set in (X, μ_X) for any su. $\hat{\omega}$ -compact (resp. su. $\hat{\eta}$ -compact) set K in (Y, μ_Y) , hence $f : X \longrightarrow Y$ is called su*. $\hat{\omega}^{**}$ -compact (resp. su*. $\hat{\eta}^{**}$ -compact) function.

Example 2.74. 1- If X is a countable set, then $I_X : (X, \mu_{ind}) \longrightarrow (X, \mu_{ind})$ is a su*. $\hat{\omega}^{**}$ -compact function.

2- The function $f : (X, \mu_{ind}) \longrightarrow (X, \mu_X)$, where X is a finite set, is su*. $\hat{\eta}^{**}$ -compact function.

Proposition 2.75. Every su. $\hat{\omega}$ -closed (resp. su. closed, su. $\hat{\eta}$ -closed) set in a su. $\hat{\omega}$ -compact (resp. su. compact, su. $\hat{\eta}$ -compact) space (X, μ_X) is a su. $\hat{\omega}$ -compact (resp. su. compact [11], su. $\hat{\eta}$ -compact) set.

Proof. Consider (X, μ_X) is a su. $\hat{\omega}$ -compact space and M is a su. $\hat{\omega}$ -closed set in X , let $\{W_\alpha\}_{\alpha \in \Lambda}$ be a su. $\hat{\omega}$ -open cover for $M \implies M \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha$, but $X = M \cup M^c \implies X \subseteq \{\bigcup_{\alpha \in \Lambda} W_\alpha\} \cup M^c$ and since M is su. $\hat{\omega}$ -closed set in X , then M^c is su. $\hat{\omega}$ -open, this means $\{W_\alpha | \alpha \in \Lambda, M^c\}$ is a su. $\hat{\omega}$ -open cover for X , but X is su. $\hat{\omega}$ -compact, then any su. $\hat{\omega}$ -open cover for X possesses a finite sub cover, so $X \subseteq (\bigcup_{i=1}^n W_{\alpha_i}) \cup M^c$, but $M \subseteq X \implies M \subseteq \bigcup_{i=1}^n W_{\alpha_i} \cup M^c$, since $M \cap M^c = \phi \implies M \subseteq \bigcup_{i=1}^n W_{\alpha_i}$, then $\{W_{\alpha_i}\}_{i=1}^n$ is a finite sub cover from the su. $\hat{\omega}$ -open cover $\{W_\alpha\}_{\alpha \in \Lambda}$ for M , therefore M is a su. $\hat{\omega}$ -compact.

Example 2.76. Every su. $\hat{\omega}$ -closed set in the su. space (X, μ_{EX}) is su. $\hat{\omega}$ -compact set.

Remark 2.77. There is no relation between su^* . $\hat{\eta}^{**}$ -compact and su^* . $\hat{\omega}^{**}$ -compact function.

Example 2.78. The function $f : (Z, \mu_{ind}) \longrightarrow (Z, \mu_Z)$ is su^* . $\hat{\eta}^{**}$ -compact but not su^* . $\hat{\omega}^{**}$ -compact function.

Definition 2.79. If for each two non-equal points a, d in the su . space (X, μ) there are disjoint su . $\hat{\omega}$ -open sets W, B in X with $a \in W$ and $d \in B$, then the su . space (X, μ) is called su . $\hat{\omega}T_2$ -space.

Remark 2.80. Every su . T_2 -space is su . $\hat{\omega}T_2$ -space, since if X is su . T_2 -space then for each non-equal points a, d in X there are two su . open sets W, B in X with $W \cap B = \phi$ and $a \in W, d \in B$, then by remark (2.7) X will be su . $\hat{\omega}T_2$ -space. This remark is irreversible.

Example 2.81. The excluded point su . space (X, μ_{EX}) , where X is finite set is su . $\hat{\omega}T_2$ -space, but not su . T_2 -space.

Definition 2.82. Let $a \neq d$ be two points in the su . space (X, μ) , if there are disjoint su . $\hat{\eta}$ -open sets W, B in X with $a \in W$ and $d \in B$, then (X, μ) is called su . $\hat{\omega}T_2$ -space.

Example 2.83. The discrete su . space (X, μ_D) is su . $\hat{\eta}T_2$ -space.

Remark 2.84. Every su . $\hat{\eta}T_2$ -space is su . $\hat{\omega}T_2$ -space, because if $a \neq d$ are points in X , so there exist two su . $\hat{\eta}$ -open sets W, B in X with $a \in W, d \in B$ and $W \cap B = \phi$, but by remark (2.8) every su . $\hat{\eta}$ -open set is su . $\hat{\omega}$ -open, hence X is su . $\hat{\omega}T_2$ -space.

Example 2.85. The indiscrete su . space (X, μ_{ind}) is su . $\hat{\omega}T_2$ -space, but not su . $\hat{\omega}T_2$ -space, where X is countable set.

Proposition 2.86. If the function $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is su^* . $\hat{\omega}$ -continuous (resp. strongly su^* . $\hat{\omega}$ -continuous, su^* . $\hat{\omega}$ -irresolute)-function, so the image of each su . $\hat{\omega}$ -compact (resp. su . compact, su . $\hat{\omega}$ -compact) set in (X, μ_X) is a su . compact (resp su . $\hat{\omega}$ -compact) set in (Y, μ_Y) .

Proof. Let f be a su^* . $\hat{\omega}$ -continuous function and W be a su . $\hat{\omega}$ -compact set in X , take $\{V_\alpha\}_{\alpha \in \Lambda}$ to be a su . open cover to $f(W)$, where each $V_\alpha \in \mu_Y, \alpha \in \Lambda$. then $f(W) \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$, but f is su^* . $\hat{\omega}$ -continuous, hence $W \subseteq f^{-1}(f(W)) \subseteq f^{-1}(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$, then $\{f^{-1}(V_\alpha)\}_{\alpha \in \Lambda}$ is su . $\hat{\omega}$ -open cover for W , since W is su . $\hat{\omega}$ -compact, so each su . $\hat{\omega}$ -open cover to it possesses a finite sub cover, hence $W \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$ and by take the image of both sides we get, $f(W) \subseteq \bigcup_{i=1}^n f(f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^n V_{\alpha_i}, \alpha \in \Lambda$ which means that $\{V_{\alpha_i}\}_{i=1}^n, \alpha \in \Lambda$ is a sub cover from the su . open cover $\{V_\alpha\}_{\alpha \in \Lambda}$, so $f(W)$ is a su . compact subset of Y .

The rest of the possibilities can be proved by the same way.

Example 2.87. Let $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ in which Y is finite, hence the image of any su. $\hat{\omega}$ -compact set in the space X is also su. $\hat{\omega}$ -compact set in the space Y .

Definition 2.88. Whenever $f(M)$ is su. closed [12] (resp. su. $\hat{\omega}$ -closed, su. $\hat{\eta}$ -closed) set in (Y, μ_Y) for every su. closed set M in (X, μ_X) , thus the function $f : X \longrightarrow Y$ is called a su*. closed (su*. $\hat{\omega}$ -closed, su*. $\hat{\eta}$ -closed) function.

Example 2.89. The function $f : (R, \mu_R) \longrightarrow (R, \mu_D)$ is a su*. $\hat{\omega}$ -closed and su*. closed function.

Example 2.90. The function $f : (X, \mu_X) \longrightarrow (Y, \mu_D)$ is su*. $\hat{\eta}$ -closed function.

Remark 2.91. Every su*. $\hat{\eta}$ -closed function is su*. $\hat{\omega}$ -closed, that is if $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is a su*. $\hat{\eta}$ -closed function, and if M is a su. closed subset of X , so $f(M)$ is su. $\hat{\eta}$ -closed subset of Y (since f is su*. $\hat{\eta}$ -closed function), follows that $f(M)$ is su. $\hat{\omega}$ -closed (by remark (2.57)), so f is su*. $\hat{\omega}$ -closed function.

Example 2.92. The identity function $I_Z : (Z, \mu_D) \longrightarrow (Z, \mu_{ind})$ is su*. $\hat{\omega}$ -closed function but not su*. $\hat{\eta}$ -closed function.

Definition 2.93. 1- If the image of every su. $\hat{\omega}$ -closed (resp. su. $\hat{\eta}$ -closed) set M in (X, μ_X) is su. closed set in (Y, μ_Y) , then $f : X \longrightarrow Y$ is called totally su*. $\hat{\omega}$ -closed (resp. totally su*. $\hat{\eta}$ -closed) function.

2- If the image of every su. $\hat{\omega}$ -closed (resp. su. $\hat{\eta}$ -closed) set M in (X, μ_X) is su. $\hat{\omega}$ -closed (resp. su. $\hat{\eta}$ -closed) set in (Y, μ_Y) , then $f : X \longrightarrow Y$ is called strongly su*. $\hat{\omega}$ -closed (resp. strongly su*. $\hat{\eta}$ -closed) function.

Example 2.94. The function $f : (X, \mu_{EX}) \longrightarrow (Y, \mu_D)$ is totally su*. $\hat{\omega}$ -closed function.

Example 2.95. The function $f : (X, \mu_X) \longrightarrow (X, \mu_D)$ is totally su*. $\hat{\eta}$ -closed function, also it is strongly su*. $\hat{\eta}$ -closed function.

Example 2.96. The identity $I_X : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is strongly su*. $\hat{\omega}$ -closed.

Remark 2.97. Any totally su*. $\hat{\omega}$ -closed function is totally su*. $\hat{\eta}$ -closed, that is if $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ is a totally su*. $\hat{\omega}$ -closed function, and M is a su. $\hat{\eta}$ -closed set in (X, μ_X) hence it is su. $\hat{\omega}$ -closed (by remark (2.57)), then $f(M)$ is su. closed set in Y (since f is totally su*. $\hat{\omega}$ -closed function), therefore f is totally su*. $\hat{\eta}$ -closed function.

Example 2.98. The identity function $I_R : (R, \mu_{cof}) \longrightarrow (R, \mu_U)$ is totally su^* . $\hat{\eta}$ -closed function, while it is not totally su^* . $\hat{\omega}$ -closed.

Remark 2.99. there is no relation between strongly su^* . $\hat{\omega}$ -closed and strongly su^* . $\hat{\eta}$ -closed function.

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Hesitant fuzzy soft sets over UP-algebras by means of anti-type

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Abstract. This paper aims to extend the concept of anti-type of hesitant fuzzy sets on UP-algebras to anti-type of hesitant fuzzy soft sets over UP-algebras by merging the concept of anti-type of hesitant fuzzy sets and soft sets. Further, we discuss the concepts of anti-hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft UP-filters, and anti-hesitant fuzzy soft UP-subalgebras of UP-algebras and provide some properties.

Keywords: UP-algebra, anti-hesitant fuzzy soft UP-subalgebra, anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal.

1. Introduction

The branch of the logical algebra, a UP-algebra was introduced by Iampan [6] in 2017, and it is known that the class of KU-algebras [16] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [23] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [10], Kaijae et al. [9] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q -fuzzy sets in UP-algebras was introduced by Tanamoon et al. [26], Sripaeng et al. [25] introduced the notion anti Q -fuzzy UP-ideals and anti Q -fuzzy UP-subalgebras of UP-algebras, the notion of $\mathcal{N}$ -fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [24], Senapati et al. [21, 22] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [17] introduced the notion of proper UP-filters in UP-algebras, etc.

A soft set over a universe set is a parametrized family of subsets of the universe set. Molodtsov [11] introduced the concept of soft sets over a universe set in 1999.

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A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The concept of a hesitant fuzzy set on a set was first considered by Torra [27] in 2010. In UP-algebras, Mosrija et al. [14] extended the concept of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [20] considered level subsets of a hesitant fuzzy set on UP-algebras in 2017. The concept of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrija et al. [15] afterwards. Mosrija and Iampan [12] introduced the notion of anti-type of hesitant fuzzy sets on UP-algebras in 2018.

The concept of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets is introduced by Babitha and John [3] in 2013. There exists some researchers, such as Jun et al. [8], applied hesitant fuzzy soft set theory to some algebraic structures, which are BCK and BCI algebras. Alshehri and Alshehri [1] introduced the notion of hesitant anti-fuzzy sets over BCK-algebras later. In UP-algebras, Mosrija and Iampan [13] extended the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras.

In this paper, we extend the concept of anti-type of hesitant fuzzy sets on UP-algebras to anti-type of hesitant fuzzy soft sets over UP-algebras by merging the concept of anti-type of hesitant fuzzy sets and soft sets. Further, we discuss the concept of anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy soft strongly UP-ideals of UP-algebras, and provide some properties.

2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 2.1 ([6]). An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A , and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

$$(UP-1) \quad (\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0),$$

$$(UP-2) \quad (\forall x \in A)(0 \cdot x = x),$$

$$(UP-3) \quad (\forall x \in A)(x \cdot 0 = 0), \text{ and}$$

$$(UP-4) \quad (\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y).$$

From [6], we know that the notion of UP-algebras is a generalization of KU-algebras (see [16]).

Example 2.2 ([19]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to Ω* .

In particular, $(\mathcal{P}_\emptyset(X), \cdot, \emptyset)$ is the power UP-algebra of type 1.

Example 2.3 ([19]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to Ω* .

In particular, $(\mathcal{P}^X(X), *, X)$ is the power UP-algebra of type 2.

Example 2.4 ([4]). Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right)$$

and

$$(\forall x, y \in \mathbb{N}) \left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

For more examples of UP-algebras, see [2, 7, 18, 19].

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.5 ([6, 7]). In a UP-algebra A , the following properties hold:

- (1) $(\forall x \in A)(x \cdot x = 0)$,
- (2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,
- (3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,
- (4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,
- (5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,
- (6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,
- (7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,
- (8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,
- (9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,
- (10) $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot z) = 0)$,
- (11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,
- (12) $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (x \cdot (y \cdot z)) = 0)$, and

$$(13) (\forall a, x, y, z \in A)((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0).$$

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A [6] as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Definition 2.6 ([6, 23, 5]). A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

- (1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$.
- (2) a *UP-filter* of A if
 - (i) the constant 0 of A is in S , and
 - (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.
- (3) a *UP-ideal* of A if
 - (i) the constant 0 of A is in S , and
 - (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.
- (4) a *strongly UP-ideal* of A if
 - (i) the constant 0 of A is in S , and
 - (ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [5] proved that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Basic results on hesitant fuzzy sets

Definition 3.1 ([27]). Let X be a reference set. A *hesitant fuzzy set* on X is defined in term of a function h_H that when applied to X return a subset of $[0, 1]$, that is, $h_H: X \rightarrow \mathcal{P}([0, 1])$. A hesitant fuzzy set h_H can also be viewed as the following mathematical representation:

$$H := \{(x, h_H(x)) \mid x \in X\}$$

where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the elements $x \in X$ to the set H .

Definition 3.2 ([14]). A hesitant fuzzy set H on A is called

- (1) a *hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(h_H(x \cdot y) \supseteq h_H(x) \cap h_H(y))$.
- (2) a *hesitant fuzzy UP-filter* of A if

- (1) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$, and
 - (2) $(\forall x, y \in A)(h_H(y) \supseteq h_H(x \cdot y) \cap h_H(x))$.
- (3) a *hesitant fuzzy UP-ideal* of A if
- (1) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$, and
 - (2) $(\forall x, y, z \in A)(h_H(x \cdot z) \supseteq h_H(x \cdot (y \cdot z)) \cap h_H(y))$.
- (4) a *hesitant fuzzy strongly UP-ideal* of A if
- (1) $(\forall x \in A)(h_H(0) \supseteq h_H(x))$, and
 - (2) $(\forall x, y, z \in A)(h_H(x) \supseteq h_H((z \cdot y) \cdot (z \cdot x)) \cap h_H(y))$.

Mosrijai et al. [14] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

Definition 3.3 ([12]). A hesitant fuzzy set H on A is called

- (1) an *anti-hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(h_H(x \cdot y) \subseteq h_H(x) \cup h_H(y))$.
- (2) an *anti-hesitant fuzzy UP-filter* of A if
 - (1) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$, and
 - (2) $(\forall x, y \in A)(h_H(y) \subseteq h_H(x \cdot y) \cup h_H(x))$.
- (3) an *anti-hesitant fuzzy UP-ideal* of A if
 - (1) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$, and
 - (2) $(\forall x, y, z \in A)(h_H(x \cdot z) \subseteq h_H(x \cdot (y \cdot z)) \cup h_H(y))$.
- (4) an *anti-hesitant fuzzy strongly UP-ideal* of A if
 - (1) $(\forall x \in A)(h_H(0) \subseteq h_H(x))$, and
 - (2) $(\forall x, y, z \in A)(h_H(x) \subseteq h_H((z \cdot y) \cdot (z \cdot x)) \cup h_H(y))$.

Mosrijai and Iampan [12] proved that the notion of anti-hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of anti-hesitant fuzzy UP-filters, the notion of anti-hesitant fuzzy UP-filters of UP-algebras is a generalization of anti-hesitant fuzzy UP-ideals, and the notion of anti-hesitant fuzzy UP-ideals of UP-algebras is a generalization of anti-hesitant fuzzy strongly UP-ideals.

Proposition 3.4 ([12]). *Let H be an anti-hesitant fuzzy UP-filter (and also anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of A . Then*

$$(\forall x, y \in A)(x \leq y \Rightarrow h_H(x) \supseteq h_H(y) \supseteq h_H(x \cdot y)).$$

Definition 3.5 ([13]). Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called a *hesitant fuzzy soft UP-subalgebra* (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-subalgebra* (resp., *p-hesitant fuzzy soft UP-filter*, *p-hesitant fuzzy soft UP-ideal*, *p-hesitant fuzzy soft strongly UP-ideal*)) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-subalgebra (resp., hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of A . If $(\tilde{H}, Y)$ is a *p-hesitant fuzzy soft UP-subalgebra* (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is a *hesitant fuzzy soft UP-subalgebra* (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A .

4. Anti-hesitant fuzzy soft UP-subalgebras

Definition 4.1 ([3]). Let X be a reference set (or an initial universe set) and P be a set of parameters. Let $HFS(X)$ be the set of all hesitant fuzzy sets on X and Y be a nonempty subset of P . A pair $(\tilde{H}, Y)$ is called a *hesitant fuzzy soft set* over X where $\tilde{H}$ is a mapping given by

$$\tilde{H}: Y \rightarrow HFS(X), p \mapsto \tilde{H}[p].$$

Definition 4.2. Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called an *anti-hesitant fuzzy soft UP-subalgebra* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-subalgebra*) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-subalgebra of A . If $(\tilde{H}, Y)$ is a *p-anti-hesitant fuzzy soft UP-subalgebra* of A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is an *anti-hesitant fuzzy soft UP-subalgebra* of A .

Theorem 4.3. If $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A , then it satisfies the property:

$$(4.1) \quad (\forall p \in Y \forall x \in A)(h_{\tilde{H}[p]}(0) \subseteq h_{\tilde{H}[p]}(x)).$$

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A . Let $p \in Y$ and $x \in A$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A . Therefore, $h_{\tilde{H}[p]}(0) = h_{\tilde{H}[p]}(x \cdot x) \subseteq h_{\tilde{H}[p]}(x) \cup h_{\tilde{H}[p]}(x) = h_{\tilde{H}[p]}(x)$. $\square$

Example 4.4. For any objects a and b , let $(\mathcal{P}_\emptyset(\{a, b\}), \cdot, \emptyset)$ be the power UP-algebra of type 1 with the following Cayley table:

$\cdot$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{b\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{a, b\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over $\mathcal{P}_\emptyset(\{a, b\})$ by the following table:

$\tilde{H}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
p_1	$\emptyset$	$\{0.1\}$	$\{0.2\}$	$\{0.1, 0.2\}$
p_2	$(0.6, 0.7)$	$(0.6, 0.7)$	$[0.6, 0.7]$	$[0.6, 0.7]$
p_3	$\{0.5\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$(0.2, 0.7)$
p_4	$\{0.9\}$	$[0.8, 0.9]$	$\{0.8, 0.9\}$	$\{0.8, 0.9\}$

Then $(\tilde{H}, Y)$ satisfies the property (4.1), but not an anti-hesitant fuzzy soft UP-subalgebra of $\mathcal{P}_\emptyset(\{a, b\})$ based on parameter p_4 . Indeed,

$$\begin{aligned} h_{\tilde{H}[p_4]}(\{b\} \cdot \{a, b\}) &= h_{\tilde{H}[p_4]}(\{a\}) = [0.8, 0.9] \not\subseteq \{0.8, 0.9\} \\ &= \{0.8, 0.9\} \cup \{0.8, 0.9\} \\ &= h_{\tilde{H}[p_4]}(\{b\}) \cup h_{\tilde{H}[p_4]}(\{a, b\}). \end{aligned}$$

Theorem 4.5. Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(4.2) \quad (\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(z) \cup h_{\tilde{H}[p]}(x)).$$

Then $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A .

Proof. Let $p \in Y$ and $x, y \in A$. By Proposition 2.5 (5) and (UP-3), we have $x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0$ and thus $x \leq y \cdot (x \cdot y)$. It follows from (4.2) that

$$h_{\tilde{H}[p]}(x \cdot y) \subseteq h_{\tilde{H}[p]}(x) \cup h_{\tilde{H}[p]}(y).$$

Therefore, $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A . Hence, $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft UP-subalgebra of A . Since p is arbitrary, we know that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A . $\square$

Corollary 4.6. If $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (4.2), then it satisfies the property (4.1).

Proof. It is straightforward from Theorems 4.5 and 4.3. $\square$

Theorem 4.7. *If $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A and N is a nonempty subset of Y , then $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A .*

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A and $\emptyset \neq N \subseteq Y$. Since $N \subseteq Y$, we have $(\tilde{H}|_N, N)$ is a p -anti-hesitant fuzzy soft UP-subalgebra of A for all $p \in N$. Therefore, $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A . $\square$

By Example 4.4, we have $(\tilde{H}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A . But if we choose $N = \{p_1, p_2, p_3\}$, then $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A . We can conclude that there exists a nonempty subset N of Y such that $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A , but $(\tilde{H}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A .

5. Anti-hesitant fuzzy soft UP-filters

Definition 5.1. Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-filter* based on $p \in Y$ (we shortly call a *anti- p -hesitant fuzzy soft UP-filter*) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-filter of A . If $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft UP-filter of A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is an *anti-hesitant fuzzy soft UP-filter* of A .

From [12], we know that every anti-hesitant fuzzy UP-filter of A is an anti-hesitant fuzzy UP-subalgebra. Then we have the following Theorem:

Theorem 5.2. *Every p -anti-hesitant fuzzy soft UP-filter of A is a p -anti-hesitant fuzzy soft UP-subalgebra.*

The following example shows that the converse of Theorem 5.2 is not true in general.

Example 5.3. For any objects a and b , let $(\mathcal{P}_\emptyset(\{a, b\}), \cdot, \emptyset)$ be the power UP-algebra of type 1 with the Cayley table from Example 4.4, and let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over $\mathcal{P}_\emptyset(\{a, b\})$ by the following table:

$\tilde{H}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
p_1	$\emptyset$	$\{0\}$	$\{0.1\}$	$\{0, 0.1\}$
p_2	$\{0.2\}$	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$
p_3	$\{0.4\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$[0.4, 0.6)$
p_4	$\{0.8\}$	$[0.7, 0.9]$	$[0.7, 0.9]$	$\{0.8\}$

Then $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A , but not an anti-hesitant fuzzy soft UP-filter of $\mathcal{P}_0(\{a, b\})$ based on parameters p_3 and p_4 . Indeed,

$$\begin{aligned} h_{\tilde{H}[p_3]}(\{a, b\}) &= [0.4, 0.6] \not\subseteq [0.4, 0.5] \\ &= [0.4, 0.5] \cup \{0.4, 0.5\} \\ &= h_{\tilde{H}[p_3]}(\{b\}) \cup h_{\tilde{H}[p_3]}(\{a\}) \\ &= h_{\tilde{H}[p_3]}(\{a\} \cdot \{a, b\}) \cup h_{\tilde{H}[p_3]}(\{a\}) \end{aligned}$$

and

$$\begin{aligned} h_{\tilde{H}[p_4]}(\{b\}) &= [0.7, 0.9] \not\subseteq \{0.8\} \\ &= \{0.8\} \cup \{0.8\} \\ &= h_{\tilde{H}[p_4]}(\emptyset) \cup h_{\tilde{H}[p_4]}(\{a, b\}) \\ &= h_{\tilde{H}[p_4]}(\{a, b\} \cdot \{b\}) \cup h_{\tilde{H}[p_4]}(\{a, b\}). \end{aligned}$$

Theorem 5.4. *A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is an anti-hesitant fuzzy soft UP-filter of A if and only if it satisfies the condition (4.2).*

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A . Let $p \in Y$ and let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-filter of A . By Proposition 3.4, we have $h_{\tilde{H}[p]}(z) \supseteq h_{\tilde{H}[p]}(x \cdot y)$. Therefore,

$$h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(x \cdot y) \cup h_{\tilde{H}[p]}(x) \subseteq h_{\tilde{H}[p]}(z) \cup h_{\tilde{H}[p]}(x).$$

Conversely, assume that $(\tilde{H}, Y)$ satisfies the condition (4.2). Let $p \in Y$ and let $x \in A$. By Corollary 4.6, we have $h_{\tilde{H}[p]}(0) \subseteq h_{\tilde{H}[p]}(x)$. Let $x, y \in A$. By Proposition 2.5 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq x \cdot y$. It follows from (4.2) that

$$h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(x \cdot y) \cup h_{\tilde{H}[p]}(x).$$

Therefore, $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-filer of A . Hence, $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft UP-filter of A . Since p is arbitrary, we know that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A . $\square$

Theorem 5.5. *Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(5.1) \quad (\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y)).$$

Then it is an anti-hesitant fuzzy soft UP-filter of A .

Proof. Assume that $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1). Let $p \in Y$ and let $x, y \in A$. By Proposition 2.5 (1) and (UP-2), we have $0 \cdot ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0$ and thus $0 \leq (x \cdot y) \cdot (x \cdot y)$. It follows from (5.1) that

$$h_{\tilde{H}[p]}(y) = h_{\tilde{H}[p]}(0 \cdot y) \subseteq h_{\tilde{H}[p]}(x \cdot y) \cup h_{\tilde{H}[p]}(x).$$

Therefore, $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A . $\square$

Corollary 5.6. *If $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1), then it satisfies the condition (4.2).*

Proof. It is straightforward from Theorems 5.5 and 5.4. $\square$

Example 5.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over A by the following table:

$\tilde{H}$	0	1	2	3
p_1	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.4\}$	$\{0.4\}$
p_2	$\{0\}$	$\{0, 1\}$	$\{0, 1\}$	$\{0\}$
p_3	$\{0.9\}$	$\{0.9\}$	$[0.9, 1]$	$[0.9, 1]$
p_4	$(0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$

Then $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A , But it does not satisfy the condition (5.1) because $3 \cdot (0 \cdot (1 \cdot 2)) = 0$ implies

$$\begin{aligned} h_{\tilde{H}[p_3]}(3 \cdot 2) &= h_{\tilde{H}[p_3]}(2) \\ &= [0.9, 1] \\ &\not\subseteq \{0.9\} \\ &= h_{\tilde{H}[p_3]}(0) \cup h_{\tilde{H}[p_3]}(1). \end{aligned}$$

Theorem 5.8. *Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(5.2) \quad (\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y)).$$

Then it is an anti-hesitant fuzzy soft UP-filter of A .

Proof. Assume that $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2). Let $p \in Y$ and let $x, y \in A$. By Proposition 2.5 (1) and (UP-2), we have $(x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq 0 \cdot (x \cdot y)$. It follows from (5.2) that

$$h_{\tilde{H}[p]}(y) = h_{\tilde{H}[p]}(0 \cdot y) \subseteq h_{\tilde{H}[p]}(x \cdot y) \cup h_{\tilde{H}[p]}(x).$$

Therefore, $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A . $\square$

Corollary 5.9. *If $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2), then it satisfies the condition (4.2).*

Proof. It is straightforward from Theorems 5.8 and 5.4. $\square$

6. Anti-hesitant fuzzy soft UP-ideals

Definition 6.1. Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-ideal* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-ideal*) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-ideal of A . If $(\tilde{H}, Y)$ is a *p-anti-hesitant fuzzy soft UP-ideal* of A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is an *anti-hesitant fuzzy soft UP-ideal* of A .

From [12], we know that every anti-hesitant fuzzy UP-ideal of A is an anti-hesitant fuzzy UP-filter. Then we have the following Theorem:

Theorem 6.2. *Every p-anti-hesitant fuzzy soft UP-ideal of A is a p-anti-hesitant fuzzy soft UP-filter.*

The following example shows that the converse of Theorem 6.2 is not true in general.

Example 6.3. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over A by the following table:

$\tilde{H}$	0	1	2	3	4
p_1	$\{0.7\}$	$[0.7, 0.8)$	$[0.7, 0.9]$	$[0.6, 1]$	$[0.6, 1]$
p_2	$\emptyset$	$\{0.6\}$	$\{0.6\}$	$\{0.6, 0.9\}$	$\{0.6, 0.9\}$
p_3	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$
p_4	$\{0.3\}$	$\{0.3\}$	$[0.2, 0.4]$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$

Then $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A , but not an anti-hesitant fuzzy soft UP-ideal of A based on parameters p_1 and p_2 . Indeed,

$$\begin{aligned}
 h_{\tilde{H}[p_1]}(3 \cdot 4) &= h_{\tilde{H}[p_1]}(3) = [0.7, 0.9] \not\subseteq [0.7, 0.8) \\
 &= \{0.7\} \cup [0.7, 0.8) \\
 &= h_{\tilde{H}[p_1]}(0) \cup h_{\tilde{H}[p_1]}(2) \\
 &= h_{\tilde{H}[p_1]}(3 \cdot (2 \cdot 4)) \cup h_{\tilde{H}[p_1]}(2)
 \end{aligned}$$

and

$$\begin{aligned}
 h_{\tilde{H}[p_2]}(3 \cdot 4) &= h_{\tilde{H}[p_2]}(3) = \{0.6, 0.9\} \not\subseteq \{0.6\} \\
 &= \emptyset \cup \{0.6\} \\
 &= h_{\tilde{H}[p_2]}(0) \cup h_{\tilde{H}[p_2]}(2) \\
 &= h_{\tilde{H}[p_2]}(3 \cdot (2 \cdot 4)) \cup h_{\tilde{H}[p_2]}(2).
 \end{aligned}$$

Theorem 6.4. *If $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A , then it satisfies the condition (5.1).*

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A . Let $p \in Y$ and let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-ideal of A and $x \cdot (w \cdot (y \cdot z)) = 0$. Thus $h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \subseteq h_{\tilde{H}[p]}(x \cdot (w \cdot (y \cdot z))) \cup h_{\tilde{H}[p]}(w) = h_{\tilde{H}[p]}(0) \cup h_{\tilde{H}[p]}(w) = h_{\tilde{H}[p]}(w)$. Therefore, $h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cup h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y)$. $\square$

Theorem 6.5. *A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is an anti-hesitant fuzzy soft UP-ideal of A if and only if it satisfies the condition (5.2).*

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A . Let $p \in Y$ and $w, x, y, z \in A$ be such that $w \leq x \cdot (y \cdot z)$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-ideal of A . By Proposition 3.4, we have $h_{\tilde{H}[p]}(w) \supseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z))$. Therefore,

$$h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cup h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y).$$

Conversely, assume that $(\tilde{H}, Y)$ satisfies the condition (5.2). Let $p \in Y$ and let $x \in A$. By Corollaries 5.9 and 4.6, we have $h_{\tilde{H}[p]}(0) \subseteq h_{\tilde{H}[p]}(x)$. Let $x, y, z \in A$. By Proposition 2.5 (1), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$ and thus $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (5.2) that

$$h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cup h_{\tilde{H}[p]}(y).$$

Therefore, $\tilde{H}[p]$ is an anti-hesitant fuzzy UP-ideal of A . Hence, $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft UP-ideal of A . Since p is arbitrary, we know that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A . $\square$

Theorem 6.6. *Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(6.1) \quad (\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow h_{\tilde{H}[p]}(x) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y)).$$

Then it is an anti-hesitant fuzzy soft UP-ideal of A .

Proof. Assume that $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (6.1). Let $p \in Y$ and let $x, y, z \in A$. By Proposition 2.5 (1) and (UP-3), we have $(x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))) = (x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot 0) = (x \cdot (y \cdot z)) \cdot 0 = 0$ and thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))$. It follows from (6.1) that

$$h_{\tilde{H}[p]}(x \cdot z) \subseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cup h_{\tilde{H}[p]}(y).$$

Therefore, $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A . $\square$

Corollary 6.7. *If $(\tilde{H}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (6.1), then it satisfies the conditions (5.1) and (5.2).*

Proof. It is straightforward from Theorems 6.6, 6.4 and 6.5. $\square$

7. Anti-hesitant fuzzy soft strongly UP-ideals

Definition 7.1. Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called a *anti-hesitant fuzzy soft strongly UP-ideal* based on $p \in Y$ (we shortly call a p -anti-hesitant fuzzy soft strongly UP-ideal) of A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy strongly UP-ideal of A . If $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft strongly UP-ideal of A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is an *anti-hesitant fuzzy soft strongly UP-ideal* of A .

From [12], we know that every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal. Then we have the following Theorem:

Theorem 7.2. *Every p -anti-hesitant fuzzy soft strongly UP-ideal of A is a p -anti-hesitant fuzzy soft UP-ideal.*

The following example shows that the converse of Theorem 7.2 is not true in general.

Example 7.3. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over A by the following table:

$\tilde{H}$	0	1	2	3
p_1	$[0.3, 0.8]$	$[0.3, 0.8]$	$[0.3, 0.8]$	$[0.3, 0.8]$
p_2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
p_3	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$
p_4	$(0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$

Then $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A , but not an anti-hesitant fuzzy soft strongly UP-ideal of A based on parameters p_1 and p_4 . Indeed,

$$\begin{aligned}
 h_{\tilde{H}[p_1]}(3) &= [0.3, 0.8] \\
 &\not\subseteq [0.3, 0.8) \\
 &= [0.3, 0.8) \cup [0.3, 0.8] \\
 &= h_{\tilde{H}[p_1]}(0) \cup h_{\tilde{H}[p_1]}(1) \\
 &= h_{\tilde{H}[p_1]}((3 \cdot 1) \cdot (3 \cdot 3)) \cup h_{\tilde{H}[p_1]}(1)
 \end{aligned}$$

and

$$\begin{aligned}
 h_{\tilde{H}[p_4]}(2) &= [0, 0.1] \\
 &\not\subseteq (0, 0.1] \\
 &= (0, 0.1] \cup (0, 0.1] \\
 &= h_{\tilde{H}[p_4]}(0) \cup h_{\tilde{H}[p_4]}(0) \\
 &= h_{\tilde{H}[p_4]}((2 \cdot 0) \cdot (2 \cdot 2)) \cup h_{\tilde{H}[p_4]}(0).
 \end{aligned}$$

By Theorems 5.2, 6.2, and 7.2 and Examples 5.3, 6.3, and 7.3, we have that the notion of p -anti-hesitant fuzzy soft UP-subalgebras is a generalization of

p -anti-hesitant fuzzy soft UP-filters, the notion of p -anti-hesitant fuzzy soft UP-filters is a generalization of p -anti-hesitant fuzzy soft UP-ideals, and the notion of p -anti-hesitant fuzzy soft UP-ideals is a generalization of p -anti-hesitant fuzzy soft strongly UP-ideals.

Theorem 7.4. *A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if it satisfies the condition (6.1).*

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A . Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A . By Proposition 3.4, we have $h_{\tilde{H}[p]}(w) \supseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x))$. Therefore,

$$h_{\tilde{H}[p]}(x) \subseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cup h_{\tilde{H}[p]}(y) \subseteq h_{\tilde{H}[p]}(w) \cup h_{\tilde{H}[p]}(y).$$

Conversely, assume that $(\tilde{H}, Y)$ satisfies the condition (6.1). Let $p \in Y$ and let $x \in A$. By Corollaries 6.7, 5.9 and 4.6, respectively, we have $h_{\tilde{H}[p]}(0) \subseteq h_{\tilde{H}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. It follows from (6.1) that

$$h_{\tilde{H}[p]}(x) \subseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cup h_{\tilde{H}[p]}(y).$$

Therefore, $\tilde{H}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A . Hence, $(\tilde{H}, Y)$ is a p -anti-hesitant fuzzy soft strongly UP-ideal of A . Since p is arbitrary, we know that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A . $\square$

Theorem 7.5. *Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over A and N be a nonempty subset of Y . Then the following statements are hold:*

- (1) *If $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A , then $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A , and*
- (2) *there exists $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A , but $(\tilde{H}, Y)$ is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A .*

Proof. (1) Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A . In the same way as Theorem 4.7, we can show that $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A .

(2) Example 7.3 (resp., Example 6.3, Example 5.3), if we choose $N = \{p_2, p_3\}$ (resp., $\{p_3, p_4\}$, $\{p_1, p_2\}$), then $(\tilde{H}|_N, N)$ is an anti-hesitant fuzzy soft strongly

UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A , but $(\tilde{H}, Y)$ is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A . $\square$

Definition 7.6 ([13]). Let Y be a nonempty subset of P . A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is called a *constant hesitant fuzzy soft set* based on $p \in Y$ (we shortly call a *p-constant hesitant fuzzy soft set*) over A if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on A is a constant hesitant fuzzy set on A . If $(\tilde{H}, Y)$ is a p -constant hesitant fuzzy soft set over A for all $p \in Y$, we state that $(\tilde{H}, Y)$ is a *constant hesitant fuzzy soft set* over A .

Theorem 7.7 ([13]). A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is a hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A .

Theorem 7.8 ([12]). A hesitant fuzzy set H on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A .

Theorem 7.9. A hesitant fuzzy soft set $(\tilde{H}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A .

Proof. Assume that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A and let $p \in Y$. Then $\tilde{H}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A . By Theorem 7.8, we obtain $\tilde{H}[p]$ is a constant hesitant fuzzy set on A . Thus $(\tilde{H}, Y)$ is a p -constant hesitant fuzzy soft set over A . Since p is arbitrary, we know that $(\tilde{H}, Y)$ is a constant hesitant fuzzy soft set over A .

Conversely, let $p \in Y$. Assume that $(\tilde{H}, Y)$ is a constant hesitant fuzzy soft set over A . Then $\tilde{H}[p]$ is a constant hesitant fuzzy set on A . By Theorem 7.8, we have $\tilde{H}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A . Since p is arbitrary, we state that $(\tilde{H}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A . $\square$

Corollary 7.10. For UP-algebras, we can conclude that the notions of anti-hesitant fuzzy soft strongly UP-ideals and hesitant fuzzy soft strongly UP-ideals coincide.

Proof. It is straightforward by Theorems 7.7 and 7.9. $\square$

8. Conclusions and future work

In this paper, we have introduced the notion of anti-hesitant fuzzy soft sets over UP-algebras which is a new extension of anti-type of hesitant fuzzy sets on UP-algebras and the notions of anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy

soft strongly UP-ideals of UP-algebras and investigated some of its important properties. Then we have the diagram of anti-type of hesitant fuzzy soft sets over UP-algebras below.

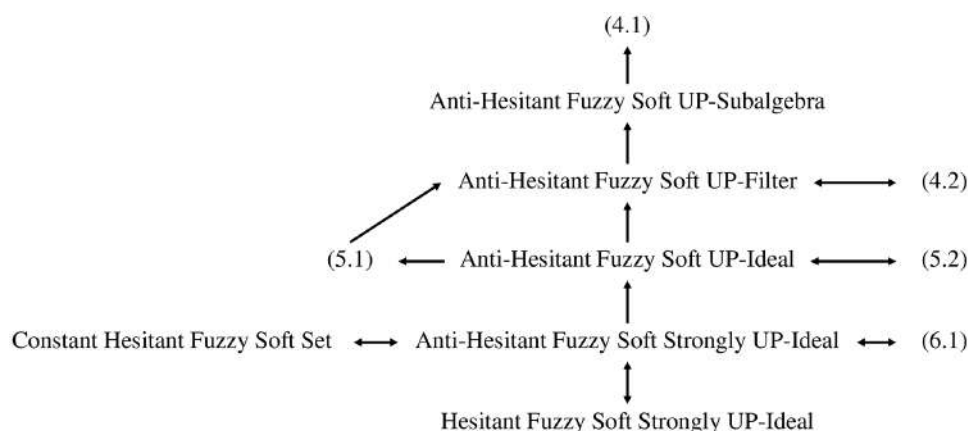


Figure 1: Anti-type of hesitant fuzzy soft sets over UP-algebras

In our future study of UP-algebras, the following objectives considered:

- To get more results in anti-type of hesitant fuzzy soft sets over UP-algebras.
- To define operations of anti-type of hesitant fuzzy soft sets over UP-algebras.
- To study anti-type of hesitant fuzzy soft sets over UP-algebras used in decision making.

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Nonexistence results of global solutions for fractional order integral equations and systems

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Abstract. The main objective of this study is to demonstrate the absence of the nontrivial global solutions of some fractional evolution integral equations and systems. We show that the critical exponent of Fujita type coincides with that obtained by S. Fuqin and S. Peihu in the classical case.

Keywords: fractional derivatives and integrals, weak solutions, nonlinear evolution equations.

1. Introduction

The most development of the fractional calculus was into the turn of the 20th century, because it has been used to describe diverse phenomenon in sciences and engineering for its special fundamental properties. Caputo [23] reformulated the most definitions of Riemann derivatives in order to solve some fractional order differential equations and systems from the point of view of various considerations, mainly to ensure that the problem is well posed, then in the majority of cases, the obtained results generalize or at least extend those obtained in the classical case.

In the first part of this study, we deal with the following Cauchy problem:

$$(1) \quad \begin{cases} \mathbf{D}_{0|t}^\beta u + (-\Delta)^m u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} |u|^p d\tau, \\ u(0, x) = u_0(x), \end{cases}$$

where $m > 1$, $p > 1$, $0 < \beta < 1$ and $0 < \gamma < 1$. $\mathbf{D}_{0|t}^\beta$ denotes the Caputo fractional derivative of order β .

The fractional Laplacien operator is defined as

$$(-\Delta)^{\beta/2} u(t, x) = \mathcal{F}^{-1}(|\xi|^\beta \mathcal{F}(u)(\xi))(x), \quad u \in H^\beta,$$

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where H^β is the homogeneous Sobolev defined by

$$\begin{aligned} H^\beta(\mathbb{R}^N) &= \{u \in S' : (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N)\}, \quad \beta \in \mathbb{N}, \\ H^\beta(\mathbb{R}^N) &= \{u \in L^2(\mathbb{R}^N) : (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N)\}, \quad \beta \notin \mathbb{N}, \end{aligned}$$

S' is the Schwartz space. $\mathcal{F}$ is the standard Fourier transform and $\mathcal{F}^{-1}$ its inverse.

We observe that the last problem can also be reformulated as follow

$$(2) \quad \begin{cases} \mathbf{D}_{0|t}^\beta u + (-\Delta)^m u = D_{0|t}^{-\alpha} |u|^p; \alpha = 1 - \gamma \\ u(0, x) = u_0(x), \end{cases}$$

and we was motived mathematically by many interesting works which deal with nonexistence and blow up of solutions of differential integral equations and systems, we cite particularly the problem treated by establishment of a critical exponent, which has been used as a very effective tool for studying the nonexistence of solutions for evolution equations and systems. Our interest is to prove the absence of nontrivial global weak solutions for the problem (1) by looking at critical exponents. The found result is more general than the one established by Sun and Shi [15], when they treated the following Cauchy problem

$$(3) \quad \begin{cases} u_t + (-\Delta)^m u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} |u|^p d\tau, \\ u(0, x) = u_0(x), \end{cases}$$

they proved that if

$$1 < p \leq \max \left\{ 1 + \frac{2m(2-\gamma)}{(N-2m+2m\gamma)_+}, \frac{1}{\gamma} \right\}$$

then the problem does not accept any nontrivial solution. We cite also the paper of Fino and Kirane [3], the authors consider the following problem

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} |u|^{p-1} d\tau, \\ u(0, x) = u_0(x), \end{cases}$$

where $0 < \beta \leq 2$ and $0 < \gamma < 1$. They proved that if

$$1 < p \leq p^* = 1 + \frac{\beta(2+\gamma)}{N-\beta+\beta\gamma} \text{ or } p < \frac{1}{\gamma},$$

where in the case $p = p^*$ and $\beta \in (0, 2)$, they take

$$p \geq \frac{N}{N-\beta}, \quad N > \beta,$$

then the solution of the posed problem blows up in a finite time for $u_0 \in C_0(\mathbb{R}^N)$.

The absence of the nontrivial global solutions was also considered for a time-space fractional evolution systems, in [29] Xu and Tan shows that any solution (u, v) to the next problem

$$\begin{cases} u_t + (-\Delta)^{\beta_1/2} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} |v|^{p-1} d\tau, \\ v_t + (-\Delta)^{\beta_2/2} v = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\delta} |u|^{p-1} d\tau, \\ u(0, x) = u_0(x), v(0, x) = v_0(x), \end{cases}$$

blows up in a finite time for $u_0, v_0 \in C_0(\mathbb{R}^N)$ (with $u_0, v_0 \geq 0, u_0, v_0 \not\equiv 0$) under appropriate conditions as

$$N < \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}, \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\},$$

or

$$p < \frac{1}{\delta} \text{ and } q < \frac{1}{\gamma},$$

where $0 < \gamma, \delta < 1$, $p, q > 1$ and $0 < \beta_1, \beta_2 \leq 2$. In the second part of this study, we extend the above result by replacing u_t and v_t respectively by $\mathbf{D}_{0|t}^\beta u$ and $\mathbf{D}_{0|t}^\beta v$ where $0 < \beta < 1$.

The paper is organized as follows. In section 1, we present some definitions of fractional calculus that will be used throughout this paper. In section 2, we present the our first main result concerning the absence of nontrivial global weak solutions for the problem (1). In section 3, we extend the first result to studying the blow-up of solutions for a time-space fractional evolution systems.

2. Preliminaries

In this section, we present some basic notions of fractional derivatives and integrations that will be required in later sections and which can be founded in [23].

Definition 1. For $0 < \alpha < 1$, the Riemann-Liouville derivatives of order α for a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, are defined by

$$D_{0|t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau,$$

$$D_{t|T}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{-\alpha} f(\tau) d\tau.$$

Definition 2. For $0 < \alpha < 1$, the Caputo derivative of order α for a differentiable function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$$\mathbf{D}_{0|t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau.$$

It is clear that

$$\mathbf{D}_{0|t}^\alpha f(t) = D_{0|t}^\alpha [f(t) - f(0)].$$

On the other hand, for every differentiable functions $f, g : [0, \infty) \rightarrow \mathbb{R}$, we have the following formula of integration by parts

$$\int_0^T D_{0|t}^\alpha f \cdot g(t) = \int_0^T D_{t|T}^\alpha g \cdot f(t),$$

and for $f \in C^2[0, \infty)$, we have

$$-DD_{t|T}^\alpha f(t) = D_{t|T}^{\alpha+1} f(t).$$

For $\eta \gg 1$ and $0 < \alpha < 1$. Let

$$(4) \quad f(t) = \begin{cases} (1 - \frac{t}{T})^\eta, & 0 < t \leq T, \\ 0, & t \geq T, \end{cases}$$

then

$$D_{t|T}^\alpha f(t) = \frac{(1 - \alpha + \eta)B(1 - \alpha; \eta - 1)}{\Gamma(1 - \alpha)} T^{-\alpha} (1 - \frac{t}{T})^{\eta-\alpha},$$

and

$$\int_0^T f(t)^{-p'/p} |D_{t|T}^\alpha f(t)|^{p'} = CT^{1-p'\alpha}.$$

3. Nonexistence results

Definition 3. Let $\beta = 2, u_0 \in C_0(\mathbb{R}^N)$. We call u is a mild solution of (2), if

$$(5) \quad u = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) D_{0|t}^{-(1-\gamma)} |u|^p.$$

for more detail see [28]

Definition 4 (Weak solution). Let $u \in C[(0, T), L^p(\mathbb{R}) \cap L_{loc}^1((0, T) \times \mathbb{R}^n) \cap L_{loc}^p((0, T) \times \mathbb{R}^n)]$ is a local weak solution to problem (2), if

$$\begin{aligned} & \int_{Q_T} u_0 D_{t|T}^\beta \varphi(t, x) dx dt + \int_{Q_T} \varphi(t, x) D_{0|t}^{-\alpha} |u|^p \\ &= \int_{Q_T} u D_{t|T}^\beta \varphi(t, x) dx dt + \int_{Q_T} (-\Delta)^m u \varphi(t, x) \end{aligned}$$

where $Q_T = [0, T] \times \Omega$; $\Omega = \text{supp } \varphi$, the test function $\varphi \in C_{t,x}^{1,2}(Q_T)$, $\varphi(t, x) = 0$, if $T = \infty$ the solution is global.

Theorem 1. Let $p > 1, 0 \leq \alpha + \beta \leq 1$ and $u_0 > 0$, assume that $p > \frac{2m}{N\beta-2m} + 1$ with $N > \frac{2m}{\beta}$ is satisfied if

$$p \leq \max \left\{ \frac{2m(\alpha + \beta)}{2m + N\beta - 2m(\alpha + \beta)} + 1, \frac{1}{\gamma} \right\}.$$

The problem (2) does not admit nontrivial global weak solution.

Proof. On the contrary, suppose that u is nontrivial nonnegative solution for all time $t > 0$. Choose the test function

$$\varphi(t, x) = D_{t|T}^\alpha \tilde{\varphi} = \varphi_1^l(x) D_{t|T}^\alpha \varphi_2(t), \quad l > \frac{p}{p-1},$$

where

$$\varphi_1(x) = \Phi\left(\frac{|x|}{T^{\frac{\beta}{2m}}}\right) = \Phi(\xi) = \begin{cases} 1, & 0 < \xi \leq 1, \\ 0, & \xi \geq 2, \end{cases}$$

and

$$\varphi_2(t) = \left(1 - \frac{t}{T}\right)^\eta, \quad \eta > \max \left\{ \frac{p(\alpha + \beta)}{p-1} - 1, \alpha + \beta + 1 \right\}.$$

Using the Definition 4, we obtain

$$\begin{aligned} & \int_{Q_T} u_0 D_{t|T}^\beta D_{t|T}^\alpha \tilde{\varphi} dx dt + \int_{Q_T} D_{t|T}^\alpha \tilde{\varphi} D_{0|t}^{-\alpha} |u|^p \\ &= \int_{Q_T} u D_{t|T}^\beta D_{t|T}^\alpha \tilde{\varphi} dx dt + \int_{Q_T} (-\Delta)^m u D_{t|T}^\alpha \tilde{\varphi}. \end{aligned}$$

A simple computation yields $D_{t|T}^\beta (D_{t|T}^\alpha \tilde{\varphi}) = D_{t|T}^{\alpha+\beta} \tilde{\varphi}$, we obtain

$$\begin{aligned} & cT^{1-(\alpha+\beta)} \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_{Q_T} \tilde{\varphi} |u|^p \\ &= \int_{Q_T} u \varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2(t) dx dt + \int_{Q_T} u (-\Delta)^m \varphi_1^l(x) D_{t|T}^\alpha \varphi_2. \end{aligned}$$

The application of Ju inequality

$$(-\Delta)^m \varphi_1^l \leq l \varphi_1^{l-1} (-\Delta)^m \varphi_1,$$

implies that

$$\int_{Q_T} \tilde{\varphi} |u|^p \leq c \int_{Q_T} u \varphi_1^{l-1} (-\Delta)^m \varphi_1(x) D_{t|T}^\alpha \varphi_2 + \int_{Q_T} u \varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2(t) dx dt.$$

For estimating the second member of the above inequality, we write

$$\int_{Q_T} u \varphi_1^{l-1} (-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2 = \int_{Q_T} u \tilde{\varphi}^{\frac{1}{p}} \varphi_1^{l-1} (-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2 \tilde{\varphi}^{\frac{-1}{p}}.$$

According to ε -Young inequality

$$XY \leq \varepsilon X^p + c(\varepsilon) Y^{p'}, p + p' = p.p',$$

we have

$$\int_{Q_T} u(-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2 \leq \varepsilon \int_{Q_T} |u|^p \tilde{\varphi} + c_1(\varepsilon) \int_{Q_T} \varphi_1^{l-1} |(-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2|^{p'} \tilde{\varphi}^{-\frac{p'}{p}}.$$

In the same way, we get

$$\int_{Q_T} u \varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2 \leq \varepsilon \int_{Q_T} |u|^p \tilde{\varphi} + c_2(\varepsilon) \int_{Q_T} |\varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2|^{p'} \tilde{\varphi}^{-\frac{p'}{p}}.$$

When ε is small, we obtain

$$\int_{Q_T} |u|^p \tilde{\varphi} \leq c \left(\int_{Q_T} \varphi_1^{l-p'} \varphi_2^{-\frac{1}{p-1}} |(-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2|^{p'} + \int_{Q_T} |D_{t|T}^{\alpha+\beta} \varphi_2|^{p'} \varphi_1^l \varphi_2^{\frac{-1}{p-1}} \right).$$

The change of variables

$$\tau = \frac{t}{T}, \xi = \frac{x}{T^{\frac{\beta}{2m}}}.$$

Implies that

$$\int_{Q_T} |u|^p \tilde{\varphi} \leq c T^{-(1+(\alpha+\beta)p' - \frac{N\beta}{2m})} \left[\int_{|\xi| < 2} \Phi |(-\Delta)^m \Phi(\xi)|^{p'} d\xi + \int_{|\xi| < 2} \Phi^l d\xi \right].$$

Then, we have the estimate

$$\int_{Q_T} |u|^p \tilde{\varphi} < C T^{-((\alpha+\beta)p' - 1 - \frac{N\beta}{2m})}.$$

We put $\delta = -((\alpha + \beta)p' - 1 - \frac{N\beta}{2m})$, then

In the case $\delta < 0$, let $T \rightarrow +\infty$, we obtain $\int_{Q_T} |u|^p \tilde{\varphi} < 0$. This implies that $u \equiv 0$.

In the case $\delta = 0$, let $\varphi_1(x) = \Phi\left(\frac{|x|}{(\frac{T}{\mu})^{\frac{\beta}{2m}}}\right)$ and using $p > \frac{2m}{N\beta-2m} + 1$, we get

$$\int_{Q_T} |u|^p \tilde{\varphi} < c \mu^{p' - \frac{N\beta}{2m}} + c \mu^{-\frac{N\beta}{2m}}.$$

This implies that $u \equiv 0$ when $\mu \rightarrow \infty$. □

Remark 2. If $\beta = 1$ we go back to classical problem that was treated by Sun and Shi [15].

4. Nonexistence result for fractional evolution systems

This section is devoted to the study of the nonexistence of a solution in a fractional evolution system that is the core of our study.

$$(6) \quad \begin{cases} D_{0|t}^\beta(u - u_0) + (-\Delta)^m u = D_{0|t}^{-\alpha}|v|^p, \alpha = 1 - \gamma \\ D_{0|t}^{\delta'}(v - v_0) + (-\Delta)^{m'} v = D_{0|t}^{-\delta}|u|^q, \delta = 1 - \beta'. \end{cases}$$

Theorem 3. *Let $p, q > 1$. The system (6) does not admit nontrivial global weak solution for all*

$$N < \left\{ \frac{(\frac{\delta'+\delta}{q}) + (\beta + \alpha) - (1 - \frac{1}{pq})}{(\frac{\delta'}{2p'm'q}) + (\frac{\beta}{2mq})}, \frac{(\frac{\beta+\alpha}{p}) + (\delta' + 1 - \delta) - (1 - \frac{1}{pq})}{(\frac{\beta}{2q'm'p}) + (\frac{\delta'}{2m'p'})} \right\}.$$

Proof. The proof proceeds by contradiction. Let

$$\xi_1(t, x) = D_{t|T}^\alpha \varphi' = D_{t|T}^\alpha(\varphi_1^l(x)\varphi_2(t)),$$

and

$$\xi_2(t, x) = D_{t|T}^\delta \varphi' = D_{t|T}^\delta(\varphi_1^l(x)\varphi_2(t))$$

such that

$$\varphi_1(x) = \Phi(y) \quad \text{and} \quad \varphi_2(t) = \left(1 - \frac{t}{T}\right)^\eta,$$

where $l > \max\{p', q'\}$ and $\eta > \max\left\{\frac{q'(\alpha+\beta)-1}{q'-\frac{1}{q-1}}, \frac{p'(\delta+\delta')-1}{p'-\frac{1}{p-1}}\right\}$.

The weak solution to system (6) such that

$$\begin{aligned} & cT^{1-(\alpha+\beta)} \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_{Q_T} \varphi' |v|^p \\ &= \int_{Q_T} u \varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2(t) dx dt + \int_{Q_T} u (-\Delta)^m \varphi_1^l(x) D_{t|T}^\alpha \varphi_2, \end{aligned}$$

and

$$\begin{aligned} & cT^{1-(\delta+\delta')} \int_{\mathbb{R}^N} v_0 \varphi_1^l(x) dx + \int_{Q_T} \varphi' |u|^q \\ &= \int_{Q_T} v \varphi_1^l D_{t|T}^{\delta+\delta'} \varphi_2(t) dx dt + \int_{Q_T} v (-\Delta)^{m'} \varphi_1^l(x) D_{t|T}^\delta \varphi_2. \end{aligned}$$

By the Hölder inequality, we may write

$$\int_{Q_T} u \varphi_1^l D_{t|T}^{\alpha+\beta} \varphi_2(t) dx dt \leq \left[\int_{Q_T} |u|^q \varphi' \right]^{\frac{1}{q}} \left[\int_{Q_T} \varphi_1^l(x) \varphi_2^{\frac{-1}{q-1}} |D_{t|T}^{\alpha+\beta} \varphi_2|^{q'} \right]^{\frac{1}{q'}},$$

and

$$\int_{Q_T} u(-\Delta)^m \varphi_1^l(x) D_{t|T}^\alpha \varphi_2 \leq \left[\int_{Q_T} |u|^q \varphi' \right]^{\frac{1}{q}} \left[\int_{Q_T} \varphi_1^{l-q'} \varphi_2^{\frac{-1}{q'-1}} |(-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2|^{q'} \right]^{\frac{1}{q'}}.$$

Then

$$\int_{Q_T} |v|^p \varphi' \leq \left[\int_{Q_T} |u|^q \varphi' \right]^{\frac{1}{q}} A,$$

where

$$A = \left[\int_{Q_T} \varphi_1^l(x) \varphi_2^{\frac{-1}{q'-1}} |D_{t|T}^{\alpha+\beta} \varphi_2|^{q'} \right]^{\frac{1}{q'}} + [C \int_{Q_T} \varphi_1^{l-q'} \varphi_2^{\frac{-1}{q'-1}} |(-\Delta)^m \varphi_1 D_{t|T}^\alpha \varphi_2|^{q'}]^{\frac{1}{q'}}.$$

Similarly, we obtain the estimate

$$\int_{Q_T} |u|^q \varphi' \leq \left[\int_{Q_T} |v|^p \varphi' \right]^{\frac{1}{p}} B,$$

where

$$B = \left[\int_{Q_T} \varphi_1^l(x) \varphi_2^{\frac{-1}{p'-1}} |D_{t|T}^{\delta+\delta'} \varphi_2|^{p'} \right]^{\frac{1}{p'}} + [C \int_{Q_T} \varphi_1^{l-p'} \varphi_2^{\frac{-1}{p'-1}} |(-\Delta)^{m'} \varphi_1 D_{t|T}^\delta \varphi_2|^{p'}]^{\frac{1}{p'}}.$$

Consequently we can write

$$\begin{aligned} \left[\int_{Q_T} |v|^p \xi_1 \right]^{1-\frac{1}{pq}} &\leq B^{\frac{1}{q}} A, \\ \left[\int_{Q_T} |u|^q \xi_2 \right]^{1-\frac{1}{pq}} &\leq A^{\frac{1}{p}} B. \end{aligned}$$

If we use the change of variables as

$$t = R\tau \quad \text{and} \quad x = R^{\frac{\beta}{2m}} y \quad \text{in} \quad A,$$

and

$$t = R\tau \quad \text{and} \quad x = R^{\frac{\delta'}{2m'}} y \quad \text{in} \quad B.$$

We find the following estimation

$$\left[\int_{Q_T} |v|^p \xi_1 \right]^{1-\frac{1}{pq}} \leq C(R^{-l_1})^{\frac{1}{q}} R^{-l_2}$$

where

$$l_1 = (\delta' + \delta) - \frac{1}{p'} \left(\frac{N\delta'}{2m'} + 1 \right),$$

$$l_2 = (\beta + \alpha) - \frac{1}{q'} \left(\frac{N\beta}{2m} + 1 \right).$$

The case $\frac{l_1}{q} + l_2 > 0$ is equivalent to

$$N < \frac{(\frac{\delta'+\delta}{q}) + (\beta + \alpha) - (1 - \frac{1}{pq})}{\frac{\delta'}{2p'm'q} + \frac{\beta}{2mq'}}.$$

Similarly

$$N < \frac{(\frac{\beta+\alpha}{p}) + (\delta' + 1 - \delta) - (1 - \frac{1}{pq})}{\frac{\beta}{2q'm'p} + \frac{\delta'}{2m'p'}}.$$

That's why to obtain contradiction it suffices to assume

$$N < \max \left\{ \frac{(\frac{\delta'+\delta}{q}) + (\beta + \alpha) - (1 - \frac{1}{pq})}{\frac{\delta'}{2p'm'q} + \frac{\beta}{2mq'}}; \frac{(\frac{\beta+\alpha}{p}) + (\delta' + 1 - \delta) - (1 - \frac{1}{pq})}{\frac{\beta}{2q'm'p} + \frac{\delta'}{2m'p'}} \right\}$$

□

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Further properties of Hurwitz series rings and Hurwitz polynomials rings

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Abstract. For a commutative ring with unity R , a commutative ring is defined called the ring of Hurwitz series, HR . As a subring of this ring, HR , the ring of Hurwitz polynomials is defined, hR . In this paper, we characterize pure ideals in the ring hR . Then we characterize when the ring hR is an almost PP -ring and a PF -ring. Finally, for the ring R satisfying $x^n = x$ for a fixed positive integer n , we prove that every prime ideal of the rings $HR(hR)$ is maximal and so their spectrum is completely characterized.

Keywords: Hurwitz series ring, Hurwitz polynomial ring, pure ideal, PF -ring, almost PP -ring, spectrum of Hurwitz series ring, spectrum of Hurwitz polynomial ring, Zariski spectrum.

1. Introduction

All rings considered in this paper are commutative with unity $1 \neq 0$. For a ring R , a new ring, HR , was defined in Keigher [13] as follows:

HR is the set of all infinite sequences of elements of R , (a_n) , together with operations

$$\begin{aligned}(a_n) + (b_n) &= (a_n + b_n), \\ (a_n) \cdot (b_n) &= (c_n),\end{aligned}$$

where $c_n = \sum_{h+k=n}^n \binom{n}{h} a_h b_k$, where $\binom{n}{h}$ is the binomial coefficient.

Now, HR is a commutative ring with unity $(1, 0, 0, 0, \dots)$. This ring is called the ring of Hurwitz series over the ring R . Let

$$hR = \{(a_n) : a_n = 0 \text{ for all } n \geq k \text{ for some } k\}.$$

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Then this is a subring of HR , called the ring of Hurwitz polynomials over R . The algebraic structure of this ring HR was studied by several authors, e.g Keigher [13], Benhissi [6, 7, 8], Abu Dayah [1]. Our aim is to study some properties of the ring of Hurwitz polynomials, hR .

Notice that two ring homomorphisms can be defined : $\varepsilon : HR \rightarrow R$ defined by $\varepsilon((a_n)) = a_0$ and $\lambda : R \rightarrow HR$ defined by $\lambda(a) = (a, 0, 0, 0, \dots)$.

Also, R is called a differential ring with a derivation $d : R \rightarrow R$ if the derivation satisfies Leibniz rule $d(ab) = a d(b) + d(a) b$ for all $a, b \in R$. An ideal I is differential ideal of R , if for every $x \in I$, $d(x) \in I$. HR is a differential ring with a derivation $d : HR \rightarrow HR$, $d((a_n)) = (a_{n+1})$ for every $(a_n) \in HR$.

Recall that an ideal I of a ring R is called pure ideal if for $a \in I$, there exists $b \in I$ such that $a.b = a$. Pure ideals are interesting because they classify important families of rings, e.g. Von Neumann rings and PF -rings, see Al-Ezeh [2, 3, 4, 5], Jondrup [12], DeMarco [10]. Also, a ring $R \in \mathcal{C}$, if for any pure ideal I in R , I is generated by a family of idempotents. Moreover, every pure ideal is a differential ideal.

In this paper we characterize pure ideals in the ring hR and give its relationship to pure ideals of the ring R .

A ring R is called a PF -ring if every principal ideal of R is a flat R -module. These rings were studied by several authors. Al-Ezeh proved in [4] that a ring R is a PF -ring if and only if the annihilator of each $a \in R$, $\text{ann}_R(a)$, is a pure ideal in R . We characterize in this paper when the ring hR is a PF -ring. Recall that a ring R is called an almost PP -ring if the annihilator of each $a \in R$, $\text{ann}_R(a)$, is generated by a set of idempotents. These rings were studied in literature as a generalization of PP -rings, i.e. rings in which every principal ideal is a projective R -module, see Al-Ezeh [5] and Burgess [9]. In this paper, we prove that hR is an almost PP -ring if and only if R is an almost PP -ring and R is a torsion free. Moreover, we prove that $hR \in \mathcal{C}$.

Finally, for the ring R satisfying $x^n = x$ for some positive integer $n \geq 2$, we characterize the Zariski spectrum of the ring, hR .

2. Pure ideals in hR ring

Note that $HR(hR)$ means the ring HR or hR . Abu Deyah [1] has given a partial characterization of the pure ideals in HR . In this section we characterize the pure ideals in a ring hR .

Theorem 2.1. *Let I be an ideal in a ring hR . Then I is pure ideal in hR if and only if $I = [\lambda(\varepsilon(I))]$ and $\varepsilon(I)$ is pure ideal in R .*

Proof. Suppose that I is pure ideal in hR . Let $(a_n) = (a_0, a_1, \dots, a_n) \in I$. But I is a pure ideal, so there exists $(b_m) = (b_0, b_1, \dots, b_m) \in I$ such that

$(a_n)(b_m) = (a_n)$. Now, a_0 and b_0 in $\varepsilon(I)$, so we have inductively

$$\begin{cases} a_1 &= a_0 b_1 + a_1 b_0 \in \varepsilon(I) \\ a_2 &= a_0 b_2 + 2a_1 b_1 + a_2 b_0 \in \varepsilon(I) \\ &\vdots \\ a_n &= a_0 b_n + \dots + a_n b_0 \in \varepsilon(I). \end{cases}$$

Thus, $\lambda(a_0), \lambda(a_1), \dots, \lambda(a_n) \in \lambda(\varepsilon(I))$. Hence, we have

$$\begin{cases} \lambda(a_0) &= \lambda(a_0)(1) \in [\lambda(\varepsilon(I))] \\ (0, a_1) &= \lambda(a_1)(0, 1) \in [\lambda(\varepsilon(I))] \\ &\vdots \\ (0, 0, \dots, a_n) &= \lambda(a_n)(0, \dots, 1) \in [\lambda(\varepsilon(I))]. \end{cases}$$

So, $(a_n) = \lambda(a_0) + (0, a_1) + \dots + (0, 0, \dots, a_n) \in [\lambda(\varepsilon(I))]$ and hence $I \subseteq [\lambda(\varepsilon(I))]$. Now let $\alpha \in \varepsilon(I)$ such that $\lambda(\alpha)$ is any generator for the ideal $[\lambda(\varepsilon(I))]$. So there exist $t_1, t_2, \dots, t_n \in R$ such that $(\alpha, t_1, \dots, t_n) \in I$. Let $\overline{\alpha_n} = (\alpha, t_1, \dots, t_n)$. However, since I is pure ideal, I is a differential ideal. Thus, inductively we get

$$\begin{cases} d(\overline{\alpha_n}) &= (t_1, \dots, t_n) \in I \\ d^2(\overline{\alpha_n}) &= (t_2, \dots, t_n) \in I \\ &\vdots \\ d^n(\overline{\alpha_n}) &= \lambda(t_n) \in I. \end{cases}$$

So, $(0, 0, \dots, t_n) = \lambda(t_n)(0, 0, \dots, 1) \in I$. Hence, $\overline{\alpha_{n-1}} = \overline{\alpha_n} - (0, 0, \dots, t_n) \in I$. In the same way we can get $\overline{\alpha_{n-2}}, \overline{\alpha_{n-3}}, \dots, \overline{\alpha_0}$ in I . But $\overline{\alpha_0} = \lambda(\alpha) \in I$. Therefore $I = [\lambda(\varepsilon(I))]$.

The other side of the theorem can be proved using the fact if $S \subseteq T$ two rings such that I is a pure ideal in S , then the ideal generated by I , $[I]$, is pure in T . $\square$

Corollary 2.2. *Suppose that I is an ideal in hR such that for any $(a_n) = (a_0, a_1, \dots, a_n) \in I$ we have $a_0 = 0$. Then*

1. $I = \{0\}$ if I is a pure ideal.
2. I is never pure ideal if there exist at least $a_i \neq 0, i = 1, 2, \dots, n$.

Proof. Clear from previous theorem. $\square$

Lemma 2.3. *Let $(a_n) = (a_0, a_1, \dots, a_n) \in hR$. Then $I = \text{ann}_{hR}((a_n))$ is pure in hR if and only if $J = \text{ann}_R(\{a_0, \dots, a_n\})$ is pure in R and for any $(b_m) \in I$, $\lambda(b_i) \in I, i = 0, \dots, m$.*

Proof. Let $b \in J$, so $ba_i = 0, i = 0, \dots, n$. This implies $\lambda(b) \in I$. But I is pure ideal, so there exists $(c_0, \dots, c_m) \in I$ such that $\lambda(b)(c_0, \dots, c_m) = \lambda(b)$. Hence $bc_0 = b$. From Theorem 2.1, $c_0 \in \varepsilon(I)$ and $\lambda(c_0) \in I$. Thus $c_0a_i = 0, i = 0, 1, \dots, n$. Therefore $c_0b = b$ and $c_0 \in J$.

Conversely, let $(b_m) = (b_0, \dots, b_m) \in I$. By assumption $\lambda(b_i) \in I, i = 0, \dots, m$. So $b_0, b_1, \dots, b_m \in J$. But J is pure, so there exist $w_i \in J$ such that $b_iw_i = b_i, i = 0, 1, \dots, m$. So, there exists $\hat{w} \in J$ such that $b_i\hat{w} = b_i, i = 0, 1, \dots, m$. Also, $\lambda(\hat{w}) \in I$ and $(b_0, \dots, b_m)\lambda(\hat{w}) = (b_0, \dots, b_m)$. $\square$

3. When HR and hR are almost PP-rings (PF-rings)

Theorem 3.1. *Suppose that $hR(HR)$ is an almost PP-ring. Then R is an almost PP-ring and R is a torsion free as $\mathbb{Z}$ -module.*

Proof. Let $a \in R$. Suppose that $r \in \text{ann}_R(a)$. Now $\text{ann}_R(a) \subseteq \text{ann}_{hR}(\lambda(a))$, so $\lambda(r) \in \text{ann}_{hR}(\lambda(a))$. However hR is an almost PP-ring, so from [2] there exist $e \in \text{Bool}(hR)$ with $re = e$ and hence R is an almost PP-ring. Also, since hR is an almost PP-ring, then hR is a PF-ring. From [7] R is torsion free.

Consequently, the same proof works for the ring HR . $\square$

Theorem 3.2. *The ring hR is an almost PP-ring if and only if R is an almost PP-ring and R is a torsion free as $\mathbb{Z}$ -module.*

Proof. Let $f = (a_0, \dots, a_n) \in hR$ and suppose that $g = (b_0, \dots, b_m) \in \text{ann}_{hR}(f)$. From [7] we get $a_ib_j = 0, i = 0, \dots, n, j = 0, \dots, m$, so $\lambda(b_i) \in \text{ann}_{hR}(f)$. Hence $b_j \in \text{ann}_R(a_i), i = 0, \dots, n, j = 0, \dots, m$. But R is an almost PP-ring, so there exist

$$\begin{aligned} e_{0i} &\in \text{ann}_R(a_i) \text{ with } b_0e_{0i} = b_0, i = 0, \dots, n, \\ e_{1i} &\in \text{ann}_R(a_i) \text{ with } b_1e_{1i} = b_1, i = 0, \dots, n, \\ &\vdots \\ e_{mi} &\in \text{ann}_R(a_i) \text{ with } b_me_{mi} = b_m, i = 0, \dots, n. \end{aligned}$$

and $(e_{ki})^2 = e_{ki}, i = 0, \dots, n, k = 0, \dots, m$.

Let

$$\begin{aligned} \bar{e}_0 &= e_{00}e_{01} \dots e_{0n}, \\ \bar{e}_1 &= e_{10}e_{11} \dots e_{1n}, \\ &\vdots \\ \bar{e}_m &= e_{m0}e_{m1} \dots e_{mn}. \end{aligned}$$

Then $\bar{e}_j \in \text{ann}_R(a_i)$ and $\bar{e}_jb_j = b_j$, where $i = 0, \dots, n, j = 0, \dots, m$. So, there exist $\hat{e} \in \text{ann}_R(a_i), i = 0, \dots, n$ and $b_j\hat{e} = b_j, j = 0, \dots, m$. Hence $\lambda(\hat{e}) \in \text{ann}_{hR}(f)$ with $g\lambda(\hat{e}) = g$. Therefore, hR is an almost PP-ring. $\square$

Theorem 3.3. *The ring hR is a PF-ring if and only if for any $f = (a_0, \dots, a_n) \in hR$, we have $\text{ann}_R\{a_0, \dots, a_n\}$ is pure ideal in R and R is torsion free.*

Proof. Since for any $r \in R$, $\lambda(r) \in hR$, then $\text{ann}_R(r)$ is pure ideal in R . From [7], hR is a *PF*-ring. Conversely, suppose that hR is a *PF*-ring. Let $f = (a_0, \dots, a_n) \in hR$ and assume $r \in \text{ann}_R\{a_0, \dots, a_n\}$, so $\lambda(r) \in I = \text{ann}_{hR}(f)$. But I is pure ideal, so there exists $(b_0, \dots, b_m) \in I$ such that $\lambda(r)(b_0, \dots, b_m) = \lambda(r)$. This implies $rb_0 = r$ and $rb_1 = rb_2 = \dots = rb_m = 0$. From Lemma 2.3, we have $\lambda(b_i) \in I$ and hence $b_0 \in \text{ann}_R\{a_0, \dots, a_n\}$ with $b_0r = r$. $\square$

Theorem 3.4. *Suppose that R be a Notherian ring. Then HR is a *PF*-ring if and only if $\text{ann}_R\{a_0, a_1, \dots\}$ is pure ideal and R is a torsion free as $\mathbb{Z}$ -moduel, for every $(a_0, a_1, \dots) \in HR$.*

Proof. Let $b \in \text{ann}_R\{a_0, a_1, \dots\}$, so $\lambda(b) \in \text{ann}_{HR}((a_0, a_1, \dots))$. But, HR is a *PF*-ring, so by [7] R is reduced and torsion free. Moreover, there exists $c \in R$ such that, $cb = c$ and $ca_i = 0, i = 0, 1, 2, \dots$, which implies $\text{ann}_R\{a_0, a_1, \dots\}$ is pure ideal.

Conversely, let $r \in R$, so $\text{ann}_R\{r\}$ is pure ideal. Since R is a torsion free and R is Notherian, from [7] HR is a *PF*-ring. $\square$

Finally, we prove the following theorem :

Theorem 3.5. *If $R \in \mathcal{C}$, then $hR \in \mathcal{C}$.*

Proof. Let I be a pure in hR , so $\varepsilon(I)$ is pure ideal in R . However $R \in \mathcal{C}$. So $\varepsilon(I)$ is generated by idempotents say $\{e_i : i \in \Delta\}$. Thus $\{\lambda(e_i) : i \in \Delta\}$ is the set of idempotent generators for the ideal $[\lambda(\varepsilon(I))]$ in hR . But, $I = [\lambda(\varepsilon(I))]$ using Theorem 2.1. Hence I is generated by idempotents in hR which implies $hR \in \mathcal{C}$ $\square$

4. Some results on the ring $hR(HR)$ over a ring R satisfying $x^n = x$

In this section we only consider rings for every $x \in R$ we have $x^n = x$ for some fixed n unless otherwise stated. It is well known that R has a finite characteristic, R is reduced, and every prime ideal is maximal ideal, see [11].

The next theorem was mentioned without proof in [13]. The theorem was proved by Abu Dayah [1]. His proof is also true for the ring hR

Theorem 4.1. *Let R be a ring with $ch(R) = m$. So,*

1. *If P is prime ideal of $HR(hR)$, then $\varepsilon(P)$ is prime ideal of R .*
2. *If P is prime ideal of $HR(hR)$, then $P = \varepsilon^{-1}(\varepsilon(P))$.*

Lemma 4.2. *Let $x = (a_0, a_1, \dots, a_t, \dots) \in HR$ (or $x = (a_0, a_1, \dots, a_t) \in hR$). Then x is nilpotent if and only if $a_0 = 0$.*

Proof. Suppose $x^r = 0$ for some positive integer r . Then $a_0^r = 0$ and since R is reduced, $a_0 = 0$. Conversely, let $\text{char}(R) = m$, so $x^m = 0$ and hence x is nilpotent. $\square$

Theorem 4.3. *Every prime ideal in $hR(HR)$ is maximal ideal.*

Proof. Assume that $\text{char}(R) = m$ and suppose that P is a prime ideal in hR with

$$P \subseteq M \subseteq hR,$$

where M is an ideal in hR .

Case 1 : Every $x \in M$ has the form $x = (0, x_1, x_2, \dots, x_t)$. Now,

$$\varepsilon(P) \subseteq \varepsilon(M) \subseteq \varepsilon(hR).$$

However, $\varepsilon(M) = \{0\}$, so $\varepsilon(P) = \{0\}$. Since every prime ideal in R is maximal, then by Theorem 4.1, we have $\varepsilon(P) = \{0\}$ is a maximal ideal in R . So, R is a field and again using Theorem 4.1 we have $P = \varepsilon^{-1}(\{0\}) = M$.

Case 2 : M has an element $x = (x_0, x_1, \dots, x_t)$ with $x_0 \neq 0$ and $x \notin P$.

So, $x^m = (x_0^m, 0, \dots, 0) \notin P$ and $x^m \in M$, hence $x_0^m \neq 0$. Let $y = \lambda(x_0^m)$, so $y^n = y$ and $y \notin P$. Thus,

$$0 = y - y^n = y(1 - y^{n-1}) \in P \subseteq M.$$

So, $1 - y^{n-1} \in P$. Hence $1 - y^{n-1} \in M$ and $y^{n-1} \in M$, so $M = hR$. Therefore P is maximal ideal in hR .

Case 3 : For each element $x = (x_0, x_1, \dots, x_t)$ in M with $x_0 \neq 0$, we have $x_0 \in P$.

From the assumption, we get $\varepsilon(P) = \varepsilon(M)$. By Theorem 4.1, we have

$$M = \varepsilon^{-1}(\varepsilon(M)) = \varepsilon^{-1}(\varepsilon(P)) = P.$$

Consequently, the same proof works for the ring HR . □

Theorem 4.4.

$$\text{Spec}(HR) = \{\varepsilon^{-1}(P) : P \text{ is prime in } R\}.$$

Proof. Let P be a prime ideal in R . So, P is maximal ideal in R . Using [6], we get $\varepsilon^{-1}(P)$ is maximal in HR and hence it is prime in HR . Conversely, let Q be a prime ideal in $\text{Spec}(HR)$. By Theorem 4.1, $\varepsilon(Q)$ is a prime ideal in R . But every prime ideal in R is maximal, so using Theorem 4.1 and [6], $Q = \varepsilon^{-1}(\varepsilon(Q))$ and so it is maximal in HR . Therefore, $Q = \varepsilon^{-1}(P)$, $P = \varepsilon(Q)$ is prime in R . □

Theorem 4.5.

$$\text{Spec}(hR) = \{\varepsilon^{-1}(P) : P \text{ is prime in } R\}.$$

Proof. Note that if M is a maximal ideal in R . Then $\varepsilon^{-1}(M)$ is maximal in hR . Because, it is easy to see that $hR/\varepsilon^{-1}(M) \simeq R/M$. So we can use the same proof of Theorem 4.4. □

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On the power of simulation map for almost $\mathbb{Z}$ -contraction in $\mathcal{G}$ -metric space with applications to the solution of the integral equation

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Abstract. In this manuscript, we explore the presence and uniqueness of a fixed point of almost $\mathbb{Z}$ -contraction by means of simulation map in the framework of $\mathcal{G}$ -metric spaces. Also, an illustrative example and an application to solve integral equation are given to help accessibility of the got outcomes.

Keywords: $\mathcal{G}$ -metric space, almost $\mathbb{Z}$ -contraction, simulation map.

1. Introduction

The Banach contraction rule is a one of the predominant outcomes in analysis and has continuously been at the front line of making and providing remarkable speculations for its researchers. Numerous authors have summed up and used the Banach principle in their relevant research ([1-4]). Along these lines, we can without much of a stretch presume that the biggest part of the fixed point theory has been involved by different speculations of the Banach contraction rule. Further, Khojasteh and Shukla [6] presented an alternate thought of simulation map by utilizing an idea of [3] and explored some fundamental

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properties. Subsequently, Argoubi [7] demonstrated fixed point results for non linear contraction in the casing of metric space.

Throughout the paper, $\mathcal{N}$ denotes set of natural numbers, $\mathcal{R}$ denotes set of real numbers.

Next, we review some essential definitions about contraction and the outcomes from the writing. Wardowski [1] defined the F -contraction as follows:

Definition 1.1. Let (X, d) be a metric space and let $f : X \rightarrow X$ be a self-mapping. Then f is called an F -contraction on (X, d) if there exist $F \in \mathfrak{F}$ for all $x, y \in X$, $d(fx, fy) > 0 \Rightarrow \gamma + F(d(fx, fy)) \leq F(d(x, y))$, where $\mathfrak{F}$ be class of all mappings $F : (0, \infty) \rightarrow \mathcal{R}$ such that:

(F1) F is strictly increasing function, that is, for all $a, b \in (0, \infty)$, if $a < b$, then $F(a) < F(b)$.

(F2) For every sequence $\{a_n\}$ of natural numbers, $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(a_n) = -\infty$.

(F3) There exists $q \in (0, 1)$ such that $\lim_{a \rightarrow 0^+} (a^q F(a)) = 0$.

F -weak contraction was established by Wardowski and Dung [2] in 2014 which is defined as follows:

Definition 1.2. [2] Let (X, d) be a metric space and $\mathcal{T} : X \rightarrow X$ be a function. $\mathcal{T}$ is known as F -weak contraction on (X, d) if there exist $F \in \mathfrak{F}$ and $\gamma > 0$ such that for all $x, y \in X$, $d(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \gamma + F(d(\mathcal{T}x, \mathcal{T}y)) \leq F(\max\{d(x, y), d(\mathcal{T}x, x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\})$.

Definition 1.3 ([3, 4]). Let (X, d) be a metric space and let $\sigma : X \rightarrow X$ be a self-mapping. Then σ is called an almost $\mathbb{Z}$ -contraction on (X, d) if there exist $\beta \geq 0$ and $\beta_1 < 1$ such that for all $x, y \in X$, $d(\sigma x, \sigma y) \leq \beta_1 d(x, y) + \beta f(x, y)$.

Definition 1.4 ([5]). The mapping $\theta : [0, \infty) \times [0, \infty) \rightarrow \mathcal{R}$ is said to be a simulation function, if the following properties hold: $(\theta_1)\theta(0, 0) = 0$, $(\theta_2)\theta(a, b) < b - a$, for all $a, b > 0$, (θ_3) if $\{a_n\}, \{b_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} \{a_n\} = \lim_{n \rightarrow \infty} \{b_n\} = \ell \in (0, \infty)$, then $\lim_{n \rightarrow \infty} \sup \theta(a_n, b_n) < 0$.

The family of all simulation functions $\theta : [0, \infty) \times [0, \infty) \rightarrow \mathcal{R}$ is denoted by $\mathbb{Z}$.

Definition 1.5 ([6]). Let (X, d) be a metric space and $\theta \in \mathbb{Z}$. The self map $\sigma : X \rightarrow X$ is $\mathbb{Z}$ -contraction with respect to θ if for each $x, y \in X$,

$$(1.1) \quad \theta(d(\sigma x, \sigma y), d(x, y)) \geq 0.$$

Definition 1.6 ([8]). Let (X, d) be a metric space and $\theta \in \mathbb{Z}$. Then, the self map $\sigma : X \rightarrow X$ is said to be almost $\mathbb{Z}$ -contraction if for each $x, y \in X$, we can find a positive constant β such that

$$(1.2) \quad \theta(d(\sigma x, \sigma y), d(x, y) + \beta m(x, y)) \geq 0,$$

where

$$m(x, y) = \min\{d(x, \sigma x), d(y, \sigma y), d(x, \sigma y), d(y, \sigma x)\}.$$

Definition 1.7 ([9]). Let X be a non void set and $\mathcal{G} : X^3 \rightarrow [0, \infty)$ be a map which fulfils the accompanying conditions:

- (i) $\mathcal{G}(x_1, y_1, z_1) = 0$ if $x_1 = y_1 = z_1$;
- (ii) $0 < \mathcal{G}(x_1, x_1, y_1)$ whenever $x_1 \neq y_1$, for all $x_1, y_1 \in X$;
- (iii) $\mathcal{G}(x_1, x_1, y_1) \leq \mathcal{G}(x_1, y_1, z_1)$, $y_1 \neq z_1$;
- (iv) $\mathcal{G}(x_1, y_1, z_1) = \mathcal{G}(x_1, z_1, y_1) = \mathcal{G}(y_1, x_1, z_1) = \mathcal{G}(z_1, x_1, y_1) = \mathcal{G}(y_1, z_1, x_1) = \mathcal{G}(z_1, y_1, x_1)$;
- (v) $\mathcal{G}(x_1, y_1, z_1) \leq \mathcal{G}(x_1, a_1, a_1) + \mathcal{G}(a_1, y_1, z_1)$;

for every $x_1, y_1, z_1, a_1 \in X$, then the function $\mathcal{G}$ is said to be $\mathcal{G}$ -metric on X and $(X, \mathcal{G})$ is known as $\mathcal{G}$ -metric space.

In this paper, we consolidate the view of simulation map, $\mathcal{G}$ -metric space and indispensable nature of contractive mappings to construct fixed point hypotheses in the casing of generalized metric space.

The principle point of our examination is to talk about the reasonability of application to solve the integral equation with the assistance of our main theorem.

2. Main results

Definition 2.1. Let $(X, \mathcal{G})$ be a $\mathcal{G}$ -metric space and $\theta \in \mathbb{Z}$. Then, the self map $\sigma : X \rightarrow X$ is said to be almost $\mathbb{Z}$ -contraction if for each $x, y, z \in X$, we can find a positive constant β such that

$$(2.1) \quad \theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \mathcal{G}(x, y, z) + \beta m(x, y, z)) \geq 0,$$

where

$$(2.2) \quad m(x, y, z) = \min\{\mathcal{G}(x, \sigma y, \sigma y), \mathcal{G}(y, \sigma x, \sigma x), \mathcal{G}(y, \sigma z, \sigma z), \mathcal{G}(z, \sigma y, \sigma y), \mathcal{G}(z, \sigma x, \sigma x), \mathcal{G}(x, \sigma z, \sigma z)\}.$$

Remark 2.2. If σ is almost $\mathbb{Z}$ -contraction with respect to $\theta \in \mathbb{Z}$, then we have $\mathcal{G}(\sigma x, \sigma y, \sigma z) < \mathcal{G}(x, y, z) + \beta m(x, y, z)$, where $x, y, z \in X$.

Theorem 2.3. Let $(X, \mathcal{G})$ be a complete $\mathcal{G}$ metric space and $\sigma : X \rightarrow X$ be an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$. Then, σ has a fixed point. Moreover, the sequence $\{\sigma^s z_0\}$ converges to fixed point of σ for each $z_0 \in X$.

Proof. Step 1. Suppose that, $z_{s+1} = \sigma^s z_0 = \sigma z_s$, where $s \in \mathcal{N}$ and $z_0 \in X$.

If $\exists s \in \mathcal{N}$ such that $z_{s+1} = z_s$, then $\sigma z_s = z_s$. So, z_s is fixed point of σ .

Let us suppose that $z_s \neq z_{s+1}$ for each $s \in \mathcal{N}$. Then, $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) > 0$ for all $s \in \mathcal{N}$.

Firstly, we prove that $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0$. On account of (2.2), we get that

$$\begin{aligned}
 f(z_{s-1}, z_s, z_s) &= \min\{\mathcal{G}(z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_s, \sigma z_{s-1}, \sigma z_{s-1}), \mathcal{G}(z_s, \sigma z_s, \sigma z_s), \\
 &\quad \mathcal{G}(z_s, \sigma z_s, \sigma z_s), \mathcal{G}(z_s, \sigma z_{s-1}, \sigma z_{s-1}), \mathcal{G}(z_{s-1}, \sigma z_s, \sigma z_s)\} \\
 &= \min\{\mathcal{G}(z_{s-1}, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_s, z_s), \\
 &\quad \mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_s, z_s), \mathcal{G}(z_{s-1}, z_{s+1}, z_{s+1})\} \\
 (2.3) \quad &= 0.
 \end{aligned}$$

On account of inequality (2.1), we get that

$$\theta(\mathcal{G}(\sigma z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_{s-1}, z_s, z_s)) + \beta f(z_{s-1}, z_s, z_s) \geq 0.$$

Taking (2.3) into account, we obtain

$$\begin{aligned}
 0 &\leq \theta(\mathcal{G}(\sigma z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_{s-1}, z_s, z_s)) \\
 &= \theta(\mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_{s-1}, z_s, z_s)) \\
 &< \mathcal{G}(z_{s-1}, z_s, z_s) - \mathcal{G}(z_s, z_{s+1}, z_{s+1})
 \end{aligned}$$

which implies that

$$\mathcal{G}(z_s, z_{s+1}, z_{s+1}) < \mathcal{G}(z_{s-1}, z_s, z_s).$$

Therefore, $\{\mathcal{G}(x_{s+1}, x_s, x_s)\}$ is non negative decreasing sequence of real numbers where $s \in \mathcal{N}$. So, $\exists \lambda \geq 0$ such that

$$\lim_{s \rightarrow \infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = \lambda.$$

Now, we will indicate that

$$\lim_{s \rightarrow \infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0.$$

Suppose that $\lambda > 0$. Let $\{c_s\}$ and $\{d_s\}$ be sequences such that $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) = c_s$ and $\mathcal{G}(z_{s-1}, z_s, z_s) = d_s$.

Now,

$$\lim_{s \rightarrow \infty} c_s = \lim_{s \rightarrow \infty} d_s = \lambda.$$

Accordingly, from θ_3 we deduce that

$$0 \leq \lim_{n \rightarrow \infty} \sup \theta((\mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_{s-1}, z_s, z_s))) < 0,$$

which is a contradiction. Thus, we have

$$(2.4) \quad \lim_{n \rightarrow \infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0.$$

Step 2. We assert that $\{z_s\}$ is bounded.

Suppose, on the contrary, that $\{z_s\}$ is unbounded. So, there exists a subsequence $\{z_{s_u}\}$ so that $s_1 = 1$ and s_{u+1} is the smallest integer larger than s_u such that $\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) > 1$ and $\mathcal{G}(z_u, z_r, z_r) \leq 1$, for every $r \in [s_u, s_{u+1} - 1]$.

With the assistance of triangle inequality, we obtain

$$\begin{aligned} 1 &< \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \\ &\leq \mathcal{G}(z_{s_{u+1}}, z_{s_{u+1}-1}, z_{s_{u+1}-1}) + \mathcal{G}(z_{s_{u+1}-1}, z_{s_u}, z_{s_u}) \\ &\leq 1 + \mathcal{G}(z_{s_{u+1}}, z_{s_{u+1}-1}, z_{s_{u+1}-1}). \end{aligned}$$

Making $u \rightarrow \infty$ and using (2.4), we get

$$(2.5) \quad \lim_{u \rightarrow \infty} \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) = 1.$$

Since, σ is an almost $\mathbb{Z}$ contraction with respect to θ , we conclude that

$$\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}).$$

On account of triangle inequality, we obtain

$$\begin{aligned} 1 &< \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \\ &\leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) \\ &\leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u}, z_{s_u}) + \mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1}) \\ &\leq 1 + \mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1}). \end{aligned}$$

Letting $u \rightarrow \infty$ and using (2.4), we get

$$(2.6) \quad \lim_{u \rightarrow \infty} \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = 1.$$

Now, taking (2.2) into account

$$\begin{aligned} f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) &= \min\{\mathcal{G}(z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\quad \mathcal{G}(z_{s_u-1}, \sigma z_{s_{u+1}-1}, \sigma z_{s_{u+1}-1}), \\ &\quad \mathcal{G}(z_{s_u-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \mathcal{G}(z_{s_u-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\quad \mathcal{G}(z_{s_u-1}, \sigma z_{s_{u+1}-1}, \sigma z_{s_{u+1}-1}), \mathcal{G}(z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1})\}. \end{aligned}$$

Letting $u \rightarrow \infty$ and using (2.4), we get

$$(2.7) \quad f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = 1.$$

With the assistance of (2.1) and θ_2 , we obtain

$$\begin{aligned} 0 &\leq \theta(\mathcal{G}(\sigma z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\quad \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, \\ &\quad z_{s_u-1}, z_{s_u-1})) < \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) \\ &\quad + \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) - \mathcal{G}(\sigma z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}) \\ &= \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) - \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \end{aligned}$$

which yields that

$$\mathcal{G}(z_{s_u+1}, z_{s_u}, z_{s_u}) < \mathcal{G}(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}).$$

Let $\{w_u\}$ and $\{v_u\}$ be sequences such that $\mathcal{G}(z_{s_u+1}, z_{s_u}, z_{s_u}) = w_u$ and $\mathcal{G}(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}) = v_u$.

Accordingly, from (2.5), (2.6) and (2.7), we get $\lim_{u \rightarrow \infty} \mathcal{G}(z_{s_u+1}, z_{s_u}, z_{s_u}) = 1 = \lim_{u \rightarrow \infty} \mathcal{G}(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1})$. On account of assumption of θ_3 , we obtain $\lim_{u \rightarrow \infty} \sup \theta(\mathcal{G}(z_{s_u+1}, z_{s_u}, z_{s_u}), \mathcal{G}(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_u+1-1}, z_{s_u-1}, z_{s_u-1})) < 0$ which is a contradiction. Therefore, $\{z_s\}$ is bounded.

Step 3. We assert that $\{z_s\}$ is a cauchy sequence. Let us suppose

$$\mathcal{O}_s = \sup\{\mathcal{G}(z_c, z_d, z_d) : c, d \geq s\}.$$

Now, $\{\mathcal{O}_s\}$ is decreasing sequence of positive entries, therefore, there exists $\mathcal{O} \geq 0$ such that

$$\lim_{s \rightarrow \infty} \mathcal{O}_s = \mathcal{O}.$$

When $\mathcal{O} > 0$, then applying the definition of $\mathcal{O}_s$, there exists r_u, s_u satisfying $r_u > s_u \geq u$ and $\mathcal{O}_s - \frac{1}{s} < \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) \leq \mathcal{O}_s$ which yields that

$$(2.8) \quad \mathcal{O} = \lim_{s \rightarrow \infty} \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u})$$

Again, with the assistance of triangle inequality, we obtain $\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) \leq \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) \leq \mathcal{G}(z_{r_u-1}, z_{r_u}, z_{r_u}) + \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) + \mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1})$. Letting $u \rightarrow \infty$ and taking (2.6), (2.8) into account, we obtain

$$(2.9) \quad \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = \mathcal{O}.$$

With the aid of remark(2.2), we have

$$(2.10) \quad \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) < \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}).$$

Accordingly, from (2.4), we get

$$(2.11) \quad \lim_{u \rightarrow \infty} f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = 0.$$

Let $\{w_u\}$ and $\{v_u\}$ be sequences such that $\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) = w_u$ and $\mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = v_u$.

Using (2.8) to (2.11), we obtain

$$\lim_{u \rightarrow \infty} \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) = \mathcal{O} = \lim_{u \rightarrow \infty} \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}).$$

On account of assumption of θ_3 , we obtain

$$\lim_{u \rightarrow \infty} \sup \theta(\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}), \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1})) < 0.$$

which is a contradiction. Therefore, $\mathcal{O} = 0$, which yields that $\{z_s\}$ is a Cauchy sequence. Due to completeness of $\mathcal{G}$ metric space, there exists $\varrho \in X$ so that $\lim_{s \rightarrow \infty} z_s = \varrho$.

Step 4. In the last step, we assert that ϱ is fixed point of σ .

Let $\sigma\varrho \neq \varrho$. Thus, $\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho) > 0$. With the assistance of (2.1), θ_2 and θ_3 , we obtain

$$\begin{aligned} 0 &\leq \limsup_{u \rightarrow \infty} \theta[\mathcal{G}(\sigma z_s, \sigma\varrho, \sigma\varrho), \mathcal{G}(z_s, \varrho, \varrho) + \beta f(z_s, \varrho, \varrho)] \\ &< \limsup_{u \rightarrow \infty} [\mathcal{G}(z_s, \varrho, \varrho) + \beta f(z_s, \varrho, \varrho) - \mathcal{G}(\sigma z_s, \sigma\varrho, \sigma\varrho)] = -\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho), \end{aligned}$$

which yields that $\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho) = 0$. Thus, $\sigma\varrho = \varrho$, which proves that ϱ is fixed point of σ . $\square$

Theorem 2.4. Let $(X, \mathcal{G})$ be a complete $\mathcal{G}$ metric space and $\sigma : X \rightarrow X$ be an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$. If, σ has a fixed point, then it is unique.

Proof. In view of Theorem 2.3, we ensure the presence of settled point of map σ , that is $\varrho = \sigma\varrho$. Next, we claim that if ϱ_1 and ϱ_2 are the fixed points of σ , then $\varrho_1 = \varrho_2$. On account of (2.1), we acquire that

$$\begin{aligned} 0 &\leq \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2) + \beta f(\varrho_1, \varrho_2, \varrho_2)] \\ &= \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2) + \beta \min\{\mathcal{G}(\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_1, \sigma\varrho_1), \\ &\quad \mathcal{G}(\varrho_2, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_1, \sigma\varrho_1), \mathcal{G}(\varrho_1, \sigma\varrho_2, \sigma\varrho_2)\}] \\ &= \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2)] \\ &< \mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2) - \mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2) = 0, \end{aligned}$$

which is contradiction. Consequently, $\varrho_1 = \varrho_2$, which indicates that fixed point of σ is unique. $\square$

Illustrative Example 2.5. Let $X = [0, 1]$ and $\mathcal{G}$ be defined as $\mathcal{G}(x_1, y_1, z_1) = |x_1 - y_1| + |y_1 - z_1| + |z_1 - x_1|$.

Now, we define $\sigma : X \rightarrow X$ as $\sigma z = \frac{1}{2} - z$, for each $z \in X$. We shall prove that $\sigma : X \rightarrow X$ is an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, but σ is not $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, where for each $a, b > 0$, $\theta(a, b) = \delta b - a$, where $\delta \in [0, 1]$. For each distinct elements z_1, z_2 of X ,

$$\begin{aligned} \theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2)) &= \delta \mathcal{G}(z_1, z_2, z_2) - \mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2) \\ &= \delta 2 |z_1 - z_2| - 2 \left| \frac{1}{2} - z_1 - \left(\frac{1}{2} - z_2 \right) \right| \\ &= 2(\delta |z_1 - z_2| - |z_1 - z_2|) \\ &< 2(|z_1 - z_2| - |z_1 - z_2|) \\ &= 2 \times 0 = 0. \end{aligned}$$

which yields that σ is not $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, but σ has a unique fixed point $z = \frac{1}{4}$. Now, $\mathcal{G}(z_1, \sigma z_2, \sigma z_2) = 2|z_1 - \sigma z_2| = 2|z_1 - (\frac{1}{2} - z_2)| = 2|z_1 + z_2 - \frac{1}{2}|$. Thus,

$$\begin{aligned} f(z_1, z_2, z_3) &= \min\{\mathcal{G}(z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_2, \sigma z_1, \sigma z_1), \mathcal{G}(z_2, \sigma z_3, \sigma z_3), \\ &\quad \mathcal{G}(z_3, \sigma z_2, \sigma z_2), \mathcal{G}(z_3, \sigma z_1, \sigma z_1), \mathcal{G}(z_1, \sigma z_3, \sigma z_3)\} \\ &= \min\{2|z_1 + z_2 - \frac{1}{2}|, 2|z_1 + z_2 - \frac{1}{2}|, 2|z_3 + z_2 - \frac{1}{2}|, \\ &\quad 2|z_3 + z_2 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|\}. \end{aligned}$$

Further,

$$\begin{aligned} &\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) \\ &= \delta[\mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)] - \mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2) \\ &= \delta[2|z_1 - z_2| + \beta \min\{2|z_1 + z_2 - \frac{1}{2}|, 2|z_1 + z_2 - \frac{1}{2}|, \\ &\quad 2|z_3 + z_2 - \frac{1}{2}|, 2|z_3 + z_2 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|, \\ &\quad 2|z_1 + z_3 - \frac{1}{2}|\}] - 2|z_1 - z_2|. \end{aligned}$$

Two cases arise:

Case 1. When $z_1 = z_2 = z_3$, then

$$\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 2\delta\beta|2z_1 - \frac{1}{2}| \geq 0.$$

Case 2. Let $z_1 > z_2 > z_3$, then $\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 2\delta|z_1 - z_2| + 2\delta\beta|z_2 + z_3 - \frac{1}{2}| - 2|z_1 - z_2|$. Now, choose $\beta = \frac{1}{3}$ and $\delta = 15$, we acquire $\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 30|z_1 - z_2| + 2 \times \frac{1}{3} \times 15|z_2 + z_3 - \frac{1}{2}| - 2 \times |z_1 - z_2| = 28|z_1 - z_2| + 10|z_2 + z_3 - \frac{1}{2}| \geq 0$. Therefore, all the assumptions of Theorem (2.3) are satisfied. Consequently, σ has a unique fixed point $\sigma = \frac{1}{4}$.

Corollary 2.6 ([10]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space where $\theta \in \mathbb{Z}$, $\beta < 1$ and the self map σ satisfies the following condition*

$$(2.12) \quad \theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \mathcal{G}(x, y, z)) \geq 0,$$

for each $x, y, z \in X$, Then, σ has a unique fixed pint.

Proof. The result follows if $\delta = 0$ and $\theta(a, b) = \beta b - a$ in Theorem 2.3. $\square$

Corollary 2.7 ([10]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space where $\theta \in \mathbb{Z}$, α is a self map which is upper semi continuous fulfilling $\alpha(s) < s$ and the self map σ satisfies the following condition*

$$(2.13) \quad \theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \alpha(\mathcal{G}(x, y, z))) \geq 0,$$

for each $x, y, z \in X$. Then, σ has a unique fixed pint.

3. Application

Let self map σ be defined as

$$(3.1) \quad \sigma\alpha(\gamma) = \kappa(\gamma) + \beta \int_m^n S(\gamma, x, x) \eta(x, \alpha(x), \alpha(x)) dx,$$

where $\gamma \in [m, n]$. Let X be equipped with the metric $\mathcal{G}$ which is defined as $\mathcal{G}(\alpha, \kappa, \kappa) = 2 \sup |\alpha(\gamma) - \kappa(\gamma)|$.

Theorem 3.1. *The integral equation (3.1) has a unique solution if the following conditions are fulfilled:*

- (i) $\sup \int_m^n S(\gamma, x, x) dx \leq \frac{1}{2n-2m}$;
- (ii) $\mathcal{G}(x, \alpha, \alpha) - \mathcal{G}(x, \kappa, \kappa) \leq \tau(|\alpha - \kappa|)$;
- (iii) $|\beta| \leq 1$,

where τ is non decreasing continuous map having $\tau(s) < s$, for each $s > 0$.

Proof. Now,

$$\begin{aligned} \mathcal{G}(\sigma\alpha_1, \sigma\alpha_2, \sigma\alpha_2) &= 2 \sup |\sigma\alpha_1(\gamma) - \sigma\alpha_2(\gamma)| \\ &= 2 \sup |\kappa(\gamma) + \beta \int_m^n S(\gamma, x, x) \eta(x, \alpha_1(x), \alpha_1(x)) dx - \kappa(\gamma) - \\ &\quad \beta \int_m^n S(\gamma, x, x) \eta(x, \alpha_2(x), \alpha_2(x)) dx| \\ &= 2|\beta| \sup \left| \int_m^n S(\gamma, x, x) (\eta(x, \alpha_1(x), \alpha_1(x)) - \eta(x, \alpha_2(x), \alpha_2(x))) dx \right| \\ &\leq 2|\beta| \sup \left[\int_m^n S(\gamma, x, x) dx \int_m^n (\eta(x, \alpha_1(x), \alpha_1(x)) - \eta(x, \alpha_2(x), \alpha_2(x))) |dx \right] \\ &\leq 2|\beta| \times \frac{1}{2n-2m} \left[\int_m^n \tau(|\alpha_1(x) - \alpha_2(x)|) dx \right] \\ &\leq |\beta| \times \frac{1}{n-m} \left[\int_m^n \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)) dx \right] \\ &= |\beta| \times \frac{1}{n-m} \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)) \times n - m \\ &= |\beta| \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)) \leq \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)). \end{aligned}$$

Therefore, σ has a unique solution in X , which means that (3.1) has a unique solution in X . $\square$

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A note on dimensions in N-groups

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Abstract. The concepts essential and finite dimension played an important role in the development of the dimension theory of modules over rings. Finite dimension, essential, strictly essential, and some related concepts were studied in nearrings and N-groups by Reddy-Satyanarayana [11], Satyanarayana [14, 20], Satyanarayana-Syam Prasad [22, 23, 24, 25]. In this paper, the authors introduced the concept finite 1-dimension and studied the relationship between finite dimension and finite 1-dimension. Some related examples are also provided.

Keywords: nearring, N-Group, essential ideal, uniform ideal, finite dimension.

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1. Introduction

A nearring is a set N together with two binary operations $+$ and $\cdot$ such that

- (1) $(N, +)$ is a group (not necessarily abelian)
- (2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- (3) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in N$.

In view of (3), N satisfies the right distributive law, and so it is called a right nearring. It is evident that $0 \cdot n = 0$ for all $n \in N$. However, $n \cdot 0$ need not be equal to 0, in general. We denote $N_0 = \{n \in N : n \cdot 0 = 0\}$, the zero-symmetric part of the right nearring. If $N = N_0$ then we say that the nearring N is zero-symmetric.

Let $(G, +)$ be a group. By an N -group, we mean a mapping $N \times G \rightarrow G$ (the image of $(n, g) \in N \times G$ is denoted by ng), satisfying the following two conditions

- (1) $(n + n^1)g = ng + n^1g$, and
- (2) $(nn^1)g = n(n^1g)$ for all $g \in G$ and $n, n^1 \in N$.

We will denote this N -group by ${}_N G$ or simply by G . For preliminary definitions and results we refer to Pilz [8], and Satyanarayana-Syam Prasad [29]. An ideal I of G is said to be essential (strictly essential, resp.) in an ideal J of G if it satisfies the condition: K is an ideal (N -subgroup of G , resp.) of G , $I \cap K = (0)$, $K \subseteq J$ imply $K = (0)$.

2. Finite dimension in N -Groups

In this section we introduce the concept “ N -group with finite dimension” and provide few examples for such type of N -group.

Definition 2.1. Let H be an ideal of G . Then H is said to have finite dimension if for any increasing chain $A_1 \subseteq A_2 \subseteq \dots$ of ideals of G contained in H , there corresponds a positive integer k such that A_i is essential in A_{i+1} for all $i \geq k$.

Definition 2.2 ([14]). Let H be an ideal of G . Then H is said to have Finite Goldie dimension (FGD, in short) if H do not contain an infinite number of non-zero ideals of G whose sum is direct.

Theorem 2.1 ([14]). Let H be an ideal of G . Then the following are equivalent:

- (i) H has finite Goldie dimension.
- (ii) For any increasing sequence $H_0 \subseteq H_1 \subseteq \dots$ of ideals of G with each $H_i \subseteq H$, there exists an integer k such that H_i is essential in H_{i+1} for $i \geq k$.

Note 2.1. From Theorem 2.1 we conclude that the following two conditions are equivalent for an ideal H .

- (i) H have finite dimension.
- (ii) H has finite Goldie dimension.

Proposition 2.1. *Every finite N -group G must have finite dimension.*

Proof. Let G be a finite N -group. Then the number of elements in G is finite. Let $k = |G|$. In a contrary way, suppose that G does not have finite dimension. Then there exists an ascending infinite sequence $A_1 \subseteq A_2 \subseteq \dots$ of ideals of G such that for any positive integer n , there exist an integer i such that $i \geq n$ and A_i is not essential in A_{i+1} . Now we use mathematical induction to create an infinite number of elements in G . For $n = 1$, there exists an integer i_1 such that A_{i_1} is not essential in A_{i_1+1} . Now it is clear that $A_{i_1} \subsetneq A_{i_1+1}$ and so there exists $a_1 \in A_{i_1+1} \setminus A_{i_1}$. For $n = i_1 + 1$ there exist an integer i_2 such that $i_2 \geq i_1 + 1$ and A_{i_2} is not essential in A_{i_2+1} . It is clear that $A_{i_2} \subsetneq A_{i_2+1}$ and so there exist $a_2 \in A_{i_2+1} \setminus A_{i_2}$. Note that $i_2 \geq i_1 + 1 \geq i_1$. So $A_{i_1} \subseteq A_{i_1+1} \subseteq A_{i_2} \subseteq A_{i_2+1}$. Since $a_2 \in A_{i_2+1} \setminus A_{i_2}$ and $a_1 \in A_{i_1+1} \subseteq A_{i_2}$, it follows that $a_1 \neq a_2$. After selecting $i_1 \leq i_2 \leq \dots \leq i_s$, we select i_{s+1} as follows: For $n = i_s$, there exists an integer $i_{(s+1)}$ such that $i_{(s+1)} \geq n = i_s$ and $A_{i_{(s+1)}}$ is not essential in $A_{i_{(s+1)}+1}$. It is clear that $A_{i_{(s+1)}} \subsetneq A_{i_{(s+1)}+1}$ and so there exist $a_{s+1} \in A_{i_{(s+1)}+1} \setminus A_{i_{(s+1)}}$ with $a_{s+1} \notin \{a_1, a_2, \dots, a_s\}$. By mathematical induction we get an infinite number of elements $a_1, a_2, \dots, a_s, \dots$ in G , a contradiction to the fact that G is a finite set. $\square$

Proposition 2.2. *Every N -group with ascending chain condition on its ideals must have finite dimension.*

Proof. Let G be an N -group with ACC on its ideals. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing chain of ideals of G . Since G have ACC on its ideals, this chain $A_1 \subseteq A_2 \subseteq \dots$ must be stationary after some stage. That is, there exists a positive integer s such that $A_s = A_{s+1} = A_{s+2} = \dots$. Now it is clear that A_i is essential in A_{i+1} for $i \geq s$. Hence G has finite dimension. $\square$

Definition 2.3 ([11]). An N -group G is said to be completely reducible if G is equal to the sum of all of its simple ideals. If G contains infinite number of simple ideals whose sum is direct, then G has no finite dimension. In this case dimension of G is infinite.

Proposition 2.3. *Every N -group G which is completely reducible and which can be written a finite sum of simple ideals of G , must have finite dimension.*

Proof. Let G be an N -group which is completely reducible and which can be written as finite sum of simple ideals of G . Suppose $G = H_1 \oplus H_2 \oplus \dots \oplus H_s$ where s is a positive integer and $H_i, 1 \leq i \leq s$ are ideals of G .

Write $B_i = H_1 \oplus H_2 \oplus \dots \oplus H_s$ for $1 \leq i \leq s$. Since the sum is direct and by Theorem 2.3.5 of Satyanarayana-Syam Prasad [29] we get $B_{i+1}/B_i =$

$\frac{(H_1 + \dots + H_{i+1})}{(H_1 + \dots + H_i)} \cong \frac{H_{i+1}}{(H_1 + \dots + H_i) \cap H_{i+1}} \cong \frac{H_{i+1}}{(0)} \cong H_{i+1}$. Hence by Proposition 2.38 of Pilz [8], we get that $(0) \subseteq B_1 \subseteq B_2 \subseteq \dots \subseteq B_s = G$ is a composition series of G . Since this composition series is finite, by Theorem 2.41 of Pilz [8], G has ACC and DCC on its ideals. Now by Proposition 2.2, we have that G have finite dimension. $\square$

Note 2.2. Every module M over a ring R with finite Goldie dimension is an N -group G with finite dimension, by considering R as a nearring N and M as G .

Proposition 2.4. *If G has finite dimension, then every ideal H of G must have finite dimension.*

Proof. Suppose G has finite dimension. let H be an ideal of G . To verify that H have finite dimension, let $A_1 \subseteq A_2 \subseteq \dots$ be an infinite sequence of ideals of G contained in H . Since $A_1 \subseteq A_2 \subseteq \dots$ is an ascending chain of ideals in G and G have finite dimension, we can get a positive integer s such that A_i is essential in A_{i+1} for all $i \geq s$. This sows that H have finite dimension. $\square$

3. Finite 1-dimension in N -groups

In this section, we study the concept strictly essential ideal of an N -group G and study finite 1-dimension in N -groups. We also observe the relationship between finite dimension studied in section-1 and finite 1-dimension.

Definition 3.1 ([17]). An ideal H of G is said to be strictly essential in G if $H \cap A = (0)$, A is an N -subgroup of G imply $A = (0)$.

(ii) Let H_1 and H_2 be two ideals of G such that $H_1 \subseteq H_2$. Then H_1 is said to be strictly essential (with respect to H_2) in H_2 if $H_1 \cap A = (0)$, $A \subseteq H_2$, A is an N -subgroup of H_2 imply $A = (0)$.

Lemma 3.1. *Suppose that N is a zero-symmetric nearring.*

(i) *Every strictly essential ideal of G is an essential ideal of G .*

(ii) *Let H_1 and H_2 be two ideals of G such that $H_1 \subseteq H_2$. Then “ H_1 is strictly essential in H_2 ” (with respect to H_2) imply that “ H_1 is essential in H_2 ” (with respect to H_2) when we consider H_2 as N -group. Hence H_1 is also essential in H_2 .*

Proof. (i) Let H be a strictly essential ideal of G . To verify that H is an essential ideal, take an ideal A of G such that $H \cap A = (0)$. Since N is a zero symmetric, by Proposition 1.34 (b) of Pilz [8], we have A is an N -subgroup of G . Since $H \cap A = (0)$, A is an N -subgroup, and H is strictly essential, it follows that $A = (0)$. Therefore, H is an essential ideal of G .

(ii) Suppose that H_1 is strictly essential in H_2 . Now we wish to verify that H_1 is essential in H_2 . Suppose $H_1 \cap A = (0)$, A is an ideal of H_2 (consider H_2 as N -group). Note that N is a zero symmetric nearring, A is an N -subgroup

of H_2 . Now, since $H_1 \cap A = (0)$, A is an N -subgroup of H_2 and H_1 is strictly essential in H_2 , it follows that $A = (0)$. This shows that “ H_1 is essential in H_2 ”. $\square$

The following example shows that every essential ideal of G need not be strictly essential.

Example 3.1. Let G be the symmetric group (written additively) on three elements. Then G can be considered as a N -group, where $N = Z$, the nearring of integers.

(i) Let P be the alternating subgroup of G . Then P is only the proper ideal of G . So P is essential in G .

(ii) Note that $P = \{f, f^2, e\}$, where $f = (123) = (12)(13)$; $f^2 = (132) = (13)(12)$; $f^3 = e = (1)$. Write $Q = \{g, e\}$ where $g = (12)$. Then $g^2 = e$. So Q is an N -subgroup of G .

(iii) Since $P \cap Q = (0)$, Q is an N -subgroup of G such that $Q \neq (0)$, it follows that P is not strictly essential in G .

Note 3.1. From the Example 3.1, we conclude that, in general, every essential ideal need not be a strictly essential ideal.

Definition 3.2. An ideal H of G is said to have finite 1-dimension, if for any increasing chain $A_1 \subseteq A_2 \subseteq \dots$ of ideals of G contained in H there exists a positive integer i such that A_i is strictly essential in A_{i+1} .

Theorem 3.1. Let $N = N_0$ and H be an ideal of G . If H have finite 1-dimension, then H must have finite dimension.

Proof. Suppose H is having finite 1-dimension. To verify that H have finite dimension, take an increasing sequence $A_1 \subseteq A_2 \subseteq \dots$ of ideals of G contained in H . Since H have finite 1-dimension, we can conclude that there exists a positive integer k such that A_i is strictly essential in A_{i+1} for all $i \geq k$. By Lemma 3.1 (ii), we conclude that A_i is essential in A_{i+1} for all $i \geq k$. Thus we have proved that given an increasing sequence $A_1 \subseteq A_2 \subseteq \dots$ of ideals of G contained in H , there corresponds a positive integer k such that A_i is essential in A_{i+1} for all $i \geq k$. Hence H has finite dimension. $\square$

Example 3.2. As in Example 3.1, let G be the symmetric group (written additively) on three symbols 1, 2 and 3. Then G can be considered as an N -group where $N = Z$, the nearring of integers. Let P be the alternating subgroup and Q be the subgroup $\{g, e\}$, as given in Example 3.1.

(i) Write $G_i = G, P_i = P, Q_i = Q$ for all positive integers i . Consider the direct sum G of the N -groups $\{G_i\}_{i \in \mathbb{Z}^+}$.

Then $G = G_1 \oplus G_2 \oplus \dots$

(ii) $\sum_{i=1}^{\infty} P_i \subseteq G_1 \oplus \sum_{i=2}^{\infty} P_i \subseteq G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i \subseteq \dots$ is an increasing sequence of ideals of G .

(iii) Since P_1 is essential in G_1 , we have that $\sum_{i=1}^{\infty} P_i = P_1 \oplus \sum_{i=2}^{\infty} P_i$ is essential in $G_1 \oplus \sum_{i=1}^{\infty} P_i$. Since P_2 is essential in G_2 , we have that $\sum_{i=2}^{\infty} P_i = P_2 \oplus \sum_{i=3}^{\infty} P_i$ is essential in $G_2 \oplus \sum_{i=3}^{\infty} P_i$, and so $G_1 \oplus \sum_{i=2}^{\infty} P_i$ is essential in $G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i$.

In this way we can verify that for all $n \geq 1$, $G_1 \oplus G_2 \oplus \dots \oplus G_n \oplus \sum_{i=n+1}^{\infty} P_i$ is essential in the next term $G_1 \oplus G_2 \oplus \dots \oplus G_n \oplus G_{n+1} \oplus \sum_{i=n+2}^{\infty} P_i$. So the sequence considered satisfies the property given in the definition “finite dimension”.

(iv) Since $P_1 \cap Q_1 = (0)$, Q_1 is an N -subgroup of G_1 and $Q_1 \neq (0)$, we have that P_1 is not strictly essential in G_1 .

Since $(P_1 \oplus \sum_{i=2}^{\infty} P_i) \cap Q_1 = (0)$, $Q_1 \neq (0)$, Q_1 is an N -subgroup of $G_1 \oplus \sum_{i=2}^{\infty} P_i$, it follows that $P_1 \oplus \sum_{i=2}^{\infty} P_i = \sum_{i=1}^{\infty} P_i$, is not essential in $G_1 \oplus \sum_{i=2}^{\infty} P_i$.

(v) Since $P_2 \cap Q_2 = (0)$, Q_2 is an N -subgroup of G_2 and $Q_2 \neq (0)$, we have that P_2 is not strictly essential in G_2 .

Since $(P_2 \oplus \sum_{i=3}^{\infty} P_i) \cap Q_2 = (0)$, $Q_2 \neq (0)$, Q_2 is an N -subgroup of $G_2 \oplus \sum_{i=3}^{\infty} P_i$, it follows that $P_2 \oplus \sum_{i=3}^{\infty} P_i$, is not essential in $G_2 \oplus \sum_{i=3}^{\infty} P_i$, and hence $G_1 \oplus \sum_{i=2}^{\infty} P_i = G_1 \oplus P_2 \oplus \sum_{i=3}^{\infty} P_i$ is not essential in $G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i$.

(vi) If we continue as in (iv) and (v), we can understand that for all $n \geq 1$, $G_1 \oplus \dots \oplus G_n \oplus \sum_{i=n+1}^{\infty} P_i$ is not strictly essential in the next term $G_1 \oplus \dots \oplus G_n \oplus G_{n+1} \oplus \sum_{i=n+2}^{\infty} P_i$.

So this sequence considered does not satisfy the property given in the definition “finite 1-dimension”.

(vii) From the discussion in (vi), we can conclude that G does not have finite 1-dimension.

(viii) Now we verify that G does not have finite dimension.

Consider the increasing sequence $P_1 \subseteq P_1 \oplus P_2 \subseteq P_1 \oplus P_2 \oplus P_3 \subseteq \dots$ of ideals of G . It is clear that $P_1 \oplus \dots \oplus P_s$ is not essential in $P_1 \oplus \dots \oplus P_s \oplus P_{s+1}$, since $(P_1 \oplus \dots \oplus P_s) \cap P_{s+1} = (0)$ and $P_{s+1} \neq (0)$. Thus G does not have finite dimension.

Example 3.3. If G is an N -group satisfying *DCC* on its ideals, then G must have finite 1-dimension. Let G be an N -group satisfying *DCC* on its ideals. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing sequence of ideals of G . Since G has *DCC* on its ideals, there exists a positive integer s such that $A_s = A_{s+1} = \dots$. Now A_{s+t} is strictly essential in A_{s+t+1} for all $t \geq 1$. Now we verified that for any increasing sequence of ideals $A_1 \subseteq A_2 \subseteq \dots$, there exists a positive integer s such that A_i is strictly essential in A_{i+1} for all $i \geq s$. So we can conclude that G have the finite 1-dimension.

Example 3.4. A routine verification shows that every finite N -group G must have the finite 1-dimension.

Example 3.5. Every module with finite Goldie dimension must have finite 1-dimension.

For this, take a module M over a ring R . The ideals and N -subgroups of $G = M$ (as an N -group over $N = R$) coincides (as M is a module). So

the concepts “finite dimension” and “finite 1-dimension” coincides for modules. By Note 2.2, we know that M have finite dimension. Hence M has finite 1-dimension.

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Different characterizations of a game-theoretical solution and its application on sports management

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Abstract. In this paper, we firstly propose some axiomatic results of a game-theoretical solution, and further investigate the relations among these game-theoretical results and the field of sports management. The main investigation methods are as follows. Different from the pre-existing results, we provide different axioms and reduction to characterize of a game-theoretical solution. By applying some rules of management sciences in the real-world, we assign several reinterpretations to these axioms and related game-theoretical results. Finally, these axioms and related game-theoretical results are applied to the framework of sports management.

Keywords: game-theoretical solution, axiom, sports management.

1. Introduction

Due to the constant renovation of the trend of real-world management systems, management methods of a combination of different theoretical results and related applications, including sports management, have become the main notion

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in the field of management science. In the framework of sports management, the use of several management methods could promote the management efficiency no matter for training of sports skills or management of sports industry. Related results could be found in as Chen and Su (2011), Mullin (1980), and so on. On the other hand, the axiomatic results of game-theoretical solutions could be always adopted to analyze various interaction relationship and related models among agents and coalitions by applying mathematical results. In addition to theoretical analysis, game-theoretical results also could be applied to provide optimal results or equilibrium conditions for some real-word models.

In a traditional coalitional transferable-utility (TU) game, each player is either fully involved or not involved at all in participation with some other agents, while in a multi-choice TU game, each player is allowed to participate with finite many different activity levels. The solutions on multi-choice TU games could be applied in many fields such as economics, political sciences and even sports management. The *core* is, perhaps, the most intuitive solution concept in game theory. There are several extended core for multi-choice TU games in literature. Here we apply the *unit-level-core* proposed by Hwang and Liao (2010).

Consistency, originally introduced by Harsanyi (1959) under the name of bilateral equilibrium, is a crucial characteristic of viable solutions in the axiomatic formulation of cooperative games. The idea behind this kind of consistency is as follows: for a given game, agents may elaborate expectations of the game and may be willing to allow the computation of their payments to be based upon these expectations. The solution concept is consistent if it gives the same payments to players in the original game as it does to players of the imaginary reduced game. Thus, consistency is a requirement of the internal “*robustness*” of compromises. The fundamental property of solutions has been investigated in various classes of problems by applying *reduced games* always. Various definitions of a reduced game have been proposed, depending upon exactly how the agents outside of the subgroup should be paid off. Hwang and Liao (2010) extended the reduction introduced by Davis and Maschler (1965) to multi-choice games and provided related consistency property to characterize the unit-level-core. Later, Liao (2009) extended the reduction introduced by Moulin (1985) and Voorneveld and van den Nouweland (1998) to multi-choice TU games respectively and provided related consistency properties to characterize the unit-level-core.

In this paper, we firstly propose new game-theoretical results of the unit-level-core, and further investigate several relations and applications among these game-theoretical results and sports management. Here we build on the results of Hwang and Liao (2010) and Liao (2009). By considering both game-theoretical approach and sports management, the main results of this paper are as follows.

1. Different from the results of Hwang and Liao (2010) and Liao (2009), several characterizations are proposed in this paper.

- Inspired by Peleg (1986), we adopt the reduction proposed by Liao (2009) to show that the unit-level-core is the only solution satisfying *non-emptiness*, *individually rationality*, *consistency* and *superadditivity* on the domain of all balanced multi-choice TU games.
 - By restricting the application of the non-emptiness property to balanced games, we adopt a weakening of non-emptiness to extend the results of Peleg (1986) to the domain of all multi-choice TU games.
 - Inspired by Serrano and Volij (1998), we propose an alternative reduction to characterize the unit-level-core on the domain of all multi-choice TU games.
2. In section 4, we adopt the framework of multi-choice TU games and the characterizations of the unit-level-core to build the model of sports management systems, and thus to apply related results to the field of sports management. Further, some more relations among the unit-level-core and sports management would be proposed in detail.

2. Notions and definitions

Let U be the universe of players. For $i \in U$ and $m_i \in \mathbb{N}$, we set $M_i = \{0, 1, \dots, m_i\}$ as the action space of player i , where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U \setminus \{\emptyset\}$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N and $L^N = \{(i, j) \mid i \in N, j \in M_i^+\}$. Denote the zero vector in $\mathbb{R}^N$ by 0_N .

A **multi-choice TU game** is a pair (N, v) , where N is a non-empty and finite set of players, $m = (m_i)_{i \in N}$ is a vector that describes the number of activity levels for each player, and $v : M^N \rightarrow \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$.

Denote the class of all multi-choice TU games by Γ . Given $(N, v) \in \Gamma$, $\alpha \in M^N$ and $T \subseteq N$, we define $S(\alpha) = \{i \in N \mid \alpha_i \neq 0\}$ and denote $\alpha_T \in \mathbb{R}^T$ to be the restriction of α to T . Let $i \in N$. For convenience we introduce the substitution notation α_{-i} to stand for $\alpha_{N \setminus \{i\}}$ and let $\beta = (\alpha_{-i}, j) \in \mathbb{R}^N$ be defined by $\beta_{-i} = \alpha_{-i}$ and $\beta_i = j$.

Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if no confusion can arise $e^S(N)$ will be denoted by e^S .

Given $(N, v) \in \Gamma$. A **payoff vector** of (N, v) is a function $x : N \rightarrow \mathbb{R}$ where the number x_i represents the per unit payoff that player i receives for all

$i \in N$, hence $m_i \cdot x_i$ is the total payoff that player i receives at (N, v) . Given $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$, $\alpha \in M^N$ and $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$, and $x(\alpha) = \sum_{i \in N} \alpha_i \cdot x_i$. Then

- a payoff vector x of $(N, v) \in \Gamma$ is **efficient (EFF)** if $x(m) = v(m)$.
- a payoff vector x of $(N, v) \in \Gamma$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in M_i$, $x(je^{\{i\}}) \geq v(je^{\{i\}})$.

Moreover, x is an **imputation** of (N, v) if it is EFF and IR. And the set of **feasible payoff vectors** of (N, v) is denoted by

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(m) \leq v(m)\},$$

whereas

$$X(N, v) = \{x \in \mathbb{R}^N \mid x \text{ is EFF}\}$$

is the set of **preimputations** of (N, v) and the set of imputations of (N, v) is denoted by $I(N, v)$.

A **solution on** Γ is a function σ which associates with each $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of $X^*(N, v)$.

Definition 1. *The **unit-level-core (Hwang and Liao, 2010)** $C(N, v)$ of $(N, v) \in \Gamma$ consists of all $x \in X(N, v)$ that satisfy for all $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$, i.e.,*

$$C(N, v) = \{x \in X(N, v) \mid x(\alpha) \geq v(\alpha) \text{ for all } \alpha \in M^N\}.$$

3. Game-theoretical results: alternative characterizations of the unit-level-core

In this section, we adopt the reduced game proposed by Liao (2009) to characterize the unit-level-core. Further, an alternative reduction and related characterizations are also proposed.

Liao (2009) extended the reduced game introduced by Voorneveld and van den Nouweland (1998) to multi-choice TU games as follows.

Definition 2 ([5]). *Let $(N, v) \in \Gamma$ and $S \in 2^N \setminus \{\emptyset\}$. The **reduced game with respect to S** and x is the game $(S, v_{S,x})$ where*

$$v_{S,x}(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ v(m) - \sum_{i \in N \setminus S} m_i \cdot x_i, & \alpha = m_S, \\ \max_{\beta \in M^{N \setminus S} \setminus \{0_{N \setminus S}\}} \{ v(\alpha, \beta) - \sum_{i \in N \setminus S} \beta_i \cdot x_i \}, & \text{otherwise.} \end{cases}$$

In the reduced game $(S, v_{S,x})$, the players in S are regathered to play. The players in $N \setminus S$ are required to cooperate in at least one participation level when some players in S take nonzero actions. Of course, if each player in S does

not participate in playing, the account S can achieve is zero. If all the players in S play at full steam, the players in $N \setminus S$ also cooperate at full steam. Assume that some players in S do not play at full steam, all the coalitions comprised by the participation levels of $N \setminus S$ should be considered to cooperate so that the maximal benefit can be reached. Since the players in $N \setminus S$ accept the payoff according to x , they receive the deserved payoff and leave then.

Consistency requires that if x is prescribed by σ for a game (N, v) , then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S . Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game. *Converse consistency* requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Consistency (CON)**: For all $(N, v) \in \Gamma$, for all $S \in 2^N \setminus \{\emptyset\}$, and for all $x \in \sigma(N, v)$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$.
- **Converse consistency (CCON)**: For all $(N, v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in X(N, v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$.
- **Weak converse consistency (WCCON)**¹: For all $(N, v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in I(N, v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$.

Liao (2009) characterized the unit-level-core by means of CON and WCCON. Subsequently, we provide alternative characterizations of the unit-level-core. We will make use of the following axioms. We say that the multi-choice game (N, v) is *balanced*² if $C(N, v) \neq \emptyset$. Let Γ_c denote the set of all balanced multi-choice games. Let $\Gamma' \subseteq \Gamma$ and σ be a solution on Γ' . σ satisfies **non-emptiness (NE)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \neq \emptyset$. σ satisfies **non-emptiness for balanced games (NEB)** if for all $(N, v) \in \Gamma_c$, $\sigma(N, v) \neq \emptyset$. σ satisfies **individually rationality (IR)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq I(N, v)$. σ satisfies **efficiency (EFF)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq X(N, v)$. σ satisfies **superadditivity (SUPA)**³ if for all $(N, v) \in \Gamma'$, $\sigma(N, v + w) \supseteq \sigma(N, v) + \sigma(N, w)$, where for all $\alpha \in M^N$, $(v + w)(\alpha) = v(\alpha) + w(\alpha)$.

Lemma 1 ([5]). 1. On both Γ_c and Γ , the unit-level-core satisfies CON and WCCON.

2. Let $\Gamma' \subset \Gamma$ and let σ be a solution on Γ' . If σ satisfies IR and CON, then it also satisfies EFF.

1. The following axiom is a weakening of the previous axiom, since it requires that x be individually rational as well.

2. The balancedness of the unit-level-core was proposed by Hwang and Liao (2010).

3. If $N \subseteq U$ and $A, B \subset \mathbb{R}^N$, then $A + B = \{ a + b \mid a \in A \text{ and } b \in B \}$.

Lemma 2. *Let $\Gamma' \subset \Gamma$ and let σ be a solution on Γ' . If σ satisfies IR and CON, then for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq C(N, v)$.*

Proof. Let $(N, v) \in \Gamma'$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$, then by IR of σ and C , $\sigma(N, v) \subseteq C(N, v)$. Assume that $\sigma(N, v) \subseteq C(N, v)$ if for $|N| \leq k - 1$, where $k \geq 2$.

The case $|N| = k$:

Since σ satisfies IR and CON, by Lemma 1, σ satisfies EFF. Hence, $\sigma(N, v) \subseteq I(N, v)$. Let $x \in \sigma(N, v)$. Since σ satisfies CON, for all $S \subseteq N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, v_{S,x})$. Hence, by the induction hypotheses, $x_S \in \sigma(S, v_{S,x}) \subseteq C(S, v_{S,x})$. Since C satisfies WCCON, $x \in C(N, v)$. $\square$

Theorem 1.

1. *On Γ_c , the unit-level-core is the only solution satisfying NE, IR, SUPA and CON.*
2. *Let $\Gamma' \subset \Gamma$. On Γ' , the unit-level-core is the only solution satisfying NEB, IR, SUPA and CON.*

Proof. By Lemma 1, the unit-level-core satisfies CON. Clearly, the unit-level-core satisfies NE, NEB, IR and SUPA.

To prove uniqueness of (1), assume that a solution σ satisfies NE, IR, SUPA and CON on Γ_c . Let $(N, v) \in \Gamma_c$. Two cases may be distinguished:

Case 1: Assume that $|N| \geq 3$. Let $x \in C(N, v)$. Define $(N, w) \in \Gamma$ by the following rule:

$$w(t) = \begin{cases} v(t), & t = je^{\{i\}} \text{ for all } (i, j) \in L^N, \\ x(t), & \text{otherwise.} \end{cases}$$

As the reader can easily verify that $C(N, w) = \{x\}$. Thus, by NE of σ and Lemma 2, $\sigma(N, w) = \{x\}$. Now let $u = v - w$. Clearly, for all $(i, j) \in L^N$, $u(je^{\{i\}}) = 0$. And for all $t \in M^N$, $u(t) = 0$ and $u(t) \leq 0$. Hence, $C(N, u) = \{0_N\}$. By NE of σ and Lemma 2, $\sigma(N, u) = \{0_N\}$. Since $v = u + w$ and σ satisfies SUPA,

$$\sigma(N, v) \supseteq \sigma(N, u) + \sigma(N, w) = \{x\}.$$

Thus, $\sigma(N, v) \supseteq C(N, v)$. By Lemma 2, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.

Case 2: Assume that $|N| \leq 2$. If $|N| = 1$, then by NE and IR of σ and C , $\sigma(N, v) = C(N, v)$. Thus, let $|N| = 2$. Denote that $N = \{i, k\}$ and let $p \in U \setminus N$. Define $(H, u) \in \Gamma$ with $H = \{i, k, p\}$ in the following rule. For all $t \in M^H$, $u(t) = v(t_N)$. Let $x \in C(N, v)$. By definitions of u and $u_{N,(x,0)}$, it is easy to verify that $(x, 0) \in C(H, u)$ and $(N, u_{N,(x,0)}) = (N, v)$. Since $|H| = 3$, by Case 1, $\sigma(H, u) = C(H, u)$. Thus, $(x, 0) \in \sigma(H, u)$. By CON of σ , $x = (x, 0)_N \in \sigma(N, u_{N,(x,0)}) = \sigma(N, v)$. Hence, $C(N, v) \subseteq \sigma(N, v)$. By Lemma 2, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.

This proof of uniqueness of (2) is a copy of (1) except “NEB and Γ' ” instead of “NE and Γ_c ”; hence, we omit it. $\square$

Different from the reductions proposed by Hwang and Liao (2010) and Liao (2009), we propose an extension of the reduction proposed by Serrano and Volij (1998) as follows.

Definition 3. Let $(N, v) \in \Gamma$ and $S \in 2^N \setminus \{\emptyset\}$. The **revised reduced game with respect to S and x** is the game $(S, v_{S,x}^R)$ where

$$v_{S,x}^R(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ \max_{\beta \in M^{N \setminus S} \setminus \{0_{N \setminus S}\}} \{ v(\alpha, \beta) - \sum_{i \in N \setminus S} \beta_i \cdot x_i \}, & \text{otherwise.} \end{cases}$$

The only difference between this and the reduced game in Definition 2 is the fact that the coalition S is also allowed to imagine potential interaction with any of the subset of $N \setminus S$. Informally, in order to reach the maximal benefit, all the coalitions comprised by the members of $N \setminus S$ should to be considered to cooperate with S . “Revised reduction” instead of “reduction”, we also introduce **R-consistency (RCON)** and **converse R-consistency (CRCON)**.

Lemma 3. On both Γ_c and Γ , the unit-level-core satisfies CRCON.

Proof. We first show that on Γ , the unit-level-core satisfies CRCON. And the proof is also applied to the domain Γ_c . Let $(N, v) \in \Gamma$ with $|N| \geq 2$ and let $x \in X(N, v)$. Suppose for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}^R) \in \Gamma$ and $x_S \in C(S, v_{S,x}^R)$. We will show that $x \in C(N, v)$, i.e., for all $\alpha \in M^N$, $\sum_{i \in N} \alpha_i x_i \geq v(\alpha)$. Let $i \in N$ and $\alpha \in M^N$. Consider the reduced game $(\{i\}, v_{\{i\},x}^R)$. Then

$$\left\{ \begin{array}{l} \alpha_i x_i \geq v_{\{i\},x}^R(\alpha_i) \quad \left(\text{Since } x_i \in C(\{i\}, v_{\{i\},x}^R) \right) \\ = \max_{\beta \in M^{N \setminus \{i\}} \setminus \{0_{N \setminus \{i\}}\}} \{ v(\alpha_i, \beta) - \sum_{j \in N \setminus \{i\}} \beta_j x_j \} \\ \geq v(\alpha_i, \alpha_{N \setminus \{i\}}) - \sum_{j \in N \setminus \{i\}} \alpha_j x_j \\ = v(\alpha) - \sum_{j \in N \setminus \{i\}} \alpha_j x_j. \end{array} \right.$$

Hence, $\sum_{k \in N} \alpha_k x_k \geq v(\alpha)$. $\square$

Lemma 4.

1. On both Γ_c and Γ , the unit-level-core satisfies RCON.
2. Let $\Gamma' \subset \Gamma$ and let σ be a solution on Γ' . If σ satisfies IR and RCON, then it also satisfies EFF.

3. Let $\Gamma' \subset \Gamma$ and let σ be a solution on Γ' . If σ satisfies IR and RCON, then for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq C(N, v)$.

Proof. The proof is similar to Lemmas 1, 2, hence we omit it. □

Next, we provide alternative characterizations by means of RCON.

Theorem 2.

1. On Γ_c , the unit-level-core is the only solution satisfying NE, IR, SUPA and RCON.
2. Let $\Gamma' \subset \Gamma$. On Γ' , the unit-level-core is the only solution satisfying NEB, IR, SUPA and RCON.

Proof. The proof is similar to Theorem 1, hence we omit it. □

The following examples show that each of the axioms used in Theorems 1, 2 is logically independent of the others.

Example 1. Let $\sigma(N, v) = \emptyset$ for all $(N, v) \in \Gamma_c$. Then σ satisfies IR, SUPA and CON (RCON), but it violates NE (NEB).

Example 2. Let $\sigma(N, v) = X(N, v)$ for all $(N, v) \in \Gamma_c$. Then σ satisfies NE (NEB), SUPA and CON (RCON), but it violates IR.

Example 3. Let $\sigma(N, v) = I(N, v)$ for all $(N, v) \in \Gamma_c$. Then σ satisfies NE (NEB), IR and SUPA, but it violates CON (RCON).

Example 4. For all $(N, v) \in \Gamma_c$, we define a solution σ on Γ_c to be

$$\sigma(N, v) = \begin{cases} C(N, v), & \text{if for all } x \in C(N, v) \text{ with } x_i \neq 0 \\ \{x \in C(N, v) \mid x_i = 0\}, & \text{otherwise.} \end{cases}$$

Then σ satisfies NE (NEB), IR and CON (RCON), but it violates SUPA.

4. Application on sports management

In this section, we provide an application of multi-choice TU games and the unit-level-core in the setting of “allocation of the utilities for sports management systems”, such as the National Basketball Association (NBA), the Major League Baseball (MLB) and so on.

In a sports organization, such as the NBA, each department of the sports organization may take several operation strategies to manage. Besides competing in sports games, all departments should develop to raise entire utilities of whole the sports organization also, such as security, box office, affiliated products and so on. This kind of problem could be formulated as follows: Let $N = \{1, 2, \dots, n\}$ be a set of all departments of the sports organization that

could be provided jointly by some coalitions and let $v(\alpha)$ be the profit of providing the strategical vector $\alpha = (\alpha_i)_{i \in N}$ in N jointly. For all department i in this sports organization, the activity level α_i could be treated as one of the operation strategies of the department i . The function v could be treated as an utility function which assigns to each operation strategical vector α the worth that the departments can obtain when each department i takes at operation strategy $\alpha_i \in M_i$. Modeled in this way, the sports management system of a sports organization could be considered as a cooperative multi-choice TU game, with v being its characteristic function. The unit-level-core could provide “optimal utilities” among all departments, in the sense that the sports organizations could get utilities from each combination of strategies of all departments under sports management systems. An illustrative example as follows.

Example 5. Let $(N, v) \in \Gamma$, where $N = \{a, b, c\}$ is a sports organization with departments a, b, c , $m = (2, 2, 1)$ is a vector that describes the number of operational strategies for departments a, b, c respectively, and $v : M^N \rightarrow \mathbb{R}$ is the utility function which assigns to each strategy vector $\alpha = (\alpha_a, \alpha_b, \alpha_c)$ the worth that the departments can obtain when each department i operates at strategy α_i with $v(2, 2, 1) = 9$, $v(2, 1, 1) = 4$, $v(1, 2, 1) = 2$, $v(1, 1, 1) = 3$, $v(0, 2, 1) = 4$, $v(0, 1, 1) = 2$, $v(2, 0, 1) = 1$, $v(1, 0, 1) = -2$, $v(2, 2, 0) = 5$, $v(2, 1, 0) = 3$, $v(1, 2, 0) = 2$, $v(1, 1, 0) = 0$, $v(0, 0, 1) = -2$, $v(0, 2, 0) = -1$, $v(0, 1, 0) = -7$, $v(2, 0, 0) = 1$, $v(1, 0, 0) = -2$, $v(0, 0, 0) = 0$. By definition of the unit-level-core, it is easy to determine that

$$C(N, v) = \{(x_a, x_b, x_c) \mid \frac{1}{2} \leq x_a \leq \frac{3}{2}, \frac{5-2x_a}{2} \leq x_b \leq 4, x_c = -2x_a - 2x_b + 9\} \\ \cup \{(x_a, x_b, x_c) \mid \frac{3}{2} < x_a \leq \frac{5}{2}, \frac{5-2x_a}{2} \leq x_b \leq 7 - 2x_a, x_c = -2x_a - 2x_b + 9\}.$$

Pick $x = (1, 4, -1)$, $y = (2, 1, 3)$ and $z = (2, 2, 1)$. Clearly, $x, y, z \in C(N, v)$ respectively.

Due to chances of the rise of sports competence and its additional values such as commercial profits, social resources and nation images, the arrangements of sports management systems meets have developed diversely under this context. On the other hand, the existence of globalization has brought shocks to the world and cities' politics, economics, and culture to a significant extent. However, it also led to increase the association as well as dependence among sports organizations. Several researches have found that the critical factor of sports management systems successfully should be that concern of “strategies and effectiveness”. More precisely, the crucial notions of success in order are drafting the conditions of “efficient strategic operation”, “balanced cooperation”, “stable processes”, and “resource sharing”.

In order to promote the effectiveness of sports management systems, we firstly propose some relations among the game-theoretical axioms and sports management systems for sports organizations follows.

1. The property **non-emptiness (non-emptiness for balanced games)** asserts that *the efficient cooperation doesn't come to naught*: There might

be several coalitions in a sports organizations. In order to successfully operate sports management systems, a efficient strategic plan should be cohered with all coalitions simultaneously.

2. The property **individually rationality** asserts that *the efficiency could be promoted by balanced cooperation.*: The payoff (or utility) of each coalition assigned by balanced cooperation may be better than the utility of each coalition when it works alone.
3. The property **consistency (R-consistency)** asserts that *the operational processes should be stable and identical*: Consistency allows us to deduce, from the desirability of an outcome for some utility allocation problem, the desirability of its restriction to each coalition of a sports organization for the associated reduced condition the coalition faces.
4. The property **superadditivity** asserts that *the effectiveness could be promoted by resource sharing*: In order to successfully operate sports management systems, the utilities of all the cooperative coalitions or groups must be integrated successfully by resource sharing.

Based on Theorems 1, 2, the unit-level-core is the only solution satisfying these game-theoretical axioms. Based previous relations and statements among these game-theoretical axioms and sports management systems for sports organizations, the unit-level-core could be considered as an useful theoretical technique for utilities allocation in sports management systems. The conclusion of this paper also could point out that other game-theoretical methods could be considered to applied in the field of sports management.

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Asymptotic behavior of conformable fractional impulsive partial differential equations

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Abstract. In this article, we discuss the asymptotic behavior of conformable fractional impulsive partial differential equations. Some new sufficient conditions possessing a prescribed asymptotic behavior at infinity are derived by using riccati transform and impulsive differential inequalities. Our results extend a number of results reported in the literature. An example is also given to illustrate the validity of our results.

Keywords: oscillation, asymptotic, conformable differential equations, impulse, partial differential equations, forcing term.

1. Introduction

In recent years, fractional calculus has received increasing attention due to its applications in a variety of disciplines such as mechanics, physics, chemistry, biology, electrical engineering, control theory, material science, mathematical psychology. For more details, we refer the reader to the monographs. The field of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real (or complex) order [1, 4, 5, 6, 8, 13, 17, 21]. Many events in diverse fields of engineering can be portrayed better and more accurately by differential equations of non-integer order.

Recently, a new definition of fractional derivative [12] that prominently compatible with the classical derivative was made by Khalil et al. Unlike other def-

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initions, this new definition satisfies formulas of the derivative of product and quotient of two functions and have a simpler chain rule.

In order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulsive differential systems to describe the model since the last century. For the basic theory on impulsive differential equations, the reader can refer to the monographs and references [2, 14, 18, 22, 25, 26]. In particular, the problem of asymptotic behavior of impulsive differential equations has been investigated by few authors, the references [3, 7, 9, 10, 11, 15, 19, 20, 23] cited therein. However, to the best of authors knowledge, no work has been reported on the asymptotic behavior of conformable fractional impulsive partial differential equations. Motivated by the above considerations we consider the following model of the form

$$(1) \quad \left. \begin{aligned} & \frac{\partial^\alpha}{\partial t^\alpha} \left[p(t) \frac{\partial^\alpha}{\partial t^\alpha} (u(x, t) + c(t)u(x, \sigma(t))) \right] + q(x, t)f(u(x, \delta(t))) \\ & = a(t)\Delta u(x, t) + \sum_{i=1}^n b_i(t)\Delta u(x, \rho_i(t)) + F(x, t), \\ & t \neq t_k, (x, t) \in \Omega \times \mathbb{R}_+ \equiv G, \\ & u(x, t_k^+) = c_k(x, t_k, u(x, t_k)), \\ & \frac{\partial^\alpha}{\partial t^\alpha} (u(x, t_k^+)) = d_k(x, t_k, \frac{\partial^\alpha}{\partial t^\alpha} (u(x, t_k))), \quad k = 1, 2, \dots, \end{aligned} \right\}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+ = [0, +\infty)$.

Equation (1) is enhancement with the boundary condition

$$(2) \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

This work is planed as follows: In Section 2, we recall some basic definitions and preliminary results which will be used throughout this paper. In Section 3, we discussed the asymptotic behavior of the all solutions of the problem (1) and (2). In Section 4, we present an example is to illustrate our main results.

2. Preliminaries

In this paper, we assume that the following assumptions hold:

$$(C_1) \quad p(t) \in C^\alpha(\mathbb{R}_+, (0, +\infty)) \text{ with } T_\alpha(p(t)) > 0 \text{ and } \int_{t_0}^{+\infty} s^{\alpha-1} \frac{1}{p(s)} ds = +\infty, \\ q(t) \in C(\mathbb{R}_+, \mathbb{R}), q(t) = \min_{x \in \bar{\Omega}} q(x, t), c(t) \in C^{2\alpha}(\mathbb{R}_+, \mathbb{R}_+), \sigma(t), \delta(t), \rho_i(t) \\ \in C^\alpha(\mathbb{R}_+, \mathbb{R}) \text{ and } \lim_{t \rightarrow +\infty} \sigma(t) = \lim_{t \rightarrow +\infty} \delta(t) = \lim_{t \rightarrow +\infty} \rho_i(t) = +\infty, \\ i = 1, 2, \dots, n.$$

$$(C_2) \quad f \in C(\mathbb{R}, \mathbb{R}) \text{ is convex in } \mathbb{R}_+ \text{ with } uf(u) > 0 \text{ and } \frac{f(u)}{u} \geq \epsilon > 0 \text{ for } u \neq 0, \\ F \in C(\bar{G}, \mathbb{R}) \text{ with } \int_{\Omega} F(x, t) dx < 0.$$

(C₃) $a(t), b_i(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, and left continuous at $t = t_k$, $k = 1, 2, \dots$.

(C₄) $u(x, t)$ and its derivative $\frac{\partial^\alpha}{\partial t^\alpha} u(x, t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \dots$, and left continuous at $t = t_k$, $u(x, t_k) = u(x, t_k^-)$, $\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k) = \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k^-)$, $k = 1, 2, \dots$.

(C₅) $c_k, d_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ for $k = 1, 2, \dots$, and there exist positive constants g_k, g_k^*, h_k, h_k^* such that $g_k^* \leq g_k \leq h_k^* \leq h_k$ for $k = 1, 2, \dots$,

$$g_k^* \leq \frac{c_k(x, t_k, u(x, t_k))}{u(x, t_k)} \leq g_k, \quad h_k^* \leq \frac{d_k\left(x, t_k, \frac{\partial^\alpha}{\partial t^\alpha}(u(x, t_k))\right)}{\frac{\partial^\alpha}{\partial t^\alpha}(u(x, t_k))} \leq h_k.$$

Definition 2.1 ([26]). A solution u of the problem (1) and (2) is a function $u \in C^{2\alpha}(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1), where

$$t_{-1} := \min \left\{ 0, \left\{ \inf_{t \geq 0} \sigma(t) \right\}, \min_{1 \leq i \leq n} \left\{ \inf_{t \geq 0} \rho_i(t) \right\} \right\}, \quad \hat{t}_{-1} := \min \left\{ 0, \inf_{t \geq 0} \delta(t) \right\}.$$

Definition 2.2. The solution u of the problem (1) – (2) is said to be oscillatory in the domain G , if it has arbitrary large zeros. Otherwise it is non-oscillatory.

Definition 2.3 ([12]). Let $f : [0, \infty) \rightarrow \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$$

Definition 2.4. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.

Definition 2.5 (Atangana et al. [1]). f be a function with n variables $x_1, x_2, \dots, x_n$. Then the conformable partial derivative of f of order $0 < \alpha \leq 1$ in x_i is defined as follows

$$\begin{aligned} & \frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, x_2, \dots, x_n) \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(x_1, x_2, \dots, x_{i-1}, x_i + \epsilon x_i^{1-\alpha}, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\epsilon}. \end{aligned}$$

Conformable fractional derivatives have the following properties :

Theorem 2.1. *Let $\alpha \in (0, 1]$ and f, g be α -differentiable at some point $t > 0$. Then*

$$(i) \quad T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \text{ for all } a, b \in \mathbb{R}.$$

$$(ii) \quad T_\alpha(t^p) = pt^{p-\alpha} \text{ for all } p \in \mathbb{R}.$$

$$(iii) \quad T_\alpha(\lambda) = 0 \text{ for all constant functions } f(t) = \lambda.$$

$$(iv) \quad T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f).$$

$$(v) \quad T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}.$$

$$(vi) \quad \text{If } f \text{ is differentiable, then } T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t).$$

It is identified that [24] the smallest eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\begin{aligned} \Delta\omega(x) + \lambda\omega(x) &= 0, & \text{in } \Omega \\ \omega(x) &= 0, & \text{on } \partial\Omega, \end{aligned}$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω .

For convenience, we introduce the following notations:

$$Y(t) = K_\Phi \int_\Omega u(x, t) \Phi(x) dx, \quad \text{where } K_\Phi = \left(\int_\Omega \Phi(x) dx \right)^{-1}.$$

We begin with some lemmas that will be used to prove our main results.

Lemma 2.1. *If X and Y are non negative, then*

$$\begin{aligned} X^\lambda - \lambda XY^{\lambda-1} + (\lambda - 1)Y^\lambda &\geq 0, & \lambda > 1 \\ X^\lambda - \lambda XY^{\lambda-1} - (1 - \lambda)Y^\lambda &\leq 0, & 0 < \lambda < 1 \end{aligned}$$

where the equality holds if and only if $X = Y$.

Lemma 2.2. *If the impulsive conformable fractional differential inequality*

$$(3) \quad \left. \begin{aligned} &T_\alpha(p(t)T_\alpha(z(t))) + c_0q(t)z(\delta(t)) \leq 0, \quad t \neq t_k \\ &g_k^* \leq \frac{Y(t_k^+)}{Y(t_k)} \leq g_k, \quad h_k^* \leq \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \leq h_k, \quad k = 1, 2, \dots \end{aligned} \right\}$$

has no eventually positive solution, then every solution of the problem (1)-(2) is oscillatory in G .

Proof. Assume that there exists a non oscillatory solution $u(x, t)$ of the problem (1) – (2) and $u(x, t) > 0$, $(x, t) \in \Omega \times [t_0, +\infty)$, $t_0 \geq 0$. By assumption that there exists a $t_1 > t_0$ such that

$$\begin{aligned} u(x, \sigma(t)) &> 0, & (x, t) &\in \Omega \times [t_1, \infty) \\ u(x, \delta(t)) &> 0, & (x, t) &\in \Omega \times [t_1, \infty) \\ u(x, \rho_i(t)) &> 0, & (x, t) &\in \Omega \times [t_1, \infty), \quad i = 1, 2, \dots, n. \end{aligned}$$

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \dots$, multiplying both sides of equation (1) by $K_\Phi \Phi(x)$ and integrating with respect to x over the domain Ω , we obtain

$$(4) \quad \left. \begin{aligned} &t^{1-\alpha} \frac{d}{dt} \left[p(t) t^{1-\alpha} \frac{d}{dt} \left(K_\Phi \int_\Omega u(x, t) \Phi(x) dx + c(t) K_\Phi \int_\Omega u(x, \sigma(t)) \Phi(x) dx \right) \right] \\ &+ K_\Phi \int_\Omega q(x, t) f(u(x, \delta(t))) \Phi(x) dx = a(t) K_\Phi \int_\Omega \Delta u(x, t) \Phi(x) dx \\ &+ \sum_{i=1}^n b_i(t) K_\Phi \int_\Omega \Delta u(x, \rho_i(t)) \Phi(x) dx + K_\Phi \int_\Omega F(x, t) \Phi(x) dx. \end{aligned} \right\}$$

Using Green's formula and boundary condition (2), we see that

$$(5) \quad \begin{aligned} K_\Phi \int_\Omega \Delta u(x, t) \Phi(x) dx &= K_\Phi \int_{\partial\Omega} \left[\Phi(x) \frac{\partial u}{\partial \gamma} - u \frac{\partial \Phi(x)}{\partial \gamma} \right] dS \\ &+ K_\Phi \int_\Omega u(x, t) \Delta \Phi(x) dx = -\lambda_0 Y(t) \leq 0 \end{aligned}$$

and for $i = 1, 2, \dots, n$, we have

$$(6) \quad \begin{aligned} K_\Phi \int_\Omega \Delta u(x, \rho_i(t)) \Phi(x) dx &= K_\Phi \int_{\partial\Omega} \left[\Phi(x) \frac{\partial u(x, \rho_i(t))}{\partial \gamma} - u(x, \rho_i(t)) \frac{\partial \Phi(x)}{\partial \gamma} \right] dS \\ &+ K_\Phi \int_\Omega u(x, \rho_i(t)) \Delta \Phi(x) dx \\ &= -\lambda_0 Y(\rho_i(t)) \leq 0, \end{aligned}$$

where dS is surface element on $\partial\Omega$. Applying Jensen's inequality and from (C_2) , it follows that

$$(7) \quad K_\Phi \int_\Omega q(x, t) f(u(x, \delta(t))) \Phi(x) dx \geq \epsilon q(t) K_\Phi \int_\Omega u(x, \delta(t)) \Phi(x) dx.$$

Also, we have

$$(8) \quad F(t) = K_\Phi \int_\Omega F(x, t) \Phi(x) dx \leq 0.$$

In view of (4)-(8), we obtain

$$(9) \quad T_\alpha(p(t)T_\alpha(z(t))) + \epsilon q(t)Y(\delta(t)) \leq 0,$$

where $z(t) = Y(t) + c(t)Y(\sigma(t))$. It is easy to obtain that $z(t) > 0$ for $t \geq t_1$. Next we prove that $T_\alpha(z(t)) > 0$ for $t \geq t_2$. Assume the contrary that there exists $K \geq t_2$ such that $T_\alpha(z(t)) \leq 0$, so we get

$$(10) \quad T_\alpha(p(t)T_\alpha(z(t))) \leq 0, \quad t \geq K.$$

From (10), we have

$$p(t)T_\alpha(z(t)) \leq p(K)T_\alpha(z(K)), \quad t \geq K.$$

Then we get

$$(11) \quad z(t) \leq z(K) + p(K)T_\alpha(z(K)) \int_K^t \frac{ds}{s^{1-\alpha}p(s)}, \quad t \geq K.$$

From the hypothesis (C_1) , we get $\lim_{t \rightarrow +\infty} z(t) = -\infty$. This contradicts that $z(t) > 0$ for $t \geq 0$. Thus $T_\alpha(z(t)) > 0$ and $\sigma(t) \leq t$ for $t \geq t_1$, we have

$$\begin{aligned} Y(t) &= z(t) - c(t)Y(\sigma(t)) \geq z(t) - c(t)z(t), \\ Y(t) &\geq z(t)(1 - c(t)) \geq c_0 z(t), \end{aligned}$$

where $c_0 = 1 - c(t)$ is a positive constant. Therefore from (9), we have

$$T_\alpha(p(t)T_\alpha(z(t))) + k_0 q(t)z(\delta(t)) \leq 0,$$

where $k_0 = \epsilon c_0$. For $t \geq t_0, t = t_k, k = 1, 2, \dots$, multiplying both sides of the equation (1) by $K_\Phi \Phi(x)$, integrating with respect to x over the domain Ω , and from (C_5) , we obtain

$$g_k^* \leq \frac{Y(t_k^+)}{Y(t_k)} \leq g_k, \quad h_k^* \leq \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \leq h_k$$

and

$$g_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq g_k, \quad h_k^* \leq \frac{T_\alpha(z(t_k^+))}{T_\alpha(z(t_k))} \leq h_k.$$

Hence we obtain that $z(t)$ is a positive solution of impulsive conformable fractional inequality (3). This completes the proof. $\square$

Lemma 2.3. Assume that conditions $(C_1) - (C_5)$ hold and let $u(x, t)$ be a positive solution of (1) and (2). Then for sufficiently large t , either

- (I) $z(t) > 0, T_\alpha(z(t)) > 0, T_\alpha(p(t)T_\alpha(z(t))) < 0$ or
- (II) $z(t) > 0, T_\alpha(z(t)) < 0, T_\alpha(p(t)T_\alpha(z(t))) < 0.$

Lemma 2.4. Assume that conditions $(C_1) - (C_5)$ hold and let $u(x, t)$ be an eventually positive solution of (1) and (2) with $z(t)$ satisfying case (II) of Lemma 2.3. If

$$(12) \quad \int_{t_1}^{\infty} \frac{1}{p(y)y^{1-\alpha}} \int_y^{\infty} \frac{q(s)}{s^{1-\alpha}} z(\delta(s)) ds dy = \infty,$$

then $\lim_{t \rightarrow \infty} z(t) = 0$.

Proof. Let $u(x, t)$ be an eventually positive solution of (1) and (2). Then $z(t)$ satisfies the inequality (3) and

$$T_{\alpha}(p(t)T_{\alpha}(z(t))) \leq -k_0 q(t)z(\delta(t)) \leq 0, \quad t \geq t_2.$$

By Lemma 2.3, there exists a constant m such that

$$(13) \quad \lim_{t \rightarrow \infty} z(t) = m < \infty.$$

Integrating the above inequality from t to ∞ , we get

$$\begin{aligned} p(t)T_{\alpha}(z(t)) &\geq \int_t^{\infty} k_0 \frac{q(s)}{s^{1-\alpha}} z(\delta(s)) ds \\ T_{\alpha}(z(t)) &\geq \frac{k_0}{p(t)} \int_t^{\infty} \frac{q(s)}{s^{1-\alpha}} z(\delta(s)) ds. \end{aligned}$$

Again integrating from t_1 to ∞ , we get

$$z(t) \geq k_0 \int_{t_1}^{\infty} \frac{1}{p(y)y^{1-\alpha}} \int_y^{\infty} \frac{q(s)}{s^{1-\alpha}} z(\delta(s)) ds dy,$$

which contradicts (13), and so we have $m = 0$. Therefore $\lim_{t \rightarrow \infty} z(t) = 0$. This complete the proof. $\square$

3. Main results

In this section, by using Riccati transformation and impulsive differential inequality, we investigate the asymptotic behavior of all solutions of nonlinear partial differential equations with impulse effects and obtained the following two theorems.

Theorem 3.1. Assume that $(C_1) - (C_5)$ holds and there exists $\eta(t) \in C^{\alpha}([t_0, \infty), \mathbb{R})$ such that for all sufficiently large K and for $t_1 \geq K$,

$$(14) \quad \limsup_{t \rightarrow \infty} \int_{t_1 t_0 \leq t_k \leq s}^t \prod \left(\frac{h_k}{g_k^*} \right)^{-1} \left\{ k_0 s^{\alpha-1} \eta(s) q(s) - \frac{s^{1-\alpha} T_{\alpha}(\eta(s))^2 p(s)}{4\eta(s)} \right\} ds = \infty,$$

then any solution $u(x, t)$ of (1) and (2) is either oscillatory or converges to zero as $t \rightarrow \infty$.

Proof. Let $u(x, t)$ be a non oscillatory solution of (1) and (2). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $u(x, t) > 0$, $u(x, \sigma(t)) > 0$ and $u(x, \delta(t)) > 0$ for $t \geq t_1$. Also $z(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

Assume that case (I) and define

$$(15) \quad \begin{aligned} w(t) &= \eta(t) \frac{p(t)T_\alpha(z(t))}{z(\delta(t))} \quad \text{for } t \geq t_1, \\ T_\alpha(w(t)) &\leq -k_0\eta(t)q(t) - \frac{w^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)}w(t). \end{aligned}$$

Also

$$(16) \quad w(t_k^+) \leq \frac{h_k}{g_k^*} w(t_k).$$

Define

$$V(t) = \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} w(t).$$

In fact, $w(t)$ is continuous on each interval $(t_k, t_{k+1}]$ and it follows that for $t \geq t_0$,

$$V(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left(\frac{h_k}{g_k^*} \right)^{-1} w(t_k^+) \leq \prod_{t_0 \leq t_j < t} \left(\frac{h_k}{g_k^*} \right)^{-1} w(t_k) = V(t_k),$$

and for all $t \geq t_0$

$$V(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{h_k}{g_k^*} \right)^{-1} w(t_k^-) \leq \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} w(t_k) = V(t_k),$$

which implies that $V(t)$ is continuous on $[t_0, +\infty)$, from (15), we get

$$(17) \quad \begin{aligned} &\prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) T_\alpha(V(t)) \leq -k_0\eta(t)q(t) \\ &- \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^2 \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) V(t), \\ T_\alpha(V(t)) &\leq - \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} V(t) \\ &- \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} k_0\eta(t)q(t). \end{aligned}$$

Applying Lemma 2.1, we have

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) \frac{1}{\eta(t)p(t)}} V(t) \quad \text{and} \quad Y = \frac{T_\alpha(\eta(t))}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} \frac{p(t)}{\eta(t)}}.$$

We have

$$\prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} V(t) \leq \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} \frac{T_\alpha^2(\eta(t))p(t)}{4\eta(t)}.$$

Thus

$$T_\alpha(V(t)) \leq - \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} \left[k_0 \eta(t) q(t) - \frac{T_\alpha^2(\eta(t))p(t)}{4\eta(t)} \right].$$

Integrating both sides from t_1 to t , we have

$$V(t) \leq V(t_1) - \int_{t_1}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*} \right)^{-1} \left[k_0 s^{\alpha-1} \eta(s) q(s) - \frac{s^{\alpha-1} T_\alpha(\eta(s))^2 p(s)}{4\eta(s)} \right] ds.$$

Letting $t \rightarrow \infty$, from (14), we have $\lim_{t \rightarrow \infty} V(t) = -\infty$, which is a contradiction. If case (II) holds, then using Lemma 2.4, we have $\lim_{t \rightarrow \infty} z(t) = 0$. Since $0 < Y(t) < z(t)$ on (t_1, ∞) , we get $\lim_{t \rightarrow \infty} u(x, t) = 0$. The proof of the theorem is complete. $\square$

Next we obtain some new asymptotic results for (1) and (2), by using integral average condition of Philos type [16]. Let $D = \{(t, s) : t_0 \leq s \leq t\}$, $H \in C^\alpha(D, \mathbb{R})$. If $H \in \mathbb{H}$, then $H(t, t) = 0$ and $H(t, s) > 0$ for $t > s$ and $h \in L_{loc}(D, \mathbb{R})$ such that

$$\frac{\partial H(t, s)}{\partial t} = h(t, s) \sqrt{H(t, s)}, \quad \frac{\partial H(t, s)}{\partial s} = -h(t, s) \sqrt{H(t, s)}.$$

Theorem 3.2. Assume that conditions (C_1) – (C_5) hold and there exist $\phi(t), \psi(t) \in C^\alpha([0, \infty), (0, +\infty))$, if

$$\limsup_{t \rightarrow +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*} \right)^{-1} k_0 \eta(s) q(s) \psi(s) \left[(1 - \alpha) s^{-\alpha} \sqrt{H(t, s)} \right. \\ \left. - s^{1-\alpha} h(t, s) + s^{1-\alpha} \sqrt{H(t, s)} \frac{\psi'(s)}{\psi(s)} \right]^2 ds = +\infty, \quad (18)$$

then every solution of the boundary value problem (1) and (2) is oscillatory or converges to zero as $t \rightarrow \infty$.

Proof. Assume that the boundary value problem (1) and (2) has a non oscillatory solution $u(x, t)$. Without loss of generality, assume that $u(x, t) > 0$, $(x, t) \in \Omega \times [0, +\infty)$. As in the proof of the Theorem 3.1, we obtain

$$T_\alpha(V(t)) \leq - \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right) \frac{V^2(t)}{\eta(t)p(t)} + \frac{T_\alpha(\eta(t))}{\eta(t)} V(t) - \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*} \right)^{-1} k_0 \eta(t) q(t).$$

Multiplying the above inequality by $H(t, s)\psi(s)$ for $t \geq s \geq K$, and integrating from K to t , we have

$$\begin{aligned} \int_K^t T_\alpha(V(t))H(t, s)\psi(s)ds &\leq - \int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(s)}{\eta(s)p(s)} H(t, s)\psi(s)ds \\ &\quad + \int_K^t \frac{T_\alpha(\eta(s))}{\eta(s)} V(t)H(t, s)\psi(s)ds \\ &\quad - \int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0 \eta(s)q(s)H(t, s)\psi(s)ds, \end{aligned}$$

Consider

$$\begin{aligned} &\int_K^t \prod_{t_0 \leq t_k < t} \left(\frac{h_k}{g_k^*}\right)^{-1} k_0 \eta(s)q(s)H(t, s)\psi(s)ds \\ &\leq H(t, K)\psi(K)T^{1-\alpha}V(K) + \int_K^t [(1-\alpha)s^{-\alpha}H(t, s)\psi(s) \\ &\quad - s^{1-\alpha}h(t, s)\sqrt{H(t, s)}\psi(s) + s^{1-\alpha}H(t, s)\psi'(s) - \frac{T_\alpha(\eta(s))}{\eta(s)}H(t, s)\psi(s)]V(s)ds \\ &\quad - \int_K^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right) \frac{V^2(s)}{\eta(s)p(s)} H(t, s)\psi(s)ds. \end{aligned} \tag{19}$$

From this,

$$\begin{aligned} &\int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} \left(k_0 \eta(s)q(s)H(t, s)\psi(s) \right. \\ &\quad \left. - \frac{\eta(s)p(s)\psi(s)}{4} \left[(1-\alpha)s^{-\alpha}\sqrt{H(t, s)} - s^{1-\alpha}h(t, s) \right. \right. \\ &\quad \left. \left. + s^{1-\alpha}\sqrt{H(t, s)}\frac{\psi'(s)}{\psi(s)} \right]^2 \right) ds \leq V(t)H(t, K)\psi(K)K^{1-\alpha}. \end{aligned} \tag{20}$$

Letting $t \rightarrow \infty$, we have

$$\begin{aligned} &\limsup_{t \rightarrow +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*}\right)^{-1} \left\{ k_0 \eta(s)q(s)H(t, s)\psi(s) \right. \\ &\quad \left. - \frac{\eta(s)p(s)\psi(s)}{4} \left[(1-\alpha)s^{-\alpha}\sqrt{H(t, s)} - s^{1-\alpha}h(t, s) \right. \right. \\ &\quad \left. \left. + s^{1-\alpha}\sqrt{H(t, s)}\frac{\psi'(s)}{\psi(s)} \right]^2 \right\} ds < +\infty \end{aligned} \tag{21}$$

which is a contradiction with (18).

Let $z(t)$ satisfies case (II) of Lemma 2.3, then using Lemma 2.4, we have $\lim_{t \rightarrow \infty} z(t) = 0$. Since $0 < Y(t) < z(t)$ on (t_1, ∞) , we get $\lim_{t \rightarrow \infty} u(x, t) = 0$. The proof of the theorem is complete. $\square$

4. Example

In this section, we present an example to illustrate our main results established in Section 3.

Example 4.1. Consider the following impulsive neutral partial differential equation is of the form

$$(22) \quad \left. \begin{aligned} & \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \left[\sqrt{t} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \left(u(x, t) + \frac{1}{3} u \left(x, \frac{t}{2} \right) \right) \right] + \frac{2}{15} u \left(x, \frac{t}{3} \right) \\ &= \frac{2}{3} \Delta u(x, t) + \frac{1}{5} \Delta u \left(x, \frac{t}{3} \right) + F(x, t), \quad t \neq 2^k, (x, t) \in \Omega \times \mathbb{R}_+ \equiv G, \\ & u(x, t_k^+) = \frac{k}{k+1} u(x, 2^k), \\ & \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(x, t_k^+) = \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(x, 2^k), \quad k = 1, 2, \dots, \end{aligned} \right\}$$

for $(x, t) \in (0, \pi) \times \mathbb{R}_+$, with the boundary condition

$$(23) \quad u(0, t) = u(\pi, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

Here $p(t) = \sqrt{t}$, $\alpha = 1/2$, $f(u) = 2u$, $q(t) = \frac{1}{15}$, $a(t) = \frac{2}{3}$, $b_1(t) = \frac{1}{5}$, $c(t) = 1/3$, $\sigma(t) = \frac{t}{2}$, $\delta(t) = \rho_1(t) = \frac{t}{3}$, $F(x, t) = \frac{5}{3} \left(1 + \frac{1}{\sqrt{t}} \right) \frac{\sin x}{t}$. Let $g_k = g_k^* = \frac{k}{k+1}$, $h_k = h_k^* = 1$, $t_0 = 1$, $t_k = 2^k$, $\eta(t) = \frac{t}{2}$, $\epsilon = \frac{1}{2}$. Then hypotheses $(C_1) - (C_5)$ hold, and moreover

$$\begin{aligned} \lim_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{h_k}{g_k^*} \right)^{-1} ds &= \int_1^{+\infty} \prod_{1 < t_k < s} \frac{k}{k+1} ds \\ &= \int_1^{t_1} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \dots \\ &= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n+1} = +\infty. \end{aligned}$$

Thus,

$$\int_1^{+\infty} \prod_{1 < t_k < s} \frac{k}{k+1} \left[\frac{s}{90} - \frac{s^{1/2}}{4} \right] ds = +\infty.$$

Here all the conditions of Theorem 3.1 are satisfied. Hence every solution $u(x, t)$ of the problem (22) and (23) is either oscillatory or converges to zero as $t \rightarrow \infty$. In fact $u(x, t) = \frac{\sin x}{t}$ is such a solution.

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On Roman domination stability in some simple graphs

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Abstract. The Roman domination stability of a graph G , denoted by $st_{\gamma_R}(G)$, is the minimum number of vertices whose removal changes the Roman domination number of G . In this paper, we continue the study of this concept, and determine the Roman domination stability of some classes of graphs, including paths, cycles, complete bipartite graphs and some Cartesian products of paths and complete graphs.

Keywords: domination, Roman domination, Roman domination stability.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u | uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v is $|N(v)|$. We refer to $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* among the vertices of G , respectively. With K_n we denote the *complete graph* on n vertices, with C_n the *cycle* on n vertices, and with P_n the *path* on n vertices. The *cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. An *r-partite* graph G is a graph whose vertex set $V(G)$ can be partitioned into r sets of pairwise non-adjacent vertices. For positive integers $p_1, p_2, \dots, p_r$, the complete *r-partite* graph $K_{p_1, p_2, \dots, p_r}$ is the *r-partite* graph with partition $V(G) = V_1 \cup V_2 \cup \dots \cup V_r$ such $|V_i| = p_i$ for $1 \leq i \leq r$ and such that every two vertices belonging to different partite sets are adjacent to each other. A set $S \subseteq V(G)$ is a *domination set* of G , if every vertex not in S is adjacent to a vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a domination set in G . A domination set of cardinality $\gamma(G)$ is called a $\gamma(G)$ -set.

The concept of domination stability in graphs has been introduced by Bauer et al [2]. The *domination stability* of a graph G is the minimum number of ver-

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tices whose removal changes the domination number. We denote the domination stability of a graph G by $st_\gamma(G)$.

A function $f : V \rightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted $\gamma_R(G)$. An RDF on G with weight $\gamma_R(G)$ is called a γ_R -function of G . For an RDF f in a graph G , we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f . Thus an RDF f can be represented by a triple (V_0, V_1, V_2) , and we can use the notation $f = (V_0, V_1, V_2)$. The concept of Roman domination is studied in, for example, [3, 5, 6].

Amraee et al. [1] studied the concept of Roman domination stability in graphs. The *Roman domination stability* of a graph G , denoted by $st_{\gamma_R}(G)$, is the minimum number of vertices whose removal changes the Roman domination number of G . In this paper, we continue the study of this concept, and determine the Roman domination stability of some classes of graphs, including paths, cycles, complete bipartite graphs and cartesian products $P_2 \square P_n$ and $K_p \square K_q$.

The following are useful.

Theorem 1 ([3]). *For paths and cycles,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil.$$

Theorem 2 ([4]). *For $n \geq 2$, $\gamma_R(P_2 \square P_n) = n + 1$.*

2. Results

We begin with determining the Roman domination stability of paths.

Theorem 3. *For a path P_n , $st_{\gamma_R}(P_n) = \begin{cases} 2, & n \equiv 0 \pmod{3} \\ 1, & n \equiv 1 \text{ or } 2 \pmod{3} \end{cases}$.*

Proof. Assume first that $n \equiv 0 \pmod{3}$. Let $n = 3k$, for some integer $k \geq 1$. By Theorem 1, $\gamma_R(P_n) = \lceil \frac{2(3k)}{3} \rceil = 2k$. Clearly removal of a leaf and a support vertex of P_n produces the path P_{n-2} , and by Theorem 1, $\gamma_R(P_{n-2}) = \lceil \frac{2(3k-2)}{3} \rceil = \lceil \frac{6k-4}{3} \rceil = 2k - 1 < \gamma_R(P_n)$. Thus $st_{\gamma_R}(P_n) \leq 2$. Let v be a vertex of P_n . If v is a leaf of P_n , then $P_n - v = P_{n-1}$, and so by Theorem 1, we obtain that $\gamma_R(P_n - v) = \gamma_R(P_n)$. Thus assume that v is not a leaf of P_n . Then $P_n - v$ has two components P_{n_1} and P_{n_2} , where $n_1 + n_2 = n - 1$. Then $\gamma_R(P_n - v) = \gamma_R(P_{n_1}) + \gamma_R(P_{n_2})$, and it is straightforward to see that $\gamma_R(P_n - v) = \gamma_R(P_n)$. We deduce that $st_{\gamma_R}(P_n) \geq 2$. Consequently $st_{\gamma_R}(P_n) = 2$.

Next assume that $n \equiv 1 \pmod{3}$. Let $n = 3k + 1$, where k is a positive integer. By Theorem 1, $\gamma_R(P_n) = 2k + 1$. Since removal of a leaf of P_n produces

the path P_{n-1} , and $\gamma_R(P_{n-1}) = 2k$ by Theorem 1, we conclude that $st_{\gamma_R}(P_n) = 1$.

The proof for the case $n \equiv 2 \pmod{3}$ is similar. $\square$

We next determine the Roman domination stability of a cycle C_n .

Theorem 4. For a cycle C_n , $st_{\gamma_R}(C_n) = \begin{cases} 2 & n \equiv 0 \pmod{3} \\ 1 & n \equiv 1 \text{ or } 2 \pmod{3} \end{cases}$.

Proof. Assume that $n \equiv 0 \pmod{3}$. Let $n = 3k$ for some integer k . Then by Theorem 1, $\gamma_R(C_n) = 2k$. Clearly removal of a vertex leaves a path P_{n-1} , and $\gamma_R(P_{n-1}) = 2k$ by Theorem 1. Thus $st_{\gamma_R}(C_n) \geq 2$. On the other hand removal of two adjacent vertices of C_n leaves a path P_{n-2} with Roman domination number $2k - 1$ by Theorem 1. Thus $st_{\gamma_R}(C_n) \leq 2$. Consequently, $st_{\gamma_R}(C_n) = 2$.

Next assume that $n \equiv 1$ or $2 \pmod{3}$. Then Removal of a vertex of C_n produces a path P_{n-1} with Roman domination number less than $\gamma_R(C_n)$, by Theorem 1. Consequently, $st_{\gamma_R}(C_n) = 1$. $\square$

Theorem 5. For complete bipartite graph $K_{m,n}$ with $m \leq n$, we have:

$$St_{\gamma_R}(K_{m,n}) = \begin{cases} m - 2, & m > 3 \\ 2, & m = 1, n = 2 \\ 1, & o.w \end{cases}$$

Proof. Let X and Y be the partite sets of $K_{m,n}$, where $|X| = m$ and $|Y| = n$. Assume that $|X| = 1$. Clearly $\gamma_R(K_{m,n}) = 2$. If $|Y| = 1$, then removal of the vertex of X produces a K_1 with Roman domination number one, and so $st_{\gamma_R}(K_{m,n}) = 1$. If $|Y| \geq 3$, then removal of the vertex of X produces a graph $\overline{K_n}$ with $n \geq 3$, and note that $\gamma_R(\overline{K_n}) > 2$. Consequently, $st_{\gamma_R}(K_{m,n}) = 1$. It remains to assume that $|Y| = 2$. It can be easily seen that removal of any vertex of $K_{m,n}$ does not change the Roman domination number, and so $st_{\gamma_R}(K_{m,n}) \geq 2$. On the other hand, removal of Y leaves the graph K_1 with Roman domination number one, and thus $st_{\gamma_R}(K_{m,n}) \leq 2$. Consequently, $st_{\gamma_R}(K_{m,n}) = 2$.

Next assume that $|X| = 2$. Then $\gamma_R(K_{m,n}) = 3$, while $\gamma_R(K_{m,n} - v) = 2$, where v is a vertex of X . Consequently, $st_{\gamma_R}(K_{m,n}) = 1$.

If $|X| = 3$, then $\gamma_R(K_{m,n}) = 4$, while $\gamma_R(K_{m,n} - v) = 3$, where v is a vertex of X . Consequently, $st_{\gamma_R}(K_{m,n}) = 1$.

Now we assume that $|X| \geq 4$. Clearly, $\gamma_R(K_{m,n}) = 4$. Note that removal of $m - 2$ vertices of X produces the graph $K_{2,n}$ with Roman domination number 3, and so $st_{\gamma_R}(K_{m,n}) \leq m - 2$. On the other hand removal of at most $m - 3$ vertices of $K_{m,n}$ leaves at least three vertices in X and at least three vertices in Y , and thus the remaining graph has Roman domination number at least 4. Thus, $st_{\gamma_R}(K_{m,n}) \geq m - 2$. Consequently, $st_{\gamma_R}(K_{m,n}) = m - 2$. $\square$

Theorem 6. For every $n \in \mathbb{N}$, $st_{\gamma_R}(P_2 \square P_n) = 1$.

Proof. By Theorem 2, $\gamma_R(P_2 \square P_n) = n + 1$. Since $P_2 \square P_1 \cong K_2$, we see that $st_{\gamma_R}(P_2 \square P_1) = 1$. Since $P_2 \square P_2 \cong C_4$, we can see that $st_{\gamma_R}(P_2 \square P_2) = 1$. Thus assume that $n \geq 3$. Assume that n is even. Since $\gamma_R(P_2 \square P_n) = n + 1$ is odd, for any $\gamma_R(P_2 \square P_n)$ -function f , there is a vertex v with $f(v) = 1$. Then Removal of v produces a graph with Roman domination number at most n , and thus $st_{\gamma_R}(P_2 \square P_n) = 1$. Next assume that n is odd. By re-labeling of the vertices of $P_2 \square P_n$, assume that $v_0, v_1, \dots, v_{n-1}$ are the vertices on the top row and $w_0, w_1, \dots, w_{n-1}$ are the vertices on the bottom row. Let $H = P_2 \square P_n - v_0$. We show that $\gamma_R(H) \leq n$. If $n \equiv 1 \pmod{4}$, so, $n = 4k + 1$, $k \in \mathbb{N}$, then it is straightforward to see that the function $f : V(H) \rightarrow \{0, 1, 2\}$ defined by $f(v_i) = \begin{cases} 2, & i \equiv 3 \pmod{4}, \\ 0, & o.w \end{cases}$

and $f(w_i) = \begin{cases} 2, & i \equiv 1 \pmod{4}, \\ 1, & i = n, \\ 0, & o.w \end{cases}$ is an RDF for H , implying that

$$\gamma_R(H) \leq \sum_{v \in V(H)} f(v) = 2\left(\frac{4k}{4}\right) + 2\left(\frac{4k}{4}\right) + 1 = 4k + 1 = n.$$

If $n \equiv 3 \pmod{4}$, so, $n = 4k + 3$, $k \in \mathbb{N}$, then the function $f : V(H) \rightarrow \{0, 1, 2\}$ defined

by $f(v_i) = \begin{cases} 2, & i \equiv 3 \pmod{4}, \\ 1, & i = n, \\ 0, & o.w \end{cases}$ and $f(w_i) = \begin{cases} 2, & i \equiv 1 \pmod{4}, \\ 0, & o.w \end{cases}$ is an RDF for H , implying

that

$$\gamma_R(H) \leq \sum_{v \in V(H)} f(v) = 2\left(\frac{4k}{4}\right) + 1 + 2\left(\frac{4(k+1)}{4}\right) = 4k + 3 = n.$$

Consequently, $st_{\gamma_R}(P_2 \square P_n) = 1$. □

Theorem 7. If $q \leq p$, then $st_{\gamma_R}(K_p \square K_q) = \begin{cases} 1, & |p - q| \leq 1, \\ p - q, & p - q \geq 2 \end{cases}$.

Proof. By Proposition 3 of [4] we have $\gamma_R(K_p \square K_q) = \begin{cases} 2q, & p \neq q, \\ 2q - 1, & p = q \end{cases}$. If $v \in V(K_p \square K_q)$ then according to the proof of Proposition 3 of [4],

$$\gamma_R(K_p \square K_q - v) = \begin{cases} 2q, & p - q \geq 2, \\ 2q - 1, & p - q = 1, \\ 2q - 2, & p = q \end{cases}.$$

Thus, $st_{\gamma_R}(K_p \square K_q) = 1$ if $p = q$ or $|p - q| = 1$. Next assume that $|p - q| \geq 2$. We will show that $st_{\gamma_R}(K_p \square K_q) = p - q$.

By re-labeling the vertices of $K_p \square K_q$, assume that

$$V(K_p \square K_q) = \{v_1^{(1)}, \dots, v_p^{(1)}, \dots, v_1^{(\alpha)}, \dots, v_p^{(\alpha)}, \dots, v_1^{(q)}, \dots, v_p^{(q)}\}.$$

For an integer α with $1 \leq \alpha \leq q$, it is straightforward to check that the function f defined by

$$f(v_i^{(j)}) = \begin{cases} 2, & 1 \leq i, j \leq q, i = j, i \neq \alpha, \\ 1, & i = j = \alpha, \\ 0, & \text{o.w} \end{cases}$$

is an RDF for $K_p \square K_q - \{v_{q+1}^{(\alpha)}, \dots, v_p^{(\alpha)}\}$. Since $w(f) = 2(q-1) + 1 = 2q-1$, we obtain that $\gamma_R(K_p \square K_q - \{v_{q+1}^{(\alpha)}, \dots, v_p^{(\alpha)}\}) \leq 2q-1$ and so $st_{\gamma_R}(K_p \square K_q) \leq p-q$. We show that $st_{\gamma_R}(K_p \square K_q) = p-q$. For this purpose, we show that the removal of less than $p-q$ vertices of $K_p \square K_q$ does not change $\gamma_R(K_p \square K_q)$. Let H be a graph obtained from $K_p \square K_q$ by removing x vertices, where $x \leq p-q-1$. Clearly each column has at least two vertices. For column 1, we assign 2 to a vertex of this column and 0 to any other vertex of this column. Assume the vertex of the column 1 valued 2 is in the row i_1 . For column 2, we assign 2 to a vertex of this column on a row $i_2 \leq i_1$ and 0 to any other vertex of this column. For column 3, we assign 2 to a vertex of this column on a row $i_3 \notin \{i_1, i_2\}$ and 0 to any other vertex of this column. We continue this process to obtain an RDF of weight $2q$ in H , and so $\gamma_R(H) \leq 2q$. Suppose that $\gamma_R(H) < 2q$. Let f be a $\gamma_R(H)$ -function. Then there is a row i such that the sum of labels of vertices of this row under f is at most one. Using this row, since any vertex valued 0 is adjacent to a vertex valued 2, we obtain that $w(f) \geq 2q$, a contradiction. We thus deduce that $\gamma_R(H) = 2q$. Consequently, $st_{\gamma_R}(K_p \square K_q) = p-q$. $\square$

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Ordered hypervector spaces

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Abstract. In this paper, we first introduce the concept of ordered hypervector spaces, then we present several examples to better explain the introduced concepts. We finally prove that although ordered hypervector spaces do not necessarily satisfy the antisymmetric condition, the positive weak linear functionals on cofinal weak subhypervector spaces can be extended to the whole weak hypervector space.

Keywords: ordered hypervector spaces, positive weak linear functional, order unit element, cofinal set.

1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934. During the recent decades, many researchers have focused on this field and applied it as a corner stone to construct some other structures such as hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions have been

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applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, etc. A wealth of applications of these concepts are given in [1, 2, 4] and [18].

In 1988, Scafati-Tallini first introduced the concept of hypervector spaces. She later considered more properties of such spaces. In [7, 13], authors considered hypervector spaces in the viewpoint of analysis and introduced some concepts such as normed hypervector spaces and the dimension of hypervector spaces. They also introduced operators on these space as well as some other important concepts.

In this paper we introduce the concept of ordered hypervector spaces and prove some Han-Banach results about it. we prove that each positive weak linear functional on cofinal weak subhypervector spaces can be extended to a positive weak linear functional on the whole weak hypervector space.

2. Preliminaries

Definition 2.1 ([16]). A weak or weakly distributive hypervector space over a field F ($\mathbb{C}$ or $\mathbb{R}$) is a quadruple $(X, +, \circ, F)$ such that $(X, +)$ is an abelian group and $\circ : F \times X \rightarrow P_*(X)$ is a multivalued product such that $(P_*(X) = \{A \subseteq X \mid A \neq \emptyset\})$

- 1- $\forall a \in F, \forall x, y \in X, [ao(x + y)] \cap [aox + aoy] \neq \emptyset;$
- 2- $\forall a, b \in F, \forall x \in X, [(a + b)ox] \cap [aox + box] \neq \emptyset;$
- 3- $\forall a, b \in F, \forall x \in X, ao(box) = (ab)ox;$
- 4- $\forall a \in F, \forall x \in X, ao(-x) = (-a)ox = -(aox);$
- 5- $\forall x \in X, x \in 1ox.$

The properties 1 and 2 are called weak right and left distributive laws respectively. Note that the set $ao(box)$ in 3 is of the form $\bigcup_{y \in box} aoy$.

Throughout this article, by "weak hypervector space" we mean "weak distributive hypervector space" on the field of real numbers.

Definition 2.2 ([7]). Let X be a weak hypervector space over F , $a \in F$ and $x \in X$. An essential point of $a \circ x$, denoted by $e_{a \circ x}$, for $a \neq 0$ is an element of $a \circ x$ such that $x \in a^{-1} \circ e_{a \circ x}$. For $a = 0$ we define $e_{a \circ x} = 0$.

Remark 2.3. As shown in the examples in the following sections, $e_{a \circ x}$ is not necessarily unique. Hence we denote the set of all essential points by $E_{a \circ x}$. In this note, $e_{a \circ x}$ represents an element in $E_{a \circ x}$.

Definition 2.4. A weak hypervector space X over the field F is said to be normal if for every $a, b \in F$ and $x, y \in X$ the following conditions hold:

1. $(E_{a \circ x} + E_{a \circ y}) \cap E_{a \circ (x+y)} \neq \emptyset$;
2. $E_{a \circ x} + E_{b \circ x} \subseteq E_{(a+b) \circ x}$.

Lemma 2.5 ([7]). *Let X be a weak hypervector space over F , $a, b \in F$ and $x \in X$. Then the following properties hold:*

- 1- $x \in E_{1 \circ x}$.
- 2- If $b \neq 0$, then $a \circ e_{b \circ x} = ab \circ x$.
- 3- $E_{-a \circ x} = -E_{a \circ x}$.
- 4- If $a \neq 0$, then there exists a $y \in X$ such that $x \in E_{a \circ y}$.
- 5- If X is normal, then $E_{a \circ x}$ is a singleton.

Definition 2.6. A subset E of a weak hypervector space X is said to be hyperconvex if $E_{t \circ x} + E_{(1-t) \circ x} \subseteq E$, for every $x, y \in E$ and $t \in [0, 1]$.

Definition 2.7 ([6]). A subset E of a weak hypervector space X is said to be hyperabsorbing if for each $x \in X$, there is an $\varepsilon > 0$ such that $E_{\varepsilon \circ x} \subseteq E$.

Example 2.8. Consider $X = \mathbb{R}^2$ with usual addition and the following multi-valued product:

$$\begin{cases} \circ : \mathbb{R} \times X \rightarrow P_*(X) \\ a \circ (x, y) = \{ax\} \times ([0, +\infty) \cup \{y\}), \quad a \in \mathbb{R}, (x, y) \in X. \end{cases}$$

Then $(X, +, \circ, \mathbb{R})$ is a weak hypervector space. The strip $E = [-1, 1] \times \mathbb{R}$ is hyperabsorbing and hyperconvex, but the set

$$A = \{(x, y) \in X \mid x^2 + y^2 \leq 1, x > 0, y > 0\}$$

is not. In fact, for $(x, y) \in X$, we choose $\varepsilon > 0$ such that $\varepsilon|x| < 1$. Since

$$E_{\varepsilon \circ (x, y)} = \begin{cases} \{\varepsilon x\} \times [0, \infty), & y \geq 0 \\ \{(\varepsilon x, y)\}, & y < 0 \end{cases}$$

then $E_{\varepsilon \circ (x, y)} \subseteq E$, therefore E is hyperabsorbing.

Definition 2.9 ([7]). An additive subgroup M of a weak hypervector space X is called a weak subhypervector space of X when $e_{a \circ x} \in M$ for every $a \in F$ and $x \in M$.

Definition 2.10 ([7]). Let X be a weak hypervector space over F . A map $f : X \rightarrow F$ is called a weak linear functional if f is additive and satisfies $f(e_{a \circ x}) = af(x)$, for every $a \in F$ and $x \in X$.

3. Ordered hypervector space

In this section, we introduce the concept of ordered weak hypervector spaces and give diverse examples of such hyperstructures. Moreover, we present the definition of positive weak linear functional, together with some examples of such functions. It is interesting that the order relation discussed in this hyperstructure is not necessarily antisymmetric, however, the corollaries of Han-Banach theorem can be proved for these theorems. We recall that Han-Banach theorem on weak hypervector spaces was first proved by Taghavi et al [8]. As a corollary of Han-Banach theorem, we here prove that each positive weak linear functional on a cofinal weak subhypervector space of an ordered weak hypervector space can be extended to the whole hypervector space.

Definition 3.1. An ordered hypervector space is a pair $(X, \leq)$, where X is a weak hypervector space over the real line $\mathbb{R}$ and $\leq$ is a relation on X satisfying the following: $(x, y, z \in X, \alpha \in \mathbb{R})$

1. $x \leq x$.
2. If $x \leq y$ and $y \leq z$, then $x \leq z$.
3. If $x \leq y$ and $z \in X$, then $x + z \leq y + z$.
4. If $x \leq y$ and $\alpha \geq 0$, then $e_{\alpha \circ x} \leq e_{\alpha \circ y}$.

Note that it is not assumed that $\leq$ is antisymmetric. In other words, it is not assumed that if $x \leq y$ and $y \leq x$, then $x = y$.

From now on, for each x, y in the ordered weak hypervector space $(X, \leq)$, $x \geq y$ is equivalent to $y \leq x$.

Example 3.2. 1. Consider $X = \mathbb{R}^2$ with usual addition. For $0 \neq \lambda \in \mathbb{R}$, we define the hyperoperation of $\circ_\lambda : \mathbb{R} \times X \rightarrow P_*(X)$ as follows:

$$\forall \alpha \in \mathbb{R}, x = (x_1, x_2) \in X, \alpha \circ_\lambda x := {}_\lambda L_{\alpha(x_2 - \lambda x_1)},$$

where for $b \in \mathbb{R}$, ${}_\lambda L_b$ is a line in $\mathbb{R}^2$ whose slope and y -intercept are equal to λ and b respectively. In other words,

$${}_\lambda L_b = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda x + b\}.$$

Then $(X, +, \circ_\lambda, \mathbb{R})$ is a weak hypervector space in which

$$\forall 0 \neq \alpha, x \in X, E_{\alpha \circ_\lambda x} = \alpha \circ_\lambda x.$$

On this hyperstructure, we now define the relation $\leq_\lambda$ as follows:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points of X , we set

$$x \leq_\lambda y \Leftrightarrow x_2 - \lambda x_1 \leq y_2 - \lambda y_1.$$

Intuitively, $x \leq_\lambda y$, if x is on or under the line $\lambda L_{(y_2 - \lambda y_1)}$. This is not an antisymmetric relation but satisfies the condition of Definition 3.1.

1, 2 and 3 are obvious. For 4. Let $\alpha > 0$, then if we suppose

$$e_{\alpha \circ_\lambda x} = (a_1, a_2), \quad e_{\alpha \circ_\lambda y} = (b_1, b_2)$$

then from $x \leq_\lambda y$ and $\alpha > 0$ we have

$$a_2 - \lambda a_1 = \alpha(x_2 - \lambda x_1) \leq \alpha(y_2 - \lambda y_1) = b_2 - \lambda b_1$$

which shows that $e_{\alpha \circ_\lambda x} \leq_\lambda e_{\alpha \circ_\lambda y}$.

2. Consider $\mathbb{C}$, the set of complex numbers with usual addition and the following scalar hyperproduct:

$$a \circ z := \{re^{i\theta} \mid 0 \leq r \leq |a||z|, \theta = \arg(z)\}, \quad a \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Then according to (Example 4.2) in [7], $(\mathbb{C}, +, \circ, \mathbb{R})$ is a hypervector space with $E_{a \circ z} = \{|a|z\}$ for each $a \in \mathbb{R}$ and $z \in \mathbb{C}$. Together with each of the relations defined in the following, this hyperstructure is an ordered hypervector space.

Let $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$ be two elements in $\mathbb{C}$, we define:

$$\begin{aligned} z_1 \leq_1 z_2 &\Leftrightarrow x_1 \leq y_1, x_2 \leq y_2 \\ z_1 \leq_2 z_2 &\Leftrightarrow x_1 < y_1 \text{ or } (x_1 = y_1, x_2 < y_2) \\ z_1 \leq_3 z_2 &\Leftrightarrow (x_1 = y_1, x_2 = y_2) \text{ or } (x_1 < y_1, x_2 < y_2). \end{aligned}$$

3. The weak hypervector space defined in Example 2.8 is an ordered hypervector space with the following defined relation:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points in X , we define

$$x \preceq y \Leftrightarrow x_1 \geq y_1.$$

This relation is not antisymmetric but satisfies the conditions of Definition 3.1. To check condition 4 of Definition 3.1, let $\alpha > 0$ be given, then $x \preceq y$ implies that $\alpha x_1 \geq \alpha y_1$. Then for each $t, s \in \mathbb{R}$, $(\alpha x_1, t) \preceq (\alpha y_1, s)$, which proves that $e_{\alpha \circ x} \preceq e_{\alpha \circ y}$.

Definition 3.3. If X is a real weak hypervector space, a hypercone is a nonempty subset P of X such that

1. If $x, y \in P$ then $x + y \in P$.
2. If $x \in P$ and $\alpha \geq 0$ then $E_{\alpha \circ x} \subseteq P$.

For instance, the set $P = \{re^{i\theta} \mid r \geq 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ is a hypercone of the hypervector space given in Example 2.8.

Proposition 3.4. 1. If $(X, \leq)$ is an ordered hypervector space such that

$$(3.1) \quad \forall x, y \in X, a \in \mathbb{R}, \quad E_{a \circ x} + E_{a \circ y} \subseteq E_{a \circ (x+y)}$$

and

$$P = \{x \in X | x \geq 0\},$$

then P is a hypercone.

2. If P is a hypercone in a real weak hypervector space X that satisfies (3.1) and $\leq$ is defined on X declared as " $x \leq y$ if and only if $y - x \in P$ ", then $(X, \leq)$ is an ordered hypervector space.

Proof. 1. Let $x \in P$ and $\alpha \geq 0$ be given, since X satisfies (3.1), then $e_{\alpha \circ x} \geq e_{\alpha \circ 0} = 0$ and hence $e_{\alpha \circ x} \in P$.

(Note that $0 \in E_{a \circ 0} - E_{a \circ 0} \subseteq E_{a \circ 0}$).

2. Let $x, y \in X, \alpha \geq 0$ and $x \leq y$, then $y - x \geq 0$, since P is a hypercone then (3.1) implies that:

$$e_{\alpha \circ y} - e_{\alpha \circ x} \in E_{\alpha \circ y} - E_{\alpha \circ x} \subseteq E_{\alpha \circ (y-x)} \subseteq P,$$

therefore $e_{\alpha \circ x} \leq e_{\alpha \circ y}$. □

If $(X, \leq)$ is an ordered hypervector space, $P = \{x \in X | x \geq 0\}$ is called the hypercone of positive elements of X .

For instance, for the ordered weak hypervector space given in Example 3.2-1, $P = \{(x, y) \in \mathbb{R}^2 | y \geq \lambda x\}$. Furthermor, if we let $-P := \{x \in X | -x \geq 0\}$, then in Example 3.2-1 we have $P \cap (-P) = {}_{\lambda}L_0$.

The following proposition gives a condition equivalent to antisymmetric property according to the point of hypercone. The proof has been omitted due to simplicity.

Proposition 3.5. If $(X, \leq)$ is an ordered normal hypervector space and P is the hypercone of positive elements of X , then $\leq$ is antisymmetric if and only if $P \cap (-P) = \{0\}$.

Definition 3.6. If $(X, \leq)$ is an ordered hypervector space, a subset A of X is said to be cofinal if for every $x \geq 0$ in X , there is an $a \in A$ such that $x \leq a$. An element e of X is said to be an order unit if for every $x \in X$ there exists a positive integer n such that $e_{(-n) \circ e} \leq x \leq e_{n \circ e}$.

If e is an order unit, clearly the set $\{e_{n \circ e} : n \geq 1\}$ is cofinal.

The set $\{(r, s) | r > 0, s \in \mathbb{R}\}$ is a set of order unit elements for the ordered hypervector space given in Example 3.2-3. Moreover, the set $\{(0, r) | r > 0\}$ is also a set of order unit elements for the ordered hypervector space given in Example 3.2-1.

Definition 3.7. If $(X, \leq)$ is an ordered hypervector space, then a weak linear functional f on X is said to be positive if $f(x) \geq 0$ for every $x \geq 0$.

Example 3.8. 1. Let $(X, \leq_\lambda)$ be the ordered hypervector space given in Example 3.2-1. Then the additive map

$$\begin{cases} f : X \rightarrow \mathbb{R} \\ f(x, y) = y - \lambda x \end{cases}$$

is a positive weak linear functional on X . Let $\alpha \in \mathbb{R}$ and $(x, y) \in X$, then for $\alpha \neq 0$,

$$\begin{aligned} f(e_{\alpha \circ_\lambda(x, y)}) &\in f(E_{\alpha \circ_\lambda(x, y)}) = f(\lambda L_{\alpha(y - \lambda x)}) = \{\alpha(y - \lambda x)\} \\ \Rightarrow f(e_{\alpha \circ(x, y)}) &= \alpha(y - \lambda x) = \alpha f(x, y). \end{aligned}$$

Moreover, if $(x, y) \geq_\lambda 0$, then $f(x, y) = y - \lambda x \geq 0$.

2. Let X be the ordered hypervectore space given in Example 3.2-3. The additive map

$$\begin{cases} f : X \rightarrow \mathbb{R} \\ f(x, y) = -x \end{cases}$$

is a positive weak linear functional on X .

The following theorem is the main result of this paper.

Theorem 3.9. *Let $(X, \leq)$ be an ordered normal hypervector space and let M be a weak subhypervector space of X , and let f be a weak linear functional on M . Then the following are equivalent:*

1. *The functional f can be extended to a positive weak linear functional on X .*
2. *There is a hyperconvex and hyperabsorbing set $A \subseteq X$ such that*

$$(3.2) \quad f(x) \leq 1 \quad \text{whenever } x \in M \text{ and } (\exists y \in A, x \leq y).$$

Furthermore, when the statement 2 is satisfied, an extension $\tilde{f}$ can be chosen such that $\tilde{f}(x) \leq 1$ whenever $x \leq y$ for some $y \in A$.

Proof. 1 $\rightarrow$ 2: Assume that a positive weak linear functional $\tilde{f}$ is an extension of f and set $A = \{x \in X \mid \tilde{f}(x) \leq 1\}$. Then A is hyperconvex and hyperabsorbing. Moreover, A satisfies the condition (3.2), for if $x \in M$ such that $x \leq y$ for some $y \in A$, then $\tilde{f}(x) = \tilde{f}(x) \leq \tilde{f}(y) \leq 1$.

2 $\rightarrow$ 1: Consider the set $C = A - P$, where P is the hypercone of positive elements of X . since A and P are hyperconvex, then C is hyperconvex. Furthermore, C is hyperabsorbing too, because if $x \in X$ then since A is hyperabsorbing, $e_{\varepsilon \circ x} \in A$, for some $\varepsilon > 0$. Since $0 \in P$, then $e_{\varepsilon \circ x} = e_{\varepsilon \circ x} - 0 \in A - P = C$. Now since $0 \in C$, Lemma 3.1 of [3] implies that the functional

$$\begin{cases} q : X \rightarrow \mathbb{R} \\ q(x) = \inf\{\alpha > 0 \mid e_{\alpha^{-1} \circ x} \in C\} \end{cases}$$

have the following properties,

$$(3.3) \quad q(x+y) \leq q(x) + q(y), \quad x, y \in X$$

$$(3.4) \quad q(e_{\lambda \circ x}) = \lambda q(x), \quad x \in X, \lambda > 0.$$

By the hypothesis, $f(x) \leq 1$ whenever $x \in M \cap C$, because if $x \in M \cap C$ then $x = a - p$ for some $a \in A$, $p \in P$, but $p \geq 0$ implies $x = a - p \leq a$, therefore $x \in M$ and $x \leq a$ for $a \in A$, hence $f(x) \leq 1$.

Now let $x \in M$ be given, if $\alpha > 0$ and $e_{\alpha^{-1} \circ x} \in C$, then since $e_{\alpha^{-1} \circ x} \in M$, $f(e_{\alpha^{-1} \circ x}) \leq 1$, hence $f(x) \leq \alpha$, so $f(x) \leq q(x)$. Therefore, by (Theorem 3.8) of [8] there is a weak linear functional $\tilde{f}$ on X which is an extension of f , such that $\tilde{f}(x) \leq 1$ for all $x \in C$. The functional $\tilde{f}$ is necessarily positive. This is because if we take $x \in P$, then for all positive numbers t , $e_{(-t) \circ x} = 0 - e_{t \circ x} \in A - P = C$, hence $1 \geq \tilde{f}(e_{(-t) \circ x}) = -t\tilde{f}(x)$, for all $t > 0$, therefore

$$\forall t > 0 \quad \tilde{f}(x) \geq \frac{-1}{t},$$

letting $t \rightarrow \infty$, we have $\tilde{f}(x) \geq 0$. This completes the proof. $\square$

Corollary 3.10. *Let M be a cofinal weak subhypervector space of an ordered normal hypervector space X . Then each positive weak linear functional on M can be extended to a positive weak linear functional on X .*

Proof. without loss of generality we can assume that $X = M + P - P$, where P is the hypercone of positive elements of X . Because if we let $X_1 = M + P - P$, then X_1 is a weak subhypervector space of X , Now if there is a positive weak linear functional $g : X_1 \rightarrow \mathbb{R}$ that extends the positive weak linear functional $f : M \rightarrow \mathbb{R}$, then each weak linear extension $\tilde{f} : X \rightarrow \mathbb{R}$ of g is positive, that is because if $x \geq 0$, then $x \in P \subseteq X_1$, which implies $\tilde{f}(x) = g(x) \geq 0$.

Thus we may assume that $X = M + P - P$. Assume that f is a positive weak linear functional on M , set

$$A = \{x \in X \mid \exists y \in M \quad x \leq y \text{ and } f(y) \leq 1\},$$

then we claim that

- (I) A is hyperconvex,
- (II) A is hyperabsorbing,
- (III) (3.2) is satisfied.

(III) is trivial. For (I), let $x_1, x_2 \in A$ and $t \in [0, 1]$ be given, then there are $y_1, y_2 \in M$ such that $x_1 \leq y_1$, $x_2 \leq y_2$ and $f(y_1) \leq 1$, $f(y_2) \leq 1$. Since M is a weak subhypervector space of X , then $y = e_{t \circ y_1} + e_{(1-t) \circ y_2} \in M$ and also $e_{t \circ x_1} + e_{(1-t) \circ x_2} \leq y$. Furthermore,

$$f(y) = tf(y_1) + (1-t)f(y_2) \leq 1.$$

Therefore $e_{tox_1} + e_{(1-t)ox_2} \in A$.

For (II), first note that we have $X = M - P$, for if $x \in X$, then $x = m + p_1 - p_2$ for some $m \in M$, $p_1, p_2 \in P$. Since M is cofinal, there is $m_1 \in M$ such that $p_1 \leq m_1$, hence $p_1 = m_1 - (m_1 - p_1) \in M - P$, thus $x = m - p_2 + p_1 \in (M - P) + (M + P) \subseteq M - P$.

Now let $x \in X$ be given, then $x = m - p$ for some $m \in M$, $p \in P$. If $e_{\lambda om} \in P \cap M$ for some $\lambda > 0$, then $m = e_{\lambda^{-1}oe_{\lambda om}} \in P \cap M$ and hence $m \geq 0$, therefore, $f(m) \geq 0$ ($f \geq 0$ on M). So $\varepsilon = \frac{1}{f(m)+1} > 0$ and since $x \leq m$, then $e_{\varepsilon ox} \leq e_{\varepsilon om}$, and $e_{\varepsilon om} \in M$ and

$$f(e_{\varepsilon om}) = \varepsilon f(m) = \frac{f(m)}{1 + f(m)} \leq 1.$$

If $e_{\lambda om} \in (-P) \cap M$ for some $\lambda > 0$, then $-m \geq 0$ and hence $x \leq m \leq 0 \leq -m$, $\varepsilon = \frac{1}{1-f(m)} > 0$ and $e_{\varepsilon ox} \leq e_{\varepsilon o(-m)}$ and $e_{\varepsilon o(-m)} \in M$ and

$$f(e_{\varepsilon o(-m)}) = \frac{-f(m)}{1 - f(m)} \leq 1.$$

So $e_{\varepsilon ox} \in A$ and therefore A is hyperabsorbing.

Hence the assertion of this corollary is a consequence of Theorem 3.9. $\square$

Corollary 3.11. *Let $(X, \leq)$ be an ordered normal hypervector space and let M be a weak subhypervector space of X . If X has an order unit $e \in M$, then each positive weak linear functional on M can be extended to a positive weak linear functional on X .*

Proof. It's enough to prove that M is cofinal. Let $x \geq 0$ be given in X , since e is an order unit, then $x \leq e_{noe}$ for some integer $n \geq 1$, but since $e \in M$, then $e_{noe} \in M$ and hence M is cofinal. Now the assertion follows from the last corollary. $\square$

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Duality of St -closed submodules and semi-extending modules

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Abstract. The main goal of this paper is to dualize the two concepts St -closed submodule and semi-extending module which were given by Ahmed and Abbas in 2015. These dualizations are called CSt -closed submodule and cosemi-extending module. Many important properties of these dualizations are investigated, as well as some others useful results which mentioned by those authors are dualized. Furthermore, the relationships of cosemi-extending and other related modules are considered.

Keywords: CSt -closed submodules, St -closed submodules, Cosemi-essential submodules, semi-essential submodules, Cosemi-extending modules, semi-extending modules.

1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary left R -module. "A submodule V of an R -module U is called essential if every non-zero submodule of an R -module U has a non-zero intersection with V " ([10], P.15). "A submodule V of U is called closed if V has no proper essential extensions inside U " ([10], P.18). "A submodule V of U is called semi-essential if every non-zero prime submodule of U has a non-zero intersection with V " ([6]). The concept of St -closed submodules is strongly than closed submodules, where "a submodule V of U is said to be St -closed, if V has no proper semi-essential extensions inside U " ([8]). "A submodule V of U is called small in U (denoted by $V \ll U$), if for every proper submodule K of U , $V + K \neq U$ " ([10], P.20). "A submodule W is called coessential of V in U (denoted by $W \leq_{ce} V$ in U) if whenever $V/W \ll U/W$ then $V = W$ " ([14]), and "V is called coclosed in U (simply $V \leq_{cc} U$) if V has no proper coessential submodule in U " ([14]). Hadi and Ibrahiem introduced P -small submodules as an extension to the concept of small submodules, where "a proper submodule V of an R -module U is called P -small (simply $V \ll_P U$) if $V + P \neq U$ for every prime submodule P of U " ([13]). "Let U be an R -module and $W \leq V \leq U$, if $\frac{V}{W} \ll_P \frac{U}{W}$ then W is called a cosemi-essential submodule of V in U " ([5]). "An R -module U is called extending if every closed submodule of U is a direct summand of U " ([16], P.118). Ahmed and Abbas introduced the concepts "semi-

extending module" as a generalization of extending modules, where "a module U is called semi-extending if every St -closed submodule of U is a direct summand of U " ([7]).

In this paper, CSt -closed submodule and cosemi-extending modules are defined as dualizations of the classes of St -closed submodules and semi-extending module respectively. This paper consists of four sections, in section 2; we introduce the concept of CSt -closed submodule as a dualization of St -closed submodule, so we present the properties of this concept, some of them are dualizations of the results which appeared in [7, 8]. Beside, we add other useful results. Among them are the following: Let $U = U_1 \oplus U_2$ R -module, where U_1 and U_2 be R -modules and V be CSt -closed in U_1 , then V is CSt -closed in U , see Proposition 2.9. Also; Let U be a finitely generated faithful and multiplication R -module, and V be a submodule of U , then $V \leq_{CSt} U$ if and only if $[V : U] \leq_{CSt} R$, see Proposition 2.13. Furthermore, we introduce the concept of Pr -supplemented and prove in Proposition 2.22 the following: Let U be a Noetherian (or multiplication) module, and $V \leq U$. Consider the following statements:

1. V is a Pr -supplement submodule of U .
2. V is CSt -closed submodule of U .
3. *Condition**: "For each submodule X of V ; if $X \ll_P V$ then $X \ll_P U$ ". Then $(1) \Rightarrow (2) \Rightarrow (3)$, and if U is weakly Pr -supplemented then $(3) \Rightarrow (1)$.

Section 3 is devoted to dualize the concept of semi-extending module, we call it cosemi-extending modules, we give conditional characterization for cosemi-extending module, see Theorem 3.4. In addition, we show in Proposition 3.9, that every free multiplication R -module is cosemi-extending if and only if every projective R -module cosemi-extending. Moreover, we discuss the direct sum of cosemi-extending modules for example we prove that if $U = X \oplus Y$ is a duo module with $\text{ann}_R X + \text{ann}_R Y = R$, where X and Y be R -modules, then X and Y are cosemi-extending module if and only if U is cosemi-extending module, see Theorem 3.10.

Section 4; discuss the relationships between cosemi-extending and some other related concepts such as cosemi-uniform, Pr -hollow, semisimple and Pr -lifting modules, see Propositions 4.2, 4.3, and Theorem 4.8. Moreover, we discuss in 4.9 and 4.10 the relationships of a cosemi-extending module with certain kind of rings.

2. CSt -closed submodules

In this section we introduce the following concept which is a dualization of St -closed submodules.

Definition 2.1. A submodule V of U is said to be CSt -closed (simply $V \leq_{CSt} U$), if V has no proper cosemi-essential extensions inside U . That is if $V/A \ll_P U/A$, then $V = A$ for all submodules A of U contained in V .

Remarks and examples 2.2.

- i. Every R -module is CSt -closed submodule of itself.
- ii. $(\bar{0})$ may not be CSt -closed submodule of a non-zero module, for example: consider the Z -module Z_4 , $(\bar{0}) \not\leq_{CSt} Z_4$ since $(\bar{0}) \leq_{cosm} (\bar{2}) \leq Z_4$ ([5], Ex.(2.3)(1)).
- iii. Consider the Z -module of rational number Q , since Z is P -small submodule of Q , then $Z/A \ll_P Q/A$, for every submodule A with $A \leq Z \leq Q$ ([13]). Thus $Z \not\leq_{CSt} Q$.
- iv. Every CSt -closed submodule is coclosed. To show that; let $V \leq_{CSt} U$, then V has no proper cosemi-essential submodule in U , and by the direct implication between coessential and cosemi-essential submodules in [5]; V has no proper coessential submodule inside U , that is V is coclosed in U .
- v. Consider the Z_{P^∞} as Z -module. The only proper CSt -closed submodule in Z_{P^∞} is zero. In fact, Z_{P^∞} is an almost finitely generated module, where "an R -module U is called almost finitely generated if U is not finitely generated and every proper submodule of U is finitely generated" ([13]). Note that in an almost finitely generated module; every P -small is small ([13]), so if V is a submodule of Z_{P^∞} with $V \leq_{CSt} Z_{P^\infty}$, then clearly V is coclosed submodule of Z_{P^∞} . But, the only coclosed submodule in Z_{P^∞} is zero, therefore $V = 0$.
- vi. For the Z -module Z_6 ; $(\bar{2}) \leq_{CSt} Z_6$, since $(\bar{2})$ has only two submodules $(\bar{0})$ and $(\bar{2})$, and clearly $(\bar{0}) \leq_{cosm} (\bar{2})$ in Z_6 . In fact, there is a prime submodule $(\bar{3})$ in Z_6 such that that $(\bar{2})/(\bar{0}) + (\bar{3})/(\bar{0}) = Z_6/(\bar{0})$. Thus the only submodule contained $(\bar{2})$ such that $(\bar{2})/W \leq_P Z_6/W$ is $(\bar{2})$ itself, hence $(\bar{2}) \leq_{CSt} Z_6$. Note that, every submodule of Z_6 is CSt -closed.
- vii. For the Z -module Z_{12} , since the only submodule W contained in $(\bar{3})$ such that $(\bar{3})/W \ll_P Z_{12}/W$ is $(\bar{3})$ itself, thus $(\bar{3}) \leq_{CSt} Z_{12}$.
- viii. If W is cosemi-essential submodule of V in U , and V is CSt -closed in U , then $V = U$.

Proof. Since W is cosemi-essential submodule of V in U then $V/W \ll_P U/W$, but V is CSt -closed, therefore $V = W$. $\square$

Remark 2.3. A direct summand of an R -module U may not be CSt -closed submodule, for example $(\bar{0})$ is a direct summand of Z_4 , but $(\bar{0}) \not\leq_{CSt} Z_4$ as was showed in Example 2.2(ii). $\square$

"An R -module U is called multiplication, if every submodule V of U can be written in the form $V = IU$ for some ideal I of R " ([9]).

Proposition 2.4. *If a module U is multiplication (finitely generated) then every non-zero direct summand of U is CSt -closed.*

Proof. Let $U = V \oplus W$ where both of V and W be submodules of U , with $V \neq (0)$. We have to show that $V \leq_{CSt} U$. Assume that X is a submodule of V with $V/X \ll_P U/X$. Now, $U/X = V/X + (W + X)/X$. On the other hand, U is multiplication module (finitely generated), therefore $V/X \ll U/X$ ([13]). This implies that $U/X = (W + X)/X$. To complete the proof, we must show that $V = X$. Let $a \in V$, then $a + X \in U/X = (V + X)/X$. So $a + X = b + X$ for some $b \in W$. This implies that $a - b \in X$, hence $a - b = c$ for some $c \in X$. Therefore, $a - c = b \in W \cap V = 0$, thus $a = c$, hence $a \in X$. So $V = X$, that is $V \leq_{CSt} U$. $\square$

Corollary 2.5. *If U is a multiplication (finitely generated) module, then the concept of CSt -closed submodules coincide with the coclosed submodules.*

Proof. The proof follows by the equivalence between "small" and "P-small" submodules in the class of multiplication (finitely generated) modules ([13] Prop.(1.4)). $\square$

Proposition 2.6. *Let U be an R -module, and X, Y be submodules of U such that $X \leq Y \leq U$, then If X is CSt -closed submodule in U , then X is CSt -closed submodule in Y .*

Proof. Suppose that V is a submodule of U with $X \leq_{cosm} V \leq Y$. So that $X \leq_{cosm} V \leq U$. Since X is CSt -closed submodule of U , then $X = V$. $\square$

Corollary 2.7. *For any submodules X and Y of an R -module U , the following are hold:*

1. *If $X \cap Y \leq_{CSt} U$, then $X \cap Y \leq_{CSt} X$ and $X \cap Y \leq_{CSt} Y$.*
2. *If $X \leq_{CSt} U$ and $Y \leq_{CSt} U$, then $X \leq_{CSt} X + Y$ (also $Y \leq_{CSt} X + Y$). The proof of (1) and (2) follows directly by Proposition 2.6.*
3. *If V is a CSt -closed submodule of U , and W is a submodule of U such that $V \cong W$, then it is not necessary that W is CSt -closed in U . For example, the Z -module Z is CSt -closed in itself, and $Z \cong 2Z$, but $2Z$ is not CSt -closed submodule in Z , since $4Z$ is a cosemi-essential submodule of $2Z$ in Z .*

Remark 2.8. A submodule of CSt -closed need not be St -closed. That is if Y is CSt -closed submodule of U and $X \leq Y$, then X may not be CSt -closed submodule of U (or Y). In fact for the Z -module Z , if $U = Y = Z$ and $X = 4Z$, then $Z \leq_{CSt} Z$ but $4Z \leq_{CSt} Z$, since $2Z/4Z \ll_P Z/4Z$.

Proposition 2.9. *Let $U = U_1 \oplus U_2$, where U_1 and U_2 be R -modules and V be CSt -closed in U_1 , then V is CSt -closed in U .*

Proof. Suppose that V is not CSt -closed in U , so there exists a proper submodule L of V such that $V/L \ll_P U/L$. Let $f : U/L \rightarrow U_1/L$ be a projection epimorphism defined by $f(u_1 + u_2 + L) = u_1 + L$ where $u_1 \in U_1$ and $u_2 \in U_2$. Since $V/L \ll_P U/L$, then $f(V/L) = V/L \ll_P U_1/L$ ([13], Prop.(1.3)). But V is CSt -closed in U_1 , therefore $V = L$ which is a contradiction, since L is a proper submodule of V . Thus, V is CSt -closed of U . $\square$

Proposition 2.10. *Every submodule of semisimple module is CSt -closed.*

Proof. Since a semisimple module has no cosemi-essential ([5], Rem.(2.3)(9)), then we are done. $\square$

"Recall that an R -module U is called *Pr*-hollow if each prime submodule of U is small" ([3]).

Remark 2.11. The only proper CSt -closed submodule of *Pr*-hollow module is a zero submodule.

Proof. Let U be a *Pr*-hollow module, and V be a proper CSt -closed in U . Since U is *Pr*-hollow, then $V \ll_P U$, hence $V/(0) \ll_P U/(0)$. But V is CSt -closed, thus $V = (0)$. $\square$

We need to give the following Lemmas.

Lemma 2.12. *Let U be a finitely generated faithful and multiplication R -module, and let V be a submodule of U . Then $V \ll_P U$ if and only if $[V : U] \ll_P R$.*

Proof. Assume that $V \ll_P U$, and I be an ideal of R with $I + [I : U] = R$, then $(I + [V : U])U = IU + [V : U]U = IU + V = U$. Since U is finitely generated, then $V \ll_P U$ implies to $V \ll U$ ([13]), so we deduce that $IU = U = RU$. Since U is finitely generated multiplication and faithful, then $I = R$ ([9]), hence $[V : U] \ll_P R$. Conversely, Suppose that $[V : U] \ll_P R$. Let W be a submodule of U with $V + W = U$. Since U is multiplication, then $[V : U]U + [W : U]U = U$, hence $([V : U] + [W : U])U = RU$. Since U is finitely generated and multiplication, so $[V : U] + [W : U] = R$ ([9]). On the other hand, since $[V : U] \ll_P R$, and R is finitely generated R -module, then $[V : U] \ll R$ ([13]). Thus $[W : U] = R$. Now, $W = [W : U]U = RU = U$, implies to $V \ll U$. But U is a finitely generated module thus $V \ll_P U$ ([13], Prop.(1.4)). $\square$

Lemma 2.13. *Let U be a finitely generated and multiplication R -module, and $W \leq V \leq U$, then $V/W \ll_P U/W$ if and only if $[V/W : U/W] \ll_P R/([W : U])$, where $[V/W : U/W]$ considered as an ideal of $R/([W : U])$.*

Proof. We can easily show that U/W is finitely generated faithful and multiplication $R/[W : U]$ module. So U/W satisfies the conditions of Lemma 2.11, hence the result follows. $\square$

Proposition 2.14. *Let U be a finitely generated faithful and multiplication R -module, and let V be a submodule of U . Then $V \leq_{CSt} U$ if and only if $[V : U] \leq_{CSt} R$.*

Proof. Assume that $V \leq_{St} U$, and let $I[V : U]R$ with $([V : U])/I \ll_P R/I$. Since $[IU : U]U = I$, then $([V : U])/([IU : U]) \ll_P R/([IU : U])$. We can easily show that:

$$([V : U])/([IU : U]) = ([V : U]U)/([IU : U]U) : U/([IU : U]U).$$

Thus:

$$([V : U]U)/([IU : U]U) : U/([IU : U]U) \ll_P R/([IU : U]).$$

Hence:

$$V = [V : U]U = [IU : U]U = IU.$$

But $V \leq_{CSt} U$, therefore $[V : U] = I$. Conversely, let W be a submodule of V such that $V/W \ll_P U/W$. By Lemma 2.12:

$$[V/W : U/W] \ll_P R/([W : U]).$$

It is clear that:

$$[V/W : U/W] = ([V : U])/([W : U]).$$

This implies that:

$$([V : U])/([W : U]) \ll_P R/([W : U]).$$

Since $[V : U] \leq_P R$, then $[V : U] = [W : U]$. Thus $V = W$, that is $V \leq_{CSt} U$. $\square$

Now, we can give the following result.

Proposition 2.15. *If V is CSt -closed submodule of an R -module U , then V/L is CSt -closed in U/L for any submodule L of V .*

Proof. Let L be a submodule of U , and W be a submodule of V containing L such that $\frac{V/L}{W/L} \ll_P \frac{U/L}{W/L}$. So that $V/W \ll_P U/W$. Since V is CSt -closed of U , then $V = W$, hence $V/L = W/L$, and we are done.

It is known that if $X \leq V \leq U$ with $X \ll_P U$ then X may not be P -small submodule of V ([13]), so we have the following.

Proposition 2.16. *Let V be a CSt -closed submodule of an R -module U . For each submodule X of V with $X \leq V \leq U$; if $X \ll_P U$ then $X \ll_P V$.*

Proof. Let X be a submodule of V with $X \ll_P U$. Suppose that $V = X + Y$ for some prime submodule Y of V . We claim that $V/Y \ll_P U/Y$. To prove this; assume that $V/Y + W/Y = U/Y$ where W/Y is a prime submodule of U/Y . This implies that W is prime submodule of U ([15]). On the other hand, $V + W = U$, hence $U = X + Y + W = X + W$. So $U = X + W$ which is a contradiction since $X \ll_P U$, thus $U \neq X + W$, hence $U/Y \neq X/Y + W/Y$, thus $V/Y \ll_P U/Y$. Now, we have $V \leq_{CSt} U$ and $V/Y \ll_P U/Y$, then $V = Y$, which is a contradiction since Y is proper, therefore $V \neq X + Y$, that is $X \ll_P V$. $\square$

From Proposition 2.16, we conclude the following.

Corollary 2.17. *Let X and V be submodules of an R -module U such that $X \leq V \leq U$. If $X \leq_{CSt} V$ and $V \leq_{CSt} U$ then $X \leq_{CSt} U$.*

Proof. Let $W \leq X$ with $X/W \ll_P U/W$. Since $V \leq_{CSt} U$, then by Proposition 2.14, $V/W \leq_{CSt} U/W$, and by Proposition 2.15, $X/W \ll_P V/W$. But $X \leq_{CSt} V$, therefore $X = W$, that is $X \leq_{CSt} U$. $\square$

"Recall that a submodule V of an R -module is called supplement (weakly supplement) of W in U , if V is minimal with the property $U = W + V$ equivalently, $U = W + V$ and $W \cap V \ll V$ (resp. $W \cap V \ll U$)" ([14]). "A submodule V of U is called a supplement submodule of U if V is a supplement of some submodule of U , and an R -module U is called supplemented (weakly supplemented) if every submodule of U has a supplement (resp. weakly supplement) in U " ([14]). Now we need to give generalizations for these classes.

Definition 2.18. *A submodule V of an R -module U is said to be Pr -supplement (weakly Pr -supplement) of W in U if $U = W + V$ and $W \cap V \ll_P V$ (resp. $W \cap V \ll_P U$). A submodule V of U is said to be Pr -supplement (weakly Pr -supplement) submodule of U if V is a Pr -supplement (weakly Pr -supplement) of some submodule of U . An R -module U is called Pr -supplemented (weakly Pr -supplemented) if every submodule of U has a Pr -supplement (weakly Pr -supplement) in U .*

It is clear that every supplement is Pr -supplement submodule.

Examples 2.19.

- i. Z_4 is a Pr -supplemented module, since every submodule of Z_4 has a Pr -supplement.
- ii. Z_{P^∞} is Pr -supplemented, since Z_{P^∞} is a Pr -supplement of every proper submodule of the Z -module Z_{P^∞} .
- iii. Z is not Pr -supplemented, since $2Z$ has no Pr -supplement submodule in Z .

"An R -module U is called Noetherian if every submodule of U is finitely generated" ([10], P.7).

Proposition 2.20. *If U be a Noetherian module, then every Pr -supplement submodule of U is CSt -closed submodule of U .*

Proof. Let V be a Pr -supplement of a submodule L in U , then $U = V + L$ and $V \cap L \ll_P V$. Let $W \leq V \leq U$ such that $V/W \ll_P U/W$. Since U is Noetherian, then $V/W \ll U/W$ ([13], Prop.(1.7)), We can write U/W as follows:

$$U/W = (V + L)/W = V/W + (L + W)/W.$$

This implies that:

$$U/W = (L + W)/W$$

hence $U = L + W$, and by minimality of V , we conclude that $W = V$, that is $V \leq_{CSt} U$. $\square$

Remark 2.21. If we replace the condition "Noetherian" in Proposition 2.20 by "finitely generated" or "multiplication", then we deduce the same result, since Hadi and Ibrahim in ([13], Prop.(1.4)) showed that under these conditions the concepts small and P -small submodules are equivalent.

From Propositions 2.16 and 2.20, we have the following

Theorem 2.22. *Let U be a Noetherian (or multiplication) module, and $V \leq U$. Consider the following statements:*

1. V is a Pr -supplement submodule of U .
2. V is CSt -closed submodule of U .
3. Condition*: "For each submodule X of V ; if $X \ll_P U$ then $X \ll_P V$ ".

Then $(1) \Rightarrow (2) \Rightarrow (3)$, and if U is weakly Pr -supplemented then $(3) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ It is just Proposition 2.20.

$(2) \Rightarrow (3)$ Proposition 2.16.

$(3) \Rightarrow (1)$ Suppose the condition*, since U is weakly Pr -supplemented module, then there exists a submodule L of U such that $U = V + L$ and $V \cap L \ll_P U$. But $V \cap L \subseteq V$, so by assumption $V \cap L \ll_P V$, hence V is Pr -supplement of U . $\square$

We need the following lemma.

Lemma 2.23 ([5], Prop. (2.4)). "For a chained of submodules $A \leq B \leq C \leq U$ of an R -module U ; if $A \leq_{cosm} B$ in U and $B \leq_{ce} C$ in U then $A \leq_{cosm} C$ in U ".

Proposition 2.24. *Let U be a multiplication (finitely generated) module. For a chain of a submodules $A \leq B \leq C \leq U$, if $A \leq_{cosm} B$ in U and $B \leq_{cosm} C$ in U , then $A \leq_{cosm} C$ in U .*

Proof. The expression $B \leq_{cosm} C$ in U , means $C/B \ll_P U/B$. Since U is multiplication (finitely generated) then $C/B \ll U/B$ ([13], Prop. (1.4)), hence $B \leq_{ce} C$ in U . Now, we have $A \leq_{cosm} B$ in U and $B \leq_{ce} C$ in U , by Lemma 2.23 we get $A \leq_{cosm} C$ in U . $\square$

Proposition 2.25. *Let U be a multiplication (finitely generated) R -module, then for every non-zero submodule V of U , there exists a CSt -closed submodule K of U with $V \leq_{cosm} K$ in U .*

Proof. Consider the set: $F = \{A \mid A \text{ is a submodule of } U \text{ such that } V \leq_{\text{cosm}} A\}$. Note that $F \neq \Phi$, so by Zorn's Lemma, F has a maximal element say K . In order to prove that K is an CSt -closed submodule in U ; assume that there exists a submodule X of U such that $K \leq_{\text{cosm}} X \leq U$. Since $V \leq_{\text{cosm}} K$ and $K \leq_{\text{cosm}} X$, and U is multiplication so by Proposition 2.24, $V \leq_{\text{cosm}} X$. But this contradicts the maximality of K , thus $K = X$. Therefore $K \leq_{CSt} U$ with $V \leq_{\text{cosm}} K$ in U . $\square$

Proposition 2.26. *Let $U = X \oplus Y$ where X and Y be two R -modules and $\text{ann}_R X + \text{ann}_R Y = R$. Assume that $C = A \oplus B$ where $A \leq X$ and $B \leq Y$. If $C \leq_{CSt} U$ then $A \leq_{CSt} X$ and $B \leq_{CSt} Y$.*

Proof. Suppose that $(A/S) \ll_P (X/S)$ and $(B/W) \ll_P (Y/W)$, where $S \leq A$ and $W \leq B$. Since $\text{ann}_R X + \text{ann}_R Y = R$, then by [13]:

$$(A/S) \oplus (B/W) \ll_P (X/W) \oplus (Y/W).$$

Hence:

$$(A \oplus B)/(S \oplus W) \ll_P (X \oplus Y)/(S \oplus W)$$

and so that:

$$(C/S \oplus W) \ll_P (U/S \oplus W).$$

But $C \leq_{CSt} U$, therefore $C = S \oplus W$. This implies that $(A \oplus B) = (S \oplus W)$, but $S \leq A$ and $W \leq B$, therefore $A = S$ and $B = W$. Thus $A \leq_{CSt} X$ and $B \leq_{CSt} Y$. $\square$

Proposition 2.27. *Let U be a multiplication module with $U = X \oplus Y$ where X and Y be two R -modules and $\text{ann}_R X + \text{ann}_R Y = R$. Then a submodule $C \leq_{CSt} U$ if and only if there exist CSt -closed submodules A, B of X and Y respectively such that $C = A \oplus B$.*

Proof. Assume that $C \leq_{CSt} U$. since $\text{ann}_R X + \text{ann}_R Y = R$, then by ([1], Prop.(4.2)) there exist submodules A, B of X and Y respectively such that $C = A \oplus B$. By Proposition 2.26 both of A and B are CSt -closed submodules X and Y respectively. Conversely, in order to prove that $C \leq_{CSt} U$; suppose that $C/W \ll_P U/W$, where $W \leq C \leq U$. Since $\text{ann}_R X + \text{ann}_R Y = R$, so by the same proof of ([1], Prop(4.2)), there exist $W_1 \leq X$ and $W_2 \leq Y$ such that $W = W_1 \oplus W_2$. Now,

$$C/W = (A \oplus B)/(W_1 + W_2) \ll_P (X \oplus Y)/(W_1 + W_2)$$

implies to:

$$A/W_1 \oplus B/W_2 \ll_P X/W_1 \oplus Y/W_2.$$

But U is multiplication, thus by [13]:

$$A/W_1 \oplus B/W_2 \ll X/W_1 \oplus Y/W_2.$$

Hence:

$$A/W_1 \ll X/W_1$$

and $B/W_2 \ll Y/W_2$ ([10], P.20) Again, U is multiplication, so that:

$$A/W_1 \ll_P X/W_1$$

and

$$B/W_2 \ll_P Y/W_2.$$

Since A and B are CSt -closed submodules of X and Y respectively, thus $A = W_1$ and $B = W_2$, and hence $C = A \oplus B$, thus $C = W_1 \oplus W_2 = W$, hence $C \leq_{CSt} U$. $\square$

3. Cosemi-extending module

In this section, we dualize the concept of semi-extending modules which is appeared in [7]. We start by the following definition.

Definition 3.1. *An R -module U is called cosemi-extending if every CSt -closed submodule of U is direct summand.*

Remarks and examples 3.2.

- i. It is clear that every coextending module is cosemi-extending, where "an R -module U is called coextending (or CCS -module), if every coclosed submodule of U is a direct summand of U " ([12]). This is follows from the direct implication between CSt -closed and coclosed submodules.
- ii. Z is cosemi-extending Z -module. In fact, the only CSt -closed submodule of Z is (0) which is a direct summand of Z .
- iii. Hollow modules is cosemi-extending, since it is coextending ([12]). In particular Z_{P^∞} is cosemi-extending module.
- iv. $Z_2 \oplus Z_4$ is a cosemi-extending Z -module, since it is coextending ([12]).
- v. Every simple module is cosemi-uniform.

Proposition 3.3. *Let U be an R -module, if every submodule of U is cosemi-essential in a direct summand of U , then U is a cosemi-extending module.*

Proof. Suppose that V is a CSt -closed submodule of U . By hypothesis, $V \leq_{cosm} K$ in U , where K is a direct summand of U . But V is CSt -closed in U , thus $V = K$, that is V is a direct summand of U . $\square$

By using Proposition 2.25, the following gives a partial characterization of cosemi-extending module.

Theorem 3.4. *A multiplication (finitely generated) module U is cosemi-extending if and only if every submodule is cosemi-essential in a direct summand of U .*

Proof. Assume that U is a cosemi-extending module, and let V be a submodule of U . In case $V = (0)$, then clearly V is a cosemi-essential submodule in a direct summand of U . Otherwise; since U is multiplication (finitely generated) so by Proposition 2.25, there exists an St -closed submodule K in U such that $V \leq_{cosm} K$ in U . By hypotheses, K is a direct summand of U , therefore $V \leq_{cosm} K$ in U . The converse is just Proposition 3.3. $\square$

Proposition 3.5. *Let U be a multiplication (finitely generated) and cosemi-extending module. For every submodules X and Y of U . If $X \cap Y \leq_{CSt} U$, then $X \cap Y$ is a direct summand of X and Y .*

Proof. We have to show that $X \cap Y$ is a direct summand of X . Since U is a multiplication (finitely generated) and cosemi-extending module, then by Theorem 3.4, $X \cap Y$ is cosemi-essential in a direct summand of U . But $X \cap Y \leq X \leq U$, so clearly $X \cap Y$ is a direct summand of X . $\square$

Proposition 3.6. *A direct summand of multiplication cosemi-extending module is a cosemi-extending module.*

Proof. Let U be an R -module, and V be a direct summand of U . Assume that K is a CSt -closed submodule of V . Since U is multiplication and V is a direct summand of U , then by Proposition 2.4, V is a CSt -closed submodule of U , and by Corollary 2.16, K is a CSt -closed submodule of U . But U is cosemi-extending, then $U = L \oplus K$ for some submodule L of U . Now, $V = U \cap V = (K \oplus L) \cap V = K \oplus (L \cap V)$ by Modular Law. Thus K is a direct summand of V , that is, i.e V is a cosemi-extending module. $\square$

As a consequence of Proposition 3.6 we have the following. Before that, an R -module U is said to be projective if every short exact sequence of the form:

$$(0) \rightarrow W \rightarrow V \rightarrow U \rightarrow (0)$$

splits ([16], P.23).

Corollary 3.7. *Let $f : U_1 \rightarrow U_2$ be an epimorphism from an R -module U_1 to a projective R -module U_2 . If U_1 is a multiplication and cosemi-extending module, then U_2 is cosemi-extending.*

Proof. Consider the following short exact sequence:

$$(0) \rightarrow \ker f \xrightarrow{i} U_1 \xrightarrow{f} U_2 \rightarrow (0),$$

where i is the inclusion homomorphism. Since U_2 is a projective, then the sequence splits. This implies that $U_1 \cong \ker f \oplus U_2$, so U_2 is isomorphic to a direct summand of U_1 . Since U_1 is multiplication and cosemi-extending, so by proposition 3.6, U_2 is cosemi-extending. $\square$

Corollary 3.8. *Let U be a multiplication and cosemi-extending R -module, and V is a CSt -closed submodule of U , then U/V is cosemi-extending.*

Proof. Since V is a CSt -closed submodule of U , and U is cosemi-extending, then V is a direct summand of U , so $U = V \oplus L$ for some submodule W of U . This implies that $U/V \cong L$. But L is a direct summand of U , so by Proposition 3.6, U/V is cosemi-extending. $\square$

Recall that an R -module U is called free if it has a basis ([16], P.21).

Proposition 3.9. *Let U be a multiplication R -module, then every free R -module is cosemi-extending if and only if every projective R -module is cosemi-extending.*

Proof. For the necessity; let U be a projective R -module, then U is an epimorphic image of a free R -module say F ([16], P.23). By the hypothesis, F is a cosemi-extending module. But U is multiplication, then by Corollary 3.7, U is cosemi-extending. The converse is straightforward. $\square$

"Recall that an R -module U is called duo, if every submodule of U is a fully invariant" ([2]). The following theorem deals with the direct sum of two cosemi-extending modules.

Theorem 3.10. *Let $U = X \oplus Y$ be a duo module, where X and Y are R -modules. Assume that $\text{ann}_R X + \text{ann}_R Y = R$. If X and Y are cosemi-extending module then U is cosemi-extending. the converse is true when U is multiplication*

Proof. Assume that X and Y are cosemi-extending, and let $V \leq_{CSt} U$. Since U is a duo module, then V is fully invariant, hence $V = (V \cap X) \oplus (V \cap Y)$ ([2]). By Proposition 2.8(1), $V \cap X \leq_{CSt} X$ and $V \cap Y \leq_{CSt} Y$ respectively. But X and Y are cosemi-extending, so $V \cap X \oplus S = X$ and $V \cap Y \oplus T = Y$ for some $S \leq X$ and $T \leq Y$. This implies that:

$$X \oplus Y = (V \cap X \oplus S) \oplus (V \cap Y \oplus T),$$

$$U = [(V \cap X) \oplus (V \cap Y)] \oplus (S \oplus T).$$

If we put $W = S \oplus T$, then $U = V \oplus W$, thus V is a direct summand of U , hence U is cosemi-extending. For the converse; since U is multiplication, then the result follows by Proposition 3.6. $\square$

Since every multiplication module is duo, then from Theorem 3.10 we deduce the following.

Corollary 3.11. *Let U be a multiplication module such that $U = X \oplus Y$, where X and Y be R -modules. If $\text{ann}_R X + \text{ann}_R Y = R$, then U is a cosemi-extending module if and only if X and Y are cosemi-extending.*

4. Cosemi-extending module related concepts

This section deals with the relationships between cosemi-extending module and other related concepts such as semisimple, Pr -hollow module, cosemi-uniform, Pr -lifting and St -semisimple modules.

Remark 4.1. Every semisimple module is cosemi-extending. This follows by every submodule of semisimple module is CSt -closed. The converse is not true in general, for example: Z_{12} is cosemi-extending Z -module, but it is not semisimple.

"An R -module U is said to be Pr -hollow if every prime submodule of U is small submodule" ([3]).

Proposition 4.2. *Every Pr -hollow module is cosemi-extending module.*

Proof. Assume that U is a Pr -hollow module. By Remark 2.10; the only CSt -closed submodule of Pr -hollow module is zero, hence the result follows. $\square$

The converse of Proposition 4.2 is not true in general, for example; Z is a cosemi-extending Z -module, but not Pr -hollow module ([3], (1.2)(2)).

"Recall that a non-zero R -module U is called cosemi-uniform, if every proper submodule V of U is either zero or there exists a proper submodule S of V such that $V/S \ll_P U/S$ " ([5]).

Proposition 4.3. *Every cosemi-uniform module is cosemi-extending module.*

Proof. Let U be a cosemi-extending module, and V be a submodule of U . If $V = (0)$ then either V is CSt -closed or not, in each case V is a direct summand of U . If $V \neq (0)$, since U is a cosemi-uniform module so there exists a proper W of V such that $V/W \ll_P U/W$, therefore, V is not CSt -closed in U . So U has a non-zero CSt -closed submodule, thus U is cosemi-extending.

The converse of Proposition 4.3 is not true in general, for example the Z -module Z_{10} is a cosemi-extending module because it is a semisimple module, but not a cosemi-uniform module, see ([5], Rem (3.2)(3)).

"A non-zero module U is called couniform, if every proper submodule V of U is either zero or there exists a proper submodule W of V such that $V/W \ll U/W$. That is for each proper submodule V of U , either $N = (0)$ or there exists a proper submodule W of V such that $W \leq_{ce} V$ in U " ([11]).

Since every couniform module is cosemi-uniform, then we have the following.

Corollary 4.4. *Every couniform module is cosemi-extending.*

The converse of Corollary 4.4 is not true in general. In fact the Z -module Z_6 is not couniform module ([11], Rem.(1.2)(2)), while Z_6 is cosemi-extending because it is semisimple.

"Recall that a module U is called lifting, if for every submodule V of U there exists a direct summand W of U such that $W \leq_{ce} V$ in U " ([14]). This motivated us to define the following.

Definition 4.5. An R -module U is called *Pr-lifting*, if for every submodule V of U there exists a direct summand W of U such that $W \leq_{cosm} V$ in U .

This concept is clearly a proper subclass of lifting module, and we can prove the following.

Proposition 4.6. If a module U is a *Pr-lifting* module, then U is cosemi-extending.

Proof. Let V be a CSt -closed submodule of U . Since U is a *Pr-lifting* module, so there exists a direct summand W of U , such that $W \leq_{cosm} V$ in U , that is $V/W \ll_P U/W$. But $V \leq_{CSt} U$, then $V = W$. That is U is a cosemi-extending module. $\square$

The converse of Proposition 4.6 is not true in general, for example: Z as Z -module is a cosemi-extending module, but it is not *Pr-lifting*.

"An R -module U is called *St-semisimple* if every submodule of U is *St-closed*" ([4]). As a dual of this concept, we introduce the following.

Definition 4.7. An R -module U is called *CSt-semisimple*, if every submodule of U is *CSt-closed*.

The following theorem gives some useful relationships of a cosemi-essential module with some related concepts.

Theorem 4.8. If U is *CSt-semisimple*, then the following statements are equivalent.

1. U is a *Pr-lifting* module.
2. U is a cosemi-extending module.
3. U is a semisimple module.

Proof. (1) $\Rightarrow$ (2) It is just Proposition 4.6.

(2) $\Rightarrow$ (3) Let V be a submodule of U , since U is *CSt-semisimple*, then $V \leq_{CSt} U$. But U is cosemi-extending, therefore V is a direct summand of U .

(3) $\Rightarrow$ (1) Let V be a submodule of U , by (3), V is a direct summand of U . On the other hand, $V \leq_{cosm} V$ in U ([5], Rem (2.3)(6)). So U satisfies the definition of *Pr-lifting*, and we are done. $\square$

Hadi and Ibrahim in [13] defined $P\text{-Rad}(U)$ as a summation of all P -small submodules of U , so we have the following.

Theorem 4.9. Let R be a ring such that $P - \text{Rad}(R/A) = 0$, for each ideal A of R . Then every R -module is a cosemi-extending module if and only if R is a semisimple ring.

Proof. Assume that every R -module is cosemi-extending, then R is a cosemi-extending R -module. So if I is a CSt -closed ideal of R , then I is a direct summand of R . Assume that I is not CSt -closed ideal in R , then there exists a proper ideal A of I such that $I/A \ll_P R/A$. This implies that $I/A \leq P - \text{Rad}(R/A) = 0$, thus $I/A = 0$, hence $A = I$ which a contradiction, therefore $I \leq_{CSt} R$. Since R is cosemi-extending R -module, then I is a direct summand of R . i.e R is a semisimple ring. Conversely; since R is a semisimple ring, then every R -module U is semisimple, hence every submodule of U is direct summand. In particular; every CSt -closed submodule of U is direct summand. $\square$

If the condition "semisimple ring" in Theorem 4.9 is replaced by "St-semisimple", then we need to add another condition as the following theorem shows.

Theorem 4.10. *Let R be a ring such that $P - \text{Rad}(R/A) = 0$, for each ideal A of R . Then every finitely generated R -module is a cosemi-extending module if and only if R is an St-semisimple ring.*

Proof. $\Rightarrow$) It is as the same proof of Theorem 4.9.

$\Leftarrow$) Since R is finitely generated, then the concept of a CSt -closed submodule coincide with coclosed submodule, see Corollary 2.5, and according that, there is no difference between St-semisimple and semisimple modules, thus by Theorem 4.9, every R -module (hence every finitely generated) is cosemi-extending. $\square$

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Computation of edge metric dimension of barycentric subdivision of Cayley graphs

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Abstract. Let $G = (V, E)$ be a connected graph, let $x \in V(G)$ be a vertex and $e = yz \in E(G)$ be an edge. The distance between the vertex x and the edge e is given by $d_G(x, e) = \min\{d_G(x, y), d_G(x, z)\}$. A vertex $t \in V(G)$ distinguishes two edges $e, f \in E(G)$ if $d_G(t, e) \neq d_G(t, f)$. A set $R \subseteq V(G)$ is an edge metric generator for G if every two edges of G are distinguished by some vertex of R . The minimum cardinality of R is called the edge metric dimension and is denoted by $edim(G)$. In this paper, we compute the edge metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.

Keywords: metric dimension, edge metric dimension, resolving set, barycentric subdivision, Cayley graph

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1. Introduction

The concept of metric dimension was introduced by Slater [1] and studied independently by Harary and Melter [2]. This problem has been investigated widely since then. The metric dimension has a lot of applications in different areas of science and technology. The concept of the edge metric dimension is a recent advancement in this line of research. Next we reveal some of the applications of metric dimension in various subjects. The metric dimension arises in many diverse areas, including navigation of robots [3], telecommunications [4], combinatorial optimization [5] and sonar and coast guard Loran [1] and applications to chemistry in [6, 7, 8]. Furthermore, this topic has some applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [9]. Metric dimension of several interesting classes of graphs can be seen in [10, 11, 12, 13, 14, 15, 16, 17]. Let $G = (V, E)$ be a simple and connected graph. A vertex $x \in V(G)$ distinguishes two vertices $y, z \in V(G)$ if $d_G(y, x) \neq d_G(z, x)$, where $d_G(x, y)$ denotes the length of the shortest path between the vertices x and y in G . The minimum cardinality of any metric generator for G is the metric dimension of G , denoted by $\dim(G)$. Let $R_1 = \{r_1, r_2, \dots, r_s\}$ be an ordered set of vertices of G and let $x \in V(G)$, then the *representation* $r(x|R_1)$ of x with respect to R_1 is the s -tuple $(d_G(x, r_1), d_G(x, r_2), d_G(x, r_3), \dots, d_G(x, r_s))$. Since the set R_1 has the minimum cardinality, therefore this is also known as the basis of G , and its cardinality is called the metric dimension or location number [18].

Similarly, let $x \in V(G)$ be a vertex and $e = yz \in E(G)$ be an edge. The distance between the vertex x and the edge e is given by $d_G(x, e) = \min\{d_G(x, y), d_G(x, z)\}$. A vertex $t \in V(G)$ distinguishes two edges $e, f \in E(G)$ if $d_G(t, e) \neq d_G(t, f)$. A set $R \subseteq V(G)$ is an edge metric generator for G if every two edges of G are distinguished by some vertex of R . The minimum cardinality of R is called the edge metric dimension and is denoted by $\text{edim}(G)$ [19]. Let $R = \{r_1, r_2, \dots, r_t\}$ be an ordered set of vertices of G and let $e \in E(G)$, then the *representation* $r(e|R)$ of e with respect to R is the t -tuple $(d_G(e, r_1), d_G(e, r_2), d_G(e, r_3), \dots, d_G(e, r_t))$. In addition, combined (mixed) form of these two parameters depicted above is of fascinate. A vertex $x \in V(G)$ distinguishes two elements (vertices or edges) $u, v \in V(G) \cup E(G)$ if $d_G(x, u) \neq d_G(x, v)$. A set $R^m \subseteq V(G)$ is a mixed metric generator for G if every two distinct elements (vertices or edges) of G are distinguished by some vertex of R^m . The smallest cardinality of R^m is the mixed metric dimension and is denoted by $\text{mdim}(G)$ [20].

Geometrically, an operation that splits an edge into two edges by inserting a new vertex into the interior of an edge is known as subdividing an edge. If we are performing a sequence of edge-subdivision operations, then it is called *Subdividing a graph G* and resulting graph is called a *subdivision of the graph G* . The subdivision of graph can be used to convert a general graph into a simple graph. If we subdividing each edge of the graph G , then this subdivision is

called the *barycentric subdivision* of G . Gross and Yellen [21] proved the results that the barycentric subdivision of any graph is a simple and bipartite graph. A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivision of graphs play a very important role in characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be homeomorphic if they are subdivisions of the same graph G . The next theorem gives a nice characterization of planar graphs.

Theorem 1.1. [21] *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

In this paper, we study the edge metric dimension of barycentric subdivision of Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. We prove that these subdivisions of Cayley graphs have constant edge metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.

2. Results and discussions

As expressed, there are a several graphs in which metric generator and edge metric generator are the same. In this sense, one could believe that most likely any edge metric generator R is likewise a standard metric generator. In any case, this is again further far from the truth, despite the fact that there are a few families of graphs in which such actualities happen. Kelenc *et al.* [19] explained some comparison between the edge metric generator and standard metric generator in details. We show a few results concerning the edge metric dimension of graphs. The first importance result about the complexity is as follows:

Theorem 2.1 ([19]). *Computing the edge metric dimension of graphs is NP-hard.*

The edge metric dimension of Cartesian product of two paths P_r and P_t with r and t vertices is determined in the following proposition.

Proposition 2.2 ([19]). *Let G be the grid graph $G = P_r \square P_t$, with $r \geq t \geq 2$. Then $\text{edim}(G) = \dim(G) = 2$.*

Kelenc *et al.* [19] proved in the next proposition the edge metric dimension of wheel graphs and observe it is strictly larger than the value for the metric dimension, except in the case $W_{1,3}$. The wheel graph $W_{1,n}$ is the graph obtained from a cycle $C_n, n \geq 3$ by joining all vertices of C_n to an additional vertex. In [19], they determined the edge metric dimension of wheel graph $W_{1,n}$ in the following proposition:

Proposition 2.3 ([19]). *Let $W_{1,n}$ be a wheel graph. Then*

$$\text{edim}(W_{1,n}) = \begin{cases} n, & \text{for } n = 3, 4 \\ n - 1, & \text{for } n \geq 5 \end{cases}$$

The fan graph $F_{1,n}$ is the graph obtained by joining each vertices of a path P_n , to an additional vertex. In the next proposition, the edge metric dimension of fan graph $F_{1,n}$ is determined.

Proposition 2.4 ([19]). *Let $F_{1,n}$ be a fan graph. Then*

$$\text{edim}(F_{1,n}) = \begin{cases} n, & \text{for } n = 1, 2, 3 \\ n - 1, & \text{for } n \geq 4 \end{cases}$$

Kelenc *et al.* [19] also determined the edge metric dimension of path, cycle, complete graph, complete bipartite, cartesian product of cycles and bounds for some families of graphs.

3. The edge metric dimension of barycentric subdivision of Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$

Let G be a semigroup, and let H be a nonempty subset of G . The Cayley graph $\text{Cay}(G, H)$ of G relative to H is defined as the graph with vertex set G and edge set $E(H)$ consisting of those ordered pairs (a, b) such that $ha = b$ for some $h \in H$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $\text{Cay}(G, H)$ of a group G is symmetric or undirected if and only if $H = H^{-1}$.

The Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$, $n \geq 3$, is a 3-regular graph which is also known as the cartesian product $C_n \square P_2$ of a cycle of order n with a path of order 2. The Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$, $n \geq 3$ consists of an inner n -cycle $a_1 a_2 a_3 \dots a_n$, an outer n -cycle $x_1 x_2 x_3 \dots x_n$ and n spokes $a_i x_i$, $1 \leq i \leq n$. This implies that the order and size of $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ is $2n$ and $3n$, respectively. The metric dimension of Cayley graphs $\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has been determined in [14] while the metric dimension of Cayley graphs $\text{Cay}(\mathbb{Z}_n : H)$ for all $n \geq 7$ and $H = \{\pm 1, \pm 3\}$ has been determined in [22].

The barycentric subdivision graph $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ can be obtained by splitting edges $a_i a_{i+1}$ by inserting a new vertices b_i , splitting edges $a_i x_i$ by inserting a new vertices c_i and splitting edges $x_i x_{i+1}$ by inserting a new vertices y_i . From this we observe that, $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ contains $5n$ vertices among of these $3n$ vertices of degree 2 and $2n$ vertices of degree 3 and $6n$ edges. In the next theorem, we prove that the edge metric dimension of the barycentric subdivision of $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$.

Theorem 3.1. *Let $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ be the barycentric subdivision of Cayley graphs $(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$. Then $\text{edim}(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities.

Case 1. When n is even.

Let $R = \{a_1, a_{\frac{n}{2}+1}, a_n\} \subset V(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$. We show that R is a resolving set for $BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ in this case. For this we give representations of any edge of $E(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ with respect to R .

Representations for the edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_i b_i | R) = \begin{cases} (2i - 2, n - 2i + 1, 2i), & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ (n - 2, 1, n - 1), & \text{for } i = \frac{n}{2} \\ (2n - 2i + 1, 2i - n - 2, 2n - 2i - 1), & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1 \\ (1, n - 2, 0), & \text{for } i = n \end{cases}$$

and

$$r(b_i a_{i+1} | R) = \begin{cases} (2i - 1, n - 2i, 2i + 1), & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ (n - 1, 0, n - 2), & \text{for } i = \frac{n}{2} \\ (2n - 2i, 2i - n - 1, 2n - 2i - 2), & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1 \\ (0, n - 1, 1), & \text{for } i = n \end{cases}$$

Representations for the set of interior edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_i c_i | R) = \begin{cases} (2i - 2, n - 2i + 2, 2i), & \text{for } 1 \leq i \leq \frac{n}{2} \\ (2n - 2i + 2, 2i - n - 2, 2n - 2i), & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

and

$$r(c_i x_i | R) = \begin{cases} (2i - 1, n - 2i + 3, 2i + 1), & \text{for } 1 \leq i \leq \frac{n}{2} \\ (2n - 2i + 3, 2i - n - 1, 2n - 2i + 1), & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

Representations for the edges on the outer cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(x_i y_i | R) = \begin{cases} (2i, n - 2i + 3, 2i + 2), & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ (n, 3, n + 1), & \text{for } i = \frac{n}{2} \\ (2n - 2i + 3, 2i - n, 2n - 2i + 1), & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1 \\ (3, n, 2), & \text{for } i = n \end{cases}$$

and

$$r(y_i x_{i+1} | R) = \begin{cases} (2i + 1, n - 2i + 2, 2i + 3), & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ (n + 1, 2, n), & \text{for } i = \frac{n}{2} \\ (2n - 2i + 2, 2i - n + 1, 2n - 2i), & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1 \\ (2, n + 1, 3), & \text{for } i = n \end{cases}$$

We note that there are no two edges having the same representations implying that $\text{edim}(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \leq 3$.

On the other hand, we show that $\text{edim}(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \geq 3$. Suppose on contrary that $\text{edim}(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 2$, then there are the following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Here are the following subcases.

- Both vertices belong to the set $\{a_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1 . Suppose that the second resolving vertex is a_k ($2 \leq k \leq \frac{n}{2} + 1$). Then for $2 \leq k \leq \frac{n}{2}$, we have $r(a_1 c_1 | \{a_1, a_k\}) = r(a_1 b_n | \{a_1, a_k\}) = (0, 2k - 2)$, and for $k = \frac{n}{2} + 1$, we have $r(a_1 b_1 | \{a_1, a_{\frac{n}{2}+1}\}) = r(a_1 b_n | \{a_1, a_{\frac{n}{2}+1}\}) = (0, n - 1)$, a contradiction.

- Both vertices belong to the set $\{b_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1 . Suppose that the second resolving

vertex is b_k ($2 \leq k \leq \frac{n}{2} + 1$). Then for $2 \leq k \leq \frac{n}{2}$, we have $r(a_1c_1|\{b_1, b_k\}) = r(a_1b_n|\{b_1, b_k\}) = (1, 2k - 1)$, and for $k = \frac{n}{2} + 1$, we have $r(a_1b_1|\{b_1, b_{\frac{n}{2}+1}\}) = r(a_2b_1|\{b_1, b_{\frac{n}{2}+1}\}) = (0, n - 1)$, a contradiction.

- One vertex belong to the set $\{a_i : 1 \leq i \leq n\}$ and the second vertex belong to the set $\{b_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1 . Suppose that the second resolving vertex is b_k ($1 \leq k \leq \frac{n}{2} + 1$). Then for $1 \leq k \leq \frac{n}{2}$, we have $r(a_1b_n|\{a_1, b_k\}) = r(a_1c_1|\{a_1, b_k\}) = (0, 2k - 1)$, and for $k = \frac{n}{2} + 1$, we have $r(a_1b_1|\{a_1, b_{\frac{n}{2}+1}\}) = r(a_1c_1|\{a_1, b_{\frac{n}{2}+1}\}) = (0, n - 1)$, a contradiction.

(2) Both vertices are in the interior vertices. Without loss of generality, we can suppose that one resolving vertex is c_1 . Suppose that the second resolving vertex is c_k ($2 \leq k \leq \frac{n}{2} + 1$). Then for $2 \leq k \leq \frac{n}{2} + 1$, we have $r(x_1c_1|\{c_1, c_k\}) = r(a_1c_1|\{c_1, c_k\}) = (0, 2k - 1)$, a contradiction.

(3) Both vertices are in the outer cycle. Due to the symmetry of the graph, this case is analogous to case (1).

(4) One vertex is in the inner cycle and the other one is in the set of interior vertices. Here are the two subcases.

- One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices $\{c_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1 . Suppose that the second resolving vertex is c_k ($1 \leq k \leq \frac{n}{2} + 1$). Then for $k = 1$, we have $r(a_1b_1|\{a_1, c_1\}) = r(a_1b_n|\{a_1, c_1\}) = (0, 1)$. For $2 \leq k \leq \frac{n}{2}$, we have $r(a_1b_n|\{a_1, c_k\}) = r(a_1c_1|\{a_1, c_k\}) = (0, 2k - 1)$ and for $k = \frac{n}{2} + 1$, we have $r(a_1b_n|\{a_1, c_{\frac{n}{2}+1}\}) = r(a_1b_1|\{a_1, c_{\frac{n}{2}+1}\}) = (0, 2n)$, a contradiction.

- One vertex is in the set $\{b_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices $\{c_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1 . Suppose that the second resolving vertex is c_k ($1 \leq k \leq \frac{n}{2} + 1$). Then for $k = 1$, we have $r(x_1y_1|\{b_1, c_1\}) = r(x_1y_n|\{b_1, c_1\}) = (3, 1)$. For $2 \leq k \leq \frac{n}{2}$, we have $r(a_1c_1|\{b_1, c_k\}) = r(a_1b_n|\{b_1, c_k\}) = (1, 2k - 1)$ and for $k = \frac{n}{2} + 1$, we have $r(c_2x_2|\{b_1, c_{\frac{n}{2}+1}\}) = r(a_nb_n|\{b_1, c_{\frac{n}{2}+1}\}) = (2, n - 1)$, a contradiction.

(5) One vertex is in the outer cycle and the other one is in the set of interior vertices. Due to the symmetry of the graph, this case is analogous to case (4).

(6) One vertex is in the inner cycle and the other one is in the outer cycle. Here are the following subcases.

- One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set $\{x_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1 . Suppose that the second resolving vertex is x_k ($1 \leq k \leq \frac{n}{2} + 1$). Then for $k = 1$, we have $r(a_1b_1|\{a_1, x_1\}) = r(a_1b_n|\{a_1, x_1\}) = (0, 2)$. For $2 \leq k \leq \frac{n}{2} + 1$, we have $r(a_1b_1|\{a_1, x_k\}) = r(a_1c_1|\{a_1, x_k\}) = (0, 2k - 1)$, a contradiction.

- One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set $\{y_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1 . Suppose that the second resolving vertex is y_k ($1 \leq k \leq \frac{n}{2} + 1$). Then

for $k = 1$, we have $r(a_1b_1|\{a_1, y_1\}) = r(a_1b_n|\{a_1, y_1\}) = (0, 3)$. For $2 \leq k \leq \frac{n}{2}$, we have $r(a_1b_1|\{a_1, y_k\}) = r(a_1c_1|\{a_1, y_k\}) = (0, 2k)$ and for $k = \frac{n}{2} + 1$, we have $r(a_1b_1|\{a_1, y_{\frac{n}{2}+1}\}) = r(a_1c_1|\{a_1, y_{\frac{n}{2}+1}\}) = (0, n+1)$, a contradiction.

• One vertex is in the set $\{b_i : 1 \leq i \leq n\}$ and the other one is in the set $\{y_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1 . Suppose that the second resolving vertex is y_k ($1 \leq k \leq \frac{n}{2} + 1$). Then for $k = 1$, we have $r(a_1b_1|\{b_1, y_1\}) = r(a_2b_1|\{b_1, y_1\}) = (0, 3)$. For $k = 2$, we have $r(x_1y_1|\{b_1, y_2\}) = r(a_3c_3|\{b_1, y_2\}) = (3, 2)$. For $3 \leq k \leq \frac{n}{2} + 1$, we have $r(x_2y_2|\{b_1, y_k\}) = r(a_3c_3|\{b_1, y_k\}) = (3, 2k - 4)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ implying that $\text{edim}(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \neq 2$ in this case. Therefore, $\text{edim}(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$.

Case 2. When n is odd.

Let $R = \{a_1, b_{\lceil \frac{n}{2} \rceil}, a_n\} \subset V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$. We show that R is a resolving set for $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ in this case. For this we give representations of any edge of $E(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ with respect to R .

Representations for the edges of the inner cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_i b_i | R) = \begin{cases} (2i - 2, n + 1 - 2i, 2i), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ (2i - 2, n + 1 - 2i, 2n - 2i - 1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n - 2i + 1, 2i - n - 2, 2n - 2i - 1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n - 1 \\ (1, n - 2, 0), & \text{for } i = n. \end{cases}$$

and

$$r(b_i a_{i+1} | R) = \begin{cases} (2i - 1, n - 2i, 2i + 1), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 2 \\ (2i - 1, n - 2i, 2n - 2i - 2), & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \\ (2n - 2i, 2i - n - 1, 2n - 2i - 2), & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n - 1 \\ (0, n - 1, 1), & \text{for } i = n \end{cases}$$

Representations for the edges on the outer cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(x_i y_i | R) = \begin{cases} (2i, n + 3 - 2i, 2i + 2), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ (2i, n - 2i + 3, 2n - 2i + 1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n - 2i + 3, 2i - n, 2n - 2i + 1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n - 1 \\ (3, n, 2), & \text{for } i = n \end{cases}$$

and

$$r(y_i x_{i+1} | R) = \begin{cases} (2i + 1, n - 2i + 2, 2i + 3), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 2 \\ (2i + 1, n - 2i + 2, 2n - 2i), & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \\ (2n - 2i + 2, 3, 2n - 2i), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n - 2i + 2, 2i - n + 1, 2n - 2i), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n - 1 \\ (2, n + 1, 3), & \text{for } i = n \end{cases}$$

Representations for the set of interior edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_i c_i | R) = \begin{cases} (2i - 2, n - 2i + 2, 2i), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ (2i - 2, n - 2i + 2, 2n - 2i), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n - 2i + 2, 2i - n - 2, 2n - 2i), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

$$r(c_i x_i | R) = \begin{cases} (2i - 1, n - 2i + 3, 2i + 1), & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ (2i - 1, n - 2i + 3, 2n - 2i + 1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n - 2i + 3, 2i - n - 1, 2n - 2i + 1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $\text{edim}(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \leq 3$.

On the other hand, suppose that $\text{edim}(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 2$, then there are the same possibilities as in Case (1) and contradictions can be deduced analogously. This implies that $\text{edim}(BS(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$ in this case, which completes the proof. $\square$

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Ordered LA-groups and ideals in ordered LA-semigroups

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Abstract. Kehayopulu et. al [7], have established a relation between the ideals in ordered semigroups and ordered groups. In this study, we extend these notions for a class of non-associative and non-commutative algebraic structures.

Keywords: Ordered (LA-semigroups, LA*-semigroups, LA-groups), left (right, interior, quasi-, bi-, generalized bi-) ideals.

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1. Introduction

A generalization of abelian semigroups has been established by Kazim et. al [5]. They introduced the braces on the left side of the ternary commutative law: $abc = cba$ and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This they called the left invertive law. A groupoid S is a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid. Mushtaq et. al [8], investigated the concept of ideals of LA-semigroups.

A groupoid S is medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$) in [2] (resp. [1]). An LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial in [5]. Every LA-semigroup with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$ by Protic et. al [9].

The concept of ordered semigroups and ideals have been discussed in [6]. A non-empty subset A of an ordered semigroup S is an interior ideal of S if 1. $SAS \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A non-empty subset A of S is a quasi-ideal of S if 1. $(AS] \cap (SA] \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A subsemigroup A of S is a bi-ideal of S if 1. $ASA \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). A non-empty subset A of S is a generalized bi-ideal of S if 1. $ASA \subseteq A$. 2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $(A] \subseteq A$). Every bi-ideal of S is a generalized bi-ideal of S .

We give the concept of ordered (LA-semigroups, LA^* -semigroups, LA-groups) and left (resp. right) simple ordered LA-semigroups. We investigate a relation between the ideals of ordered (LA-semigroups, LA^* -semigroup) and (ordered LA-groups, left simple, right simple). Specifically, we show that: 1. If an ordered LA^* -semigroup $(S, \cdot, \leq)$ with left identity e , is an ordered LA-group, then $Sa = aS = S$ for all $a \in S$. 2. An ordered LA^* -semigroup $(S, \cdot, \leq)$ with left identity e is a left simple and a right simple if and only if it contains no proper quasi-ideal.

2. Ordered LA-groups and ideals in ordered LA-semigroups

An ordered LA-semigroup $(S, \cdot, \leq)$, is a poset (partially ordered set or simply ordered set), at the same time an LA-semigroup such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$.

For $\emptyset \neq A \subseteq S$, we define $(A] = \{s \in S \mid s \leq a \text{ for some } a \in A\}$ and obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, then $AB = \{ab \mid a \in A, b \in B\}$, $((A]) = (A]$, $(A](B] \subseteq (AB]$, $((A](B]) = (AB]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

Now we define an ideal theory for a class of non-associative and non-commutative algebraic structures (ordered AG-groupoid), which are similar to the ideal theory of associative algebraic structure (ordered semigroup), but not identical.

A non-empty subset A of an ordered LA-semigroup S is an LA-subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of an ordered LA-semigroup S is a left (resp. right) ideal of S if

1. $SA \subseteq A$ (resp. $AS \subseteq A$).
2. If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or $[A] \subseteq A$).

A is called an ideal of S if A is both a left and a right ideal of S . Every ideal (whether left, right, two-sided) is an LA-subsemigroup, but the converse is not in general.

A non-empty subset A of S is an interior (resp. quasi-) ideal of S if $(SA)S \subseteq A$ (resp. $(AS] \cap (SA] \subseteq A$) and $[A] \subseteq A$.

An LA-subsemigroup A of S is a $(1, 2)$ -ideal of S if $(AS)A^2 \subseteq A$ and $[A] \subseteq A$. An LA-subsemigroup A of S is a bi-ideal of S if $(AS)A \subseteq A$ and $[A] \subseteq A$. A non-empty subset A of S is a generalized bi-ideal of S if $(AS)A \subseteq A$ and $[A] \subseteq A$. Every bi-ideal of S is a generalized bi-ideal of S .

In [3], a groupoid $(G, \cdot)$ is a left almost group (abbreviated LA-group) if

1. $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.
2. There exists $e \in G$ such that $e \cdot a = a$ for every $a \in G$.
3. For every $a \in G$ there exists $a' \in G$ such that $a' \cdot a = e$.

It is very easy to see that the left identity ' e ' and the left inverse are unique. Left inverse is also right inverse as $aa' = (ea)a' = (a'a)e = ee = e$. This implies that a' is also a right inverse of a .

An ordered LA-group $(G, \cdot, \leq)$, is a poset, at the same time an LA-group such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in G$.

An ordered LA-group is also called a po-LA-group for short. One of the immediate properties of a po-LA-group is this if $a \leq b$, then $b^{-1} \leq a^{-1}$. Now, let

$$\begin{aligned}
 a &\leq b \\
 \Rightarrow a^{-1}a &\leq a^{-1}b \\
 \Rightarrow (a^{-1}a)b^{-1} &\leq (a^{-1}b)b^{-1} \\
 \Rightarrow eb^{-1} &\leq (b^{-1}b)a^{-1} \text{ by the left invertive law} \\
 \Rightarrow b^{-1} &\leq ea^{-1} \\
 \Rightarrow b^{-1} &\leq a^{-1}
 \end{aligned}$$

Example 1. Consider a set $S = \{e, a, b, c, d\}$ with the following multiplication “ $\cdot$ ” and order relation “ $\leq$ ”

$\cdot$	e	a	b	c	d
e	e	a	b	c	d
a	d	e	a	b	c
b	c	d	e	a	b
c	b	c	d	e	a
d	a	b	c	d	e

$\leq = \{(e, e), (e, a), (e, b), (e, c), (e, d), (a, a), (b, b), (c, c), (d, d)\}$, then $(S, \cdot, \leq)$ is an ordered LA-group.

Lemma 1. Every right ideal of an ordered LA-semigroup S with left identity e , is an ideal of S .

Proof. Let I be a right ideal of S , this implies that $(I] \subseteq I$. Let $i \in I$ and $s \in S$. Now $si = (es)i = (is)e \in IS \subseteq I$. Hence I is an ideal of S . $\square$

Lemma 2. Every two-sided ideal of an ordered LA-semigroup S is an interior ideal of S .

Proof. Let I be a two sided ideal of S , this means that $(I] \subseteq I$. Now $(SI)S \subseteq IS \subseteq I$. Therefore I is an interior ideal of S . $\square$

Proposition 1. Let S be an ordered LA-semigroup with left identity e . Then any non-empty subset I of S is an interior ideal of S if and only if I is an ideal of S .

Proof. Suppose that I is an interior ideal of S , this implies that $(I] \subseteq I$. Let $i \in I$ and $s \in S$. Now $is = (ei)s \in (SI)S \subseteq I$, thus I is a right ideal of S . Hence I is an ideal of S by Lemma 1. Converse is true by Lemma 2. $\square$

Lemma 3. Every right (two-sided) ideal of S is a bi-ideal of S .

Proof. Let I be a right ideal of S , this means that $(I] \subseteq I$. Now $(IS)I \subseteq IS \subseteq I$. Hence I is a bi-ideal of S . $\square$

Lemma 4. Every left (right, two-sided) ideal of S is a quasi-ideal of S .

Proof. Let I be a right ideal of S , this implies that $(I] \subseteq I$. Now $(IS] \cap (SI] \subseteq (IS] \subseteq (I] \subseteq I$. Therefore I is a quasi-ideal of S . $\square$

Proposition 2. Every quasi-ideal of S is an LA-subsemigroup of S .

Proof. Let I be a quasi-ideal of S . Now $II \subseteq IS \subseteq (I](S] \subseteq (IS]$ and $II \subseteq SI \subseteq (S](I] \subseteq (SI]$. Thus $I^2 = II \subseteq (IS] \cap (SI] \subseteq I$. So I is an LA-subsemigroup of S . $\square$

Proposition 3. *Let I and G be a right and a left ideal of an ordered LA-semigroup S , respectively. Then $I \cap G$ is a quasi-ideal of S .*

Proof. $((I \cap G)S] \cap (S(I \cap G))] \subseteq (IS] \cap (SG] \subseteq (I] \cap (G] \subseteq I \cap G$, since I is a right ideal and G is a left ideal of S . Also $(I \cap G] \subseteq I \cap G$. Consequently $I \cap G$ is a quasi-ideal of S . $\square$

Lemma 5. *If an ordered LA-semigroup $(S, \cdot, \leq)$ with left identity e , is an ordered LA-group, then S does not contain any proper interior (resp. bi-, generalized bi-, quasi-) ideal.*

Proof. Let A be an interior ideal of S . Let $a \in S$ and $b \in A$, i.e., $b \in S$. This implies that there exists $b^{-1} \in S$ such that $bb^{-1} = b^{-1}b = e$ (e is the left identity of S). Now $a = ea = (b^{-1}b)a = (ab)b^{-1} \in (SA)S \subseteq A$. Thus $S \subseteq A$, i.e., $S = A$.

Suppose that A is a bi-ideal of S . Let $a \in S$ and $b \in A$, i.e., $b \in S$. This means that there exists $b^{-1} \in S$ such that $bb^{-1} = b^{-1}b = e$ (e is the left identity of S). Now

$$\begin{aligned} a &= (ee)a = (ae)e = (a(bb^{-1}))(bb^{-1}) \\ &= (b(ab^{-1}))(bb^{-1}) = ((bb^{-1})(ab^{-1}))b \\ &= ((ba)(b^{-1}b^{-1}))b = ((b^{-1}a)(b^{-1}b))b \\ &= ((b^{-1}a)(bb^{-1}))b \text{ since } b^{-1}b = bb^{-1} \\ &= (b((b^{-1}a)b^{-1}))b \in (AS)A \subseteq A. \end{aligned}$$

Thus $S \subseteq A$, i.e., $S = A$.

Assume that Q is a quasi-ideal of S and $0 \neq a \in Q$, then $e = aa^{-1} = a^{-1}a \in QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, i.e., $e \in Q$. Let $x \in S$, and $x = ex = e(ex) = (xe)e \in QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, i.e., $S \subseteq Q$. Hence $Q = S$. $\square$

Proposition 4. *Let S be a right cancellative ordered LA-semigroup with left identity e . Then S is an ordered LA-group if and only if $Sa = S$ for all $a \in S$.*

Proof. Suppose that S is an ordered LA-group. Let $x \in S$ and $x = ex = (aa^{-1})x = (xa^{-1})a \in Sa$, i.e., $S \subseteq Sa$. Thus $Sa = S$ for all $a \in S$.

Conversely, assume that S is a right cancellative ordered LA-semigroup such that $Sa = S$ for all $a \in S$. From the given relation it follows that $ya = e$, for some $y \in S$. This implies that there exists left inverse of a in S , where a is an arbitrary element of S . For uniqueness, let $za = e$, $z \in S$. Now $ya = za$, i.e., $y = z$, by right cancellative. Hence S is an ordered LA-group. $\square$

Definition 1. *An ordered LA-semigroup satisfies one the following conditions*

1. $(ab)c = b(ac)$,
2. $(ab)c = b(ca)$ for all $a, b, c \in S$.

then it is called weak associative ordered LA-semigroup and such that an ordered LA-semigroup is denoted by ordered LA-semigroup. Note that condition 1. and 2. are equivalent.*

Importance of definition 1, is that, when our need not fulfill by the left identity, then we involve the definition 1. That is why, we can prove this property, $Sa = S = aS$ for all $a \in S$, in the Proposition 5, but we cannot prove this property $Sa = S = aS$ for all $a \in S$, in the Proposition 4.

Proposition 5. *If an ordered LA^* -semigroup $(S, \cdot, \leq)$ with left identity e , is an ordered LA^* -group, then $Sa = aS = S$ for all $a \in S$.*

Proof. Assume that S is an ordered LA^* -group and $Sa = S$. Since $aS \subseteq S$, let $x \in S$ and $x = ex = (a^{-1}a)x = a(a^{-1}x) \in aS$, i.e., $S \subseteq aS$. Thus $aS = S$. $\square$

Converse of this statement is not true because associative law does not hold in an LA -group.

Suppose that S is an ordered LA^* -semigroup such that $Sa = aS = S$ for all $a \in S$. From the given relation, it follows that $ax = e$ and $ya = e$, for some $x, y \in S$. This implies that x is the right inverse of a and y is the left inverse of a . Now we have to show that $x = y$.

$$\begin{aligned} y(ax) &= ye = e(ye) = (ee)y = ey = y \text{ and } (ya)x = ex = x \\ y(ax) &= (ay)x = ((ee)(ay))x = ((ya)(ee))x \\ &= ((ya)e)x = e((ya)x) = (ya)x. \\ \Rightarrow x &= y. \\ \Rightarrow y(ax) &= (ya)x. \end{aligned}$$

This means that associative law holds in S , which is a contradiction to fact that S is an LA -semigroup.

An ordered LA -semigroup S is said to be left (resp. right) simple if it does not contain any proper left (resp. right) ideal of S or equivalently, if for every left (resp. right) ideal A of S , we have $A = S$.

Equivalent definition: An ordered LA -semigroup S is a left (resp., right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.

Theorem 1. *If an ordered LA -semigroup $(S, \cdot, \leq)$ with left identity e is a left (resp. right) simple. Then S does not contain any proper interior ideal.*

Proof. Suppose that S is a right simple, this implies that $(aS] = S$ for every $a \in S$. Let A be a right ideal of S , then $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now

$$\begin{aligned} a &\in (bS] = (b(bS]) \subseteq (b(bS)) = (b((eb)S)) \\ &= (b((Sb)e)) = ((Sb)(be)) \subseteq ((SA)S] \subseteq (A] = A, \end{aligned}$$

since every right ideal of S is an interior ideal of S , (because every right ideal of S is an ideal of S by the Lemma 1). Thus $S \subseteq A$, so $A = S$. Similarly, for left simple. $\square$

Theorem 2. *If an ordered LA-semigroup $(S, \cdot, \leq)$ is a left and a right simple. Then S does not contain any proper bi-ideal.*

Proof. Assume that S is a left and right simple, then $(Sa] = S$ and $(aS] = S$ for every $a \in S$. Let A be an ideal of S , this implies that A is a left and a right ideal of S . Thus $(Sb] = S$ and $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now $a \in (Sb] = ((bS]b] \subseteq ((bS)b] \subseteq ((AS)A] \subseteq (A] = A$, since every ideal of S is a bi-ideal of S by Lemma 3. Thus $S \subseteq A$, so $A = S$. $\square$

Remark 1. If an ordered LA-semigroup $(S, \cdot, \leq)$ is a left and a right simple. Then S does not contain any proper generalized bi-ideal.

Theorem 3. *If an ordered LA*-semigroup $(S, \cdot, \leq)$ with left identity e is a left and a right simple if and only if it contains no proper quasi-ideal.*

Proof. Let S be a left and a right simple, then $(Sa] = S$ and $(aS] = S$ for every $a \in S$. Suppose that A is an ideal of S , this implies that A is a left and a right ideal of S . Thus $(Sb] = S$ and $(bS] = S$ for every $b \in A \subseteq S$. As $a \in S$ and $b \in A$. Now $a \in (bS] \cap (Sb] \subseteq (AS] \cap (SA] \subseteq A$, since A is quasi-ideal of S (because every ideal of S is a quasi-ideal of S by Lemma 4). Hence $S = A$. Conversely, assume that S contains no proper quasi-ideal. So S contains no proper left ideal and no proper right ideal, because every left and right ideal of S is a quasi ideal of S by Lemma 4. Let $a \in S$. First we have to show that $(Sa]$ is a left ideal of S and $(aS]$ is a right ideal of S . Now

$$\begin{aligned} S(Sa] &\subseteq (S](Sa] \subseteq (S(Sa)] \\ &= ((Se)(Sa)] = ((SS)(ea)] \subseteq (Sa]. \end{aligned}$$

and $((Sa)] \subseteq (Sa]$. Now

$$\begin{aligned} (aS]S &\subseteq (aS](S] \subseteq ((aS)S] = ((aS)(eS)] = ((ae)(SS)] \\ &\subseteq ((ae)S] = (e(aS)] = (aS], \text{ by LA}^*\text{-semigroup.} \end{aligned}$$

and $((aS)] \subseteq (aS]$. Therefore $(Sa]$ is a left ideal of S and $(aS]$ is a right ideal of S . Now we have to show that $(aS] = S = (Sa]$. Obviously, $(Sa] \subseteq S$ and $(aS] \subseteq S$. Now $a = ea \in Sa \subseteq (Sa]$ and $a = ea = (ee)a = e(ae) = ae \in aS \subseteq (aS]$, by LA*-semigroup, i.e., $S \subseteq (Sa]$ and $S \subseteq (aS]$. Thus $(aS] = (Sa] = S$ for all $a \in S$. Hence S is a left and a right simple. $\square$

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Ideals in LA-rings

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Abstract. Our aim is to encourage research and maturity of associative algebraic structures by studying a class of non-associative and non-commutative algebraic structures (LA-ring).

Keywords: (left, right, interior, quasi-, bi-,generalized bi-) ideals, regular (intra-regular) LA-rings.

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1. Introduction

In ternary operations, the commutative law is given by $abc = cba$. Kazim et al [6] have generalized this notion by introducing the paranthesis on the left side of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is the left invertive law. A groupoid S is a left almost semigroup (abbreviated as LA-semigroup), if it satisfies the left invertive law. An LA-semigroup is a midway structure between a commutative semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al [7].

In [2] (resp. [1]), a groupoid S is to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [6], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al [7] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [3], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is a left almost group, if there exists a left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et al [8], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

In this paper, we will define the left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R . We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings in terms of left (resp. right, quasi-, bi-, generalized bi-) ideals.

2. Ideals in LA-rings

We initiate the concept of LA-subrings and left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R .

A non-empty subset A of an LA-ring R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R , if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R .

Example 1. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define $+$ and $\cdot$ in R as follows:

$+$	0	1	2	3	4	5	6	7		$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	2	0	3	1	6	4	7	5		1	0	4	4	0	0	4	4	0
2	1	3	0	2	5	7	4	6		2	0	4	4	0	0	4	4	0
3	3	2	1	0	7	6	5	4	and	3	0	0	0	0	0	0	0	0
4	4	5	6	7	0	1	2	3		4	0	3	3	0	0	3	3	0
5	6	4	7	5	2	0	3	1		5	0	7	7	0	0	7	7	0
6	5	7	4	6	1	3	0	2		6	0	7	7	0	0	7	7	0
7	7	6	5	4	3	2	1	0		7	0	3	3	0	0	3	3	0

Let $A = \{0, 4\}$ be a subset of R . Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned} A^2 &= AA = \{0, 4\}\{0, 4\} = \{0\} \subseteq A. \\ AR &= \{0, 4\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 3\} \not\subseteq A. \end{aligned}$$

Thus A is an LA-subring of R , but not right ideal of R .

A non-empty subset A of an LA-ring R is an interior ideal of R , if $(A, +)$ is an LA-group and $(RA)R \subseteq A$. A non-empty subset A of R is a quasi-ideal of R , if $(A, +)$ is an LA-group and $AR \cap RA \subseteq A$. An LA-subring A of an LA-ring R is a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of an LA-ring R is a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. An ideal I of an LA-ring R is an idempotent ideal of R if $A^2 = A$.

Now we give the central properties of such ideals of an LA-ring R , which will be very helpful for further sections.

Lemma 2.1. *Let R be an LA-ring with left identity e , then $RR = R$ and $eR = R = Re$.*

Proof. Since $RR \subseteq R$ and $x = ex \in RR$, i.e., $R \subseteq RR$, thus $RR = R$. Obviously, $eR = R$ and $Re = (RR)e = (eR)R = RR = R$. $\square$

Lemma 2.2 ([4, Lemma 8]). *Let R be an LA-ring with left identity e and $a \in R$. Then Ra is the smallest left ideal of R containing a .*

Lemma 2.3 ([4, Lemma 9]). *Let R be an LA-ring with left identity e and $a \in R$. Then Ra is a left ideal of R .*

Proposition 2.1 ([4, Proposition 5]). *Let R be an LA-ring with left identity e and $a \in R$. Then $aR \cup Ra$ is the smallest right ideal of R containing a .*

Lemma 2.4. *Let R be an LA-ring with left identity e . Then every right ideal of R is an ideal of R .*

Proof. Let I be a right ideal of R . Let $a \in I$ and $r \in R$. Now $ra = (er)a = (ar)e \in IR \subseteq I$. Thus I is an ideal of R . $\square$

Lemma 2.5. *Every two-sided ideal of R is an interior ideal of R , but the converse is not true.*

Proof. Straight forward. $\square$

Example 2. Let $A = \{0, 1, 2, 3\}$ be a subset of R , defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned} (RA)R &= (\{0, 1, 2, 3, 4, 5, 6, 7\}\{0, 1, 2, 3\})\{0, 1, 2, 3, 4, 5, 6, 7\} \\ &= \{0, 3, 4, 7\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 3\} \subseteq A. \\ RA &= \{0, 1, 2, 3, 4, 5, 6, 7\}\{0, 1, 2, 3\} = \{0, 3, 4, 7\} \not\subseteq A. \\ AR &= \{0, 1, 2, 3\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 4\} \not\subseteq A. \end{aligned}$$

Thus A is an interior ideal of R , but not an ideal of R .

Proposition 2.2. *Let R be an LA-ring with left identity e . Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R .*

Proof. Let I be an interior ideal of R . Let $a \in I$ and $r \in R$. Now $ar = (ea)r \in (RI)R \subseteq I$, i.e., I is a right ideal of R . Thus I is an ideal of R by the Lemma 2.4. Converse is true by the Lemma 2.5. $\square$

Lemma 2.6. *Every left (resp. right, two-sided) ideal of R is a bi-ideal of R .*

Proof. Straight forward. $\square$

Example 3. Let $A = \{0, 4\}$ be a subset of R defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned} A^2 &= AA = \{0, 4\}\{0, 4\} = \{0\} \subseteq A. \\ (AR)A &= (\{0, 4\}\{0, 1, 2, 3, 4, 5, 6, 7\})\{0, 4\} \\ &= \{0, 3\}\{0, 4\} = \{0\} \subseteq A. \\ AR &= \{0, 4\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 3\} \not\subseteq A. \end{aligned}$$

Thus A is a bi-ideal of R , but not right ideal of R .

Lemma 2.7. *Every bi-ideal of R is a generalized bi-ideal of R .*

Proof. Obvious. $\square$

Lemma 2.8. *Every left (resp. right, two-sided) ideal of R is a quasi-ideal of R .*

Proof. Let I be a left ideal of R and $IR \cap RI \subseteq RI \subseteq I$, i.e., I is a quasi-ideal of R . $\square$

Proposition 2.3. *Let I be a right ideal and L be a left ideal of an LA-ring R , respectively. Then $I \cap L$ is a quasi-ideal of R .*

Proof. Since $(I \cap L)R \cap R(I \cap L) \subseteq IR \cap RL \subseteq I \cap L$, i.e., $I \cap L$ is a quasi-ideal of R . $\square$

Lemma 2.9. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then every quasi-ideal of R is a bi-ideal of R .*

Proof. Let Q be a quasi-ideal of R . Now $(QR)Q \subseteq RQ$ and $(QR)Q \subseteq (QR)R = (QR)(eR) = (Qe)(RR) = (Qe)R = QR$, thus $(QR)Q \subseteq QR \cap RQ \subseteq Q$. Hence Q is a bi-ideal of R . $\square$

3. Regular LA-rings

An LA-ring R is regular, if for every $a \in R$, there exists an element $x \in R$ such that $a = (ax)a$. We characterize regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1. *Every right ideal of a regular LA-ring R is an ideal of R .*

Proof. Suppose that I is a right ideal of R . Let $a \in I$ and $r \in R$. This implies that there exists an element $x \in R$, such that $r = (rx)r$. Now $ra = ((rx)r)a = (ar)(rx) \in IR \subseteq I$. Hence I is an ideal of R . $\square$

Lemma 3.2. *Every ideal of a regular LA-ring R is an idempotent.*

Proof. Let I be an ideal of R . Since $I^2 \subseteq I$ and $a \in I$. This means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II = I^2$, i.e., $I \subseteq I^2$. Thus $I^2 = I$. $\square$

Remark 1. Every right ideal of a regular LA-ring R is an idempotent.

Proposition 3.1. *Let R be a regular LA-ring. Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R .*

Proof. Assume that I is an interior ideal of R . Let $a \in I$ and $r \in R$. Then there exists an element $x \in R$, such that $a = (ax)a$. Now $ar = ((ax)a)r = (ra)(ax) \in (RI)R \subseteq I$, i.e., $IR \subseteq I$. Thus I is an ideal of R by the Lemma 3.1. Converse is true by the Lemma 2.5. $\square$

Proposition 3.2. *Let R be a regular LA-ring with left identity e . Then $IR \cap RI = I$ for every right ideal I of R .*

Proof. Let I be a right ideal of R . This implies that $IR \cap RI \subseteq I$, because every right ideal of R is a quasi-ideal of R . Let $a \in I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II \subseteq IR$, i.e., $I \subseteq IR$ and $a = (ax)a = (ax)(ea) = (ae)(xa) \in I(RI) = R(II) = RI$, i.e., $I \subseteq RI$. Thus $I \subseteq IR \cap RI$, hence $IR \cap RI = I$. $\square$

Lemma 3.3. *Let R be a regular LA-ring. Then $DL = D \cap L$ for every right ideal D and for every left ideal L of R .*

Proof. Since $DL \subseteq D \cap L$ is obvious. Let $a \in D \cap L$, then there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (DR)L \subseteq DL$, i.e., $D \cap L \subseteq DL$. Hence $DL = D \cap L$. $\square$

Theorem 1. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $D \cap L = DL$ for every right ideal D and for every left ideal L of R .
- (3) $Q = (QR)Q$ for every quasi-ideal Q of R .

Proof. Suppose that (1) holds. Let Q be a quasi-ideal of R and $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (QR)Q$, i.e., $Q \subseteq (QR)Q \subseteq Q$, because every quasi-ideal of R is a bi-ideal of R . Hence $Q = (QR)Q$, i.e., (1) $\Rightarrow$ (3). Assume that (3) holds, let D be a right ideal and L be a left ideal of R . Then D and L be quasi-ideals of R by the Lemma 2.8, so $D \cap L$ be also a quasi-ideal of R . Now $D \cap L = ((D \cap L)R)(D \cap L) \subseteq (DR)L \subseteq DL$. Since $DL \subseteq D \cap L$, so $DL = D \cap L$, i.e., (3) $\Rightarrow$ (2). Suppose that (2) is true, let $a \in R$, then Ra is a left ideal of R containing a by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 2.1. By our supposition

$$\begin{aligned}(aR \cup Ra) \cap Ra &= (aR \cup Ra)(Ra) = (aR)(Ra) \cup (Ra)(Ra). \\ \text{Now } (Ra)(Ra) &= ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra).\end{aligned}$$

Thus

$$\begin{aligned}(aR \cup Ra) \cap Ra &= (aR)(Ra) \cup (Ra)(Ra) \\ &= (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).\end{aligned}$$

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (aR)(Ra)$. Then $a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = ((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a \in (aR)a$ for any $x, y \in R$, i.e., $a \in (aR)a$. Hence a is a regular, so R is a regular, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 2. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $Q = (QR)Q$ for every quasi-ideal Q of R .
- (3) $B = (BR)B$ for every bi-ideal B of R .
- (4) $G = (GR)G$ for every generalized bi-ideal G of R .

Proof. (1) $\Rightarrow$ (4), is obvious. (4) $\Rightarrow$ (3), since every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. (3) $\Rightarrow$ (2), since every quasi-ideal of R is bi-ideal of R by the Lemma 2.9. (2) $\Rightarrow$ (1), by the Theorem 1. $\square$

Theorem 3. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $Q \cap I = (QI)Q$ for every quasi-ideal Q and for every ideal I of R .
- (3) $B \cap I = (BI)B$ for every bi-ideal B and every for ideal I of R .
- (4) $G \cap I = (GI)G$ for every generalized bi-ideal G and for every ideal I of R .

Proof. Suppose that (1) holds. Let G be a generalized bi-ideal and I be an ideal of R . Now $(GI)G \subseteq (RI)R \subseteq I$ and $(GI)G \subseteq (GR)G \subseteq G$, thus $(GI)G \subseteq G \cap I$. Let $a \in G \cap I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (GI)G$, thus $G \cap I \subseteq (GI)G$. Hence $G \cap I = (GI)G$, i.e., (1) $\Rightarrow$ (4). (4) $\Rightarrow$ (3), since every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. (3) $\Rightarrow$ (2), since every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Assume that (2) is true. Now $Q \cap R = (QR)Q$, i.e., $Q = (QR)Q$, where Q is a quasi-ideal of R . Hence R is a regular by the Theorem 1, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 4. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $D \cap Q \subseteq DQ$ for every quasi-ideal Q and for every right ideal D of R .
- (3) $D \cap B \subseteq DB$ for every bi-ideal B and for every right ideal D of R .
- (4) $D \cap G \subseteq DG$ for every generalized bi-ideal G and for every right ideal D of R .

Proof. Since (1) $\Rightarrow$ (4), is obvious. (4) $\Rightarrow$ (3), since every bi-ideal of R is a generalized bi-ideal of R . (3) $\Rightarrow$ (2), since every quasi-ideal of R is a bi-ideal of R . Suppose that (2) is true. Now $D \cap Q \subseteq DQ$, where Q is a left ideal and D is right ideal of R , because every left ideal of R is a quasi-ideal of R . Since $DQ \subseteq D \cap Q$, thus $D \cap Q = DQ$. Hence R is a regular by the Theorem 1, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 5. *Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $Q \cap D \cap L \subseteq (QD)L$ for every quasi-ideal Q , every right ideal D and for every left ideal L of R .
- (3) $B \cap D \cap L \subseteq (BD)L$ for every bi-ideal B , every right ideal D and for every left ideal L of R .
- (4) $G \cap D \cap L \subseteq (GD)L$ for every generalized bi-ideal G , every right ideal D and for every left ideal L of R .

Proof. Suppose that (1) holds. Let $x \in G \cap D \cap L$, where G is a generalized bi-ideal, D is a right ideal and L is a left ideal of R . Let $x \in R$, this implies that

there exists an element $a \in R$ such that $x = (xa)x$. Now

$$\begin{aligned}
 x &= (xa)x. \\
 xa &= ((xa)x)a = (ax)(xa) = x((ax)a). \\
 (ax)a &= (a((xa)x))a = ((xa)(ax))a = (a(ax))(xa) \\
 &= x((a(ax))a) = x(((ea)(ax))a) = x(((xa)(ae))a) \\
 &= x(((ae)a)x)a = x((nx)a) = x((nx)(ea)) = x((ae)(xn)) \\
 &= x(x((ae)n)) = x(xm). \\
 \Rightarrow xa &= x((ax)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe).
 \end{aligned}$$

Thus $x = (xa)x = (((xm)x)(xe))x \in (GD)L$, i.e., $G \cap D \cap L \subseteq (GD)L$. Hence (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) is true. Then $Q \cap R \cap L \subseteq (Q \circ R) \circ L$, where Q is a right ideal of R , i.e., $Q \cap L \subseteq Q \circ L$. Since $Q \circ L \subseteq Q \cap L$, so $Q \circ L = Q \cap L$. Therefore R is a regular by the Theorem 1, i.e., (2) $\Rightarrow$ (1). $\square$

4. Intra-regular LA-rings

An LA-ring R is an intra-regular, if for every $a \in R$, there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. We characterize intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 4.1. *Every left (right) ideal of an intra-regular LA-ring R is an ideal of R .*

Proof. Suppose that I is a right ideal of R . Let $i \in I$ and $a \in R$. This implies that there exist elements $x_i, y_i \in R$, such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now $ai = ((x_i a^2) y_i) i = (iy_i)(x_i a^2) \in IR \subseteq I$. Hence I is an ideal of R . $\square$

Lemma 4.2. *Every ideal of an intra-regular LA-ring R with left identity e is an idempotent.*

Proof. Let I be an ideal of R . Since $I^2 \subseteq I$ and $a \in I$, this means that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now

$$\begin{aligned}
 a &= (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i \\
 &= (a(x_i a))(e y_i) = (ae)((x_i a) y_i) = (x_i a)((ae) y_i) \in II.
 \end{aligned}$$

Thus $I \subseteq I^2$, i.e., $I^2 = I$. $\square$

Proposition 4.1. *Let R be an intra-regular LA-ring with left identity e . Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R .*

Proof. Assume that I is an interior ideal of R . Let $i \in I$ and $a \in R$. Then there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now

$$\begin{aligned} ia &= i((x_i a^2) y_i) = i((x_i (aa)) y_i) \\ &= i((a(x_i a)) y_i) = i((a(x_i a))(e y_i)) \\ &= i((ae)((x_i a) y_i)) = i((x_i a)((ae) y_i)) \\ &= (x_i a)(i((ae) y_i)) = (x_i i)(a((ae) y_i)) \in (RI)R \subseteq I. \end{aligned}$$

Thus I is a right ideal of R . Therefore I is an ideal of R by the Lemma 4.1. Converse is obvious. $\square$

Lemma 4.3. *Let R be an intra-regular LA-ring. Then $L \cap D \subseteq LD$ for every left ideal L and every right ideal D of R .*

Proof. Let $a \in L \cap D$, where L is a left ideal and D is a right ideal of R . This implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now

$$\begin{aligned} a &= (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i = (a(x_i a))(e y_i) \\ &= (ae)((x_i a) y_i) = (x_i a)((ae) y_i) \in LD. \end{aligned}$$

Thus $L \cap D \subseteq LD$. $\square$

Theorem 6. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is an intra-regular.
- (2) $L \cap D \subseteq LD$ for every left ideal L and for every right ideal D of R .

Proof. (1) $\Rightarrow$ (2) is true by the Lemma 4.3. Suppose that (2) holds and $a \in R$, then Ra is a left ideal of R containing a by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 2.1. By our supposition

$$\begin{aligned} Ra \cap (aR \cup Ra) &\subseteq (Ra)(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra). \\ \text{Now } (Ra)(aR) &= (Ra)((ea)R) = (Ra)((Ra)e) = (Ra)((Ra)(ee)) \\ &= (Ra)((Re)(ae)) = (Ra)(R(ae)) = (Ra)(Ra). \end{aligned}$$

Thus

$$\begin{aligned} (aR \cup Ra) \cap Ra &\subseteq (Ra)(aR) \cup (Ra)(Ra) = (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = R^2 a^2 = (RR)(a^2 e) \\ &= (ea^2)(RR) = (Ra^2)(Re) = (Ra^2)R. \end{aligned}$$

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (Ra^2)R$, thus a is an intra regular. Hence R is an intra-regular, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 7. Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is an intra-regular.
- (2) $Q \cap I = (QI)Q$ for every quasi-ideal Q and for every ideal I of R .
- (3) $B \cap I = (BI)B$ for every bi-ideal B and for every ideal I of R .
- (3) $G \cap I = (GI)G$ for every generalized bi-ideal G and for every ideal I of R .

Proof. Suppose that (1) holds. Let $a \in G \cap I$, where G is a generalized bi-ideal and I is an ideal of R , this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now

$$\begin{aligned}
 a &= (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i = (y_i (x_i a)) a. \\
 y_i (x_i a) &= y_i (x_i ((x_i a^2) y_i)) = y_i ((x_i a^2) (x_i y_i)) = (x_i a^2) (y_i (x_i y_i)) \\
 &= (x_i a^2) (x_i y_i^2) = (x_i (aa)) m_i, \text{ say } x_i y_i^2 = m_i \\
 &= (a(x_i a)) m_i = (m_i (x_i a)) a. \\
 m_i (x_i a) &= m_i (x_i ((x_i a^2) y_i)) = m_i ((x_i a^2) (x_i y_i)) = (x_i a^2) (m_i (x_i y_i)) \\
 &= (x_i (aa)) n_i, \text{ say } m_i (x_i y_i) = n_i \\
 &= (a(x_i a)) n_i = (n_i (x_i a)) a \\
 &= v_i a, \text{ say } n_i (x_i a) = v_i. \\
 \Rightarrow y_i (x_i a) &= (m_i (x_i a)) a = (v_i a) a = (v_i a) (ea) = (v_i e) (aa) = a((v_i e) a).
 \end{aligned}$$

Thus $a = (x_i a^2) y_i = (y_i (x_i a)) a = (a((v_i e) a)) a \in (GI)G$, i.e., $G \cap I \subseteq (GI)G$. Now $(GI)G \subseteq (RI)R \subseteq I$ and $(GI)G \subseteq (GR)G \subseteq G$, thus $(GI)G \subseteq G \cap I$. Hence $G \cap I = (GI)G$, i.e., (1) $\Rightarrow$ (4). (4) $\Rightarrow$ (3), every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. (3) $\Rightarrow$ (2), every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Assume that (2) is true and let Q be a right ideal and I be a two-sided ideal of R . Now $I \cap Q = (QI)Q \subseteq (RI)Q \subseteq IQ$, since every right ideal of R is a quasi-ideal of R . Therefore R is an intra-regular by the Theorem 6, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 8. Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is an intra-regular.
- (2) $L \cap Q \subseteq LQ$ for every quasi-ideal Q and for every left ideal L of R .
- (3) $L \cap B \subseteq LB$ for every bi-ideal B and for every left ideal L of R .
- (4) $L \cap G \subseteq LG$ for every generalized bi-ideal G and for every left ideal L of R .

Proof. Suppose that (1) holds. Let $a \in L \cap G$, where L is a left ideal and G is a generalized bi-ideal of R , this means that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now $a = (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i =$

$(y_i(x_i a))a \in LG$, i.e., $a \in LG$. Thus $L \cap G \subseteq LG$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of R is a generalized bi-ideal of R . $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R . Assume that (2) is true and let Q be a right ideal and L be a left ideal of R . Now $L \cap Q \subseteq LQ$, where Q is a quasi-ideal of R . Hence R is an intra-regular by the Theorem 6, i.e., $(2) \Rightarrow (1)$. $\square$

Theorem 9. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is an intra-regular.
- (2) $L \cap Q \cap D \subseteq (LQ)D$ for every quasi-ideal Q , every right ideal D and for every left ideal L of R .
- (3) $L \cap B \cap D \subseteq (LB)D$ for every bi-ideal B , every right ideal D and for every left ideal L of R .
- (4) $L \cap G \cap D \subseteq (LG)D$ for every generalized bi-ideal G , every right ideal D and for every left ideal L of R .

Proof. Suppose that (1) holds. Let $a \in G \cap L \cap D$, where G is a generalized bi-ideal, L is a left ideal and D is a right ideal of R , this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Now

$$\begin{aligned} a &= (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i = (y_i(x_i a)) a. \\ y_i(x_i a) &= y_i(x_i((x_i a^2) y_i)) = y_i((x_i a^2)(x_i y_i)) = (x_i a^2)(y_i(x_i y_i)) \\ &= (x_i a^2)(x_i y_i^2) = (x_i(aa)) m_i, \text{ say } x_i y_i^2 = m_i \\ &= (a(x_i a)) m_i = (m_i(x_i a)) a. \end{aligned}$$

Thus $a = (x_i a^2) y_i = (y_i(x_i a)) a = ((m(x_i a)) a) a \in (LG)R$, i.e., $a \in (LG)D$. Hence $G \cap L \cap D \subseteq (LG)D$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of R is a generalized bi-ideal of R . $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R . Assume that (2) is true. Now

$$\begin{aligned} L \cap R \cap D &\subseteq (LR)D = ((eL)R)D = ((RL)e)D \subseteq (Le)D \\ &= (e(Le))D \subseteq (R(Le))D \subseteq (RL)D \subseteq LD. \\ &\Rightarrow L \cap D \subseteq LD. \end{aligned}$$

Hence R is an intra-regular by the Theorem 6, i.e., $(2) \Rightarrow (1)$. $\square$

5. Regular and intra-regular LA-rings

We characterize both regular and intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Theorem 10. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular and an intra-regular.
- (2) $B^2 = B$ for every bi-ideal B of R .
- (3) $B_1 \cap B_2 = B_1 B_2 \cap B_2 B_1$ for all bi-ideals B_1, B_2 of R .

Proof. Suppose that (1) holds. Let B be a bi-ideal of R and $B^2 \subseteq B$. Let $a \in B$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y_i, z_i \in R$ such that $a = (y_i a^2)z_i$. Now

$$\begin{aligned}
 a &= (ax)a = (ax)((y_i a^2)z_i) = (((y_i a^2)z_i)x)a. \\
 ((y_i a^2)z_i)x &= (xz_i)(y_i a^2) = m_i(y_i a^2), \text{ say } m_i = xz_i \\
 &= m_i(y_i(aa)) = m_i(a(y_i a)) = a(m_i(y_i a)) \\
 &= ((ax)a)(m_i(y_i a)) = ((ax)m_i)(a(y_i a)) \\
 &= ((m_i x)a)(a(y_i a)) = (n_i a)(a(y_i a)), \text{ say } n_i = m_i x \\
 &= ((en_i)a)(a(y_i a)) = ((an_i)e)(a(y_i a)) \\
 &= ((an_i)a)(e(y_i a)) = ((an_i)a)(y_i a) = (s_i a)(y_i a), \text{ say } s_i = an_i \\
 &= (aa)(y_i s_i) = (aa)t_i, \text{ say } t_i = y_i s_i \\
 &= (((ax)a)a)t_i = ((aa)(ax))t_i = (t_i(ax))(aa) \\
 &= (a(t_i x))(aa) = (aw_i)(aa), \text{ say } w_i = t_i x.
 \end{aligned}$$

Thus $a = (((y_i a^2)z_i)x)a = ((aw_i)(aa))a \in ((BR)B)B \subseteq B^2$, i.e., $B \subseteq B^2$. Hence $B^2 = B$, i.e., (1) $\Rightarrow$ (3). Assume that (2) is true. Let B_1, B_2 be bi-ideals of R , then $B_1 \cap B_2$ be also a bi-ideal of R . Now $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 B_2$ and $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_2 B_1$, thus $B_1 \cap B_2 \subseteq B_1 B_2 \cap B_2 B_1$. Now we show that $B_1 B_2$ is a bi-ideal of R . It is enough to show that $((B_1 B_2)R)(B_1 B_2) \subseteq B_1 B_2$. Now

$$\begin{aligned}
 ((B_1 B_2)R)(B_1 B_2) &= ((B_1 B_2)(RR))(B_1 B_2) \\
 &= ((B_1 R)(B_2 R))(B_1 B_2) \\
 &= ((B_1 R)B_1)((B_2 R)B_2) \subseteq B_1 B_2. \\
 \Rightarrow ((B_1 B_2)R)(B_1 B_2) &\subseteq B_1 B_2.
 \end{aligned}$$

Thus $B_1 B_2$ is a bi-ideal of R , similarly $B_2 B_1$ is also a bi-ideal of R . Then $B_1 B_2 \cap B_2 B_1$ is also a bi-ideal of R . Now

$$\begin{aligned}
 B_1 B_2 \cap B_2 B_1 &= (B_1 B_2 \cap B_2 B_1)(B_1 B_2 \cap B_2 B_1) \\
 &\subseteq (B_1 B_2)(B_2 B_1) \subseteq (B_1 R)(RB_1) \\
 &= ((RB_1)R)B_1 = (((Re)B_1)R)B_1 \\
 &= (((B_1 e)R)R)B_1 = ((B_1 R)R)B_1 \\
 &= ((RR)B_1)B_1 = (RB_1)B_1 \\
 &= ((Re)B_1)B_1 = ((B_1 e)R)B_1 \\
 &= (B_1 R)B_1 \subseteq B_1. \\
 \Rightarrow B_1 B_2 \cap B_2 B_1 &\subseteq B_1.
 \end{aligned}$$

Similarly, we have $B_1 B_2 \cap B_2 B_1 \subseteq B_2$, thus $B_1 B_2 \cap B_2 B_1 \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = B_1 B_2 \cap B_2 B_1$, i.e., (2) $\Rightarrow$ (3). Suppose that (3) holds,

let D be right ideal and I be an ideal of R . Then D and I be bi-ideals of R , because every right ideal and two-sided ideal of R is bi-ideal of R by the Lemma 2.6. Now $D \cap I = DI \cap ID$, this implies that $D \cap I \subseteq DI \cap ID$. Thus $D \cap I \subseteq DI$ and $D \cap I \subseteq ID$, where I is also a left ideal of R . Since $DI \subseteq D \cap I$, i.e., $DI = D \cap I$. Thus R is regular by the Theorem 1. Also, $D \cap I \subseteq ID$, Thus R is an intra-regular by the Theorem 6. Hence (3) $\Rightarrow$ (1). $\square$

Theorem 11. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular and an intra-regular.
- (2) Every quasi-ideal of R is an idempotent.

Proof. Suppose that (1) holds. Let Q be a quasi-ideal of R and $Q^2 \subseteq Q$. Let $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y_i, z_i \in R$ such that $a = (y_i a^2)z_i$. Now $a = (ax)a = (ax)((y_i a^2)z_i) = ((y_i a^2)z_i)x a = ((aw_i)(aa))a$, where $((y_i a^2)z_i)x = (aw_i)(aa)$, by the Theorem 10. Thus $a = ((aw_i)(aa))a \in ((QR)Q)Q \subseteq QQ \subseteq Q^2$, i.e., $Q \subseteq Q^2$, because every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Thus $Q^2 = Q$, i.e., (1) $\Rightarrow$ (2). Assume that (2) is true. Let $a \in R$, then Ra is a left ideal of R containing a , i.e., Ra is a quasi-ideal of R , because every left ideal of R is a quasi-ideal of R . Now $Ra = (Ra)^2 = (Ra)(Ra)$, i.e., $a \in (Ra)(Ra)$, thus R is an intra-regular by the Theorem 6. Now $Ra = (Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$, i.e., $a \in (aR)(Ra)$, thus R is regular by the Theorem 1. Hence (2) $\Rightarrow$ (1). $\square$

Theorem 12. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular and an intra-regular.
- (2) Every quasi-ideal of R is an idempotent.
- (3) Every bi-ideal of R is an idempotent.

Proof. (1) $\Rightarrow$ (3), by the Theorem 10. (3) $\Rightarrow$ (2), every quasi-ideal of R is a bi-ideal of R , by the Lemma 2.9. (2) $\Rightarrow$ (1), by the Theorem 11. $\square$

Theorem 13. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular and an intra-regular.
- (2) $Q_1 \cap Q_2 \subseteq Q_1 Q_2$ for all quasi-ideals Q_1, Q_2 of R .
- (3) $Q \cap B \subseteq QB$ for every quasi-ideal Q and for every bi-ideal B of R .
- (4) $B \cap Q \subseteq BQ$ for every bi-ideal B and for every quasi-ideal Q of R .
- (5) $B_1 \cap B_2 \subseteq B_1 B_2$ for all bi-ideals B_1, B_2 of R .

Proof. Suppose that (1) holds. Let B_1, B_2 be bi-ideals of R , then $B_1 \cap B_2$ be also a bi-ideal of R . Since every bi-ideal of R is an idempotent by the Theorem 10, then $B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 B_2$, i.e., (1) $\Rightarrow$ (5). Since (5) $\Rightarrow$ (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3) $\Rightarrow$ (2), because every quasi-ideal of R is

a bi-ideal of R by the Lemma 2.9. Assume that (2) is true. Now $D \cap L \subseteq DL$, where D is a right ideal and L is a left ideal of R . Since $DL \subseteq D \cap L$, i.e., $D \cap L = DL$, thus R is regular. Again by (2) $L \cap D \subseteq LD$, thus R is an intra-regular. Hence (2) $\Rightarrow$ (1). $\square$

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Fuzzy bi-ideals in LA-rings

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Abstract. In this paper, we give the characterizations of different classes of LA-ring in terms of fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$ -) ideals.

Keywords: Fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$ -) ideals.

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1. Introduction

In 1972, a generalization of abelian semigroups initiated by Naseerdin et al [6]. In ternary commutative (abelian) law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid. Ideals in LA-semigroup have been investigated by [14].

In [4] (resp. [1]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [6], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Stevanovic et al [14] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [5], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup S is said to be a left almost group, if there exists left identity $e \in S$ such that $ea = a$ for all $a \in S$ and for every $a \in S$ there exists $b \in S$ such that $ba = e$.

Rehman et al [16], initiated the concept of left almost ring (abbreviated as LA-ring) of finitely nonzero functions which is a generalization of a commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

A non-empty subset A of an LA-ring R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R .

An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is called a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. Every bi-ideal of R is a generalized bi-ideal of R . An LA-subring A of R is called a $(1, 2)$ -ideal of R if $(AR)A^2 \subseteq A$.

We will initiate the concept of regular (resp. left regular, right regular, $(2, 2)$ -regular, left weakly regular, right weakly regular, intra-regular) LA-rings. We will also define the concept of fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$ -) ideals.

We will describe a study of regular (resp. left regular, right regular, $(2, 2)$ -regular, left weakly regular, right weakly regular, intra-regular) LA-rings by the properties of fuzzy left (right, bi-, generalized bi-) ideals. In this regard, we will prove that in regular (resp. left weakly regular) LA-rings, the concept of fuzzy

(right, two-sided) ideals coincides. We will also show that in right regular (resp. $(2, 2)$ -regular, right weakly regular, intra-regular) LA-rings, the concept of fuzzy (left, right, two-sided) ideals coincides. Also in left regular LA-rings with left identity, the concept of fuzzy (left, right, two-sided) ideals coincides. We will also characterize left weakly regular LA-rings in terms of fuzzy right (two-sided, bi-, generalize bi-) ideals.

2. Basic definitions and preliminary results

First time concept of fuzzy set introduced by Zadeh in his classical paper [19]. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems like (set, group, semigroup, ring, near-ring, semiring, measure) theory, groupoids, real analysis, topology, differential equations and so forth. Rosenfeld [?], was the first, who introduced the concept of fuzzy set in a group. The study of fuzzy set in semigroups was established by Kuroki [9]. He studied fuzzy ideals and fuzzy (interior, quasi-, bi-, generalized bi-, semiprime) ideals of semigroups. Liu [11], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [3, 10, 12, 13, 18]). Gupta et al [3], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [10], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (resp. right, quasi, bi-) ideals.

By a fuzzy subset μ of an LA-ring R , we mean a function $\mu : R \rightarrow [0, 1]$ and the complement of μ is denoted by μ' , is also a fuzzy subset of R defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in R$.

A fuzzy subset μ of an LA-ring R is a fuzzy LA-subring of R , if $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Equivalent definition: A fuzzy subset μ of an LA-ring R is a fuzzy LA-subring of R , if $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(-x) \geq \mu(x)$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

μ is a fuzzy left (resp. right) ideal of R , if $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$) for all $x, y \in R$.

μ is a fuzzy ideal of R , if it is both a fuzzy left and a fuzzy right ideal of R . Every fuzzy ideal (whether left, right, two-sided) of R is a fuzzy LA-subring of R , here we are giving the example which show that the converse is not true.

Example 1. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define $+$ and $\cdot$ in R as follows :

$+$	0	1	2	3	4	5	6	7		$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	2	0	3	1	6	4	7	5		1	0	4	4	0	0	4	4	0
2	1	3	0	2	5	7	4	6		2	0	4	4	0	0	4	4	0
3	3	2	1	0	7	6	5	4	and	3	0	0	0	0	0	0	0	0
4	4	5	6	7	0	1	2	3		4	0	3	3	0	0	3	3	0
5	6	4	7	5	2	0	3	1		5	0	7	7	0	0	7	7	0
6	5	7	4	6	1	3	0	2		6	0	7	7	0	0	7	7	0
7	7	6	5	4	3	2	1	0		7	0	3	3	0	0	3	3	0

Then R is an LA-ring and μ be a fuzzy subset of R . We define $\mu(0) = \mu(4) = 0.7$, $\mu(1) = \mu(2) = \mu(3) = \mu(5) = \mu(6) = \mu(7) = 0$. Then μ is a fuzzy LA-subring of R , but not fuzzy right ideal of R as

$$\begin{aligned}\mu(41) &= \mu_A(3) = 0. \\ \mu(4) &= 0.7. \\ \Rightarrow \mu(41) &\not\geq \mu(4).\end{aligned}$$

A fuzzy LA-subring μ of an LA-ring R is a fuzzy bi-ideal of R if $\mu((xa)y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, a \in R$.

A fuzzy subset μ of an LA-ring R is a fuzzy generalized bi-ideal of R if $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu((xa)y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, a \in R$. Every fuzzy bi-ideal of R is a fuzzy generalized bi-ideal of R .

A fuzzy LA-subring μ of an LA-ring R is called a fuzzy $(1, 2)$ -ideal of R if $\mu((xa)(yz)) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z, a \in R$.

Let A be a non-empty subset of an LA-ring R . Then the characteristic function of A is denoted by χ_A and defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The product of two fuzzy subsets μ and ν is denoted by $\mu \circ \nu$ and defined as:

$$(\mu \circ \nu)(x) = \begin{cases} \bigvee_{x=\sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{\mu(a_i) \wedge \nu(b_i)\} \}, & \text{if } x = \sum_{i=1}^n a_i b_i, \ a_i, b_i \in R \\ 0, & \text{if } x \neq \sum_{i=1}^n a_i b_i. \end{cases}$$

Now we are giving the some fundamental properties, which will be very helpful in the next section.

Theorem 1. *Let A and B be two non-empty subsets of an LA-ring R . Then the following conditions hold.*

- (1) *If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.*
- (2) *$\chi_A \circ \chi_B = \chi_{AB}$.*
- (3) *$\chi_A \cup \chi_B = \chi_{A \cup B}$.*
- (4) *$\chi_A \cap \chi_B = \chi_{A \cap B}$.*

Proof. Straight forward. □

Example 2. Let $R = \{a, b, c, d\}$. Define $+$ and $\cdot$ in R as follows :

$+$	a	b	c	d		$\cdot$	a	b	c	d
a	a	b	c	d		a	a	a	a	a
b	d	a	b	c	and	b	a	b	a	b
c	c	d	a	b		c	a	a	c	c
d	b	c	d	a		d	a	b	c	d

Then R is an LA-ring and μ be a fuzzy subset of R . We define $\mu(a) = \mu(c) = 0.7$, $\mu(b) = \mu(d) = 0$. Then μ is a fuzzy ideal of R .

Lemma 2.1. *Every fuzzy left (resp. right, two-sided) ideal of an LA-ring R is a fuzzy bi-ideal of R . But the converse is not true in general.*

Proof. Straight forward. □

Example 3. Above μ in example 1 is also a fuzzy bi-ideal of R , but not right ideal of R .

Lemma 2.2. *Every fuzzy bi-ideal of an LA-ring R is a fuzzy $(1, 2)$ -ideal of R .*

Proof. Straight forward. □

Remark 1. Every fuzzy left (resp. right, two-sided) ideal of an LA-ring R is a fuzzy $(1, 2)$ -ideal of R .

Proposition 2.1. *Let R be an LA-ring having the property $a = a^2$ for every $a \in R$. Then every fuzzy $(1, 2)$ -ideal of R is a fuzzy bi-ideal of R .*

Proof. Suppose that μ is a fuzzy $(1, 2)$ -ideal of R and $a, x, y \in R$. Thus

$$\mu((xa)y) = \mu((xa)(yy)) \geq \min\{\mu(x), \mu(y), \mu(y)\} = \min\{\mu(x), \mu(y)\}.$$

Therefore μ is a fuzzy bi-ideal of R . □

Lemma 2.3. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is an LA-subring of R if and only if the characteristic function χ_A of A is a fuzzy LA-subring of R .*

Proof. Straight forward. □

Lemma 2.4. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a left (resp. right) ideal of R if and only if the characteristic function χ_A of A is a fuzzy left (resp. right) ideal of R .*

Proof. Straight forward. $\square$

Proposition 2.2. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a $(1, 2)$ -ideal of R if and only if the characteristic function χ_A of A is a fuzzy $(1, 2)$ -ideal of R .*

Proof. Let A be a $(1, 2)$ -ideal of R , this implies that A is an LA-subring of R . Then χ_A is a fuzzy LA-subring of R , by the Lemma 2.3. Let $a, x, y, z \in R$. If $x, y, z \in A$, then by definition $\chi_A(x) = 1 = \chi_A(y) = \chi_A(z)$. Since $(xa)(yz) \in A$, A being a $(1, 2)$ -ideal, so $\chi_A((xa)(yz)) = 1$. Thus $\chi_A((xa)(yz)) \geq \min\{\chi_A(x), \chi_A(y), \chi_A(z)\}$. Similarly, we have $\chi_A((xa)(yz)) \geq \min\{\chi_A(x), \chi_A(y), \chi_A(z)\}$, when $x, y, z \notin A$. Hence the characteristic function χ_A of A is a fuzzy $(1, 2)$ -ideal of R .

Conversely, assume that the characteristic function χ_A of A is a fuzzy $(1, 2)$ -ideal of R , this implies that χ_A is a fuzzy LA-subring of R . Then A is an LA-subring of R by the Lemma 2.3. Let $t \in (AR)A^2$, so $t = (xa)(yz)$, where $x, y, z \in A$, $a \in R$. So by definition $\chi_A(x) = 1 = \chi_A(y) = \chi_A(z)$. Since $\chi_A((xa)(yz)) \geq \chi_A(x) \wedge \chi_A(y) \wedge \chi_A(z) = 1$, χ_A being a fuzzy $(1, 2)$ -ideal of R . Thus $\chi_A((xa)(yz)) = 1$, i.e., $(xa)(yz) \in A$. Hence A is a $(1, 2)$ -ideal of R . $\square$

Remark 2. Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a bi-ideal of R if and only if the characteristic function χ_A of A is a fuzzy bi-ideal of R .

Zadeh [19], introduced the concept of level set. Das [2], studied the fuzzy groups, level subgroups and gave the proper definition of a level set such that: let μ be a fuzzy subset of a non-empty set S , for $t \in [0, 1]$, the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$, is called a level subset of the fuzzy subset μ .

Let μ be a fuzzy subset of an LA-ring R , then for all $t \in (0, 1]$, we define a set $U(\mu; t) = \{x \in R \mid \mu(x) \geq t\}$, which is called an upper t -level of μ .

Lemma 2.5. *Let μ be a fuzzy subset of an LA-ring R . Then μ is a fuzzy LA-subring of R if and only if upper t -level $U(\mu; t)$ of μ is an LA-subring of R for all $t \in (0, 1]$.*

Proof. Straight forward. $\square$

Lemma 2.6. *Let μ be a fuzzy subset of an LA-ring R . Then μ is a fuzzy left (resp. right) ideal of R if and only if upper t -level $U(\mu; t)$ of μ is a left (resp. right) ideal of R for all $t \in (0, 1]$.*

Proof. Straight forward. $\square$

Proposition 2.3. *Let μ be a fuzzy subset of an LA-ring R . Then μ is a fuzzy $(1, 2)$ -ideal of R if and only if upper t -level $U(\mu; t)$ of μ is a $(1, 2)$ -ideal of R for all $t \in (0, 1]$.*

Proof. Suppose that μ is a fuzzy $(1, 2)$ -ideal of R , this implies that μ is a fuzzy LA-subring of R . Then $U(\mu; t)$ is an LA-subring of R by the Lemma 2.5. Let $x, y, z \in U(\mu; t)$ and $a \in R$, then by definition $\mu(x), \mu(y), \mu(z) \geq t$. Now $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \geq t$, μ being a fuzzy $(1, 2)$ -ideal of R , i.e., $(xa)(yz) \in U(\mu; t)$. Therefore $U(\mu; t)$ is a $(1, 2)$ -ideal of R .

Conversely, assume that $U(\mu; t)$ is a $(1, 2)$ -ideal of R , this means that $U(\mu; t)$ is an LA-subring of R . Then μ is a fuzzy LA-subring of R by the Lemma 2.5. We have to show that $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$. We suppose a contradiction $\mu((xa)(yz)) < \mu(x) \vee \mu(y) \vee \mu(z)$. Let $\mu(x) = t = \mu(y) = \mu(z)$, so $\mu(x), \mu(y), \mu(z) \geq t$, i.e., $x, y, z \in U(\mu; t)$. But $\mu((xa)(yz)) < t$, i.e., $(xa)(yz) \notin U(\mu; t)$, which is a contradiction. Therefore $\mu((xa)(yz)) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$. $\square$

Remark 3. Let μ be a fuzzy subbset of an LA-ring R . Then μ is a fuzzy bi-ideal of R if and only if upper t -level $U(\mu; t)$ of μ is a bi-ideal of R for all $t \in (0, 1]$.

3. Characterizations of LA-rings

In this section, we characterize different classes of LA-ring in terms of fuzzy left (right, bi-, generalized bi-) ideals. An LA-ring R is a regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = (xa)x$. An LA-ring R is an intra-regular, if for every element $x \in R$, there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$.

An LA-ring R is a left (resp. right) regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = ax^2$ (resp. $x^2 a$). An LA-ring R is completely regular if it is regular, left regular and right regular.

An LA-ring R is a $(2, 2)$ -regular if for every element $x \in R$, there exists an element $a \in R$ such that $x = (x^2 a)x^2$.

An LA-ring R is a locally associative LA-ring if $(a.a).a = a.(a.a)$ for all $a \in R$.

A ring R is a left (resp. right) weakly regular if $I^2 = I$ for every left (resp. right) ideal I of R , equivalently $x \in RxRx$ ($x \in xRxR$) for every $x \in R$. An LA-ring R is called a weakly regular if it is both a left weakly regular and a right weakly regular [15].

Now we define this notion in a class of non-associative and non-commutative rings (LA-ring).

An LA-ring R is called a left (resp. right) weakly regular if for every element $x \in R$, there exist elements $a, b \in R$ such that $x = (ax)(bx)$ (resp. $x = (xa)(xb)$). An LA-ring R is called a weakly regular if it is both a left weakly regular and a right weakly regular.

Lemma 3.1. *Every fuzzy right ideal of an LA-ring R with left identity e , is a fuzzy ideal of R .*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$. Thus $\mu(xy) = \mu((ex)y) = \mu((yx)e) \geq \mu(yx) \geq \mu(y)$. Hence μ is a fuzzy ideal of R . $\square$

Lemma 3.2. *Every fuzzy right ideal of a regular LA-ring R is a fuzzy ideal of R .*

Proof. Suppose that μ is a fuzzy right ideal of R . Let $x, y \in R$, this implies that there exists $a \in R$, such that $x = (xa)x$. Thus $\mu(xy) = \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(yx) \geq \mu(y)$. Therefore μ is a fuzzy ideal of R . $\square$

Proposition 3.1. *Let R be a regular LA-ring having the property $a = a^2$ for every $a \in R$, with left identity e . Then every fuzzy generalized bi-ideal of R is a fuzzy bi-ideal of R .*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = (xa)x$. We have to show that μ is a fuzzy LA-subring of R . Thus

$$\begin{aligned}\mu(xy) &= \mu(((xa)x)y) = \mu(((xa)x^2)y) = \mu(((xa)(xx))y) \\ &= \mu(x((xa)x)y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Hence μ is a fuzzy LA-subring of R . $\square$

Lemma 3.3 ([7, Lemma 8]). *Let R be an LA-ring with left identity e . Then Ra is the smallest left ideal of R containing a .*

Lemma 3.4 ([7, Lemma 9]). *Let R be an LA-ring with left identity e . Then aR is a left ideal of R .*

Proposition 3.2 ([7, Proposition 5]). *Let R be an LA-ring with left identity e . Then $aR \cup Ra$ is the smallest right ideal of R containing a .*

Lemma 3.5. *Let R be an LA-ring. Then $\mu \circ \nu \subseteq \mu \cap \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R .*

Proof. Let μ be a fuzzy right and ν be a fuzzy left ideal of R and $x \in R$. If x cannot be expressible as $x = \sum_{i=1}^n a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obviously $\mu \circ \nu \subseteq \mu \cap \nu$, otherwise we have

$$\begin{aligned}(\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n \{\mu(a_i) \wedge \nu(b_i)\} \right\} \\ &\leq \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n \{\mu(a_i b_i) \wedge \nu(a_i b_i)\} \right\} \\ &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n (\mu \cap \nu)(a_i b_i) \right\} = (\mu \cap \nu)(x). \\ \Rightarrow \mu \circ \nu &\subseteq \mu \cap \nu.\end{aligned}$$

$\square$

Theorem 2. *Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a regular.
- (2) $\mu \cap \nu = \mu \circ \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R .

Proof. Suppose that (1) holds. Since $\mu \circ \nu \subseteq \mu \cap \nu$, for every fuzzy right ideal μ and every fuzzy left ideal ν of R by the Lemma 3.5. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \{\wedge_{i=1}^n \{\mu(a_i) \wedge \nu(b_i)\}\} \\ &\geq \min\{\mu(xa), \nu(x)\} \geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \wedge \nu)(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu. \end{aligned}$$

Hence $\mu \cap \nu = \mu \circ \nu$, i.e., (1) $\Rightarrow$ (2). Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. So χ_{Ra} is a fuzzy left ideal and $\chi_{aR \cup Ra}$ is a fuzzy right ideal of R , by the Lemma 2.4. By our assumption $\chi_{aR \cup Ra} \cap \chi_{Ra} = \chi_{aR \cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR \cup Ra) \cap Ra} = \chi_{(aR \cup Ra)Ra}$. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. Now $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$. This implies that

$$(aR)(Ra) \cup (Ra)(Ra) = (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).$$

Thus $a \in (aR)(Ra)$. Then

$$\begin{aligned} a &= (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a \\ &= ((xe)(ay))a = (a((xe)y))a \in (aR)a, \text{ for any } x, y \in R. \end{aligned}$$

This means that $a \in (aR)a$, i.e., a is regular. Hence R is a regular, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 3. *Let R be a regular locally associative LA-ring having the property $a = a^2$ for every $a \in R$. Then for every fuzzy bi-ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (a^2x)a^2$, because $a = a^2$. Thus

$$\begin{aligned} \mu(a) &= \mu((a^2x)a^2) \geq \min\{\mu(a^2), \mu(a^2)\} = \mu(a^2) \\ &= \mu(aa) \geq \min\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^2). \end{aligned}$$

Now $a^2 = aa = ((a^2x)a^2)((a^2x)a^2) = (a^4x^2)a^4$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^ka = ((a^{2k}x^k)a^{2k})((a^2x)a^2) = (a^{2(k+1)}x^{k+1})a^{2(k+1)}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu((a^{2(k+1)}x^{k+1})a^{2(k+1)}) \geq \min\{\mu(a^{2(k+1)}), \mu(a^{2(k+1)})\} \\ &= \mu(a^{2(k+1)}) = \mu(a^{k+1}a^{k+1}) \\ &\geq \min\{\mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}). \\ \Rightarrow \mu(a^{k+1}) &= \mu(a^{2(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. $\square$

Lemma 3.6. *Let R be a $(2, 2)$ -regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R .*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exists an element $a \in R$ such that $x = (x^2a)x^2$. Thus $\mu(xy) = \mu((x^2a)x^2)y) = \mu((y^2a)y^2)x) \geq \mu(y^2a) \geq \mu(y)$. Hence μ is a fuzzy ideal of R . Similarly, for left ideal. $\square$

Remark 4. The concept of fuzzy (left, right, two-sided) ideals coincides in $(2, 2)$ -regular LA-rings.

Proposition 3.3. *Every fuzzy generalized bi-ideal of $(2, 2)$ -regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Suppose that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists an element $a \in R$ such that $x = (x^2a)x^2$. We have to show that μ is a fuzzy LA-subring of R . Thus

$$\begin{aligned}\mu(xy) &= \mu(((x^2a)x^2)y) = \mu(((x^2a)(xx))y) \\ &= \mu((x((x^2a)x))y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore μ is a fuzzy LA-subring of R . $\square$

Theorem 4. *Let R be a $(2, 2)$ -regular locally associative LA-ring. Then for every fuzzy bi-ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. Same as Theorem 3. $\square$

Lemma 3.7. *Let R be a right regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R .*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = x^2a$. Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \\ &= \mu((yx)(ax)) \geq \mu(yx) \geq \mu(y).\end{aligned}$$

Hence μ is a fuzzy ideal of R . Similarly, for left ideal. $\square$

Remark 5. The concept of fuzzy (left, right, two-sided) ideals coincides in right regular LA-rings.

Proposition 3.4. *Every fuzzy generalized bi-ideal of a right regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Suppose that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = x^2a$. We have to show that μ is a fuzzy LA-subring of R . Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)(ea))y) = \mu(((ae)(xx))y) \\ &= \mu((x((ae)x))y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore μ is a fuzzy LA-subring of R . $\square$

Lemma 3.8. *Let R be a left regular LA-ring with left identity e . Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R .*

Proof. Assume that μ is a fuzzy right ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = ax^2$. Thus

$$\begin{aligned}\mu(xy) &= \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \\ &= \mu(y(ax)x) \geq \mu(y(ax)) \geq \mu(y).\end{aligned}$$

So μ is a fuzzy ideal of R . Similarly, for left ideal. $\square$

Remark 6. The concept of fuzzy (left, right, two-sided) ideals coincides in left regular LA-rings with left identity.

Proposition 3.5. *Every fuzzy generalized bi-ideal of a left regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = ax^2$. We have to show that μ is a fuzzy LA-subring of R . Thus $\mu(xy) = \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \geq \min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy LA-subring of R . $\square$

Theorem 5. *Let R be a regular and right regular locally associative LA-ring. Then for every fuzzy right ideal μ of R , $\mu(a^n) = \mu(a^{3n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this means that there exists an element $x \in R$ such that $a = (ax)a$ and $a = a^2x$. Now $a = (ax)a = (ax)(a^2x) = a^3x^2$. Thus

$$\begin{aligned}\mu(a) &= \mu(a^3x^2) \geq \mu(a^3) = \mu(aa^2) \geq \min\{\mu(a), \mu(a^2)\} \\ &\geq \min\{\mu(a), \mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^3).\end{aligned}$$

Here $a^2 = aa = (a^3x^2)(a^3x^2) = a^6x^4$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{3k})$. Now $a^{k+1} = a^ka = (a^{3k}x^{2k})(a^3x^2) = a^{3(k+1)}x^{2(k+1)}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu(a^{3(k+1)}x^{2(k+1)}) \geq \mu(a^{3(k+1)}) = \mu(a^{3k+3}) \\ &= \mu(a^{k+1}a^{2k+2}) \geq \min\{\mu(a^{k+1}), \mu(a^{2k+2})\} \\ &\geq \min\{\mu(a^{k+1}), \mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{3(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. $\square$

Theorem 6. Let R be a right regular locally associative LA-ring. Then for every fuzzy right ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.

Proof. For $n = 1$. Let $a \in R$, then there exists an element $x \in R$ such that $a = a^2x$. Thus

$$\begin{aligned}\mu_A(a) &= \mu_A(a^2x) \geq \mu_A(a^2) = \mu_A(aa) \\ &\geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a) \Rightarrow \mu(a) = \mu(a^2).\end{aligned}$$

Now $a^2 = aa = (a^2x)(a^2x) = a^4x^2$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^ka = (a^{2k}x^k)(a^2x) = a^{2(k+1)}x^{(k+1)}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu(a^{2(k+1)}x^{(k+1)}) \geq \mu(a^{2(k+1)}) \\ &= \mu(a^{2k+2}) = \mu(a^{k+1}a^{k+1}) \\ &\geq \min\{\mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}) \Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. $\square$

Lemma 3.9. Let R be a right regular locally associative LA-ring with left identity e . Then for every fuzzy right ideal μ of R , $\mu(ab) = \mu(ba)$ for all $a, b \in R$.

Proof. Let $a, b \in R$. By using Theorem 6 (for $n = 1$). Now

$$\begin{aligned}\mu(ab) &= \mu((ab)^2) = \mu((ab)(ab)) \\ &= \mu((ba)(ba)) = \mu((ba)^2) = \mu(ba).\end{aligned}$$

$\square$

Remark 7. It is easy to see that, if R is a left regular locally associative LA-ring with left identity e . Then for every fuzzy left ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer. And also for every fuzzy left ideal μ of R , $\mu(ab) = \mu(ba)$ for all $a, b \in R$.

Lemma 3.10. *Let R be a right weakly regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R .*

Proof. Suppose that μ is a fuzzy right ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (xa)(xb)$. Thus

$$\begin{aligned}\mu(xy) &= \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \\ &= \mu((((ab)x)x)y) = \mu((yx)((ab)x)) \\ &= \mu((yx)(nx)) \text{ say } ab = n \\ &\geq \mu(yx) \geq \mu(y).\end{aligned}$$

Therefore μ is a fuzzy ideal of R . Similarly, for left ideal. $\square$

Remark 8. The concept of fuzzy (left, right, two-sided) ideals coincides in right weakly regular LA-rings.

Proposition 3.6. *Every fuzzy generalized bi-ideal of a right weakly regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Assume that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (xa)(xb)$. We have to show that μ is a fuzzy LA-subring of R . Thus $\mu(xy) = \mu(((xa)(xb))y) = \mu((x((xa)b))y) \geq \min\{\mu(x), \mu(y)\}$. So μ is a fuzzy LA-subring of R . $\square$

Lemma 3.11. *Let R be a left weakly regular LA-ring. Then every fuzzy right ideal of R is a fuzzy ideal of R .*

Proof. Let μ be a fuzzy right ideal of R and $x, y \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus $\mu(xy) = \mu(((ax)(bx))y) = \mu((y(bx))(ax)) \geq \mu(y(bx)) \geq \mu(y)$. Hence μ is a fuzzy ideal of R . $\square$

Lemma 3.12. *Let R be a left weakly regular LA-ring with left identity e . Then every fuzzy left ideal of R is a fuzzy ideal of R .*

Proof. Suppose that μ is a fuzzy left ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus

$$\begin{aligned}\mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) = \mu((y((ab)x))x) \geq \mu(x).\end{aligned}$$

Therefore μ is a fuzzy ideal of R . $\square$

Proposition 3.7. *Every fuzzy generalized bi-ideal of a left weakly regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Assume that μ is a fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. We have to show that μ is a fuzzy LA-subring of R . Thus

$$\begin{aligned}\mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) \geq \min\{\mu(x), \mu(y)\}.\end{aligned}$$

So μ is a fuzzy LA-subring of R . $\square$

Remark 9. It is easy to see that, if R is a left (resp. right) weakly regular locally associative LA-ring. Then for every fuzzy left (resp. right) ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.

Theorem 7. *Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a left weakly regular.
- (2) $\mu \cap \nu = \mu \circ \nu$ for every fuzzy right ideal μ and for every fuzzy left ideal ν of R .

Proof. Suppose that (1) holds. Since $\mu \circ \nu \subseteq \mu \cap \nu$, for every fuzzy right ideal μ and every fuzzy left ideal ν of R by the Lemma 3.5. Let $x \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Now

$$\begin{aligned}(\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n \{\mu(a_i) \wedge \nu(b_i)\} \right\} \\ &\geq \mu(x) \wedge \nu((ab)x) \geq \mu(x) \wedge \nu(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu.\end{aligned}$$

Hence $\mu \cap \nu = \mu \circ \nu$, i.e., (1) $\Rightarrow$ (2). Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. So χ_{Ra} is a fuzzy left ideal and $\chi_{aR \cup Ra}$ is a fuzzy right ideal of R , by the Lemma 2.4. Then by our assumption $\chi_{aR \cup Ra} \cap \chi_{Ra} = \chi_{aR \cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR \cup Ra) \cap Ra} = \chi_{(aR \cup Ra)Ra}$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that $a \in (aR)(Ra)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(Ra)$, then R is a left weakly regular. If $a \in (aR)(Ra)$, then

$$\begin{aligned}(aR)(Ra) &= ((ea)(RR))(Ra) = ((RR)(ae))(Ra) \\ &= (((ae)R)R)(Ra) = ((aR)R)(Ra) \\ &= ((RR)a)(Ra) = (Ra)(Ra).\end{aligned}$$

Hence R is a left weakly regular, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 8. *Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a left weakly regular.
- (2) $\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal μ and for every fuzzy ideal γ of R .
- (3) $\nu \cap \gamma \subseteq \nu \circ \gamma$ for every fuzzy generalized bi-ideal ν and for every fuzzy ideal γ of R .

Proof. Assume that (1) holds. Let ν be a fuzzy generalized bi-ideal and γ be a fuzzy ideal of R . Let $x \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Thus

$$\begin{aligned} (\mu \circ \gamma)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n \{\mu(a_i) \wedge \gamma(b_i)\} \right\} \\ &\geq \mu(x) \wedge \gamma((ab)x) \\ &\geq \mu(x) \wedge \gamma(x) = (\mu \cap \gamma)(x). \\ &\Rightarrow \nu \cap \gamma \subseteq \nu \circ \gamma. \end{aligned}$$

Therefore (1) $\Rightarrow$ (3). It is clear that (3) $\Rightarrow$ (2). Suppose that (2) holds. Then $\mu \cap \gamma \subseteq \mu \circ \gamma$, where μ is a fuzzy right ideal of R . Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \circ \gamma = \mu \cap \gamma$. Therefore R is a left weakly regular by the Theorem 7, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 9. *Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is a left weakly regular.
- (2) $\mu \cap \gamma \cap \delta \subseteq (\mu \circ \gamma) \circ \delta$ for every fuzzy bi-ideal μ , every fuzzy ideal γ and for every fuzzy right ideal δ of R .
- (3) $\nu \cap \gamma \cap \delta \subseteq (\nu \circ \gamma) \circ \delta$ for every fuzzy generalized bi-ideal ν , every fuzzy ideal γ and for every fuzzy right ideal δ of R .

Proof. Suppose that (1) holds. Let ν be a fuzzy generalized bi-ideal, γ be a fuzzy ideal and δ be a fuzzy right ideal of R . Let $x \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. Now

$$\begin{aligned} x &= (ax)(bx) = (xb)(xa) \\ xb &= ((ax)(bx))b = ((xx)(ba))b \\ &= (b(ba))(xx) = c(xx) = x(cx) \quad \text{say } c = b(ba) \end{aligned}$$

Thus

$$\begin{aligned} ((\nu \circ \gamma) \circ \delta)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left\{ \bigwedge_{i=1}^n \{(\nu \circ \gamma)(a_i) \wedge \delta(b_i)\} \right\} \\ &\geq (\nu \circ \gamma)(xb) \wedge \delta(xa) \end{aligned}$$

$$\begin{aligned}
&\geq (\nu \circ \gamma)(xb) \wedge \delta(x) \\
&= \bigvee_{xb=\sum_{i=1}^n p_i q_i} \left\{ \bigwedge_{i=1}^n \{\nu(p_i) \wedge \gamma(q_i)\} \right\} \wedge \delta(x) \\
&\geq \nu(x) \wedge \gamma(cx) \wedge \delta(x) \\
&\geq \nu(x) \wedge \gamma(x) \wedge \delta(x) = (\mu \cap \gamma \cap \delta)(x). \\
&\Rightarrow \mu \cap \gamma \cap \delta \subseteq (\nu \circ \gamma) \circ \delta.
\end{aligned}$$

Hence (1) $\Rightarrow$ (3). Since (3) $\Rightarrow$ (2), every fuzzy bi-ideal of R is a fuzzy generalized bi-ideal of R . Assume that (2) is true. Then $\mu \cap \gamma \cap R \subseteq (\mu \circ \gamma) \circ R$, where μ is a fuzzy right ideal of R , i.e., $\mu \cap \gamma \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \circ \gamma = \mu \cap \gamma$. Hence R is a left weakly regular by the Theorem 7, i.e., (2) $\Rightarrow$ (1). $\square$

Lemma 3.13. *Let R be an intra-regular LA-ring. Then every fuzzy left (resp. right) ideal of R is a fuzzy ideal of R .*

Proof. Suppose that μ is a fuzzy right ideal of R . Let $x, y \in R$, this implies that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Thus $\mu(xy) = \mu((a_i x^2) b_i y) = \mu((y b_i)(a_i x^2)) \geq \mu(y b_i) \geq \mu(y)$. Hence μ is a fuzzy ideal of R . Similarly, for left ideal. $\square$

Proposition 3.8. *Every fuzzy generalized bi-ideal of an intra-regular LA-ring R with left identity e , is a fuzzy bi-ideal of R .*

Proof. Let μ be a fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. We have to show that μ is a fuzzy LA-subring of R . Now

$$\begin{aligned}
x &= (a_i x^2) b_i = (a_i x^2)(e b_i) = (a_i e)(x^2 b_i) \\
&= (a_i e)((x x) b_i) = (a_i e)((b_i x) x) = (x(b_i x))(e a_i) \\
&= (x(b_i x)) a_i = (a_i(b_i x)) x = (a_i(b_i x))(e x) \\
&= (x e)((b_i x) a_i) = (b_i x)((x e) a_i) = (b_i x)((a_i e) x) \\
&= (x(a_i e))(x b_i) = x((x(a_i e)) b_i) = x n, \quad \text{say } n = (x(a_i e)) b_i
\end{aligned}$$

Thus $\mu(xy) = \mu((x n) y) \geq \min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy LA-subring of R . $\square$

Theorem 10. *Let R be an LA-ring with left identity e , such that $(x e) R = x R$ for all $x \in R$. Then the following conditions are equivalent.*

- (1) R is an intra-regular.
- (2) $\mu \cap \nu \subseteq \mu \circ \nu$ for every fuzzy right ideal ν and for every fuzzy left ideal μ of R .

Proof. Assume that (1) holds. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Now

$$\begin{aligned} x &= (a_i x^2) b_i = (a_i (x x)) b_i = (x (a_i x)) (e b_i) \\ &= (x e) ((a_i x) b_i) = (a_i x) ((x e) b_i). \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \nu)(x) &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \mu(a_i) \wedge \nu(b_i) \} \\ &\geq \min\{\mu(a_i x), \nu((x e) b_i)\} \geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \wedge \nu)(x) = (\mu \cap \nu)(x). \\ &\Rightarrow \mu \cap \nu \subseteq \mu \circ \nu. \end{aligned}$$

Hence (1) $\Rightarrow$ (2). Suppose that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.3 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.2. This means that χ_{Ra} is a fuzzy left ideal and $\chi_{aR \cup Ra}$ is a fuzzy right ideal of R , by the Lemma 2.4. By our supposition $\chi_{aR \cup Ra} \cap \chi_{Ra} \subseteq \chi_{Ra} \circ \chi_{aR \cup Ra}$, i.e., $\chi_{(aR \cup Ra) \cap Ra} \subseteq \chi_{(Ra)(aR \cup Ra)}$. Thus $(aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra)$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. Now

$$\begin{aligned} (Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)((ae)R)R = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra). \end{aligned}$$

This implies that

$$\begin{aligned} (Ra)(aR) \cup (Ra)(Ra) &= (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((Re)(aa))R \\ &= (Ra^2)R. \end{aligned}$$

Thus $a \in (Ra^2)R$, i.e., a is an intra regular. Therefore R is an intra-regular, i.e., (2) $\Rightarrow$ (1). $\square$

Theorem 11. Let R be an intra-regular locally associative LA-ring. Then for every fuzzy right ideal μ of R , $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.

Proof. For $n = 1$. Let $a \in R$, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^n (x_i a^2) y_i$. Thus

$$\begin{aligned} \mu(a) &= \mu((x_i a^2) y_i) \geq \mu(x_i a^2) \geq \mu(a^2) \\ &= \mu(aa) \geq \min\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^2). \end{aligned}$$

Now $a^2 = aa = ((x_i a^2)y_i)((x_i a^2)y_i) = (x_i^2 a^4)y_i^2$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^k a = ((x_i^k a^{2k})y_i^k)((x_i a^2)y_i) = (x_i^{k+1} a^{2(k+1)})y_i^{k+1}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu((x_i^{k+1} a^{2(k+1)})y_i^{k+1}) \geq \mu(x_i^{k+1} a^{2(k+1)}) \\ &\geq \mu(a^{2(k+1)}) = \mu(a^{k+1} a^{k+1}) \\ &\geq \min\{\mu(a^{(k+1)}), \mu(a^{(k+1)})\} = \mu(a^{(k+1)}) \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. $\square$

Proposition 3.9. *Let R be an intra-regular locally associative LA-ring with left identity e . Then for every fuzzy right ideal μ of R , $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Proof. Same as Lemma 3.9. $\square$

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Characterizations of new open and closed mappings in topological spaces

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Abstract. In this paper we present θ -semigeneralized pre-open maps, θ -semigeneralized pre-closed maps, pre- θ -semigeneralized pre-open maps, pre- θ -semigeneralized pre-closed maps by employing θ -sgp-closed sets. We also introduce higher separation axioms θ -semigeneralized pre-regular spaces and θ -semigeneralized pre-normal spaces using θ -sgp-closed sets in topological spaces and we investigate some new characterizations of these mappings and higher separation axioms.

Keywords: θ -sgp-open map, θ -sgp-closed map, pre- θ -sgp-open map, pre- θ -sgp-closed map, θ -sgp-regular space, θ -sgp-normal space.

1. Introduction

Malghan [10] defined generalized closed functions and obtained the preservation theorems of normality and regularity. Recently Arya and Nour [2] defined generalized semi-open sets and used them to obtain some characterizations of s-normal spaces due to Maheshwari and Prasad [9]. Dorsett ([5], [6]) defined and studied the concepts of semi-normal spaces and semi-regular spaces respectively. Quite recently, Devi et al. [4] have introduced generalized semi-closed functions and shown that the continuous generalized semi-closed surjective image of a normal space is s-normal. Present authors [13] introduced the notion of θ -sgp-closed set. Object of this article is to continue the study of θ -sgp-closed sets by intercalating new classes of mappings called θ -sgp-open map, θ -sgp-closed map, pre- θ -sgp-open map, pre- θ -sgp-closed map, contra pre- θ -sgp-open map and contra pre- θ -sgp-closed map using θ -sgp-open and θ -sgp-closed sets and also defined

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new spaces called θ -sgp-regular and θ -sgp-normal utilizing θ -sgp-closed set. We also investigated properties and characterizations of above mappings and spaces.

2. Preliminaries

In this paper, no separation axioms are taken up on the notations (X, τ) and (Y, σ) (or simply X and Y) which stands for topological spaces except clearly stated. If $M \subset X$ then $Cl(M)$ and $Int(M)$ stands for closure of M and the interior of M in X respectively.

We recall some definitions used in this paper to make it self-contained.

Definition 2.1. A subset A of a space X is called

- (i) a semi-open set [8] if $A \subset Cl(Int(A))$.
- (ii) a semi-closed set [2] if $Int(Cl(A)) \subset A$.
- (iii) a pre-open [11] if $A \subseteq (Int(Cl(A)))$

Definition 2.2. The pre- θ -closure denoted by $pCl_\theta(A)$, is the set of all pre- θ -cluster points of A . A subset A is called pre- θ -closed set [1] if $A = pCl_\theta(A)$. The complement of pre- θ -closed set is pre- θ -open set.

Definition 2.3. A subset A of a topological space X is called θ -semigeneralized pre-closed set [14] (briefly, θ -sgp-closed) if $pCl_\theta(A) \subset U$ whenever $A \subset U$ and U is semi-open in X . The complement of θ -semigeneralized pre-closed set is called θ -semigeneralized pre-open (briefly, θ -sgp-open).

Definition 2.4. A space X is said to be $T_{\theta sgp}$ [16] if every θ -sgp-closed set is semi-closed set.

Definition 2.5. A function $f : X \rightarrow Y$ is called θ -sgp-irresolute [16] (resp., semi-totally semi-continuous [7], θ -sgp-continuous [16], irresolute [3]) if $f^{-1}(F)$ is θ -sgp-closed (resp., both semi-open and semi-closed, θ -sgp-closed, semi-closed) in X for every θ -sgp-closed set (resp., semi-open set, semi-closed set, semi-closed set) F of Y .

Definition 2.6. For a subset A of X , θ -semigeneralized pre-closure [15] of A , denoted by $\theta\text{-sgpCl}(A)$ and is defined as $\theta\text{-sgpCl}(A) = \cap \{G : A \subseteq G, G \text{ is } \theta\text{-sgp-closed in } X\}$.

Definition 2.7. [18] A space X is said to be:

- (i) θ -sgp- T_0 , if for each pair $x, y \in X$ and $x \neq y$, there is a θ -sgp-open set containing one point but not the other.
- (ii) θ -sgp- T_1 , if for each pair of unequal points $x, y \in X$, there are disjoint θ -sgp-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
- (iii) θ -sgp- T_2 , if for each pair of unequal points $x, y \in X$, there are disjoint θ -sgp-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

3. θ -sgp-open and θ -sgp-closed mappings

Definition 3.1. Let $f : X \rightarrow Y$ be a map. Then f is called θ -sgp-open (resp., θ -sgp-closed) if $f(M)$ is θ -sgp-open set (resp., θ -sgp-closed set) in Y for every semi-open set (resp., semi-closed set) M in X .

Theorem 3.2. A map $f : X \rightarrow Y$ is θ -sgp-closed if and only if for each subset P of Y and for each semi-open set Q containing $f^{-1}(P)$ there is a semi-open set R of Y such that $P \subseteq R$ and $f^{-1}(R) \subseteq Q$.

Proof. Assume that f is θ -sgp-closed map. Let P be a subset of Y and Q be a semi-open set of X such that $P \subseteq f(Q)$, that is $f^{-1}(P) \subseteq Q$. Now Q^c is semi-closed set in X . Then $f(Q^c)$ is θ -sgp-closed in Y , since f is θ -sgp-closed map. So, $Y - f(Q^c)$ is θ -sgp-open in Y . Thus $R = Y - f(Q^c)$ is a θ -sgp-open set containing P such that $f^{-1}(R) \subseteq Q$.

Conversely, suppose that f is θ -sgp-closed map. Let F be a semi-closed set in X . Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is semi-open. By hypothesis, there is a θ -sgp-open set R of Y such that $Y - f(F) \subseteq R$ and $f^{-1}(R) \subseteq X - F$ and so $F \subseteq X - f^{-1}(R)$. Hence $Y - R \subseteq f(F) \subseteq f(X - f^{-1}(R)) \subseteq Y - R$ which implies $f(F) = Y - R$. Since $Y - R$ is θ -sgp-closed, $f(F)$ is θ -sgp-closed and thus f is θ -sgp-closed map.

Theorem 3.3. If $f : X \rightarrow Y$ is irresolute, pre- θ -closed and B is θ -sgp-closed subset of X , then $f(B)$ is θ -sgp-closed in Y .

Proof. Let M be a semi-open set in Y such that $f(B) \subseteq M$. Since f is irresolute, $f^{-1}(M)$ is a semi-open set containing B , that is $B \subseteq f^{-1}(M)$. Hence $pCl_\theta(B) \subseteq f^{-1}(M)$, as B is θ -sgp-closed in X . Since f is pre- θ -closed, $f(pCl_\theta(B))$ is pre- θ -closed. So, $f(pCl_\theta(B))$ is a θ -sgp-closed set contained in the semi-open set M , that is $f(pCl_\theta(B)) \subseteq M$. Now $pCl_\theta(f(B)) \subseteq pCl_\theta(f(pCl_\theta(B))) = f(pCl_\theta(B)) \subseteq M$. Hence $pCl_\theta(f(B)) \subseteq M$. Therefore $f(B)$ is θ -sgp-closed set in Y .

Remark 3.4. The composition of two θ -sgp-closed maps need not be θ -sgp-closed as seen from the following example.

Example 3.5. Let $X=Y=Z=\{p, q, r\}$, $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}$, $\sigma = \{\phi, \{p\}, \{q, r\}, Y\}$ and $\eta = \{\phi, \{p, q\}, Z\}$. Let $f : X \rightarrow Y$ be the identity map and define $g : Y \rightarrow Z$ by $g(p) = g(q) = r$ and $g(r) = p$. Then both f and g are θ -sgp-closed maps but their composition $g \circ f : X \rightarrow Z$ is not θ -sgp-closed map. Since for semi-closed set $\{r\}$ in X , $(g \circ f)(\{r\}) = g(f(\{r\})) = \{p\}$ which is not θ -sgp-closed set in Z .

Corollary 3.6. Let $f : X \rightarrow Y$ be a θ -sgp-closed map and $g : Y \rightarrow Z$ be pre- θ -closed and irresolute, then their composition $g \circ f : X \rightarrow Z$ is θ -sgp-closed.

Proof. Let B be a semi-closed set of X . Then by hypothesis $f(B)$ is θ -sgp-closed in Y . Since g is pre- θ -closed and irresolute, by theorem 3.3, $g(f(B)) = (g \circ f)(B)$ is θ -sgp-closed in Z . Hence $(g \circ f)$ is θ -sgp-closed.

Theorem 3.7. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be θ -sgp-closed maps and Y be a $T_{\theta sgp}$. Then their composition $(g \circ f)$ is θ -sgp-closed.

Proof. Let D be a semi-closed set of X . Then by hypothesis $f(D)$ is a θ -sgp-closed in Y . Since Y is a $T_{\theta sgp}$, $f(D)$ is semi-closed in Y . Also by assumption $g(f(D))$ is θ -sgp-closed in Z . Hence $(g \circ f)$ is θ -sgp-closed.

Theorem 3.8. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two maps such that their composition $g \circ f : X \rightarrow Z$ is a θ -sgp-open. Then the following statements are true.

- (i) If f is semi-totally semi-continuous and surjective, then g is θ -sgp-open.
- (ii) If g is θ -sgp-irresolute and injective, then f is θ -sgp-open.

Proof. (i) Let D be a semi-open set in Y . Then $f^{-1}(D)$ is both semi-open and semi-closed in X as f is semi-totally semi-continuous. Since $(g \circ f)$ is θ -sgp-open map and f is surjective, $(g \circ f)(f^{-1}(D)) = g(D)$ is θ -sgp-open set in Z . Therefore g is a θ -sgp-open in X .

(ii) Let H be a semi-open set of X . Since $(g \circ f)$ is θ -sgp-open, $(g \circ f)(H)$ is a θ -sgp-open set in Z . Since g is θ -sgp-irresolute, $g^{-1}((g \circ f)(H)) = g^{-1}(g(f(H))) = f(H)$ is θ -sgp-open set in Y , since g is injective. Thus f is a θ -sgp-open in X .

Theorem 3.9. For any bijection $f : X \rightarrow Y$ the following statements are identical:

- (i) f^{-1} is θ -sgp-continuous.
- (ii) f is a θ -sgp-open map.
- (iii) f is a θ -sgp-closed map.

Proof. (i) $\rightarrow$ (ii): Let Q be a semi-open set of X . By assumption $(f^{-1})^{-1}(Q) = f(Q)$ is θ -sgp-open set in Y and so f is θ -sgp-open map.

(ii) $\rightarrow$ (iii): Let P be a semi-closed set of X . Then P^c is semi-open in X . By assumption $f(P^c)$ is θ -sgp-open set in Y , that is $f(P^c) = (f(P))^c$ θ -sgp-open set in Y and therefore $f(P)$ is θ -sgp-closed set in Y . Hence f is θ -sgp-closed map.

(iii) $\rightarrow$ (i): Let P be a semi-closed set of X . By assumption $f(P)$ is θ -sgp-closed set in Y . But $(f^{-1})^{-1}(P) = f(P)$ and therefore f is θ -sgp-continuous.

Theorem 3.10. If $f : X \rightarrow Y$ is semi-open, θ -sgp-irresolute onto and M is a θ -sgp-closed set in Y , then $f^{-1}(M)$ is a θ -sgp-closed set in X .

Proof. Let J be any semi-open set of X containing $f^{-1}(M)$. Then $M \subset f(J)$ and $f(J)$ is semi-open in Y . Since M is θ -sgp-closed set in Y , $pCl_{\theta}(M) \subset f(J)$ and hence $f^{-1}(M) \subset f^{-1}(pCl_{\theta}(M)) \subset J$. Since f is θ -sgp-irresolute, $f^{-1}(pCl_{\theta}(M))$ is θ -sgp-closed in X and hence we have $pCl_{\theta}(f^{-1}(M)) \subset f^{-1}(pCl_{\theta}(M)) \subset J$. This shows that $f^{-1}(M)$ is θ -sgp-closed set in X .

Theorem 3.11. If $f : X \rightarrow Y$ is onto, θ -sgp-open. If X is θ -sgp- T_1 and $T_{\theta sgp}$, then Y is θ -sgp- T_1 .

Proof. Let m_1 and m_2 be two points of Y . Since f is onto, there exist distinct points n_1 and n_2 of X such that $f(n_1) = m_1$ and $f(n_2) = m_2$. Since X is a θ -sgp- T_1 , there exist θ -sgp-open sets J and K such that $n_1 \in J$ and $n_2 \notin J$ and $n_2 \in K$ and $n_1 \notin K$. Again since X is $T_{\theta sgp}$, J and K are semi-open sets in X . As f is θ -sgp-open map, $f(J)$ and $f(K)$ are θ -sgp-open sets such that $m_1 = f(n_1) \in f(J)$, $m_2 = f(n_2) \notin f(J)$ and $m_1 = f(n_1) \notin f(K)$, $m_2 = f(n_2) \in f(K)$. Hence Y is θ -sgp- T_1 .

4. Pre- θ -sgp-closed and pre- θ -sgp-open mappings

Definition 4.1. A map $f : X \rightarrow Y$ is called pre- θ -sgp-closed (resp., pre- θ -sgp-open) if $f(M)$ is θ -sgp-closed (resp., θ -sgp-open) in Y for every θ -sgp-closed (resp., θ -sgp-open) set M of X .

Theorem 4.2. If $f : X \rightarrow Y$ is semi-totally semi-continuous, pre- θ -sgp-closed and N is a θ -sgp-closed subset of X , then $f(N)$ is θ -sgp-closed set in Y .

Proof. Let M be a semi-open set such that $f(N) \subseteq M$. Since f is semi-totally semi-continuous $f^{-1}(M)$ is a semi-open set containing N , that is $N \subseteq f^{-1}(M)$. Hence $pCl_{\theta}(N) \subseteq f^{-1}(M)$ as N is θ -sgp-closed set in X . Since f is pre- θ -sgp-closed, $f(pCl_{\theta}(N))$ is θ -sgp-closed set. So $f(pCl_{\theta}(N))$ is θ -sgp-closed set contained in semi-open set M , that is $f(pCl_{\theta}(N)) \subseteq M$. Now $pCl_{\theta}(f(N)) \subseteq pCl_{\theta}(f(\theta - sgpCl(N))) \subseteq pCl_{\theta}(f(pCl_{\theta}(N))) = f(pCl_{\theta}(N)) \subseteq M$. Hence $pCl_{\theta}(f(N)) \subseteq M$. Therefore $f(N)$ is a θ -sgp-closed set in Y .

Theorem 4.3. The property of a space being θ -sgp- T_0 is preserved under one-one, onto, pre- θ -sgp-open map and hence is a topological property.

Proof. Let X be a θ -sgp- T_0 -space and Y be any other topological space. Let $f : X \rightarrow Y$ be one-one, onto, pre- θ -sgp-open map from X to Y . Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and since f is one-one, onto, there exist distinct points $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$. Since X is θ -sgp- T_0 , there exists a θ -sgp-open set J in X such that $x_1 \in J$ but $x_2 \notin J$. Since f is pre- θ -sgp-open map, $f(J)$ is θ -sgp-open set containing $f(x_1)$ but not containing $f(x_2)$. Thus there exist a θ -sgp-open set $f(J)$ in Y such that $y_1 \in f(J)$ but $y_2 \notin f(J)$ and hence Y is θ -sgp- T_0 . Again as the property of being θ -sgp- T_0 is preserved under one-one, onto map, it is also preserved under homeomorphism and hence is a topological property.

Theorem 4.4. Let $f : X \rightarrow Y$ be a θ -sgp-closed map and $g : Y \rightarrow Z$ be a pre- θ -sgp-closed map and semi-totally semi-continuous, then their composition $g \circ f : X \rightarrow Z$ is θ -sgp-closed.

Proof. Let K be a semi-closed set in X . Then by hypothesis $f(K)$ is θ -sgp-closed set Y . Since g is pre- θ -sgp-closed map and semi-totally semi-continuous, by Theorem 4.2 $g(f(K)) = (g \circ f)(K)$ is θ -sgp-closed set in Z . Hence $(g \circ f)$ is θ -sgp-closed.

Definition 4.5. A map $f : X \rightarrow Y$ is called contra pre- θ -sgp-open if for every θ -sgp-open set O of X , $f(O)$ is θ -sgp-closed set in Y .

Definition 4.6. A map $f : X \rightarrow Y$ is called contra pre- θ -sgp-closed if for every θ -sgp-closed set F of X , $f(F)$ is θ -sgp-open set in Y .

The following examples show that contra pre- θ -sgp-closedness and contra pre- θ -sgp-openness are independent.

Example 4.7. Let $X=Y= \{p, q, r\}$, $\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$ and $\sigma = \{Y, \phi, \{p\}, \{q\}, \{p, q\}\}$. We have θ -sgp-open sets in X are $\{X, \{p\}, \{q, r\}\}$ and θ -sgp-open sets in Y are $\{Y, \{p\}, \{q\}, \{p, r\}, \{q, r\}\}$. Let $f : X \rightarrow Y$ is defined by $f(p) = f(r) = r$, $f(q) = q$ and $g : X \rightarrow Y$ is by $g(p) = g(q) = p$, $g(r) = q$. Then f is contra pre- θ -sgp-open but not contra pre- θ -sgp-closed and g is contra pre- θ -sgp-closed but not contra pre- θ -sgp-open.

Remark 4.8. Contra pre- θ -sgp-closedness and contra pre- θ -sgp-openness are equivalent if the map is onto.

Theorem 4.9. For a map $f : X \rightarrow Y$ the following properties are equivalent:

- (i) f is contra pre- θ -sgp-open.
- (ii) For every subset D of Y and for every θ -sgp-closed subset E of X with $f^{-1}(D) \subseteq E$, there exists a θ -sgp-open subset G of Y with $D \subseteq G$ and $f^{-1}(G) \subseteq E$.
- (iii) For every point $q \in Y$ and every θ -sgp-closed subset E of X with $f^{-1}(q) \subseteq E$, there exists a θ -sgp-open subset G of Y with $q \in G$ and $f^{-1}(G) \subseteq E$.

Proof. (i) $\rightarrow$ (ii): Let D be a subset of Y and let E be a θ -sgp-closed subset of X with $f^{-1}(D) \subseteq E$. For the case $D \neq \phi$, put $G = [f(E^c)]^c$. Then $f^{-1}(G) = [f^{-1}(E^c)]^c \subseteq E$ and G is a θ -sgp-open subset of Y . We claim that $D \subseteq G$. There are two cases to be considered.

Case 1: $f^{-1}(D) \neq \phi$. Since $f^{-1}(D) \subseteq E$, we have $f(E^c) \subseteq D^c$ and $D \subseteq G$.

Case 2: $f^{-1}(D) = \phi$. Since $f^{-1}(D) = \phi \subseteq E$, we have $f(E^c) \subseteq f(X)$. We have $D \cap f(X) = \phi$, because $f^{-1}(D) = \phi$ and $D \neq \phi$. Thus $D \subseteq [f(X)]^c \subseteq [f(E^c)]^c = G$. For the case $D = \phi$, put $G = \phi$. Then the set G is required θ -sgp-open of Y .

(ii) $\rightarrow$ (iii): It suffices to put $D=q$, in (ii).

(iii) $\rightarrow$ (i): Let H be a θ -sgp-open subset of X . Then $q \in [f(H)]^c$ and $E = H^c$. First we claim that $f^{-1}(q) \subset E$. For non empty set $f^{-1}(q)$, let $p \in f^{-1}(q)$; $f(p) = q \notin f(H)$. Suppose that $p \notin H$ and so $q = f(p) \in f(H)$. By contradiction, $p \in E$ for any $p \in f^{-1}(q)$, that is $f^{-1}(q) = \phi \subseteq E$. For the case where $f^{-1}(q) = \phi$, we have $f^{-1}(q) = \phi \subseteq E$. For both cases, we can use (iii) and get the following: there exists a θ -sgp-open set $G_q \subseteq Y$ such that $q \in G_q$ and $f^{-1}(G_q) \subseteq E = H^c$. Namely, $(\star) f^{-1}(G_q) \cap H = \phi$ holds for each $q \in [f(H)]^c$. Finally we claim that $[f(H)]^c \subseteq \bigcup \{G_q : q \in [f(H)]^c\}$. Obviously, we have that $[f(H)]^c \subseteq \bigcup \{G_q : q \in [f(H)]^c\}$

Conversely, let $p \in \bigcup \{G_q : q \in [f(H)]^c\}$. Then there exists a point $r \in [f(H)]^c$ such that $p \in G_r$. Suppose that $p \notin [f(H)]^c$. Then $p \in f(H)$ and there exists a point $b \in H$ such that $f(b)=p$. Thus we have that $f(b) \in G_r$ and so $b \in f^{-1}(G_r)$. We have a contradiction to $(\star)$ above, that is $b \in f^{-1}(G_r) \cap H$. Hence, we show that $\bigcup \{G_q : q \in [f(H)]^c\} \subseteq [f(H)]^c$ and so $[f(H)]^c = \bigcup \{G_q : q \in [f(H)]^c\}$. Hence $f(H)$ is a θ -sgp-closed subset of Y .

5. θ -sgp-regular and θ -sgp-normal spaces

Definition 5.1. A topological space X is said to be θ -sgp-regular if for every θ -sgp-closed set P and each point x which is not in P , there exist disjoint semi-open sets M and N such that $x \in M, P \subseteq N$ and $M \cap N = \phi$.

Definition 5.2. A topological space X is said to be θ -sgp-normal if for every pair of disjoint θ -sgp-closed sets P_1 and P_2 in X , there exist disjoint semi-open sets M and N such that $P_1 \subset M, P_2 \subset N$ and $M \cap N = \phi$.

Theorem 5.3. Let X be a topological space. Then the following statements are identical: (i) X is θ -sgp-regular.

(ii) For each point $x \in X$ and for each θ -sgp-open nbd Z of x , there exists a semi-open set Q of x such that $pCl_\theta(Q) \subseteq Z$.

(iii) For each point $x \in X$ and for each θ -sgp-closed not containing x , there exists a semi-open set Q of x such that $pCl_\theta(Q) \cap F = \phi$.

Proof. Let Z be a θ -sgp-open nbd of x . Then there exists a θ -sgp-open set K such that $x \in K \subseteq Z$. Since K^c is θ -sgp-closed set and $x \notin K^c$, by hypothesis there exist semi-open sets P and Q such that $K^c \subseteq P, x \in Q$ and $P \cap Q = \phi$ and so $Q \subset P^c$. Now $pCl_\theta(Q) \subseteq pCl_\theta(P^c) = P^c$ and $K^c \subseteq P$ implies $P^c \subseteq K \subseteq Z$. Therefore $pCl_\theta(Q) \subseteq Z$. Hence (i) implies (ii).

Let F be any θ -sgp-closed set and $x \notin F$. Then $x \notin F^c$ and F^c is θ -sgp-open and so F^c is a θ -sgp-open nbd of x . By hypothesis, there exists a semi-open set Q of x such that $x \in Q$ and $pCl_\theta(Q) \subseteq F^c$, which implies $F \subseteq (pCl_\theta(Q))^c$. Then $(pCl_\theta(Q))^c$ is a semi-open set containing F and $Q \cap (pCl_\theta(Q))^c = \phi$. Therefore X is θ -sgp-regular. Hence (ii) implies (i).

Let $x \in X$ and F be a θ -sgp-closed set such that $x \notin F$. Then F^c is a θ -sgp-open nbd of x and by hypothesis there exists a semi-open set Q of x such that $pCl_\theta(Q) \subseteq F^c$ and therefore $pCl_\theta(Q) \cap F = \phi$. Hence (ii) implies (iii).

Let $x \in X$ and Z be a θ -sgp-open nbd of x . Then there exists a θ -sgp-open set K such that $x \in K \subseteq Z$. Since K^c is θ -sgp-closed set and $x \notin K^c$, by hypothesis there exist semi-open set Q of x such that $pCl_\theta(Q) \cap K^c = \phi$. Therefore $pCl_\theta(Q) \subseteq K \subseteq Z$. Hence (iii) implies (ii).

Theorem 5.4. A space X is a θ -sgp-regular if and only if given any $x \in X$ and any semi-open set B of X there is θ -sgp-open set C such that $x \in C \subset \theta - sgpCl(C) \subseteq B$.

Proof. Let B be a semi-open set, $x \in B$. So B^c is semi-closed set such that $x \notin B^c$. Since X is a θ -sgp-regular then there exist θ -sgp-open sets C_1 and C_2 such that $C_1 \cap C_2 = \phi$, $B^c \subset C_2$, $x \in C_1$. Since $C_1 \cap C_2 = \phi$, we have $\theta - \text{sgpCl}(C_1) \subset \theta - \text{sgpCl}(C_2^c) = C_2^c$. Since $B^c \subset C_2$, we have $C_2^c \subset B$. Hence we have $x \in C_1 \subset \theta - \text{sgpCl}(C_1) \subset C_2^c \subseteq B$.

Conversely, let F be a semi-closed set in X and $x \in X - F$. So F^c is a semi-open set such that $x \in F^c$. Hence there exist a θ -sgp-open set B such that $x \in B \subset \theta - \text{sgpCl}(B) \subseteq F^c$. Let $C = X - \theta - \text{sgpCl}(B)$. So C is a θ -sgp-open set which contains F and $B \cap C = \phi$. Hence X is θ -sgp-regular.

Theorem 5.5. Let X be a topological space. If X is a θ -sgp-regular and T_1 then X is a θ -sgp- T_2 .

Proof. Suppose $x, y \in X$ such that $x \neq y$. Since X is T_1 then there is an open set M such that $x \in M$, $y \notin M$. Since X is θ -sgp-regular and M is semi-open set which contains x , then there is θ -sgp-open set N such that $x \in N \subset \theta - \text{sgpCl}(N) \subseteq M$. Since $y \notin M$, hence $y \notin \theta - \text{sgpCl}(N)$. Therefore $y \in X - (\theta - \text{sgpCl}(N))$. Hence there are θ -sgp-open sets N and $X - (\theta - \text{sgpCl}(N))$ such that $(X - (\theta - \text{sgpCl}(N))) \cap N = \phi$. Hence X is a θ -sgp- T_2 .

Theorem 5.6. Every subspace of θ -sgp-regular space is θ -sgp-regular.

Proof. Let X be a θ -sgp-regular space and Y be a subspace of X . Let $x \in Y$ and F be a θ -sgp-closed set of Y such that $x \notin F$. Then there is a semi-closed set and so θ -sgp-closed set A of X with $F = Y \cap A$ and $x \notin A$. Since X is a θ -sgp-regular, there exist semi-open sets K and L such that $x \in K$, $A \subseteq L$ and $K \cap L = \phi$. Note that $Y \cap K$ and $Y \cap L$ are semi-open sets in Y . Also $x \in K$ and $x \in Y$ which implies $x \in Y \cap K$ and $A \subseteq L$. This implies $Y \cap A \subseteq Y \cap K$, $F \subseteq Y \cap L$. Also $(Y \cap K) \cap (Y \cap L) = \phi$. Hence Y is a θ -sgp-regular.

Theorem 5.7. If $f : X \rightarrow Y$ is onto, θ -sgp-irresolute and semi-open from θ -sgp-regular space X into a topological space Y , then Y is θ -sgp-regular.

Proof. Let $y \in Y$ and F be a θ -sgp-closed set of Y with $y \notin F$. Since f is θ -sgp-irresolute, $f^{-1}(F)$ is θ -sgp-closed set in X . As f is onto, let $f(x)=y$ and $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again since X is θ -sgp-regular space, there exist semi-open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$ and $U \cap V = \phi$. Since f is semi-open and onto, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence Y is θ -sgp-regular.

Theorem 5.8. If $f : X \rightarrow Y$ is one-to-one, θ -sgp-closed map from a topological space X into θ -sgp-regular space Y . If X is $T_{\theta \text{sgp}}$, then X is θ -sgp-regular.

Proof. Let $x \in X$ and F be a θ -sgp-closed set in X such that $x \notin F$. Since X is $T_{\theta \text{sgp}}$ -space, F is semi-closed in X . Then $f(F)$ is θ -sgp-closed set with $f(x) \notin f(F)$ in Y as f is θ -sgp-closed. Again Y is θ -sgp-regular, there exist disjoint semi-open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subset f^{-1}(V)$. Hence X is θ -sgp-regular.

Theorem 5.9. A topological space X is said to be θ -sgp-normal if and only if for every semi-closed set F and for every semi-open set G contain F there exists θ -sgp-open set U such that $F \subset U \subset \theta - \text{sgpCl}(U) \subset G$.

Proof. Let F be a semi-closed set in X and G be semi-open set in X such that $F \subset G$, G^c is a semi-closed set and $G^c \cap F = \phi$. Since X is θ -sgp-normal then there exist θ -sgp-open sets U and V of X such that $U \cap V = \phi$, $G^c \subset V$ and $F \subset U$, $U \subset V^c$. We have $\theta\text{-sgpCl}(U) \subset \theta - \text{sgpCl}(V^c) = V^c$. Hence $F \subset U \subset \theta - \text{sgpCl}(U) \subset V^c \subset G$.

Theorem 5.10. A θ -sgp-closed subspace of a θ -sgp-normal space is θ -sgp-normal.

Proof. Let X be a θ -sgp-normal space. Let Y be a θ -sgp-closed subspace of X . Let A and B be a pair of disjoint θ -sgp-closed sets in Y . Then A and B are disjoint θ -sgp-closed sets in X . Since X is θ -sgp-normal, there exist disjoint semi-open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Since G and H are semi-open in X , then $Y \cap G$ and $Y \cap H$ are semi-open in Y . Also we have $A \subseteq G$ and $B \subseteq H$ which implies $Y \cap A \subseteq Y \cap G$, $Y \cap B \subseteq H$. So $A \subseteq Y \cap G$, $B \subseteq Y \cap H$ and $(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = \phi$. Thus for each pair of disjoint θ -sgp-closed sets A and B in Y , there exist disjoint semi-open sets $Y \cap G$ and $Y \cap H$ such that $A \subseteq Y \cap G$ and $B \subseteq Y \cap H$. Hence Y is θ -sgp-normal.

Theorem 5.11. Let X be a topological space. The following statements are equivalent: (i) X is θ -sgp-normal.

(ii) For each θ -sgp-closed set A and each θ -sgp-open set U such that $A \subseteq U$, there exists a semi-open set V such that $A \subseteq V \subseteq pCl_\theta(V) \subseteq U$.

(iii) For each pair A, B of distinct θ -sgp-closed sets there exists a semi-open set V such that $A \subseteq V$ and $pCl_\theta(V) \cap B = \phi$.

(iv) For each pair A, B of distinct θ -sgp-closed sets there exists a semi-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$.

Proof. (i) $\rightarrow$ (ii) : Let A be a θ -sgp-closed set and U be a θ -sgp-open set such that $A \subseteq U$. Then A and $X - U$ are disjoint θ -sgp-closed sets in X . Since X is θ -sgp-normal, there exist disjoint semi-open sets V and W in X such that $A \subseteq V$ and $X - U \subseteq W$. Now $X - W \subseteq X - (X - U)$ that is $X - W \subseteq U$. Also $V \cap W = \phi$, which implies $V \subseteq X - W$ implies that $pCl_\theta(V) \subseteq pCl_\theta(X - W)$ that is $pCl_\theta(V) \subseteq X - W$. Therefore $pCl_\theta(V) \subseteq X - W \subseteq U$. So $pCl_\theta(V) \subseteq U$. Hence $A \subseteq V \subseteq pCl_\theta(V) \subseteq U$.

(ii) $\rightarrow$ (iii): Let A and B be a pair of disjoint θ -sgp-closed sets in X . Now $A \cap B = \phi$, so $A \subseteq X - B$, where A is θ -sgp-closed and $X - B$ is θ -sgp-open. Then by (ii), there exists semi-open set V such that $A \subseteq V \subseteq pCl_\theta(V) \subseteq X - B$. Now $pCl_\theta(V) \subseteq X - B$ implies $pCl_\theta(V) \cap B = \phi$. Thus $A \subseteq V$ and $pCl_\theta(V) \cap B = \phi$.

(iii) $\rightarrow$ (iv): Let A and B be a pair of disjoint θ -sgp-closed sets in X . From (iii), there exists a semi-open set U such that $A \subseteq U$ and $pCl_\theta(U) \cap B = \phi$. Now

$pCl_\theta(V)$ is θ -sgp-closed set and B is θ -sgp-closed set. Therefore again from (iii), there exists a semi-open set V such that $B \subseteq V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$.

(iv) $\rightarrow$ (i) : Let A and B be a pair of disjoint θ -sgp-closed sets in X . Then from (iv), there exists a semi-open sets U and V in X such that $A \subset U, B \subset V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$. So $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$. Hence X is θ -sgp-normal.

6. Conclusion

The notions θ -sgp-open, θ -sgp-closed maps, θ -sgp-regular and θ -sgp-normal spaces have been introduced and investigated their properties. Compact and connected spaces may be discussed in future using θ -sgp-closed set.

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New separation axioms in binary soft topological spaces

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Abstract. The concept of binary soft topological spaces defined on two universal sets and a parameter set was initiated by Benchalli et al. [2]. In the present paper, new separation axioms in binary soft topological spaces are introduced and their properties, characterizations are studied. Further, the interrelationship between these new binary soft topological spaces are investigated.

Keywords: binary soft n - T_0 spaces, binary soft n - T_1 spaces, binary soft n - T_2 spaces.

1. Introduction

In the real world situations, we often come across more than one universal sets. Therefore, the structures dealing with two universal sets, simultaneously, constitutes an important role in mathematical tooling. Nithyanantha Jothi and Thangavelu [5] constructed a single structure called binary structure, which gives the information about two universal sets and initiated the concepts of binary topological spaces. Also, they have introduced the notions of binary separation axioms namely binary T_0 , binary T_1 and binary T_2 -spaces [6]. To solve the problems with vague and uncertainties, Molodtsov [4] proposed the concept of soft set theory. Acikgoz et al. [1] initiated the concept of binary soft set theory and defined the notions of binary soft subset, binary soft equality, binary null soft set, binary absolute soft set, union and intersection of binary soft sets. With these background, Benchalli et al. [2] initiated the notions of binary soft topological spaces and studied the basic properties such as binary soft open and binary soft closed sets, binary soft closure, binary soft interior and binary soft boundary of binary soft set. Further, they have extended their

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work on binary soft topological spaces by introducing the notions of binary soft subspaces and binary soft separation axioms [3].

In the present paper, we defined points in binary soft sets, based on which new separation axioms are defined. Also, we studied the properties, characterizations and interrelationship between these new binary soft topological spaces. Further, we obtained the relationship between binary soft topology and general topology.

2. Basic notions and notations

Definition 2.1 ([5]). Let U and V be any two non-empty sets with the power sets $P(U)$ and $P(V)$ respectively. A binary topology on U, V is a binary structure $M \subseteq P(U) \times P(V)$ that satisfies the following axioms:

1. (ϕ, ϕ) and $(U, V) \in M$.
2. $(G_1 \cap G_2, H_1 \cap H_2) \in M$ whenever $(G_1, H_1), (G_2, H_2) \in M$.
3. If $\{(G_\alpha, H_\alpha) : \alpha \in \Delta\}$ is a family of members of M , then

$$\left(\bigcup_{\alpha \in \Delta} G_\alpha, \bigcup_{\alpha \in \Delta} H_\alpha \right) \in M.$$

Then, the triplet (U, V, M) is called a binary topological space and the elements of M are called the binary open subsets and if $G \subseteq U, H \subseteq V$ and $(G, H)' \in M$ then (G, H) is binary closed in (U, V, M) .

Definition 2.2 ([1]). Let U_1, U_2 be two initial universe sets with power sets $P(U_1), P(U_2)$ respectively and E be a set of parameters. A pair (F, E) is said to be a binary soft (briefly BS) set over U_1, U_2 , where F is defined as $F : E \rightarrow P(U_1) \times P(U_2)$, $F(e) = (X, Y)$ for each $e \in E$ such that $X \subseteq U_1, Y \subseteq U_2$.

Definition 2.3 ([2]). Let U_1, U_2 be two initial universe sets with power sets $P(U_1), P(U_2)$ respectively and E be a set of parameters. A collection τ_b of BS sets over U_1, U_2 is said to be a BS topology on U_1, U_2 if

1. $\tilde{\phi}, \tilde{E} \in \tau_b$.
2. arbitrary union of members of τ_b is again a member of τ_b .
3. finite intersection of members of τ_b is again a member of τ_b .

Then, (U_1, U_2, τ_b, E) is a binary soft topological space (briefly BSTS) over U_1, U_2 . (G, E) is BS open in U_1, U_2 if $(G, E) \in \tau_b$ and (F, E) is BS closed in U_1, U_2 if $(F, E)' \in \tau_b$.

Throughout this paper, let (U_1, U_2, τ_b, E) denotes a BSTS unless any separation axioms are imposed on it.

Definition 2.4 ([3]). $(\tilde{Y}, \tau_{b_Y}, E)$ is called a binary soft subspace of (U_1, U_2, τ_b, E) , if $\tau_{b_Y} = \{^Y(G, E) = (G, E) \cap (Y, E) : (G, E) \in \tau_b\}$. Then, τ_{b_Y} is a binary soft relative topology on $\tilde{Y}$.

Definition 2.5 ([3]). A BSTS (U_1, U_2, τ_b, E) is said to be a BS normal space if for every pair of disjoint BS closed sets (F, E) and (H, E) , there exist disjoint BS open sets (A, E) and (B, E) such that $(F, E) \subseteq (A, E)$ and $(H, E) \subseteq (B, E)$.

3. Main results

In this section, we have introduced new separation axioms namely binary soft n - T_i ($i = 0, 1, 2, 3, 4$) spaces and BS n - T_i^* ($i = 0, 1, 3, 4$) spaces.

Definition 3.1. In a BSTS (U_1, U_2, τ_b, E) , for a point $(x, y) \in U_1 \times U_2$ and a BS subset (A, E) over U_1, U_2 , where $A(e) = (X, Y)$ for each $e \in E$, we say that

- i. (x, y) belongs to (A, E) , denoted by $(x, y) \in (A, E)$, whenever $x \in X, y \in Y$ for each $e \in E$.
- ii. (x, y) does not belong to (A, E) , denoted by $(x, y) \notin (A, E)$ if $x \notin X$ for some $e \in E$ or $y \notin Y$ for some $e \in E$ or both.
- iii. (x, y) strictly does not belong to (A, E) , denoted by $(x, y) \notin^s (A, E)$ if $x \notin X$ and $y \notin Y$ for each $e \in E$.

Note that (iii) implies (ii).

Definition 3.2. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n - T_0 if for any pair $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$ of distinct points (that is, $x_1 \neq x_2$ and $y_1 \neq y_2$), there exists at least one BS open set (F, E) or (G, E) such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$.

Definition 3.3. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n - T_0^* if for any pair of distinct points $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$, there exists at least one BS open set (F, E) or (G, E) such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \in (F, E)'$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$.

Theorem 3.4. Every BS n - T_0^* space is BS n - T_0 .

Proof. Let $(x_1, y_1), (x_2, y_2)$ be any two distinct points of a BS n - T_0^* space (U_1, U_2, τ_b, E) . Then, there exists at least one BS open set (F, E) or (G, E) such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \in (F, E)'$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$. This implies $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$.

Example 3.5. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$ and $E = \{p_1, p_2\}$ with $\tau_b = \{\tilde{\phi}, \tilde{E}, \{(p_1, (\{a_1\}, \{b_1\})), (p_2, (\{a_1\}, \{b_1\}))\}, \{(p_1, (\{a_2\}, \{b_1\})), (p_2, (\{a_2\}, \{b_1\}))\}, \{(p_1, (\{a_1, a_2\}, \{b_1\})), (p_2, (\{a_1, a_2\}, \{b_1\}))\}, \{(p_1, (\phi, \{b_1\})), (p_2, (\phi, \{b_1\}))\}\}$.

Then, (U_1, U_2, τ_b, E) is BS n - T_0^* as well as BS n - T_0 .

Remark 3.6. Converse of the Theorem 3.4 need not be true in general.

Example 3.7. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$ and $E = \{p_1, p_2\}$ with $\tau_b = \{\tilde{\phi}, \tilde{E}, \{(p_1, (\{a_1\}, U_2)), (p_2, (U_1, \{b_1\}))\}, \{(p_1, (\{a_2\}, \{b_1\})), (p_2, (U_1, U_2))\}, \{(p_1, (\phi, \{b_1\})), (p_2, (U_1, \{b_1\}))\}\}$. Then, (U_1, U_2, τ_b, E) is BS n - T_0 but not BS n - T_0^* .

Theorem 3.8. Let (U_1, U_2, τ_b, E) be a BS n - T_0 space and $\tilde{Y}$ be a non-empty subset of $\tilde{E}$ with $Y = (A, B)$ such that $A \subseteq U_1$ and $B \subseteq U_2$ for each $e \in E$. Then $(\tilde{Y}, \tau_{b_Y}, E)$ is BS n - T_0 . That is, the property of BS n - T_0 is hereditary.

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. Then, there exists at least one BS open set (F, E) or (G, E) in (U_1, U_2, τ_b, E) such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$ as (U_1, U_2, τ_b, E) is BS n - T_0 . Therefore, there exists at least one BS open set ${}^Y(F, E)$ or ${}^Y(G, E)$ in $(\tilde{Y}, \tau_{b_Y}, E)$ such that $(x_1, y_1) \in {}^Y(F, E)$, $(x_2, y_2) \notin {}^Y(F, E)$ or $(x_2, y_2) \in {}^Y(G, E)$, $(x_1, y_1) \notin {}^Y(G, E)$.

Theorem 3.9. The property of BS n - T_0^* is hereditary.

Definition 3.10. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n - T_1 if for any pair of distinct points $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$, there exist $(F, E), (G, E) \in \tau_b$ such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$.

Definition 3.11. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n - T_1^* if for any pair of distinct points $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$, there exist $(F, E), (G, E) \in \tau_b$ such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \in (F, E)'$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$.

Theorem 3.12. Every BS n - T_1^* space is BS n - T_1 .

Proof. Let $(x_1, y_1), (x_2, y_2)$ be any two distinct points of a BS n - T_1^* space (U_1, U_2, τ_b, E) . Then, there exist $(F, E), (G, E) \in \tau_b$ such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \in (F, E)'$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$. This implies $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$.

Example 3.13. Every discrete BSTS is BS n - T_1^* as well as BS n - T_1 .

Remark 3.14. The converse of the Theorem 3.12 need not be true in general.

Example 3.15. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$ and $E = \{p_1, p_2\}$ with $\tau_b = \{\tilde{\phi}, \tilde{E}, (\Psi_1, E), (\Psi_2, E), (\Psi_3, E), (\Psi_4, E), (\Psi_5, E), (\Psi_6, E), (\Psi_7, E)\}$ where, $(\Psi_1, E) = \{(p_1, (U_1, \{b_1\})), (p_2, (U_1, \{b_1\}))\}$, $(\Psi_2, E) = \{(p_1, (\{a_2\}, U_2)), (p_2, (\{a_2\}, U_2))\}$, $(\Psi_3, E) = \{(p_1, (U_1, \{b_2\})), (p_2, (U_1, \{b_2\}))\}$, $(\Psi_4, E) = \{(p_1, (\{a_2\}, \{b_1\})), (p_2, (\{a_2\}, \{b_1\}))\}$, $(\Psi_5, E) = \{(p_1, (\{a_2\}, \{b_2\})), (p_2, (\{a_2\}, \{b_2\}))\}$, $(\Psi_6, E) = \{(p_1, (U_1, \phi)), (p_2, (U_1, \phi))\}$, $(\Psi_7, E) = \{(p_1, (\{a_2\}, \phi)), (p_2, (\{a_2\}, \phi))\}$. Then, (U_1, U_2, τ_b, E) is BS $n\text{-}T_1$ but not BS $n\text{-}T_1^*$.

Theorem 3.16. *The property of BS $n\text{-}T_1$ is hereditary.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. Then, there exist $(F, E), (G, E) \in \tau_b$ such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \notin (F, E)$ and $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \notin (G, E)$. Thus, there exist BS open sets ${}^Y(F, E)$ and ${}^Y(G, E)$ in $(\tilde{Y}, \tau_{b_Y}, E)$ such that $(x_1, y_1) \in {}^Y(F, E)$, $(x_2, y_2) \notin {}^Y(F, E)$ and $(x_2, y_2) \in {}^Y(G, E)$, $(x_1, y_1) \notin {}^Y(G, E)$.

Theorem 3.17. *The property of BS $n\text{-}T_1^*$ is hereditary.*

Definition 3.18. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft $n\text{-}T_2$ if for any pair of distinct points $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$, there exist disjoint BS open sets (F, E) and (G, E) such that $(x_1, y_1) \in (F, E)$ and $(x_2, y_2) \in (G, E)$.

Example 3.19. Every discrete BSTS is BS $n\text{-}T_2$.

Theorem 3.20. *The property of BS $n\text{-}T_2$ is hereditary.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$ be a pair of distinct points. Now, $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$ are distinct. Since (U_1, U_2, τ_b, E) is BS $n\text{-}T_2$, there exist disjoint BS open sets (F, E) and (G, E) in (U_1, U_2, τ_b, E) such that $(x_1, y_1) \in (F, E)$ and $(x_2, y_2) \in (G, E)$. Therefore, there exist disjoint BS open sets ${}^Y(F, E)$ and ${}^Y(G, E)$ in $(\tilde{Y}, \tau_{b_Y}, E)$ such that $(x_1, y_1) \in {}^Y(F, E)$ and $(x_2, y_2) \in {}^Y(G, E)$.

Definition 3.21. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n -regular if for every point $(x, y) \in U_1 \times U_2$ and for every BS closed set (F, E) with $(x, y) \notin^s (F, E)$, there exist disjoint BS open sets (A, E) and (B, E) such that $(x, y) \in (A, E)$ and $(F, E) \subseteq (B, E)$.

Theorem 3.22. *Binary soft n -regularity is a hereditary property.*

Proof. Let $(x, y) \in A \times B$ and (F, E) be BS closed in $(\tilde{Y}, \tau_{b_Y}, E)$ such that $(x, y) \notin^s (F, E)$. Then, there exists a BS closed set (H, E) in (U_1, U_2, τ_b, E) , so that $(F, E) = {}^Y(H, E)$, which implies $(x, y) \notin^s (H, E)$. Since, (U_1, U_2, τ_b, E) is BS n -regular, there exist disjoint BS open sets (U, E) and (V, E) such that

$(x, y) \in (U, E)$ and $(H, E) \subseteq (V, E)$. Thus, there exist disjoint BS open sets ${}^Y(U, E)$ and ${}^Y(V, E)$ in $(\tilde{Y}, \tau_{b_Y}, E)$ such that $(x, y) \in {}^Y(U, E)$ and $(F, E) \subseteq {}^Y(V, E)$.

Theorem 3.23. *For any BSTS (U_1, U_2, τ_b, E) , the following statements are equivalent:*

- i. (U_1, U_2, τ_b, E) is BS n -regular.
- ii. For any $(x, y) \in U_1 \times U_2$ and $(G, E) \in \tau_b$ with $(x, y) \in (G, E)$, there exists BS open set (A, E) such that $(x, y) \in (A, E) \subseteq \overline{(A, E)} \subseteq (G, E)$.
- iii. For any $(x, y) \in U_1 \times U_2$ and BS closed set (F, E) with $(x, y) \notin^s (F, E)$, there exists BS open set (B, E) such that $(x, y) \in (B, E)$ and $\overline{(B, E)} \cap (F, E) = \tilde{\phi}$.

Proof. $(i) \Rightarrow (ii)$:

Let $(x, y) \in U_1 \times U_2$ and $(G, E) \in \tau_b$ with $(x, y) \in (G, E)$. Then, $(G, E)'$ is BS closed with $(x, y) \notin^s (G, E)'$. Since (U_1, U_2, τ_b, E) is BS n -regular, there exist $(A, E), (B, E) \in \tau_b$ such that $(x, y) \in (A, E)$, $(G, E)' \subseteq (B, E)$ and $(A, E) \cap (B, E) = \tilde{\phi}$. Now, $(A, E) \subseteq (B, E)'$, implies $\overline{(A, E)} \subseteq \overline{(B, E)'} = (B, E)'$ as $(B, E)'$ is BS closed. Also, $(B, E)' \subseteq (G, E)$. Therefore, $(x, y) \in (A, E) \subseteq \overline{(A, E)} \subseteq (G, E)$.

$(ii) \Rightarrow (iii)$:

Let $(x, y) \in U_1 \times U_2$ and (F, E) be a BS closed set with $(x, y) \notin^s (F, E)$. Then $(F, E)'$ is BS open with $(x, y) \in (F, E)'$. By (ii), there exists $(B, E) \in \tau_b$ such that $(x, y) \in (B, E) \subseteq \overline{(B, E)} \subseteq (F, E)'$. Therefore, $(x, y) \in (B, E)$ and $\overline{(B, E)} \cap (F, E) = \tilde{\phi}$.

$(iii) \Rightarrow (i)$: Let $(x, y) \in U_1 \times U_2$ and (F, E) be a BS closed set with $(x, y) \notin^s (F, E)$. By (iii), there exists $(B, E) \in \tau_b$ such that $(x, y) \in (B, E)$ and $\overline{(B, E)} \cap (F, E) = \tilde{\phi}$, implies $(F, E) \subseteq \overline{(B, E)'}'$. Further, (B, E) and $\overline{(B, E)'}'$ are disjoint BS open sets. Therefore, (U_1, U_2, τ_b, E) is BS n -regular.

Thus, all the above three statements are equivalent.

Definition 3.24. A BSTS (U_1, U_2, τ_b, E) is said to be binary soft n - T_3 (respectively binary soft n - T_3^*) if it is both binary soft n -regular and binary soft n - T_1 (respectively binary soft n -regular and binary soft n - T_1^*).

Remark 3.25. Every BS n - T_3^* space is BS n - T_3 , but not conversely, by Theorem 3.12 and Remark 3.14.

Theorem 3.26. *Binary soft n - T_3 (respectively binary soft n - T_3^*) is a hereditary property.*

Proof. Follows from the Theorems 3.16 and 3.22 (respectively Theorems 3.17 and 3.22).

Definition 3.27. A BS normal space (U_1, U_2, τ_b, E) which is also BS $n-T_1$ (respectively BS $n-T_1^*$) is said to be a BS $n-T_4$ space (respectively BS $n-T_4^*$ space).

Remark 3.28. Every BS $n-T_4^*$ space is BS $n-T_4$, but converse need not be true in general, follows from the Theorem 3.12 and Remark 3.14.

Theorem 3.29. *Binary soft $n-T_4$ (respectively binary soft $n-T_4^*$) is a closed hereditary property.*

Proof. Follows from the Theorem 4.12 [2], and Theorem 3.16 (respectively Theorem 3.17).

Theorem 3.30. *Every BS $n-T_1$ space is BS $n-T_0$.*

Remark 3.31. The converse of the Theorem 3.30 need not be true in general, by Example 3.7.

Theorem 3.32. *Every BS $n-T_2$ space is BS $n-T_1^*$.*

Proof. Let (U_1, U_2, τ_b, E) be a BS $n-T_2$ space. Therefore, for every pair of distinct points $(x_1, y_1), (x_2, y_2) \in U_1 \times U_2$, there exist $(F, E), (G, E) \in \tau_b$ such that $(x_1, y_1) \in (F, E), (x_2, y_2) \in (G, E)$ and $(F, E) \cap (G, E) = \emptyset$, which implies $(x_2, y_2) \in (F, E)', (x_1, y_1) \in (G, E)'$.

Corollary 3.33. *Every BS $n-T_2$ space is BS $n-T_1$.*

Remark 3.34. The converse of Corollary 3.33 need not be true in general, by Example 3.15.

Remark 3.35. A BS $n-T_1^*$ space need not be a BS $n-T_2$ space.

Remark 3.36. A BS $n-T_1$ space need not be a BS $n-T_0^*$ space.

Example 3.37. Let $U_1 = \{a_1, a_2\}, U_2 = \{b_1, b_2\}$ and $E = \{p_1, p_2\}$ with $\tau_b = \{\emptyset, \tilde{E}, \{(p_1, (U_1, \{b_1\})), (p_2, (U_1, \{b_1\}))\}, \{(p_1, (U_1, \{b_2\})), (p_2, (U_1, \{b_2\}))\}, \{(p_1, (U_1, \phi)), (p_2, (U_1, \phi))\}\}$. Then, (U_1, U_2, τ_b, E) is BS $n-T_1$ but not BS $n-T_0^*$.

Definition 3.38. A BS set (F, E) such that $F(e) = (X, Y)$ for each $e \in E$ is said to be BS singleton if both X and Y are singleton sets.

Remark 3.39. From Example 3.15, it is clear that in a BS $n-T_1$ space, every BS singleton set need not be BS closed.

Remark 3.40. If every BS singleton set is BS closed then the BSTS need not be BS $n-T_1$.

Example 3.41. Let $U_1 = \{a\}$, $U_2 = \{b_1, b_2\}$, $E = \{p_1, p_2\}$ and $\tau_b = \{\tilde{\phi}, \tilde{E}, \{(p_1, (\phi, \{b_1\})), (p_2, (\phi, \phi))\}, \{(p_1, (\phi, \{b_1\})), (p_2, (\phi, \{b_1\}))\}, \{(p_1, (\phi, \{b_1\})), (p_2, (\phi, \{b_2\}))\}, \{(p_1, (\phi, \{b_2\})), (p_2, (\phi, \phi))\}, \{(p_1, (\phi, \{b_1\})), (p_2, (\phi, U_2))\}, \{(p_1, (\phi, \{b_2\})), (p_2, (\phi, \phi))\}, \{(p_1, (\phi, \{b_2\})), (p_2, (\phi, \{b_1\}))\}, \{(p_1, (\phi, \{b_2\})), (p_2, (\phi, U_2))\}, \{(p_1, (\phi, U_2)), (p_2, (\phi, \phi))\}, \{(p_1, (\phi, U_2)), (p_2, (\phi, \{b_1\}))\}, \{(p_1, (\phi, U_2)), (p_2, (\phi, \{b_2\}))\}, \{(p_1, (\phi, U_2)), (p_2, (\phi, U_2))\}, \{(p_1, (\phi, \phi)), (p_2, (\phi, \{b_1\}))\}, \{(p_1, (\phi, \phi)), (p_2, (\phi, \{b_2\}))\}, \{(p_1, (\phi, \phi)), (p_2, (\phi, U_2))\}\}$. Then, (U_1, U_2, τ_b, E) is not BS $n-T_1$ in which all the BS singleton sets are BS closed.

Proposition 3.42. In (U_1, U_2, τ_b, E) , for any $e \in E$,

1. $\tau_{b_e} = \{F(e) : (F, E) \in \tau_b\}$ is a binary topology on U_1, U_2 .
2. $\tau_e = \{X \subseteq U_1 : (G, E) \in \tau_b \text{ and } G(e) = (X, Y)\}$ is a topology on U_1 .
3. $\tau_e = \{Y \subseteq U_2 : (G, E) \in \tau_b \text{ and } G(e) = (X, Y)\}$ is a topology on U_2 .

Example 3.43. In Example 3.7, $\tau_{b_{e_1}} = \{(\phi, \phi), (U_1, U_2), (\{a_1\}, U_2), (\{a_2\}, \{b_1\}), (\phi, \{b_1\})\}$ is a binary topology on U_1, U_2 . $\tau_{e_1} = \{\phi, U_1, \{a_1\}\{a_2\}\}$ is a topology on U_1 and $\tau_{e_1} = \{\phi, U_2, \{b_1\}, \{b_2\}\}$ is a topology on U_2 .

Theorem 3.44. If (U_1, U_2, τ_b, E) is BS $n-T_0^*$, then for any $e \in E$, (U_1, U_2, τ_{b_e}) is binary T_0 .

Proof. Since (U_1, U_2, τ_b, E) is BS $n-T_0^*$, for any pair of distinct points $(x_1, y_1), (x_2, y_2)$, there exists at least one BS open set (F, E) or (G, E) such that $(x_1, y_1) \in (F, E)$, $(x_2, y_2) \in (F, E)'$ or $(x_2, y_2) \in (G, E)$, $(x_1, y_1) \in (G, E)'$. This implies, for any $e \in E$, there exists at least one binary open set $F(e)$ or $G(e)$ such that $(x_1, y_1) \in F(e)$, $(x_2, y_2) \in F(e)'$ or $(x_2, y_2) \in G(e)$, $(x_1, y_1) \in G(e)'$.

Theorem 3.45. If (U_1, U_2, τ_b, E) is BS $n-T_1^*$ (respectively BS $n-T_2$), then for any $e \in E$, (U_1, U_2, τ_{b_e}) is binary T_1 (respectively binary T_2).

Remark 3.46. From Example 3.7, it follows that if (U_1, U_2, τ_b, E) is BS $n-T_0$, then for $e \in E$, (U_1, U_2, τ_{b_e}) may not be a binary T_0 -space.

Remark 3.47. If (U_1, U_2, τ_b, E) is BS $n-T_1$, then for $e \in E$, (U_1, U_2, τ_{b_e}) may not be binary T_1 .

Example 3.48. Let $U_1 = \{a_1, a_2, a_3\}$, $U_2 = \{b_1, b_2\}$, $E = \{p_1, p_2\}$ and $\tau_b = \{\tilde{\phi}, \tilde{E}, \{(p_1, (U_1, \{b_1\})), (p_2, (U_1, \{b_1\}))\}, \{(p_1, (U_1, \{b_2\})), (p_2, (U_1, \{b_2\}))\}\}$,

$\{(p_1, (U_1, \phi)), (p_2, (U_1, \phi))\}$. Then (U_1, U_2, τ_b, E) is BS n-T₁ but (U_1, U_2, τ_{b_e}) is not binary T₁ for any $e \in E$.

Remark 3.49. Let (U_1, U_2, τ_b, E) be a BSTS with initial universal sets U_1, U_2 and parameter set E . Then the number of all possible BS subsets over U_1, U_2 is given by $(|P(U_1)| \times |P(U_2)|)^{|E|}$, where, $|P(U_1)|$, $|P(U_2)|$ and $|E|$ are the cardinalities of the power set of U_1 , power set of U_2 and E respectively.

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On the $(2 + 2)$ -Einstein warped product manifolds with f -curvature-base

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Abstract. We study the $(2 + 2)$ -Einstein warped product manifolds, where the scalar curvature of the Base is a multiple of the warping function, and we called this condition (inside a warped product manifold) *f-curvature-Base* (R_{f_B}). The aim of this paper is to check if there are Base-manifolds with non-flat metrics that satisfy this condition, and this was done in cases where M and Fiber-manifold are not both non-Ricci-flat. As a results of our cases we find that the “*f-curvature-Base*” is equivalent to requesting a flat metric on the Base-manifold.

Keywords: Special Einstein warped product manifolds, multiple of the warping function, *f-curvature-Base*, 2-dimensional Base, 2-dimensional Fiber, $(2 + 2)$ signature.

Introduction and preliminaries

The warped product manifolds are very interesting for Riemannian Geometry and also for General Relativity, as a model of space-time, and in the same way the Einstein warped product manifolds are object of study for mathematics and physics. In the last few years several authors, have dealt with the study of the Einstein warped product manifolds in which the Base-manifold was a 2-dimensional manifold (see for example [2], [7] and [9]).

In [2] the author shows that the nonconstant warping functions f , for 2-dimensional Base-manifold, exists only if the metrics of the Base-manifold is of the form: $g_B = dt^2 + f'(t)^2 du^2$. The aim of this paper is to study the Einstein warped product manifolds $(M, \bar{g})$ with constrained scalar curvature of the Base-manifold (B, g) , namely R_B is a multiply of the warping function f . We try to see if this condition admits (B, g) with non-trivial metrics which must be of the form considered in [2]. We have called this condition, inside the warped product manifold, *f-curvature-Base* (R_{f_B}) just to indicate that the curvature of (B, g) is bounded to f . We have studied this condition in the following cases:

- or only $(M, \bar{g})$ is Ricci-flat (i.e. $\bar{Ric} = \lambda \bar{g}$, with $\lambda = 0$, case (1b)),
- or only Fiber-manifold $(F, \check{g})$ is Ricci-flat (i.e. $\check{Ric} = \mu \check{g}$, with $\mu = 0$, case (2b)),
- or both are Ricci-flat at the same time (case (1a)=case (2a)).

Note that the condition "*f-curvature-Base*" means to find a metric for the Base-manifold (we are interested in the non-flat case) that also allows the solution for a particular nonlinear elliptic pde of the form: $\Delta f + af^2 = 0$ or $\Delta f + af^2 + bf = 0$ (our equations are well known pde, for example in [14] the author studied the non-homogeneous case $\Delta f + af^2 = v$, see [5] for a basic pde theory).

This paper addresses the $(2+2)$ -Einstein warped product manifolds, a type of signatures that, over the years, has been the subject of many studies both from mathematical point of view ([1], [15]) and from the physical point of view ([3], [4], [6], [11] and [12]).

We have found that there are no Base-manifolds with non-trivial metrics for the cases examined. In our analysis we exclude a priori the case where the warping function f is null ($f = 0$) as a trivial case. In fact it will not be considered in the initial hypotheses.

Definition 1. From [10] (see also [2], [8] and [13]) a warped product manifold is Einstein if only if:

$$(1) \quad \bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric - \frac{m}{f} \nabla^2 f = \lambda g \\ \ddot{Ric} = \mu \ddot{g} \\ f \Delta f + (m-1) |\nabla f|^2 + \lambda f^2 = \mu \end{cases}.$$

where λ and μ are constants, m is the dimension of F , $\nabla^2 f$, Δf and ∇f are, respectively, the Hessian, the Laplacian and the gradient of f for g , with $f : (B) \rightarrow (0, \infty)$ a smooth positive function.

It follows from (1):

$$(2) \quad R_B f^2 - m f \Delta f = n f^2 \lambda,$$

where n and R_B are the dimension and the scalar curvature of B respectively; and from third equation, considering $m > 1$, we have:

$$(3) \quad m f \Delta f + m(m-1) |\nabla f|^2 + m \lambda f^2 = m \mu.$$

Now, from (2) and (3) we obtain:

$$(4) \quad |\nabla f|^2 + \left[\frac{\lambda(m-n) + R_B}{m(m-1)} \right] f^2 = \frac{\mu}{(m-1)}.$$

Definition 2. Let $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ be an Einstein warped-product manifold with $\bar{g} = g + f^2 \ddot{g}$. We define the scalar curvature of the Base-manifold (B, g) as *f-curvature-Base* (R_{f_B}), if it is a multiple of the warping function f (i.e. $R_{f_B} = cf$ for c an arbitrary constant $\in \mathbb{R}$).

Case 1: Einstein warped-product manifold Ricci-flat ($\lambda = 0$)**Case 1a:** $\mu = 0$.

Theorem 1. Let (B^2, g) be a smooth surface with R_{f_B} , and $(F^2, \ddot{g})$ a smooth Ricci-flat surface (i.e. $\bar{Ric} = \mu \ddot{g}$, with $\mu = 0$).

If $(M^{2+2}, \bar{g}) = (B^2, g) \times_f (F^2, \ddot{g})$ is an Einstein warped-product manifold Ricci-flat (i.e. $\bar{Ric} = \lambda \bar{g}$, where $\lambda = 0$), then f can only be a constant function.

Proof. In our case, we have $m = n = 2$, $\mu = 0$ and $R_B = R_{f_B}$. Then (2) and (3) become:

$$(5) \quad \Delta f - hf^2 + \lambda f = 0 \quad (\text{with } h = c/2),$$

$$(6) \quad f\Delta f + |\nabla f|^2 + \lambda f^2 = 0.$$

Then (4) becomes:

$$(7) \quad |\nabla f|^2 + hf^3 = 0.$$

If we consider $h = 0$, it is easy to see that f must be a constant. So we assume $h \neq 0$. By setting $u = -hf$, the above equations (5) and (6) (with $\lambda = 0$) becomes:

$$(8) \quad \Delta u + u^2 = 0,$$

$$(9) \quad u\Delta u + |\nabla u|^2 = 0.$$

Then (7) becomes:

$$(10) \quad |\nabla u|^2 - u^3 = 0.$$

Let g be the metric on B and assume that u is a nonzero (and hence necessarily positive) solution, to the above system on a simply-connected open subset $B' \subset B$.

The equation (10) implies that $\omega_1 = u^{-3/2}du$ is a 1-form with g -norm 1 on B' and hence g can be written in the form $g = \omega_1^2 + \omega_2^2$ on B' for some ω_2 , which is also a unit 1-form. Fix an orientation by requiring that $\omega_1 \wedge \omega_2$ be the g -area form on B' , then $\star du = u^{3/2}\omega_2$ and since $d(\star du) = \Delta u \omega_1 \wedge \omega_2$, it follows that:

$$\frac{3}{2}u^{1/2}du \wedge \omega_2 + u^{3/2}d\omega_2 = d(u^{3/2}\omega_2) = -u^2\omega_1 \wedge \omega_2 = -u^{1/2}du \wedge \omega_2$$

or $d\omega_2 = -\frac{5}{2}u^{-1}du \wedge \omega_2$, which can be written as: $d(u^{5/2}\omega_2) = 0$.

Since B' is simply connected, it follows that there exists a function v on B' such that $u^{5/2}\omega_2 = dv$. Consequently, the metric g has the form:

$$(11) \quad g = \omega_1^2 + \omega_2^2 = u^{-3}du^2 + u^{-5}dv^2.$$

This metric can be expressed in polar form by setting $u = 4r^{-2}$ and $v = 32\vartheta$, in which case, it becomes:

$$(12) \quad g = dr^2 + r^{10}d\vartheta^2.$$

This is a singular and incomplete metric at $r = 0$, though it is complete at $r = \infty$ and its Gauss curvature is $K = -\frac{(r^5)''}{r^5} = -20r^{-2} = -5u < 0$, the original f takes the form: $f = \frac{-u}{h} = \frac{-4}{hr^2}$.

From the initial hypothesis, where we had set $R_B = R_{f_B}$, we have that K is incompatible, then there is no solutions if the background metric g is not flat, so the only solution that we have is for g flat and this implies $R_{f_B} = 0$, i.e. $h = 0$ and $f = \text{constant}$. $\square$

Case 1b: $\mu \neq 0$.

Theorem 2. Let (B^2, g) be a smooth surface with R_{f_B} , and $(F^2, \ddot{g})$ a smooth Einstein-surface (i.e. $\ddot{Ric} = \mu\ddot{g}$).

If $(M^{2+2}, \bar{g}) = (B^2, g) \times_f (F^2, \ddot{g})$ is an Einstein warped-product manifold Ricci-flat (i.e. $\bar{Ric} = \lambda\bar{g}$), then f can only be an affine function.

Proof. From (1) and considering $\lambda = 0$, the equations (5) and (7) become:

$$(13) \quad \Delta f - hf^2 = 0.$$

$$(14) \quad |\nabla f|^2 + hf^3 - \mu = 0.$$

Now, if we consider the flat-metric ($h = 0$), it is easy to see that f must be an affine function, while if f is a constant, this implies $h = \mu = 0$ (and $\mu = 0$ is not the case in our interest); so we assume $h \neq 0$ with f nonconstant and setting $u = -hf$, for an open set where u nonzero, then:

$$(15) \quad \Delta u + u^2 = 0,$$

$$(16) \quad |\nabla u|^2 - u^3 - h^2\mu = 0.$$

Now, for simplicity we replace the constant $h^2\mu$ with constant A .

The equation (16) implies that $\omega_1 = (u^3 + A)^{-\frac{1}{2}}du$, and this implies that we have to assume $(u^3 + A)$ nonzero, then $g = \omega_1^2 + \omega_2^2$ for some ω_2 .

Following the procedure used for *Theorem 1*, we have: $\star du = (u^3 + A)^{\frac{1}{2}}\omega_2$, and $d(\star du) = \Delta u \omega_1 \wedge \omega_2$. Now we have:

$$\begin{aligned} & \frac{3}{2}(u^3 + A)^{-\frac{1}{2}}u^2du \wedge \omega_2 + (u^3 + A)^{\frac{1}{2}}d\omega_2 \\ &= d[(u^3 + A)^{\frac{1}{2}}\omega_2] = -u^2\omega_1 \wedge \omega_2 = -u^2(u^3 + A)^{-\frac{1}{2}}du \wedge \omega_2 \\ &= -\frac{3}{2}(u^3 + A)^{-1}u^2du \wedge \omega_2 - u^2(u^3 + A)^{-1}du \wedge \omega_2 = d\omega_2. \end{aligned}$$

Then $-\frac{5}{2}(u^3 + A)^{-1}u^2 du \wedge \omega_2 = d\omega_2$ and we have $\omega_2 = (u^3 + A)^{-\frac{5}{6}} dv$. Thus the metric $g = (u^3 + A)^{-1} du^2 + (u^3 + A)^{-\frac{5}{3}} dv^2$ has a singularity in $u^3 = -A$.

Note 1. The purpose of this analysis is to check if there is an incompatibility between the initial hypothesis and the Gaussian curvature built by the obtained metric and therefore does not concern the study, in depth, of singularities.

So, only some considerations will be made.

Given that we had to assume that $u^3 + A$ was nonzero, see ω_1 obtained from (16), we also consider that $u^3 + A$ is never zero, and to study what happens near this locus we need to insert the cases according to whether $A = 0$ or not.

If $A = 0$ then we have $\mu = 0$ (and we have the same situation of the "case 1a"), but for the initial assumptions we must have $\mu \neq 0$, which implies that we can not consider the case $A = 0$.

Now, we assume that A is not zero. By hypothesis, $u^3 + A = |\nabla u|^2$ is always non-negative. Write $A = -a^3$ for some unique constant a . Then, we have:

$$|\nabla u|^2 = (u - a)(u^2 + ua + a^2)$$

and the metric is: $g = [(u - a)(u^2 + ua + a^2)]^{-1} du^2 + (u - a)(u^2 + ua + a^2)^{-\frac{5}{3}} dv^2$. Since a is non-zero, it follows that $(u^2 + ua + a^2)$ is always positive, so if $u - a$ vanishes along a gradient line (which must be a geodesic), then it must vanish exactly to order 2 along that gradient line. There are two cases: First, $u - a$ vanishes at an isolated point, in which case our surface is rotationally symmetric about that point. This means that our coordinate system will break down (just as polar coordinates do) or else, we rewrite $u = a + s^2$ for some function s whose gradient along where it vanishes is nonzero.

At the end of these considerations, being that for the purposes of our work we have considered an open set for which $(u^3 + A)$ is nonzero and the Gaussian curvature will be given by: $K = 5u - 10u^4(u^3 + A)^{-1}$.

Also in this case, it is easy to verify that for the initial hypothesis, where we had set $R_B = R_{f_B}$ (i.e. $K = -u$), we have that K is incompatible with our analysis. In fact this implies that f is constant (i.e. $h = \mu = 0$). Then the only solution is a flat metric g , (i.e. $h = 0$) which in this case implies that f is an affine function.

Case 2: Flat fiber surface ($\mu = 0$).

Case 2a: We point out that if we consider $\lambda = 0$ we are obviously in the same situation treated in the case (1a), so we will not have to dwell further.

Case 2b: $\lambda \neq 0$.

Note 2. If we set $h = 0$ we force $\lambda = 0$ and this is not the case in questions.

Theorem 3. Let (B^2, g) be a smooth surface with R_{f_B} , and $(F^2, \ddot{g})$ a smooth Ricci-flat surface (i.e. $\ddot{Ric} = \mu \ddot{g}$, with $\mu = 0$).

If $(M^{2+2}, \bar{g}) = (B^2, g) \times_f (F^2, \ddot{g})$ is an Einstein warped-product manifold (i.e. $\bar{Ric} = \lambda \bar{g}$), then $(M^{2+2}, \bar{g})$ exists if and only if $\lambda = 0$.

Proof. The analysis is essentially the same as seen so far, so we assume $h \neq 0$ and set $u = -hf$. Now our equations become:

$$(17) \quad \Delta u + u^2 + \lambda u = 0,$$

$$(18) \quad u\Delta u + |\nabla u|^2 + \lambda u^2 = 0.$$

Then, from (17) and (18) we have:

$$(19) \quad |\nabla u|^2 - u^3 = 0.$$

On the open set where $u \neq 0$ it is positive, so if we set $u = 4/r^2$ we find that $\omega_1 = dr$ and $\omega_2 = r^5 e^{(\lambda/4)r^2} d\vartheta$ for some angular coordinate ϑ . Then the metric:

$$(20) \quad g = dr^2 + r^{10} e^{(\lambda/2)r^2} d\vartheta^2.$$

The above case is singular and incomplete only at $r = 0$ and its Gauss curvature is $K = -20/r^2 - (11/2)\lambda - (\lambda^2/4)r^2$.

Also, in this case for the initial hypothesis, $2K = R_{f_B} = cf$, we have $f = \frac{(11+\sqrt{57})\lambda}{8c}$ and $f = \frac{(11-\sqrt{57})\lambda}{8c}$. This means that f is constant and this implies $h = 0$. This case is not admitted in our analysis (we have set $h \neq 0$), so there is not solution for this case. Since if we consider a flat metric (i.e. $h = 0$) this force $\lambda = 0$ that is not the case in our interest.

Remark. It is easy to verify that the results obtained in this paper are the same if we consider the Base-manifold with semi-Riemannian metrics.

Conclusions

Under the examined conditions of M and Fiber, the f -curvature-Base is equivalent to requesting a flat metric on the Base-manifold. In fact for what we have seen, we can consider M in only two types of Ricci-flat manifolds:

- Einstein-Fiber (i.e. $\bar{Ric} = \mu \bar{g}$) with Flat-Base and f affine;
- Flat-Fiber (i.e. $\bar{Ric} = \mu \bar{g}$, with $\mu = 0$) with Flat-Base and f constant.

while cannot exist non-Ricci-flat M with R_{f_B} and Ricci-flat Fiber.

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Boundedness in a nonlinear gradient chemotaxis model with logistic source

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Abstract. This paper deals with a nonlinear gradient chemotaxis system with logistic source motivated by the model of tumor lymphangiogenesis in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. By using the iterative method and the test-function argument, we prove that the problem possesses a unique global solution which is uniformly bounded.

Keywords: nonlinear gradient chemotaxis model, global existence, boundedness, logistic source.

1. Introduction

In this paper we study the nonlinear gradient chemotaxis system with logistic source

$$(1.1) \quad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \mathbf{F}(\nabla v)) + f(u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\ (\nabla u - \chi u \mathbf{F}(\nabla v)) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, ν denotes the outward normal on $\partial\Omega$, $\chi > 0$, $\mathbf{F}$ is a vector-valued function and the initial data $u_0(x)$, $v_0(x)$ are given nonnegative functions. The unknown function u denotes cell density and v describes the concentration of the chemical signal.

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The problem (1.1) is a variation of the so-called minimal model of the Keller-Segel chemotaxis model which is obtained when $\mathbf{F}(\nabla v) = \nabla v$ and $f \equiv 0$. Such a model in which the chemotactic component to motion depends nonlinearly on signal gradient provides a more realistic depiction of individual cell migration [2]. Various versions of the model for chemotaxis have attracted much attention in recent years; see the survey articles [2, 3, 9] and the references therein for more information.

Throughout this paper we assume that the vector-valued function $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ belongs to $C^{1+\delta}(\mathbb{R}^n)$ for some $\delta > 0$ and that

$$(1.2) \quad |\mathbf{F}(\mathbf{a})| \leq A, \quad \text{for all } \mathbf{a} \in \mathbb{R}^n.$$

The logistic function $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth satisfying $f(0) = 0$ and

$$(1.3) \quad f(s) \leq \kappa s - \mu s^2, \quad \text{for all } s \geq 0$$

with $\kappa > 0$ and $\mu > 0$. We note that there exist some important models in which the assumption (1.2) is satisfied. For instance,

$$\mathbf{F}(\nabla v) = \frac{1}{c} \left(\tanh\left(\frac{c v_{x_1}}{1+c}\right), \dots, \tanh\left(\frac{c v_{x_n}}{1+c}\right) \right)$$

for describing migration of the flagella bacteria *Escherichia coli* [2]. Another specific choice of the function

$$\mathbf{F}(\nabla v) = \frac{\nabla v}{\sqrt{1+c|\nabla v|^2}}$$

represents chemotactic factor in model of tumor lymphangiogenesis [1].

For the case $f(s) = 0$, the global existence of solutions has been shown by Hillen and Painter [2, Lemma 3] if $\mathbf{F}$ is uniformly bounded. The main purpose of this paper is to prove the global existence and the boundedness of solutions of (1.1) with nontrivial logistic source for all dimensions $n \geq 1$. The proof is based on a test-function argument and an iterative technique on L^p norms [4, 5].

2. Preliminary lemmas and statements of the main result

In this paper we need the following well-known facts concerning the Laplacian on Ω supplemented with homogeneous Neumann boundary conditions (for instance, see [4, 6]). Firstly, the operator $-\Delta + 1$ is sectorial in $L^p(\Omega)$ and therefore possesses closed fractional powers $(-\Delta + 1)^\theta$, $\theta \in (0, 1)$, with dense domain $D((-\Delta + 1)^\theta)$. If $m \in \{0, 1\}$, $p \in [1, \infty]$ and $q \in (1, \infty)$ with $m - \frac{n}{p} < 2\theta - \frac{n}{q}$, then we have

$$(2.1) \quad \|\omega\|_{W^{m,p}(\Omega)} \leq c \|(-\Delta + 1)^\theta \omega\|_{L^q(\Omega)},$$

for all $\omega \in D((-\Delta + 1)^\theta)$, where c is a positive constant. Moreover, for $p < \infty$ the associated heat semigroup $(e^{t\Delta})_{t \geq 0}$ maps $L^p(\Omega)$ into $D((-\Delta + 1)^\theta)$ in any of the space $L^q(\Omega)$ for $q \geq p$, and there exist $c > 0$ and $\mu > 0$ such that

$$(2.2) \quad \|(-\Delta + 1)^\theta e^{t(\Delta-1)} \omega\|_{L^q(\Omega)} \leq ct^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\mu t} \|\omega\|_{L^p(\Omega)}$$

for all $\omega \in L^p(\Omega)$.

In the following proof, we shall use the ODE comparison principle repeatedly.

Lemma 2.1. *Let the function $y \in C^1[0, \infty)$ be positive and satisfy*

$$\begin{cases} y' + \gamma y^p \leq \delta y^q, & t > 0, \\ y(0) = y_0 \end{cases}$$

with $\gamma > 0$, $p > q \geq 0$ and $\delta \geq 0$. Then, for all $t \geq 0$

$$y(t) \leq \max \left\{ y_0, \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}} \right\}.$$

Proof. If $y_0 \leq \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}}$, then $y(t) \leq \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}}$ for all $t \geq 0$. If $y_0 > \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}}$, then there exists $t_0 \in (0, \infty)$ such that

$$(2.3) \quad \begin{aligned} y(t) &\geq \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}}, & t \in [0, t_0], \\ y(t) &\leq \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-q}}, & t \in [t_0, \infty). \end{aligned}$$

From (2.3), we have $y'(t) \leq 0$ in $(0, t_0)$, which implies $y(t) \leq y_0$ in $(0, t_0)$. Hence we complete the proof. $\square$

In the proof of boundedness of u , we also use the following Lemma (see [8, 5] for details).

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. For each $\varepsilon > 0$, there exists $C > 0$ which depends only on n and Ω with the property that*

$$(2.4) \quad \|\varphi\|_{L^2(\Omega)}^2 \leq \varepsilon \|\nabla \varphi\|_{L^2(\Omega)}^2 + C(1 + \varepsilon^{-\frac{n}{2}}) \|\varphi\|_{L^1(\Omega)}^2, \quad \varphi \in W^{1,2}(\Omega).$$

The following lemma asserts that the system (1.1) has a unique local-in-time existence classical solution.

Lemma 2.3. *Let u_0 and v_0 be nonnegative functions such that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,r}(\Omega)$ for some $r > n$. Suppose that $\mathbf{F}$ satisfies (1.2) and f is given by (1.3). Then there exist the maximal existence time $T_{\max} \in (0, \infty]$ and a unique*

local-in-time classical nonnegative solution $(u(x, t), v(x, t))$ to the problem (1.1) such that

$$\begin{aligned} u &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap L_{loc}^\infty([0, T_{\max}); W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})). \end{aligned}$$

If $T_{\max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \quad \text{as } t \nearrow T_{\max}.$$

In addition, there exists a constant $M > 0$ such that

$$(2.5) \quad \int_{\Omega} u(x, t) dx \leq M, \quad \text{for all } t \in (0, T_{\max}).$$

Proof. The proof of local existence of a classical solution to (1.1) is similar to that of [6, Lemma 2.1] and [7, Lemma 1.1] and so is omitted. We now prove (2.5). Integrating the first equation in (1.1) over Ω , we obtain

$$\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} \Delta u - \chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) + \int_{\Omega} f(u).$$

Here the Green's formulas ensures that

$$\int_{\Omega} \Delta u = \int_{\partial\Omega} \nabla u \cdot \nu$$

and

$$-\chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) = -\chi \int_{\partial\Omega} u \mathbf{F}(\nabla v) \cdot \nu.$$

We next recall the boundary condition in (1.1) to see that

$$\int_{\Omega} \Delta u - \chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) = \int_{\partial\Omega} (\nabla u - \chi u \mathbf{F}(\nabla v)) \cdot \nu = 0.$$

Hence, the assumption (1.3) implies that

$$(2.6) \quad \frac{d}{dt} \int_{\Omega} u dx \leq \int_{\Omega} (\kappa u - \mu u^2) dx.$$

From Hölder's inequality, we get the inequality

$$(2.7) \quad - \int_{\Omega} u^2 dx \leq -\frac{1}{|\Omega|} \left(\int_{\Omega} u dx \right)^2.$$

It follows from (2.6) and (2.7) that

$$\frac{d}{dt} \int_{\Omega} u dx \leq \left(\kappa - \frac{\mu}{|\Omega|} \int_{\Omega} u dx \right) \int_{\Omega} u dx.$$

Let $y(t) = \int_{\Omega} u(x, t) dx$, then we have

$$y' \leq \left(\kappa - \frac{\mu}{|\Omega|} y \right) y.$$

Hence, we apply Lemma 2.1 to obtain

$$\int_{\Omega} u dx \leq M := \max \left\{ \int_{\Omega} u_0 dx, \frac{\kappa}{\mu} |\Omega| \right\}.$$

This completes the proof. $\square$

The main result of this paper reads as follows:

Theorem 2.1. *Under the assumptions of Lemma 2.2, problem (1.1) possesses a unique global classical solution (u, v) such that both u and v are nonnegative and bounded in $\Omega \times (0, \infty)$.*

3. Proof of Theorem 2.1

We first prove uniform boundedness of the first component of the solution $u(\cdot, t)$ in the space $L^k(\Omega)$ for all $2 < k < \infty$. In the sequel we shall denote by C various positive constants which may vary from line to line. Also $u(\cdot, t)$ will be denoted sometimes by u . Multiplying the first equation in (1.1) by u^{k-1} and integrating over Ω , we obtain

$$\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k dx \leq \int_{\Omega} \Delta u u^{k-1} dx - \chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) u^{k-1} dx + \int_{\Omega} u^{k-1} (\kappa u - \mu u^2) dx,$$

where by the Green's formulas

$$\int_{\Omega} \Delta u u^{k-1} dx = -(k-1) \int_{\Omega} u^{k-2} |\nabla u|^2 dx + \int_{\partial\Omega} u^{k-1} \nabla u \cdot \nu dS$$

and

$$-\chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) u^{k-1} dx = \chi(k-1) \int_{\Omega} u^{k-1} \nabla u \cdot \mathbf{F}(\nabla v) dx - \chi \int_{\partial\Omega} u^k \mathbf{F}(\nabla v) \cdot \nu dS.$$

The boundary condition in (1.1) implies that

$$\begin{aligned} & \int_{\Omega} \Delta u u^{k-1} dx - \chi \int_{\Omega} \nabla \cdot (u \mathbf{F}(\nabla v)) u^{k-1} dx \\ &= -(k-1) \int_{\Omega} u^{k-2} |\nabla u|^2 dx + \chi(k-1) \int_{\Omega} u^{k-1} \nabla u \cdot \mathbf{F}(\nabla v) dx \\ & \quad + \int_{\partial\Omega} u^{k-1} (\nabla u - \chi u \mathbf{F}(\nabla v)) \cdot \nu dS \\ &= -(k-1) \int_{\Omega} u^{k-2} |\nabla u|^2 dx + \chi(k-1) \int_{\Omega} u^{k-1} \nabla u \cdot \mathbf{F}(\nabla v) dx. \end{aligned}$$

Hence, we get the estimate

$$\begin{aligned} & \frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k dx + (k-1) \int_{\Omega} u^{k-2} |\nabla u|^2 dx \\ & \leq \chi(k-1) \int_{\Omega} u^{k-1} \nabla u \cdot \mathbf{F}(\nabla v) dx + \int_{\Omega} u^{k-1} (\kappa u - \mu u^2) dx. \end{aligned}$$

Applying (1.2) and Cauchy's inequality, we get

$$\chi(k-1) \int_{\Omega} u^{k-1} \nabla u \cdot \mathbf{F}(\nabla v) dx \leq \frac{k-1}{2} \int_{\Omega} u^{k-2} |\nabla u|^2 dx + \frac{k-1}{2} (A\chi)^2 \int_{\Omega} u^k dx.$$

In view of Hölder's inequality

$$\int_{\Omega} u^k dx \leq \left(\int_{\Omega} u^{k+1} dx \right)^{\frac{k}{k+1}} |\Omega|^{\frac{1}{k+1}},$$

we have

$$-\int_{\Omega} u^{k+1} dx \leq -|\Omega|^{-\frac{1}{k}} \left(\int_{\Omega} u^k dx \right)^{\frac{k+1}{k}}.$$

It follows that

$$\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k dx \leq -\mu |\Omega|^{-\frac{1}{k}} \left(\int_{\Omega} u^k dx \right)^{\frac{k+1}{k}} + \left(\frac{k-1}{2} (A\chi)^2 + \kappa \right) \int_{\Omega} u^k dx.$$

Therefore, Lemma 2.1 yields

$$(3.1) \quad \int_{\Omega} u^k dx \leq C, \quad t \in (0, T_{\max}),$$

for all $2 < k < \infty$.

In order to complete the proof of Theorem 2.1, by means of Lemma 2.2, it is sufficient to prove that for any fixed $\tau \in (0, T_{\max})$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\tau), \quad t \in (\tau, T_{\max})$$

holds with some constant $C(\tau) > 0$. To do so, we let $\tau \in (0, T_{\max})$ such that $\tau < 1$ and pick $k > \max\{2, \frac{n}{2}\}$, so that we can choose $\theta \in (\frac{n}{2k}, 1)$. From the variation-of-constant formula, we get

$$v(\cdot, t) = e^{t(\Delta-1)} v_0 + \int_0^t e^{(t-s)(\Delta-1)} u(\cdot, s) ds, \quad t \in (0, T_{\max}).$$

By using (2.1), (2.2), (1.2) and (3.1), we have

$$\begin{aligned} \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} & \leq C \|(-\Delta + 1)^\theta v(\cdot, t)\|_{L^k(\Omega)} \\ & \leq C t^{-\theta} e^{-\mu t} \|v_0\|_{L^k(\Omega)} + C \int_0^t (t-s)^{-\theta} e^{-\mu(t-s)} \|u(\cdot, s)\|_{L^k(\Omega)} ds \\ & \leq C t^{-\theta} + C \int_0^t (t-s)^{-\theta} e^{-\mu(t-s)} ds \\ (3.2) \quad & \leq C(\tau^{-\theta} + \Gamma(1-\theta)), \quad t \in (\tau, T_{\max}), \end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

We now turn to establish a uniform bound of u . For any $p \geq 2$, using the first equation in (1.1), (1.2) and (1.3), we compute

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p dx &= p \int_{\Omega} u^{p-1} u_t dx \\ &= p \int_{\Omega} u^{p-1} \left(\Delta u - \chi \nabla(u \mathbf{F}(\nabla v)) + f(u) \right) dx \\ &\leq -p(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 dx + \chi A p(p-1) \int_{\Omega} u^{p-1} |\nabla u| dx + p\kappa \int_{\Omega} u^p dx \\ &= -\frac{4(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + 2\chi A(p-1) \int_{\Omega} u^{\frac{p}{2}} |\nabla u^{\frac{p}{2}}| dx + p\kappa \int_{\Omega} u^p dx \\ &\leq -\frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + \left(\frac{(\chi A)^2}{2} p(p-1) + p\kappa \right) \int_{\Omega} u^p dx, \end{aligned}$$

where we have used Young's inequality. Therefore,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p dx + p(p-1) \int_{\Omega} u^p dx \\ (3.3) \quad \leq -\frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + C_1 p(p-1) \int_{\Omega} u^p dx, \end{aligned}$$

where $C_1 = \frac{(\chi A)^2}{2} + \frac{\kappa}{p-1} + 1$. Using (2.4) with $\varphi = u^{\frac{p}{2}}$ and $\varepsilon = \frac{2}{C_1 p^2}$, we obtain

$$\begin{aligned} C_1 p(p-1) \int_{\Omega} u^p dx &= C_1 p(p-1) \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\ &\leq \frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + C_2 p(p-1)(1+p)^n \left(\int_{\Omega} u^{\frac{p}{2}} dx \right)^2, \end{aligned}$$

where $C_2 = CC_1 \max\{1, (\frac{C_1}{2})^{\frac{n}{2}}\}$. Substituting in (3.3) we get

$$\frac{d}{dt} \int_{\Omega} u^p dx + p(p-1) \int_{\Omega} u^p dx \leq C_2 p(p-1)(1+p)^n \left(\int_{\Omega} u^{\frac{p}{2}} dx \right)^2.$$

Taking supremum for $t \in [0, T_{\max}]$ on the right hand-side, we get

$$\frac{d}{dt} \int_{\Omega} u^p dx + p(p-1) \int_{\Omega} u^p dx \leq C_2 p(p-1)(1+p)^n \sup_{0 \leq t \leq T_{\max}} \left(\int_{\Omega} u^{\frac{p}{2}} dx \right)^2.$$

Therefore, it follows from Lemma 2.1 that

$$(3.4) \quad \int_{\Omega} u^p dx \leq \max \left\{ \int_{\Omega} u_0^p dx, \quad C_2 (1+p)^n \sup_{0 \leq t \leq T_{\max}} \left(\int_{\Omega} u^{\frac{p}{2}} dx \right)^2 \right\}.$$

Denoting

$$F(p) := \max \left\{ \|u_0\|_{L^\infty(\Omega)}, \quad \sup_{0 \leq t \leq T_{\max}} \|u(t)\|_{L^p(\Omega)} \right\}$$

and hence (3.4) yields

$$F(p) \leq (C_3(1+p)^n)^{\frac{1}{p}} F\left(\frac{p}{2}\right),$$

where $C_3 = \max\{|\Omega|, C_2\}$. Taking $p = 2^j$ ($j \in \mathbb{N}^+$), we obtain

$$\begin{aligned} F(2^j) &\leq C_3^{\frac{1}{2^j}} (1+2^j)^{\frac{n}{2^j}} F(2^{j-1}) \\ &\leq \dots \\ &\leq C_3^{\sum_{i=1}^j 2^{-i}} \cdot \prod_{i=1}^j (1+2^i)^{\frac{n}{2^i}} F(1) \\ &\leq C_3 2^{n \sum_{i=1}^j \frac{i}{2^i}} \cdot 2^{n \sum_{i=1}^j \frac{1}{2^i}} F(1) \\ &\leq C_3 2^{n(2 - \frac{j}{2^{j+1}})} F(1). \end{aligned}$$

Letting $j \rightarrow \infty$, we infer that

$$\max \left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_{0 \leq t \leq T_{\max}} \|u(t)\|_{L^\infty(\Omega)} \right\} =: F(\infty) \leq C_3 2^{2n} F(1)$$

according to (2.5). This along with (3.2) yields

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} < \infty.$$

This is a contradiction with Lemma 2.3.

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Laplace transform of the product of two functions

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Abstract. In this paper, we prove many propositions of Laplace transform. In particular, we prove that the Laplace transform of product of two functions, under certain conditions, satisfies

$$\mathcal{L}_s(f(t)h(t)) = \int_s^\infty (\mathcal{L}^{-1}h)(\xi - s)(\mathcal{L}f)(\xi)d\xi = \int_0^\infty (\mathcal{L}^{-1}h)(\xi)(\mathcal{L}f)(\xi + s)d\xi.$$

Then we find interesting relation between the Laplace of the product of two functions and their convolution. Also, using these results, we prove a generalized result that the Laplace transform of the convolution of two function is the product of their Laplace transforms. Finally, we find particular solutions of integral, differential and difference equations as application for these results.

Keywords: Laplace transform, Laplace transform of the product of two functions, integro-differential equations, integro-difference equations.

1. Introduction

Suppose that f is a real-valued function of the (time) variable $t > 0$ and s is a real variable. We define the Laplace transform of $f(t)$ as

$$(1.1) \quad \mathcal{L}_s(f(t)) = (\mathcal{L}f)(s) = F(s) = \int_0^\infty e^{-st}f(t)dt.$$

One of the disappointments of the Laplace transform is that the Laplace transform of the product of two functions is not the product of their Laplace transforms. In fact, the Laplace transform of the convolution of two functions is the product of their Laplace transforms.

$$\mathcal{L}_s((f * g)(t)) = \mathcal{L}_s\left(\int_0^t f(u)g(t-u)du\right) = \mathcal{L}_s(f(t))\mathcal{L}_s(g(t)).$$

In this paper we answer the following question:

In [6], it is proved that the Laplace transform of the product of $f(t)$ and $\frac{1}{t}$ is

$$\mathcal{L}_s\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\xi)d\xi.$$

Now, what are the Laplace transforms of the product of $f(t)$ and $\frac{1}{t^r}$ ($r > 0$)? of the product of $f(t)$ and $\frac{1}{t-a}$? of the product of $f(t)$ and $\frac{1}{t^2+a^2}$? or even the the product of $f(t)$ and $\frac{1}{t^2(t+3)(t^2+1)}$?

2. Main results

Theorem 2.1. Assume that $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$. If $\int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) dt d\xi$ converges absolutely for $s > \alpha$, then $\mathcal{L}_s(f(t)G(t))$ is given as

$$\begin{aligned} \mathcal{L}_s(f(t)G(t)) &= \mathcal{L}_s(f(t)\mathcal{L}(g(t))) \\ (2.1) \quad &= \int_0^\infty g(\xi)F(\xi+s)d\xi = \int_s^\infty g(\xi-s)F(\xi)d\xi \text{ for } s > \alpha. \end{aligned}$$

Proof. By Fubini's Theorem (see [5]), since $\int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) dt d\xi$ converges absolutely for $s > \alpha$ then

$$\int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) dt d\xi = \int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) d\xi dt.$$

Therefore, for $s > \alpha$

$$\begin{aligned} \mathcal{L}_s(f(t)G(t)) &= \int_0^\infty e^{-st} f(t) \int_0^\infty e^{-\xi t} g(\xi) d\xi dt \\ &= \int_0^\infty g(\xi) \int_0^\infty e^{-(s+\xi)t} f(t) dt d\xi \\ &= \int_0^\infty g(\xi) F(s+\xi) d\xi \\ &= \int_s^\infty g(\xi-s) F(\xi) d\xi. \end{aligned}$$

□

This theorem can be read as: if $\int_0^\infty \int_0^\infty e^{-(s+\xi)t} (\mathcal{L}^{-1}h)(\xi) f(t) dt d\xi$ converges absolutely for $s > \alpha$. Then

$$\begin{aligned} \mathcal{L}_s(f(t)h(t)) &= \int_0^\infty (\mathcal{L}^{-1}h)(\xi) (\mathcal{L}f)(\xi+s) d\xi \\ (2.2) \quad &= \int_s^\infty (\mathcal{L}^{-1}h)(\xi-s) (\mathcal{L}f)(\xi) d\xi \text{ for } s > \alpha. \end{aligned}$$

Example 2.2. Using the fact $\mathcal{L}(e^{-at}) = \frac{1}{s+a}$ we get

$$\mathcal{L}_s\left(\frac{f(t)}{t+a}\right) = \mathcal{L}(f(t)\mathcal{L}(e^{-at})) = e^{as} \int_s^\infty e^{-a\xi} F(\xi) d\xi.$$

Example 2.3.

$$\begin{aligned} \mathcal{L}_s\left(\frac{1-\cos(t)}{t^2}\right) &= \int_s^\infty (u-s)\left(\frac{1}{u} - \frac{u}{u^2+1}\right) du \\ &= \int_s^\infty \frac{du}{u^2+1} - s \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+1}\right) du \\ &= \arctan\left(\frac{1}{s}\right) + s \ln\left(\frac{s}{\sqrt{s^2+1}}\right). \end{aligned}$$

Example 2.4. For $s > 1$

$$\begin{aligned} \mathcal{L}_s\left(\frac{e^t - t - 1}{t^2}\right) &= \int_s^\infty (\xi - s)\left(\frac{1}{\xi-1} - \frac{1}{\xi^2} - \frac{1}{\xi}\right) d\xi \\ &= (s-1) \ln\left(\frac{s-1}{s}\right) - 1. \end{aligned}$$

Using this result, we will solve equations contain mixture of differential, difference and integral operators

Example 2.5. To solve the integral equation that

$$(2.3) \quad \int_x^\infty y(u-x)\left(\frac{2}{u^3} + \frac{3}{u^2} + \frac{2}{u}\right) du = \frac{7}{x} - \frac{3}{x^2}, x > 0.$$

The left side can be written as $\mathcal{L}_x(Y(t)(t^2 + 3t + 2))$, where $Y(t) = \mathcal{L}_t(y(x))$, and the right side is $\mathcal{L}_x(7 - 3t)$. Therefore, $Y(t) = \mathcal{L}_t(y(x)) = \frac{7-3t}{t^2+3t+2}$. This means that $y(x) = 10e^{-x} - 13e^{-2x}$ is a particular solution of the equation (2.3). To verify our answer, use interchanging the integration orders and the fact

$$Y(w) = \mathcal{L}_w(10e^{-x} - 13e^{-2x}) = \frac{10}{w+1} - \frac{13}{w+2} = \frac{7-3w}{(w+1)(w+2)}$$

to get that

$$\begin{aligned} \int_x^\infty y(u-x)\left(\frac{2}{u^3} + \frac{3}{u^2} + \frac{2}{u}\right) du &= \int_x^\infty y(u-x) \int_0^\infty e^{-wu}\left(w^2 + 3w + 2\right) dw du \\ &= \int_0^\infty \left(w^2 + 3w + 2\right) \int_x^\infty e^{-wu} y(u-x) du dw \\ &= \int_0^\infty e^{-wx}\left(w^2 + 3w + 2\right) \int_0^\infty e^{-wv} y(v) dv dw \\ &= \int_0^\infty e^{-wx}\left(w^2 + 3w + 2\right) Y(w) dw \\ &= \int_0^\infty e^{-wx}\left(7 - 3w\right) dw \\ &= \frac{7}{x} - \frac{3}{x^2}. \end{aligned}$$

Example 2.6. In this example, we find a solution for the difference-integral equation

$$y(x+1) + \int_x^\infty (\xi-x)^2 y(\xi) d\xi = \frac{6}{(x+1)^4} + \frac{2}{x}, x > -1.$$

The left side can be written as $\mathcal{L}_x(e^{-t}f(t) + \frac{2f(t)}{t^3})$, where $y(x) = \mathcal{L}_x(f(t))$, and the right side is $\mathcal{L}_x(t^3e^{-t} + 2)$. Therefore, for $t > 0$, we have $f(t) = t^3$. This means $y(x) = \mathcal{L}_x(f(t)) = \frac{6}{x^4}$ is a solution of this equation.

Example 2.7. We wish to find a solution for the integro-differential equation

$$(2.4) \quad Y''(x) + Y(x+1) + \int_x^\infty \sin(u-x)Y(u)du = e^{-3x}.$$

This equation is resulting by taking the Laplace transform of

$$t^2y(t) + e^{-t}y(t) + \frac{y(t)}{t^2+1} = \delta(t-3),$$

where $Y(x) = \mathcal{L}_x(y(t))$. Hence,

$$y(t) = \frac{(t^2+1)\delta(t-3)}{(t^2+1)(t^2+e^{-t})+1}.$$

Therefore,

$$Y(x) = \mathcal{L}_x(y(t)) = \frac{10e^{-3x}}{10(9+e^{-3})+1}$$

is a particular solution for the equation (2.4).

The following result gives the Laplace transform for "the weighted convolution"

Corollary 2.8. *The Laplace transform of the product of $h(t)$ and the convolution of $f * g$ is given as*

$$\mathcal{L}_s(h(t)(f * g)(t)) = \int_s^\infty (\mathcal{L}^{-1}h)(\xi-s)F(\xi)G(\xi)d\xi.$$

In particular, if $h(t) = 1$ then we get

$$\mathcal{L}_s((f * g)(t)) = \int_s^\infty \delta(\xi-s)F(\xi)G(\xi)d\xi = F(s)G(s),$$

which is the well-known result that the Laplace transform of the convolution of two function is the product of their Laplace transforms.

Example 2.9. Use $f(t) = g(t) = \sin t$ and $h(t) = \frac{2}{t^3}$ to get that

$$\begin{aligned} \mathcal{L}_s\left(\frac{\sin t - t \cos t}{t^3}\right) &= \mathcal{L}_s\left(\frac{2(f * g)(t)}{t^3}\right) \\ &= \int_s^\infty \frac{(\xi-s)^2}{(\xi^2+1)^2}d\xi = \frac{1}{4}(\pi s^2 - 2(s^2+1)\arctan(s) - 2s + \pi). \end{aligned}$$

One of the application of this result is the following interesting relations between convolution and the Laplace transform of the product of two functions

Corollary 2.10.

$$\int_0^\infty h(u) \mathcal{L}_u(f(t)(\mathcal{L}g)(t)) du = \int_0^\infty (\mathcal{L}f)(\xi)(g * h)(\xi) d\xi.$$

In particular,

$$\mathcal{L}_u(f(t)(\mathcal{L}g)(t))(a) = \int_0^\infty (\mathcal{L}f)(\xi)(e^{-at} * h(t))(\xi) d\xi.$$

Proof.

$$\begin{aligned} \int_0^\infty h(u) \mathcal{L}_u(f(t)(\mathcal{L}g)(t)) du &= \int_0^\infty h(u) \left(\int_u^\infty g(\xi - u) F(\xi) d\xi \right) du \\ &= \int_0^\infty h(u) \left(\int_u^\infty g(\xi - u) F(\xi) d\xi \right) du \\ &= \int_0^\infty F(\xi) \left(\int_0^\xi g(\xi - u) h(u) du \right) d\xi \\ &= \int_0^\infty F(\xi)(g * h)(\xi) d\xi. \end{aligned}$$

□

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Numerical blow-up time and growth rate of a reaction-diffusion equation

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Abstract. This paper is concerned with the numerical blow-up solutions of a homogeneous Dirichlet problem for a semilinear heat equation with a nonlinear reaction term, defined in one dimensional-space. Namely, we compute the approximate blow-up time and the blow-up growth rate constant for a numerical experiment of this problem by using a finite difference method. The numerical results confirm and support the known theoretical blow-up results.

Keywords: blow-up solution, semilinear heat equation, growth rate, finite difference.

1. Introduction

We consider the initial-boundary problem which takes the following form:

$$(1) \quad \left\{ \begin{array}{ll} u_t = u_{xx} + \lambda u^p e^{qu}, & (x, t) \in (-1, 1) \times (0, T), \\ u(x, t) = 0, & x = \mp 1, \quad t \in (0, T), \\ u(x, 0) = u_0(x), & x \in [-1, 1] \end{array} \right\},$$

where $p \geq 1$; $q, \lambda > 0$, $u_0 \in C^2(R)$, nonzero, nonnegative, radially non increasing function, such that $u_0(x) = 0$, $x = \mp 1$.

The blow-up phenomena in the homogeneous Dirichlet problem of semilinear heat equations, defined in bounded domains, has been studied, theoretically and numerically, with different types of nonlinear reaction term, by many authors, see for instance [1, 2, 3, 4, 5, 6]. The blow-up properties of problem (1) were studied in [4]. It has been shown that, with a large value initial function, the classical solution of problem (1) blows up in a finite time. Moreover, it has been shown that the blow-up can only occur at $x = 0$ and the upper point wise

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estimate takes the forms:

$$(2) \quad u(x, t) \leq \log A + \frac{2}{\alpha} \log \left(\frac{1}{|x|} \right), \quad x \in (-1, 1) \setminus \{0\}, t \in (0, T)$$

where $A > 0, \alpha \in (0, 1)$.

For blow-up rate estimate of this problem, it has been shown that there exist $C > 0$, such that:

$$(3) \quad u(x, t) \leq \frac{1}{q} (\log C - \log [q\lambda(T-t)]), \quad x \in (-1, 1), \quad t \in (0, T).$$

Over the last decades, finite difference methods have been used to solve many parabolic problems, see for instance [7, 8, 9, 10]. In this paper, we aim to estimate blow-up time and blow-up growth rate constant, C , for some examples of problem (1), using explicit Euler finite difference method.

2. The discrete problem

In this section, we derive the explicit Euler finite difference formula for problem (1).

To derive the discrete problem of (1), we set $h = 2/J$, for J a positive integer, and define the grids $x_0 = -1$, $x_J = 1$, and $x_j = x_{j-1} + h$ for $j = 1, 2, \dots, J-1$. Also, we introduce the time step $k > 0$ and the discrete time levels $t_0 = 0$, $t_{n+1} = t_n + k = nk$, for $n = 0, 1, 2, \dots$. We shall denote by U_j^n the approximate value of $u(x_j, t_n)$, for $1 \leq j \leq J-1$, and $n > 0$, obtained by numerical methods.

We approximate $u_{xx}(x_j, t_n)$ by the standard second order finite difference operator, while $u_t(x_j, t_n)$ is approximated by the forward finite difference operator. Thus, the discrete equation of the semilinear equation in (1) becomes:

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + f(U_j^n), \quad \text{for } 1 \leq j \leq J-1, n > 0,$$

where $f(U_j^n) = \lambda(U_j^n)^p e^{qU_j^n}$ i.e. the discrete problem of problem (1), becomes:

$$(4) \quad \left\{ \begin{array}{l} U_j^{n+1} = (1-2r)U_j^n + r(U_{j+1}^n + U_{j-1}^n) + kf(U_j^n), \\ \quad \quad \quad \text{for } 1 \leq j \leq J-1, n > 0, \\ U_0^n = U_J^n = 0, \\ U_j^0 = u(x_j, 0) = u_0(x_j), \quad \text{for } 0 \leq j \leq J \end{array} \right\}$$

where $r = k/h^2$.

It is well known that, $2r \leq 1$, is the stability condition of the explicit Euler method for the heat equation. Therefore, h and k must be chosen such that this condition is satisfied. Moreover, the rate of convergence for this method is $O(k + h^2)$.

The discrete formula (4) has been used by some authors to compute the numerical solution of the semilinear heat equation in one dimensional space, where the nonlinear term is of exponential or power type functions, for instance, see [9, 10].

3. Convergence of numerical blow-up solution and time

The next theorem summaries some results, those have been proved in [10], for more general nonlinear terms f (including power and exponential types), which guarantees that for small h , the solution of the discrete problem (4) convergences to the exact solution of problem (1)

Theorem 3.1. *Let $U^n = [U_0^n, U_1^n, \dots, U_J^n]$ is the solution of (4).*

If $0 < k \leq \frac{h^2}{2}$ then

1- The solution of (4) is nonnegative, and

$$U_j^{n+1} \geq U_j^n, \quad U_{J-j}^n = U_j^n \quad \text{for} \quad 0 \leq j \leq J, n \geq 0$$

$$2 - \|U^n - u^n\| \leq Ch^2, \quad n \geq 0,$$

where $u^n = [u_0(x_j), u(x_1, t_n), \dots, u(x_J, t_n)]$. i.e. $U^n \rightarrow u^n$, where $h \rightarrow 0$ which means, at each point (x_j, t_n) , the numerical solution given by (4) convergences to the exact solution of problem (1).

We will see, the solutions of (4) do not exist for all $n \in N$, because they become unbounded for some n .

Definition 3.2 ([10]). Let $\{U^n\} = (U_0^n, U_1^n, \dots, U_J^n)$ $n \geq 0$ is the numerical solution of problem (4), we say that $\{U^n\}$ blows up in the finite time, if there exists $m \in N$ such that $\|U^m\|_\infty$ is unbounded, where $\|U^n\|_\infty = \max_{j=0, \dots, J} |U_j^n|$.

Moreover, $T_J^m = mk$ is called the numerical blow-up time of problem (1).

In fact, the size of spatial grid h and the choice of time steps k play an important role in determining the numerical blow-up time.

The next theorem, which has been proved in [10], studies the convergence of the blow-up time of the discrete problem (4) to the blow-up time of the of problem (1), for more general types of f (including power and exponential types).

Theorem 3.3. *Let $\{U^n\} = (U_0^n, U_1^n, \dots, U_J^n)$ $n \geq 0$ be the solution of (4), such that $0 < k \leq \frac{h^2}{2}$.*

- 1. There is $m \in N$ such that $\{U^n\}$ blows up in the finite time T_J^m*
- 2. Let T be the blow-up time of (1), then $T_J^m \rightarrow T$, where $J \rightarrow \infty$, which means, the numerical blow-up time approaches to the theoretical blow-up time, for h sufficiently small.*

4. Numerical experiment

In this section, we use the Euler discrete formula, to study the numerical solutions of problem (1) with $u_0(x) = 5(1 - x^2)$.

It is clear that u_0 satisfies all our assumptions in (1), and takes its maximum value at the point $x = 0$, therefore according to the blow-up results of problem (1), which we have proved in [4], the blow-up in equation (1) should only occur at $x = 0$.

We aim to confirm these theoretical results and find the approximate values to the blow-up time. Moreover, we estimate the blow-up growth rate constant depending on the numerical results.

Our aim is to use the discrete equation (4) to compute numerically, the blow-up time and solutions of problem (1), and the computational codes are written in Matlab software.

In fact, the blow-up time will be taken experimentally, at the first time that $\|U^n\|_\infty \geq 10^6$. For converging, we will choose, $k = h^2/2$, and we will get a symmetric numerical solution which takes its maximum at $x_{J/2} = 0$, with respect to the meshes $J=150, 200, 250, 300$ and 350 .

Moreover, we estimate the constant of the blow-up growth rate in the blow-up estimates (3), at each fixed value of J , as follows $C_J = \max_{0 \leq n \leq m-1} C(n)$, where $C(n) = q\lambda(T_J^m - t_n) \exp(qU_{J/2}^n)$.

The numerical solutions will be carried out for three cases:

- Case1: $p = q = \lambda = 1$,
- Case 2: $p = 1, q = \lambda = 1/2$
- Case 3: $p = 2, q = 1, \lambda = 1/2$.

In Tables 4, 4 and 4, we present the numerical blow-up times and the numerical values of the constant of blow-up growth rate C_J , for Cases 1, 2 and 3 respectively, with respect to the meshes $J=150, 200, 250, 300$ and 350 , while in Table 4, we present the iterative errors obtained by using the error form $E_J = |T_J^m - T_{J-50}^m|$, for Cases 1, 2 and 3, where $m = m(J)$ refers to the number of iteration, when numerical blow-up occurs.

J	k	m	$T = mk$	C
150	8.8889e-05	18	0.00151	1.4539e+05
200	5.0000e-05	29	0.00140	5.6757e+97
250	3.2000e-05	42	0.00131	3.8499e+12
300	2.2222e-05	58	0.00126	7.6391e+05
350	1.6327e-05	78	0.00125	9.0910e+04

Table 1: Case 1.

J	k	m	$T = mk$	C
150	8.8889e-05	837	0.07431	3.1238e+02
200	5.0000e-05	1482	0.07405	1.1749e+17
250	3.2000e-05	2312	0.07395	4.1404e+124
300	2.2222e-05	3326	0.07388	5.9799e+104
350	1.6327e-05	4524	0.07384	1.7548e+12

Table 2: Case 2.

J	k	m	$T = mk$	C
150	8.8889e-05	9	7.11111e-04	1.7108e+05
200	5.0000e-05	13	6.00000e-04	2.1918e+15
250	3.2000e-05	18	5.44000e-04	4.7385e+115
300	2.2222e-05	24	5.11111e-04	2.7181e+02
350	1.6327e-05	30	4.93469e-04	7.7754e+09

Table 3: Case 3.

J	E_J , Case 1	E_J , Case 2	E_J , Case 3
200	0.00011	0.00026	1.11111e-04
250	0.00009	0.00010	5.59999e-05
300	0.00005	0.00007	e-053.28890
350	0.00001	0.00004	e-051.76420

Table 4: Errors bounds.

The next figures show the evolutions in time, of the numerical blow-up solutions of problem (1), for cases 1, 2 and 3, with respect to $J = 350$ and $0 \leq n \leq m$.

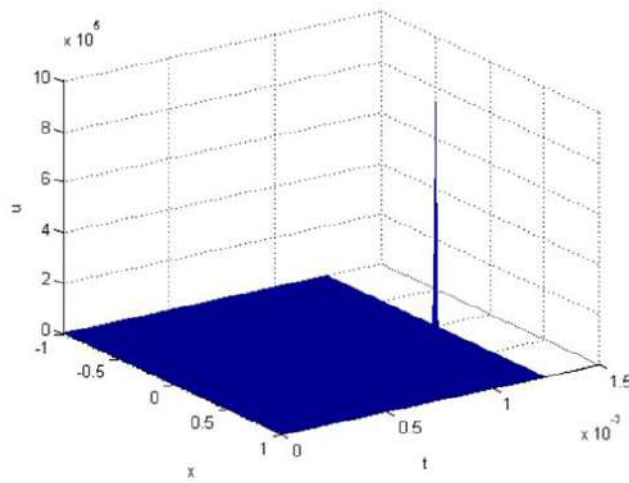


Figure 1: Case 1

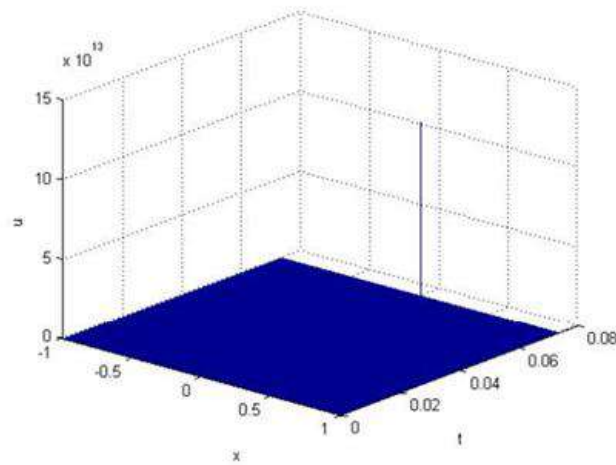


Figure 2: Case 2

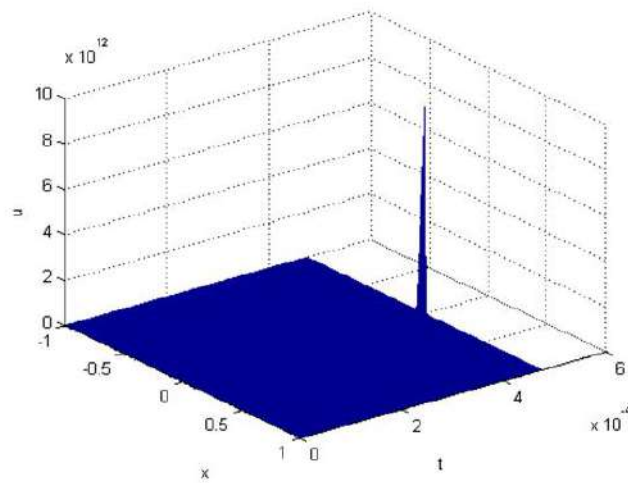


Figure 3: Case 3

From our numerical results Tables 4–4 and Figures 1-3, we can point out the following observations:

1. The numerical solution blows up only at the centre, $x = 0$, and that confirms the theoretical results proved in [4].
2. Decreasing the space-step and time-step, leads to decreasing the numerical blow-up time, which means the numerical blow-up occurs faster, as we refine the grid points.
3. The error bound, E_J , decreases as J increases, which indicates that: the numerical solutions converge as the space-step and time-step tend to zero.
4. With large values to the one or more of p, λ , or q , the solutions divergences radially, while decreasing at least one of them, leads to increasing the numerical blow-up time. And that support the know blow-up properties for problem (1).
5. The numerical constant C_J plays an important role in estimating the growth rate of the corresponding numerical blow-up solution and deriving a discrete formula to the blow-up rate estimate (3) as follows:

$$U_j^n \leq \frac{1}{q} (\log C_J - \log [q\lambda (T_J^m - t_n)]) ,$$

$$\text{for } 0 \leq n \leq m - 1, i = 1, 2, \dots J - 1$$

5. Conclusion

In this paper, the numerical blow-up solutions of problem (1) are studied. Namely, we compute the approximate blow-up time and the blow-up growth rate constant for a numerical experiment of this problem by using explicit Euler difference method. The numerical results obtained by the proposed methods are analyzed, simulated and presented in the form of tables and figures. We have observed that the numerical result confirms and supports the known theoretical blow-up results. Moreover, the proposed method is successfully implemented with good efficiency and high order of convergence.

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Edge-to-vertex and edge-to-edge D -distances

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Abstract. In a (simple and connected) graph G , the concept of distance is one of the important concept. Earlier we introduced the concept of D -distance between vertices of a graph.

In this article we study edge-to-vertex and edge-to-edge distances w.r.t. D -distance. We obtain relation between their eccentrics and also determine centers of some graphs. We prove the relation between the edge-to-vertex and edge-to-edge D -eccentricities w.r.t. D -distance. We also prove that for any graph G either $C_2^D(G) \subseteq C_3^D(G)$ or $C_3^D(G) \subseteq C_2^D(G)$.

Keywords: D -distance, radius, diameter, center, eccentricity.

1. Introduction

The distance between vertices of a graph plays an important role in many places. The adjacency problems can be converted into distance problems and vice versa. This concept has been extended to distance between vertex-to-edge distance edge-to-vertex distance and edge-to-edge distance.

The first two authors have introduced the concept of D -distance between vertices of a graph in [2] and extended it vertex-to-edge D -distance in [3].

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In the present work, we extend the concept of D -distance to edge-to-vertex and edge-to-edge D -distance and study some properties. We give a result connecting edge-to-vertex D -centers and edge-to-edge D -centers. Also we prove a result on edge-to-vertex center of a tree.

2. Edge to vertex and edge to edge distances

By a graph $G(V, E)$ or simply G , we mean a non-trivial, finite, undirected graph without multiple edges and loops. Further unless otherwise specified all graphs will be connected.

In this section we introduce some definitions by extending the vertex to vertex concepts to edge to vertex and edge to edge. We begin with D -distance in graphs introduced the authors in (see [2]).

Definition 1. If u, v are vertices of a connected graph G . The D -length of a connected $u - v$ path s is defined as $l^D(s) = l(s) + \deg(v) + \deg(u) + \sum \deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

Definition 2 (D -distance). The D -distance $d^D(u, v)$ between two vertices u, v of a connected graph G is defined as $d^D(u, v) = 0$ if $u = v$ and $d^D(u, v) = \min \{l^D(s)\}$ if u, v are distinct, where the minimum is taken over all $u - v$ paths s in G . In other words,

$$d^D(u, v) = \begin{cases} \min_s \{l^D(s) + \deg(u) + \deg(v) + \sum \deg(w)\}, & \text{if } u \neq v \\ 0, & \text{if } u = v \end{cases}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G .

Now we define the edge to vertex D -distance.

Definition 3. Let G be a connected graph taken an edge $e = xy \in E(G)$ and vertex v in G . Then the D -distance between edge (e) and vertex (v) is defined as $d^D(e, v) = \min \{d^D(x, v), d^D(y, v)\}$.

Definition 4. For any edge f of G in a connected graph G , the edge-to-vertex D -eccentricity of $e_2^D(f)$, is given by $e_2^D(f) = \max \{d^D(e, v) : v \in V\}$.

Definition 5. Any edge f for which $e_2^D(f)$ is minimum is called edge-to-vertex central edge of G . The set of all edge-to-vertex central edges of G is the edge-to-vertex center and is denoted as $C_2^D(G)$.

Next, we extend the distance concept to edge to edge D -distance.

Definition 6. Let G be a connected graph and $e = xy, f = uv$ be two edges of G . Then the edge-to-edge D -distance is defined as

$$d^D(e, f) = \min \{d^D(x, u), d^D(x, v), d^D(y, u), d^D(y, v)\}.$$

Definition 7. For any edge f in a connected graph G , the edge-to-edge D -eccentricity of $e_3^D(f)$, is defined as $e_3^D(f) = \max \{d^D(f, e) : e \in E(G)\}$.

One more definition.

Definition 8. Any edge f of G , for which e_3^D is minimum is called central edge of G . The set of all central edges of G is the edge to edge D -distance center of G and is denoted by $C_3^D(G)$. Any edge e for which $e_3^D(e) = d^D(f, e)$ is called an eccentric edge of f .

Remark 9. Observe that $d^D(e, f) = 0 \Leftrightarrow e, f$ are neighbor edges i.e., they have one common vertex

Below we give an example.

Example 10. Consider the $(6, 6)$ graph G , shown in the figure 1. In this graph, the edge-to-vertex and edge-to-edge D -distances, are as shown in the tables 1 and 2. From these tables, we write various eccentricities of edges which are presented in table 3.

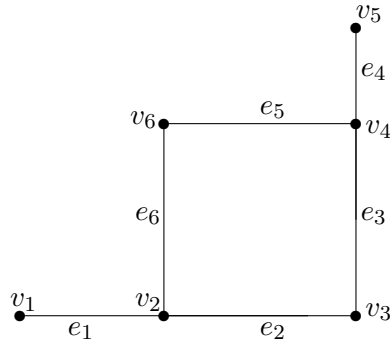


Figure 1: Graph G

The following table shows the edge-to-vertex D -distances of the graph G .

	v_1	v_2	v_3	v_4	v_5	v_6
e_1	0	0	6	10	12	6
e_2	5	0	0	6	8	6
e_3	8	6	0	0	5	6
e_4	12	10	6	0	0	6
e_5	8	6	6	0	5	0
e_6	5	0	6	6	8	0

Table 1: D -distance between edges and vertices of G .

The following table shows D -distances between edges of graph G .

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	0	6	10	6	0
e_2	0	0	0	6	6	0
e_3	6	0	0	0	0	6
e_4	10	6	0	0	0	6
e_5	6	6	0	0	0	0
e_6	0	0	6	6	0	0

Table 2: D -distance between edges of G .

From the above two tables we write down the D -eccentricities of edges which are as follows:

	e_1	e_2	e_3	e_4	e_5	e_6
$e_2^D(e)$	12	8	8	12	8	8
$e_3^D(e)$	10	6	6	10	6	6

Table 3: D -eccentricities of G

Remark 11. In the above example observe that $C_2^D(G) = \{e_2, e_3, e_5, e_6\}$ and $C_3^D(G) = \{e_2, e_3, e_5, e_6\}$ and hence $C_2^D(G) = C_3^D(G)$.

3. Edge-to-vertex and edge-to-edge centers

We begin this with a result on the edge to vertex and edge to edge eccentricities.

Theorem 12. For any edge e in a connected graph G , we have $e_2^D(e) - \deg(e) - 1 \leq e_3^D(e) \leq e_2^D(e)$. Further, $e_3^D(e) = e_2^D(e)$ if and only if both the vertices of any eccentric edge of e are eccentric vertices of e .

Proof. Let e be any edge of G and let f be an eccentric edge of e . Let $p = u = v_0, v_1, v_2, \dots, v_n = v$ be a $u - v$ minimum D -path and let $e = v_{n-1}v_n$. Then

$$\begin{aligned} d^D(e, v) &= k + \deg(v_0) + \deg(v_1) + \deg(v_2) + \dots + \deg(v_n) \\ &= e_2^D(e), \end{aligned}$$

where k is length of the path.

Let $f = v_{m-1}v_m$, since

$$\begin{aligned} d^D(e, f) &= k - 1 + \deg(v_0) + \deg(v_1) + \deg(v_2) + \dots + \deg(v_{n-1}) \\ &= e_2^D(e) - \deg(v) - 1, \end{aligned}$$

it follows that $e_3^D(e) \geq e_2^D(e) - \deg(v) - 1$. Also since $e_3^D(e) = \min \{d^D(e, f) : e, f \in E\}$, it follows that $e_3^D(e) \leq d^D(e, v) = e_2^D(e)$. Thus $e_2^D(e) - \deg(v) - 1 \leq e_3^D(e) \leq e_2^D(e)$. $\square$

Next, we prove a result on edge to vertex and edge to edge centers.

Theorem 13. *For any graph G , either $C_2^D(G) \subseteq C_3^D(G)$ or $C_3^D(G) \subseteq C_2^D(G)$.*

Proof. Suppose the result is false. Then there exist two edges e and f such that $e \in C_3^D(G) - C_2^D(G)$ and $f \in C_2^D(G) - C_3^D(G)$ i.e., $e \in C_3^D(G)$, $e \notin C_2^D(G)$ and $f \in C_2^D(G)$, $f \notin C_3^D(G)$. Then

$$(3.1) \quad e_2^D(f) < e_2^D(e)$$

and

$$(3.2) \quad e_3^D(e) < e_3^D(f).$$

If $e_3^D(e) = e_2^D(e)$ we have $e_2^D(f) < e_2^D(e) = e_3^D(e) < e_3^D(f) \Rightarrow e_2^D(f) < e_3^D(f)$ which is contradiction to theorem(12). If $e_3^D(f) = e_2^D(f) - \deg(f) - 1$ we have $e_3^D(e) - e_3^D(f) = e_2^D(f) - \deg(f) - 1 < e_2^D(e) - \deg(e) - 1 \Rightarrow e_3^D(e) < e_2^D(e) - \deg(e) - 1$ which is again contradiction to theorem(12). Thus

$$(3.3) \quad e_3^D(e) = e_2^D(e) - \deg(e) - 1$$

and

$$(3.4) \quad e_3^D(f) = e_2^D(f).$$

From equation (3.2) we have $e_3^D(e) < e_3^D(f) = e_2^D(f) \Rightarrow e_3^D(e) < e_2^D(f) \Rightarrow e_2^D(e) - \deg(e) - 1 < e_2^D(f) \Rightarrow e_2^D(e) < e_2^D(f)$ which is contradiction to equation (3.1). Hence the theorem. $\square$

Below we construct a class of graphs for which both of these centers are equal.

Theorem 14. *Let $G = (K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \dots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 4$, $n_i \geq 2$ ($1 \leq i \leq r$) and $n_1 + n_2 + n_3 + \dots + n_r + k = p - 1$, then $C_2^D(G) = C_3^D(G)$.*

Proof. Let $r \geq 2$. For any edge $g = xy$ incident at vertex v then $e_2^D(g) = e_3^D(g) = \deg(v) + \deg(w)$, $\forall w \in K_{n_r}$, $n_1 = n_2 = n_3 = \dots = n_r$, $e_2^D(g) = e_3^D(g) = 1 + \deg(v) + \deg(w)$, $\forall w \in K_{n_r}$, $n_1, n_2, n_3, \dots, n_r$ are different and $e_2^D(f) = e_3^D(f) = 2 + \deg(v) + \deg(w_1) + \deg(w_2)$. Thus $C_2^D(G) = C_3^D(G) =$ the set of all edges incident at v . $\square$

Example 15. Consider the (5,5) graph kite (K), shown in the figure 2. For this, we see $C_3^D \subseteq C_2^D$.

Example 16. Consider the (5,6) graph Hut (H), shown in the figure 3. For this, we see $C_2^D \subseteq C_3^D$.

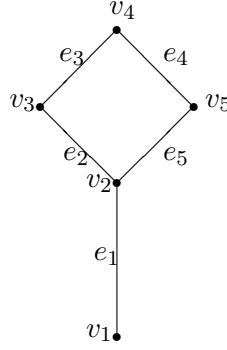


Figure 2: Kite (K)

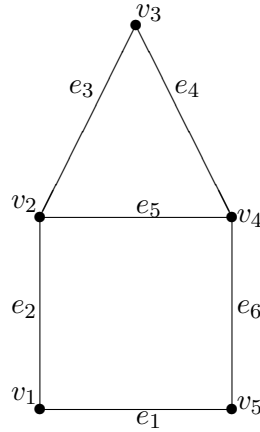


Figure 3: Hut (H)

Next, we present a result on edge to vertex center of a tree.

Theorem 17. *The edge-to-vertex center $C_2^D(T)$ of a non trivial tree T induces a star.*

Proof. If T is a tree and $T \neq K_2$, then $e_2^D(e) = n + 1$ for any edge e . If $T = K_2$ then $e_2^D(e) = 0$ thus the subgraph induced by $C_2^D(T)$ is T it self. If T is not a star, we first prove that the subgraph induced by $C_2^D(T)$ is connected other wise, there exists two edges $e = v_1v_2$ and $f = v_3v_4$ in $C_2^D(T)$. Such that the path connecting e and f contains an edge, say $g = xy$ with g not in $C_2^D(T)$. Let $e_2^D(e) = e_2^D(f) = m$ and $e_2^D(g) = n$ then $n > m$. Let z be an eccentric vertex of g so that $e_2^D(g) = d^D(g, z) = n$. If this D -distance attained at x , then $e_2^D(f) > n > m$ which is contradiction. If the D -distance attained at y then $e_2^D(e) > n > m$ which is again a contradiction thus $C_2^D(T)$ induces a star other wise, the subgraph induced by $C_2^D(T)$ contains a path $p : v_1, v_2, v_3, v_4$ of length $\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) + 3$. Let $e = v_1v_2$, $f = v_2v_3$ and $g = v_3v_4$ since $e, f, g \in C_2^D(T)$. We have $e_2^D(e) = e_2^D(f) = e_2^D(g) = m$ (say). Let x be an eccentric vertex of f so that $e_2^D(f) = d^D(f, x) = m$. If

the D -distance attained at v_2 it is easy to see that $d^D(g, x) = m + \deg(x) + 1$. So that $e_2^D(g) > m + \deg(x) + 1$ which a contradiction and if the D -distance $d^D(g, x)$ is attained at v_3 it is easy to see that $d^D(e, x) = m + \deg(x) + 1$, so that $e_2^D(e) > m + \deg(x) + 1$ which is again contradiction. Hence $C_2^D(T)$ induces a star. $\square$

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Hyers-Ulam stability of second order difference equations

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Abstract. In this paper, we study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous and non-homogeneous linear difference equations of second order with constant co-efficients by applying Z -Transforms method.

Keywords: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, homogeneous and non-homogeneous, linear difference equation, Z -Transforms method.

1. Introduction

The study of stability problem for various functional equations originated from a famous talk of S.M. Ulam [23]. In 1940, Ulam [23] raised the question concerning the stability of functional equations: “Give Conditions in order for a linear function near an approximately linear function to exist” (see [10, 15]). Since then, this question has attracted the attention of many researchers. The first partial solution to this problem was given by D.H. Hyers [10] in 1941. He gave an affirmative answer to the question of Ulam’s for additive Cauchy equation in Banach Spaces. Thereafter, Aoki [2], Bourgin [4] and Rassias [20] have generalized and also improved the result reported in [10]. Since then, several mathematicians have been extensively established the stability results to the various functional equations in different directions (see [5, 8, 12, 25, 28, 29, 30]).

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A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi(f, x, x', x'', \dots x^{(n)}) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left| \phi(f, x, x', x'', \dots x^{(n)}) \right| \leq \epsilon,$$

there exists a solution x_a of the differential equation

$$\phi(f, x, x', x'', \dots x^{(n)}) = 0$$

such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \rightarrow 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

Oblaza [16, 17] were among the first to contribute to deal with the Hyers-Ulam stability of differential equations. Thereafter Alsina and Ger [1] were the first authors, who investigated the Hyers-Ulam stability of differential equations of the form $x'(t) = x(t)$. They proved in [1] the following Theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \rightarrow \mathbb{R}$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \rightarrow \mathbb{R}$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [1] has been generalized by Takahasi [22]. They proved in [22] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. These previous results are extended to the Hyers-Ulam stability of linear differential equation of first order, second order and higher order in [3, 9, 11, 12, 13, 15, 21, 24, 26] by different approaches.

Now a days, the investigation of Hyers-Ulam stability of difference equations has been given attention. Recently, some authors are studied the Hyers-Ulam stability of linear and non-linear recurrences, linear recurrence with constant coefficients, linear recurrence of higher order in [6, 7, 14, 18, 19]. In this paper, we are going to prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous second order linear difference equation

$$(1.1) \quad x(n+2) + a x(n+1) + b x(n) = 0$$

and the non-homogeneous linear difference equation

$$(1.2) \quad x(n+2) + a x(n+1) + b x(n) = p(n)$$

with initial conditions

$$(1.3) \quad x(0) = 0 \text{ and } x(1) = 1$$

by using Z -Transforms for all $n \in \mathbb{N}$, where $a, b \neq 0$ are constants.

2. Preliminaries

Z -transforms plays an important role in discrete analysis as Laplace and Fourier transforms in continuous system. It has many properties similar to those of the Laplace transforms. The Z -transforms operate on sequences of the discrete integer valued arguments, not on functions of continuous arguments. For every operational rule or Laplace transforms, there is a corresponding operational rule of Z -transforms and for every application of the Laplace transforms, there is a corresponding application of Z -transforms.

Firstly, we introduce some basic concepts of difference equations and Z -Transforms. Throughout this paper, $\mathbb{F}$ will denote the field of Complex numbers $\mathbb{C}$. If the function f_n is defined for discrete values ($n = 0, 1, 2, 3, \dots$) and $f_n = 0$ for $n < 0$, then the Z -Transform is defined to be

$$Z(f_n) = F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n}$$

whenever the infinite series converges. Here z is a complex number. Z is an operator and $F(z)$ is the Z -Transform of $\{f(n)\}$. It is well known that Z is one-to-one and linear. If $Z[\{f(n)\}] = F(z)$, $n > 0$, then $f(0) = \lim_{z \rightarrow \infty} F(z)$ is known as the initial value problem of Z -Transform. If $Z[\{f(n)\}] = F(z)$, $n > 0$, then

$$\lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow 1} (z - 1) F(z)$$

is known as the Final value problem of Z -Transform. Let $\{f(n)\}$ and $\{g(n)\}$ are any two sequences and let the convolution of $\{f(n)\}$ and $\{g(n)\}$ be $\{h(n)\}$, where $\{h(n)\} = \{f(n)\} * \{g(n)\}$ then $\{h(n)\}$ is defined as

$$\{f(n)\} * \{g(n)\} = \sum_{n=-\infty}^{\infty} f(n) g(n - k) = \sum_{n=-\infty}^{\infty} g(n) f(n - k) = \{g(n)\} * \{f(n)\}.$$

Also, if $Z[\{f(n)\}] = F(z)$ and $Z[\{g(n)\}] = G(z)$, then

$$Z[\{h(n)\}] = Z[\{f(n)\}] * Z[\{g(n)\}] = F(z) G(z) = H(z),$$

where the region of convergence of $Z[\{h(n)\}]$ is the common region of convergence of $F(z)$ and $G(z)$.

If $F(z)$ is the Z -Transform of the sequence $\{f(k)\}$ then $\{f(k)\}$ is called the inverse Z -Transform of $F(z)$. The operator for inverse Z -Transform is denoted by Z^{-1} . Thus, if $Z[\{f(k)\}] = F(z)$, then $Z^{-1}[F(z)] = \{f(k)\}$.

Now, we will give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability for (1.1) and (1.2).

Definition 2.1. We say that the homogeneous linear difference equation (1.1) has the Hyers-Ulam stability, if there is a positive constant K such that for every $\epsilon > 0$ and $x(n)$ satisfying the inequality

$$|x(n+2) + a x(n+1) + b x(n)| \leq \epsilon$$

with (1.3) then there exists $y(n)$ satisfying (1.1) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K \epsilon$.

Definition 2.2. We say that the non-homogeneous linear difference equation (1.2) has the Hyers-Ulam stability, if there is a positive constant K such that for every $\epsilon > 0$ and $x(n)$ satisfying the inequality

$$|x(n+2) + a x(n+1) + b x(n) - p(n)| \leq \epsilon$$

with (1.3) then there exists $y(n)$ satisfying (1.2) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K \epsilon$.

Definition 2.3. We say that the homogeneous linear difference equation (1.1) has the Hyers-Ulam-Rassias stability if there is a positive constant K such that for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfying the inequality

$$|x(n+2) + a x(n+1) + b x(n)| \leq \phi(n) \epsilon$$

with (1.3) then there is a $y(n)$ satisfying (1.1) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K(\epsilon) \phi(n)$.

Definition 2.4. We say that the non-homogeneous linear difference equation (1.2) has the Hyers-Ulam-Rassias stability if there is a positive constant K such that for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfying the inequality

$$|x(n+2) + a x(n+1) + b x(n) - p(n)| \leq \phi(n) \epsilon$$

with (1.3) then there exists $y(n)$ satisfying (1.2) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K(\epsilon)\phi(n)$.

3. Hyers-Ulam stability of (1.1) and (1.2)

In this section, we are going to prove the Hyers-Ulam stability of the second order homogeneous and non-homogeneous linear difference equation (1.1) and (1.2) by applying Z -Transforms. Now, we prove the Hyers-Ulam stability of the homogeneous linear difference equation (1.1) with initial conditions.

Theorem 3.1. For any $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. For every $\epsilon > 0$, we can find a positive constant K such that a positive sequence $\{x(n)\}$ satisfies the inequality

$$(3.1) \quad |x(n+2) + a x(n+1) + b x(n)| \leq \epsilon,$$

for each value of $n \in \mathbb{N}$ with (1.3), then there exists a solution $\{y(n)\}$ of the second order linear difference equation satisfies (1.1) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K \epsilon$.

Proof. Let us define a function $h : (0, \infty) \rightarrow \mathbb{F}$ such that

$$h(n) = x(n+2) + a x(n+1) + b x(n).$$

Also, in view of (3.1), we have $|h(n)| \leq \epsilon$. Now, taking Z -Transform to $h(n)$, then we have

$$(3.2) \quad Z[h(n)] = H(z) = (z^2 + az + b) X(z) - z(z+a)x(0) - z x(1).$$

In view of (3.2), a sequence $\{x_0(n)\}$ is a solution of (1.1) if and only if

$$(3.3) \quad (z^2 + az + b) X(z) - z(z+a)x_0(0) - z x_0(1) = 0.$$

Applying (1.3) in (3.2), we have

$$(3.4) \quad H(z) = (z^2 + az + b) X(z) - z.$$

Since, we have $z^2 + az + b = (z - \alpha)(z - \beta)$, then (3.4) becomes

$$(3.5) \quad X(z) - \frac{z}{(z - \alpha)(z - \beta)} = \frac{H(z)}{(z - \alpha)(z - \beta)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} x(1)$, then applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z x(1)}{(z - \alpha)(z - \beta)}.$$

Hence

$$(3.6) \quad (z - \alpha)(z - \beta) Y(z) - z(z+a)x(0) - z x(1) = 0.$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a y(n+1) + b y(n)] &= Z[y(n+2)] + a Z[y(n+1)] + b Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z+a)y(0) - zy(1) \\ &= 0. \quad [\text{form (3.6)}] \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a y(n+1) + b y(n) = 0$, therefore $y(n)$ is a solution of (1.1). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z - \alpha)(z - \beta)} = Z[r(n) * h(n)],$$

where $r(n) = \frac{1}{z} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}$. Since, Z -Transforms is linear and one-to-one, we have $x(n) - y(n) = (r(n) * h(n))$. Now, taking modulus on both sides we have,

$$\begin{aligned} |x(n) - y(n)| &= |r(n) * h(n)| = \left| \sum_{n=-\infty}^{\infty} r(n-k) h(n) \right| \\ &\leq \sum_{n=-\infty}^{\infty} |r(n-k)| |h(n)| \leq K \epsilon. \end{aligned}$$

for each n , where $K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left| \frac{1}{z(\alpha-\beta)} \right| \sum_{n=-\infty}^{\infty} |\{\alpha^{n-k} - \beta^{n-k}\}|$ occurs for each value of n . Then by the virtue of Definition 2.1, the linear difference equation (1.1) has the Hyers-Ulam stability. $\square$

Now, we illustrate the Theorem 3.1 by the following example.

Example 3.2. For every $\epsilon > 0$, we can find a positive constant K such that a sequence $\{x(n)\}$ satisfies the inequality

$$(3.7) \quad |2x(n+2) + 5x(n+1) - 3x(n)| \leq \epsilon,$$

for each value of $t > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon$.

Let $h(n) = 2x(n+2) + 5x(n+1) - 3x(n)$, for all n . Also, in view of (3.7), we have $|h(n)| \leq \epsilon$. Now, taking Z -Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (2z^2 + 5z - 3)X(z) - z(2z + 5)x(0) - 2zx(1).$$

Since, we have $2z^2 + 5z - 3 = 2(z - 1/2)(z + 3)$, then we have

$$X(z) - \frac{2z}{2(z - 1/2)(z + 3)} = \frac{H(z)}{2(z - 1/2)(z + 3)}.$$

Thus

$$X(z) = \frac{H(z) + 2z}{2(z - 1/2)(z + 3)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \left(\frac{(1/2)^n - (-3)^n}{7/2} \right) x(1)$, then applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{2zx(1)}{2(z - 1/2)(z + 3)}.$$

Hence $2(z - 1/2)(z + 3)Y(z) - z(2z + 5)x(0) - 2zx(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[2y(n+2) + 5y(n+1) - 3y(n)] = 0$. Since, Z is one-to-one operator, it holds that $2y(n+2) + 5y(n+1) - 3y(n) = 0$, therefore $y(n)$ is a solution of the difference equation. Then by using the Theorem 3.1, the difference equation $2x(n+2) + 5x(n+1) - 3x(n) = 0$ has the Hyers-Ulam stability.

Example 3.3. For every $\epsilon > 0$, we can find a positive constant K such that a sequence $\{x(n)\}$ satisfies the inequality

$$(3.8) \quad |x(n+2) + 3x(n+1) + 2x(n)| \leq \epsilon,$$

for each value of $n > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon$. Let $h(n) = x(n+2) + 3x(n+1) +$

2 $x(n)$. Also, in view of (3.8), we have $|h(n)| \leq \epsilon$. Now, taking Z -Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + 3z + 2) X(z) - z(z + 3)x(0) - z x(1).$$

Since, we have $z^2 + 3z + 2 = (z + 1)(z + 2)$, then we have

$$X(z) - \frac{z}{(z + 1)(z + 2)} = \frac{H(z)}{(z + 1)(z + 2)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \{(-1)^n - (-2)^n\} x(1)$, then applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z x(1)}{(z + 1)(z + 2)}.$$

Hence $(z + 1)(z + 2)Y(z) - z(z + 3)x(0) - z x(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[y(n + 2) + 3y(n + 1) + 2y(n)] = 0$. Since, Z is one-to-one operator, it holds that $y(n + 2) + 3y(n + 1) + 2y(n) = 0$, therefore $y(n)$ is a solution. Then by using the Theorem 3.1, the difference equation $x(n + 2) + 3x(n + 1) + 2x(n) = 0$ has the Hyers-Ulam stability.

Now, we would like to prove the Hyers-Ulam stability of the non-homogeneous linear difference equation (1.2) with initial conditions.

Theorem 3.4. *For any $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. For every $\epsilon > 0$, we can find a positive constant K such that a positive sequence $\{x(n)\}$ satisfies the inequality*

$$(3.9) \quad |x(n + 2) + a x(n + 1) + b x(n) - p(n)| \leq \epsilon,$$

for each value of n with (1.3), then there is a solution $\{y(n)\}$ of the second order linear difference equation satisfies (1.2) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K \epsilon$, with $p(0) = p(1) = 0$.

Proof. If we define a function $h : (0, \infty) \rightarrow \mathbb{F}$ such that

$$h(n) = x(n + 2) + a x(n + 1) + b x(n) - p(n).$$

Also, in view of (3.9), we have $|h(n)| \leq \epsilon$. Now, taking Z -Transforms to $h(n)$, then we have

$$(3.10) \quad Z[h(n)] = H(z) = (z^2 + az + b) X(z) - z(z + a)x(0) - z x(1) - P(z).$$

In view of (3.10), a sequence $\{x_0(n)\}$ is a solution of (1.2) if and only if

$$(3.11) \quad (z^2 + az + b) X(z) - z(z + a)x_0(0) - z x_0(1) = P(z).$$

Applying (1.3) in (3.10), we have

$$(3.12) \quad H(z) = (z^2 + az + b) X(z) - z - P(z).$$

Since, we have $z^2 + az + b = (z - \alpha)(z - \beta)$, then (3.12) gives that

$$(3.13) \quad X(z) - \frac{z}{(z - \alpha)(z - \beta)} = \frac{H(z) + P(z)}{(z - \alpha)(z - \beta)}.$$

Now, we define a function $y(n)$ as

$$y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} x(1) + [r(n) * p(n)],$$

where $r(n) = \frac{1}{z} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}$. Now, applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z x(1)}{(z - \alpha)(z - \beta)} + \frac{P(z)}{(z - \alpha)(z - \beta)}.$$

Hence

$$(3.14) \quad (z - \alpha)(z - \beta) Y(z) - z(z + a) x(0) - z x(1) = P(z).$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a y(n+1) + b y(n)] &= Z[y(n+2)] + a Z[y(n+1)] + b Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z + a)y(0) - zy(1) \\ Z[y(n+2) + a y(n+1) + b y(n)] &= P(z) = Z[p(n)]. \quad [\text{form (3.14)}] \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a y(n+1) + b y(n) = p(n)$, therefore $y(n)$ is a solution of (1.2). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z - \alpha)(z - \beta)} = Z[r(n) * h(n)],$$

Since, Z -Transforms is linear and one-to-one, so $x(n) - y(n) = (r(n) * h(n))$.

Now, taking modulus on both sides we have,

$$\begin{aligned} |x(n) - y(n)| &= |r(n) * h(n)| = \left| \sum_{n=-\infty}^{\infty} r(n-k) h(n) \right| \\ &\leq \sum_{n=-\infty}^{\infty} |r(n-k)| |h(n)| \\ &\leq K \epsilon. \end{aligned}$$

for each n , where

$$K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left| \frac{1}{z(\alpha - \beta)} \right| \sum_{n=-\infty}^{\infty} \left| \left\{ \alpha^{n-k} - \beta^{n-k} \right\} \right|$$

occurs for each value of n . Then by the virtue of Definition 2.2, the linear difference equation (1.2) has the Hyers-Ulam stability. $\square$

4. Hyers-Ulam-Rassias stability of (1.1) and (1.2)

In this section, we prove the Hyers-Ulam-Rassias stability of the second order homogeneous and non-homogeneous linear difference equation (1.1) and (1.2) with initial conditions.

Theorem 4.1. *For any $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. If there exist a positive constant K , for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfies the inequality*

$$(4.1) \quad |x(n+2) + a x(n+1) + b x(n)| \leq \epsilon \phi(n),$$

where $\phi(n)$ is bounded for each value of n with (1.3) then there is a solution $\{y(n)\}$ of (1.1) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon \phi(n)$.

Proof. Let us define a function $h : (0, \infty) \rightarrow \mathbb{F}$ such that

$$h(n) = x(n+2) + a x(n+1) + b x(n),$$

for all n . Also, in view of (4.1), we have $|h(n)| \leq \epsilon \phi(n)$. Now, taking Z -Transform to $h(n)$, thus

$$(4.2) \quad Z[h(n)] = H(z) = (z^2 + az + b) X(z) - z(z+a)x(0) - zx(1).$$

In view of (4.2), a sequence $\{x_0(n)\}$ is a solution of (1.1) if and only if

$$(4.3) \quad (z^2 + az + b) X(z) - z(z+a)x_0(0) - zx_0(1) = 0.$$

Applying (1.3) in (4.2), we obtain that

$$(4.4) \quad H(z) = (z^2 + az + b) X(z) - z.$$

Since, we have $z^2 + az + b = (z - \alpha)(z - \beta)$, then (4.4) gives that

$$(4.5) \quad X(z) - \frac{z}{(z - \alpha)(z - \beta)} = \frac{H(z)}{(z - \alpha)(z - \beta)}.$$

Now, let us choose $y(n)$ as $y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} x(1)$, then applying Z -Transform to $y(n)$,

$$Z[y(n)] = Y(z) = \frac{zx(1)}{(z - \alpha)(z - \beta)}.$$

$$(4.6) \quad (z - \alpha)(z - \beta) Y(z) - z(z+a)x(0) - zx(1) = 0.$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a y(n+1) + b y(n)] &= (z^2 + az + b) Y(z) - z(z+a)y(0) - zy(1) \\ &= 0. \quad [\text{form (4.6)}] \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a y(n+1) + b y(n) = 0$, therefore $y(n)$ is a solution of (1.1). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z-\alpha)(z-\beta)} = Z[r(n) * h(n)]$$

where, $r(n) = \frac{1}{z} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}$. Since, Z -Transforms is linear and one-to-one, so $x(n) - y(n) = (r(n) * h(n))$. Now,

$$\begin{aligned} |x(n) - y(n)| &= |r(n) * h(n)| = \left| \sum_{n=-\infty}^{\infty} r(n-k) h(n) \right| \\ &\leq \sum_{n=-\infty}^{\infty} |r(n-k)| |h(n)| \leq K \phi(n) \epsilon. \end{aligned}$$

for each n , where $K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left| \frac{1}{z(\alpha-\beta)} \right| \sum_{n=-\infty}^{\infty} |\{\alpha^{n-k} - \beta^{n-k}\}|$ occurs for each value of n . Then by the virtue of Definition 2.3, the linear difference equation (1.1) has the Hyers-Ulam-Rassias stability. $\square$

Now, we illustrate the Theorem 4.1 by the following examples.

Example 4.2. For every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences, if there exists a constant $K > 0$ such that

$$(4.7) \quad |x(n+2) + 3x(n+1) + 2x(n)| \leq \epsilon \phi(n),$$

with (1.3) then there is a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K \epsilon \phi(n)$. Let $h(n) = x(n+2) + 3x(n+1) + 2x(n)$, for all n . Also, in view of (4.7), we have $|h(n)| \leq \epsilon \phi(n)$. Now, taking Z -Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (z^2 + 3z + 2) X(z) - z(z+3)x(0) - zx(1).$$

Since, we have $z^2 + 3z + 3 = (z+1)(z+2)$, then we have

$$X(z) - \frac{z}{(z+1)(z+2)} = \frac{H(z)}{(z+1)(z+2)}.$$

Now, let us choose $y(n) = \{(-1)^n - (-2)^n\} x(1)$, then applying Z -Transform, we get

$$Z[y(n)] = Y(z) = \frac{zx(1)}{(z+1)(z+2)}.$$

Hence

$$(z+1)(z+2)Y(z) - z(z+3)x(0) - zx(1) = 0.$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then

$$Z[y(n+2) + 3 y(n+1) + 2 y(n)] = 0.$$

Since, Z is one-to-one operator, it holds that $y(n+2) + 3 y(n+1) + 2 y(n) = 0$, therefore $y(n)$ is a solution. Then by using the Theorem 4.1, the difference equation $x(n+2) + 3 x(n+1) + 2 x(n) = 0$ has the Hyers-Ulam-Rassias stability.

Example 4.3. For every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences, if there exists a constant $K > 0$ such that

$$(4.8) \quad |2 x(n+2) + 5 x(n+1) - 3 x(n)| \leq \phi(n)\epsilon,$$

for each value of $t > 0$ with (1.3), then there exists a solution $y(n)$ with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\phi(n)\epsilon$.

Let $h(n) = 2 x(n+2) + 5 x(n+1) - 3 x(n)$, for all n . Also, in view of (4.8), we have $|h(n)| \leq \phi(n)\epsilon$. Now, taking Z -Transform to $h(n)$, then we have

$$Z[h(n)] = H(z) = (2z^2 + 5z - 3) X(z) - z(2z + 5) x(0) - 2z x(1).$$

Since, we have $2z^2 + 5z - 3 = 2(z - 1/2)(z + 3)$, then we have

$$X(z) - \frac{2z}{2(z - 1/2)(z + 3)} = \frac{H(z)}{2(z - 1/2)(z + 3)}.$$

Thus,

$$X(z) = \frac{H(z) + 2z}{2(z - 1/2)(z + 3)}.$$

Now, let us choose the function $y(n)$ such that $y(n) = \left(\frac{(1/2)^n - (-3)^n}{7/2}\right) x(1)$, then applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{2z x(1)}{2(z - 1/2)(z + 3)}.$$

Hence $2(z - 1/2)(z + 3) Y(z) - z(2z + 5) x(0) - 2z x(1) = 0$. Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Then $Z[2 y(n+2) + 5 y(n+1) - 3 y(n)] = 0$. Since, Z is one-to-one operator, it holds that $2 y(n+2) + 5 y(n+1) - 3 y(n) = 0$, therefore $y(n)$ is a solution of the difference equation. Then by using the Theorem 4.1, the difference equation $2 x(n+2) + 5 x(n+1) - 3 x(n) = 0$ has the Hyers-Ulam-Rassias stability.

Theorem 4.4. For every $a, b \in \mathbb{F}$ there is a $\alpha, \beta \in \mathbb{F}$ such that $\alpha + \beta = -a$ and $\alpha\beta = b$ and $\alpha \neq \beta$. If there exist a positive constant K , for every $\epsilon > 0$, $\phi(n)$ and $x(n)$ be a positive sequences satisfies the inequality

$$(4.9) \quad |x(n+2) + a x(n+1) + b x(n) - p(n)| \leq \epsilon \phi(n),$$

where $\phi(n)$ is bounded for each value of n with (1.3) then there exists a solution $y(n)$ of (1.2) with $y(0) = 0$ and $y(1) = 1$ such that $|x(n) - y(n)| \leq K\epsilon\phi(n)$ with $p(0) = p(1) = 0$.

Proof. If we define a function $h : (0, \infty) \rightarrow \mathbb{F}$ such that

$$h(n) = x(n+2) + a x(n+1) + b x(n) - p(n).$$

Also, in view of (4.9), we have $|h(n)| \leq \epsilon \phi(n)$. Now, taking Z -Transform to $h(n)$, then we have

$$(4.10) \quad Z[h(n)] = H(z) = (z^2 + az + b) X(z) - z(z+a)x(0) - z x(1) - P(z).$$

In view of (4.10), a sequence $\{x_0(n)\}$ is a solution of (1.2) if and only if

$$(4.11) \quad (z^2 + az + b) X(z) - z(z+a)x_0(0) - z x_0(1) = P(z).$$

Applying (1.3) in (4.10), we obtain that

$$(4.12) \quad H(z) = (z^2 + az + b) X(z) - z - P(z).$$

Since, we have $z^2 + az + b = (z - \alpha)(z - \beta)$, then (4.12) gives that

$$(4.13) \quad X(z) - \frac{z}{(z - \alpha)(z - \beta)} = \frac{H(z) + P(z)}{(z - \alpha)(z - \beta)}.$$

Now, we define a function $y(n)$ such that

$$y(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} x(1) + [r(n) * p(n)],$$

where $r(n) = \frac{1}{z} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}$. Now, applying Z -Transform to $y(n)$, we get

$$Z[y(n)] = Y(z) = \frac{z x(1)}{(z - \alpha)(z - \beta)} + \frac{P(z)}{(z - \alpha)(z - \beta)}.$$

Hence, we have,

$$(4.14) \quad (z - \alpha)(z - \beta) Y(z) - z(z+a)y(0) - z y(1) = P(z)$$

Since, we have $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$. Now,

$$\begin{aligned} Z[y(n+2) + a y(n+1) + b y(n)] &= Z[y(n+2)] + a Z[y(n+1)] + b Z[y(n)] \\ &= (z^2 + az + b)Y(z) - z(z+a)y(0) - z y(1) \\ Z[y(n+2) + a y(n+1) + b y(n)] &= P(z) = Z[p(n)]. \quad [\text{form (4.14)}] \end{aligned}$$

Since, Z is one-to-one operator, it holds that $y(n+2) + a y(n+1) + b y(n) = p(n)$, therefore $y(n)$ is a solution of (1.2). Then we have

$$Z[x(n)] - Z[y(n)] = X(z) - Y(z) = \frac{H(z)}{(z - \alpha)(z - \beta)} = Z[r(n) * h(n)],$$

Since, Z -Transforms is linear and one-to-one, thus $x(n) - y(n) = (r(n) * h(n))$. Now,

$$\begin{aligned} |x(n) - y(n)| &= |r(n) * h(n)| = \left| \sum_{n=-\infty}^{\infty} r(n-k) h(n) \right| \\ &\leq \sum_{n=-\infty}^{\infty} |r(n-k)| |h(n)| \leq K\epsilon \phi(n). \end{aligned}$$

for each n , where $K = \sum_{n=-\infty}^{\infty} |r(n-k)| = \left| \frac{1}{z(\alpha-\beta)} \right| \sum_{n=-\infty}^{\infty} |\{\alpha^{n-k} - \beta^{n-k}\}|$ occurs for each value of n . Then by the virtue of Definition 2.4, the linear difference equation (1.2) has the Hyers-Ulam-Rassias stability. $\square$

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Schur convexity of the dual form of complete symmetric function involving exponent parameter

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Abstract. In this paper, we generalize the dual form of complete symmetric function by introducing an exponent parameter, we also study the Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of this class of functions. As applications, we establish three new inequalities, which are associated with the arithmetic mean, geometric mean and harmonic mean.

Keywords: Schur-convexity, Schur-geometric convexity, Schur-harmonic convexity, complete symmetric function, dual form, inequality.

1. Introduction

The notion of majorization originates from characterizing the diversity of the components of an n -dimensional vector, it has led to large number of applications in different fields of pure and applied mathematics [1]. A famous monograph [2], published by Marshall and Olkin in 1979, covers many of the fundamental properties and research results in the majorization theory. Since then, the theory of majorization has received growing attention from mathematicians and has emerged as fascinating field.

In recent years, the Schur-convexity, Schur-geometric and Schur-harmonic convexity of various symmetric functions are hot topic in the field of majorization and inequalities. Among these investigations, the Schur convexities of the

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complete symmetric function and the dual form of complete symmetric function are of special interest and have attracted many researchers (see [3-17]). For being convenient to understand the subsequent contents, let us briefly recall the background of the relevant topic.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n -tuple (n -dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \dots, n\},$$

$$\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i > 0, i = 1, 2, \dots, n\},$$

$$\mathbb{R}_-^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i < 0, i = 1, 2, \dots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}_+^1$, respectively.

In 1978, Menon [18] investigated certain inequalities related to the symmetric function, this kind of symmetric function is now known in the literature as complete symmetric function, which is defined by

$$(1) \quad c_n(\mathbf{x}, r) = \sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $r \in \{1, 2, \dots, n\}$, $i_1, i_2, \dots, i_n$ are non-negative integers, $c_0(\mathbf{x}, r) = 1$.

Guan [19] discussed the Schur-convexity of the complete symmetric function $c_n(\mathbf{x}, r)$ and proved the following assertion:

Proposition 1. *For $r = 1, 2, \dots, n$, $c_n(\mathbf{x}, r)$ is increasing and Schur-convex on $\mathbb{R}_+^n$.*

Inspired by the work of Guan in [19], Chu et al. [20] proved the following result:

Proposition 2. *For $r = 1, 2, \dots, n$, $c_n(\mathbf{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.*

In 2016, Shi et al. [21] further considered the Schur-convexity of $c_n(\mathbf{x}, r)$ on $\mathbb{R}_-^n$, they drew the conclusion asserted by Proposition 3 below.

Proposition 3. *If r is even integer (or odd integer, respectively), then $c_n(\mathbf{x}, r)$ is decreasing and Schur-convex (or increasing and Schur-concave, respectively) on $\mathbb{R}_-^n$.*

The dual form of the complete symmetric function $c_n(\mathbf{x}, r)$ is defined by $c_n^*(\mathbf{x}, r)$, i.e.,

$$(2) \quad c_n^*(\mathbf{x}, r) = \prod_{i_1+i_2+\dots+i_n=r} \sum_{j=1}^n i_j x_j,$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $r \in \{1, 2, \dots, n\}$, $i_1, i_2, \dots, i_n$ are non-negative integers, $c_0^*(\mathbf{x}, r) = 1$.

Zhang and Shi [22] proved the following results:

Proposition 4. *For $r = 1, 2, \dots, n$, the following assertions are valid.*

- (i) $c_n^*(\mathbf{x}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$.
- (ii) $c_n^*(\mathbf{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.

By introducing an exponent parameter, Guan [23] studied the following generalized form of the complete symmetric function,

$$(3) \quad c_n\left(\mathbf{x}^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\dots+i_n=r} \left(\prod_{j=1}^n x_j^{i_j} \right)^{\frac{1}{r}}.$$

Guan [23] proved a result asserted by the Theorem A below.

Theorem A. *For $r = 1, 2, \dots, n$, $c_n(\mathbf{x}^{\frac{1}{r}}, r)$ is an increasing Schur-concave function on $\mathbb{R}_+^n$.*

Chu and Sun [24] proved the assertion of Theorem B, i.e.,

Theorem B. *For $r = 1, 2, \dots, n$, $c_n(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur-harmonically concave on $\mathbb{R}_+^n$.*

Wang and Yang [25] obtained more extensive results described in Theorem C below.

Theorem C. *For $r = 1, 2, \dots, n$, $c_n(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur m -power concave on $\mathbb{R}_+^n$ when $m > 1$, and $c_n(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur m -power convex on $\mathbb{R}_+^n$ when $m \leq \frac{1}{r}$.*

Motivated by the work of [23], [24] and [25], in this paper, we generalize the dual form of complete symmetric function by introducing an exponent parameter, the generalized version of $c_n^*(\mathbf{x}, r)$, written as $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$, is defined by

$$(4) \quad c_n^*\left(\mathbf{x}^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\dots+i_n=r} \sum_{j=1}^n i_j \left(x_j^{\frac{1}{r}} \right).$$

Also, we will study the Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of the function $c_n^*(\mathbf{x}, r)$. Our main results are stated in the Theorem 1 below.

Theorem 1. *For $r = 1, 2, \dots, n$, the following assertions are valid.*

- (i) *The symmetric function $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$;*
- (ii) *The symmetric function $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur-geometrically convex on $\mathbb{R}_+^n$;*
- (iii) *The symmetric function $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur-harmonically convex on $\mathbb{R}_+^n$.*

2. Definitions and lemmas

In this section we quote some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 1 ([2, 3]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$.
- (ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2 ([2, 3]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.
- (ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω .

Definition 3 ([2, 3]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\mathbf{x}, \mathbf{y} \in \Omega, 0 \leq \alpha \leq 1$, implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.
- (ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\varphi(\mathbf{x}) + (1-\alpha)\varphi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$, and all $\alpha \in [0, 1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω .

Definition 4 ([2, 3]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$.

- (i) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $\mathbf{x} \in \Omega$ implies $\mathbf{x}P \in \Omega$ for every $n \times n$ permutation matrix P .
- (ii) A function $\varphi: \Omega \rightarrow \mathbb{R}$ is called symmetric if for every permutation matrix P , $\varphi(\mathbf{x}P) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

Lemma 1 ([2, 3]). Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω° is the interior of Ω . $\varphi: \Omega \rightarrow \mathbb{R}$ is continuous on Ω and differentiable in Ω° . Then φ is the Schur-convex (or Schur-concave, respectively) function if and only if φ is symmetric on Ω and

$$(5) \quad (x_1 - x_2) \left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\text{or } \leq 0, \text{ respectively})$$

holds for any $\mathbf{x} \in \Omega^\circ$.

Definition 5 ([4]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}_+^n$.

- (i) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \dots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$.
- (ii) Let $\Omega \subset \mathbb{R}_+^n$. The function $\varphi: \Omega \rightarrow \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\log x_1, \log x_2, \dots, \log x_n) \prec (\log y_1, \log y_2, \dots, \log y_n)$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function on Ω .

Lemma 2 ([4]). Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω° . Let $\varphi: \Omega \rightarrow \mathbb{R}_+$ be continuous on Ω and differentiable in Ω° . If φ is symmetric on Ω and

$$(6) \quad (\log x_1 - \log x_2) \left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\text{or } \leq 0, \text{ respectively})$$

holds for any $\mathbf{x} \in \Omega^\circ$, then φ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function.

Definition 6 ([24]). Let $\Omega \subset \mathbb{R}_+^n$ or $\Omega \subset \mathbb{R}_-^n$.

- (i) A set Ω is said to be harmonically convex if $\frac{\mathbf{xy}}{\lambda \mathbf{x} + (1-\lambda) \mathbf{y}} \in \Omega$, for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\frac{\mathbf{xy}}{\lambda \mathbf{x} + (1-\lambda) \mathbf{y}} = \left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \dots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n} \right)$.
- (ii) A function $\varphi: \Omega \rightarrow \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$, where $\frac{1}{\mathbf{x}} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n} \right)$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function.

Lemma 3 ([2, 3]). Let $\mathbb{A}, \mathbb{B} \subset \mathbb{R}$, $\varphi: \mathbb{B}^n \rightarrow \mathbb{R}$, $f: \mathbb{A} \rightarrow \mathbb{B}$ and $\psi(x_1, x_2, \dots, x_n) = \varphi(f(x_1), f(x_2), \dots, f(x_n)): \mathbb{A}^n \rightarrow \mathbb{R}$.

- (i) If f is convex and φ is increasing and Schur-convex, then ψ is Schur-convex;
- (ii) If f is concave, φ is increasing and Schur-concave, then ψ is Schur-concave.

Lemma 4 ([2, 3]). Let $\Omega \subset \mathbb{R}_+^n$. The function $\varphi: \Omega \rightarrow \mathbb{R}_+$ is differentiable.

- (i) If φ is increasing and Schur-geometrically convex, then φ is Schur-harmonically convex.
- (ii) If φ is decreasing and Schur-geometrically concave, then φ is Schur-harmonically concave.

Lemma 5 ([2, 3]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then

$$(7) \quad (A(\mathbf{x}), A(\mathbf{x}), \dots, A(\mathbf{x})) \prec \mathbf{x} = (x_1, x_2, \dots, x_n),$$

where $A(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i$.

3. Proof of main results

Proof of Theorem 1.

(i) Define a function $g(t) = t^{\frac{1}{r}}$, $t \in (0, +\infty)$. Then

$$(8) \quad g'(t) = \frac{1}{r} t^{\frac{1}{r}-1}, \quad g''(t) = \frac{1}{r} \left(\frac{1}{r} - 1 \right) t^{\frac{1}{r}-2}.$$

For $r = 1, 2, \dots, n$, $g(t)$ is increasing and concave on $\mathbb{R}_+$. By using Proposition 4, Lemma 3 and Definition 1, it follows that $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$.

(ii) For $r = 1$ and $r = 2$, it is easy to prove that the symmetric function $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$ is Schur-geometrically convex on $\mathbb{R}_+^n$.

Now consider the case of $r \geq 3$. By the symmetry of $c_n^*(\mathbf{x}^{\frac{1}{r}}, r)$, without loss of generality, we set $x_1 > x_2$. Note that

$$\begin{aligned} c_n^*\left(\mathbf{x}_j^{\frac{1}{r}}, r\right) &= \prod_{\substack{i_1+i_2+\dots+i_n=r \\ i_1 \neq 0, i_2=0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \times \prod_{\substack{i_1+i_2+\dots+i_n=r \\ i_1=0, i_2 \neq 0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \\ &\times \prod_{\substack{i_1+i_2+\dots+i_n=r \\ i_1 \neq 0, i_2 \neq 0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \times \prod_{\substack{i_1+i_2+\dots+i_n=r \\ i_1=0, i_2=0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial c_n^*\left(\mathbf{x}_j^{\frac{1}{r}}, r\right)}{\partial x_1} &= c_n^*\left(\mathbf{x}_j^{\frac{1}{r}}, r\right) \\ &\times \left(\sum_{\substack{i_1+i_2+\dots+i_n=r \\ i_1 \neq 0, i_2=0}} \frac{i_1 \frac{1}{r} x_1^{\frac{1}{r}-1}}{\sum_{j=1}^n i_j x_j^{\frac{1}{r}}} + \sum_{\substack{i_1+i_2+\dots+i_n=r \\ i_1=0, i_2 \neq 0}} \frac{i_1 \frac{1}{r} x_1^{\frac{1}{r}-1}}{\sum_{j=1}^n i_j x_j^{\frac{1}{r}}} \right) \\ &= c_n^*\left(\mathbf{x}_j^{\frac{1}{r}}, r\right) \left(\sum_{\substack{k+k_3+\dots+k_n=r \\ k \neq 0}} \frac{k \frac{1}{r} x_1^{\frac{1}{r}-1}}{k x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right. \\ (9) \quad &+ \left. \sum_{\substack{k+m+i_3+\dots+i_n=r \\ k \neq 0, m \neq 0}} \frac{k \frac{1}{r} x_1^{\frac{1}{r}-1}}{k x_1^{\frac{1}{r}} + m x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right). \end{aligned}$$

By the same arguments, we have

$$\begin{aligned}
 (10) \quad & \frac{\partial c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)}{\partial x_2} = c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right) \left(\sum_{\substack{k+k_3+\dots+k_n=r \\ k \neq 0}} \frac{k \frac{1}{r} x_2^{\frac{1}{r}-1}}{k x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right. \\
 & \left. + \sum_{\substack{k+m+i_3+\dots+i_n=r \\ k \neq 0, m \neq 0}} \frac{k \frac{1}{r} x_2^{\frac{1}{r}-1}}{k x_2^{\frac{1}{r}} + m x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right). \\
 & x_1 \frac{\partial c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)}{\partial x_1} - x_2 \frac{\partial c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)}{\partial x_2} = c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right) (A_1 + A_2),
 \end{aligned}$$

where

$$\begin{aligned}
 (11) \quad & A_1 = \sum_{\substack{k+k_3+\dots+k_n=r \\ k \neq 0}} \left(\frac{k \frac{1}{r} x_1^{\frac{1}{r}}}{k x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} - \frac{k \frac{1}{r} x_2^{\frac{1}{r}}}{k x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right) \\
 & = k \sum_{\substack{k+k_3+\dots+k_n=r \\ k \neq 0}} \frac{\frac{1}{r} (x_1^{\frac{1}{r}} - x_2^{\frac{1}{r}}) \sum_{j=3}^n i_j x_j^{\frac{1}{r}}}{(k x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}) (k x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}})}
 \end{aligned}$$

and

$$\begin{aligned}
 (12) \quad & A_2 = \sum_{\substack{k+m+i_3+\dots+i_n=r \\ k \neq 0, m \neq 0}} \left(\frac{k \frac{1}{r} x_1^{\frac{1}{r}}}{k x_1^{\frac{1}{r}} + m x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} - \frac{k \frac{1}{r} x_2^{\frac{1}{r}}}{k x_2^{\frac{1}{r}} + m x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \right) \\
 & = k \sum_{\substack{k+m+i_3+\dots+i_n=r \\ k \neq 0, m \neq 0}} \frac{\delta}{(k x_1^{\frac{1}{r}} + m x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}) (k x_2^{\frac{1}{r}} + m x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}})}
 \end{aligned}$$

where

$$\delta = \frac{m}{r} (x_1^{\frac{2}{r}} - x_2^{\frac{2}{r}}) + \frac{1}{r} (x_1^{\frac{1}{r}} - x_2^{\frac{1}{r}}) \sum_{j=3}^n i_j x_j^{\frac{1}{r}}.$$

For $r = 1, 2, \dots, n$ and $\mathbf{x} \in \mathbb{R}_+^n$, it is easy to see that $A_1 \geq 0$ and $A_2 \geq 0$, and then we have

$$x_1 \frac{\partial c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)}{\partial x_1} - x_2 \frac{\partial c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)}{\partial x_2} \geq 0,$$

by Lemma 2, it follows that $c_n^* \left(\mathbf{x}_j^{\frac{1}{r}}, r \right)$ is Schur-geometrically convex on $\mathbb{R}_+^n$.

(iii) Since $c_n^*(\mathbf{x}_j^{\frac{1}{r}}, r)$ is increasing and Schur-geometrically convex on $\mathbb{R}_+^n$, by Lemma 4, it follows that $c_n^*(\mathbf{x}_j^{\frac{1}{r}}, r)$ is Schur-harmonically convex on $\mathbb{R}_+^n$.

The proof of Theorem 1 is completed.

4. Applications

As an application of Theorem 1, we establish three new inequalities, which are associated with the arithmetic mean, geometric mean and harmonic mean.

Theorem 2. For $\mathbf{x} \in \mathbb{R}_+^n$, $r = 1, 2, \dots, n$, we have the following inequalities

$$(13) \quad \prod_{i_1+i_2+\dots+i_n=r} \sum_{j=1}^n i_j \left(x_j^{\frac{1}{r}} \right) \leq \left(r(A_n(\mathbf{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}},$$

$$(14) \quad \prod_{i_1+i_2+\dots+i_n=r} \sum_{j=1}^n i_j \left(x_j^{\frac{1}{r}} \right) \geq \left(r(G_n(\mathbf{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}},$$

$$(15) \quad \prod_{i_1+i_2+\dots+i_n=r} \sum_{j=1}^n i_j \left(x_j^{\frac{1}{r}} \right) \geq \left(r(H_n(\mathbf{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}},$$

where $A(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i$, $G_n(\mathbf{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$, $H_n(\mathbf{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n+r-1-r)!}$.

Proof. By using the assertion (i) of Theorem 1 and the majorization relation (7), we get $c_n^*(\mathbf{x}^{\frac{1}{r}}, r) \leq c_n^*(\mathbf{y}^{\frac{1}{r}}, r)$, $\mathbf{y} = (A(\mathbf{x}), A(\mathbf{x}), \dots, A(\mathbf{x}))$, which is equivalent to the inequality (13).

A direct application of the majorization relation (7), it is easy to observe that

$$(\log G_n(\mathbf{x}), \log G_n(\mathbf{x}), \dots, \log G_n(\mathbf{x})) \prec (\log x_1, \log x_2, \dots, \log x_n).$$

Now, from the above majorization relation and the assertion (ii) of Theorem 1, we deduce the inequality (14).

From the majorization relation (7), it follows that

$$\left(\frac{1}{H_n(\mathbf{x})}, \frac{1}{H_n(\mathbf{x})}, \dots, \frac{1}{H_n(\mathbf{x})} \right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n} \right).$$

Utilizing the above majorization relation and the assertion (iii) of Theorem 1 leads to the desired inequality (15).

This completes the proof of Theorem 2. $\square$

Remark 1. Taking $r = 1$ in Theorem 2, from the inequalities (13) and (15), we obtain the inequalities

$$A(\mathbf{x}) \geq G_n(\mathbf{x}) \geq H_n(\mathbf{x}),$$

where $A(\mathbf{x})$, $G_n(\mathbf{x})$, $H_n(\mathbf{x})$ are the classical means known as arithmetic mean, geometric mean and harmonic mean respectively.

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Testing the equality of two covariance matrices for non-normal high-dimensional data

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Abstract. This paper focuses on testing the equality of two high-dimensional covariance matrices without the normality assumption. Two new test statistics are proposed by taking the ratio between the unbiased and consistent estimators of the trace of the covariance matrices. Under some mild assumptions, the proposed test statistics are proved to be asymptotically normal. Furthermore, numerical simulations demonstrate that the proposed tests have good size and power with varying dimensions and sample sizes.

Keywords: high-dimensional covariance matrices, hypothesis test, equality, non-normality.

1. Introduction

Let $X_{ij} = (x_{ij1}, \dots, x_{ijp})'$, $j = 1, \dots, N_i$, be independent and identically distributed vectors coming from a p -dimensional population $\mathfrak{F}_i$ with the mean vector μ_i and covariance matrix Σ_i , where N_i denotes the sample size of the i -th population, $i = 1, 2$. We devote to checking on the equality of two covariance matrices, that is, testing the following hypotheses:

$$(1) \quad H_0 : \Sigma_1 = \Sigma_2 = \Sigma \quad \text{vs.} \quad H_1 : \Sigma_1 > \Sigma_2 \text{ or } \Sigma_2 > \Sigma_1,$$

where $A > B$ means that $A - B$ is positive definite. Here we just pay attention to the case $p > N_i$, i.e., so-called high-dimensional situation, because data with the dimension larger than the sample sizes are increasing encountered in recent statistical studies, such as gene data analyses, see [1, 4] for a broader introduction.

As noted in [7], if we know that $\mathfrak{F}_1$ has a larger covariance matrix than $\mathfrak{F}_2$, one may advise us to take larger sample from $\mathfrak{F}_1$ to offset the largeness to some

degree; therefore, it is interesting to consider the testing problem (1), in which H_1 is so-called one-side alternative. By using a lower bound on a measure of distance between the null and alternative hypotheses, Srivastava and Yanagihara [7] constructed their test procedure based on the unbiased estimators of $\text{tr}\Sigma_i$ and $\text{tr}\Sigma_i^2$.

It is worth pointing out that H_1 is a special case of $H'_1 : \Sigma_1 \neq \Sigma_2$. Thus, the test procedures, designed to test the hypotheses $H_0 : \Sigma_1 = \Sigma_2$ vs. $H'_1 : \Sigma_1 \neq \Sigma_2$, still work for testing the hypotheses (1). We also mention that there exist many strategies to test the hypotheses $H_0 : \Sigma_1 = \Sigma_2$ vs. $H'_1 : \Sigma_1 \neq \Sigma_2$. For instance, based on the Frobenius norm of the difference of two covariances, Schott [5] proposed a test statistic which is an unbiased estimator of $\text{tr}(\Sigma_1 - \Sigma_2)^2$.

Note that the test procedures, both in [5] and [7] mentioned above, heavily relied on the normality assumption. They used an estimator of $\text{tr}\Sigma_i^2$ proposed by [2, 6], which is given by

$$(2) \quad \widehat{\text{tr}\Sigma_{i(1)}^2} = \frac{(N_i - 1)^2}{(N_i - 2)(N_i + 1)} \left(\text{tr}S_i^2 - \frac{1}{N_i - 1}(\text{tr}S_i)^2 \right),$$

where $S_i = \frac{1}{N_i - 1} \sum_{j=1}^{N_i} Y_{ij} Y'_{ij}$ with $Y_{ij} = X_{ij} - \bar{X}_i$ and $\bar{X}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij}$, $i = 1, 2$. The estimator $\widehat{\text{tr}\Sigma_{i(1)}^2}$ given in (2) is unbiased under the normality assumption, but is biased under the non-normality assumption. Therefore, Srivastava et al. [8] defined an unbiased estimator of $\text{tr}\Sigma_i^2$ dispensing with the normality condition, which is given by

$$(3) \quad \widehat{\text{tr}\Sigma_{i(2)}^2} = \frac{1}{f} \left((N_i - 2)(N_i - 1)^3 \text{tr}S_i^2 - N_i(N_i - 1) \text{tr}D_i^2 + (N_i - 1)^2 (\text{tr}S_i)^2 \right),$$

where $f = N_i(N_i - 1)(N_i - 2)(N_i - 3)$ and $D_i = \text{diag}(Y'_{i1}Y_{i1}, \dots, Y'_{iN_i}Y_{iN_i})$, $i = 1, 2$. Moreover, Srivastava et al. [8] proposed a test procedure by employing $\widehat{\text{tr}\Sigma_{i(2)}^2}$ to modify the test statistic proposed in [5].

Inspired by [8], we aim to propose some new and more powerful test procedures for testing the hypotheses (1) without the normal condition. The test statistics are constructed by taking the ratio between the unbiased and consistent estimators of the trace of the covariance matrices. The rest paper is organized as follows. Section 2 reviews the estimators of $\text{tr}\Sigma_i^l/p$, $l = 1, 2, 3, 4$, and the properties of these estimators for general populations under some mild assumptions. In Section 3, two new test statistics are proposed, whose asymptotic distributions are derived to be normal. Simulation results are reported in Section 4. A conclusion is made in Section 5.

2. Notations and preliminaries

In this section, we collect some important results needed in the later sections. We begin with the description of the general multivariate model, including

the multivariate normal distribution as a special case. Assume that the data $\{X_{ij}\}_{j=1}^{N_i}$ satisfy

$$(4) \quad X_{ij} = \Lambda_i Z_{ij} + \mu_i, \quad i = 1, 2, j = 1, \dots, N_i.$$

Here, Λ_i is a $p \times p$ matrix such that $\Lambda_i \Lambda_i' = \Sigma_i = (\sigma_{ist})_{p \times p}$, and $Z_{ij} = (z_{ij1}, \dots, z_{ijp})'$ has independent and identically distributed elements, yielding that $\mathbb{E}Z_{ij} = \mathbf{0}$, $\text{Cov}(Z_{ij}) = I_p$, where I_p denotes a $p \times p$ identity matrix. Note that the model (4) is widely used in the high dimensional multivariate analysis, such as [2, 8, 3].

Throughout the sequel we abbreviate $a_l = \frac{1}{p} \text{tr} \Sigma^l$, $a_{li} = \frac{1}{p} \text{tr} \Sigma_i^l$, $l = 1, 2, 3, 4$, $i = 1, 2$. Then the estimators of a_{li} can be stated as follows:

$$(5) \quad \hat{a}_{1i} = \frac{1}{p} \text{tr} S_i,$$

$$(6) \quad \hat{a}_{2i} = \frac{1}{p} \widehat{\text{tr} \Sigma_i^2(2)},$$

$$(7) \quad \hat{a}_{3i} = \frac{\tau_1}{p} \left(\text{tr} S_i^3 - \frac{3}{n_i} \text{tr} S_i^2 \text{tr} S_i + \frac{2}{n_i^2} (\text{tr} S_i)^3 \right),$$

$$(8) \quad \hat{a}_{4i} = \frac{\tau_2}{p} \left(\text{tr} S_i^4 - \frac{4}{n_i} \text{tr} S_i^3 \text{tr} S_i - \frac{2n_i^2 + 3n_i - 6}{n_i^3 + n_i^2 + 2n_i} (\text{tr} S_i^2)^2 \right. \\ \left. + \frac{10n_i + 12}{n_i^3 + n_i^2 + 2n_i} \text{tr} S_i^2 (\text{tr} S_i)^2 - \frac{5n_i + 6}{n_i^4 + n_i^3 + 2n_i^2} (\text{tr} S_i)^4 \right),$$

where $n_i = N_i - 1$, and

$$\tau_1 = \frac{n_i^4}{(n_i - 1)(n_i - 2)(n_i + 2)(n_i + 4)},$$

$$\tau_2 = \frac{n_i^7 + n_i^6 + 2n_i^5}{(n_i + 1)(n_i + 2)(n_i + 4)(n_i + 6)(n_i - 1)(n_i - 2)(n_i - 3)}.$$

One can see [8, 9] and references therein for details. To consider the unbiasedness, consistence and asymptotic normality of these estimators, the following assumptions are needed:

Assumption 1. Both N_i and p tend to infinity with $N_i = O(p^\varepsilon)$, $1/2 < \varepsilon < 1$.

Assumption 2. For $N_1 \leq N_2$, $0 < N_1/N_2 \leq 1$.

Assumption 3. $0 < a_{2i} < \infty$, $a_{4i}/p = o(1)$, and $\sum_{s,t=1}^p \sigma_{ist}^4/p^2 = o(1)$.

Assumption 4. $\mathbb{E}(z_{ijk}^4) = \Delta_i + 3$ with $\Delta_i < \infty$, where z_{ijk} is the k -th component of the vector $Z_{ij} = (z_{ij1}, \dots, z_{ijp})'$, and $\mathbb{E} \left(\prod_{k=1}^p z_{ijk}^{\gamma_k} \right) = \prod_{k=1}^p \mathbb{E}(z_{ijk}^{\gamma_k})$ for integers $\gamma_k \geq 0$ with $\sum_{k=1}^p \gamma_k \leq 8$.

Assumption 5. $\text{tr}(\Sigma_i^s \odot \Sigma_i^t) = o(\text{tr} \Sigma_i^{s+t})$ for integers s, t with $1 \leq s, t \leq 2$, where $\odot$ denotes the Hadamard product of two matrices.

Assumptions 1 ~ 4 are used frequently, see for instance [8]. Indeed, Assumption 4 indicates that z_{ijk} has the finite fourth moment and the existence of the moments of z_{ijk} are up to the order eight. Assumption 5 is also practical and holds for many common covariances. For example, let Σ_i be compound symmetric structure, i.e., $\Sigma_i = (1 - \rho)I_p + \rho J_p$, where $0 < \rho < 1$ and J_p is the $p \times p$ matrix with all elements being one. It is easy to show that $\text{tr}(\Sigma_i^s \odot \Sigma_i^t) = o(p)$ and $\text{tr}\Sigma_i^{s+t} = O(p^{s+t})$ for integers $1 \leq s, t \leq 2$, which indicates that Assumption 5 is reasonable.

Furthermore, by employing the results of [8, 9], the properties of $\hat{a}_{li}$, $l = 1, 2, 3, 4$, can be summarized immediately by the following lemmas.

Lemma 2.1. *For $i = 1, 2$, under Assumptions 1 ~ 5,*

(i) *when the underlying distribution is normal, $\hat{a}_{li}$ given above is unbiased and consistent estimator of a_{li} , $l = 1, 2, 3, 4$;*

(ii) *when the underlying distribution is not normal, $\hat{a}_{1i}$ and $\hat{a}_{2i}$ are still unbiased and consistent estimators of a_{1i} and a_{2i} , respectively, while $\hat{a}_{3i}$ and $\hat{a}_{4i}$ are consistent but biased estimators of a_{3i} and a_{4i} , respectively.*

Lemma 2.2. *For $i = 1, 2$, under Assumptions 1 ~ 5, one has*

$$U^{-\frac{1}{2}} (\hat{a}_{1i} - a_{1i}, \hat{a}_{2i} - a_{2i})' \xrightarrow{D} N_2(\mathbf{0}, I_2),$$

where

$$U = \begin{pmatrix} 2a_{2i}/(N_i p) & 4a_{3i}/(N_i p) \\ 4a_{3i}/(N_i p) & (4pa_{2i}^2 + 8N_i a_{4i} - 8a_{4i})/(N_i^2 p) \end{pmatrix}.$$

3. Two test procedures

Based on the consistent estimators of two distance functions β_1 and β_2 given in (9) and (14) respectively, we accordingly propose two test statistics for the testing problem (1) in this section.

3.1 The first test procedure

We firstly notice that a measure of distance between H_0 and H_1 can be given by

$$(9) \quad \beta_1 = \frac{\text{tr}\Sigma_1^2}{\text{tr}\Sigma_2^2} - 1,$$

where β_1 equals zero if H_0 is true, meanwhile, β_1 does not equal zero if H_1 holds. Then, the corresponding test statistic, based on the consistent estimator of β_1 , can be constructed by

$$(10) \quad \hat{\beta}_1 = \frac{\hat{a}_{21}}{\hat{a}_{22}} - 1,$$

where $\hat{a}_{2i}$ is given in (6), $i = 1, 2$. A straightforward calculation, embedded in the proof of Theorem 3.1, shows that the asymptotic variance of $\hat{\beta}_1$ is

$$(11) \quad \delta_1 = \frac{4pa_{21}^2 + 8(N_1 - 1)a_{41}}{N_1^2 pa_{22}^2} + \frac{4pa_{21}^2 a_{22}^2 + 8(N_2 - 1)a_{21}^2 a_{42}}{N_2^2 pa_{22}^4}.$$

Obviously, under $H_0 : \Sigma_1 = \Sigma_2 = \Sigma$, we have $a_{21} = a_{22} = a_2$ and $a_{41} = a_{42} = a_4$. Thus δ_1 reduces to

$$(12) \quad \delta_{10} = \frac{4}{N_1^2} + \frac{4}{N_2^2} + \left(\frac{8N_1 - 8}{N_1^2 p} + \frac{8N_2 - 8}{N_2^2 p} \right) \frac{a_4}{a_2^2}.$$

The following theorem establishes the asymptotic normality of $\hat{\beta}_1$.

Theorem 3.1. *Under Assumptions 1 ~ 5, we have*

$$(\delta_1)^{-\frac{1}{2}}(\hat{\beta}_1 - \beta_1) \xrightarrow{D} N(0, 1).$$

In particular, under H_0 , we have

$$(\delta_{10})^{-\frac{1}{2}}\hat{\beta}_1 \xrightarrow{D} N(0, 1),$$

where δ_1 and δ_{10} are given in (11) and (12), respectively.

Proof. Since two samples are drawn from two independent populations, by employing Lemma 2.2, we have the distribution of $(\hat{a}_{21}, \hat{a}_{22})'$ as follows:

$$V^{-\frac{1}{2}}(\hat{a}_{21} - a_{21}, \hat{a}_{22} - a_{22})' \xrightarrow{D} N_2(\mathbf{0}, I_2),$$

where $V = \text{diag}(\xi_1, \xi_2)$ with $\xi_i = (4pa_{2i}^2 + 8N_i a_{4i} - 8a_{4i})/(N_i^2 p)$, $i = 1, 2$. Note that $\hat{\beta}_1 = \hat{a}_{21}/\hat{a}_{22} - 1$ is a function of $\hat{a}_{21}$ and $\hat{a}_{22}$. Hence the partial derivatives of $\hat{\beta}_1$ with respect to $\hat{a}_{21}$ and $\hat{a}_{22}$ are

$$\frac{\partial \hat{\beta}_1}{\partial \hat{a}_{21}} = \frac{1}{\hat{a}_{22}}, \quad \frac{\partial \hat{\beta}_1}{\partial \hat{a}_{22}} = -\frac{\hat{a}_{21}}{\hat{a}_{22}^2}.$$

By employing the delta method, the asymptotic distribution of $\hat{\beta}_1$ can be derived to be normal with mean $\beta_1 = a_{21}/a_{22} - 1$ and variance

$$\begin{aligned} \delta_1 &= \left(\frac{1}{a_{22}}, -\frac{a_{21}}{a_{22}^2} \right) V \left(\frac{1}{a_{22}}, -\frac{a_{21}}{a_{22}^2} \right)' \\ &= \frac{1}{a_{22}^2} \xi_1 + \frac{a_{21}^2}{a_{22}^4} \xi_2 \\ &= \frac{4pa_{21}^2 + 8(N_1 - 1)a_{41}}{N_1^2 pa_{22}^2} + \frac{4pa_{21}^2 a_{22}^2 + 8(N_2 - 1)a_{21}^2 a_{42}}{N_2^2 pa_{22}^4}, \end{aligned}$$

which completes the proof of Theorem 3.1. □

To use $(\delta_{10})^{-\frac{1}{2}}\hat{\beta}_1$ in practice, we should estimate δ_{10} , which involves unknown terms a_2 and a_4 . In fact, by Lemma 2.1, we can obtain that a consistent estimator of δ_{10} is

$$\hat{\delta}_{10} = \frac{4}{N_1^2} + \frac{4}{N_2^2} + \left(\frac{8N_1 - 8}{N_1^2 p} + \frac{8N_2 - 8}{N_2^2 p} \right) \frac{\hat{a}_4}{\hat{a}_2^2},$$

where $\hat{a}_4 = (N_1\hat{a}_{41} + N_2\hat{a}_{42})/(N_1 + N_2)$ and $\hat{a}_2 = (N_1\hat{a}_{21} + N_2\hat{a}_{22})/(N_1 + N_2)$, with $\hat{a}_{4i}$ and $\hat{a}_{2i}$ given in (8) and (6), respectively, $i = 1, 2$. By employing Slutsky's theorem, under $H_0 : \Sigma_1 = \Sigma_2$ and Assumptions 1 ~ 5, we have

$$(13) \quad T_1 := (\hat{\delta}_{10})^{-\frac{1}{2}}\hat{\beta}_1 \xrightarrow{D} N(0, 1).$$

Hence, we can test the hypotheses (1) at significance level α by

$$\text{rejecting } H_0 \Leftrightarrow |T_1| > Z_{\frac{\alpha}{2}},$$

where $Z_{\frac{\alpha}{2}}$ is upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

3.2 The second test procedure

We have proposed a test statistic based merely on $\text{tr}\Sigma_1^2$ and $\text{tr}\Sigma_2^2$ in the subsection above. However, we note that $\Sigma_1 = \Sigma_2$ meaning that $\text{tr}\Sigma_1^2 = \text{tr}\Sigma_2^2$ and $\text{tr}\Sigma_1 = \text{tr}\Sigma_2$. Realizing the difference between $\text{tr}\Sigma_1$ and $\text{tr}\Sigma_2$ may also have influence on the difference between Σ_1 and Σ_2 . Therefore, $\text{tr}\Sigma_i^2$ in conjunction with $\text{tr}\Sigma_i$ may provide us another idea to construct the measure of distance between the null and alternative hypotheses. In fact, we can define another distance function

$$(14) \quad \beta_2 = \frac{\text{tr}\Sigma_1^2/(\text{tr}\Sigma_1)^2}{\text{tr}\Sigma_2^2/(\text{tr}\Sigma_2)^2} - 1.$$

Similar to β_1 , β_2 also takes the value zero if H_0 is true, nonzero if H_1 is true, since $\text{tr}\Sigma_i^2/(\text{tr}\Sigma_i)^2$ is a monotone increasing function of the ordered eigenvalues of Σ_i . Thus, we can construct the second test statistic for testing the hypotheses (1) by

$$(15) \quad \hat{\beta}_2 = \frac{\hat{a}_{21}/(\hat{a}_{11})^2}{\hat{a}_{22}/(\hat{a}_{12})^2} - 1,$$

where $\hat{a}_{2i}$ and $\hat{a}_{1i}$ are given by (6) and (5), respectively. Mention that $\hat{\beta}_2$ is a consistent estimator of β_2 . A simple calculation in the proof of Theorem 3.2 indicates that the asymptotic variance of $\hat{\beta}_2$ is

$$(16) \quad \delta_2 = \frac{a_{12}^4}{a_{22}^2}\eta_1 + \frac{a_{21}^2 a_{12}^8}{a_{11}^4 a_{22}^4}\eta_2,$$

where

$$\eta_i = \frac{8a_{2i}^3}{N_i p a_{1i}^6} - \frac{16a_{2i}a_{3i}}{N_i p a_{1i}^5} + \frac{4pa_{2i}^2 + 8(N_i - 1)a_{4i}}{N_i^2 p a_{1i}^4}, \quad i = 1, 2.$$

Obviously, under $H_0 : \Sigma_1 = \Sigma_2 = \Sigma$, we have $a_{l1} = a_{l2} = a_l$, $l = 1, 2, 3, 4$. Thus δ_2 reduces to

$$(17) \quad \delta_{20} = \frac{a_1^4(\eta_{10} + \eta_{20})}{a_2^2},$$

with $\eta_{i0} = \frac{8a_2^3}{N_i p a_1^6} - \frac{16a_2a_3}{N_i p a_1^5} + \frac{4pa_2^2 + 8(N_i - 1)a_4}{N_i^2 p a_1^4}$. Furthermore, the asymptotic normality of $\hat{\beta}_2$ can be obtained by the following theorem.

Theorem 3.2. *Under Assumptions 1 ~ 5, we have*

$$(\delta_2)^{-\frac{1}{2}}(\hat{\beta}_2 - \beta_2) \xrightarrow{D} N(0, 1).$$

In particular, under H_0 , we have

$$(\delta_{20})^{-\frac{1}{2}}\hat{\beta}_2 \xrightarrow{D} N(0, 1),$$

where δ_2 and δ_{20} are given in (16) and (17), respectively.

Proof. To simplify the presentation, we set $\omega_i = \hat{a}_{2i}/\hat{a}_{1i}^2$, $i = 1, 2$. First, we derive the distribution of (ω_1, ω_2) . Note that ω_i is a function of $\hat{a}_{1i}$ and $\hat{a}_{2i}$, hence the partial derivatives of ω_i with respect to $\hat{a}_{1i}$ and $\hat{a}_{2i}$ are

$$\frac{\partial \omega_i}{\partial \hat{a}_{1i}} = -\frac{2\hat{a}_{2i}}{\hat{a}_{1i}^3}, \quad \frac{\partial \omega_i}{\partial \hat{a}_{2i}} = \frac{1}{\hat{a}_{1i}^2}.$$

By employing the delta method and Lemma 2.2, the asymptotic distribution of ω_i can be derived to be normal with mean a_{2i}/a_{1i}^2 and variance

$$\begin{aligned} \eta_i &= \left(-\frac{2a_{2i}}{a_{1i}^3}, \frac{1}{a_{1i}^2} \right) U \left(-\frac{2a_{2i}}{a_{1i}^3}, \frac{1}{a_{1i}^2} \right)' \\ &= \frac{8a_{2i}^3}{N_i p a_{1i}^6} - \frac{16a_{2i}a_{3i}}{N_i p a_{1i}^5} + \frac{4pa_{2i}^2 + 8(N_i - 1)a_{4i}}{N_i^2 p a_{1i}^4}. \end{aligned}$$

Thus, the distribution of (ω_1, ω_2) is obtained as follows:

$$W^{-\frac{1}{2}}(\omega_1 - a_{21}/a_{11}^2, \omega_2 - a_{22}/a_{12}^2)' \xrightarrow{D} N_2(\mathbf{0}, I_2),$$

where $W = \text{diag}(\eta_1, \eta_2)$.

Next, we can proceed along the lines of the proof of Theorem 3.1 and then one gives us the distribution of $\hat{\beta}_2 = \omega_1/\omega_2 - 1$. This finishes the proof of Theorem 3.2. $\square$

To use $(\delta_{20})^{-\frac{1}{2}}\hat{\beta}_2$ in practice, we consider a consistent estimator of δ_{20} given by

$$\hat{\delta}_{20} = \frac{\hat{a}_1^4 (\hat{\eta}_{10} + \hat{\eta}_{20})}{\hat{a}_2^2},$$

where

$$\hat{\eta}_{i0} = \frac{8\hat{a}_2^3}{N_i p \hat{a}_1^6} - \frac{16\hat{a}_2 \hat{a}_3}{N_i p \hat{a}_1^5} + \frac{4p\hat{a}_2^2 + 8(N_i - 1)\hat{a}_4}{N_i^2 p \hat{a}_1^4}, \quad i = 1, 2,$$

with $\hat{a}_l = (N_1 \hat{a}_{l1} + N_2 \hat{a}_{l2}) / (N_1 + N_2)$, $l = 1, 2, 3, 4$. We have

$$(18) \quad T_2 := (\hat{\delta}_{20})^{-\frac{1}{2}} \hat{\beta}_2 \xrightarrow{D} N(0, 1).$$

Hence, we can test the hypotheses (1) at significance level α by

$$\text{rejecting } H_0 \Leftrightarrow |T_2| > Z_{\frac{\alpha}{2}},$$

where $Z_{\frac{\alpha}{2}}$ is upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

4. Numerical simulations

In this section, the performance of the proposed tests T_1 and T_2 given in (13) and (18) respectively, are evaluated by Monte Carlo simulation. For the purpose of comparison, we also consider two existing methods: one was discussed in [5], referred as T_{SC} , and the other was proposed by [8], referred as T_{SR} . Throughout the simulation, we set $\mu_1 = \mu_2 = 0$ and $N_1 = N_2$ without loss of generality.

We generate the samples from two scenarios of distributions:

$$(I) \quad z_{ija} \sim N(0, 1), \quad (II) \quad z_{ija} \sim \Gamma(4, 0.5) - 2.$$

In the scenario (I), z_{ija} is a standard normal distributed random variable, and in the scenario (II), z_{ija} is actually a centralized Gamma distributed random variable.

To investigate the empirical sizes of these tests, we set the null hypothesis as follows:

$$(19) \quad H_0 : \Sigma_1 = \Sigma_2 = \Sigma = (\sigma_{st})_{p \times p},$$

where $\sigma_{st} = I(s = t) + \exp(-|s - t|/2)$, $s, t = 1, \dots, p$, and $I(\cdot)$ denotes the indicator function.

To discuss the empirical powers of T_1 , T_2 , T_{SC} and T_{SR} , we design the following two alternative hypotheses:

$$H_1^a : \Sigma_1 = \Sigma, \quad \Sigma_2 = \Sigma + \text{banded}(5, 0.4)$$

and

$$H_1^b : \Sigma_1 = \Sigma, \quad \Sigma_2 = \Sigma + 0.8I_p,$$

where Σ is given in (19) and $\text{banded}(5, 0.4)$ denotes a banded matrix with the bandwidth 5, i.e., we can generate $\tilde{X}_{ij} = (\tilde{x}_{ij1}, \dots, \tilde{x}_{ijp})'$ from a moving average model of order 5: $\tilde{x}_{ija} = \tilde{z}_{ija} + \sum_{l=1}^5 0.4\tilde{z}_{ija-l}$, with $\tilde{z}_{ija}$ generated from the scenario (I) or (II). The nominal significance level is set at $\alpha = 0.05$ and the number of replications is 1000.

Begin by sampling $N_1 = N_2 = 120$ observations generated from the scenario (II) with $p = 240$. We first show the QQ plots of the proposed tests T_1 and T_2 , under H_0 , H_1^a and H_1^b in Figures 1, 2 and 3, respectively. These figures indicate that the normality result appears to be satisfied, which validates the theoretical results of Theorem 3.1 and Theorem 3.2. We also obtain some QQ plots when the samples come from the scenario (I). However, there are not substantially different from Figures 1 ~ 3 and thus we do not list those here.

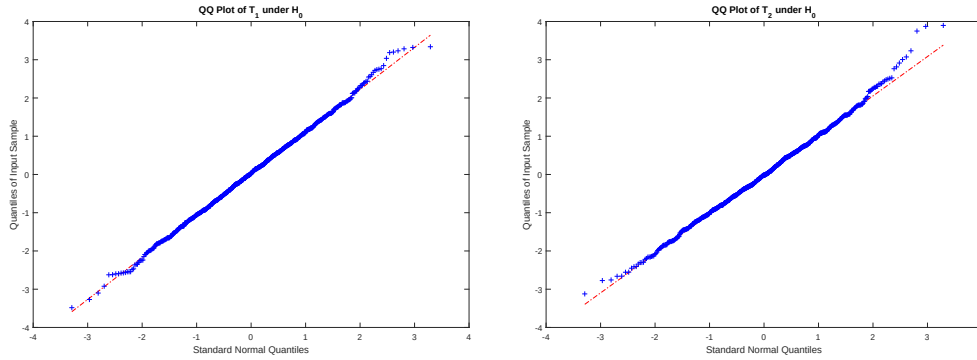


Figure 1: QQ Plots of T_1 (the left panel) and T_2 (the right panel) under H_0 , with the sample generated from the scenario (II) of Gamma distribution and $p = 240$, $N_1 = N_2 = 120$.

Table 1 reports the sizes of the tests T_1 , T_2 , T_{SC} and T_{SR} . The results show that under the scenario (I) of normal distribution, the sizes of these tests are similar and reasonable, which are close to the nominal level 0.05. However, under the scenario (II) of Gamma distribution, T_{SC} encounters serious size distortion, while the proposed two tests T_1 and T_2 as well as T_{SR} still have reasonable sizes. This is understandable because T_{SC} is designed based on the normality assumption, while others are not.

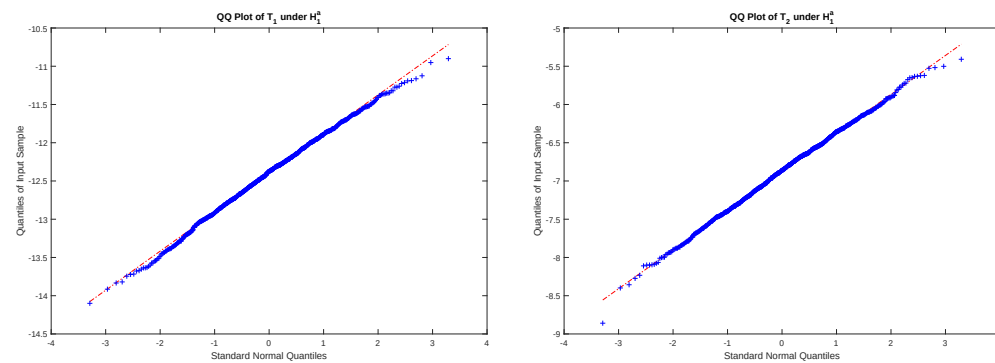


Figure 2: QQ Plots of T_1 (the left panel) and T_2 (the right panel) under H_1^a , with the sample generated from the scenario (II) of Gamma distribution and $p = 240$, $N_1 = N_2 = 120$.

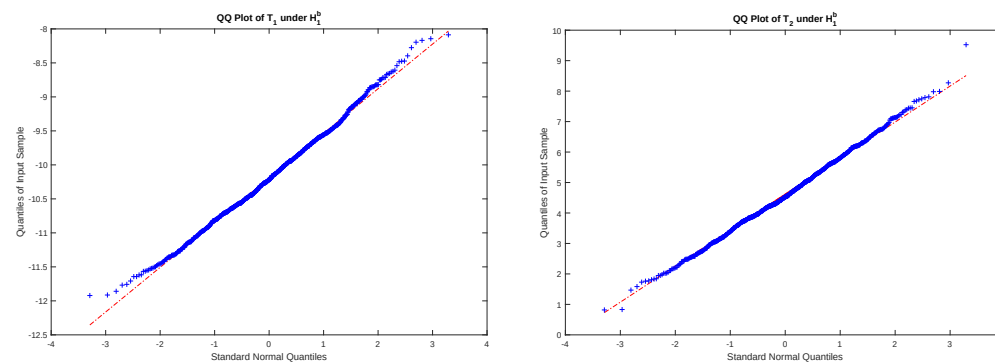


Figure 3: QQ Plots of T_1 (the left panel) and T_2 (the right panel) under H_1^b , with the sample generated from the scenario (II) of Gamma distribution and $p = 240$, $N_1 = N_2 = 120$.

Tables 2 and 3 provide the powers of these tests under the alternative hypotheses H_1^a and H_1^b , respectively. We observe from Table 2 that under H_1^a , many entries of the powers of these tests approach to 1 in both scenarios (I) and (II), which is especially obvious for T_1 . Despite the fact that the power of T_2 is lower than other tests when N_i is small, it increases rapidly and approaches to 1 as N_i and p increase. From Table 3, we find that under the alternative hypothesis H_1^b , the proposed tests T_1 and T_2 are more powerful than T_{SC} and T_{SR} . For instance, under the scenario (I) of normal distribution, when $p = 240$ and $N_1 = N_2 = 80$, the powers of T_1 and T_2 are 1 and 0.848, respectively, while the powers of T_{SC} and T_{SR} are 0.589 and 0.586, respectively.

Table 3: Empirical powers of the tests T_1 , T_2 , T_{SC} and T_{SR} under H_1^b

p	$N_1 = N_2$	Scenario (I)				Scenario (II)			
		T_{SC}	T_{SR}	T_1	T_2	T_{SC}	T_{SR}	T_1	T_2
60	20	0.088	0.093	0.572	0.226	0.210	0.096	0.490	0.219
	40	0.232	0.223	0.960	0.366	0.424	0.230	0.931	0.326
	80	0.523	0.531	1.000	0.688	0.719	0.521	1.000	0.680
	120	0.839	0.839	1.000	0.886	0.923	0.836	1.000	0.878
120	20	0.099	0.103	0.708	0.211	0.213	0.100	0.675	0.236
	40	0.230	0.225	0.998	0.393	0.390	0.222	0.998	0.416
	80	0.558	0.567	1.000	0.785	0.744	0.538	1.000	0.776
	120	0.832	0.834	1.000	0.967	0.938	0.833	1.000	0.965
180	20	0.092	0.105	0.795	0.217	0.207	0.094	0.752	0.214
	40	0.224	0.229	1.000	0.405	0.380	0.214	0.999	0.389
	80	0.554	0.556	1.000	0.856	0.769	0.561	1.000	0.801
	120	0.842	0.845	1.000	0.984	0.935	0.848	1.000	0.973
240	20	0.093	0.093	0.839	0.215	0.189	0.101	0.802	0.199
	40	0.200	0.208	1.000	0.428	0.389	0.215	1.000	0.423
	80	0.589	0.586	1.000	0.848	0.773	0.575	1.000	0.852
	120	0.839	0.838	1.000	0.976	0.939	0.817	1.000	0.991

5. Conclusion

In this concluding section, we provide a short summary of our main results on the hypothesis test about the homogeneity of covariance matrices for non-normal high-dimensional data. Firstly, two new test statistics are proposed using the ratio between the unbiased and consistent estimators of the trace of covariance matrices. Secondly, the asymptotic normality of the proposed test statistics are proved. Finally, numerical simulations demonstrate that the proposed tests, which are flexible to the underlying distribution, perform well and better in several cases than some existing tests and are thus recommended.

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Some common fixed point theorems for four maps in fuzzy metric-like spaces using α - ϕ and β - φ -fuzzy contractions

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Abstract. In this paper, some fixed point theorems for α - ϕ -fuzzy contractive and β - ψ -fuzzy contractive mappings in fuzzy metric-like spaces are proved. The results of this paper generalize, unify and improve the recent results of Shukla and Abbas [12] and Gopal and Vetro [7]. Some examples are provided which justify the importance of the results and notions introduced herein.

Keywords: common fixed point, fuzzy metric like space, α - ϕ -contractive maps, β - φ -contractive maps.

1. Introduction

The fuzzy sets were first introduced by Zadeh [9] in 1965 which explain the vagueness in daily life problems. This concept is used to interpret and analyze several problems where system possesses fuzzy rather than stochastic nature. The concept of fuzzy metric was introduced by Kramosil and Michalek [8]. The fuzzy metric can be considered as the degree of nearness of points x and y of space. George and Veeramani [1] modified the notion of fuzzy metric spaces in such a way that the modified definition of fuzzy metric generates a Hausdorff and first countable topology. The Cauchy sequences in fuzzy metric spaces and the completeness of fuzzy metric spaces are defined in two ways. First, Grabiec [10] defined the notions of a Cauchy sequence in a fuzzy metric space and the completeness of a fuzzy metric space, which are often called the G -Cauchy sequences and the G -completeness respectively. The G -completeness defined by Grabiec [10] was a very strong notion so that even $\mathbb{R}$ fails to be G -complete (see, George and Veeramani [1]). Therefore, George and Veeramani

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[1] redefined the notions of Cauchy sequence and completeness which are called M -Cauchy sequence and M -completeness respectively.

On the other hand, Harandi [2] introduced the notion of a metric-like space which generalizes the notion of partial metric spaces (see, [2] and the references therein). Shukla and Abbas [12] unified the notions of the metric-like spaces and the fuzzy metric spaces by introducing the notion of fuzzy metric-like spaces. In fuzzy metric-like spaces, the degree of nearness of points x and y of space, when $x = y$ is not perfect, and so, the self fuzzy distance may not be unity. Shukla and Abbas [12] defined the notions of Cauchy sequences and completeness of fuzzy metric-like spaces which were the extensions of the G -Cauchy sequence and G -completeness respectively, in fuzzy metric-like spaces. Recently, Shukla et al. [11] introduced various notions of Cauchy sequences and completeness of a fuzzy metric-like spaces. They showed that among the various notions 1- M -completeness is most general notion and is an extension and generalization of the M -completeness defined by George and Veeramani [1].

Gopal and Vetro [7] introduced the notions of α - ϕ -fuzzy contractive mapping and β - ψ -fuzzy contractive mapping and proved some fixed point theorems which ensure the existence and uniqueness of the fixed point for the classes of these two types of mappings. They extended and unified the fixed point results of Gregori and Sapena [13], Samet et al. [4] and several others (see, Gopal and Vetro [7]). The purpose of this paper is to extend and generalize and improve the recent results of Shukla and Abbas [12] and Gopal and Vetro [7] by proving fixed point theorem for α - ϕ -fuzzy contractive mappings and β - ψ -fuzzy contractive mappings in 1- M -complete fuzzy metric-like spaces. Some examples are given which verifies the generalization of this paper.

2. Preliminaries

We start with some known concepts and results.

Definition 2.1 (Zadeh [9]). A fuzzy set A in a nonempty set X is a function with domain X and values in $[0, 1]$.

Definition 2.2 (Schweizer and Sklar [6]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if $\{[0, 1], *\}$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Three typical examples of t -norms are $a *_m b = \min\{a, b\}$ (minimum t -norm), $a *_p b = ab$ (product t -norm), and $a *_l b = \max\{a + b - 1, 0\}$ (Lukasiewicz t -norm).

Definition 2.3 (George and Veeramani [1]). The triplet $(X, M, *)$ is a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t -norm, and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(FM1) $M(x, y, t) > 0$;

- (FM2) $M(x, y, t) = 1$ if and only if $x = y$;
- (FM3) $M(x, y, t) = M(y, x, t)$;
- (FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (FM5) $M(x, y, \cdot): (0, \infty) \rightarrow [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$.

Here M (endowed with $*$) is called a fuzzy metric on X . It is known that, for all $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function. Several examples of fuzzy metric spaces can be found in [1, 3, 5, 14].

Definition 2.4 (Shukla and Abbas [12]). The triplet $(X, F, *)$ is a fuzzy metric-like space if X is an arbitrary set, $*$ is a continuous t -norm, and F is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

- (FML1) $F(x, y, t) > 0$;
- (FML2) if $F(x, y, t) = 1$ then $x = y$;
- (FML3) $F(x, y, t) = F(y, x, t)$;
- (FML4) $F(x, y, t) * F(y, z, s) \leq F(x, z, t + s)$;
- (FML5) $F(x, y, \cdot): (0, \infty) \rightarrow [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$. Here F (endowed with $*$) is called a fuzzy metric-like on X .

If we choose $F(x, x, t) = 1$ for all $x \in X$ and $t > 0$, then a fuzzy metric-like space reduces into a fuzzy metric space. Therefore, the class of fuzzy metric spaces are a particular case of the class of fuzzy metric-like spaces (with unit self fuzzy distance). We observe that $F(x, x, t)$ may be less than or equal to 1 for some or may be for all $x \in X$ which is not possible in fuzzy metric spaces. We illustrate this fact by the following example:

Example 2.5. Let $X = [0, 1]$. Then, the triplet $(X, F, *_l)$ is a fuzzy metric-like space, where the fuzzy set F is defined by

$$F(x, y, t) = \begin{cases} 1, & \text{if } x = y = 0; \\ \frac{\max\{x, y\}}{a}, & \text{otherwise} \end{cases} \quad \text{for all } t > 0$$

where $a \geq 2$. Notice that, if $0 < x \leq 1$, then we have $F(x, x, t) = \frac{x}{a} < 1$.

Remark 2.6. If $x \neq y$, then we must have $F(x, y, t) < 1$, for all $t > 0$.

For various examples of fuzzy metric-like spaces, the reader is referred to Shukla and Abbas [12] and Shukla et al. [11].

Definition 2.7 (Shukla and Abbas [12] and Shukla et al. [11]). Let $(X, F, *)$ be a fuzzy metric-like space and $\{x_n\}$ be a sequence in X . Then:

(i) $\{x_n\}$ is said to be convergent to $x \in X$ and x is called a limit of $\{x_n\}$ if,

$$\lim_{n \rightarrow \infty} F(x_n, x, t) = F(x, x, t) \quad \text{for all } t > 0.$$

(ii) $\{x_n\}$ is said to be G -Cauchy if, for all $t > 0$ and each $p \geq 1$, the limit $\lim_{n \rightarrow \infty} F(x_{n+p}, x_n, t)$ exists.

(iii) $(X, F, *)$ is said to be G -complete if every G -Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n \rightarrow \infty} F(x_n, x, t) = F(x, x, t) = \lim_{n \rightarrow \infty} F(x_{n+p}, x_n, t)$ for all $t > 0$ and each $p \geq 1$.

Definition 2.8 (Shukla et al. [11]). Let $(X, F, *)$ be a fuzzy metric-like space and let $\{x_n\}$ be a sequence in X . The sequence $\{x_n\}$ is called a 1- G -Cauchy sequence if $\lim_{n \rightarrow \infty} F(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and each $p \geq 1$. The space $(X, F, *)$ is called 1- G -complete if every 1- G -Cauchy sequence in X converges to some $x \in X$ such that $F(x, x, t) = 1$ for all $t > 0$.

It is obvious that every 1- G -Cauchy sequence is a G -Cauchy sequence and every G -complete fuzzy metric-like space is 1- G -complete, while the converse of these facts do not hold (see, Shukla et al. [11]).

Definition 2.9 (Shukla et al. [11]). Let $(X, F, *)$ be a fuzzy metric-like space and let $\{x_n\}$ be a sequence in X . The sequence $\{x_n\}$ is called a 1- M -Cauchy sequence if $\lim_{n, m \rightarrow \infty} F(x_n, x_m, t) = 1$ for all $t > 0$. The space $(X, F, *)$ is called 1- M -complete if every 1- M -Cauchy sequence in X converges to some $x \in X$ such that $F(x, x, t) = 1$ for all $t > 0$.

Every complete fuzzy metric space in the sense of Grabiec [10] is 1- G -complete as a fuzzy metric-like space, and every complete fuzzy metric space in the sense of George and Veeramani [1] is 1- M -complete as a fuzzy metric-like space. Also, 1- G -completeness implies 1- M -completeness and the notion of 1- M -completeness is more general than that of 1- G -completeness, for detail see, Shukla et al. [11].

We now state the main results of this paper.

3. Main results

Suppose, Φ denotes the family of all right continuous functions $\phi: [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(r) < r$ for all $r > 0$. Then it is obvious that $\phi(0) = 0$.

Proposition 3.1. Let $\phi \in \Phi$ and $\{a_n\}$ be a sequence of positive numbers such that $a_{n+1} \leq \phi(a_n)$ for all $n \in \mathbb{N}$. Then, the sequence $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Suppose, $\phi \in \Phi$ and $\{a_n\}$ be a sequence of positive numbers such that $a_{n+1} \leq \phi(a_n)$ for all $n \in \mathbb{N}$. Since $a_n > 0$ for all $n \in \mathbb{N}$, we have

$$a_{n+1} \leq \phi(a_n) < a_n \text{ for all } n \in \mathbb{N}.$$

It shows that $\{a_n\}$ is decreasing sequence of positive numbers, and so, it converges to some $a \in [0, \infty)$. Suppose $a > 0$. Again, since $a_n \leq \phi(a_{n-1})$ for all $n \in \mathbb{N}$ by the right continuity of ϕ we have

$$a = \lim_{n \rightarrow \infty} a_n \leq \phi(a) < a.$$

This contradiction shows that $a = 0$, i.e., $\lim_{n \rightarrow \infty} a_n = 0$. $\square$

The following definition is an analogue of [7] in fuzzy metric-like spaces.

Definition 3.2. Let $(X, F, *)$ be a fuzzy metric-like space. A mappings $T: X \rightarrow X$ is said to be an α - ϕ -fuzzy contractive mapping if there exist two functions $\alpha: X^2 \times (0, \infty) \rightarrow [0, \infty)$ and $\phi \in \Phi$ such that

$$(1) \quad \alpha(x, y, t) \left(\frac{1}{F(Tx, Ty, t)} - 1 \right) \leq \phi \left(\frac{1}{F(x, y, t)} - 1 \right)$$

for all $x, y \in X$ and $t > 0$.

Remark 3.3. If $\alpha(x, y, t) = 1$ for all $x, y \in X, t > 0$, and $\phi(r) = kr$ for all $r > 0$ and for some $k \in (0, 1)$, then Definition 3.2 reduces into the definition of fuzzy contractive mapping given by Shukla and Abbas [12]. It shows that every fuzzy contractive mapping is an α - ϕ -fuzzy contractive mapping; but the converse is not true. The following example verifies this fact.

Example 3.4. Let $X = [0, \infty)$ and define $F: X^2 \times (0, \infty) \rightarrow [0, 1]$ by

$$F(x, y, t) = \frac{1}{1 + \max\{x, y\}} \text{ for all } x, y \in X, t > 0.$$

Then $(X, F, *_m)$ is a fuzzy metric-like space. Define the mappings $T: X \rightarrow X$, $\phi: [0, \infty) \rightarrow [0, \infty)$ and $\alpha: X^2 \times (0, \infty) \rightarrow [0, \infty)$ by:

$$Tx = \begin{cases} x - 1, & \text{if } x \in [1, \infty) \cap \mathbb{Q}^c; \\ x + 1, & \text{otherwise} \end{cases}, \quad \phi(r) = \begin{cases} \frac{r}{a}, & \text{if } r \leq \frac{1}{2}; \\ r - \frac{1}{b}, & \text{if } r > \frac{1}{2} \end{cases} \text{ and}$$

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, y \in \mathbb{Q}^c \text{ and } x + 1 \leq y - 1; \\ 0, & \text{otherwise} \end{cases}$$

where $a > 1, b \geq 2$ are such that $b = \frac{2a}{a-1}$. Then T is an α - ϕ -fuzzy contractive mapping, but not a fuzzy contractive mapping in the sense of Shukla and Abbas [12].

Notice that, although T is an α - ϕ -fuzzy contractive mapping, but it is a fixed point free mapping, i.e., has no fixed point. We denote by $\text{Fix}(T)$ the set of all fixed points of T , i.e., $\text{Fix}(T) = \{x \in X : Tx = x\}$.

Definition 3.5. Let X be a nonempty set. A mappings $T : X \rightarrow X$ is said to be an α -admissible mapping if there exists a function $\alpha : X^2 \times [0, \infty) \rightarrow [0, \infty)$ such that for all $x, y \in X$ and $t > 0$ the following condition is satisfied:

$$(2) \quad \alpha(x, y, t) \geq 1 \implies \alpha(Tx, Ty, t) \geq 1.$$

Example 3.6. Let X be a nonempty set and $T : X \rightarrow X$ be defined by $Tx = x$ for all $x \in X$. Then T is α -admissible for every function $\alpha : X^2 \times [0, \infty) \rightarrow [0, \infty)$.

Example 3.7. Let X be a nonempty set and $\alpha : X^2 \times [0, \infty) \rightarrow [0, \infty)$ be a function such that $\alpha(c, c, t) \geq 1$ for all $t > 0$ for some $c \in X$. Then, the mapping $T : X \rightarrow X$ defined by $Tx = c$ for all $x \in X$ is an α -admissible mapping.

Example 3.8. Let $X = [0, \infty)$, $T : X \rightarrow X$ be defined by $Tx = x + 1$ for all $x \in X$ and $\alpha : X^2 \times [0, \infty) \rightarrow [0, \infty)$ be defined by $\alpha(x, y, t) = \frac{xy}{t}$ for all $x, y \in X, t > 0$. Then T is an α -admissible mapping.

We next state a fixed point result for α - ϕ -fuzzy contractive mappings in 1- G -complete fuzzy metric-like spaces.

Theorem 3.9. Let $(X, F, *)$ be a 1- G -complete fuzzy metric-like space and $T : X \rightarrow X$ be an α - ϕ -fuzzy contractive mapping. Suppose, the following conditions are satisfied:

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x , then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.
- (iv) for all $x, y \in X$ and $t > 0$ there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.

Proof. Let $x_0 \in X$ be the point such that $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$. Let us define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T . Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, and so, we must have $F(x_n, x_{n-1}, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$. Since T is α -admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1 \implies \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \geq 1.$$

Proceeding in the same way further we get by induction:

$$(3) \quad \alpha(x_{n-1}, x_n, t) \geq 1 \text{ for all } n \in \mathbb{N}, t > 0.$$

By (3) we have

$$\frac{1}{F(x_n, x_{n+1}, t)} - 1 = \frac{1}{F(Tx_{n-1}, Tx_n, t)} - 1 \leq \alpha(x_{n-1}, x_n, t) \left[\frac{1}{F(x_{n-1}, x_n, t)} - 1 \right].$$

Using (1) with $x = x_{n-1}$ and $y = x_n$ in the above inequality and using the fact that $\phi(r) < r$, we have

$$\frac{1}{F(x_n, x_{n+1}, t)} - 1 \leq \phi \left(\frac{1}{F(Tx_{n-1}, Tx_n, t)} - 1 \right) < \frac{1}{F(x_{n-1}, x_n, t)} - 1.$$

Suppose, $F_n(t) = \frac{1}{F(x_n, x_{n+1}, t)} - 1$ for $t > 0$ and $n \in \mathbb{N}$. We shall show that $\lim_{n \rightarrow \infty} F_n(t) = 0$ for all $t > 0$. Then, since $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ we have $F_n(t) > 0$ for all $t > 0$ and $n \in \mathbb{N}$. Therefore, by the above inequality we have

$$F_n(t) \leq \phi(F_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

The above inequality with Proposition 3.1 yields

$$\lim_{n \rightarrow \infty} F_n(t) = 0, \text{ i.e., } \lim_{n \rightarrow \infty} \left[\frac{1}{F(x_n, x_{n+1}, t)} - 1 \right] = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} F(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0.$$

Thus, if $n \in \mathbb{N}$ and $p \geq 1$, then we have

$$F(x_n, x_{n+p}, t) \geq F\left(x_n, x_{n+1}, \frac{t}{p}\right) * F\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right) * \cdots * F\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right).$$

Letting $n \rightarrow \infty$ in the above inequality we obtain

$$\lim_{n \rightarrow \infty} F(x_n, x_{n+p}, t) = 1 \text{ for all } t > 0.$$

Hence, $\{x_n\}$ is a G -Cauchy sequence in $(X, F, *)$. Since $(X, F, *)$ is G -complete, there exists $u \in X$ such that

$$(4) \quad \lim_{n \rightarrow \infty} F(x_n, u, t) = \lim_{n \rightarrow \infty} F(x_{n+p}, x_n, t) = F(u, u, t) = 1 \text{ for all } t > 0, p \geq 1.$$

Let us show that u is the fixed point of T . For all $t > 0, n \in \mathbb{N}$, we obtain from (1) and (iii) that $\alpha(x_n, u, t) \geq 1$. Without loss of generality we can assume that $F(x_n, u, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$, and then, by (1) we have

$$\begin{aligned} \frac{1}{F(x_{n+1}, Tu, t)} - 1 &\leq \alpha(x_n, u, t) \left(\frac{1}{F(Tx_n, Tu, t)} - 1 \right) \\ &\leq \phi \left(\frac{1}{F(x_n, u, t)} - 1 \right) \\ &< \frac{1}{F(x_n, u, t)} - 1. \end{aligned}$$

The above inequality shows that $F(x_n, u, t) < F(x_{n+1}, Tu, t)$. Therefore, by (4) we obtain:

$$(5) \quad \lim_{n \rightarrow \infty} F(x_{n+1}, Tu, t) = 1 \text{ for all } t > 0.$$

By (FML4) we have

$$F(u, Tu, t) \geq F(u, x_{n+1}, t/2) * F(x_{n+1}, Tu, t/2).$$

Using (4) and (5) in the above inequality we obtain $F(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Hence, u is a fixed point of T and $F(u, u, t) = 1$ for all $t > 0$.

For uniqueness, let v be the another fixed point of T and $u \neq v$. Then $F(u, v, t) < 1$ for all $t > 0$. It follows from (iv) that there exists $z \in X$ such that

$$\alpha(u, z, t) \geq 1 \text{ and } \alpha(v, z, t) \geq 1 \text{ for all } t > 0.$$

We shall show that

$$\lim_{n \rightarrow \infty} F(u, T^n z, t) = \lim_{n \rightarrow \infty} F(v, T^n z, t) = 1 \text{ for all } t > 0.$$

Since $\alpha(u, z, t) \geq 1$ and T is α -admissible, we get $\alpha(Tu, Tz, t) = \alpha(u, Tz, t) \geq 1$. Similarly, we obtain

$$\alpha(u, T^n z, t) \geq 1 \text{ and } \alpha(v, T^n z, t) \geq 1, \text{ for all } n \in \mathbb{N}, t > 0.$$

Therefore, using (1) we obtain

$$\begin{aligned} \frac{1}{F(u, T^n z, t)} - 1 &\leq \alpha(u, T^{n-1} z, t) \left(\frac{1}{F(u, T^{n-1} z, t)} - 1 \right) \\ &= \alpha(u, T^{n-1} z, t) \left(\frac{1}{F(Tu, TT^{n-1} z, t)} - 1 \right) \\ &\leq \phi \left(\frac{1}{F(u, T^{n-1} z, t)} - 1 \right). \end{aligned}$$

Suppose, $z_n(t) = \frac{1}{F(u, T^n z, t)} - 1$ for all $n \in \mathbb{N}$, $t > 0$. The it follows from the above inequality that

$$z_n(t) \leq \phi(z_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

If $z_{n_0}(t) = 0$ for any $n_0 \in \mathbb{N}$, then it follows from the above inequality that $z_n(t) = 0$ for all $n > n_0$, and so, we obtain $\lim_{n \rightarrow \infty} F(u, T^n z, t) = 1$. If it is not the case, we have $F(u, T^n z, t) < 1$ for all $n \in \mathbb{N}$, and the above inequality with Proposition 3.1 yields

$$\lim_{n \rightarrow \infty} z_n(t) = 0, \text{ i.e., } \lim_{n \rightarrow \infty} \left[\frac{1}{F(u, T^n z, t)} - 1 \right] = 0.$$

It shows that $\lim_{n \rightarrow \infty} F(u, T^n z, t) = 1$.

Following similar arguments and replacing u by v in the above we have

$$(6) \quad \lim_{n \rightarrow \infty} F(u, T^n z, t) = \lim_{n \rightarrow \infty} F(v, T^n z, t) = 1 \text{ for all } t > 0.$$

Now, by (FML4) we have

$$F(u, v, t) \geq F(u, T^n z, t/2) * F(T^n z, v, t/2).$$

Letting $n \rightarrow \infty$ and using (6) in the above inequality we obtain $F(u, v, t) = 1$. This contradiction shows that $u = v$ and the uniqueness follows. $\square$

The following corollary was the main result of Shukla and Abbas [12].

Corollary 3.10. *Let $(X, F, *)$ be a G -complete fuzzy metric-like space and $T: X \rightarrow X$ be a fuzzy contractive mapping with fuzzy contractive constant k , then T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.*

Proof. Let us define two functions $\alpha: X^2 \times (0, \infty) \rightarrow [0, \infty)$ and $\phi: [0, \infty \rightarrow [0, \infty)$ by $\alpha(x, y, t) = 1$ and $\phi(r) = kr$ for all $x, y \in X$, $t > 0$ and $r \geq 0$. Then, it is easy to see that T is α - ϕ -fuzzy contractive mapping. We can show easily that all the hypotheses of Theorems 3.9 are satisfied. Consequently, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$. $\square$

Remark 3.11. Gopal and Vetro [7] proved the existence and uniqueness of fixed of an α - ϕ -fuzzy contractive mapping (see, Theorem 3.6 of [7]). We notice that, they assumed that the fuzzy metric M is triangular. In the following corollary, we prove that this assumption is superfluous and Theorem 3.6, and so, Theorem 3.9 of [7] can be proved without this assumption.

Remark 3.12. It is obvious that the conditions (i), (ii) and (iii) are sufficient for the existence of fixed point of the mapping T . The condition (iv) is needed only when proving the uniqueness of fixed point.

Corollary 3.13. *Let $(X, M, *)$ be a G -complete fuzzy metric space. Let $T: X \rightarrow X$ be an α - ϕ -fuzzy contractive mapping satisfying the following conditions:*

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x , then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.
- (iv) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$.

Proof. Because every G -complete fuzzy metric spaces is a G -complete fuzzy metric-like space with unit self-fuzzy distance, therefore the proof follows from Theorem 3.9. $\square$

Definition 3.14. Let $(X, \preceq)$ be a partially ordered set such that $(X, F, *)$ is a fuzzy metric-like space. A mapping $T: X \rightarrow X$ is called an ordered ϕ -fuzzy contractive mapping if there exists $\phi \in \Phi$ such that the following implication holds:

$$x, y \in X, x \preceq y \implies \frac{1}{F(Tx, Ty, t)} - 1 \leq \phi \left(\frac{1}{F(x, y, t)} - 1 \right), \text{ for all } t > 0.$$

The next theorem is the ordered version of the main result of Shukla and Abbas [12].

Theorem 3.15. Let $(X, \preceq)$ be a partially ordered set such that $(X, F, *)$ is a 1- G -complete fuzzy metric-like space and $T: X \rightarrow X$ be an ordered ϕ -fuzzy contractive mapping. Suppose, the following conditions hold:

- (i) T is a non-decreasing mapping with respect to $\preceq$;
- (ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;
- (iv) for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Then, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.

We next introduce the notion of continuity and 1-continuity in fuzzy metric-like spaces.

Definition 3.16. Let $(X, F_X, *_X)$ and $(Y, F_Y, *_Y)$ be two fuzzy metric-like spaces and $T: X \rightarrow Y$ be a mapping. Then, T is called:

- (A) continuous at $x \in X$ if for every sequence $\{x_n\} \subset X$ converging to x the sequence $\{Tx_n\} \subset Y$ converges to Tx , i.e., T is continuous at x if for every sequence $\{x_n\} \subset X$ such that $\lim_{n \rightarrow \infty} F_X(x_n, x, t) = F_X(x, x, t)$ for all $t > 0$, we have $\lim_{n \rightarrow \infty} F_Y(Tx_n, Tx, t) = F_Y(Tx, Tx, t)$ for all $t > 0$. T is called continuous on X if it is continuous at each $x \in X$.
- (B) 1-continuous at $x \in X$ if for every sequence $\{x_n\} \subset X$ such that $\lim_{n \rightarrow \infty} F_X(x_n, x, t) = F_X(x, x, t) = 1$ for all $t > 0$, we have $\lim_{n \rightarrow \infty} F_Y(Tx_n, Tx, t) = F_Y(Tx, Tx, t) = 1$ for all $t > 0$.

The notions of continuity and 1-continuity are independent of each other which can be seen in the following examples:

Example 3.17. Let $X = [0, 1]$ and $F: X^2 \times (0, \infty) \rightarrow [0, 1]$ be defined by

$$F(x, y, t) = \frac{t}{t + \max\{x, y\}} \text{ for all } x, y \in X, t > 0.$$

Then $(X, F, *_p)$ is a fuzzy metric-like space. Suppose $T: X \rightarrow X$ be defined by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 0, & \text{if } x = 1. \end{cases}$$

Then, T is a 1-continuous mapping. Indeed, if $\{x_n\}$ be any sequence in X such that $\lim_{n \rightarrow \infty} F(x_n, x, t) = F(x, x, t) = 1$ for all $t > 0$, then we have:

$$\lim_{n \rightarrow \infty} \frac{t}{t + \max\{x_n, x\}} = \frac{t}{t + x} = 1 \text{ for all } t > 0.$$

It shows that $x = 0$ and the sequence $\{x_n\}$ must converge to 0 (in usual metric sense of $[0, 1]$). Also, $T0 = 0$ and so by the above inequality we must have

$$\begin{aligned} \lim_{n \rightarrow \infty} F(Tx_n, T0, t) &= \lim_{n \rightarrow \infty} \frac{t}{t + \max\{Tx_n, T0\}} \\ &= \lim_{n \rightarrow \infty} \frac{t}{t + x_n/2} = 1 \\ &= F(T0, T0, t) \text{ for all } t > 0. \end{aligned}$$

Thus, T is 1-continuous. On the other hand, T is not a continuous mapping. To see this choose a sequence $\{x_n\}$ in X , where $x_n = 1 - \frac{1}{2n}$, then we have

$$\lim_{n \rightarrow \infty} F(x_n, 1, t) = \frac{t}{t + 1} = F(1, 1, t).$$

So, $\{x_n\}$ converges to 1 in $(X, F, *_p)$. But

$$\lim_{n \rightarrow \infty} F(Tx_n, T1, t) = \lim_{n \rightarrow \infty} \frac{t}{t + \frac{1}{2} - \frac{1}{4n}} \neq F(T1, T1, t).$$

Therefore, $Tx_n = \frac{1}{2} - \frac{1}{4n}$ does not converge to $T1 = 0$ in $(X, F, *_p)$.

Example 3.18. Let $X = [0, 1]$ and $F: X^2 \times (0, \infty) \rightarrow [0, 1]$ be defined by

$$F(x, y, t) = \begin{cases} 1, & \text{if } x = y = 1; \\ \frac{t}{t + \max\{x, y\}}, & \text{otherwise} \end{cases}$$

for all $x, y \in X, t > 0$. Then $(X, F, *_p)$ is a fuzzy metric-like space. Suppose $T: X \rightarrow X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$. Then, T is a continuous

mapping. Indeed, if $\{x_n\}$ be any sequence in X such that $\lim_{n \rightarrow \infty} F(x_n, x, t) = F(x, x, t)$ for all $t > 0$, then we have $Tx = \frac{x}{2}$. Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} F(Tx_n, Tx, t) &= \lim_{n \rightarrow \infty} \frac{t}{t + \max\{Tx_n, Tx\}} \\ &= \lim_{n \rightarrow \infty} \frac{t}{t + \max\{x_n/2, x/2\}} \\ &= F(T(x/2), T(x/2), t) \text{ for all } t > 0. \end{aligned}$$

Thus, T is continuous. On the other hand, T is not a 1-continuous mapping. To see this choose a sequence $\{x_n\}$ in X , where $x_n = 1 - \frac{1}{2n}$, then we have

$$\lim_{n \rightarrow \infty} F(x_n, 1, t) = \frac{t}{t+1} = F(1, 1, t) = 1.$$

So, $\{x_n\}$ converges to 1 in $(X, F, *_p)$ with $F(1, 1, t) = 1$ for all $t > 0$. But

$$\lim_{n \rightarrow \infty} F(Tx_n, T1, t) = \lim_{n \rightarrow \infty} \frac{t}{t + \frac{1}{2}} = F(T1, T1, t) \neq 1.$$

Therefore, $Tx_n = \frac{1}{2} - \frac{1}{4n}$ does not converge to $T1$ in $(X, F, *_p)$ such that $F(T1, T1, t) = 1$ for all $t > 0$.

In the next theorem, we show that the condition (iii) of Theorem 3.9 can be replaced by 1-continuity of the mapping T and thus, we find a generalization of Theorem 3.5 of Gopal and Vetro [7].

Theorem 3.19. *Let $(X, F, *)$ be a 1- G -complete fuzzy metric-like space and $T : X \rightarrow X$ be an α - ϕ -fuzzy contractive mapping. Suppose, the following conditions are satisfied:*

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (iii) T is 1-continuous on X ;
- (iv) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.

Proof. Following the lines of the proof of Theorem 3.9 we obtain: for the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ is convergent and converges to $u \in X$ such that

$$(7) \quad \lim_{n \rightarrow \infty} F(x_n, u, t) = \lim_{n \rightarrow \infty} F(x_{n+p}, x_n, t) = F(u, u, t) = 1 \text{ for all } t > 0, p \geq 1.$$

Let us show that u is the fixed point of T . By 1-continuity of T and the equation (7) we have

$$(8) \quad \lim_{n \rightarrow \infty} F(Tx_n, Tu, t) = \lim_{n \rightarrow \infty} F(x_{n+1}, Tu, t) = F(Tu, Tu, t) = 1 \text{ for all } t > 0.$$

By (FML4) we have

$$F(u, Tu, t) \geq F(u, x_{n+1}, t/2) * F(x_{n+1}, Tu, t/2).$$

Using (7) and (8) in the above inequality we obtain $F(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Thus, u is a fixed point of T .

The proof for uniqueness of fixed point is similar to the proof of uniqueness in Theorem 3.9. $\square$

Again, one can see that in the proof of the above theorems conditions (i), (ii) and (iii) are sufficient for the existence of fixed point, and the condition (iv) is required only when proving the uniqueness of fixed point. In other words, if condition (iv) is not satisfied, then the fixed point of T may not be unique as shown in the following example.

Example 3.20. Let $X = [0, 1]$ and $\mathbb{Q}_{[0,1]} = [0, 1] \cap \mathbb{Q}$. Define $F: X^2 \times (0, \infty) \rightarrow [0, 1]$ by

$$F(x, y, t) = \begin{cases} 1, & \text{if } x = y = 2; \\ \frac{1}{1 + \max\{x, y\}}, & \text{otherwise} \end{cases}$$

for all $x, y \in X, t > 0$. Then $(X, F, *_p)$ is a G -complete fuzzy metric-like space. Suppose, the mappings $T: X \rightarrow X$ and $\alpha: X^2 \times (0, \infty) \rightarrow [0, \infty)$ be defined by

$$Tx = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}_{[0,1]}; \\ x, & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in \mathbb{Q}_{[0,1]}; \\ 0, & \text{otherwise.} \end{cases}$$

Define the function $\phi: [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(r) = \begin{cases} r^2, & \text{if } 0 \leq r < 1; \\ \frac{r}{2}, & \text{otherwise.} \end{cases}$$

Then, ϕ is a right continuous function such that $\phi(r) < r$ for all $r > 0$. Further, T is an α -admissible, 1-continuous mapping and it is α - ϕ -fuzzy contractive mapping. Also, if $x_0 \in \mathbb{Q}_{[0,1]}$ then $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$. Thus, all the conditions (except the condition (iv)) of Theorem 3.19 are satisfied and T has a fixed point $u (= 0) \in X$ such that $F(u, u, t) = 1$ for all $t > 0$. Notice that, only the condition (iv) of Theorem 3.19 is not satisfied because there exists no $z \in X$ such that $\alpha(0, z, t) \geq 1$, $\alpha(i, z, t) \geq 1$, where $i \in [0, 1]$ is irrational. So, fixed point of T is not unique. Indeed, $\text{Fix}(T) = \{u: u \in X \setminus \mathbb{Q}_{[0,1]}\} \cup \{0\}$.

One can see easily that the continuity of mapping T assumed by Gopal and Vetro [7] in fuzzy metric spaces is a particular case of 1-continuity (as well as of continuity) of T in fuzzy metric-like spaces. Therefore, we obtain an improvement of Theorem 3.6 of [7] as a corollary of the above theorem.

Corollary 3.21. *Let $(X, M, *)$ be a G -complete fuzzy metric space and $T : X \rightarrow X$ be an α - ϕ -fuzzy contractive mapping. Suppose, the following conditions are satisfied:*

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (iii) T is continuous on X ;
- (iv) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$.

We next define the β - ψ -contractions in fuzzy metric-like spaces.

Let Ψ be the class of all functions $\psi : [0, 1] \rightarrow [0, 1]$ satisfying the following properties:

- (i) ψ is non-decreasing and left continuous;
- (ii) $r < \psi(r)$ for all $r \in (0, 1)$.

It can be seen easily that $\psi(1) = 1$ and $\lim_{n \rightarrow \infty} \psi^n(r) = 1$ for all $r \in (0, 1)$.

Proposition 3.22. Let $\psi \in \Psi$ and $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \geq \phi(a_{n-1})$ for all $n \in \mathbb{N}$. Then, the sequence $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = 1$.

Proof. Suppose, $\psi \in \Psi$ and $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \geq \phi(a_{n-1})$ for all $n \in \mathbb{N}$. By given condition we have

$$a_n \geq \psi(a_{n-1}) > a_{n-1} \text{ for all } n \in \mathbb{N}.$$

It shows that $\{a_n\}$ is increasing bounded above sequence of positive numbers, and so, it converges to some $a \in (0, 1]$. Suppose $a < 1$. Again, since $a_n \geq \psi(a_{n-1})$ for all $n \in \mathbb{N}$ by the left continuity of ψ we have

$$a = \lim_{n \rightarrow \infty} a_n \geq \psi(a) > a.$$

This contradiction shows that $a = 1$, i.e., $\lim_{n \rightarrow \infty} a_n = 1$. □

Definition 3.23. Let $(X, F, *)$ be a fuzzy metric-like space and $T: X \rightarrow X$ be a mapping. Then, the mapping T is called a β - ψ -contractions if there exist two functions $\psi \in \Psi$ and $\beta: X^2 \times (0, \infty) \rightarrow (0, \infty)$ such that

$$(9) \quad \beta(x, y, t)F(Tx, Ty, t) \geq \psi(F(x, y, t))$$

for all $t > 0$ and for all $x, y \in X$ with $x \neq y$.

Definition 3.24. Let $(X, F, *)$ be a fuzzy metric-like space and $T: X \rightarrow X$ be a mapping. Suppose $\beta: X^2 \times (0, \infty) \rightarrow (0, \infty)$ be function. Then, the mapping T is called β -admissible if the following condition is satisfied:

$$x, y \in X, \beta(x, y, t) \leq 1 \implies \beta(Tx, Ty, t) \leq 1.$$

for all $t > 0$ and for all $x, y \in X$.

The identity mapping on X is a trivial example of β -admissible mapping on X for any arbitrary function $\beta: X^2 \times (0, \infty) \rightarrow (0, \infty)$.

We now state some fixed point results for β - ψ -contractions.

Theorem 3.25. Let $(X, F, *)$ be a 1-M-complete fuzzy metric-like space such that $\inf_{x, y \in X} F(x, y, t) > 0$ for all $t > 0$. Suppose that $T: X \rightarrow X$ be an β - ψ -fuzzy contractive mapping and the following conditions are satisfied:

- (i) T is β -admissible;
- (ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $m > n > n_0$ and $t > 0$;
- (iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x , then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$.
- (v) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.

Then, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.

Proof. Let $x_0 \in X$ be the point such that $\beta(x_0, Tx_0, t) \leq 1$, for all $t > 0$. Let us define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T . Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, and so, we must have $F(x_n, x_{n-1}, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$. Since T is β -admissible, we have

$$\beta(x_0, x_1, t) = \beta(x_0, Tx_0, t) \leq 1 \implies \beta(Tx_0, Tx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

Proceeding in the same way further we get by induction:

$$(10) \quad \beta(x_{n-1}, x_n, t) \leq 1 \text{ for all } n \in \mathbb{N}, t > 0.$$

Therefore, by the condition (iii) there exists $n_0 \in \mathbb{N}$ such that

$$(11) \quad \beta(x_n, x_m, t) \leq 1 \text{ for all } n, m \geq n_0, t > 0.$$

By (9) we have

$$F(x_n, x_{n+1}, t) = F(Tx_{n-1}, Tx_n, t) \geq \beta(x_{n-1}, x_n, t)F(Tx_{n-1}, Tx_n, t).$$

Using (9) with $x_n = x_{n-1}$ and $y = x_n$ in the above inequality and using the fact that $\psi(r) > r$, we have $F(x_n, x_{n+1}, t) \geq \beta(x_{n-1}, x_n, t)F(Tx_{n-1}, Tx_n, t) \geq \psi(F(x_{n-1}, x_n, t)) > F(x_{n-1}, x_n, t)$. Suppose, $F_n(t) = F(x_n, x_{n+1}, t)$ for $t > 0$ and $n \in \mathbb{N}$. We shall show that $\lim_{n \rightarrow \infty} F_n(t) = 1$ for all $t > 0$. Then, since $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ we have $F_n(t) < 1$ for all $t > 0$ and $n \in \mathbb{N}$. Therefore, by the above inequality we have

$$F_n(t) \geq \phi(F_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

The above inequality with Proposition 3.22 yields $\lim_{n \rightarrow \infty} F_n(t) = 1$, i.e.,

$$\lim_{n \rightarrow \infty} F(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0.$$

We shall show that $\{x_n\}$ is a Cauchy sequence. Suppose

$$a_n(t) = \inf_{m > n} F(x_m, x_n, t) \text{ for all } t > 0.$$

Using (10) we obtain from (9) that

$$F(x_{n+1}, x_{m+1}, t) \geq \beta(x_m, x_n, t)F(Tx_m, Tx_n, t) \geq \psi(F(x_m, x_n, t)) > F(x_m, x_n, t)$$

for all $n, m > n_0$ and $t > 0$. It shows that $a_{n+1}(t) \geq a_n(t)$ for all $n > n_0$. Therefore, the sequence $\{a_n(t)\}$ is convergent. Suppose, $\lim_{n \rightarrow \infty} a_n(t) = a(t) < 1$. Then from the above inequality and the properties of ψ we have

$$a(t) = \lim_{n \rightarrow \infty} a_n(t) \geq \psi(a(t)) > a(t).$$

This contradiction shows that $a(t) = 1$ for all $t > 0$, i.e., $\lim_{n \rightarrow \infty} a_n(t) = 1$, and so, for every given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$F(x_m, x_n, t) \geq a_n(t) > 1 - \varepsilon \text{ for all } m > n > \max\{n_0, N\} \text{ and for all } t > 0$$

or $\lim_{n, m \rightarrow \infty} F(x_m, x_n, t) = 1$. Therefore, $\{x_n\}$ is a 1- M -Cauchy sequence in X . By 1- M -completeness of X , there exists $u \in X$ such that

$$(12) \quad \lim_{n \rightarrow \infty} F(x_n, u, t) = F(u, u, t) = 1 \text{ for all } t > 0.$$

We shall show that u is the fixed point of T .

For all $t > 0, n \in \mathbb{N}$, we obtain from (iii) that $\beta(x_n, u, t) \leq 1$. Without loss of generality we can assume that $F(x_n, u, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$, and then, by (9) we have

$$\begin{aligned} F(x_{n+1}, Tu, t) &= \beta(x_n, u, t)F(Tx_n, Tu, t) \\ &\geq \psi(F(x_n, u, t)) \\ &> F(x_n, u, t). \end{aligned}$$

Therefore, $F(x_n, u, t) < F(x_{n+1}, Tu, t)$, and so, by (12) we obtain:

$$(13) \quad \lim_{n \rightarrow \infty} F(x_{n+1}, Tu, t) = 1 \text{ for all } t > 0.$$

By (FML4) we have

$$F(u, Tu, t) \geq F(u, x_{n+1}, t/2) * F(x_{n+1}, Tu, t/2).$$

Using (12) and (13) in the above inequality we obtain $F(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Hence, u is a fixed point of T and $F(u, u, t) = 1$ for all $t > 0$.

For uniqueness, let v be the another fixed point of T and $u \neq v$, then $F(u, v, t) < 1$ for all $t > 0$. It follows from (v) that there exists $z \in X$ such that

$$\beta(u, z, t) \leq 1 \text{ and } \beta(v, z, t) \leq 1 \text{ for all } t > 0.$$

Since $\beta(u, z, t) \leq 1$ and T is β -admissible, we get $\beta(Tu, Tz, t) = \beta(u, Tz, t) \leq 1$. Similarly, we obtain

$$\beta(u, T^n z, t) \leq 1 \text{ and } \beta(v, T^n z, t) \leq 1, \text{ for all } n \in \mathbb{N}, t > 0.$$

Therefore, using (9) we obtain

$$\begin{aligned} F(u, T^n z, t) &\geq \beta(u, T^{n-1} z, t)F(u, T^n z, t) \\ &= \beta(u, T^{n-1} z, t)F(Tu, TT^{n-1} z, t) \\ &\geq \psi(F(u, T^{n-1} z, t)). \end{aligned}$$

Suppose, $z_n(t) = F(u, T^n z, t)$ for all $n \in \mathbb{N}, t > 0$. Then it follows from the above inequality that

$$z_n(t) \geq \psi(z_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

If $z_{n_0} = 1$ for some $n_0 \in \mathbb{N}$, then it follows from the above inequality that $z_n(t) = 1$ for all $n > n_0$, and so, $\lim_{n \rightarrow \infty} F(u, T^n z, t) = 1$ for all $t > 0$. If this is not the case, we have $F(u, T^z, t) = 1$ for all $t > 0$ and the above inequality with Proposition 3.22 yields

$$\lim_{n \rightarrow \infty} z_n(t) = 1, \text{ i.e., } \lim_{n \rightarrow \infty} F(u, T^n z, t) = 1.$$

Following similar arguments and replacing u by v in the above we have

$$(14) \quad \lim_{n \rightarrow \infty} F(u, T^n z, t) = \lim_{n \rightarrow \infty} F(v, T^n z, t) = 1 \text{ for all } t > 0.$$

Now, by (FML4) we have

$$F(u, v, t) \geq F(u, T^n z, t/2) * F(T^n z, v, t/2).$$

Letting $n \rightarrow \infty$ and using (14) in the above inequality we obtain $F(u, v, t) = 1$. This contradiction shows that $u = v$ and the uniqueness follows. $\square$

The following corollary is an improvement of Theorem 4.4 and Theorem 4.6 of Gopal and Vetro [7] (see, Remark 3.27 below).

Corollary 3.26. *Let $(X, M, *)$ be an M -complete fuzzy metric-like space such that $\inf_{x, y \in X} M(x, y, t) > 0$ for all $t > 0$. Suppose that $T: X \rightarrow X$ be an β - ψ -fuzzy contractive mapping and the following conditions are satisfied:*

- (i) T is β -admissible;
- (ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $m > n > n_0$ and $t > 0$;
- (iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x , then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$.
- (v) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.

Then, T has a unique fixed point $u \in X$.

Remark 3.27. In Theorem 3.25 we have proved the existence and uniqueness of the fixed point of a β - ψ -fuzzy contractive mapping in 1- M -fuzzy metric-like spaces. Note that, Gopal and Vetro [7] proved a similar result in M -complete fuzzy metric spaces, but with an additional assumption that the fuzzy metric space is *non-Archimedean*. Here, we have removed this superfluous assumption, as well as, we have extended and generalize their result in the setting of 1- M -fuzzy metric-like spaces.

Again, if we assume the 1-continuity of the mapping T , then we can omit the condition (iv) in Theorem 3.25. In this case, one can prove the following result by following the lines of proof of Theorem 3.25, and so, we omit the proof.

Theorem 3.28. *Let $(X, F, *)$ be a 1- M -complete fuzzy metric-like space such that $\inf_{x, y \in X} F(x, y, t) > 0$ for all $t > 0$. Suppose that $T: X \rightarrow X$ be an β - ψ -fuzzy contractive mapping and the following conditions are satisfied:*

- (i) T is β -admissible;
- (ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;
- (iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $n, m > n_0$ and $t > 0$;
- (iv) T is 1-continuous on X ;
- (v) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.

Then, T has a unique fixed point $u \in X$ and $F(u, u, t) = 1$ for all $t > 0$.

Again, in the above theorem the condition (v) is required only when we prove uniqueness of fixed point.

Example 3.29. Let $X = [0, 1]$ and define $F: X^2 \times (0, \infty) \rightarrow [0, 1]$ by $F(x, y, t) = e^{-\max\{x, y\}}$ for all $x, y \in X$ and $t > 0$. Then, $(X, F, *_p)$ is a 1- M -complete fuzzy metric-like space. Define the mappings $T: X \rightarrow X$, $\beta: X^2 \times (0, \infty) \rightarrow (0, \infty)$ and $\psi: [0, 1] \rightarrow [0, 1]$ by:

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 1, & \text{if } x = 1; \end{cases} \quad \beta(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1) \text{ with } y \leq x; \\ 2, & \text{otherwise} \end{cases}$$

and $\psi(r) = \sqrt{r}$ for all $r \in [0, 1]$.

Then, ψ is a left continuous function such that $\psi(r) > r$ for all $r \in [0, 1]$. Further, T is an β -admissible, 1-continuous mapping and it is β - ψ -fuzzy contractive mapping. Also, if $x_0 \in [0, 1)$ then $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$. Thus, all the conditions (except the condition (iv)) of Theorem 3.28 are satisfied and T has a fixed point $u(= 0) \in X$ such that $F(u, u, t) = 1$ for all $t > 0$. Notice that, only the condition (iv) of Theorem 3.28 is not satisfied because there exists no $z \in X$ such that $\beta(0, z, t) \leq 1$, $\beta(1, z, t) \leq 1$. So, fixed point of T is not unique. Indeed, $\text{Fix}(T) = \{0, 1\}$.

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On orbit reflexive tuple of operators and weak orbit reflexivity

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Abstract. In this paper we give a various conditions for which the tuple $\mathcal{T} = (T_1, T_2, \dots, T_n)$ of commuting bounded linear operators on an infinite dimensional (real, complex) Banach space X is orbit reflexive. After we introduce the notion of weak orbit reflexive operator and we show some results.

Keywords: tuple of operators, orbit, orbit reflexive, weak orbit reflexive, C_0 -semi group.

1. Introduction

By an n -tuple of operators we mean a finite sequence of length n of commuting bounded linear operators on a Banach space X . The orbit of a point $x \in X$ under an operator $T \in B(X)$ is the sequence,

$$Orb(T, x) := \{T^n x : n = 0, 1, \dots\}.$$

The notion of orbit-reflexive operators on a Hilbert space was introduced and studied in [4]. While the reflexivity of operators is connected to the invariant subspace problem, its natural analogue of orbit reflexivity is in the same way connected to the problem of existence of closed invariant subsets.

Recall that if X is a Banach space and $B(X)$ the set of all bounded linear operators acting on X , an operator $T \in B(X)$ is said to be reflexive if every operator $A \in B(X)$, if $Ax \in \overline{\{p(T)x : p \text{ polynomial}\}}$ (the norm closure of the set $\{p(T) : p \text{ polynomial}\}$) for each $x \in X$, then A belongs to $\overline{\{p(T) : p \text{ polynomial}\}}^{SOT}$ (the closure in the strong operator topology).

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Similarly an operator T is said to be orbit reflexive if every operator $A \in B(X)$, if $Ax \in \overline{Orb(T, x)}$ for each $x \in X$ (or equivalently, such that every closed subset of X invariant for T is invariant for A), then A belongs to $\overline{Orb(T)}^{SOT}$.

The orbit reflexivity of many classes of Hilbert space operators was shown in [4], e.g. for normal operators, contractions, algebraic operators, weighted shifts and compact operators. Among others, each operator whose spectrum does not intersect the unit circle is orbit reflexive, and a various conditions which insure that an operator acting on a Banach space is orbit reflexive are given in [10]. Other authors are introduce a similar notions (Null-orbit reflexive operators [5], $\mathbb{C}$ -orbit reflexive operators [6], $\mathbb{R}$ -orbit Reflexive Operators [7]).

In this work we focus on the orbit reflexivity.

2. Main results

2.1 Operator orbit reflexive

In this part, we will introduce the conditions for which the adjoint and the power of an orbit reflexive operator are also orbit reflexive. We will answer the open question posed in the thesis of Jan Versovsky under the direction of Vladimir Muller supported in 2013 [11] in relation to the reflexivity of the orbit of the inverse of an orbit reflexive operator.

Theorem 2.1 ([4, Theorem 5]). *Let H be a Hilbert space and $T \in B(H)$. Then T is orbit reflexive in any of the following cases hold:*

1. *there is a non empty open subset $U \subset H$ such that for each $x \in U$, the orbit $Orb(T, x)$ is closed;*
2. *there is a non empty open subset $U \subset H$ such that for each $x \in U$, $\|T^n x\| \rightarrow \infty$, as $n \rightarrow \infty$;*
3. *for each $x \in X$, $\|T^n x\| \rightarrow 0$;*
4. *the set $\overline{Orb(T)}^{SOT}$ is countable and strongly compact;*
5. $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$.

Theorem 2.2 ([10, Theorem 7]). *Let X be a Banach space and $T \in B(X)$ satisfying $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$, or when $X = H$ is a complex Hilbert space $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$. Then T is orbit reflexive.*

Corollary 2.1 ([10, Corollary 8]). *Let X be a Banach space and $T \in B(X)$. If $r(T) \neq 1$, then T is orbit reflexive.*

Corollary 2.2. *Let H be a Hilbert space and $T \in B(H)$. If $r(T) \neq 1$, then T^* is orbit reflexive.*

Proof. Follows from, $r(T) = r(T^*)$. $\square$

Corollary 2.3. *Let X be a Banach space and $T \in B(X)$. If T is self-adjoint and $r(T) \neq 1$, Then T^n is orbit reflexive for all $n \geq 1$.*

Proof. Follows from $r(T^n) = r(T)^n$ for all $n \geq 2$ and Corollary 2.1. $\square$

Let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ the spectrum, the point spectrum and the approximate point spectrum of T , where

- $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}.$
- $\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective}\}.$
- $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective or } Rg(\lambda I - T) \text{ is not closed in } X\}.$

Theorem 2.3 ([8, Theorem 3.1]). *Let X be an infinite dimensional reflexive Banach space and $T, S \in B(X)$. If $\lambda \in \sigma_{ap}(T) \setminus \sigma_p(T)$ and $\mu \in \sigma_{ap}(S) \setminus \sigma_p(S)$, then for any two positive sequences $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$ with $\sum_{n \geq 1} \alpha_n < +\infty$ and $\sum_{n \geq 1} \beta_n < +\infty$, in every open ball in X with radius $\sum_{n \geq 1} (\alpha_n + \beta_n)$, there is $x \in X$ satisfying*

$$\|T^n x\| \geq \alpha_n \frac{|\lambda|^n}{2}, \|S^n x\| \geq \beta_n \frac{|\mu|^n}{2}.$$

Corollary 2.4. *Let $T, S \in B(X)$, if the sets $\sigma_{ap}(T) \setminus \sigma_p(T)$ and $\sigma_{ap}(S) \setminus \sigma_p(S)$ both have a non-empty intersection with the domain $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ then, T and S are orbit reflexive.*

Proof. From theorem 2.3 there is a non empty open subset $U \subset X$ such that for each $x \in U$,

$$\|T^n x\| \longrightarrow \infty, \|S^n x\| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

and from (2) of theorem 2.1 then, T and S are orbit reflexive. $\square$

Let Ω be a non empty subset of the complex plane whose boundary consists of finite number of rectifiable Jordan curves, oriented in the positive sense and $Hol(\Omega)$ the set of all holomorphic functions on some open neighborhood of the closure of Ω .

To obtain the reflexivity of the orbit of the inverse operator we need the following theorem.

Theorem 2.4. *If $T \in B(X)$, $\sigma(T) \subset \Omega$ and $f \in \Omega$ is injective and non-constant function on each of the components of Ω and the set $\sigma_{ap}(T) \setminus \sigma_p(T)$ has a non-empty intersection with the both domains $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\{\lambda \in \mathbb{C} : |f(\lambda)| > 1\}$ then, the both T and $f(T)$ are orbit reflexive.*

Proof. Follows from corollary 2.4. $\square$

If $T \in B(X)$ is an invertible operator, then

$$\sigma(T) \subset \{\lambda \in \mathbb{C} / [r(T^{-1})]^{-1} \leq |\lambda| \leq r(T)\}.$$

Since $f(\lambda) = \lambda^{-1}$ is holomorphic, non-constant and injective function on $\Omega = \{\lambda \in \mathbb{C} : m \leq |\lambda| \leq M\} \supset \sigma(T)$, where $0 < m < [r(T^{-1})]^{-1}$ and $M > r(T)$.

So, according to theorem 4.4 in [8] and corollary 2.4, we deduce that the both T and T^{-1} are orbit reflexive if the set $\sigma_{ap}(T) \setminus \sigma_p(T)$ has a non-empty intersection with both domains $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

2.2 n-tuple of operators orbit reflexive

Definition 2.1 ([2]). Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a n -tuple of operators acting on an infinite dimensional Banach space X . We will let,

$$\mathcal{F} = \{T_1^{k_1} T_2^{k_2}, \dots, T_n^{k_n} : k_i \geq 0, i = 1, \dots, n\}$$

be the semi-group generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$\text{Orb}(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

Definition 2.2 ([13, Definition 1.2]). The orbit of x under the tuples $\mathcal{T}$ tending to infinity if:

$$\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| \rightarrow \infty \text{ as } k_i \rightarrow \infty \text{ with } k_i \geq 0, \forall i = 1, \dots, n.$$

Lemma 2.1 ([4]). Let $A, S_1, S_2, \dots \in B(X)$.

If the set of vectors $u \in X$ for which $Au \in \{S_1 u, S_2 u, \dots\}$ is of second category, then $A \in \{S_1, S_2, \dots\}$

Proposition 2.1. Let $A, T_1, T_2, \dots, T_n \in B(X)$ and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of commuting operators. If the set of vectors $x \in X$ for which $Ax \in \text{Orb}(\mathcal{T}, x)$ is of second category, then $A \in \text{Orb}(\mathcal{T})$.

Proof. Follows from Lemma 2.1. □

Theorem 2.5. Let $T_1, T_2, \dots, T_n \in B(X)$ and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of commuting operators. $\mathcal{T}$ is orbit reflexive in any of the following cases:

1. there is a non empty open subset $U \subset X$ such that for each $x \in U$, the orbit $\text{Orb}(\mathcal{T}, x)$ is closed;
2. there is a non empty open subset $U \subset X$ such that for each $x \in U$, $\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| \rightarrow \infty$ as $k_i \rightarrow \infty$ with $k_i \geq 0$, for all $i = 1, \dots, n$;
3. The set $\overline{\text{Orb}(\mathcal{T})}^{SOT}$ is countable and strongly compact.

Proof. Throughout the proof, suppose that $Ax \in \overline{Orb(\mathcal{T}, x)}$ for every $x \in X$. We need to show that $A \in \overline{Orb(\mathcal{T})}^{SOT}$ in each of the cases.

1. Follows from Lemma 2.1,
2. Since $\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| \rightarrow \infty$ as $k_i \rightarrow \infty$ then $Orb(\mathcal{T}, x)$ is closed, it follows from (1) that $\mathcal{T}$ is orbit reflexive.
3. Let $S \in \mathcal{F}$ and suppose that $Ax = \lim S^{n_k} x$ for some sequence $(n_k)_{k \geq 0}$ of positive integers.
Since $\overline{Orb(\mathcal{T})}^{SOT}$ is strongly compact, the sequence $(S^{n_k})_{k \geq 0}$ has a strongly convergent subsequence. Therefore $Ax \in \left\{ Bx : B \in \overline{Orb(\mathcal{T})}^{SOT} \right\}$ for each $x \in X$, and according to Lemma 2.1, we have $A \in \overline{Orb(\mathcal{T})}^{SOT}$. $\square$

Example 1. Let S be the unilateral forward shift on $\ell^2(\mathbb{N})$:

$$Se_i = e_{i+1}; i \geq 1,$$

where $\{e_i : i \in \mathbb{N}^*\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$.

Given a sequence of positive numbers $(a_i)_{i \geq 1}$ so that $a_i > 1$ for all $i \geq 1$ and $a_i \rightarrow 1$ as $i \rightarrow \infty$ and let

$$T_i = a_i S; i = 1, \dots, n.$$

So, of after [13, Example 1.1] there is a non empty open subset $U \subset X$ such that for each $x \in U$, the orbit of x under the tuple $(T_1, T_2, \dots, T_n)$ tend strongly to infinity, then follows from (2) of Theorem 2.5, the tuple $(T_1, T_2, \dots, T_n)$ it orbit reflexive.

Proposition 2.2. *Let X be an infinite dimensional reflexive Banach space and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be the n -tuple of operators in $B(X)$ bounded below for all $n \geq 1$. If there is $x \in X$ such that the orbit of x under T_i for all $i \in \{1, 2, \dots, n\}$ tend strongly to infinity then $\mathcal{T}$ is orbit reflexive.*

Proof. From [13, Corollary 1.1] the orbit of x under the tuple $\mathcal{T}$ tend strongly to infinity and from (2) of Theorem 2.5, $\mathcal{T}$ is orbit reflexive. $\square$

2.3 Operator weakly orbit reflexive

By a weak orbit of an operator $T \in B(X)$ we mean a sequence of the form $(\langle T^n x, x^* \rangle)_{n=0}^\infty$, where $x \in X$ and $x^* \in X^*$, and we write:

$$W - Orb(T, x) = (\langle T^n x, x^* \rangle)_{n=0}^\infty.$$

Definition 2.3. *Let X be a Banach space, an operator $T \in B(X)$ is said to be weakly orbit reflexive if every operator $A \in B(X)$ belongs to $\overline{W - Orb(T)}^{WOT}$ (the closure in the weak operator topology), whenever $\langle Ax, x^* \rangle \in \overline{W - Orb(T, x)}^W$ (the closure in the weak topology) for each $x \in X$ and $x^* \in X^*$.*

Example 2. Since the norm topology is strictly stronger than the weak topology, then every operator orbit reflexive is weakly orbit reflexive. So in a Hilbert space a normal operators, contractions, algebraic operators, weighted shifts and compact operators are weak orbit-reflexive.

Each Banach space operator have spectral radius different from 1, is weak orbit-reflexive.

Theorem 2.6. 1. If T is weakly orbit reflexive and S is invertible operator then STS^{-1} is also weakly orbit reflexive.

2. If T is weakly orbit reflexive then $T \oplus T$ is also weakly orbit reflexive.

Proof. 1. The image by an operator of a weakly convergent sequence is a sequence weakly convergent. Hence $\langle SAS^{-1}x, x^* \rangle \in \overline{W - Orb(STS^{-1}, x)}^W$ for all $x \in X$ and $x^* \in X^*$ is equivalent to $\langle Ax, x^* \rangle \in \overline{W - Orb(T, x)}^W$ for all $x \in X$ and $x^* \in X^*$, or T is weakly orbit reflexive, then $A \in \overline{W - Orb(T)}^{WOT}$, so $SAS^{-1} \in \overline{W - Orb(STS^{-1})}^{WOT}$.

2. Let $A \in B(X \oplus X)$ and suppose that $\langle Ax, x^* \rangle \in \overline{W - Orb(T \oplus T, x)}^W$ for all $x \in X \oplus X$ and $x^* \in X^* \oplus X^*$.

In particular, for any $a \in X$ we have:

$$\langle A(a, 0), x^* \rangle \in \overline{W - Orb(T \oplus T, (a, 0))}^W \subset \mathbb{C} \times \{0\}.$$

And similarly for the second component, so that both copies of X are A -invariant. Hence A can be written as $B \oplus C$ where $B, C \in B(X)$. Moreover, if $a \in X$ then: $\langle (Ba, Ca), x^* \rangle = \langle A(x, x), x^* \rangle \in \overline{W - Orb(T \oplus T, x)}^W \subset \{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$ So, $B = C$.

Now, since T is weakly orbit reflexive and $Ba \in \overline{W - Orb(T, x)}^W$ for all $a \in X$, we have $B \in \overline{W - Orb(T)}^{WOT}$, and therefore:

$$A = B \oplus B \in \overline{W - Orb(T \oplus T)}^{WOT}.$$

□

Proposition 2.3. Let X be a Banach space and $T \in B(X)$ satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^{1/2}} < \infty,$$

or when $X = H$ is a complex Hilbert space $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Then T is weakly orbit reflexive.

Proof. Let $A \in B(X)$ and suppose that $\langle Ax, x^* \rangle \in \overline{W - Orb(T, x)}^W$, for all $x \in X$ and $x^* \in X^*$.

Suppose, that $A \neq T^n$ for all $n \in \mathbb{N}$, otherwise there is nothing to prove.
 Since $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^{1/2}} < \infty$, then $\|T^n\|^{1/2} \rightarrow \infty$ and we have:

$$\|T^n - A\| \geq \|T^n\| - \|A\| \geq \frac{1}{2} \|T^n\| \text{ for all } n \text{ large enough.}$$

So, $\sum_{n=1}^{\infty} \frac{1}{\|T^n - A\|^{1/2}} < \infty$. Therefore, the operators $S_n := T_n - A$ satisfy the conditions in [1, Theorem 6]. So there exists a dense set of pairs $x \in X$; $x^* \in X$ with:

$$|\langle (T_n - A)x, x^* \rangle| > 0, \forall n \geq 1, |\langle (T_n - A)x, x^* \rangle| \rightarrow \infty.$$

Thus there is a constant $C > 0$ such that $\inf_n |\langle (T_n - A)x, x^* \rangle| > 0$ and we have a contradiction with the assumption that $\langle Ax, x^* \rangle \in \overline{W - \text{Orb}(T, x)}^{-W}$.

The Hilbert space case can be proved similarly. $\square$

We finish this paper by posing the open question related to existence of operator weakly orbit reflexive but not orbit reflexive.

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Numerical solution of fractional order differential equation with different methods

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Abstract. Various motion phenomena in natural environment and engineering is regular, but it is often impossible to describe them accurately with linear theory, which promotes the development of nonlinear theory. Nonlinear differential equation is one of them. Solving equations is an important part of theory. In this study, the exponential function expansion method, the first integral method and the wavelet operator matrix algorithm were introduced firstly. Then the steps of the three methods were explained. Finally, the concrete equations were solved by the three methods respectively. The fractional order STO equation was solved by the exponential function expansion method and the fractional order cahn-allen equation was solved by the first integral method. The wavelet operator matrix algorithm was used to solve the fractional order differential equations. The results showed that the exponential function expansion method could obtain the exact solution of STO smoothly and easily, but the amount of calculation was determined by the equilibrium principle. The first integral method could obtain the exact solution of cahn-allen equation smoothly and effectively, and there were many kinds of exact solutions when m was equal to 1. The wavelet operator matrix algorithm could get the exact solution easily and quickly, and with the increase of the value of m' , the absolute error between the numerical solution and the exact solution decreased.

Keywords: fractional order differential equation, fractional order STO equation, fractional order cahn-allen equation, numerical solution.

1. Introduction

Different natural phenomena show regular forms, but they often can not be described by linear mathematics, which promotes the emergence of nonlinear theory [1]. Nonlinear partial differential equation is a part of nonlinear theory. Mathematical modeling based on nonlinear partial differential equation can effectively help analyze the mathematical essence of nonlinear natural phenomena [2]. Integral order differential equation has a wide application, and there are relevant studies. Albzeirat et al. [3] proposed a solution which was based on reproducing kernel Hilbert space tool for the numerical solution of second-order integer and fractional differential equations under fuzzy conditions and verified the effectiveness and flexibility of the method using an example. Saadatmandi et al. [4] put forward a method based on Tau method and Legendre polynomial

to solve a class of time fractional telegraph equations. The Legendre matrix of fractional derivatives was used to appropriately represent the solution, and the numerical was simplified to the solution of a set of linear algebraic equations. However, integer differential equations often have several problems in simulating complex systems and phenomena [5]: model parameters setting is out of practice, model is susceptible to external interference, and the solution is complex. Therefore, fractional differential equations was proposed. There is no universal solution for nonlinear differential equations at present, so there are many exact solutions or approximate solutions. Muthukumar et al. [6] studied the numerical solution of fractional order delay differential equations and verified its applicability by several examples like a mathematical model of houseflies and a model based on the effect of noise on light that reflected from laser to mirror. Ray et al. [7] calculate the numerical solutions of parabolic fractional partial differential equation (PDE) and nonlinear time fractional partial differential equation with two-dimensional Legendre wavelet method and two-dimensional Haar wavelet method respectively and compared them with the exact solutions obtained by homotopy perturbation method. In this paper, the exponential function expansion method, the first integral method and the wavelet operator matrix algorithm were introduced, and then the steps of the three methods were explained. Finally, the exponential function expansion method was used to solve the fractional order STO equation, the first integral method was used to solve the fractional order cahn-allen equation, and the wavelet operator matrix algorithm was used to solve the fractional order differential equation.

2. Fractional order partial differential equation

Nonlinear partial differential equations include integer-order differential equations and fractional order differential equations, in which fractional order differential equations can describe complex nonlinear systems concisely and accurately; hence it is very important to solve the theoretical and numerical solutions of fractional-order differential equations. The related definitions of fractional differential equations are as follows.

Definition 1 ([8]). $f(x)$ is real function and $x > 0$, and moreover $f(x)$ is contained in space $C_\alpha (\alpha \in R)$. If $\exists k \in R$ and $k > \alpha$, making $f_1(x)$ contained in $C \in [0, \infty)$ and $f(x) = x^k f_1(x)$, then it can be considered that $f(x) \in C_a^b$ when and only when $f^{(b)}$ belongs to space C_α and $b \in N$.

Definition 2. β -order R-L fractional calculus is defined as:

$$(1) \quad (I^\beta f)(x) = \begin{cases} f(x), & \beta = 0 \\ \frac{1}{\Gamma(\beta)} \int_0^x (x-a)^{\beta-1} f(a) da, & \beta > 0, x > 0 \end{cases}$$

Its derivative is:

$$(2) \quad \left(D_l^\beta f\right)(x) = \begin{cases} \frac{d^n f(x)}{dt^n}, & \beta = n \in \mathbb{N} \\ \frac{d^n}{dt^n} \frac{1}{\Gamma(m-\beta)} \int_\beta^x \frac{f(y)}{(x-y)^{\beta-m+1}} dy, & 0 \leq n-1 < \beta < n \end{cases}$$

Definition 3: caputo frictional order calculus is defined as:

$$(3) \quad {}^C D_x^\beta [f(x)] = \frac{1}{\Gamma(n-p)} \int_\beta^x (x-y)^{n-p-1} f^{(n)}(y) dy, \\ n = \lfloor \beta \rfloor + 1, n-1 < \beta \leq n, x > \beta.$$

3. Two solutions of fractional order differential equation

3.1 Exponential function expansion method

Exponential function expansion method is used for solving integer order differential equation initially. The equation is processed by travelling wave exchange, and then nonlinear ordinary differential equation is introduced. The forms of the solution are diverse.

$$(4) \quad \Phi'(\delta) = g\left(e^{-\Phi(\delta)}\right) + h\left(e^{\Phi(\delta)}\right) + o$$

was selected as the nonlinear ordinary differential equation [9].

The general form of fractional order partial differential equation is:

$$(5) \quad f\left(s, s_y, s_x, D_x^\beta s, D_y^\beta s, D_y^\gamma s, D_x^\beta D_x^\gamma s, D_y^\beta D_y^\gamma s, \dots\right) = 0, 0 < \beta, \gamma < 1.$$

The solution steps of exponential function expansion method are as follows.

Firstly a new variable which is composed of y and x was set, and original function

$u(y, x)$ is transformed to new function $U(\delta)$ [10], then

$$(6) \quad s(y, x) = S(\delta),$$

$$(7) \quad \delta = \frac{ky^\gamma}{\Gamma(1+\gamma)} \pm \frac{dx^\beta}{\Gamma(1+\beta)}$$

where k and c are constants. Equation (6) and (7) are substituted to equation (5). A new nonlinear ordinary differential equation with regard to U and derivatives of U can be obtained based on the nature of R-L fractional order derivative.

$$(8) \quad G(S, S', S'', \dots) = 0.$$

Then the form of the solution of equation (8) [11] is supposed as:

$$(9) \quad S(\delta) = \sum_{i=0}^N B_i \left(e^{-\Phi(\delta)}\right)^i, B_N \neq 0,$$

where B_i is an undetermined constant and Φ satisfied equation (4).

The third step is to solve the value of N according to principle of equilibrium [12] and high-order derivative and the highest order nonlinear items in equation (8). The last step is to substitute to equation (4) and (9) to equation (8), then the right side of the equal sign of equation (8) remains to be 0, and the left side is a $e^{-\Phi(\delta)}$ related polynomial. The coefficients of different powers of $e^{-\Phi(\delta)}$ in the polynomial are set as 0, and then equation (8) becomes an algebraic equation set. Unknown numbers in equation (8) including B_i, k, g, h, o, d can be calculated using mathematica software [13].

Nonlinear ordinary differential equation (4) needs to be used in the solution of equation (5), i.e., the solution of equation (5) is related to the solution of equation (4). But the solution of equation (4) has several special circumstances.

Special circumstance 1: $g = 1$

$$(10) \quad \Phi(\delta) = \begin{cases} \ln \left(-\frac{\sqrt{o^2 - 4h} \tanh \left\{ 0.5\sqrt{o^2 - 4h}(\delta + \delta_0) \right\}}{2h} - \frac{o}{2h} \right), \\ h \neq 0, o^2 - 4h > 0, \\ \ln \left(-\frac{\sqrt{o^2 - 4h} \tan \left\{ 0.5\sqrt{o^2 - 4h}(\delta + \delta_0) \right\}}{2h} - \frac{o}{2h} \right), \\ h \neq 0, o^2 - 4h < 0, \\ -\ln \left(\frac{o}{e^{o(\delta + \delta_0)} - 1} \right), h = 0, o \neq 0, o^2 - 4h > 0, \\ \ln \left(\frac{2(o(\delta + \delta_0) + 2)}{o^2(\delta + \delta_0)} \right), h \neq 0, o \neq 0, o^2 - 4h = 0. \end{cases}$$

Special circumstance 2: $o = 0$

$$(11) \quad \Phi(\delta) = \begin{cases} \ln \left(\sqrt{\frac{g}{h}} \tan [\sqrt{gh}(\delta + \delta_0)] \right), g > 0, h > 0 \\ \ln \left(-\sqrt{\frac{g}{h}} \tan [\sqrt{gh}(\delta - \delta_0)] \right), g < 0, h < 0 \\ \ln \left(\sqrt{\frac{-g}{h}} \tanh [\sqrt{-gh}(\delta - \delta_0)] \right), g > 0, h < 0 \\ \ln \left(-\sqrt{\frac{-g}{h}} \tanh [\sqrt{-gh}(\delta + \delta_0)] \right), g < 0, h > 0. \end{cases}$$

Special circumstance 3: $h = 0$ and $o = 0$, $\Phi(\delta) = \ln \{g(\delta + \delta_0)\}$ (12). δ_0 in equation (10) ~ (12) is integration constant. The exact solution of equation (5) can be obtained according to the above steps and equation (7), (9) and (10) ~ (12).

3.2 First integral method

First integral method [14] is also one of the methods for solving nonlinear differential equations. Similar to the exponential function expansion method, it is

applied to solving integer order differential equations initially. In order to better adapt to the solution of fractional differential equations, the method was a little different. In order to better adapt to the solution of fractional differential equations. The theoretical basis of first integral method is exact division of polynomial, i.e., binary polynomial $F(x, y)$ and $G(x, y)$ are both in complex number field and $F(x, y)$ is irreducible. If $G(x, y)$ is 0 at the zero point of $F(x, y)$, then there must have $H(x, y)$ in complex number field, and $G(x, y) = F(x, y)H(x, y)$ holds.

According to the theory of polynomial integral division [15], first integral method was used to solve equation (5), and its procedures are as follows:

The first step was the same with the step of exponential expansion function. New variable δ which is composed of y and x is introduced. Function $U(\delta)$ after transformation is substituted into equation (5) and transformed into non-linear differential equation (8) based on the properties of R-L fractional order derivative. If every item in equation (8) has derivative, then the integral of variable δ is taken, and the integration constant is set as 0.

Then the solution of equation (8) is supposed as

$$(12) \quad S(\delta) = X(\delta).$$

Moreover new variable is introduced to construct an equation set with equation (13).

$$(13) \quad \begin{cases} X'(\delta) = Y(\delta) \\ Y'(\delta) = H(X(\delta), Y(\delta)) \end{cases}.$$

The third step is to suppose that polynomial $X(\delta)$ and $Y(\delta)$ in complex number field satisfying the following equation according to first integral method.

$$(14) \quad H(X(\delta), Y(\delta)) = \sum_{i=0}^m b_i(X(\delta))Y(\delta)^i = 0.$$

where $b_i(X)$ is the polynomial of X and $b_m(X) \neq 0$. According to the principle of polynomial integral division, there must be polynomial $p(X) + q(X)Y$ in complex number field, which can make tenable.

$$(15) \quad \frac{dH}{d\delta} = \frac{\partial H}{\partial X} \frac{dX}{d\delta} + \frac{\partial H}{\partial Y} \frac{dY}{d\delta} = (p(X) + q(X)Y) \sum_{i=0}^m b_i(X)Y^i.$$

Then $b_i(X)$ in equation (16) is obtained using method of undetermined coefficients [16] and substituted into $H(X, Y) = \sum_{i=0}^m b_i(X)Y^i = 0$; then $X(\delta)$ is obtained, i.e., $S(\delta)$. $S(\delta)$ is substituted into equation (6) and (7), and then the solution of equation (5) is obtained.

The following equation is obtained in the process of solution:

$$(16) \quad S'(\delta) = c_0 + c_1 S(\delta) + c_2 S^2(\delta).$$

The solution of equation (5) is related to equation (16). c_0, c_1, c_2 in equation (16) are constants, and their relationships will affect the solution of equation (5) and (16).

Relationship 1: when $c_1^2 - 4c_0c_2 > 0$,

$$(17) \quad S(\delta) = \begin{cases} -\frac{\sqrt{c_1^2 - 4c_0c_2}}{2c_2} \tanh\left(\frac{\sqrt{c_1^2 - 4c_0c_2}}{2}(\delta + \delta_0)\right) - \frac{c_1}{2c_2} \\ -\frac{\sqrt{c_1^2 - 4c_0c_2}}{2c_2} \coth\left(\frac{\sqrt{c_1^2 - 4c_0c_2}}{2}(\delta + \delta_0)\right) - \frac{c_1}{2c_2} \end{cases}.$$

Relationship 2: When $c_1^2 - 4c_0c_2 < 0$,

$$(18) \quad S(\delta) = \begin{cases} \frac{\sqrt{-(c_1^2 - 4c_0c_2)}}{2c_2} \tan\left(\frac{\sqrt{-(c_1^2 - 4c_0c_2)}}{2}(\delta + \delta_0)\right) - \frac{c_1}{2c_2} \\ -\frac{\sqrt{-(c_1^2 - 4c_0c_2)}}{2c_2} \cot\left(\frac{\sqrt{-(c_1^2 - 4c_0c_2)}}{2}(\delta + \delta_0)\right) - \frac{c_1}{2c_2} \end{cases}.$$

Relationship 3: when $c_1^2 - 4c_0c_2 = 0$,

$$(19) \quad S(\delta) = -\frac{1}{c_2(\delta + \delta_0)} - \frac{c_1}{2c_2}.$$

3.3 Wavelet operator matrix solution method

Wavelet operator matrix solution method is based on wavelet transform and wavelet function. Wavelet transform was first proposed by an engineer who engaged in processing oil signal in 1974. At first, it was an empirical formula derived from the intuitive actual signal, but it was gradually improved in the later research and solved the problem that Fourier transform is difficult to solve. The basic principle of wavelet function is to represent $f(t)$ which needs to be solved using multiple approximated smooth expressions and regard $f(t)$ as an image, and the expression group is the resolution of the image. As a new mathematical branch, wavelet analysis has been widely applied in fields such as data compression, signal recognition and solution of differential equation.

he computation of fractional differential equation is large and complex, and the property of wavelet analysis which decomposes function into multi-resolution expression group can reduce the dimension of matrix in fractional differential equation, and the reduced expression group can be flexibly expressed according to different resolution requirements, which greatly reduces the computational difficulty. Its specific solving principle is shown below.

The second-class Chebyshev wavelet is defined as:

$$(20) \quad \psi_{nm}(x) = \begin{cases} 2^{\frac{k}{2}} \tilde{S}_m(2^k x - 2n + 1), & x \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}), \\ 0, & \text{otherwise,} \end{cases}$$

$$(21) \quad \tilde{S}_m(x) = \sqrt{\frac{2}{\pi}} S_m(x),$$

where $n = 1, \dots, 2^{k-1}, (k \in N^+)$ and $S_m(x)$ is the m -order second-class Chebyshev polynomials

The fractional integral operator matrix is composed of:

$$(22) \quad (I^\beta \Psi)(x) \approx \mathbf{Q}_{m' \times m'}^\beta \Psi(x),$$

$$(23) \quad \mathbf{Q}_{m' \times m'}^\beta = \mathbf{\Phi}_{m' \times m'} \mathbf{F}^\beta \mathbf{\Phi}_{m' \times m'}^{-1},$$

$$(24) \quad \mathbf{\Phi}_{m' \times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \dots, \Psi\left(\frac{2m'-1}{2m'}\right) \right],$$

$$(25) \quad \mathbf{F}^\beta = \frac{1}{m'^\beta} \frac{1}{\Gamma(\beta+2)} \begin{bmatrix} 1 & \delta_1 & \delta_2 & \cdots & \delta_{m'-1} \\ 0 & 1 & \delta_1 & \cdots & \delta_{m'-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \delta_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where I^β refers to fractional order integral operator and $\varphi_{m' \times m'}$ is Chebyshev wavelet matrix of the second class; the expression of δ_k in equation (25) is $\delta_k = (k+1)^{\beta+1} - 2k^{\beta+1} + (k-1)^{\beta+1}$

The solution steps of fractional order differential equation are as follows.

Suppose $\frac{\partial^2 s}{\partial y \partial x} \approx \Psi^T(y) \mathbf{S} \psi(x)$, then

$$(26) \quad \frac{s(y, x) \approx \psi^T(y) [\mathbf{Q}_{m' \times m'}^1]^T S \mathbf{Q}_{m' \times m'}^1 \psi(x) + \int_0^y \varphi_2(y) dy + s(0, x) = \psi^T(y) [\mathbf{Q}_{m' \times m'}^1]^T S \mathbf{Q}_{m' \times m'}^1 \psi(x) + \int_0^y \varphi_2(y) dy + \varepsilon_1(x)}{.}$$

It is obtained from equation (26) that

$$(27) \quad \begin{cases} \frac{\partial^\beta s}{\partial y^\beta} = I^{1-\beta} \left(\frac{\partial s}{\partial y} \right) \approx I^{1-\beta} (\psi^T(y) S \mathbf{Q}_{m' \times m'}^1 \psi(x) + \varphi_2(y)), \\ \frac{\partial^y s}{\partial x^y} = I^{1-y} \left(\frac{\partial s}{\partial x} \right) \approx I^{1-y} (\psi^T(y) [\mathbf{Q}_{m' \times m'}^1]^T S \psi(x) + \varphi_1(x)). \end{cases}$$

Equation (27) is substituted into $f(x) = \sum_{n=0}^\infty \sum_{m \in Z} d_{nm} \psi_{mm}$, then

$$(28) \quad \begin{aligned} & \psi^T(y) [\mathbf{Q}_{m' \times m'}^{1-\beta}] S \mathbf{Q}_{m' \times m'}^1 \psi(x) + \psi^T(y) [\mathbf{Q}_{m' \times m'}^1]^T S [\mathbf{Q}_{m' \times m'}^{1-y}] \psi(x) = g(y, x) \\ & = f(y, x) - I^{1-\beta} \varphi_2(y) - I^{1-y} \varphi_1(x) \approx \psi^T(y) G \psi(x), \end{aligned}$$

where $G = [g_{ij}]_{m \times n}$. Then collocation point (y_i, x_j) is substituted to equation (28), and i and j are both positive integer and the largest value of them is m' .

Then we have

$$(29) \quad [\mathbf{Q}_{m' \times m'}^{-1}]^T [\mathbf{Q}_{m' \times m'}^{1-\beta}]^T \mathbf{S} + \mathbf{S} \mathbf{Q}_{m' \times m'}^{1-\gamma} \mathbf{Q}_{m' \times m'}^{-1} = [\mathbf{Q}_{m' \times m'}^{-1}] \mathbf{G} \mathbf{Q}_{m' \times m'}^{-1}.$$

$\mathbf{s}$ is obtained after equation (29) is solved. The numerical solution of $s(y, x)$ can be calculated according to equation (26).

4. Application of different solutions

4.1 Solution of fractional order STO equation based on Exponential expansion

The object equation is:

$$(30) \quad D_x^\beta s + 3bs_y^2 + 3bs^2s_y + 3bss_{yy} + bs_{yyy} = 0, x > 0, 0 < \beta \leq 1.$$

Solution: According to the above steps, equation (30) is transformed firstly. $s(y, x)$ is replaced by $S(\delta)$, $\delta = k \left(y + \frac{dx^\beta}{\Gamma(1+\beta)} \right)$, where k and d are constants. Then equation (30) transformed from a fractional order differential equation into an integral differential equation. As there is derivative in every item, both sides of the equation is integrated and integration constant is set as 0. Finally the following equation is obtained.

$$(31) \quad dS + 3bkSS' + bS^3 + bk^2S'' = 0.$$

According to principle of equilibrium, $N = 1$. Therefore the form of solution of equation (31) is set as

$$(32) \quad S(\delta) = B_0 + B_1 e^{-\Phi(\delta)}$$

according to equation (9). The solution is substituted into equation (31), and an algebraic equation set with regard to B_i, k, g, h, o, d is obtained:

$$(33) \quad \begin{cases} -3bkgB_1^2 + bB_1^3 + 2bk^2g^2B_1 = 0, \\ -3bk(gB_0B_1 + oB_1^2) + 3bB_0B_1^2 + 3bogk^2B_1 = 0, \\ dB_1 - 3bk(oB_0B_1 + hA_1^2) + 3bB_0^2B_1 + 2bghk^2B_1 + bk^2o^2B_1 = 0, \\ dB_0 - 3bkhB_0B_1 + bB_0^3 + bk^2hoB_1 = 0. \end{cases}$$

The equation set is solved using mathematica. k, g, h, o are constants. Then the groups of solution is obtained:

Solution 1:

$$(34) \quad \begin{cases} B_0 = \frac{ko}{2}, \\ B_1 = kg, \\ k = k, \\ d = \frac{bk^2(4gh - o^2)}{4}. \end{cases}$$

Solution 2:

$$(35) \quad \begin{cases} B_0 = \frac{k(o \pm \sqrt{o^2 - 4gh})}{2}, \\ B_1 = kg, \\ k = k, \\ d = bk^2(4gh - o^2). \end{cases}$$

According to solution 1, equation (32), the initial equation transformation and the value of $\Phi(\delta)$ given in section 3.1, the solution of the fractional order STO equation under solution 1 is as follows.

When $g = 1$, then

$$(36) \quad s(y, x) = \begin{cases} \frac{k_0}{2} - \frac{2kh}{\sqrt{o^2 - 4h} \tanh\left(0.5\sqrt{o^2 - 4h}(\delta + \delta_0) + o\right)}, \\ h \neq 0, o^2 - 4h > 0 \\ \frac{k_0}{2} + \frac{2kh}{\sqrt{4h - o^2} \tanh\left(0.5\sqrt{4h - o^2}(\delta + \delta_0) - o\right)}, \\ h \neq 0, o^2 - 4h < 0 \\ \frac{k_0}{2} + \frac{k_0}{e^{o(\delta + \delta_0)} - 1}, h = 0, o^2 - 4h > 0 \\ \frac{k_0}{2} + \frac{ko^2(\delta + \delta_0)}{2o(\delta + \delta_0) + 4}, h \neq 0, o \neq 0, o^2 - 4h = 0 \end{cases},$$

where $\delta = ky + \frac{bk^3x^\beta(4h-o^2)}{4\Gamma(1+\beta)}$; when $o = 0$, then

$$(37) \quad s(y, x) = \begin{cases} k\sqrt{gh} \cot[\sqrt{gh}(\delta + \delta_0)], g > 0, h > 0, \\ k\sqrt{gh} \cot[\sqrt{gh}(\delta - \delta_0)], g < 0, h < 0, \\ k\sqrt{-gh} \coth[\sqrt{-gh}(\delta - \delta_0)], g > 0, h < 0, \\ k\sqrt{-gh} \coth[\sqrt{-gh}(\delta + \delta_0)], g < 0, h > 0, \end{cases}$$

where $\delta = ky + \frac{bk^3ghx^8}{\Gamma(1+\beta)}$;

When $h = 0, o = 0$,

$$(38) \quad s(y, x) = \frac{k}{ky + \delta_0}.$$

The solution of the fractional order STO equation under solution 2 is as follows.

When $g = 1$,

$$(39) \quad s(y, x) = \begin{cases} k\left(\frac{o \pm \sqrt{o^2 - 4h}}{2}\right) - \frac{2h}{\sqrt{o^2 - 4h} \tanh(0.5\sqrt{o^2 - 4h}(\delta + \delta_0) + o)}, \\ h \neq 0, o^2 - 4h > 0 \\ k\left(\frac{o \pm \sqrt{4h - o^2}}{2}\right) - \frac{2h}{\sqrt{4h - o^2} \tanh(0.5\sqrt{o^2 - 4h}(\delta + \delta_0) - o)}, \\ h \neq 0, o^2 - 4h < 0 \\ k_0 + \frac{k_0}{e^{o(\delta + \delta_0)} - 1}, h = 0, o^2 - 4h > 0 \\ \frac{k_0}{2} - \frac{ko^2(\delta + \delta_0)}{2o(\delta + \delta_0) + 4}, h \neq 0, o \neq 0, o^2 - 4h = 0 \end{cases}$$

where $\delta = ky + \frac{bk^3x^\beta(4h-o^2)}{4\Gamma(1+\beta)}$;

When $o = 0$,

$$(40) \quad s(y, x) = \begin{cases} k\sqrt{gh}(\pm 1 + \cot \lfloor \sqrt{gh}(\delta + \delta_0) \rfloor), & g > 0, h > 0, \\ k\sqrt{gh}(\pm 1 + \cot \lfloor \sqrt{gh}(\delta - \delta_0) \rfloor), & g < 0, h < 0, \\ k\sqrt{-gh}(\pm 1 + \coth \lfloor \sqrt{-gh}(\delta - \delta_0) \rfloor), & g > 0, h < 0, \\ k\sqrt{-gh}(\pm 1 + \coth \lfloor \sqrt{-gh}(\delta + \delta_0) \rfloor), & g < 0, h > 0, \end{cases}$$

where $\delta = ky + \frac{bk^3ghx^\beta}{\Gamma(1+\beta)}$;

When $h = 0, o = 0$, it is the same with equation (38).

4.2 Solution of fractional order cahn-allen equation with first integral method

Object equation

$$(41) \quad D_x^\beta s - s_{yv} + s^3 - s = 0, (x > 0, 0 < \beta \leq 1)$$

is solved. It is usually used for describing fluid dynamics and reaction diffusion, and its energy has dissipativity.

Solution: According to the procedures of section 3.2, the equation is processed by complex transformation firstly. $s(y, x)$ is replaced by $S(\delta)$, where $\delta = ky - \frac{dx^\beta}{\Gamma(1+\beta)}$. After the simplification of equation (41), new variables $X = S(\delta)$ and $Y = S'(\delta)$ are introduced, and an equation set is formed by the new variables and the simplified equation:

$$(42) \quad \begin{cases} X'(\delta) = Y(\delta) \\ Y'(\delta) = \frac{X^3(\delta) - X(\delta) - dY(\delta)}{k^2} \end{cases}.$$

According to the procedures, suppose that equation (33) is tenable in complex number field:

$$(43) \quad H(X, Y) = \sum_{i=0}^m b_i(X)Y^i = 0.$$

Moreover $b_i(X)$ and $H(X, Y)$ satisfy the condition of polynomial integral division theorem; therefore

$$(44) \quad \frac{dH}{d\delta} = \frac{\partial H}{\partial X} \frac{dX}{d\delta} + \frac{\partial H}{\partial Y} \frac{dY}{d\delta} = (P(X) + q(Y)Y) \sum_{i=0}^m b_i(X)Y^i$$

is tenable.

Let $m = 1$, $b_0(X)$ and $b_1(X)$ are obtained using method of undetermined coefficients and substituted into equation (33), and the following algebraic equations are obtained:

$$(45) \quad \begin{cases} \frac{1}{k^2} - \frac{B_1^2}{2} = 0 \\ \frac{3B_0B_1}{2} + \frac{dB_1}{k^2} = 0 \\ \frac{1}{k^2} + C_0B_1 + B_0^2 + \frac{dB_0}{k^2} = 0 \\ B_0C_0 = 0 \end{cases}.$$

The following groups of solutions are obtained:

$$(1) \text{ When } B_0 = d = 0, \begin{cases} C_0 = \frac{1}{\sqrt{2k}}, B_1 = -\frac{\sqrt{2}}{k} \\ C_0 = -\frac{1}{\sqrt{2k}}, B_1 = \frac{\sqrt{2}}{k} \end{cases};$$

$$(2) \text{ When } C_0 = 0, B_0 = -\frac{\sqrt{2}}{k}, d = \frac{3k}{\sqrt{2}}, B_1 = \pm \frac{\sqrt{2}}{k};$$

$$(3) \text{ When } C_0 = 0, B_0 = \frac{\sqrt{2}}{k}, d = \frac{3k}{\sqrt{2}}, B_1 = \pm \frac{\sqrt{2}}{k}.$$

According to (1) and equation (43) and (42), an exact solution can be obtained:

$$(46) \quad s_1(y, x) = \begin{cases} \pm \tanh \left[\frac{\sqrt{2}(ky + \delta_0)}{2k} \right], \\ \pm \coth \left[\frac{\sqrt{2}(ky + \delta_0)}{2k} \right]. \end{cases}$$

In the same way, according to (2) and equation (43) and (42), an exact solution can be obtained:

$$(47) \quad s_2(y, x) = \begin{cases} \pm \frac{\tanh \left[\frac{\sqrt{2}}{4k} \left(ky - \frac{3k}{\sqrt{2}\Gamma(1+\beta)} \right) x^\beta + \delta_0 \right] - 1}{2}, \\ \pm \frac{\coth \left[\frac{\sqrt{2}}{4k} \left(ky - \frac{3k}{\sqrt{2}\Gamma(1+\beta)} \right) x^\beta + \delta_0 \right] - 1}{2}. \end{cases}$$

In the same way, according to (3) and equation (43) and (42), an exact solution can be obtained:

$$(48) \quad s_3(y, x) = \begin{cases} \pm \frac{\tanh \left[\frac{\sqrt{2}}{4k} \left(ky + \frac{3k}{\sqrt{2}\Gamma(1+\beta)} \right) x^\beta + \delta_0 \right] + 1}{2}, \\ \pm \frac{\coth \left[\frac{\sqrt{2}}{4k} \left(ky + \frac{3k}{\sqrt{2}\Gamma(1+\beta)} \right) x^\beta + \delta_0 \right] + 1}{2}. \end{cases}$$

4.3 The solution of fractional order differential equation based on wavelet operator matrix

The object equation is $\frac{\partial^\beta s}{\partial y^\beta} + \frac{\partial^\beta s}{\partial x^\beta} = f(y, x), (0 \leq y, x \leq 1)$ (17).

Solution: As $\frac{\partial s}{\partial x}\big|_{y=0} = 2x$ and $\frac{\partial s}{\partial y}\big|_{x=0} = 2y$, $\begin{cases} s(0, x) = x^2 + 1 \\ s(y, 0) = y^2 + 1 \end{cases}$. Moreover as $f(y, x) = \frac{\Gamma(3)y^{2-\beta}(x^2+1)}{\Gamma(3-\beta)} + \frac{\Gamma(3)(y^2+1)x^{2-\gamma}}{\Gamma(3-\gamma)}$, the exact solution of equation (17) is $s(y, x) = (y^2 + 1)(x^2 + 1)$.

Suppose $\beta = \frac{1}{2}, \gamma = \frac{1}{3}$, then the changes of absolute error under different values of m' are shown in Figure 1. The absolute error between the numerical solution and exact solution decreased with the increase of m' .

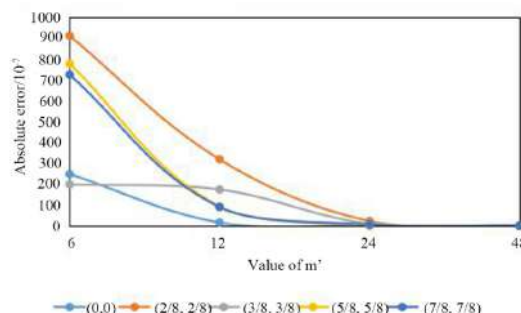


Figure 1: The absolute error between the numerical solution and exact solution under different values of m'

5. Conclusion

Fractional calculus is derived from integer calculus. At present, most of the solutions of fractional differential equations are derived from the solutions of integer differential equations. Because of the variety of nonlinear systems, the forms of nonlinear fractional calculus are various, so there is no general solution to solve all fractional differential equations. In this paper, the exponential function expansion method, first integral method and wavelet operator matrix method were introduced. Then the solving steps of the three methods were explained. Finally, the concrete equations were solved by the three methods respectively. Fractional order STO equation was solved by the exponential function expansion method, fractional cahn-allen equation was solved by the first integral method, and the fractional differential equation was solved by wavelet operator matrix method. Two groups of solutions were obtained in the solution of fractional order STO equation based on exponential expansion method. There was no special limitation in the solution. But suppose that N in $S(\delta) = \sum_{i=0}^N B_i (e^{-\Phi(\delta)})^i$, $B_N \neq 0$ is determined by principle of equilibrium, the calculation quantity will be large if the value of N is too large. Three groups of solutions were obtained in the solution of fractional order cahn-allen equation using the first integral method, two solutions each group; the value of m was set as 1 for the convenience of calculation, and the number of solutions was rich. The exact solution of the

equation was obtained by wavelet operator matrix algorithm, and the absolute error decreased with the increase of m' value.

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Secure communication applications of the Chinese remainder theorem

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Abstract. In this paper, we extend the Chinese remainder theorem in the number theory to several applications: (1) Lagrange interpolation formula is proved to be an extension of the Chinese remainder theorem in the polynomial ring. (2) By applying The Chinese Remainder Theorem in distributive lattice, a communication scheme is proposed. (3) Adjusting and combining the Chinese remainder theorem with RSA public-key cryptosystem, a dynamic secure communication on identity is proposed.

Keywords: chinese remainder theorem, secure communication, polynomial ring, distributive lattice.

1. Introduction

With the widely application of the computer and the continuous development of network, communication system security problems cannot be ignored. The inherent openness and limitations of network makes the security problem increasingly obvious [2,3].

To protect information security has become the consensus of the whole society. Great attention and commitment have been given to the communication security system.

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In order to prevent information security risks, new security technology and standard constantly appear. Cryptography also get great development as the key technology in recent years [4,5,10-11].

The Chinese remainder theorem in number theory is essentially solving the congruence equations. In the modern number theory, the Chinese remainder theorem theory is of great importance, and also gains some applications in several different algebras [1,6,13]. Besides the theoretical applications, the Chinese remainder theorem theory also gains some applications in information security. For example, [10] constructed a secret sharing schemes based on Chinese Remainder Theorem.

Ring and lattice theory play a vital role as branches of algebra and in recent years, and significance of ring and lattice theory also become gradually apparent as their applications occur in many fields [9,12,14-15].

Lagrange's interpolation formula includes interpolation by vector polynomials and by rational vector functions with prescribed pole characteristics. The formula is applied to obtain representations of the inverses of CauchyVandermonde matrices generalizing former results. The shamir threshold scheme and some password authentication schemes in cryptography are essentially based on Lagrange interpolation formula [2,3].

Just as normal subgroups play a crucial role in the theory of groups, ideals play an analogous role in the study of rings and lattice. The theory of ideals functions well not only in algebras, but also in Computer Science. Nowadays, ideals of different algebras were further studied.

In this paper, after Lagrange interpolation formula is proved just by extending the Chinese remainder theorem to the polynomial ring, we further construct a dynamic secure communication scheme on identity by extending the Chinese remainder theorem to distributive lattice.

The outline of this paper is organized as follows. In Section 2, preliminaries of definitions and results are given. In Section 3, based on the Chinese Remainder Theorem in Number Theory, Lagrange interpolation formula is proved to be as an expression of the Chinese Remainder Theorem in the polynomial ring. In Section 4, the Chinese Remainder Theorem in distributive lattice are constructed and applied to secure communication application. In Section 5, an identity-based dynamic secure communication scheme is proposed.

2. Preliminaries

Definition 2.1 ([12]). A ring is a nonempty set R together with two binary operations (usually denoted as addition $(+)$ and multiplication) such that

- (1) $(R, +)$ is an abelian group;
- (2) $(ab)c = a(bc)$ for all $a, b, c \in R$ (associative multiplication);
- (3) $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ (left and right distributive laws).

If in addition: $ab = ba$ for all $a, b \in R$, then R is said to be a commutative ring. If R contains an element 1_R such that $1_R a = a 1_R = a$ for all $a \in R$, then R is said to be with identity.

Definition 2.2 ([8], the Chinese remainder theorem). Suppose $n \geq 2$, and $m_1, m_2, \dots, m_n$ are n positive mutually prime integers. Let $M = m_1 m_2 \dots m_n = m_1 M_1 = m_2 M_2 = \dots = m_n M_n$, here $M_i = \frac{M}{m_i}, i = 1, 2, \dots, n$, then for the following congruence equations group

$$\begin{aligned} x &\equiv b_1 \pmod{m_1}, \\ x &\equiv b_2 \pmod{m_2}, \\ &\vdots \\ x &\equiv b_n \pmod{m_n}, \end{aligned}$$

the minimum solution is $x_0 = b_1 M'_1 M_1 + b_2 M'_2 M_2 + \dots + b_n M'_n M_n \pmod{M}$, here positive integer M'_i satisfying: $M'_i M_i \equiv 1 \pmod{m_i}, (i = 1, \dots, n)$ i.e., M'_i is the inverse element of M_i with respect to m_i .

Definition 2.3 ([12]). Let R be a ring and S a nonempty subset of R that is closed under the operations of addition and multiplication in R . If S is itself a ring under these operations then S is called a subring of R . A subring I of a ring R is a left ideal provided $r \in R$ and $x \in I \Rightarrow rx \in I$. I is a right ideal provided $r \in R$ and $x \in I \Rightarrow xr \in I$. I is an ideal if it is both a left and right ideal.

Theorem 2.4 ([12], Chinese Remainder Theorem on ring). Let $A_1, \dots, A_n$ be ideals in a ring R , such that $R^2 + A_i = R$ for all i , and $A_i + A_j = R$ for all $i \neq j$. If $b_1, \dots, b_n \in R$, then there exists $b \in R$ such that $b \equiv b_i \pmod{A_i} (i = 1, \dots, n)$. Furthermore b is uniquely determined up to congruence modulo ideal $A_1 \cap A_2 \cap \dots \cap A_n$.

Remark 2.5. If R has an identity, then $R^2 = R$, whence $R^2 + A = R$ for every ideal A of R .

Definition 2.6 ([15]). A nonempty set L with binary operations $\wedge$ and $\vee$ is called a lattice if for $x, y, z \in L$:

- (1) $x \wedge x = x \vee x = x$,
- (2) $x \wedge y = y \wedge x, x \vee y = y \vee x$,
- (3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$,
- (4) $(x \wedge y) \vee x = (x \vee y) \wedge x = x$.

A binary relation $\leq$ is defined as for $x, y \in L, x \leq y$ if $x \wedge y = x$ or $x \vee y = y$. Then we can find that binary relation $\leq$ is a partially ordered relation.

A lattice L is called bounded if there exists $0, 1 \in L$, such that $0 \leq x \leq 1$ for any $x \in L$.

Definition 2.7 ([14]). For any $x, y, z \in L$, a lattice L is called distributive if:

- (5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
- (6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.8 ([5], RSA public-key system). RSA is one of the best secure algorithms till now [6].

RSA public-key system:

- (1) Choose two large different prime numbers p and q .
- (2) Compute $n = pq$, $\varphi(n) = (p-1)(q-1)$.
- (3) Find e ($1 < e < \varphi(n)$), such that $\text{g.c.d.}(e, \varphi(n)) = 1$, set e as public encryption key.
- (4) Compute d , such that $de \equiv 1 \pmod{\varphi(n)}$, set d as confidential decryption key.
- (5) Encryption process: for plain text $m \in Z_n$, cipher text c is $c = m^e \pmod{n}$.
- (6) Decryption process: for cipher text $c \in Z_n$, plain text m is $m = c^d \pmod{n}$.

Definition 2.9 ([12], Lagrange interpolation formula). Suppose $X_i = (x_{i1}, x_{i2}, \dots, x_{in})$ ($i = 1, 2, \dots, m$), then there exists a polynomial $f_i(x)$, such that $f_i(x_j) = x_{ij}$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$). If we define so-called Lagrange basic polynomial $l_p(x)$ as $l_p(x) = \prod_{p \neq q} \frac{x-q}{p-q}$, ($p = 1, 2, \dots, n$). Let $f_i(x) = \sum_p l_p(x)x_{ip}$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$), then we can find that $f_i(j) = x_{ij}$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$). $f_i(x)$ is called Lagrange interpolation polynomial.

3. The Chinese remainder theorem of polynomials ring-Lagrange interpolation formula

Lagrange interpolation formula is constructed usually from a Lagrange basic polynomial. In fact, Lagrange interpolation formula can be proved to be as an expression of the Chinese Remainder Theorem in the polynomial ring as the following.

Theorem 3.1. Suppose $n \geq 2$, and $x - x_1, x - x_2, \dots, x - x_n$ are n positive mutually prime polynomials, Let $X = (x - x_1)(x - x_2) \dots (x - x_n) = (x - x_1)X_1 = (x - x_2)X_2 = \dots = (x - x_n)X_n$, then for the following congruence equations group:

$$\begin{aligned} F(x) &\equiv F(x_1) \pmod{x - x_1}, \\ F(x) &\equiv F(x_2) \pmod{x - x_2}, \\ &\vdots \\ F(x) &\equiv F(x_n) \pmod{x - x_n}, \end{aligned}$$

the minimum degree polynomial solution is $F(x) = F(x_1)X'_1X_1 + F(x_2)X'_2X_2 + \dots + F(x_n)X'_nX_n = \sum_i F(x_i) \frac{\prod_{j,j \neq i}(x-x_j)}{\prod_{j,j \neq i}(x_i-x_j)}$, here polynomial solution X'_i satisfied $X'_iX_i \equiv 1 \pmod{x - x_i}$, ($i = 1, 2, \dots, n$) i.e., X'_i is the inverse element of X_i with respect to $x - x_i$.

Lemma 3.2. Let $X_i = \prod_{j,j \neq i}(x - x_j)$, then the corresponding inverse element X'_i of X_i is $X'_i = \frac{1}{\prod_{j,j \neq i}(x_i - x_j)}$ ($i = 1, 2, \dots, n$).

Proof. $X_iX'_i \pmod{x - x_i} = X_iX'_i|_{x=x_i} = \frac{\prod_{j,j \neq i}(x-x_j)}{\prod_{j,j \neq i}(x_i-x_j)}|_{x=x_i} = \frac{\prod_{j,j \neq i}(x_i-x_j)}{\prod_{j,j \neq i}(x_i-x_j)} = 1$
The theorem holds by the uniqueness of X'_i . $\square$

Remark 3.3. Lagrange interpolation formula can be proved to be as an expression of the Chinese Remainder Theorem in the polynomial ring. If the modules are set properly, some other important interpolation formula can also obtained as an expression of the Chinese Remainder Theorem in the polynomial ring.

4. The Chinese remainder theorem in distributive lattice

4.1 Congruence relation on ideals of distributive lattice

Definition 4.1 ([15]). A nonempty subset I of a lattice L is called an ideal of L if it satisfies

- (1) $x \in L, y \in I, x \leq y \Rightarrow y \in I$,
- (2) $x \in I, y \in I \Rightarrow x \vee y \in I$.

Theorem 4.2. Suppose I be an ideal of a distributive lattice L . A binary relation $\equiv$ is defined as for $\forall a, b \in L$, $a \equiv b(\text{mod } I)$ if for some $d \in I$, $a \vee d = b \vee d$, then the binary relation $\equiv$ is a congruence relation on L .

Proof. It is easy to see that binary relation $\equiv$ has reflexivity and symmetry.

Suppose for $a, b, c \in L$, $a \equiv b(\text{mod } I)$ and $b \equiv c(\text{mod } I)$, then there exist $d, e \in I$, such that $a \vee d = b \vee d$, $b \vee e = c \vee e$. So $a \vee (d \vee e) = b \vee d \vee e = b \vee e \vee d = c \vee (d \vee e)$, and $d \vee e \in I$, then $a \equiv c(\text{mod } I)$. So binary relation $\equiv$ is an equivalent relation.

Suppose $a \equiv b(\text{mod } I)$, then there exists some $d \in I$, such that $a \vee d = b \vee d$, then $\forall c \in L$, $(a \vee d) \vee c = (b \vee d) \vee c$, i.e., for some $d \in I$, $(a \vee c) \vee d = (b \vee c) \vee d$, then $a \vee c \equiv b \vee c(\text{mod } I)$.

In the same way, $a \equiv b(\text{mod } I)$, then there exists some $d \in I$, such that $a \vee d = b \vee d$, then $\forall c \in L$, $(a \wedge c) \vee d = (a \vee d) \wedge (c \vee d) = (b \vee d) \wedge (c \vee d) = (b \wedge c) \vee d$, then $a \wedge c \equiv b \wedge c(\text{mod } I)$.

Suppose $a \equiv b(\text{mod } I)$ and $c \equiv d(\text{mod } I)$, then $a \vee c \equiv b \vee c(\text{mod } I)$ and $b \vee c \equiv b \vee d(\text{mod } I)$, so $a \vee c \equiv b \vee d(\text{mod } I)$.

In the same way, we can get $a \wedge c \equiv b \wedge d(\text{mod } I)$. □

Remark 4.3. Suppose I be an ideal of a distributive lattice L . A congruence relation $\equiv$ can be induced by I . If we use $[x]_I$ to denote the equivalent class of $x \in L$, i.e., $L/I = \{[x]_I | x \in L\}$. If we define $[x]_I \vee [y]_I = [x \vee y]_I$, $[x]_I \wedge [y]_I = [x \wedge y]_I$, then $(L/I, \wedge, \vee)$ is a distributive lattice.

Theorem 4.4. Suppose I be an ideal of a lattice L . If $a \in I$, then $[a]_I = I$.

Proof. Suppose for $x \in [a]_I$, there exist $d \in I$, such that $x \vee d = a \vee d$, since $x \leq x \vee d = a \vee d \in I$. So $x \in I$. On the other hand, if $x \in I$, then $x \vee a \in I$, since $x \vee (x \vee a) = a \vee (x \vee a)$, then $x \equiv a(\text{mod } I)$, i.e., $x \in [a]_I$. □

Corollary 4.5. Suppose I be an ideal of a lattice L . If $a \in I$, then $\forall d \in L$, $a \equiv a \wedge d(\text{mod } I)$.

Proof. $\forall d \in L$ and $a \in I$, $a \vee a = (a \wedge d) \vee a$, so $a \equiv a \wedge d \pmod{I}$. $\square$

Definition 4.6. Suppose I_1, I_2 be two ideals of a lattice L . An ideal I generated by $I_1 \cup I_2$ is called the sum of ideals I_1 and I_2 , denoted by $I = I_1 + I_2$.

Lemma 4.7. Suppose I_1, I_2 be two ideals of a lattice L . Then $I_1 + I_2 = \{x \mid \text{for some } a \in I_1, b \in I_2, x \leq a \vee b\}$.

Proof. Suppose $I = \{x \mid \text{for some } a \in I_1, b \in I_2, x \leq a \vee b\}$, then any ideal J containing I_1, I_2 must contain I : if $x \in I$, then for some $a \in I_1, b \in I_2, x \leq a \vee b$, $x \leq a \vee b \in J$, we get $x \in J$. i.e., $I \subseteq J$. And we have $I_1 + I_2 \supseteq I$.

Next we prove that I is an ideal. Suppose $x \leq y$ and $y \in I$, then for some $a \in I_1, b \in I_2, x \leq y \leq a \vee b$, so $x \in I$. If $x, y \in I$, then for some $a, a_1 \in I_1, b, b_1 \in I_2$, we have $x \leq a \vee b, y \leq a_1 \vee b_1$, so $x \vee y \leq (a \vee b) \vee (a_1 \vee b_1) = (a \vee a_1) \vee (b \vee b_1)$, and $a \vee a_1 \in I_1, b \vee b_1 \in I_2$, we have $x \vee y \in I$.

It is easy to see that $I_1 \subseteq I, I_2 \subseteq I$, so $I_1 + I_2 \subseteq I$. $\square$

Corollary 4.8. Suppose I_1, I_2 be two ideals of a distributive lattice L . Then $I_1 + I_2 = \{a \vee b \mid \text{for some } a \in I_1, b \in I_2\}$.

Proof. Suppose $x \in I_1 + I_2$, then for some $a \in I_1, b \in I_2$, we have $x \leq a \vee b$. And $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$, since $x \wedge a \leq a \in I_1, x \wedge b \leq b \in I_2$, we get $x \wedge a \in I_1, x \wedge b \in I_2$. i.e., $I_1 + I_2 \subseteq \{a \vee b \mid \text{for some } a \in I_1, b \in I_2\}$. And we have $I_1 + I_2 \supseteq \{a \vee b \mid \text{for some } a \in I_1, b \in I_2\}$. $\square$

4.2 The Chinese Remainder Theorem in distributive lattice

Based on the above results, we can construct the Chinese Remainder Theorem in distributive lattice.

Theorem 4.9. Suppose $I_1, I_2, \dots, I_n$ be ideals of a distributive lattice L , such that $L = I_k + \bigcap_{i \neq k} I_i, k = 1, \dots, n$. If $b_1, b_2, \dots, b_n \in L$, then there exist $b \in L$, such that $b \equiv b_i \pmod{I_i}, i = 1, 2, \dots, n$. And b is uniquely determined with respect to module ideal $I_1 \cap I_2 \cap \dots \cap I_n$.

Proof. For every $k, b_k \in L = I_k + \bigcap_{i \neq k} I_i, k = 1, \dots, n$. When $a_k \in I_k, r_k \in \bigcap_{i \neq k} I_i, b_k = a_k \vee r_k$. On the other hand, $a_k \in I_k$, then $b_k \vee a_k = a_k \vee r_k \vee a_k = r_k \vee a_k$, so $b_k \equiv r_k \pmod{I_k}, k = 1, \dots, n$. $r_k \in \bigcap_{i \neq k} I_i$, then for all $d \in L, r_k \equiv r_k \wedge d \pmod{I_i}, (i \neq k)$. Let $d = r_i$, then $r_k \equiv r_k \wedge r_i \pmod{I_i}, (i \neq k)$. Let $b = r_1 \vee r_2 \vee \dots \vee r_k \vee \dots \vee r_n \equiv r_1 \vee r_2 \vee \dots \vee r_k \vee \dots \vee r_n \equiv (r_1 \wedge r_k) \vee (r_2 \wedge r_k) \vee \dots \vee r_k \vee \dots \vee (r_n \wedge r_k) \pmod{I_k} \equiv r_k \pmod{I_k}, (k = 1, 2, \dots, n)$. So we get $b \equiv b_k \pmod{I_k}, (k = 1, \dots, n)$.

Then we prove the uniqueness.

Suppose there exist $c \in L$, such that $c \equiv b_k \pmod{I_k}, (k = 1, \dots, n)$. Then $b \equiv c \pmod{I_k}, (k = 1, \dots, n)$. So there exists $d_k \in I_k$, such that $b \vee d_k = c \vee d_k, (k = 1, \dots, n)$. $(b \vee d_1) \wedge (b \vee d_2) \wedge \dots \wedge (b \vee d_n) = (c \vee d_1) \wedge (c \vee d_2) \wedge \dots \wedge (c \vee d_n)$, i.e., $b \vee (d_1 \wedge d_2 \wedge \dots \wedge d_n) = c \vee (d_1 \wedge d_2 \wedge \dots \wedge d_n)$, then we get $b \equiv c \pmod{\bigcap_i I_i}$. $\square$

4.3 A new communication scheme based on the Chinese remainder theorem in distributive lattice

Based on the Chinese Remainder Theorem in distributive lattice, we can further propose a secure communication scheme as follows.

Theorem 4.10. Suppose $L = \{ \text{the polynomial space on } GF(2) \}$. A binary relation $\leq$ is defined as for $x \leq y, (x, y \in L) \Leftrightarrow y|x$, then $(L, \leq)$ is a partial ordered set.

Proof. $\forall f(x), g(x), h(x) \in L$,

- (1) $f(x)|f(x) \Leftrightarrow f(x) \leq f(x)$;
- (2) $f(x)|g(x), g(x)|f(x) \Rightarrow f(x) = g(x) \Leftrightarrow f(x) \leq g(x), g(x) \leq f(x) \Rightarrow f(x) = g(x)$;
- (3) $f(x)|g(x), g(x)|h(x) \Rightarrow f(x)|h(x) \Leftrightarrow g(x) \leq f(x), h(x) \leq g(x) \Rightarrow h(x) \leq f(x)$.

$(L, \leq)$ is a partial ordered set. $\square$

Theorem 4.11. Suppose $L = \{ \text{the polynomial space on } GF(2) \}$. For $\forall f(x), g(x) \in L$, $f(x) + g(x)$ and $f(x)g(x)$ exist, i.e. L is a commutative ring. The binary operations $\vee, \wedge$ are defined as

$$(1) f(x) \vee g(x) = g.c.d.(f(x), g(x));$$

$$(2) f(x) \wedge g(x) = l.c.m.(f(x), g(x)),$$

then $f(x) \vee g(x)$ and $f(x) \wedge g(x)$ exist, $(L, \vee, \wedge)$ is a distributive lattice.

Lemma 4.12. Suppose I be an ideal of a lattice L . Then $I = \{m(x)p(x) | m(x) \in L\}$, here $p(x)$ is the irreducible polynomial on L .

Proof. Suppose $f(x) \in L, g(x)|f(x)$ and $g(x) \in I$, then $g(x) = m_1(x)p(x), f(x) = h(x)g(x) \Rightarrow f(x) = h(x)m_1(x)p(x) \Rightarrow f(x) \in I$, i.e., $f(x) \in L, f(x) \leq g(x)$ and $g(x) \in I$, then we get $f(x) \in I$. On the other hand, $\forall f(x), g(x) \in I, f(x) \vee g(x) \in I$. $\square$

Theorem 4.13. The unique solution determined by the module

$$\bigcap_i I_i = \{m(x) \bigcap_i p_i(x), m(x) \in L\}$$

is $F(x) = \sum_j f_j(x) \prod_{i \neq j} p_i(x)$.

Proof. Let $p_i(x) \in I_i$, since $(f_j(x) \prod_{i \neq j} p_i(x)) \vee p_i(x) = 0 \vee p_i(x) = p_i(x)$, we get $f_j(x) \prod_{i \neq j} p_i(x) \equiv 0 \pmod{I_k}$, and from $(\prod_{i \neq j} p_i(x), p_k(x)) = 1$, we have $u(x)(\prod_{i \neq j} p_i(x)) + v(x)p_k(x) = 1$, then $u(x)(f_k(x) \prod_{i \neq j} p_i(x)) + v(x)f_k(x)p_k(x) = f_k(x)$, so $(f_k(x) \prod_{i \neq j} p_i(x), p_k(x)) = f_k(x)$, by $(f_k(x) \prod_{i \neq j} p_i(x)) \vee p_k(x) = f_k(x) \vee p_k(x)$, we get $(f_k(x) \prod_{i \neq j} p_i(x)) \pmod{I_k} = f_k(x)$, for any $k, (k = 1, 2, \dots, n)$, $F(x) \pmod{I_k} = \sum_j f_j(x) \prod_{i \neq j} p_i(x) \pmod{I_k} = (f_1 p_2 p_3 \dots p_n + p_1 f_2 p_3 \dots p_n + p_1 p_2 p_3 \dots f_n) \pmod{I_k} = f_k(x)$, then $F(x)$ is the unique solution. $\square$

Based on the above analysis, a secure communication scheme can be designed as follows: after the sender and the receiver of the communication chose n modules on L : $I_1, I_2, \dots, I_n$, the sender can send solution of the congruence equations group $F(x)$ directly through the channel, the receiver can obtain $f_i(x) (i = 1, 2, \dots, n)$ by $F(x) \bmod I_i$, then the receiver can get original information flow safely and effectively, so as to achieve the requirements of the secure communication.

System only need to transfer $F(x)$ secretly, transfer volume decreases. Even if $F(x)$ is obtained by an intruder, since he couldn't know the module and order, it is difficult to use $F(x)$ to get the original sequence; When the receiver needs to restore sequence, he only needs to perform modular operations, which is simpler and faster.

Remark 4.14. The Chinese Remainder Theorem can be constructed in distributive lattice, not in lattice, because the congruence relation is defined just based on the properties of distributive lattice.

5. Identity based dynamic secure communication scheme

There has been a growing interest in the use of chaotic techniques for enabling secure communication in recent years. The use of Chaotic techniques can enhance communication security, while it is inferior in low bit error rates (BER) performance as compared to conventional communication schemes.

In this section, by constructing the Chinese remainder theorem with respect to ID_j combining RSA public-key cryptosystem, a new kind of secure communication is obtained with the same secure level and less complexity compared with the techniques mentioned above.

Assume that there is n users $u_i, (i = 1, 2, \dots, n)$ in the system, ID_i is the identifier of u_i (such that $i \neq j, (ID_i, ID_j) = 1$). System Center(SC) chooses two different large prime numbers p and q , and calculate $n = pq$ (public), $\varphi(n) = (p - 1)(q - 1)$ (confidential by SC). The system selects encryption key e_i (public) and decryption key d_i (confidential by u_i) for each user u_i , such that $e_i d_i \equiv 1 \pmod{\varphi(n)}$, and $1 < d_i < ID_i$.

For congruence equation:

$$x \equiv d_i \pmod{ID_i}$$

SC get the solution x_0 (public) based on the Chinese Remainder Theorem. SC chooses Two-way function f (public).

Communication process: Suppose there is a user u_i who will want to communicate with u_j , then u_i get the decryption key d_j (confidential by u_j) by computing $d_j = x_0 \bmod ID_j$, and compute $f^{d_j}(m, T)$, T is the sending time, m is the message. When u_j received $f^{d_j}(m, T)$, he computes $(f^{d_j}(m, T))^{e_j} \bmod n$, then gets T and m . If T meets the requirements, accepts m , otherwise refuses m .

Remark 5.1. The improvement of the secure communication system over the RSA is that on one hand by combining RSA public-key cryptosystem and the Chinese remainder theorem the scheme just stores public x_0 instead of public key list, store volume decreases; on the other hand the scheme keeps the same level security of RSA, while avoids complicated power operations.

6. Conclusion

Based on the Chinese remainder theorem of polynomials, Lagrange interpolation formula is found to be an extension of the Chinese remainder theorem in the polynomial ring. And two new communication schemes based on the Chinese Remainder Theorem in distributive lattice and combined with RSA are proposed.

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Ideal theory on bounded semihoops

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Abstract. In this paper, we investigate some types of ideals on bounded semihoops. First, we introduce the notion of ideals on bounded semihoops and give some characterizations of ideals on bounded semihoops. Furthermore, we study primary ideals, prime ideals and maximal ideals. We discuss some relations between these ideals. In particular, every primary ideal is a prime ideal on bounded semihoops with $x^2 = x$. Also, we introduce the concept of perfect ideals and find that every perfect ideal is a primary ideal. In addition, we prove that I is a perfect ideal if and only if A/I is a perfect semihoop. Moreover, we introduce the notion of local semihoops. We characterize some equivalent conditions of local semihoops. In fact, A/I is local if and only if I is a primary ideal. Finally, we define the concept of locally finite, and show that I is a maximal ideal if and only if A/I is locally finite.

Keywords: semihoop, ideal, local semihoop.

1. Introduction

Various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, MV-algebras[3], BL-algebras [12], MTL-algebras [5], residuated lattices [10]. In fact, semihoops [6] are the fundamental residuated structures and contain all logic algebras based on residuated lattices. Semihoops are generalization of hoops. The algebraic hoop were investigated by Buchi and Owens in an unpublished manuscript in 1975. They have been studied by Blok and Ferreirim [15], Aglianò [11]et.al. In particular, from the structure theorem of finite basic hoops [11], one obtains an elegant short proof of

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the completeness theorem for propositional basic logic [11], introduced by Hájek [12]. It is proved that a hoop is a meet-semilattice ordered residuated, commutative monoid, integral, divisible. Juntao Wang [8] investigated monadic bounded hoops and proved the completeness of the monadic hoop logic. A. Namdar et.al [1] introduced some results in hoop algebras. R.T. Khorami et. al [14] introduced the notions of multiplier, $\odot$ -closure modal operator on hoop-algebras. As a more general structure, a semihoop is a hoop without the divisibility. It follows that a semihoop does not satisfy the divisibility condition. Therefore, semihoops play an important role in fuzzy logics and the related algebraic structures.

From a logic point of view, various filters corresponding to various sets of provable formulae. Ideal theory is a very effective tool for investigating these various algebraic and logic systems. The notion of ideal has been introduced in many algebraic structure such as lattices, rings, MV-algebras and so on. For residuated lattices, Yi Liu and Ya Qin [17] introduced the notion of ideals in residuated lattices. Also, Lele and Nganou [4] introduced the notion of ideals in BL-algebras as a natural generalization of that of ideals in MV-algebras. Akbar Paad [2] introduced the notion of integral ideals and maximal ideals in BL-algebras. Furthermore, Wenjuan Chen [16] mainly investigated ideals and congruences in a quasi-pseudo-MV algebras. Forouzesh [7] presented fuzzy semi-maximal and fuzzy radical ideals in MV-algebras. However, the study of semihoops main focus on filters, the notion of ideal is missing.

In this paper, we will extend ideals from MV-algebras to semihoops for obtaining the most general results on logic algebraic ideals. For semihoops, there is no addition operation, so we introduce the definition of addition operation to define ideals on bound semihoops. One of our aims is to show the relationships among different ideals on bound semihoops, such as prime ideal, primary ideal and perfect ideal. On the other hand, we portray perfect semihoops, local semihoops and locally finite semihoops using perfect ideals, primary ideal and maximal ideals respectively. Above arguments motivate us to investigate ideals on bounded semihoops.

The papers organized as follows: In section 2, we give some basic results on bounded semihoops, which are needed in the rest papers. In section 3, we discuss characterizations of ideals and the relations between ideals and filters on a bounded semihoop. Also, we prove that every primary ideal with the condition of $x^2 = x$ is a prime ideal. In section 4, we study that every perfect ideal is a primary ideal and I is a perfect ideal if and only if A/I is a perfect semihoop. We prove that A is local if and only if it has unique maximal ideal, and A/I is local if and only if I is a primary ideal. Further, we obtain that $I \in \mathcal{M}(A)$ if and only if A/I is locally finite.

2. Preliminaries

In this section, we summarize some definitions and results on semihoops which will be used in the following sections.

Definition 2.1 ([6]). *An algebra $(A, \odot, \rightarrow, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a semihoop if it satisfies the following conditions:*

- (1) $(A, \wedge, 1)$ is a $\wedge$ -semilattice with upper bound 1,
- (2) $(A, \odot, 1)$ is a commutative monoid,
- (3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$, for all $x, y, z \in A$.

On a semihoop $(A, \odot, \rightarrow, \wedge, 1)$, we define $x \leq y$ if and only if $x \rightarrow y = 1$ for any $x, y \in A$. It is easy to check that $\leq$ is a partial order relation on A and for all $x \in A, x \leq 1$.

For any $x \in A$, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any natural number n .

Proposition 2.2 ([6]). *In any semihoop $(A, \odot, \rightarrow, \wedge, 1)$, the following properties hold for all $x, y, z \in A$:*

- (1) $x \odot y \leq z$ iff $x \leq y \rightarrow z$,
- (2) $x \odot y \leq x, y$,
- (3) $1 \rightarrow x = x, x \rightarrow 1 = 1$,
- (4) $x^n \leq x$,
- (5) $y \leq x \rightarrow y$,
- (6) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y$ and $x \odot z \leq y \odot z$,
- (7) $x \leq (x \rightarrow y) \rightarrow y$,
- (8) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

A semihoop $(A, \odot, \rightarrow, \wedge, 1)$ is called a bounded semihoop if there exists an element $0 \in A$ such that $0 \leq x$ for any $x \in A$.

In a bounded semihoop $(A, \odot, \rightarrow, \wedge, 0, 1)$, we define $*$: $x^* = x \rightarrow 0$ for any $x \in A$. If $x^{**} = x$, for any $x \in A$, then the bounded semihoop $(A, \odot, \rightarrow, \wedge, 0, 1)$ is said to have a double negation property (DNP, for short).

We denote a bounded semihoop $(A, \odot, \rightarrow, \wedge, 0, 1)$ by A .

Proposition 2.3 ([13]). *In a bounded semihoop A , the following properties hold for all $x, y \in A$:*

- (1) $1^* = 0, 0^* = 1$,
- (2) $x \leq x^{**}$,
- (3) $x \odot x^* = 0$,
- (4) $y^* \leq y \rightarrow x$,
- (5) $x \leq y$ implies that $y^* \leq x^*$,
- (6) if A has (DNP), then $x \rightarrow y = y^* \rightarrow x^*$.

Proposition 2.4. *In a bounded semihoop A , the following properties hold for all $x, y \in A$:*

- (1) $x \rightarrow y \leq y^* \rightarrow x^*$,
- (2) if A has (DNP), then $x^* \rightarrow y = y^* \rightarrow x$.

Theorem 2.5 ([9]). *Let A be a hoop and X be a nonempty subset of A . Then $(X) = \{a \in A : \exists n \in \mathbb{N} : a \leq x_1 \ominus (x_2 \ominus \cdots \ominus (x_{n-1} \ominus x_n) \cdots)\}$, for $x_1, x_2, \dots, x_n \in X$.*

Proposition 2.6 ([9]). *Let A be a $\vee$ -hoop with (DNP) and let P be a proper ideal of A . Then P is a prime ideal if and only if, for any $I, J \in \mathcal{ID}(A)$ such that $I \cap J \subseteq P$, we get $I \subseteq P$ or $J \subseteq P$.*

Proposition 2.7 ([13]). *Let A be a semihoop and for all x, y , we define $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. Then the following conditions are equivalent:*

- (1) $\vee$ is an associative on A ;
- (2) $x \leq y$ implies $\vee \sqcup z \leq y \vee z$, for all $x, y, z \in A$;
- (3) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$, for all $x, y, z \in A$;
- (4) $\vee$ is the join operation on A .

Definition 2.8 ([13]). *A $\vee$ -semihoop is a semihoop if it satisfies one of the equivalent conditions of Proposition 2.7.*

Remark: A semihoop with (DNP) may not be a hoop.

Example 2.9. Let $A = \{0, a, b, c, 1\}$ be a chain with $0 < a < b < c < 1$. Defined $\odot$ and $\rightarrow$ on A as follows,

$\odot$	0	a	b	c	1	$\rightarrow$	0	a	b	c	1
0	0	0	0	0	0	0	1	1	1	1	1
a	0	0	0	0	a	a	c	1	1	1	1
b	0	0	0	a	b	b	b	c	1	1	1
c	0	0	a	a	c	c	a	c	c	1	1
1	0	a	b	c	1	1	0	a	b	c	1

We can check A is a bounded semihoop with (DNP). But it is not a hoop, because $b \odot (b \rightarrow c) = b \neq c \odot (c \rightarrow b) = a$.

3. Ideals on bounded semihoops

In this section, we introduce the notion of ideals and some important ideals such as primary ideals, prime ideals and maximal ideals on bounded semihoops. We discuss the relations between them on bounded semihoops. We will use these ideals in the next sections.

Definition 3.1. *In a bounded semihoop A , the binary operation $\oplus$ is defined by $x \oplus y = x^* \rightarrow y$ for any $x, y \in A$.*

Example 3.2. Let $A = \{0, a, b, 1\}$ be a chain with $0 < a < b < 1$. Defined $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	a	a	a	a	0	1	1	1
b	0	a	a	b	b	a	b	1	1
1	0	a	b	1	1	0	a	b	1

Then A is a bounded semihoop. Since $a \oplus 0 = 1$ and $0 \oplus a = a$, we get the operation $\oplus$ is not commutative in general.

In a bounded semihoop A with (DNP), the operation $\oplus$ is commutative. Since $x \oplus y = x^* \rightarrow y = y^* \rightarrow x = y \oplus x$, for any $x, y \in A$.

In a bounded semihoop A with (DNP), the operation $\oplus$ is associative. Since $(x \oplus y) \oplus z = (x^* \rightarrow y)^* \rightarrow z = (x^* \rightarrow y^{**})^* \rightarrow z = ((x^* \odot y^*) \rightarrow 0)^* \rightarrow z = (x^* \odot y^*)^{**} \rightarrow z = x^* \rightarrow (y^* \rightarrow z) = x \oplus (y \oplus z)$.

Proposition 3.3. *In a bounded semihoop A , the following properties hold for any $x, y, z \in A$:*

- (1) if $x \leq y$, then $x \oplus z \leq y \oplus z$,
- (2) $x \leq x \oplus y$,
- (3) $x \oplus x^* = 1$,
- (4) $0 \oplus x = x, x \oplus 0 = x^{**}$,
- (5) $x \oplus y = 1$ if and only if $x^* \leq y$,
- (6) $(x \oplus y) \oplus z \leq x \oplus (y \oplus z)$,
- (7) if A has (DNP), then $x^* \odot y^* = (x \oplus y)^*$,
- (8) if A has (DNP), then $x^* \oplus y^* = (x \odot y)^*$.

Proof. The proof of (1),(2),(3),(4),(5) are clear.

(6) $(x \oplus y) \oplus z \leq x \oplus (y \oplus z) \Leftrightarrow (x^* \rightarrow y)^* \rightarrow z \leq x^* \rightarrow (y^* \rightarrow z) \Leftrightarrow x^* \odot ((x^* \rightarrow y)^* \rightarrow z) \leq y^* \rightarrow z \Leftrightarrow x^* \leq ((x^* \rightarrow y)^* \rightarrow z) \rightarrow (y^* \rightarrow z) \Leftrightarrow x^* \rightarrow (((x^* \rightarrow y)^* \rightarrow z) \rightarrow (y^* \rightarrow z)) = 1$. Since $x^* \rightarrow (((x^* \rightarrow y)^* \rightarrow z) \rightarrow (y^* \rightarrow z)) \geq x^* \rightarrow (y^* \rightarrow (x^* \rightarrow y)^*) \geq x^* \rightarrow ((x^* \rightarrow y) \rightarrow y) = 1$, we have $x^* \rightarrow (((x^* \rightarrow y)^* \rightarrow z) \rightarrow (y^* \rightarrow z)) = 1$.

(7) Since $(x^* \odot y^*) \rightarrow (x \oplus y)^* = x^* \rightarrow (y^* \rightarrow (x \oplus y)^*) \geq x^* \rightarrow ((x \oplus y) \rightarrow y) = x^* \rightarrow ((x^* \rightarrow y) \rightarrow y) \geq x^* \rightarrow x^* = 1$, we have $(x^* \odot y^*) \rightarrow (x \oplus y)^* = 1$, and so $x^* \odot y^* \leq (x \oplus y)^*$.

On the other hand, $(x \oplus y)^* \leq x^* \odot y^* \Leftrightarrow (x^* \odot y^*)^* \leq (x^* \rightarrow y)^{**} \Leftrightarrow x^* \rightarrow y^{**} \leq x^* \rightarrow y \Leftrightarrow x^* \rightarrow y \leq x^* \rightarrow y$.

Therefore, $x^* \odot y^* = (x \oplus y)^*$.

(8) Since $(x^* \oplus y^*) \rightarrow (x \odot y)^* = (x^{**} \rightarrow y^*) \rightarrow (x \rightarrow y^*) \geq x \rightarrow x^{**} = 1$, we have $(x^* \oplus y^*) \rightarrow (x \odot y)^* = 1$, and so $x^* \oplus y^* \leq (x \odot y)^*$.

On the other hand, $(x \odot y)^* \leq x^* \oplus y^* \Leftrightarrow x \rightarrow y^* \leq x^{**} \rightarrow y^* \Leftrightarrow x \rightarrow y^* \leq x \rightarrow y^*$.

Therefore, $x^* \oplus y^* = (x \odot y)^*$. □

Definition 3.4. *Let A be a bounded semihoop and I be a non-empty subset of A . Then I is said to be an ideal of A , if it satisfies:*

- (I1) $x \leq y$ and $y \in I$ imply $x \in I$, for any $x, y \in A$,
- (I2) $x \oplus y \in I$, for any $x, y \in I$.

We can see that $\{0\}$ and A are ideals of A .

We denote the set of all ideals of A by $\mathcal{I}(A)$.

Definition 3.5. Let A be a bounded semihoop. An ideal I of A is called a proper ideal if $1 \notin I$.

Example 3.6. Let $A = \{0, a, b, c, d, 1\}$ with $0 < b, d < a < 1, 0 < d < c < 1$, where a and c are incomparable, b and d are incomparable. Defined $\odot$ and $\rightarrow$ on A as follows,

$\odot$	0	a	b	c	d	1	$\rightarrow$	0	a	b	c	d	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1
a	0	b	b	d	0	a	a	d	1	a	c	c	1
b	0	b	b	0	0	b	b	c	1	1	c	c	1
c	0	d	0	c	d	c	c	b	a	b	1	a	1
d	0	0	0	d	0	d	d	a	1	a	1	1	1
1	0	a	b	c	d	1	1	0	a	b	c	d	1

We can check A is a bounded semihoop, where $x \wedge y = x \odot (x \rightarrow y)$, for any $x, y \in A$ ([13]). It is routine to verify that $I_1 = \{0, b\}$ and $I_2 = \{0, c, d\}$ are ideals of A .

Theorem 3.7. Let A be a bounded semihoop and I be a non-empty subset of A . Then the following conditions are equivalent:

- (1) I is an ideal of A ;
- (2) $0 \in I$, and $x \oplus y \in I, x^* \odot y \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in A$;
- (3) $0 \in I$, and $x \oplus y \in I, (x^* \rightarrow y^*)^* \in I$ and $x \in I$ imply $y \in I$, for any $x, y \in A$;
- (4) $x \oplus y \in I$, for any $x, y \in I$, and $x \in I$ implies $x \wedge y \in I$, for any $x, y \in A$.

Proof. (1) $\Rightarrow$ (2) Suppose that I is an ideal of A . It follows from (I1) that $0 \in I$. Let $x, y \in A$ such that $x^* \odot y \in I$ and $x \in I$. Then $y \rightarrow (x \oplus (x^* \odot y)) = y \rightarrow (x^* \rightarrow (x^* \odot y)) = (y \odot x^*) \rightarrow (y \odot x^*) = 1$, and so $y \leq x \oplus (x^* \odot y)$. Since $x^* \odot y \in I$ and $x \in I$, it follows from (I2) that $x \oplus (x^* \odot y) \in I$. By (I1), we have $y \in I$. Therefore (2) holds.

(2) $\Rightarrow$ (3) Let $(x^* \rightarrow y^*)^* \in I$ and $x \in I$. Since $x^* \odot y^{**} \leq (x^* \odot y^{**})^{**} = (x^* \rightarrow y^{***})^* = (x^* \rightarrow y^*)^* \in I$ and $(x^* \rightarrow y^*)^* \in I$, we have $x^* \odot y^{**} \in I$. It follows from (2), we get $y^{**} \in I$. Since $y \leq y^{**}$, we have $y \in I$.

(3) $\Rightarrow$ (1) Suppose that $x, y \in A, x \leq y$ and $y \in I$, for any $x, y \in A$. Then $y^* \leq x^*$, and so $(y^* \rightarrow x^*)^* = 0 \in I$. Since $y \in I$, and it follows from (3), we have $x \in I$. Thus, I is an ideal of A .

(1) $\Rightarrow$ (4) Suppose that I is an ideal of A . It is clear that I satisfies (4).

(4) $\Rightarrow$ (1) Suppose that I satisfies (4). Let $x \in I, y \in L$ such that $y \leq x$, then $0 = x \wedge 0 \in I$ and $y = x \wedge y \in I$. Therefore I is an ideal of A . $\square$

Let A be a bounded semihoop and X be a non-empty subset of A . The ideal generated by X means the smallest ideal containing X , which is denoted by $\langle X \rangle$.

Corollary 3.8. Let A be a bounded semihoop and I be an ideal of A . Then $x \in I$ if and only if $x^{**} \in I$.

Proof. Suppose that $x \in I$. By Theorem 3.7, taking $y = x^{**}$ in (3), then $(x^* \rightarrow x^{***})^* = 0 \in I$. Thus $x^{**} \in I$. Conversely, suppose that $x^{**} \in I$. Since $x \leq x^{**}$ and I is an ideal, we have $x \in I$. $\square$

Theorem 3.9. *Let A be a bounded semihoop and X be a non-empty subset of A . Then $\langle X \rangle = \{a \in A \mid a \leq x_n \oplus (\cdots \oplus (x_3 \oplus (x_2 \oplus x_1)) \cdots), x_i \in X, i = 1, 2 \cdots n\}$.*

Proof. The proof is similar to Theorem 2.5. $\square$

Proposition 3.10. *Let A be a bounded semihoop and $x \in A$. Then $\langle I \cup \{x\} \rangle = \{a \in A \mid a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \cup \{a \in A \mid a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$.*

Proof. Let $B = \{a \in A \mid a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \cup \{a \in A \mid a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$. It is clear that $\langle I \cup \{x\} \rangle \subseteq B$ and $0 \in B$.

Suppose that $b, c \in A$ such that $b \leq c$ and $c \in B$. Then there exist $y_i \in I (i = 1, 2 \cdots n)$ and $y \in I$ such that $c \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))$ or $c \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$. Since $b \leq c$, we have $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$. Thus $b \in B$.

If for any $b, c \in B$, then there exist $y_i \in I, z_j \in I (i = 1, 2 \cdots n; j = 1, 2 \cdots m)$ and $y \in I, z \in I$ such that $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$ and $c \leq z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx))$ or $c \leq x \oplus (x \oplus \cdots \oplus (x \oplus z))$.

Case 1: if $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))$ and $c \leq z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx))$, By Proposition 3.3(6), we have $b \oplus c$

$$\begin{aligned} &\leq [y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx))] \\ &\leq y_1 \oplus \{[y_2 \oplus \cdots \oplus (y_n \oplus nx)] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx))]\} \\ &\leq y_1 \oplus (y_2 \oplus \{[y_3 \oplus (\cdots \oplus (y_n \oplus nx))] \oplus [z_1 \oplus (z_2 \oplus \cdots \oplus (z_n \oplus nx))]\}) \\ &\leq \cdots \leq y_1 \oplus (y_2 \oplus (\cdots (y_n \oplus (nx \oplus (z_1 \oplus (z_2 \oplus (\cdots \oplus (z_n \oplus nx) \cdots)))))) \cdots)). \end{aligned}$$

Hence, $b \oplus c \in B$.

The proof of other cases is similar to case 1.

Therefore, B is an ideal. Next, we prove that B is the smallest ideal that containing $\langle I \cup \{x\} \rangle$. Let C be an ideal such that $\langle I \cup \{x\} \rangle \subseteq C$. Then for any $b \in B$, $b \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx))$ or $b \leq x \oplus (x \oplus \cdots \oplus (x \oplus y))$, $y_i \in I, y \in I$. Since C is an ideal and $\langle I \cup \{x\} \rangle \subseteq C$, we have $y_i \in C, y \in C$ and $y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)) \in C$, or $x \oplus (x \oplus \cdots \oplus (x \oplus y)) \in C$, and so $b \in C$. Hence $B \subseteq C$. Therefore $\langle I \cup \{x\} \rangle = \{a \in A \mid a \leq y_1 \oplus (y_2 \oplus \cdots \oplus (y_n \oplus nx)), y_i \in I, i = 1, 2 \cdots n\} \cup \{a \in A \mid a \leq x \oplus (x \oplus \cdots \oplus (x \oplus y)), y \in I\}$. $\square$

We know if A has (DNP), then the operation $\oplus$ is commutative and associative. So we have the following corollary.

Corollary 3.11. *Let A be a bounded semihoop with (DNP). Then $\langle I \cup \{x\} \rangle = \{a \in A \mid a \leq y \oplus nx, y \in I\}$.*

Let A be a bounded semihoop. We define a binary relation $\sim_I$ on A as follows:

$x \sim_I y$ if and only if $(x \rightarrow y)^* \in I$ and $(y \rightarrow x)^* \in I$.

Proposition 3.12. *Let A be a bounded semihoop and I be an ideal of A . Then $\sim_I$ is a congruence relation on a bounded semihoop A .*

Proof. It is easy to check that $\sim_I$ is an equivalence relation on A .

Suppose that $x \sim_I y$. Then $(x \rightarrow y)^* \in I$ and $(y \rightarrow x)^* \in I$. Since $(x \wedge z) \rightarrow (y \wedge z) \geq x \rightarrow y$ for any $z \in I$, we have $((x \wedge z) \rightarrow (y \wedge z))^* \leq (x \rightarrow y)^*$. Since I is an ideal, we have $((x \wedge z) \rightarrow (y \wedge z))^* \in I$. By the similar way, we obtain $((y \wedge z) \rightarrow (x \wedge z))^* \in I$. Therefore, $x \wedge z \sim_I y \wedge z$.

By the similar way, we obtain $x \odot z \sim_I y \odot z$ and $x \rightarrow z \sim_I y \rightarrow z$.

Therefore, $\sim_I$ is a congruence relation on A . $\square$

Let $A/I = \{[x] | x \in A\}$, where $[x] = \{y \in A | x \sim_I y\}$. Then the binary relation $\leq$ on A/I which is defined by, $[x] \leq [y]$ if and only if $(x \rightarrow y)^* \in I$, is an order relation on A/I . Now, $(A/I, \wedge_{A/I}, \otimes, \rightsquigarrow, 0_{A/I})$ is a bounded semihoop, where for any $x, y \in A$:

$$0_{A/I} = [0], [x] \wedge_{A/I} [y] = [x \wedge_A y], [x] \otimes [y] = [x \odot y], [x] \rightsquigarrow [y] = [x \rightarrow y].$$

Definition 3.13. *Let A be a bounded semihoop and P be a proper ideal of A . P is called a primary ideal, if for all $x, y \in A$, $x \odot y \in P$ implies $x^n \in P$ or $y^n \in P$, for some $n \in N$.*

Example 3.14. (1) Let $A = \{0, a, b, 1\}$ be a chain. Defined $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	0	a	a	b	1	1	1
b	0	0	a	b	b	a	b	1	1
1	0	a	b	1	1	0	a	b	1

We can check A is a bounded semihoop ([13]). It is routine to verify that $I = \{0\}$ is a primary ideal of A .

(2) In the example 3.6, $I_1 = \{0, b\}$ is a primary ideal.

Proposition 3.15. *Let A be a bounded semihoop and P be a proper ideal of A . Then P is a primary ideal if and only if $[x] \otimes [y] = [0]$ implies that there exists $n \in N$ such that $[x]^n = [0]$ or $[y]^n = [0]$, for any $[x], [y] \in A/P$.*

Proof. ($\Rightarrow$) Let P be a primary ideal of A and $[x] \otimes [y] = [0]$, for some $[x], [y] \in A/P$. Since $[x] \otimes [y] = [x \odot y] = [0]$, we have $((x \odot y) \rightarrow 0)^* \in P$ and so $(x \odot y)^{**} \in P$. By Corollary 3.8, $x \odot y \in P$. Since P is a primary ideal, there exists $n \in N$ such that $x^n \in P$ or $y^n \in P$. If $x^n \in P$, then $(x^n \rightarrow 0)^* = (x^n)^{**}$. By Corollary 3.8, $(x^n)^{**} \in P$. Since $(0 \rightarrow x^n)^* = 0 \in P$, by definition of quotient, $[x]^n = [0]$. By the similar way, we have $[y]^n = [0]$.

($\Leftarrow$) Let $x \odot y \in P$, for some $x, y \in A$. Then $((x \odot y) \rightarrow 0)^* = (x \odot y)^{**}$. By Corollary 3.8, we have $(x \odot y)^{**} \in P$. Since $0 \rightarrow (x \odot y)^* = 0 \in P$, we get $x \odot y \equiv_P 0$. Hence $[x] \otimes [y] = [x \odot y] = [0]$. So, there exists $n \in N$ such that $[x]^n = [0]$ or $[y]^n = [0]$. If $[x]^n = [0]$, then $(x^n \rightarrow 0)^* = (x^n)^{**} \in P$ and so $x^n \in P$. By the similar way, we can prove that $y^n \in P$. Therefore P is a primary ideal. $\square$

Definition 3.16. A proper ideal I of a bounded semihoop A is called a prime ideal of A , if $H \cap G \subseteq I$ implies $H \subseteq I$ or $G \subseteq I$, for any $H, G \in \mathcal{I}(A)$.

Proposition 3.17. Let A be a bounded $\vee$ -semihoop with (DNP) and I be a proper ideal of A . Then the following conditions are equivalent:

- (1) I is a prime ideal;
- (2) if $x \wedge y \in I$, for any $x, y \in A$, then $x \in I$ or $y \in I$.

Proof. The proof is similar to Theorem 2.6. $\square$

Proposition 3.18. Let A be a bounded semihoop and $x^2 = x$, for any $x \in A$. Then every primary ideal of A is a prime ideal.

Proof. Let P be a primary ideal of A and $x \wedge y \in P$, for any $x, y \in P$. Since $x \odot y \leq x \wedge y$, then $x \odot y \in P$. Since P is a primary ideal, there exists $n \in N$ such that $x^n \in P$ or $y^n \in P$. Since $x^2 = x$, if $x^n \in P$, we get $x^n = x \in P$. If $y^n \in P$, then by the similar way, we have $y \in P$. Therefore P is a prime ideal. $\square$

Open problem: Is there a prime ideal, which is not primary on a bounded semihoop?

Definition 3.19. Let A be a bounded semihoop and I be a proper ideal of A . I is called maximal ideal of A , if it is not properly contained in the any other proper ideal of A . We denote the set of all maximal ideal of bounded semihoops A by $\mathcal{M}(A)$.

Proposition 3.20. Let A be a bounded semihoop and I be a proper ideal of A . Then the following conditions are equivalent:

- (1) $I \in \mathcal{M}(A)$;
- (2) if $x \notin I$, then $\langle I \cup \{x\} \rangle = A$, for any $x \in A$.

Proof. (1) $\Rightarrow$ (2) If $x \notin I$, then $I \subsetneq \langle I \cup \{x\} \rangle$. Since $I \in \mathcal{M}(A)$, we have $\langle I \cup \{x\} \rangle = A$.

(2) $\Rightarrow$ (1) Let G be an ideal of A such that $I \subseteq G$ and $I \neq G$, then there exists $x \in G$ such that $x \notin I$. It follows from (2), $\langle I \cup \{x\} \rangle = A$. Since $A = \langle I \cup \{x\} \rangle \subseteq G$, then we get $A = G$, which is a contradiction. Therefore, $I \in \mathcal{M}(A)$. $\square$

Proposition 3.21. *Let A be a bounded semihoop and M be a proper ideal of A . If M is a maximal ideal of A , then for all $x \notin M$ if and only if there exists $n \in N$ such that $(nx)^* \in M$.*

Proof. ($\Rightarrow$) Suppose that M is a maximal ideal of A . If $x \notin M$, by Proposition 3.20, we get $\langle M \cup \{x\} \rangle = A$. Since $1 \in A$, we have $1 \in \langle M \cup \{x\} \rangle$. By Corollary 3.11, there exist $n \in N, y \in M$ such that $1 \leq y \oplus (nx)$. Since $1 \leq y^* \rightarrow (nx)$, we have $1 \odot y^* \leq nx$, and so $(nx)^* \leq y^{**}$. Since M is an ideal and $y \in M$, we have $y^{**} \in M$. Hence $(nx)^* \in M$.

($\Leftarrow$) Suppose that $x \in M$. Then we have $nx \in M$. Since $(nx)^* \in M$, we have $(nx) \oplus (nx)^* = 1 \in M$, which is contradiction. Therefore, $x \notin M$. $\square$

4. Perfect ideals of bounded semihoops

In this section, we introduce the notion of perfect ideals and discuss the relation of perfect ideals on perfect semihoops. In addition, we study the relation ideals on local semihoops and give some characterizations of local semihoops.

Definition 4.1. *The order of $x \in A$, in symbols $\text{ord}(x)$, is the smallest $n \in N$ such that $nx = 1$. If no such n exists, then $\text{ord}(x) = \infty$.*

Proposition 4.2. *The following conditions hold, for any $x, y \in A$:*

- (1) $\langle x \rangle$ is a proper ideal of A if and only if $\text{ord}(x) = \infty$,
- (2) if $x \leq y$ and $\text{ord}(y) = \infty$, then $\text{ord}(x) = \infty$,
- (3) if $x \leq y$ and $\text{ord}(x) < \infty$, then $\text{ord}(y) < \infty$,
- (4) for every $M \in \mathcal{M}(A)$, $M \subseteq \{x \in A \mid \text{ord}(x) = \infty\}$,
- (5) $\text{ord}(1) < \infty$ and $\text{ord}(0) = \infty$.

Proof. (1) Let $\langle x \rangle$ be a proper ideal of A . If there exists $n \in N$ such that $nx = 1$, then $1 \in \langle x \rangle$, which is a contradiction. Conversely, let $\langle x \rangle = A$. Since $1 \in A$, we have $1 \in \langle x \rangle$ and so there exists $n \in N$ such that $nx = 1$. Thus $\text{ord}(x) < \infty$, which is a contradiction.

(2) Let $x \leq y$, for some $x, y \in A$. Then $nx \leq ny$, for all $n \in N$. Since $\text{ord}(y) = \infty$, we have $ny \neq 1$, for all $n \in N$. Therefore $\text{ord}(x) = \infty$.

(3) Let $x \leq y$ and $\text{ord}(x) < \infty$, for some $x, y \in A$. Then there exists $n \in N$ such that $nx = 1$ and so $ny = 1$. Hence $\text{ord}(y) < \infty$.

(4) Let $x \in M$ and $\text{ord}(x) < \infty$. Then exists $n \in N$ such that $nx = 1$. Thus $1 \in M$, which is a contradiction.

(5) The proof is clear. $\square$

Definition 4.3. *A is called a perfect semihoop, if $\text{ord}(x) < \infty$, then $\text{ord}(x^*) = \infty$, and if $\text{ord}(x) = \infty$, then $\text{ord}(x^*) < \infty$, for any $x \in A$.*

Definition 4.4. *Let A be a bounded semihoop and P be a proper ideal of A . P is called perfect ideal of A , if P satisfies the condition, for any $x \in A, x^n \in P$, for some $n \in N$ if and only if $(x^*)^m \notin P$ for all $m \in N$.*

Example 4.5. Let $A = \{0, a, b, c, 1\}$ be a chain with $0 < a < b < c < 1$. Define $\odot$ and $\rightarrow$ on A as follows,

$\odot$	0	a	b	c	1	$\rightarrow$	0	a	b	c	1
0	0	0	0	0	0	0	1	1	1	1	1
a	0	a	0	a	a	a	b	1	1	1	1
b	0	0	b	b	b	b	a	a	1	1	1
c	0	a	b	c	c	c	0	a	b	1	1
1	0	a	b	c	1	1	0	a	b	c	1

We can check A is a bounded semihoop. It is routine to verify that $I = \{0, a\}$ is a perfect ideal.

Proposition 4.6. *Let A be a bounded semihoop. Then every perfect ideal of A is a primary ideal.*

Proof. Let A be a bounded semihoop and P be a perfect ideal of A . Suppose that $x \odot y \in P$, for some $x, y \in A$ such that $x^m \notin P$, for all $m \in N$. Since $x \odot y \in P$ and P is an ideal, we have $(x \odot y)^{**} \in P$. Since $(x \odot y)^{**} = (y \odot x)^{**} = (y \rightarrow x^*)^* \in P$, by definition of quotient, $y/P \leq x^*/P$. Thus $y^n/P \leq (x^*)^n/P$, for all $n \in N$. Since $x^m \notin P$ and P is a perfect ideal, we have for some $n \in N$ such that $(x^*)^n \in P$ and so $(x^*)^n/P = 0/P$. Thus $y^n/P = 0/P$. Hence $y^n \in P$. Therefore, P is a primary ideal. $\square$

The following example shows that the converse of the proposition 4.6 may not be true.

Example 4.7. In the example 3.14(1), $I = \{0\}$ is a primary ideal, but it is not a perfect ideal, because $a^2 = 0 \in I$, but $(a^*)^3 = 0 \in I$.

Theorem 4.8. *Let A be a bounded semihoop with (DNP) and I be a proper ideal of A . Then I is a perfect ideal if and only if A/I is a perfect semihoop.*

Proof. Let I be a proper ideal. Suppose that $x/I \in A/I$ such that $\text{ord}(x/I) < \infty$. Then there exists $n \in N$ such that $n(x/I) = 1/I$, and so $(n(x/I))^* = 0/I$. By Proposition 3.3(7), $((x/I)^*)^n = (n(x/I))^*$, then $((x/I)^*)^n = 0/I$. Hence $(x^*)^n \in I$. Since I is a perfect ideal, we have $x^m \notin I$, for any $x \in A$ and for any $m \in N$. Thus $x^m/I \neq 0/I$, hence $(x^m/I)^* \neq 1/I$. By Proposition 3.3(8), $m(x^*/I) = (x^m/I)^*$, then $m(x^*/I) \neq 1/I$. Thus $\text{ord}(x^*/I) = \infty$. If there exists $n \in N$ such that $n(x^*/I) = 1/I$, then $(n(x^*/I))^* = 0/I$. Since $(x^{**}/I)^n = (n(x^*/I))^*$, we have $(x^{**})^n/I = 0/I$, then $(x^{**})^n \in I$ and so $x^n \in I$, which is a contradiction with $x^m \notin I$, for any $m \in N$. Therefore, A/I is a perfect. Let $\text{ord}(x/I) = \infty$. Then $n(x/I) \neq 1/I$, for all $n \in N$, and so $(n(x/I))^* \neq 0/I$. By Proposition 3.3(7), $((x/I)^*)^n = (n(x/I))^*$, then $((x/I)^*)^n \neq 0/I$. Thus $(x^*)^n \notin I$. Since I is a perfect ideal, we have there exists $m \in N$, $x^m \in I$, then $x^m/I = 0/I$. Thus $(x^m/I)^* = 1/I$. By Proposition 3.3(8), $m(x^*/I) = (x^m/I)^*$, $m(x^*/I) = 1/I$, we have $\text{ord}(x^*/I) < \infty$. Therefore, A/I is a perfect.

Conversely, let I be a proper ideal of A such that $x^n \in I$, for some $n \in N$ and for any $x \in A$. Then $x^n/I = 0/I$ and so $(x^n/I)^* = 1/I$. By Proposition 3.3(8), $n(x^*/I) = (x^n/I)^*$, then $n(x^*/I) = 1/I$. Hence $\text{ord}(x^*/I) < \infty$. Since A/I is a perfect, we get $\text{ord}(x/I) = \infty$, that is $m(x/I) \neq 1/I$, for all $m \in N$. Hence $(m(x/I))^* \neq 0/I$. By Proposition 3.3(7), $(x^*/I)^m = (m(x/I))^*$, then $(x^*)^m/I \neq 0/I$, and so $(x^*)^m \notin I$, for all $m \in N$. The proof of the other implication is similar. Therefore, I is a perfect ideal. $\square$

Definition 4.9. Let A be a bounded semihoop. A is called a local semihoop if $\text{ord}(x) < \infty$ or $\text{ord}(x^*) < \infty$, for any $x \in A$.

Definition 4.10. The radical of A , denoted by $\text{Rad}(A)$, is defined by $\text{Rad}(A) = \bigcap_{M \in \mathcal{M}(A)} M$.

Theorem 4.11. Let A be a bounded semihoop with (DNP). Then the following conditions are equivalent:

- (1) A is local;
- (2) A has unique maximal ideal;
- (3) $\text{Rad}(A) = \{x \in A \mid \text{ord}(x) = \infty\}$.

Proof. (1) $\Rightarrow$ (2) Let $M \in \mathcal{M}(A)$. Then we prove $M = \{x \in A \mid \text{ord}(x) = \infty\}$. By Proposition 4.2(4), $M \subseteq \{x \in A \mid \text{ord}(x) = \infty\}$. Next, we prove that $\{x \in A \mid \text{ord}(x) = \infty\} \subseteq M$. Suppose that there exists $x \in \{x \in A \mid \text{ord}(x) = \infty\}$ such that $x \notin M$, then $\text{ord}(x) = \infty$. By Proposition 3.21, there exists $n \in N$ such that $(nx)^* \in M$, then $\text{ord}(nx)^* = \infty$. Since A is local, we have $\text{ord}(nx) < \infty$. Hence $\text{ord}(x) < \infty$, which is a contradiction, then $x \in M$. Thus $M = \{x \in A \mid \text{ord}(x) = \infty\}$. Therefore, A has unique maximal ideal.

(2) $\Rightarrow$ (1) Suppose that A has unique maximal ideal M . Let $x \in A$ such that $\text{ord}(x) = \text{ord}(x^*) = \infty$. By Proposition 4.2(1), $\langle x \rangle$ and $\langle x^* \rangle$ are proper ideal of A . By Zorn's Lemma, they are contained in a maximal ideal. Since A has unique maximal ideal M , we get $\langle x \rangle \subseteq M$ and $\langle x^* \rangle \subseteq M$, then $x, x^* \in M$. By Proposition 3.3(3), $x \oplus x^* = 1 \in M$, which is a contradiction. Hence $\text{ord}(x) < \infty$ or $\text{ord}(x^*) < \infty$. Therefore, A is local.

(2) $\Rightarrow$ (3) Let A has unique maximal ideal M such that $M = \{x \in A \mid \text{ord}(x) = \infty\}$. Since $\text{Rad}(A) = \bigcap_{M_i \in \mathcal{M}(A)} M_i$ and A has one maximal ideal M , we have $\text{Rad}(A) = M = \{x \in A \mid \text{ord}(x) = \infty\}$.

(3) $\Rightarrow$ (2) Let $\text{Rad}(A) = \{x \in A \mid \text{ord}(x) = \infty\}$. Since $\text{Rad}(A) = \bigcap_{M \in \mathcal{M}(A)} M$, for all $M \in \mathcal{M}(A)$, we have $\text{Rad}(A) \subseteq M$, so $\{x \in A \mid \text{ord}(x) = \infty\} \subseteq M$. On the other side, by Proposition 4.2(4), $M \subseteq \{x \in A \mid \text{ord}(x) = \infty\}$, for all $M \in \mathcal{M}(A)$. Thus, $M = \{x \in A \mid \text{ord}(x) = \infty\}$, for all $M \in \mathcal{M}(A)$. Therefore, A has unique maximal ideal. $\square$

Theorem 4.12. Let A be a bounded semihoop with (DNP) and I be a proper ideal of A . Then A/I is local if and only if I is a primary ideal.

Proof. ($\Rightarrow$) Let $x \odot y \in I$, for some $x, y \in A$. Then $x \odot y/I = 0/I$, and so $x/I \leq y^*/I$. Suppose that $x^n \notin I$, for any $n \in N$. Then $x^n/I \neq 0/I$. Hence $(x^n/I)^* \neq 1/I$. By Proposition 3.3(8), $n(x^*/I) = (x^n/I)^*$. Then $n(x^*/I) \neq 1/I$, for any $n \in N$. Since A/I is local, there exists $m \in N$ such that $m(x/I) = 1/I$. Since $x/I \leq y^*/I$, we have $m(y^*/I) = 1/I$ and so $(m(y^*/I))^* = 0/I$. By Proposition 3.3(7), $(y^{**}/I)^m = (m(y^*/I))^*$, then $(y^{**}/I)^m = 0/I$, that is $(y^{**})^m \in I$ and so $y^m \in I$. Therefore, I is a primary ideal.

($\Leftarrow$) Let I be a primary ideal. Suppose that $x/I \in A/I$. Since $x \odot x^* = 0 \in I$, we have $x^n \in I$ or $(x^*)^n \in I$, for some $n \in N$. Hence $x^n/I = 0/I$ or $(x^*)^n/I = 0/I$. If $x^n/I = 0/I$, then $(x^n/I)^* = 1/I$. By Proposition 3.3(8), $n(x^*/I) = ((x^n/I)^*)^*$. Then $n(x^*/I) = 1/I$ and so $\text{ord}(x^*/I) < \infty$. If $(x^*)^n/I = 0/I$, then by the similar way, we have $\text{ord}(x/I) < \infty$. Therefore, A/I is local. $\square$

Corollary 4.13. *If any proper ideals of A are primary, then A is local.*

Proof. Suppose that every proper ideal of A is primary. Since $\{0\}$ is a proper ideal of A , we have $\{0\}$ is primary. Thus, by Theorem 4.12, $A/\{0\}$ is local. Moreover, since $A/\{0\} \cong A$, we get A is local. $\square$

Definition 4.14. *A is said to be local finite if for any $0 \neq x \in A$, $\text{ord}(x) < \infty$.*

Theorem 4.15. *Let A be a bounded semihoop with(DNP) and I be a proper ideal of A . Then $I \in \mathcal{M}(A)$ if and only if A/I is locally finite.*

Proof. ($\Rightarrow$) Let $I \in \mathcal{M}(A)$. Since I is a proper ideal of A , we have $A/I \neq \emptyset$. Let for any $x/I \in A/I$, but $x/I \neq 0/I$. Since $x \notin I$, by Proposition 3.21, there exists $n \in N$ such that $(nx)^* \in I$. Since $(nx \rightarrow 1)^* = 0 \in I$ and $(1 \rightarrow nx)^* = (nx)^* \in I$, we have $[nx] = [1]$. Therefore, A/I is locally finite.

($\Leftarrow$) Let I be a proper ideal of A and A/I be a locally finite. Suppose that $x \notin I$. Then $x/I \in A/I$. Since A/I is a locally finite, we have $\text{ord}(x/I) < \infty$, that is there exists $n \in N$ such that $n(x/I) = 1/I$. Hence $(n(x/I))^* = 0/I$ and so $(nx)^* \in I$. By Proposition 3.21, $I \in \mathcal{M}(A)$. $\square$

5. Conclusions

In this paper, motivated by the research of previous ideals on MV-algebras, BL-algebras and residuated lattices, we extend the concept of ideals to more general fuzzy structures, namely bounded semihoops. We introduce the notion of perfect semihoops and portray perfect semihoops by applying perfect ideals. Furthermore, we use primary ideals to characterize local semihoops. Finally, we give the concept of locally finite bounded semihoops and characterize them with maximal ideals. In the future, we will continue our discussion of the relationship between primary ideals and prime ideals. Also, we try to use the prime ideals and maximal ideals to research topological space on bounded semihoops.

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Design of university fixed assets file management system and its security mechanism

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Abstract. In the 21st century, the rapid development of economy in China promotes the development of higher education. The expansion of university enrollment and school scale increases the number and type of fixed assets in colleges and universities, which generates large pressure on fixed assets management in colleges and universities. Therefore, a stable and reliable fixed assets files informatization management system is needed to help universities and colleges to solve the management problems.

A university fixed assets file management system was designed from the perspective of informatization management of fixed assets. Firstly the functional structure of the system was designed, and the technical structure of the system was constructed using JavaEE tool and MVC structure. Then system logic model was established using Unified Modeling Language (UML) tool, and relevant database was set up to support the system. Security mechanism was introduced in the aspects of concept, physical security, network security and database security.

Finally the university fixed assets file management system was developed, and its functions and performance were tested. The results demonstrated that the system had stable performance and normal log-in and functions. Hence it was a relatively perfect fixed assets file management system, which can solve a series problems of university fixed assets file management and provide assistance for university fixed assets file management.

Keywords: university, fixed assets, file management, security mechanism, JavaEE.

1. Introduction

Fixed asset is an important factor reflecting the economic development of enterprises. Fixed assets file management is based on the collection, modification and updating of fixed assets file information. Scholars in China and abroad have studied the subject. Peng [1] introduced Internet of Things into assets management, described the advantages of radio frequency identification devices (RFID) compared to bar code tag, and analyzed the changes of university fixed assets file management after the application of Internet of Things as well as the limitations of Internet of Things in assets management. The research results demonstrated that Internet of Things was useful in enhancing the efficiency of university fixed assets management. You [2] investigated the application of RFID technology in

university fixed assets management and considered RFID could accelerate fixed assets accounting and strengthen the dynamic management of fixed assets. Wu et al. [3] put forward an open-type architecture for fixed assets management.

The communication between different business modules and seamless integration of heterogeneous system were realized through enterprise service bus. Safe communication channel was established to ensure the reliable transmission of sensitive data on the Internet. Fragments were introduced to ensure fluent exchange of large documents and reduce bandwidth demand. The system could manage all fixed assets including intangible assets. Zhang et al. [4] applied MVC design based Struts framework into the development of fixed assets management system, studied how to realize MVC structure using Struts framework, and verified the effective coordination, favorable maintainability and extendibility of the system framework. Shvets Natalja V et al. [5] considered the problems of fixed assets analysis and the essential conditions of management in practice and found the physical wear and utilization degree of fixed assets could affect the estimation of management cost. In this study, a safe and reliable fixed assets file management system was developed using JavaEE tool to enhance the efficiency of university fixed assets management.

2. Problems existing in university fixed assets file management

With the rapid development of Chinese universities, the management and use of fixed assets are more and more standardized, but inevitably, there are some problems in the process of management. The first problem is the loss of fixed assets. Fixed assets of university include teaching equipment, books, livelihood facilities, etc.; the amount is large, and fixed asset structure is lack of [6], which brings great difficulties to management. Moreover such a large amount of fixed assets generate large pressure on account keeping. For example, subsidiary accounts may be inconsistent with general ledger, which can cause the loss of fixed asset files.

The second problem is the incompleteness of part of fixed assets files. The collection of fixed asset files is complex and some files may lose because of the infrequent communication and poor coordination between departments such as secondary schools, accounting department and logistics department and different standards for file management. The last problem is the slow updating of fixed assets files. The updating of fixed assets files is extremely difficult because of the heavy workload of filing, unified filing standard and large span of service time of fixed assets. The authors of this study considered that developing a fixed assets informatization management system could effectively enhance the management efficiency of fixed assets in universities, standardize file management and bring the values of fixed assets into full play.

3. Design of fixed assets file management system

3.1 Design of system architecture

(1) **Functional framework.** As shown in Figure 1, the functional structure of the system was divided into four sections, i.e. fixed assets file setting, fixed assets management, assets information query and user management. Fixed assets file setting included recording of the basic information of fixed assets including the category, producer and amount [7], the affiliated department, storage place, person in charge and assets disposal way including allocation, sale, breakage and scrap. Fixed assets management includes addition, elimination, borrowing, returning and depreciation of fixed assets [8] and recording of alteration of fixed assets related information including users, values, storage place and amount. Fixed assets information query included query of the basic information, borrowing and repairing conditions of fixed assets and other information [9]. Most of the users of the system were from the administrative departments or secondary schools of universities. System administrator can add, modify, delete and query user information to enhance the security of the fixed assets file management system.

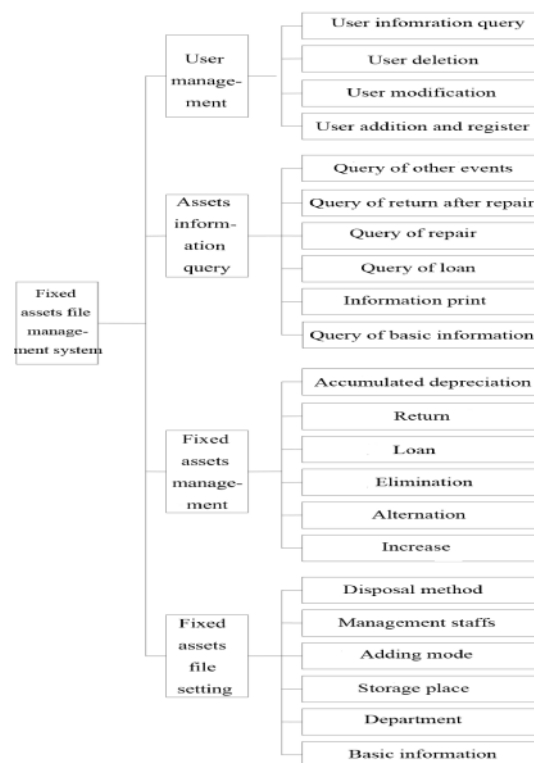


Figure 1: System function design

(2) Technical framework. The system was developed using JavaEE and java server. MVC structure [10], i.e. model-vision-controller, was used, which was transformed to presentation layer, business logic layer and data access layer. The presentation layer was responsible for designing interactive interface with Cascading Style Sheets (CSS), Extensible Markup Language (XML) and JavaApplat to display users the functions and interaction effect of the system. The business logic layer was used to do logic identification on function commands, analyze the categories of data which need to be processed, deliver instructions such as addition, modification and query of data to database. The data access layer was responsible for connecting database using Mybatis framework technology, call Structured Query Language (SQL) sentence in the database using Mapper port, conduct operations such as addition and modification according to data command, and feedback via the business logic layer and representation layer. The technical framework is shown in Figure 2.

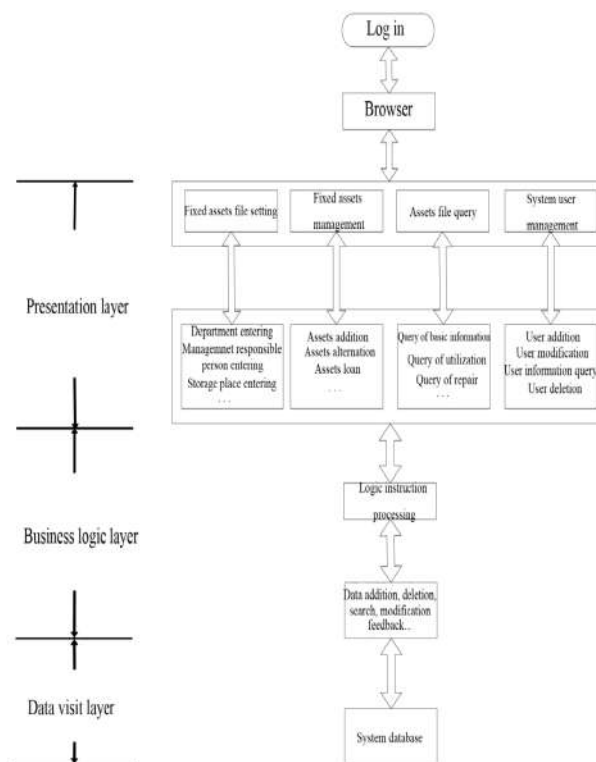


Figure 2: The technical framework of the system

3.2 Model building

Unified Modeling Language (UML) [11] was used to build model. Hierarchical design was adopted. Firstly a large class, for example, assets file, was designed,

followed by the static modeling of small class. The design of large class established the overall framework, while the design of small class added many content and details into the system framework. Figure 3 is one of the static models of fixed assets file management system. In Figure 3, fixed assets file, fixed assets, assets delivered for repairing, scrapped assets, system administrator, system user and fixed assets classification were all a class of the system. Corresponding detailed attributes were set for each class. There was a connection between every class. For example, assets delivered for repairing could be regarded as a subset of fixed asset class, fixed assets class could be regarded as a subset of fixed assets file class. System user class and system administrator class can transfer instructions to fixed assets class. Some rights of system users were limited [12], and administrator with the largest authority could query and revise assets file information.

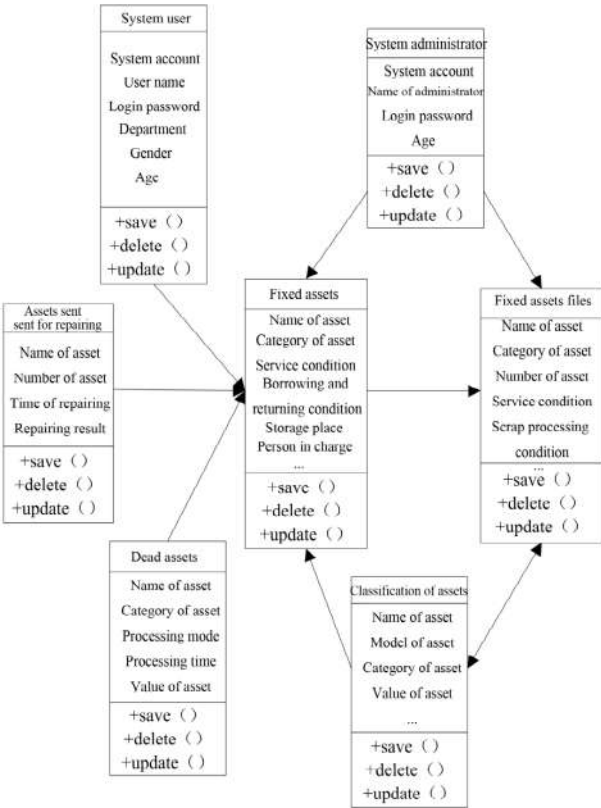


Figure 3: The static model of fixed assets file management system

A complete system generally includes many dynamic behaviors. Here those dynamic behaviors were represented by a dynamic diagram. Figure 4 suggests the dynamic model for the borrowing and returning of fixed assets. The flow

started from log-in. Then instructions were released. Next a command control node encountered. The node had two branches, one was query of borrowing and returning conditions of fixed assets and the other was query of ranking and idle amount of fixed assets. The grades of the two commands were the same. One command could be exerted only when the other has stopped. For example, a user needed to apply to the system for borrowing fixed assets. Then he was allowed to query the condition of fixed assets. Only fixed assets that meet the condition in the database could be borrowed. After all instructions finished, the user exited the system.

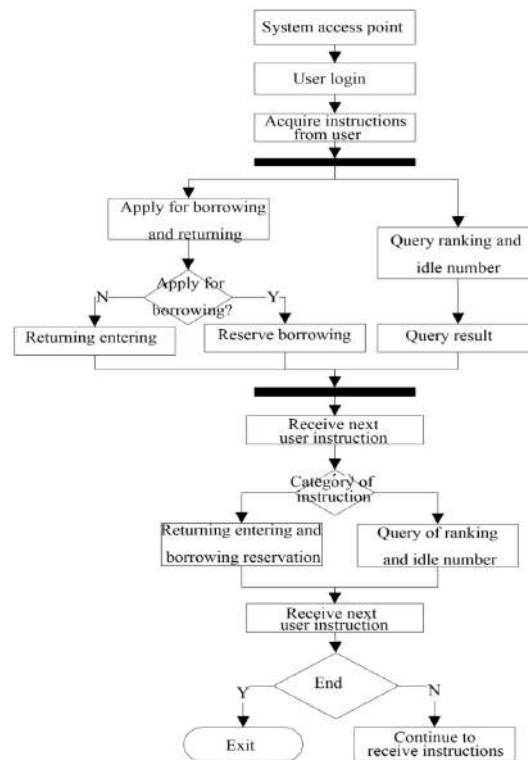


Figure 4: The dynamic model for the borrowing and returning of fixed assets

3.3 Design of database

Design of database is quite important for a system. SQL [13] was used for construction of database. Database was composed of several tables [14]. Tables were designed according to functional requirements such as user administration, fixed assets file management and analytic statistics. Some database tables were displayed. Table 1 and 2 are user table and fixed assets table respectively.

Character field	Data type	Description	Field length
Login	Name varchar	Login name	10
User	Name varchar	User name	10
Login	Password varchar	Login password	20
Department ID	INT	Department	8

Table 1: User table

Character field	Data type	Description	Field length
Declarant	Varchar	Staff responsible for declaring assets	10
Auditor	Varchar	Staff responsible for auditing assets	10
Assets name	Varchar	Name of assets	20
Assets class	Varchar	Class of assets (education equipment, living equipment, etc.)	12
Assets model	Varchar	Model of assets	8
Assets price	Float	Purchase price of assets	8
Manufacture factory	Varchar	Manufacture factory of assets	8
Date of registration	Datetime	Date of assets entering university	10
Assets utilization department	Varchar	Department which utilizes assets	10
Spare 1	Varchar	Spare content to be added	50
Spare 2	Varchar	Spare content to be added	50

Table 2: Fixed assets table

3.4 Design of security mechanism

Security mechanism was designed in the aspects of system concept, physical security, network security and database security.

System concept: The management concept of assets file managers should be strengthened. Moreover, managers are required to be more interactive and coordinative, put an example to solve problems, establish the awareness of security and secrecy, and trained to be more careful and responsible.

Physical security: Stored files and transmission equipments should be guaranteed not to be easily destroyed by natural disasters. Moreover fixed assets files should be stored using multiple storage tools such as light disk, magnetic disk, cloud disk and electronic equipment. The security of information carrier and normal operation of system should be ensured. Generally light disk is used for long-term storage, while other storage tools are used for temporary storage.

Network security: Network security threats [15] mainly come from network attack and system bugs. Network security is protected using firewall and antivirus program.

Database security: Database access right is set as database level, record level and field level. Access right and content of users are controlled. System privileges are divided into four levels. The first level is system administrator who has the highest authority. The second level is file manager who is allowed to check and download files but needs to be authorized by system administrator. The third level is user manager who can add, delete and revise user but also needs to be authorized by system administrator. The fourth level is general user who can check field-level data of some fixed assets and can access to files of higher levels after the authorization of system administrator.

4. System implementation and test

4.1 System implementation

The implementation environment of the designed university fixed assets file management system was follows.

As to the server, HP DL180Gen9 which was equipped with InteXreon 1.9GHz (CPU), 8 GB internal storage, 1 TB hard drive capacity, 192.168.70.253 (local area network), Microsoft windows 2003 Server, Microsoft SQL Server 2012, 360 Security Guard antivirus program and CISCO ASA5545-K9 (FireWall Gateway) was used.

As to the client, Hewlett-Packard personal computer which was equipped with Hard drive capacity larger than 80 GB, Intel dual-core processor, internal storage larger than 1 GB and local area network was used.

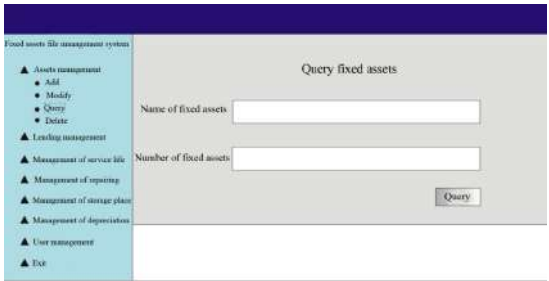
The obtained system interface is shown in Figure 5.



a. Login interface



b. User addition interface



c. Assets query interface

Figure 5: The interface of the fixed assets file management system

4.2 System test and results

Possible bugs in the system were managed and controlled using mantis v1.2.17. The stability of the system was tested using ORTHOS software. Valgrindv5.0 was used to test whether there was memory leak. The load capacity of the system was tested using LoadRunner 8.1. The utilization condition of system functions was tested using modules. Table 3 demonstrates the results of system test. As there are many module testing examples, part of test examples and test results were displayed (Table 3). The results suggested that the system operated normally and stably, users could log in the system successfully and use function as expected, no bugs were found, and there was also no storage leakage. The requirements on functions and performance were all satisfied. The stable, reliable and perfect fixed assets file management system can help universities to sort out the information of fixed asset files to make the information more completed, solve complicated archiving issues, timely update the conditions of fixed assets, and improve the management efficiency of fixed assets.

No. of test examples	Operation	Input data	Expectation	Test results
1	Input correct name of account	Account name: admin Password: admin	Login successfully and enter main interface of system	Login normally
2	Not input name of account	Account name: Password:1111	Hint name of user and password cannot be empty	Meet expectation
3	Input correct name of account and wrong password	Account name: admin Password: 1111	Hint name of user and password are wrong	Meet expectation
4	Enter user addition interface, input information, and click Save button	ID: 110 Name of user: xx Department: office of academic affairs Password: 123456 Confirm password: 123456	User addition succeeds	Meet expectation
5	Input correct asset information and click Save button	Asset information	Save successfully	Asset information is successfully saved.
6	Input existing number of asset and click Query button	Asset number	Hint corresponding asset information	Meet expectation
7	Input inexistent number of asset and click Query button	Asset number	The queried information is not found	Meet expectation
8	Revise existing asset information and click Confirm button		Hint successful information edit	Meet expectation

Table 3: Some test examples and test results

5. Conclusion

In this study, the problems existing in fixed asset management in universities were summarized, and then a management system was designed against the problems. Firstly the functions of the system were designed, and the technical architecture of the system was established using Java EE and MVC model. Secondly the logistical model and database were established to support the system framework. Then security mechanisms which involved system concept, physical security, network security and database security were introduced to enhance the security of the system. Finally the system was implemented and tested using ORTHOS software. The testing suggested that the system operated stably, had no security bugs, and had perfect functions. The file management system basically satisfies the working requirements and can coordinate with managers in universities to comprehensively and timely file information of fixed assets and timely grasp the changes of fixed assets files to effectively manage fixed assets in universities. For employees and society, the research results can help file management staffs in universities process fixed assets more efficiently and make the information management in universities more orderly, which can ensure the stable development and construction of universities and promote the development of higher education.

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The maximal hyperrings of quotients

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Abstract. We show that the maximal quotient hyperring $Q_{mr}(R)$ of a semiprime hyperring R can be obtained in a similar way to a maximal quotient ring. In this regard we introduce and study some basic notions of hyperrings such as dense hyperideal, essential hyperideal, singular hyperideal and prove some results satisfying them. Finally, we show that if R is a semiprime hyperring and $Q = Q_{mr}(R)$, then Q is regular (in the sense of von Neumann) if and only if R has a zero right singular hyperideal.

Keywords: hyperring, dense hyperideal, essential hyperideal, good homomorphism.

1. Introduction

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the same set. The theory of hyperstructures was introduced by Marty in 1934 at the 8th Congress of the Scandinavian Mathematicians [8]. Marty defined hypergroups, began to analyze their properties and applied them to groups. Hyperstructures have many applications to several sectors of both pure and applied mathematics [2]. The notion

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of hyperrings is investigated and studied by Krasner [7], Nakasis [13], Vougiouklis [17], Davvaz [3], Davvaz and his research group [5, 6, 9, 10, 11, 12, 14, 15], and many others. Also, see Davvaz and Leoreanu-Fotea book [4]. A well-known type of a hyperring is called the Krasner hyperring [7]. Krasner hyperring is essentially ring with approximately modified axioms in which addition is hyperoperation, while the multiplication is an operation. This type of hyperrings has been studied by a variety of authors.

The notion of left quotient ring for a ring without right zero divisors was introduced by Utumi [16]. In his paper, Utumi proved that every ring without total right zero divisors has a unique maximal quotient ring. This ring, denoted by $Q_{\max}^l(R)$, is called the maximal left quotient ring of R . The many properties of this ring were investigated in [1].

In this paper, the maximal quotient hyperring $Q = Q_{mr}$ is structured. We define and study dense hyperideal, essential hyperideal and we give the relationship between essential and dense right hyperideals. Furthermore, we study singular hyperideal and some properties of its. Finally, we show that R has zero right singular hyperideal if and only if $Q = Q_{mr}$ is a von Neumann regular hyperring.

2. Preliminaries

In this section we give some definitions and results of hyperstructures which we need to develop our paper.

A mapping $\circ : H \times H \rightarrow P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the set of all non-empty subsets of H . An algebraic system $(H, \circ)$ is called a hypergroupoid.

For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hyperoperation “ $\circ$ ” is called associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c.$$

A hypergroupoid with the associative hyperoperation is called a semihypergroup.

A hypergroupoid $(H, \circ)$ is a quasihypergroup, whenever $a \circ H = H = H \circ a$ for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called a hypergroup.

Definition 2.1 ([4]). *A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:*

(1) $(R, +)$ is a canonical hypergroup, i.e.,

- (i) for every $x, y, z \in R$, $(x + y) + z = x + (y + z)$,
- (ii) for every $x, y \in R$, $x + y = y + x$,
- (iii) there exists $0 \in R$ such that $0 + x = \{x\}$ for all $x \in R$,
- (iv) for every $x \in R$ there exists a unique element denoted by $-x \in R$ such that $0 \in x + (-x)$,
- (v) for every $x, y, z \in R$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$;
- (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e.,
 - (i) for every $x, y, z \in R$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
 - (ii) $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$;
- (3) The multiplication is distributive with respect to the hyperoperation $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$, where $-A = \{-a \mid a \in A\}$. In definition, for simplicity of notations we write sometimes xy instead of $x \cdot y$ and in (iii), $0 + x = x$ instead of $0 + x = \{x\}$.

In a hyperring R , if there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in A$, then the element 1 is called the identity element of the hyperring R . If $ab = ba$ for every $a, b \in R$ then the hyperring R is called a commutative hyperring.

A hyperring R is called a hyperdomain if R does not have zero divisors. In other words, for $x, y \in R$ if $xy = 0$ then either $x = 0$ or $y = 0$.

A Krasner hyperring is called a Krasner hyperfield, if $(R \setminus \{0\}, \cdot)$ is a group.

Throughout this paper, by a hyperring we mean that Krasner hyperring.

Let R be a hyperring. A non-empty subset S of R is called a subhyperring of R , if $x - y \subseteq S$ and $xy \in S$ for all $x, y \in S$.

A subhyperring I of a hyperring R is a left (resp. right) hyperideal of R if $ra \in I$ (resp. $ar \in I$) for all $r \in R$, $a \in I$. A hyperideal of R is both a left and a right hyperideal.

Lemma 2.1 ([4]). *A non-empty subset A of a hyperring R is a left (right) hyperideal if and only if;*

- (1) $a, b \in A$ implies $a - b \subseteq A$,
- (2) $a \in A, r \in R$ imply $ra \in A$ ($ar \in A$).

Let A and B be non-empty subsets of a hyperring R

$$A + B = \{x \mid x \in a + b \text{ for some } a \in A, b \in B\}$$

and

$$AB = \left\{ x \mid x \in \sum_{i=1}^n a_i b_i, a_i \in A, b_i \in B, n \in \mathbb{Z}^+ \right\}.$$

If A and B are hyperideals of R , then $A + B$ and AB are also hyperideals of R .

Definition 2.2 ([18]). *The hyperideal I said to be the direct sum of its hyperideals J and K , denoted by $I = J \oplus K$ if for every element $x \in I$ there exist unique elements a, b such that $a \in J$, $b \in K$ and $x \in a + b$.*

A hyperideal P of a hyperring R is called a prime hyperideal, if for hyperideals I and J of R , satisfying $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. A hyperideal I of a hyperring R is called semiprime hyperideal if for any hyperideal H of R , satisfying $H^2 \subseteq I$ implies $H \subseteq I$. Prime hyperideals are surely semiprime hyperideals. A hyperring R is called a prime hyperring if $aRb = 0$ for all $a, b \in R$ implies $a = 0$ or $b = 0$. Also, R is called semiprime if $aRa = 0$ implies that $a = 0$. Clearly, every prime hyperring is a semiprime hyperring but the converse is not always true.

Definition 2.3 ([4]). *Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a good (strong) homomorphism if for all $a, b \in R_1$,*

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b) \quad \text{and} \quad \varphi(0) = 0.$$

A good homomorphism φ is an isomorphism if φ is one to one and onto. If there exists isomorphism between hyperrings R_1 and R_2 , we write $R_1 \cong R_2$.

Since R_1 is a hyperring, $0 \in a - a$ for all $a \in R_1$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies that $\varphi(-a) \in -\varphi(a) + 0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in R_1$. Moreover, if φ is a good homomorphism from R_1 into R_2 , then the kernel of φ is the set $\{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $\ker \varphi$ is a hyperideal of R_1 and $\Im(\varphi) = \{\varphi(r) \mid r \in R\}$ is a subhyperring of R_2 .

Corollary 2.1. *Let φ be a good homomorphism from R_1 into R_2 . Then φ is one to one if and only if $\ker \varphi = \{0\}$.*

Let R be a hyperring. A canonical hypergroup $(M, +)$ together with the map $\cdot : R \times M \rightarrow M$ is called a left hypermodule over R if for all $r_1, r_2 \in R$, $m_1, m_2 \in M$ the following axioms hold:

- (1) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,
- (2) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,
- (3) $(r_1r_2)m_1 = r_1(r_2m_1)$,
- (4) $0_Rm_1 = 0_M$.

A subhypermodule N of M is a subhypergroup of M which is closed under multiplication by elements of R .

Definition 2.4. *Let M and N be two R -hypermodules. A function $f : M \rightarrow N$ that satisfies the conditions:*

- (1) $f(x + y) \subseteq f(x) + f(y)$,
- (2) $f(xr) = f(x)r$, for all $r \in R$ and all $x, y \in M$

is called to be a right R -homomorphism from M into N .

In Definition 2.4, if the equality holds, then f is called a good (strong) right R -homomorphism.

(ii) Let $l_a : R \rightarrow R$, $x \mapsto ax$. Then

$$(a : J) = \{x \in R \mid ax \in J\} = \{x \in R \mid l_a(x) \in J\} = l_a^{-1}(J).$$

Let $x, y \in (a : J)$, hence $ax, ay \in J$. For some $z \in x - y$ we have $az \in a(x - y) = ax - ay$. Thus $az \in J$ and so $z \in (a : J)$. Therefore, $x - y \subseteq (a : J)$. Let $r \in R$ and $x \in (a : J)$, hence $ax \in J$. Then $a(xr) = (ax)r \in J$. Since $a(xr) \in J$, $xr \in (a : J)$. Let $r_1 \neq 0$, $r_2 \in R$. Hence $r_1 \neq 0$, $ar_2 \in R$. Since J is a dense right hyperideal of R , $r_1r \neq 0$ and $(ar_2)r \in J$ for some $r \in R$. Then $r_1r \neq 0$ and $r_2r \in (a : J)$, i.e.; $(a : J) \in D(R)$ for all $a \in R$.

(iii) If $i : I \rightarrow R$ is the inclusion map, then $i^{-1}(J) = I \cap J$. If the option (i) is applied, the proof ends.

(iv) Let K is a right hyperideal of R , $I \subseteq K$ and $r_1 \neq 0$, $r_2 \in R$. Since I is a dense right hyperideal of R , we have $r_1r \neq 0$, $r_2r \in I$ for some $r \in R$. Since $I \subseteq K$, we obtain $r_1r \neq 0$, $r_2r \in K$. That is, $K \in D(R)$.

(v) We suppose that $Ia = 0$ for some $0 \neq a \in R$. Setting $r_1 = a = r_2$, we have there exists $r \in R$ such that $0 \neq ar \in I$. Hence $arRar \subseteq Iar = 0$. By semiprimeness of R , we have $ar = 0$. Then we have a contradiction and so $r(I) = 0$. Now we suppose $l(I) \neq 0$. Since R is semiprime hyperring, there exists $a, b \in l(I)$ such that $ab \neq 0$. Since I is a dense hyperideal, we can find $r \in R$ such that $abr \neq 0$ and $br \in I$. Then $abr \in aI = 0$ and again we have a contradiction.

(vi) Let $r_1 \neq 0$, $r_2 \in R$. Since $I \in D(R)$, there exists an element $r' \in R$ such that $r_1r' \neq 0$ and $r_2r' \in I$. Hence by hypothesis, $(r_2r' : K) \in D(R)$. From (v), we have $l((r_2r' : K)) = 0$ and so $r_1r'r'' \neq 0$ and $r_2r'r'' \in K$ for some $r'' \in (r_2r' : K)$. Therefore $K \in D(R)$.

(vii) Let $x \in L$ and $r \in R$. By (ii), $(r : S)_R \in D(R)$ and so by (iii), $M = (r : S)_S \cap S \in D(R)$. For all $y \in M \subseteq S$ we get $ry \in S$ and

$$0 \in g(xry) - g(xry) = g(xr)y - g(x)ry = (g(xr) - g(x)r)y.$$

Hence $ay = 0$ for some $a \in g(xr) - g(x)r$. Since $y \in M$ and $M \in D(R)$, it follows from (v) that $0 \in g(xr) - g(x)r$, which implies that $-g(x)r$ is the inverse of $g(xr)$ in the canonical hypergroup $(R, +)$. Hence $-g(x)r = -g(xr)$ and so $g(xr) = g(x)r$.

(viii) Let $r_1 \neq 0$, $r_2 \in R$. By (ii), $L = (r_2 : I) \in D(R)$ and by (v) there exists $r' \in L$ such that $r_1r' \neq 0$ and there exists $r'' \in J$ such that $r_1r'r'' \neq 0$. Getting $r = r'r''$ we have $r_1r \neq 0$ and $r_2r = r_2(r'r'') = (r_2r')r'' \in IJ$.

Now, we give an alternative definition of dense right hyperideals:

Corollary 3.1. *Let J be a right hyperideal of R . Then $J \in D(R)$ if and only if $l_R((a : J)) = 0$ for all $a \in R$.*

Proof. Let $J \in D(R)$. According to Proposition 3.1 (ii) and (v), $(a : J) \in D(R)$ and $l_R((a : J)) = 0$. Conversely, let $r_1 \neq 0$, $r_2 \in R$. By hypothesis, we have $r_1(r_2 : J) \neq 0$. Then we may choose $r \in (r_2 : J)$ such that $r_1r \neq 0$. Since $r \in (r_2 : J)$, we also have $r_2r \in J$. This completes the proof.

Definition 3.2. A right hyperideal I of R is essential if for every nonzero right hyperideal K of R we have $I \cap K \neq 0$.

Example 3.2. Define the hyperoperation $\oplus$ on the $R = [0, 1]$ by

$$x \oplus y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y \\ [0, x], & \text{if } x = y \end{cases}.$$

Then $([0, 1], \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on real numbers. Let $I = [0, 0.3]$. Hence I is an essential hyperideal of R .

The following remark give the relationship between essential and dense right hyperideals:

Remark 3.2. Let J be a dense right hyperideal of R . Then J is an essential right hyperideal of R .

Proof. For $0 \neq a \in R$, pick $r \in R$ such that $0 \neq ar \in J$. Then $0 \neq ar \in J \cap aR$.

Remark 3.3. Let I be a 2-sided hyperideal of R . Then the following conditions are equivalent:

- (i) $l(I) = 0$;
- (ii) I is a dense right hyperideal;
- (iii) I is an essential right hyperideal;
- (iv) I is essential as a 2-sided hyperideal (i.e., for any hyperideal $J \neq 0$, $I \cap J \neq 0$).

Remark 3.4. Let I be a 2-sided hyperideal of R . Then:

- (i) $l(I) = r(I)$;
- (ii) $l(I) \cap I = 0$;
- (iii) $I + l(I)$ is dense right hyperideal of R .

Remark 3.5. Let J be a right hyperideal of R and $f : J \rightarrow R$ be a good right R -hypermodule homomorphism. Then:

- (i) If $a \in R$ and $r(a) \in D(R)$, then $a = 0$;
- (ii) If $\ker f \in D(R)$, then $f = 0$.

Proof. (i) It follows from Proposition 3.1 (v).

(ii) We assume that $\ker f \in D(R)$. Then we have $f(b)(b : \ker f) = 0$ for all $b \in J$. By Proposition 1 (ii), $(b : \ker f) \in D(R)$. According to the first statement we obtain $f(b) = 0$. Thus $f = 0$.

Let R is a semiprime hyperring $\Psi = \{f_U \mid f : U \rightarrow R \text{ is a good right } R\text{-homomorphism and } U \in D\}$.

Define a relation " $\approx$ " on Ψ by $f_U \approx g_V : \Leftrightarrow$ there exists $K \in D$ and $K \subseteq U \cap V$ such that $f = g$ on K .

One readily checks that " $\approx$ " is an equivalence relation. This gives a chance for us to get a partition of Ψ . We denote the equivalence class by $\overline{f_U}$, where

$\bar{f}_U := \{g : V \rightarrow R \mid f_U \approx g_V\}$ and denote by Q_{mr} set of all equivalence classes. We define a hyperaddition " + " on Q_{mr} as follows:

$$\bar{f}_U + \bar{g}_V := \overline{f + g_{U \cap V}}$$

where $f + g : U \cap V \rightarrow R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1 (\in D) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2 (\in D) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2 . Taking $K = K_1 \cap K_2$ and so $K \in D$. For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = \bigcup \{t(x) \mid t(x) \in f_1(x) + g_1(x)\} = \bigcup \{t(x) \in f_2(x) + g_2(x)\} = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ on K . Therefore $f_1 + g_{1U_1 \cap V_1} \approx f_2 + g_{2U_2 \cap V_2}$, which means that the addition in Q_{mr} is well-defined.

Now we will prove that Q_{mr} is a canonical hypergroup. Let $\bar{f}_U, \bar{g}_V, \bar{h}_H$ be elements of Q_{mr} . Since $U \cap (V \cap H) = (U \cap V) \cap H$, for all $x \in U \cap (V \cap H)$

$$\begin{aligned} [(f + g) + h](x) &= (f + g)(x) + h(x) = \bigcup_{t(x) \in (f+g)(x)} t(x) + h(x) \\ &= \bigcup_{t(x) \in (f+g)(x)} \{k(x) \mid k(x) \in t(x) + h(x)\} \\ &= \bigcup \{k(x) \mid k(x) \in (f(x) + g(x)) + h(x)\} \\ &= \bigcup \{k(x) \mid k(x) \in f(x) + (g(x) + h(x))\} \\ &= \bigcup_{p(x) \in g(x) + h(x)} \{k(x) \mid k(x) \in f(x) + p(x)\} \\ &= \bigcup_{p(x) \in (g+h)(x)} f(x) + p(x) = f(x) + (g + h)(x) = [f + (g + h)](x). \end{aligned}$$

Hence $(f + g) + h = f + (g + h)$ on $U \cap (V \cap H)$. That is $(\bar{f}_U + \bar{g}_V) + \bar{h}_H = \bar{f}_U + (\bar{g}_V + \bar{h}_H)$. One can easily check that $\bar{f}_U + \bar{g}_V = \bar{g}_V + \bar{f}_U$. Taking $\theta_R \in Q_{mr}$ where $\theta : R \rightarrow R, x \mapsto 0$ for all $x \in R$. Since $U \subseteq U \cap R$, $(\theta + f)(x) = \theta(x) + f(x) = 0 + f(x) = f(x)$ for all $x \in U$. Then we have $\bar{\theta}_R + \bar{f}_U = \bar{f}_U$ and similarly $\bar{f}_U + \bar{\theta}_R = \bar{f}_U$ for all $\bar{f}_U \in Q_{mr}$. Hence $\bar{\theta}_R$ is the additive identity in Q_{mr} . Let $-\bar{f}_U \in Q_{mr}$, where $-f : U \rightarrow R, x \mapsto -f(x) = (-f)(x)$ for all $x \in U$. Since $-f(x)$ is the unique inverse of $f(x)$ in R , we have $\theta(x) \in f(x) - f(x) = f(x) + (-f)(x)$ for all $x \in U$. So $\bar{\theta}_R \in \bar{f}_U + (-\bar{f}_U)$. Finally, let $\bar{f}_U, \bar{g}_V, \bar{h}_H$ be elements of Q_{mr} and $\bar{h}_H \in \bar{f}_U + \bar{g}_V$. So there exists a $f_1 \in \bar{f}_U$ and a $g_1 \in \bar{g}_V$ such that $h = f_1 + g_1$. For any $x \in K (\in D) \subseteq U \cap V$, we get $h(x) = (f_1 + g_1)(x) = f_1(x) + g_1(x) \subseteq f(x) + g(x)$. Since R is a hyperring, $h(x) \in f(x) + g(x)$ implies $g(x) \in -f(x) + h(x)$ and $f(x) \in h(x) - g(x)$. Thus

we have $g(x) \in (-f + h)(x)$ and $f(x) \in (h - g)(x)$. That is, $\bar{g}_V \in -\bar{f}_U + \bar{h}_H$ and $\bar{f}_U \in \bar{h}_H - \bar{g}_V$. Therefore $(Q_{mr}, +)$ a canonical hypergroup.

Now we define a multiplication " ." on Q_{mr} as follows: for all $\bar{f}_U, \bar{g}_V \in Q_{mr}$

$$\bar{f}_U \bar{g}_V := \overline{f g_{g^{-1}(U)}}$$

where $f g : g^{-1}(U) \rightarrow R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1(\in D) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2(\in D) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2 . Taking $K := g_1^{-1}(K_1) \cap K_2$ and so $K \in D$. For any $x \in K$,

$$(f_1 g_1)(x) = f_1(g_1(x)) = f_1(g_2(x)) = f_2(g_2(x)) = (f_2 g_2)(x).$$

Thus $f_1 g_1 = f_2 g_2$ on K . Hence " ." is well-defined. Now we will prove that $(Q_{mr}, .)$ is a semigroup having zero as a bilaterally element. Let $\bar{f}_U, \bar{g}_V, \bar{h}_H \in Q_{mr}$. Since $h^{-1}(g^{-1}(U)) = (gh)^{-1}(U)$, we get for all $x \in h^{-1}(g^{-1}(U))$,

$$[(f g) h](x) = (f g)(h(x)) = f(g(h(x))) = f((gh)(x)) = (f(gh))(x).$$

Hence $(f g) h = f(gh)$ on $h^{-1}(g^{-1}(U))$. That is, $(\bar{f}_U \bar{g}_V) \bar{h}_H = \bar{f}_U (\bar{g}_V \bar{h}_H)$. Now we prove that $\bar{f}_U \bar{\theta}_R = \bar{\theta}_R = \bar{\theta}_R \bar{f}_U$ for all $\bar{f}_U \in Q_{mr}$. Since $\theta^{-1}(U) \subseteq \theta^{-1}(U) \cap R$ and $f \theta = \theta$ on $\theta^{-1}(U)$, we get $\bar{f}_U \bar{\theta}_R = \bar{\theta}_R$. Similarly $\bar{\theta}_R \bar{f}_U = \bar{\theta}_R$.

Let $\bar{f}_U, \bar{g}_V, \bar{h}_H$ be elements of Q_{mr} . Since $h^{-1}(U \cap V) = h^{-1}(U) \cap h^{-1}(V)$, we get for all $x \in h^{-1}(U \cap V)$,

$$\begin{aligned} [(f + g) h](x) &= (f + g)(h(x)) = f(h(x)) + g(h(x)) \\ &= (fh)(x) + (gh)(x) = (fh + gh)(x). \end{aligned}$$

Then $(f + g) h = fh + gh$ on $h^{-1}(U \cap V)$. That is, $(\bar{f}_U + \bar{g}_V) \bar{h}_H = \bar{f}_U \bar{h}_H + \bar{g}_V \bar{h}_H$. Also since $(g + h)^{-1}(U) = g^{-1}(U) \cap h^{-1}(U)$, for all $x \in (g + h)^{-1}(U)$

$$\begin{aligned} [f(g + h)](x) &= f((g + h)(x)) = f(g(x) + h(x)) \\ &= f(g(x)) + f(h(x)) = (fg + fh)(x). \end{aligned}$$

Hence $f(g + h) = fg + fh$ on $(g + h)^{-1}(U)$. That is, $\bar{f}_U (\bar{g}_V + \bar{h}_H) = \bar{f}_U \bar{g}_V + \bar{f}_U \bar{h}_H$.

Therefore, $(Q_{mr}, +, .)$ is a hyperring.

Taking $\bar{1}_R \in Q_{mr}$ where $1 : R \rightarrow R, x \mapsto x$ for all $x \in R$. Let $\bar{f}_U \in Q_{mr}$. Since $1^{-1}(U) \subseteq U$, we get for all $x \in 1^{-1}(U)$, $(f1)(x) = f(1(x)) = f(x)$ and

$(1f)(x) = 1(f(x)) = f(x)$. Thus, $\bar{f}_U \bar{1}_R = \bar{1}_R \bar{f}_U = \bar{f}_U$. Hence $\bar{1}_R$ is the multiplicative identity in Q_{mr} .

We denote the hyperring constructed by $Q_{mr} = Q_{mr}(R)$ and we call it the maximal right quotient hyperring of R . One can of course, characterize Q_{ml} , the left quotient hyperring of R in similar manner. For purpose of convenience, we use q instead of $\bar{q}_V \in Q_{mr}$.

Proposition 3.2. *The maximal right quotient hyperring Q_{mr} of R satisfies:*

- (i) R is a subhyperring of Q_{mr} ;
 - (ii) For all $q \in Q_{mr}$ there exists $U \in D$ such that $qU \subseteq R$;
 - (iii) If $q \in Q_{mr}$ and $U \in D$, then $qU = 0$ if and only if $q = \theta$;
 - (iv) If $U \in D$ and $f : U \rightarrow R$ is a good right R -hypermodule homomorphism, then there exists $q \in Q_{mr}$ such that $f(x) = qx$ for all $x \in U$.
- Furthermore, if Q' is any hyperring satisfying (i) – (iv), then Q' is isomorphic to Q_{mr} .

Proof. (i) For a fixed element a in R , consider a mapping $\lambda_a : R \rightarrow R$ by $\lambda_a(r) = ar$ for all $r \in R$. It is easy to prove that the mapping λ_a is a good right R homomorphism. Define a mapping $\varphi : R \rightarrow Q_{mr}$ by $\varphi(a) = \lambda_a$ for $a \in R$. Clearly the mapping φ is injective homomorphism and so R is a subring of Q_{mr} .

(ii) Let $q = f_U \in Q_{mr}$. One sees that $f_U \lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$, i.e., $q\varphi(U) \subseteq \varphi(R)$. Since $R \cong \varphi(R)$, we can write $qU \subseteq R$.

(iii) Let $q = f_U \in Q_{mr}$ and $U \in D$ such that $q\varphi(U) = 0$. Then $q = \theta$. Indeed, we have $\theta = f_U \lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$. Then $f(a) = 0$ and so $f(U) = 0$. By Remark 3.5, $q = \theta$.

(iv) Let $U \in D$ and $f : U \rightarrow R$ is a good right R -hypermodule homomorphism. Hence $f_U \lambda_{a_R} = \lambda_{f(a)_R}$ for all $a \in U$. That is, $q\varphi(a) = \varphi(f(a))$ for all $a \in U$, where $q = f_U$. Thus we write $qx = f(x)$ for all $x \in U$.

Now suppose $Q' \supseteq R$ is a hyperring having properties (i) – (iv). Define the mapping $\alpha : Q' \rightarrow Q_{mr}$, $q \mapsto \lambda_{q(q \cdot R)_R}$. One can easily check that α is an isomorphism of hyperrings.

Lemma 3.1. *Let $q_1, q_2, \dots, q_n \in Q_{mr}$ and $I, J \in D(R)$. Then there exists $L \in D(R)$ such that $L \subseteq J$ and $q_i L \subseteq I$ for all $i = 1, 2, \dots, n$.*

Proof. Let $J_i = (q_i : R)_R$. Then $J_i \in D$ for all $i = 1, 2, \dots, n$. Consider the mapping $f_i : J_i \rightarrow R$, $f_i = \lambda_{q_i}$. By Proposition 3.1, $K_i = f_i^{-1}(I) = \{x \in J_i \mid q_i x \in I\} \in D$. Then $L = (\bigcap_{i=1}^n K_i) \cap J$ we have the desired dense right hyperideal.

Lemma 3.2. *Let K be a dense right hyperideal of a semiprime hyperring R and S a subhyperring of $Q_{mr}(R)$ such that $K \subseteq S$. Then:*

- (i) S is a semiprime hyperring;
- (ii) A right hyperideal J of S is dense if and only if $(J \cap R)K \in D(R)$ (in particular, if $I \in D(R)$ then $IS \in D(S)$);

(iii) A right hyperideal J of S is essential if and only if $(J \cap R)K$ is an essential right hyperideal of R .

Since the proof is similar to the proof of the corresponding lemma in ring theory, we omitted it.

Proposition 3.3. Let K be a dense right hyperideal of R and S a subhyperring of $Q_{mr}(R)$ such that $K \subseteq S$. Then $Q_{mr}(S) = Q_{mr}(R)$.

The following result is an immediate corollary of Proposition 3.3.

Theorem 3.1. Let R be a semiprime hyperring and $Q = Q_{mr}(R)$. Then $Q_{mr}(Q) = Q$.

Corollary 3.2. Let R be a semiprime hyperring, I be an hyperideal of R and $J = l_R(I)$. Then $Q_{mr}(R) = Q_{mr}(I) \oplus Q_{mr}(J)$.

Proof. According to Remark 3.4, $I \oplus J \in D(R)$ and by Proposition 3.3, $Q_{mr}(R) = Q_{mr}(I \oplus J)$. From equality $Q_{mr}(I \oplus J) = Q_{mr}(I) \oplus Q_{mr}(J)$, the proof is obtained.

Definition 3.3. Let R be a hyperring. The set

$$Z_r(R) = \{x \in R \mid r_R(x) \text{ is an essential right hyperideal}\}$$

is called the right singular hyperideal of R .

Example 3.3. Let R be a hyperring and $H(R) = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in R \right\}$ be a collection of 2×2 matrices over R . A hyperoperation “+” and a multiplication “.” are defined on $H(R)$ as follows:

$$\begin{aligned} \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} &= \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \mid x \in a + c, y \in b + d \right\} \\ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} &= \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \in H(R). \end{aligned}$$

Clearly, $(H(R), +, \cdot)$ is a Krasner hyperring. We will show that $Z_r(H(R)) = 0$. Indeed, for each $0 \neq A \in H(R)$, $r_{H(R)}(A) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in H(R) \mid AB = 0 \right\}$ but $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \in H(R) \mid y \in R \right\}$ is a nonzero right hyperideal of $H(R)$ and $r_{H(R)}(A) \cap I = 0$. Thus $r_{H(R)}(A)$ is not an essential right hyperideal of $H(R)$.

Lemma 3.3. Let R be a semiprime hyperring, K an essential right hyperideal of R and $r \in R$. Then:

- (i) $(r : K)_R$ is an essential right hyperideal of R ;
- (ii) $Z_r(R)$ is a hyperideal of R ;
- (iii) $Z_r(R) = 0$ if and only if every essential right hyperideal is dense;
- (iv) For any subhyperring $R \subseteq S \subseteq Q_{mr}(R)$, $Z_r(R) = R \cap Z_r(S)$.

Proof. (i) Clearly, $(r : K)_R = \{x \in R \mid rx \in K\}$ is a right hyperideal of R . Let $L \neq 0$ be a right hyperideal of R . If $rL = 0$, then $L \subseteq (r : K)_R$. Hence $0 \neq L = L \cap (r : K)_R$. Now, suppose that $rL \neq 0$. Since rL is a right hyperideal of R and K an essential right hyperideal of R , $rL \cap K \neq 0$. But $rL \cap K = r[L \cap (r : K)_R]$. Thus $L \cap (r : K)_R \neq 0$ and so $(r : K)_R$ is essential.

(ii) Let $x, y \in Z_r(R)$ and $r \in R$. We first verify that $x - y \subseteq Z_r(R)$. Let $a \in x - y$. We shall show that $r_R(a)$ is essential. Since $r_R(x) \cap r_R(y) \subseteq r_R(x - y)$ and $r_R(x) \cap r_R(y)$ is an essential right hyperideal, $r_R(x - y)$ is essential as well. Further $r_R(x - y) \subseteq r_R(a)$. Indeed, for any $b \in r_R(x - y)$, $(x - y)b = 0$. Since $a \in x - y$, $ab \in (x - y)b = 0$. Thus $ab = 0$ and so $b \in r_R(a)$. Hence $r_R(a)$ is an essential right hyperideal of R . As $r_R(x) \subseteq r_R(rx)$ and $r_R(x)$ is essential, $rx \in Z_r(R)$. By the above result, the right hyperideal $(r : r_R(x))$ is essential. From $(r : r_R(x)) \subseteq r_R(xr)$ it follows that $xr \in Z_r(R)$. Therefore $Z_r(R)$ is a hyperideal of R .

(iii) Suppose that $Z_r(R) = 0$. Let J be an essential right hyperideal of R . Taking into account (i), we have $l_R((a : J)) = 0$ for all $a \in R$. By Corollary 3.1, we obtain $J \in D(R)$. The converse statement follows from Proposition 3.1 (v).

(iv) Note that $r_R(x) = r_S(x) \cap R$ for all $x \in R$. Thus, by Lemma 3.2, $r_R(x)$ is an essential right hyperideal of S . Hence $Z_r(R) = R \cap Z_r(S)$.

Lemma 3.4. *Let R be a semiprime hyperring, $Q = Q_{mr}(R)$ and K a subhypermodule of the right R -hypermodule Q . Suppose that $\alpha : K \rightarrow Q$ is a good homomorphism of right R -hypermodules. Then:*

(i) *The rule $\hat{\alpha} : KQ \rightarrow Q$, $\hat{\alpha}(\sum_{i=1}^n k_i q_i) = \sum_{i=1}^n \alpha(k_i) q_i$, where $k_i \in K$, $q_i \in Q$ defines a good homomorphism of right Q -hypermodules.*

(ii) *If K is a right hyperideal of the hyperring Q , then α is a good homomorphism of right Q -hypermodules.*

Proof. (i) It is enough to check that $\hat{\alpha}$ is well-defined. Indeed, let $\sum_{i=1}^n k_i q_i = 0$, where $k_i \in K$, $q_i \in Q$. By Lemma 3.1, there exists a dense right hyperideal L of R such that $q_i L \subseteq R$ for all i . For any $x \in L$ we have

$$\left(\sum_{i=1}^n \alpha(k_i) q_i \right) x = \sum_{i=1}^n \alpha(k_i) (q_i x) = \alpha \left(\sum_{i=1}^n k_i q_i x \right) = 0.$$

That is, $\sum_{i=1}^n \alpha(k_i) q_i = 0$ and $\hat{\alpha}$ is well-defined.

(ii) If K is a right hyperideal of the hyperring Q , then $\alpha = \hat{\alpha}$ which means that α is a good homomorphism of right Q -hypermodules.

Definition 3.4. *An element $a \in R$ is said to be regular if $a \in aRa$. That is, there exists an element $b \in R$ such that $a = aba$. A hyperring R is said to be regular (in the sense of von Neumann) if every element of R is regular.*

Theorem 3.2. *Let R be a semiprime hyperring and $Q = Q_{mr}(R)$. Then the following conditions are equivalent:*

- (i) *Q is a von Neumann regular hyperring;*
- (ii) *$Z_r(R) = 0$.*

Proof. (i) $\Rightarrow$ (ii). We suppose that Q is a von Neumann regular hyperring. Let $0 \neq q \in Q$. Hence $qpq = q$ for some $p \in Q$. Clearly, $r_Q(pq) = r_Q(q)$ and $(pq)^2 = pq$. Then $r_Q(pq) = (1 - pq)Q$. Indeed, for any $x \in r_Q(pq)$, we write $pqx = 0$. Since Q is a hyperring, $0 = pqx \in pqx' - pqx' = pqx' - pqpqx' = pq(1 - pq)x'$ where $x' \in Q$. Thus, $x \in (1 - pq)Q$, i.e.; $r_Q(pq) \subseteq (1 - pq)Q$. Similarly, $(1 - pq)Q \subseteq r_Q(pq)$. Since $(1 - pq)Q \cap pqQ = 0$, $(1 - pq)Q$ is not essential. Therefore $Z_r(Q) = 0$.

By Lemma 3.3, $Z_r(R) = 0$.

(ii) $\Rightarrow$ (i). Suppose that $Z_r(R) = 0$. Then, according to Lemma 3.3 (iii), the set $D(R)$ coincides with the set of all essential right hyperideals of R . Let $q = f_U \in Q = Q_{mr}$ and $K = \ker f$. Choosing L to be a right hyperideal of R maximal with the properties $L \subseteq U$ and $L \cap K = 0$, we note that $L \cong qL$. Then $K + L$ is an essential right hyperideal of R and so $K + L \in D(R)$. We choose M to be a right hyperideal of R maximal with the property $M \cap qL = 0$. Obviously, $M \oplus qL$ is an essential right hyperideal of R . Thus $M \oplus qL \in D(R)$. The mapping $g : M \oplus qL \rightarrow L$, $g(x) = l$ where $x \in m + ql$, $m \in M$ and $l \in L$. Hence $p = g_{M \oplus qL} \in Q$ and $(fgf)(k + l) = f(k + l)$ for all $k \in K$, $l \in L$. Thus $ppp = p$ and so Q is von Neumann regular hyperring.

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Study on the optimization selection of cross-border E-commerce logistics distribution path under the O2O mode

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Abstract. The rapid development of cross-border e-commerce in online to offline (O2O) mode has brought more opportunities and challenges to the development of logistics industry. Aviation logistics is the most important way of cross-border logistics, the optimization of which plays an important role in e-commerce enterprises and logistics enterprises. This study studied the types of aircraft and the arrangement of air routes in the process of aviation logistics. The route optimization of passenger cabin and full cargo transportation was discussed respectively. A model was established and the optimization method was designed. Finally example analysis was performed. The results showed the designed method could achieve the maximum profit and the optimization of distribution routes. This study provides some references for the optimization of logistics distribution of cross-border e-commerce and is beneficial to the further development of the cross-border e-commerce logistics industry.

Keywords: O2O, logistics distribution, path optimization.

1. Introduction

Online to offline (O2O) mode can improve the efficiency of commodity sales and the consumption experience of users [1]. It breaks the barrier between traditional enterprises and the Internet and promotes the integration of online and offline businesses. Logistics distribution is a very important part of the O2O mode, and its optimization can effectively improve the competitive edge of enterprises. Site selection of experience shops and logistics distribution path planning are important parts of the O2O mode. Whether the distribution path is reasonable will directly affect the delivery time and efficiency [2]. Therefore, the problem of route optimization is very important in logistics distribution [3].

Deng et al. [4] proposed the hybrid genetic tabu search algorithm for the minimization of total cost of logistics transportation and route optimization and found that the method had high convergence speed and calculation efficiency through simulation experiments. Wang et al. [5] studied cold-chain logistics

distribution, analyzed route optimization based on hybrid genetic algorithm, and proved the effectiveness of the method through numerical simulation.

Jiang et al. [6] proposed a logistics distribution path optimization method based on genetic algorithm and ant colony optimization algorithm, and verified the effectiveness of the method through simulation experiments, indicating that this method can effectively optimize the logistics distribution path. Chen et al. [7] proposed a hybrid cuckoo search (HCS) algorithm for vehicle distribution path optimization, compared it with other existing algorithms through MATLAB.

The results showed that the algorithm was significantly superior to other traditional algorithms in terms of optimal solution and average solution, and could effectively optimize the logistics distribution path. In the O2O cross-border e-commerce, the traditional vehicle delivery can no longer meet the needs of cross-border logistics. The cross-border logistics industry includes aviation logistics, overseas warehouses and so on. Compared to the traditional land transportation, aviation logistics is faster and more convenient in picking up the goods, which can satisfy the requirements of cross-border logistics better.

As the main form of cross-border logistics, the optimization of aviation logistics is of great significance to the e-commerce enterprises and logistics enterprises. In the field of air cargo transportation, price is an important part that affects corporate attitude [8]. Only by reducing their own logistic transportation costs can the logistics enterprises provide the client companies with lower prices and attract more business. In order to reduce the transportation cost and improve the transportation efficiency, aviation logistics needs to consider the aspects of air transportation, aircraft model selection, route arrangement and ground distribution. Aiming at the problems of aircraft type loading and route arrangement, this study studied two transportation modes of passenger plane belly compartment carrying and full cargo plane carrying, and verified the validity of the model through model construction and concrete example analysis.

2. Overview of O2O mode

O2O mode refers to the combination of online and offline. Through online purchase and offline consumption, merchants can publicize their products through the advantages of the Internet and provide more convenient services for consumers. As a new business mode, the O2O mode has developed rapidly with the development of economy and Internet technology. It combines online and offline perfectly, which not only provides consumers with a convenient and fast platform, but also provides offline experience stores that can be experienced personally, helping consumers to feel the quality of products better.

3. Logistics distribution under O2O mode

3.1 Distribution mode classification

3.1.1 Direct distribution

Direct distribution means that the e-commerce company sets up logistics distribution system for distribution in order to meet the needs of its own products. The advantages of direct distribution are that it can not only provide consumers with better distribution services and make the enterprise control the distribution system better, but also can protect the safety of enterprise information and avoid information leakage. The disadvantage is that the investment cost of the direct distribution is relatively high, which requires a large amount of manpower, material and financial input. These investments aggravate the cost burden for enterprises, especially in the case of small distribution demand, which is easier to drag up the cost and reduce the benefits. The flow chart of direct distribution mode is shown in Figure 1.

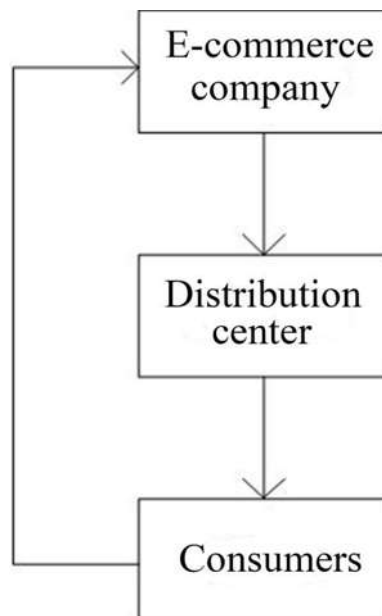


Figure 1: Flow chart of direct distribution mode

3.1.2 Third party distribution

Third party distribution means that the e-commerce enterprise cooperates with the third party logistics enterprise, entrust the logistics business of the company to logistics enterprises, and let the logistics companies complete the distribution of products of the e-commerce enterprise. The advantages of third party distribution are that it can not only reduce the expenditure in logistics and logistics

cost, but also increase the core competitiveness of the enterprise and improve the service efficiency of the enterprise and the customer service experience. The disadvantage is the lack of logistics control. It is difficult for the enterprise to timely obtain information under the third party distribution, and the enterprise is faced with the problems of fund risks and information security. The flow chart of third party distribution is shown in Figure 2.

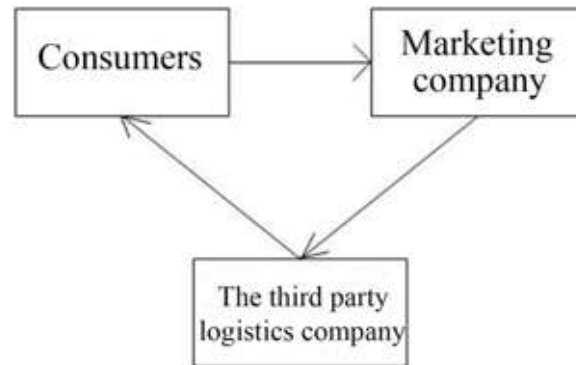


Figure 2: Flow chart of third party distribution

3.1.3 Mixed distribution

The mixed distribution mode refers to the combination of direct distribution and third-party distribution, which develops strengths and avoids weaknesses, and achieve the effect of strong alliance. The comprehensive use of both methods can not only minimize the investment cost in logistics, but also meet the needs of logistics distribution, so as to gain competitive advantages. The disadvantage of mixed distribution lies in the difficulty of unified management and the problem of information security. However, compared with its advantages of high efficiency and low cost, mixed distribution is still favored by many e-commerce enterprises.

3.2 Development status of cross-border e-commerce logistics and distribution

The development prospect of O2O cross-border e-commerce is very promising, but there are also some problems in logistics distribution, such as:

(1) Low consumer satisfaction. Most cross-border e-commerce enterprises use international express to deliver goods to overseas consumers. The high delivery cost and long delivery time, as well as the adverse factors such as customs seizure, bring unpleasant shopping experience to consumers.

(2) Difficulties in goods replace and return. Once the need to return and replace goods purchased across the border occurs, it is a relatively difficult thing to deal with. High logistics costs and long time have caused inconvenience to

return and replace goods, which has done more harm than good to both sellers and buyers.

(3) Goods are easy to lose. Due to the large number of logistics operation procedures and long distance, problems can easily occur in the delivery process, especially in some less developed countries and regions. Once the goods are lost, it is difficult to regain them according to the logistics information.

4. Distribution path optimization model construction

Cross-border logistics distribution is mainly based on aviation logistics. Aviation logistics is a high speed and efficient method of transportation. It transports goods by cargo plane or passenger plane, flies on the established routes and realizes the spatial displacement of goods. Aviation logistics includes the whole process of goods transportation from the beginning to the end, which also means the whole process of air transport, logistics transit and ground distribution. This study only focus on the optimization of the air transport route.

The arrangement of cargo transport routes can greatly affect the cost and time of cargo transport. On the premise of meeting the transportation requirements of e-commerce enterprises and delivering goods within the specified time, airlines can obtain the maximum transportation profits through reasonable arrangement of routes.

At present, there are mainly two ways of air transportation: one is passenger plane belly compartment carrying, the other is full cargo plane carrying. These two transportation methods include different types of aircraft, and the transportation capacity of different aircraft types is quite different. Therefore, it is necessary to discuss these two methods separately.

4.1 Path optimization of passenger plane belly compartment carrying

4.1.1 Model Construction

Before establishing the model, it is assumed that all the goods can be loaded into the aircraft, and the transportation is carried under ideal conditions without considering the loss of the goods or other interference factors in the transportation process.

The parameters involved in model construction are as follows:

A stands for airport set, $a \in A$; T stands for time set, t_j stands for a specific time, $t_j \in T$; M stands for the aircraft type geometry, $m \in M$, u^m stands for total number of aircraft type m , and d^m stands for the transportation capacity of m . Space-time network node is: $n = (e, s, t)$; h_{mat} stands for the number of aircraft type m in the airport a at t_j ; L stands for segment set, $l \in L$; $I(n)$ stands for the segment of flying into the space-time network node n ; $O(n)$ stands for the segment of flying out of node n ; F_l stands for the total flight time of segment l ; f^m stands for maximum daily utilization of aircraft type m ; G stands

for goods set, $g \in G$, B^g stands for the total demand for cargo g ; C_i^m stands for the total operating costs to segment l of aircraft type m ; $\mathcal{V}_Q$ stands for unit variable cost of cargo transshipment in airport a ; $r_l^{g,m}$ stands for unit revenue of cargo g in aircraft type m and segment l ; p_i^m stands for passenger revenue of aircraft type m in segment l ; $y_a^{g,m}$ stands for transshipment volume of cargo g of aircraft type m in hub airport a , $a' \in A$; $y_i^{g,m}$ stands for the volume of cargo g transported by aircraft type m on segment l ; x_i^m stands for whether to use aircraft type m to fly on segment l , $x_1^m \in \{0, 1\}$; y_i^g stands for the transportation volume of cargo g on segment l .

From the above parameters specifications, the following equations can be obtained.

The maximum return is:

$$(4.1) \quad \max Z = \sum_{m \in M} \left[\sum_{g \in G} \left(\sum_{l \in L} y_l^{g,m} r_l^{g,m} - \sum_{a \in A} y_{a'}^{g,m} v_{a'} \right) + \sum_{l \in L} \sum_{m \in M} p_l^m x_l^m - \sum_{l \in L} \sum_{m \in M} c_l^m x_l^m \right].$$

Aircraft flow balance constraint is:

$$(4.2) \quad h_{m,a,t_{j-1}} + \sum_{l \in I(n)} x_l^m = h_{m,a,t_{jt}} + \sum_{l \in O(n)} x_l^m, \\ n = (m, a, t_j), a \in A, m \in M, t_j \in T.$$

Aircraft quantity constraint is:

$$(4.3) \quad \sum_{a \in A} h_{m,a,z_n} + \sum_{l \in O(ct)} x_l^m \leq u^m, m \in M.$$

Aircraft utilization rate is:

$$(4.4) \quad \sum_{l \in \mathcal{L}^m} F_l x_l^m \leq 7u^m f^m, m \in M.$$

Flight coverage constraint is:

$$(4.5) \quad \sum_{m \in M} x_l^m = 1, l \in L, \sum_{g \in G} \sum_{l \in L} y_l^{g,m} \leq d^m x_l^m, m \in M.$$

The cargo volume of aircraft type m is smaller than the total volume of the airplane:

$$(4.6) \quad \sum_{m \in M} \sum_{l \in L} y_l^{g,m} \leq B^g, g \in G.$$

Volume of cargo g in aircraft type m cannot exceed the maximum demand:

$$(4.7) \quad y_l^{g,m} \geq 0, x_l^m \in \{0, 1\}, m \in M, l \in L, g \in G.$$

4.1.2 Benders decomposition algorithm

The Benders decomposition algorithm was used for calculation, the aircraft type selection was taken as the main problem, and the path optimization is solved as the sub-problem respectively.

Suppose the variable $\bar{x}_l^m$ satisfies the constraint condition of type selection, then

$$\begin{aligned} \max \sum_{g \in G} \left(\sum_{l \in L} r_l^g y_l^g - \sum_{d' \in A} y_{a'}^{g,a} v_{a'} \right), \\ \sum_{g \in G} \sum_{l \in L} y_l^g \leq d^m \bar{x}_l^m, m \in M, \\ \sum_{m \in M} \sum_{l \in L} y_l^g \leq B^g, g \in G, \\ y_l^g \geq 0, g \in G, l \in L. \end{aligned}$$

Suppose $\pi_l, l \in L$ and $\delta^g, g \in G$ are dual variables of the sub-problem, then the following equations can be obtained:

$$\begin{aligned} \min \left[\sum_{l \in L} \sum_{m \in M} (d^e \bar{x}_l^m) \pi_l + \sum_{g \in G} B^g \delta^g \right]_{t^*}, \\ \sum_{l \in L} \pi_l + \delta^k \geq r_l^m, g \in G, \\ \pi_l, l \in L, \delta^g \geq 0, g \in G. \end{aligned}$$

The following equation can be obtained according to the dual theorem:

$$\text{Max} \left\{ \min \left[\sum_{l \in L} \sum_{m \in M} (d^e \bar{x}_l^m) \pi_l + \sum_{g \in G} B^g \delta^g \right] + \sum_{l \in L} \sum_{m \in M} p_l^m x_l^m - \sum_{l \in L} \sum_{m \in M} c_l^m x_l^m \right\}.$$

Suppose the optimal vertex of feasible domain is , then the following equations can be obtained:

$$\begin{aligned} \text{Max} \left(\eta + \sum_{l \in L} \sum_{m \in M} p_l^m x_l^m - \sum_{l \in L} \sum_{m \in M} c_l^m x_l^m \right), \\ \eta \leq \sum_{l \in L} \sum_{m \in M} (d^m \bar{x}_l^m) \pi_l + \sum_{g \in G} B^g \delta^g, \\ h_{m,a,t_{j-1}} + \sum_{l \in I(n)} x_l^m = h_{m,a,t_{j+1}} + \sum_{l \in O(n)} x_l^m, n = (m, a, t_j), a \in A, m \in M, t_j \in T, \\ \sum_{a \in A} h_{m,a,t_n} + \sum_{l \in O(ct)} x_l^m \leq u^m, m \in M, \end{aligned}$$

$$\sum_{l \in L^n} F_l x_l^m \leq 7u^m f^m, m \in M,$$

$$\sum_{m \in M} x_l^m = 1, l \in L,$$

$$x_l^m \in \{0, 1\}, l \in L, m \in M.$$

4.1.3 Case analysis

Some flights of an airline were randomly selected for the study. Five sets of data are shown in Table 1.

Table 1: Related data

No.	A	B	C	D	E
Passenger segment	60	100	500	1400	1400
Cargo segment	0	0	200	200	200
Passenger aircraft type	6	6	6	6	6
Cargo aircraft type	0	0	1	1	1
Cargo carrying capacity (piece)	6300	10000	30000	55000	64000

The cost of aircraft shipment, transport and passenger revenue are based on the average level of domestic airports. Assume that the average passenger seating rate capacity of an aircraft is 70 percent, and that passengers carry 10 kilograms of luggage. Aircraft type parameters are shown in Table 2.

Table 2: Aircraft type parameters

Aircraft type	A318	B737-300	B737-500	B737-900	B757-300	B777-200
Cargo loading volume (cubic metre)	21.21	30.2	23.3	52.5	114.1	162.0
Number of passengers	107	149	108	215	280	0
Payload (ton)	3	3	2.5	6.5	9	100

Five sets of data were calculated according to the model and algorithm, and the results are shown in Table 3.

Table 3: Related data

No.	A	B	C	D	E
Calculated time (second)	2.3	3.1	15.6	100.7	116.3
Error rate (%)	0.01	0.05	0.06	0.07	0.08
Objective function value (yuan)	2705600	4306000	19500000	48657200	55042100

As can be seen from the above table, the algorithm in this study could complete the calculation of the data in a relatively short time, and the error

rate was also small, which proved the reliability of the algorithm. By comparing the data, the fifth group got the greatest benefit. It showed that this method can significantly improve the income of airlines in the premise of fully considering the cargo transportation demand.

4.2 Path optimization of full cargo plane carrying

(1) Model construction

x_{ij} stands for the number of planes in the aircraft flow network arc (i, j) ; y_{ij}^n stands for transport volume of the n -th batch goods transported on the cargo flow network arc (i, j) ; N stands for the goods set, n stands for the goods of the specified batch, $n \in N$; C_{ij} stands for the cost on the arc (i, j) ; T_{ij}^n stands for the transportation cost of the n -th batch goods on the arc (i, j) ; E_{ij}^n stands for unit revenue of the n -th batch goods on the arc (i, j) ; U_i stands for unit handling cost of goods i ; AF stands for total available flights; FF stands for set of festival of flight; CF stands for flight ring set; BF_n stands for the total demand for the n -th batch goods; K stands for the maximum capacity of an aircraft; SA stands for airport set; NF stands for aircraft flow network arc set; NP_n stands for the set of arcs for the transport of the n -th batch goods.

From the above parameters specifications, the following equations can be obtained.

The maximum return is:

$$\begin{aligned} \max Z = & \sum_{n \in N} \sum_{ij \in NP_n} E_{ij}^n y_{ij}^n \\ & - \left[\sum_{ij \in NF} C_{ij} x_{ij} + \sum_{n \in N} \sum_{ij \in NP_n} T_{ij}^n y_{ij}^n + \sum_{n \in N} \sum_{ij \in NP_n} y_{ij}^n (U_i + U_j) \right]. \end{aligned}$$

Aircraft flow balance constraint is:

$$\sum_{j \in NF} x_{ij} - \sum_{k \in NF} x_{ki} = 0, \forall i \in NF.$$

Cargo flow balance constraint is:

$$\sum_{j \in NP_n} y_{ij}^n - \sum_{k \in NP_n} y_{ki}^n = 0, \forall i \in NP_n, \forall n \in N.$$

Operating aircraft cannot exceed the total number of aircraft:

$$\sum_{ij \in CF} x_{ij} \leq AF.$$

The total amount of transported cargo must be less than the maximum load capacity of the aircraft:

$$\sum_{n \in N} y_{ij}^n \leq K x_{ij}, \forall ij \in FF.$$

The volume of transported cargo cannot exceed the total demand:

$$\sum_{n \in N} \sum_{ij \in NP_n} y_{ij}^n \leq \sum_{n \in N} BF_n.$$

$$x_{ij} \in 1, \forall ij \in NF, y_{ij}^n \geq 0, \forall ij \in NP_n.$$

CPLEX optimization software was used to obtain the results.

(2) Case analysis

Suppose there are three Boeing 747-200s that carry out 30 times cargo transportation between 8 cities. Each aircraft has a load capacity of 100 tons. The routes are short-distance routes of 1,500 kilometers and long-distance routes of 3,000 kilometers. The transportation modes were divided into short-distance transportation, medium-distance transportation and long-distance transportation. Short-distance transportation means direct transportation. Medium-transportation can be carried out directly or in one transshipment. Long-distance transportation can be carried out directly, in one transshipment or multiple times transshipment, i.e., a mixed transportation method, which is also called the hybrid scheme. According to the model and software, the four schemes were calculated, and the results are shown in Table 4.

Table 4: Related data

Scheme	The number of aircraft	Flight frequency	Average loading rate	Transshipment rate	Optimal value (yuan)	Calculated time (second)
Direct transportation	3	28	60.1%	0	21007000	4.1
One transshipment	3	28	65.3%	5.8%	22080600	13.5
Multiple times transshipment	3	28	67.8%	9.7%	22809700	1100
Hybrid scheme	3	28	66.9%	9.3%	23001200	402

According to the calculation results, the benefit of direct transportation was the worst. Although the efficiency was high, it was not conducive to the full utilization of the aircraft. The loading rate and income of one transshipment were improved, but the effect was slightly lower than that of multiple times transshipment and hybrid scheme. The hybrid scheme had the most advantages. According to the hybrid scheme, the flight routes of the three aircraft are shown in Figure 3.

5. Discussion

With the development of economic globalization, cross-border e-commerce is developing rapidly, which is an important means for enterprises to carry out international trade [9], and is also an important driving force for the growth of China's foreign trade [10, 11]. In cross-border e-commerce enterprises, cross-border logistics is an important part. Only by reducing the cost of international logistics and improving the satisfaction of consumers can the development of cross-border e-commerce enterprises be better promoted.

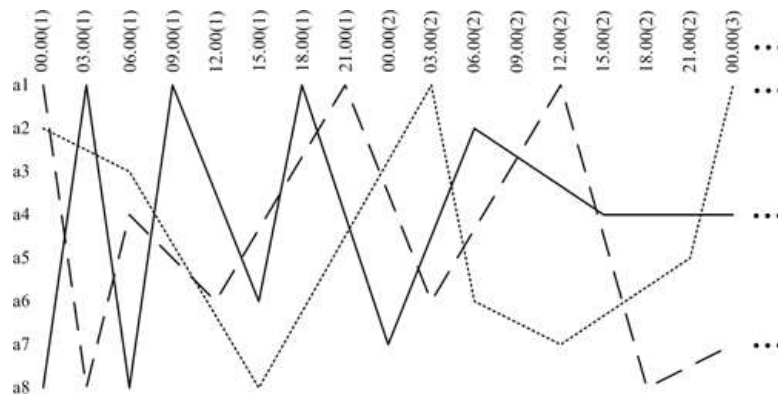


Figure 3: The flight routes of aircraft in the hybrid scheme

In the O2O model, cross-border e-commerce directly faces consumers, reduces the intermediate links of transactions through Internet communication, reduces the cost of transactions [12], and the transaction speed is also relatively fast. However, there are many problems in logistics transportation, such as long transportation time, high logistics cost, difficulties in return and replacement, etc. The current cross-border e-commerce mainly adopt the international express delivery, overseas warehouse and other modes. International express delivery is mainly by air and sea. Aviation logistics is a fast and convenient logistics mode, which is favored by cross-border e-commerce enterprises. In order to meet the requirements of e-commerce enterprises, aviation logistics enterprises must vigorously improve customer satisfaction and gain advantages in the fierce competition. Air logistics not only refers to the transportation of goods in the air, but also to the transshipment, ground distribution and other links. The optimization of route paths is of great significance to reduce cost and improve efficiency. The optimization of distribution path is aimed at reducing transportation cost [13]. Li et al. [14] combined GA and ACA to build a logistics distribution model, which achieved good results. Wang et al. [15] improved the ant colony algorithm, solved the problem of collaborative allocation of forward logistics and reverse logistics, and verified its effectiveness in the optimization of distribution path.

In this study, two cargo transport methods of passenger plane belly compartment carrying and passenger plane belly compartment carrying were used for optimization research respectively. Through the model construction of two cargo transport methods, the calculation method of maximum benefits was obtained, and then different schemes were used for calculation in the actual analysis. The case analysis showed that the proposed method in this study has shorter calculation time and lower error rate, which proved the effectiveness of the method. Then, in terms of the aircraft type selection and route arrangement, it was also proved that different aircraft types and route selection will greatly affect the

earnings of airlines. Therefore, calculating the two and finding the best solution will play an important role in reducing transportation costs and increasing logistics revenue.

6. Conclusion

In the O2O model, cross-border e-commerce cannot leave the support of aviation logistics. The aircraft type selection and route arrangement of the passenger plane belly compartment carrying and the full cargo plane carrying are studied in this study. The validity of the method was proved by the model construction and calculation. The method in this paper can produce results in a short period of time and has good practicability, which also demonstrates the impact of aircraft type selection and route arrangement on revenue. Through reasonable aircraft type selection and route arrangement, it is possible to reduce transportation costs, improve transportation efficiency, and effectively improve the competitive advantage of aviation logistics enterprises in the premise of meeting the logistics needs of e-commerce enterprises.

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Falling fuzzy hyper deductive systems of hyper residuated lattices

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Abstract. This paper introduces and applies the notion of fuzzy hyper deductive systems of hyper residuated lattices as a generalization of the notion of fuzzy deductive systems. It considers hyper deductive systems of hyper residuated lattices, and the relationship between hyper deductive systems and fuzzy hyper deductive systems. Several properties and characterizations of fuzzy hyper deductive systems are given. Based on the falling shadow theory, we establish a theoretical approach by means of exploring the relationships between hyper residuated lattices and probability spaces, and tend to ascertain a falling fuzzy hyper deductive system of a hyper residuated lattice as a generalization of a fuzzy hyper deductive system. We also show some conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system.

Keywords: hyper residuated lattice, falling fuzzy hyper deductive system, falling fuzzy hyper implicative deductive system.

1. Introduction

Residuated lattices as generalization of ideal lattices of rings with identity were introduced by Ward and Dilworth [1] in 1939. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered

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in a frame of residuated lattices. The main examples of residuated lattices related to logic are MV-algebras and BL-algebras. Apart from their logical interest, residuated lattices have interesting algebraic properties, they have long been considered of interest by algebraists starting from the classical example of the lattice-ordered monoid [2, 3, 4]. In order to provide a unified algebraic foundation for probabilities of fuzzy events in substructural logics, Zhao and He [5] enlarged the language of noncommutative residuated lattices by adding an internal state. Since various filters correspond to various sets of provable formulae from a logical point of view, they become an important tool in studying logical algebras and the completeness of non-classical logics [6, 7]. Some filters and their fuzzification of pseudo BCI-algebras [8, 9] and equality algebras [10] which closely related to residual lattices have been widely studied, and some important results have been published.

Algebraic hyperstructures were introduced in 1934 by Marty [11] at 8th Congress of Scandinavian Mathematicians, which represent a natural extension of classical algebraic structures. Nowadays, algebraic hyperstructure theory has a multiplicity of applications to other disciplines such as geometry, graphs and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, artificial intelligence and probability theory [12]. It seems that one of important applications is in logic where the logical operations is not uniquely determined, i.e., they give some (set) possibilities. This provides sufficient motivations for researchers to study hyperstructures of various logical algebras. Ghorbani et al. applied hyperstructures to MV-algebras and introduced the concept of hyper MV-algebras which is a generalization of MV-algebras [13]. Moreover, some types of hyper MV-ideals [14], deductive systems [15], fuzzy deductive systems [16] and the state theory [17] of hyper MV-algebras were introduced and investigated. It is well know that the class of MV-algebras, BL-algebras, and Heyting algebras are proper subclass of residuated lattices. As an application of hyperstructures to residuated lattices, Zahiri et al. put forward the notion of hyper residuated lattices, and constructed quotient hyper residuated lattices by the concept of a regular compatible congruence on hyper residuated lattices [18]. Continue the works in hyper residuated lattices, Borzooei et al. proposed the notions of hyper filters and hyper deductive systems in hyper residuated lattices, and obtained some properties of them [19]. hyperstructures were also applied other algebraic structures, and many kinds of hyper algebras were introduced in recent years, such as hyper EQ-algebras [20], hyper equality algebras [21] and hyper hoop-algebras [22].

The theory of falling shadows is an important tool in the theoretical developments and practical applications of fuzzy sets, and some of their properties and notions are displayed in [23]. On the basis of the theory of a falling shadow, Jun and Kang [24] gave a theoretical approach of a fuzzy positive implicative ideal of a BCK-algebra, and a fuzzy subalgebraic system was considered as the falling shadow of the cloud of the subalgebraic system. Inspired by the work in [25], Yang et al. investigated falling fuzzy (implicative) ideals of MV-algebras

[26] based on the theory of falling shadows and fuzzy sets, which provides a theoretical approach for the further studying of fuzzy ideals in MV-algebras. The falling shadow theory was also applied to study Gödel ideals of BL-algebras [27], hemirings [28] and prefilters of EQ-algebras [29]. These studies have presented a preliminary, but potentially interesting research direction.

Based on the aforementioned, in the present paper we further consider some characterizations of hyper deductive systems of hyper residuated lattices based on the fuzzy set theory and the falling shadow theory. The notion of fuzzy hyper deductive systems of hyper residuated lattices is introduced and some related properties are investigated. We also establish a theoretical approach by means of exploring the relationships between hyper residuated lattices and probability spaces, and tend to use hyper deductive systems of falling shadows to ascertain a falling fuzzy hyper deductive systems in an exceedingly hyper residuated lattices as a generalization of a fuzzy hyper hyper deductive system. Some relationships between fuzzy hyper deductive systems and falling fuzzy hyper deductive systems are offered, and conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system are presented. Related research methods in this paper may provide a useful tool for the research of fuzzy mathematics and hyper algebraic structures, and enrich the research contents of probability theory to some extent.

2. Preliminaries

To facilitate our discussion, we give some primary notions and previous results about MV-algebras which are necessary for the subsequent discussions.

Definition 2.1 ([18]). *A hyper residuated lattice is a non-empty set L endowed with four binary hyperoperations $\vee, \wedge, \otimes, \rightarrow$ and two constants 0 and 1 satisfying the following conditions:*

- (1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded superlattice,
- (2) $(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity,
- (3) $a \otimes b \ll c$ if and only if $a \ll b \rightarrow c$.

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for any nonempty subsets A and B of L . We also define $x * A = \bigcup_{a \in A} x * a$, $A * x = \bigcup_{a \in A} a * x$, and $A * B = \bigcup_{a \in A, b \in B} a * b$, where $*$ $\in \{\vee, \wedge, \otimes, \rightarrow\}$.

L is called nontrivial if $0 \neq 1$. An element $a \in L$ is called scalar if $|a \otimes x| = 1$, for any $x \in L$. In what follows, unless otherwise specified, we denote a hyper residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ by L . Some primary properties of hyper residuated lattices are collected in the following proposition.

Proposition 2.2 ([19]). *Given a hyper residuated lattice L , for any $x, y, z \in L$ and for any non-empty subsets A, B and C of L , the following hold:*

- (1) $x \leq y$ implies $1 \in x \rightarrow y$, and if 1 is a scalar, the converse hold,
- (2) $1 \in x \rightarrow x$, $1 \in x \rightarrow 1$, $1 \in 0 \rightarrow x$, $0 \in x \otimes 0$, $x \ll \neg\neg x$, where $\neg x = x \rightarrow 0$,
- (3) $x \otimes y \ll y$, $x \otimes y \ll x$, $x \ll y \rightarrow (x \otimes y)$, $x \otimes (x \rightarrow y) \ll x$ and $x \otimes (x \rightarrow y) \ll y$,
- (4) $x \rightarrow (y \rightarrow z) \ll (x \otimes y) \rightarrow z \ll x \rightarrow (y \rightarrow z)$,
- (5) $x \leq y$ implies that $x \otimes z \ll y \otimes z$, $z \rightarrow x \ll z \rightarrow y$ and $y \rightarrow z \ll x \rightarrow z$,
- (6) $x \leq y$ and $x \leq z$ imply $x \ll y \wedge z$,
- (7) $y \leq x$ and $z \leq x$ imply $y \vee z \ll x$,
- (8) $A \ll B \rightarrow C$ if and only if $A \otimes B \ll C$,
- (9) $A \ll x \ll B$ implies $A \ll B$.
- (10) $A \cap B \neq \emptyset$ implies $A \ll B$ and $B \ll A$,
- (11) $A \otimes B \ll A$, $A \otimes B \ll B$,
- (12) $A \subseteq B$ implies $A \ll B$.

Definition 2.3 ([19]). A nonempty subset D of L is called a hyper deductive system if it satisfies the conditions: (d_1) $1 \in D$, (d_2) $D \ll x \rightarrow y$ and $x \in D$ imply $y \in D$ for any $x, y \in L$.

Proposition 2.4 ([19]). Let L be a hyper residuated lattice and D be a nonempty subset of L . Then D is a hyper deductive system of L if and only if it satisfying (d_1) and (d_3) $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$.

Definition 2.5 ([19]). Let L be a hyper residuated lattice and D be a nonempty subset D of L containing 1 . Then D is called a hyper implicative deductive system of L if $x \rightarrow (y \rightarrow z) \cap D \neq \emptyset$ and $(x \rightarrow y) \cap D \neq \emptyset$ imply $(x \rightarrow z) \cap D \neq \emptyset$ for any $x, y, z \in L$.

For purpose of convenience, let $\emptyset$ be a hyper deductive system or a hyper implicative deductive system of L in the the rest of the sections.

Let $(P, \leq)$ be a partially ordered set and θ be an equivalence relation on P . Then θ is called regular if the set $P/\theta = \{[x] | x \in P\}$ can be ordered in such a way that the natural map $\pi : P \rightarrow P/\theta$ is order preserving. Let θ be an equivalence relation on L and $A, B \subseteq L$. Then

- (1) $A\theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\theta b$,
- (2) $A\bar{\theta}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\theta b$ and for all $b \in B$, there exists $a \in A$ such that $a\theta b$.

Definition 2.6 ([18]). *An equivalence relation θ on L is called a congruence relation if for all $x, y, w, v \in L$, $x\theta y$ and $w\theta v$ imply $(x * w)\bar{\theta}(y * v)$, where $*$ $\in \{\vee, \wedge, \otimes, \rightarrow\}$.*

Let θ be a regular congruence relation on L , $L/\theta = \{[x] | x \in L\}$ and $\leq_\theta$ be the relation on L/θ (for more details, see [18]). For any $x, y \in L$,

$$[x]\bar{\vee}[y] = [x \vee y], [x]\bar{\wedge}[y] = [x \wedge y], [x]\bar{\otimes}[y] = [x \otimes y], [x] \rightsquigarrow [y] = [x \rightarrow y],$$

where $[A] = \{[a] | a \in A\}$, for all $A \subseteq L$.

A regular congruence relation θ on L if it satisfies the conditions: (i) $[x] \in [x]\bar{\vee}[y]$ iff $[x] \leq_\theta [y]$, (ii) $[x] \in [x]\bar{\wedge}[y]$ iff $[x] \leq_\theta [y]$, then it said to be a regular compatible congruence relation on L .

Theorem 2.7. *Let θ be a regular compatible congruence relation on L . Then $(L/\theta, \bar{\vee}, \bar{\wedge}, \bar{\otimes}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.*

Definition 2.8 ([18]). *Let $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ and $(L', \vee', \wedge', \otimes', \rightarrow', 0', 1')$ be two hyper residuated lattices. A map $f: L \rightarrow L'$ is said to be a homomorphism if $f(0) = 0'$, $f(1) = 1'$ and $f(x * y) = f(x) *' (y)$ for any $x, y \in L$, where $*$ $\in \{\vee, \wedge, \otimes, \rightarrow\}$.*

3. Fuzzy hyper deductive systems

In this section, we consider fuzzy hyper deductive systems of hyper residuated lattices and investigate some properties of them.

Theorem 3.1. *Let D be a nonempty subset of L . Then D is a hyper deductive system of L if and only if*

- (1) $x \leq y$ and $x \in D$ imply $y \in D$,
- (2) $x \in D$ and $y \in D$ imply $x \otimes y \subseteq D$,

for any $x, y \in L$.

Proof. $\Rightarrow$ Let $x, y \in L$ be such that $x \leq y$ and $x \in D$. Then $1 \in x \rightarrow y$. From $1 \in D$, we get that $D \ll x \rightarrow y$, and so $y \in D$ by hypothesis. Now let $x \in D$ and $y \in D$. For any $u \in x \otimes y$, we have $x \otimes y \ll u$, it follows that $x \ll y \rightarrow u$ by Proposition 2.2 (8). Since $x \in D$, then $D \ll y \rightarrow u$, and so $u \in D$ by hypothesis. Thus $x \otimes y \subseteq D$.

$\Leftarrow$ Let $x, y \in L$ be such that $D \ll x \rightarrow y$ and $x \in D$. Then there exist $u \in D$ and $v \in x \rightarrow y$ such that $u \leq v$, it follows that $v \in D$, and so $x \otimes v \subseteq D$. Notice that $v \in x \rightarrow y$, we have $v \ll x \rightarrow y$, that is, $v \otimes x \ll y$. Thus there exists $t \in v \otimes x \subseteq D$ such that $t \leq y$, therefore we get $y \in D$. $\square$

Proposition 3.2. *Let D be a hyper deductive system of L . For any nonempty subset B of L , and $a \in D$, if $(a \rightarrow B) \cap D \neq \emptyset$, then $B \cap D \neq \emptyset$.*

Proof. Let $B \subseteq L$ and $a \in D$ be such that $(a \rightarrow B) \cap D \neq \emptyset$. Then there exists $x \in D$ such that $x \in a \rightarrow b$ for some $b \in B$. Thus $(a \rightarrow b) \cap D \neq \emptyset$. Since D is a hyper deductive system of L , it follows from Proposition 2.4 that $b \in D$, and so $B \cap D \neq \emptyset$. $\square$

Proposition 3.3. *Let D be a hyper deductive system of L . Then the following hold: for any $x, y, z \in L$,*

- (1) $(x \rightarrow y) \subseteq D$ and $x \in D$ imply $y \in D$,
- (2) if $x, y \in D$ and $x \ll y \rightarrow z$, then $z \in D$,
- (3) if $(x \rightarrow y) \subseteq D$ and $x \otimes z \subseteq D$, then $y \otimes z \subseteq D$.

Proof. (1) Let $x, y \in L$ be such that $(x \rightarrow y) \subseteq D$ and $x \in D$. Then $x \otimes (x \rightarrow y) = \bigcup_{u \in x \rightarrow y} x \otimes u \subseteq D$. On the other hand, from Proposition 2.2 (3), we know that $x \otimes (x \rightarrow y) \ll y$. Hence, there exists $v \in x \otimes (x \rightarrow y)$ such that $v \leq y$. Notice that $v \in D$, we get $y \in D$.

(2) Suppose that $x, y \in D$ and $x \ll y \rightarrow z$ for any $x, y, z \in L$, then $x \otimes y \subseteq D$ and $x \otimes y \ll z$. Then there exists $u \in x \otimes y \subseteq D$ such that $u \leq z$. By hypothesis that D is a hyper deductive system of L , we get $z \in D$.

(3) According to Proposition 2.2 (3), we have $x \otimes z \ll x, z$, then there exists $u \in x \otimes z \subseteq D$ such that $u \leq x$, and so $x \in D$. Similarly, $z \in D$. Now, since $x \in D$ and $(x \rightarrow y) \subseteq D$, then we get $y \in D$ by (1). From Theorem 3.1 and $y, z \in D$, it follows that $y \otimes z \subseteq D$. $\square$

A fuzzy subset μ of L satisfies the sup-property if for any nonempty subset T of L there exists $x_0 \in T$ such that $\mu(x_0) = \sup_{x \in T} \mu(x)$. For any fuzzy subset μ of L the set $\mu_t := \{x \in L \mid \mu(x) \geq t\}$ is called a level set of μ . In order to facilitate the discussion, we shall use the notations “ $\bigvee \mu(A)$ ” and “ $\bigwedge \mu(A)$ ” to mean “ $\sup_{a \in A} \mu(a)$ ” and “ $\inf_{a \in A} \mu(a)$ ”, respectively, for any nonempty set A . Moreover, for any $a, b \in A$, $a \wedge b$ and $a \vee b$ mean $\min\{a, b\}$ and $\max\{a, b\}$, respectively.

Definition 3.4. *A fuzzy set μ of L is called a fuzzy hyper deductive system if it satisfies: for any $x, y \in L$,*

- (1) $\mu(1) \geq \mu(x)$,
- (2) $\mu(y) \geq \bigvee \mu(x \rightarrow y) \wedge \mu(x)$.

For better understanding the definition of fuzzy hyper deductive systems, we illustrate it by the following example.

Example 3.5. Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$. Consider hyperoperations $\vee$ and $\wedge$ as given in the following tables,

$\vee$	0	a	b	1
0	$\{0, a, b, 1\}$	$\{a, b, 1\}$	$\{b, 1\}$	$\{1\}$
a	$\{a, b, 1\}$	$\{a, b, 1\}$	$\{b, 1\}$	$\{1\}$
b	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{1\}$
1	$\{0, 1\}$	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{0\}$	$\{0, a\}$	$\{0, a, b\}$	$\{0, a, b\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$

Then $(L, \vee, \wedge, 0, 1)$ is a bound hyper lattice. Define hyper hyperoperations $\otimes$ and $\rightarrow$ as follows,

$\otimes$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

$\rightarrow$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{0\}$	$\{0, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

it is easy to verify that $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a hyper residuated lattice [19]. Define a fuzzy set μ of L as follows:

$$\mu(x) = \begin{cases} 0.3, & x = 0, \\ 0.3, & x = a, \\ 0.5, & x = b, \\ 0.5, & x = 1. \end{cases}$$

Route calculations show that μ is a fuzzy hyper deductive system of L .

Proposition 3.6. Let μ be a fuzzy hyper deductive system of L . Then $\emptyset \neq \mu_t$ is a hyper deductive system of L for any $t \in [0, 1]$.

Proof. Let $t \in [0, 1]$ such that $\emptyset \neq \mu_t$. Then there exists $x \in \mu_t$, and so $\mu(x) \geq t$. Since μ is a fuzzy hyper deductive system of L , then $\mu(1) \geq \mu(x)$, so $1 \in \mu_t$. Let $(x \rightarrow y) \cap \mu_t \neq \emptyset$ and $x \in \mu_t$. Then $\mu(x) \geq t$ and there exists $w \in (x \rightarrow y) \cap \mu_t$, therefore $\bigvee \mu(x \rightarrow y) \geq \mu(w) \geq t$. It follows that $\mu(y) \geq \bigvee \mu(x \rightarrow y) \wedge \mu(x) \geq t$, hence $y \in \mu_t$. According to Proposition 2.4, we know that μ_t is a hyper deductive system of L . $\square$

Proposition 3.7. *Let μ be a fuzzy subset of L with the sup-property. If $\emptyset \neq \mu_t$ is a hyper deductive system of L for any $t \in [0, 1]$, then μ is a fuzzy hyper deductive system of L .*

Proof. For any $x \in L$, $x \in \mu_{\mu(x)}$. Since $\mu_{\mu(x)}$ is a hyper deductive system of L , then $1 \in \mu_{\mu(x)}$, and so $\mu(1) \geq \mu(x)$. For every $x, y \in L$, Now let $t_1 = \bigvee \mu(x \rightarrow y)$, $t_2 = \mu(x)$, and $t = t_1 \wedge t_2$, then $x \in \mu_t$. Notice that μ satisfies the sup-property, there exists $w \in x \rightarrow y$ such that $t_1 = \bigvee \mu(x \rightarrow y) = \mu(w)$, and so $w \in \mu_t$. It follows that $(x \rightarrow y) \cap \mu_t \neq \emptyset$ and $x \in \mu_t$, we get $y \in \mu_t$ by Proposition 2.4, hence $\mu(y) \geq t = \bigvee \mu(x \rightarrow y) \wedge \mu(x)$. Thus, μ is a fuzzy hyper deductive system of L . $\square$

Corollary 3.8. *Let $\emptyset \subsetneq D \subseteq L$ and $\alpha \in (0, 1]$. Define a fuzzy set $\mu_D : L \rightarrow [0, 1]$ by*

$$\mu_D(x) = \begin{cases} \alpha, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Then μ_D is a fuzzy hyper deductive system of L if and only if D is a hyper deductive system of L .

Proposition 3.9. *Let μ be a fuzzy hyper deductive system of L . Then for any $x, y \in L$, the following statements hold:*

- (1) *if $x \leq y$, then $\mu(x) \leq \mu(y)$,*
- (2) *$\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$.*

Proof. (1) Assume that $x \leq y$, then $1 \in x \rightarrow y$. Notice that μ is a fuzzy hyper deductive system of L , we get $\mu(1) \geq \mu(z)$ for any $z \in L$, and so, $\bigvee \mu(x \rightarrow y) = \mu(1)$. Moreover, $\mu(y) \geq \bigvee \mu(x \rightarrow y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$.

(2) For any $x, y \in L$, let $\mu(x) = t_1$, $\mu(y) = t_2$ and $t_1 \wedge t_2 = t$. It follows that $x, y \in \mu_t$, and μ_t is a hyper deductive system of L by Proposition 3.6. Therefore, $x \otimes y \subseteq \mu_t$, and so $\bigwedge \mu(x \otimes y) \geq t = \mu(x) \wedge \mu(y)$. $\square$

Proposition 3.10. *Let μ be a fuzzy subset of L with the sup-property. If μ satisfies the following conditions: for any $x, y \in L$, (1) $x \leq y$ implies $\mu(x) \leq \mu(y)$, (2) $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$, then μ is a fuzzy hyper deductive system of L .*

Proof. Let $t \in [0, 1]$ and $\mu_t \neq \emptyset$. Let $x, y \in L$ be such that $x \leq y$ and $x \in \mu_t$. By assumption, we obtain that $t \leq \mu(x) \leq \mu(y)$, thus $y \in \mu_t$. For any $x, y \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows that $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y) \geq t$. For any $w \in x \otimes y$, we get $\mu(w) \geq t$, therefore $w \in \mu_t$, and so $x \otimes y \subseteq \mu_t$. According to Proposition 3.7, we conclude that μ is a fuzzy hyper deductive system of L . $\square$

Theorem 3.11. *Let μ be a fuzzy subset of L with the sup-property. Then μ is a fuzzy hyper deductive system of L if and only if $x \otimes y \ll z$ implies $\mu(z) \geq \bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$, for any $x, y, z \in L$.*

Proof. The sufficiency is obvious by Proposition 3.9, now we give the proof of the necessity. Since $x \otimes y \ll x$ for any $x, y \in L$, then $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$. Notice that $x \otimes x \ll x \ll 1$, we get $x \otimes x \ll 1$, thus $\mu(1) \geq \mu(x) \wedge \mu(x) = \mu(x)$. If $x \leq y$, then $x \otimes 1 \ll x \ll y$, and so $x \otimes 1 \ll y$, it follows that $\mu(y) \geq \mu(x) \wedge \mu(1) = \mu(x)$. From Proposition 3.10, we get that μ is a fuzzy hyper deductive system of L . $\square$

Proposition 3.12. *Let μ, ν be two fuzzy hyper deductive systems of L . Then we have*

- (1) $H_\mu := \{x \in L \mid \mu(x) = \mu(1)\}$ is a hyper deductive system of L ,
- (2) if $\nu \leq \mu$ and $\nu(1) = \mu(1)$, then $H_\mu \otimes H_\nu = \bigcup_{a \in H_\mu, b \in H_\nu} a \otimes b$ is a hyper deductive system of L , where $\nu \leq \mu$ is defined by $\nu(x) \leq \mu(x)$ for any $x \in L$.

Proof. (1) Let $x, y \in L$ be such that $x \leq y$ and $x \in H_\mu$. Since μ is a fuzzy hyper deductive system of L , then $\mu(1) = \mu(x) \leq \mu(y)$, and so $\mu(y) = \mu(1)$, that is, $y \in H_\mu$. Now let $x \in H_\mu$ and $y \in H_\mu$. Then $\mu(x) = \mu(1)$ and $\mu(y) = \mu(1)$. For any $t \in x \otimes y$, $\mu(t) \geq \bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y) = \mu(1)$ by Proposition 3.9. It follows that $\mu(t) = \mu(1)$, that is, $t \in H_\mu$, and so $x \otimes y \subseteq H_\mu$. According to the above discussion, we get that H_μ is a hyper deductive system of L by Theorem 3.1.

(2) Obviously, $1 \in H_\mu \otimes H_\nu$. Let $x, y \in L$ be such that $(x \rightarrow y) \cap (H_\mu \otimes H_\nu) \neq \emptyset$ and $x \in H_\mu \otimes H_\nu$. Then there exist $a \in H_\mu$ and $b \in H_\nu$ such that $x \in a \otimes b$. It follows that $\mu(a) = \mu(1)$, $\nu(b) = \nu(1) = \mu(1)$ and $\mu(y) \geq \bigvee \mu(x \rightarrow y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x) \geq \bigwedge \mu(a \otimes b) \geq \mu(a) \wedge \mu(b) = \mu(b) \geq \nu(b) = \mu(1)$. Therefore $\mu(y) = \mu(1)$, and so $y \in y \otimes 1 \subseteq H_\mu \otimes H_\nu$. Hence $H_\mu \otimes H_\nu$ is a hyper deductive system of L . $\square$

Let $(L_1, \vee_1, \wedge_1, \otimes_1, \rightarrow_1, 0_1, 1_1)$ and $(L_2, \vee_2, \wedge_2, \otimes_2, \rightarrow_2, 0_2, 1_2)$ be two hyper residuated lattices. Then $L_1 \times L_2$ is also a hyper residuated lattice with respect to the point-wise hyper operations given by:

$$(a, b) * (u, v) = (a *_1 u, b *_2 v),$$

for any $(a, b), (u, v) \in L_1 \times L_2$, where $*$ $\in \{\vee, \wedge, \otimes, \rightarrow\}$.

Definition 3.13. *Let μ and ν be fuzzy sets of L_1 and L_2 , respectively, where L_1 and L_2 are two hyper residuated lattices. The cartesian product $\mu \times \nu$ of μ and ν is defined by*

$$(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y),$$

for any $(x, y) \in L_1 \times L_2$. Obviously, $\mu \times \nu$ is a fuzzy set of $L_1 \times L_2$.

Proposition 3.14. *Let L_1 and L_2 be two hyper residuated lattices. If μ and ν are fuzzy hyper deductive systems of L_1 and L_2 , respectively, then $\mu \times \nu$ is a fuzzy hyper deductive system of $L_1 \times L_2$.*

Proof. For any $(x, y), (a, b) \in L_1 \times L_2$, we have $(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y) \leq \mu(1_1) \wedge \nu(1_2) = (\mu \times \nu)(1_1, 1_2)$. And

$$\begin{aligned} (\mu \times \nu)(a, b) &= \mu(a) \wedge \nu(b) \\ &\geq \left(\bigvee \mu(x \rightarrow_1 a) \wedge \mu(x) \right) \wedge \left(\bigvee \nu(y \rightarrow_2 b) \wedge \nu(y) \right) \\ &= \left(\bigvee \mu(x \rightarrow_1 a) \wedge \bigvee \nu(y \rightarrow_2 b) \right) \wedge (\mu(x) \wedge \nu(y)) \\ &= \bigvee (\mu \times \nu)((x, y) \rightarrow (a, b)) \wedge (\mu \times \nu)((x, y)). \end{aligned}$$

Therefore, $\mu \times \nu$ is a fuzzy hyper deductive system of $L_1 \times L_2$. $\square$

Proposition 3.15. *Let μ be a fuzzy set of L . Then μ is a fuzzy hyper deductive system of L if and only if $\mu \times \mu$ is a fuzzy hyper deductive system of $L \times L$.*

Proof. The sufficiency is very clear by Proposition 3.14, we only need to give the proof of the necessity. We first show that $\mu(x) \leq \mu(1)$ for any $x \in L$. In fact that $\mu(x) = \mu(x) \wedge \mu(x) = (\mu \times \mu)(x, x) \leq (\mu \times \mu)(1, 1) = \mu(1) \wedge \mu(1) = \mu(1)$, hence $\mu(x) \leq \mu(1)$. Moreover, due to $1 \in 1 \rightarrow 1$, we get that $\bigvee \mu(1 \rightarrow 1) = \mu(1)$.

For $x, y \in L$, it follows that

$$\begin{aligned} \mu(y) &= \mu(y) \wedge \mu(1) \\ &= (\mu \times \mu)(y, 1) \\ &\geq \bigvee (\mu \times \mu)((x, 1) \rightarrow (y, 1)) \wedge (\mu \times \mu)(x, 1) \\ &= \bigvee (\mu \times \mu)(x \rightarrow y, 1 \rightarrow 1) \wedge (\mu \times \mu)(x, 1) \\ &= \bigvee \mu(x \rightarrow y) \wedge \bigvee \mu(1 \rightarrow 1) \wedge \mu(x) \wedge \mu(1) \\ &= \bigvee \mu(x \rightarrow y) \wedge \mu(x). \end{aligned}$$

Therefore, μ is a fuzzy hyper deductive system of L . $\square$

Definition 3.16. *Let μ and ν be fuzzy sets of L . Then we define the fuzzy product $\mu \circledast \nu$ as*

$$(\mu \circledast \nu)(t) = \begin{cases} \bigvee_{t \in x \otimes y} (\mu(x) \wedge \nu(y)), & \text{if } \exists x, y \in L \text{ such that } t \in x \otimes y, \\ 0, & \text{otherwise,} \end{cases}$$

for any $t \in L$.

Theorem 3.17. *Let μ be a fuzzy set of L with the sup-property. If μ is order preserving, that is, $x \leq y$ implies $\mu(x) \leq \mu(y)$ for any $x, y \in L$, then μ is a fuzzy hyper deductive system of L if and only if $\mu \circledast \mu \leq \mu$.*

Proof. Let μ be a fuzzy hyper deductive system of L and $x \in L$. If there do not exist $y, z \in L$ such that $x \in y \otimes z$, then $(\mu \circledast \mu)(x) = 0 \leq \mu(x)$, and so

$\mu \circledast \mu \leq \mu$. Otherwise, notice that μ is a fuzzy hyper deductive system of L , we get that $\mu(x) \geq \bigwedge_{x \in y \otimes z} \mu(x) \geq \mu(y) \wedge \mu(z)$ by Proposition 3.9. Hence

$$(\mu \circledast \mu)(x) = \bigvee_{x \in y \otimes z} (\mu(y) \wedge \mu(z)) \leq \bigvee_{x \in y \otimes z} \mu(x) = \mu(x),$$

and so $\mu \circledast \mu \leq \mu$ is valid.

Conversely, assume that $\mu \circledast \mu \leq \mu$ hold. Then for every $x \in y \otimes z$, we have $\mu(x) \geq (\mu \circledast \mu)(x) = \bigvee_{x \in y \otimes z} (\mu(y) \wedge \mu(z)) \geq \mu(y) \wedge \mu(z)$. Thus $\bigwedge_{x \in y \otimes z} \mu(x) \geq \mu(y) \wedge \mu(z)$, from Proposition 3.10 we get that μ is a fuzzy hyper deductive system of L . $\square$

Theorem 3.18. *Let L_1 and L_2 be two hyper residuated lattices, $f : L_1 \rightarrow L_2$ be a surjective homomorphism. Then the fuzzy set $\nu : L_2 \rightarrow [0, 1]$ is a fuzzy hyper deductive system of L_2 if and only if $\nu_f : L_1 \rightarrow [0, 1]$ is a fuzzy hyper deductive system of L_1 , where $\nu_f : L_1 \rightarrow [0, 1]$ is defined by $\nu_f(x) = \nu(f(x))$ for any $x \in L_1$.*

Proof. Suppose that ν is a fuzzy hyper deductive system of L_2 , then

$$\nu_f(1_1) = \nu(f(1_1)) = \nu(1_2) \geq \nu(f(x)) = \nu_f(x).$$

For any $x, y \in L_1$,

$$\begin{aligned} \nu_f(y) &= \nu(f(y)) \\ &\geq \bigvee \nu(f(x) \rightarrow_2 f(y)) \wedge \nu(f(x)) \\ &= \bigvee \nu(f(x \rightarrow_1 y)) \wedge \nu(f(x)) \\ &= \bigvee \nu_f(x \rightarrow_1 y) \wedge \nu_f(x). \end{aligned}$$

Therefore, ν_f is a fuzzy hyper deductive system of L_1 .

Conversely, assume that ν_f is a fuzzy hyper deductive system of L_1 . Since f is surjective, then there exists $x \in L_1$ such that $f(x) = y$ for any $y \in L_2$. We get that

$$\nu(1_2) = \nu(f(1_1)) = \nu_f(1_1) \geq \nu_f(x) = \nu(f(x)) = \nu(y).$$

Let $a, b \in L_2$. Then there exists $x, y \in L_1$ such that $f(x) = a$ and $f(y) = b$. It follows that

$$\begin{aligned} \nu(b) &= \nu(f(y)) \\ &= \nu_f(y) \\ &\geq \bigvee \nu_f(x \rightarrow_1 y) \wedge \nu_f(x) \\ &= \bigvee \nu(f(x \rightarrow_1 y)) \wedge \nu(f(x)) \\ &= \bigvee \nu(f(x) \rightarrow_2 f(y)) \wedge \nu(f(x)) \\ &= \bigvee \nu(a \rightarrow_2 b) \wedge \nu(a). \end{aligned}$$

Hence, ν is a fuzzy hyper deductive system of L_2 . $\square$

Proposition 3.19. *Let L_1 and L_2 be two hyper residuated lattices, $f : L_1 \rightarrow L_2$ be a homomorphism, the fuzzy set $\nu : L_2 \rightarrow [0, 1]$ is a fuzzy hyper deductive system of L_2 . Then we have*

- (1) *if $x \in \ker(f) := \{x \in L_1 | f(x) = 1_2\}$, then $\nu_f(x) \geq \nu(y)$ for any $y \in L_2$,*
- (2) *if μ is a fuzzy hyper deductive system of L_1 and f is an epimorphism, then $f(\mu) : L_2 \rightarrow [0, 1]$ is a fuzzy hyper deductive system of L_2 , where $f(\mu)(y) = \bigvee \{\mu(x) | f(x) = y\}$ for any $y \in L_2$.*

Proof. (1) For any $x \in \ker(f)$, we get $f(x) = 1_2$. Notice that ν is a fuzzy hyper deductive system of L_2 , then $\nu_f(x) = \nu(f(x)) = \nu(1_2) \geq \nu(y)$ for any $y \in L_2$.

(2) For any $y \in L_2$, we have

$$f(\mu)(y) = \bigvee \{\mu(x) | f(x) = y\} \leq \mu(1_1) = \bigvee \{\mu(x) | f(x) = 1_2\} = f(\mu)(1_2).$$

For any $a, b \in L_2$,

$$\begin{aligned} f(\mu)(a) \wedge f(\mu)(a \rightarrow_1 b) &= \bigvee \{\mu(x) | f(x) = a, x \in L_1\} \wedge \bigvee \{\mu(x \rightarrow_1 y) | f(x \rightarrow_1 y) = a \rightarrow_2 b, x, y \in L_1\} \\ &= \bigvee \{\mu(x) | f(x) = a, x \in L_1\} \wedge \bigvee \{\mu(x \rightarrow y) | f(x) \rightarrow_2 f(y) = a \rightarrow_2 b, x, y \in L_1\} \\ &\geq \bigvee \{\mu(x) \wedge \bigvee \mu(x \rightarrow_1 y) | f(x) = a, f(y) = b, x, y \in L_1\} \\ &\geq \bigvee \{\mu(y) | f(y) = b, y \in L_1\} \\ &= f(\mu)(b), \end{aligned}$$

thus $f(\mu)$ is a fuzzy hyper deductive system of L_2 . □

Theorem 3.20. *Let μ be a fuzzy hyper deductive system of L . Then there exists a fuzzy hyper deductive system $\bar{\mu}$ of L/θ such that $\bar{\mu} \circ \pi \geq \mu$.*

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L/\theta \\ & \searrow \mu & \downarrow \bar{\mu} \\ & & [0, 1] \end{array}$$

Proof. Define $\bar{\mu} : L/\theta \rightarrow [0, 1]$ by $\bar{\mu}([x]) = \bigvee_{t \in [x]} \mu(t)$, for any $x \in L$. Obviously, $\bar{\mu}$ is well defined. Let $x, y \in L$. Then for any $a \in [x]$ and $b \in [y]$, we get $\bigvee_{b \in [x]} \mu(b) \geq \mu(b) \geq \bigvee \mu(a \rightarrow b) \wedge \mu(a)$, and so

$$\begin{aligned} \bar{\mu}([y]) &= \bigvee_{b \in [y]} \mu(b) \\ &= \bigvee_{a \in [x]} \bigvee_{b \in [y]} \mu(b) \\ &\geq \bigvee_{a \in [x]} \bigvee_{b \in [y]} \left(\bigvee \mu(a \rightarrow b) \wedge \mu(a) \right) \end{aligned}$$

$$\begin{aligned}
&\geq \bigvee_{a \in [x]} \bigvee_{b \in [y]} \left(\bigvee \mu(a \rightarrow b) \right) \wedge \bigvee_{a \in [x]} \bigvee_{b \in [y]} \mu(a) \\
&\geq \bigvee_{a \rightarrow b \in [x \rightarrow b]} \left(\bigvee \mu(a \rightarrow b) \right) \wedge \bigvee_{a \in [x]} \mu(a) \\
&= \bigvee \bar{\mu}([a] \rightsquigarrow [b]) \wedge \bar{\mu}([a]).
\end{aligned}$$

Moreover, $\bar{\mu}([1]) = \bigvee_{t \in [1]} \mu(t) = \mu(1) \geq \bigvee_{t \in [x]} \mu(t) = \bar{\mu}([x])$, thus $\bar{\mu}$ is a fuzzy hyper deductive system of L/θ and clearly $\bar{\mu} \circ \pi \geq \mu$. $\square$

4. Hyper deductive systems based on falling shadows

Given an universal set U , and let $\mathcal{P}(U)$ denote the power set of U . For any $u \in U$ and $D \subseteq U$, let $\dot{u} = \{E | u \in E, E \subseteq U\}$ and $\dot{D} = \{\dot{u} | u \in D\}$. An order pair $(\mathcal{P}(U), \mathcal{B})$ is called to be a hyper-measurable structure on U if $\mathcal{B}$ is a σ -field in $\mathcal{P}(U)$, and $\dot{U} \subseteq \mathcal{B}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{P}(U), \mathcal{B})$ be a hyper-measurable structure on U . If a map $\xi : \Omega \rightarrow \mathcal{P}(U)$ is a random set on U , and for any $C \in \mathcal{B}$,

$$\xi^{-1}(C) = \{\omega \in \Omega | \xi(\omega) \in C\} \in \mathcal{A},$$

then ξ is called a $\mathcal{A} \rightarrow \mathcal{B}$ measurable.

Given a random set ξ on U . Consider a function $\tilde{H}(x) := P\{\omega | x \in \xi(\omega)\}$ for any $x \in U$, then $\tilde{H}$ is a fuzzy set of U , and we call it a falling shadow of the random set ξ , and ξ is called a cloud of $\tilde{H}$.

Definition 4.1. Consider a probability space $(\Omega, \mathcal{A}, P)$, and let $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a random set. If $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ , that is, for any $x \in L$,

$$\tilde{H}(x) = P(\omega | x \in \xi(\omega)),$$

is called a falling fuzzy hyper deductive system of L .

Example 4.2. Let L be a hyper residuated lattice defined in Example 3.5, and $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. The mapping $\xi : \Omega \rightarrow \mathcal{P}(L)$ is defined by

$$\xi(t) = \begin{cases} \{1\}, & t \in [0, 0.4), \\ \{b, 1\}, & t \in [0.4, 0.6), \\ \{0, a, b, 1\}, & t \in [0.6, 1], \end{cases}$$

then $\xi(t)$ is a hyper deductive system of L for any $t \in [0, 1]$. Thus $\tilde{H}$ is a falling fuzzy hyper deductive system of L , where $\tilde{H}(x) = P(t | x \in \xi(t))$ is represented

as follows:

$$\tilde{H}(x) = \begin{cases} 0.4, & x = 0, \\ 0.4, & x = a, \\ 0.6, & x = b, \\ 1, & x = 0. \end{cases}$$

Proposition 4.3. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure, then every fuzzy hyper deductive system of a hyper residuated lattice is a falling fuzzy hyper deductive system.*

Proof. Let μ is a fuzzy hyper deductive system of L , then μ_t is a hyper deductive system of L for any $t \in [0, 1]$ by Proposition 3.6. Define the random set $\xi : [0, 1] \rightarrow \mathcal{P}(A)$ by $\xi(t) = \mu_t$ for any $t \in [0, 1]$, then μ is a falling fuzzy hyper deductive system of L . $\square$

Given a probability space $(\Omega, \mathcal{A}, P)$, and let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \rightarrow \mathcal{P}(M)$. We define

$$\Omega(K; \xi) := \begin{cases} \{\omega \in \Omega | K \in \xi(\omega)\}, & K \in M, \\ \{\omega \in \Omega | K \cap \xi(\omega) \neq \emptyset\}, & K \subseteq M, \end{cases}$$

then $\Omega(K; \xi) \in \mathcal{A}$. In what follows, we give a number of equivalent conditions of falling fuzzy hyper deductive systems for further discussion.

Theorem 4.4. *Let $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a random set and $\tilde{H}$ a falling shadow of ξ . Then $\tilde{H}$ is a falling fuzzy hyper deductive system of L if and only if for any $x, y \in L$,*

- (1) $\Omega(x; \xi) \subseteq \Omega(1; \xi)$;
- (2) $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy hyper deductive system of L , then $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega(x; \xi)$. It follows that $1 \in \xi(\omega)$, that is, $\omega \in \Omega(1; \xi)$, and so $\Omega(x; \xi) \subseteq \Omega(1; \xi)$. For any $\omega \in \Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi)$, we get that $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$. From Proposition 2.4 we have $y \in \xi(\omega)$, i.e., $\omega \in \Omega(y; \xi)$, therefore $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)$.

Conversely, assume that (1) and (2) hold. Let $\omega \in \Omega$ be such that $\xi(\omega) \neq \emptyset$. Then there exists $x \in \xi(\omega)$, that is, $\omega \in \Omega(x; \xi) \subseteq \Omega(1; \xi)$, and so $1 \in \xi(\omega)$. Let $x, y \in L$ be such that $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$, that is, $\omega \in \Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi)$ and $\omega \in \Omega(x; \xi)$. It follows that $\omega \in \Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)$, therefore $y \in \xi(\omega)$. According to the above arguments, we get that $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, thus $\tilde{H}$ is a falling fuzzy hyper deductive system of L . $\square$

Proposition 4.5. *Consider a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \rightarrow \mathcal{P}(L)$. If $\tilde{H}$ is a falling fuzzy hyper deductive system of L , then the following statements hold: for any $x, y, z \in L$,*

- (1) $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \otimes y; \xi)$,
- (2) $x \leq y$ implies $\Omega(x; \xi) \subseteq \Omega(y; \xi)$,
- (3) $x \otimes y \ll z$ implies $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(z; \xi)$,
- (4) for any nonempty subset A of L , if $x \ll A$, then $\Omega(x; \xi) \subseteq \Omega(A; \xi)$,
- (5) for any nonempty subset A of L , $\Omega(x \rightarrow A; \xi) \cap \Omega(x; \xi) \subseteq \Omega(A; \xi)$,
- (6) $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) = \Omega(x; \xi) \cap \Omega(y; \xi)$,
- (7) $\Omega(1 \rightarrow y; \xi) \cap \Omega(1; \xi) = \Omega(y; \xi)$,
- (8) if $x \leq y$, then $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) = \Omega(x; \xi)$,
- (9) if $\Omega(x \rightarrow y; \xi) = \Omega(1; \xi)$, then $\Omega(x; \xi) \subseteq \Omega(y; \xi)$,

Proof. (1) Assume that $\tilde{H}$ is a falling fuzzy hyper deductive system of L , then $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega(x; \xi)$. For any $x, y \in L$, if $x \in \xi(\omega)$, $y \in \xi(\omega)$, then $x \otimes y \subseteq \xi(\omega)$ by Theorem 3.1, and so $(x \otimes y) \cap \xi(\omega) \neq \emptyset$, this yields $\omega \in \Omega(x \otimes y; \xi)$. Therefore, $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \otimes y; \xi)$.

(2) Let $x \leq y$. For any $\omega \in \Omega(x; \xi)$, that is, $x \in \xi(\omega)$, then $y \in \xi(\omega)$. Hence $\omega \in \Omega(y; \xi)$, and so $\Omega(x; \xi) \subseteq \Omega(y; \xi)$.

(3) Let $x, y, z \in M$ be such that $x \otimes y \ll z$. For any $\omega \in \Omega(x; \xi) \cap \Omega(y; \xi)$, we have $x \in \xi(\omega)$ and $y \in \xi(\omega)$. By hypothesis that $\tilde{H}$ is a falling fuzzy hyper deductive system of L , we get that $\xi(\omega)$ is a hyper deductive system of L . From Theorem 3.1, it follows that $x \otimes y \subseteq \xi(\omega)$. Notice that $x \otimes y \ll z$, there exists $u \in x \otimes y \subseteq \xi(\omega)$ such that $u \leq z$, therefore $z \in \xi(\omega)$, that is, $\omega \in \Omega(z; \xi)$. Hence $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(z; \xi)$.

(4) For any $\omega \in \Omega(x; \xi)$, we have $x \in \xi(\omega)$. Since $x \ll A$, then there exists $a \in A$ such that $x \leq a$. By hypothesis, $\xi(\omega)$ is a hyper deductive system of L , therefore $a \in \xi(\omega)$ by Theorem 3.1. Hence $A \cap \xi(\omega) \neq \emptyset$, and so $\omega \in \Omega(A; \xi)$, it follows that $\Omega(x; \xi) \subseteq \Omega(A; \xi)$.

(5) For any $\omega \in \Omega(x \rightarrow A; \xi) \cap \Omega(x; \xi)$, we get that $(x \rightarrow A) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$. Then there exists $a \in A$ such that $(x \rightarrow a) \cap \xi(\omega) \neq \emptyset$, it follows from Proposition 2.4 that $a \in \xi(\omega)$. And therefore, $A \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(A; \xi)$, hence $\Omega(x \rightarrow A; \xi) \cap \Omega(x; \xi) \subseteq \Omega(A; \xi)$.

(6) From Theorem 4.4, we get $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(y; \xi)$, and so $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(x; \xi) \cap \Omega(y; \xi)$. To obtain the reverse inclusion, observe that $y \ll x \rightarrow y$, we get $\Omega(y; \xi) \subseteq \Omega(x \rightarrow y; \xi)$, moreover $\Omega(y; \xi) \cap \Omega(x; \xi) \subseteq \Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi)$. And thus $\Omega(x \rightarrow y; \xi) \cap \Omega(x; \xi) = \Omega(x; \xi) \cap \Omega(y; \xi)$.

(7)–(9) The proofs directly follow from (6) and Theorem 4.4. $\square$

Definition 4.6. Give a probability space $(\Omega, \mathcal{A}, P)$, and let $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a random set. For any $\omega \in \Omega$, if $(x \otimes y) \cap \xi(\omega) \neq \emptyset$ implies $x \otimes y \subseteq \xi(\omega)$ for any $x, y \in L$, then ξ is said to be a $S_{\otimes}$ -reflexive random set.

Proposition 4.7. Let $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a $S_{\otimes}$ -reflexive random set, and $\tilde{H}$ be a falling shadow of ξ . If the following conditions are valid: for any $x, y, z \in L$,

- (1) $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \otimes y; \xi)$,
- (2) $x \leq y$ implies $\Omega(x; \xi) \subseteq \Omega(y; \xi)$,

then $\tilde{H}$ is a falling fuzzy hyper deductive system of L .

Proof. Assume that (1) and (2) hold. For any $\omega \in \Omega$, if $x \leq y$ and $x \in \xi(\omega)$, then $\omega \in \Omega(x; \xi) \subseteq \Omega(y; \xi)$, and so $y \in \xi(\omega)$. Now let $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x; \xi)$ and $\omega \in \Omega(y; \xi)$. By hypothesis, $\omega \in \Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x \otimes y; \xi)$, and so $(x \otimes y) \cap \xi(\omega) \neq \emptyset$. Due to the fact that ξ is a $S_{\otimes}$ -reflexive random set, we get $x \otimes y \subseteq \xi(\omega)$. Thus $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, so $\tilde{H}$ is a falling fuzzy hyper deductive system of L . $\square$

In the following, we give the notion of falling fuzzy hyper implicative deductive systems, and present some properties of it.

Definition 4.8. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a random set. If $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ , that is, for any $x \in L$,

$$\tilde{H}(x) = P(\omega | x \in \xi(\omega)),$$

is called a falling fuzzy hyper implicative deductive system of L .

Example 4.9. Let $L = \{0, a, b, c, 1\}$ be a set such that $0 < a < b < 1$ and $0 < c < 1$. The hyperoperations $\otimes$ and $\rightarrow$ are given in the following Cayley tables.

$\otimes$	0	a	b	c	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a\}$	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{a, b\}$	$\{0\}$	$\{a, b\}$
c	$\{0\}$	$\{0\}$	$\{0\}$	$\{c\}$	$\{c\}$
1	$\{0\}$	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c\}$

$\rightarrow$	0	a	b	c	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{c\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{c\}$	$\{a, b, c\}$	$\{1\}$	$\{c\}$	$\{1\}$
c	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{1\}$

Let $x \wedge y = \{u \in L | u \leq x, u \leq y\}$ and $x \vee y = \{u \in L | x \leq u, y \leq u\}$ for any $x, y \in L$. It is easy to check that $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a hyper residuated lattice [19]. Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. The mapping $\xi : \Omega \rightarrow \mathcal{P}(L)$ is defined by

$$\xi(t) = \begin{cases} \{a, 1\}, & t \in [0, 0.2), \\ \{c, 1\}, & t \in [0.2, 0.6), \\ \{a, b, 1\}, & t \in [0.6, 0.7), \\ \{0, a, b, c, 1\}, & t \in [0.7, 1], \end{cases}$$

then $\xi(t)$ is a hyper implicative deductive system of L for any $t \in [0, 1]$. Thus $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L , where $\tilde{H}(x) = P(t | x \in \xi(t))$ is represented as follows:

$$\tilde{H}(x) = \begin{cases} 0.3, & x = 0, \\ 0.6, & x = a, \\ 0.4, & x = b, \\ 0.7, & x = c, \\ 1, & x = 1. \end{cases}$$

Theorem 4.10. *Let $\xi : \Omega \rightarrow \mathcal{P}(L)$ be a random set and $\tilde{H}$ be a falling shadow of ξ . Then $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L if and only if for any $x, y, z \in L$,*

- (1) $\Omega(x; \xi) \subseteq \Omega(1; \xi)$,
- (2) $\Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$.

Proof. Assume that $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L , then $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega(x; \xi)$. Therefore $1 \in \xi(\omega)$, that is, $\omega \in \Omega(1; \xi)$, and so $\Omega(x; \xi) \subseteq \Omega(1; \xi)$. For any $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi)$, we have $(x \rightarrow (y \rightarrow z)) \cap \xi(\omega) \neq \emptyset$ and $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$. From the definition of hyper implicative deductive system, it follows that $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$, i.e., $\omega \in \Omega(x \rightarrow z; \xi)$, therefore $\Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$.

Conversely, suppose that (1) and (2) hold. Let $\omega \in \Omega$ be such that $\xi(\omega) \neq \emptyset$. Then there exists $x \in \xi(\omega)$, that is, $\omega \in \Omega(x; \xi) \subseteq \Omega(1; \xi)$, and so $1 \in \xi(\omega)$. Let $x, y, z \in L$ be such that $x \rightarrow (y \rightarrow z) \cap \xi(\omega) \neq \emptyset$ and $x \rightarrow y \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi)$ and $\omega \in \Omega(x \rightarrow y; \xi)$. It follows that $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$, therefore $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$. Hence $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega$, thus $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L . $\square$

Due to the fact that if a hyper implicative deductive system D of L is an upset, then D is a hyper deductive system [19], it is easy to obtain the following result.

Proposition 4.11. *Give a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \rightarrow \mathcal{P}(L)$. If $\xi(\omega)$ is an upset for any $\omega \in \Omega$, then a falling fuzzy hyper implicative deductive system $\tilde{H}$ of L is a falling fuzzy hyper deductive system.*

For any nonempty subsets A, B of L , $A \leq B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leq b$.

Lemma 4.12 ([19]). *Let L be a hyper residuated lattice. Then the following hold: for any $x, y, z \in L$,*

- (1) $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$,
- (2) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$,
- (3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Proposition 4.13. *Let L be a hyper residuated lattice. Then we have:*

- (1) $x \ll A$ implies $y \rightarrow x \leq y \rightarrow A$, for any nonempty subset $A \subseteq L$ and $x, y \in L$,
- (2) let D be a hyper deductive system of L . For any nonempty subsets A and B of L , if $A \cap D \neq \emptyset$ and $A \leq B$, then $B \cap D \neq \emptyset$.

Proof. (1) Since $x \ll A$, then there exists $a \in A$ such that $x \leq a$. From Lemma 4.12 (2), we get that $y \rightarrow x \leq y \rightarrow a$, and so $y \rightarrow x \leq y \rightarrow A$.

(2) Let $A \cap D \neq \emptyset$. Then there exists $a \in A \cap D$, that is, $a \in A$ and $a \in D$. Notice that $A \leq B$, then there exists $b \in B$ such that $a \leq b$. It follows from Theorem 3.1 that $b \in D$, and therefore $B \cap D \neq \emptyset$. $\square$

In the following, we give some conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system.

Theorem 4.14. *Let $\tilde{H}$ be a falling fuzzy hyper deductive system of L . Then the following statements are equivalent: for any $x, y, z \in L$,*

- (1) $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L ,
- (2) $\Omega(y \rightarrow (y \rightarrow x); \xi) \subseteq \Omega(y \rightarrow x; \xi)$,
- (3) $\Omega(z \rightarrow (y \rightarrow (y \rightarrow x)); \xi) \cap \Omega(z; \xi) \subseteq \Omega(y \rightarrow x; \xi)$.

Proof. (1) $\Rightarrow$ (2) We first show the fact that $\Omega(y \rightarrow y; \xi) = \Omega$. For any $\omega \in \Omega$, we get $1 \in \xi(\omega)$. In view that $1 \in y \rightarrow y$, we obtain $(y \rightarrow y) \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(y \rightarrow y; \xi)$. Therefore $\Omega \subseteq \Omega(y \rightarrow y; \xi)$, and so $\Omega(y \rightarrow y; \xi) = \Omega$. According to Theorem 4.10, we get that $\Omega(y \rightarrow (y \rightarrow x); \xi) = \Omega(y \rightarrow (y \rightarrow x); \xi) \cap \Omega(y \rightarrow y; \xi) \subseteq \Omega(y \rightarrow x; \xi)$.

(2) $\Rightarrow$ (3) Notice that $\tilde{H}$ is a falling fuzzy hyper deductive system of L , we get that $\Omega(z \rightarrow (y \rightarrow (y \rightarrow x)); \xi) \cap \Omega(z; \xi) \subseteq \Omega(y \rightarrow (y \rightarrow x); \xi) \subseteq \Omega(y \rightarrow x; \xi)$ by hypothesis and Proposition 4.5 (5).

(3) $\Rightarrow$ (1) Let $x, y, z \in L$ be such that $(x \rightarrow (y \rightarrow z)) \cap \xi(\omega) \neq \emptyset$ and $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$. Since $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$, then $y \rightarrow (x \rightarrow z) \cap \xi(\omega) \neq \emptyset$ by Proposition 4.13, and so there exists $t \in y \rightarrow (x \rightarrow z) \cap \xi(\omega)$. Moreover, $t \ll y \rightarrow (x \rightarrow z) \cap \xi(\omega)$, therefore $y \ll t \rightarrow (x \rightarrow z)$. Using Proposition 4.13, we get that $x \rightarrow y \leq x \rightarrow (t \rightarrow (x \rightarrow z)) \leq t \rightarrow (x \rightarrow (x \rightarrow z))$. In view that $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$, we get that $(t \rightarrow (x \rightarrow (x \rightarrow z))) \cap \xi(\omega) \neq \emptyset$. By the above discussing, we know that $\omega \in \Omega(t \rightarrow (x \rightarrow (x \rightarrow z)); \xi) \cap \Omega(t; \xi) \subseteq \Omega(x \rightarrow z; \xi)$. It follows that $\omega \in \Omega(x \rightarrow z; \xi)$, i.e. $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$. Hence we conclude that $(x \rightarrow (y \rightarrow z)) \cap \xi(\omega) \neq \emptyset$ and $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$ imply $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$, so $\xi(\omega)$ is a hyper implicative deductive system of L , and therefore $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L . $\square$

Theorem 4.15. *Let $\tilde{H}$ be a falling fuzzy hyper deductive system of L . Then $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L if and only if $(x \rightarrow u) \cap \xi(\omega) \neq \emptyset$ for any $u \in x \otimes x$ and $\omega \in \Omega$.*

Proof. It is directly from Theorem 4.8 of [19]. $\square$

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Robust sparse coding for subspace learning

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Abstract. Conventional sparse coding fails to learn a robust subspace when the data is corrupted by noises or outliers. To remedy this problem, matrix completion is considered into sparse coding to recover the corrupted data by the normal data and the robust sparse representations can be learned from the recovered data and the normal data. Therefore, a robust sparse coding method, called RSC, is proposed to learn a low-dimensional subspace from the corrupted data. Experiments are carried out on the image dataset which is contaminated by noises or outliers. It is demonstrated that our proposed RSC is more effective and robust in subspace learning and image clustering than other dimensionality reduction methods.

Keywords: sparse coding, matrix completion, robustness, dimensionality reduction.

1. Introduction

Many applications including community detection, recommender system, semi-supervised learning and clustering [1, 2, 3] suffer from the processing of high-dimensional data. Actually, only a few data are meaningful and the remaining data are redundant or noisy. It is necessary to find a low-dimensional space to represent images. For this purpose, matrix factorization methods have been paid great attention in the last two decades[4, 5, 6, 7].

Conventional matrix factorization methods are to seek several matrices such that can best approximate the original matrix. Principal component analysis (PCA) [8] is a famous unsupervised dimensionality reduction method, which require the basis vectors to be orthogonal. The non-negative matrix factorization (NMF) [4] method decomposes the non-negative data matrix into two non-negative low-dimensional matrices, which can achieve parts-based representations. Sparse coding (SC) [9] requires the leaned representations to be more sparser. This means that each image can be represented by a fewer features. However, above-mentioned approaches have a common drawback that they fail

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to learn a robust subspace when the original data are corrupted by noises or outliers.

In this paper, a novel robust sparse coding (RSC) method is proposed based on matrix completion for dimensionality reduction from the corrupted dataset. Due to the use of matrix completion, the corrupted data matrix can be firstly recovered. Then we decompose the repaired matrix into two low-dimensional matrices by utilizing sparse coding. Finally we repeat above two steps until a robust subspace is obtained. The contributions of this paper are summarized as follows:

- A robust sparse coding framework called RSC is proposed, and we develop an iterative optimization algorithm to solve RSC.
- We evaluate the performance of RSC by clustering on the ORL dataset [10] with Salt and Pepper noise or contiguous occlusion. Experiments demonstrate that the proposed RSC is more effective to learn a robust subspace.

2. Robust sparse coding

Suppose that each image $\vec{\xi} \in R^m$ can be represented by sets of basis vectors $\vec{b}_1, \dots, \vec{b}_r \in R^m$. Hence, we obtain

$$(1) \quad \vec{\xi} = \sum_{i=1}^r \vec{b}_i s_i,$$

where $\vec{s} = [s_1 \dots s_r] \in R^r$ is a coefficient vector. Given an image dataset $V = [\vec{\xi}_1, \dots, \vec{\xi}_n] \in R^{m \times n}$, the basis matrix $W = [\vec{b}_1, \dots, \vec{b}_r] \in R^{m \times r}$ and the coefficient matrix $H = [\vec{s}_1, \dots, \vec{s}_n] \in R^{r \times n}$ are expected to represent V . Based on the previous analysis, sparse coding is mathematically summarized into the following optimization problem:

$$(2) \quad \min_{W, H} F(W, H) = \|V - WH\|_F^2 + \lambda \sum_{i=1}^n \|H_i\|_1,$$

where $\|\cdot\|_F$ and $\|\cdot\|_1$ denote the Frobenius norm and the L1 norm, separately. Each column of W is a basis vector that captures the effective features in the dataset, and each column of H can be viewed as the r -dimensional representation with respect to the new bases. Hence, sparse coding is also a dimensionality reduction method because the dimension can be reduced from m to r ($r \ll m$). Utilizing sparse coding, a satisfactory subspace can be achieved from the image dataset without noises or outliers. However, it fails when the dataset is contaminated by Salt and Pepper noise or contiguous occlusion. To address this problem, matrix completion is proposed to recover the corrupted data by the normal data.

Suppose that $M_i \in R^m$ and $V_i \in R^m$ are the corrupted image vector and the recovered image vector, separately. The conventional matrix completion to minimize the error between the corrupted dataset $M = [M_1, \dots, M_n] \in R^{m \times n}$ and the recovered dataset $V = [V_1, \dots, V_n] \in R^{m \times n}$ [11] can be summarized as

$$(3) \quad \begin{aligned} \min_V \quad & \text{rank}(V) \\ \text{s.t.} \quad & P_\Omega(M) = P_\Omega(V), \end{aligned}$$

where P is the projection operator and Ω is the corrupted area. If $(i, j) \in \Omega$, we obtain that

$$(4) \quad [P_\Omega(M)]_{ij} = \begin{cases} M_{ij}, & \text{if } (i, j) \in \Omega, \\ 0, & \text{otherwise,} \end{cases}$$

where M_{ij} is the (i, j) -th element of M . It is obvious that problem (3) cannot be solved directly. Hence, many researchers proposed some related matrix completion methods [11, 12, 13, 14, 15]. According to previous researches, (3) can be rewritten into the following optimization problem:

$$(5) \quad \begin{aligned} \min_V \quad & \|V\|_F^2 \\ \text{s.t.} \quad & P_\Omega(M) = P_\Omega(V). \end{aligned}$$

Euclidean distance is proposed to measure $P_\Omega(M)$ and $P_\Omega(V)$. Problem (5) can be transformed into the following form:

$$(6) \quad \begin{aligned} \min_V \quad & R_2 = \beta \|V\|_F^2 + \alpha \|(V - M) \otimes S\|_F^2, \\ & S_{ij} = \begin{cases} 0, & \text{if } (i, j) \in \Omega, \\ 1, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha \geq 0$ and $\beta \geq 0$ are the regularization parameters, and S is the weight matrix that denotes the contaminated or uncontaminated positions.

Combining sparse coding and matrix completion, a novel robust sparse coding method (RSC) is proposed to handle Salt and Pepper noise and contiguous occlusion. Therefore, RSC can be summarized as follows:

$$(7) \quad \begin{aligned} \min_{V, W, H} \quad & F(V, W, H) = \frac{1}{2} (\|V - WH\|_F^2 + \beta \|V\|_F^2 \\ & + \alpha \|(V - M) \otimes S\|_F^2) + \lambda \sum_{i=1}^n \|H_i\|_1. \end{aligned}$$

3. Optimization schemes

In this section, we present necessary optimization theory to solve (7). Because of non-smooth and non-convex properties of (7), it is too complicated to obtain

the global solution. Therefore, we transform (7) into several convex smooth problems and alternately solve them until convergence.

Problem (7) can be transformed into the following three convex problems:

$$(8) \quad \min_V F(V) = \frac{1}{2} (\|V - WH\|_F^2 + \beta \|V\|_F^2 + \alpha \|(V - M) \otimes S\|_F^2)$$

and

$$(9) \quad \min_W F(W) = \frac{1}{2} \|V - WH\|_F^2$$

and

$$(10) \quad \min_H F(H) = \frac{1}{2} \|V - WH\|_F^2 + \lambda \sum_{i=1}^n \|H_i\|_1.$$

For problem (8), we obtain that its optimum solution as

$$(11) \quad V = (\alpha M \otimes S + WH) \oslash ((1 + \beta) \otimes I + \alpha S),$$

where $X \otimes Y$ and $X \oslash Y$ are the element-wise product and division, respectively. Similarly, the solution of (9) is

$$(12) \quad W = W - \frac{1}{L} \nabla F(H) = W - \frac{1}{L} (WHH^T - VH^T),$$

where the Lipschitz constant $L_W = \|H^T H\|$ and $\|\cdot\|$ denotes the L2 norm. The objective function of (10) is non-smooth, therefore, it should be transformed into smooth functions. Problem (10) can be re-written as

$$(13) \quad \min_{P, Q} F(P, Q) = \frac{1}{2} \|V - W(P - Q)\|_F^2 + \lambda \sum_{i,j} p_{ij} + \lambda \sum_{i,j} q_{ij}$$

subject to $P \geq 0, Q \geq 0,$

where $P = [p_{ij}] = \max(H, 0)$, $Q = [q_{ij}] = \max(-H, 0)$ and $H = P - Q$. To minimize (13), we transform it into the following two problems:

$$(14) \quad \min_P F(P) = \frac{1}{2} \|V + WQ - WP\|_F^2 + \beta \sum_{i,j} p_{ij}$$

subject to $P \geq 0$

and

$$(15) \quad \min_Q F(Q) = \frac{1}{2} \|V - WP - (-W)(Q)\|_F^2 + \beta \sum_{i,j} q_{ij}$$

subject to $Q \geq 0.$

The solution of (14) and (15) can be achieved by

$$(16) \quad P = [P - \frac{1}{L_H} \nabla F(P)]^+ = [P - \frac{1}{L_H} (W^T W P - (W^T V + W^T W Q) + \lambda)]^+$$

and

$$(17) \quad Q = [Q - \frac{1}{L_H} \nabla F(Q)]^+ = [Q - \frac{1}{L_H} (W^T W Q - (-W^T V + W^T W P) + \lambda)]^+,$$

where the Lipschitz $L_H = \|W^T W\|$ and $[\cdot]^+$ makes the negative data zeros.

By (11), (12), (16) and (17), the local solution of (7) can be easily searched. The related algorithm is summarized in Algorithm 1.

Algorithm 1 RSC

Input: $V \in R^{m \times n}$, $M \in R^{m \times n}$, $W \in R^{m \times r}$, $H \in R^{r \times n}$, α , β , λ , *iter*, *number*

Oupt: $V \in R^{m \times n}$, $W \in R^{m \times r}$, $H \in R^{r \times n}$

1: Calculate S by (6)

2: **for** $it = 0$ to *number* **do**

3: $V = (\alpha M \otimes S + WH) \oslash ((1 + \beta) \otimes I + \alpha S)$

4: $W = W - \frac{1}{L} \nabla F(H) = W - \frac{1}{L_W} (W H H^T - V H^T)$

5: **for** $k = 0$ to *iter* **do**

6: $P = [P - \frac{1}{L_H} \nabla F(P)]^+ = [P - \frac{1}{L_H} (W^T W P - (W^T V + W^T W Q) + \lambda)]^+$

7: $Q = [Q - \frac{1}{L_H} \nabla F(Q)]^+ = [Q - \frac{1}{L_H} (W^T W Q - (-W^T V + W^T W P) + \lambda)]^+$

8: **end for**

9: **end for**

10: $V = WH$

11: **return** V, W, H

4. Experiments

We explore the clustering performance of RSC on the ORL dataset, by comparing with three dimensionality reduction methods: NMF [4], PCA [8] and SC [9]. In this experiment, two corruptions, i.e., Salt and Pepper noise and contiguous occlusion are proposed to test the effectiveness and robustness of above-mentioned methods.

The ORL dataset contains 400 images of different 40 individuals. Each person has 10 different images with different facial expressions, facial details (without-glasses or with-glasses) and lighting. Each face image is a grayscale image with the 32×32 pixel array.

We complete the cluttering experiments by the following steps:

- 90% face images are corrupted by Salt and Pepper noise or contiguous occlusion. The 360 normal images and 40 corrupted images can be represented by a matrix $M \in R^{1024 \times 400}$.

- Given the factorization rank r , M can be decomposed into two low-dimensional matrices $W \in R^{1024 \times 100}$ and $H \in R^{100 \times 400}$ by different dimensionality methods.
- We utilize the K-means algorithm to cluster the subspace H , and present the clustering performances measured by two metrics including Accuracy (AC) and Normalized Mutual Information (NMI) [16].
- All methods are implemented in matlab R2012b and their related algorithms are all terminated in 200 iterations.
- Because the objective function of RSC, NMF and SC is non-convex, the related algorithms cannot search the global solution easily. Each algorithm is implemented 50 times and we report the average clustering results.

4.1 Salt and Pepper noise

For each image, Salt and Pepper noise is utilized to corrupt some pixels, and the corrupted pixel value is 0 or 255. In this subsection, we verify whether RSC can obtain a robust subspace from the ORL dataset contaminated by Salt and Pepper noise. To demonstrate the effectiveness and robustness of RSC, we propose different corrupted percentages for the ORL dataset, i.e., $p=10\%$, 15% , 20% , 25% and 30% . Let $r = 100$, $\alpha = 100$, $\lambda = 100$, $\beta = 0$, $iter = 20$ and $number = 200$.

Table 1 shows the clustering results in different corrupted cases. From the comparisons, we observe that: (1) Our proposed RSC achieves the best clustering results than other methods. (2) SC performs better than NMF and PCA as the corrupted percentage increases. The clustering results demonstrate that RSC can learn a robust and effective subspace on the ORL dataset contaminated by Salt and Pepper noise.

Table 1: Clustering AC and NMI with different corrupted percentages from 10% to 30%(the best PSNR in bold).

$p(\%)$	10	15	20	25	30
RSC	63.62 (79.93)	60.03 (77.71)	57.18 (77.16)	61.57 (77.48)	61.70 (78.40)
NMF	48.65 (67.47)	33.70 (56.10)	25.70 (47.14)	21.85 (43.79)	21.95 (43.05)
PCA	37.75 (58.28)	26.50 (45.75)	22.75 (42.60)	22.25 (42.27)	18.00 (40.66)
SC	55.07 (72.76)	43.95 (64.90)	37.60 (56.75)	30.10 (52.03)	22.92 (45.36)

4.2 Contiguous occlusion

In this subsection, contiguous occlusion is applied to simulate extreme outliers. We randomly corrupt a $b \times b$ pixel block on each image, and the corrupted pixel blocks are filled with the pixel value 255. To discuss the influence of different block sizes, we propose the block size b from 10 to 14 with the step size 1.

Table 2 present the clustering results measured by AC and NMI. According to Table 2, RSC achieves the best clustering performances in different block sizes, while other algorithms fail to clustering. It is obvious that RSC is more effective to learn a robust subspace in large number of outliers on the ORL dataset.

Table 2: Clustering AC and NMI with different block sizes from 10 to 14 (the best PSNR in bold).

b	10	11	12	13	14
RSC	60.80 (77.80)	56.68 (77.34)	63.05 (78.45)	61.95 (78.80)	64.60 (79.23)
NMF	17.62 (40.38)	16.97 (39.30)	16.92 (38.97)	17.47 (40.43)	17.40 (39.90)
PCA	18.75 (39.09)	16.75 (37.25)	16.50 (38.84)	15.50 (38.29)	16.00 (38.09)
SC	17.92 (39.62)	16.50 (39.58)	16.95 (39.5)	17.17 (40.14)	17.20 (40.03)

5. Conclusion and future work

This paper proposed a novel robust sparse coding method (RSC) for dimensionality reduction. Based on matrix completion, RSC is more effective to handle Salt and Pepper noise and contiguous occlusion than NMF, PCA and SC. Experiments demonstrate that our proposed RSC can learn more robust low-dimensional representations and achieves better clustering results from the corrupted ORL dataset.

Two topics can be discussed in the future work:

- RSC, NMF, PCA and SC are unsupervised methods, hence semi-supervised or supervised methods should be considered to handle noises or outliers.
- Based on matrix completion, most of dimensionality reduction models can be improved to achieve a robust subspace.

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Exponential stabilization of semi-linear wave equation

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Abstract. In this article, we establish the stabilization of a class of second order semi-linear hyperbolic systems obtained by nonlinear feedback using the observability of the corresponding uncontrollable systems. We obtain the well-posedness of the semi-linear system by standard argument of Ball ([3]), Our technique of proof relies on an appropriate decomposition of the solution, and the energy method. Our result generalizes an earlier one by Haraux [5] who studied the same type of problem for linear systems, and by Louis Tebou [9] for de case of semi-linear systems. Application of our result are provided.

Keywords: semi-linear systems, nonlinear feedback stabilization, decay estimate.

1. Introduction and statements of main results

Control and stability of distributed parameter systems can be reformulated as problems of analysis of nonlinear (semi-linear, bi-linear, ...) systems in many real problems, see [2, 5, 6, 7, 10, 11, 12, 13] for instance. Within this broad heading there are many different concepts for example exponential stabilization.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. let H be a Hilbert space and A be an unbounded coercive operator on H with $A = A^*$. Also let $B : H \rightarrow H$ be a bounded non-negative linear operator. Denote $\langle \cdot, \cdot \rangle$, the scalar product on H , and by $\|\cdot\|_H$ the corresponding norm on H , also we denote $\|\cdot\|_V$ the norms on V with $V = D(A^{\frac{1}{2}})$ respectively such that for every $v \in V$, set $\|v\|_V = |A^{\frac{1}{2}}v|_H$ and $Z = V \times H$ is the state space. In this paper we are concerned with the question of feedback stabilization of the

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following distributed semi-linear control system:

$$(1) \quad \begin{cases} y_{tt} = -Ay + Ny + uBy_t & \text{in } Q \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x) & \text{in } \Omega \\ y(\xi, t) = 0 & \text{on } \Sigma, \end{cases}$$

where $Q = \Omega \times]0, +\infty[$, $\Sigma = \partial\Omega \times]0, +\infty[$ and A generates a C_0 -semigroup $(S(t))_{t \geq 0}$ on Hilbert H and N is a nonlinear dissipative operator. For well-posed linear systems, we refer to [8]. In this work we take nonlinear state-feedback for (1), that is, $u(t) = -f_\rho(y_t(t))$, where

$$(2) \quad f_\rho(y) = \rho \frac{\langle y, By \rangle}{1 + |\langle y, By \rangle|}, \quad \rho > 0.$$

or

$$(3) \quad \begin{cases} f_\rho(y) = \rho \frac{\langle y, By \rangle}{\|y\|^2}, \quad \rho > 0, & y \neq 0, \\ f_\rho(y) = 0, & y = 0. \end{cases}$$

Then, we obtain the following closed-loop system

$$(4) \quad \begin{cases} y_{tt} = -Ay + Ny - f_\rho By_t & \text{in } Q \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x) & \text{in } \Omega \\ y(\xi, t) = 0 & \text{on } \Sigma. \end{cases}$$

We first consider the well-posedness of (5), which is, we prove that (5) admits a unique mild solution $y(t, x_0)$ for all $x_0 \in Z$.

2. Well-posedness problem

In this section, we show the well-posedness of the system

$$(5) \quad \begin{cases} y_{tt} = -Ay + Ny - f_\rho By_t & \text{in } Q \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x) & \text{in } \Omega \\ y(\xi, t) = 0 & \text{on } \Sigma, \end{cases}$$

where A generates a C_0 -semigroup $(S(t))_{t \geq 0}$ on Hilbert H , and $N(\cdot) : H \rightarrow H$ is a locally Lipschitz, that is, there exists a positive constant L such that

$$(6) \quad \|N(x) - N(y)\| \leq L\|x - y\|,$$

for all $x, y \in H$, and $N(0) = 0$. Since A is a strictly positive operator, then there exists the best positive constant such that:

$$(7) \quad \|v\|_V^2 = |(A)^{1/2}v|_H^2 \geq \mu^2 \|v\|_H^2.$$

Theorem 2.1. Assume that A generates a C_0 -semigroup $(S(t))_{t \geq 0}$ on Hilbert H , and that $N(\cdot) : H \rightarrow H$ is a locally Lipschitz. Then, for any $z_0 = (y_0, y_1) \in Z$. It can be shown that a function $Z \in C([0, T], Z), \forall T > 0$. (5) has a mild solution if and only if z satisfies the variation formula ([2])

$$(8) \quad z(t) = T(t)z_0 + \int_0^t T(t-s)g(z(s))ds, \forall t \in [0, T].$$

Where g and $T(t), t > 0$ are given in proof.

Proof of Theorem 2.1. We have Z form a Hilbert space under the inner product

$$(\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle)_Z = \langle v_1, v_2 \rangle_V + \langle w_1, w_2 \rangle_H.$$

The equation 1 can be written in the form

$$(9) \quad z_t = \tilde{A}z + g(z), z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

$$(10) \quad \tilde{A} : D(\tilde{A}) \longrightarrow Z \text{ with } \tilde{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix},$$

where

$$(11) \quad D(\tilde{A}) = \{(x, y) \in V \times H, -Ax \in H, y \in V\},$$

and

$$z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix}, g(z(t)) = \begin{pmatrix} 0 \\ Ny(t) - f_\rho(y_t(t))By_t(t) \end{pmatrix}.$$

Our assumptions imply that $\tilde{A}$ generates a C_0 -semigroup of linear contraction $(T(t))_{t \geq 0}$ on Hilbert Z , and that $g : Z \rightarrow Z$ is locally lipschitz. Then according to Ball ([2]) we have the inial data $z(0) = z_0$, that z is a weak solution of (5), if and only if z satisfies (8).

Theorem 2.2. A standard argument shows that for a given $z_0 \in Z$, there exists a unique solution $z(t) \in Z$ to system (8). Moreover, if $z_0 \in D(\tilde{A})$ then $z \in C(0, \infty; D(\tilde{A})) \cap C^1(0, \infty; Z)$.

Proof of Theorem 2.2. Since $\tilde{A}$ is m-dissipative see [1] it generates a contraction semigroup on Z , denoted by $(T(t))$. Then $\tilde{A}$ is also the generator of a strongly continuous semigroup on Z and using Ball and Slemrod ([3]), which completes the proof.

3. Exponential stabilization using observability

In this section we consider the following semi-linear equations:

$$(12) \quad \begin{cases} y_{tt} = -Ay + Ny - f_\rho By_t & \text{in } Q \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x) & \text{in } \Omega \\ y(\xi, t) = 0 & \text{on } \Sigma, \end{cases}$$

and

$$(13) \quad \begin{cases} \phi_{tt} = -A\phi + N\phi & \text{in } Q \\ \phi(x, 0) = \phi_0(x), \phi_t(x, 0) = \phi_1(x) & \text{in } \Omega \\ \phi(\xi, t) = 0 & \text{on } \Sigma. \end{cases}$$

Our result concerns the equivalence between the observability of system 13 and stabilization of the equilibrium state of 12. We will construct a nonlinear control serving primarily to bring the nonlinearity in the system generated by $\psi = y - \phi$ to zero, and secondly to ensure the exponential decay of the system energy. More precisely, we will prove the following results:

Theorem 3.1. *We prove that, (a) implies (b) such that:*

(a) *Assume that there is a time T and constant $C > 0$ such that*

$$\int_0^T | \langle B\phi_t, \phi_t \rangle |_H dt \geq C \| (\phi_0, \phi_1) \|_Z^2 \quad \forall (\phi_0, \phi_1) \in Z,$$

where $\phi \in C([0, +\infty[, H) \cap C^1([0, +\infty[, V)$ is the mild solution of 13.

(b) *There exist $M > 0$, and $\sigma > 0$ such that for every $(y_0, y_1) \in Z$, the solution of 12 with $y(0) = y_0$ and $y_t(0) = y_1$ satisfies:*

$$\|y(t)\|_V^2 + |y_t|_H^2 \leq M \exp(-\sigma t) (\|y_0\|_V^2 + |y_1|_H^2).$$

Proof of Theorem 3.1. Firstly, if $u(t) = 0$ the idea (see [5]) is to achieve the equivalent between observability and stabilization concerned the case of linear systems. Secondly, if $u(t) \neq 0$, we prove that $(a) \Rightarrow (b)$. Then, we consider the solution of (13) with $\phi(0) = y_0$ and $\phi_t(0) = y_1$ and let us consider the following energy functional

$$(14) \quad E_\phi(t) = \frac{1}{2} (\|\phi(t)\|_V^2 + |\phi_t(t)|_H^2) + F(\phi(t)),$$

where

$$F(u) = \int_0^u N(s) ds,$$

and set $\psi = y - \phi$, then we obtain the following system

$$(15) \quad \begin{cases} \psi_{tt} = -A\psi - f_\rho B y_t - N(y) + N(\phi) & \text{in } Q \\ \psi(x, 0) = 0, \frac{\partial \psi}{\partial t}(x, 0) = 0 & \text{in } \Omega \\ \psi(\xi, t) = 0 & \text{on } \Sigma. \end{cases}$$

We consider

$$(16) \quad E_\psi(t) = \frac{1}{2}(\|\psi(t)\|_V^2 + |\psi_t(t)|_H^2)$$

multiplying (15) by ψ and integrating over Ω we have

$$(17) \quad \frac{dE_\psi}{dt} = \int_\Omega -(N(y) - N(\phi))\psi_t dx - \int_\Omega \langle f_\rho B y_t, \psi_t \rangle dx.$$

Thanks to 6, and the Cauchy-Schwarz inequality, one derives from $0 \leq t \leq T$

$$(18) \quad E_\psi(t) \leq \int_0^T L \|\psi\|_H |\psi_t|_H ds + \int_0^T \|f_\rho B y_t\|_H |\psi_t|_H ds$$

then

$$(19) \quad E_\psi(t) \leq \frac{L}{\delta} \int_0^T \|\psi\|_V \|\psi_t\|_H ds + \frac{1}{2} \left(\int_0^T \|f_\rho B y_t\|_H^2 + \|\psi_t\|_H^2 ds \right)$$

from the boundedness of the operator B and 3 we obtain

$$(20) \quad E_\psi(t) \leq \frac{L}{\mu} \int_0^T E_\psi(s) ds + \frac{\rho}{2} \left(\int_0^T K^2 |\langle B y_t, y_t \rangle|_H ds \right) + \int_0^T E_\psi(s) ds.$$

Applying Gronwall lemma we derive:

$$(21) \quad E_\psi(t) \leq K' \exp(\mu_1) \int_0^T |\langle B y_t, y_t \rangle|_H ds$$

with $\mu_1 = \frac{TL + T\mu}{\mu}$, and $K' = \rho.K^2$.

Now we set

$$(22) \quad E_y(t) = \frac{1}{2}(\|y(t)\|_H^2 + \|y_t(t)\|_V^2) + F(y(t))$$

and we observe that

$$E_\phi(0) = E_y(0),$$

using the observability estimate provided by (a) we have

$$(23) \quad E_y(0) \leq C \int_0^T |\langle B \phi_t, \phi_t \rangle|_H ds$$

then for $C' > 2C$ we have

$$(24) \quad E_y(0) \leq C' \int_0^T | \langle B\psi_t, \psi_t \rangle |_H ds + C' \int_0^T | \langle By_t, y_t \rangle |_H ds.$$

Thanks to 21 and the boundedness of the operator B gives the existence of $K'' > 0$ such that

$$(25) \quad E_y(0) \leq K'' \int_0^T | \langle By_t, y_t \rangle |_H ds$$

we multiply (15) by ψ_t and integrate over Ω we have

$$(26) \quad E_y(T) - E_y(0) = - \int_0^T | \langle By_t, y_t \rangle |_H ds$$

then E_y is a non-increasing function of the time variable. we deduce that

$$E(0) \leq \beta E_y(T) - \beta E_y(0),$$

and the semigroup property gives

$$\|y(t)\|_V^2 + |y_t|_H^2 \leq M \exp(-\sigma t) (\|y_0\|_V^2 + |y_1|_H^2)$$

with

$$M = \frac{\beta}{1+\beta} \text{ and } \sigma = \frac{1}{T} \log\left(\frac{\beta+1}{\beta}\right),$$

which gives the exponential stabilization.

4. Application to wave equation

We consider the following semi-linear equation and $\Omega =]0, 1[$:

$$(27) \quad \begin{cases} \frac{\partial^2 y(x, t)}{\partial t^2} = \Delta y(x, t) - y(x, t)(\beta + \sin(|y(x, t)|)) \\ + v(t) \frac{\partial y(t)}{\partial t}, (x, t) \in \Omega \times]0, +\infty[, \\ y(x, 0) = y_0, y_t(x, 0) = y_1, & x \in \Omega, \\ y(0, t) = y(1, t) = 0, y_t(0, t) = y_t(1, t) = 0, & t > 0, \end{cases}$$

we take the state space $Z = V \times H = H_0^1(\Omega) \times L^2(\Omega)$ and the operators B and A are defined by $B = I$, $A = -\Delta$ which A generates a C_0 -semigroup of linear contraction where $\Delta y(t) = \frac{\partial^2 y(t)}{\partial x^2}$ with $\mathcal{D}(A) = (H^2(\Omega) \cap H_0^1(\Omega))$. In [4], we consider the operator $A_1 z = \Delta z$ in $H = L^2(\Omega)$, (Ω is a bounded open set in $\mathcal{R}^n$, $n \geq 1$) and domain

$$\mathcal{D}(A_1) = \{z \in L^2(\Omega) \mid z \text{ is absolutely continuous, } \Delta z \in L^2(\Omega) \text{ et } z = 0 \text{ sur } \partial\Omega\}.$$

We have $\overline{\mathcal{D}(A_1)} = L^2(\Omega)$, $A_1^* = A_1$ and $\operatorname{Re}(\langle A_1 z, z \rangle) = -\|\nabla z\|^2 \leq 0$. Then A_1 is dissipative.

5. Conclusion

In this work, we have developed the partial exponential stabilization using the proof of non uniform observability estimates for some semi-linear problems with superlinear nonlinearities. The idea of the non-linearity cost of the optimal stabilization is very interesting and constitutes a new issue in the applications. In addition. Various questions remain open for instance, the case of stabilization of semi-linear systems in Banach space, a confrontation to more realistic situations remain done. This leads us to consider the stabilization problem for stochastic semi-linear systems.

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More on some generalized soft mappings in soft topological spaces

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Abstract. In this paper, we continue to study the properties of soft pu-semi-continuous and soft pu-semi-open(closed) functions[7] and initiate the notions of soft pu-irresolute function, soft pre-semi-open(closed) functions and explore it properties. Moreover, we develop the characterization, which gives us the relationship between these mappings.

Keywords: Soft topology, soft semi-open(closed) sets, soft semi-interior(closure), soft pu-semi-continuous function, soft pu-semi-open(closed) functions, soft pu-irresolute function, soft pre-semi-open(closed) functions.

1. Introduction

To deal with complex problems involving imprecisions, vagueness and uncertainty, it is observed that the tools of classical methods are not always successful. One of the reasons of failure of classical methods to solve the problems having uncertainties is having not enough information. Many researches proceeding towards new theories and ideas are developed in recent days and a lot of material is available in the literature.

Molodtsov et. al [30-31] introduced soft sets theory as a new general mathematical approach to deal with uncertain data and not clear objects and applied successfully this approach for modelling the problems having uncertainties. Maji et. al initiated and studied soft sets and explored its applications in decision making problems [28-29]. Xiao et. al [35] and Pei et. al [33] discussed the relationship among soft sets and information systems. Kostek [26] introduced the criteria to measure sound quality using approach of soft sets. Mushrif et. al [32] developed the remarkable method for the classification of natural textures using soft sets. Many researches have been done to add different notions in terms of weak and strong forms and fuzzification in the tool box of soft sets in terms of algebraic structures of soft sets, fuzzy soft sets, intuitionistic fuzzy soft sets, weak structures of soft sets, soft topological spaces, fuzzy soft topological

spaces, intuitionistic fuzzy soft topological spaces as well as its applications [1], [4-24], [27], [34] and [36].

In [25], Kharral and Ahmad and then Zorlutana [36] explored and studied the mappings on soft classes in soft topological spaces. Later, Hussain [6] established fundamental and important characterizations of soft pu-continuous functions, soft pu-open functions and soft pu-closed functions.

Chen [2-3] presented and discussed soft semi-open(closed) sets in soft topological spaces. Hussain [4] continued to study the properties of soft semi-open sets and soft semi-closed sets in soft topological spaces.

In 2016, Hussain [7] generalized the soft functions studied in [6], [36] and introduced and explored new form of continuity called soft pu-semi-continuity via soft semi-open set in soft topological spaces. Moreover the concepts of soft-pu-semi-open and soft pu-semi-closed functions introduced and many of their characterizations and properties were discussed.

In this paper, we continue to study the properties of soft pu-semi-continuous and soft pu-semi-open(closed) functions [7] and initiate the notions of soft pu-irresolute function, soft pre-semi-open (closed) functions and explore its properties. Moreover, we develop the characterization, which gives us the relationship between these mappings.

2. Preliminaries

To make our paper self contained, first we recall some definitions and results defined and discussed in [1], [2-7], [19-20], [28-31], [34] and [36].

Definition 2.1. Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E . A pair (F, A) is called a soft set over X , where F is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X . For $e \in A$, $F(e)$ may be considered as the set of e -approximate elements of the soft set (F, A) .

Definition 2.2. For two soft sets (F, A) and (G, B) over a common universe X , we say that (F, A) is a soft subset of (G, B) , if

- (1) $A \subseteq B$ and
- (2) for all $e \in A$, $F(e)$ and $G(e)$ are identical approximations. We write $(F, A) \tilde{\subseteq} (G, B)$.

(F, A) is said to be a soft super set of (G, B) , if (G, B) is a soft subset of (F, A) . We denote it by $(F, A) \tilde{\supset} (G, B)$.

Definition 2.3. Two soft sets (F, A) and (G, B) over a common universe X are said to be soft equal, if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A) .

Definition 2.4. The union of two soft sets of (F, A) and (G, B) over the common universe X is the soft set (H, C) , where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A. \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Definition 2.5. The intersection (H, C) of two soft sets (F, A) and (G, B) over a common universe X , denoted $(F, A) \tilde{\cap} (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \tilde{\cap} G(e)$, for all $e \in C$.

Definition 2.6. The difference (H, E) of two soft sets (F, E) and (G, E) over X , denoted by $(F, E) \tilde{\setminus} (G, E)$, is defined as $H(e) = F(e) \setminus G(e)$, for all $e \in E$.

Definition 2.7. Let (F, A) be a soft set over X and Y be a non-empty subset of X . Then the sub soft set of (F, A) over Y denoted by (Y_F, A) , is defined as follows: $F_Y(\alpha) = Y \tilde{\cap} F(\alpha)$, for all $\alpha \in A$. In other words $(Y_F, A) = \tilde{Y} \tilde{\cap} (F, A)$.

Definition 2.8. The relative complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F' : A \rightarrow P(U)$ is a mapping given by $F'(\alpha) = U \setminus F(\alpha)$, for all $\alpha \in A$.

Definition 2.9. Let τ be the collection of soft sets over X , then τ is said to be a soft topology on X if:

- (1) $\Phi, \tilde{X}$ belong to τ .
- (2) the union of any number of soft sets in τ belongs to τ .
- (3) the intersection of any two soft sets in τ belongs to τ .

The triplet (X, τ, E) is called a soft topological space over X .

Definition 2.10. Let (X, τ, E) be a soft topological space over X , then soft interior of soft set (F, E) over X is denoted by $(F, E)^\circ$ and is defined as the union of all soft open sets contained in (F, E) . Thus $(F, E)^\circ$ is the largest soft open set contained in (F, E) . A soft set (F, E) over X is said to be a soft closed set in X , if its relative complement $(F, E)'$ belongs to τ .

Definition 2.11. Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X . Then the soft closure of (F, E) , denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E) . Clearly $\overline{(F, E)}$ is the smallest soft closed set over X which contains (F, E) .

Definition 2.12. Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X . Then (F, E) is called soft semi-open set if and only if there exists a soft open set (G, E) such that $(G, E) \subseteq (F, E) \subseteq \overline{(G, E)}$. The set of all

soft semi-open sets is denoted by $S.S.O(X)$. Note that every soft open set is soft semi-open set.

A soft set (F, E) is said to be soft semi-closed if its relative complement is soft semi-open. Equivalently there exists a soft closed set (G, E) such that $(G, E)^\circ \tilde{\subseteq} (F, E) \tilde{\subseteq} (G, E)$. Note that every soft closed set is soft semi-closed set.

Definition 2.13. Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X . The soft set (F, E) is called a soft point in $\tilde{X}$, denoted by e_F , if for the element $e \in E$, $F(e) \neq \phi$ and $F(e') = \phi$, for all $e' \in E - \{e\}$.

Definition 2.14. The soft point e_F is said to be in the soft set (G, E) , denoted by $e_F \tilde{\in} (G, E)$, if for the element $e \in E$, $F(e) \subseteq G(e)$.

Proposition 2.15. Let $e_F \tilde{\in} \tilde{X}$ and (G, E) be a soft set. If $e_F \tilde{\in} (G, E)$, then $e_F \tilde{\notin} (G, E)^c$.

Definition 2.16. A soft set (F, A) over X is said to be a null soft set, denoted by Φ_A , if for all $e \in A$, $F(e) = \phi$.

Definition 2.17. A soft set (F, A) over X is said to be an absolute soft set, denoted by $\tilde{X}_A$, if for all $e \in A$, $F(e) = X$. Clearly, $\tilde{X}_A^c = \Phi_A$ and $\Phi_A^c = \tilde{X}_A$.

Definition 2.18. Let (X, τ, E) be a soft topological space over X and $A \subseteq E$. Then:

- (i) soft semi-interior of soft set (F, A) over X is denoted by $\text{sint}^s(F, A)$ and is defined as the union of all soft semi-open sets contained in (F, A) .
- (ii) soft semi-closure of (F, A) over X is denoted by $\text{scl}^s(F, A)$ is the intersection of all soft semi-closed super sets of (F, A) .

Definition 2.19. Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. $u : X \rightarrow Y$ and $p : A \rightarrow B$ be mappings. Then a soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ defined as:

(1) Let (F, A) be a soft set in $SS(X)_A$. The image of (F, A) under f_{pu} , written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is a soft set in $SS(Y)_B$ such that

$$f_{pu}(F)(y) = \begin{cases} \bigcup_{x \in p^{-1}(y) \cap A} u(F(x)), & p^{-1}(y) \cap A \neq \phi \\ \phi, & \text{otherwise} \end{cases},$$

for all $y \in B$.

(2) Let (G, B) be a soft set in $SS(Y)_B$. Then the inverse image of (G, B) under f_{pu} , written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SS(X)_A$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B \\ \phi, & \text{otherwise} \end{cases},$$

for all $x \in A$.

The soft function f_{pu} is called soft surjective, if p and u are surjective. The soft function f_{pu} is called soft injective, if p and u are injective. The soft function f_{pu} is called soft bijective, if p and u are bijective.

Definition 2.20. Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces over X and Y respectively and $u : X \rightarrow Y$ and $p : A \rightarrow B$ be mappings. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-semi-continuous if and only if for any soft open set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is a soft semi-open set in $SS(X)_A$.

Clearly it follows from the definition that the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-semi-continuous if and only if for any soft closed set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is a soft semi-closed set in $SS(X)_A$.

Definition 2.21. A soft set (F, A) in $SS(X)_A$ is said to be a soft semi-nbd of a soft point $e_F \in \tilde{X}_A$, if there exists a soft semi-open set (H, A) such that $e_F \in (H, A) \subseteq (F, A)$.

Definition 2.22. Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces over X and Y respectively and $u : X \rightarrow Y$ and $p : A \rightarrow B$ be mappings. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-semi-open if and only if for any soft open set (F, A) in $SS(X)_A$, $f_{pu}(F, A)$ is soft semi-open in $SS(Y)_B$.

Definition 2.23. Let (X, τ, A) be a soft topological spaces over X , (F, A) be a soft set in $SS(X)_A$ and soft point $e_F \in \tilde{X}_A$. Then e_F is called a soft semi-limit point of a soft set (F, A) , if $(H, A) \cap ((F, A) \setminus \{e_F\}) \neq \emptyset$, for any soft semi-open set (H, A) such that $e_F \in (H, A)$. The set of all soft semi-limit point of (F, A) is called as soft semi-derived set of (F, A) and is denoted by $sd^s(F, A)$. Note that if $(F, A) \subseteq (H, A)$ then $sd^s(F, A) \subseteq sd^s(H, A)$. Clearly e_F is a soft semi-limit point of (F, A) if and only if $e_F \in scl^s((F, A) \setminus \{e_F\})$.

Theorem 2.24. Let (X, τ, A) be a soft topological spaces over X and (F, A) be a soft set in $SS(X)_A$. Then

- (1) $scl^s(F, A) \subseteq (F, A) \cup sd^s(F, A)$.
 - (2) $sd^s((F, A) \cup (H, A)) \subseteq sd^s(F, A) \cup sd^s(H, A)$.
- In general,
- (3) $\bigcup_i sd^s(F, A_i) \subseteq sd^s(\bigcup_i (F, A_i))$.
 - (4) $sd^s(sd^s(F, A)) \subseteq sd^s(F, A)$.
 - (5) $scl^s(sd^s(F, A)) \subseteq sd^s(F, A)$.

3. Soft pu-irresolute mappings

Definition 3.1. Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively and $u : X \rightarrow Y$ and $p : A \rightarrow B$ are mappings. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-irresolute if and only if for any soft semi-open set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is a soft semi-open set in $SS(X)_A$.

Theorem 3.2. *Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively and e_F be a soft point in $SS(X)_A$. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-irresolute if and only if for each soft point e_F in $SS(X)_A$, the soft inverse of every soft semi-nbd of $f_{pu}(e_F)$ is a soft semi-nbd of e_F .*

Proof. Let e_F be a soft point in $SS(X)_A$ and (F, B) a soft semi-nbd of $f_{pu}(e_F)$. By definition of soft semi-nbd, there exists soft semi-open set (G, B) in $S.S.O(Y)$ such that $f_{pu}(e_F) \tilde{\in} (G, B) \tilde{\subseteq} (F, B)$. This follows that $e_F \tilde{\in} f_{pu}^{-1}(G, B) \tilde{\subseteq} f_{pu}^{-1}(F, B)$. Since f_{pu} is soft pu-irresolute, then $f_{pu}^{-1}(G, B)$ is a soft semi-open set in $SS(X)_A$. Therefore $f_{pu}^{-1}(F, B)$ is a soft semi-nbd of e_F .

Conversely, suppose that (F, B) be a soft semi-open sets in $S.S.O(Y)$. Take $(G, B) \tilde{=} f_{pu}^{-1}(F, B)$.

Let $e_H \tilde{\in} (G, B)$, then $f_{pu}(e_H) \tilde{\in} (F, B)$. Clearly, (F, B) (being soft semi-open) is a soft semi-nbd of $f_{pu}(e_H)$. So by hypothesis, $(G, B) \tilde{=} f_{pu}^{-1}(F, B)$ is a soft semi-nbd of e_H . Therefore, by definition, there exists soft semi-open set $(G, B)_{e_H}$ in $S.S.O(X)$ such that $e_H \tilde{\in} (G, B)_{e_H} \tilde{\subseteq} (G, B)$. Thus, $(G, B) \tilde{=} \bigcup_{e_H \tilde{\in} (G, B)} (G, B)_{e_H}$. It follows that (G, B) is soft semi-open in $S.S.O(X)$. Therefore the soft function f_{pu} is soft pu-irresolute. This completes the proof.

Remark 3.3. Note that soft semi-nbd of e_F may be replaced by soft semi-open-nbd of e_F in above Theorem 3.2.

Theorem 3.4. *Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces over X and Y respectively and e_F be a soft point in $SS(X)_A$. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-irresolute if and only if for each soft point e_F in $SS(X)_A$ and each soft semi-nbd (G, B) of $f_{pu}(e_F)$, there exists a soft semi-nbd (F, A) of e_F such that $f_{pu}(F, A) \tilde{\subseteq} (G, B)$.*

Proof. Suppose e_F be a soft point in $SS(X)_A$ and (G, B) be a soft semi-nbd of $f_{pu}(e_F)$. Then there exists soft semi-open set $(H, B)_{f_{pu}(e_F)}$ in $SS(Y)_B$ such that $f_{pu}(e_F) \tilde{\in} (H, B)_{f_{pu}(e_F)} \tilde{\subseteq} (G, B)$.

This follows that $e_F \tilde{\in} f_{pu}^{-1}((H, B)_{f_{pu}(e_F)}) \tilde{\subseteq} f_{pu}^{-1}(G, B)$.

By hypothesis, $f_{pu}^{-1}((H, B)_{f_{pu}(e_F)})$ is soft semi-open in $SS(X)_A$. Consider we take $(F, A) = f_{pu}^{-1}(G, B)$. Then it follows that (F, A) is soft semi-nbd of e_F and $f_{pu}(F, A) = f_{pu}(f_{pu}^{-1}(G, B)) \tilde{\subseteq} (G, B)$.

Conversely, let (U, B) be a soft semi-open set in $SS(Y)_B$. Take $(H, A) \tilde{=} f_{pu}^{-1}(U, B)$. Let $e_F \tilde{\in} (H, A)$, then $f_{pu}(e_F) \tilde{\in} (U, B)$. Thus (U, B) is a soft semi-nbd of $f_{pu}(e_F)$. So by hypothesis, there exists a soft semi-nbd $(V, A)_{e_F}$ of e_F such that $f_{pu}((V, A)_{e_F}) \tilde{\subseteq} (U, B)$. Thus it follows that $e_F \tilde{\in} (V, A)_{e_F} \tilde{\subseteq} f_{pu}^{-1}(f_{pu}(V, A)_{e_F}) \tilde{\subseteq} f_{pu}^{-1}(U, B) \tilde{=} (H, A)$. Therefore, $(V, A)_{e_F}$ is a soft semi-nbd of e_F , which follows that there exists a soft semi-open set $(H, A)_{e_F}$ in $SS(X)_A$ such that $e_F \tilde{\in} (H, A)_{e_F} \tilde{\subseteq} (H, A)$. Thus $(H, A) \tilde{=} \bigcup_{e_F \tilde{\in} (H, A)} (H, A)_{e_F}$. It follows that (H, A)

is soft semi-open in $SS(X)_A$ [4]. Therefore, f_{pu} is soft pu-irresolute. Hence the proof.

The following theorem gives the characterization of soft pu-irresolute function in terms of soft semi-derived sets.

Theorem 3.5. *Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-irresolute if and only if $f_{pu}(sd^s(G, A)) \subseteq_{\tilde{scl}^s} (f_{pu}(G, A))$, for all soft sets (G, A) in $SS(X)_A$.*

Proof. Suppose soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pu-irresolute. Let (G, A) be a soft set in $SS(X)_A$, and $e_F \in sd^s(G, A)$.

Assume that $f_{pu}(e_F) \notin f_{pu}(G, A)$ and let (V, B) denote a soft semi-nbd of $f_{pu}(e_F)$. Since f_{pu} is soft pu-irresolute, then by Theorem 3.4, there exists a soft semi-nbd (U, A) of e_F such that $f_{pu}(U, A) \subseteq_{\tilde{scl}^s} (V, B)$. Now $e_F \in sd^s(G, A)$ follows that $(U, A) \cap (G, A) \neq \tilde{\Phi}$. Therefore, there exists at least one soft point $e_H \in (U, A) \cap (G, A)$ such that $f_{pu}(e_H) \in f_{pu}(G, A)$ and $f_{pu}(e_H) \in f_{pu}(U, A)$. Since $f_{pu}(e_F) \notin f_{pu}(G, A)$, we have $f_{pu}(e_F) \neq f_{pu}(e_H)$. Thus every soft semi-nbd of $f_{pu}(e_F)$ contains a soft point $f_{pu}(e_H)$ of $f_{pu}(G, A)$ different from $f_{pu}(e_F)$.

Consequently, $f_{pu}(e_H) \in sd^s(f_{pu}(G, A))$. This proves the necessity.

Conversely, suppose that f_{pu} is not soft pu-irresolute. Then by Theorem 3.4, there exists a soft point e_F in $SS(X)_A$ and a soft semi-nbd (V, B) of $f_{pu}(e_F)$ such that every soft semi-nbd (U, A) of e_F contains at least one soft element e_H in (U, A) for which $f_{pu}(e_H) \notin (V, B)$. Put $(G, A) = \{e_H \in SS(X)_A : f_{pu}(e_H) \notin (V, B)\}$. Since $f_{pu}(e_F) \in (V, B)$, therefore $e_F \notin (G, A)$ and hence $f_{pu}(e_F) \notin f_{pu}(G, A)$. Since $f_{pu}(G, A) \cap ((V, B) \setminus f_{pu}(e_F)) = \tilde{\Phi}$, therefore $f_{pu}(e_F) \notin sd^s(f_{pu}(G, A))$. This implies that $f_{pu}(e_F) \in f_{pu}(sd^s(G, A)) \setminus ((f_{pu}(G, A)) \cup sd^s(f_{pu}(G, A))) \neq \tilde{\Phi}$. This is a contradiction to the given condition. This proves sufficiency. This completes the proof.

Theorem 3.6. *Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively and soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft one to one. Then the soft function f_{pu} is soft pu-irresolute if and only if $f_{pu}(sd^s((G, A))) \subseteq_{\tilde{scl}^s} (sd^s(f_{pu}(G, A)))$, for all soft sets (G, A) in $SS(X)_A$.*

Proof. Consider soft sets (G, A) in $SS(X)_A$, $e_F \in sd^s(G, A)$ and (V, B) be a soft semi-nbd of $f_{pu}(e_F)$. Since f_{pu} is soft pu-irresolute, then by Theorem 3.4, there exists a soft semi-nbd (U, A) of e_F such that $f_{pu}(U, A) \subseteq_{\tilde{scl}^s} (V, B)$. But $e_F \in sd^s(G, A)$ gives there exists a soft element $e_F \in ((U, A) \cap (G, A))$ such that $e_F \neq e_H$. Clearly $f_{pu}(e_H) \in f_{pu}(G, A)$ and since f_{pu} is soft one to one, $f_{pu}(e_F) \neq f_{pu}(e_H)$. Thus every soft semi-nbd (V, B) of $f_{pu}(e_F)$ contains an element $f_{pu}(e_H)$ of $f_{pu}(G, A)$ different from $f_{pu}(e_F)$. Consequently, $f_{pu}(e_F) \in sd^s(f_{pu}(G, A))$.

Therefore, we have $f_{pu}(sd^s(G, A)) \subseteq_{\tilde{scl}^s} (sd^s(f_{pu}(G, A)))$. This proves the necessity.

Sufficiency follows from Theorem 3.5. Hence the proof.

4. Soft pre-semi-open mappings

Definition 4.1. Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively and $u : X \rightarrow Y$ and $p : A \rightarrow B$ are mappings. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-open if and only if for each soft semi-open set (F, A) in $SS(X)_A$, $f_{pu}(F, A)$ is a soft semi-open set in $SS(Y)_B$.

Remark 4.2. The class of soft pre-semi-open functions is a subclass of class of soft semi-open functions defined in [7].

The proof of following theorem is easy and therefore omitted:

Theorem 4.3. A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-open if and only if for each soft point e_F in $SS(X)_A$ and for every soft semi-open set (H, A) in $S.S.O(X)$ such that $e_F \tilde{\in} (H, A)$, there exists soft semi-open set (G, B) in $S.S.O(Y)$ such that $f_{pu}(e_F) \tilde{\in} (G, B)$ and $(G, B) \tilde{\subseteq} f_{pu}(H, A)$.

Theorem 4.4. A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-open if and only if for each soft point e_F in $SS(X)_A$ and for every soft semi-nbd (U, A) of e_F in $SS(X)_A$, there exists a soft semi-nbd (V, B) of $f_{pu}(e_F)$ in $SS(Y)_B$ such that $(V, B) \tilde{\subseteq} f_{pu}(U, A)$.

Proof. Suppose (U, A) be a soft semi-nbd of e_F in $SS(X)_A$. Then by definition there exists soft semi-open set (W, A) in $S.S.O(X)$ such that $e_F \tilde{\in} (W, A) \tilde{\subseteq} (U, A)$. Then $f_{pu}(e_F) \tilde{\in} f_{pu}(W, A) \tilde{\subseteq} f_{pu}(U, A)$. Since soft function f_{pu} is soft pre-semi-open, therefore $f_{pu}(W, A)$ is soft semi-open sets in $S.S.O(Y)$. Hence, $(V, B) \tilde{=} f_{pu}(W, A)$ is a soft semi-nbd of $f_{pu}(e_F)$ and $(V, B) \tilde{\subseteq} f_{pu}(U, A)$.

Conversely, let (U, A) be a soft semi-open set in $S.S.O(X)$ and e_F be a soft point in (U, A) . Then (U, A) is a soft semi-nbd of e_F . So by hypothesis, there exists a soft semi-nbd (V, B) of $f_{pu}(e_F)$ such that $f_{pu}(e_F) \tilde{\in} (V, B) \tilde{\subseteq} f_{pu}(U, A)$. That is, $f_{pu}(U, A)$ is a soft semi-nbd of $f_{pu}(e_F)$. Thus $f_{pu}(U, A)$ is a soft semi-nbd of each of its soft points. Thus $f_{pu}(U, A)$ is soft semi-open. Hence f_{pu} is soft pre-semi-open. This completes the proof.

Theorem 4.5. A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-open if and only if $f_{pu}(sint^s(F, A)) \tilde{\subseteq} sint^s(f_{pu}(F, A))$, for all soft sets (F, A) in $SS(X)_A$.

Proof. Suppose $e_F \tilde{\in} sint^s(F, A)$. Then there exists soft semi-open set (U, A) in $S.S.O(X)$ such that $e_F \tilde{\in} (U, A) \tilde{\subseteq} (F, A)$. This implies that $f_{pu}(e_F) \tilde{\in} f_{pu}(U, A) \tilde{\subseteq} f_{pu}(F, A)$. Since f_{pu} is soft pre-semi-open, then $f_{pu}(U, A)$ is soft semi-open in $SS(Y)_B$.

Hence $f_{pu}(e_F) \tilde{\in} sint^s(f_{pu}(F, A))$. Thus, $f_{pu}(sint^s(F, A)) \tilde{\subseteq} sint^s(f_{pu}(F, A))$. This proves necessity.

Conversely, let (U, A) be a soft semi-open sets in $S.S.O(X)$. Then by hypothesis, $f_{pu}(U, A) \tilde{=} f_{pu}(sint^s(U, A)) \tilde{\subseteq} sint^s(f_{pu}(U, A)) \tilde{\subseteq} f_{pu}(U, A)$ or $f_{pu}(U, A) \tilde{\subseteq}$

$\text{sint}^s(f_{pu}(U, A)) \subseteq f_{pu}(U, A)$. This follows that $f_{pu}(U, A)$ is soft semi-open in $SS(Y)_B$. So the soft function f_{pu} is soft pre-semi-open. Hence the proof.

We use Theorem 4.5 and prove:

Theorem 4.6. *A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft -pre-semi-open if and only if $\text{sint}^s(f_{pu}^{-1}(G, B)) \subseteq f_{pu}^{-1}(\text{sint}^s(G, B))$, for all soft set (G, B) in $SS(Y)_B$.*

Proof. Let (G, B) be any soft set in $SS(Y)_B$. Clearly, $\text{sint}^s(f_{pu}^{-1}(G, B))$ is soft semi-open in $SS(Y)_B$. Also $f_{pu}(\text{sint}^s(f_{pu}^{-1}(G, B))) \subseteq f_{pu}(f_{pu}^{-1}(G, B)) \subseteq (G, B)$.

Since f_{pu} is soft pre-semi-open, by Theorem 4.5, we have $f_{pu}(\text{sint}^s(f_{pu}^{-1}(G, B))) \subseteq \text{sint}^s(G, B)$.

Therefore, $\text{sint}^s(f_{pu}^{-1}(G, B)) \subseteq f_{pu}^{-1}(\text{sint}^s(G, B))$ or $\text{sint}^s(f_{pu}^{-1}(G, B)) \subseteq f_{pu}^{-1}(\text{sint}^s(G, B))$. This proves necessity.

Conversely, let (F, A) be any soft subset in $SS(X)_A$. By hypothesis, we have $\text{sint}^s(F, A) \subseteq \text{sint}^s(f_{pu}(F, A)) \subseteq f_{pu}^{-1}(\text{sint}^s(f_{pu}(F, A)))$. This implies that

$$\begin{aligned} f_{pu}(\text{sint}^s(F, A)) &\subseteq f_{pu}(\text{sint}^s(f_{pu}(F, A))) \\ &\subseteq f_{pu}(f_{pu}^{-1}(\text{sint}^s(f_{pu}(F, A)))) \subseteq \text{sint}^s(f_{pu}(F, A)). \end{aligned}$$

Consequently, $f_{pu}(\text{sint}^s(F, A)) \subseteq \text{sint}^s(f_{pu}(F, A))$, for all soft set (F, A) in $SS(X)_A$. By Theorem 4.5, f_{pu} is soft pre-semi-open. This completes the proof.

We use Theorem 4.6 and prove:

Theorem 4.7. *A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft -pre-semi-open if and only if $f_{pu}^{-1}(scl^s(G, B)) \subseteq scl^s(f_{pu}^{-1}(G, B))$, for all soft set (G, B) in $SS(Y)_B$.*

Proof. Let (G, B) be a soft sets in $SS(Y)_B$.

By Theorem 4.6, $\text{sint}^s(f_{pu}^{-1}((G, B)^c)) \subseteq f_{pu}^{-1}(\text{sint}^s((G, B)^c))$. This implies that $\text{sint}^s((f_{pu}^{-1}(G, B))^c) \subseteq f_{pu}^{-1}(\text{sint}^s((G, B)^c))$. As $\text{sint}^s(G, B) = (scl^s((G, B)^c))^c$ [8], therefore $(scl^s(f_{pu}^{-1}(G, B)))^c \subseteq f_{pu}^{-1}((scl^s(G, B))^c)$, or

$$(scl^s(f_{pu}^{-1}(G, B)))^c \subseteq f_{pu}^{-1}((scl^s(G, B))^c).$$

Hence

$$f_{pu}^{-1}(scl^s(G, B)) \subseteq scl^s(f_{pu}^{-1}(G, B)).$$

This proves necessity.

Conversely, (G, B) be a soft sets in $SS(Y)_B$. By hypothesis,

$$f_{pu}^{-1}(scl^s((G, B)^c)) \subseteq scl^s(f_{pu}^{-1}((G, B)^c)).$$

This implies $(scl^s(f_{pu}^{-1}((G, B)^c)))^c \subseteq (f_{pu}^{-1}(scl^s((G, B)^c)))^c$. Hence,

$$(scl^s((f_{pu}^{-1}((G, B)^c)))^c) \subseteq f_{pu}^{-1}((scl^s((G, B)^c))^c).$$

This gives $\text{sint}^s(f_{pu}^{-1}(G, B)) \subseteq f_{pu}^{-1}(\text{sint}^s(G, B))$. Now from Theorem 4.6, it follows that f_{pu} is soft pre-semi-open. This completes the proof.

5. Soft pre-semi-closed mappings

Definition 5.1. Let (X, τ, A) and (Y, τ^*, B) are soft topological spaces over X and Y respectively and $u : X \rightarrow Y$ and $p : A \rightarrow B$ are mappings. Then the soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-closed if and only if for each soft semi-closed set (F, A) in $SS(X)_A$, $f_{pu}(F, A)$ is a soft semi-closed set in $SS(Y)_B$.

Theorem 5.2. A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-closed if and only if $scl^s(f_{pu}(F, A)) \subseteq f_{pu}(scl^s(F, A))$, for every soft subset (F, A) of $SS(X)_A$.

Proof. Suppose f_{pu} is soft pre-semi-closed and let (F, A) be a soft subset in $SS(X)_A$. Since f_{pu} is soft pre-semi-closed, then $f_{pu}(scl^s(F, A))$ is soft semi-closed in $SS(Y)_B$. Since $f_{pu}(F, A) \subseteq f_{pu}(scl^s(F, A))$, then we have $scl^s(f_{pu}(F, A)) \subseteq f_{pu}(scl^s(F, A))$. This proves necessity.

Conversely, suppose (F, A) is a soft semi-closed set in $SS(X)_A$. Using hypothesis, we have $f_{pu}(F, A) \subseteq scl^s(f_{pu}(F, A)) \subseteq f_{pu}(scl^s(F, A)) \cong f_{pu}(F, A)$. Hence, $f_{pu}(F, A) \cong scl^s(f_{pu}(F, A))$. Thus $f_{pu}(F, A)$ is soft semi-closed set in $SS(Y)_B$. This proves that f_{pu} is soft pre-semi-closed. This completes the proof.

Theorem 5.3. A soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-closed if and only if $\{\overline{f_{pu}(F, A)}\}^0 \subseteq f_{pu}(scl^s(F, A))$, for every soft subset (F, A) of $SS(X)_A$.

Proof. Suppose f_{pu} is soft pre-semi-closed and (F, A) is any soft subset of $SS(X)_A$. Then $f_{pu}(scl^s(F, A))$ is soft semi-closed in $SS(Y)_B$. This implies that there exists a soft closed subset (G, B) of $SS(Y)_B$ such that

$$(G, B)^0 \subseteq f_{pu}(scl^s(F, A)) \subseteq (G, B).$$

This gives that $\{\overline{f_{pu}(scl^s(F, A))}\}^0 \subseteq (G, B)^0 \subseteq f_{pu}(scl^s(F, A))$.

Then $\{\overline{(f_{pu}(F, A))}\}^0 \subseteq (\overline{(f_{pu}(scl^s(F, A)))})^0$ gives $\{\overline{f_{pu}(F, A)}\}^0 \subseteq f_{pu}(scl^s(F, A))$. This proves necessity.

Conversely, suppose (F, A) is a soft semi-closed set in $SS(X)_A$. Then by hypothesis, $\{\overline{(f_{pu}(F, A))}\}^0 \subseteq f_{pu}(scl^s(F, A)) \cong f_{pu}(F, A)$. Put $(G, B) \cong \overline{(f_{pu}(F, A))}$. Clearly, (G, B) is soft closed in $SS(Y)_B$. This implies that $(G, B)^0 \subseteq f_{pu}(F, A) \subseteq (G, B)$. Hence $f_{pu}(F, A)$ is soft semi-closed in $SS(Y)_B$. This implies f_{pu} is soft pre-semi-closed.

Theorem 5.4. A soft bijective soft function $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ is soft pre-semi-closed if and only if for each subset (G, B) of $SS(Y)_B$ and each soft semi-open set (F, A) in $SS(X)_A$ containing $f_{pu}^{-1}(G, B)$, there exists a soft semi-open set (H, B) in $SS(Y)_B$ soft containing (G, B) such that $f_{pu}^{-1}(H, B) \subseteq (F, A)$.

Proof. Let $(H, B) \cong (f_{pu}((F, A)^c))^c$. Then $(H, B)^c \cong f_{pu}((F, A)^c)$. Since f_{pu} is soft pre-semi-closed, so (H, B) is soft semi-open. Since $f_{pu}^{-1}(G, B) \subseteq (F, A)$, we have $(H, B)^c \cong f_{pu}((F, A)^c) \subseteq f_{pu}(f_{pu}^{-1}((G, B)^c)) \subseteq (G, B)^c$. Hence, $(G, B) \subseteq (H, B)$, and thus (H, B) is a soft semi-open nbd of (G, B) .

Further, $(F, A)^c \subseteq f_{pu}^{-1}(f_{pu}((F, A)^c)) \cong f_{pu}^{-1}((H, B)^c) \cong (f_{pu}^{-1}((H, B)))^c$. This proves that $f_{pu}^{-1}((H, B)) \subseteq (F, A)$.

Conversely, suppose (F, A) is a soft semi-closed set in $SS(Y)_B$. Let $e_G \in (f_{pu}(F, A))^c$. Then $f_{pu}^{-1}(e_G) \subseteq (f_{pu}^{-1}(f_{pu}(F, A)))^c \subseteq (F, A)^c$ and $(F, A)^c$ is soft semi-open in $SS(X)_A$. Hence by hypothesis, there exists a soft semi-open set (H, B) soft containing e_G such that $f_{pu}^{-1}((H, B)) \subseteq (F, A)^c$. Since f_{pu} is soft one-one, we have $e_G \in (H, B) \subseteq (f_{pu}(F, A))^c$. Thus $(f_{pu}(F, A))^c \cong \bigcup_{e_G \in (f_{pu}(F, A))^c} (H, B)$. Hence, $(f_{pu}(F, A))^c$ is soft semi-open set [4]. This proves that f_{pu} is soft pre-semi-closed. Hence the proof.

We use Theorem 4.7 and give the following characterizations:

Theorem 5.5. Let $f_{pu} : SS(X)_A \rightarrow SS(Y)_B$ be a soft bijective soft function. Then the following are equivalent:

- (1) f_{pu} is soft pre-semi-closed.
- (2) f_{pu} is soft pre-semi-open.
- (3) f_{pu}^{-1} is soft pu-irresolute.

Proof. (1) $\Rightarrow$ (2). It is straightforward.

(2) $\Rightarrow$ (3) Let (F, A) be soft set in $SS(X)_A$. Since f_{pu} is soft pre-semi-open, by Theorem 4.7, $f_{pu}^{-1}(scl^s(f_{pu}(F, A))) \subseteq scl^s(f_{pu}^{-1}(f_{pu}(F, A)))$ implies

$$scl^s(f_{pu}(F, A)) \subseteq f_{pu}(scl^s(F, A)).$$

Thus $scl^s((f_{pu}^{-1})^{-1}(F, A))$ is soft contained in $(f_{pu}^{-1})^{-1}(scl^s(F, A))$, for every soft subset (F, A) of $SS(X)_A$. Then again by Theorem 4.7, it follows that f_{pu}^{-1} is soft pu-irresolute.

(3) $\Rightarrow$ (1). Let (F, A) be a soft semi-closed set in $SS(X)_A$. Then $(F, A)^c$ is soft semi-open in $SS(X)_A$. Since f_{pu}^{-1} is pu-irresolute, $(f_{pu}^{-1})^{-1}((F, A)^c)$ is soft semi-open in $SS(Y)_B$. But $(f_{pu}^{-1})^{-1}((F, A)^c) \cong f_{pu}((F, A)^c) \cong (f_{pu}(F, A))^c$. Thus $f_{pu}(F, A)$ is soft semi-closed in $SS(Y)_B$. This proves that f_{pu} is soft pre-semi-closed. This completes the proof.

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Estimating loss given default based on time of default

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Abstract. The Basel II capital structure requires a minimum capital to cover the exposures of credit, market, and operational risks in banks. The Basel Committee gives three methodologies to estimate the required capital; standardized approach, Internal Ratings-Based (IRB) approach, and Advanced IRB approach. The IRB approach is typically favored contrasted with the standard approach because of its higher accuracy and lower capital charges. The loss given default (LGD) is a key parameter in credit risk management. The models are fit to a sample data of credit portfolio obtained from a bank in Jordan for the period of January 2010 until December 2014. The best parametric models are selected using several goodness-of-fit criteria. The results show that LGD fitted with Gamma distribution. After that, the financial variables as a covariate that affect on two parameters in Gamma regression will find.

Keywords: Credit risk, LGD, Survival model, Gamma regression

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1. Introduction

Survival analysis is a statistical method whose outcome variable of interest is the time to the occurrence of an event which is often referred to as failure time, survival time, or event time. Survival data can be divided into three categories; complete, censored and truncated. Complete data is the ideal data that contains the beginning and end dates of which the event time is determined. Censored and truncated data is also called missing data due to the unavailability of information on the beginning or end dates (Klugman et al. 2012). Right censored data can be divided into three types: type I, type II and type III. Type I and type II are also called the singly censored data, while type III is also called the progressively censored data (Cohen 1965). Another commonly used name for the type III censoring is random censoring.

The credit risk is a crucial issue for financial institutions as well as for non-financial companies. Banks are allowed to estimate their own credit risk parameters dependent on the Basel II regulation and under the internal rate approach (IRB) and thus more accurate for the capital requirement with the underlying risk in the credit portfolio. In the past, Jordanian banks had to use standardized approaches to estimate minimum capital requirement under Basel I. Indeed, Basel II leads to a better differentiation of risks and takes into account the diversification of the bank's portfolio to estimate minimum capital requirement for credit risk (Kollar & Klietnik 2014).

In the related literature, there is numerous growths in various academic research and publications focused on credit risk models namely; probability default (PD) and loss given default (LGD) that can be formulated by using survival analysis techniques. It was initiated with the research paper by (Altman 1989) who used actuarial analysis to investigate the mortality rates of US corporate bonds. This was followed by various empirical studies on PD and LGD. Narain in 1992 defined the PD as the complement of the conditional survival function evaluated at the forecast horizon. Later, this technique was developed by (Carling et al. 1998). They used a semi-parametric duration model (Cox's proportional hazards model) with a data set consisting of 4733 individuals who were granted credit by a Swedish lending institution between 1993 and 1995 to analyze the factors that determine the time to maturity on a loan and to evaluate loan applicants by their expected duration and profits. In addition, (Stepanova & Thomas 2002) and (Malik & Thomas 2010) modeled the lifetime of individual credit with Cox's proportional hazards model. Furthermore, (Glennon & Nigro 2005) measured the default risk of small business loans by survival analysis techniques with Cox's proportional hazards models. They found that the default behavior of the loans is time sensitive; the likelihood of default increases initially, peaks in the second year, and declines thereafter. However, (Beran & Djaïdja 2007) proposed statistical modelling of credit risk for retail clients based on survival analysis under extreme censoring for the time-to-default variable. (Cao et

al. 2009) proposed three different mechanisms to estimate PD by using survival analysis techniques.

However, the survival analysis for predicting LGD has been presented by many researchers. (Dermine & De Carvalho 2006) used mortality approach to measure the percentage of the bad and doubtful loan of corporate bonds that are recovered n -months after the default date. The actuarial-based mortality approach is appropriate because the population sample changes over time. The dataset of this study obtained from micro-data on defaulted bank loans of a private bank in Portugal, Banco Commercial Portugues (BCP). It consists of 10000 short-term loans granted to small and medium-sized companies over the period from June 1995 to December 2000 (66 months). They identify the LGD by the following: $LGD = \prod_{t=1}^T SPULB_t$, Where, $SPULB_t$ is a sample (weighted) percentage unpaid loan balance at period t , $SPULB_t = 1 - SMRR_t$. Furthermore, $SMRR_t$ is a sample (weighted) marginal recovery rate at time t ,

$$SMRR_t = \frac{\sum_{i=1}^m \text{Cashflow received}_{it}}{\sum_{i=1}^m \text{Loan outstanding}_{it}},$$

i stands for each of the m loan balances outstanding in the sample, t periods after default. The empirical results show that marginal recovery rate is high in the first five months and heavy tail in the last months. Also, the cumulative recovery rate n -months after default increase gradually from 20% to 70% over the time. Furthermore, (Chen 2018) used the right censored survival techniques for predicting bank loan LGD. The data contains 2644 defaulted loan over eight years (quarterly) with start dates between January 2007 and December 2014 for real estate and non-real estate for a U.S financial services company. The datasets tested by five parametric models namely, gamma, lognormal, Weibull, exponential, and log-logistic. In the figure we can see the LGD is high in the first months then decreases over time to become heavy tail in the last months. Moreover, (Tanoue et al. 2017) study forecasts loss given default of bank loans with multi-stage model. And, (Krüger & Rösch 2017) estimated downturn LGD modeling using quantile regression. Finally, (Thompson & Brandenburger 2019) constructed a model to predict the risk of a cardholder for the lifetime of the account and the survival analysis methodologies applied to a case study from capital card services.

The empirical studies on credit risk have depended mostly on the corporate bond market to gauge losses in the case of default. The purpose behind this is that, as bank loans are private instruments, little information on loan losses are freely accessible. The researchers use parametric, non-parametric, semi-parametric and transformation regression models to estimate LGD. In the other words, the ordinal least squares regression (OLS), Ridge regression (RiR), Fractional response regression, Tobit model, Decision trees model, Beta distribution and Normal distribution for parametric models. However, regression tree, neural networks, Multivariate adaptive regression spline, Least squares support vector machine for non-parametric models. In the semi-parametric models applied

the joint beta additive model. However, inverse Gaussian regression, inverse Gaussian regression with beta transformation, Box-Cox transformation, Beta transformation, Fractional logit transformation, and Log transform for Transformation regression see for instance, (Bellotti & Crook 2008, 2012; Bruche & González-Aguado 2010; Calabrese 2012; Giese 2005; Huang & Oosterlee 2011; Loterman et al. 2012; Pykhtin 2003; Qi & Zhao 2011).

Theoretically, this study is of crucial importance for several reasons. First, this study used hybrid model for estimating LGD with new data which get from corporate credit portfolio from Jordanian bank. Second, it uses the actuarial analysis of the progressive right censored data for estimating the loss given default $S(t)$ and recovery rate $F(t)$ of a sample of corporate loans obtained from a bank in Jordan. Third, this study are to fit LGD data to several parametric distributions related with time such as Gama distribution, in order to estimate LGD each month which reduce capital requirement to meet default obligation each month. Finally, the financial variables which significantly affects on two parameters for Gamma regression can be explored. Indeed, the LGD is fixed under Basel I, but it changes over time (monthly) based on Gamma distribution under Basel II.

The primary explanations behind using the progressive censoring are: the time of study is constant, the borrowers can join the study any time during the fixed period of investigation, and the borrowers could default or not before the end of the study. Practically, the objective of this paper is to estimate the Loss Given Default (LGD) using parametric models based on survival time. The used parametric models are fitted to the right-censored data obtained from a bank in Jordan for the period from January 2010 until December 2014. The portfolio capacity is 4393, and the overall number of defaults during the 5-year period is 495. The sample size is same as number of default. A borrower is declared as default if he is unable to pay his cash installment in a period of 3 months. The estimated LGD is then used for predicting the performance of credit risk of a corporate portfolio under Basel II. For estimating LGD default used this proxy:

$$LGD \text{ for each borrower} = \frac{\text{outstanding amount}}{\text{Amount of borrowing}}$$

But with the effect of survival time, the used hybrid LGD monthly approach is:

$$(1.1) \quad LGD \text{ for each month} = \frac{\sum_{i=1}^n LGD \text{ for each borrower}_i}{n}$$

where i represents the number of borrowers in the same month. For more details to calculate LGD, see the following example a company borrows 1 million dollars for 5 years with interest rate about 15 % yearly and payments monthly see Table 1.

Table 1. Example for calculate LGD

Months	1	2	3	—	58	59	60
Beginning Amount	1000000	988710	977279	—	69616	46696	23490
Interest	12500	12359	12216	—	870	584	294
Cash paid	(23790)	(23790)	(23790)	—	(23790)	(23790)	(23790)
Outstand	988710	977279		—	46696	23490	0
LGD	0.9887 (=988710/ 1000000)	0.9773 (=977279/ 1000000)	0.9657 (=965705/ 1000000)	—	0.0467 (=46696/ 1000000)	0.0235 (=23490/ 1000000)	0.0000 (=0000/ 1000000)

In the previous example, the LGD is higher in the beginning months then decreased gradually to become lower in the last months same as (Chen 2017). In this study, we consider LGD at time of default ($T=3$) equals 0.9657, LGD at time of default ($T=58$) equals 0.0467, and so on. Then the average of LGD takes in each time.

Theoretically, LGD is defined as the key component for expected loss for credit portfolio, which is central to credit risk management. The expected loss (EL) or risk weight asset (Bauer & Agarwa 2014) for a credit portfolio is defined as:

$$\text{The expected loss (EL)} = \sum_{i=1}^n \text{Probability of default} * \text{Loss given default} * \text{Exposure value at the time of default}$$

LGD is used to find the expected loss that is key to determine minimum capital required based on IRB approach in the Basel committee. Moreover, it may be used to calculate the premium in credit insurance for corporations.

The rest of this paper proceeds as follows. In the next section, the framework of estimating LGD will be discussed. Section 3 presents parametric distributions for fitting the data, and Section 4 describes regression models for determining which financial variables significantly affect the LGD for our sample data. We present the sample data and the results in Section 5, and the final section will be the conclusion.

2. Proposed framework for estimating LGD

In figure 1.1, LGD data related with time extracted from credit portfolio from Jordanian bank. The right censored data approach is used to extracted LGD data for default borrowers during the 5-year period. The sample size of default which is declared as unable to pay financial obligations in a period of 3 months is 495 from 4393 borrowers. Then LGD data is fitted probability density function (pdf) with parametric models associated with time such as exponential, Weibull, Gamma, Gompertz models. Gamma distribution is better than other parametric models.

The LGD is assumed to be survival function $S(t)$ as a result of heavy tail. We will consider Gamma distribution that has two parameters namely; shape 2 and rate. To find the financial variables that affect on LGD as covariate, Gamma regression model is used to estimate equation of rate parameter with constant shape 1. In this case, we consider time as dependent variables and financial variables as independent variable to estimate mean in Gamma regression.

Then we used Gamma distribution with two parameters; the rate parameter from Gamma regression model and the shape 2 parameter from a simulation. The gamma distribution has the cumulative distribution, survival distribution, and other characteristics more than Gamma regression model. Finally, Gamma distribution with the best link function for rate from Gamma regression selected based on statistical criteria. The methodology of this paper can be summarized in the following flow chart.

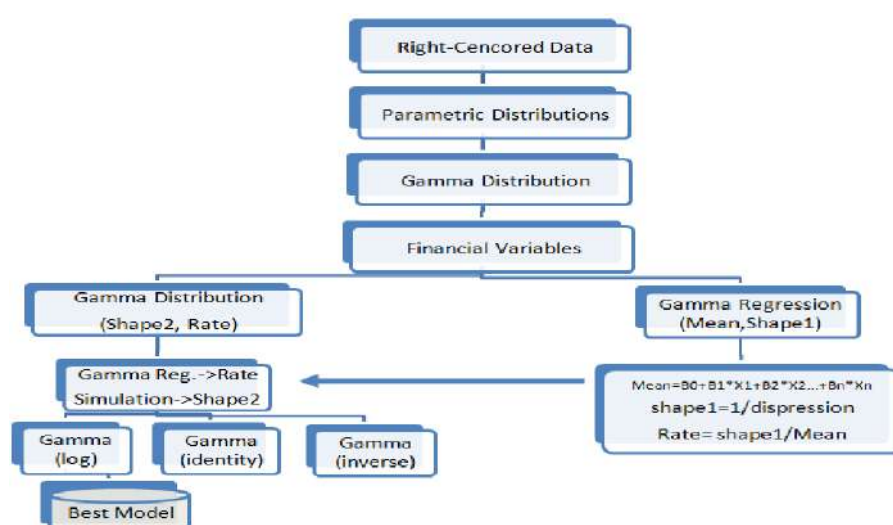


Figure 1: The flowchart of the WT forecasting.

3. Modelling LGD with parametric distributions

Common parametric distributions for survival analysis are considered. Table 1 provides the density and survival function for the distributions considered in this study, which are Exponential, Gamma, Weibull, Gompertz. Generalize Gompertz, and Gompertz-Makeham see Table 1 in next section.

We use five types of accuracy criteria to select the best model; mean square error (MSE), root mean square deviation (RMSD), mean absolute error (MAE), Nash–Sutcliffe model efficiency coefficient (NSE), and mean absolute percentage

error loss (MAPE). The mean square error

$$(\text{Ramsey 1999}) = \frac{\sum_{i=1}^n (\text{actual} - \text{predicted})^2}{n},$$

where n is the sample size. The root mean square deviation (RMSD) $= \sqrt{MSE}$. The RMSE values can be used to distinguish model performance in a calibration period with that of a validation period as well as to compare the individual model performance to that of other predictive models. The mean absolute error

$$(MAE) = \frac{\sum_{i=1}^n |\text{actual} - \text{predicted}|}{n},$$

where n is the sample size. The MAE is the average vertical and horizontal distance between each actual and predicted points. The other criteria is Nash–Sutcliffe model efficiency coefficient

$$(NSE) = 1 - \left[\frac{\sum_{i=1}^n (\text{actual}_i - \text{predicted}_i)^2}{\sum_{i=1}^n (\text{actual}_i - E(\text{actual}))^2} \right]$$

where n is the sample size. The NSE is a normalized statistic that determines the relative magnitude of the residual variance “noise” compared to the measured data variance “information” (Nash & Sutcliffe 1970). The Nash–Sutcliffe efficiency can range between $-\infty$ and 1.0 (1 inclusive), with $NSE = 1$ being the optimal value. Values between 0.0 and 1.0 are generally viewed as acceptable levels of performance, whereas values ≤ 0.0 indicates that the mean observed value is a better predictor than the simulated value, which indicates unacceptable performance. (Servat & Dezetter 1991) also approved NSE to be the best objective function for reflecting the overall fit of a hydrograph. (Legates & McCabe Jr 1999) suggested a modified NSE that is less sensitive to high extreme values due to the squared differences, but that modified version was not selected because of its limited use and resulting relative lack of reported values. Mean absolute percentage error loss

$$(MAPE) = \frac{100\%}{n} \sum_{t=1}^n \left| \frac{\text{actual}_t - \text{predicted}_t}{\text{actual}_t} \right|,$$

the MAPE can be indicated on how much error in predicting compared with the actual value.

4. Modeling LGD with Gamma Regression

In our study we use Gamma regression for fitting the LGD data with covariates. The reason for fitting Gamma regression is that the distribution is well-known with time. Gamma distribution is the best model for fitting the sample data of our case study. Therefore, we consider several Gamma regressions, by using different link functions, for fitting the LGD data with covariates.

Table 2: parametric models for LGD

Models	Parameters	Probability density function $f(t)$	Survival function $S(T)$
Exponential distribution	Rate(λ)	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$
Gamma distribution	Shape (α) Rate (β)	$\frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}$	$1 - \frac{\gamma(\alpha, \beta t)}{\Gamma(\alpha)} = \frac{\gamma(\alpha, \beta t)}{\Gamma(\alpha)}$
Weibull (proportional hazards)	Shape (α) Rate (β)	$\beta t^{\alpha-1} e^{-\beta t^\alpha}$	$e^{-\beta t^\alpha}$
Gompertz distribution	Shape (α) Rate(λ)	$\lambda e^{\alpha t} e^{-\frac{\lambda}{\alpha}(e^{\alpha t}-1)}$	$e^{-\frac{\lambda}{\alpha}(e^{\alpha t}-1)}$
Generalize Gompertz distribution (GGD)	Shape (α) Shape (c) Rate (λ)	$\theta \lambda e^{\alpha t} e^{-\frac{\lambda}{c}(e^{\alpha t}-1)} \left[1 - e^{-\frac{\lambda}{c}(e^{\alpha t}-1)} \right]^{\theta-1}$ $\lambda, \theta > 0, c \geq 0; t \geq 0$	$1 - \left[1 - e^{-\frac{\lambda}{c}(e^{\alpha t}-1)} \right]^\theta$
Gompertz-Makeham	Shape (α) Shape (β) Rate(λ)	$(\alpha e^{\beta t} + \lambda) e^{-(\lambda t + \frac{\alpha}{\beta}(e^{\beta t}-1))}$ $\alpha, \beta, \lambda > 0; t \in \mathcal{R}$	$e^{-(\lambda t + \frac{\alpha}{\beta}(e^{\beta t}-1))}$

Let random variable Y follows a Gamma distribution $Gamma(\alpha, \beta)$, where the parameters $\alpha, \beta > 0$. The mean and variance of Y are, respectively $E(y) = \alpha/\beta$ and $Var(y) = \frac{\alpha}{\beta^2}$. Joint modeling of the mean and the shape parameters in gamma regressions were proposed by (Cepeda-Cuervo 2001). With the re-parameterization of the gamma distribution as a function of the mean, $\mu = E(Y)$, and the shape parameter, α , as proposed in (Cepeda-Cuervo 2001) and (Cepeda & Gamerman 2005), setting $\beta = \frac{\alpha}{\mu}$, the gamma density function can be written as

$$f(y) = \frac{1}{y \Gamma(\alpha)} \left(\frac{\alpha y}{\mu} \right)^\alpha e^{-\frac{\alpha}{\mu} y}$$

Under this re-parameterization, we use $Y \sim Gamma(\mu, \alpha)$ to denote that Y follows a gamma distribution with $E(y) = \mu$, $Var(y) = \frac{\mu^2}{\alpha}$ and α as a shape parameter. From this re-parameterization of the gamma distribution, the joint mean and shape gamma regression was proposed in (Cepeda-Cuervo 2001), under classic and Bayesian methodologies.

Let $y = (y_1, \dots, y_n)^T$ be a random sample, where $y_i \sim Gamma(\mu_i, \alpha)$, $i = 1, \dots, n$. In the gamma regression model with a constant shape parameter, the mean regression structure is defined by

$$(4.2) \quad g(\mu_i) = \eta_i = f(x_i^T; \beta)$$

where g is the link function, $\beta = (\beta_1, \dots, \beta_n)^T$ a vector of covariate unknown regression parameters which are assumed to be functionally independent, $\beta \in \mathbb{R}^n$, η_i is a predictor. Some usual link functions in the gamma regression are:

$\log g(\mu) = \log(\mu)$; identity $g(\mu) = \mu$; and inverse $g(\mu) = 1/\mu$. A rich discussion of link function can be found in (McCullagh 1989).

The parameter of gamma regression can get from maximum likelihood estimator (MLE). The log-likelihood function for this class of gamma regression models has to form

$$(4.3) \quad \ell(\beta, \alpha) = \sum_{i=1}^n \ell_i(\mu_i, \alpha)$$

where $\ell_i(\mu_i, \alpha) = (\alpha - 1) \log y - \log \Gamma(\alpha) - \alpha \log \mu + \alpha \log \alpha - \left(\frac{\alpha}{\mu}\right) y$;

$\mu_i = g^{-1}(\eta_i)$, as defined in equations (3), are functions of β and α , respectively. The estimated parameters can be obtained by maximizing the likelihood in (4). The derivatives $U_k(\beta, \alpha) = \partial \ell(\beta, \alpha) / \partial \beta_k$ obtained parameters beta regressions β 's. For more detail about a prove gamma regression refer to (Cepeda-Cuervo 2001).

5. Results

The sample data of credit portfolio obtained for this study are collected from a bank in Jordan. The monthly data of the credit portfolio were collected from January 2010 until December 2014. The size of the portfolio is 4393, while the total number of defaults throughout the 5-year period is 495. The sample size is same as number of default. In this section, LGD data related with time. For the sample data, a borrower is declared default when his/her cash installment is not paid within 3 months. In this study, LGD is estimated by the default sample only.

Table 3 shows the risk exposures (number of loans at risk) and the number of defaults in each year. The highest number of defaults occurred in the second year, and the highest number of defaults per exposure also occurred in the same year (168 defaults from 1125 exposures). Table 4 provides summary statistics for the monthly data. The average monthly exposure is 73, while the average number of defaults per month is 8.

Table 3: Number of exposures and defaults in each year

Year	exposure	# of defaults	% (# of defaults per exposure)
2010	1265	137	10.83
2011	1125	168	14.93
2012	783	67	8.56
2013	652	41	6.29
2014	568	82	14.44
Total	4393	495	-

Table 4: Summary statistics for credit data (monthly)

	Exposure per month	# of defaults per month	# of censored per month
Min	29	0	23
Max	272	33	254
Mean	73.22	8.25	64.97
Std. dev.	42.88	6.35	38.93
Total (N)	4393	495	3898

Table 5 provides the estimated parameters, together with the MSE, RMSD, MAE, NSE, and MAPE. The best model is chosen based on the smallest MSE, RMSD, MAE, MAPE and the largest NSE. The results in Table 8 show that the Gamma distribution is the best parametric model since it has the smallest MSE, RMSD, MAE, MAPE and the largest NSE.

Table 5: Estimated parameters and goodness-of-fit criteria for parametric models

Models	Parameters est.	No. of iter- ation	MSE	RMSD	MAE	NSE	MAPE
Exponential	rate (λ)=0.00901	1000	0.00766	0.08755	0.06751	0.73329	0.11290
Gamma	shape (α)=1.59 rate (β)=0.02	50	0.00662	0.08138	0.05804	0.76956	0.09999
Weibull (propor- tional haz- ards)	Shape (α)=1.4 rate (β)=0.0021	50	0.00669	0.08181	0.05857	0.76708	0.10059
Gompertz	shape (α)=0.00680 rate (λ)=0.01380	50	0.00702	0.08376	0.06318	0.75584	0.10615
Generalize Gompertz distribution (GGD)	shape (α)=0.0180 shape (c)=0.99 rate (λ)=0.006	50	0.00702	0.08381	0.06194	0.75558	0.10497
Gompertz- Makeham	shape (α)=0.0301 shape (β)=0.00510 shape (λ)=0.00200	50	0.00712	0.08436	0.06367	0.75236	0.10689

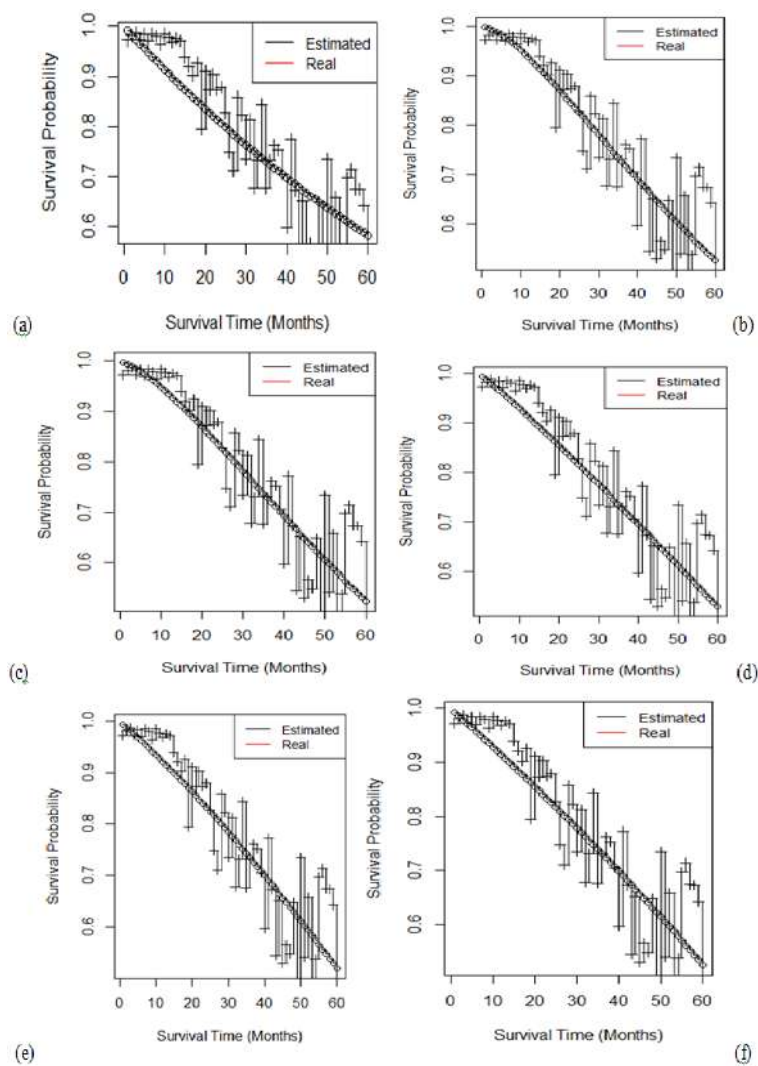


Figure 2: Fitted survival function for parametric models. (a) Exponential distribution, (b) Gamma distribution, (c) Weibull distribution, (d) Gompertz distribution, (e) Generalize Gompertz distribution, (f) Gompertz -Makeham.

For further comparison, Figure 2 shows the curve of survival function for all of the fitted parametric models. The graphs illustrate that the survival curve of Gamma model is closest compared to other models.

Table 6. Summary statistics for explanatory variables as covariate

Explanatory variables	Mean	Std. dev.	Minimum	Maximum
The service pricing policy(O.E/ T.L)	2.3063	.6324	.9578	3.1188
Operating efficiency (T.O.E/T.O.R)	.9208	.1318	.6048	1.2214
Liquidity ratio (Current ratio)	1.0168	.1408	.7783	1.5508
Profitability ratio (ROE)	.5839	.5838	-1.1598	1.7039

Table 7: Gamma regression distributions with different link function $\text{Gamma}(\mu, \alpha_1)$,

Models	parameters	Equations	Beta	Std. Error	t value
Gamma regression (log)	μ	(Intercept)	3.2932	0.0611	53.8850***
		(O.E/T.L)	-0.2989	0.0726	-4.1150***
		(T.O.E/T.O.R)	-0.2481	0.0627	-3.9550***
		Current ratio	-0.1495	0.0760	-1.9670.
		ROE	0.3796	0.0750	5.0630***
	α_1	=1/dispression	=1/0.2241019		
		Chi.test	12.3256	AIC	490.02
		p-value	0.0155	BIC	502.5905
				Iteration	10
Gamma regression (Identity)	μ	(Intercept)	36.1	3.4	10.8***
		(O.E/T.L)	0.7	0.7	1
		(T.O.E/T.O.R)	-15.5	2.6	-6***
		Current ratio	8.8	3.5	2.5.
		ROE	17.8	2.4	7.5***
	α_1	=1/dispression	1/0.416465		
		Chi.test	215.8659	AIC	538.7
		p-value	0.001397218	BIC	551.2635
				Iteration	25
Gamma regression (Inverse)	μ	(Intercept)	-0.0558	0.0225	-2.5.
		(O.E/T.L)	0.0121	0.0042	2.8*
		(T.O.E/T.O.R)	0.0368	0.0126	2.9*
		Current ratio	0.0458	0.0157	2.9*
		ROE*	-0.0222	0.005	-4.5***
	α_1	=1/dispression	1/0.2527023		
		Chi.test	13.89824	AIC	499.6
		p-value	0.009505567	BIC	512.1716
				Iteration	5

Table 6 explains the descriptive statistics for explanatory variables. The mean of service pricing policy (operating expenses (O.E) to total liability (T.L)) is 2.31 and standard deviation is 0.63. In addition, the mean of operating

efficiency (Total operating expenses (T.O.E)/Total operating revenue (T.O.R)) is 0.92 with standard deviation is 0.13. Furthermore, the mean of liquidity ratio (Current ratio=current asset/current liability) is 1.02 with standard deviation is 0.14. Finally, the mean of profitability ratio (return on equity ROE=Net Income/Owner Equity) is 0.58 with standard deviation is 0.58.

Estimating LGD based on $\text{Gamma}(\alpha_2, \beta)$ in this section. The parameters rate β estimated based on $\text{Gamma}(\mu, \alpha_1)$, where mean μ estimated from financial variables namely; standardized profitability ratio, standardized liquidity ratio, standardized operating efficiency ratio, and standardized service pricing policy. The different link function used to estimate this parameter μ such as log, identity, and inverse with gamma regression explain in table in next table.

Conclusion

In the context of credit portfolio losses, loss given default (LGD) is the proportion of the exposure that will be lost if a default occurs. Uncertainty, regarding the actual LGD is an important source of credit portfolio risk in addition to default risk. In this study, we have used several parametric models to estimate LGD with time variables in the aim to evaluate the performance of a credit risk portfolio samples. The financial variables models are fitted to a sample data of credit portfolio obtained from a bank in Jordan for the period from 2010 - 2014. The used models were selected using several goodnesses of fitting criteria to compare the performance of varied distributions. The used models were selected using several goodnesses of fitting criteria to compare the performance of the varied distributions. The results show that the Gamma distribution and Gamma regression are the best parametric models for estimating LGD with time of default based on the following tests: MSE, RMSD, MAE, NSE, and MAPE.

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Closed linear transformations of complex space-time endowed with Euclidean or Lorentz metric

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Abstract. Linear transformations (LTs) are essential for the development of Relativity Theory. Special Relativity is based on Lorentz Boost (LB). This cancels the transitive attribute in parallelism (which is equivalent to the 5th Euclidean Postulate), when three observers are related (successive transformations), because LB is not closed LT. So, LB is combined with Euclidean spatial rotation, in order to obtain Lorentz transformation (which is closed LT) and the corresponding Lorentz group. In this paper, a new closed isometric LT in spaces (V^4) of dimension four ($n = 4$), with Euclidean or Lorentz metric (Minkowski Space), is presented (correlating frames with parallel spatial axes). This LT is represented by a matrix (Λ_B) containing real and imaginary numbers. Thus, V^4 is based on the field of complex numbers (C), by using real 0-(temporal) and complex 1, 2, 3- (spatial) coordinates.

Keywords: 5th Euclidean postulate, Euclidean metric, Euclidean space, linear transformation, spacetime, Lorentz Boost, Lorentz Transformation, Minkowski space, special relativity, Thomas rotation.

Abbreviations and annotations

E^3 ► three-dimensional Euclidean

E^3 ► four-dimensional Euclidean Space

LB ► Lorentz Boost

LT ► Linear Transformation

M^4 ► Minkowski Space

RT ► Relativity Theory

SR ► Special Relativity

V^4 ► Four-dimensional Space

1. Introduction

Linear transformations (LTs) are essential for the development of Relativity Theory (RT), Quantum Mechanics (QMs) and generally in Modern Physics. In

Special Relativity (SR), Hermann Minkowski combined [1] (pp. 39-53) the one-dimensional time (T) with the *Three-dimensional Euclidean space* (E^3) endowed with Euclidean metric [2] (p. 14):

$$(1.1) \quad g_{E^3} = \text{diag}(1, 1, 1)$$

and he produced a four-dimensional space M^4 (since known as *Minkowski space-time*), endowed with *Lorentz metric* [2] (p. 8):

$$(1.2) \quad g_L = \text{diag}(-1, 1, 1, 1).$$

In this space, the position four-vector is written as

$$(1.3) \quad \vec{X} = x^0 \vec{e}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu,$$

where

$$(1.4) \quad [\vec{e}_\mu] = [\vec{e}_0 \ \vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

are the basis of M^4 and the coordinates of the position four-vector, respectively. The *Einstein's summation convention* [2] (p. 3) has been used in (1.3ii) and the following equations. Besides the *Lorentz length* $|\vec{X}|$ of the position four-vector is defined [2] (p. 17):

$$(1.5) \quad |\vec{X}|^2 = x_\mu g_{L\mu\nu} x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2.$$

Correspondingly in E^3 , we have

$$(1.6) \quad \vec{X} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \vec{e}_i x^i.$$

The common choice is the usage of real coordinates

$$(1.7) \quad x^1, x^2, x^3 \in R,$$

in order to be easily perceived by human senses.

We shall see that this field is not enough, in case that we wish to produce closed isometric LTs in Four-dimensional Spaces (V^4). So, we prefer complex coordinates

$$(1.8) \quad x^1, x^2, x^3 \in C.$$

For simplicity reasons, wherever we write i (the imaginary unit), we mean $\pm i$:

$$(1.9) \quad i \rightarrow \pm i ; \quad -i \rightarrow \mp i.$$

Besides, V^4 endowed with *Euclidean metric* [2] (p. 8):

$$(1.10) \quad g_{E^4} = \text{diag}(1, 1, 1, 1),$$

is called *Euclidean Four-dimensional Space* (E^4). In this space, the position four-vector is written as

$$(1.11) \quad \vec{X} = X^0 \vec{E}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu,$$

where

$$(1.12) \quad [\vec{e}_\mu] = [\vec{E}_0 \ \vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] \ ; \ X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

are the basis of E^4 and the coordinates of the position four-vector, respectively. Moreover, the *Euclidean length* of the position four-vector is

$$(1.13) \quad |\vec{X}|^2 = x_\mu g_{E^4 \mu\nu} x^\nu = (X^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2.$$

2. Connection between spaces endowed with Euclidean and Lorentz metric and their transformations

From (1.2) and (1.10), we respectively have

$$(2.1) \quad \vec{e}_0 \cdot \vec{e}_0 = -1 \ ; \ \vec{E}_0 \cdot \vec{E}_0 = 1,$$

where dot “.” is *Euclidean inner product* [2] (p. 7).

Thus, we understand that

$$(2.2) \quad \vec{E}_0 = i\vec{e}_0.$$

After replacing the above to (1.11), we have

$$(2.3) \quad \vec{X} = iX^0 \vec{e}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu.$$

Comparing this to (1.3), we obtain

$$(2.4) \quad X^0 = \frac{1}{i}x^0.$$

The above procedure shows that the difference between *Euclidean* and *Lorentz metric* is caused by the different 0-four-vector of the used basis: $\vec{E}_0$ and $\vec{e}_0$, respectively. So, E^4 and M^4 are related via (2.2) and (2.4).

The corresponding LTs are also easily related. For instance, the *active interpretation* of LT [2] (p. 6) is

$$(2.5) \quad X' = \Lambda_{(\beta)} X,$$

where

$$(2.6) \quad \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} \quad ; \quad \beta^i = \frac{dx^i}{dx^0} \quad ; \quad i = 1, 2, 3,$$

is called β -factor. Eqn (2.5) expresses *proper Lorentz Boost* (LB) in M^4 [2] (p. 30) and E^4 , correspondingly:

$$(2.7) \quad A_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ -\gamma_{(\beta)}\beta & I_3 + \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix},$$

$$(2.8) \quad A_{L(\beta)}^E = \begin{bmatrix} \gamma_{(\beta)} & i\gamma_{(\beta)}\beta^T \\ -i\gamma_{(\beta)}\beta & I_3 + \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} \quad ; \quad X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{i}x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix},$$

where I_3 is the unitary 3×3 matrix and *Lorentz γ -factor* is

$$(2.9) \quad \gamma_{(\beta)} = \frac{1}{\sqrt{1 - \beta^T\beta}}.$$

The *typical proper LB* along x -axis in M^4 [2] (p. 21) and E^4 has, correspondingly:

$$(2.10) \quad A_{L(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix},$$

$$(2.11) \quad A_{L(x)(\beta)}^E = \begin{bmatrix} \gamma_{(\beta)} & i\gamma_{(\beta)}\beta & 0 & 0 \\ -i\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{i}x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

It is noted that transformation matrices (2.8) and (2.11) are rotation matrices. So, LB in M^4 becomes rotation in E^4 (*Wick Rotation*) [3].

The physical content of the four-dimensional space is obtained by the replacement

$$(2.12) \quad x^0 = ct,$$

where

$$(2.13) \quad c = 299,792,458 \text{ms}^{-1}$$

is the speed of light in vacuum. Then β -factor is called *velocity factor*

$$(2.14) \quad \beta^i = \frac{1}{c} \frac{dx^i}{dt} = \frac{v^i}{c}.$$

3. Derivation of the proper closed isometric linear transformation in four-dimensional space endowed with Euclidean or Lorentz metric

3.1 Motion in x -direction

We consider one unmoved observer (frame) $Oxyz$, who measures real space-time and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving to the right, along x -axis with velocity

$$(3.1.1) \quad v = \beta c$$

wrt the observer (frame) $Oxyz$ (Figure 1).

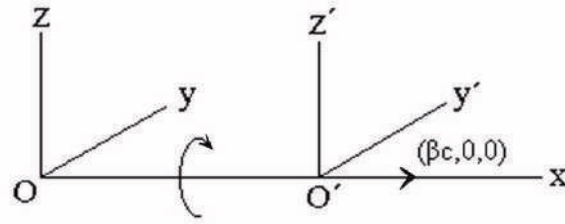


Figure 1. Two frames $Oxyz$ and $O'x'y'z'$, which initially coincide. The second is moving with velocity $(\beta c, 0, 0)$ wrt to $Oxyz$.

Supposing the next linear transformation:

$$(3.1.2) \quad ct' = bct + ax + ky + \nu z$$

$$(3.1.3) \quad x' = gct + fx + \delta y + \theta z$$

$$(3.1.4) \quad y' = g_1 ct + f_1 x + h y + \lambda z$$

$$(3.1.5) \quad z' = g_2 ct + f_2 x + \xi y + \mu z,$$

we determine the coefficients with the following conditions:

- (i) Isotropy: We postulate the transformation to be invariant to the spatial rotation. Rotating the frame about x -axis, through one right angle (Figure 1), we correspond the new axes to the initial axes:

$$(3.1.6) \quad \begin{aligned} ct &\rightarrow ct; ct' \rightarrow ct'; x \rightarrow x; x' \rightarrow x'; y \rightarrow -z; \\ y' &\rightarrow -z'; z \rightarrow y; z' \rightarrow y'. \end{aligned}$$

Thus, from (3.1.2), we have

$$(3.1.7) \quad ct' = bct + ax - kz + \nu y.$$

Comparing (3.1.2) and (3.1.7), it emerges $k = \nu = 0$. Besides, from (3.1.3) we have

$$(3.1.8) \quad x' = gct + fx - \delta x + \theta y.$$

Comparing (3.1.3) and (3.1.8), it emerges $\delta = \theta = 0$. Besides, from (3.1.4) we have

$$(3.1.9) \quad -z' = g_1 ct + f_1 x - hz + \lambda y.$$

Comparing (3.1.5) and (3.1.9), it emerges $g_2 = -g_1$, $f_2 = -f_1$, $\xi = -\lambda$ and $\mu = h$. Besides, from (3.1.5), we have

$$(3.1.10) \quad y' = g_2 ct + f_2 x - \xi z + \mu y.$$

Comparing (3.1.4) and (3.1.10), it emerges $g_2 = g_1$, $f_2 = f_1$, $\xi = -\lambda$ and $\mu = h$. So, $k = \nu = \delta = \theta = g_1 = g_2 = f_1 = f_2 = 0$; $\xi = -\lambda$; $\mu = h$ and the transformation becomes:

$$(3.1.11) \quad ct' = bct + ax$$

$$(3.1.12) \quad x' = gct + fx$$

$$(3.1.13) \quad y' = hy + \lambda z$$

$$(3.1.14) \quad z' = -\lambda y + hz.$$

- (ii) The frame $O'x'y'z'$ is moving with velocity $(\beta c, 0, 0)$ wrt to $Oxyz$: for $x' = 0$, it is $x = \beta ct$. Replacing these to (3.1.12), we obtain

$$(3.1.15) \quad 0 = gct + f\beta ct,$$

for any value of t . This emerges

$$(3.1.16) \quad g = -\beta f$$

and the transformation becomes:

$$(3.1.17) \quad ct' = bct + ax$$

$$(3.1.18) \quad x' = -\beta fct + fx$$

$$(3.1.19) \quad y' = hy + \lambda z$$

$$(3.1.20) \quad z' = -\lambda y + hz.$$

- (iii) Maintenance of Lorentz length (spacetime interval) $|\vec{X}| = S : S'^2 = S^2$.
Thus,

$$(3.1.21) \quad x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2,$$

or equivalently,

$$(3.1.22) \quad \begin{aligned} & (-\beta c f t + f x)^2 + (h y + \lambda z)^2 + (-\lambda y + h z)^2 - (b c t + a x)^2 \\ & = x^2 + y^2 + z^2 - c^2 t^2. \end{aligned}$$

From the terms $x^2; y^2; z^2; c^2 t^2; c t x$, we obtain:

$$(3.1.23) \quad f^2 - a^2 = 1$$

$$(3.1.24) \quad h^2 + \lambda^2 = 1$$

$$(3.1.25) \quad \beta^2 f^2 - b^2 = -1$$

$$(3.1.26) \quad -\beta f^2 - a b = 0.$$

Combining (3.1.26) with (3.1.23), we have

$$(3.1.27) \quad b = \frac{-\beta a^2 - \beta}{a}.$$

The Combination of (3.1.25) with (3.1.23) and (3.1.27) gives

$$(3.1.28) \quad a = \pm \frac{\beta}{\sqrt{1 - \beta^2}} = \pm \beta \gamma.$$

Replacing the above to (3.1.27), we obtain

$$(3.1.29) \quad b = \mp \frac{1}{\sqrt{1 - \beta^2}} = \mp \gamma.$$

Now, we must choose the sign in the above equations. We observe that for $\beta = 0$ the *upper sign* ($\uparrow$) gives $a = 0$ and $b = -1$. This transforms (3.1.11) to $t' = -t$, producing *time inversion*. The *lower sign* ($\downarrow$) gives $t' = t$, corresponding to the *proper transformation* and we have:

$$(3.1.30) \quad a = -\frac{\beta}{\sqrt{1 - \beta^2}} = -\beta \gamma ; \quad b = \frac{1}{\sqrt{1 - \beta^2}} = \gamma.$$

The replacement of the above to (3.1.23) gives

$$(3.1.31) \quad f = \pm \frac{1}{\sqrt{1 - \beta^2}} = \pm \gamma.$$

We also observe that for $\beta = 0$ the *upper sign* ($\uparrow$) gives $f = 1$. This transforms (3.1.18) to $x' = x$. Thus, the *proper transformation* has

$$(3.1.32) \quad f = \frac{1}{\sqrt{1 - \beta^2}} = \gamma.$$

On the other hand the *lower sign* ($\downarrow$) for $\beta = 0$, gives $f = -1$ and (3.1.18) is transformed to $x' = -x$, producing *space inversion*. So, the *proper transformation* ($\uparrow\downarrow$) becomes:

$$(3.1.33) \quad ct' = \gamma(ct - \beta x)$$

$$(3.1.34) \quad x' = \gamma(-\beta ct + x)$$

$$(3.1.35) \quad y' = hy + \lambda z$$

$$(3.1.36) \quad z' = -\lambda y + hz,$$

with condition (3.1.24). Using matrices, LT (2.5) has

$$(3.1.37) \quad A_{B(x)} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix}.$$

Besides, the differential form of the transformation is

$$(3.1.38) \quad cdt' = \gamma(cdt - \beta dx)$$

$$(3.1.39) \quad dx' = \gamma(-\beta cdt + dx)$$

$$(3.1.40) \quad dy' = hdy + \lambda dz$$

$$(3.1.41) \quad dz' = -\lambda dy + hdz.$$

Thus, the velocities are related as following:

$$(3.1.42) \quad v'_x = \frac{-\beta c + u_x}{c - \beta u_x} c, \quad v'_y = \frac{hu_y + \lambda u_z}{\gamma(c - \beta u_x)} c, \quad v'_z = \frac{-\lambda u_y + hu_z}{\gamma(c - \beta u_x)} c.$$

3.2 General linear transformation (motion in random direction)

We consider an unmoved observer (frame) $Oxyz$ and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving with velocity (v_x, v_y, v_z) wrt the observer (frame) $Oxyz$ (Figure 2).

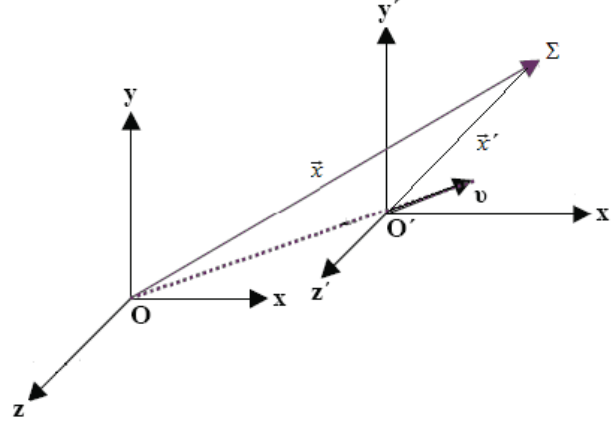


Figure 2. Two frames $Oxyz$ and $O'x'y'z'$, which initially coincide. The second is moving with random velocity (v_x, v_y, v_z) wrt to $Oxyz$.

We rotate the initial frame $Oxyz$, in order to parallelize the unitary vector $\hat{x}$ to the velocity vector $\vec{v}$ of the observer $O'x'y'z'$. This is sequentially achieved as following: We firstly rotate the coordinate system $Oxyz$ about z -axis, through an angle θ [$O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k})$]. We then rotate the coordinate system $O(\hat{i}, \hat{j}, \hat{k})$ about $\hat{j}$ through an angle ω [$O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$] (Figure 3). The corresponding matrices are:

$$(3.2.1) \quad R_{xy(\theta)} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} ; R_{xz(\omega)} = \begin{bmatrix} \cos \omega & 0 & \sin \omega \\ 0 & 1 & 0 \\ -\sin \omega & 0 & \cos \omega \end{bmatrix} .$$

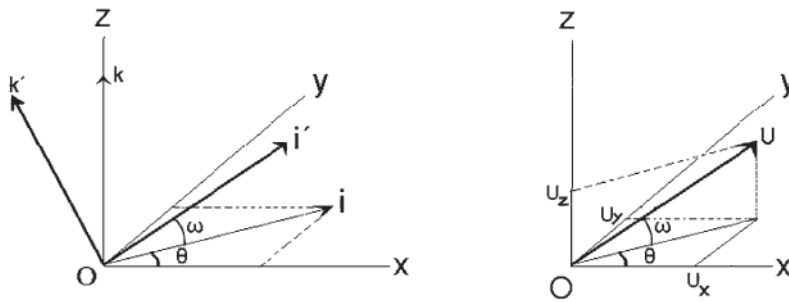


Figure 3. Rotation of the initial frame $Oxyz$, in order to achieve parallelization of vector $\hat{x}$ to the velocity vector $\vec{v}$ of the moving observer $O'x'y'z'$ [$O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$].

Thus, we have the transformation:

$$(3.2.2) \quad \begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} = \begin{bmatrix} \cos \omega \cos \theta & \cos \omega \sin \theta & \sin \omega \\ -\sin \theta & \cos \theta & 0 \\ -\sin \omega \cos \theta & -\sin \omega \sin \theta & \cos \omega \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where (x_R, y_R, z_R) are the coordinates wrt the frame $O(\hat{i}', \hat{j}', \hat{k}')$ and

$$(3.2.3) \quad \sin \theta = \frac{v_y}{\sqrt{v_x^2 + v_y^2}} ; \quad \cos \theta = \frac{v_x}{\sqrt{v_x^2 + v_y^2}},$$

$$(3.2.4) \quad \sin \omega = \frac{v_z}{|v|} ; \quad \cos \omega = \frac{\sqrt{v_x^2 + v_y^2}}{|v|}.$$

As a result, the above 3×3 matrix becomes

$$(3.2.5) \quad R = \begin{bmatrix} \frac{\beta_x}{|\beta|} & \frac{\beta_y}{|\beta|} & \frac{\beta_z}{|\beta|} \\ -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & 0 \\ -\frac{\beta_x \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_y \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \end{bmatrix}$$

and we define

$$(3.2.6) \quad \tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}.$$

The unit means that time is not affected by the spatial rotation.

Moreover, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence of successive transformations:
 $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$; $O(\hat{i}', \hat{j}', \hat{k}') \rightarrow O'(\hat{i}', \hat{j}', \hat{k}')$; $O'(\hat{i}', \hat{j}', \hat{k}') \rightarrow O'(\hat{x}, \hat{y}, \hat{z})$.
 The above simple transformations have active interpretations, respectively:

$$(3.2.7) \quad X_R = \tilde{R}X ; \quad X'_R = \Lambda_{B(x)}X_R ; \quad X' = \tilde{R}^T X'_R,$$

where $\tilde{R}^T$ is the transpose matrix of $\tilde{R}$.

Thus, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$(3.2.8) \quad X' = \tilde{R}^T \Lambda_{B(x)(\beta)} \tilde{R}X,$$

and LT (2.5) has

$$(3.2.9) \quad \Lambda_{B(\beta)} = \tilde{R}^T \Lambda_{B(x)(\beta)} \tilde{R}.$$

It is $\beta > 0$. So, $\beta = |\vec{\beta}|$ and we calculate:

$$(3.2.10) \quad \Lambda_{B(\beta)} = \tilde{R}^T \begin{bmatrix} \gamma & -|\beta|\gamma & 0 & 0 \\ -|\beta|\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_x}{|\beta|} & \frac{\beta_y}{|\beta|} & \frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & 0 \\ 0 & -\frac{\beta_x \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_y \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \end{bmatrix},$$

or equivalently,

$$(3.2.11) \quad \Lambda_{B(\beta)} = \tilde{R}^T \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma|\beta| & \gamma\frac{\beta_x}{|\beta|} & \gamma\frac{\beta_y}{|\beta|} & \gamma\frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y h}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_x \beta_z \lambda}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x h}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_y \beta_z \lambda}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\lambda \sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \\ 0 & \frac{\beta_y \lambda}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_x \beta_z h}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_x \lambda}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_y \beta_z h}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{h \sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \end{bmatrix}.$$

Furthermore, we have

$$(3.2.12) \quad \Lambda_{B(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_x}{|\beta|} & -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_x \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} \\ 0 & \frac{\beta_y}{|\beta|} & \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_y \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} \\ 0 & \frac{\beta_z}{|\beta|} & 0 & \frac{\sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \end{bmatrix} \cdot \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma|\beta| & \gamma\frac{\beta_x}{|\beta|} & \gamma\frac{\beta_y}{|\beta|} & \gamma\frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y h}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_x \beta_z \lambda}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x h}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_y \beta_z \lambda}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\lambda \sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \\ 0 & \frac{\beta_y \lambda}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_x \beta_z h}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_x \lambda}{\sqrt{\beta_x^2 + \beta_y^2}} - \frac{\beta_y \beta_z h}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{h \sqrt{\beta_x^2 + \beta_y^2}}{|\beta|} \end{bmatrix}.$$

So, we obtain

$$(3.2.13) \quad \Lambda_{B(\beta)} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & (\gamma - h)\frac{\beta_x^2}{|\beta|^2} + h & (\gamma - h)\frac{\beta_x \beta_y}{|\beta|^2} + \frac{\beta_z \lambda}{|\beta|} & (\gamma - h)\frac{\beta_x \beta_z}{|\beta|^2} - \frac{\beta_y \lambda}{|\beta|} \\ -\gamma\beta_y & (\gamma - h)\frac{\beta_x \beta_y}{|\beta|^2} - \frac{\beta_z \lambda}{|\beta|} & (\gamma - h)\frac{\beta_y^2}{|\beta|^2} + h & (\gamma - h)\frac{\beta_y \beta_z}{|\beta|^2} + \frac{\beta_x \lambda}{|\beta|} \\ -\gamma\beta_z & (\gamma - h)\frac{\beta_x \beta_z}{|\beta|^2} + \frac{\beta_y \lambda}{|\beta|} & (\gamma - h)\frac{\beta_y \beta_z}{|\beta|^2} - \frac{\beta_x \lambda}{|\beta|} & (\gamma - h)\frac{\beta_z^2}{|\beta|^2} + h \end{bmatrix}$$

and we have the transformation

$$(3.2.14) \quad \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & (\gamma-h)\frac{\beta_x^2}{|\beta|^2} + h & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2} + \frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2} - \frac{\beta_y\lambda}{|\beta|} \\ -\gamma\beta_y & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2} - \frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y^2}{|\beta|^2} + h & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2} + \frac{\beta_x\lambda}{|\beta|} \\ -\gamma\beta_z & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2} + \frac{\beta_y\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2} - \frac{\beta_x\lambda}{|\beta|} & (\gamma-h)\frac{\beta_z^2}{|\beta|^2} + h \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.$$

3.3 The solution of proper closed isometric linear transformation (correlation of two perpendicular moving observers)

We consider one unmoved observer (frame $Oxyz$, another observer (frame $O'x'y'z'$ with parallel spatial axes, moving to the right, along x -axis with velocity $(\beta c, 0, 0)$ wrt $Oxyz$ and also a third observer (frame) $O''x''y''z''$ with parallel spatial axes, moving upward, along y -axis with velocity $(0, \beta c, 0)$ wrt $Oxyz$ (Figure 4).

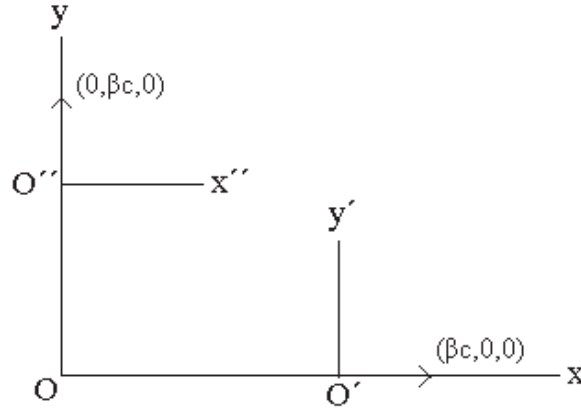


Figure 4. Two frames $O'x'y'z'$ and $O''x''y''z''$ moving with corresponding velocities $(\beta c, 0, 0)$ and $(0, \beta c, 0)$ wrt $Oxyz$.

Now, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence: $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{x}, \hat{y}, \hat{z})$; $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$. The above simple transformations have active interpretations, respectively:

$$(3.3.1) \quad X = \Lambda_{B(x)(\beta)}^{-1} X' ; \quad X'' = \Lambda_{B(y)(\beta)} X.$$

Thus, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$(3.3.2) \quad X'' = \Lambda_{B(y)(\beta)} \Lambda_{B(x)(\beta)}^{-1} X'$$

and LT (2.5) has

$$(3.3.3) \quad \Lambda_3 = \Lambda_{B(y)(\beta)} \Lambda_{B(x)(\beta)}^{-1}.$$

According to (3.2.13), it is

$$(3.3.4) \quad \Lambda_{B(x)(\beta)} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix}; \Lambda_{B(y)(\beta)} = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & h & 0 & -\lambda \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & \lambda & 0 & h \end{bmatrix}.$$

Besides, the inverse of matrix $\Lambda_{B(x)(\beta)}$ is

$$(3.3.5) \quad \Lambda_{B(x)(\beta)}^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & -\lambda \\ 0 & 0 & \lambda & h \end{bmatrix}.$$

With that

$$(3.3.6) \quad \Lambda_3 = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & h & 0 & -\lambda \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & \lambda & 0 & h \end{bmatrix} \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & -\lambda \\ 0 & 0 & \lambda & h \end{bmatrix} = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma h & \gamma h & -\lambda^2 & -h\lambda \\ -\beta\gamma^2 & -\beta^2\gamma^2 & \gamma h & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & h\lambda & h^2 \end{bmatrix}.$$

Now, we calculate velocity factor $\vec{\beta}'_4$ of observer $O''x''y''z''$ wrt observer (frame) $O'x'y'z'$. Eqn (3.1.41) can be applied, because observer $O'x'y'z'$ is moving in x -direction and observer O'' can be considered as observed body:

$$(3.3.7) \quad \beta'_{4x} = \frac{-\beta c + 0}{c - 0} = -\beta$$

$$(3.3.8) \quad \beta'_{4y} = \frac{h\beta c + \lambda \cdot 0}{\gamma(c - 0)} = \frac{h\beta}{\gamma}$$

$$(3.3.9) \quad \beta'_{4z} = \frac{-\lambda\beta c + 0}{\gamma(c - 0)} = \frac{-\lambda\beta}{\gamma}.$$

As a result, it is

$$(3.3.10) \quad |\vec{\beta}'_4|^2 = \beta^2 + \beta^2 h^2 \frac{1}{\gamma^2} + \frac{\beta^2 \lambda^2}{\gamma^2} = \frac{\beta^2 \gamma^2 + \beta^2 h^2 + \beta^2 \lambda^2}{\gamma^2},$$

or equivalently,

$$(3.3.11) \quad |\vec{\beta}'_4|^2 = \beta^2 \frac{\gamma^2 + h^2 + \lambda^2}{\gamma^2} = \beta^2 \frac{\gamma^2 + 1}{\gamma^2}.$$

Thus, it emerges

$$(3.3.12) \quad |\vec{\beta}'_4| = \beta \frac{\sqrt{\gamma^2 + 1}}{\gamma} \quad ; \quad \gamma_{(\vec{\beta}'_4)} = \gamma^2.$$

According to (3.2.13), the matrix corresponding to the velocity factor $\vec{\beta}'_4$ is

$$(3.3.13) \quad \Lambda_4 = \Lambda_{B(\vec{\beta}'_4)} = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma^2 & \cdot & \cdot & \cdot \\ -\beta\gamma h & \cdot & \cdot & \cdot \\ \beta\gamma\lambda & \cdot & \cdot & \cdot \end{bmatrix}$$

We postulate the transformation to be closed:

$$(3.3.14) \quad \Lambda_3 = \Lambda_{B(y)(\beta)} \Lambda_{B(x)(\beta)}^{-1} = \Lambda_{(\vec{\beta}'_4)} = \Lambda_4.$$

Comparing the matrices, element by element, we have:

$$(3.3.15) \quad (\Lambda_3)_{10} = (\Lambda_4)_{10},$$

or equivalently,

$$(3.3.16) \quad h = \gamma.$$

Applying the foregoing equation in (3.1.24), we obtain

$$(3.3.17) \quad \lambda^2 = -\beta^2\gamma^2 \quad ; \quad \lambda = i\beta\gamma = i|\vec{\beta}'_4|\gamma.$$

Thus, (3.3.16), (3.3.12ii), (3.3.17ii) and (3.3.12i) emerge

$$(3.3.18) \quad h_{(\vec{\beta}'_4)} = \gamma_{(\beta'_4)} = \gamma^2$$

and

$$(3.3.19) \quad \lambda_{(\vec{\beta}'_4)} = i|\vec{\beta}'_4|\gamma_{(\vec{\beta}'_4)} = i\beta \frac{\sqrt{\gamma^2 + 1}}{\gamma} \gamma^2 = i\beta\gamma\sqrt{\gamma^2 + 1} = \lambda\sqrt{\gamma^2 + 1}.$$

Thus, the matrix (3.2.13) for the velocity factor $\vec{\beta}'_4$ [see also (3.3.13)] is written:

$$(3.3.20) \quad \Lambda_4 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma^2 & \beta\gamma\lambda \\ \beta\gamma^2 & h_{(\vec{\beta}'_4)} & \frac{\beta'_{4z}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & -\frac{\beta'_{4y}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} \\ -\beta\gamma^2 & -\frac{\beta'_{4z}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & h_{(\vec{\beta}'_4)} & +\frac{\beta'_{4x}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} \\ \beta\gamma\lambda & \frac{\beta'_{4y}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & -\frac{\beta'_{4x}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & h_{(\vec{\beta}'_4)} \end{bmatrix}.$$

Replacing only (3.3.16) to (3.3.8), we rewrite the velocity factor components

$$(3.3.21) \quad \beta'_{4x} = -\beta \quad ; \quad \beta'_{4y} = \beta \quad ; \quad \beta'_{4z} = -\frac{\lambda\beta}{\gamma}.$$

Now, we calculate the following quotients contained in matrix Λ_4 , by using the above and also (3.3.12):

$$(3.3.22) \quad \frac{\beta'_{4x}}{|\vec{\beta}'_4|} = -\frac{\gamma}{\sqrt{\gamma^2 + 1}}; \quad \frac{\beta'_{4y}}{|\vec{\beta}'_4|} = \frac{\gamma}{\sqrt{\gamma^2 + 1}}; \quad \frac{\beta'_{4z}}{|\vec{\beta}'_4|} = -\frac{\lambda}{\sqrt{\gamma^2 + 1}}.$$

The replacement of the above, (3.3.18) and (3.3.19) to (3.3.20) gives

$$(3.3.23) \quad \Lambda_4 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma^2 & \beta\gamma\lambda \\ \beta\gamma^2 & \gamma^2 & -\lambda^2 & -\gamma\lambda \\ -\beta\gamma^2 & \lambda^2 & \gamma^2 & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & \gamma\lambda & \gamma^2 \end{bmatrix},$$

while from (3.3.6) it is

$$(3.3.24) \quad \Lambda_3 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma h & \gamma h & -\lambda^2 & -h\lambda \\ -\beta\gamma^2 & -\beta^2\gamma^2 & \gamma h & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & h\lambda & h^2 \end{bmatrix}.$$

We validate the equation of the matrices: $\Lambda_3 = \Lambda_4$, because of (3.3.16) and (3.3.17i).

Finally, we replace (3.3.16) and (3.3.17ii) to (3.2.14) and (3.2.13) and we obtain the *proper closed isometric LT*:

$$(3.3.25) \quad \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix} \cdot \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

and the corresponding matrix

$$(3.3.26) \quad \Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix}.$$

We have preferred the physical approach (spacetime) for the derivation of the *proper isometric LT* in M^4 , because SR is the main application [4]. The pure mathematical approach is simply obtained, by replacing $ct \rightarrow x^0$, according to (2.11):

$$(3.3.27) \quad \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

and the corresponding matrix is

$$(3.3.28) \quad \Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^T \\ -\beta & I_3 + iA_{(\beta)} \end{bmatrix},$$

where

$$(3.3.29) \quad \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} ; \quad A_{(\beta)} = \begin{bmatrix} 0 & \beta^3 & -\beta^2 \\ -\beta^3 & 0 & \beta^1 \\ \beta^2 & -\beta^1 & 0 \end{bmatrix}.$$

The matrix (Λ_B) of the *proper isometric LT* has determinant equal to the unit ($\det \Lambda_B = 1$). Besides, the *typical transformation* along x -axis, has

$$(3.3.30) \quad \Lambda_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix}$$

It is noted that antisymmetric matrix $A_{(\beta)}$ is related to the *cross product* (*external product*) because:

$$(3.3.31) \quad \begin{aligned} A_{(\vec{e}_1, \vec{e}_2, \vec{e}_3)} = [\vec{e}_i \times \vec{e}_j] &= \begin{bmatrix} \vec{e}_1 \times \vec{e}_1 & \vec{e}_1 \times \vec{e}_2 & \vec{e}_1 \times \vec{e}_3 \\ \vec{e}_2 \times \vec{e}_1 & \vec{e}_2 \times \vec{e}_2 & \vec{e}_2 \times \vec{e}_3 \\ \vec{e}_3 \times \vec{e}_1 & \vec{e}_3 \times \vec{e}_2 & \vec{e}_3 \times \vec{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} \vec{0} & \vec{e}_3 & -\vec{e}_2 \\ -\vec{e}_3 & \vec{0} & \vec{e}_1 \\ \vec{e}_2 & -\vec{e}_1 & \vec{0} \end{bmatrix}. \end{aligned}$$

So,

$$(3.3.32) \quad \begin{aligned} \vec{x} \times \vec{y} &= (x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3) \times (y^1 \vec{e}_1 + y^2 \vec{e}_2 + y^3 \vec{e}_3) \\ &= [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] \cdot \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}, \end{aligned}$$

written in compact form:

$$(3.3.33) \quad \vec{x} \times \vec{y} = [\vec{e}_i] \cdot [A_{(-\vec{x})^{ij}}] \cdot [y^j] = -[\vec{e}_i] \cdot [A_{(\vec{x})^{ij}}] \cdot [y^j].$$

On the other hand, the proper isometric transformation in E^4 is obtained as following: we initially divide (3.3.27) with i

$$(3.3.34) \quad \begin{bmatrix} \frac{x'^0}{i} \\ \frac{x'^1}{i} \\ \frac{x'^2}{i} \\ \frac{x'^3}{i} \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{x^0}{i} \\ \frac{x^1}{i} \\ \frac{x^2}{i} \\ \frac{x^3}{i} \end{bmatrix}.$$

This is equivalent to

$$(3.3.35) \quad \begin{bmatrix} \frac{x'^0}{i} \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{x^0}{i} \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

The above is written by using (2.4):

$$(3.3.36) \quad \begin{bmatrix} X'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

So, the corresponding matrix in E^4 is

$$(3.3.37) \quad \begin{aligned} \Lambda_{B(\beta)}^E &= \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \\ &= \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^T \\ -i\beta & I_3 + iA_{(\beta)} \end{bmatrix}. \end{aligned}$$

This is rotation matrix, because it is orthogonal (unitary) matrix with determinant equal to the unit ($\det \Lambda_B^E = 1$). Besides, the *typical transformation* along x -axis, has

$$(3.3.38) \quad \Lambda_{B(x)(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta & 0 & 0 \\ -i\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix}$$

4. Improper isometric linear transformations in four-dimensional space endowed with Euclidean or Lorentz metric

In the derivation of proper closed isometric Linear Transformation ($\downarrow\uparrow$), we have chosen the *lower sign* ($\downarrow$) in (3.1.28) and (3.1.29), but the *upper* ($\uparrow$) in (3.1.31). So, they have remained three (3) improper non-closed isometric Linear Transformations (which does not contain the *identity transformation*):

- (i) *Space inversion non-closed isometric Linear Transformation* ($\downarrow\downarrow$) [*lower sign* ($\downarrow$) in (3.1.28) and (3.1.29) as well as *lower sign* ($\downarrow$) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda_B^E = -1$):

$$(4.1) \quad \Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^T \\ \beta & -I_3 - iA_{(\beta)} \end{bmatrix}; \quad \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^T \\ i\beta & -I_3 - iA_{(\beta)} \end{bmatrix}.$$

The respective *typical transformations* along x -axis, have

$$(4.2) \quad \Lambda_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -i\beta \\ 0 & 0 & i\beta & -1 \end{bmatrix} ; \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta & 0 & 0 \\ i\beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -i\beta \\ 0 & 0 & i\beta & -1 \end{bmatrix}.$$

- (ii) *Time inversion non-closed isometric Linear Transformation* ($\uparrow\uparrow$) [*upper sign* ($\uparrow$) in (3.1.28) and (3.1.29) as well as *upper sign* ($\uparrow$) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda_B^E = -1$):

$$(4.3) \quad \Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} -1 & \beta^T \\ -\beta & I_3 + iA_{(\beta)} \end{bmatrix} ; \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} -1 & -i\beta^T \\ -i\beta & I_3 + iA_{(\beta)} \end{bmatrix}.$$

The respective *typical transformations* along x -axis, have

$$(4.4) \quad \Lambda_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} -1 & \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix} ; \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} -1 & -i\beta & 0 & 0 \\ -i\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix}.$$

- (iii) *Spacetime inversion closed isometric Linear Transformation* ($\uparrow\downarrow$) [*upper sign* ($\uparrow$) in (3.1.28) and (3.1.29), but *lower sign* ($\downarrow$) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda_B^E = 1$):

$$(4.5) \quad \Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} -1 & \beta^T \\ \beta & -I_3 - iA_{(\beta)} \end{bmatrix} ; \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} -1 & -i\beta^T \\ i\beta & -I_3 - iA_{(\beta)} \end{bmatrix}.$$

The respective *typical transformations* along x -axis, have

$$(4.6) \quad \Lambda_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} -1 & \beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -i\beta \\ 0 & 0 & i\beta & 1 \end{bmatrix} ; \Lambda_{B(\beta)}^E = \gamma_{(\beta)} \begin{bmatrix} -1 & -i\beta & 0 & 0 \\ i\beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -i\beta \\ 0 & 0 & i\beta & -1 \end{bmatrix}.$$

These matrices are exactly the opposite of the corresponding *proper closed isometric LT*.

In case of *Lorentz Boost*, we have [2] (pp. 30-31):

- (i) *Space inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = -1$):

$$(4.7) \quad \Lambda_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)}\beta^T \\ -\gamma_{(\beta)}\beta & -I_3 - \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} ; \Lambda_{L(\beta)}^E = \begin{bmatrix} \gamma_{(\beta)} & -i\gamma_{(\beta)}\beta^T \\ -i\gamma_{(\beta)}\beta & -I_3 - \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix}.$$

The respective *typical transformations* along x -axis, have

$$(4.8) \quad \begin{aligned} \Lambda_{L(x)(\beta)} &= \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \\ \mathbb{L}_{L(x)(\beta)}^E &= \begin{bmatrix} \gamma_{(\beta)} & -i\gamma_{(\beta)}\beta & 0 & 0 \\ -i\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

(ii) *Time inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = 1$):

$$(4.9) \quad \begin{aligned} \Lambda_{L(\beta)} &= \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta^T \\ \gamma_{(\beta)}\beta & I_3 - \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}; \\ \Lambda_{L(\beta)}^E &= \begin{bmatrix} -\gamma_{(\beta)} & -i\gamma_{(\beta)}\beta^T \\ i\gamma_{(\beta)}\beta & I_3 - \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \end{aligned}$$

The respective *typical transformations* along x -axis, have

$$(4.10) \quad \begin{aligned} \Lambda_{L(x)(\beta)} &= \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \\ L_{L(x)(\beta)}^E &= \begin{bmatrix} -\gamma_{(\beta)} & -i\gamma_{(\beta)}\beta & 0 & 0 \\ i\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(iii) *Spacetime inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = -1$):

$$(4.11) \quad \begin{aligned} \Lambda_{L(\beta)} &= \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ \gamma_{(\beta)}\beta & -I_3 + \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}; \\ \Lambda_{L(\beta)}^E &= \begin{bmatrix} -\gamma_{(\beta)} & i\gamma_{(\beta)}\beta^T \\ i\gamma_{(\beta)}\beta & -I_3 + \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \end{aligned}$$

The respective *typical transformations* along x -axis, have

$$(4.12) \quad \begin{aligned} A_{L(x)(\beta)} &= \begin{bmatrix} -\gamma(\beta) & -\gamma(\beta)\beta & 0 & 0 \\ \gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \\ \mathbf{L}_{L(x)(\beta)}^E &= \begin{bmatrix} -\gamma(\beta) & i\gamma(\beta)\beta & 0 & 0 \\ i\gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

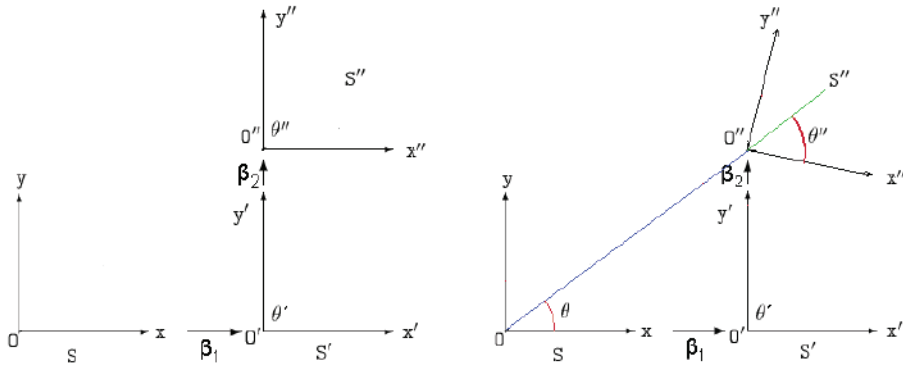


Figure 5. Correlation of three successive observers (frames), by using Lorentz Boost. The frame $O'x'y'z'$ has parallel axes to the corresponding of frame $Oxyz$, moving with velocity $(\beta_1 c, 0, 0)$ wrt $Oxyz$. The frame $O''x''y''z''$ has parallel axes to the corresponding of frame $O'x'y'z'$, moving with velocity $(0, \beta_2 c, 0)$ wrt $O'x'y'z'$. The correlation of the observers, by using Lorentz Boost, cancels the absolute character of parallelism. Thus, the axes of frame $O''x''y''z''$ are not parallel to the corresponding of frame $Oxyz$ (Thomas Rotation).

5. Conclusions

The closed isometric linear transformation which maintains *Lorentz length* (S^2) is represented by a matrix (A_B) containing real and imaginary numbers. Under this transformation, the real spacetime of the initial rest observer (frame) is transformed to real time and complex space for one moving observer (frame) ($R^4 \rightarrow R \times C^3$). Subsequently, the axes rotation (*Thomas Rotation* [5]) that happens from the correlation of three observers related by using *Lorentz Boost* (Figure 5) [2] (pp. 177-183), in this approach is avoided. Thus, the validation of the transitive attribute in parallelism of unmoved straight lines (which is equivalent to the 5th Euclidean postulate), is extended to the case moving straight lines (for any observer). This is achieved, by working in the domain of complex numbers, validating one more time, the words of J. Hadamard: “*It has been*

written that the shortest and best way between two truths of the real domain often passes through the imaginary one” [6] (p. 123).

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On periodicities in cluster algebras

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Abstract. The aim of this paper is to study periodicities in cluster algebras. Firstly, we give a proof of Restriction and Extension Theorem for cluster algebras without coefficients. Then, we show that the periodicity of a labeled seed with coefficients from an arbitrary semifield depends only on the corresponding extended exchange matrix, which implies the independence of periodicity relative to coefficients.

Keywords: cluster algebra, periodicity, labeled seed, coefficient.

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky in a series of papers. They are defined as commutative $\mathbb{Z}\mathbb{P}$ -algebras generated by cluster variables in which $\mathbb{P}$ is a semifield. Many relations between cluster algebras and other branches of mathematics have been discovered, such as periodicities of T -systems and Y -systems, representation theory of quivers, combinatorics, Poisson geometry and higher Teichmüller spaces.

Cluster algebras of finite type were classified by Fomin and Zelevinsky in [5], which is the same as the Cartan-Killing classification of semisimple Lie algebras and finite root systems. Periodicities in cluster algebras were firstly studied by Fomin and Zelevinsky in [6]. They proved Zamolodchikovs periodicity conjecture on Y -systems from indecomposable Cartan matrices of finite type. The periodicity conjecture for a pair of Dynkin diagrams was proved by Keller using additive categorification of cluster algebras in [15]. Another related concept closed with cluster algebras and periodicities is T -systems. Periodicity conjecture and half-periodicity conjecture for T -systems and Y -systems associated with the quantum affine algebras of Dynkin type at any level were proved by In-

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oue, Iyama, Keller, Kunima, and Nakanishi [11, 12]. The theory of periodicities in cluster algebras studied above has also important applications in dilogarithm identities. Indeed for any period of a seed, there exists a dilogarithm identity.

Nakanishi studied periodicities in cluster algebras in general cases rather than finite type in [16]. Nakanishi also proved an interesting theorem (Restriction and Extension Theorem) for cluster algebras with coefficients from subtraction-free semifields. Even though it is hard to determine all periods of labeled seeds, it is possible to find a large number of periods by Extension Theorem. He also formulated dilogarithm identities associated to periodicities of labeled seeds. These results are extended to more general cases, in [17] for generalized cluster algebras and in [18] for quantum generalized cluster algebras.

In this paper, we prove Restriction and Extension Theorem for cluster algebras without coefficients by using positive property in cluster algebras and sign-coherence of g -vectors and c -vectors, see Proposition 3.2. Following from Proposition 3.2, Restriction and Extension Theorem holds for cluster algebras with coefficients from an arbitrary semifield, and it also implies periods of labeled seeds only depend on the extended exchange matrix and thus are irrelevant to coefficients in the labeled seeds.

The organization of this paper is as follows. In Section 2, we give the notions and basic properties for cluster algebras including c -vectors, g -vectors and F -polynomials. In Section 3, we introduce periodicities of labeled seeds and exchange matrices following from [16]. Firstly, we give a proof of Restriction and Extension Theorem for cluster algebras without coefficients. Then, we show that the periodicity of a labeled seed with coefficients from an arbitrary semifield depends only on the corresponding extended exchange matrix. This consequence implies the independence of periodicity relative to coefficients.

2. Preliminaries

In this section, we recall basic concepts and important properties of cluster algebras. Let $\mathbb{P}$ be a semifield, that is an abelian multiplicative group equipped with a binary operation $\oplus$ which is associative, commutative and distributive with respect to the multiplication in $\mathbb{P}$. Note that $\mathbb{P}$ is a torsion-free multiplicative group and thus $\mathbb{Z}\mathbb{P}$ is a domain. Let $\mathbb{Q}\mathbb{P}$ denote the quotient field of $\mathbb{Z}\mathbb{P}$. And take a field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{Q}\mathbb{P}$. In the following, for two positive integers $a < b$ we use $[a, b]$ to denote the subset $\{a, a+1, \dots, b\}$ of natural numbers for simplicity.

A *labeled seed* is a triplet $(\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x}$ is an n -tuple of free generators of $\mathcal{F}$, $\mathbf{y}$ is an n -tuple of elements of $\mathbb{P}$, and B is an $n \times n$ skew-symmetrizable integer matrix. Note that B is said to be skew-symmetrizable if there exists an positive definite diagonal matrix D such that DB is skew-symmetric. Moreover we call $(\mathbf{y}, B)$ an Y -seed. For $k \in [1, n]$, define another triplet $(\mathbf{x}', \mathbf{y}', B') = \mu_k(\mathbf{x}, \mathbf{y}, B)$ which is called the *mutation* of $(\mathbf{x}, \mathbf{y}, B)$ at k and obtained by the following rules:

- (1) $\mathbf{x}' = (x'_1, \dots, x'_n)$ is given by $x'_k x_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{y_k \oplus 1}$ and $x'_i = x_i$ for $i \neq k$;
- (2) $\mathbf{y}' = (y'_1, \dots, y'_n)$ is given by $y'_i = y_k^{-1}$ for $i = k$; and otherwise $y'_i = y_i y_k^{[b_{ki}]_+} (y_k \oplus 1)^{-b_{ki}}$;
- (3) $b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik}) \max[b_{ik} b_{kj}]_+, & \text{otherwise.} \end{cases}$

where $[x]_+ = \max\{x, 0\}$. Note that $(\mathbf{x}', \mathbf{y}', B')$ is also a labeled seed and μ_k is an involution for any $k \in [1, n]$. In a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, $\mathbf{x}$ is called a *labeled cluster*, the elements in $\mathbf{x}$ are called *cluster variables*, $\mathbf{y}$ is the coefficient tuple and B is called an *exchange matrix*.

Remark 2.1. There is a bijection between skew-symmetric integer matrices and finite quivers without loops or 2-cycles. Such quivers are called *cluster quivers*. This bijection is kept under mutations of matrices and quiver mutations, see [15] for details.

Definition 2.2 ([7]). Let T_n be a n -regular tree and valencies emitting from each vertex are labelled by $1, 2, \dots, n$. A cluster pattern is a n -regular tree T_n such that for each vertex $t \in T_n$, there is a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ and for each edge labelled by k , two labeled seeds in the endpoints are obtained from each other by seed mutation at k . And $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ are written as follows:

$$\mathbf{x}_t = (x_{1,t}, x_{2,t}, \dots, x_{n,t}), \mathbf{y}_t = (y_{1,t}, y_{2,t}, \dots, y_{n,t}), B_t = (b_{ij}^t).$$

Note that a cluster pattern is uniquely determined by one labeled seed, thus for a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, we may associate with a cluster pattern $T_n(\mathbf{x}, \mathbf{y}, B)$.

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ associated to the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ is a $\mathbb{ZP}$ -subalgebra of $\mathcal{F}$ generated by cluster variables appeared in $T_n(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. For simplicity, we always denote the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ of the cluster algebra with principle coefficients by

$$\mathbf{x}_t = (x_1, x_2, \dots, x_n), \mathbf{y}_t = (y_1, y_2, \dots, y_n), B_t = (b_{ij}).$$

One of most important properties in cluster algebras is the Laurent phenomenon, that is, any cluster variable can be expressed as a Laurent polynomial in terms of cluster variables in the initial labeled seed with coefficients in $\mathbb{ZP}$. It was conjectured that these Laurent polynomials always have positive coefficients in $\mathbb{NP}$, which has been proved in the skew-symmetrizable case in [8].

Let I be a finite set. The tropical semifield $\mathbb{P} = \text{trop}(y)$ of $y = (y_i)_{i \in I}$ is a multiplicative group generated freely by the elements $y_i (i \in I)$. And the binary operation $\oplus$ is defined as follows:

$$\prod_{i \in I} y_i^{a_i} \oplus \prod_{i \in I} y_i^{b_i} = \prod_{i \in I} y_i^{\min(a_i, b_i)}.$$

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ is said to have *principal coefficients*, if the semifield is given by $\mathbb{P} = \text{trop}(y_{1,t_0}, y_{2,t_0}, \dots, y_{n,t_0})$. If $\mathbb{P} = \mathbf{1}$, the cluster algebra is said to have *no coefficients*. In the rest of this section, we concentrate on cluster algebras with principal coefficients.

Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a cluster algebra with principle coefficients. In this case, for any $t \in T_n$ and $i \in [1, n]$, the coefficient $y_{i,t}$ is actually a Laurent monomial in $y_j (j \in [1, n])$, i.e. $y_{i,t} = \prod_{j=1}^n y_j^{c_{ji}^t}$. The vector $\mathbf{c}_{i,t} = (c_{1i}^t, c_{2i}^t, \dots, c_{ni}^t)^T$ is called the *c-vector* corresponding to $y_{i,t}$, and $C_t = (\mathbf{c}_{1,t}, \mathbf{c}_{2,t}, \dots, \mathbf{c}_{n,t})$ is called *c-matrix* at t . For a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$, we write $\mathbf{v} \geq 0$ to denote $v_i \geq 0$ for any $i \in [1, n]$ and $\mathbf{v} \leq 0$ to denote $v_i \leq 0$ for any $i \in [1, n]$. For row vectors, we have similar statements.

Proposition 2.3 ([7, 8]). *The c-vectors have the following properties:*

(1) (recurrence relations) for any edge $t \xrightarrow{k} t'$ in T_n , the c-vectors satisfy that

$$c_{ij}' = \begin{cases} -c_{ik}^t, & \text{if } j = k; \\ c_{ij}^t + \text{sgn}(c_{ik}^t)[c_{ik}^t b_{jk}^t]_+, & \text{otherwise.} \end{cases}$$

(2) (sign-coherence) for any $t \in T_n, i \in [1, n]$, the c-vector $\mathbf{c}_{i,t}$ satisfies either $\mathbf{c}_{i,t} \geq 0$ or $\mathbf{c}_{i,t} \leq 0$.

It is natural to consider extended matrices $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$, and they are called extended exchange matrices at t . Note that $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$ is also obtained from $\begin{bmatrix} B_0 \\ I \end{bmatrix}$ by the mutation rule of matrices.

Remark 2.4. If $B = B_0$ is skew-symmetric, its framed quiver is obtained from its cluster quiver by adding n (frozen) vertices $\{n+1, n+2, \dots, 2n\}$ and arrows from $n+i$ to i for each $i \in [1, n]$, which is the corresponding quiver of $\begin{bmatrix} B \\ I \end{bmatrix}$. The corresponding quiver of $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$ is also obtained from the framed quiver of B by quiver mutations.

For a cluster algebra with principle coefficients, the Laurent phenomenon and positive property are stated as follows.

Proposition 2.5 ([7]). *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra with principle coefficients. Each cluster variable $x_{i,t}$ is contained in $\mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, y_1, y_2, \dots, y_n]$.*

Let $F_{i,t} = x_{i,t}(1, 1, \dots, 1, y_1, y_2, \dots, y_n) \in \mathbb{Z}_{\geq 0}[y_1, y_2, \dots, y_n]$ be polynomials with nonnegative coefficients and they are called *F-polynomials*.

Proposition 2.6 ([7, 8]). *The F -polynomials satisfy the following properties:*

(1) (recurrence relations) *for any edge $t^k t'$ in T_n , the c -vectors satisfy that*

$$F_{i,t'} = \begin{cases} F_{i,t}^t, & \text{if } i \neq k; \\ (F_{k,t})^{-1} (\prod_{j=1}^n y_j^{c_{jk}^t} (F_{j,t})^{[b_{jk}^t]_+} + \prod_{j=1}^n y_j^{-c_{jk}^t} (F_{j,t})^{[-b_{jk}^t]_+}), & \text{otherwise.} \end{cases}$$

(2) *each F -polynomial $F_{i,t}$ has a constant term 1.*

(3) *let $\mathcal{A}_{\mathbb{P}_1} = \mathcal{A}_{\mathbb{P}_1}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra over an arbitrary semifield $\mathbb{P}_1$, and $x_{i,t,\mathbb{P}_1}$ be the corresponding cluster variable in the i -th position of the labeled cluster at t . Then, we have*

$$x_{i,t,\mathbb{P}_1} = \frac{x_{i,t}|_{\mathcal{F}_1}(x_1, x_2, \dots, x_n; y_1, \dots, y_n)}{F_{i,t}|_{\mathbb{P}_1}(y_1, y_2, \dots, y_n)}.$$

In addition, any cluster variable $x_{i,t}$ is homogeneous with respect to a given $\mathbb{Z}^n$ -grading in $\mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, y_1, y_2, \dots, y_n]$, which is given by

$$\deg(x_i) = \mathbf{e}_i, \quad \deg(y_i) = -\mathbf{b}_i,$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{Z}_n$, and $\mathbf{b}_i$ is the i -th column vector of B . Let $\mathbf{g}_{i,t}$ denote the degree (column) vector of $x_{i,t}$. They are called g -vectors and the matrix $G_t = (\mathbf{g}_{1,t}, \mathbf{g}_{2,t}, \dots, \mathbf{g}_{n,t})$ is called the g -matrix at t .

Proposition 2.7 ([7, 8, 19]). *The g -vectors and g -matrices satisfy the following properties:*

(1) (recurrence relations) *for any edge $t^k t'$ in T_n , the g -vectors satisfy that*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t, & \text{if } j \neq k; \\ -g_{ik}^t + \sum_{s=1}^n g_{is}^t [-b_{sk}^t]_+ - \sum_{s=1}^n b_{is}^t [-c_{sk}^t]_+, & \text{otherwise.} \end{cases}$$

(2) (sign-coherence) *any row vector $\mathbf{v}$ of the g -matrix G_t satisfies either $\mathbf{v} \geq 0$ or $\mathbf{v} \leq 0$.*

(3) *for any $t \in T_n$, we have*

$$G_t^T D(C_t) D^{-1} = I,$$

where D is the skew-symmetrizer of B .

(4) *let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra with coefficients from an arbitrary semifield $\mathbb{P}$. For any $t \in T_n, i \in [1, n]$, the cluster variable can be expressed by*

$$x_{i,t} = \left(\prod_{j=1}^n x_j^{g_{j,t}} \right) \frac{F_{i,t}|_{\mathcal{F}}(\widehat{y}_1, \widehat{y}_2, \dots, \widehat{y}_n)}{F_{i,t}|_{\mathbb{P}}(y_1, y_2, \dots, y_n)},$$

where $\widehat{y}_j = y_j \prod_i x_i^{b_{ij}}$ ($1 \leq j \leq n$).

3. Periodicities in cluster algebras

In this section, we recall basic concepts on periodicities, and we prove main results in this section. Let I be a subset of $[1, n]$, a sequence $\mathbf{i} = (i_1, i_2, \dots, i_s)$ is called an I -sequence if $i_p \in I$ for any $p \in [1, s]$. Moreover, $\mathbf{i}$ is called an essential I -sequence if $i_p \neq i_{p+1}$ for any $p \in [1, s-1]$. Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed and $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n . Then we define $(\mathbf{x}, \mathbf{y}, B)^\sigma := (\mathbf{x}^\sigma, \mathbf{y}^\sigma, B^\sigma)$, where $\mathbf{x}^\sigma, \mathbf{y}^\sigma, B^\sigma$ are given by

$$\mathbf{x}^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}), \quad \mathbf{y}^\sigma = (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)}), \quad B^\sigma = (b_{\sigma(i)\sigma(j)}).$$

Moreover if $B = (b_{ij})_{m \times n}$ is an $m \times n$ matrix with $m > n$ and $\sigma \in \mathfrak{S}_n$, we define $B^\sigma = (b_{\sigma(i)\sigma(j)})_{m \times n}$ where $\sigma(k) = k$ for any $k \in [n+1, m]$.

Definition 3.1. Let $(\mathbf{x}_t, \mathbf{y}_t, B_t), (\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'})$ be two labeled seeds of the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, $\mathbf{i} = (i_1, i_2, \dots, i_s)$ be an I -sequence, and $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n .

(i) If $B_{t'} = \mu_{\mathbf{i}}(B_t)$ and $B_{t'} = B_t^\sigma$, we call $\mathbf{i}$ a σ -period of the exchange matrix B_t ; furthermore, if $\sigma = \text{id}$, we simply call it a period of B_t .

(ii) If $(\mathbf{y}_{t'}, B_{t'}) = \mu_{\mathbf{i}}(\mathbf{y}_t, B_t)$ and $(\mathbf{y}_{t'}, B_{t'}) = (\mathbf{y}_t^\sigma, B_t^\sigma)$, we call $\mathbf{i}$ a σ -period of the Y -seed $(\mathbf{y}_t, B_t)$; furthermore, if $\sigma = \text{id}$, we simply call it a period of $(\mathbf{y}_t, B_t)$.

(iii) If $(\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'}) = \mu_{\mathbf{i}}(\mathbf{x}_t, \mathbf{y}_t, B_t)$ and $(\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'}) = (\mathbf{x}_t, \mathbf{y}_t, B_t)^\sigma$, we call $\mathbf{i}$ a σ -period of the labeled seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$; furthermore, if $\sigma = \text{id}$, we simply call it a period of $(\mathbf{x}_t, \mathbf{y}_t, B_t)$.

The definition of σ -period is a little different from the one in [16]. Note that it is well-known that labeled seeds are determined by clusters, thus to show $\mathbf{i}$ is a σ -period of $(\mathbf{x}_t, \mathbf{y}_t, B_t)$, it is enough to show $(\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'}) = \mu_{\mathbf{i}}(\mathbf{x}_t, \mathbf{y}_t, B_t)$ and $\mathbf{x}_{t'} = \mathbf{x}_t^\sigma$.

Let $\mathbf{i} = (i_1, i_2, \dots, i_s)$ and $\mathbf{j} = (j_1, j_2, \dots, j_p)$ be two arbitrary I -sequences. Suppose that $(\mathbf{x}_t, \mathbf{y}_t, B_t) = \mu_{\mathbf{j}}(\mathbf{x}, \mathbf{y}, B)$ and $\mathbf{i}$ is a period of $(\mathbf{x}, \mathbf{y}, B)$. Then we have

$$\mu_{\mathbf{j}}\mu_{\mathbf{i}}\mu_{\mathbf{j}^{-1}}(\mathbf{x}_t, \mathbf{y}_t, B_t) = \mu_{\mathbf{j}}\mu_{\mathbf{i}}\mu_{\mathbf{j}^{-1}}\mu_{\mathbf{j}}(\mathbf{x}, \mathbf{y}, B) = \mu_{\mathbf{j}}\mu_{\mathbf{i}}(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}_t, \mathbf{y}_t, B_t),$$

where $\mathbf{j}^{-1} = (j_p, \dots, j_2, j_1)$. Thus, the sequence $\mathbf{j}^{-1}\mathbf{i}\mathbf{j}$ is a period of $(\mathbf{x}_t, \mathbf{y}_t, B_t) = \mu_{\mathbf{j}}(\mathbf{x}, \mathbf{y}, B)$, which induces a natural bijection between the sets of periods of two labeled seeds. Thus we may, without loss of generality, mainly consider periods of the initial labeled seeds.

The Extension Theorem was partially formulated by Keller in [14] and was generalized by Plamondon. Nakanishi proved Restriction/Extension Theorem of periodicities of labeled seeds for cluster algebras with coefficients from the universal semifield, see [16]. We now prove this theorem for cluster algebras without coefficients in Proposition 3.2.

Let $I, \tilde{I}$ be two index sets with $I \subset \tilde{I}$, and $B = (b_{ij})_{i,j \in I}$ is the principal submatrix of a skew-symmetrizable matrix $\tilde{B} = (\tilde{b}_{ij})_{i,j \in \tilde{I}}$ such that $B = \tilde{B}|_I$ under the restriction of the index set I . In this case, B is called the I -restriction of $\tilde{B}$ and $\tilde{B}$ is called the $\tilde{I}$ -extension of B .

Proposition 3.2. For $I \subset \tilde{I}$, let B be the I -restriction of the skew-symmetrizable matrix $\tilde{B}$, $\tilde{B}$ be the $\tilde{I}$ -extension of B , and $\sigma \in \mathfrak{S}_n$.

(i) (Restriction) Assume that an I -sequence $\mathbf{i} = (i_1, i_2, \dots, i_s)$ is a σ -period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$, then $\mathbf{i}$ is also an σ -period of the labeled seed $(\mathbf{x}, B)$ in $\mathcal{A}(\mathbf{x}, B)$.

(ii) (Extension) Assume that an I -sequence $\mathbf{i} = (i_1, i_2, \dots, i_s)$ is a σ -period of the labeled seed $(\mathbf{x}, B)$ in $\mathcal{A}(\mathbf{x}, B)$, then $\mathbf{i}$ is also an σ -period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$.

Proof. (i) For any I -sequence $\mathbf{j}$, the labeled seed $\mu_{\mathbf{j}}(\mathbf{x}, B)$ can be obtained from $\mu_{\mathbf{j}}(\tilde{\mathbf{x}}, \tilde{B})$ by specializing all cluster variables $x_k (k \in \tilde{I} \setminus I)$ to 1. Then the result follows.

(ii) It is enough to prove this result for $I = \{1, 2, \dots, n\}$ and $\tilde{I} = \{1, 2, \dots, n+1\}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_n, x_{n+1})$. Since $\mathbf{i}$ is an I -sequence and a σ -period of $(\mathbf{x}, B)$, we have $\mu_{\mathbf{i}}(\mathbf{x}, B) = (\mathbf{x}^\sigma, B^\sigma)$. Suppose that $(\tilde{\mathbf{x}}', \tilde{B}') = \mu_{\mathbf{i}}(\tilde{\mathbf{x}}, \tilde{B})$, where $\tilde{\mathbf{x}}' = (x'_1, \dots, x'_n, x'_{n+1})$. Note that $x'_{n+1} = x_{n+1}$ follows from the mutation rules and the fact that $\mathbf{i}$ is an I -sequence. Since every cluster variable is a Laurent polynomial in terms of the initial seed with nonnegative coefficients, for each $i \in [1, n]$, suppose that

$$x'_i = f_i(x_1, x_2, \dots, x_{n+1})$$

be the corresponding Laurent polynomial in terms of $x_1, x_2, \dots, x_{n+1}$ with positive coefficients. Then, we have

$$x_{\sigma(i)} = f_i(x_1, x_2, \dots, x_n, 1)$$

for any $i \in [1, n]$. Since $x_1, x_2, \dots, x_n$ are algebraically independent, $f_i(x_1, x_2, \dots, x_{n+1})$ must be a Laurent monomial in the form $x_{\sigma(i)} x_{n+1}^{a_i}$ for some $a_i \in \mathbb{Z}$, that is,

$$x'_i = x_{\sigma(i)} x_{n+1}^{a_i}$$

for each $i \in [1, n]$ and some $a_i \in \mathbb{Z}$.

Using terms of F -polynomials and g -vectors, we have another expression of x'_i . By Proposition 2.7(4), for each $i \in [1, n]$, we have

$$x_{\sigma(i)} x_{n+1}^{a_i} = x'_i = F_{i, \mathbf{i}}^{\tilde{B}} \left(\prod_i x_i^{\tilde{b}_{i1}}, \dots, \prod_i x_i^{\tilde{b}_{in}}, \prod_i x_i^{\tilde{b}_{i(n+1)}} \right) x_1^{g_{1i}} \dots x_n^{g_{ni}} x_{n+1}^{g_{n+1,i}}.$$

Note that $F_{i, \mathbf{i}}^{\tilde{B}} \in \mathbb{Z}_{\geq 0}[y_1, \dots, y_n, y_{n+1}]$ is a polynomial with nonnegative integer coefficients, and $F_{i, \mathbf{i}}^{\tilde{B}}$ has a constant term 1. Then we must have

$$x_{\sigma(i)} x_{n+1}^{a_i} = x_1^{g_{1i}} \dots x_n^{g_{ni}} x_{n+1}^{g_{n+1,i}}.$$

Then $g_{\sigma(i)i} = 1, g_{n+1,i} = a_i, g_{qi} = 0$ for $q \in [1, n+1] \setminus \{\sigma(i), n+1\}$. The corresponding G -matrix has the form

$$(1) \quad G = \begin{bmatrix} P & 0 \\ \alpha & 1 \end{bmatrix},$$

where P is the permutation matrix corresponding to σ (note that $P^T = P^{-1}$), and $\alpha = (a_1, a_2, \dots, a_n)$.

By the property of sign-coherence of g -vectors, the vector $\alpha \geq 0$. By Proposition 2.7(3), the transpose of DCD^{-1} is inverse to G , that is

$$(2) \quad G^T DCD^{-1} = I_{n+1},$$

where D is the positive definite diagonal integer matrix such that $D\tilde{B}$ is skew-symmetric.

Assume that

$$(3) \quad DCD^{-1} = \begin{bmatrix} A & \beta \\ \gamma & w \end{bmatrix},$$

we have

$$(4) \quad G^T DCD^{-1} = \begin{bmatrix} P^T A + \alpha^T \gamma & P^T \beta + \alpha^T w \\ \gamma & w \end{bmatrix}.$$

Replacing (4) into (2), since $P^T = P^{-1}$, we obtain that $A = P$, $\beta = -P\alpha^T$, $\gamma = 0$, $w = 1$. Thus,

$$(5) \quad DCD^{-1} = \begin{bmatrix} P & -P\alpha^T \\ 0 & 1 \end{bmatrix}.$$

By the property of sign-coherence of c -vectors, $-P\alpha^T \geq 0$ which implies $a_i \leq 0$ for each $i \in [1, n]$. Then we have $a_i = 0$ for each $i \in [1, n]$, and $x'_i = x_{\sigma(i)}$ follows. Thus $\mathbf{i}$ is also an σ -period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ $\square$

Even though Restriction Theorem holds for exchange matrices, in general, Extension Theorem does not hold. For exchange matrices, we observe the following lemma.

Lemma 3.3. *Let $I = \{1, 2, \dots, n\}$ and $\tilde{I} = \{1, 2, \dots, m\}$ with $m \geq n$, and B be the I -restriction of the skew-symmetrizable matrix $\tilde{B} = \begin{bmatrix} B & H \\ C & M \end{bmatrix}$, and $\mathbf{i}$ be an I -sequence. Suppose that $\mu_{\mathbf{i}}(\tilde{B}) = \begin{bmatrix} B' & H' \\ C' & M' \end{bmatrix}$, then $\mu_{\mathbf{i}} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} B' \\ C' \end{bmatrix}$.*

Proof. Note that $\mathbf{i}$ is an I -sequence. The statement follows easily from the definition of mutation of matrices. $\square$

As an application of Proposition 3.2, we prove the following theorem.

Theorem 3.4. *An I -sequence $\mathbf{i}$ is a σ -period of the initial labeled seed $(\mathbf{x}, B)$ if and only if it is a σ -period of the $\begin{bmatrix} B \\ I_n \end{bmatrix}$. In particular, if B is skew-symmetric, then $\mathbf{i}$ is a σ -period of the initial labeled seed $(\mathbf{x}, B)$ if and only if $\mathbf{i}$ is a σ -period of the corresponding framed quiver.*

Proof. Let $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_{2n})$, and $\tilde{B} = \begin{bmatrix} B & -I_n \\ I_n & 0 \end{bmatrix}$. If $\mu_{\mathbf{i}}(\mathbf{x}, B) = (\mathbf{x}^\sigma, B^\sigma)$, by Extension Theorem, $\mu_{\mathbf{i}}(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}^\sigma, \tilde{B}^\sigma)$. Then the result follows from Lemma 3.3.

Conversely, suppose $\mu_{\mathbf{i}} \begin{bmatrix} B \\ I_n \end{bmatrix} = \begin{bmatrix} B \\ I_n \end{bmatrix}^\sigma$, we consider the cluster algebra with principal coefficients and the initial labeled seed (X, Y, B) . It is easy to see $\mu_{\mathbf{i}}(Y) = Y^\sigma$. Since labeled clusters are determined by their coefficients, i.e. C -matrices, thus we have $\mu_{\mathbf{i}}(X, Y, B) = (X^\sigma, Y^\sigma, B^\sigma)$. Then the conclusion follows from Proposition 2.6(3). $\square$

Note that by [19] exchange matrices are determined by c -matrices, Corollary 3.4 essentially says a σ -period of the initial labeled seed is exactly the σ -period of the corresponding c -matrices. Actually Theorem 3.4 holds for cluster algebras with coefficients from arbitrary semifields.

Theorem 3.5. Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a cluster algebra over an arbitrary semifield $\mathbb{P}$, with the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ given by

$$\mathbf{x}_{t_0} = (x_1, x_2, \dots, x_n), \mathbf{y}_{t_0} = (y_1, y_2, \dots, y_n), B_{t_0} = (b_{ij})_{n \times n}.$$

Let $I = \{1, 2, \dots, n\}$ and $\mathbf{i}$ be an I -sequence and $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed of $\mathcal{A}$. Then $\mathbf{i}$ is a σ -period of $(\mathbf{x}, \mathbf{y}, B)$ if and only if $\mathbf{i}$ is a σ -period of $\begin{bmatrix} B \\ I_n \end{bmatrix}$.

Proof. Without loss of generality, assume that $(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$.

Suppose that $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0}) = (\mathbf{x}_{t_0}^\sigma, \mathbf{y}_{t_0}^\sigma, B_{t_0}^\sigma)$. Since $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, B_{t_0})$ is obtained from $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ by specializing all $y_i (1 \leq i \leq n)$ to 1. Thus, we know that $\mathbf{i}$ is a σ -period of $(\mathbf{x}_{t_0}, B_{t_0})$ in $\mathcal{A}(\mathbf{x}_{t_0}, B_{t_0})$, which is also a σ -period of $\begin{bmatrix} B_{t_0} \\ I_n \end{bmatrix}$ by Theorem 3.4.

Conversely, assume that $\mathbf{i}$ is a σ -period of $\begin{bmatrix} B_{t_0} \\ I_n \end{bmatrix}$. Let $\mathcal{A}(X_{t_0}, Y_{t_0}, B_{t_0})$ be the corresponding cluster algebra of $\mathcal{A}$ with principal coefficients and with the initial labeled seed $(X_{t_0}, Y_{t_0}, B_{t_0})$. We have $\mathbf{i}$ is a σ -period of $(X_{t_0}, Y_{t_0}, B_{t_0})$ in $\mathcal{A}(X_{t_0}, Y_{t_0}, B_{t_0})$, since labeled seeds are determined by c -matrices by [[1], Theorem 2.5]. In particular, $\mathbf{i}$ is a σ -period of X_{t_0} . By Proposition 2.6, $\mathbf{i}$ is also a σ -period of $\mathbf{x}_{t_0}$ and thus is a σ -period of $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. $\square$

Corollary 3.6. Periods of labeled seeds are independent of coefficients.

Remark 3.7. Corollary 3.6 is also proved by Peigen Cao and the first author in a different way, see e.g. [2] for details. In this paper, we emphasise that it can also be shown by Extension Theorem.

Example 3.8. Let $\mathcal{A}$ be a cluster algebra with the initial labeled seed $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, x_2)$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the exchange matrix corresponding to

the cluster quiver of type A_2 . Notice that $\mathcal{A}$ is of finite type, all labeled seeds of $\mathcal{A}$ are shown in the following graph. There are ten various labeled seeds and five unlabeled seeds.

Let $I = [1, 2]$, then the two I -sequences $\mathbf{i} = (1, 2, 1, 2, 1)$ and $\mathbf{j} = (2, 1, 2, 1, 2)$ are (12)-periods of the labeled seed $(\mathbf{x}, B)$, and the two I -sequences

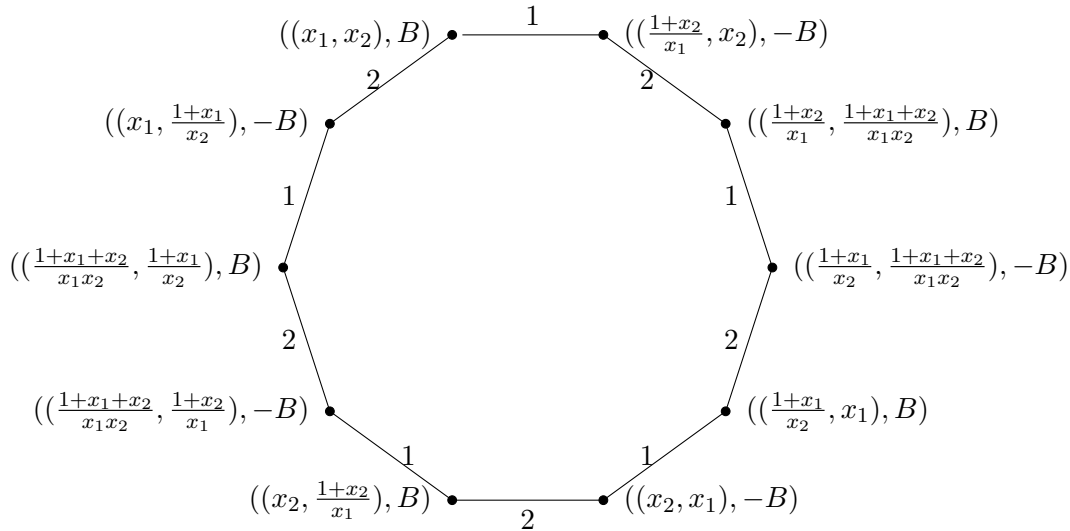
$$\mathbf{p} = (1, 2, 1, 2, 1, 2, 1, 2, 1, 2) \text{ and } \mathbf{q} = (2, 1, 2, 1, 2, 1, 2, 1, 2, 1)$$

are periods of $(\mathbf{x}, B)$.

Since $\mu_{\mathbf{p}}\mu_{\mathbf{q}} = id = \mu_{\mathbf{q}}\mu_{\mathbf{p}}$, any period of $(\mathbf{x}, B)$ is copy of $\mathbf{p}$ or $\mathbf{q}$. The two I -sequences $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (2, 1)$ are two periods of the exchange matrix B , and any period of B is copy of $\mathbf{a}$ or $\mathbf{b}$.

On the other hand, by the Extension Theorem, for any cluster quiver which has a simple edge connecting two vertices i and j , the actions of two mutation sequences $\mu_i\mu_j\mu_i\mu_j\mu_i$ and $\mu_j\mu_i\mu_j\mu_i\mu_j$ on the labeled seed $(\mathbf{x}, B)$ are both equivalent to the action of the permutation (ij) .

In general, for any skew-symmetrizable matrix B , let $B[i, j]$ be the principal submatrix obtained from B by removing its rows or columns indexed by $[1, n] \setminus I$, where $I = \{i, j\}$. Then there exists an essential I -sequence $\mathbf{i}$ such that $\mathbf{i}$ is a period of $(\mathbf{x}, B)$ if and only if $B[i, j]$ is of finite type, i.e. $|b_{ij}b_{ji}| \leq 3$.



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Logarithm and space BMO

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Abstract. Functions with bounded mean oscillation create so called BMO space or John Nirenberg space. They have considerable applications in differential equations theory and harmonic analysis. Logarithm presents an important example of function which belongs to this space BMO but which is not bounded. The norm of this function can be obtained by simple means of infinitesimal calculus. The calculations of it can be viewed as a suitable contribution to the teaching/learning of functional analysis.

Keywords: Morrey Campanato spaces, John Nirenberg space, BMO, bounded mean oscillation, logarithm.

1. Introduction

Functional analysis is enrolled among special subjects which are taught only in some specialized courses of several universities of science or technology. During such lessons students are usually acquainted with the theoretical knowledge by the form of sequence of definitions and theorems, often without sufficient number of suitable supporting isolated models. We would like to offer one useful example which calculates BMO-norm of function logarithm and which can serve as a good training of limits.

Theory of function spaces is a special part of functional analysis which is developing significantly and which has extensive applications in other branches of mathematics, in particular in differential equations (see e.g. [5, 14, 15]). Many types of function spaces are very important for investigation of plenty of the various properties of many differential equations solutions. Also the so called Morrey Campanato spaces have proven to be a very useful tools in this way. They can be considered as a generalization of Lebesgue spaces $L^p(\Omega)$ of integrable functions (see [13, 18]). Space of special type of such functions is called BMO — functions with bounded mean oscillation. The bounded functions create the subset of BMO spaces but no any function with bounded means oscillation is bounded. Well known example of such function is logarithm. We would like to offer one useful calculation of BMO-norm of logarithm which can serve as a good training for calculations of limits.

2. Morrey Campanato spaces

Lets us recall the basic definitions and notations of Morrey Campanato spaces (see e.g. [16]).

We have a function $f : \Omega \rightarrow \mathbb{R}$, which is defined and locally integrable on area $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary.

For any measurable set $Q \subset \mathbb{R}^n$ we can denote mean value

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

The open ball with center $a \in \mathbb{R}^n$ and diameter $r > 0$ is usually denoted by

$$B(a; r) = \{x \in \mathbb{R}^n; |x - a| < r\}.$$

At first, for any $\lambda > 0$ and $p \geq 1$ we shall define norms or pseudo-norm

$$\begin{aligned} \|f\|_{L_M^{p,\lambda}(\Omega)} &= \sup_{a \in \Omega, r > 0} \left(\frac{1}{r^\lambda} \int_{B(a;r) \cap \Omega} |f|^p dx \right)^{\frac{1}{p}}, \\ [f]_{L_C^{p,\lambda}(\Omega)} &= \sup_{a \in \Omega, r > 0} \left(\frac{1}{r^\lambda} \int_{B(a;r) \cap \Omega} |f - f_{B(a;r)}|^p dx \right)^{\frac{1}{p}} \quad \text{and} \\ \|f\|_{L_C^{p,\lambda}(\Omega)} &= [f]_{L_C^{p,\lambda}(\Omega)} + \|f\|_{L^p(\Omega)}. \end{aligned}$$

Function $f \in L^p(\Omega)$ belongs to Morrey space $L_M^{p,\lambda}(\Omega)$ if

$$\|f\|_{L_M^{p,\lambda}(\Omega)} < \infty$$

and to Campanato space $L_C^{p,\lambda}(\Omega)$ if

$$\|f\|_{L_C^{p,\lambda}(\Omega)} < \infty.$$

The definition can be formulated by means of the supremum over n -dimensional cubes Q (their sides are usually supposed to be parallel with axes) with the same result, too.

It holds

$$L_M^{p,\lambda}(\Omega) \subset L_C^{p,\lambda}(\Omega)$$

and for $1 \leq p \leq q < \infty$ and $\lambda, \nu > 0$, where $\frac{\lambda-n}{p} \leq \frac{\nu-n}{q}$, we have following embedding relations

$$L_M^{q,\nu}(\Omega) \subset L_M^{p,\lambda}(\Omega) \quad \text{and} \quad L_C^{q,\nu}(\Omega) \subset L_C^{p,\lambda}(\Omega)$$

for suitable Ω .

3. John Nirenberg spaces

A special type of these spaces is represented by functions with bounded mean oscillation. Such space called and denoted by BMO was first introduced by F. John and L. Nirenberg in 1961 (see [12]).

Let us present a possible definition of space $BMO(\Omega)$ where Ω is a cube (with sides parallel with axes) in $\mathbb{R}^n$. The function (class of equivalent functions, more precisely) f is said to in $BMO(\Omega)$ if the norm

$$(1) \quad \begin{aligned} \|f\|_{BMO(\Omega)} &= \sup_{a \in \Omega, r > 0} \frac{1}{r^n} \int_{B(a;r) \cap \Omega} |f - f_{B(a;r) \cap \Omega}| \, dx \\ &= \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_{Q \cap \Omega} |f - f_Q| \, dx \end{aligned}$$

is finite, the second supremum is taken over all cubes $Q \subset \Omega$ (with sides parallel with axes).

This space BMO is very interesting and it is equal to space $L_C^{1,n}$. Its use is remarkable in harmonic analysis, especially for investigation of Calderon Zygmund type operators and interpolation (see [1, 2, 3, 17]). For instance, if we have a linear operator T continuous from L^2 to L^2 and continuous from L^∞ to BMO then it is also continuous from L^p to L^p for any $2 < p < \infty$.

The following diagram expresses the relationship for given $p \geq 1$ where $C^{0,\alpha}$ denotes a space of Holder functions.

$$\begin{array}{ccccccc} L^p & \approx & L_M^{p,0} & \approx & L_C^{p,0} & \approx & L^p \\ & & \cup & & \cup & & \\ & & L_M^{p,\lambda} & \approx & L_C^{p,\lambda} & & \\ & & \cup & & \cup & & \\ L^\infty & \approx & L_M^{p,n} & \subsetneq & L_C^{p,n} & \approx & BMO \\ & & \cup & & \cup & & \\ & & \{0\} & & L_C^{p,\lambda} & \approx & C^{0, \frac{\lambda-n}{p}} \\ & & & & \cup & & \\ & & & & L_C^{p,n+p} & \approx & C^{0,1} \\ & & & & \cup & & \\ & & & & \{\text{const.}\} & & \end{array}$$

It was proved by C. Fefferman in 1971 that the space BMO is a dual of the Hardy space H^1 (see [8]).

There are plenty of generalizations of this space or alternative definitions (see [4, 6, 7, 9, 10, 11]).

We can simply see that the space $L^\infty(\Omega)$ is a subspace of the space $BMO(\Omega)$. But not every function from $BMO(\Omega)$ belongs to the space $L^\infty(\Omega)$.

Logarithm on $\Omega = \langle 1; \infty \rangle$ is well known example of such function. Many proofs that logarithm belongs to the space $BMO(\Omega)$ are presented in many publications. We would like to offer a straight calculation of BMO norm. We

are going to use only the results of simple calculus, in particular limits. The calculations used can also serve as a useful contribution to the teaching process of functional analysis and as an example of an interesting application of calculus.

Notice that $\log |x|$ is an element of the space $BMO(\mathbb{R})$ but function

$$g(x) = \begin{cases} \log x, & \text{on } (0; \infty) \\ 0, & \text{on } (-\infty; 0) \end{cases}$$

is not in $BMO(\mathbb{R})$.

4. Calculations

We consider $\Omega = \langle 1; \infty \rangle$ and a positive function $f : \Omega \rightarrow \mathbb{R}$. We denote by F the antiderivative to f on $(1; \infty)$.

For arbitrary interval $Q = \langle a; b \rangle$, $1 \leq a < b$, we have by the mean value theorem that there is some point $c \in (a; b)$ such that

$$f(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} \int_a^b f \, dx = \frac{1}{|Q|} \int_Q f \, dx = f_Q.$$

Lemma 4.1. *For f increasing we have*

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dx = \frac{1}{b - a} (F(a) + F(b) - 2F(c) + f(c)(2c - a - b)).$$

Proof. The result can be reached by simple calculations

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx &= \frac{1}{b - a} \int_a^b |f(x) - f(c)| \, dx \\ &= \frac{1}{b - a} \left(\int_a^c (f(c) - f(x)) \, dx + \int_c^b (f(x) - f(c)) \, dx \right) \\ &= \frac{1}{b - a} \left([xf(c) - F(x)]_a^c + [F(x) - xf(c)]_c^b \right) \\ &= \frac{1}{b - a} (F(a) + F(b) - 2F(c) + f(c)(2c - a - b)). \end{aligned}$$

Now we apply this lemma to the case of natural logarithm $f(x) = \log x$ and interval $Q = \langle a; b \rangle$, $1 \leq a < b$. This function is increasing on $\Omega = \langle 1; \infty \rangle$ and its antiderivative is $F(x) = x \log x - x = x(\log x - 1)$.

We have mean value

$$f_Q = \frac{1}{|Q|} \int_Q f \, dx = \frac{1}{b - a} \int_a^b \log x \, dx = \frac{1}{b - a} [x \log x - x]_a^b = \frac{b \log b - a \log a}{b - a} - 1.$$

Inverse function of logarithm is exponential function, hence the point $c \in (a; b)$ for which $f(c) = f_Q$ can be expressed as

$$c = e^{f_Q} = \frac{1}{e} \cdot e^{\frac{b \log b - a \log a}{b-a}}, \quad f(c) = \log c = \frac{b \log b - a \log a}{b-a} - 1.$$

Then the ratio from definition of the space BMO is according to the lemma 4.1

$$(2) \quad \frac{1}{|Q|} \int_Q |f - f_Q| \, dx = 2 \left(\frac{e^{\frac{b \log b - a \log a}{b-a}}}{e(b-a)} - ab \frac{\log b - \log a}{(b-a)^2} \right),$$

because

$$\begin{aligned} & \frac{1}{b-a} (F(a) + F(b) - 2F(c) + f(c)(2c - a - b)) \\ &= \frac{1}{b-a} (a \log a - a + b \log b - b - 2c \log c + 2c + \log c(2c - a - b)) \\ &= \frac{1}{b-a} (a \log a + b \log b + 2c - (a+b)(\log c + 1)) \\ &= \frac{1}{b-a} \left(\frac{(b-a)(a \log a + b \log b) - (a+b)(b \log b - a \log a)}{b-a} + 2c \right) \\ &= \frac{1}{b-a} \left(2c - 2ab \frac{\log b - \log a}{b-a} \right) = 2 \left(\frac{e^{\frac{b \log b - a \log a}{b-a}}}{e(b-a)} - ab \frac{\log b - \log a}{(b-a)^2} \right). \end{aligned}$$

Now we define

$$(3) \quad \phi(a, b) = \frac{e^{\frac{b \log b - a \log a}{b-a}}}{b-a} - abe \frac{\log b - \log a}{(b-a)^2}$$

and for arbitrary $a \geq 1$ we define on $\langle a; \infty \rangle$

$$(4) \quad \phi_1(b) = \phi(a, b).$$

We would like to prove that this function (for any a) is increasing,

$$(5) \quad \lim_{b \rightarrow a^+} \phi_1(b) = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} \phi_1(b) = 1.$$

Lemma 4.2. *It holds*

$$\phi_1(b) = \phi_2\left(\frac{b}{a}\right), \quad \text{where} \quad \phi_2(x) = \frac{e^{\frac{x \log x}{x-1}} - e^{\frac{x \log x}{x-1}}}{x-1}.$$

Proof. At first we try to adjust the exponent of power

$$\begin{aligned}\frac{b \log b - a \log a}{b - a} &= \frac{b \log b - b \log a + b \log a - a \log a}{b - a} \\ &= b \frac{\log b - \log a}{b - a} + \log a = \frac{\frac{b}{a} \log \frac{b}{a}}{\frac{b}{a} - 1} + \log a.\end{aligned}$$

Then, we can calculate

$$\begin{aligned}\phi_1(b) &= \frac{e^{\frac{b \log b - a \log a}{b - a}} - a b e^{\frac{\log b - \log a}{b - a}}}{b - a} = \frac{e^{\frac{\frac{b}{a} \log \frac{b}{a}}{\frac{b}{a} - 1}} \cdot a - a b e^{\frac{\log \frac{b}{a}}{\frac{b}{a} - 1}}}{b - a} \\ &= \frac{e^{\frac{\frac{b}{a} \log \frac{b}{a}}{\frac{b}{a} - 1}} - e^{\frac{b}{a} \frac{\log \frac{b}{a}}{\frac{b}{a} - 1}}}{\frac{b}{a} - 1} = \phi_2\left(\frac{b}{a}\right).\end{aligned}$$

This relation between functions ϕ_1 and ϕ_2 is a consequence of linear transformation $x = \frac{b}{a}$. It enables us to investigate the properties of function ϕ_2 instead of function ϕ_1 .

Lemma 4.3. *It holds $\lim_{x \rightarrow 1^+} \phi_2(x) = 0$.*

Proof. At first we will realize that

$$\lim_{x \rightarrow 1^+} \frac{x \log x}{x - 1} = 1.$$

Using l'Hospital rule we calculate

$$\lim_{x \rightarrow 1^+} \frac{x - \log x - 1}{(x - 1)^2} = \lim_{x \rightarrow 1^+} \frac{1 - \frac{1}{x}}{2(x - 1)} = \lim_{x \rightarrow 1^+} \frac{1}{2x} = \frac{1}{2}.$$

Then, we use again l'Hospital rule for calculation

$$\begin{aligned}&\lim_{x \rightarrow 1^+} \frac{e^{\frac{x \log x}{x-1}} - e^{\frac{x \log x}{x-1}}}{x - 1} \\ &= \lim_{x \rightarrow 1^+} \frac{\left(e^{\frac{x \log x}{x-1}} - e\right) \cdot \left(\frac{x \log x}{x-1}\right)'}{1} = \lim_{x \rightarrow 1^+} \left(e^{\frac{x \log x}{x-1}} - e\right) \cdot \frac{x - \log x - 1}{(x - 1)^2} \\ &= \underbrace{\lim_{x \rightarrow 1^+} \left(e^{\frac{x \log x}{x-1}} - e\right)}_{=0} \cdot \underbrace{\lim_{x \rightarrow 1^+} \frac{x - \log x - 1}{(x - 1)^2}}_{=\frac{1}{2}} = 0 \cdot \frac{1}{2} = 0.\end{aligned}$$

Lemma 4.4. *It holds*

$$\lim_{x \rightarrow \infty} \phi_2(x) = 1.$$

Proof. Again, at first we realize that

$$\lim_{x \rightarrow \infty} \frac{\log x}{x-1} = 0$$

and using relation

$$\frac{x \log x}{x-1} = \log x + \frac{\log x}{x-1}$$

we calculate

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^{\frac{x \log x}{x-1}} - e^{\frac{x \log x}{x-1}}}{x-1} &= \lim_{x \rightarrow \infty} \frac{x e^{\frac{\log x}{x-1}} - e x^{\frac{\log x}{x-1}}}{x-1} \\ &= \lim_{x \rightarrow \infty} \frac{x}{x-1} \cdot \left(\lim_{x \rightarrow \infty} e^{\frac{\log x}{x-1}} - e \lim_{x \rightarrow \infty} \frac{\log x}{x-1} \right) = 1 \cdot (e^0 - e \cdot 0) = 1. \end{aligned}$$

Lemma 4.5. *Function ϕ_2 is increasing on interval $\langle 1; \infty \rangle$.*

Proof. For one of the possible proofs of monotony of this function we can use derivative

$$\begin{aligned} \phi_2'(x) &= \left(\frac{e^{\frac{x \log x}{x-1}} - e^{\frac{x \log x}{x-1}}}{x-1} \right)' \\ &= \frac{\left(e^{\frac{x \log x}{x-1}} - e \right) \cdot \frac{x-1-\log x}{(x-1)^2} \cdot (x-1) - \left(e^{\frac{x \log x}{x-1}} - e^{\frac{x \log x}{x-1}} \right)}{(x-1)^2} \\ &= \frac{e(x \log x + \log x + 1 - x) - e^{\frac{x \log x}{x-1}} \cdot \log x}{(x-1)^3} = \frac{g_1(x)}{(x-1)^3}, \end{aligned}$$

where $g_1(x) = e(x \log x + \log x + 1 - x) - e^{\frac{x \log x}{x-1}} \cdot \log x$.

In order to prove that function ϕ_2' is positive on $\langle 1; \infty \rangle$ and so function ϕ_2 is increasing it is necessary to show that function g_1 is positive on this interval.

Function g_1 can be split into two parts

$$g_1(x) = \underbrace{\left(ex + e - 2e^{\frac{x \log x}{x-1}} \right) \cdot \frac{\log x}{2}}_{=g_3(x)} + \frac{e}{2} \cdot \underbrace{(x \log x + \log x + 2 - 2x)}_{=g_2(x) \geq 0}.$$

Function g_2 is positive on $\langle 1; \infty \rangle$ because $g_2(1) = g_2'(1) = 0$ and $g_2''(x) = \frac{x-1}{x^2} \geq 0$ on $\langle 1; \infty \rangle$.

We shall prove that function g_3 is positive, which is equivalent with inequality

$$g_4(x) = \frac{e^{\frac{x \log x}{x-1}}}{x+1} \leq \frac{e}{2}.$$

This can be shown similarly. Derivative of g_4 is negative and $\lim_{x \rightarrow 1^+} g_4(x) = \frac{e}{2}$. Let us calculate derivative

$$\begin{aligned} g_4'(x) &= \frac{e^{\frac{x \log x}{x-1}} \cdot \frac{x-1-\log x}{(x-1)^2} \cdot (x+1) - e^{\frac{x \log x}{x-1}}}{(x+1)^2} \\ &= -\frac{e^{\frac{x \log x}{x-1}}}{(x^2-1)^4} \cdot \underbrace{(x \log x + \log x + 2 - 2x)}_{=g_2(x) \geq 0} \leq 0. \end{aligned}$$

Properties (5) of function ϕ_1 are proved by lemmas 4.3, 4.4 and 4.5.

5. BMO norm of logarithm

Now we can calculate BMO-norm of logarithm by definition (1) above using supremum over all $Q = \langle a; b \rangle$, $1 \leq a < b$,

$$\begin{aligned} \|\log\|_{BMO(1;\infty)} &= \sup_Q \int_Q |f - f_Q| dx = 2 \sup_{1 \leq a < b} \left(\frac{e^{\frac{b \log b - a \log a}{b-a}}}{e(b-a)} - ab \frac{\log b - \log a}{(b-a)^2} \right) \\ &= \frac{2}{e} \sup_{1 \leq a < b} \phi(a, b) = \frac{2}{e} \lim_{b \rightarrow \infty} \phi(a, b) = \frac{2}{e} \lim_{b \rightarrow \infty} \phi_1(b) = \frac{2}{e}, \end{aligned}$$

where we have used relation (2), definitions (3), (4) and properties (5).

We can conclude that function logarithm has bounded mean oscillation (i.e. it is an element of $BMO(1; \infty)$) and its norm is equal to $\|\log\|_{BMO(1;\infty)} = \frac{2}{e}$.

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The minimum vv-coloring Laplacian energy of a graph

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Abstract. Let $B(G)$ denote the set of all blocks of a graph G . Two vertices are vv-adjacent if they incident on the same block. Then vv-degree of a vertex u , $d_{vv}(u)$ is the number vertices vv-adjacent to the vertex u . In this paper we introduce new kind of graph energy, the minimum vv-coloring Laplacian energy of a graph denoting it as $LE_{cuv}(G)$. It depends both on underlying graph of G and its particular colors on its vertices of G . We studied some of the properties of $LE_{cuv}(G)$ and bounds for $LE_{cuv}(G)$ are established.

Keywords: energy of a graph, Laplacian energy of a graph, Color energy of a graph, vv-coloring of a graph.

1. Introduction

The terminologies and notations used here are as in [10]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q . A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non-separable. A maximal non-separable subgraph is a block of G . Let $B(G)$ denote the set of all blocks of G with $|B(G)| = m$. The concept of energy of a graph was introduced by I. Gutman [5] in 1978. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \dots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^p |\lambda_i(G)|$. In connection with graph energy, energy-like quantities were considered for other matrices such as distance [7], covering [1], incidence [8] and vb-dominating [9]. Gutman and

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Zhou [6] defined the Laplacian energy of a graph G in 2006. Let G be a graph with p vertices and q edges. The Laplacian matrix of the graph G denoted by $L = [l_{ij}]$ is the square matrix of order p where

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i, & \text{if } i = j. \end{cases}$$

where d_i is the degree of the vertex v_i . Let $\mu_1, \mu_2, \dots, \mu_p$ be the eigenvalues of Laplacian matrix $L = [l_{ij}]$, which are called Laplacian eigenvalues of G . Then the Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^p |\mu_i(G) - \frac{2q}{p}|$.

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$. Chandrashekar adiga et al. [2] defined color energy of a graph in 2013. The color matrix of the graph G denoted by $A_L = [a_{ij}]$ is the square matrix of order p where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j) \\ 0, & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of v_i . Let $\lambda_1, \lambda_2, \dots, \lambda_p$ be the eigen values of color matrix $A_L = [a_{ij}]$, which are called color eigenvalues of G . Then the color energy of G is defined as $E_c(G) = \sum_{i=1}^p |\lambda_i(G)|$.

2. Minimum vv-coloring Laplacian energy of a graph

In 2013, P. G. Bhat et al. [4] defined the vv-adjacency. Two vertices are vv-adjacent if they incident on the same block. vv-degree of a vertex u , $d_{vv}(u)$ is the number vertices vv-adjacent to the vertex u . The point graph $P_G(G)$ is a graph with vertex set same as that of G and any two vertices in $P_G(G)$ are adjacent if, and only if, they are vv-adjacent in G . Number of edges in the point graph is denoted by q_p .

Motivated by the definition of coloring and Laplacian energy of the graph G , we define the following.

vv-coloring of graph G is the coloring of vertices such that no two vertices in the same block receives same color. Then vv-chromatic number of a graph G denoted by $\chi_{cvv}(G)$ is the minimum number of colors needed for vv-coloring the graph G . In this paper, we consider the vv-colored graph. The vv-colored

Laplacian matrix of G is the $p \times p$ matrix $L_{cvv}(G) = [l_{ij}]$, where

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are vv-adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not vv-adjacent but } c(v_i) = c(v_j) \\ d_{vv}(v_i), & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of v_i . The characteristic polynomial of $L_{cvv}(G)$ is denoted by $f_p(G, \mu) = \det(\mu I - L_{cvv}(G))$.

The minimum vv-coloring Laplacian eigenvalues of the graph G are the eigenvalues of $L_{cvv}(G)$. Since $L_{cvv}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$. The minimum vv-coloring Laplacian energy of G is then defined as $LE_{cvv}(G) = \sum_{i=1}^p |\mu_i(G) - \frac{2q_p}{p}|$. In this paper we discuss some basic properties of minimum vv-coloring Laplacian energy of the graph $LE_{cvv}(G)$ and derive an upper and lower bound for $LE_{cvv}(G)$. The minimum vv-coloring Laplacian matrix is independent of the internal structure of the blocks of the graph.

3. Some basic properties of minimum vv-coloring Laplacian energy of a graph

First we compute the minimum vv-coloring Laplacian energy of the graph shown in Figure 1.

Example 3.1.

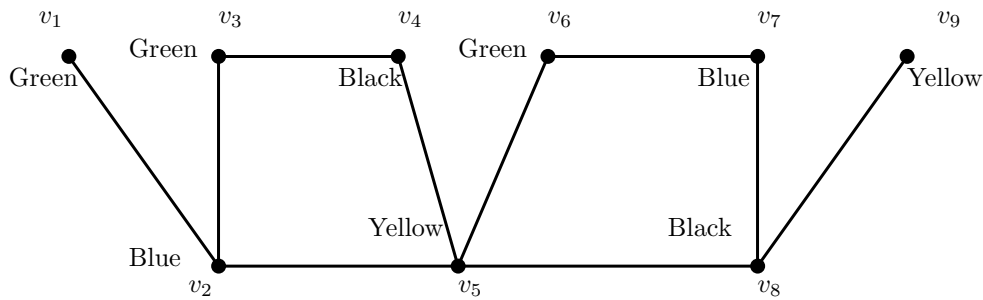


Figure 1: Graph G

Let G be the graph with 9 vertices which are vv-colored using colors black, blue, yellow and green (see Figure 1). Then

$$L_{vv}(G) = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 6 & -1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The characteristic polynomial of $L_{cvv}(G)$ is $\mu^9 - 28\mu^8 + 319\mu^7 - 1903\mu^6 + 6366\mu^5 - 11849\mu^4 + 11373\mu^3 - 4613\mu^2 + 245\mu + 105$. Then the vv-colored eigenvalues are $-0.1130, 0.2778, 0.9229, 1.3257, 3.1845, 3.5438, 5.5648, 5.8480$ and 7.4455 . Therefore the minimum vv-coloring Laplacian energy of the graph G is $LE_{cvv}(G) = 20.0619$. Note that the minimum vv-coloring Laplacian energy of the graph G depends on its color set.

Theorem 3.1. *If $\mu_1(G), \mu_2(G), \dots, \mu_p(G)$ are the eigenvalues of $L_{cvv}(G)$, then*

$$(1) \quad \sum_{i=1}^p \mu_i = 2q_p,$$

$$(2) \quad \sum_{i=1}^p \mu_i^2 = 2(q_p + V_s) + \sum_{i=1}^p (d_{vv}(v_i))^2,$$

where q_p is the number of edges in the point graph of G and V_s is the number of pairs of vertices receive the same color.

Proof. (1) We know that the sum of the eigenvalues of $L_{cvv}(G)$ is equal to trace of $L_{cvv}(G)$. Therefore $\sum_{i=1}^p \mu_i = \sum_{i=1}^p a_{ii} = \sum_{i=1}^p d_{vv}(v_i) = 2q_p$.

(2) The sum of the squares of the Laplacian eigenvalues of $L_{cvv}(G)$ is just the trace of $L_{cvv}(G)^2$.

Therefore

$$\begin{aligned} \sum_{i=1}^p \mu_i^2 &= \sum_{i=1}^p \sum_{j=1}^p l_{ij} l_{ji} \\ &= 2 \sum_{i < j} (l_{ij})^2 + \sum_{i=1}^p (l_{ii})^2 \\ &= 2(q_p + V_s) + \sum_{i=1}^p (d_{vv}(v_i))^2. \end{aligned}$$

□

Observation 3.1 ([4]). For any graph G with p vertices

$$\sum_{i=1}^p d_{vv}(v_i) = 2q_p.$$

Theorem 3.2. If $\mu_1(G)$ is the largest eigenvalue of the minimum vv-coloring Laplacian matrix $L_{cvv}(G)$, then

$$(3) \quad \mu_1(G) \geq \frac{2V_s}{p}.$$

Proof. Let Y be any non zero vector, then we have

$$\begin{aligned} \mu_1(L_{cvv}(G)) &= \max_{Y \neq 0} \left[\frac{Y' L_{cvv}(G) Y}{Y' Y} \right] \quad (\text{see [3]}), \\ \mu_1(L_{cvv}(G)) &\geq \left[\frac{J' L_{cvv}(G) J}{J' J} \right] = \frac{2V_s - 2q_p + \sum_{i=1}^p d_{vv}(i)}{p} = \frac{2V_s}{p}, \text{ where } J \text{ is the} \\ &\text{all one's vector.} \end{aligned}$$

$$(4) \quad \text{Let } N = q_p + V_s + \frac{1}{2} \sum_{i=1}^p (d_{vv}(v_i))^2,$$

where q_p is the number of edges in the point graph of G and V_s is the number of pairs of vertices receive the same color. $\square$

Theorem 3.3. Let G_1 and G_2 be two vv-colored graphs with p vertices, q_p and $q_{p'}$ be the number of edges in the point graph of G_1 and G_2 respectively and V_s and $V_{s'}$ be the number of pairs of vertices in G_1 and G_2 receives same color respectively. If $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ and $\mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_p$ are the Laplacian eigenvalues of G_1 and G_2 respectively. Then

$$(5) \quad \sum_{i=1}^p \mu_i \mu'_i \leq 2\sqrt{(N(G_1)N(G_2))},$$

where N is the expression defined as in (4).

Proof. Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^p a_i b_i \right)^2 \leq \left(\sum_{i=1}^p a_i^2 \right) \left(\sum_{i=1}^p b_i^2 \right).$$

Taking $a_i = \mu_i, b_i = \mu'_i$, we get

$$\begin{aligned} \left(\sum_{i=1}^p \mu_i \mu'_i \right)^2 &\leq \left(\sum_{i=1}^p \mu_i^2 \right) \left(\sum_{i=1}^p (\mu'_i)^2 \right), [\text{From the Theorem 3.1}] \\ &= 2N(G_1)2N(G_2). \end{aligned}$$

$\square$

4. Some bounds for minimum vv-coloring Laplacian energy of a graph

Theorem 4.1. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G . Then*

$$(6) \quad LE_{cvv}(G) \leq \sqrt{2Np} + 2q_p,$$

where N is the expression defined as in (4).

Proof. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ be the Laplacian eigenvalues of $L_{cvv}(G)$.

Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^p a_i b_i \right)^2 \leq \left(\sum_{i=1}^p a_i^2 \right) \left(\sum_{i=1}^p b_i^2 \right)$$

choose $a_i = 1$ and $b_i = |\mu_i|$,

$$\begin{aligned} \left(\sum_{i=1}^p |\mu_i| \right)^2 &\leq \left(\sum_{i=1}^p 1 \right) \left(\sum_{i=1}^p |\mu_i|^2 \right) = p2N, \text{ [From the Theorem 3.1].} \\ \sum_{i=1}^p |\mu_i| &\leq \sqrt{2Np}. \end{aligned}$$

Using Triangular inequality $|a - b| \leq |a| + |b|$, we have

$$\begin{aligned} \left| \mu_i - \frac{2q_p}{p} \right| &\leq |\mu_i| + \frac{2q_p}{p}, \\ \sum_{i=1}^p \left| \mu_i - \frac{2q_p}{p} \right| &\leq \sum_{i=1}^p |\mu_i| + \sum_{i=1}^p \frac{2q_p}{p}, \\ LE_{cvv}(G) &\leq \sum_{i=1}^p |\mu_i| + 2q_p \\ &\leq \sqrt{2Np} + 2q_p. \end{aligned}$$

□

Theorem 4.2. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G . Then*

$$(7) \quad LE_{cvv}(G) \leq \sqrt{2Np - 4q_p^2},$$

where N is the expression defined as in (4).

Proof. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ be the Laplacian eigenvalues of $L_{cvv}(G)$.

Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^p a_i b_i \right)^2 \leq \left(\sum_{i=1}^p a_i^2 \right) \left(\sum_{i=1}^p b_i^2 \right)$$

choose $a_i = 1$ and $b_i = |\mu_i - \frac{2q_p}{p}|$,

$$\begin{aligned} \left(\sum_{i=1}^p \left| \mu_i - \frac{2q_p}{p} \right| \right)^2 &\leq \left(\sum_{i=1}^p 1 \right) \left(\sum_{i=1}^p \left| \mu_i - \frac{2q_p}{p} \right|^2 \right), \\ (LE_{cvv}(G))^2 &\leq p \left(\sum_{i=1}^p \mu_i^2 + \sum_{i=1}^p \frac{4q_p^2}{p^2} - \frac{4q_p}{p} \sum_{i=1}^p \mu_i \right) \\ &= p \left(2N + \frac{4q_p^2}{p} - \frac{4q_p}{p} 2q_p \right), \text{ [From the Theorem 3.1]} \\ &= 2Np + 4q_p^2 - 8q_p^2. \end{aligned}$$

Therefore

$$LE_{cvv}(G) \leq \sqrt{2Np - 4q_p^2}.$$

□

Theorem 4.3. Let G be a graph with p vertices. If D is the determinant of $L_{cvv}(G)$, then

$$(8) \quad LE_{cvv}(G) \geq \sqrt{2N + p(p-1)D^{\frac{2}{p}}} - 2q_p,$$

where N is the expression defined as in (4).

Proof.

$$\begin{aligned} \left(\sum_{i=1}^p |\mu_i| \right)^2 &= \left(\sum_{i=1}^p |\mu_i| \right) \left(\sum_{j=1}^p |\mu_j| \right) \\ &= \sum_{i=1}^p |\mu_i|^2 + \sum_{i \neq j} |\mu_i| |\mu_j|, \\ \sum_{i \neq j} |\mu_i| |\mu_j| &= \left(\sum_{i=1}^p |\mu_i| \right)^2 - \sum_{i=1}^p |\mu_i|^2, \\ \frac{1}{p(p-1)} \sum_{i \neq j} |\mu_i| |\mu_j| &\geq \left(\prod_{i \neq j} |\mu_i| |\mu_j| \right)^{\frac{1}{p(p-1)}}, \\ \sum_{i \neq j} |\mu_i| |\mu_j| &\geq p(p-1) \left(\prod_{i \neq j} |\mu_i| |\mu_j| \right)^{\frac{1}{p(p-1)}}. \end{aligned}$$

Thus

$$\begin{aligned} \left(\sum_{i=1}^p |\mu_i| \right)^2 - \sum_{i=1}^p |\mu_i|^2 &\geq p(p-1) \left(\prod_{i=1}^p |\mu_i|^{2(p-1)} \right)^{\frac{1}{p(p-1)}}, \\ \left(\sum_{i=1}^p |\mu_i| \right)^2 - 2N &\geq p(p-1) \left(\prod_{i=1}^p |\mu_i| \right)^{\frac{2}{p}}, \\ \left(\sum_{i=1}^p |\mu_i| \right)^2 &\geq 2N + p(p-1) D^{\frac{2}{p}}, \\ \left(\sum_{i=1}^p |\mu_i| \right) &\geq \sqrt{2N + p(p-1) D^{\frac{2}{p}}}. \end{aligned}$$

Using Triangular inequality $|a| - |b| \leq |a - b|$,

$$\begin{aligned} |\mu_i| - \left| \frac{2q_p}{p} \right| &\leq \left| \mu_i - \frac{2q_p}{p} \right|, \quad \forall i = 1, 2, \dots, p, \\ \sum_{i=1}^p |\mu_i| - \sum_{i=1}^p \frac{2q_p}{p} &\leq \sum_{i=1}^p \left| \mu_i - \frac{2q_p}{p} \right|, \\ \sum_{i=1}^p |\mu_i| - 2q_p &\leq LE_{cvv}(G), \\ LE_{cvv}(G) &\geq \sum_{i=1}^p |\mu_i| - 2q_p \geq \sqrt{2N + p(p-1) D^{\frac{2}{p}}} - 2q_p. \end{aligned}$$

□

Theorem 4.4. *Let G be a graph with p vertices, and let q_p be the number of edges in the point graph of G . If D is the determinant of $L_{cvv}(G)$, then*

$$(9) \quad LE_{vv}(G) \leq \sqrt{2N(p-1) + pD^{\frac{2}{p}}} + 2q_p,$$

where N is the expression defined as in (4).

Proof. Using Kober's inequality,

$$p \left[\frac{1}{p} \sum_{i=1}^p a_i - \left(\prod_{i=1}^p a_i \right)^{\frac{1}{p}} \right] \leq p \sum_{i=1}^p a_i - \left(\sum_{i=1}^p \sqrt{a_i} \right)^2.$$

Putting $a_i = |\mu_i|^2$

$$\begin{aligned} p \left[\frac{1}{p} \sum_{i=1}^p |\mu_i|^2 - \left(\prod_{i=1}^p |\mu_i|^2 \right)^{\frac{1}{p}} \right] &\leq p \sum_{i=1}^p |\mu_i|^2 - \left(\sum_{i=1}^p |\mu_i| \right)^2, \\ 2N - pD^{\frac{2}{p}} &\leq p(2N) - \left(\sum_{i=1}^p |\mu_i| \right)^2. \end{aligned}$$

Thus

$$\left(\sum_{i=1}^p |\mu_i| \right) \leq \sqrt{2N(p-1) + pD^{\frac{2}{p}}}.$$

Now, using triangular inequality,

$$\begin{aligned} \left| \mu_i - \frac{2q_p}{p} \right| &\leq |\mu_i| + \left| \frac{2q_p}{p} \right|, \\ \sum_{i=1}^p \left| \mu_i - \frac{2q_p}{p} \right| &\leq \sum_{i=1}^p |\mu_i| + \sum_{i=1}^p \frac{2q_p}{p}, \\ LE_{cuv}(G) &\leq \sum_{i=1}^p |\mu_i| + 2q_p. \end{aligned}$$

Therefore

$$LE_{cuv}(G) \leq \sqrt{2N(p-1) + pD^{\frac{2}{p}}} + 2q_p.$$

□

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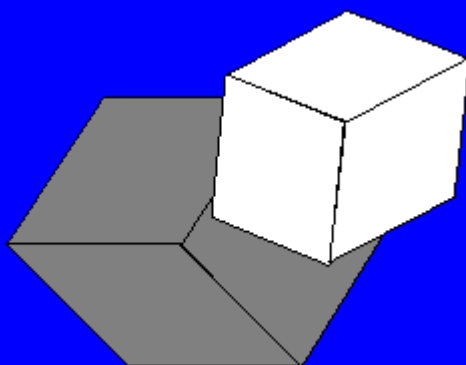
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