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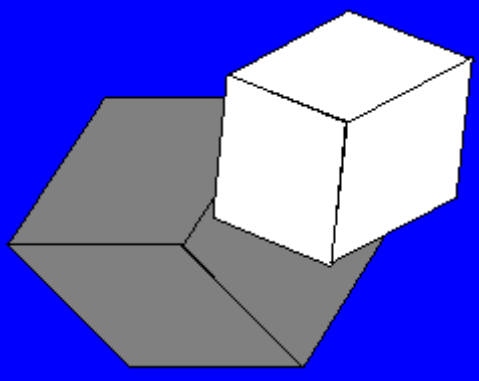
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Approximate approach for solving two points fuzzy boundary value problems

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Abstract. In this paper the Homotopy Perturbation Method (HPM) is employed to solve n'thorder ($n \geq 2$) non linear two point fuzzy boundary value problems (TPFBVP).

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The homotopy perturbation method can be used for solving n'th order fuzzy differential equations directly without reduction to first order system. The convergence theorem of this method in fuzzy case is presented and proved. We illustrate the method in numerical experiment including second order linear TPFBVP to show the accuracy and capabilities of HPM.

Keywords: fuzzy numbers, fuzzy differential equations, two point fuzzy boundary value problems, homotopy perturbation method.

1. Introduction

Many dynamical real life problems may be formulated as a mathematical model. Many of them can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equations (FDEs) are a useful tool to model a dynamical system when information about its behavior is inadequate. FDE appears when the modeling of these problems was imperfect and its nature is under uncertainty. FDEs are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including, population models [1, 2, 3], mathematical physics [4], and medicine [5, 6]. In recent year's semi -analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), Optimal Homotopy asymptotic method (OHAM) and Homotopy Analysis Method (HAM) have been used to solve fuzzy first and $n'th$ order ordinary differential equations. For $n'th$ order fuzzy initial value problems, The ADM was employed in [7] to solve second order linear fuzzy initial value problems. Abbasbandy et al. [8] used the VIM to solve linear system of first order fuzzy initial value problems. Moreover, some of these methods have been also used to obtain a semi-analytical solution of TPFBVP. VIM, OHAM and ADM have been used in [9, 10, 11] to solve linear TPFBVP. Other method like undetermined fuzzy coefficients method has been introduced in [12] in order to obtain an approximate solution of second order linear TPFBVP. The major disadvantage in the traditional perturbation technique is the over dependence on the existence of small parameter. This condition is greatly affects the applications of perturbation techniques because most nonlinear equations do not even contain the small parameter. Moreover, the determination of small parameter is a complicated process and requires special techniques. It is, in fact, a coupling of the traditional perturbation method and homotopy in topology [13]. The HPM, as the previous methods that have been described, does not involve discretization of the variables and linearization, hence is free from rounding of errors and does not require large computer memory or time [14, 15]. The method also provides the solution in a rapidly convergent series with components that are elegantly computed [16].

The Homotopy Perturbation Method has been introduced by He [17] and has been applied to a wide class of deterministic and stochastic problems of mathematical and physical sciences [18, 19, 20]. This method provides the solution

as a rapidly convergent series with components that are elegantly computed. This method can be used to solve all types of linear and nonlinear equations such as differential and integral equations, so it is known as a powerful method. Another important advantage of this method is that it can reduce the size of computations, while increases the accuracy of the approximate solutions so it is known as a powerful method.

In this paper, our aim is to formulate HPM from crisp into fuzzy case in order to solve nonlinear $n'th$ order TPFBVP directly. To the best of our knowledge, this is the first attempt at solving the $n'th$ order TPFBVP using the HPM. The structure of this paper is as follows: In section 2, some basic definitions and notations are given about fuzzy numbers that will be used in other sections we discussed. In section 3, the structure of HPM is formulated for solving high order TPFBVP. In section 4, we present a numerical example and finally, in section 5, we give the conclusion of this study.

2. Preliminaries

Definition 2.1 ([21])**.** The r-level (or r-cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e.

$$\tilde{A}_r = \left\{ x \in X \mid \mu_{\tilde{A}} > r, \;\; x, r \in [0,1] \right\}.$$

Definition 2.2. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number μ is called a triangular fuzzy number [22] if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$ and its membership function has the following form:

$$\mu(x; \alpha, \beta, \gamma) = \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\ \dfrac{\gamma-x}{\gamma-\beta}, & \text{if } \beta \leq x \leq y \\ 0, & \text{if } if \; x > \gamma \end{cases}$$

and its r-level is: $[\mu]_r = [\alpha + \; r \;\; (\beta - \alpha), \;\; \gamma \;\; - \;\; r \;\; (\gamma - \beta)], \;\; r \in \;\; [0, \;\; 1]$.

In this paper the class of all fuzzy subsets of $\mathbb{R}$ will be denoted by $\tilde{E}$ and satisfy the following properties [22, 23]:

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$,
2. $\mu(t)$ Is convex fuzzy set, i. e. $\mu(\lambda t + (1-\lambda)s) \geq \min\{\mu(t), \mu(s)\} \;\; \forall t, s \in \mathbb{R}, \lambda[0,1]$,
3. μ upper semi-continuous on $\mathbb{R}$,

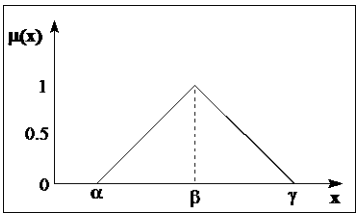


Figure 1: Triangular Fuzzy Number

4. $\overline{\{t \in \mathbb{R}:\mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\mathbb{R}$ is a proper subset of $\tilde{E}$. Define the r-level set $x \in \mathbb{R}$, $[\mu]_r = \{x \setminus (x) \geq r\}$, $0 \leq r \leq 1$, where$[\mu]_0 = \{x \setminus \mu(x) > 0\}$ is compact [28] which is a closed bounded interval and denoted by $[\mu]_r = \left(\underline{\mu}(t), \overline{\mu}(t)\right)$. In the parametric form [19], a fuzzy number is represented by an ordered pair of functions $\left(\underline{\mu}(t), \overline{\mu}(t)\right)$, $r \in [0,1]$ which satisfies:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0,1]$.
2. $\overline{\mu}(t)$ is a bounded left continuous non-increasing function over $[0,1]$.
3. $\underline{\mu}(t) \leq \overline{\mu}(t)$, $r \in [0,1]$. A crisp number r is simply represented by $\underline{\mu}(r) = \overline{\mu}(r) = r$, $r \in [0,1]$.

Definition 2.3 ([24])**.** If $\tilde{E}$ be the set of all fuzzy numbers, we say that $f(t)$ is a fuzzy function if $f : \mathbb{R} \to \tilde{E}$

Definition 2.4 ([25])**.** A mapping $f : T \to \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r-level set by:

$$[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right], t \in T, r \in [0,1].$$

The r-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem.

Definition 2.5 ([26, 27])**.** Each function $f : X \to Y$ induces another function $\tilde{f} : F(X) \to F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} U(x), & \text{if } y \in range\ (f) \\ 0, & \text{if } y \notin range\ (f). \end{cases}$$

This is called the Zadeh extension principle.

Definition 2.6 ([28])**.** Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.7 ([28]). If $\tilde{f} : I \to \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara Differentiable at y_0, if there exists an element $\left[\widetilde{f'}\right]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric$(\tilde{E}\ ,\ D)$

$$\lim_{h\to 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $[\widetilde{f'}(y_0)]_r$ is called the Hukuhara derivative of $[\widetilde{f'}]_r$ at y_0.

These limits are taken in the space $(\tilde{E}\ ,\ D)$ if t_0 or T, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r, \forall\ r \in [0, 1]$. By consideration of definition of the metric D all the r-level set $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0, with Hukuhara derivatives $[\widetilde{f'}(y_0)]_r$, when $\tilde{f} : Iarrow\tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\widetilde{f'}(y_0)]_r$ it' lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0, 1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r-levels $[\widetilde{f'}(t)]_r$ are Hukuhara differentiable at t_0.

Definition 2.8 ([23]). Define the mapping $\widetilde{f'} : I \to \tilde{E}$ and $y_0 \in$I , where $I \in [t_0, T]$. We say that $\widetilde{f'}$ Hukuhara differentiable $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\widetilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)$, $\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric$(\tilde{E}\ ,\ D)$

$$\lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative.

Theorem 2.9 ([23]). *Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote by*

$$\left[\widetilde{f'}(t)\right]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t; r), \overline{f}'(t; r)\right].$$

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable

$$\left[\widetilde{f'}(t)\right]_r = \left[\left(\underline{f}(t; r)\right)', \left(\overline{f}(t; r)\right)'\right], \forall\ r \in [0, 1].$$

Theorem 2.10 ([23]). *Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote$[\widetilde{f'}(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)]$.*

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable we can write for $n'th$ order fuzzy derivative

$$\left[\tilde{f}^{(n)}(t)\right]_r = \left[\left(\underline{f}^{(n)}(t; r)\right)', \left(\overline{f}^{(n)}(t; r)\right)'\right], \forall\ r \in [0, 1].$$

3. Fuzzification and defuzzification of HPM

The basic idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to1. When p = 0, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution [29].
The general structure of HPM for solving crisp $n'th$ order two point boundary value problems involving ordinary differential equations are mentioned in [29, 30, 31].To solve $n'th$ order TPFBVP , we need to fuzzify HPM and then defuzzify it.

Consider the following general $n'th$ order TPFBVP

$$\tilde{y}^{(n)}(t) = f\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \quad t \in [t_0, T] \tag{1}$$

subject to two point boundary conditions

$$\begin{cases} \tilde{y}(t_0) = \widetilde{\alpha}^{(0)}, \tilde{y}'(t_0) = \widetilde{\alpha}^{(1)}, \dots . \tilde{y}^{(k)}(t_0) = \widetilde{\alpha}^{(k)}, \\ \tilde{y}(T) = \widetilde{\beta}^{(0)}, \tilde{y}'(T) = \widetilde{\beta}^{(1)}, \ \dots ., \tilde{y}^{(n-k-2)}(T) = \widetilde{\beta}^{(n-k-2)}, \end{cases} \tag{2}$$

where $0 \leq k \leq n-2$ is an integer(For cases when $n = 4$ and $n = 6$, see [32, 33], $\tilde{y}(t)$ is a fuzzy function of the crisp variable$t$. Also, f is fuzzy function of the crisp variable t and the fuzzy variable$\tilde{y}$. Here $\tilde{y}^{(n)}$ is the $n'th$ order fuzzy H-derivative of $\tilde{y}(t)$ and the H-derivatives $\tilde{y}'(t), \tilde{y}''(t), \dots \tilde{y}^{(n-1)}(t)$. Moreover the fuzzy boundary conditions $\tilde{y}(t_0), \tilde{y}'(t_0), \dots \tilde{y}^{(k)}(t_0), \tilde{y}(T), y', \dots . \tilde{y}^{(n-k-s)}(T)$ are convex fuzzy numbers as in section 2. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$, for $t \in [t_0, T]$and $r \in [0, 1]$ it means that the r-level set of $y(t)$ can be defined as:

$$[\tilde{y}(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right],$$

$$\left[\tilde{y}'(t)\right]_r = \left[\underline{y}'(t;r), \overline{y}'(t_0;r)\right], \dots \left[\tilde{y}^{(n-1)}(t)\right]_r = \left[\underline{y}^{(n-1)}(t;r), \overline{y}^{(n-1)}(t;r)\right],$$

$$[\tilde{y}(t_0)]_r = \left[\underline{y}(t_0;r), \overline{y}(t_0;r)\right],$$

$$\left[\tilde{y}'(t_0)\right]_r = \left[\underline{y}'(t_0;r), \overline{y}'(t_0;r)\right], \dots \left[\tilde{y}^{(k)}(t_0)\right]_r = \left[\underline{y}^{(k)}(t_0;r), \overline{y}^{(k)}(t_0;r)\right],$$

$$\left[\tilde{y}'(T)\right]_r = \left[\underline{y}'(T;r), \overline{y}'(T;r)\right], \dots \left[\tilde{y}^{(k)}(T)\right]_r = \left[\underline{y}^{(k)}(T;r), \overline{y}^{(k)}(T;r)\right],$$

where $w(t)$ is crisp or fuzzy inhomogeneous term such that $[\tilde{w}(t)]_r = [\underline{w}(t;r), \overline{w}(t;r)]$. Since

$$y^{(n)}(t) = f\left(t, y(t), y'(t), y''(t), \dots y^{(n-1)}(t)\right) + w(t).$$

If we let $\mathcal{Y}(t) = y(t), y'(t), y''(t), \dots y^{(n-1)}(t)$, such that

$$\begin{aligned} \widetilde{\mathcal{Y}}(t;r) &= \left[\underline{\mathcal{Y}}(t;r), \overline{\mathcal{Y}}(t;r)\right] \\ &= \left[\underline{y}(t;r), \underline{y}'(t;r), \dots, \underline{y}^{(n-1)}(t;r), \overline{y}(t;r), \overline{y}'(t;r), \dots, \overline{y}^{(n-1)}(t;r)\right]. \end{aligned}$$

Also, we can write

$$\left[\tilde{f}\left(t,\widetilde{\mathcal{Y}}\right)\right]_r=\left[\underline{f}\left(t,\widetilde{\mathcal{Y}};r\right),\overline{f}\left(t,\widetilde{\mathcal{Y}};r\right)\right] \tag{3}$$

by using Zadeh extension principles we have

$$\tilde{f}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)=\left[\underline{f}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right),\underline{f}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)\right],$$

such that

$$\begin{cases}\underline{f}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)=\mathcal{F}\left(t,\underline{\mathcal{Y}}\left(t;r\right),\overline{\mathcal{Y}}\left(t;r\right)\right)=\mathcal{F}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right),\\ \overline{f}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)=\mathcal{G}\left(t,\underline{\mathcal{Y}}\left(t;r\right),\overline{\mathcal{Y}}\left(t;r\right)\right)=\mathcal{G}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right).\end{cases}$$

Then, we have

$$\begin{cases}\underline{y}^{(n)}\left(t;r\right)=\mathcal{F}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)+\underline{w}\left(t;r\right),\\ \overline{y}^{(n)}\left(t;r\right)=\mathcal{G}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)+\overline{w}\left(t;r\right),\end{cases} \tag{4}$$

where the membership function of $\mathcal{F}(t,\widetilde{\mathcal{Y}}(t;r))+\underline{w}(t;r)$ and $\mathcal{G}(t,\widetilde{\mathcal{Y}}(t;r))+\overline{w}(t;r)$ can be defined as

$$\begin{aligned}&\mathcal{F}(t,\widetilde{\mathcal{Y}}(t;r))+\underline{w}(t;r)=\min\{\tilde{y}^{(n)}(t,\widetilde{\mu}(r)):.\mu|\mu\in[\widetilde{\mathcal{Y}}(t;r)]_r\},\\ &\mathcal{G}(t,\widetilde{\mathcal{Y}}(t;r))+\overline{w}(t;r)=\max\{\tilde{y}^{(n)}(t,\widetilde{\mu}(r)):.\mu|\mu\in[\widetilde{\mathcal{Y}}(t;r)]_r\}.\end{aligned}$$

Now for all $r\in[0,1]$ we rewrite Eq. 1 in the following forms

$$\begin{cases}\underline{\mathcal{L}}_n\underline{y}\left(t;r\right)=\mathcal{F}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)+\underline{w}\left(t;r\right),\quad t\in[t_0,T],\\ \underline{y}\left(t_0;r\right)=[\underline{\alpha}_0]_r,\underline{y}'\left(t_0;r\right)=[\underline{\alpha}_1]_r\ ,\ldots,\underline{y}^{(k)}\left(t_0;r\right)=[\underline{\alpha}_k]_r,\\ \underline{y}\left(T;r\right)=\left[\underline{\beta}_0\right]_r,\underline{y}'\left(T;r\right)=\left[\underline{\beta}_1\right]_r\ ,\ldots,\underline{y}^{(n-k-2)}\left(T;r\right)=\left[\underline{\beta}_{n-k-2}\right]_r,\end{cases} \tag{5}$$

$$\begin{cases}\overline{\mathcal{L}}_n\overline{y}\left(t;r\right)=\mathcal{G}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right)+\overline{w}\left(t;r\right),\quad t\in[t_0,T],\\ \overline{y}\left(t_0;r\right)=[\overline{\alpha}_0]_r,\overline{y}'\left(t_0;r\right)=[\overline{\alpha}_1]_r\ ,\ldots,\overline{y}^{(k)}\left(t_0;r\right)=[\overline{\alpha}_k]_r,\\ \overline{y}\left(T;r\right)=\left[\overline{\beta}_0\right]_r,\overline{y}'\left(T;r\right)=\left[\overline{\beta}_1\right]_r\ ,\ldots,\overline{y}^{(n-k-2)}\left(T;r\right)=\left[\overline{\beta}_{n-k-2}\right]_r,\end{cases} \tag{6}$$

where $\widetilde{\mathcal{L}}_n=[\underline{\mathcal{L}}_n,\ \overline{\mathcal{L}}_n]$ are the linear operators with $\widetilde{\mathcal{L}}_n=\frac{d^{(n)}}{dt^{(n)}}$, followed by inverse operator $\widetilde{\mathcal{L}}_n^{-1}=\int_0^t\int_0^t\int_0^t\ldots\widetilde{[.]}_r d\tau d\tau\ldots d\tau$ and $\mathcal{F}$, $\mathcal{G}$ are nonlinear operators. One of the advantages of the HPM is that HPM solution series is only dependent upon two factors: the auxiliary linear operator and initial guess. In fact, the proper choice of the initial condition and the auxiliary linear operator will guarantee the convergence of the HPM solution series [17]. According to the

HPM [29], we construct a homotopy form into Eq. (5) and (6) which satisfies the following relation

$$
\begin{aligned}
\underline{\mathcal{H}}(t,p;r) &= (1-p)\underline{\mathcal{L}}_n[\underline{y}(t;r) - \underline{y}_0(t;r;\underline{a}_s(r))] \\
&\quad + p[\underline{\mathcal{L}}_n\underline{y}_0(t;r;\underline{a}_s(r)) - \mathcal{F}(t,\widetilde{\mathcal{Y}}(t;r)) - \underline{w}(t;r)] = 0, \\
\overline{\mathcal{H}}(t,p;r) &= (1-p)\overline{\mathcal{L}}_n[\overline{y}(t;r) - \overline{y}_0(t;r;\overline{a}_s(r))] + p[\overline{\mathcal{L}}_n\overline{y}_0(t;r;\overline{a}_s(r)) \\
&\quad - \mathcal{G}(t,\widetilde{\mathcal{Y}}(t;r)) - \overline{w}(t;r)],
\end{aligned} \tag{7}
$$

where $p \in [0,\ 1]$ is an embedding parameter and $\underline{y}_0\left(t;r;\underline{a}_s(r)\right)$, and $\overline{y}_0\left(t;r;\overline{a}_s\left(r\right)\right)$are initials guessing and can be defining as follows:

$$
\underline{y}_0\left(t;r;\underline{a}_s(r)\right) = \underline{a}_1\left(r\right) + \underline{a}_2\left(r\right)t + \frac{\underline{a}_3\left(r\right)}{2!}t^2 + \cdots + \frac{\underline{a}_n\left(r\right)}{(n-1)!}t^{n-1}, \tag{8}
$$

$$
\overline{y}_0\left(t;r;\overline{a}_s(r)\right) \ = \overline{a}_1\left(r\right) + \overline{a}_2\left(r\right)t + \frac{\overline{a}_3\left(r\right)}{2!}t^2 + \cdots + \frac{\overline{a}_n\left(r\right)}{(n-1)!}t^{n-1}, \tag{9}
$$

where $\underline{a}_s\left(r\right)$ and $\overline{a}_s\left(r\right)$ are fuzzy constants to be determined. From Eq. (2) we will have

$$
\begin{cases}
\underline{\mathcal{H}}\left(t,0;r\right) = \underline{\mathcal{L}}_n\left[\underline{y}\left(t;r\right) - \underline{y}_0\left(t;r;\underline{a}_s(r)\right)\right] \ = 0, \\
\underline{\mathcal{H}}\left(t,1;r\right) = \left[\underline{\mathcal{L}}_n\underline{y}_0\left(t;r;\underline{a}_s(r)\right) - \mathcal{F}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right) - \underline{w}\left(t;r\right)\right] = 0.
\end{cases} \tag{10}
$$

Also for the upper bound we have

$$
\begin{cases}
\overline{\mathcal{H}}\left(t,0;r\right) = \overline{\mathcal{L}}_n\left[\overline{y}\left(t;r\right) - \overline{y}_0\left(t;r;\overline{a}_s(r)\right)\right] = 0, \\
\overline{\mathcal{H}}\left(t,1;r\right) = \left[\overline{\mathcal{L}}_n\overline{y}_0\left(t;r;\overline{a}_s(r)\right) - \mathcal{G}\left(t,\widetilde{\mathcal{Y}}\left(t;r\right)\right) - \overline{w}\left(t;r\right)\right] = 0.
\end{cases} \tag{11}
$$

As in crisp case we called this deformation, From Eq.(10) and (11) we can represent each y in these equations as follows:

$$
\begin{cases}
\tilde{y}(t;r;\sum_{i=1}^{n}\tilde{a}_s(r)) = \sum_{k=0}^{\infty}p^k\tilde{y}_k(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)), \\
\tilde{y}'(t;r;\sum_{i=1}^{n}\tilde{a}_s(r)) = \sum_{k=0}^{\infty}p^k\tilde{y}'_k(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)), \\
\vdots \\
\tilde{y}^{(n)}(t;r;\sum_{i=1}^{n}\tilde{a}_s(r)) = \sum_{k=0}^{\infty}p^k\tilde{y}^{(n)}_k(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)).
\end{cases} \tag{12}
$$

Now substituting equation the above equations into Eq.(10) and (11) and collecting terms of the same powers of p, we have:

$$
p^0 : \begin{cases}
\underline{\mathcal{L}}_n\left[\underline{u}_0\left(t;r\right) - \underline{y}_0\left(t;r;\sum_{s=1}^{n}\underline{a}_s\left(r\right)\right)\right] = 0, \\
\underline{y}\left(t_0;r\right) = \left[\underline{\alpha}_0\right]_r, \underline{y}'\left(t_0;r\right) = \left[\underline{\alpha}_1\right]_r, \ldots, \underline{y}^{(k)}\left(t_0;r\right) = \left[\underline{\alpha}_k\right]_r, \\
\underline{y}\left(T;r\right) = \left[\underline{\beta}_0\right]_r, \underline{y}'\left(T;r\right) = \left[\underline{\beta}_1\right]_r, \ldots, \underline{y}^{(n-k-2)}\left(T;r\right) = \left[\underline{\beta}_{n-k-2}\right]_r,
\end{cases}
$$

$$p^1:\begin{cases}\underline{\mathcal{L}}_n\left[\underline{u}_1(t;r)+\underline{y}_0(t;r;\sum_{s=1}^n\underline{a}_s(r))\right]\\ -\mathcal{F}\left(t,\widetilde{\mathcal{Y}}_0(t;r,\sum_{i=1}^n\underline{a}_s(r))\right)-\underline{w}(t;r)=0,\\ \underline{y}_1(t_0;r)=0_r,\underline{y}_1{}'(t_0;r)=0\ ,\ldots,\underline{y}_1{}^{(k)}(t_0;r)=0,\\ \underline{y}_1(T;r)=0_r,\underline{y}_1{}'(T;r)=0\ ,\ldots,\underline{y}_1{}^{(n-k-2)}(T;r)=0,\end{cases}$$

$$p^2:\begin{cases}\underline{\mathcal{L}}_n\underline{u}_2(t;r)-\mathcal{F}\left(t,\widetilde{\mathcal{Y}}_1(t;r,\sum_{i=1}^n\underline{a}_s(r))\right)=0,\\ \underline{y}_2(t_0;r)=0_r,\underline{y}_2{}'(t_0;r)=0\ ,\ldots,\underline{y}_2{}^{(k)}(t_0;r)=0,\\ \underline{y}_2(T;r)=0_r,\underline{y}_2{}'(T;r)=0\ ,\ldots,\underline{y}_2{}^{(n-k-2)}(T;r)=0,\end{cases}$$

$$\vdots$$

$$p^{n+1}:\begin{cases}\underline{\mathcal{L}}_n\underline{u}_n(t;r)-\mathcal{F}\left(t,\widetilde{\mathcal{Y}}_n(t;r,\sum_{i=1}^n\underline{a}_s(r))\right)=0,\\ \underline{y}_n(t_0;r)=0_r,\underline{y}_n{}'(t_0;r)=0\ ,\ldots,\underline{y}_n{}^{(k)}(t_0;r)=0,\\ \underline{y}_n(T;r)=0_r,\underline{y}_n{}'(T;r)=0\ ,\ldots,\underline{y}_n{}^{(n-k-2)}(T;r)=0.\end{cases}$$

Similarly for the upper bound

$$p^0:\begin{cases}\overline{\mathcal{L}}_n\left[\overline{u}_0(t;r)-\overline{y}_0(t;r;\sum_{s=1}^n\overline{a}_s(r))\right]=0,\\ \overline{y}(t_0;r)=[\overline{\alpha}_0]_r,\overline{y}'(t_0;r)=[\overline{\alpha}_1]_r\ ,\ldots,\overline{y}^{(k)}(t_0;r)=[\overline{\alpha}_k]_r,\\ \overline{y}(T;r)=\left[\overline{\beta}_0\right]_r,\overline{y}'(T;r)=\left[\overline{\beta}_1\right]_r\ ,\ldots,\overline{y}^{(n-k-2)}(T;r)=\left[\overline{\beta}_{n-k-2}\right]_r,\end{cases}$$

$$p^1:\begin{cases}\overline{\mathcal{L}}_n\left[\overline{y}_1(t;r)+\overline{y}_0(t;r;\sum_{s=1}^n\overline{a}_s(r))\right]\\ -\mathcal{G}\left(t,\widetilde{\mathcal{Y}}_0(t;r,\sum_{i=1}^n\overline{a}_s(r))\right)-\overline{w}(t;r)=0,\\ \overline{y}_1(t_0;r)=0,\overline{y}_1{}'(t_0;r)=0\ ,\ldots,\overline{y}_1{}^{(k)}(t_0;r)=0,\\ \overline{y}_1(T;r)=0,\overline{y}_1{}'(T;r)=0\ ,\ldots,\overline{y}_1{}^{(n-k-2)}(T;r)=0,\end{cases}$$

$$p^2:\begin{cases}\overline{\mathcal{L}}_n\overline{y}_2(t;r)-\mathcal{G}\left(t,\widetilde{\mathcal{Y}}_n(t;r,\sum_{i=1}^n\overline{a}_s(r))\right)=0,\\ \overline{y}_2(t_0;r)=0,\overline{y}_2{}'(t_0;r)=0\ ,\ldots,\overline{y}_2{}^{(k)}(t_0;r)=0,\\ \overline{y}_2(T;r)=0,\overline{y}_2{}'(T;r)=0\ ,\ldots,\overline{y}_2{}^{(n-k-2)}(T;r)=0,\end{cases}$$

$$\vdots$$

$$p^{n+1}:\begin{cases}\overline{\mathcal{L}}_n\overline{y}_2(t;r)-\mathcal{G}\left(t,\widetilde{\mathcal{Y}}_n(t;r,\sum_{i=1}^n\overline{a}_s(r))\right)=0,\\ \overline{y}_n(t_0;r)=0,\overline{y}_n{}'(t_0;r)=0\ ,\ldots,\overline{y}_n{}^{(n-1)}(t_0;r)=0,\\ \overline{y}_n(T;r)=0,\overline{y}_n{}'(T;r)=0\ ,\ldots,\overline{y}_{nn}{}^{(n-1)}(T;r)=0,\end{cases}$$

for all $r\in[0,1]$. Then the approximate solution is given by setting p=1as follows:

$$\tilde{y}\left(t;r;\sum_{s=1}^n\tilde{a}_s(r)\right)=\varphi_m\left(t;r;\sum_{s=1}^n\tilde{a}_s(r)\right)=\sum_{i=0}^{m-1}\widetilde{\mathcal{Y}}_i\left(t;r;\sum_{s=1}^n\tilde{a}_s(r)\right).\tag{13}$$

Now the exact solution of Eq.(1), therefore, can be obtained by setting $p = 1$ as follows:

$$\tilde{Y}(t;r) = \lim_{p\to 1} \tilde{y}\left(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)\right) = \lim_{p\to 1}\left\{\sum_{i=0}^{\infty} p^i \widetilde{\mathcal{Y}}_i\left(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)\right)\right\}$$

$$= \sum_{i=0}^{\infty}\widetilde{\mathcal{Y}}_i\left(t;r;\sum_{s=1}^{n}\tilde{a}_s(r)\right). \tag{14}$$

Convergence of HPM in fuzzy environment

According to [34, 35] we can write Eq.(1) in the following forms

$$\mathcal{L}\left([\tilde{u}(t)]_r\right) = \mathcal{L}[\tilde{y}_0(t)]_r + p\left[\tilde{f}(s;r) - \mathcal{N}\left([\tilde{u}(t)]_r\right) - \mathcal{L}[\tilde{y}_0(t)]_r\right], \tag{15}$$

where $\tilde{f}(s;r)$is known as analytic fuzzy function Applying the inverse operator, $\mathcal{L}^{-1}$ as mentioned in section 4 to both sides of Eq.(15), we obtain

$$[\tilde{u}(x;t)]_r = [\tilde{y}_0(t)]_r + p\left[\mathcal{L}^{-1}\tilde{f}(s;r) - \mathcal{L}^{-1}\mathcal{N}\left([\tilde{u}(t)]_r\right) - [\tilde{y}_0(t)]_r\right]. \tag{16}$$

Suppose that

$$\tilde{u}(t;r) = \sum_{1=0}^{\infty} p^i \tilde{u}_i(t;r). \tag{17}$$

Substituting Eq.(17) into the right-hand side of Eq.(16), we have

$$[\tilde{u}(t)]_r = [\tilde{y}_0(t)]_r + p\left[\mathcal{L}^{-1}\tilde{f}(s;r) - \mathcal{L}^{-1}\mathcal{N}\left(\sum_{1=0}^{\infty} p^i\tilde{u}_i(t;r)\right) - [\tilde{y}_0(t)]_r\right]. \tag{18}$$

If $p \to 1$, the exact solution may be obtained by using the following formula $\tilde{y}(t;r) = \lim_{p\to 1}\tilde{u}(t;r) = \mathcal{L}^{-1}\tilde{f}(s;r) - \mathcal{L}^{-1}\mathcal{N}(\sum_{1=0}^{\infty} p^i\tilde{u}_i(t;r)) = \mathcal{L}^{-1}\tilde{f}(s;r) - (\sum_{1=0}^{\infty}\mathcal{L}^{-1}\mathcal{N}\tilde{u}_i(t;r))\}$.

To study the convergence of the method let us state the following Theorem.

Theorem 1. Suppose that $[\tilde{u}]_r$ and $[\tilde{w}]_r$ be Banach spaces and $N : [\tilde{u}]_r \to [\tilde{w}]_r$be a contraction mapping such that for all $r \in [0,1]$

$$[\tilde{u}]_r, [\tilde{u}^*]_r \in \left[\tilde{U}\right]_r : \|N([\tilde{u}]_r) - N([\tilde{u}^*]_r)\| \le \varepsilon\|[\tilde{u}]_r - [\tilde{u}^*]_r\|, \varepsilon \in [0,1]. \tag{19}$$

Then according to Banach's fixed point theorem [36] N has a unique fixed point $[\tilde{y}]_r$ (say) such that $N([\tilde{y}]_r) = [\tilde{y}]_r$.The sequence generated by the homotopy perturbation method will be assumed in the following form as

$$\tilde{U}_n(t;r) = N(\tilde{U}_{n-1}(t;r)), \tilde{U}_{n-1}(t;r) = \sum_{1=0}^{n-1}\tilde{U}_i(t;r), n = 1,2,3,\ldots$$

and suppose that $\tilde{U}_0(t;r) = \tilde{u}_0(t;r) \in B_s(\tilde{u}_0(t;r))$ where $B_s(\tilde{u}_0(t;r)) = \{[\tilde{u}^*]_r \in [\tilde{U}]_r : \|[\tilde{u}^*]_r - [\tilde{u}]_r\| < s\}$, then we have

1. $\left\|\tilde{U}_1(t;r) - \tilde{V}(t;r)\right\| \leq \varepsilon^n \left\|\tilde{u}_0(t;r) - \tilde{u}(t;r)\right\|,$
2. $\tilde{U}_n(t;r) \in B_s\left(\tilde{u}(t;r)\right),$
3. $\lim_{p\to 1} \tilde{U}_n(t;r) = \tilde{u}(t;r).$

Proof. As mentioned in section 2 all fuzzy sets are subsets of real numbers $\mathbb{R}$ and the r-level sets is crisp sets for each value of $r \in [0,1]$, then we can define:

$$\tilde{U}_n(t;r) = \left[\underline{U}_n(t;r), \overline{U}_n(t;r)\right] = \left[\underline{U}_n, \overline{U}_n\right]_r, \tag{20}$$

$$\tilde{V}(t;r) = \left[\underline{V}(t;r), \overline{V}(t;r)\right] = \left[\underline{V}, \overline{V}\right]_r. \tag{21}$$

For the lower bound of Eq.(1), according to [36] part (i) of the above theorem is proved by using method of induction.

For $n = 1.$, $\|[\underline{U}_1]_r - [\underline{u}]_r\| = \|N([\underline{U}_0]_r) - N([u]_r)\| \leq \varepsilon\|[\underline{u}_0]_r - [\underline{u}]_r\|$ (from Eq.(34)). Assume that$\|[\underline{U}_{n-1}]_r - [\underline{u}]_r\| \leq \varepsilon^{n-1}\|[\underline{u}_0]_r - [\underline{u}]_r\|$as induction hypothesis, then $\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \|N([\underline{U}_{n-1}]_r) - N([\underline{u}]_r)\| \leq \varepsilon\|[\underline{U}_{n-1}]_r - [\underline{u}]_r\| \leq \varepsilon\varepsilon^{n-1}\|[\underline{u}_0]_r - [\underline{u}]_r\| \leq \varepsilon^n\|[\underline{u}_0]_r - [\underline{u}]_r\|$ (from induction hypothesis).

Hence $\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \varepsilon^n\|[\underline{u}_0]_r - [u]_r\|$.

(ii) Using (i), we have

$$\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \varepsilon^n\|[\underline{u}_0]_r - [\underline{u}]_r\| \leq \varepsilon^n s \leq s.$$

This implies $[\underline{U}_n]_r \in B_s([\underline{u}]_r)$.

(iii) From (i) we know that $\|[\underline{U}_n]_r - [u]_r\| \leq \varepsilon^n\|[\underline{u}_0]_r - [\underline{u}]_r\|$.

Since $0 < \varepsilon < 1$, $\lim_{n\to\infty} \varepsilon^n = 0$. So $\lim_{n\to\infty} \|[\underline{U}_n]_r - [\underline{u}]_r\| = 0$. This yield $\lim_{n\to\infty} [\underline{U}_n]_r = [\underline{u}]_r$.

Similarly in the same way we can prove (i), (ii), (iii) for the upper bound convergence of Eq.(14). Thus the series in Eq. (**??**) such that $[\underline{y}_i(t;r), \overline{y}_i(t;r)]$ converges for each r-level set.

Numerical example

Let us consider the following fuzzy two-point boundary value problem [9]

$$y''(t) = y'(t) + \gamma_0,\ y(0) = 0, y(1) = \gamma_1, \tag{22}$$

where γ_0 and γ_1 are triangular fuzzy numbers having r-level sets $[r, 2 - r]$, $[r - 1, 1 - r]$ for all$r \in [0,1]$. It can be verified the exact solution of Eq.(22) is given in [9]. According to section 3 the linear operator of Eq.(17) is $\widetilde{\mathcal{L}}_2$ with the inverse operator $\widetilde{\mathcal{L}}_2^{-1}$, and then the initial approximation guesses for all $r \in [0,1]$ are given by:

$$\underline{y}_0(t;r) = \underline{a}_1(r) + \underline{a}_2(r)\,t, \overline{y}_0(t;r) = \overline{a}_1(r) + \overline{a}_2(r)\,t$$

According to the boundary conditions in Eq.(22) the values of $\tilde{a}_1(r) = 0$. According to the HPM in section 3, the n components of $\tilde{y}_k(t;r)$ for $k = 1, 2, .., n$ and $r \in [0,1]$ can be determined by solving the following

$$p^0 : \left\{ \underline{y}_0(t;r) = \underline{a}_2(r)\, t\,, \tag{23}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = \underline{\mathcal{L}}_2^{-1}\left[\underline{y}_0{}'(t;r;\underline{a}_2(r)) + r\right], \\ \underline{y}_1(0;r) = 0, \underline{y}_1(1;r) = 0, \end{cases} \tag{24}$$

$$p^2 : \begin{cases} \underline{y}_2(t;r) = \underline{\mathcal{L}}_2^{-1}\left[\underline{y}_1{}'(t;r;\underline{a}_2(r))\right], \\ \underline{y}_2(0;r) = 0, \underline{y}_2{}'(1;r) = 0. \end{cases} \tag{25}$$

And so on till

$$p^{k+1} : \begin{cases} \underline{y}_{k+1}(t;r) = \underline{\mathcal{L}}_2{}^{-1}\underline{y}_k{}'(t;r;\underline{a}_2(r)), \\ \underline{y}_k(0;r) = 0, \underline{y}_k{}'(1;r) = 0. \end{cases} \tag{26}$$

Similarly for the upper bound

$$p^0 : \{\overline{y}_0(t;r) = \overline{a}_2(r)\, t\,, \tag{27}$$

$$p^1 : \begin{cases} \overline{y}_1(t;r) = \overline{\mathcal{L}}_2{}^{-1}\left[\overline{y}_0{}'(t;r;\overline{a}_2(r)) + (2-r)\right], \\ \overline{y}_1(0;r) = 0, \overline{y}_1{}'(1;r) = 0, \end{cases} \tag{28}$$

$$p^2 : \begin{cases} \overline{y}_2(t;r) = \overline{\mathcal{L}}_2{}^{-1}\left[\overline{y}_1{}'(t;r;\overline{a}_2(r))\right], \\ \overline{y}_2(0;r) = 0, \overline{y}_2{}'(1;r) = 0. \end{cases} \tag{29}$$

And so on till

$$p^{k+1} : \begin{cases} \overline{y}_{k+1}(t;r) = \overline{\mathcal{L}}_2{}^{-1}\left[\overline{y}_k{}'(t;r;\overline{a}_2(r))\right], \\ \overline{y}_k(0;r) = 0, \overline{y}_k{}'(1;r) = 0. \end{cases} \tag{30}$$

Evaluating Eqs.(23)-(30) to obtain 10-order HPM series solution in the following form

$$\widetilde{\varphi}_{10}(t;r;\tilde{a}_2(r)) = \tilde{y}_0(t;r) + \sum_{i=1}^{9} \tilde{y}_i(t;r;\tilde{a}_2(r)). \tag{31}$$

From the boundary conditions in Eq.(21) and Eq.(31), we obtain the values of $\tilde{a}_2(r)$ for all $r \in [0,1]$ such that:

$$\begin{cases} \underline{a}_2(r) = -0.5819767076349684 + 0.16395341526993693r, \\ \overline{a}_2(r) = -0.2540698770950946 - 0.16395341526993693r. \end{cases}$$

Table 1: Approximate solution and Absolut error of 10-order HPM at r=0 and $t\in[0,1]$

t	$\underline{\varphi}_{10}(t;0)$	$[\underline{E}(t)]_0$	$\overline{\varphi}_{10}(t;0)$	$[\overline{E}(t)]_0$
0	0	0	0	0
0.2	−0.128851248255356	1.695150×10^{-10}	−0.01344625523392	5.085456×10^{-10}
0.4	−0.286230518266809	3.765406×10^{-10}	0.058691554800427	1.129621×10^{-9}
0.6	−0.478453992733306	6.266763×10^{-10}	0.235361978199918	1.880029×10^{-9}
0.8	−0.713236274547007	8.493847×10^{-10}	0.539708823641023	2.548154×10^{-9}
1	−1	1.110223×10^{-16}	1	4.440892×10^{-16}

Table 2: Approximate solution Absolut error of 10-order HPM at t=0.5 and $r\in[0,1]$

r	$\underline{\varphi}_{10}(0.5;r)$	$[\underline{E}(0.5)]_r$	$\overline{\varphi}_{10}(0.5;r)$	$[\overline{E}(0.5)]_r$
0	−0.3775406692945253	4.963798×10^{-10}	0.1326220078835757	1.489139×10^{-9}
0.2	−0.3265244015767151	2.978278×10^{-10}	0.0816057401657656	1.290587×10^{-9}
0.4	−0.2755081338589050	9.927592×10^{-11}	0.0305894724479555	1.092035×10^{-9}
0.6	−0.2244918661410949	9.927592×10^{-11}	−0.0204267952698545	8.934834×10^{-10}
0.8	−0.1734755984232848	2.978279×10^{-10}	−0.0714430629876646	6.949317×10^{-10}
1	−0.1224593307054747	4.963796×10^{-10}	−0.1224593307054747	4.963797×10^{-10}

Substitute these values again in Eq.(31) to get 10- order HPM approximate solution series, and then we can define the absolute error of approximate solution as follows

$$\left[\tilde{E}(t)\right]_r = \left|\tilde{Y}(t;r) - \widetilde{\varphi}_{10}(t;r)\right|. \tag{32}$$

It can be noted from Figure 2; the exact and 10-order HPM approximate solutions of Eq.(22) when $r = 0$ for all $t \in [0,1]$ satisfy the fuzzy numbers properties in the form of triangular fuzzy number as all the values of $r \in [0,1]$. Also Tables 1-2 and Figures 3-4 shows the accuracy of the HPM with 10 order of the approximate solution.

4. Conclusions

In this study, an approximate-analytical method HPM was applied to obtain an approximate solution $n'th$ order nonlinear TPFBVP. A scheme based HPM approximate the solution of $n'th$ order TPFBVP has been formulated to obtain the approximate solution directly with our reduced in to first order system. The convergence theorem of HPM has been presented and proved. From the

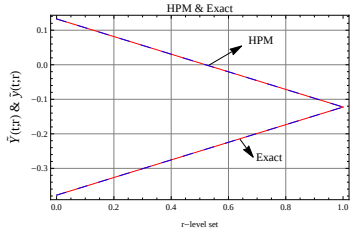


Figure 2: Approximate solution of 10-order of Eq. (25) at r=0 and $t\in[0,1]$

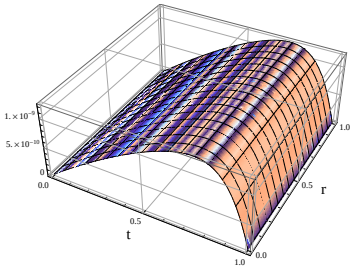


Figure 3: Approximate solution of 10-order of Eq. (25) at r=0 and $t\in[0,1]$

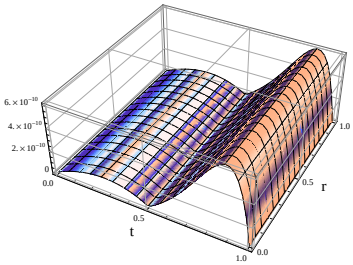


Figure 4: Absolut error of 10-order HPM solution of Eq. (25) $\forall\ r\in[0,1]$ and $t\in[0,1]$

nonlinear TPFIVP the accuracy of HPM can be determined even these equations without exact analytical solution. HPM give accurate solution even with low number of terms of the series solution. Numerical example which included linear TPFBVP show the efficiency of the implemented of this approximate- analytical method. The results of the solved problem satisfied the fuzzy numbers properties by taking the triangular fuzzy numbers shape.

References

[1] A. Omer and O. Omer, *A pray and pretdour model with fuzzy intial values*, Hacettepe Journal of Mathematics and Statistics, 41 (2013), 387-395.

[2] S. Tapaswini and S. Chakraverty, *Numerical solution of fuzzy arbitrary order predator-prey equations*, Applications and Applied Mathematics, 8 (2013), 647-673.

[3] M. Zaini, Ahmad Bernard De Baets, *A predator-prey model with fuzzy initial populations*, IFSA-EUSFLAT, 2009.

[4] M. S. El Naschie, *From experimental quantum optics to quantum gravity via a fuzzy kahler manifold*, Chaos Solution and Fractals, 25 (2005), 969-977.

[5] M. F. Abbod, D.G. Von Keyserlingk, D.A. Linkens and M. Mahfouf, *Survey of utilization of fuzzy technology in medicine and healthcare,* Fuzzy Sets and System, 120 (2001), 331-3491.

[6] S. Barro and R. Marin, *Fuzzy logic in medicine*, Heidelberg: Physica-Verlag, 2002.

[7] L. Wang and S. Guo, *Adomian method for second-order fuzzy differential equation,* World Academy of Science, Engineering and Technology, 5 (2011), 783-786.

[8] S. Abbasbandy, T. Allahviranloo, P. Darabi and O. Sedaghatfar, *Variational iteration method for solving N-th order fuzzy differential equations*, Mathematical and Computational Applications, 16 (2011), 819-829.

[9] A. Armand and Z. Gouyandeh, *Solving two-point fuzzy boundary value problem using variational iteration method,* Communications on Advanced Computational Science with Applications, (2013), 1-10.

[10] A. F. Jameel and A. I. Ismail, *Approximate solution of N'th order two point fuzzy boundary value problems by optimal homotopy asymptotic method,* International Journal of Modern Mathematical Sciences, 6 (2013), 107-120.

[11] A. F. Jameel, *Semi analytical-solution of nonlinear two points fuzzy boundary value problems by adomian decomposition method,* Applied & Computational Mathematics, 4 (2015), 1-5.

[12] X. Guo, D.Shang and X. Lu, *Fuzzy approximate solutions of second-order fuzzy linear boundary value problems,* Journal of Boundary Value Problems, (2013), 1-17.

[13] J.-H. He., *A coupling method of a homotopy technique and a perturbation technique for non-linear problems,* Internat. J. Non-Linear Mech., 35 (2000), 7-43.

[14] L. Jin, *Analytical approach to the sine-Gordon equation using homotopy perturbation method*, Int. J. Contemp. Math. Sci., 4(5-8) (2009), 225-231.

[15] C. Chun, H. Jafari and Y.-I. Kim, *Numerical method for the wave and nonlinear diffusion equations with the homotopy perturbation method*, Comput. Math. Appl., 57 (2009), 1226-1231.

[16] Y. Khan and Q. Wu, *Homotopy perturbation transform method for nonlinear equations using He's polynomials*, Comput. Math. Appl., 61 (2011), 1963-1967.

[17] J.-H. He, *Homotopy perturbation technique,* Comput. Methods Appl. Mech. Engrg., 178 (1999), 257-262.

[18] J.-H. He., *The homotopy perturbation method for nonlinear oscillators with discontinuities*, Applied Mathematics and Computation, 151 (2004), 287-292.

[19] J.-H. He., *Application of homotopy perturbation method to nonlinear wave equations*, Chaos, Solitons and Fractals, 26 (2005), 695-700.

[20] J.-H. He, *Limit cycle and bifurcation of nonlinear problems*, Chaos, Solitons and Fractals, 26 (2005), 827-833.

[21] S. Bodjanova, *Median alpha-levels of a fuzzy number*, Fuzzy. Set. Syst., 157 (2006), 879-891.

[22] D. Dubois, H. Prade, *Towards fuzzy differential calculus, Part 3: Differentiation*, Fuzzy Sets and Systems, 8 (1982), 225-233.

[23] S. Mansouri. S , N. Ahmady, *A numerical method for solving n'th-order fuzzy differential equation by using characterization theorem,* Communication in Numerical Analysis, (2012), 1-12.

[24] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets Syst, 24 (1987), 301-317.

[25] O. S. Fard, *An Iterative Scheme for the Solution of Generalized System of Linear Fuzzy Differential Equations*, World. Appl. Sci. J., 7 (2009), 1597-11604.

[26] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[27] L. A. Zadeh, *Toward a generalized theory of uncertainty*, Information Sciences, 172 (2005), 1-40.

[28] S. Salahshour, *N'th-order fuzzy differential equations under generalized differentiability,* Journal of Fuzzy Set Value Analysis, (2011), 1-14.

[29] A. Saadatmandi, M. Dehghan and A. Eftekhari, *Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems,* Nonlinear Anal. Real World Appl., 10 (2009), 1912-1922.

[30] Z. Shun-dong, *Homotopy perturbation method for two point boundary value problems,* Journal of the Juliusz Schauder Cente, 31 (2008), 369-372.

[31] S. S. Shahid and M. Iftikhar, *Use of homotopy perturbation method for solving multi-point boundary value problems*, Research Journal of Applied Sciences, Engineering and Technology, 7 (2014), 778-785.

[32] S. Liang and D. Jeffrey, *Approximate solutions to a parameterized sixth order boundary value problem*, Comput. Math. Appl, 59 (2010), 247-253.

[33] S. Liang and D. Jeffrey, *An efficient analytical approach for solving fourth order boundary value problems*, Comput. Phys. Commun, 180 (2009), 2034-2040.

[34] J. Biazar, and H. Aminikhah, *Study of convergence of homotopy perturbation method for systems of partial differential equations*, Computers & Mathematics with Applications, 58 (2009), 2221-2230.

[35] J. Biazar, and H. Ghazvini, *Convergence of the homotopy perturbation method for partial differentialequations,* Nonlinear Analysis, 10 (2009), 2633–2640.

[36] A. K. Gupta, and S. Saha Ray, *Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the solution of Boussinesq–Burgers equations*, Comput. Fluids, 103 (2014), 34-41.

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On the sum of element orders of $\mathrm{PSL}(2,p)$ for some p

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Abstract. Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. In this paper, we show that if G is a group of order $|\mathrm{PSL}(2,p)|$, then $\psi(G) \geq \psi(\mathrm{PSL}(2,p))$, where $p \in \{11, 13, 19, 23, 29, 37, 61\}$. Also we prove that $|G| = |\mathrm{PSL}(2,p)|$ and $\psi(G) = \psi(\mathrm{PSL}(2,p))$ if and only if $G \cong \mathrm{PSL}(2,p)$, where $p \in \{11, 13, 19, 23, 29, 37, 61\}$. Furthermore, we prove that $\mathrm{PSL}(2,17)$ is determined by its order and the sum of element orders.

Keywords: finite groups, simple group, element orders, sum of element orders.

1. Introduction

In this paper all groups are finite. Let $\psi(G) = \sum_{g\in G} o(g)$, the sum of element orders in a group G. The function $\psi(G)$ was introduced by Amiri, Jafarian and Isaacs [2]. They proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(\mathbb{Z}_n)$. In fact $\mathbb{Z}_n$ is characterized by $\psi(\mathbb{Z}_n)$ and $|\mathbb{Z}_n|$. In [7], Jafarian proved that $\mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,7)$ are uniquely determined by their orders and the sum of the element orders.

In [1, 8, 9, 12], finite groups G of order n with the second largest value of $\psi(G)$ were studied. Recently, in [6], Marcel Herzog et al. proved that $\psi(G) \leq 7/11\psi(\mathbb{Z}_n)$ and $\psi(G) < \psi(\mathbb{Z}_n)/(q-1)$, where G is a non-cyclic finite group of order n and q is the least prime divisor of n.

In [1], the following conjecture was stated: *Let S be a simple group. If G is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.* In fact, in [7], Jafarian proved that this conjecture holds for $\mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,7)$. In [11], Marefat et al. by providing groups $\mathrm{PSL}(2,64)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathrm{Sz}(8)$ showed that the conjecture of minimality of $\psi(G)$ in simple groups is incorrect.

In [7, 10], the following general question was proposed: *What information about a group G can be obtained from $\psi(G)$ and $|G|$?* In general, the invariants $\psi(G)$ and $|G|$ do not characterize G. For example, there exist two non-

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isomorphic p-groups of order p^3 and exponent p with sum of element orders $p^4 - p + 1$, where p is an odd prime number.

In this paper, we prove that if $p \in \{11, 13, 17, 19, 23, 29, 37, 61\}$, then $|\mathrm{PSL}(2, p)|$ is uniquely determined by its order and the sum of element orders.

2. Preliminary results

Let s_n be the number of elements of order n and $\omega(G)$ denotes the set of element orders of G. We note that

$$\psi(G) = \sum_{i \in \omega(G)} i s_i, \qquad |G| = \sum_{i \in \omega(G)} s_i. \tag{1}$$

Lemma 2.1 ([3]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

We know that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \omega(G)$, then by the above lemma we have

$$\phi(n) \mid s_n, \qquad n \mid \sum_{d|n} s_d. \tag{2}$$

Lemma 2.2 ([5]). *An integer $n = {p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k}$ is the number of Sylow p-subgroups of a finite solvable group G if and only if ${p_i}^{\alpha_i} \equiv 1 \pmod{p}$ for $i = 1, \cdots, k$.*

Lemma 2.3 ([13, Lemma 1]). *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 2.4 ([2, Corollary B]). *Let $P \in \mathrm{Syl}_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 2.5 ([1, Lemma 2.1]). *If G and H are finite groups, then $\psi(G \times H) \leq \psi(G)\psi(H)$. Also $\psi(G \times H) = \psi(G)\psi(H)$ if and only if $\gcd(|G|, |H|) = 1$.*

3. Main results

Theorem 3.1. *Let G be a finite group of order $|\mathrm{PSL}(2, p)|$, where*

$$p \in \{11, 13, 19, 23, 29, 37, 61\}.$$

Then $\psi(\mathrm{PSL}(2, p)) \leq \psi(G)$. Also $|G| = |\mathrm{PSL}(2, p)|$ and $\psi(G) = \psi(\mathrm{PSL}(2, p))$ if and only if $G \cong \mathrm{PSL}(2, p)$.

Proof. We prove the theorem in the following steps.

Step 1. Let $p = 11$.

Let G be a group such that $\psi(G) = 3741$ and $|G| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$. First, we show that if G is a solvable group, then $\psi(G) > 3741$. By Lemma 2.2, we have $n_5 \in \{1, 11\}$ and $n_{11} = 1$. By NC-theorem, we have $s_{22} \neq 0$ and $s_{33} \neq 0$. Let Q_{22} and Q_{33} be cyclic subgroups of orders 22 and 33, respectively. If $n_5 = 1$, then $s_{55} \neq 0$ and we have

$$\psi(G) \geq \psi(\mathbb{Z}_{55}) + \phi(33) \cdot 33 + \phi(22) \cdot 22 + (|G| - 85) \cdot 2 = 4361, \tag{3}$$

and we get the result. If $n_5 = 11$, then $|N_G(P_5)| = 2^2 \cdot 3 \cdot 5$, where $P_5 \in \mathrm{Syl}_5(G)$. By NC-theorem, $s_{15} \neq 0$. Let Q_{15} be a cyclic subgroup of order 15.

• If $11 \mid |N_G(Q_{15})|$, then $11 \cdot 15 \in \omega(G)$ and similarly to (3), we get the result.

• If $11 \nmid |N_G(P_{15})|$, then $s_{15} = \phi(15) \cdot 11 \cdot k_{15}$, where $k_{15} \in \mathbb{N}$.

If $3 \mid |N_G(Q_{22})|$, then $3 \cdot 22 \in \omega(G)$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{66}) + 88 \cdot 15 + (|G| - (66+88)) \cdot 2 = 4663$, as required. If $3 \nmid |N_G(Q_{22})|$, then $s_{22} = \phi(22) \cdot 3 \cdot k_{22}$, where $k_{22} \in \mathbb{N}$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{33}) + 30 \cdot 22 + 88 \cdot 15 + (|G| - (33+30+88)) \cdot 2 = 3775$, as wanted.

If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,11)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$. If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 11$ and $G = K$. Therefore G is an extension of $\mathbb{Z}_{11}$ by $\mathrm{PSL}(2,5)$. We know that $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$ and $(G/H)/(C_G(H)/H) \cong G/C_G(H)$. So G is a central extension of H by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{11} \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(\mathbb{Z}_{11})\psi(\mathrm{PSL}(2,5)) = 111 \cdot 211 = 23421$ and the result holds. If $K/H \cong \mathrm{PSL}(2,11)$, then $G \cong \mathrm{PSL}(2,11)$ and the proof is complete.

Step 2. Let $p = 13$.

In this case, $\psi(G) = 7281$ and $|G| = 1092 = 2^2 \cdot 3 \cdot 7 \cdot 13$. If G is a solvable group, then by Lemma 2.2, $n_7 = 1$ and $n_{13} = 1$. Therefore $s_{7\cdot 13} \neq 0$ and we have

$$\psi(G) \geq \psi(\mathbb{Z}_{91}) + (|G| - 91) \cdot 2 = 6751 + (1092 - 91) \cdot 2 = 8753,$$

as required.

If G is non-solvable, then $G \cong \mathrm{PSL}(2,13)$. This completes the proof.

Step 3. Let $p = 19$.

We know that $\psi(G) = 28843$ and $|G| = 3420 = 2^2 \cdot 3^2 \cdot 5 \cdot 19$. First, we show that if G is a solvable group, then $\psi(G) > 28843$.

If G is a solvable group, then there exists a Hall subgroup H of order $3^2 \cdot 5 \cdot 19$. By Lemma 2.2, we get that $n_5(H) = 1$. NC-theorem implies that P_5 is central in H, where $P_5 \in \mathrm{Syl}_5(H)$. Using Lemma 2.4, we have $\psi(H) = \psi(P_5)\psi(A)$, where A is a group of order $3^2 \cdot 19$. By [4], we get that $\psi(A) \geq 1483$. Hence

$$\psi(G) > \psi(H) \geq \psi(P_5)\psi(A) \geq 21 \cdot 1483 = 31143,$$

and we get the result. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,19)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 3 \cdot 19$ and $G = K$. We know that $H \cong \mathbb{Z}_{3\cdot 19}$ or $19 : 3$. If $H \cong \mathbb{Z}_{3\cdot 19}$, then G is a central extension of $\mathbb{Z}_{3\cdot 19}$ by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{3\cdot 19} \times \mathrm{PSL}(2,5)$. If $H \cong 19 : 3$, then by [4], $|\mathrm{Out}(19 : 3)| = 6$ and $Z(19 : 3) = 1$. NC-theorem implies that $C_G(H) \neq 1$. Therefore $H \cap C_G(H) = 1$ and it follows that $HC_G(H) \cong H \times C_G(H)$. Also $C_G(H) \cong HC_G(H)/H \trianglelefteq G/H \cong \mathrm{PSL}(2,5)$ which implies that $G \cong (19 : 3) \times \mathrm{PSL}(2,5)$. In both cases, G has a subgroup isomorphic to $\mathbb{Z}_{19} \times \mathrm{PSL}(2,5)$. Hence $\psi(G) > \psi(\mathbb{Z}_{19} \times \mathrm{PSL}(2,5)) = \psi(\mathbb{Z}_{19})\psi(\mathrm{PSL}(2,5)) = 72373$, and we get the result.

If $K/H \cong \mathrm{PSL}(2,19)$, then $G \cong \mathrm{PSL}(2,19)$ and the proof is complete.

Step 4. Let $p = 23$.

Let G be a group such that $\psi(G) = 61733$. By assumption $|G| = 6072 = 2^3 \cdot 3 \cdot 11 \cdot 23$. We claim that $\psi(G) > 61733$, when G is a solvable group. Let G be a solvable group. Then by Lemma 2.2, $n_3 \in \{1, 4\}$, $n_{11} \in \{1, 23\}$ and $n_{23} = 1$. By NC-theorem, we have $s_{69} \neq 0$ and $s_{46} \neq 0$. If $n_{11} = 1$, then $s_{11\cdot 23} \neq 0$ and we have

$$\psi(G) \geq \psi(\mathbb{Z}_{11\cdot 23}) + (|G| - 11 \cdot 23) \cdot 2 = 67915, \tag{4}$$

as required. If $n_{11} = 23$, then $s_{11} = 230$ and $|N_G(P_{11})| = 2^3 \cdot 3 \cdot 11$, where $P_{11} \in \mathrm{Syl}_{11}(G)$. By NC-theorem, we get that $s_{22} \neq 0$ and $s_{33} \neq 0$. Let Q_{22} and Q_{33} be cyclic subgroups of orders 22 and 33, respectively.

If $23 \mid |N_G(Q_{33})|$, then $23 \cdot 33 \in \omega(G)$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{23\cdot 33}) = 393939$. If $23 \nmid |N_G(Q_{33})|$, then $s_{33} = \phi(33) \cdot 23 \cdot k_{33}$, where $k_{33} \in \mathbb{N}$, and similarly $s_{22} = \phi(22) \cdot 23 \cdot k_{22}$, where $k_{22} \in \mathbb{N}$. If $k_{33} \geq 3$, then $\psi(G) \geq \psi(\mathbb{Z}_{69}) + \phi(33) \cdot 23 \cdot 3 \cdot 33 + \phi(22) \cdot 23 \cdot 22 + s_{11} \cdot 11 + (|G| - (69 + 20 \cdot 23 \cdot 3 + 10 \cdot 23 + 230)) \cdot 2 = 65005$, as required. We assume that $k_{33} \in \{1, 2\}$, therefore $s_{33} \in \{460, 920\}$. Since $n_3 \in \{1, 4\}$, therefore $s_3 \in \{2, 8\}$. On the other hand, by (2), we have $33 \mid (1 + s_3 + s_{11} + s_{33})$. Hence $s_3 = 2$ and $s_{33} = 460$. Therefore $|G : N_G(Q_{33})| = 23$ and $|N_G(Q_{33})| = 2^3 \cdot 3 \cdot 11$. By NC-theorem for Q_{33}, we get that $s_{66} \neq 0$. Let Q_{66} be a cyclic subgroup of order 66. If $23 \mid |N_G(Q_{66})|$, then $23 \cdot 66 \in \omega(G)$ and the result holds. If $23 \nmid |N_G(Q_{66})|$, then $s_{66} = \phi(66) \cdot 23 \cdot k_{66}$, where $k_{66} \in \mathbb{N}$. Hence

$$\begin{aligned}
\psi(G) &\geq \psi(\mathbb{Z}_{69}) + s_{66}66 + s_{33}33 + s_{22}22 + s_{11}11 \\
&\quad + 2(|G| - (69 + s_{66} + s_{33} + s_{22} + s_{11})) \\
&\geq 3549 + 20 \cdot 23 \cdot 66 + 460 \cdot 33 + 230 \cdot 22 + 230 \cdot 11 + 2(6072 - 1449) \\
&= 65925,
\end{aligned}$$

as required. If G is non-solvable, then by Lemma 2.3, $G \cong \mathrm{PSL}(2,23)$ and the proof is complete.

Step 5. Let $p = 29$.

We have $\psi(G) = 139317$ and $|G| = 12180 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$. If G is a solvable group, then by Lemma 2.2, $n_5 = 1$ and $n_{29} = 1$. By NC-theorem, $s_{3\cdot5\cdot29} \neq 0$ and so $\psi(G) \geq \psi(\mathbb{Z}_{3\cdot5\cdot29}) + (|G| - 3 \cdot 5 \cdot 29) \cdot 2 = 143001$, as required. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,29)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 7 \cdot 29$ and $G = K$. We know that $H \cong \mathbb{Z}_{7\cdot29}$ or $29 : 7$. If $H \cong \mathbb{Z}_{7\cdot29}$, then G is a central extension of $\mathbb{Z}_{7\cdot29}$ by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{7\cdot29} \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(\mathbb{Z}_{7\cdot29})\psi(\mathrm{PSL}(2,5)) = 34959 \cdot 211$, as required. If $H \cong 29 : 7$, then by [4], we get that $|\mathrm{Out}(H)| = 4$ and $Z(H) = 1$. NC-theorem implies that $C_G(H) \neq 1$. Therefore $H \cap C_G(H) = 1$ and it follows that $HC_G(H) \cong H \times C_G(H)$. Also $C_G(H) \cong HC_G(H)/H \trianglelefteq G/H \cong \mathrm{PSL}(2,5)$ which implies that $G \cong (29 : 7) \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(29 : 7)\psi(\mathrm{PSL}(2,5)) = 2031 \cdot 211$, as required. If $K/H \cong \mathrm{PSL}(2,29)$, then $G \cong \mathrm{PSL}(2,29)$ and the proof is complete.

Step 6. Let $p = 37$.

Note that $\psi(G) = 406335$ and $|G| = 25308 = 2^2 \cdot 3^2 \cdot 19 \cdot 37$. If G is a solvable group, then by Lemma 2.2, $n_{19} = 1$ and $n_{37} = 1$. By NC-theorem, we have $s_{19\cdot37} \neq 0$ and so $\psi(G) \geq \psi(\mathbb{Z}_{19\cdot37}) = 457219$, as required. If G is non-solvable, then by Lemma 2.3, $G \cong \mathrm{PSL}(2,37)$ and the proof is complete.

Step 7. Let $p = 61$.

Let G be a group such that $\psi(G) = 2760861$ and $|G| = 2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61 = 113460$. If G is a solvable group, then by Lemma 2.2, $n_{31} = 1$ and $n_{61} = 1$. By NC-theorem, we have $s_{31\cdot61} \neq 0$ and so

$$\psi(G) \geq \psi(\mathbb{Z}_{31\cdot61}) = 3408391, \tag{5}$$

as required. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,61)$. If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 31{\cdot}61$ and similarly to (5) we obtain the result. If $K/H \cong \mathrm{PSL}(2,61)$, then $G \cong \mathrm{PSL}(2,61)$ and the proof is complete. □

Lemma 3.2. *If G is a finite group with a normal subgroup N, then*

$$|L_n(G)| \leq |N||L_n(G/N)|. \tag{6}$$

Proof. The proof is straightforward. □

Theorem 3.3. *Let G be a finite group such that $|G| = 2448 = 2^4 \cdot 3^2 \cdot 17$ and $\psi(G) = 19483$. Then $G \cong \mathrm{PSL}(2,17)$.*

Proof. If G is a solvable group, then $C_G(\mathrm{Fit}(G)) \leq \mathrm{Fit}(G)$. By NC-theorem, we have

$$|G| \mid |\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))| \tag{7}$$

and by Lemma 2.2, we get that $n_{17} = 1$. Obviously,

$$\psi(\mathrm{Fit}(G)) + 2 \cdot (|G| - |\mathrm{Fit}(G)|) \leq \psi(G). \tag{8}$$

As $n_{17} = 1$, using (7) and (8), we have the following cases: $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{17}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. Now we consider each possibility:

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{17}$. As $n_3 \in \{1, 4, 16\}$, therefore $s_3 \leq 16 \cdot 8 = 128$. Also, since $O_2(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Sylow 2-subgroup of G, we have $s_2 = 15$ and $s_4 = 0$. Hence $\psi(G) \geq \psi(\mathrm{Fit}(G)) + 3 \cdot 128 + 6 \cdot (|G| - (|\mathrm{Fit}(G)| + 128)) = 21135$, which is a contradiction.

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. In Lemma 3.2, if we consider $N \cong O_2(G) \times O_{17}(G)$, then by [4], $|L_2(G)| \leq |N||L_2(G/N)| = 68 \cdot 19 = 1292$ and so $s_2 \leq 1291$. Similarly, for $N \cong O_2(G) \times O_3(G)$, we get that $s_3 \leq 107$. Therefore $\psi(G) \geq \psi(\mathrm{Fit}(G)) + 2 \cdot 1291 + 3 \cdot 107 + 4 \cdot (|G| - (|\mathrm{Fit}(G)| + 1291 + 107)) = 19664$, and we get a contradiction.

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. Let c be a number such that $c \mid |G|$, $s_{17c} \neq 0$ and $\gcd(c, 17) = 1$. If $c \geq 9$, then $\psi(G) \geq \psi(\mathbb{Z}_{17c}) + 2 \cdot (|G| - 17c)$, a contradiction. If $c = 8$, then $s_{8 \cdot 17} = s_{136} = \phi(136) k_{136}$, where $k_{136} \in \mathbb{N}$. If $k_{136} > 2$, then we get a contradiction. If $k_{136} \leq 2$, then $s_{136 \cdot 3} \neq 0$, a contradiction. Similarly, we get a contradiction for $c = 4$. If $c = 6$, then $s_{6 \cdot 17} = s_{102} = \phi(102) k_{102}$, where $k_{102} \in \mathbb{N}$. We assume that Q_{102} is a cyclic subgroup of order 102. If $k_{102} > 2$, then $\psi(G) \leq \psi(\mathrm{Fit}(G)) + 3 \cdot \phi(102) \cdot 102 + (|G| - (|\mathrm{Fit}(G)| + 3 \cdot \phi(102))) \cdot 2 = 21015$, which is a contradiction. If $k_{102} \leq 2$, then $\mathrm{Fit}(N_G(Q_{102})) \cong O_2(N_G(Q_{102})) \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$, where $O_2(N_G(Q_{102})) \neq 1$. Therefore $\psi(G) \geq \psi(N_G(Q_{102})) \geq \psi(\mathrm{Fit}(N_G(Q_{102}))) \geq 3 \cdot 25 \cdot 273 = 20475$, which is a contradiction.

Now if $c = 3$, then by Lemma 3.2, $|L_{51}(G)| \leq |\mathrm{Fit}(G)||L_{51}(G/\mathrm{Fit}(G))| = 153$ and since $s_{51}(G) = 32 \cdot k_{51} \geq s_{51}(\mathrm{Fit}(G)) = 128$, we get that $s_{51}(G) = 128$. Also, we know that $s_3 = 8, s_{17} = 16$.

For $c = 2$, we claim that $s_{34} = 0$ or 144. Let $s_{34} \neq 0$. We assume that Q_{34} is a cyclic subgroup of order 34. Since we proved that $s_{102} = 0$, so we get that $3 \nmid |N_G(Q_{34})|$, and so $s_{34} = \phi(34) \cdot 9 \cdot k_{34}$, where $k_{34} \in \mathbb{N}$. If $k_{34} > 1$, then we get a contradiction. Therefore $s_{34} = 0$ or $s_{34} = 144$. Similarly we get that $s_d = \phi(d) \cdot 17 \cdot k_d$, where $d \in \omega(G)$, $\gcd(d, 17) = 1$ and $d \notin \{1, 2, 3\}$.

By (1) and the above restrictions, we have the following system of equations:

$$\begin{cases} 1 + s_2 + 8 + 16 + s_{34} + 128 + 34 \cdot t_1 = 2448, \\ 1 + 2 \cdot s_2 + 3 \cdot 8 + 17 \cdot 16 + 34 \cdot s_{34} + 51 \cdot 128 + 2 \cdot 34 \cdot t_2 = 19483. \end{cases}$$

The solution of these equations shows that $t_2 - t_1 \notin \mathbb{Z}$.

Hence, we showed that if G is solvable, then $\psi(G) \neq 19483$. If G is a non-solvable group, then by Lemma 2.3, $G \cong \mathrm{PSL}(2, 17)$. This completes the proof. □

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References

[1] H. Amiri and S. M. Jafarian Amiri, *Sum of element orders on finite groups of the same order,* J. Algebra Appl., 10 (2011), 187–190.

[2] H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs, *Sums of element orders in finite groups,* Comm. Algebra, 37 (2009), 2978–2980.

[3] G. Frobenius, *Verallgemeinerung des Sylow'schen Satzes*, volume 1895, Königlich Preussische Akademie der Wissenschaften, Berlin, 1895.

[4] GAP Group et al, *Gap–groups, algorithms, and programming,* version 4.7. 5; 2014, 2012.

[5] P. Hall, *A note on soluble groups, J. Lond. Math. Soc.*, 3 (1928), 98–105.

[6] M. Herzog, P. Longobardi and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*, J. Pure Appl. Algebra, 2017.

[7] S. M. Jafarian Amiri, *Characterization of* A_5 *and* $\mathrm{PSL}(2,7)$ *by sum of element orders,* Int. J. Group Theory, 2 (2013), 35–39.

[8] S. M. Jafarian Amiri, *Second maximum sum of element orders of finite nilpotent groups,* Comm. Algebra, 41 (2013), 2055–2059.

[9] S. M. Jafarian Amiri and M. Amiri, *Second maximum sum of element orders on finite groups,* J. Pure Appl. Algebra, 218 (2014), 531–539.

[10] S. M. Jafarian Amiri and M. Amiri, *Sum of the element orders in groups of the square-free orders,* Bull. Malays. Math. Sci. Soc., 1–10, 2016.

[11] Y. Marefat, A. Iranmanesh and A. Tehranian, *On the sum of element orders of finite simple groups,* J. Algebra Appl., 12 (2013), 4 pages.

[12] R. Shen, G. Chen and C. Wu, *On groups with the second largest value of the sum of element orders,* Comm. Algebra, 43 (2015), 2618–2631.

[13] H. Xu, G. Chen and Y. Yan, *A new characterization of simple* K_3*-groups by their orders and large degrees of their irreducible characters,* Comm. Algebra, 42 (2014), 5374–5380.

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The non-coprime graph of a finite group with respect to a subgroup

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Abstract. The non-coprime graph $\Pi_{G,H}$ of a finite group G with respect to a subgroup H is a simple graph with $G \setminus \{e\}$ as the vertex set. Two vertices u and v of this graph are adjacent if and only if $(|u|, |v|) \neq 1$ and ($u \in H$ or $v \in H$). In this paper, the main properties of this graph are obtained. The isomorphism problem of this graph is also investigated for nilpotent groups.
Keywords: non-coprime graph, nilpotent group, Mathieu group.

1. Introduction

Throughout this paper, we will consider only simple graphs which are undirected, without loops and multiple edges. For any graph Π, the sets of all vertices and edges of Π are denoted by $V(\Pi)$ and $E(\Pi)$, respectively. We also denote the order and the length of a shortest cycle in Π by $|V(\Pi)|$ and girth(Π), respectively. The last quantity is called the girth of Π.

The study of an algebraic structures is a part of algebraic graph theory. In this topic, the researchers associate a graph structure to an algebraic system

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and then study the relationship between them. In this paper, a graph structure $\Pi_{G,H}$ will be associated to a pair (G, H) of a group G and a subgroup H of G. Then we will prove that some important algebraic properties of this pair has a graph theory meaning. In an exact phrase, the non-coprime graph $\Pi_{G,H}$ of G with respected to H is a simple graph with $G \setminus \{e\}$ as its vertex set and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) \neq 1$ and ($x \in H$ or $y \in H$). If $G = H$ then we use the notation Π_G as $\Pi_{G,G}$. The last graph first studied by Ma, Wei and Yang. [4]. In the mentioned paper, it was proved that this graph has diameter ≤ 2 and its automorphism group is studied in some special cases that G is cyclic or a $p-$group. In [2], Dorbidi proved that the chromatic and clique numbers of this graph are equal. He also classified all the groups for which this graph is complete $r-$partite or planar.

The non-coprime graph of a finite group G is a simple graph Π_G with vertex set $G \setminus \{e\}$ such that two distinct vertices are adjacent whenever their orders are relatively non-coprime. This graph was introduced by Mansoori, Erfanian and Tolue [5]. In the mentioned paper, the authors computed some graph invariants like diameter, girth, dominating number, independence and chromatic numbers. They also proved that the non-coprime graph of a group G is planar if and only if $|G| < 7$. The non-coprime graph of a group G with respect to a subgroup H, $\Pi_{G,H}$, is a generalization of the non-coprime graph Π_G presented in [5].

For a graph Π and a subset S of the vertex set $V(\Pi)$, we denote $N_\Pi[S]$ to be the set of all vertices in Π which are in S or adjacent to a vertex in S. If $N_\Pi[S] = V(\Pi)$, then S is said to be a dominating set in Π. The domination number of Π, $\gamma(\Pi)$, is the minimum size of a dominating set in Π. A planar graph is a graph that can be embedded in the plane so that two edges intersect geometrically except at a vertex which both are incident. The Kuratowski's theorem states that a graph is planar if and only if it contains no subgraph isomorphic with a subdivision of K_5 or $K_{3,3}$. A clique in the graph Π is a complete subgraph of Π and the number of vertices in a largest clique of Π is called the clique number of Π, denoted by $\omega(\Pi)$. An independence set of Π is a subset of the vertex set of Π such that no two vertices are adjacent in Π. The independence number of Π, denoted by $\alpha(\Pi)$, is the cardinality of a largest independent set in Π. The minimum number $\chi(\Pi)$ of colors which can be assigned to the vertices of Π in such a way that every two adjacent vertices have different colors is said to be the chromatic number of Π. The covering number of Π, denoted by $\theta(\Pi)$, is the minimum number of the cliques of Π that covering the vertex set of Π.

In this paper, our notations are almost standard and taken mainly from [1, 3] and our aim is to extend some results of [6, 7] to non-coprime graph of a group with respect to a subgroup.

2. Non-coprime graph of Mathieu groups

In this section the non-coprime graph of all Mathieu groups are obtained. The number of elements which is divisible by a prime divisor p of a group G is denoted by $m_p(G)$.

We start by the Mathieu group M_{11}. We first note that this is a group of order $8920 = 2^4 \times 3^2 \times 5 \times 11$. Apply Gap [8] to calculate the orders of elements and the number of elements of each order in M_{11}. These numbers are:

$$[[1, 1], [2, 165], [3, 440], [4, 990], [5, 1584], [6, 1320], [8, 1980], [11, 1440]].$$

Our calculations show that this graph has exactly three components. The first and second components are complete graph of orders 1584 and 1440, respectively. The orders of elements in the third component are $2, 3, 4, 6$ and 8. By our calculations, $m_2 = 4455$, $m_3 = 1760$, $m_5 = 1584$ and $m_{11} = 1440$. It can be calculated by Gap that the diameter of the third connected components is 2. We record our calculations in the following proposition:

Proposition 2.1. *The non-coprime graph of the Mathieu group M_{11} has three components K_{1440}, K_{1584} and a subgraph of diameter 2 with elements of orders $2, 3, 4, 6, 8$. Moreover, $\Pi_{M_{11}}$ has minimum degree 1439, maximum degree 4894, clique number 4895 and $\chi(\Pi_{M_{11}}) = \omega(\Pi_{M_{11}}) = 4895$.*

The Mathieu group M_{12} has order $95040 = 2^6 \times 3^2 \times 5 \times 11$. Furthermore, the orders of elements and the number of elements of each order in the Mathieu group M_{12} are as follows $[[1, 1], [2, 891], [3, 4400], [4, 5940], [5, 9504], [6, 23760], [8, 23760], [10, 9504], [11, 17280]]$. Thus, the number of elements that are divisible by $2, 3, 5, 11$ are

$$[[2, 63855], [3, 28160], [5, 19008], [11, 17280]].$$

Hence this graph has two connected components which one of them is isomorphic to K_{17280} and the second one is a subgraph with elements of orders $2, 3, 4, 5, 6, 8, 10$. The elements of order 10 have maximum degree and the elements of order 11 have minimum degree. It is obvious that the diameter of the second component is 3, the clique number of this component is 63855 and since $17280 < 63855$, $\omega(\Pi_{M_{12}}) = 63855$. We record our calculations in the following proposition:

Proposition 2.2. *The non-coprime graph $\Pi_{M_{12}}$ has two connected components, $\delta(M_{12}) = 17279$, $\Delta(M_{12}) = 73359$, one of the component is isomorphic to K_{17280} and another component has diameter 3. Moreover, $\chi(\Pi_{M_{12}}) = \omega(\Pi_{M_{12}}) = 63855$.*

We are now ready to calculate the graph invariants related to the non-coprime graph of the Mathieu group M_{22} of order $443520 = 2^7 \times 3^2 \times 5 \times 7 \times 11$. The order of elements and the number of each element in M_{22} are as follows:

$[[1,1],[2,1155],[3,12320],[4,41580],[5,88704],[6,36960],[7,126720],[8,55440],[11,80640]]$.

To do this, we first notice that the number of elements that can be divided by $2,3,5,11$ are

$$[[2,135135],[3,49280],[5,88704],[7,126720],[11,80640]].$$

So, this graph has four connected components K_{88704} , K_{126720} , K_{80640} and a subgraph with elements of orders $2,3,4,6,8$. The maximum degree are attained by elements of order 6 and the minimum degree of vertices are related to elements of order 11. It is obvious that the diameter of non-complete component is 2 and by [1, Theorem 1.7.2], this component is a line graph. Since the clique number of non-complete component of M_{12} is 135135 and by a simple calculation $\omega(\Pi_{M_{22}}) = 135135$. We record our calculations in the following proposition:

Proposition 2.3. *The non-coprime graph $\Pi_{M_{22}}$ has four connected components which three of them are complete graphs of orders* $88704, 126720, 80640$*, respectively. The last component has diameter four,* $\delta(M_{22}) = 49279$*,* $\Delta(M_{22}) = 147454$*, the fourth component has diameter* 3*,* $\chi(\Pi_{M_{22}}) = \omega(\Pi_{M_{22}}) = 135135$*.*

By [8], $| M_{23} | = 10200960 = 2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$. The orders of elements and the number of elements of each orders in the Mathieu group M_{23} are:

$[[1,1],[2,3795],[3,56672],[4,318780],[5,680064],[6,850080],$
$[7,1457280],[8,1275120],[11,1854720],[14,1457280],[15,1360128],[23,887040]]$

and number of elements that their orders are divisible by $2,3,5,7,11,23$ are

$$[[2,3905055],[3,2266880],[5,2040192],[7,2914560],[11,1854720],[23,887040]].$$

Therefore, this graph has three connected components $K_{1854720}$, K_{887040} and a subgraph with elements of orders $2,3,4,5,6,7,8,14,15$. Hence we prove the following proposition:

Proposition 2.4. *The non-coprime graph $\Pi_{M_{23}}$ has three component with minimum degree* 887039 *and maximum degree* 5362334*. This graph has two complete components $K_{1854720}$ and K_{887040} and the non-complete component has diameter* 3*. Moreover,* $\chi(\Pi_{M_{23}}) = \omega(\Pi_{M_{23}}) = 3905055$*.*

By similar calculations by Gap such as our calculations in Propositions 2.1, 2.2, 2.3 and 2.4, one can easily prove the following result:

Proposition 2.5. *The non-coprime graph $\Pi_{M_{24}}$ has three connected components. Two complete of these components are isomorphic to the complete graphs $K_{21288960}$ and $K_{22256640}$ and the third one is a non-complete graph of diameter* 3*. The minimum and maximum degree of vertices in this graph are* 21288959 *and* 247003646*, respectively. Moreover,* $\chi(\Pi_{M_{24}}) = \omega(\Pi_{M_{24}}) = 128866815$*.*

3. Non-coprime graphs of nilpotent groups

Suppose G is a finite group of order $n = p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$, where $p_i, 1 \le i \le r$, are distinct primes. The number of elements of orders divisible by $p_{i_1}, p_{i_2}, \dots, p_{i_t}$ which is not divisible by p_j, $1 \le j \le r$ and $j \ne i_1, i_2, \dots, i_t$, is denoted by $\Delta_G(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j)$. The aim of this section is to study the main properties of the non-coprime graph. We first prove a result for nilpotent groups which is crucial throughout this paper.

Theorem 3.1. *Let G be a nilpotent group of order $n = p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$, where $p_i, 1 \le i \le r$, be distinct primes. Then $\Delta_G(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j) = \Delta_{Z_n}(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j)$.*

Proof. Suppose G is a nilpotent group, P_{i_1}, P_{i_2}, …, P_{i_t} are Sylow subgroups of G and $H = P_{i_1}P_{i_2}\dots P_{i_t}$. It is enough to prove that $x \in H \setminus (P_{i_2}P_{i_3}\dots P_{i_t} \cup P_{i_1}P_{i_3}\dots P_{i_t} \cup \dots \cup P_{i_1}P_{i_2}\dots P_{i_{t-1}})$ if and only if $p_{i_1}p_{i_2}\dots p_{i_t} \mid o(x)$ and for each j, $j \ne i_1, i_2, \dots, i_t$ and $1 \le j \le r$, we have $p_j \nmid o(x)$. We first note that if $p_{i_1}p_{i_2}\dots p_{i_t} \mid o(x)$ and $p_j \nmid o(x)$ then $x \notin P_{i_2}P_{i_3}\dots P_{i_t}$, $x \notin P_{i_1}P_{i_3}\dots P_{i_t}$, …, $x \notin P_{i_1}P_{i_2}\dots P_{i_{t-1}}$ which shows that

$$x \in H \setminus (P_{i_2}P_{i_3}\dots P_{i_t} \cup P_{i_1}P_{i_3}\dots P_{i_t} \cup \dots \cup P_{i_1}P_{i_2}\dots P_{i_{t-1}}).$$

On the other hand, we assume that $x \in H \setminus (P_{i_2}P_{i_3}\dots P_{i_t} \cup P_{i_1}P_{i_3}\dots P_{i_t} \cup \dots \cup P_{i_1}P_{i_2}\dots P_{i_{t-1}})$. Then $x \in H$ and $x \notin P_{i_2}P_{i_3}\dots P_{i_t} \cup P_{i_1}P_{i_3}\dots P_{i_t} \cup \dots \cup P_{i_1}P_{i_2}\dots P_{i_{t-1}}$. This proves that $p_{i_1}p_{i_2}\dots p_{i_t} \mid o(x)$ and $p_j \nmid o(x)$.

On the other hand,

$$\begin{aligned}
t &= |H \setminus (P_{i_2}P_{i_3}\dots P_{i_t} \cup P_{i_1}P_{i_3}\dots P_{i_t} \cup \dots \cup P_{i_1}P_{i_2}\dots P_{i_{t-1}})| \\
&= |H| - (|P_{i_2}P_{i_3}\dots P_{i_t}| + |P_{i_1}P_{i_3}\dots P_{i_t}| + \dots + |P_{i_1}P_{i_2}\dots P_{i_{t-1}}| - |P_{i_3}P_{i_4}\dots P_{i_t}| \\
&\quad - |P_{i_1}P_{i_4}\dots P_{i_{t-1}}| - \dots - |P_{i_1}P_{i_2}\dots P_{i_{t-2}}| + 1) \\
&= p_{i_1}^{k_{i_1}}p_{i_2}^{k_{i_2}}\dots p_{i_t}^{k_{i_t}} - p_{i_2}^{k_{i_2}}p_{i_3}^{k_{i_3}}\dots p_{i_t}^{k_{i_t}} - p_{i_1}^{k_{i_1}}p_{i_3}^{k_{i_3}}\dots p_{i_t}^{k_{i_t}} - \dots - p_{i_1}^{k_{i_1}}p_{i_2}^{k_{i_2}}\dots p_{i_{t-1}}^{k_{i_{t-1}}} \\
&\quad + p_{i_3}^{k_{i_3}}p_{i_4}^{k_{i_4}}\dots p_{i_t}^{k_{i_t}} + p_{i_1}^{k_{i_1}}p_{i_4}^{k_{i_4}}\dots p_{i_t}^{k_{i_t}} + \dots + p_{i_1}^{k_{i_1}}p_{i_2}^{k_{i_2}}\dots p_{i_{t-2}}^{k_{i_{t-2}}} - 1 \\
&= (p_{i_1}^{k_{i_1}} - 1)(p_{i_2}^{k_{i_2}} - 1)\dots(p_{i_t}^{k_{i_t}} - 1)
\end{aligned}$$

For every j, $1 \le j \le t$, we define:

$$A_{p_{i_j}} = \left\{ \frac{n}{{p_{i_j}}^{k_{i_j}}}, \frac{2n}{{p_{i_j}}^{k_{i_j}}}, \frac{3n}{{p_{i_j}}^{k_{i_j}}}, \dots, \frac{({p_{i_j}}^{k_{i_j}} - 1)n}{{p_{i_j}}^{k_{i_j}}} \right\}.$$

Then $|A_{p_{i_1}} + A_{p_{i_2}} + \dots + A_{p_{i_t}}| = (p_{i_1}^{k_{i_1}} - 1)(p_{i_2}^{k_{i_2}} - 1)\dots(p_{i_t}^{k_{i_t}} - 1)$, proving the result. □

It is clear that if G_1 and G_2 are isomorphic then $\Pi_{G_1} \cong \Pi_{G_2}$. In what follows the converse of this result will considered into account.

Theorem 3.2. *Let G_1 and G_2 be finite nilpotent groups. Then $\Pi_{G_1} \cong \Pi_{G_2}$ if and only if $|G_1| = |G_2|$.*

Proof. If $\Pi_{G_1} \cong \Pi_{G_2}$ then by definition of non-coprime graph one can easily be seen that $|G_1| = |G_2|$. Suppose G is an arbitrary nilpotent group of order n and $n = |G| = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where p_i, $1 \leq i \leq r$, are distinct prime numbers. We will prove that $\Pi_G \cong \Pi_{Z_n}$. Choose $P_i, 1 \leq i \leq r$, to be the Sylow p_i-subgroup of G. Since G is nilpotent, $G \cong P_1 \times P_2 \times \ldots \times P_r$. Define:

$$\begin{aligned}
A_{P_i} &= \{(e, e, \ldots, a_i, e, \ldots, e) \mid a_i \in P_i, a_i \neq e\},\\
B_{P_i} &= \left\{ \frac{n}{{p_i}^{k_i}}, \frac{2n}{{p_i}^{k_i}}, \frac{3n}{{p_i}^{k_i}}, \ldots, \frac{({p_i}^{k_i} - 1)n}{{p_i}^{k_i}} \right\},\\
A_{P_iP_j} &= \{(e, e, \ldots, a_i, e, \ldots, e, a_j, e, \ldots, e) \mid a_i \in P_i, a_j \in P_j, a_i \neq e, a_j \neq e\},\\
B_{P_iP_j} &= B_{p_i} + B_{p_j} = \{x + y \mid x \in B_{p_i}, y \in B_{p_j}\},\\
&\vdots = \vdots\\
A_{P_1P_2\ldots P_r} &= \{(a_1, a_2, \ldots, a_r) \mid 1 \leq i \leq r-1; a_i \in P_i; a_i \neq e\},\\
B_{P_1P_2\ldots P_r} &= B_{p_1} + B_{p_2} + \ldots + B_{p_r}.
\end{aligned}$$

Define $\varphi : v(\Pi_{P_1 \times P_2 \times \ldots \times P_r}) \longrightarrow v(\Pi_{Z_n})$ by $\varphi(a_{1_{j_1}}, a_{2_{j_2}}, \ldots, a_{r_{j_r}}) = j_1 \frac{n}{{p_1}^{k_1}} + j_2 \frac{n}{{p_2}^{k_2}} + \ldots j_r \frac{n}{{p_1}^{k_1}}$, where $a_{i_{j_i}} \in P_i$, $1 \leq j_i \leq {p_i}^{k_i}$ and if $a_{i_{j_i}} = 0$ then $j_i = 0$. A simple argument shows that φ is a graph isomorphism, as desired. □

Corollary 3.3. *Suppose G_1 and G_2 are nilpotent groups of the same order, $H_1 \leq G_1$ and $H_2 \leq G_2$. Then $\Pi_{G_1,H_1} \cong \Pi_{G_2,H_2}$ if and only if $|H_1| = |H_2|$.*

Theorem 3.4. *Suppose G is a finite $p-$group and $\{e\} \neq H \leq G$. Then,*

1) *$\Pi_{G,H}$ is connected,*

2) *$\mathrm{diam}\Pi_{G,H} \leq 2$ and $\Pi_{G,H}$ is a complete graph if and only $H = G$.*

Proof. Suppose G has order p^k, where $k \in \mathbb{N}$. We first notice that $V(\Pi_{G,H}) = V(\Pi_G)$. By definition of $\Pi_{G,H}$, Π_H is an induced subgraph of $\Pi_{G,H}$ and all vertices of $V(H \setminus \{e\})$ are adjacent with all vertices of $V(G \setminus H)$. Furthermore, there is no edge connecting vertices of $V(G \setminus H)$. If $x, y \in V(G)$, then we consider the following three cases:

a *$x \in H$ and $y \in G \setminus H$.* Since $|x|, |y| \mid p^k$, x and y are adjacent;

b *$x \in H$ and $y \in H$.* Since $H \cong K_{p^h - 1}$, $d(x, y) = 1$;

c *$x \in G \setminus H$ and $y \in G \setminus H$.* Then $x \nsim y$ and by definition of $\Pi_{G,H}$, for all $z \in H$, $x \sim z$, $y \sim z$ and so $d(x, y) = 2$.

Thus $\Pi_{G,H}$ is connected and $\mathrm{diam}\Pi_{G,H} \leq 2$. Now it is clear that $\Pi_{G,H}$ is complete if and only if $G = H$. □

Theorem 3.5. *Suppose G is a nilpotent group of order $n = p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$ and H is a subgroup of G of order $h = p_1^{h_1}p_2^{h_2}\dots p_r^{h_r}$, where $p_1, p_2, \dots, p_r$ are distinct prime numbers, $h_i, k_i \in \mathbb{N}$ with $1 \leq i \leq r$. Then,*

1) *$\Pi_{G,H}$ is connected and* $\mathrm{diam}\Pi_{G,H} \leq 2$.

2) $\gamma(\Pi_{G,H}) = 1$

Proof. We consider each part separately.

1) By Theorem 3.2, $\Pi_H \cong \Pi_{Z_n}$ and by [5, Theorem 2.4], Π_H is a connected graph with $\mathrm{diam}\Pi_H \leq 2$. We now consider the following three cases.

   a $x, y \in H$. If $x \nsim y$ then there is $z \in H$ such that $x \sim z \sim y$;

   b $x \in H$, $y \in G \setminus H$ and $x \nsim y$. Then there is an element $x_1 \in V(H)$ such that $p_1p_2\dots p_r$ divides $\mid x_1 \mid$ and so $x \sim x_1 \sim y$;

   c $x \in G \setminus H$ and $y \in G \setminus H$. So, there is an element $x_1 \in V(H)$ of order $p_1p_2\dots p_r$, where $x \sim x_1 \sim y$.

   This proves that $\mathrm{diam}\Pi_{G,H} \leq 2$.

2) Since there is an element of order $p_1p_2\dots p_r \in H$ as x_1 that is adjacent to all vertices of $\Pi_{G,H}$, $\gamma(\Pi_{G,H}) = 1$, as desired.

Hence the result. □

Theorem 3.6. *Let G be a finite group of order n, $n \geq 4$, and $H \leq G$ with $|H| > 2$. Then* $\mathrm{girth}(\Pi_{G,H}) = 3$.

Proof. If $|H| \geq 4$ then by [5, Theorem 2.2], $\mathrm{girth}\Pi_{G,H} = 3$. If $|H| = 3$ then $\Pi_H \cong K_2$, where each element of $V(H)$ has order 3. Since there exists at least an element a of order 3 in $G \setminus H$ such that $3 \big| |a|$, $a \sim x$, $a \sim y$ and $x \sim y$. Hence $\mathrm{girth}(\Pi_{G,H}) = 3$. □

Corollary 3.7. *Let G be a group and $H \leq G$ has order 2. Then $\Pi_{G,H}$ is acyclic and if $\mid G \mid = 2^k$ then $\Pi_{G,H}$ is tree.*

Theorem 3.8. *Let G be a nilpotent group of order ≥ 4, $\{e\} \neq H \leq G$ and $H \ncong Z_2$. If $\Pi_{G,H}$ is connected then it has Eulerian spanning subgraph.*

Proof. We first prove that if H is a subgroup of G of order $\neq 2$ then Π_H has Eulerian subgraph. It is a well-known fact in graph theory that if each edge of Π_H belong to a triangle then Π_H has Eulerian spanning subgraph. Let H be a subgroup of G of order $h = p_1^{h_1}p_2^{h_2}\dots p_r^{h_r}$, where p_i's are primes and r_i's are non-negative integers. It is enough to prove that Π_{Z_h} has Eulerian subgraph. If h is a prime power then by Theorem 3.4 Π_H is complete and so each edge of Π_G belong to a triangle. Suppose $r \geq 2$. If e is an edge of Π_G then we will consider the following three cases:

a) $e = \{x, y\}$ *and* $\langle x\rangle, \langle y\rangle \neq Z_h$. Then there exists an element $z \in Z_h$ such that $\langle z\rangle = H$. Thus $x \sim z$ and $y \sim z$, as desired. Choose generators $a, b \in Z_h$. Then a and b are adjacent to all vertices of Π_{Z_H} that leads to our result.

b) $e = \{x, y\}$, $\langle x\rangle = Z_h$ *and* $\langle y\rangle \neq Z_h$. Choose another generator z for Z_h. By definition, $z \sim x$ and $z \sim y$ and so the edge e belongs to a triangle.

c) $e = \{x, y\}$, $Z_h = \langle x\rangle\ \langle y\rangle$. Since each generator of Π_{Z_h} is adjacent to all vertices and $h \geq 4$, by choosing an arbitrary non-identity element $z \in Z_h$, we have $z \sim x$ and $z \sim y$. Hence e belongs to a triangle.

This proves that the graph Π_G has Eulerian spanning subgraph. Next we assume that $Z_2 \ncong H < G$ and choose an edge $e = \{x, y\}$ of $\Pi_{G,H}$. If $x, y \in V(H)$ then $e \in \Pi_H$ and so it is an edge of a triangle, and if $x \in V(H)$, $y \in V(G)\backslash V(H)$ then there exists an element $z \in H$ which is adjacent to all vertices of $V(G)$. This completes our argument. □

Theorem 3.9. *Let G be a group and $H \leqslant G$. $\Pi_{G,H}$ is planar if and only if $|G| < 7$ or $|G| \geq 7$ and $H \cong Z_2$.*

Proof. If $G = H$ Then by [5, Theorem 2.11], Π_G is planar if and only if $|G| < 7$. Suppose $H \neq G$. If $|G| < 7$ then $\Pi_{G,H}$ is a subgraph of a planar graph and so it is planar. In the case that $|G| \geq 7$ and $H \cong Z_2$, $\Pi_{G,H}$ is a subgraph of a star graph and so it is planar.

Next we assume that $\Pi_{G,H}$ is planar. Since Π_H is an induced subgraph of $\Pi_{G,H}$, $|H| < 7$. We now consider the following three cases for the subgroup H:

1) $|H| = 2$. In this case $\Pi_{G,H}$ is a subgraph of a star graph and the result is obvious.

2) $|H| = 3$. If $|G| = 3$ or 6 then by [5, Theorem 2.11], $\Pi_{G,H}$ is planar, and if $|G| > 6$ then we have at least three vertices in $G \setminus H$ adjacent to all vertices of H. Hence $K_{3,3} \leq \Pi_{G,H}$ which is not possible.

3) $|H| = r, r \geq 4$. Then $|G| > 6$ and there exist at least r vertices in $G \setminus H$ adjacent to all vertices in H. Again $\Pi_{G,H}$ has a subgraph isomorphic to $K_{3,3}$, that is impossible.

Hence the result. □

Theorem 3.10. *$\Pi_{G,H}$ has perfect matching if and only if $G = H$ or $|G|$ is odd.*

Proof. Suppose $H < G$. Since $|V(\Pi_H)| < |V(G \setminus H)|$, there is not perfect matching in $\Pi_{G,H}$. So $H = G$ and the proof follows from [5, Theorem 2.12]. □

Theorem 3.11. *If G is a non-trivial $p-$group and $H \leq G$ then $\Pi_{G,H}$ is not Eulerian.*

Proof. Suppose $|G| = p^n$, where n is a positive integer. If $p = 2$ then each vertex of $G \setminus H$ is adjacent to all vertices of $V(\Pi_H)$. Since the number of such vertices are $2^n - 1$, $\Pi_{G,H}$ is not Eulerian. Next we assume that p is odd. Then $|V(G \setminus H)| = p^n - p^k$, where $|H| = p^k$. Since Π_H is a complete graph isomorphic to K_{p^k-1}, each vertex in $V(H)$ is adjacent to $p^k - 2$ vertices in H and all vertices in $G \setminus H$. Hence each vertex in $V(H)$ is adjacent to $p^k - 2 + p^n - p^k = p^n - 2$ vertices in $\Pi_{G,H}$. Hence $\Pi_{G,H}$ cannot be Eulerian. □

Theorem 3.12. *Let G be a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where p_i's are distinct primes, r_i's are positive integers, $p_r^{k_r} < \dots < p_2^{k_2} < p_1^{k_1}$ and put*

$$a_1 = \sum_{\alpha_1 \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = \frac{n(p_r^{k_r} - 1)}{p_r^{k_r}}$$

$$a_2 = \sum_{\forall i_1 \neq i_2 \in \{1,2,3\},\ \alpha_{i_1}\alpha_{i_2} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})$$

$$a_3 = \sum_{\forall i_1 \neq i_2 \neq i_3 \in \{1,2,3,4,5\},\ \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})$$

$$\vdots$$

$$a_{[\frac{r}{2}]+1} = \sum_{\forall i_1 \neq i_2 \dots\dots \neq i_{[\frac{r}{2}]+1} \in \{1,2,\dots,r\},\ \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_{[\frac{r}{2}]+1}} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}).$$

Then $w(\Pi_G) = max\{a_1, a_2, \dots, a_{[\frac{r}{2}]+1}\}$.

Proof. Let A_1 be the set of all elements of G that p_1 divides its order; A_2 be the set of all elements of G that p_1p_2, p_1p_3 or p_2p_3 divides its order; …; A_t be the set of all elements of G that at least a product of t distinct primes in $\{p_1, p_2, p_3, \dots p_{2t-1}\}$ divides its order, $1 \leq t \leq [\frac{r}{2}] + 1$. We also assume that $|A_i| = a_i$. It is now obvious that $w(\Pi_G) = max\{a_1, a_2, \dots, a_{[\frac{r}{2}]+1}\}$, as desired. □

Corollary 3.13. *Let G be a nilpotent group, $\{e\} \neq H < G$, $n_1 = |H| = p_1^{h_1} p_2^{h_2} \dots p_s^{h_s}$ be the prime factorization of n_1 and $p_1^{h_1} < p_2^{k_2} < \dots < p_s^{h_s}$. Then $\omega(\Pi_{G,H}) = \omega(\Pi_H) + 1$.*

Proof. Since $H \neq G$, there exists $g \in G \setminus H$ such that $p_1p_2 \dots p_r \big| |g|$. Thus, all vertices of $V(\Pi_H)$ are adjacent to g. Since the induced subgraph of $G \setminus H$ is empty, $\omega(\Pi_{G,H}) = \omega(H) + 1$ and the proof follows from Theorem 3.12. □

Theorem 3.14. *Let G be a group of order n containing a non-trivial subgroup H of order n_1. Then $\alpha(\Pi_G) = \pi(|G|)$ and if $G \neq H$, $\alpha(\Pi_{G,H}) = n - n_1$.*

Proof. The case of $H = G$ follows from [5, Theorem 2.10]. Suppose $H \neq G$. Then by definition of $\Pi_{G,H}$, there is no edge connecting two vertices of $G \setminus H$ and so $\alpha(\Pi_{G,H}) \leq n - n_1$. On the other hand, for each vertex h of H, there exists a vertex in $G \setminus H$ adjacent to h. Hence $\alpha(\Pi_{G,H}) = n - n_1$. □

Theorem 3.15. *Let G be a group and $\{e\} \neq H \leqslant G$. If $G = H$ then $\theta(\Pi_G) = \alpha(\Pi_G) = \pi(|G|)$ and if $G \neq H$ and $\Pi_{G,H}$ is connected then $\theta(\Pi_{G,H}) = \pi(|H|) + (|G| - |H|)$.*

Proof. Suppose $G = H$. Then the size of a minimum clique covering vertices of Π_G is equal to $\pi(|G|)$. Hence $\theta(\Pi_G) = \alpha(\Pi_G)$. If $G \neq H$ then the size of a minimum clique covering all vertices of $\Pi_{G,H}$ is the sum of the size of a minimum clique covering all vertices of $V(\Pi_H)$ and the size of a minimum clique that covers all vertices of $G \setminus H$. If $G \neq H$ and $\Pi_{G,H}$ is connected then each vertex of $G \setminus H$ is adjacent to at least a vertex of H and there is no edge connecting vertices of $V(G \setminus H)$. Hence $\theta(\Pi_{G,H}) = \pi(H) + (|G| - |H|)$. □

Theorem 3.16. *Let G be a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$. Then $\chi(\Pi_G) = \omega(\Pi_G)$.*

Proof. It is well-known that $\chi(\Pi_G) \geq \omega(\Pi_G)$ and so it is enough to prove that $\chi(\Pi_G) \leq \omega(\Pi_G)$. Let $p_r^{k_r} < \dots < p_2^{k_2} < p_1^{k_1}$ and for $i \in \{1, 2, r\}$ define a subset of G that p_i divides its order by B_i and let A_i, $i \in \{1, 2, \dots [\frac{r}{2}] + 1\}$, are subset of G that are defined in Theorem 3.12. If $\omega(\Pi_G) = |B_1| = |A_1|$, since $|B_1| > |B_2| > \dots |B_r|$ by definition of Π_G vertices of Π_G are colorable with $|B_1|$ colors. If $\omega(\Pi_G) = |A|$ and $|A| \neq |A_1|$, then $|A| > |B_1| > |B_2| > \dots |B_r|$ and so vertices of Π_G are colorable with $|A|$ colors. □

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier, New York, 1976.

[2] H. R. Dorbidi, *A note on the coprime graph of a group*, Int. J. Group Theory, 5 (2016), 17–22.

[3] C. Godsil and G. F. Royle, *Algebraic graph theory*, Springer Science, Business Media, 2013.

[4] X. Ma, H. Wei and L. Yang, *The coprime graph of a group*, Int. J. Group Theory, 3 (2014), 13–23.

[5] F. Mansoori, A. Erfaniana and B. Tolue, *Non-coprime graph of a finite group*, AIP Conference Proceedings, 1750 (2016), 050017.

[6] Z. Mehranian, A. Gholami and A. R. Ashrafi, *A note on the power graph of a finite group*, Int. J. Group Theory, 5 (2016), 1-10.

[7] M. Mirzargar, A. R. Ashrafi and M. J. Nadjafi-Arani, *On the power graph of a finite group*, Filomat, 26 (2012), 1201-1208.

[8] The Gap Team, GAP – Groups, Algorithms and Programing.

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Some fixed soft point theorems on a new soft topology related to a self soft mapping

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Abstract. In the present paper, we introduce a new soft topology related to a self soft mapping and study its some basic properties. Also, we give the notion of a soft orbit and support it with examples. Moreover, we present the notion of a soft b-metric space, which is a soft version of b-metric space of Czerwik [7]. Next, by using these notions, we establish some fixed soft point theorems, which are the main results of our paper.

Keywords: soft set, fixed soft point, soft orbit, soft b-metric space.

1. Introduction

In 1999, Molodtsov [22] initiated the concept of a soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. In [22], Molodtsov successfully applied the soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration and theory of measurement. After presentation of the operations of soft sets [20], the properties and applications of this theory have been studied increasingly [2, 23, 27].

Maji et al. [21] gave the first practical application of soft sets in decision making problems. Pei and Miao [27] showed that soft sets are a class of special information systems. Ali et al. [2] presented some new algebraic operations on soft sets. Das and Samanta [9] introduced the notions of soft real sets and soft real numbers and studied their properties. Shabir and Naz [31] initiated the study of soft topological spaces. Recently, many papers concerning the soft set theory have been published [5, 10, 11, 15, 18, 19, 24, 25, 29, 34].

Fixed point theory plays a fundamental role in mathematics and applied sciences, such as optimization, mathematical models and economic theories. Also, this theory have been applied to show the existence and uniqueness of the solutions of diferential equations, integral equations and many other branches of mathematics [13, 26, 28]. A basic result in fixed point theory is the Banach

contraction principle. Since the appearance of this principle, there has been a lot of activity in this area. Czerwik [7, 8] introduced the notion of a b-metric space and generalized the Banach contraction principle in the context of complete b-metric spaces. After that, some authors such as Boriceanu [6], Khamsi [17], Aydi [4], Roshan et al. [30], Arab [3] and Kamran et al. [16] studied the fixed point theory for single-valued or multivalued operators in b-metric spaces.

Extensions of fixed point theorems to the soft set theory have been studied by some authors. Wardowski [32] defined a new notion of soft element of a soft set and established some fixed point results. Later, Abbas et al. [1] studied soft metric versions of several important fixed point theorems for metric spaces. Afterwards, many authors obtained various existence theorems of fixed points using this theory [12, 14, 33].

In this work, we define a new soft topology with the help of a self soft mapping and investigate its related properties. Also, we prove some fixed soft point theorems on this soft topology. Then, we introduce the notion of a soft b-metric space and study its relation with a soft metric space. Finally, we give the concept of a soft orbit and obtain a fixed soft point theorem using this concept.

2. Preliminaries

In this section, we recollect some basic notions regarding soft sets. Throughout this work, let X be an initial universe, $P(X)$ be the power set of X and E be a set of parameters for X.

Definition 2.1 ([22]). *A soft set F on the universe X with the set of parameters E is defined by the set of ordered pairs*

$$F = \{(e, F(e)) : e \in E, F(e) \in P(X)\}$$

where F is a mapping given by $F : E \to P(X)$.

Throughout this paper, the family of all soft sets over X is denoted by $S(X, E)$ [5].

Definition 2.2 ([2, 20, 27]). *Let $F, G \in S(X, E)$. Then:*

(i) The soft set F is called a null soft set, denoted by $\widetilde{\emptyset}$, if $F(e) = \emptyset$ for every $e \in E$.

(ii) The soft set F is called an absolute soft set, denoted by $\widetilde{X}$, if $F(e) = X$ for every $e \in E$.

(iii) F is a soft subset of G if $F(e) \subseteq G(e)$ for every $e \in E$. It is denoted by $F \sqsubseteq G$.

(iv) The complement of F is denoted by F^c, where $F^c : E \to P(X)$ is a mapping defined by $F^c(e) = X - F(e)$ for every $e \in E$. Clearly, $(F^c)^c = F$.

(v) The union of F and G is a soft set H defined by $H(e) = F(e) \cup G(e)$ for every $e \in E$. H is denoted by $F \sqcup G$.

(*vi*) *The intersection of* F *and* G *is a soft set* H *defined by* $H(e) = F(e) \cap G(e)$ *for every* $e \in E$. H *is denoted by* $F \sqcap G$.

Definition 2.3 ([10, 19, 24]). *A soft set* P *over* X *is said to be a soft point if there exists an* $e \in E$ *such that* $P(e) = \{x\}$ *for some* $x \in X$ *and* $P(e') = \emptyset$ *for every* $e' \in E \backslash \{e\}$. *This soft point is denoted as* x^e.

A soft point x^e *is said to belongs to a soft set* F, *denoted by* $x^e \widetilde{\in} F$, *if* $x \in F(e)$.

From now on, let $SP(X)$ be the family of all soft points over X.

Definition 2.4 ([18]). *Let* $S(X, E)$ *and* $S(Y, K)$ *be the families of all soft sets over* X *and* Y, *respectively. Let* $\varphi : X \to Y$ *and* $\psi : E \to K$ *be two mappings. Then, the mapping* φ_ψ *is called a soft mapping from* X *to* Y, *denoted by* $\varphi_\psi : S(X, E) \to S(Y, K)$.

(i) *Let* $F \in S(X, E)$. *Then,* $\varphi_\psi(F)$ *is the soft set over* Y *defined as follows:*

$$\varphi_\psi(F)(k) = \begin{cases} \bigcup_{e \in \psi^{-1}(k)} \varphi(F(e)), & \text{if } \psi^{-1}(k) \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases},$$

for all $k \in K$. $\varphi_\psi(F)$ *is called a soft image of a soft set* F.

(ii) *Let* $G \in S(Y, K)$. *Then,* $\varphi_\psi^{-1}(G)$ *is the soft set over* X *defined as follows:*

$$\varphi_\psi^{-1}(G)(e) = \varphi^{-1}(G(\psi(e)))$$

for all $e \in E$. $\varphi_\psi^{-1}(G)$ *is called a soft inverse image of a soft set* G.

Theorem 2.5 ([18]). *Let* $F, F_i \in S(X, E)$ *and* $G, G_i \in S(Y, K)$ *for all* $i \in J$, *where* J *is an index set. Then, for a soft mapping* $\varphi_\psi : S(X, E) \to S(Y, K)$, *the following conditions are satisfied.*

(1) *If* $F_1 \sqsubseteq F_2$, *then* $\varphi_\psi(F_1) \sqsubseteq \varphi_\psi(F_2)$.
(2) *If* $G_1 \sqsubseteq G_2$, *then* $\varphi_\psi^{-1}(G_1) \sqsubseteq \varphi_\psi^{-1}(G_2)$.
(3) $F \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(F))$ *and* $\varphi_\psi(\varphi_\psi^{-1}(G)) \sqsubseteq G$.
(4) $\varphi_\psi\big(\bigsqcup_{i \in J} F_i\big) = \bigsqcup_{i \in J} \varphi_\psi(F_i)$ *and* $\varphi_\psi\big(\bigsqcap_{i \in J} F_i\big) \sqsubseteq \bigsqcap_{i \in J} \varphi_\psi(F_i)$.
(5) $\varphi_\psi^{-1}\big(\bigsqcup_{i \in J} G_i\big) = \bigsqcup_{i \in J} \varphi_\psi^{-1}(G_i)$ *and* $\varphi_\psi^{-1}\big(\bigsqcap_{i \in J} G_i\big) = \bigsqcap_{i \in J} \varphi_\psi^{-1}(G_i)$.

Proposition 2.6 ([25]). *Let* $\varphi_\psi : S(X, E) \to S(Y, K)$ *be a soft mapping and* $x^e \in SP(X)$. *Then, we have* $\varphi_\psi(x^e) = \varphi(x)^{\psi(e)} \in SP(Y)$.

Definition 2.7 ([31]). *Let* τ *be a collection of soft sets over* X. *Then,* τ *is said to be a soft topology on* X *if*

(st_1) $\widetilde{\emptyset}, \widetilde{X}$ *belong to* τ.
(st_2) *the union of any number of soft sets in* τ *belongs to* τ.
(st_3) *the intersection of any two soft sets in* τ *belongs to* τ.

The triplet (X, τ, E) is called a soft topological space. The members of τ are called soft open sets in X. A soft set F over X is called a soft closed in X if $F^c \in \tau$.

Definition 2.8 ([31]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. The soft closure of F is the soft set $\overline{F} = \bigsqcap\{G : G \; is \; soft \; closed \; set \; and \; F \sqsubseteq G\}$.*

Definition 2.9 ([24]). *A collection $\mathcal{B}$ of soft open sets of a soft topological space (X, τ, E) is called a soft base for τ if for any $F \in \tau$ and every soft point x^e contained in F, there exists a $G \in \mathcal{B}$ such that $x^e \,\widetilde{\in}\, G \sqsubseteq F$.*

Definition 2.10 ([24]). *A soft set F in a soft topological space (X, τ, E) is called a soft neighborhood of the soft point x^e if there exists a soft open set G such that $x^e \,\widetilde{\in}\, G \sqsubseteq F$.*

Definition 2.11 ([31]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. Then, $\tau_F = \{F \sqcap G : G \in \tau\}$ is called the soft relative topology on F and (F, τ_F, E) is called a soft subspace of (X, τ, E).*

Definition 2.12 ([25]). *Let (X, τ_1, E) and (Y, τ_2, K) be two soft topological spaces. A soft mapping $\varphi_\psi : (X, \tau_1, E) \to (Y, \tau_2, K)$ is called a soft continuous mapping at $x^e \,\widetilde{\in}\, \widetilde{X}$ if for every soft neighborhood G of $\varphi_\psi(x^e)$ in Y, there exists a soft neighborhood F of x^e in X such that $\varphi_\psi(F) \sqsubseteq G$.*

A soft mapping φ_ψ is called soft continuous on X if it is soft continuous at each $x^e \,\widetilde{\in}\, \widetilde{X}$.

Theorem 2.13 ([25]). *Let (X, τ_1, E) and (Y, τ_2, K) be two soft topological spaces and $\varphi_\psi : (X, \tau_1, E) \to (Y, \tau_2, K)$ be a soft mapping. Then, the following conditions are equivalent:*

(1) *φ_ψ is a soft continuous mapping on X.*
(2) *For every soft open set G over Y, $\varphi_\psi^{-1}(G)$ is a soft open set over X.*
(3) *For every soft closed set F over Y, $\varphi_\psi^{-1}(F)$ is a soft closed set over X.*
(4) *For every $F \in S(X, E)$, $\varphi_\psi(\overline{F}) \sqsubseteq \overline{\varphi_\psi(F)}$.*

Theorem 2.14 ([19]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. A soft point $x^e \,\widetilde{\in}\, \overline{F}$ if and only if every soft neighborhood of x^e intersects F.*

Definition 2.15 ([9]). *Let $\mathbb{R}$ be the set of real numbers, $\mathfrak{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be a set of parameters. Then,*

$$F = \{(e, F(e)) : e \in E, F(e) \in \mathfrak{B}(\mathbb{R})\}$$

is called a soft real set.

If specifically F is a singleton soft set, then after identifying F with the corresponding soft element, it will be called a soft real number.

Definition 2.16 ([9]). *Let F, G be soft real numbers. Then:*

(i) The sum is defined by $(F+G)(e) = F(e) + G(e)$ for every $e \in E$.

(ii) The difference is defined by $(F-G)(e) = F(e) - G(e)$ for every $e \in E$.

(iii) The product is defined by $(F.G)(e) = F(e).G(e)$ for every $e \in E$.

From the above definition of soft real numbers it follows that $F+G, F-G$ and $F.G$ are soft real numbers.

Definition 2.17 ([9]). *Let F be a soft real number. Then, F is said to be a non-negative soft real number if $F(e)$ is a non-negative real number for every $e \in E$.*

Let $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{k}$ to denote soft real numbers whereas $\bar{r}$, $\bar{s}$, $\bar{k}$ will denote a particular type of soft real numbers such that $\bar{r}(e) = r$ for every $e \in E$ etc. For example, $\overline{0}$ is the soft real number, where $\overline{0}(e) = 0$ for every $e \in E$ [10].

Definition 2.18 ([10]). *Let $\tilde{r}$ and $\tilde{s}$ be two soft real numbers. Then:*

(i) $\tilde{r} \widetilde{\leq} \tilde{s}$ if $\tilde{r}(e) \leq \tilde{s}(e)$ for every $e \in E$.

(ii) $\tilde{r} \widetilde{\geq} \tilde{s}$ if $\tilde{r}(e) \geq \tilde{s}(e)$ for every $e \in E$.

(iii) $\tilde{r} \widetilde{<} \tilde{s}$ if $\tilde{r}(e) < \tilde{s}(e)$ for every $e \in E$.

(iv) $\tilde{r} \widetilde{>} \tilde{s}$ if $\tilde{r}(e) > \tilde{s}(e)$ for every $e \in E$.

Definition 2.19 ([10]). *A mapping $d : SP(X) \times SP(X) \to \mathbb{R}(E)^*$ is called a soft metric on X if it satisfies the following axioms:*

(sm_1) $d(x_1^{e_1}, x_2^{e_2}) = \overline{0}$ if and only if $x_1^{e_1} = x_2^{e_2}$ for every $x_1^{e_1}, x_2^{e_2} \widetilde{\in} \widetilde{X}$.

(sm_2) $d(x_1^{e_1}, x_2^{e_2}) = d(x_2^{e_2}, x_1^{e_1})$ for every $x_1^{e_1}, x_2^{e_2} \widetilde{\in} \widetilde{X}$.

(sm_3) $d(x_1^{e_1}, x_2^{e_2}) \widetilde{\leq} d(x_1^{e_1}, x_3^{e_3}) + d(x_3^{e_3}, x_2^{e_2})$ for every $x_1^{e_1}, x_2^{e_2}, x_3^{e_3} \widetilde{\in} \widetilde{X}$.

The triplet (X, d, E) is called a soft metric space on X.

Definition 2.20 ([11]). *A soft topological space (X, τ, E) is called a soft Hausdorff space or a soft T_2-space if for any two distinct soft points $x_1^{e_1}$, $x_2^{e_2} \in SP(X)$ there exist soft open sets F and G such that $x_1^{e_1} \widetilde{\in} F, x_2^{e_2} \widetilde{\in} G$ and $F \sqcap G = \widetilde{\emptyset}$.*

Definition 2.21 ([5, 34]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$.*

(i) *A family of soft sets $\mathcal{C} = \{F_i : i \in I\}$ over X is called a cover of F if it satisfies $F \sqsubseteq \bigsqcup_{i \in I} F_i$. It is called a soft open cover if each member of $\mathcal{C}$ is a soft open set. A subfamily of $\mathcal{C}$ is called a subcover of $\mathcal{C}$ if it is also a cover of F.*

(ii) *(X, τ, E) is called a soft compact space if every soft open cover of $\widetilde{X}$ has a finite subcover.*

Theorem 2.22 ([29]). *Let (X, τ, E) be a soft topological space and $F \in S(X, E)$. Then, (F, τ_F, E) is a soft compact space if and only if every cover of F by soft open sets over X contains a finite subcover.*

Theorem 2.23 ([29]). *Every soft compact subspace of a soft Hausdorff space is soft closed.*

3. A new soft topology related to a self soft mapping

In this section, using a self soft mapping, we introduce a new soft topology and prove some fixed soft point theorems in this soft topology. Also, we give the concept of a soft b-metric space and establish its connection with a soft metric space.

Let φ_ψ be a soft mapping from $S(X,E)$ to itself, which is called a self soft mapping. We define

$$\tau_{\varphi_\psi} = \{F \in S(X,E) : \varphi_\psi(F) \sqsubseteq F\}.$$

Then, we get the following theorem.

Theorem 3.1. *The family τ_{φ_ψ} is a soft topology on X.*

Proof. It is clear from Theorem 2.5 (4).

Example 3.2. Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2\}$. Let us define a soft mapping $\varphi_\psi : S(X,E) \to S(X,E)$ by

$$\varphi(x_1) = x_2, \varphi(x_2) = x_1 \quad \textit{and} \quad \psi(e_1) = e_2, \psi(e_2) = e_2.$$

Then, it is easy to see that $\tau_{\varphi_\psi} = \{\widetilde{X}, \widetilde{\emptyset}, F_1, F_2, F_3\}$ is a soft topology on X since $\varphi_\psi(F_1) \sqsubseteq F_1$, $\varphi_\psi(F_2) \sqsubseteq F_2$, $\varphi_\psi(F_3) \sqsubseteq F_3$, $\varphi_\psi(\widetilde{X}) \sqsubseteq \widetilde{X}$, $\varphi_\psi(\widetilde{\emptyset}) \sqsubseteq \widetilde{\emptyset}$, where $F_1 = \{(e_1, \{x_1\}), (e_2, X)\}$, $F_2 = \{(e_1, \{x_2\}), (e_2, X)\}$ and $F_3 = \{(e_1, \emptyset), (e_2, X)\}$.

Lemma 3.3. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space. Then:*
(1) *If $F \in \tau_{\varphi_\psi}$, then $F \sqsubseteq \varphi_\psi^{-1}(F)$.*
(2) *If $F^c \in \tau_{\varphi_\psi}$, then $\varphi_\psi^{-1}(F) \sqsubseteq F$.*

Proof. It follows immediately from Theorem 2.5 (2) and (3).

Theorem 3.4. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space, $F \in S(X,E)$ and $x^e \in SP(X)$. If $x^e \,\widetilde{\in}\, \overline{F}$, then there exists a soft point $y^t \,\widetilde{\in}\, F$ such that $\varphi_\psi(y^t) \,\widetilde{\in}\, G$ for every $G \in \tau_{\varphi_\psi}$ containing x^e.*

Proof. Let $x^e \,\widetilde{\in}\, \overline{F}$ and take a soft open set G such that $x^e \,\widetilde{\in}\, G$. Suppose that $\varphi_\psi(y^t) \,\widetilde{\notin}\, G$ for every soft point $y^t \,\widetilde{\in}\, F$. Then, we have $\varphi_\psi(y^t) \,\widetilde{\in}\, G^c$. By Lemma 3.3(2), we obtain $y^t \,\widetilde{\in}\, \varphi_\psi^{-1}(G^c) \sqsubseteq G^c$. Thus, from the fact that $\overline{F} \sqsubseteq G^c$ it follows that $x^e \,\widetilde{\in}\, G^c$, which is a contradiction.

The following example shows that the converse of the above theorem does not hold in general.

Example 3.5. Consider Example 3.2. Let $F = x_2^{e_1} \in S(X, E)$ and let us take a soft point $x_1^{e_1}$. Then, we have soft open sets F_1 and $\widetilde{X}$ containing $x_1^{e_1}$. One can readily verify that $\varphi_\psi(x_2^{e_1}) = x_1^{e_2} \,\widetilde{\in}\, F_1$. The same also holds for the soft set $\widetilde{X}$. But, $x_1^{e_1} \,\widetilde{\notin}\, \overline{F}$.

Definition 3.6. *Let $\varphi_\psi : S(X, E) \to S(X, E)$ be a soft mapping. A soft point x^e is called a fixed soft point of φ_ψ if $\varphi_\psi(x^e) = x^e$.*

Remark 3.7. The converse of Theorem 3.4 is true if for a soft set F, each soft point in F is a fixed soft point of φ_ψ.

Proof. Suppose that $x^e \,\widetilde{\notin}\, \overline{F}$. Then, there exits a soft closed set H such that $F \sqsubseteq H$ and $x^e \,\widetilde{\notin}\, H$. Therefore, we have $H^c \in \tau_{\varphi_\psi}$ with $x^e \,\widetilde{\in}\, H^c$. On the other hand, since $F \sqcap H^c = \widetilde{\emptyset}$, we obtain $\varphi_\psi(y^t) = y^t \,\widetilde{\notin}\, H^c$ for every $y^t \,\widetilde{\in}\, F$, which yields a contradiction.

Lemma 3.8. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space. Then, a soft mapping $\varphi_\psi : (X, \tau_{\varphi_\psi}, E) \to (X, \tau_{\varphi_\psi}, E)$ is a soft continuous mapping.*

Proof. We can easily prove it by using the definition of τ_{φ_ψ}.

Theorem 3.9. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft Hausdorff space. Then, the soft mapping $\varphi_\psi : (X, \tau_{\varphi_\psi}, E) \to (X, \tau_{\varphi_\psi}, E)$ has a fixed soft point.*

Proof. Suppose that $\varphi_\psi(x^e) \neq x^e$ for every $x^e \,\widetilde{\in}\, \widetilde{X}$. By hypothesis, there exist soft open sets F and G such that $x^e \,\widetilde{\in}\, F$, $\varphi_\psi(x^e) \,\widetilde{\in}\, G$ and $F \sqcap G = \widetilde{\emptyset}$. From Lemma 3.8, there exists a soft open set H such that $x^e \,\widetilde{\in}\, H$ and $\varphi_\psi(H) \sqsubseteq G$. Therefore, since

$$\varphi_\psi(H \sqcap F) \sqsubseteq \varphi_\psi(H) \sqcap \varphi_\psi(F) \sqsubseteq G \sqcap F = \widetilde{\emptyset}$$

we have $\varphi_\psi(H \sqcap F) = \widetilde{\emptyset}$, so $H \sqcap F = \widetilde{\emptyset}$. This is a contradiction. Thus, there exists a fixed soft point of φ_ψ.

Lemma 3.10. *A soft topological space $(X, \tau_{\varphi_\psi}, E)$ is soft compact space if and only if every family of soft closed sets over X satisfying the finite intersection property has non-empty intersection.*

Proof. We can easily prove it by using the definition of soft compact space.

Lemma 3.11. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft compact space. Then, $\varphi_\psi(\widetilde{X})$ is a soft compact set.*

Proof. Let $\{F_i : i \in I\}$ be a family of soft open sets over X such that $\varphi_\psi(\widetilde{X}) \sqsubseteq \bigsqcup_{i\in I} F_i$. By Lemma 3.8, the family $\{\varphi_\psi^{-1}(F_i) : i \in I\}$ is a soft open cover of $\widetilde{X}$. Then, there exists a finite set $\{i_1, i_2, ..., i_n\} \subseteq I$ such that

$$\widetilde{X} = \varphi_\psi^{-1}(F_{i_1}) \sqcup \varphi_\psi^{-1}(F_{i_2}) \sqcup ... \sqcup \varphi_\psi^{-1}(F_{i_n})$$

and this implies that $\varphi_\psi(\widetilde{X}) \sqsubseteq F_{i_1} \sqcup F_{i_2} \sqcup ... \sqcup F_{i_n}$. Thus, $\varphi_\psi(\widetilde{X})$ is a soft compact set.

Theorem 3.12. *Let (F, τ_F, E) be a soft open compact Hausdorff subspace of a soft topological space $(X, \tau_{\varphi_\psi}, E)$ with $\varphi_\psi(F) \neq F$. If every non-empty soft set $H \neq F$ being both soft open and soft closed over F contains only one soft point of F, then there exists a soft point $x^e \widetilde{\in} F$ such that $\varphi_\psi(x^e) = x^e$.*

Proof. Let us consider a family of soft sets of the form

$$G_1 = \varphi_\psi(F),\ G_2 = \varphi_\psi(G_1) = \varphi_\psi^2(F),\ ...\ , G_n = \varphi_\psi(G_{n-1}) = \varphi_\psi^n(F),\ ...$$

where $n \in \mathbb{N}$. Since $F \in \tau_{\varphi_\psi}$, we have $G_n \sqsubseteq G_{n-1}$ for all $n \in \mathbb{N}$. By Lemma 3.11 and Theorem 2.23, G_n is a soft closed set over F for all $n \in \mathbb{N}$. Then, from Lemma 3.10 it follows that $H = \bigsqcap_{n\in\mathbb{N}} G_n$ is a non-empty soft closed set over F. Also, since

$$\varphi_\psi(H) \sqsubseteq \bigsqcap_{n\in\mathbb{N}} \varphi_\psi^{n+1}(F) \sqsubseteq \bigsqcap_{n\in\mathbb{N}} \varphi_\psi^n(F) = H$$

H is a soft open set over F. Therefore, by hypothesis, the soft set H contains only one soft point of F. Let us choose such a soft point x^e. Thus, from the fact that

$$\varphi_\psi(x^e) \sqsubseteq \varphi_\psi(H) \sqsubseteq \bigsqcap_{n\in\mathbb{N}} \varphi_\psi^n(F) = x^e$$

it follows that $\varphi_\psi(x^e) = x^e$.

Czerwik [7] defined the notion of a soft b-metric. We introduce this notion in soft setting as follows:

Definition 3.13. *Let X be a non-empty set, E be a non-empty parameter set and $\tilde{s} \widetilde{\geq} \overline{1}$. A mapping $d_b : SP(X) \times SP(X) \to \mathbb{R}(E)^*$ is called a soft b-metric on X if it satisfies the following axioms:*

(sbm_1) $d_b(x_1^{e_1}, x_2^{e_2}) = \overline{0}$ if and only if $x_1^{e_1} = x_2^{e_2}$ for all $x_1^{e_1}, x_2^{e_2} \widetilde{\in} \widetilde{X}$.

(sbm_2) $d_b(x_1^{e_1}, x_2^{e_2}) = d_b(x_2^{e_2}, x_1^{e_1})$ for all $x_1^{e_1}, x_2^{e_2} \widetilde{\in} \widetilde{X}$.

(sbm_3) $d_b(x_1^{e_1}, x_2^{e_2}) \widetilde{\leq} \tilde{s}\left(d_b(x_1^{e_1}, x_3^{e_3}) + d_b(x_3^{e_3}, x_2^{e_2})\right)$ for all $x_1^{e_1}, x_2^{e_2}, x_3^{e_3} \widetilde{\in} \widetilde{X}$.

The triplet (X, d_b, E) is called a soft b-metric space on X.

It is seen that the above definition coincides with that of the soft metric when $\tilde{s} = \overline{1}$. Thus, the class of the soft b-metric spaces is larger than that of the soft metric spaces, that is, every soft metric space is a soft b-metric space. But the converse is not true in general as seen in the following example.

Example 3.14. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and a mapping $d_b : SP(X) \times SP(X) \to \mathbb{R}(E)^*$ be defined by:

$$\begin{aligned}
&d_b(x_1^{e_1}, x_1^{e_1}) = d_b(x_1^{e_2}, x_1^{e_2}) = \overline{0}, \qquad d_b(x_2^{e_1}, x_2^{e_1}) = d_b(x_2^{e_2}, x_2^{e_2}) = \overline{0},\\
&d_b(x_1^{e_1}, x_1^{e_2}) = d_b(x_1^{e_2}, x_1^{e_1}) = \overline{3}, \qquad d_b(x_1^{e_1}, x_2^{e_1}) = d_b(x_2^{e_1}, x_1^{e_1}) = \overline{1},\\
&d_b(x_1^{e_1}, x_2^{e_2}) = d_b(x_2^{e_2}, x_1^{e_1}) = \overline{\tfrac{5}{2}}, \qquad d_b(x_1^{e_2}, x_2^{e_1}) = d_b(x_2^{e_1}, x_1^{e_2}) = \overline{\tfrac{1}{2}},\\
&d_b(x_1^{e_2}, x_2^{e_2}) = d_b(x_2^{e_2}, x_1^{e_2}) = \overline{\tfrac{3}{2}}, \qquad d_b(x_2^{e_1}, x_2^{e_2}) = d_b(x_2^{e_2}, x_2^{e_1}) = \overline{\tfrac{1}{8}}.
\end{aligned}$$

Then, (X, d_b, E) is a soft b-metric space with constant $\tilde{s} = \overline{2}$. However, since

$$d_b(x_1^{e_1}, x_2^{e_1}) + d_b(x_2^{e_1}, x_1^{e_2}) = \frac{\overline{3}}{2} \widetilde{<} d_b(x_1^{e_1}, x_1^{e_2}) = \overline{3},$$

it is not a soft metric space.

Definition 3.15. *Let (X, d_b, E) be a soft b-metric space, $\tilde{r}$ be a non-negative soft real number and $x^e \widetilde{\in} \widetilde{X}$. A soft b-open ball with centre x^e and radius $\tilde{r}$ is defined by*

$$B_b(x^e, \tilde{r}) = \bigsqcup \{ y^t \widetilde{\in} \widetilde{X} : d_b(x^e, y^t) \widetilde{<} \tilde{r} \}.$$

Example 3.16. Let (X, d_b, E) be a soft b-metric space which is defined in Example 3.14. Then, for $x_1^{e_1} \widetilde{\in} \widetilde{X}$

$$B_b(x_1^{e_1}, \overline{2}) = x_1^{e_1} \sqcup x_2^{e_1} = \{(e_1, X), (e_2, \emptyset)\}.$$

Definition 3.17. *Let (X, d_b, E) be a soft b-metric space. A soft mapping $\varphi_\psi : S(X, E) \to S(X, E)$ is called a soft contraction if there exists a non-negative soft real number $\tilde{k} \widetilde{<} \overline{1}$ such that $d_b\big(\varphi_\psi(x^e), \varphi_\psi(y^t)\big) \widetilde{\leq} \tilde{k}\, d_b(x^e, y^t)$ for every $x^e, y^t \widetilde{\in} \widetilde{X}$.*

Theorem 3.18. *Let (X, d_b, E) be a soft b-metric space and $\varphi_\psi : S(X, E) \to S(X, E)$ be a soft contraction mapping such that $\tilde{s}.\tilde{k} \widetilde{<} \overline{1}$. If τ_{φ_ψ} is a soft topology on X, then, for every $x^e \widetilde{\in} \widetilde{X}$, there exists a soft real number $\tilde{r}$ such that $B_b(x^e, \tilde{r}) \in \tau_{\varphi_\psi}$.*

Proof. Let $x^e \widetilde{\in} \widetilde{X}$. Let us take a soft real number $\tilde{r}$ satisfying the following condition:

$$\frac{\tilde{s}\, d_b(x^e, \varphi_\psi(x^e))}{\overline{1} - \tilde{s}.\tilde{k}} \widetilde{<} \tilde{r}.$$

Then, we have $B_b(x^e, \tilde{r}) \in \tau_{\varphi_\psi}$. Indeed, let $y^t \widetilde{\in} \varphi_\psi\big(B_b(x^e, \tilde{r})\big)$. Then, there exists a soft point $x_1^{e_1} \widetilde{\in} B_b(x^e, \tilde{r})$ such that $\varphi_\psi(x_1^{e_1}) = y^t$. Therefore, since

$$\begin{aligned}
&d_b(x^e, \varphi_\psi(x_1^{e_1})) \widetilde{\leq} \tilde{s}\, \big(d_b(x^e, \varphi_\psi(x^e)) + d_b(\varphi_\psi(x^e), \varphi_\psi(x_1^{e_1}))\big)\\
&\widetilde{\leq} \tilde{s}\, \big(d_b(x^e, \varphi_\psi(x^e)) + \tilde{k}\, d_b(x^e, x_1^{e_1})\big)\\
&\widetilde{<} \tilde{s}\, \big(d_b(x^e, \varphi_\psi(x^e)) + \tilde{k}.\tilde{r}\big) \widetilde{<} \tilde{r}
\end{aligned}$$

we obtain $\varphi_\psi(x_1^{e_1}) = y^t \widetilde{\in} B_b(x^e, \tilde{r})$. Thus, from the fact that $\varphi_\psi\big(B_b(x^e, \tilde{r})\big) \sqsubseteq B_b(x^e, \tilde{r})$ it follows that $B_b(x^e, \tilde{r}) \in \tau_{\varphi_\psi}$, which completes the proof.

4. Soft orbit

Now, we present the notion of a soft orbit and investigate some of its basic properties. Also, on the basis of this notion, we establish a fixed soft point theorem.

Definition 4.1. *A soft orbit of a soft point $x^e \widetilde{\in} \widetilde{X}$ under a soft mapping $\varphi_\psi : S(X,E) \to S(X,E)$ is defined as*

$$O_s(x^e) = \bigsqcup \{\varphi_\psi^n(x^e) : n = 0, 1, 2, ...\}$$

where φ_ψ^n is the nth iterate of φ_ψ.

Then, we get the following properties.

Theorem 4.2. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space.*

(1) *For every $x^e \widetilde{\in} \widetilde{X}$, $O_s(x^e)$ is the smallest soft open set containing x^e.*
(2) *A family $\{O_s(x^e) : x^e \widetilde{\in} \widetilde{X}\}$ of soft orbits is a soft base for τ_{φ_ψ}.*
(3) *If a soft point $x^e \widetilde{\in} \widetilde{X}$ is a fixed soft point of φ_ψ, then we have $O_s(x^e) = x^e$.*

Proof. (1) Let $x^e \widetilde{\in} \widetilde{X}$. One can readily verify that $\varphi_\psi(O_s(x^e)) \sqsubseteq O_s(x^e)$. So, we have $O_s(x^e) \in \tau_{\varphi_\psi}$. Now, we shall show that $O_s(x^e) = \bigsqcap\{F \in S(X,E) : x^e \widetilde{\in} F \text{ and } F \in \tau_{\varphi_\psi}\}$. Let $y^t \widetilde{\in} O_s(x^e)$. Then, there exists an $n \in \mathbb{N}$ such that $\varphi_\psi^n(x^e) = y^t$. Suppose that there exists an $F \in \tau_{\varphi_\psi}$ containing x^e such that $y^t \widetilde{\notin} F$. Therefore, we have $\varphi_\psi(x^e) \widetilde{\in} F$. Again, since $\varphi_\psi(F) \sqsubseteq F$, we obtain $\varphi_\psi^2(x^e) \widetilde{\in} F$. Continuing this process, we get $\varphi_\psi^n(x^e) = y^t \widetilde{\in} F$. But this result is in contradiction with the fact that $y^t \widetilde{\notin} F$. Similarly, the converse inclusion holds as well, since $x^e \widetilde{\in} O_s(x^e)$ and $O_s(x^e) \in \tau_{\varphi_\psi}$.

(2) and (3) are clear from (1) and the definition of soft orbit.

Now, we give the following example to show the validity of the properties in Theorem 4.2.

Example 4.3. Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2\}$. Let us define a soft mapping $\varphi_\psi : S(X,E) \to S(X,E)$ by

$$\varphi(x_1) = x_2, \varphi(x_2) = x_2 \quad \textit{and} \quad \psi(e_1) = e_2, \psi(e_2) = e_2.$$

One can readily verify that $\tau_{\varphi_\psi} = \{\widetilde{X}, \widetilde{\emptyset}, F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$, where $F_1 = \{(e_1, \{x_1\}), (e_2, \{x_2\})\}$, $F_2 = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$, $F_3 = \{(e_1, X), (e_2, \{x_2\})\}$, $F_4 = \{(e_1, \emptyset), (e_2, \{x_2\})\}$, $F_7 = \{(e_1, \emptyset), (e_2, X)\}$.

Therefore, we have $O_s(x_1^{e_1}) = F_1$, $O_s(x_1^{e_2}) = F_7$, $O_s(x_2^{e_1}) = F_2$ and $O_s(x_2^{e_2}) = F_4$ and so that $O_s(x_i^{e_j})$ is the smallest soft open set containing $x_i^{e_j}$ for all $i \in \{1,2\}$ and all $j \in \{1,2\}$.

Also, as it can be easily seen, $\{O_s(x_1^{e_1}), O_s(x_1^{e_2}), O_s(x_2^{e_1}), O_s(x_2^{e_2})\}$ is a soft base for τ_{φ_ψ}.

Moreover, the soft point $x_2^{e_2}$ is a fixed soft point of φ_ψ and we obtain $O_s(x_2^{e_2}) = x_2^{e_2}$.

Definition 4.4. *A soft orbit of a soft set $F \in S(X,E)$ under a soft mapping $\varphi_\psi : S(X,E) \to S(X,E)$ is defined as*

$$O_s(F) = \bigsqcup_{x^e \,\widetilde{\in}\, F} O_s(x^e).$$

Example 4.5. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Let us consider a soft mapping $\varphi_\psi : S(X,E) \to S(X,E)$ such that

$$\varphi(x_1) = x_3, \varphi(x_2) = x_1, \varphi(x_3) = x_3 \quad \textit{and} \quad \psi(e_1) = e_1, \psi(e_2) = e_2.$$

Then, the soft orbit of a soft set $F = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$ is defined as the following:

$$O_s(F) = O_s(x_1^{e_1}) \sqcup O_s(x_1^{e_2}) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_1, x_3\})\}.$$

Theorem 4.6. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space and $F, G \in S(X,E)$. Then, the following statements are satisfied.*

(1) $F \sqsubseteq O_s(F)$.
(2) $O_s(F) \in \tau_{\varphi_\psi}$.
(3) *If $F \sqsubseteq G$, then $O_s(F) \sqsubseteq O_s(G)$.*
(4) *If $F \in \tau_{\varphi_\psi}$, then $O_s(F) = F$.*
(5) $O_s(F) = \bigsqcap \{H : F \sqsubseteq H \textit{ and } H \in \tau_{\varphi_\psi}\}$.

Proof. (1), (2) and (3) are clear.

(4) Let $F \in \tau_{\varphi_\psi}$. Then, we have $O_s(x^e) \sqsubseteq F$ for every $x^e \,\widetilde{\in}\, F$. Therefore, we obtain

$$\bigsqcup_{x^e \,\widetilde{\in}\, F} O_s(x^e) = O_s(F) \sqsubseteq F.$$

From the fact that $F \sqsubseteq O_s(F)$ it follows that $O_s(F) = F$.

(5) By (1) and (2), we have $\bigsqcap \{H : F \sqsubseteq H \textit{ and } H \in \tau_{\varphi_\psi}\} \sqsubseteq O_s(F)$. For the converse, let $x^e \,\widetilde{\in}\, O_s(F)$. Then, there exists a soft point $y^t \,\widetilde{\in}\, F$ such that $x^e \,\widetilde{\in}\, O_s(y^t)$. Suppose that $x^e \,\widetilde{\notin}\, \bigsqcap \{H : F \sqsubseteq H \textit{ and } H \in \tau_{\varphi_\psi}\}$. Therefore, there exists a soft open set H such that $F \sqsubseteq H$ and $x^e \,\widetilde{\notin}\, H$. From $\varphi_\psi(F) \sqsubseteq \varphi_\psi(H) \sqsubseteq H$ it follows that $x^e \,\widetilde{\notin}\, O_s(y^t)$, which is a contradiction. Thus, the required equality is satisfied.

Lemma 4.7. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space. Then, for every $x^e \,\widetilde{\in}\, \widetilde{X}$, the soft set $\overline{O_s(x^e)}$ is both soft open and soft closed on X.*

Proof. It is clear that $\overline{O_s(x^e)}$ is a soft closed set on X. We shall show that $\overline{O_s(x^e)}$ is a soft open set over X. Let $y^t \,\widetilde{\in}\, \varphi_\psi\big(\overline{O_s(x^e)}\big)$. Then, there exists a soft

point $x_1^{e_1} \widetilde{\in} \overline{O_s(x^e)}$ such that $y^t = \varphi_\psi(x_1^{e_1})$. Now, let us take a soft open set F_{y^t} containing $\varphi_\psi(x_1^{e_1})$. Since φ_ψ is a soft continuous mapping, there exists a soft open set $G_{x_1^{e_1}}$ containing $x_1^{e_1}$ such that $\varphi_\psi(G_{x_1^{e_1}}) \sqsubseteq F_{y^t}$. From Theorem 2.14 it follows that $G_{x_1^{e_1}} \sqcap O_s(x^e) \neq \widetilde{\emptyset}$ and therefore $F_{y^t} \sqcap O_s(x^e) \neq \widetilde{\emptyset}$. Hence, we have $y^t = \varphi_\psi(x_1^{e_1}) \widetilde{\in} \overline{O_s(x^e)}$. Thus, we obtain $\varphi_\psi\big(\overline{O_s(x^e)}\big) \sqsubseteq \overline{O_s(x^e)}$, which completes the proof.

Theorem 4.8. *Let $(X, \tau_{\varphi_\psi}, E)$ be a soft topological space and $\varphi_\psi : (X, \tau_{\varphi_\psi}, E) \to (X, \tau_{\varphi_\psi}, E)$ be a soft mapping such that if x^e is not a fixed soft point of φ_ψ, then $x^e \widetilde{\notin} \overline{O_s\big(\varphi_\psi^2(x^e)\big)}$. If $\overline{O_s(x_0^{e_0})}$ is a soft compact set for some $x_0^{e_0} \widetilde{\in} \widetilde{X}$, then there exists a soft point $y^t \widetilde{\in} \overline{O_s(x_0^{e_0})}$ such that $\varphi_\psi(y^t) = y^t$.*

Proof. Let us consider a family $\mathcal{U} = \big\{F \sqsubseteq \overline{O_s(x_0^{e_0})} : F \neq \widetilde{\emptyset}$ *and* $F, F^c \in \tau_{\varphi_\psi}\big\}$. By Lemma 4.7, $\mathcal{U}$ is a non-empty family. Let $\mathcal{U}$ be partially ordered by the set inclusion and let $\mathcal{V}$ be a totally ordered subfamily of $\mathcal{U}$. Take a soft set $F_0 = \bigsqcap\{F : F \in \mathcal{V}\}$. From Lemma 3.10 it follows that F_0 is a non-empty soft set on X. Then, we have $F_0 \in \mathcal{U}$ since the other conditions are easily verified. Also, it is a lower bound of $\mathcal{V}$ and so, by Zorn's Lemma, $\mathcal{U}$ has a minimal element H. Therefore, using the minimality of H, we obtain $\overline{\varphi_\psi(H)} = H$. Now, suppose that $\varphi_\psi(x^e) \neq x^e$ for some $x^e \widetilde{\in} H$. By hypothesis about φ_ψ, we obtain $x^e \widetilde{\notin} \overline{O_s\big(\varphi_\psi^2(x^e)\big)}$. From the fact that $\overline{O_s\big(\varphi_\psi^2(x^e)\big)} \sqsubseteq H$ and H is a minimal soft set of $\mathcal{U}$ it follows that $\overline{O_s\big(\varphi_\psi^2(x^e)\big)} = H$. Hence, we get $x^e \widetilde{\notin} H$, which leads to a contradiction. Thus, since $H \sqsubseteq \overline{O_s(x_0^{e_0})}$, there exists a soft point $y^t \widetilde{\in} \overline{O_s(x_0^{e_0})}$ such that $\varphi_\psi(y^t) = y^t$.

5. Conclusion

Soft set theory is a general method for solving problems of uncertainty. Recently, many authors have already studied the notion of topology on soft sets, because topology is an important area of mathematics with many applications in the domains of computer sciences and information sciences. On the other hand, metric spaces and their various generalizations (such as b-metric spaces) and also fixed point theory have attracted the attention of many mathematicians, physicists and computer scientists. Hence, there exists a considerable literature of fixed point theory dealing with results on fixed or common fixed points in b-metric spaces. Thus, softification of these concepts is highly desirable.

In this paper, we introduce a new soft topology with the help of a self soft mapping and study some of its basic properties. Also, we prove some fixed soft point theorems on this soft topology. Then, we define the notion of a soft b-metric space and investigate its relation with a soft metric space. Later, we present the concept of a soft orbit and establish a fixed soft point theorem using this concept.

In the fields of soft topology and soft metric, the findings in this paper will open up new areas of research. Since there exist close relations between soft sets and information systems, one can use the results inferred from this research to improve these kinds of relations. Also, a soft b-metric space can be extended in fuzzy soft and intuitionistic fuzzy soft b-metric spaces in order to have more affirmative solution in decision making problems in real life situations. Moreover, one can obtain fixed point results for self soft mapping by using suitable conditions in soft b-metric spaces.

References

[1] M. Abbas, G. Murtaza and S. Romaguera, *On the fixed point theory of soft metric spaces*, Fixed Point Theory Appl., 2016 (2016).

[2] M.I. Ali, F. Feng, X. Liu, W.K. Min and M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl., 57 (2009), 1547-1553.

[3] R. Arab and K. Zare, *New fixed point results for rational type contractions in partially ordered b-metric spaces*, Int. J. Anal. Appl., 10 (2016), 64-70.

[4] H. Aydi, M. Bota, E. Karapnar and S. Mitrovic, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory Appl., 88 (2012).

[5] A. Aygünoğlu and H. Aygün, *Some notes on soft topological spaces*, Neural Comput. Appl., 21 (2012), 113-119.

[6] M. Boriceanu, *Fixed point theory for multivalued generalized contractions on a set with two b-metrics*, Stud. Univ. Babeş-Bolyai, Math. LIV, 3 (2009), 3-14.

[7] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostrav., 1 (1993), 5-11.

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena, 46 (1998), 263-276.

[9] S. Das and S.K. Samanta, *Soft real sets, soft real numbers and their properties*, J. Fuzzy Math., 20 (2012), 551-576.

[10] S. Das and S.K. Samanta, *Soft metric*, Ann. Fuzzy Math. Inform., 6 (2013), 77-94.

[11] İ. Demir and O.B. Özbakır, *Soft Hausdorff spaces and their some properties*, Ann. Fuzzy Math. Inform., 8 (2014), 769-783.

[12] İ. Demir, O.B. Özbakır and İ. Yıldız, *On fixed soft element theorems in se-uniform spaces*, J. Nonlinear Sci. Appl., 9 (2016), 1230-1242.

[13] B.C. Dhage, *Condensing mappings and applications to existence theorems for common solution of differential equations*, Bull. Korean Math. Soc., 36 (1999), 565-578.

[14] A.Ç. Güler, E.D. Yıldırım and O.B. Özbakır, *A fixed point theorem on soft G-metric spaces*, J. Nonlinear Sci. Appl., 9 (2016), 885-894.

[15] S. Hussain and B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacet. J. Math. Stat., 44 (2015), 559-568.

[16] T. Kamran, M. Postolache, A. Ghiura, S. Batul and R. Ali, *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application*, Fixed Point Theory Appl., 10 (2016).

[17] M.A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., 73 (2010), 3123-3129.

[18] A. Kharal and B. Ahmad, *Mappings on soft classes,* New Math. Nat. Comput., 7 (2011), 471-481.

[19] F. Lin, *Soft connected spaces and soft paracompact spaces*, Int. J. Math. Sci. Eng., 7 (2013), 754-760.

[20] P.K. Maji, R. Biswas and A.R. Roy, *Soft set theory*, Comput. Math. Appl., 45 (2003), 555-562.

[21] P.K. Maji, A.R. Roy and R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl., 44 (2002), 1077-1083.

[22] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl., 37 (1999), 19-31.

[23] D. Molodtsov, V.Y. Leonov and D.V. Kovkov, *Soft sets technique and its application*, Nechetkie Sist. Myagkie Vychisl., 1 (2006), 8-39.

[24] S. Nazmul and S.K. Samanta, *Neighbourhood properties of soft topological spaces*, Ann. Fuzzy Math. Inform., 6 (2013), 1-15.

[25] O.B. Özbakır and İ. Demir, *On the soft uniformity and its some properties*, J. Math. Comput. Sci., 5 (2015), 762-779.

[26] H.K. Pathak and B. Fisher, *Common fixed point theorems with application in dynamic programming*, Glas. Mat. 31 (1996) 321-328.

[27] D. Pei and D. Miao, *From soft sets to information systems*, In: Hu X, Liu Q, Skowron A, Lin TY, Yager RR, Zhang B, editors. Proceedings of Granular Computing-IEEE, 2 (2005), 617-621.

[28] H.K. Pathak, S.N. Mishra and A.K. Kalinde, *Common fixed point theorems with applications to nonlinear integral equations*, Demonstratio Math., 32 (1999), 547-564.

[29] E. Peyghan, B. Samadi and A. Tayebi, *Some results related to soft topological spaces*, Facta Universitatis Ser. Math. Inform., 29 (2014), 325-336.

[30] J.R. Roshan, N. Shobkolaei, S. Sedghi and M. Abbas, *Common fixed point of four maps in b-metric spaces*, Hacet. J. Math. Stat., 43 (2014), 613-624.

[31] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl., 61 (2011), 1786–1799.

[32] D. Wardowski, *On a soft mapping and its fixed points*, Fixed Point Theory Appl., 182 (2013).

[33] M. I. Yazar, C. G. Aras and S. Bayramov, *Fixed point theorems of soft contractive mappings*, Filomat, 30 (2016), 269-279.

[34] I. Zorlutuna, M. Akdag, W.K. Min and S. Atmaca, *Remarks on soft topological spaces*, Ann. Fuzzy Math. Inform., 3 (2012), 171-185.

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Semi-symmetry properties of the tangent bundle with a pseudo-Riemannian metric

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Abstract. In this note, we consider the tangent bundle TM equipped with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g). We investigate semi-symmetry properties of the tangent bundle TM with respect to the Levi-Civita connection $\widetilde{\nabla}$ and a metric connecton $\widetilde{\widetilde{\nabla}}$ with torsion.
Keywords: metric connection, pseudo-Riemannian metric, Ricci semi-symmetry, semi-symmetry, tangent bundle.

1. Introduction

A Riemannian manifold (M, g) is said to be locally symmetric if the Riemannian curvature tensor R is parallel with respect to the Levi-Civita connection ∇, i.e., $\nabla R = 0$. A natural generalization of the notion of local symmetry is semi-symmetry. A semi-symmetric space is a (pseudo-) Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0$$

for all vector fields X and Y on M, where $R(X, Y)$ is a linear operator acting as a derivation on the curvature tensor R. This class of spaces was first studied by E. Cartan. Nevertheless, N. S. Sinjukov first used the name "semi-symmetric spaces" for manifolds satisfying the above curvature condition [5]. Later, Z. I. Szabo gave the full local and global classification of semi-symmetric spaces

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[6, 7]. A (pseudo-)Riemannian manifold (M, g) is called Ricci semi-symmetric if the following condition is satisfied:

$$R(X, Y) \cdot Ric = 0,$$

where Ric is the Ricci tensor of (M, g). It is clear that any semi-symmetric manifold is Ricci semi-symmetric.

In this paper, we consider the tangent bundle TM with a pseudo-Riemannian metric $\widetilde{g}$. Semi-symmetry and Ricci semi-symmetry are investigated first for the Levi-Civita connection then for a metric connection with a non-zero torsion tensor which is in a special form.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : TMarrowM$ its tangent bundle with fibres the tangent spaces to M. Then TM is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i)$, $\bar{i} = n + i = n + 1, ..., 2n$, where (y^i) is the cartesian coordinates in each tangent space T_pM at $p \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^i}|_p.\}$, p being an arbitrary point in U whose coordinates are (x^i).

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X are given, with respect to the induced coordinates, by

$$ {}^VX = X^i \partial_{\bar{i}}, \tag{2.1}$$

$$ {}^HX = X^i \partial_i - y^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [9].

In each coordinate neighborhood U of M, we put $X = X_j = \frac{\partial}{\partial x^j}$ in (2.1) and (2.2), we get in each induced coordinate neighborhood $\pi^{-1}(U)$ of TM a frame field which consists of the following $2n$ linearly independent vector fields: [9]

$$\begin{cases} E_j = {}^H(X_j) = \partial_j - y^s \Gamma^h_{sj} \partial_{\bar{h}}, \\ E_{\bar{j}} = {}^V(X_j) = \partial_{\bar{j}}. \end{cases}$$

We will mark the adapted frame as $\{E_\beta\} = \{E_j, E_{\bar{j}}\}$. The Lie brackets of the adapted frame are as follows: [9]

$$\begin{cases} [E_i, E_j] = y^s R_{jis}^{\ \ h} E_{\bar{h}}, \\ \left[E_i, E_{\bar{j}}\right] = \Gamma^h_{ij} E_{\bar{h}}, \\ \left[E_{\bar{i}}, E_{\bar{j}}\right] = 0 \end{cases} \tag{2.3}$$

where $R_{hji}{}^{s}$ denote components of the curvature tensor field R of g.

3. Semi-symmetry properties of the tangent bundle with respect to the Levi-Civita connection $\widetilde{\nabla}$

Let (M, g) be a Riemannian manifold. A pseudo-Riemannian metric $\widetilde{g}$ on the tangent bundle TM over (M, g) is defined by

$$\begin{cases} \widetilde{g}({}^{H}X, {}^{H}Y) = a(X, Y) + a(Y, X), \\ \widetilde{g}({}^{V}X, {}^{H}Y) = \widetilde{g}({}^{H}X, {}^{V}Y) = g(X, Y), \\ \widetilde{g}({}^{V}X, {}^{V}Y) = 0 \end{cases}$$

for all vector fields X and Y on M, where a is a $(0,2)-$tensor field on M [4]. The pseudo-Riemannian metric $\widetilde{g}$ is expressed in the adapted local frame by

$$\widetilde{g} = (\widetilde{g}_{\alpha\beta}) = \begin{pmatrix} a_{ij} + a_{ji} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

where g_{ij} and a_{ij} respectively are the local components of g and a. If the $(0,2)-$tensor field a is symmetric, then the metric $\widetilde{g}$ becomes the metric known as synectic lift metric in the literature. The synectic lift metric was firstly considered in [8] and then studied by some authors [1, 2].

For the Levi-Civita connection $\widetilde{\nabla}$ of the pseudo-Riemannian metric $\widetilde{g}$, we have:

Proposition 1. *The Levi-Civita connection $\widetilde{\nabla}$ of $(TM, \widetilde{g})$ is given by*

$$\begin{cases} \widetilde{\nabla}_{E_i} E_j = \Gamma^h_{ij} E_h + \{y^s R_{sij}{}^{h} + g^{mh}(\nabla_i c_{mj} + \nabla_j c_{im} - \nabla_m c_{ij})\} E_{\overline{h}}, \\ \widetilde{\nabla}_{E_i} E_{\overline{j}} = \Gamma^h_{ij} E_{\overline{h}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_j = 0, \ \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0, \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij} and $R_{hji}{}^{s}$ respectively denote components of the Levi-Civita connection ∇ and the curvature tensor field R of g on M and c_{ij} denotes the symmetric part of a_{ij}, i.e., $c_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ (see [4]).

Proof. The Koszul formula for a connection $\widetilde{\nabla}$ is given by

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}(\widetilde{g}(\widetilde{Y}, \widetilde{Z})) + \widetilde{Y}(\widetilde{g}(\widetilde{Z}, \widetilde{X})) - \widetilde{Z}(\widetilde{g}(\widetilde{X}, \widetilde{Y})) \\ -\widetilde{g}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]) &+ \widetilde{g}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) + \widetilde{g}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on TM. Using the Koszul formula for pairs $\widetilde{X} = E_i, E_{\overline{i}}$ and $\widetilde{Y} = E_j, E_{\overline{j}}$ and $\widetilde{Z} = E_k, E_{\overline{k}}$, the formulas (2.3) and the first Bianchi identity for R, we obtain the Christoffel symbols of the Levi-Civita connection $\widetilde{\nabla}$ of $(TM, \widetilde{g})$. We omit standart calculations. □

For the Riemannian curvature tensor $\widetilde{R}$ of the pseudo-Riemannian metric $\widetilde{g}$, we have:

Proposition 2. *The Riemannian curvature tensor $\widetilde{R}$ of $(TM,\widetilde{g})$ is given as follows:*

$$\begin{aligned}
\widetilde{R}(E_i,E_j)E_k &= R_{ijk}{}^{h}E_h+\{y^s(\nabla_i R_{sjk}{}^{h}-\nabla_j R_{sik}{}^{h})\\
&\quad +g^{mh}(\nabla_i(\nabla_k c_{jm}-\nabla_m c_{jk})-\nabla_j(\nabla_k c_{im}-\nabla_m c_{ik})\\
&\quad -R_{ijm}{}^{l}c_{lk}-R_{ijk}{}^{l}c_{ml})\}E_{\overline{h}},\\
\widetilde{R}(E_i,E_j)E_{\overline{k}} &= R_{ijk}{}^{h}E_{\overline{h}},\\
\widetilde{R}(E_i,E_{\overline{j}})E_k &= R_{ijk}{}^{h}E_{\overline{h}},\\
\widetilde{R}(E_{\overline{i}},E_j)E_k &= R_{ijk}{}^{h}E_{\overline{h}},\\
\widetilde{R}(E_{\overline{i}},E_{\overline{j}})E_k &= 0,\ \widetilde{R}(E_{\overline{i}},E_j)E_{\overline{k}}=0,\\
\widetilde{R}(E_i,E_{\overline{j}})E_{\overline{k}} &= 0,\ \widetilde{R}(E_{\overline{i}},E_{\overline{j}})E_{\overline{k}}=0
\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$ (see also [4]).

A semi-Riemannian manifold (M,g), $n=\dim(M)\geq 3$, is said to be semi-symmetric [6] if its curvature tensor R satisfies the condition

$$\begin{aligned}
&((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^{n}=\nabla_i\nabla_j R_{klm}{}^{n}-\nabla_j\nabla_i R_{klm}{}^{n}\\
&=R_{ijp}{}^{n}R_{klm}{}^{p}-R_{ijk}{}^{p}R_{plm}{}^{n}-R_{ijl}{}^{p}R_{kpm}{}^{n}-R_{ijm}{}^{p}R_{klp}{}^{n}=0
\end{aligned}$$

and Ricci semi-symmetric if its Ricci tensor satisfies the condition

$$((R(X,Y)\cdot Ric)(Z,W))_{ijkl}=\nabla_i\nabla_j R_{kl}-\nabla_j\nabla_i R_{kl}=R_{ijk}{}^{p}R_{pl}+R_{ijl}{}^{p}R_{kp}=0.$$

Note that a locally symmetric manifold is obviously semi-symmetric, but in general the converse is not true.

Theorem 1. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M,g). We assume that $\nabla_i R_{sjk}{}^{h}-\nabla_j R_{sik}{}^{h}=0$ and $\nabla_i(\nabla_k c_{jm}-\nabla_m c_{jk})-\nabla_j(\nabla_k c_{im}-\nabla_m c_{ik})-R_{ijm}{}^{l}c_{lk}-R_{ijk}{}^{l}c_{ml}=0$, where R is the curvature tensor of the Levi-Civita connection ∇ of g. Then the tangent bundle $(TM,\widetilde{g})$ is semi-symmetric if and only if the Riemannian manifold (M,g) is semi-symmetric.*

Proof. We consider the conditions $(\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}=0$ for all vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}$ and $\widetilde{U}$ on TM. In the adapted frame $\{E_\beta\}$, the tensor $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}$ is locally expressed as follows:

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}\\
&=\widetilde{R}_{\alpha\beta\tau}{}^{\varepsilon}\widetilde{R}_{\gamma\theta\sigma}{}^{\tau}-\widetilde{R}_{\alpha\beta\gamma}{}^{\tau}\widetilde{R}_{\tau\theta\sigma}{}^{\varepsilon}-\widetilde{R}_{\alpha\beta\theta}{}^{\tau}\widetilde{R}_{\gamma\tau\sigma}{}^{\varepsilon}-\widetilde{R}_{\alpha\beta\sigma}{}^{\tau}\widetilde{R}_{\gamma\theta\tau}{}^{\varepsilon}.
\end{aligned}\tag{3.1}$$

Putting $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = \overline{m}, \varepsilon = \overline{h}$ in (3.1), it follows that

$$
\begin{aligned}
(3.2)\qquad & ((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijkl\overline{m}}{}^{\overline{h}} \\
= \;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{kl\overline{m}}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{kl\overline{m}}{}^{\overline{p}} - \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{pl\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}l\overline{m}}{}^{\overline{h}} \\
& -\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kp\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{p}\widetilde{R}_{klp}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{h}} \\
= \;& R_{ijp}{}^{h}R_{klm}{}^{p} - R_{ijk}{}^{p}R_{plm}{}^{h} - R_{ijl}{}^{p}R_{kpm}{}^{h} - R_{ijm}{}^{p}R_{klp}{}^{h} \\
= \;& ((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^{h}.
\end{aligned}
$$

Putting $\alpha = i, \beta = j, \gamma = \overline{k}, \theta = \overline{l}, \sigma = \overline{m}, \varepsilon = \overline{n}$ in (3.1), we get

$$
\begin{aligned}
(3.3)\qquad & ((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ij\overline{k}\overline{l}\overline{m}}{}^{\overline{h}} \\
= \;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{\overline{k}\overline{l}\overline{m}}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{\overline{k}\overline{l}\overline{m}}{}^{\overline{p}} - \widetilde{R}_{ij\overline{k}}{}^{p}\widetilde{R}_{p\overline{l}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{k}}{}^{\overline{p}}\widetilde{R}_{\overline{p}\overline{l}\overline{m}}{}^{\overline{h}} \\
& -\widetilde{R}_{ij\overline{l}}{}^{p}\widetilde{R}_{\overline{k}p\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{l}}{}^{\overline{p}}\widetilde{R}_{\overline{k}\overline{p}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{p}\widetilde{R}_{\overline{k}\overline{l}p}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{\overline{p}}\widetilde{R}_{\overline{k}\overline{l}\overline{p}}{}^{\overline{h}} \\
= \;& 0.
\end{aligned}
$$

Putting $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = \overline{h}$ in (3.1), we obtain

$$
\begin{aligned}
(3.4)\qquad & ((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijklm}{}^{\overline{h}} \\
= \;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{klm}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{klm}{}^{\overline{p}} - \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{plm}{}^{\overline{h}} - \widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}lm}{}^{\overline{h}} \\
& -\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kpm}{}^{\overline{h}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}m}{}^{\overline{h}} - \widetilde{R}_{ijm}{}^{p}\widetilde{R}_{klp}{}^{\overline{h}} - \widetilde{R}_{ijm}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{h}}.
\end{aligned}
$$

Let us assume that

$$
\begin{aligned}
(3.5)\qquad \widetilde{R}_{ijk}{}^{\overline{h}} \;=\; & y^{s}(\nabla_{i}R_{sjk}{}^{h} - \nabla_{j}R_{sik}{}^{h}) + g^{mh}(\nabla_{i}(\nabla_{k}c_{jm} - \nabla_{m}c_{jk}) \\
& -\nabla_{j}(\nabla_{k}c_{im} - \nabla_{m}c_{ik}) - R_{ijm}{}^{l}c_{lk} - R_{ijk}{}^{l}c_{ml}) \\
=\; & 0.
\end{aligned}
$$

On differentiating the above equation with respect to $\partial_{\overline{l}}$, we obtain

$$
(3.6)\qquad \nabla_{i}R_{ljk}{}^{h} - \nabla_{j}R_{lik}{}^{h} = 0.
$$

In the case, (3.5) reduces the following form

$$
(3.7)\qquad \nabla_{i}(\nabla_{k}c_{jm} - \nabla_{m}c_{jk}) - \nabla_{j}(\nabla_{k}c_{im} - \nabla_{m}c_{ik}) - R_{ijm}{}^{l}c_{lk} - R_{ijk}{}^{l}c_{ml} = 0.
$$

The other coefficients of $(\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}$ reduce to one of (3.2) and (3.3). Hence the proof follows from (3.2)-(3.7). □

Let $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ denote the components of Ricci tensor of $(TM, \widetilde{g})$. From the Proposition 2, we have only one non-zero component, i.e., $\widetilde{R}_{ij} = 2R_{ij}$ and other components are zero. The tensor $(\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{Ric})(\widetilde{Z},\widetilde{W})$ has components

$$
((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{Ric})(\widetilde{Z},\widetilde{W}))_{\alpha\beta\gamma\theta} = \widetilde{R}_{\alpha\beta\gamma}{}^{\varepsilon}\widetilde{R}_{\varepsilon\theta} + \widetilde{R}_{\alpha\beta\theta}{}^{\varepsilon}\widetilde{R}_{\gamma\varepsilon}
$$

with respect to the adapted frame $\{E_\beta\}$. The following corollary is a direct consequence of Theorem 1.

Corollary 1. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M,g). Then the tangent bundle $(TM,\widetilde{g})$ is Ricci semi-symmetric if and only if the Riemannian manifold (M,g) is Ricci semi-symmetric.*

4. Semi-symmetry properties of the tangent bundle with respect to the metric connection ∇ with torsion

H. A. Hayden introduced the idea of metric connection with torsion on a Riemannian manifold [3]. A linear connection ∇ on a Riemannian manifold (M,g) is called a metric connection with repect to g if $\nabla g=0$ and its torsion tensor is non-zero. In this section we consider a metric connection on $(TM,\widetilde{g})$ with a non-zero torsion tensor.

Let $\widehat{\nabla}$ be a metric connection on TM and its torsion tensor $\widehat{T}$ have only one non-zero component, i.e., $\widehat{T}\ {}^{\overline{h}}_{ij}=y^sR_{ijs}{}^h$ and other components are zero. The torsion tensor $\widehat{T}$ is very specific, which is actually related to the Lie bracket of $[E_i,E_j]$ (see the formulas (2.3)). On following the method shown in ([9], p.151-152), we get:

Proposition 3. *The metric connection $\widehat{\nabla}$ of $(TM,\widetilde{g})$ is given by*

$$\begin{cases}\widehat{\nabla}_{E_i}E_j=\Gamma^h_{ij}E_h+g^{mh}(\nabla_ic_{mj}+\nabla_jc_{im}-\nabla_mc_{ij})E_{\overline{h}},\\ \widehat{\nabla}_{E_i}E_{\overline{j}}=\Gamma^h_{ij}E_{\overline{h}},\\ \widehat{\nabla}_{E_{\overline{i}}}E_j=0,\ \widehat{\nabla}_{E_{\overline{i}}}E_{\overline{j}}=0,\end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij} denote components of the Levi-Civita connection ∇ of g on M and c_{ij} denotes the symmetric part of a_{ij}, i.e., $c_{ij}=\frac{1}{2}(a_{ij}+a_{ji})$ (see also [1]).

For the curvature tensor $\widehat{R}$ of the metric connection $\widehat{\nabla}$, we have:

Proposition 4. *The curvature tensor $\widehat{R}$ of the metric connection $\widehat{\nabla}$ is given by*

$$\begin{aligned}\widehat{R}(E_i,E_j)E_k &= R_{ijk}{}^hE_h+g^{mh}(\nabla_i(\nabla_kc_{jm}-\nabla_mc_{jk})\\ &\quad -\nabla_j(\nabla_kc_{im}-\nabla_mc_{ik})-R_{ijm}{}^lc_{lk}-R_{ijk}{}^lc_{ml})\}E_{\overline{h}},\\ \widehat{R}(E_i,E_j)E_{\overline{k}} &= R_{ijk}{}^hE_{\overline{h}},\ \widehat{R}(E_i,E_{\overline{j}})E_k=0,\ \widehat{R}(E_{\overline{i}},E_j)E_k=0,\\ \widehat{R}(E_{\overline{i}},E_{\overline{j}})E_k &= 0,\widehat{R}(E_{\overline{i}},E_j)E_{\overline{k}}=0,\ \widehat{R}(E_i,E_{\overline{j}})E_{\overline{k}}=0,\\ \widehat{R}(E_{\overline{i}},E_{\overline{j}})E_{\overline{k}} &= 0\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$, where $R_{ijk}{}^h$ denote components of the curvature tensor field R of g on M.

Theorem 2. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M,g) and $\widehat{\nabla}$ be the metric connection with*

torsion of $(TM,\widetilde{g})$. *We assume that* $\nabla_i(\nabla_k c_{jm}-\nabla_m c_{jk})-\nabla_j(\nabla_k c_{im}-\nabla_m c_{ik})-R_{ijm}{}^l c_{lk}-R_{ijk}{}^l c_{ml}=0$, *where* R *is the curvature tensor of the Levi-Civita connection* ∇ *of* g. *Then the tangent bundle* $(TM,\widetilde{g})$ *is semi-symmetric with respect to the metric connection* $\widehat{\nabla}$ *if and only if the Riemannian manifold* (M,g) *is semi-symmetric.*

Proof. We calculate

$$
\begin{aligned}
(4.1)\qquad & ((\widehat{R}(\widetilde{X},\widetilde{Y})\cdot\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}\\
= \;& \widehat{R}_{\alpha\beta\tau}{}^{\varepsilon}\,\widehat{R}_{\gamma\theta\sigma}{}^{\tau}-\widehat{R}_{\alpha\beta\gamma}{}^{\tau}\,\widehat{R}_{\tau\theta\sigma}{}^{\varepsilon}-\widehat{R}_{\alpha\beta\theta}{}^{\tau}\,\widehat{R}_{\gamma\tau\sigma}{}^{\varepsilon}-\widehat{R}_{\alpha\beta\sigma}{}^{\tau}\,\widehat{R}_{\gamma\theta\tau}{}^{\varepsilon}
\end{aligned}
$$

for all cases $\alpha=(i,\overline{i})$, $\beta=(j,\overline{j})$, $\gamma=(k,\overline{k})$, $\theta=(l,\overline{l})$, $\sigma=(m,\overline{m})$ and $\varepsilon=(h,\overline{h})$.

The cases $\alpha=i$, $\beta=j$, $\gamma=k$, $\theta=l$, $\sigma=m$, $\varepsilon=h$ and $\alpha=i$, $\beta=j$, $\gamma=k$, $\theta=l$, $\sigma=\overline{m}$, $\varepsilon=\overline{h}$ in (4.1), we respectively get

$$
\begin{aligned}
& ((\widehat{R}(\widetilde{X},\widetilde{Y})\cdot\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijklm}{}^{h}\\
= \;& \widehat{R}_{ijp}{}^{h}\,\widehat{R}_{klm}{}^{p}+\widehat{R}_{ij\overline{p}}{}^{h}\,\widehat{R}_{klm}{}^{\overline{p}}-\widehat{R}_{ijk}{}^{p}\,\widehat{R}_{plm}{}^{h}-\widehat{R}_{ijk}{}^{\overline{p}}\,\widehat{R}_{\overline{p}lm}{}^{h}\\
& -\widehat{R}_{ijl}{}^{p}\,\widehat{R}_{kpm}{}^{h}-\widehat{R}_{ijl}{}^{\overline{p}}\,\widehat{R}_{k\overline{p}m}{}^{h}-\widehat{R}_{ijm}{}^{p}\,\widehat{R}_{klp}{}^{h}-\widehat{R}_{ijm}{}^{\overline{p}}\,\widehat{R}_{kl\overline{p}}{}^{h}\\
= \;& R_{ijp}{}^{h}R_{klm}{}^{p}-R_{ijk}{}^{p}R_{plm}{}^{h}-R_{ijl}{}^{p}R_{kpm}{}^{h}-R_{ijm}R_{klp}{}^{h}\\
= \;& ((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^{h}
\end{aligned}
$$

and similarly

$$
((\widehat{R}(\widetilde{X},\widetilde{Y})\cdot\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijkl\overline{m}}{}^{\overline{h}}=((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^{h}.
$$

The case $\alpha=i$, $\beta=j$, $\gamma=k$, $\theta=l$, $\sigma=m$, $\varepsilon=\overline{h}$ in (4.1), we obtain

$$
\begin{aligned}
& ((\widehat{R}(\widetilde{X},\widetilde{Y})\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijklm}{}^{\overline{h}}\\
= \;& \widehat{R}_{ijp}{}^{\overline{h}}\,\widehat{R}_{klm}{}^{p}+\widehat{R}_{ij\overline{p}}{}^{\overline{h}}\,\widehat{R}_{klm}{}^{\overline{p}}-\widehat{R}_{ijk}{}^{p}\,\widehat{R}_{plm}{}^{\overline{h}}-\widehat{R}_{ijk}{}^{\overline{p}}\,\widehat{R}_{\overline{p}lm}{}^{\overline{h}}\\
& -\widehat{R}_{ijl}{}^{p}\,\widehat{R}_{kpm}{}^{\overline{h}}-\widehat{R}_{ijl}{}^{\overline{p}}\,\widehat{R}_{k\overline{p}m}{}^{\overline{h}}-\widehat{R}_{ijm}{}^{p}\,\widehat{R}_{klp}{}^{\overline{h}}-\widehat{R}_{ijm}{}^{\overline{p}}\,\widehat{R}_{kl\overline{p}}{}^{\overline{h}}\\
= \;& \widehat{R}_{ijp}{}^{\overline{h}}\,\widehat{R}_{klm}{}^{p}+\widehat{R}_{ij\overline{p}}{}^{\overline{h}}\,\widehat{R}_{klm}{}^{\overline{p}}-\widehat{R}_{ijk}{}^{p}\,\widehat{R}_{plm}{}^{\overline{h}}-\widehat{R}_{ijl}{}^{p}\,\widehat{R}_{kpm}{}^{\overline{h}}\\
& -\widehat{R}_{ijm}{}^{p}\,\widehat{R}_{klp}{}^{\overline{h}}-\widehat{R}_{ijm}{}^{\overline{p}}\,\widehat{R}_{kl\overline{p}}{}^{\overline{h}}.
\end{aligned}
$$

For the other cases, $(\widehat{R}(\widetilde{X},\widetilde{Y})\cdot\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U}=0$. If $\widehat{R}_{ijp}{}^{\overline{h}}=0$, i.e., $\nabla_i(\nabla_k c_{jm}-\nabla_m c_{jk})-\nabla_j(\nabla_k c_{im}-\nabla_m c_{ik})-R_{ijm}{}^l c_{lk}-R_{ijk}{}^l c_{ml}=0$, then $(\widehat{R}(\widetilde{X},\widetilde{Y})\cdot\widehat{R})(\widetilde{Z},\widetilde{W})\widetilde{U}=0$ if and only if $(R(X,Y)\cdot R)(Z,W)U=0$. This complets the proof. □

On computing the contracted curvature tensor (Ricci tensor) $\widehat{R}_{\gamma\beta}$, the only non-zero component is as follows: $\widehat{R}_{ij}=R_{ij}$ (see also [1]). As a consequence of the Theorem 3, we obtain:

Corollary 2. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M,g) and $\widehat{\nabla}$ be the metric connection of $(TM,\widetilde{g})$. Then the tangent bundle $(TM,\widetilde{g})$ is Ricci semi-symmetric with respect to the metric connection $\widehat{\nabla}$ if and only if the Riemannian manifold (M,g) is Ricci semi-symmetric.*

References

[1] M. Aras, *The metric connection with respect to the synectic metric,* Hacet. J. Math. Stat., 41 (2012), 169–173.

[2] A. Gezer, *On infinitesimal conformal transformations of the tangent bundles with the synectic lift of a Riemannian metric*, Proc. Indian Acad. Sci. Math. Sci., 119 (2009), 345–350.

[3] H. A. Hayden, *Sub-spaces of a space with torsion*, Proc. London Math. Soc., S2-34 (1932), 27-50.

[4] V. Oproiu, N. Papaghiuc, *On the geometry of tangent bundle of a (pseudo-) Riemannian manifold*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), 44 (1998), 67–83.

[5] N. S. Sinjukov, *Geodesic mappings of Riemannian spaces* (Russian), Publishing House "Nauka", Moscow, 1979.

[6] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$*, I. The local version, J. Differential Geom., 17 (1982), 531–582.

[7] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$. II*, Global version, Geom. Dedicata, 19 (1985), 65-108.

[8] N. V. Talantova, A. P. Shirokov, *A remark on a certain metric in the tangent bundle*, Izv. Vyss. Uchebn. Zaved. Math., 157 (1975), 143-146.

[9] K. Yano, S. Ishihara, *Tangent and cotangent bundles*, Marcel Dekker, Inc., New York 1973.

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Absolutely self pure modules

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Abstract. An R-module M is called absolutely self pure if for any finitely generated left ideal L of R, any map from L into M, whose kernel is in the filter generated by the set of all annihilator ideals $\mathrm{ann}(m)$ of elements m in M, is a restriction of a map from the ring R into M. Certain properties of quasi injective and absolutely pure modules are extended to absolute self-purity. Regular, left noetherian, left semihereditary and Prüfer rings are characterized using this concept, for example, a ring R is left noetherian if and only if every absolutely self pure R-module is quasi injective and R is regular if and only if every R-module is absolutely self pure.
Keywords: (self) pure submodule, absolutely (self) pure module, quasi injective module.

1. Introduction

Let R be an associative ring with identity. By a module and an R-module we mean a left unital R-module. Given two modules M and N, recall that M is said to be N-*injective* if for any submodule L of N, any map $L \to M$ is a restriction of a map $N \to M$. If M is M-injective then it is called *quasi injective*. The module M is called *injective* if it is N-injective for all R-modules N. Baer condition states that a module is injective if and only if it is R-injective. The minimal (quasi) injective module that contains M as a submodule is called the (*quasi*) *injective envelope* of M. The quasi injective envelope of an R-module M is denoted $Q(M)$. For details, the reader is referred to [2] and [4].

A submodule A of an R-module B is called a *pure submodule* [8] if for any commutative diagram with K a finitely generated submodule of a free R-module F

$$\begin{array}{ccc} K & \hookrightarrow & F \\ \downarrow & \swarrow & \downarrow \\ A & \hookrightarrow & B \end{array}$$

there is a map $F \to A$ making the upper triangle commutative. A module A is called *absolutely pure* [5] if it is pure in every module containing it as a submodule or equivalently if A is pure in some injective module or A is pure in

its injective envelope [6]. Our goal is to study modules that are pure in their quasi injective envelopes. Of course, if the module is pure in every quasi injective module then it must be absolutely pure. So we give a weaker form of purity, *self purity*, and study modules that are self pure in every module containing them as submodules. This turns out to be equivalent to saying that the module is self pure in some quasi injective module or self pure in its quasi injective envelope.

By $\Omega(M)$ we mean the set of all left ideals L of R such that $L \supseteq \text{ann}(m)$ for some $m \in M$. The filter generated by $\Omega(M)$ in the lattice of left ideals of R is denoted $\overline{\Omega}(M)$. Recall that L. Fuchs [3] has proved that a module M is quasi injective if and only if for any homomorphism f from a left ideal L of R whose kernel is in $\overline{\Omega}(M)$ there is an extension to a homomorphism $R \to M$ of f. This gives a generalization of Baer condition for injective modules.

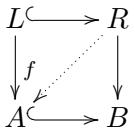
On the other hand, absolutely pure modules A are characterized by the property that for any finitely generated submodule K of a free module F, any homomorphism $K \to A$ can be extended to a homomorphism $F \to A$ [6].

Ramamurthi and Rangaswamy [7] defined an R-module M to be *finitely R-injective* if any map from a finitely generated left ideal of R into M can be extended to a map $R \to M$. So any absolutely pure R-module is finitely R-injective and the converse is true if R is left semihereditary [6].

Generalizing the above facts, we say that a module A is absolutely self pure if any map from a finitely generated left ideal of R into A whose kernel is in $\overline{\Omega}(A)$ can be extended to a map $R \to A$. Regular, left noetherian, left semihereditary and Prüfer rings are characterized using properties of absolutely self pure modules.

2. (Absolute) Self purity

Definition 2.1. A submodule A of an R-module B is called a *self pure* submodule of B (denoted $A \leq^{sp} B$) if the following condition holds: For any finitely generated left ideal L of R and any map $f : L \to A$ with $\ker f \in \overline{\Omega}(A)$, if there is a map $R \to B$ making the following diagram commutative:



then there exists a map $R \to A$ making the upper triangle commutative.

Any pure submodule of a module is self pure, but not conversely, for one may take any quasi injective module that is not absolutely pure, which must clearly be self pure in its injective envelope, but of course not pure.

More generally, one can study purity with respect to another module M. Precisely, we say that a submodule A of an R-module B is called *M-pure* submodule of B if the following condition holds: For any finitely generated left ideal

L of R and any map $f : L \to A$ with $\ker f \in \overline{\Omega}(M)$, if there is a map $R \to B$ making the above diagram commutative then there exists a map $R \to A$ making the upper triangle commutative. Therefore, A is A-pure in B exactly when A is self pure in B. However we will restrict our attention to the concept of self purity.

Proposition 2.2. *If A, B and C are R-modules such that $A \leq^{sp} B$ and $B \leq^{sp} C$ then $A \leq^{sp} C$.*

Proof. Consider the following diagram

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow \\ A & \hookrightarrow & C \end{array}$$

where L is a finitely generated left ideal of R with $\ker f \in \overline{\Omega}(A)$ and $R \to C$ is a map making the diagram commutative. Then obviously, we can consider f as a map $L \to B$ for which we have a commutative diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow \\ B & \hookrightarrow & C \end{array}$$

Since $B \leq^{sp} C$, there is an extension $g : R \to B$ of f, so that the following square commutes:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A & \hookrightarrow & B \end{array}$$

and since $A \leq^{sp} B$, there is an $h : R \to A$ extending f. □

Proposition 2.3. *If $A \subseteq B \subseteq C$ are R-modules such that $A \leq^{sp} C$ then $A \leq^{sp} B$.*

Proof. Consider the following commutative diagram with the obvious maps:

$$\begin{array}{ccccc} L & \hookrightarrow & R & & \\ \downarrow{\scriptstyle f} & & \downarrow & & \\ A & \hookrightarrow & B & \hookrightarrow & C \end{array}$$

Since $A \leq^{sp} C$, there is a $g : R \to A$ extending f. □

Definition 2.4. A module is called *absolutely self pure* if it is self pure in every module containing it as a submodule.

It is clear that quasi injective and absolutely pure modules are examples of absolutely self pure modules. If a module A is self pure in some quasi injective module then by Proposition 2.3, it must be self pure in its quasi injective envelope $Q(A)$. So if A is contained in some other module B, then as $Q(A) \leq^{sp} Q(B)$ we must have by Propositions 2.2 and 2.3 that $A \leq^{sp} B$. Therefore, a module is absolutely self pure if and only if it is self pure in some quasi injective module if and only if it is self pure in its quasi injective envelope.

Theorem 2.5. *A module A is absolutely self pure if and only if for each finitely generated left ideal L of R and each map $f : L \to A$ with $\ker f \in \overline{\Omega}(A)$ there is an extension map $R \to A$ of f.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A & \hookrightarrow & Q(A) \end{array}$$

where $Q(A)$ denotes the quasi injective envelope of A. Existence of g is guaranteed be quasi injectivity of $Q(A)$. Now A is self pure in $Q(A)$ if and only if f can be extended to a map $R \to A$. □

By Proposition 2.2, a self pure submodule of an absolutely self pure module is again absolutely self pure. In particular, direct summands of absolutely self pure modules are absolutely self pure. Examples of absolutely self pure modules now extend to finitely R-injectives.

Let K be a left ideal of the ring R. An R-module A is called *K-bounded* [3] if $\Omega(A)$ consists of left ideals that contain K. That is, K is the smallest element in $\Omega(A)$. In that case K must be a two-sided ideal [3] and we can think of A as an R/K-module. So we get:

Proposition 2.6. *Let K be a finitely generated left ideal of a ring R.Then a K-bounded R-module A is absolutely self pure if and only if it is finitely R/K-injective as an R/K-module.*

Theorem 2.7. *A module A is absolutely self pure if and only if any direct sum of copies of A is absolutely self pure.*

Proof. Suppose that A is absolutely self pure and let f be a map from a finitely generated left ideal L of R into $A^{(I)}$ for some index set I such that $\ker f \in \overline{\Omega}(A^{(I)})$, i.e. $\ker f \supseteq \bigcap_{j\in J} \operatorname{ann}((a_i)_j)$ for finite J. But the non-zero a_i's in each $(a_i)_j$ are also finite. Therefore $\ker f$ contains the intersection of annihilators of a finite set of individual a_i's and hence it belongs to $\overline{\Omega}(A)$. Let $\{l_1, \cdots, l_n\}$ be a generating set for L. Each of the images $f(l_i)$ is of finite support, so f can be considered as a map $L \to A^{(K)}$ for some finite subset K of I and hence it is the

finite direct sum of coordinate maps f_k into each factor A. Clearly $\ker f_k \supseteq \ker f$ and so $\ker f_k \in \overline{\Omega}(A)$ for each coordinate map f_k. By absolute self purity of A, each f_k is extendable to a $g_k : R \to A$. The map whose coordinate maps are the g_k's is the desired extension of f. The other implication is clear. □

3. Characterization of rings

Left noetherian rings are precisely rings over which every absolutely pure module is (quasi) injective [6, Theorem 3] and [1, Theorem 8]. It is clear that over such rings, the concepts of absolutely self pure and that of quasi injective modules are equivalent and by [1, Theorem 8] the converse is also true. So we have the following characterization of left noetherian rings:

Theorem 3.1. *A ring R is left noetherian if and only if any absolutely self pure R-module is quasi injective.* □

Recall that a ring R is called regular if every principal left ideal of R is a direct summand. Over a regular ring, all modules are absolutely pure and hence all modules are absolutely self pure. The converse is also true:

Theorem 3.2. *A ring R is regular if and only if every left R-module is absolutely self pure.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) Given a principal left ideal L of R, extend the identity map of L to the map $L \to L \oplus R$ defined by $a \mapsto (a, 0)$. This map clearly has kernel containing $\operatorname{ann}(0, 1)$. By assumption $L \oplus R$ is absolutely self pure, hence there is a $g : R \to L \oplus R$ extending f. Follow g by the projection $L \oplus R \to L$ to get an extension of the identity map $L \to L$, which shows that L is a direct summand of R. □

Combining Theorems 3.1 and 3.2 we get the well-known characterization that R is semisimple (= regular and left noetherian) if and only if every R-module is quasi injective.

By Theorem 3.1, over the ring of integers $\mathbb{Z}$ the absolutely self pure modules are exactly the quasi injective ones. So any quasi injective abelian group that is not injective serves as an example of an absolutely self pure module which is not absolutely pure. Also by Theorems 3.2 and 3.1, if a ring R is regular but not left noetherian then there must exist an absolutely self pure R-module that is not quasi injective. Another example is the following.

Example 3.3. We know that if R is a regular but not a left noetherian ring then there must exist an absolutely pure R-module A that is not quasi injective. Over the ring $\mathbb{Z}$ the module $\mathbb{Z}_2$ is quasi injective but not absolutely pure. Now let R and A be as above and consider the module $M = \begin{pmatrix} A \\ \mathbb{Z}_2 \end{pmatrix}$ over the ring $S = \begin{pmatrix} R & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. The module M is not quasi injective (nor finitely R-injective

nor absolutely pure), for otherwise the direct summand $\begin{pmatrix} A \\ 0 \end{pmatrix}$ would be quasi injective (the direct summand $\begin{pmatrix} 0 \\ \mathbb{Z}_2 \end{pmatrix}$ would be finitely R-injective). Now we proceed to show that M is absolutely self pure. Any finitely generated left ideal of S is of the form $K = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$, where I and J are finitely generated left ideals of R and $\mathbb{Z}$, respectively. Any map $K \to M$ is of the form $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$, where $f \in \mathrm{Hom}_R(I, A)$ and $g \in \mathrm{Hom}_{\mathbb{Z}}(J, \mathbb{Z}_2)$. To extend $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$ to S is to extend both f to R and g to $\mathbb{Z}$. The former is always possible because I is a finitely generated left ideal of a regular ring, hence it is a direct summand. So we focus on extending g to a map $\mathbb{Z} \to \mathbb{Z}_2$. But if $\ker(K \to M)$ contains the intersection of $\mathrm{ann}\begin{pmatrix} a_i \\ z_i \end{pmatrix}$ for some $\begin{pmatrix} a_i \\ z_i \end{pmatrix} \in M$, $i = 1, \cdots, n$, which must be equal to $\begin{pmatrix} \cap \mathrm{ann}(a_i) & 0 \\ 0 & \cap \mathrm{ann}(z_i) \end{pmatrix}$ then we need only to focus on $\cap \mathrm{ann}(z_i)$. This latter is equal to $2\mathbb{Z}$ or $\mathbb{Z}$ according to whether one of the z_i's is 1 or not. In any case we must have that $2\mathbb{Z} \subseteq \ker(g : J \to \mathbb{Z}_2)$. Hence either $J = \mathbb{Z}$ and we are done or $J = 2\mathbb{Z}$ and therefore g is the zero map.

Semihereditary rings are characterized in [6] as those rings over which absolute purity is preserved when taking homomorphic images. Here we relax this condition to absolute self purity, but first we need a lemma.

Lemma 3.4. *A left ideal K of a ring R is projective if and only if for any epimorphism $A \to A'$ from an (injective) module A, any map $K \to A'$ whose kernel contains the annihilator of some $a' \in A'$ can be lifted to a map $K \to A$.*

Proof. The nontrivial implication is to show that K is projective. So consider the following diagram:

$$\begin{array}{ccc} & & K \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & A' \end{array}$$

with the obvious maps and replace A and A' by $A \oplus E(R)$ and $A' \oplus E(R)$ respectively and α by $\alpha \oplus 1_{E(R)}$, where $E(R)$ is the injective envelope of the ring R. Now clearly the kernel of (the new) f contains $\{0\} = \mathrm{ann}(0, 1)$ and therefore f lifts to a map $K \to A \oplus E(R)$ when followed by a projection we get a lifiting of $f : K \to A'$. □

Theorem 3.5. *A ring R is left semihereditary if and only if the homomorphic image of any absolutely pure R-module is absolutely self pure.*

Proof. If R is left semihereditary then the result is clear by Theorem 2 of [6]. For the other implication, we will show that any finitely generated left ideal of R is projective. So consider the following diagram:

$$\begin{array}{ccc} R & \longleftarrow & K \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & A' \end{array}$$

with K a finitely generated left ideal of R, $K \to R$ the inclusion map ι, $f : K \to A'$ any map whose kernel contains the annihilator of some a' in A' and $\alpha : A \to A'$ is any epimorphism from an injective module A. By assumption A' is absolutely self pure. This means there is a $g : R \to A'$ extending f. But R is projective, so there is an $h : R \to A$ lifting g.

$$\begin{array}{ccc} R & \longleftarrow & K \\ & \overset{h}{\swarrow} \quad \overset{g}{\searrow} & \downarrow f \\ A & \xrightarrow{\alpha} & A' \end{array}$$

So we have, $\alpha h \iota = g\iota = f$ and K is projective by the above Lemma. □

A similar argument may be carried out to show that a ring R is left hereditary if and only if the homomorphic image of any injective module is quasi injective.

Meggibon [6] proved that absolutely pure modules are divisible and that the converse is true when and only when R is a Prüfer domain. We can relax this condition as follows.

Theorem 3.6. *Let R be a commutative integral domain then R is a Prüfer ring if and only if any divisible R-module is absolutely self pure.*

Proof. If R is Prüfer then the result is immediate by Theorem 6 of [6]. Conversely, we know that any homomorphic image of a divisible module is divisible which implies that the image of any divisible module is absolutely self pure by assumption. In particular the image of any absolutely pure R-module is absolutely self pure, and therefore R must be semihereditary by Theorem 3.5. □

A note about modules that are pure in their quasi injective envelopes is in order. Let us call them *absolutely quasi pure* modules. It is obvious that this concept lies between quasi injectivity and absolute self purity. It is not clear whether every absolutely self pure module is absolutely quasi pure. Of course over regular rings the two concepts are equivalent, and over left noetherian rings they are also equivalent to quasi injectivity. If a module A is absolutely quasi pure then so is any finite direct sum of copies of A. In fact, slight modifications can be applied to the proofs of Theorems 3.1, 3.2, 3.5 and 3.6 to replace the words “absolutely self pure” by “absolutely quasi pure”.

References

[1] M. S. Abbas and M. F. Hamid, *(Flat) modules that are fully invariant in their pure-injective (cotorsion) envelopes*, Al-Mustansiriyah J. Sci., 22 (2011), 351–359.

[2] F. W. Anderson and K. R. *Fuller, Rings and Categories of Modules*, Springer-Verlag New York, 1992.

[3] L. Fuchs, *On quasi injective modules*, Annali della Scuola Norm. Sup. Pisa, 23 (1969), 541–546.

[4] R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., 36 (1961), 260–268.

[5] B. Maddox, *Absolutely pure modules*, Proc. Amer. Math. Soc., 18 (1967), 155–158.

[6] C. Meggibon, *Absolutely pure modules*, Proc. Amer. Math. Soc., 26 (1970), 561-566.

[7] V.S. Ramamurthi, K.M. Rangaswamy, *On finitely injective modules*, J. Austral. Math. Soc., 16 (1973), 239-248.

[8] B. T. Stenström, *Pure submodules*, Arkiv for Mat., 7 (1967), 159–171.

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Cryptographic study of functional message using two chaotic models

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Abstract. In this paper we present two chaotic models used for secure transmission of a functional message which is the function $\sin \omega t$, without forgetting the role of synchronization mechanisms of chaotic systems to the success of these transmissions.
Keywords: chaos, synchronization, cryptography, CSK modulator, CSK demodulator.

1. Introduction

Hide specific information to certain persons always been a major interest of people. Thus, we have sought to establish technics called "encryption" in order to make this information incomprehensible to those who do not have access to a secret "key." In the past century, several researchers have examined the unusual behavior of chaotic dynamical systems and it was found that some systems showed very strange nature of instabilities. It was the discovery of chaotic signals whose behavior completely deterministic but are reminiscent of gaits pseudo-random. A universal definition of chaos exists not really, mathematicians who study chaotic systems use certain characteristics of stability system (Lyapunov exponents) for defining a chaotic behavior [2].

In 1990, Pecora and Carroll published a paper [1] in which they present theoretical and experimental demonstration of the ability to synchronize two chaotic systems. Here, the synchronization means that two chaotic systems with the same structure but of course with initial conditions different are led to reproduce the same chaotic signal. This phenomenon appeared to be impossible for systems chaotic [2] [3] (these are very sensitive to disturbances on their

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trajectories) was a revolution in the community scientific, thus the use of the chaos in cryptography [4].

Initially, several electronic assemblies simulating synchronization of Lorenz and Rössler systems were achieved [5]. They were followed by numerous configurations to allow transmission of signals "secure" [6], the most notably those of Cuomo [7] in 1992 that used a transmission by masking and spread spectrum. In the decade that followed there were many other configurations and improvements of existing ones [8]. In our report we will study on detail the use of two chaotic models; "Four scroll" model [10],[12],[13] and Lorenz model [11],[14] to secure a functional message which is the function $\sin \omega t$ by using Chaos Shift Keying method [9].

2. Characterization of the chaotic systems

2.1 Four-scroll model

Consider the Four-scroll pattern [12],[13]:

$$\begin{cases} \dot{x}_1 = ax_1 + y_1 z_1 \\ \dot{y}_1 = -by_1 + x_1 z_1 \\ \dot{z}_1 = -cz_1 + x_1 y_1 \end{cases} \tag{1}$$

x, y and z are state variables, and a, b and c real parameters, consistent and positives. In the following numerical data : $a = 0,4$, $b = 12$ et $c = 5$, the system four scroll (1) have the chaotic attractor of Figure 1.

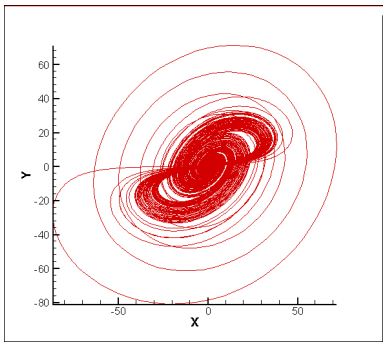


Figure 1: Four-scroll attractor

2.2 Lorenz model

Consider the Lorenz pattern [14]:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) \\ \dot{y}_1 = bx_1 - x_1 z_1 - y_1 \\ \dot{z}_1 = x_1 y_1 - cz_1 \end{cases} \tag{2}$$

x, y and z are state variables, and a, b and c real parameters, consistent and positives. In the following numerical data : $a = 10$, $b = \frac{8}{3}$ et $c = 28$, the Lorenz system (2) have the chaotic attractor shows in Figure 2.

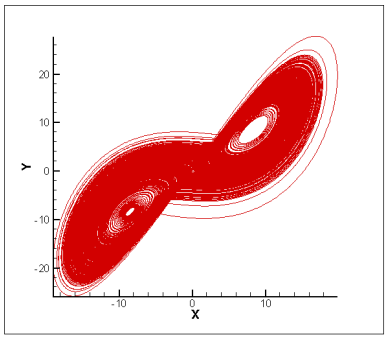


Figure 2: Lorenz attractor

3. Using the chaotic models for secure communications

The basic idea is to blur a message adequately with the chaos at the transmitter level, in order to conceal intruder before forwarding it to its destination, which will be the only able to decipher it. So in this work we will have an encrypted functional message using two chaotic systems fourscroll model [11],[12] and Lorenz model [14] by Chaos Shift Keying method [9] which noted by CSK.

3.1 CSK modulator

This method proposed for the first time by the Dedieu group [9], and its current denomination known by "Chaos shift keying: CSK". The CSK method defined as a digital modulation, it's inspired by the classical techniques of modulation such as FSK (frequency shift keying) ASK (amplitude Shift keying) and the PSK (phase shift keying). Then the masked system by CSK method is consisting CSK modulator at the transmitter and a CSK demodulator at the receiver connected by a router channel of the signal as shown in the figure (3).

So, its basic idea is the same as the classical digital modulation, that is to associate with each symbol of the message not a sinusoidal carrier, but a different chaotic carrier, moving in duration T. The elements of the digital signal message

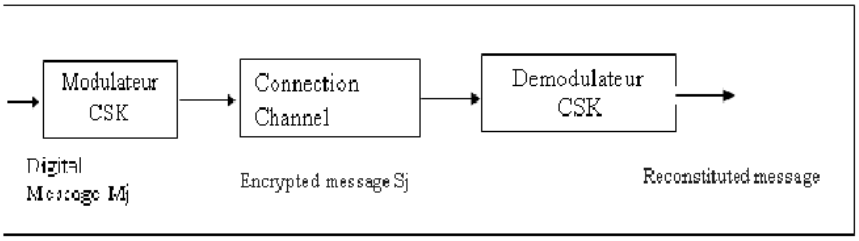


Figure 3: Simplified schema of a CSK method for encryption system

include in the area of M-level symbols modulated by CSK method are defined by:

$$S(t) = \sum_{j=1}^{n} m_j g_i(t) \tag{3}$$

where: m_j are the elements of the message signal vector and $g_i(t)$ are the chaotic carrier. And $j = 1, 2.....N$; $i = 1, 2.....M$ et $N \leq M$

$$m_j = \sin \omega t \tag{4}$$

Figure (4) shows the shape of the message signal mj before the modulation.

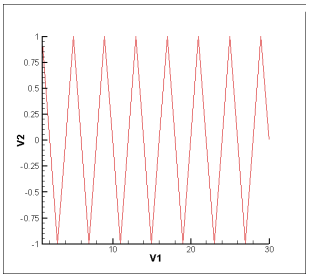


Figure 4: The appearance of the message signal before encryption (Message to the encrypted signal).

3.1.1 CSK modulator using Four-scroll model

Because the speed of the message signal (which we will mask it) is limited by the time of synchronization T second of the chaotic oscillators (5 second in our case [10]) we choose in our simulation the function $\sin \omega t$, with $\omega = \frac{\pi}{2}$ generating sequences 1 and -1, and we adjusted its flow T second equal to 15 second more

than the synchronization time T second for the probability of occurrence of 1 and -1.

It has been verified [10] that the system (1), which considered as master system synchronizes identically with a slave system whose state equations are given by :

$$\begin{cases} \dot{x}_2 = ax_2 + y_2z_2 + u \\ \dot{y}_2 = -by_2 + x_2z_2 \\ \dot{z}_2 = -cz_2 + x_2y_2 \end{cases} \tag{5}$$

where : u is the coupling control chosen by :

$$u = -y_1z_1 + y_2z_2 - (a+1)(x_2 - x_1)$$

With the parameters given in the reference [10]. At time $t = 5$ second, it was found that the timing error between the systems (1) and (5) tending towards 0 second. So we will consider the synchronization Time T second around 15 second.
In our simulation, we used two chaotic oscillators $G1$ (master system) and $G2$ (slave system) to code the 1 and -1 message signal. They are similar but statistically different. Concerning their parameters were chosen respectively:

1. System $G1$: $x_1(0) = 0,\ \ y_1(0) = 1,\ \ z_1(0) = 1$
2. System $G2$: $x_2(0) = -5,6,\ \ y_2(0) = 13,6,\ \ z_2(0) = -12,5$

At the reception, the synchronization blocks $G1$ and $G2$ whose state equations are those of the systems (1) and (5) respectively, the parameters are identical to systems $G1$ and $G2$. At the output of the CSK modulator, the signal $S(t)$ is shown in Figure (5).

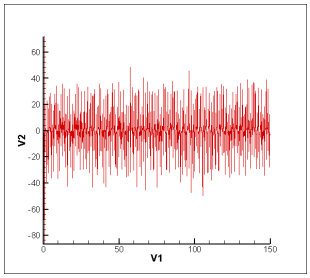


Figure 5: The appearance of the message signal after CSK encryption using Four-scroll model

3.1.2 CSK modulator using Lorenz model

By the same idea of CSK modulator, the message signal will be encrypted. In this case the synchronization time will be adjusted flow T second equal to 15 second more than the synchronization time (4 second in our case [14]) for the probability of occurrence of 1 and -1, because the speed of the message signal (which we will mask it) is limited by the time of synchronization. It has been verified [14] that the system (2), which considered as master system synchronizes identically with a slave system whose state equations are given by :

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + C_1 \\ \dot{y}_2 = bx_2 - x_2 z_2 - y_2 + C_2 \\ \dot{z}_2 = x_2 y_1 - cz_2 + C_3 \end{cases} \tag{6}$$

where : C_1, C_2 and C_3 are three coupling control functions. With properly choice of C_1, C_2 and C_3 and with the parameters given in the reference [14], at time $t = 4$ second, it was found that the timing error between the systems (2) and (6) tending towards 0 second. So we will consider the synchronization Time T second around 15 second.
In our simulation, we used two chaotic oscillators $G1$ (master system) and $G2$ (slave system) to code the 1 and -1 message signal. They are similar but statistically different. Concerning their parameters were chosen respectively:

1. System $G1$: $x_1(0) = 1,\ \ y_1(0) = 1,\ \ z_1(0) = 0$
2. System $G2$: $x_2(0) = 0,2,\ \ y_2(0) = 0,3,\ \ z_2(0) = 0,1$

At the reception, the synchronization blocks $G1$ and $G2$ whose state equations are those of the systems (2) and (6) respectively, the parameters are identical to systems $G1$ and $G2$. At the output of the CSK modulator, the signal $S(t)$ is shown in Figure (6).

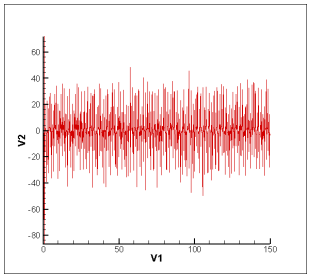


Figure 6: The appearance of the message signal after CSK encryption using Lorenz model

3.2 CSK demodulator

3.2.1 CSK demodulator using Four-scroll model

Assuming that the encrypted message signal $S(t)$ is received by the authorized receiver, shown in Figure (6), without disturbance. At this level we have a synchronization of the two oscillators $G1$ or $G2$ respectively, with the signal received $S(t)$, depending on whether the value of the message signal mj equal 1 or -1. This result is identified by canceling error signal of synchronization e_1, as shown in Figure (7).

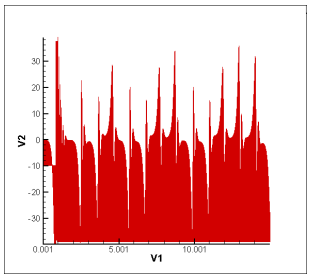


Figure 7: Detecting synchronization areas of $G1$ and $G2$ for Four-scroll model

From this synchronization error, we will take the decision of retrieve the message signal in the decision block.

The processing procedure in this block can be summarized as follow:

1. whatever the error sign of the identical synchronization between two chaotic signals, the basic rule is the zero error involves the synchronization. Figure (7) represents the detection of the synchronization areas of $G1$ and $G2$, and the signals of the function $\sin \omega t$ which are enveloping the chaotic carrier. To detect these envelopes, we proceeded as in classical demodulation, clipping our signal in the interval $[-10, 10]$ by the limiting stage. This will provide the signal shown in Figure (8).

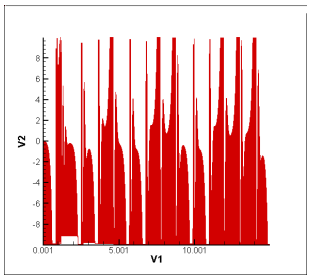


Figure 8: Limited values of synchronization error signal for Four-scroll model.

2. Then these signals are filtered by low-pass filters, to mitigate the chaotic carrier and keep the envelopes of $\sin \omega t$. The filtered signal is shown in Figure (9).

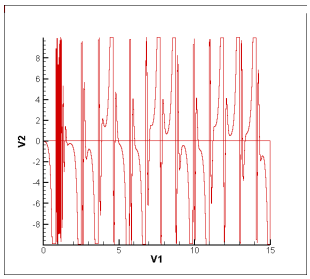


Figure 9: Filtering the error signal for Four-scroll model.

3. The signals of the function $\sin \omega t$ results of signal error, will be shown in Figure (10) by thresholding and comparison. We have used for this the comparator stage performing the following operation :

$$\begin{cases} 1 \Rightarrow signal \geq 0 \\ -1 \Rightarrow signal < 0. \end{cases}$$

We will have the signal shown in Figure (10).

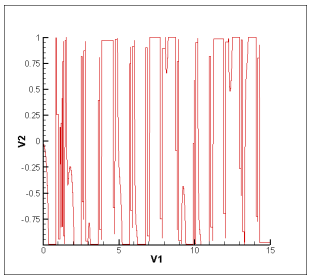


Figure 10: Thresholding and comparison of error signal for Four-scroll model.

4. To reconstruct the original signal, we have performed detecting the error signal rising edges as shown in the figure (10) by the detector rising edges stage. This signal becomes as shown in Figure (11).

5. By comparing the signal shown in Figure (11) with the original signal message mj shown in Figure (4), It was noted that the message signal was slightly tilted according to the error signal $\sin \omega t$. So to recover the original message signal, this signal will be injected into an T flip-flop.

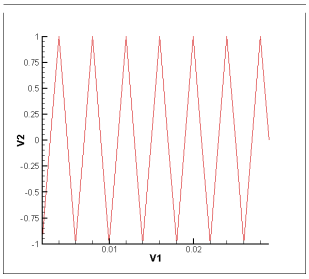


Figure 11: Detection of the rising edge for Four-scroll model.

Figures (12) represent the signal reconstructed by the decrypting operation using Four-scroll model and the original signal respectively. So they are identique.

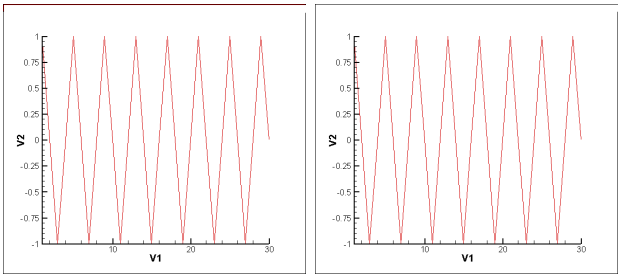


Figure 12: Message after decryption for Four-scroll model at the right and the original message at the left.

3.2.2 CSK demodulator using Lorenz model

Now, we will decrypt the signal message which we masked it using Lorenz model by the same steps of CSK demodulator. Firstly, whatever the error sign of the identical synchronization between the two chaotic signals, the basic rule is the zero error involves the synchronization. Figure (13) represents the detection of the synchronization areas of $G1$ and $G2$, and the signals of the function $\sin \omega t$ which are enveloping the chaotic carrier. To detect these envelopes, we proceeded as in classical demodulation, clipping our signal in the interval $[-10, 10]$ by the limiting stage as we done on the Four-scroll model. This will provide the signal shown in Figure (13).

Then these signals are filtered by low-pass filters, to mitigate the chaotic carrier and keep the envelopes of $\sin \omega t$. The signals of the function $\sin \omega t$ results of signal error, will be shown in Figure (14) by filtring, thresholding and comparison. We do it as the same as we done in the Four-scroll model.

We will have the signal shown in Figure (14).

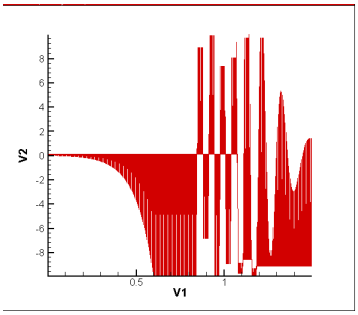


Figure 13: Detecting synchronization areas of $G1$ and $G2$, and limiting them for Lorenz model

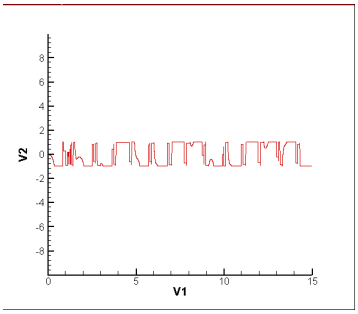


Figure 14: Filtring, thresholding and comparison of error signal for Lorenz model

To reconstruct the original signal, we have performed detecting the error signal rising edges as shown in the figure (15) by the detector rising edges stage. This signal becomes as shown in Figure (15).

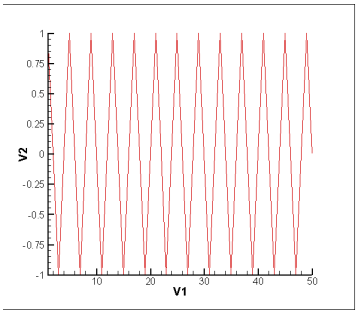


Figure 15: Detection of the rising edge for Lorenz model.

By comparing the signal shown in Figure (15) with the original signal message mj shown in Figure (4), we noted that there was a small difference between them. So, to recover the original message signal, we will have injected it into an T flip-flop. Figures (16) represent the signal reconstructed by the decrypting operation using Lorenz model and the original signal respectively. So they are identical.

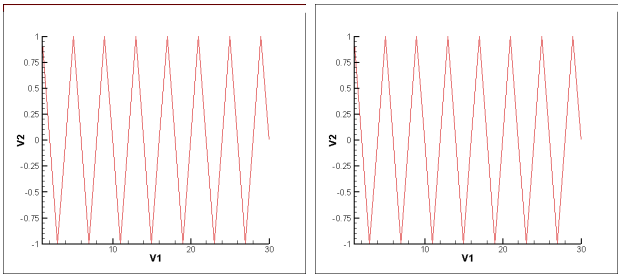


Figure 16: Message after decryption for Lorenz model at the right and the original one at the left.

4. Results discussion

Through this analysis study of the chaotic cryptosystem of the function $\sin \omega t$ using the CSK method by the both systems separately, Lorenz system and Four-scroll system, it was found that we can encrypt the functional message using the two systems. For the encryption we have obtained almost the same results, but for decryption , it was depended on the speed of the synchronization of each system. As well as the speed of the decryption in Lorenz System was slightly faster than the speed of decryption in Four-scroll System. On this basis, we conclude that the encryption and the decryption system depended on the speed of synchronization. However, in our analysis study the process of encryption and decryption using CSK method in both cases was completely successful and we have obtained a very good results.

5. Conclusion

We have located first the problems in crypto systems currently used. An emphasis has been made on the safety limits of this cryptosystems. We noticed through our study, review the cryptosystems the most used currently, and this computational security, since it is based on algebraical calculations. It's one of the reasons that triggered the need to seek alternative, the use of chaos in our work, is one of the proposed solutions.

References

[1] L. M. Pecora, T. L. Carroll, *Synchronization in chaotic systems*, Physical Review Letters, 64, 1990.

[2] H.D.I. Abarbanel, N.F. Rulkov, Mikhail M. Sushchik, *Generalized synchronization of chaos: the auxiliary system approach*, Physical Review E, 53 (1995), 4528-4535

[3] G. Alvarez, L. Hernández, J. Muñoz. Montoya, S. Li, *Security analysis of communication system based on the synchronization of different order chaotic systems*, Physics Letters A, 345 (2005), 245-250.

[4] R. Dumont, *Introduction a la cryptographie et a la sécurité informatique*, Note de cours, Université de Liége, 2006-2007.

[5] X. Bavard, *Numérisation du chaos et applications aux systèmes de communication sécurisés par chaos en longueur d'onde*, Thèse de doctorat, Université de Franche-Comté, 2004.

[6] S. Boccaletti, J. Kurths, G. Osipovd, D.L. Valladares, C.S. Zhou, *The synchronization of chaotic systems*, Physics Reports, 366 (2002), 1-101.

[7] K.M. Cuomo, A.V. Oppenheim, S.H. Isabelle, *Spread spectrum modulation and signal masking using synchronized chaotic systems*, MIT Tech. Rep, 570 (1992).

[8] K.M. Cuomo, A.V. Oppenheim, S.H. Strogatz, *Synchronization of Lorenzbased chaotic circuits with applications to communications*, IEEE Transactions on Circuits and Systems II, 40 (1993), 626-633.

[9] H. Dedieu, M.P. Kennedy, M. Hasler, *Chaos shift keying: modulation and demodulation of a chaotic carrier using self-synchronizing Chua's circuits*, IEEE Transactions on Circuits and Systems I, 40 (1993), 634-642.

[10] Ali Khan Mohammad, *Synchronization of different 3D chaotic systems by generalized active control*, ISSN 1746-7659, England, UK Journal of Information and Computing Science, 7 (2012), 272-283.

[11] E. Lorenz, *Deterministic nonperiodic flow*, J. Atmos. Sci., 20, 130-141.

[12] W. Liu, G. Chen, *Can a three-dimensional smooth autonomous quadratic chaotic system generate a single four-scroll attractor?*, Inte. J. Bifur. Chaos, 14 (2004), 1395-1403.

[13] J. Lü, G. Chen, D.Z. Chen, *A new chaotic system and beyond: The generalized Lorenz-like system*, Internat. J. Bifur. Chaos, 14 (2004), 1507-1537.

[14] G.H. Tigan, *A note on chaos synchronization between two differential three-dimensional systems*, Differential Geometry-Dynamical Systems, 7 (2005), 105-110.

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A note on bisections of graphs with girth at least 6

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Abstract. A *bisection* of a graph is a partition of its vertex set into two sets which differ in size by at most 1, and its *size* is the number of edges between the two sets. Let G be a graph with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees, then there is a constant $c > 0$ such that G admits a bisection of size at least $m/2 + c\sum_{i=1}^{n} \sqrt{d_i}$. It follows as a corollary that G has a bisection of size at least $m/2 + cm^{3/4}$.

Keywords: bisection, girth, perfect matching.

1. Introduction

All graphs considered here are finite, undirected and have no loops and no parallel edges, unless otherwise indicated. Given a graph $G = (V, E)$, the Max-Cut problem asks for a partition of V into V_1 and V_2, which maximizes the number of crossing edges between V_1 and V_2. This problem is NP-hard even when restricted to triangle-free cubic graphs and has been a very active research subject in both Combinatorics and Computer Science (see [8, 9, 15]).

It is easy to see by considering a random partition that every graph with m edges contains a cut of size at least $m/2$ edges. Edwards [5, 6] improved this lower bound to

$$\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8},$$

which is essentially best possible as evidenced by the complete graphs K_{2n+1}. Moreover, for certain range of m, Alon gave an additive improvement of order $m^{1/4}$.

Many results concerned on Max-Cut are about graphs with forbidden cycles. Let H be a graph, we say G is H-free if G contains no copy of H. After a series of papers by various researchers, Alon [1] showed that every triangle-free graph

G with m edges has a cut of size at least $m/2 + \Omega(m^{4/5})$. In [3], Alon *et al.* proved that for every odd integer $r > 5$, there is a $c(r) > 0$ such that every C_{r-1}-free graph with m edges has a bipartite cut of size at least

$$\frac{m}{2} + c(r)m^{r/(r+1)}.$$

It is further shown that this is tight, up to the value of $c(r)$, for $r \in \{5, 7, 11\}$. For more information on the Max-Cut problems in H-free graphs, we refer the reader to [1, 2, 3, 4, 12, 13] and their references.

In this note we focus on bisections of graphs. Given a graph $G = (V, E)$, the Max-Bisection problem is to find a bipartition of its vertex set $V = V_1 \cup V_2$, which maximizes the number of crossing edges, with the additional constraint that $||V_1| - |V_2|| \leq 1$. Indeed, the Max-Bisection problems tend to be much more complicated to analyze than the Max-Cut problems. As noticed in [7, 11], the Edwards bound implicitly implies that a connected graph G with n vertices and m edges admits a cut of size

$$\frac{m}{2} + \frac{n-1}{4}.$$

However, every bisection of $K_{1,n-1}$ has size $n/2$. Lee *et al.* [10] proved that every connected graph G with n vertices and m edges has a bisection of size at least

$$\frac{m}{2} + \frac{n+1-\Delta(G)}{4},$$

and the bound is tight.

Let $d \geq 1$ be an integer and n be sufficiently large. It is easy to see that the size of any bisection of a complete bipartite graph $K_{d,n}$ with m edges is at most $m/2 + O(1)$. Thus, we consider graphs with perfect matchings throughout the paper. Our main new results are as follows.

Theorem 1.1. *Let G be a graph on n vertices with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees, then there is a constant $c > 0$ such that G admits a bisection of size at least*

$$\frac{m}{2} + c\sum_{i=1}^{n}\sqrt{d_i}.$$

Corollary 1.2. *Let c be a positive constant. Every graph with m edges, perfect matchings and girth at least 6, has a bisection of size at least*

$$\frac{m}{2} + cm^{3/4}.$$

2. Proofs

Let $G = (V, E)$ be a graph on n vertices with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees. Suppose that $M = \{e_1, e_2, \ldots, e_{\frac{n}{2}}\}$ is a perfect matching in G. Consider the following randomized procedure for obtaining a bisection of G. Let $h : V \mapsto \{0, 1\}$ be a random function obtained by separating each $e_i = uu' \in M$ randomly and independently, either $h(u) = 0$ and $h(u') = 1$, or $h(u) = 1$ and $h(u') = 0$, where both choices are equally likely. Let $n_0(u)$ be the number of neighbours v of u satisfying $h(v) \neq h(u)$ where $v \neq u'$. Let $n_1(u)$ be the number of neighbours w of u satisfying $h(w) = h(u)$ where $w \neq u'$. Similarly let $n_0(u')$ be the number of neighbours v' of u' satisfying $h(v') \neq h(u')$ where $v' \neq u$, and $n_1(u')$ be the number of neighbours w' of u' satisfying $h(w') = h(u')$ where $w' \neq u$. Call an edge $uu' \in M$ *good* if $n_0(u) + n_0(u') > n_1(u) + n_1(u')$. If $n_0(u) + n_0(u') = n_1(u) + n_1(u')$, then call it *good with probability 1/2*. Otherwise call it *bad*. Let $h' : V \mapsto \{0, 1\}$ be the randomly function obtained from h as follows. For each $uu' \in M$, if uu' is *bad*, then choose randomly again, either $h'(u) = 0$ and $h'(u') = 1$, or $h'(u) = 1$ and $h'(u') = 0$, where both choices are equally likely and all choices are independent. Otherwise define $h'(u) = h(u)$ and $h'(u') = h(u')$. Finally, define $V_0 = (h')^{-1}(0)$ and $V_1 = (h')^{-1}(1)$. We obtain a bisection $V = V_0 \cup V_1$ and we denote each $uu' \in M$ by e_u (or $e_{u'}$).

We wish to calculate the expected size of this bisection. We want the probability that each edge ends up in the bisection. Note that for each $uu' \in M$, $\mathbb{P}[h'(u) \neq h'(u')] = 1$. Hence we aim to determine this probability for every edge in $E - M$.

Lemma 2.1. *Let G be a graph with perfect matchings. Let M be a perfect matching in G, and let $h, h' : V \mapsto \{0, 1\}$ be the two random functions defined by the randomized procedure described above. For a given edge $uv \in E - M$, let $p = \mathbb{P}[e_u$ and e_v are good $\mid h(u) = h(v)], q = \mathbb{P}[e_u$ and e_v are good $\mid h(u) \neq h(v)]$. Then*

$$\mathbb{P}[h'(u) \neq h'(v)] = \frac{1}{2} + \frac{1}{4}(q - p).$$

Proof. Let uv be an edge in $E - M$. By the law of total probability,

$$\begin{aligned} \mathbb{P}[h'(u) \neq h'(v)] = \frac{1}{2}\mathbb{P}[h'(u) \neq h'(v) \mid h(u) = h(v)] \\ + \frac{1}{2}\mathbb{P}[h'(u) \neq h'(v) \mid h(u) \neq h(v)]. \end{aligned} \tag{1}$$

Clearly, if $h(u) = h(v)$ and at least one of the edges e_u or e_v is bad, then the probability that $h'(u) \neq h'(v)$ is 1/2. But if both e_u and e_v are good, then the probability that $h'(u) \neq h'(v)$ is 0. Thus the conditional probability

$$\mathbb{P}[h'(u) \neq h'(v) \mid h(u) = h(v)] = 0 \cdot p + \frac{1}{2} \cdot (1 - p) = \frac{1}{2}(1 - p). \tag{2}$$

Similarly, if $h(u) \neq h(v)$ and at least one of the edges e_u or e_v is bad, then the probability that $h'(u) \neq h'(v)$ is $1/2$, whereas if they are both good, then certainly $h'(u) \neq h'(v)$. This yields that

$$\mathbb{P}[h'(u) \neq h'(v) \mid h(u) \neq h(v)] = 1 \cdot q + \frac{1}{2} \cdot (1-q) = \frac{1}{2}(1+q). \tag{3}$$

Combining (1), (2) and (3), we get the required result. □

It remains to estimate the difference $q - p$. Let $p_u = \mathbb{P}[e_u \text{ is good } \mid h(u) = h(v)], p_v = \mathbb{P}[e_v \text{ is good } \mid h(u) = h(v)]$. Similarly, let $q_u = \mathbb{P}[e_u \text{ is good } \mid h(u) \neq h(v)]$ and $q_v = \mathbb{P}[e_v \text{ is good } \mid h(u) \neq h(v)]$. For each vertex u of G, let d_u denote the degree of u in G. We then have the following lemma.

Lemma 2.2. *Let G be a graph with perfect matchings and girth at least 6. Let M be a perfect matching in G, and let $h, h' : V \mapsto \{0,1\}$ be the two random functions defined by the randomized procedure described above. For a given edge $uv \in E - M$,*

$$\mathbb{P}\big[h'(u) \neq h'(v)\big] = \frac{1}{2} + \frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right).$$

Proof. Since G is a graph with girth at least 6, the event e_u is good given $h(u) = h(v)$ is independent of the event e_v is good given $h(u) = h(v)$. Then $p = p_u p_v$. Similarly, $q = q_u q_v$. Let uu' and vv' be the edges in M. Because of the way we defined good and bad, if $d_u + d'_u$ is odd, then

$$\begin{aligned}\mathbb{P}[e_u \text{ is good }] = \left(\frac{1}{2}\right)^{d_u + d'_u - 2} \left(\binom{d_u + d'_u - 2}{0} + \binom{d_u + d'_u - 2}{1}\right. \\ \left. + \cdots + \binom{d_u + d'_u - 2}{(d_u + d'_u - 3)/2}\right) = \frac{1}{2}.\end{aligned}$$

The computation for the case that $d_u + d'_u$ is even is similar. Hence we have $\mathbb{P}[e_u \text{ is good }] = 1/2$. Similarly, $\mathbb{P}[e_v \text{ is good }] = 1/2$. It follows that $p_u + p_v = 1$, $q_u + q_v = 1$. This, together with Lemma 2.1, implies that

$$\mathbb{P}\big[h'(u) \neq h'(v)\big] = \frac{1}{2} + \frac{1}{4}(p_u p_v - q_u q_v) = \frac{1}{2} + \frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right).$$

This completes the proof of Lemma 2.2. □

Next we employ a result of Shearer [12], which shows that the truth of Theorem 1.1 implies the truth of Corollary 1.2.

Lemma 2.3 ([12])**.** *For any graph G with m edges and vertex degrees $d_1, d_2, \ldots, d_n$, we have $\sum_{i=1}^{n} \sqrt{d_i} \geq m^{3/4}$.*

Finally, we shall also need two simple facts. The first one can be found in [14].

Lemma 2.4 ([14])**.** *Let δ be a positive constant. Every graph on n vertices with m edges and perfect matchings, has a bisection of size at least $m/2 + \delta n$.*

Lemma 2.5. *Let G be a graph on n vertices with m edges and perfect matchings. Suppose that M is a perfect matching in G and $M' \subseteq M$. Let the vertex set $X = V(M')$ and $G' = G - X$ be a subgraph of G with m' edges. If G' has a bisection of size k, then G has a bisection of size at least $k + (m - m')/2$.*

Proof. Let $G = (V, E)$ be a graph on n vertices with m edges, and $G' = (V', E') = G - X$ be a subgraph of G with m' edges. Let $M' = \{u_1v_1, \cdots, u_tv_t\}$, $1 \leq t \leq n/2$. Suppose G' admits a bisection $V' = V_1 \cup V_2$ of size k. We now get a bisection of G by adding each pair $\{u_i, v_i\}$ in X into V' in turn, either $u_i \in V_1$ and $v_i \in V_2$, or $u_i \in V_2$ and $v_i \in V_1$, trying to maximize the number of crossing edges in the bisection. This immediately implies that there is a bisection in G which contains all the k edges of the initial bisection of G', and at least half of all other edges. This completes the proof. □

Having finished all the necessary preparations we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $G = (V, E)$ be a given graph and let M be a perfect matching in G. As long as there is an edge $uu' \in M$ of degrees $d_u + d_{u'} \leq 3$, delete the two vertices $\{u, u'\}$. If during this process we delete more than $\frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ vertices, by Lemma 2.4, we get the desired result, with $c = \frac{\delta}{4}$ and where δ is the constant from that lemma. Therefore we may assume that the process terminates after deleting at most $\frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ vertices. It thus terminates with an induced subgraph $G' = (V', E')$ with n' vertices, m' edges and degree sequence $d'_1, \ldots, d'_n$. Clearly, G' also has a perfect matching $M' \subseteq M$, in which every edge uu' satisfies the condition that the degree of u in G' plus the degree of u' in G' is more than 3. Note that G' is obtained from G by deleting fewer than $2 \cdot \frac{1}{8}\sum_{i=1}^{n}\sqrt{d_i} = \frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ edges. Since each deletion of an edge decreases the sum of roots of the degrees by at most 2, we have

$$\sum_{i=1}^{n'}\sqrt{d'_i} \geq \frac{1}{2}\sum_{i=1}^{n}\sqrt{d_i}. \tag{4}$$

Let uv be an edge of G'. Suppose that uu', vv' are edges in M'. Consider the randomized procedure for obtaining a bisection of G' by choosing the random fuctions $h, h' : V' \mapsto \{0, 1\}$ as described in the beginning of the section. Clearly p_u depends only on the degree d of u in G' and the degree d' of u' in G'. By Lemma 2.2, the expected size of our bisection is

$$\begin{aligned}\sum_{uv\in E'}\mathbb{P}\big[h'(u) \neq h'(v)\big] &= |M'| + \sum_{uv\in E'-M'}\mathbb{P}\big[h'(u) \neq h'(v)\big] \\ &= \frac{m}{2} + \frac{|M'|}{2} + \sum_{uv\in E'-M'}\left(\frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right)\right)\end{aligned}$$

$$= \frac{m'}{2} + \frac{|M'|}{2} + \frac{1}{4}\sum_{u\in V'}(d-1)\left(p_u - \frac{1}{2}\right).$$

Also, because of the definition of p_u, the expected size of the bisection is

$$\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{1}{4}\sum_{u\in V'}(d-1)\cdot\frac{1}{2^{d+d'-2}}\cdot\binom{d+d'-3}{\lfloor(d+d'-3)/2\rfloor}. \tag{5}$$

Claim 2.6. *For any integer $a>0$, there exist a constant $0<c_1<1$ such that*

$$\frac{1}{2^a}\binom{a}{\lfloor a/2\rfloor} \geq \frac{c_1}{\sqrt{a}}.$$

The desired result follows from Stirling's approximation.

Claim 2.7.

$$\sqrt{d+d'-3} \geq \frac{\sqrt{2}}{2}\big(\sqrt{d}+\sqrt{d'}\big) - 3.$$

If $\sqrt{d}+\sqrt{d'} \leq 3\sqrt{2}$, then $(\sqrt{2}/2)(\sqrt{d}+\sqrt{d'}) - 3 \leq 0 \leq \sqrt{d+d'-3}$. If $\sqrt{d}+\sqrt{d'} > 3\sqrt{2}$, then

$$\begin{aligned}&\big(\sqrt{d+d'-3}\big)^2 - \big[(\sqrt{2}/2)(\sqrt{d}+\sqrt{d'})-3\big]^2\\ &= \frac{1}{2}\big(d+d'-2\sqrt{dd'}\big) - 12 + 3\sqrt{2}\big(\sqrt{d}+\sqrt{d'}\big) > \frac{1}{2}\big(d-d'\big)^2 + 3\sqrt{2}\cdot 3\sqrt{2} - 12 > 0.\end{aligned}$$

This completes the proof of Claim 2.7.

Now it follows from (5) and Claim 2.6 that G' has a bisection of size at least

$$\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{1}{8}\sum_{u\in V'}(d-1)\frac{c_1}{\sqrt{d+d'-3}}. \tag{6}$$

Note that

$$\sum_{u\in V'}\frac{(d-1)}{\sqrt{d+d'-3}} = \sum_{uu'\in M'}\frac{(d+d'-2)}{\sqrt{d+d'-3}} > \sum_{uu'\in M'}\sqrt{d+d'-3} = \frac{1}{2}\sum_{u\in V'}\sqrt{d+d'-3}.$$

This together with Claim 2.7 and (6), implies that G' admits a bisection of size at least

$$\begin{aligned}&\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{c_1}{16}\cdot\frac{\sqrt{2}}{2}\Big(\sum_{u\in V'}\big(\sqrt{d}+\sqrt{d'}\big) - 3n'\Big)\\ &> \frac{m'}{2} + \frac{\sqrt{2}c_1}{16}\sum_{u\in V'}\sqrt{d} + \Big(\lfloor\frac{n'}{4}\rfloor - \frac{3\sqrt{2}c_1}{32}n'\Big)\\ &> \frac{m'}{2} + c\sum_{i=1}^{n'}\sqrt{d'_i}.\end{aligned}$$

The desired result follows from (4) and Lemma 2.5 by choosing $c = \frac{\sqrt{2}}{16}c_1$. □

3. Concluding remarks

This note is motivated by the results on Max-Cuts of triangle-free graphs given by Shearer [12]. But bisection problems tend to be much more complicated to analyze than the Max-Cut problems. The proofs in Section 2 combine combinatorial and probabilistic techniques. We also define several random events in the proofs. To ensure the mutual independence of these events, we consider the graphs with girth at least 6. However, there are still very few results on the maximum bisections of graphs.

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References

[1] N. Alon, *Bipartite subgraphs*, Combinatorica, 16 (1996), 301-311.

[2] N. Alon, B. Bollobás, M. Krivelevich and B. Sudakov, *Maximum cuts and judicious partitions in graphs without short cycles*, J. Combin. Theory Ser. B, 88 (2003), 329–346.

[3] N. Alon, M. Krivelevich and B. Sudakov, *Maxcut in H-free graphs*, Combin. Probab. Comput., 14 (2005), 629-647.

[4] B. Bollobás and A. D. Scott, *Problems and results on judicious partitions*, Random Struct. Alg., 21 (2002), 414-430.

[5] C.S. Edwards, *Some extremal properties of bipartite graphs*, Canad. J. Math., 3 (1973), 475-485.

[6] C. S. Edwards, *An improved lower bound for the number of edges in a largest bipartite subgraph*, Proceedings of the Second Czechoslovak Symposium on Graph Theory, Prague, 1975, 167-181.

[7] P. Erdős, A. Gyárfás and Y. Kohayakawa, *The size of the largest bipartite subgraphs,* Discrete Math., 177 (1997), 267-271.

[8] M. Goemans and D. Williamson, *0.878 approximation algorithms for MAX CUT and MAX 2-SAT,* in: Proc. 26th ACM Symp. Theory Comput., 1994, 422-431.

[9] J. Hőstad, *Some optimal inapproximability results,* J. ACM, 48 (2001), 798-859.

[10] C. Lee, P. Loh and B. Sudakov, *Bisections of graphs,* J. Combin. Theory Ser. B, 103 (2013), 599-629.

[11] S. Poljak and Zs. Tuza, *Bipartite subgraphs of triangle-free graphs*, SIAM J. Discrete Math., 7 (1994), 307-313.

[12] J. Shearer, *A note on bipartite subgraphs of triangle-free graphs*, Random Struct. Alg., 3 (1992), 223-226.

[13] B. Xu and X. Yu, *Triangle-free subcubic graphs with minimum bipartite density*, J. Combin. Theory Ser. B, 98 (2008), 516-537.

[14] B. Xu, J. Yan and X. Yu, *A note on balanced bipartitions*, Discrete Math., 310 (2010), 2613-2617.

[15] M. Yannakakis, *Node- and edge-deletion NP-complete problems*, STOC, 1978, 253-264.

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Common fixed point theorems for generalized contraction conditions involving rational expressions in complex valued metric spaces

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Abstract. The aim of this paper is to prove common fixed point theorems for a pair of self maps satisfying general contraction conditions involving rational expressions having a point dependent control functions as coefficients in complex valued metric spaces. Our results extend and generalize the results of Nashine et.al., [5] and some known results in the literature. To show validity of our results some illustrative examples are also furnished.

Keywords: complex valued metric spaces, self maps, common fixed points, point dependent control functions.

1. Introduction

The benefit of metric spaces is a natural growth of functional analysis. Several researchers attempted various generalizations of this notion in recent past years such as generalized metric space, D-metric space, quasi-metric space, dislocated metric space, fuzzy metric space, ordered metric spaces *etc.*

Recently Azam et.al. [1] introduced a new notion namely complex valued metric space, which is one of the most attractive research in the field of fixed point theory and established sufficient conditions for the existence of common fixed points for a contractive condition involving rational expressions for a pair of self maps. The idea of complex metric space offers numerous research activities in mathematical analysis. Many researchers have contributed with different concepts in these spaces, for example we refer [2, 4, 5, 6, 7, 8, 9, 10].

We begin this section with notations and definitions in complex valued metric space due to Azam et al. [1] which are useful to prove our main results.

Let $\mathcal{C}$ be the set of complex numbers and $z_1,\ z_2 \in \mathcal{C}$. Define a partial order on C as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \preceq Re(z_2)$ and $Im(z_1) \preceq Im(z_2)$.

Thus we can say that $z_1 \preceq z_2$ if one of the following condition holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we write $z_1 \preceq z_2$ if only (4) is satisfied. Note that $0 \preceq z_1 \preceq z_2$ implies $|z_1| \preceq |z_2|$.

Definition 1.1 (Azam et al. [1]). Let X be a nonempty set. A function $d : X \times X \to C$ is called a *complex valued metric on* X if for all $x, y, z \in X$, the following conditions are satisfied:

$(CM1)$ $0 \preceq d(x, y)$ and $d(x, y) = 0$ iff $x = y$;
$(CM2)$ $d(x, y) = d(y, x)$;
$(CM3)$ $d(x, y) \preceq d(x, z) + d(y, z)$

Then the pair (X, d) is called a complex valued metric space.

Example 1.2. Let $X = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$ and we define from $d : X \times X \to C$ by $d(z_1, z_2) = 2|x_1 - x_2| + 3i|y_1 - y_2|$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Clearly, (X, d) is a complex valued metric space.

Definition 1.3 (Azam et al. [1]). Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. We say that:

(i) the sequence $\{x_n\}$ *converges to* $x \in X$ if for each $c \in C$ with $0 \prec c$ there is a $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) the sequence $\{x_n\}$ *is a Cauchy sequence* if for each $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in N$.

(iii) (X, d) is *complete complex valued metric space* if every Cauchy sequence in X is convergent to a point in X.

Lemma 1.4 (Azam et al. [1]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Proposition 1.5 (Sithikul and Saejung [10]). *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies $a_n \leq h\ a_{n-1}$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

In [3], Jaggi and Dass generalized Banach contraction principle using rational type of contraction.

Theorem 1.6 (Jaggi and Dass ([3]). *Let T be a selfmap of a metric space (X, d) which satisfy*:

(i) for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that

$$d(Tx, Ty) \leq \alpha \frac{d(x,\ Tx)d(y,\ Ty)}{d(x,\ Ty) + d(y,\ Tx) + d(x,\ y)} + \beta d(x, y), \tag{1.6.1}$$

for all $x,\ y \in X$, $x \neq y$;

(ii) there exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X.

Then z is the unique fixed point of T in X.

Using the analog of condition (1.6.1) Nashine et al. [5] proved the following theorem in complex valued metric spaces.

Theorem 1.7 (Nashine et al. [5])**.** *Let (X, d) be a complete complex valued metric space and suppose that S and T be mappings on X satisfy either*
(i)

$$d(Sx, Ty) \preceq \alpha d(x, y) + \beta \frac{d(x, Sx)d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)}, \tag{1.7.1}$$

where $\alpha + \beta < 1$ for all $x,\ y \in X$, $x \neq y$ and $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ or

(ii) $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) + d(x, y) = 0$.

Then S and T have a unique common fixed point in X.

The following proposition was proved by Naval Singh et al. [6].

Proposition 1.8 (Naval Singh et. al. [6])**.** *Let (X, d) be a complex valued metric space and $S, T : X \to X$. Let $x_0 \in X$ and define a sequence $\{x_n\} \in X$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n = 0, 1, 2, 3, ...$*

Assume that there exists a mapping $\alpha : X \times X \times X \to [0, 1)$ such that

$$\alpha(TSx, y, a) \leq \alpha(x, y, a) \text{ and } \alpha(x, STy, a) \leq \alpha(x, y, a),$$

for all $x, y \in X$ for any fixed $a \in X$ and $n = 0, 1, 2, 3, ...$.

Then $\alpha(x_{2n}, y, a) \leq \alpha(x_0, y, a)$ and $\alpha(x, x_{2n+1}, a) \leq \alpha(x, x_1, a)$.

The aim of this paper is to prove common fixed point theorems for a pair of self maps satisfying general contraction conditions involving rational expressions having a point dependent control functions as coefficients in complex valued metric spaces. Our results extend and generalize the results of Nashine et al. [5] (Theorem 2.1) and some known results in the literature. To show validity of our results some illustrative examples are also furnished.

2. Common fixed point theorems for a pair of maps using control functions

We start this section with our main result.

Theorem 2.1. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\alpha, \beta,\ \gamma, \delta$ and $\mu : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, for a fixed $a \in X$:*

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ and $\alpha(x, STy, a) \leq \alpha(x, y, a)$;

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ and $\beta(x, STy, a) \leq \beta(x, y, a)$;

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ and $\gamma(x, STy, a) \leq \gamma(x, y, a)$;
(iv) $\delta(TSx, y, a) \leq \delta(x, y, a)$ and $\delta(x, STy, a) \leq \delta(x, y, a)$;
(v) $\mu(TSx, y, a) \leq \mu(x, y, a)$ and $\mu(x, STy, a) \leq \mu(x, y, a)$. *Also, suppose that* S *and* T *satisfy either*

$$
\begin{aligned}
(vi) \quad d(Sx, Ty) \preceq\ & \alpha(x, y, a) d(x, y) \\
& + \beta(x, y, a) \frac{d(x, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \\
& + \gamma(x, y, a) \frac{d(x, Ty) d(y, Sx)}{d(x, Sx) + d(y, Ty) + d(x, y)} \\
& + \delta(x, y, a) \frac{[d(x, Sx) d(x, Ty) + d(y, Ty) d(y, Sx)]}{d(x, Sx) + d(y, Ty) + d(x, y)} \\
& + \mu(x, y, a) \frac{[d(x, Sx) d(y, Sx) + d(y, Ty) d(x, Ty)]}{d(x, Sx) + d(y, Ty) + d(x, y)}, \qquad (2.1.1)
\end{aligned}
$$

where $|\alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) + \mu(x, y, a)| < 1$, $d(x, Sx) + d(y, Ty) + d(x, y) \neq 0$ *and* $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ *or*
(vii) $d(Sx, Ty) = 0$ *if* $d(x, Sx) + d(y, Ty) + d(x, y) = 0$ *or* $d(x, Ty) + d(y, Sx) + d(x, y) = 0$.
Then S *and* T *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\} \in X$ by $x_{2n+1} = Sx_{2n}$ and

$$x_{2n+2} = Tx_{2n+1}. \qquad (2.1.2)$$

First we suppose that

$$d(x_{2k}, x_{2k+1}) + d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) = 0, \text{ for some } k, \qquad (2.1.3)$$

therefore we have

$$x_{2k} = Sx_{2k} = Tx_{2k+1} = x_{2k+1} \qquad (2.1.4)$$

hence there exists k_1 and k_2 such that

$$k_1 = Sk_1 = k_2 = Tk_2. \qquad (2.1.5)$$

Thus from (2.1.5) it follows that k_1 is a common fixed point of S and T.

Now, we will prove that k_1 is a unique common fixed point of S and T. Suppose k_1 and k_2 be two common fixed points of S and T which implies $k_1 = Sk_1 = Tk_1$ and $k_2 = Tk_2 = Sk_2$.

Hence from (2.1.3), we have

$$d(k_1, k_2) + d(k_1, Tk_2) + d(k_2, Sk_1) = 0,$$

this implies $k_1 = k_2$.

Similarly proceeding on the same lines of the above argument S and T have a unique fixed point if $d(x_{2k}, x_{2k+1})+d(x_{2k}, Sx_{2k})+d(x_{2k+1}, Tx_{2k+1}) = 0$. Thus we suppose that $d(x, Sx)+d(y, Ty)+d(x, y) \neq 0$ and $d(x, Ty)+d(y, Sx)+d(x, y) \neq 0$. First we show that $\{x_n\}$ is a Cauchy sequence in X. Now on using the inequality (2.1.1) with $x = x_0$ $y = x_1$, we have

$$\begin{aligned} d(x_1, x_2) = d(Sx_0, Tx_1) &\leq \alpha(x_0, x_1, a)d(x_0, x_1) \\ &+ \beta(x_0, x_1, a)\frac{d(x_0, x_1)d(x_1, x_2)}{d(x_0, x_1) + d(x_0, x_2)} \\ &+ \delta(x_0, x_1, a)\frac{d(x_0, x_1)d(x_0, x_2)}{d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_1)} \\ &+ \mu(x_0, x_1, a)\frac{d(x_1, x_2)d(x_0, x_2)}{d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_1)} \end{aligned}$$

so that

$$\begin{aligned} |d(x_1, x_2)| \leq \alpha(x_0, x_1, a)|d(x_0, x_1)| &+ \beta(x_0, x_1, a)\frac{|d(x_0, x_1)||d(x_1, x_2)|}{|d(x_0, x_1) + d(x_0, x_2)|} \\ &+ \delta(x_0, x_1, a)\frac{|d(x_0, x_1)||d(x_0, x_2)|}{|d(x_0, x_1) + d(x_1, x_2)|} \\ &+ \mu(x_0, x_1, a)\frac{|d(x_1, x_2)||d(x_0, x_2)|}{|d(x_0, x_1) + d(x_1, x_2)|}. \end{aligned} \tag{2.1.6}$$

On using (CM3), we have

$$|d(x_0, x_1) + d(x_0, x_2)| \geq |d(x_1, x_2)| \tag{2.1.7}$$

and

$$|d(x_0, x_1) + d(x_1, x_2)| \geq |d(x_0, x_2)|. \tag{2.1.8}$$

Thus, from (2.1.6), (2.1.7) and (2.1.8), it follows that

$$|d(x_1, x_2)| \leq [\frac{\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \delta(x_0, x_1, a)}{1 - \mu(x_0, x_1, a)}]|d(x_0, x_1)| \tag{2.1.9}$$

Similarly, once again on using the inequality with $x = x_2, y = x_1$, we have $d(x_2, x_3) = d(STx_1, Tx_1) \leq \alpha(Tx_1, x_1, a)d(x_2, x_1)+\beta(Tx_1, x_1, a)\frac{d(x_3,x_2)d(x_1,x_2)}{d(x_1,x_3)+d(x_1,x_2)}+ \delta(Tx_1, x_1, a)\frac{d(x_1,x_2)d(x_1,x_3)}{d(x_2,x_3)+d(x_1,x_2)+d(x_1,x_2)}+\mu(Tx_1, x_1, a)\frac{d(x_2,x_3)d(x_1,x_3)}{d(x_2,x_3)+d(x_1,x_2)+d(x_1,x_2)}$. Thus

$$\begin{aligned} |d(x_2, x_3)| \leq \alpha(x_0, x_1, a)|d(x_2, x_1)| &+ \beta(x_0, x_1, a)\frac{|d(x_3, x_2)||d(x_1, x_2)|}{|d(x_1, x_3) + d(x_1, x_2)|} \\ &+ \delta(x_0, x_1, a)\frac{|d(x_1, x_2)||d(x_1, x_3)|}{|d(x_2, x_3) + d(x_1, x_2)|} \\ &+ \mu(x_0, x_1, a)\frac{|d(x_2, x_3)||d(x_1, x_3)|}{|d(x_2, x_3) + d(x_1, x_2)|}. \end{aligned} \tag{2.1.10}$$

Again using triangle inequality, we have

$$|d(x_1, x_3) + d(x_1, x_2)| \geq |d(x_2, x_3)| \tag{2.1.11}$$

and

$$|d(x_2, x_3) + d(x_1, x_2)| \geq |d(x_1, x_3)|. \tag{2.1.12}$$

Thus, from (2.1.10), (2.1.11) and (2.1.12), it follows that

$$|d(x_3, x_2)| \leq [\frac{\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \delta(x_0, x_1, a)}{1 - \mu(x_0, x_1, a)}]|d(x_2, x_1)|. \tag{2.1.13}$$

Hence from (2.1.9) , (2.1.13) and Proposition 1.8, we can conclude that

$$|d(x_n, x_{n+1})| \leq p|d(x_n, x_{n-1})|,$$

for all $n = 1, 2, 3, ...$ and $p = \frac{\alpha(x_0,x_1,a)+\beta(x_0,x_1,a)+\delta(x_0,x_1,a)}{1-\mu(x_0,x_1,a)} < 1$.

Hence

$$|d(x_n, x_{n+1})| \leq p^n|d(x_0, x_1)| \to 0 \text{ as } n \to \infty. \tag{2.1.14}$$

Thus, by Proposition 1.5, it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Therefore

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u. \tag{2.1.15}$$

We now show that u is a fixed point of S. On the contrary $u \neq Su$. In view of inequality (2.1.1) and proposition 1.8, we have

$$\begin{aligned}
d(u, Su) &\leq d(u, x_{2n+2}) + d(x_{2n+1}, Su) \leq d(u, x_{2n+2}) + d(Tx_{2n+1}, Su) \\
&\leq d(u, x_{2n+2}) + \alpha(u, x_{2n+1}, a)d(u, x_{2n+1}) \\
&+ \beta(u, x_{2n+1}, a)\frac{d(u, Su)d(x_{2n+1}, Tx_{2n+1})}{d(u, Tx_{2n+1}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)} \\
&+ \gamma(u, x_{2n+1}, a)\frac{d(u, Tx_{2n+1})d(x_{2n+1}, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\
&+ \delta(u, x_{2n+1}, a)\frac{d(u, Tx_{2n+1})d(u, Su) + d(Tx_{2n+1}, x_{2n+1})d(x_{2n+1}, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\
&+ \mu(u, x_{2n+1}, a)\frac{d(x_{2n+1}, Su)d(u, Su) + d(Tx_{2n+1}, x_{2n+1})d(u, Tx_{2n+1})}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)}. \\
&\leq d(u, x_{2n+2}) + \alpha(u, x_1, a)d(u, x_{2n+1}) \\
&+ \beta(u, x_1, a)\frac{d(u, Su)d(x_{2n+1}, Tx_{2n+1})}{d(u, Tx_{2n+1}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)} \\
&+ \gamma(u, x_1, a)\frac{d(u, Tx_{2n+1})d(x_{2n+1}, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\
&+ \delta(u, x_1, a)\frac{d(u, Tx_{2n+1})d(u, Su) + d(Tx_{2n+1}, x_{2n+1})d(x_{2n+1}, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\
&+ \mu(u, x_1, a)\frac{d(x_{2n+1}, Su)d(u, Su) + d(Tx_{2n+1}, x_{2n+1})d(u, Tx_{2n+1})}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)}.
\end{aligned}$$

Letting $n \to \infty$ and from (2.1.15), gives rise to

$$|d(u, Su)| \leq |\mu(u, x_1, a)||d(u, Su)| < |d(u, Su)|.$$

Hence $u = Su$. We now show that u is a fixed point of T. Suppose that $u \neq Tu$. Now on utilizing the inequality (2.1.1) and Proposition (1.8), we have

$$\begin{aligned}
d(u, Tu) &\leq d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \leq d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\
&\leq d(u, x_{2n+1}) + \alpha(x_{2n}, u, a)d(u, x_{2n}) \\
&+ \beta(x_{2n}, u, a)\frac{d(u, Tu)d(x_{2n}, Sx_{2n})}{d(u, Sx_{2n}) + d(x_{2n}, Tu) + d(x_{2n}, u)} \\
&+ \gamma(x_{2n}, u, a)\frac{d(u, Sx_{2n})d(x_{2n}, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \delta(x_{2n}, u, a)\frac{d(u, Sx_{2n})d(u, Tu) + d(x_{2n}, Tu)d(x_{2n}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \mu(x_{2n}, u, a)\frac{d(x_{2n}, Sx_{2n})d(Sx_{2n}, u) + d(x_{2n}, Tu)d(u, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&\leq d(u, x_{2n+1}) + \alpha(x_0, u, a)d(u, x_{2n}) \\
&+ \beta(x_0, u, a)\frac{d(u, Tu)d(x_{2n}, Sx_{2n})}{d(u, Sx_{2n}) + d(x_{2n}, Tu) + d(x_{2n}, u)} \\
&+ \gamma(x_0, u, a)\frac{d(u, Sx_{2n})d(x_{2n}, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \delta(x_0, u, a)\frac{d(u, Sx_{2n})d(u, Tu) + d(x_{2n}, Tu)d(Sx_{2n}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \mu(x_0, u, a)\frac{d(x_{2n}, Sx_{2n})d(Sx_{2n}, u) + d(x_{2n}, Tu)d(u, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)},
\end{aligned}$$

which on making $n \to \infty$, using (2.1.15), we get

$$|d(u, Tu)| \leq |\mu(x_0, u, a)||d(u, Tu)|.$$

Hence, we find that $u = Tu$. Accordingly, we conclude that u is a common fixed point of S and T. To prove uniqueness let u and $u*$ be two distinct fixed points of S and T. This implies $d(Su*, u) + d(u*, Tu) + d(u, u*) \neq 0$ and $d(Su, u) + d(u*, Tu*) + d(u, u*) \neq 0$.

$$\begin{aligned}
d(u, u*) = d(Su, Tu*) &\leq \alpha(u, u*, a)d(u, u*) \\
&+ \beta(u, u*, a)\frac{d(u, Su)d(u*, Tu*)}{d(u, Tu*) + d(u*, Su) + d(u, u*)} \\
&+ \gamma(u, u*, a)\frac{d(u, Tu*)d(u*, Su)}{d(u, Su) + d(u*, Tu*) + d(u, u*)}
\end{aligned}$$

$$+ \delta(u, u*, a)\frac{[d(u, Su)d(u, Tu*) + d(u*, Tu*)d(u*, Su)]}{d(u, Su) + d(u*, Tu*) + d(u, u*)}$$
$$+ \mu(u, u*, a)\frac{[d(u, Su)d(u*, Su) + d(u*, Tu*)d(u, Tu*)]}{d(u, Su) + d(u*, Tu*) + d(u, u*)}$$
$$\leq [\alpha(u, u*, a) + \gamma(u, u*, a)]d(u, u*),$$

a contradiction. Thus u is a common fixed point of S and T. This concludes the proof of this theorem.

Theorem 2.2. *If $\{S_i\}_1^m$ and $\{T_i\}_1^n$ are two finite pairwise commuting finite families of self maps defined on a complete complex valued metric space such that the mappings S and T satisfy condition* (2.1.1)*, then two families of maps $\{S_i\}_1^m$ and $\{T_i\}_1^n$ have a unique common fixed point in X.*

Proof. In view of Theorem 2.1, we hypothesize that S and T have a unique common fixed point z in X *i.e.*, $Sz = Tz = z$. By pairwise commutative property of $\{S_i\}_1^m$ and $\{T_i\}_1^n$ for every $(1 \leq p \leq n)$, we have

$$S_p z = S_p S z = S(S_p z)$$

and

$$S_p z = S_p T z = T(S_p z)$$

which implies that $S_p z$ is a common fixed point of S and T. On using uniqueness of fixed point of S and T, we have $S_p z = z$, for all $(1 \leq p \leq n)$. Hence it follows that z is a common fixed point of the family $\{S_i\}_1^m$.

Similar to the above argument we can show that z is a common fixed point of the family $\{T_i\}_1^n$.

Thus the conclusion of this theorem follows.

In the following we prove similar type of results for different rational expressions.

Theorem 2.3. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\alpha, \beta, \gamma, \delta$ and $\mu : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, for a fixed $a \in X$:*

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ and $\alpha(x, STy, a) \leq \alpha(x, y, a)$;

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ and $\beta(x, STy, a) \leq \beta(x, y, a)$;

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ and $\gamma(x, STy, a) \leq \gamma(x, y, a)$.

Also, suppose that S and T satisfy either

(iv)

$$d(Sx, Ty) \preceq \alpha(x, y, a)d(x, y) + \beta(x, y, a)\frac{d(x, Sx)d^2(x, Ty) + d(y, Ty)d^2(y, Sx)}{d^2(x, Ty) + d^2(y, Sx)}$$
$$+ \gamma(x, y, a)\frac{d(x, Sx)d(x, Ty) + d^2(x, y) + d(x, Sx)d(x, y)}{d(x, Sx) + d(x, Ty) + d(x, y)}, \tag{2.3.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)| < 1$ for all $x, y \in X$, $d(x,Sx)+d(x,Ty)+d(x,y) \neq 0$ and $d^2(x,Ty)+d^2(y,Sx) \neq 0$ or

(v) $d(Sx,Ty) = 0$ *if* $d(x,Sx)+d(x,Ty)+d(x,y) = 0$ or $d^2(x,Ty)+d^2(y,Sx) = 0$. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\} \in X$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}. \tag{2.3.2}$$

First we suppose that

$$d^2(x_{2k}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k}) = 0 \text{ for some } k, \tag{2.3.3}$$

then we have

$$x_{2k} = Sx_{2k} = Tx_{2k+1} = x_{2k+1}, \tag{2.3.4}$$

therefore exists k_1 and k_2 such that

$$k_1 = Sk_1 = k_2 = Tk_2. \tag{2.3.5}$$

Thus from (2.3.5) it follows that k_1 is a common fixed point of S and T.

We now prove that k_1 is a unique common fixed point of S and T. Suppose k_1 and k_2 be two common fixed points of S and T. This implies $k_1 = Sk_1 = Tk_1$ and $k_2 = Tk_2 = Sk_2$. Hence from (2.3.3), we have

$$d^2(k_1,k_2) + d^2(k_1,k_2) = 0,$$

this implies $k_1 = k_2$. Similarly proceeding on the same lines of the above argument S and T have a unique fixed point if $d(x_{2k}, x_{2k+1}) + d(x_{2k}, Sx_{2k}) + d(x_{2k}, Tx_{2k+1}) = 0$. Thus we suppose that $d(x,Sx) + d(x,Ty) + d(x,y) \neq 0$ and $d^2(x,Ty) + d^2(y,Sx) \neq 0$.

First we show that $\{x_n\}$ is a Cauchy sequence in X. On using the inequality (2.3.1) with $x = x_0$ $y = x_1$, we have

$$\begin{aligned}
d(x_1,x_2) = d(Sx_0,Tx_1) &\leq \alpha(x_0,x_1,a)d(x_0,x_1) \\
&+ \beta(x_0,x_1,a)\frac{d(x_0,Sx_0)d^2d(x_0,Tx_1) + d(x_1,Tx_1)d^2d(x_1,Sx_0)}{d^2(x_0,Tx_1) + d^2(x_1,Sx_0)} \\
&+ \gamma(x_0,x_1,a)\frac{d(x_0,Sx_0)d(x_0,Tx_1) + d^2(x_0,x_1) + d(x_0,Sx_0)d(x_0,x_1)}{d(x_0,Sx_0) + d(x_0,Tx_1) + d(x_0,x_1)} \\
&\leq \alpha(x_0,x_1,a)d(x_0,x_1) + \beta(x_0,x_1,a)\frac{d(x_0,x_1)d^2d(x_0,x_2)}{d^2(x_0,x_2)} \\
&+ \gamma(x_0,x_1,a)\frac{d(x_0,x_1)d(x_0,x_2) + d^2(x_0,x_1) + d(x_0,x_1)d(x_0,x_1)}{d(x_0,x_1) + d(x_0,x_1) + d(x_0,x_2)} \\
&\leq [\alpha(x_0,x_1,a) + \beta(x_0,x_1,a)\gamma(x_0,x_1,a)]d(x_0,x_1). \qquad (2.3.6)
\end{aligned}$$

Again on using the inequality (2.3.1) with $x = x_1$ $y = x_2$, we have

$$\begin{aligned} d(x_2, x_3) &= d(Sx_2, Tx_1) \leq \alpha(Tx_1, x_1, a)d(x_2, x_1) \\ &+ \beta(Tx_1, x_1, a)\frac{d(x_2, Sx_2)d^2d(x_2, Tx_1) + d(x_1, Tx_1)d^2(x_1, Sx_2)}{d^2d(x_2, Tx_1) + d^2(x_1, Sx_2)} \\ &+ \gamma(Tx_1, x_1, a)\frac{d(x_2, Sx_2)d(x_2, Tx_1) + d^2(x_2, x_1) + d(x_2, Sx_2)d(x_1, x_2)}{d(x_2, Sx_2) + d(x_2, Tx_1) + d(x_2, x_1)} \\ &\leq [\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \gamma(x_0, x_1, a)]d(x_2, x_1) \end{aligned} \tag{2.3.7}$$

Thus, from (2.3.6), (2.3.7)and Proposition 1.8 , it follows that

$$|d(x_n, x_{n+1})| \leq p^n |d(x_0, x_1)|, \quad \text{for} \quad \text{all } n \in N,$$

where $p = |\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \gamma(x_0, x_1, a)| < 1$.

Thus, by Proposition 1.5, it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Therefore

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u. \tag{2.3.8}$$

We now show that u is a fixed point of S. On the contrary $u \neq Su$. In view of inequality (2.3.1) and Proposition (1.8), we have

$$\begin{aligned} &d(u, Su) \leq d(u, x_{2n+2}) + d(x_{2n+2}, Su) \leq d(u, x_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\leq d(u, x_{2n+2}) + \alpha(u, x_{2n+1}, a)d(u, x_{2n+1}) \\ &+ \beta(u, x_{2n+1}, a)\frac{d(u, Su)d^2(u, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d^2(x_{2n+1}, Su)}{d^2(u, Tx_{2n+1}) + d^2(x_{2n+1}, Su)} \\ &+ \gamma(u, x_{2n+1}, a)\frac{d(u, Su)d(u, Tx_{2n+1}) + d^2(u, x_{2n+1}) + d(x_{2n+1}, u)d(u, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\ &\leq d(u, x_{2n+2}) + \alpha(u, x_1, a)d(u, x_{2n+1}) \\ &+ \beta(u, x_1, a)\frac{d(u, Su)d^2(u, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d^2(x_{2n+1}, Su)}{d^2(u, Tx_{2n+1}) + d^2(x_{2n+1}, Su)} \\ &+ \gamma(u, x_1, a)\frac{d(u, Su)d(u, Tx_{2n+1}) + d^2(u, x_{2n+1}) + d(u, Su)d(u, x_{2n+1})}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)}. \end{aligned}$$

Letting $n \to \infty$ and in view of condition (2.3.8), it follows that $|d(u, Su)| \leq 0$. Hence $u = Su$. We now show that u is a fixed point of T. Suppose that $u \neq Tu$.

Now on utilizing the inequality (2.3.1) and Proposition (1.8), we have

$$\begin{aligned}
&d(u,Tu) \le d(u,x_{2n+1}) + d(x_{2n+1},Tu) \le d(u,x_{2n+1}) + d(Sx_{2n},Tu)\\
&\le d(u,x_{2n+1}) + \alpha(x_{2n},u,a)d(u,x_{2n})\\
&+ \beta(x_{2n},u,a)\frac{d(x_{2n},Sx_{2n})d^2(x_{2n},Tu) + d(u,Tu)d^2(u,Sx_{2n})}{d^2(x_{2n},Tu) + d^2(u,Sx_{2n})}\\
&+ \gamma(x_{2n},u,a)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu) + d^2(u,x_{2n}) + d(x_{2n},Sx_{2n})d(x_{2n},u)}{d(x_{2n},Sx_{2n}) + d(x_{2n},Tu) + d(x_{2n},u)}\\
&\le d(u,x_{2n+1}) + \alpha(x_0,u,a)d(u,x_{2n+1})\\
&+ \beta(x_0,u,a)\frac{d(x_{2n},Sx_{2n})d^2(x_{2n+1},Tu) + d(u,Tu)d^2(u,Sx_{2n+1})}{d^2(x_{2n},Tu) + d^2(u,Sx_{2n})}\\
&+ \gamma(x_0,u,a)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu) + d^2(u,x_{2n}) + d(x_{2n},Sx_{2n})d(x_{2n},u)}{d(x_{2n},Sx_{2n}) + d(x_{2n},Tu) + d(x_{2n},u)}.
\end{aligned}$$

On taking limits $n \to \infty$ and using (2.3.8), it follows that $|d(u,Tu)| \le 0$. Hence u is a common fixed point of S and T. Uniqueness of the fixed point trivially follows from (2.3.1). This completes the proof of the theorem.

3. Corollaries and examples

Setting $S = T$ in Theorem 2.1, we have the following corollary.

Corollary 3.1. *Let (X,d) be a complete complex valued metric space and $T : X \to X$. If there exist mappings $\alpha, \beta, \gamma, \delta$ and $\mu : X \times X \times X \to [0,1)$ such that for all $x, y \in X$, for a fixed $a \in X$:*

(i) $\alpha(Tx,y,a) \le \alpha(x,y,a)$ and $\alpha(x,Ty,a) \le \alpha(x,y,a)$;
(ii) $\beta(Tx,y,a) \le \beta(x,y,a)$ and $\beta(x,Ty,a) \le \beta(x,y,a)$;
(iii) $\gamma(Tx,y,a) \le \gamma(x,y,a)$ and $\gamma(x,Ty,a) \le \gamma(x,y,a)$;
(iv) $\delta(Tx,y,a) \le \delta(x,y,a)$ and $\delta(x,Ty,a) \le \delta(x,y,a)$.
(v) $\mu(Tx,y,a) \le \mu(x,y,a)$ and $\mu(x,Ty,a) \le \mu(x,y,a)$.
Also, suppose that T satisfy either
(vi)

$$\begin{aligned}
d(Tx,Ty) \preceq\ & \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx) + d(x,y)}\\
&+ \gamma(x,y,a)\frac{d(x,Ty)d(y,Tx)}{d(x,Tx) + d(y,Ty) + d(x,y)}\\
&+ \delta(x,y,a)\frac{[d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)]}{d(x,Tx) + d(y,Ty) + d(x,y)}\\
&+ \mu(x,y,a)\frac{[d(x,Tx)d(y,Tx) + d(y,Ty)d(x,Ty)]}{d(x,Tx) + d(y,Ty) + d(x,y)}, \qquad (3.1.1)
\end{aligned}$$

where $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a) + \delta(x,y,a) + \mu(x,y,a)| < 1$, $d(x,Tx) + d(y,Ty) + d(x,y) \ne 0$ and $d(x,Ty) + d(y,Tx) + d(x,y) \ne 0$ or

(vi) $d(Tx,Ty)=0$ *if* $d(x,Tx)+d(y,Ty)+d(x,y)=0$ *or* $d(x,Ty)+d(y,Tx)+d(x,y)=0$.

Then T *has a unique common fixed point in* X.

If $\mu=0$ *in Theorem 2.1, we have the following corollary.*

Corollary 3.2. *Let* (X,d) *be a complete complex valued metric space and* $S,T: X\to X$. *If there exist mappings* α,β,γ *and* $\delta: X\times X\times X\to[0,1)$ *such that for all* $x,y\in X$, *for a fixed* $a\in X$:

(i) $\alpha(TSx,y,a)\le\alpha(x,y,a)$ *and* $\alpha(x,STy,a)\le\alpha(x,y,a)$;

(ii) $\beta(TSx,y,a)\le\beta(x,y,a)$ *and* $\beta(x,STy,a)\le\beta(x,y,a)$;

(iii) $\gamma(TSx,y,a)\le\gamma(x,y,a)$ *and* $\gamma(x,STy,a)\le\gamma(x,y,a)$;

(iv) $\delta(TSx,y,a)\le\delta(x,y,a)$ *and* $\delta(x,STy,a)\le\delta(x,y,a)$.

Also, suppose that S *and* T *satisfy either*

(v)

$$\begin{aligned} d(Sx,Ty) &\preceq \alpha(x,y,a)d(x,y)\\ &+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\\ &+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}\\ &+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}, \end{aligned} \tag{3.2.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)|<1$, $d(x,Sx)+d(y,Ty)+d(x,y)\neq 0$ *and* $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$ *or*

(vi) $d(Sx,Ty)=0$ *if* $d(x,Sx)+d(y,Ty)+d(x,y)=0$ *or* $d(x,Ty)+d(y,Sx)+d(x,y)=0$.

Then S *and* T *have a unique common fixed point in* X.

If $\mu=\delta=0$, *in Theorem 2.1, we have the following corollary.*

Corollary 3.3. *Let* (X,d) *be a complete complex valued metric space and* $S,T: X\to X$. *If there exist mappings* α,β *and* $\gamma: X\times X\times X\to[0,1)$ *such that for all* $x,y\in X$, *for a fixed* $a\in X$:

(i) $\alpha(TSx,y,a)\le\alpha(x,y,a)$ *and* $\alpha(x,STy,a)\le\alpha(x,y,a)$;

(ii) $\beta(TSx,y,a)\le\beta(x,y,a)$ *and* $\beta(x,STy,a)\le\beta(x,y,a)$;

(iii) $\gamma(TSx,y,a)\le\gamma(x,y,a)$ *and* $\gamma(x,STy,a)\le\gamma(x,y,a)$.

Also, suppose that S *and* T *satisfy either*

(iv)

$$\begin{aligned} d(Sx,Ty) &\preceq \alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\\ &+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}. \end{aligned} \tag{3.3.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)|<1$, $d(x,Sx)+d(y,Ty)+d(x,y)\neq 0$ *and* $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$ *or*

(*v*) $d(Sx,Ty)=0$ *if* $d(x,Sx)+d(y,Ty)+d(x,y)=0$ *or* $d(x,Ty)+d(y,Sx)+d(x,y)=0$.

Then S *and* T *have a unique common fixed point in* X.

On choosing $\mu=\delta=\gamma=0$, *in Theorem 2.1, we get the following corollary.*

Corollary 3.4. *Let* (X,d) *be a complete complex valued metric space and* $S,T: X\to X$. *If there exist mappings* α *and* $\beta: X\times X\times X\to[0,1)$ *such that for all* $x,y\in X$, *for a fixed* $a\in X$;

(*i*) $\alpha(TSx,y,a)\leq\alpha(x,y,a)$ *and* $\alpha(x,STy,a)\leq\alpha(x,y,a)$;

(*ii*) $\beta(TSx,y,a)\leq\beta(x,y,a)$ *and* $\beta(x,STy,a)\leq\beta(x,y,a)$.

Also, suppose that S *and* T *satisfy either* (*iii*)

$$d(Sx,Ty)\preceq\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} \tag{3.4.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)|<1$ *and* $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$ *or*

(*iv*) $d(Sx,Ty)=0$ *if* $d(x,Ty)+d(y,Sx)+d(x,y)=0$.

Then S *and* T *have a unique common fixed point in* X.

Remark 3.5. Theorem 1.7 follows as a corollary to Corollary 3.4 if we choose $\alpha(x,y,a)=\alpha$ and $\beta(x,y,a)=\beta$ in Corollary 3.4.

The following examples corroborates the validity of our main results.

Example 3.6. Let $X=[0,1]$ and define $d: X\times X\to C$ by $d(x,y)=i|x-y|$, we define $S,T: X\to X$ by $Sx=\frac{x}{6}$ and $Tx=\frac{y}{6}$. We define $\alpha,\beta,\gamma,\delta$ and $\mu: X\times X\times X\to[0,1)$ by $\alpha(x,y,a)=\frac{x}{6}+\frac{y}{8}+\frac{1}{3}$, $\beta(x,y,a)=\frac{x^2y^2a}{16}$, $\gamma(x,y,a)=\frac{xya}{16}$ $\delta(x,y,a)=\frac{xya}{8}$ and $\mu(x,y,a)=\frac{x^8}{8}+\frac{y^8}{8}+\frac{1}{5}$.

Clearly, $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)+\mu(x,y,a)|<1$ with $a=\frac{1}{3}$. Now consider $\alpha(TSx,y,a)=\alpha(\frac{x}{36},y,a)=\frac{x}{216}+\frac{y}{8}+\frac{1}{3}<\frac{x}{6}+\frac{y}{8}+\frac{1}{3}$ and $\alpha(x,STy,a)=\alpha(x,y,a)=x+\frac{y}{216}+\frac{1}{3}$.

We now show that the inequality (2.1.1) holds for $x,y\in X$. Consider,

$$\begin{aligned}
d(Sx,Ty)=d(\frac{x}{6},\frac{y}{6})&=i\frac{|x-y|}{6}\leq i\frac{|x-y|}{3}\\
&\leq(\frac{x}{6}+\frac{y}{8}+\frac{1}{3})i|x-y|\leq\alpha(x,y,a)d(x,y)\\
&+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\\
&+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}\\
&+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}\\
&+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}.
\end{aligned}$$

Therefore all the conditions of Theorem 2.1 are satisfied 0 is a unique fixed point of S and T.

Example 3.7. Let $X = \{2, 3, 4, 5\}$ and we write $A = \{(2,3),(3,2),(2,5),(5,2),(5,3),(3,5),(4,3),(3,4),(4,5),(5,4)\}$, $B = \{(2,4),(4,2)\}$ and $C = \{(2,2),(3,3),(4,4),(5,5)\}$.

We define $d : X \times X \to C$ by $d(x,y) = \begin{cases} 2e^{i\theta}, & \text{if } (x,y) \in A \\ e^{\theta}, & \text{if } (x,y) \in B \\ 0, & \text{if } (x,y) \in C \end{cases}$

Then (X, d) is a complex metric space for $\theta \in [0, \frac{\pi}{2}]$.

We now define S and $T : X \to X$ by

$Sx = \begin{cases} 2, & \text{if } x \in \{2,4,5\} \\ 4, & \text{if } x = 3 \end{cases}$ and $Tx = \begin{cases} 2, & \text{if } x \in \{2,3,4\} \\ 4, & \text{if } x = 5. \end{cases}$

Let $\alpha(x,y,a) = \frac{1}{4}, \beta(x,y,a) = 0, \gamma(x,y,a) = \frac{1}{21}, \delta(x,y,a) = \frac{1}{2}$ and $\mu(x,y,a) = \frac{1}{5}$.

Then clearly, $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a) + \delta(x,y,a) + \mu(x,y,a)| = .99 < 1$. Now we verify the inequality (2.1.1).

Case (i). If $(x,y) = \{(2,3),(2,4),(2,2),(3,5),(4,2),(5,2),(4,3),(4,4),(5,3),(5,4)\}$ then we have $d(Sx,Ty) = 0$ so that the inequality (2.1.1) holds.

Case (ii). If $x = 2$, $y = 5$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.123e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$, so that the inequality (2.1.1) holds.

Case (iii). If $(x,y) = (3,3)$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.44e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$, so that the inequality (2.1.1) holds.

Case (iv). If $(x,y) = (3,4)$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.64e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (v). If $(x,y) = (3,2)$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.64e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (vi). If $(x,y) = (4,5)$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.29e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (vii). If $(x,y) = (5,5)$ then $d(Sx,Ty) = d(2,4) = e^{i\theta} \leq 1.44e^{i\theta} = \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)} + \delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)} + \mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Thus all the hypotheses of Theorem 2.1 are verified. Here, 2 is the unique common fixed point of S and T.

Here we note that inequality (1.7.1) fails to hold when $x = y = 3$ and for any $\alpha, \beta \in [0,1)$. Indeed, when $(x,y) = (3,3)$, we have $d(Sx,Ty) = d(2,4) = e^{i\theta} \not\leq \beta\frac{d(3,4)d(3,2)}{d(3,2)+d(3,4)} = \beta e^{i\theta}$.

Hence Theorem 1.7 is not applicable.

Example 3.8. Let $X = \{(1,2),(3,4),(5,6),(7,8)\}$ and we define from $d : X \times X \to C$ by $d(z_1,z_2) = 2|x_1 - x_2| + 3i|y_1 - y_2|$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then (X,d) is a complex valued metric space. We now define Sz and Tz with $z = x + iy$ on X by

$$Tz = Sz = \begin{cases} |x-y| + 2i|x-y|, & \text{if } z \in \{(1,2),(3,4),(5,6)\} \\ 3|x-y| + 4i|x-y|, & \text{if } z = (7,8) \end{cases}$$

We define $\alpha(x,y,a) = \frac{1}{3}, \beta(x,y,a) = \frac{1}{43}, \gamma(x,y,a) = 0, \delta(x,y,a) = \frac{1}{2}$ and $\mu(x,y,a) = \frac{1}{7}$.

Then clearly, $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a) + \delta(x,y,a) + \mu(x,y,a)| = 0.96 < 1$. Now we verify the inequality (2.1.1).

Case (i). If $(z_1,z_2) = \{((1,2),(1,2)),((1,2),(3,4)),((1,2),(5,6)),((3,4),(1,2)),((3,4),(3,4)),((3,4),(5,6)),((5,6),(1,2)),((5,6),(3,4)),((5,6),(5,6)),((7,8),(7,8))\}$ then $d(Sz_1,Tz_2) = 0$, so that the inequality (2.1.1) holds.

Case (ii). If $z_1 = (3,4)$ and $z_2 = (7,8)$, then $d(Sz_1,Tz_2) = 4 + 6i \leq \frac{1}{3}(8+12i) + \frac{2}{215}(4+6i) + \frac{3}{5}(4+6i) + \frac{3}{35}(4+6i) = 1.29(4+6i)$, so that inequality (2.1.1) holds.

Case (iii). If $z_1 = (5,6)$ and $z_2 = (7,8)$, then $d(Sz_1,Tz_2) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{1}{43}(4+6i) + \frac{4}{5}(4+6i) + \frac{8}{35}(4+6i) = 1.3(4+6i)$, so that inequality (2.1.1) holds.

Case (iv). If $z_1 = (1,2)$ and $z_2 = (7,8)$, then $d(Sz_1,Tz_2) = 4 + 6i \leq \frac{1}{3}3(4+6i) + (4+6i) + \frac{2}{21}(4+6i) = 2.09(4+6i)$, so that inequality (2.1.1) holds.

Case (v). If $z_1 = (7,8)$ and $z_2 = (3,4)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{2}{3}(4+6i) + \frac{2}{215}(4+6i) + \frac{3}{5}(4+6i) + \frac{3}{35}(4+6i) = 1.29(4+6i)$, so that inequality (2.1.1) holds.

Case (vi). If $z_1 = (7,8)$ and $z_2 = (1,2)$, then $d(Sz_1, Tz_2) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{2}{5}(4+6i) + \frac{2}{35}(4+6i) = 1.44(4+6i)$ so that inequality (2.1.1) holds.

Case (vii). If $z_1 = (7,8)$ and $z_2 = (5,6)$, then $d(Sz_1, Tz_2) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{2}{215}(4+6i) + \frac{4}{5}(4+6i) + \frac{8}{35}(4+6i) = 1.3(4+6i)$.

Thus all the hypotheses of Theorem 2.1 are verified. Also $(1,2)$ is the unique common fixed point of S and T.

Here we note that inequality (1.7.1) fails to hold when $z_1 = (5,6)$ and $z_2 = (7,8)$, for any α and $\beta \in [0,1)$ with $\alpha + \beta < 1$. Indeed, we have

$$d(Sz_1, Tz_2) = 4 + 6i \leq \alpha(4+6i) + \beta\frac{4}{5}(4+6i) = (\alpha+\beta)(4+6i) < (4+6i).$$

Thus Theorem 1.7 is not applicable.

Example 3.9. Let $X = \{(1,2),(3,4),(5,6),(7,8)\}$ and we define $d : X \times X \to C$, $T : X \to X$ and $S : X \to X$ as in Example 3.8.

We define $\alpha(x,y,a) = \frac{1}{3}, \beta(x,y,a) = \frac{1}{2}, \gamma(x,y,a) = \frac{1}{7}$.

Then clearly, $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a)| = .97 < 1$. Now we verify the inequality (2.3.1).

Case (i). If $(z_1, z_2) = \{((1,2),(1,2)),((1,2),(3,4)),((1,2),(5,6)),((3,4),(1,2)),((3,4),(3,4)),((3,4),(5,6)),((5,6),(1,2)),((5,6),(3,4)),((5,6),(5,6)),((7,8),(7,8))\}$, then $d(Sz_1, Tz_2) = 0$, so that the inequality (2.3.1) holds.

Case (ii). If $z_1 = (3,4)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(8+12i) + \frac{1}{2}(8+12i) + \frac{8+12i}{7} = 1.864(4+6i)$, so that inequality (2.3.1) holds.

Case (iii). If $z_1 = (5,6)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{1}{2}2(4+6i) + \frac{1}{7}\frac{5}{4}(4+6i) = 1.5(4+6i)$, so that inequality (2.3.1) holds.

Case (iv). If $z_1 = (1,2)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{1}{4}\frac{9}{5}(4+6i) + \frac{1}{7}\frac{9}{4}(4+6i) = 1.77(4+6i)$, so that inequality (2.3.1) holds.

Case (v). If $z_1 = (7,8)$ and $z_2 = (3,4)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(8+12i) + \frac{1}{2}(8+6i) + \frac{1}{7}(8+6i) = 1.8(4+6i)$, so that inequality (2.3.1) holds.

Case (vi). If $z_1 = (7,8)$ and $z_2 = (1,2)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{1}{2}\ \frac{8}{5}(4+6i) + \frac{1}{7}\frac{19}{7}(4+6i) = 2.18(4+6i)$ so that inequality (2.3.1) holds.

Case (vii). If $z_1 = (7, 8)$ and $z_2 = (5, 6)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{2}{5}(4+6i) + \frac{1}{7}\frac{6}{4}(4+6i) = 1.5(4+6i)$, so that inequality (2.3.1) holds.

Thus all the hypotheses of Theorem 2.3 are verified.

Here $(1, 2)$ is a unique common fixed point of S and T.

References

[1] A. Azam , B. Fisher , M. Khan , *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim., 32 (2011), 243-253.

[2] S. Bhatt , S. Chaukiyal , R.C. Dimri, *A common fixed point theorem for weakly compatible maps in complex valued metric spaces*, Int. J. Math. Sci. Appl., 3 (2011), 1385-1389

[3] D.S. Jaggi, B.K. Dass, *An extension of Banach's fixed point through rational expression*, Bull. Cal. Math. Soc., 72 (1980), 261-262.

[4] A. Jamshaid , K. Chakkrid , A. Azam, *Common fixed points for multivalued mappings in complex valued metric spaces with applications*, Abstr. Appl. Anal., 12 (2013), (Article ID 854965).

[5] H. K. Nashine, M. Imdad, M. Hasan, *Common fixed point theorems under rational contractions in complex valued metric spaces*, J. Nonlinear Sci. Appl., 7 (2014), 42-50.

[6] N. Singh, Deepak Singh, Anil Badal and V. Joshi, *Fixed point theorems in complex valued metric spaces*, J. Egy. Math. Soc., 24 (2016), 402-409

[7] F. Rouzkard , M. Imdad , *Some common fixed point theorems on complex valued metric spaces*, Comput. Math. Appl., 64 (2012), 1866-1874.

[8] M. Sarwar, M. Bahadur Zada, S. Manro, *Common fixed point theorems for weakly compatible mappings in complex valued metric spaces*, Bull. Internat. Math. Virtual Inst., 7 (2017), 11-21.

[9] W. Sintunavarat, P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequal. Appl., 2012 (2012), 84 .

[10] K. Sitthikul, S. Saejung, *Some fixed point theorems in complex valued metric space*, Fixed Point Theory Appl., 189 (2012).

[11] W. Sintunavarat , J.C. Yeol, P. Kumam, *Urysohn integral equa- tions approach by common fixed points in complex-valued metric spaces*, Adv. Differ. Eqn., 49 (2013).

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An iterative solution for a class of optimization problems

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Abstract. In this paper, we will describe an iterative approximation method to approach the optimal solution of a linear programming problem by providing an algorithm for resolution with an analysis of its convergence and its complexity.
Keywords: linear programming, optimal solution, primal-dual problem, iterative approximation method.

1. Introduction

Linear programming models are used in all natural sciences disciplines such as biology, physics, chemistry, earth science, and engineering disciplines such as computer science, logistics, mechanics, artificial intelligence, as well as in the social sciences such as economics, sociology, political science, see [15].

We can deal with problems of management, electoral politics, molecular biology, engineering, logistics, etc.., Whether strategic (choice to invest or not, choosing a location, the design of a fleet of vehicles or real estate ...) or operational (including scheduling, inventory management, sales forecasts ..), see [15].

Linear programming is the methods of optimizations concentrate on achieving the best outcome in a mathematical model whose requirements are represented by linear relationships. That was first solved by Dantzig [3] using Simplex algorithm, how still the most widely used in the literature. The linear programming problem was first shown to be solvable in polynomial time, but a larger theoretical and practical breakthrough in the field came in 1984 when Karmarkar [8, 9] introduced a new interior point algorithm for solving this type

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of problems. After that, Many algorithms and methods have been used to solve this kind of optimization problem with the same idea, as an example, Interior point methods [1, 2], Logarithmic barrier function [7], Path-following algorithm [11], or as Interior-Exterior point algorithm [4].

In this paper, we will describe an iterative approximation method to approach the optimal solution of a linear programming problem by providing an algorithm for resolution with an analysis of its convergence and its complexity.

2. Statement problem

We study an iterative approximation method for solving the linear programming problem

$$(P): \min_{x\in\mathbb{R}^n}\{c^T x: \ Ax=b, \ x\geq 0\},$$

The logarithmic barrier function method have for but to solve the family of problems

$$\min_{x>0}\{c^T x-\mu\sum_{j=1}^{n}\ln x_j: \ Ax=b\},$$

with $\mu>0$ called the barrier penalty coefficient.

Such problem can be solved by considering the family of penalized subproblems [4, 12]

$$(P_\mu): \min_{x>0}\{f_\mu(x)=c^T x-\mu\sum_{j=1}^{n}\ln x_j+\frac{1}{2\mu}\sum_{i=1}^{m}(a_{i.}x-b_i)^2\},$$

where x_j is the component of x number j also $a_{i.}$ is the row of the matrix A number i.

The family problems (P_μ) have an optimal solution for all $\mu>0$ while they are strictly convex add to some standard assumptions see [11] .

From other side $\omega(\mu)=(x(\mu),y(\mu),z(\mu))$ of the system (P_μ) corresponding to the Karush-Kuhn-Tucker (K-K-T) conditions gives a dual and primal μ-feasible solution.

Finally, every accumulation point ω^* of the sequence $\omega(\mu)$ is such that x^* is an optimal solution of the primal problem

$$(P): \min\{c^T x: \ Ax=b, \ x\geq 0\}$$

and (y^*,z^*) is an optimal solution of the dual

$$(D): \max\{b^T y: \ A^T y=z=c, \ z\geq 0\}.$$

To solve the primal problem (P), we consider the following penalized problem:

$$(P_\mu): \min_{x>0}\{f_\mu(x)=c^T x-\mu\sum_{j=1}^{n}\ln x_j+\frac{1}{2\mu}\sum_{i=1}^{m}(a_i x-b_i)^2\}, \text{ where } \mu>0.$$

Observe that f_μ, the objective function of problem (P_μ), is a strictly convex function.

This implies that problems (P_μ) has at most one minimum, and that this minimum, if it exists, is global and completely characterized by the K-K-T optimality conditions

$$c - \mu X^{-1}e + \frac{1}{\mu}A^T(Ax - b) = 0 \quad \text{and} \quad x > 0,$$

where e denotes the $\mathbb{R}^n$ unit vector i.e. $e = (1, ..., 1)^T$ see [12].

By introducing two vectors $y \in \mathbb{R}^m$ et $z \in \mathbb{R}^n$, this system can be rewritten in an equivalent way as:

$$(S_\mu) \begin{cases} ZXe - \mu e = 0 \\ Ax + \mu y - b = 0, \ x > 0 \\ A^T y + z - c = 0 \end{cases} .$$

We denote the unique solution of (S_μ) by $\omega(\mu) = (x(\mu), y(\mu), z(\mu))$.

In the next section, we describe our methods. The fourth section provide an algorithm of solution with an analysis of its convergence and its complexity.

3. Methods

From theoretical point of view, it is very difficult to find an exact solution of (P_μ). Which justify our utilization of a constructive method leads to an ideal approximative solution of our initial problem. In the next, We will explain how to formulate a solution of the system (S_μ) which allow the best approximation of $(x(\mu), y(\mu), z(\mu))$. The obvious idea is to determine a solution of (S_μ) using the Newton method. But this idea is to be avoided, noting the Newton method necessary iterations to find a satisfactory approximative solution.

Consequently, we consider the definition

Definition 3.1. *For $\mu > 0$. A point $\omega(\mu) = (x, y, z)$ is called $\mu-$feasible if it verify:*

$$Ax + \mu y - b = 0, \ x > 0,$$
$$A^T y + z - c = 0, \ z > 0.$$

Such solution x is called the primal $\mu-$feasible in relation with y.

The main idea of this section is the following. For every iteration, among the $\mu-$feasible solutions (i.e. solutions satisfying the two last equations of the system (S_μ)), we take the ones which violate the first equation of (S_μ). This well known technique is usually used in the "path following" category of interior point algorithms. The direction in these last algorithms coincides with the Newton direction, while they are different in our case.

Proposition 3.2. *For a choose parameter $\mu > 0$, and each $(y, z) \in I\!R^m \times I\!R^n$, we consider the point*

$$x_y^\mu = \arg\min_x \left\{ \left\| \frac{Zx}{\mu} - e \right\| : \ Ax + \mu y - b = 0, \ x \in I\!R^n \right\}.$$

We have $x_y^\mu = \mu Z^{-1}e + \mu Z^{-2}A^T d_y^\mu$ where $d_y^\mu = -(\mu AZ^{-2}A^T)^{-1}(\mu AZ^{-1}e + \mu y - b)$.

Proof. First, we remark that d_y^μ is well defined because Z is positive definite and A is of full rank. Which imply that $AZ^{-2}A^T$ is invertible. From other side, first-order optimality conditions of the problem

$$\min_x \left\{ \left\| \frac{Zx}{\mu} - e \right\| : \ Ax + \mu y - b = 0, \ x \in I\!R^n \right\}$$

give $Z\left(\frac{Zx_y^\mu}{\mu} - e\right) = A^T d_y^\mu$ with $d_y^\mu \in I\!R^n$, and we deduce that

$$x_y^\mu = \mu Z^{-1}e + \mu Z^{-2}A^T d_y^\mu.$$

Since $Ax_y^\mu + \mu y - b = 0$, we have

$$\mu AZ^{-1}e + \mu AZ^{-2}A^T d_y^\mu + \mu y - b = 0$$

also

$$d_y^\mu = -(\mu AZ^{-2}A^T)^{-1}(\mu AZ^{-1}e + \mu y - b).$$

□

We use the Euclidean norm of $\frac{Zx_y^\mu}{\mu} - e$ to measure the distance between the given point y and $y(\mu)$ on the central trajectory. This norm is a dual version of the measure used by Roos and Vial [13].

We take this distance as measure noted $m(y, \mu)$ and we write

$$m(y, \mu) = \left\| \frac{Zx_y^\mu}{\mu} - e \right\|.$$

It is important to remark that $m(y(\mu), \mu) = 0$ and $y_{y(\mu)}^\mu = x(\mu)$.

Next theorem prove that if we take the direction d_y^μ at a point y such that $m(y, \mu) < 1$, we converge quadratically to $y(\mu)$.

Theorem 3.3. *Let $\widehat{y} = y + d_y^\mu$ and $\widehat{z} = c - A^T\widehat{y}$, for $m(y, \mu) < 1$, we have the following results*

1. *$\widehat{z} > 0$ i.e $\widehat{y}$ is strictly feasible solution for (D).*
2. *$m(\widehat{y}, \mu) \leq m(y, \mu)^2$.*

Proof. 1. We remark that,

$$\widehat{z} = z - Z\left(\frac{Zx_y^\mu}{\mu} - e\right) = 2z - \frac{Z^2 x_y^\mu}{\mu}$$

then $\widehat{z}_j = (2 - \frac{Zx_y^\mu}{\mu})_j z_j$ where u_j is the jth component of vector u.

From other side, $m(y,\mu) < 1 \;\Rightarrow\; \|\frac{Zx_y^\mu}{\mu} - e\| < 1 \;\Rightarrow\; 0 < \left(\frac{Zx_y^\mu}{\mu}\right) < 2$.

Consequently, since $z_j > 0$, we deduce that $\widehat{z}_j > 0 \forall j$.

2. Let the vector $u = \frac{Zx_y^\mu}{\mu}$, and the definition of x_y^μ implies that

$$m(\widehat{y},\mu) = \left\| \frac{\widehat{Z}x_{\widehat{y}}^\mu}{\mu} - e \right\| \leq \left\| \frac{\widehat{Z}x_y^\mu}{\mu} - e \right\| = \|\widehat{Z}Z^{-1}u - e\|.$$

Furthermore $\widehat{Z}Z^{-1}u = (2Z - ZU)Z^{-1}u = 2u - Uu$.

Consequently

$$\begin{aligned}
&m(\widehat{y},\mu) \leq \|2u - Uu - e\| \Rightarrow m(\widehat{y},\mu)^2 \\
&\leq \sum_{j=1}^{n}(2u_j - u_j^2 - 1)^2 = \sum_{j=1}^{n}(u_j - 1)^4 \\
&\leq \left[\sum_{j=1}^{n}(u_j - 1)^2\right]^2 = \|u - e\|^4 \\
&= \|\frac{Zx_y^\mu}{\mu} - e\|^4 \\
&= m(y,\mu)^4.
\end{aligned}$$

Which completes the second statement of the proof. □

Proposition 3.4. *For $\widehat{u} = (1-\theta)\mu$, with $0 < \theta < 1$, we have*

$$m(y,\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(m(y,\mu) + \theta\sqrt{n}\Big).$$

Proof. The definition of x_y^μ, gives

$$m(y,\widehat{\mu}) = \left\| \frac{Zx_y^{\widehat{\mu}}}{\widehat{\mu}} - e \right\| \leq \left\| \frac{Zx_y^\mu}{\widehat{\mu}} - e \right\| = \left\| \frac{u}{1-\theta} - e \right\| \leq \frac{1}{1-\theta}(\|u - e\| + \theta\|e\|).$$

Adding the fact that $\|u - e\| = m(y,\mu)$ and $\|e\| = \sqrt{n}$, complete the result of proposition. □

Theorem 3.5. *Suppose that $m(y,\mu) \leq \frac{1}{2}$ and $\theta = \frac{1}{4\sqrt{n}+2}$, then $m(\widehat{y},\widehat{\mu}) \leq \frac{1}{2}$.*

Proof. From Proposition (3.4), we have

$$m(\widehat{y},\widehat{\mu}) \leq \tfrac{1}{1-\theta}\Big(m(\widehat{y},\mu)+\theta\sqrt{n}\Big).$$

And using Theorem (3.3), we obtain

$$m(\widehat{y},\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(m(y,\mu)^2+\theta\sqrt{n}\Big).$$

Furthermore

$$m(\widehat{y},\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(\frac{1}{4}+\theta\sqrt{n}\Big) \leq \frac{1+4\theta\sqrt{n}}{4(1-\theta)} = \frac{1}{2}.$$

From theorem (3.5), if we take $(\mu_k)_{k\in I\!N}$ such that

$$\mu_{k+1} = (1-\theta)\mu_k,\ \mu_0 > 0,$$

with $\theta = \frac{1}{4\sqrt{n}+2}$ and, starting from a strictly dual feasible point y^0 such

$$m(y^0,\mu_0) \leq \frac{1}{2}.$$

We can found a sequence of strictly dual feasible points $\{y^k\}$ such that

$$m(y^k,\mu_k) \leq \frac{1}{2}.$$

□

We define the pseudo-gap at point $\omega(\mu)$ by

$$\triangle(\mu) = |c^T x(\mu) - b^T y(\mu)| + \frac{1}{\mu}\|Ax(\mu)-b\|^2.$$

The next theorem gives an upper bound on the pseudo-gap calculated at (x_y^μ, y, z). This allows us to present an algorithm for the solution and to give evaluation its complexity.

Theorem 3.6. *Suppose that $m(y,\mu) \leq 1$, we have*

1. *x_y^μ is the primal $\mu-$feasible in relation to y.*
2. *The pseudo-gap $\triangle(x_y^\mu, y)$ calculated at (x_y^μ, y, z) verify*

$$\triangle(x_y^\mu, y) \leq \mu(\beta(y,\mu)\sqrt{n}), \quad \textit{with} \quad \beta = \sup(2\alpha, 2n).$$

Proof. 1. The definition of x_y^μ, gives $Ax_y^\mu + \mu y - b = 0$. from other side,

$$\left\| \frac{Zx_y^\mu}{\mu} - e \right\| = m(y,\mu) < 1 \;\Rightarrow\; Zx_y^\mu > 0 \;\Rightarrow\; x_y^\mu > 0$$

since $z > 0$. Which prove the first statement of theorem (3.6).

2. We have

$$\begin{aligned} \triangle(x_y^\mu, y) &= |c^T x_y^\mu - b^T y| + \tfrac{1}{\mu}\|Ax_y^\mu - b\|^2 \\ &= |z^T x_y^\mu - \mu\|y(\mu)\|^2| + \mu\|y(\mu)\|^2. \end{aligned}$$

Then

$$\triangle(x_y^\mu, y) = \begin{cases} z^T x_y^\mu, & \text{if } z^T x_y^\mu \geq \mu\|y(\mu)\|^2 \\ \mu\left(2\|y(\mu)\|^2 - \frac{z^T x_y^\mu}{\mu}\right), & \text{else .} \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left| \frac{z^T x_y^\mu}{\mu} - n \right| \leq \left\| \frac{z^T x_y^\mu}{\mu} - e \right\| \|e\| = m(y,\mu)\sqrt{n}.$$

It follows that

$$\mu(n - m(y,\mu)\sqrt{n}) \leq z^T x_y^\mu \leq \mu(n + m(y,\mu)\sqrt{n}).$$

Consequently $z^T x_y^\mu \leq \mu(\beta - n + m(y,\mu))$ since $\beta \geq 2n$.

And whereas $\beta \geq 2\alpha$, we obtain

$$\begin{aligned} 2\mu\|y(\mu)\|^2 - z^T x_y^\mu - z^T x_y^\mu &\leq 2\mu\alpha - \mu(n - m(y,\mu)\sqrt{n}) \\ &\leq \mu(\beta - n + m(y,\mu)\sqrt{n}). \end{aligned}$$

This achieve the proof of theorem (3.6). □

4. Results and discussion

We establish the following algorithm for the solution.

4.1 Algorithm

Initialization

- Let $\mu_0 > 0$, $(y^0, z^0) \in I\!R^m \times I\!R^n$ with $m(y^0, \mu_0) \leq \frac{1}{2}$, and $\theta = \frac{1}{4\sqrt{n}+2}$.
- Define threshold accuracy $\epsilon > 0$ for the pseudo-gab.
- Set $k := 0$.

Main step

y^k, z^k and μ_k, are given at iteration k.

- If $(\beta - n)\mu_k < \epsilon$ calculate x^k by
  $x^k = (Z^k)^{-2}A^T(A(Z^k)^{-2}A^T)^{-1}[b - \mu_k y^k - \mu_k A(Z^k)^{-1}e] + \mu_k(Z^k)^{-1}e$
- Else $(\beta - n)\mu_k \geq \epsilon$, then
  - Compute $d^k = \frac{1}{\mu_k}(A(Z^k)^{-2}A^T)^{-1}[b - \mu_k y^k - \mu_k A(Z^k)^{-1}e]$.
  - Let $y^{k+1} := y^k + d^k$ and $z^{k+1} = c - A^T y^{k+1}$.
  - Put $\mu_{k+1} := (1 - \theta)\mu_k$ and $k := k + 1$ and repeat the main step.

4.2 Convergence and complexity

The following theorem study the convergence of the previous algorithm and its complexity.

Theorem 4.1. *1. Let $v = -\ln\varepsilon$ and $v_0 = -\ln(\beta-n)\mu_0$, the algorithm stops after $(4\sqrt{n}+2)(v-v_0)$ steps.*

2. The last generated points (y,z) and x_y^μ are respectively strictly dual feasible and strictly primal $\mu-$feasible in relation to y.

Moreover, the pseudo-gap verify

$$\triangle(x_y^\mu, y) \leq \Big(1+\frac{1}{2\sqrt{n}}\Big)\varepsilon.$$

Proof. 1. For every iteration, the generated point y by the algorithm will be strictly feasible for (D), and $\mu > 0$, while $m(y,\mu) \leq \frac{1}{2}$, due to Theorem (3.3) and Theorem (3.5).

We have $\mu_k = (1-\theta)^k\mu_0$, and the algorithm stops if

$$(1-\theta)^k e^{-v_0} < e^{-v}. \tag{1}$$

Since $-\ln(1-\theta) > \theta$, Inequality (1) will hold if $k\theta > v - v_0$ which is equivalent to

$$k > (4\sqrt{n}+2)(v-v_0).$$

2. Theorem (3.6) implies that x_y^μ is primal $\mu-$feasible in relation to y and

$$\begin{aligned}\triangle(x_y^\mu, y) &\leq \mu(\beta - n + m(y,\mu)\sqrt{n}) \leq \mu(\beta-n) + \mu m(y,\mu)\sqrt{n} \\ &\leq \varepsilon + \tfrac{1}{2}\tfrac{\beta-n}{\sqrt{n}}\mu \leq \Big(1+\tfrac{1}{2\sqrt{n}}\Big)\varepsilon.\end{aligned}$$

Which completes the proof. □

4.3 Discussion

1. The computational effort for every iteration requires $\mathcal{O}(n^3)$ arithmetic operations. Since the algorithm terminates at most in $\mathcal{O}(\sqrt{n}v)$ iterations, the complexity is $\mathcal{O}(n^{3.5}v)$.

2. To start the algorithm, we have assumed that a feasible solution (y^0, z^0), such that $m(y^0,\mu_0) \leq \frac{1}{2}$, is available.

   In general, it is not the case. However, it is necessary to provide a means for determining an $(y^0, z^0) \in I\!R^m \times I\!R^n$ so that the algorithm can be initiated.

3. For the initialization of the algorithm, the method suggested by Monteiro and Adler in [11] can be easily adapted to our approach.

5. Conclusion

The problems (P_μ) are strictly convex and under some standard assumptions, have an optimal solution for all $\mu > 0$. The solution $\omega(\mu) = (x(\mu), y(\mu), z(\mu))$ of the system corresponding to the Karush-Kuhn-Tucker conditions gives a feasible dual solution and a primal μ-feasible solution; moreover, every accumulation point of the sequence $\{\omega(\mu)\}$ is such that x^* is an optimal solution of the primal problem and (y^*, z^*) is an optimal solution of the dual problem. The presented algorithm is polynomial and its complexity is $\mathcal{O}(n^{3.5}L)$.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

Both authors worked in coordination. Both authors carried out the proof, read and approved the final version of the manuscript.

References

[1] K.M. Ansteicher and R.M. Freund, *Interior point methods in mathematical programming*, Annals of Operations Research, Baltzer Science Publishing, Vol. 62, 1996.

[2] Z.Y. Cheng and J.E. Mitchell, *A primal dual interior point method for linear programming based on a weigthed barrier function*, Journal of Optimization Theory and Application, 87 (1995), 301-321.

[3] G. Dantzig, *Maximization of a linear function of variable subject to linear inequality*, Activity Analysid of Production and Allocation, ed., John Wiley, New York, 87 (1951), 339-347.

[4] K. El Yassini, A. Benchakroun, J.P. Dussault and A. El Afia, *Approche de type intérieur-extérieur pour la programmation linéaire et quadratique convexe*, Proceedings CIRO'05, 39-54. 2005.

[5] A.V. Fiacco and N. McCormick, *Nonlinear programming: sequential unconstrained minimization techniques*, SIAM, 1990.

[6] R.M. Freund, *A potential-function reduction algorithm for solving linear program directly from an infeasible warm start*, Mathematical Programming, 52 (1991), 441-466.

[7] O. Güler, *Barrier functions in interior point methodes*, Mathematics of Operations Research, 21 (1996), 860-885.

[8] N. Karmarkar, *A new polynomial time algorithm for linear programming*, Combinatorica, 4 (1984), 373-395.

[9] N. Karmarkar, *Some comments on the significance of the new polynomial-time algorithm for the linear programming*, AT&T Bell Laboratories, Murray Hill, New Jersey, 1984.

[10] R.E. Marsten, D.F. Shanno and E.M. Simantiraki, *Interior point methods for linear programming and nonlinear programming*, Duff and Watson (ed.), Numerical Analysis: The State of the Art, Institute of Mathematical and its Applications Publisher, 1996.

[11] R.D.C. Monteiro and I. Adler, *Interior path following Primal-dual Algorithms part II: Convex quadratic Programming*, Mathematical Programming, 44 (1989), 27-41.

[12] M. Ould Sidi, *Interior-exterior polynomial path-following approach for convex quadratic programming*, International Journal of Pure and Applied Mathematics, 111 (2016), 605-612.

[13] C. Roos and J.Ph Vial, *A polynomial method of approximative centers for Linear Programming*, Mathematical Programming, 54 (1992), 295-305.

[14] C. Roos, T. Terlaky and J.Ph Vial, *Theory and algorithms for Linear Programming. An interior point approach*, John Wiley and Sons, 1997.

[15] G. Zhang, J. Lu and Y. Gao, *Multi-level decision making models, methods and applications*, Springer, 2015.

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Some fixed point theorems in complex valued b-metric spaces

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Abstract. We introduce the notion of fixed points for a mappings in complex valued b-metric space and demonstrate the existence and uniqueness of the main Banach contractive type, Kannan type, and Chatterjea type in complex valued b-metric spaces. Presented theorems in this paper extend and generalize the results derived by Mehmet and Kiziltunc in [12]. Some examples are given to illustrate the main results.

Keywords: b-metric spaces, complex valued, b-metric spaces, fixed point.

1. Introduction

Banach [9] introduced the main contraction principle for the fixed point theory which was the starting point for many researchers in nonlinear analysis. In 1989, Bakhtin [8] introduced the concept of a b-metric space as a generalization of metric spaces. In 1993, Czerwik [10, 11] extended many results related to the b-metric spaces. Close to our interest in this paper many researchers studied some fixed point theorems in the so called b-metric space [1, 2, 3, 4, 5, 12, 13, 14]. In 2011, Azam et al. [17] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings. The existence and uniqueness of the fixed points of self-mappings noted were considered in [18, 19]. In 2013, Rao et al [20] introduced the notion of complex valued b-metric spaces which is more general than the well known complex valued metric spaces and also gave common fixed point theorems.

In this paper, we generalize results of Mehmet and Kiziltunc in [12], by introducing the contractive mapping in a complete complex valued b-metric space.

Definition 1.1 ([7]). *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(*i*) $d(x, y) = 0$ *if and only if* $x = y$;

(*ii*) $d(x, y) = d(y, x)$;

(iii) $d(x,y) \leq s[d(x,z)+d(z,y)]$.

The pair (X,d) *is called a b-metric space. The number* $s \geq 1$ *is called the coefficient of* (X,d).

Example 1.1 ([6])**.** Let $X = \{x_1, x_2, x_3, x_4\}$. Define a function $d : X \times X \to [0,\infty)$ such that $d(x_1,x_2) = k \geq 2$ and $d(x_1,x_3) = d(x_1,x_4) = d(x_2,x_3) = d(x_2,x_4) = d(x_3,x_4) = 1$, $d(x_i,x_j) = d(x_j,x_i)$ for all $i,j = 1,2,3,4$ and $d(x_i,x_i) = 0$ for all $i,j = 1,2,3,4$. Then $d(x_i,x_j) \leq \frac{k}{2}[d(x_i,x_n)+d(x_n,x_j)]$ for $x_i \in X$ and $i,j = 1,2,3,4$. Therefore, (X,d) is a b-metric space with constant $s = \frac{k}{2}$. However if $k > 2$ the ordinary triangle inequality does not hold and thus (X,d) is not a metric space.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_1),\ \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$$

Thus $z_1 \precsim z_2$ if one of the following holds:

$$\begin{aligned}
&(1)\ \ \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \quad \textit{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2),\\
&(2)\ \ \mathrm{Re}(z_1) < \mathrm{Re}(z_2) \quad \textit{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2),\\
&(3)\ \ \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \quad \textit{and} \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2),\\
&(4)\ \ \mathrm{Re}(z_1) < \mathrm{Re}(z_2) \quad \textit{and} \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2).
\end{aligned}$$

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2),(3) and (4) is satisfied, also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1. We can easily check that the following statements are hold:

(i) If $a,b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$,

(ii) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$,

(iii) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2 ([17])**.** *Let* X *be a nonempty set. A function* $d : X \times X \to \mathbb{C}$ *is called a complex valued metric on* X *if for all* $x,y,z \in X$ *the following conditions are satisfied:*

(i) $0 \precsim d(x,y)$ *and* $d(x,y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x,y) = d(y,x)$*;*

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$.

The pair (X,d) *is called a complex valued metric space.*

Example 1.2 ([18])**.** Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x,y) = i|x-y|$, for all $x,y \in X$. Then (X,d) is a complex valued metric space.

Example 1.3 ([19]). Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = e^{ik}|x - y|$, where $k \in \mathbb{R}$ and for all $x, y \in X$. Then (X, d) is a complex valued metric space.

Definition 1.3 ([20]). *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:*

(*i*) $0 \precsim d(x, y)$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(*ii*) $d(x, y) = d(y, x)$;

(*iii*) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 1.4 ([20]). Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 1.4 ([20]). *Let (X, d) be a complex valued b-metric space.*

(*i*) *A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) := \{y \in X : d(x, y) \prec r\} \subseteq A$.*

(*ii*) *A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - X) \neq \emptyset$.*

(*iii*) *A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.*

(*iv*) *A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.*

(*v*) *A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) : x \in X and 0 \prec r\}$.*

Definition 1.5 ([20]). *Let (X, d) be a complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.*

(*i*) *If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > \mathbb{N}, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.*

(*ii*) *If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > \mathbb{N}, d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.*

(*iii*) *If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.*

Lemma 1.1 ([20]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2 ([20]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

2. Main result

Definition 2.1. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$. A mappings $T : X \to X$ is called contraction if there exists a constant $k \in [0, 1)$*

$$d(Tx, Ty) \precsim kd(x, y) \tag{2.1}$$

for all $x, y \in X$.

Definition 2.2. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called weak contraction if there exists a constant $\alpha \in (0, 1)$ and $\beta \geq 0$ such that*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \beta d(y, Tx) \tag{2.2}$$

for all $x, y \in X$.

Remark 2.1. Note that by using the symmetry of the distance, the weak contraction condition (2.2) implies

$$d(Tx, Ty) \precsim \alpha d(x, y) + \beta d(x, Ty) \tag{2.3}$$

for all $x, y \in X$. So, we have to verify (2.2)and (2.3) in order to check the weak contractiveness of T.

Remark 2.2. It is clear that any contraction mapping is also weak contraction in the (usual) complex b-metric space.

Our next theorem is about the generalization of the main Banach's contraction theorem in complex valued b-metric spaces.

Theorem 2.1. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \precsim kd(x, y) \tag{2.4}$$

where $k \in [0, 1)$ and $sk < 1$. Then, T has a unique fixed point in X.

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then $d(x_1, x_2) = d(Tx_0, Tx_1) \precsim kd(x_0, x_1)$, and also $d(x_2, x_3) = d(Tx_1, Tx_2) \precsim kd(x_1, x_2) \precsim k^2 d(x_0, x_1)$, By induction on n we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \precsim kd(x_{n-1}, x_n) \precsim k^n d(x_0, x_1). \tag{2.5}$$

Since $k \in [0, 1)$ we get $k^n \in [0, 1)$. Therefore, T is a contraction mapping. Now we show that $\{x_n\}$ is a Cauchy sequence in (X, d). Let $m > n > 0$. Thus,

$$\begin{aligned}
d(x_n, x_m) &\precsim sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_m) + s^3 d(x_{n+2}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_m) \\
&+ ... + s^{m-n-2} d(x_{m-3}, x_{m-2}) + s^{m-n-1} d(x_{m-2}, x_{m-1}) \\
&+ s^{m-n-1} d(x_{m-1}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_m) \\
&+ ... + s^{m-n-2} d(x_{m-3}, x_{m-2}) + s^{m-n-1} d(x_{m-2}, x_{m-1}) \\
&+ s^{m-n} d(x_{m-1}, x_m).
\end{aligned}$$

By using (2.5) we get

$$\begin{aligned}
d(x_n, x_m) &\precsim sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + s^3 k^{n+2} d(x_0, x_1) \\
&+ ... + s^{m-n-2} k^{m-3} d(x_0, x_1) + s^{m-n-1} k^{m-2} d(x_0, x_1) \\
&+ s^{m-n} k^{m-1} d(x_0, x_1) = \sum_{i=1}^{m-n} s^i k^{i+n-1} d(x_0, x_1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(x_n, x_m) &\precsim \sum_{i=1}^{m-n} s^{i+n-1} k^{i+n-1} d(x_0, x_1) \\
&= \sum_{t=n}^{m-1} s^t k^t d(x_0, x_1) \precsim \sum_{t=n}^{\infty} (sk)^t d(x_0, x_1) = \frac{(sk)^n}{1 - sk} d(x_0, x_1).
\end{aligned}$$

Taking the modulus, we get

$$|d(x_n, x_m)| \leq \frac{(sk)^n}{1 - sk} \ |d(x_0, x_1)| \to 0 \ \text{ as } \ m, n \to \infty. \tag{2.6}$$

Therefore

$$\lim_{n,m \to \infty} |d(x_n, x_m)| = 0. \tag{2.7}$$

Hence, $\{x_n\}$ is a Cauchy sequence in X.
Since X is complete, there exists some $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now we show that $Tx = x$. Assume not, then there exist $z \in X$ such that

$$|d(Tx, x)| = |z| > 0. \tag{2.8}$$

So, we have

$$\begin{aligned} z = d(x, Tx) &\precsim sd(x, x_n) + sd(x_n, Tx) = sd(x, x_n) + sd(Tx_{n-1}, Tx) \\ &\precsim sd(x, x_n) + skd(x_{n-1}, x). \end{aligned}$$

This implies that

$$|z| = |d(x, Tx)| \leq s|d(x, x_n)| + sk|d(x_{n-1}, x)| \to 0 \text{ as } n \to \infty, \tag{2.9}$$

we obtain that $|z| = |d(x, Tx)| \leq 0$, a contradiction with (2.8). So $|z| = |d(x, Tx)| = 0$, Hence $Tx = x$.
Now we show that T has a unique fixed point. For this, assume that x^* is another fixed point of T. Then by using (2.1) we get

$$d(x, x^*) = d(Tx, Tx^*) \precsim kd(x, x^*) \prec d(x, x^*).$$

This implies that $|d(x, x^*)| < |d(x, x^*)|$, a contradiction. Thus x is a unique fixed point in X. This completes the proof. □

Remark 2.3. Any mapping satisfying the contractive condition in (2.1) need not be a contraction in complex valued b-metric spaces unless some conditions on k.

In the following theorem we extend the Kannan type fixed point theorem in [15] in complex valued b-metric spaces.

Theorem 2.2. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \precsim \mu[d(x, Tx) + d(y, Ty)] \tag{2.10}$$

where $\mu \in [0, \frac{1}{2})$. Then, T has a unique fixed point in X.

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then

$$d(x_1, x_2) = d(Tx_0, Tx_1) \precsim \mu[d(x_0, Tx_0) + d(x_1, Tx_1)],$$

and also

$$d(x_2, x_3) = d(Tx_1, Tx_2) \precsim \mu[d(x_1, Tx_1) + d(x_2, Tx_2)].$$

By induction on n we get $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \precsim \mu[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] = \mu[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$. Thus we obtain

$$d(x_n, x_{n+1}) \precsim \frac{\mu}{1-\mu} d(x_{n-1}, x_n). \tag{2.11}$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\precsim \frac{\mu}{1-\mu} d(x_{n-1}, x_n) \\ &\precsim (\frac{\mu}{1-\mu})^2 d(x_{n-2}, x_{n-1}) \\ &\precsim (\frac{\mu}{1-\mu})^3 d(x_{n-3}, x_{n-2}) \\ &\precsim ... \precsim (\frac{\mu}{1-\mu})^n d(x_0, x_1). \end{aligned}$$

Since $\mu \in [0, \frac{1}{2})$, we get $(\frac{\mu}{1-\mu})^n \in [0, 1)$. Therefore, T is a contraction mapping. Easily as in the same manner in Theorem 2.1 we can show that $\{x_n\}$ is a Cauchy sequence in (X, d). Since X is complete, there exists some $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now we show that $Tx = x$. Assume not, then there exist $z \in X$ such that

$$|d(Tx, x)| = |z| > 0. \tag{2.12}$$

So, we have

$$\begin{aligned} z = d(x, Tx) &\precsim sd(x, x_n) + sd(x_n, Tx) = sd(x, x_n) + sd(Tx_{n-1}, Tx) \\ &\precsim sd(x, x_n) + s\mu[d(x_{n-1}, Tx_{n-1}) + d(x, Tx)] \\ &= sd(x, x_n) + s\mu d(x_{n-1}, x_n) + s\mu d(x, Tx). \end{aligned}$$

This implies that

$$\begin{aligned} d(x, Tx) &\precsim \frac{s}{1-s\mu} d(x, x_n) + \frac{s\mu}{1-s\mu} d(x_{n-1}, x_n) \\ &\precsim \frac{s}{1-s\mu} d(x, x_n) + \frac{s\mu}{1-s\mu} (\frac{\mu}{1-\mu})^{n-1} d(x_0, x_1). \end{aligned} \tag{2.13}$$

Taking modulus in (2.13) we get

$$\begin{aligned} |z| &= |d(x, Tx)| \\ &\leq \frac{s}{1-s\mu} |d(x, x_n)| + \frac{s\mu}{1-s\mu} (\frac{\mu}{1-\mu})^{n-1} |d(x_0, x_1)| \to 0 \ \text{ as } \ n \to \infty, \end{aligned} \tag{2.14}$$

we obtain that $|z| = |d(x, Tx)| \leq 0$, a contradiction with (2.12). So $|z| = |d(x, Tx)| = 0$, Hence $Tx = x$. As in Theorem 2.1 it is easily to show that T has a unique fixed point. This completes the proof. □

Remark 2.4. It is clear that we have no restriction on the contraction mapping (2.10) for applying the Kannan's fixed point theorem complex valued b-metric spaces.

Remark 2.5 (see also [12])**.** Any mapping satisfying the contractive condition (2.10) need not be a weak contraction in complex valued b-metric spaces unless $s\mu \in [0, \frac{1}{2})$

In the following theorem we extend the Chatterjea type fixed point theorem in [16] in complex valued b-metric spaces.

Theorem 2.3. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \precsim \lambda[d(x, Ty) + d(y, Tx)] \tag{2.15}$$

where $s\lambda \in [0, \frac{1}{2})$. Then, T has a unique fixed point in X.

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \precsim \lambda[d(x_0, Tx_1) + d(x_1, Tx_0)] \\ &= \lambda[d(x_0, x_2) + d(x_1, x_1)] = \lambda d(x_0, x_2), \end{aligned}$$

and

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \precsim \lambda[d(x_1, Tx_2) + d(x_2, Tx_1)] \\ &= \lambda[d(x_1, x_3) + d(x_2, x_2)] = \lambda d(x_1, x_3), \end{aligned}$$

By induction on n we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\precsim \lambda[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &= \lambda[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &= \lambda d(x_{n-1}, x_{n+1}) \\ &\precsim s\lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

Thus we obtain

$$d(x_n, x_{n+1}) \precsim \frac{s\lambda}{1 - s\lambda} d(x_{n-1}, x_n).$$

Since $s\lambda \in [0, \frac{1}{2})$, we get $\frac{s\lambda}{1-s\lambda} \in [0, 1)$. Therefore, T is a contraction mapping. Easily as in the same manner in Theorem 2.1 and Theorem 2.2 we can show that $\{x_n\}$ is a Cauchy sequence in X and hence it is convergent Since X is complete. Now we show that x is a fixed point of T. Assume not, then there exist $z \in X$ such that

$$|d(Tx, x)| = |z| > 0. \tag{2.16}$$

So, we have

$$\begin{aligned} z &= d(x,Tx) \\ &\precsim sd(x,x_{n+1}) + sd(x_{n+1},Tx) \\ &= sd(x,x_{n+1}) + sd(Tx_n,Tx) \\ &\precsim sd(x,x_{n+1}) + s\lambda[d(x_n,Tx) + d(x,Tx_n)] \\ &= sd(x,x_{n+1}) + s\lambda d(x_n,Tx) + s\lambda d(x,x_{n+1}). \end{aligned}$$

Taking modulus in the above inequality we get

$$|z| = |d(x,Tx)| \leq s|d(x,x_{n+1})| + s\lambda|d(x_n,Tx)| + s\lambda|d(x,x_{n+1})|. \tag{2.17}$$

Taking the limit of (2.17) as $n \to \infty$, we get

$$|z| = |d(x,Tx)| \leq s\lambda|d(x,Tx)| < \frac{1}{2}|z|, \tag{2.18}$$

The inequality (2.18) true only if $|z| = |d(x,Tx)| = 0$, Hence $Tx = x$. Finally, we show that x is the unique fixed point of T. Assume not, then there exist x^* another fixed point of T. Then, by using (2.1) we get

$$\begin{aligned} d(x,x^*) &= d(Tx,Tx^*) \\ &\precsim \lambda[d(x,Tx^*) + d(x^*,Tx)] \\ &= \lambda[d(x,x^*) + d(x^*,x)] = 2\lambda d(x,x^*). \end{aligned} \tag{2.19}$$

Taking modulus of (2.19), and since $\lambda < \frac{1}{2}$ we get

$$|d(x,x^*)| \leq 2\lambda|d(x,x^*)| < |d(x,x^*)|$$

a contradiction. we deduce that T has a unique fixed point. This completes the proof. □

Example 2.1. Let $X = \{x_1, x_2, x_3, x_4\}$. Define a function $d : X \times X \to \mathbb{C}$ such that

$$d(x,y) = \begin{cases} d(x_1,x_2) = 3i, \\ d(x_1,x_3) = d(x_2,x_3) = i, \\ d(x_1,x_4) = d(x_2,x_4) = d(x_3,x_4) = 5i, \\ d(x_m,x_n) = d(x_n,x_m), & \text{for all } \ m,n = 1,2,3,4 \\ d(x_m,x_m) = 0, & \text{for all } m = 1,2,3,4. \end{cases}$$

Then $d(x_m,x_n) \leq \frac{k}{2}[d(x_m,x_l) + d(x_l,x_n)]$ for $x_m, x_n, x_l \in X, k \geq 3$ and $m,n,l = 1,2,3,4$. Therefore, (X,d) is a complex valued b-metric space with constant $s = \frac{3}{2}$.

Let $x = x_1$, $y = x_2$, and $z = x_3$, so we can see that

$$d(x,y) = d(x_1,x_2) = 3i \succ 2i = i + i = d(x_1,x_3) + d(x_3,x_2) = d(x,z) + d(z,y).$$

So d is not a complex valued metric.

Now define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} x_1, & \text{if } x = x_4 \\ x_3, & \text{if } x \in \{x_1, x_2, x_3\} \end{cases}$$

Note that

$$d(Tx_1, Tx_2) = d(Tx_1, Tx_3) = d(Tx_2, Tx_3) = 0$$

Also, in all other cases

$$d(Tx, Ty) = i, [d(x, Tx) + d(y, Ty)] \succ 5i.$$

Therefore, for $\lambda = \frac{1}{2}$, all conditions of Theorem 2.2 are satisfied to get a unique fixed point x_3 of T.

References

[1] A. Mukheimer, *$\alpha - \psi$-contractive mappings on b-metric space*, Journal of Computational Analysis and Applications, 18 (2015), 636-644.

[2] A. Mukheimer, *Common fixed point theorems for a pair of mappings in complex valued b-metric spaces*, Advances in Fixed Point Theory, 4 (2014), 344-354.

[3] A. Mukheimer, *Some common fixed point theorems in complex valued b-metric spaces*, Hindawi Publishing Corporation, The Scientific World Journal, 2014, Article ID 587825, http://dx.doi.org/10.1155/2014/587825, (2014).

[4] A. Mukheimer, *$\alpha - \psi - \phi$-contractive mappings in ordered partial b-metric spaces*, Journal of Nonlinear Sciences and Applications, 7 (2014), 168-179.

[5] T. Abeljawad, K. Abodayeh, N. Mlaiki, *On Fixed point generalizations to partial b-metric spaces*, Journal of Computational Analysis and Applications, 19 (2015), 883-891.

[6] S.L. Singh, B. Prasad, *Some coincidence theorems and stability of iterative procedures*, Comput. Math. Appl., 55 (2008), 2512-2520.

[7] H. Aydi, M. Bota, E. Karapinar, S. Mitrovic, *A fixed point theorem for set valued quasi-contractions in b-metric spaces,* Fixed Point Theory and its Applications, 2012, 2012:212..

[8] IA. Bakhtin, *The contraction principle in quasimetric spaces*, Funct. Anal., 30 (1989), 26-37.

[9] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fundamenta Mathematicae, 3 (1922), 133-181.

[10] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces,* Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 263-276.

[11] S. Czerwik, *contraction mappings in b-metric spaces,* Acta Math. Inform. Univ. Osrav., 1 (1993), 5-11.

[12] K. Mehmet, H. Kiziltunc, *On some well known fixed point theorems in b-metric Spaces*, Turkish Journal of Analysis and Number Theory, 1.1 (2013), 13-16.

[13] S. Singh, B. Chamola, *Quasi-contractions and approximate fixed points,* J. Natur. Phys. Sci., 16 (2002), 105-107.

[14] H. Yingtaweesittikul, *Suzuki type fixed point for generalized multi-valued mappings in b-metric spaces*, Fixed Point Theory and Applications, 23 (2013), 2012:215.

[15] R. Kannan, *Some results on fixed points,* Bull. Calcutta. Math. Soc., 60 (1968), 71-76.

[16] S. Chatterjea, *Fixed point theorems,* C. R. Acad. Bulgare Sci., 25 (1972), 727-730.

[17] A. Azam, F. Brain, M. Khan, *Common fixed point theorems in complex valued metric spaces,* Numerical Functional Analysis and optimization, 32 (2011), 243-253.

[18] S. Datta, S. Ali, *A Common fixed point theorem under contractive condition in complex valued metric spaces,* International Journal of Advanced Scientific and Technical Research, 6 (2012), 467-475.

[19] W. Sintunavarat, P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, Journal of Inequalities and Applications, 2012, 2012:84.

[20] K. Rao, P. Swamy, J. Prasad, *A Common fixed point theorem in complex valued b-metric spaces,* Bulletin of Mathematics and Statistics Research, 1 (2013).

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Fuzzy H_v-semigroups

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Abstract. In this paper, we introduce the notion of fuzzy H_v-semigroups, fuzzy H_v-groups and fuzzy H_v-homomorphisms and establish connections between fuzzy H_v-semigroups and H_v-semigroups. Also, we define and analyze the concept of fuzzy (strong) hypercongruences on fuzzy H_v-semigroups.

Keywords: hyperstructure, H_v-semigroup, fuzzy H_v-semigroup, fuzzy homomorphism, fuzzy (strong) hypercongruence.

1. Introduction

The notion of a hypergroup introduced by Marty in 1934 [12] at the 8th congress of Scandinavian Mathematicians. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, (see [1-8, 17]). Vougiouklis [18] introduced a new class of hyperstructures, the so-called H_v-structures.

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The H_v-structures are hyperstructures where equality is replaced by non-empty intersection.

The notion of a fuzzy subset introduced by Zadeh in 1965 [19] as a function from a nonempty set H to unit real interval $I = [0, 1]$. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [15], there have been a number of generalizations of this fundamental concept (see [14]). The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work to do on the connections between fuzzy sets and hyperstructures (see [3, 9, 13]). Recently, Davvaz has applied fuzzy sets to the study of algebraic hyperstructures. He defined fuzzy H_v-subgroups and fuzzy H_v-submodules which are generalizations of Rosenfeld's fuzzy subgroups and fuzzy submodules (see [6, 7]).

In [16] Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups. The fuzzy hyperring notion is defined and studied in [10]. In [11] Leoreanu-Fotea defined and studied fuzzy hypermodules notion and connections with hypermodules.

The paper is organized as follows: in Section 2, some fundamental definitions on H_v-structures are explored. In Section 3, we consider the concept of fuzzy H_v-semigroups and fuzzy H_v-groups. In Section 4, we study connections between the fuzzy H_v-semigroups(H_v-groups) and H_v-semigroups(H_v-groups) and then establish some useful theorems. Finally, in Section 5, we define the notion of fuzzy (strong) hypercongruence on fuzzy H_v-semigroups and present a few results in this respect.

2. Preliminaries

First of all, we recall some notions and results (see [4, 5, 8, 17]) that we shall use in the following paragraphs.

Let H be a nonempty set and let $\wp^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid* (or hyperstructure).

If A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$.

A hyperstructure $(H, \circ)$ is called an H_v-*semigroup* if

$$((x \circ y) \circ z) \cap (x \circ (y \circ z)) \neq \emptyset,$$

for all $x, y, z \in H$.

Definition 2.1. *An H_v-semigroup $(H, \circ)$ is called a H_v-group if for all $x \in H$, we have $x \circ H = H \circ x = H$.*

A H_v-subgroup $(K, \circ)$ of $(H, \circ)$ is a nonempty set K, such that for all $k \in K$, we have $x \circ K = K \circ x = K$.

An H_v-group $(H, +)$ is called a *weak commutative H_v-group* if $(x + y) \cap (y + x) \neq \emptyset$ for all $x, y \in H$.

It is natural to speak now about homomorphisms between H_v-groups.

Definition 2.2. *Let $(H_1, \circ_1)$, $(H_2, \circ_2)$ be two H_v-groups. A map $f : H_1 \longrightarrow H_2$ is called an H_v-homomorphism or a weak homomorphism if*

$$f(x \circ_1 y) \cap f(x) \circ_2 f(y) \neq \emptyset; \text{ for all } x, y \in H_1.$$

f is called an inclusion homomorphism if

$$f(x \circ_1 y) \subseteq f(x) \circ_2 f(y); \text{ for all } x, y \in H_1.$$

Finally, f is called a strong homomorphism if

$$f(x \circ_1 y) = f(x) \circ_2 f(y); \text{ for all } x, y \in H_1.$$

If f is onto, one to one and strong homomorphism, then it is called an isomorphism.

Many examples of hypergroups and hyperrings can be found in [4], [5], [8] and [17]. We present here only two examples of H_v-groups.

Example 2.3. 1) Let $(G, \cdot)$ be a group and R an equivalence relation on G. In G/R consider the hyperoperation $\circ$ defined by $\overline{x} \circ \overline{y} = \{\overline{z} | z \in \overline{x} \cdot \overline{y}\}$, where $\overline{x}$ denotes the equivalence class of the element x. Then $(G, \circ)$ is an H_v-group which is not always a hypergroup.

2) On the set Z_{mn} consider the hyperoperation $\oplus$ defined by setting $0 \oplus m = \{0, m\}$ and $x \oplus y = x + y$ for all $(x, y) \in Z_{mn}^2 - \{(0, m)\}$. Then $(Z_{mn}, \oplus)$ is an H_v-group.

Let $(H, \circ)$ be a hypergroupoid. If $\{A, B\} \in \wp^*(H)$ and ρ is an equivalence relation on H, then we denote $A \overline{\rho} B$ if

1) $\forall a \in A, \exists b \in B$, such that $a \rho b$ and

2) $\forall b \in B, \exists a \in A$, such that $a \rho b$.

We denote $A \overline{\overline{\rho}} B$ if $\forall a \in A, \forall b \in B$, we have $a \rho b$.

3. Fuzzy H_v-semigroups

The fuzzy hypergroup notion was introduced and studied in [16]. Let S be a nonempty set and $F(S)$ denote the set of all fuzzy subsets of S. A *fuzzy hyperoperation* on S is a map $\circ : S \times S \longrightarrow F(S)$ and the image of (a, b) is denoted by $a \circ b$.

The set S together with a *fuzzy hyperoperation* " $\circ$ " is called a *fuzzy hypergroupoid*. We say that $(S, \circ)$ is commutative if for all $a, b \in S$, we have $a \circ b = b \circ a$. A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy semihypergroup* if for all $a, b, c \in S$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, where for any $\mu \in F(S)$, we have

$$(a \circ \mu)(r) = \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)) \text{ and } (\mu \circ a)(r) = \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)) \text{ for all } r \in S.$$

If A is a nonempty subset of S and $x \in S$, then for all $t \in S$ we have

$$(A \circ x)(t) = \bigvee_{a \in A} (a \circ x)(t) \text{ and } (x \circ A)(t) = \bigvee_{a \in A} (x \circ a)(t).$$

A fuzzy semihypergroup $(S, \circ)$ is called a *fuzzy hypergroup* if for all $x \in S$, we have $x \circ S = S \circ x = \chi_S$, where χ_S is the characteristic function of the set S.

We start by defining the notion of fuzzy H_v-semigroup.

Definition 3.1. *A fuzzy hypergroupoid $(S, \circ)$ is called a fuzzy H_v-semigroup if*

$$\exists x \in S,\ ((a \circ b) \circ c)(x) > 0 \text{ and } (a \circ (b \circ c))(x) > 0 \text{ for all } a, b, c \in S.$$

Now, suppose that $(H, \circ)$ is an H_v-semigroup and $a, b, c, \in H$. Then, $((a \circ b) \circ c) \cap (a \circ (b \circ c)) \neq \emptyset$ if and only if there exists $x \in H$ such that $x \in ((a \circ b) \circ c) \cap (a \circ (b \circ c))$ if and only if there exists $x \in H$ such that $(\chi_{(a \circ b) \circ c} \wedge \chi_{a \circ (b \circ c)})(x) = 1$ if and only if there exists $x \in H$ such that $(\chi_{(a \circ b) \circ c} \wedge \chi_{a \circ (b \circ c)})(x) \neq 0$ if and only if there exists $x \in H$ such that $\chi_{(a \circ b) \circ c}(x) > 0$ and $\chi_{a \circ (b \circ c)}(x) > 0$. As we show that, the notion of fuzzy H_v-semigroups generalizes the concept of H_v-semigroup.

The following two theorems provide us some examples of fuzzy H_v-semigroups:

Theorem 3.2. *Let S be a semigroup and $\mu(a) \neq 0$ for all $a \in S$ be a fuzzy semigroup on S. Let $a, b \in S$ and define a fuzzy hyperoperation $\circ$ on S by*

$$(a \circ b)(t) = \begin{cases} \mu(a) \wedge \mu(b), & \text{if } t = ab \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. According to Examples 2.7 [16], we have

$$((a \circ b) \circ c)(abc) = (a \circ (b \circ c))(abc) = \mu(a) \wedge \mu(b) \wedge \mu(c) > 0 \text{ for all } a, b, c \in S.$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

Theorem 3.3. *Let S be a semigroup and $\mu(a) \neq 0$ for all $a \in S$ be a fuzzy semigroup on S. Let $a, b \in S$ and define a fuzzy hyperoperation $\circ$ on S by*

$$(a \circ b)(t) = \begin{cases} \mu(a) \vee \mu(b), & \text{if } t = ab \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. Since $\mu(a) \vee \mu(b) \geq \mu(a) \wedge \mu(b)$, for all $a, b \in S$. By the above theorem we have $((a \circ b) \circ c)(abc) = (a \circ (b \circ c))(abc) \geq \mu(a) \wedge \mu(b) \wedge \mu(c) > 0$, for all $a, b, c \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

Example 3.4. Let S be a nonempty set and $a, b \in S$. We define a fuzzy hyperoperation $\circ$ on S by

$$(a \circ b)(t) = \begin{cases} \frac{1}{2}, & \text{if } t = \{a, b\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. Since $((a \circ b) \circ c)(a) = (a \circ (b \circ c))(a) = \frac{1}{2} > 0$ for all $a, b, c \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

We now give an example of a fuzzy H_v-semigroups that is not fuzzy semigroups.

Example 3.5. Let $S = \{a, b\}$ and $(a \circ a)(a) = 0.1$, $(a \circ a)(b) = 0.2$, $(b \circ b)(a) = 0.3$, $(b \circ b)(b) = 0.4$, $(a \circ b)(a) = 0.5$, $(a \circ b)(b) = 0.6$, $(b \circ a)(a) = 0.7$ and $(b \circ a)(b) = 0.8$. Clearly, $((a \circ b) \circ c)(d) > 0$ and $(a \circ (b \circ c))(d) > 0$ for all $a, b, c, d \in S$. It follows that $(S, \circ)$ is a fuzzy H_v-semigroup, but $0.6 = ((a \circ b) \circ a)(a) \neq (a \circ (b \circ a))(a) = 0.5$. Thus $(S, \circ)$ is not a fuzzy semigroup.

Definition 3.6. *A fuzzy H_v-semigroup $(S, \circ)$ is called a fuzzy H_v-group if*

$$a \circ S = S \circ a = \chi_S \text{ for all } a \in S.$$

Theorem 3.7. *Let $(S, \circ)$ be a fuzzy hypergroup and $(a \circ b)(x) > 0$ for all $a, b, x \in S$. Then $(S, \circ)$ is a fuzzy H_v-group.*

Proof. Straightforward. □

The following three theorems provide us some examples of fuzzy H_v-groups:

Theorem 3.8. *Let S be a nonempty set and define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{\{a,b\}}$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup as well as a fuzzy H_v-group.*

Proof. Let $a, b, c \in S$. If $t \in \{a, b, c\}$, by Examples 2.5 [16], we have $((a \circ b) \circ c)(t) = (a \circ (b \circ c))(t) = 1 > 0$. Then $(S, \circ)$ is a fuzzy H_v-semigroup and also a fuzzy H_v-group. □

According to Example 2.6 [16], we have the following result.

Theorem 3.9. *Let $(S, .)$ be a semigroup and define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{\{a.b\}}$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. If $(S, .)$ be a group, then $(S, \circ)$ is a fuzzy H_v-group.*

Theorem 3.10. *Let μ be a fuzzy subgroup of an abelian group S such that $\mu(e) \neq 0$ and let define a fuzzy hyperoperation $\circ$ on S by $(a \circ b)(t) = \mu(abt^{-1})$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-group.*

Proof. Let $a, b, c \in S$. By Theorem 5.16 [16], we have $((a \circ b) \circ c)(c^{-1}b^{-1}a^{-1}) = (a \circ (b \circ c))(c^{-1}b^{-1}a^{-1}) = \mu(e) > 0$. Then $(S, \circ)$ is a fuzzy H_v-group. □

Definition 3.11. *[16] Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$, then we define $(\mu \circ \nu)$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S}(\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.*

Theorem 3.12. *Let $(S, \circ)$ be a fuzzy H_v-semigroup. Then we have*

(*i*) $\chi_a \circ \chi_b = a \circ b$ for all $a, b \in S$,

(*ii*) $\chi_S \circ \chi_a = S \circ a$ and (*ii*)$'$ $\chi_a \circ \chi_S = a \circ S$ for all $a \in S$,

(*iii*) $\chi_S \circ \mu = S \circ \mu$ and (*iii*)$'$ $\mu \circ \chi_S = \mu \circ S$ for all $\mu \in F(S)$.

Proof. Straightforward. □

Theorem 3.13. *Let $(S, \circ)$ be a fuzzy H_v-semigroup. Then $a \circ b \neq 0$, for all $a, b \in S$.*

Proof. Let $a, b, c \in S$, and $a \circ b = 0$. Then $((a \circ b) \circ c)(r) = \bigvee_{t \in S}((a \circ b)(t) \wedge (t \circ c)(r)) = 0$, for all $r \in S$, which is absurd. Hence, $a \circ b \neq 0$, for all $a, b \in S$. □

Corollary 3.14. *Let $(S, \circ)$ be a fuzzy H_v-group. Then $a \circ b \neq 0$, for all $a, b \in S$.*

4. Connections between fuzzy H_v-semigroups and H_v-semigroups

Let S be a nonempty set, endowed with a fuzzy hyperoperation $\circ$ and for all $a, b \in S$, consider the p-cuts $(a \circ b)_p = \{t \in S |\ (a \circ b)(t) \geq p\}$ of $a \circ b$, where $p \in [0, 1]$. For all $p \in [0, 1]$, we define the following crisp hyperoperation on $S : a \circ_p b = (a \circ b)_p$.

Theorem 4.1 ([11])**.** *For all $a, b, c, t \in S$ and for all $p \in [0, 1]$ the following equivalence holds:*

$(a \circ (b \circ c))(t) \geq p \Longleftrightarrow t \in a \circ_p (b \circ_p c)$ *and* $((a \circ b) \circ c)(t) \geq p \Longleftrightarrow t \in (a \circ_p b) \circ_p c)$.

Theorem 4.2. *$(S, \circ)$ is a fuzzy H_v-semigroup if and only if $\forall p \in (0,1]$, $(S, \circ_p)$ is a H_v-semigroup.*

Proof. Let $(S, \circ)$ be a fuzzy H_v-semigroup and $p \in (0,1]$. Then for all $a, b, c \in S$ there exists $t \in S$ such that $((a \circ b) \circ c)(t) = m > 0$ and $(a \circ (b \circ c))(t) = n > 0$. Hence, if $m \wedge n = p$, then $((a \circ b) \circ c)(t) \geq p$ and $(a \circ (b \circ c))(t) \geq p$ if and only if for all $a, b, c \in S$ there exists $t \in S$ such that $t \in (a \circ_p (b \circ_p c)) \cap ((a \circ_p b) \circ_p c)$. Therefore $(S, \circ_p)$ is a H_v-semigroup.

Conversely, if $p \in (0,1]$ and $(S, \circ_p)$ be a H_v-semigroup, then for all $a, b, c \in S$ there exists $t \in S$ such that $t \in (a \circ_p (b \circ_p c)) \cap ((a \circ_p b) \circ_p c)$. Hence $((a \circ b) \circ c)(t) \geq p > 0$ and $(a \circ (b \circ c))(t) \geq p > 0$. Therefore $(S, \circ_p)$ is a fuzzy H_v-semigroup. □

Theorem 4.3 ([11])**.** *For all $a \in S$, the following equivalence holds:*

$$a \circ S = \chi_S \Longleftrightarrow p \in [0,1],\ a \circ_p S = S.$$

Corollary 4.4. *$(S, \circ)$ is a fuzzy H_v-group if and only if $\forall p \in (0,1]$, $(S, \circ_p)$ is a H_v-group.*

According to [16], we can associate a hyperoperation on a fuzzy hypergroup $(G, \circ)$, as follows:

$$\forall\ a, b \in G,\ a * b = \{x \in G |\ (a \circ b)(x) > 0\}.$$

Theorem 4.5 ([16])**.** *Let $(G, \circ)$ be a fuzzy hypergroup. Then $(G, *)$ is a hypergroup.*

Theorem 4.6. *$(S, *)$ is a fuzzy H_v-semigroup if and only if $(S, \circ)$ is a H_v-semigroup.*

Proof. Let $a, b, c \in S$. Then $(S, *)$ be a H_v-semigroup if and only if $\exists x \in S, x \in (a * (b * c)) \cap ((a * b) * c) \Longleftrightarrow \exists x, p \in S, p \in b * c,\ x \in a * p$ and $\exists x, q \in S, q \in a * b, x \in q * c \Longleftrightarrow \exists x, p \in S, (b \circ c)(p) > 0, (a \circ p)(x) > 0$ and $\exists x, q \in S, (a \circ b)(q) > 0, (q \circ c)(x) > 0 \Longleftrightarrow \exists x \in S, \bigvee_{t \in S}((b \circ c)(t) \wedge (a \circ t)(x)) > 0$ and $\bigvee_{t \in S}((a \circ b)(t) \wedge (t \circ c)(x)) > 0 \Longleftrightarrow \exists x \in S,\ ((a \circ b) \circ c)(x) > 0$ and $(a \circ (b \circ c))(x) > 0$.

If and only if $(S, \circ)$ is a fuzzy H_v-semigroup. □

According to Theorem 4.5 and Theorem 4.6, we have the following result.

Theorem 4.7. *Let $(G, \circ)$ be a fuzzy H_v-group. Then $(G, *)$ is a H_v-group.*

Theorem 4.8 ([13])**.** *Let $(G, *)$ be a H_v-group. Then $a * b \neq \emptyset$, for all $a, b \in G$.*

According to [16], we can define a fuzzy hyperoperation on a hypergroup $(G, *)$, as follows:

$$\forall\ a, b \in G,\ a \circ b = \chi_{a*b}.$$

Theorem 4.9 ([16]). *Let $(G,*)$ be a hypergroup. Then $(G,\circ)$ is a fuzzy hypergroup.*

Theorem 4.10. *Let $(G,*)$ be a H_v-semigroup. Then $(G,\circ)$ is a fuzzy H_v-semigroup.*

Proof. Let $(G,*)$ be a H_v-semigroup and $a,b,c \in G$. Then for all $a,b,c \in G$ there exists $t_0 \in G$ such that $t_0 \in (a*(b*c)) \cap ((a*b)*c)$. On the other hand, for all $t \in G$ we have

$$((a\circ b)\circ c)(t) = \begin{cases} 1, & \text{if } t \in ((a*b)*c) \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(a\circ(b\circ c))(t) = \begin{cases} 1, & \text{if } t \in (a*(b*c)) \\ 0, & \text{otherwise.} \end{cases}$$

Hence $((a\circ b)\circ c)(t_0) = (a\circ(b\circ c))(t_0) = 1 > 0$. Therefore $(S,\circ)$ is a fuzzy H_v-semigroup. □

By Theorem 4.9 and Theorem 4.10, we have the following result.

Theorem 4.11. *Let $(G,*)$ be a H_v-group. Then $(G,*)$ is a fuzzy H_v-group.*

By Theorems 4.7 and 4.11, we have the following facts.

Denote by $\mathcal{H}_v\mathcal{G}$ the class of all H_v-groups and by $\mathcal{FH}_v\mathcal{G}$ class of all fuzzy H_v-groups, then we can consider two maps:

(1) $\phi : \mathcal{H}_v\mathcal{G} \longrightarrow \mathcal{FH}_v\mathcal{G}$, $\phi(G,*) = (G,\circ)$, where for all $a,b \in G$ we have $a\circ b = \chi_{a*b}$;

(2) $\psi : \mathcal{FH}_v\mathcal{G} \longrightarrow \mathcal{H}_v\mathcal{G}$, $\psi(G,\circ) = (G,*)$, where for all $a,b \in G$ we have $a*b = \{x \in G |\ (a\circ b)(x) > 0\}$.

Let $\mu,\lambda \in F(G)$. Then we say that μ includes λ and we denote $\mu \subseteq \lambda$ if and only if for all $x \in G$, we have $\mu(x) \leq \lambda(x)$ (see [15]).

Let f be a mapping from a set X into a set Y. Let μ be a fuzzy set in X. Then the *image* $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x) | x \in X, f(x) = y\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$ (see [19]).

Now, we introduce the notion of the fuzzy H_v-group (inclusion, weak) homomorphism, as follows.

Definition 4.12. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-groups. A map $f : S \longrightarrow T$ is called an H_v-homomorphism or a weak homomorphism if*

$$f(x \circ_1 y) \cap f(x) \circ_2 f(y) \neq \emptyset; \text{ for all } x, y \in S.$$

f is called an inclusion homomorphism if

$$f(x \circ_1 y) \subseteq f(x) \circ_2 f(y); \text{ for all } x, y \in S.$$

Finally, f is called a strong homomorphism if

$$f(x \circ_1 y) = f(x) \circ_2 f(y); \text{ for all } x, y \in S.$$

Remark 4.13. If $f : S \longrightarrow T$ is a map and for all $a \in S$, then $f(\chi_a) = \chi_{f(a)}$.

The next two theorems show connections between the inclusion (weak) homomorphism of fuzzy H_v-groups and the inclusion (weak) homomorphism of corresponding H_v-groups.

Theorem 4.14. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-groups and $(S, *_1) = \psi(S, \circ_1)$, $(T, *_2) = \psi(T, \circ_2)$ be their associated H_v-groups, respectively. If $f : S \longrightarrow T$ is an inclusion (weak) homomorphism of fuzzy H_v-groups, then f is an inclusion (weak) homomorphism of the associated H_v-groups.*

Proof. Suppose that $f : S \longrightarrow T$ is an inclusion homomorphism of fuzzy H_v-groups. Then we have $f(a \circ_1 b) \subseteq f(a) \circ_2 f(b)$ for all $a, b \in S$. This means that $(f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y)$ for all $y \in T$. If $x \in a *_1 b$, then $(a \circ_1 b)(x) > 0$ and let $y = f(x)$. It follows that

$$(f(a \circ_1 b))(y) = \bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) \geq (a \circ_1 b)(x) > 0.$$

Thus $(f(a) \circ_2 f(b))(y) > 0$. Hence $y = f(x) \in f(a) *_2 f(b)$. So we obtain $f(a *_1 b) \subseteq f(a) *_2 f(b)$. Therefore, f is an inclusion homomorphism of the associated H_v-groups.

Now, let $f : S \longrightarrow T$ be a weak homomorphism of fuzzy H_v-groups. Then we have

$$\exists y \in T,\ y \in f(a \circ_1 b)(y) > 0 \text{ and } (f(a) \circ_2 f(b))(y) > 0,\ \forall a, b \in S.$$

If $y \in f(a \circ_1 b)(y) > 0$, then $\bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) > 0$, implies that there exists $x \in S$, $f(x) = y$ such that $(a \circ_1 b)(x) > 0$. Thus $x \in a *_1 b$, this means that $y = f(x) \in f(a *_1 b)$.

If $(f(a) \circ_2 f(b))(y) > 0$, then $y \in f(a) *_2 f(b)$. Hence, $y \in f(a *_1 b) \cap f(x) *_2 f(y) \neq \emptyset$; for all $a, b \in S$. Therefore, f is a weak homomorphism of the associated H_v-groups. □

Theorem 4.15. *Let $(S, *_1)$ and $(T, *_2)$ be two H_v-groups and $(S, \circ_1) = \phi(S, *_1)$, $(T, \circ_2) = \phi(T, *_2)$ be their associated fuzzy H_v-groups, respectively. Then, $f : S \longrightarrow T$ is an inclusion (weak) homomorphism of H_v-groups if and only if f is an inclusion (weak) homomorphism of the associated fuzzy H_v-groups.*

Proof. " $\Longleftarrow$ " Since f is an inclusion homomorphism of fuzzy H_v-groups, we have $f(a \circ_1 b) \subseteq f(a) \circ_2 f(b)$ for all $a, b \in S$. This means that $(f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y)$ for all $y \in T$. Then, we have $\chi_{f(a*_1b)}(y) = \bigvee_{r \in f^{-1}(y)} \chi_{a*_1b}(r) = \bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) = (f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y) = \chi_{f(a)*_2f(b)}(y)$.

This means that $f(a *_1 b) \subseteq f(a) *_2 f(b)$.

" $\Longrightarrow$ " Let f be an inclusion homomorphism of H_v-groups and $a, b \in S$. If $y \notin \mathrm{Im}f$, then $(f(a \circ_1 b))(y) = 0 \leq (f(a) \circ_2 f(b))(y)$. If $y \in \mathrm{Im}f$, then we have

$$(f(a \circ_1 b))(y) = \chi_{f(a*_1b)}(y) \leq \chi_{f(a)*_2f(b)} = (f(a) \circ_2 f(b))(y).$$

Which means that $f(a *_1 b) \subseteq f(a) *_2 f(b)$. Therefore, f is an inclusion homomorphism of the associated fuzzy H_v-groups.

Now, $f : S \longrightarrow T$ is a weak homomorphism of H_v-groups if and only if there exists $y \in T$ such that $(f(a \circ_1 b))(y) > 0$ and $(f(a) \circ_2 f(b))(y) > 0$ for all $a, b \in S$. Since $\chi_{f(a*_1b)}(y) = (f(a \circ_1 b))(y) > 0$ and $\chi_{f(a)*_2f(b)}(y) = (f(a) \circ_2 f(b))(y) > 0$, if and only if $\chi_{f(a*_1b)}(y) = 1$ and $\chi_{f(a)*_2f(b)}(y) = 1$, if and only if there exists $y \in T$ such that $y \in f(a *_1 b) \cap f(a) *_2 f(b)$ for all $a, b \in S$. If and only if f is a weak homomorphism of the associated fuzzy H_v-groups. □

Let us consider now the algebraic H_v-substructures of fuzzy H_v-groups.

Definition 4.16. *Let $(G, \circ)$ be a fuzzy H_v-group and $(G, *) = \psi(G, \circ)$ be the associated H_v-group. A nonempty subset H of G is called a fuzzy H_v-subgroup if for all $x, y \in G$, the following conditions hold:*

(i) $(x \circ y)(z) > 0$, *then* $z \in H$;

(ii) $x \circ H = H \circ x = \chi_H$.

Theorem 4.17. *(i) Let $(H, \circ)$ be a fuzzy H_v-subgroup of $(G, \circ)$. Then $(H, *) = \psi(H, \circ)$ is an H_v-subgroup of $(G, *) = \psi(G, \circ)$;*

(ii) Let $(H, *)$ be an H_v-subgroup of $(G, *)$. Then $(H, \circ) = \phi(H, *)$ is a fuzzy H_v-subgroup of $(G, \circ) = \psi(G, *)$.

5. Fuzzy (strong) hypercongruence on fuzzy H_v-semigroups

Fuzzy regular relations are introduced in the context of fuzzy semihypergroups in [16]. Now, we introduce the hypercongruence on H_v-semigroups and the fuzzy (strong) hypercongruence on fuzzy H_v-semigroups and discuss the relations of them. We recall some definitions:

Let $(G, \circ)$ be a fuzzy H_v-semigroup and ρ be an equivalence relation on G. If $\mu, \lambda \in F(G)$, then we say that $\mu \overline{\rho} \lambda$ if the following two conditions hold:

(1) $\forall a \in G$, if $\mu(a) > 0$, then there exists $b \in G$, such that $\lambda(b) > 0$ and $a\rho b$;

(2) $\forall x \in G$, if $\lambda(x) > 0$, then there exists $y \in G$, such that $\mu(y) > 0$ and $x\rho y$.

Now, we can introduce the notion of the hypercongruence on fuzzy H_v-semigroups, in the following manner.

Definition 5.1. *An equivalence relation ρ on a fuzzy H_v-semigroup $(G, \circ)$ is called a fuzzy regular relation (or a fuzzy hypercongruence) on G if, for all $a, b, c \in G$, the following implication holds:*

$$a\rho b \implies (a \circ c)\overline{\rho}(b \circ c) \text{ and } (c \circ a)\overline{\rho}(c \circ b).$$

Clearly, the condition is equivalent to $a\rho a'$and $b\rho b'$ implies $(a \circ b)\overline{\rho}(a' \circ b')$ for all $a, b, a', b' \in S$.

The next theorem shows a fuzzy homomorphism of fuzzy H_v-semigroups can induce a fuzzy hypercongruence on a fuzzy H_v-semigroup.

Theorem 5.2. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-semigroups. Let $f : S \longrightarrow T$ be a fuzzy homomorphism. Then $\rho = \ker f = \{(a, b) \in S^2 |\ f(a) = f(b)\}$ is a fuzzy hypercongruence on S.*

Proof. Clearly, ρ is an equivalence relation on S. Let $a, b, a', b' \in S$, $a\rho a'$ and $b\rho b'$. Then we have $f(a) = f(a')$ and $f(b) = f(b')$. Since f is a homomorphism of fuzzy H_v-semigroups, we have

$$f(a \circ_1 b) = f(a) \circ_2 f(b) = f(a') \circ_2 f(b') = f(a' \circ_1 b').$$

Now, let $x \in S$ and $(a \circ_1 b)(x) > 0$. Which means that

$$f(a \circ_1 b)(f(x)) = \bigvee_{x' \in S,\ f(x') = f(x)} (a' \circ_1 b')(x') \geq (a \circ_1 b)(x) > 0.$$

This means that there exists $x' \in S$ such that $(a' \circ_1 b')(x') > 0$ and $f(x') = f(x)$. Hence $(a' \circ_1 b')(x') > 0$ and $x\rho x'$.

Similarly, for any $s \in S$ and $(a' \circ_1 b')(s) > 0$, there exists $t \in S$ such that $(a \circ_1 b)(t) > 0$ and $s\rho t$. Hence $(a \circ_1 b)\overline{\rho}(a' \circ_1 b')$. Therefore, $\rho = \ker f$ is a fuzzy hypercongruence on S. □

In what follows, we shall see a relation between the fuzzy hypercongruence on fuzzy H_v-semigroups and the hypercongruence on corresponding H_v-semigroups.

Let $(S, \circ)$ be a fuzzy H_v-semigroup and $(S, *) = \psi(S, \circ)$ be the associated H_v-semigroup, where for any $a, b \in S$, we have $a * b = \{x \in |\ (a \circ b)(x) > 0\}$. Then we can obtain:

Theorem 5.3. *An equivalence relation ρ is a fuzzy hypercongruence on a fuzzy H_v-semigroup $(S, \circ)$ if and only if ρ is a hypercongruence on corresponding H_v-semigroup $(S, *)$.*

Proof. Set $a\rho b$ and $a'\rho b'$, where $a, b, a', b' \in S$. We have that $(a \circ a')\overline{\rho}(b \circ b')$ if and only if the following conditions hold:

(1) if $(a \circ a')(x) > 0$, then there exists $y \in S$, such that $(b \circ b')(y) > 0$ and $x\rho y$;

(2) if $(b \circ b')(s) > 0$, then there exists $t \in S$, such that $(a \circ a')(t) > 0$ and $s\rho t$.

These conditions are equivalent to the following ones:

(1) if $x \in a * a'$, then there exists $y \in b * b'$ and $x\rho y$;

(2) if $s \in b * b'$, then there exists $t \in a * a'$ and $s\rho t$.

Which show that $(a * a')\overline{\rho}(b * b')$. Therefore, ρ is a fuzzy hypercongruence on $(S, \circ)$ if and only if ρ is a hypercongruence on $(S, *)$. □

A hypercongruence plays an important role in studying the quotient structure of H_v-semigroups, as the following theorem shows.

Theorem 5.4. *Let $(S, *)$ be an H_v-semigroup and ρ be an equivalence relation on S. If we define a hyperoperation $\oplus$ on the quotient set S/ρ: for all $a\rho, b\rho, c\rho \in S/\rho$, $a\rho \oplus b\rho = \{c\rho : \ c \in a * b\}$, then ρ is a hypercongruence on S if and only if $(S/\rho, \oplus)$ is an H_v-semigroup.*

Proof. " $\Longrightarrow$ " Suppose that ρ is a hypercongruence on S. Let $a, b \in S$, since $a * b$ is nonempty set, implies $a\rho \oplus b\rho$ is nonempty set. Clear, $\oplus$ is well-defined. Since $(S, *)$ is an H_v-semigroup, implies there exists $x \in S$ such that $x \in (a * (b * c)) \cap ((a * b) * c)$ for all $a, b, c \in S$. Now, if $x \in a * (b * c)$, then $x \in a * d$, for some $d \in b * c$, implies that $x\rho \in a\rho \oplus d\rho$, for some $d\rho \in b\rho \oplus c\rho$. Therefore $x\rho \in a\rho \oplus (b\rho \oplus c\rho)$.

Similarly, if $x \in (a * b) * c$, we can prove that $x\rho \in (a\rho \oplus b\rho) \oplus c\rho$. Therefore, $(S/\rho, \oplus)$ is an H_v-semigroup.

" $\Longleftarrow$ " Suppose that $a, b \in S$, since $a\rho \oplus b\rho$ is nonempty set, then $a * b$ is nonempty set. In the other hand, since $(S/\rho, \oplus)$ is an H_v-semigroup, implies there exists $x \in S$ such that $x\rho \in (a\rho \oplus (b\rho \oplus c\rho)) \cap ((a\rho \oplus b\rho) * c\rho)$ for all $a, b, c \in S$. If $x\rho \in a\rho \oplus (b\rho \oplus c\rho)$, then $x\rho \in a\rho \oplus d\rho$, for some $d\rho \in b\rho \oplus c\rho$, implies that $x \in a * d$, for some $d \in b * c$. Therefore $x \in a * (b * c)$.

Similarly, if $x\rho \in (a\rho \oplus b\rho) * c\rho$, we can prove that $x \in (a * b) * c$. Therefore, ρ is a hypercongruence on S. □

The H_v-semigroup $(S/\rho, \oplus)$ is called the quotient H_v-semigroup induced by the fuzzy H_v-semigroup $(S, *)$ and the fuzzy hypercongruence ρ on $(S, *)$.

Let $(G, \circ)$ be a fuzzy H_v-semigroup and ρ be an equivalence relation on G. If $\mu, \lambda \in F(G)$, then we say that $\mu\overline{\overline{\rho}}\lambda$ if the following two conditions hold:

(1) $\forall a \in G$, if $\mu(a) > 0$, then there exists $b \in G$, such that $\lambda(b) > 0$, $\mu(a) = \lambda(b)$ and $a\rho b$;

(2) $\forall x \in G$, if $\lambda(x) > 0$, then there exists $y \in G$, such that $\mu(y) > 0$, $\mu(x) = \lambda(y)$ and $x\rho y$.

Definition 5.5. *An equivalence relation ρ on a fuzzy H_v-semigroup $(G, \circ)$ is called a fuzzy strong hypercongruence on G if, for all $a, b, c, d \in G$, the following implication holds:*

$$a\rho b,\ c\rho d \implies (a \circ c)\overline{\overline{\rho}}(b \circ d).$$

Theorem 5.6. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy strong hypercongruence on S. If we define a fuzzy hyperoperation $\odot$ on S/ρ by $(x\rho \odot y\rho)(z\rho) = \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho}(x' \circ y')(z')$, for all $x\rho, y\rho, z\rho \in S/\rho$, then S/ρ is a fuzzy H_v-semigroup.*

Proof. First, we prove that fuzzy hyperoperation $\odot$ is well-defined. Suppose that $x, y, x', y' \in S$, such that $x\rho = x'\rho, y\rho = y'\rho$. Since ρ is a fuzzy strong hypercongruence on S. Hence $(x \circ y)\overline{\overline{\rho}}(x' \circ y')$. Then, for all $z\rho \in S/\rho$, we have $(x\rho \odot y\rho)(z\rho) = \bigvee_{x_1 \in x\rho, y_1 \in y\rho, z_1 \in z\rho}(x_1 \circ y_1)(z_1) = \bigvee_{x_2 \in x'\rho, y_2 \in y'\rho, z_2 \in z'\rho}(x_2 \circ y_2)(z_2) = (x'\rho \odot y'\rho)(z'\rho)$, it follows that $x\rho \odot y\rho = x'\rho \odot y'\rho$. Therefore, $\odot$ is well-defined.

Now, let $x, y, z \in S$, since $(S, \circ)$ is a fuzzy H_v-semigroup, then there exists $t \in S$, such that $((x \circ y) \circ y)(t) > 0$ and $(x \circ (y \circ z))(t) > 0$.

In the other hand, we have

$$\begin{aligned}((x\rho \odot y\rho) \odot z\rho)(t\rho) &= \bigvee_{s\rho \in S/\rho}((x\rho \odot y\rho)(s\rho) \wedge (s\rho \odot z\rho)(t\rho)) \\ &= \bigvee_{s \in S}\Big(\bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho}((x' \circ y')(s') \wedge (s' \circ z')(t'))\Big) \\ &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho}((x' \circ y') \circ z')(t')).\end{aligned}$$

It follows that

$$\begin{aligned}((x\rho \odot y\rho) \odot z\rho)(t\rho) &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho}((x' \circ y') \circ z')(t')) \\ &\geq ((x \circ y) \circ z)(t) > 0.\end{aligned}$$

In a similar way, we can show that

$$\begin{aligned}(x\rho \odot (y\rho \odot z\rho))(t\rho) &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho}(x' \circ (y' \circ z'))(t')) \\ &\geq (x \circ (y \circ z))(t) > 0.\end{aligned}$$

Therefore, S/ρ is a fuzzy H_v-semigroup. □

Theorem 5.7. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy strong hypercongruence on S. If we define a mapping $f : S \longrightarrow S/\rho$ by $f(a) = a\rho$, for all $a \in S$, then f is a surjective strong homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$.*

Proof. Clearly, the map $f : S \longrightarrow S/\rho$ is surjective. Suppose that $a, b \in S$, since ρ is a fuzzy strong hypercongruence of fuzzy H_v-semigroups $(S, \circ)$, then for all $x\rho \in S/\rho$, we have $(f(a \circ b))(x\rho) = \bigvee_{f(t)=x\rho}(a \circ b)(t) = \bigvee_{t\rho=x\rho}(a \circ b)(t) = \bigvee_{t\in x\rho}(a \circ b)(t) = \bigvee_{t'\in x\rho, a'\in a\rho, b'\in b\rho}(a' \circ b')(t') = (a\rho * b\rho)(x\rho) = (f(a) * f(b))(x\rho)$, thus $f(a \circ b) = f(a) * f(b)$. Therefore, f is a surjective strong homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$. □

Corollary 5.8. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy hypercongruence on S. If we define a mapping $f : S \longrightarrow S/\rho$ by $f(a) = a\rho$, for all $a \in S$, then f is a surjective inclusion homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$.*

References

[1] M. Asghari-Larimi, B. Davvaz, *Hyperstructures associated to arithmetic functions*, Ars Combitoria, 97 (2010), 51-63.

[2] M. Asghari-Larimi, V. Leoreanu-Fotea, *A connection between hypergroupoids and L-fuzzy sets of type 2*, Italian J. of Pure and Appl. Math., 26 (2009), 207-216.

[3] G. Chowdhury, *Fuzzy transposition hypersemigroups,* Iranian Journal of Fuzzy Systems, 6 (2009), 37-52.

[4] P. Corsini, *Prolegomena of hypergroup theory*, Second edition, Aviani editor, 1993.

[5] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[6] B. Davvaz, *Fuzzy Hv-groups*, Fuzzy Sets Syst., 101 (1999), 191-195.

[7] B. Davvaz, *Fuzzy Hv-submodules*, Fuzzy Sets Syst., 117 (2001), 477-484.

[8] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, USA, 2007.

[9] P. He, X. Xin, *Fuzzy hyperlattices,* Computers and Mathematics with Applications, 62 (2011), 4682-4690.

[10] V. Leoreanu-Fotea, B. Davvaz, *Fuzzy hyperrings*, Fuzzy Sets and Systems, 160 (2009), 2366-2378.

[11] V. Leoreanu-Fotea, *Fuzzy hypermodules*, Computers and Mathematics with Applications, 57 (2009), 466-475.

[12] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, (1934), 45-49.

[13] Ch. G. Massouros and A. Dramalidis, *Transposition H_v-groups*, Ars Combinatoria, 106, (2012), 143-160.

[14] J.N. Mordeson, M.S. Malik, *Fuzzy commutative algebra*, Word Publ., 1998.

[15] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[16] M.K. Sen, R. Ameri, G. Chowdhury, *Fuzzy hypersemigroups*, Soft Comput., 2007.

[17] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.

[18] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield,* in: Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications (AHA 1990), World Scientific, 1991, 203-211.

[19] L.A. Zadeh, *Fuzzy sets*, Inform and Control, 8 (1965), 338-353.

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Derivations of rings and Banach algebras

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Abstract. We investigate the action of derivation d in a prime ring and semiprime ring R with center $Z(R)$ satisfying (i) $(d(x\circ y))^m-(x\circ y)^n \in Z(R)$ (ii) $(d(x)\circ d(y))^m=(x\circ y)^n$ for all $x, y \in I$, a nonzero ideal of R. Finally, as an application we apply our results to the continuous derivations on Banach algebras.

Keywords: prime and semiprime rings, derivation, martindale ring of quotient, Banach algebra, radical.

1. Introduction

Throughout this paper, unless stated otherwise, R will be a semi(-prime) ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C is called the extended centroid of R (we refer the reader to [4], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy+yx$, respectively. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds, for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$ if $d(x) = [q, x]$ holds, for all $x \in R$.

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During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [2], [3], [19] and [20] where further references can be found). In [3], Ashraf and Rehman proved that if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [1], Argaç and Inceboz generalized the above result and they proved that: If a prime ring R admits a nonzero derivation d with the property $(d(x \circ y))^n = x \circ y$, for all $x, y \in I$, a nonzero ideal of R, where n is a fixed positive integer, then R is commutative.

If R is a ring and $S \subseteq R$, a mapping $f : S \to R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$, for all $x, y \in S$. In 1994 Bell and Daif [5] initiated the study of strong commutativity-preserving maps and proved that a nonzero right ideal I of a semiprime ring R is central if R admits a derivation which is scp on I. In 2002 Ashraf and Rehman [3] proved that if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$, for all $x, y \in I$, then R is commutative. Inspired by above mention results, we here generalized the result obtained in [1] and [3]. Moreover, we continue this line of investigation by examining what happens when a ring R satisfies the identity

1. $(d(x \circ y))^m - (x \circ y)^n \in Z(R)$, for all $x, y \in I$;
2. $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$.

In the last section of this paper we will consider $\mathfrak{A}$ as a Banach algebra with Jacobson radical $rad(\mathfrak{A})$. Let us introduce some well known and elementary definitions for the sake of completeness. By Banach algebra we shall mean a complex normed algebra $\mathfrak{A}$ whose underlying vector space is a Banach space. The Jacobson radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $\mathfrak{A}$ is called semisimple. In fact any Banach algebra $\mathfrak{A}$ without a unity can be embedded into a unital Banach algebra $\mathfrak{A}_I = \mathfrak{A} \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we may identify $\mathfrak{A}$ with the ideal $\{(x, 0) : x \in \mathfrak{A}\}$ in $\mathfrak{A}_I$ via the isometric isomorphism $x \to (x, 0)$.

The classical result of Singer and Wermer in [22], says that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988 Thomas [23], verified this conjecture. It is clear that the same result of Singer and Wermer does not hold in non-commutative Banach algebras (because of inner derivations). Hence in this context a very interesting question is how to obtain the non-commutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair in [21]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable

conditions in Banach algebras. Here we will continue the investigation about the relationship between the structure of an algebra $\mathfrak{A}$ and the behaviour of derivations defined on $\mathfrak{A}$. Then we apply our results on prime rings to the study of analogous conditions for continuous derivations on Banach algebras.

2. The results in prime rings

We will make frequent use of the following result due to Kharchenko [13] (see also [15]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \cdots x_n, d(x_1), \cdots d(x_n))$ be a differential identity in I, that is

$$f(r_1, \cdots r_n, d(r_1), \cdots, d(r_n)) = 0 \text{ for all } r_1, \cdots, r_n \in I.$$

One of the following holds:

1. Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity
$$f(r_1, \cdots, r_n, [q, r_1], \cdots, [q, r_n]) = 0;$$
2. or, I satisfies the generalized polynomial identity
$$f(x_1, \cdots, x_n, y_1, \cdots, y_n) = 0.$$

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a derivation d such that $(d(x \circ y))^m = (x \circ y)^n$, for all $x, y \in I$, then R is commutative.*

Proof. Suppose that $d = 0$, then $(xy + yx)^n = 0$, for all $x, y \in I$. Now if $char(R) \neq 2$, then $(2x^2)^n = 0$, for all $x \in I$. This is a contradiction by Xu [24]. If $char(R) = 2$, then $(xy+yx)^n = 0 = [x, y]^n$, for all $x, y \in I$. Thus, by Herstein [10, Theorem 2], $I \subseteq Z(R)$ and so R is commutative by Mayne [18].

Now we assume that $d \neq 0$. Since R is a prime ring and if R admits a nonzero derivation d, by the given hypothesis, we have

$$(d(xy + yx))^m = (xy + yx)^n, \text{ for all } x, y \in I.$$

Thus I satisfies the differential identity

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n, \text{ for all } x, y \in I.$$

In the light of [13] and [15], we split the proof into two cases:

Case 1. If d is not inner, then by Kharchenko's theorem [13], I satisfies the polynomial identity

$$(sy + ys + tx + xt)^m = (xy + yx)^n, \text{ for all } x, y, s, t \in I.$$

In particular for $y = 0$, I satisfies the blended component $(tx + xt)^m = 0$, for all $x, t \in I$. Then R is commutative, by using the argument presented above.

Case 2. Let d be an inner derivation induced by an element $q \in Q$, i.e., $d(x) = [q, x]$ for all $x \in R$. Then for any $x, y \in I$,

$$([q,x]y + y[q,x] + [q,y]x + x[q,y])^m = (xy + yx)^n$$

By Chuang [7, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$([q,x]y + y[q,x] + [q,y]x + x[q,y])^m = (xy + yx)^n, \text{ for all } x, y \in Q.$$

In case the center C of Q is infinite, we have

$$([q,x]y + y[q,x] + [q,y]x + x[q,y])^m = (xy + yx)^n,$$

for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (*i.e.*, $RC = R$) which is either finite or algebraically closed and

$$([q,x]y + y[q,x] + [q,y]x + x[q,y])^m = (xy + yx)^n, \text{ for all } x, y \in R.$$

By Martindale [16, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring. Hence by Jacobson's theorem [11, p.75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$. Then the density of R on $\mathcal{V}$ implies that $R \cong \mathcal{M}_k(C)$, where $k = dim_{\mathcal{D}}\mathcal{V}$. Assume first that $dim_{\mathcal{D}}\mathcal{V} \geq 3$.

Step 1. We want to show that, for any $v \in \mathcal{V}$, v and qv are linearly $\mathcal{D}$-dependent. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Suppose v and qv are linearly $\mathcal{D}$-independent and as $dim_{\mathcal{D}}\mathcal{V} \geq 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ is also linearly $\mathcal{D}$-independent. By the density of R, there exist $x, y \in R$ such that

$$xv = 0,\ xqv = w,\ xw = 0,\ \ yv = 0,\ yqv = 0,\ yw = v.$$

These imply that $(-1)^m v = ([q,x]y + y[q,x] + [q,y]x + x[q,y])^m v = (xy + yx)^n v = 0$, a contradiction.

So we conclude that $\{v, qv\}$ is linearly $\mathcal{D}$-dependent, for all $v \in V$.

Step 2. We show here that there exists $\alpha \in \mathcal{D}$ such that $qv = v\alpha$, for any $v \in \mathcal{V}$. Note that the arguments in [6], are still valid in the present situation. For the sake of completeness and clearness we prefer to present it. In fact, choose

$v, w \in \mathcal{V}$ linearly independent. Since $dim_{\mathcal{D}}\mathcal{V} \geq 3$, there exists $u \in \mathcal{V}$ such that v, w, u are linearly independent. By Step 1, there exist $\alpha_v, \alpha_w, \alpha_u \in \mathcal{D}$ such that

$$qv = v\alpha_v, qw = w\alpha_w, qu = u\alpha_u \text{ that is } q(v+w+u) = v\alpha_v + w\alpha_w + u\alpha_u.$$

Moreover $q(v+w+u) = (v+w+u)\alpha_{v+w+u}$, for a suitable $\alpha_{v+w+u} \in \mathcal{D}$. Then

$$0 = v(\alpha_{v+w+u} - \alpha_v) + w(\alpha_{v+w+u} - \alpha_w) + u(\alpha_{v+w+u} - \alpha_u),$$

and, because v, w, u are linearly independent, $\alpha_u = \alpha_w = \alpha_v = \alpha_{v+w+u}$, that is, α does not depend on the choice of v. So there exists $\alpha \in \mathcal{D}$ such that $qv = \alpha v$ for all $v \in \mathcal{V}$. This completes the proof of Step 2. Now for any $r \in R, v \in \mathcal{V}$. By Step 2, $qv = v\alpha$, $r(qv) = r(v\alpha)$, and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$, for any $v \in \mathcal{V}$, that is $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q, r] = 0$, for all $r \in R$, i.e., $q \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $dim_{\mathcal{D}}\mathcal{V}$ must be ≤ 2. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [14, Lemma 2], it follows that there exists a suitable field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of all $k \times k$ matrices over $\mathcal{F}$, and moreover $\mathcal{M}_k(\mathcal{F})$ satisfies the same generalized polynomial identity of R. If we assume $k \geq 3$, then by the same argument as in Steps 1 and 2, we get a contradiction. Obviously if $k = 1$, then R is commutative. Thus we may assume that $k = 2$, i.e., $R \subseteq \mathcal{M}_2(\mathcal{F})$, where $\mathcal{M}_2(\mathcal{F})$ satisfies

$$([q, x]y + y[q, x] + [q, y]x + x[q, y])^m = (xy + yx)^n \text{ for all } x, y \in \mathcal{M}_2(\mathcal{F}).$$

Denote e_{ij} the usual unit matrix with 1 in (i, j)-entry and zero elsewhere. By choosing $x = e_{11}, y = e_{12}$ we get,

$$0 = ([q, x]y + y[q, x] + [q, y]x + x[q, y])^m - (xy + yx)^n = (qe_{12} - e_{12}q)^m - (e_{12})^n.$$

In any case, we can arrive $0 = e_{12}(qe_{12})^m$. Now set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation, we have $\begin{pmatrix} 0 & q_{21}^m \\ 0 & 0 \end{pmatrix} = 0$ which implies that $q_{21} = 0$. In the same manner, we can see that $q_{12} = 0$. Thus we conclude that q is a diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. Let $\chi \in Aut(\mathcal{M}_2(\mathcal{F}))$. Since

$$\begin{aligned}\big([\chi(q), \chi(x)]\chi(y) + \chi(y)[\chi(q), \chi(x)] + [\chi(q), \chi(y)]\chi(x) + \chi(x)[\chi(q), \chi(y)]\big)^m \\ = (\chi(x)\chi(y) + \chi(y)\chi(x))^n.\end{aligned}$$

So, $\chi(q)$ must be diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. In particular, let $\chi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\chi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $\mathcal{M}_2(\mathcal{F})$, which leads to $d = 0$, a contradiction. This completes the proof. □

Theorem 2.2. *Let R be a prime ring with characteristic deferent from 2, I be a nonzero ideal of R and m, n be the two fixed positive integers. If R admits a derivation d such that $(d(x \circ y))^m - (x \circ y)^n \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $(xy + yx)^n \in Z(R)$, for all $x, y \in I$. By Chuang [7], this polynomial identity is also satisfied by Q and hence R as well. Note this is a polynomial identity and hence there exists a field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of $k \times k$ matrices over field $\mathcal{F}$ where $k \geq 1$. Moreover, R and $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identity, that is $[(xy+yx)^n, w] = 0$ for all $x, y, w \in \mathcal{M}_k(\mathcal{F})$. But by choosing $x = e_{12},\ y = e_{21}$, we get $(xy + yx)^n = e_{11} + e_{22} \in Z(\mathcal{M}_k(\mathcal{F}))$, a contradiction.

Now we assume that $d \neq 0$. If $(d(xy + yx))^m = (xy + yx)^n$ for all $x, y \in I$, then R is commutative by Theorem 2.1. Otherwise we have $I \cap Z(R) \neq 0$ by our assumptions. Let J be a nonzero two sided ideal of R_Z, the ring of the central quotients of R. Since $J \cap R$ is an ideal of R, then $J \cap R \cap Z(R) \neq 0$. Hence J contains an invertible element in R_Z, and so R_Z is simple with 1. By the hypothesis for any $x, y \in I$ and $w \in I$, thus I satisfies the differential identity

$$[(d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n, w] = 0. \tag{2.1}$$

Since I and Q satisfy the same differential identities [7, Theorem 1], we may assume that Q satisfies (2.1). If d is not an inner derivation of R, then Q satisfies

$$[(sy + ys + tx + xt)^m - (xy + yx)^n, w] = 0.$$

In particular $x = 0$, we have $[(sy + ys)^m, w] = 0$, for all $y, s, w \in Q$. This is a polynomial identity and hence there exists a field $\mathcal{F}$ such that $Q \subseteq \mathcal{M}_k(\mathcal{F})$ with $k > 1$ and Q, $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identity [14]. Using same technique as above we get a contradiction. Let d be an inner derivation induced by an element $q \in Q$, that is, $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$. Then by (2.1) we have

$$[([q, x]y + y[q, x] + [q, y]x + x[q, x])^m - (xy + yx)^n, w] = 0. \tag{2.2}$$

for all $x, y \in I$ and $w \in R$. By Chuang [7], Q satisfy (2.2). By localizing R at $Z(R)$ it follows that

$$([q, x]y + y[q, x] + [q, y]x + x[q, x])^m - (xy + yx)^n \in Z(R_Z),$$

for all $x, y \in R_Z$. Since R and R_Z satisfy the same polynomial identities, by our assumption, we have that R_Z is not commutative. Thus, replacing R with R_Z, we may assume that R is a simple ring with 1. By, Martindale theorem [16], R is a primitive ring with minimal right ideal, whose commuting ring $\mathcal{D}$ is a division ring which is finite dimensional over $Z(R)$. However, since R is a simple with 1, R must be Artinian. Hence $R = \mathcal{D}_s$, the $s \times s$ matrices over $\mathcal{D}$,

for some $s \geq 1$. By [14, Lemma 2], there exists a field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of $k \times k$ matrices over field $\mathcal{F}$, with $k > 1$, and $\mathcal{M}_k(\mathcal{F})$ satisfies (2.2) that is,

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n \in Z(\mathcal{M}_k(\mathcal{F})) = \mathcal{F}.\mathcal{I}_k.$$

If $k \geq 2$, now let $q = (q_{ij})_{k\times k}$. By assumption, for all $x, y \in R$,

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n$$

is zero or invertible. We choose $x = e_{ii}, y = e_{ij}$ for any $i \neq j$. Then we have

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n = [q, e_{ij}]^m$$

Since rank of $[q, e_{ij}]^m$ is ≤ 2, it cannot be invertible in R and so $[q, e_{ij}]^m = 0$. By solving above and left multiplying by e_{ij}, one can get $0 = e_{ij}(qe_{ij})^m = e_{ij}q_{ji}^m$ implying $q_{ji} = 0$. Thus, for any $i \neq j$, $q_{ji} = 0$ that is q is diagonal. Now set $q = \sum_t q_{tt}e_{tt}$ with $q_{tt} \in \mathcal{F}$. For any $\mathcal{F}$-automorphism χ of R, we have

$$\begin{aligned}([\chi(q), \chi(x)]\chi(y) + \chi(y)[\chi(q), \chi(x)] + [\chi(q), \chi(y)]\chi(x) + \chi(x)[\chi(q), \chi(x)])^m \\ - (\chi(x)\chi(y) + \chi(y)\chi(x))^n\end{aligned}$$

is zero or invertible, for every $x, y \in R$. By above argument $\chi(q)$ must be diagonal. Therefore, for each $j \neq i$, we have $\chi(q) = (1 + e_{ij})q(1 - e_{ij}) = \sum_{i=1}^{k} q_{ii}e_{ii} + (q_{jj} - q_{ii})e_{ij}$ is diagonal. Therefore, $q_{jj} = q_{ii}$ and so $q \in \mathcal{F}.\mathcal{I}_k$, and hence $d = 0$, a contradiction. This completes the proof of theorem. □

Theorem 2.3. *Let R be a prime ring, I be a nonzero ideal of R and m, n be the two fixed positive integers. If R admits a derivation d such that $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $(xy + yx)^n = 0$, for all $x, y \in I$, and hence R is commutative as already done in Theorem 2.1. Hence, onward we will assume that $d \neq 0$ and $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$. Now we divide our proof into two cases:

Case 1. If d is not an inner derivation of R, then I satisfies the polynomial identity

$$(s \circ t)^m = (xy + yx)^n, \text{ for all } x, y, s, t \in I.$$

In particular for $s = 0$, I satisfies the blended component $(xy + yx)^n = 0$, for all $x, y \in I$, and hence R is commutative by Theorem 2.1.

Case 2. Let d is an inner derivation induced by an element $q \in Q$, that is, $d(x) = [q, x]$ for all $x \in R$. It follows that, $([q, x] \circ [q, y])^m = (x \circ y)^n$, for all $x, y \in I$. Using the same argument presented in the proof of Theorem 2.1, we see that

$$([q,x] \circ [q,y])^m = (xy + yx)^n \text{ for all } x, y \in R, \tag{2.3}$$

where R is a primitive ring with $\mathcal{D}$ as the associated division ring. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$, then the density of R implies that $R \cong \mathcal{M}_k(C)$, where $k = dim_{\mathcal{D}}\mathcal{V}$. Suppose that $dim_{\mathcal{D}}\mathcal{V} \geq 2$, otherwise we are done. Now, we want to show that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. On contrary, suppose that v and qv are linearly $\mathcal{D}$-independent for some $v \in \mathcal{D}$.

If $q^2v \notin Span_{\mathcal{D}}\{v, qv\}$, then $\{v, qv, q^2v\}$ is linearly $\mathcal{D}$-independent. By the density of R there exist $x_0, y_0 \in R$ such that

$$\begin{aligned} x_0v = v, \quad x_0qv = 0, \quad x_0q^2v = 0, \\ y_0v = 0, \quad y_0qv = v, \quad y_0q^2v = v. \end{aligned}$$

The application of (2.3) implies that $(-1)^m v = ([q, x_0] \circ [q, y_0])^m v = (xy + yx)^n v = 0$, and we arrive at a contradiction.

If $q^2v \in Span_{\mathcal{D}}\{v, qv\}$, then $q^2v = v\alpha + qv\gamma$ for some $\alpha, 0 \neq \gamma \in \mathcal{D}$. In view of the density of R, there exist $x_0, y_0 \in R$ such that

$$\begin{aligned} x_0v = v, \quad x_0qv = 0, \\ y_0v = 0, \quad y_0qv = v. \end{aligned}$$

It follows from the relation (2.3) that $0 = ([q, x_0] \circ [q, y_0])^m v = (xy + yx)^n v = (-1)^m v\gamma^m \neq 0$, and we arrive at a contradiction. So we conclude that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. Therefore, for each $v \in \mathcal{V}$, $qv = v\beta_v$ for some $\beta_v \in \mathcal{D}$. By a standard arguments, it is easy to see that β_v is independent of the choice of $v \in \mathcal{V}$. Thus we can write $qv = v\beta$ for all $v \in \mathcal{V}$ and a fixed $\beta \in \mathcal{D}$. Using the same techniques as in the proof of the Theorem 2.1, we conclude that $d = 0$, a contradiction. This completes the proof of theorem. □

The following examples shows that the main results are not true in the case of arbitrary rings.

Example 2.1. Let S be any ring.

(i) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$. We define a map $d : R \to R$ by $d(x) = e_{11}x - xe_{11}$. Then it is easy to see that d is a nonzero derivation and I is a nonzero ideal of R such that for all positive integers m, n, d satisfies the requirements of Theorems 2.1,2.2 and 2.3. However, R is not commutative. Hence, the hypothesis of primeness is crucial.

(ii) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$. Define a map $d : R \to R$ by $d(x) = [x, e_{11} + e_{12}]$. Then R is a ring under usual operations. It is easy to see that d is a nonzero derivation and I is a nonzero ideal of R. Further, for any $x, y \in I$, the following conditions (1) $(d(x \circ y))^m = (x \circ y)^n$ and (2) $(d(x) \circ d(y))^m = (x \circ y)^n$ are satisfied but R is not commutative. Hence, in Theorems 2.1,2.2 and 2.3, the hypothesis of primeness can not be omitted.

3. The results in semiprime rings

From onwards R will be a semiprime ring, U is the left Utumi quotient ring of R. In order to prove the main results of this section we will make use of the following facts:

Fact 3.1 ([4, Proposition 2.5.1])**.** Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.

Fact 3.2 ([8, p.38])**.** If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.

Fact 3.3 ([8, p.42])**.** Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a B-algebra orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.

Theorem 3.1. *Let R be a semiprime ring with center $Z(R)$ and m, n are fixed positive integers. If R admits a nonzero derivation d such that $(d(x \circ y))^m = (x \circ y)^n$, for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and we are given that $(d(x \circ y))^m = (x \circ y)^n$, which is rewritten as

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n,$$

for all $x, y \in R$. By Beidar [4], any derivation of a semiprime ring R can be defined on the whole U, the Utumi quotient ring of R. In view of Lee [15], R and U satisfy the same differential identities, hence

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n,$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Due to Chuang [8, p.42], U is an orthogonal complete B-algebra and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore $\overline{d}$ has in $\overline{U}$ the same property as d on U. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 2.1, $\overline{U}$ is commutative. This implies that, for any maximal ideal M of B, $[U, U] \subseteq MU$ and hence $[U, U] \subseteq \bigcap_M MU = 0$, where MU runs over all prime ideals of U. In particular, $[R, R] = 0$ and so R is commutative. This completes the proof of theorem. □

Using arguments similar to those used in the proof of the above theorem, we may conclude with the following (we omit the proof brevity). We can prove

Theorem 3.2. *Let R be a semiprime ring and m, n are fixed positive integers. If R admits a nonzero derivation d such that $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in R$, then R is commutative.*

We come now to our last result of this section:

Theorem 3.3. *Let R be a semiprime ring of characteristic different from 2 and m, n are fixed positive integers. If R admits a nonzero derivation d such that $d(x \circ y)^m - (x \circ y)^n \in Z(R)$, for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and we are given that $d(x \circ y)^m - (x \circ y)^n \in Z(R)$, that is

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in Z(R),$$

for all $x, y \in R$. by Fact 3.2, $Z(U) = C$, the extended centroid of R, and, by Fact 3.1, d can be uniquely defined on the whole U. By Lee [15], R and U satisfy the same differential identities. Then

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in C,$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. As already pointed out in the proof of Theorem 3.1, U is a B-algebra orthogonal complete and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Let $\overline{d}$ is the derivation induced by d on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in (C + MU)/MU,$$

for any $x, y \in \overline{U}$. Moreover $\overline{U}$ is prime, hence we may conclude, by Theorem 2.2, either $\overline{U}$ is commutative or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, we have $[U, U] \subseteq \bigcap MU$. Hence $[U, U] \subseteq_M \bigcap MU = 0$. This completes the proof. □

4. Applications on Banach algebras

In this section, we apply the purely algebraic results which are obtained in section 2 and reveal the conditions that every continuous derivation on a Banach algebra maps into the radical. The proofs of the results rely on a Sinclair's Theorem [21] which states that every continuous derivation d of a Banach algebra $\mathfrak{A}$ leaves the primitive ideals of $\mathfrak{A}$ invariant. As we have mentioned before, Thomas [23], has generalized the Singer-Wermer theorem by proving that any derivation on a commutative Banach algebra maps the algebra into its radical. This result leads to the question whether the theorem can be proven without any commutativity assumption. There are many papers that the theorem holds without commutativity assumption [16, 17, 21, 22]. We also obtain that every

derivation maps into its radical with some property, but without any commutativity assumption. Our first result in this section is about continuous derivations on Banach algebras:

Theorem 4.1. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and $d : R \to R$ be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x \circ y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then d maps into the radical of $\mathfrak{A}$.*

Proof. We have given that $(d(x \circ y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, which can be rewritten as $(d(xy + yx))^m - (xy + yx)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$. Under the assumption that d is nonzero continuous derivation with Jacobson radical $rad(\mathfrak{A})$. In [21], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Since the Jacobson radical $rad(\mathfrak{A})$ is the intersection of all primitive ideals, we have $d(rad(\mathfrak{A})) \subseteq rad(\mathfrak{A})$, which means that there is no loss of generality in assuming that $\mathfrak{A}$ is semisimple. Since d leaves all primitive ideals invariant, one can introduce for any primitive ideal $P \subseteq \mathfrak{A}$, a nonzero derivation $d_P : \mathfrak{A}/P \to \mathfrak{A}/P$ by $d_P(x + P) = d(x) + P$, $x \in \mathfrak{A}$ where $\mathfrak{A}/P = \overline{A}$ is a factor Banach algebra. Moreover, by $(d(xy + yx))^m - (xy + yx)^n = 0$ for all $x, y \in \mathfrak{A}/P$.

Thus, by Theorem 2.1, it is immediate that either $\overline{\mathfrak{A}}$ is commutative or $d_P = 0$ on $\mathfrak{A}/P$. Consequently $d(\mathfrak{A}) \subseteq P$ for any primitive ideal P. Since the radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals in $\mathfrak{A}$, we get the required conclusion. □

Using arguments similar to those used in the proof of the above theorem, we can prove

Theorem 4.2. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and $d : R \to R$ be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x) \circ d(y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then d maps into the radical of $\mathfrak{A}$.*

In order to prove our last theorem we will use the following well-known result concerning semisimple Banach algebra contained in [12].

Lemma 4.1. *Every nonzero derivation on a semisimple Banach algebra is continuous.*

In view of the above Lemma 4.1 and Theorem 4.1, we may prove the following theorem in the case when $\mathfrak{A}$ is a semisimple Banach algebra.

Corollary 4.1. *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and d be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x \circ y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then $d(\mathfrak{A}) = 0$.*

Proof. By the hypothesis d is continuous. In view of the above Lemma 4.1, every nonzero derivation on a semisimple Banach algebra is continuous. Thus, every nonzero derivation on a semisimple Banach algebra leaves the primitive ideals of the algebra invariant. Now by using the same argument as used in the proof of the Theorem 4.1 and the fact that $rad(\mathfrak{A}) = 0$, we get the required result. □

Corollary 4.2. *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and d be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x) \circ d(y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then $d(\mathfrak{A}) = 0$.*

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References

[1] Argaç, N. and Inceboz, H. G., *Derivation of prime and semiprime rings*, J. Korean Math. Soc., 46 (2009), 997-1005.

[2] Ashraf, M., Rehman, N. and Raza, M. A., *A note on commutativity of semiprime Banach algebras*, Beitr Algebra Geom., 57 (2016), 553-560.

[3] Ashraf, M. and Rehman, N., *On commutativity of rings with derivations*, Results Math., 42 (2002), 3-8.

[4] Beidar, K. I., Martindale III, W. S. and Mikhalev, A. V., *Rings with Generalized Identities*, Pure and Applied Mathematics, Marcel Dekker 196, New York, 1996.

[5] Bell, H. E. and Daif, M. N., *On commutativity and strong commutativity-preserving maps*, Cand. Math. Bull., 37 (1994), 443-447.

[6] Carini, L. and De Filippis, V., *Commuatators with power central values on a Lie ideal*, Pacific J. Math., 193 (2000), 269-278.

[7] Chuang, C. L., *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., 103 (1988), 723-728.

[8] Chuang, C. L., *Hypercentral derivations*, J. Algebra, 166 (1994), 34-71.

[9] Erickson, T. S., Martindale III, W. S. and Osborn J. M., *Prime nonassociative algebras*, Pacific J. Math., 60 (1975), 49-63.

[10] Herstein, I. N., *Center-like elements in prime rings*, J. Algebra, 60 (1979), 567-574.

[11] Jacobson, N., Structure of Rings, *Colloquium Publications 37, Amer. Math. Soc. VII, Provindence, RI*, 1956.

[12] Johnson, B. E. and Sinclair, A. M., *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math., 90 (1968), 1067-1073.

[13] Kharchenko, V. K., *Differential identities of prime rings*, Algebra i Logic, 17 (1979), 155-168.

[14] Lanski, C., *An Engel condition with derivation*, Proc. Amer. Math. Soc., 118 (1993), 731-734.

[15] Lee, T. K., *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin., 20 (1992), 27-38.

[16] Martindale III, W. S., *Prime rings satisfying a generalized polynomial identity*, J. Algebra, 12 (1969), 576-584.

[17] Mathieu, M. and Murphy, G. J., *Derivations mapping into the radical*, Arch. Math., 57 (1991), 469-474.

[18] Mayne, J. H., *Centralizing mappings of prime rings*, Canad. Math. Bull., 27 (1984), 122-126.

[19] Raza, M. A. and Rehman, N., *A note on an Engel condition with derivations in rings*, Creat. Math. Inform., 26 (2017), 19-27.

[20] Raza, M. A. and Rehman, N., *On prime and semiprime rings with generalized derivations and non-commutative Banach algebras*, Proc. Indian Acad. Sci. (Math. Sci.), 126 (2016), 389-398.

[21] Sinclair, A. M., *Continuous derivations on Banach algebras*, Proc. Amer. Math. Soc., 20 (1969), 166-170.

[22] Singer, I. M. and Wermer, J., *Derivations on commutative normed algebras*, Math. Ann., 129 (1955), 260-264.

[23] Thomas, M. P., *The image of a derivation is contained in the radical*, Ann. Math., 128 (1988), 435-460.

[24] Xu, X. W., *The power values properties of generalized derivations*, Doctor Thesis of Jilin University, Changchun, 2006.

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Coefficient estimates for a subclass of analytic functions using Faber polynomials

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Abstract. In this paper, we introduce and investigate a new subclass $Q_{\Sigma}^{**}(\alpha, \phi)$ of normalized analytic functions defined using convolution in the open unit disk U whose inverse has univalent analytic continuation to U. Estimates of the coefficients of bi-univalent functions belonging to this class are determined by using Faber polynomial techniques.

Keywords: bi-univalent, Faber polynomials, Quasi-convex.

1. Introduction

Let A be the class of all normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n, \tag{1}$$

which are analytic in the unit disk U.

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A function that is regular (holomorphic) in U is said to be univalent in U if it assumes no value more than once in U. Denote by S, the subclass of A, of all univalent functions in U.

Convolution or Hadamard is a mathematical operation on two functions f and g resulting in a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. For $f(z)$ defined by (1) and $g(z)$ defined by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n \; b_n \; z^n.$$

Polya and Schoenberg [12] conjectured that the class of convex functions C [13] is preserved under convolution with convex functions: $f, g \in C \Rightarrow f * g \in C$. In 1973, Ruscheweyh and Sheil-Small [14] see also [12] proved the Polya-Schoenberg conjecture. They also proved that the classes of starlike [13] functions and close-to-convex [7] functions are closed under convolution with convex functions. Also it is to be noted that the convolution of two univalent (or starlike) functions need not be univalent.

If $f(z) \in S$ and its inverse has an analytic continuation to $|w| < 1$, then the function $f(z) \in A$ is said to be bi-univalent in U. Let Σ represent the class of all bi-univalent functions.

The concept of bi-univalent functions was introduced by Lewin [8] who proved that if $f(z)$ is bi-univalent, then $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's [8] result to $|a_2| \leq \sqrt{2}$. There is a rich literature on the estimates of the initial coefficients of bi-univalence. However not much is known about the estimates of higher coefficients.

The classes $S_{\Sigma}^{*}(\alpha)$ of bi-starlike functions and $C_{\Sigma}(\alpha)$ of bi-convex functions of order α, where $0 \leq \alpha < 1$ were discussed by Brannan and Taha [5].

A function $f(z)$ defined by (1) is in the class $S_{\Sigma}^{*}(\alpha)$, $0 \leq \alpha < 1$ if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, |z| < 1$$

and

$$Re\left(\frac{wg'(w)}{g(w)}\right) > \alpha, |w| < 1$$

The class $C_{\Sigma}(\alpha)$, $0 \leq \alpha < 1$ is the class of all functions of the form (1) satisfying the following conditions:

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, |z| < 1$$

and

$$Re\left(1+\frac{wg''(w)}{g'(w)}\right)>\alpha, |w|<1$$

where $g(w)=(f^{-1})(w)=w-a_2w^2+(2a_2^2-a_3)w^3-\dots$

Let C^* (see [10], [11]) denote the class of Quasi-convex functions in U, that is if $f(z)\in C^*$, then there exists a function $g\in C$ so that $Re\{\frac{(zf'(z))'}{g'(z)}\}>0$ in U.

It is possible to approximate an analytic function h on K by polynomials uniformly on K, where K is a simply connected compact set in the Complex plane.

Faber polynomials can be viewed as a.r. (almost regular) formal Laurent series over the Field $L_\infty(u)$, the set of Formal Laurent series over F with an indeterminate u. Faber polynomials introduced by Faber play an important role in various areas of mathematical sciences, especially in geometric function theory. The advantage of using Faber polynomials over the other methods is that we find the nth coefficient that is the general term and use it for our computations.

Using the Faber polynomial expansion for functions $f\in S$ of the form (1), the coefficients of its inverse map $F=f^{-1}$may be expressed as (e.g. see [3], Eq.(1.33), page 185)

$$\begin{aligned}F(w)=f^{-1}(w)&=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\dots)w^n\\&=w+\sum_{n=2}^{\infty}A_nw^n,\end{aligned}\tag{2}$$

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3))!(n-4)!}a_2^{n-4}a_4\\&+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5+(-n+2)a_3^2]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]+\sum_{j\geq7}a_2^{n-j}V_j,\end{aligned}$$

where V_j is a homogeneous polynomial in the variables a_2, a_3, …, a_n (see [12] and [6]). In particular, the first few terms of K_{n-1}^{-n} are $K_1^{-2}=-2a_2$, $K_2^{-3}=3(2a_2^2-a_3)$ and $K_3^{-4}=-4(5a_2^3-5a_2a_3+a_4)$. In general, an expansion of $K_{n-1}^{p}(a_2,\, a_3\,\dots,\, a_n)$ is given by

$$K_{n-1}^{p}=pa_n+\frac{p(p-1)}{2}D_{n-1}^2+\frac{p!}{(p-3)!3!}D_{n-1}^3+\dots+\frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1},$$

where $D_{n-1}^p = D_{n-1}^p(a_2, a_3, \ldots, a_n)$,

$$D_{n-1}^m(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!} \quad \text{for } m \leq n,$$

and the sum is taken over all non-negetive integers $\mu_1, \ldots, \mu_{n-1}$ satisfying $\mu_1 + \mu_2 + \cdots + \mu_{n-1} = m$ and $\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1$. Evidently: $D_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$ (see [1], [2]).

An analytic function p of the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is called a function with positive real part in U if $\Re[p(z)] > 0$ for all $z \in U$. The class of all functions with positive real part is denoted by $\wp$.

Lemma 1.1 ([6]). *The coefficient p_n of a function $p \in \wp$ satisfies the sharp inequality $|p_n| \leq 2, n \geq 1$.*

Motivated by the definition of the class C^* of Quasi-convex functions investigated by Noor and Thomas ([10], [11]), In this paper we introduce the class $Q_\Sigma^{**}(\alpha, \phi)$, $0 \leq \alpha < 1$ and obtain the coefficient bounds using Faber polynomial techniques.

2. Main result

Coefficient bounds for functions in the class $Q_\Sigma^{**}(\alpha, \phi)$, $0 \leq \alpha < 1$.

Definition 2.1. *Let $f \in A$. Then $f \in Q_\Sigma^{**}(\alpha, \phi)$, $0 \leq \alpha < 1$ if $f \in \Sigma$,*

$$Re\left(\frac{(z(f * \phi)'(z))'}{(h * \phi)'(z)}\right) > \alpha$$

and

$$Re\left(\frac{(w((f * \phi)^{-1})'(w))'}{((h * \phi)^{-1})'(w)}\right) > \alpha,$$

where $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ are convex.

Theorem 2.1. *For $0 \leq \alpha < 1$, let the function $f \in Q_\Sigma^{**}(\alpha, \phi)$ in U. If $a_k = b_k = B_k = 0; 2 \leq k \leq n-1$ then*

$$|a_n| \leq \frac{1}{n} + \frac{2(1-\alpha)}{n^2}.$$

Proof. The Faber polynomial expansions are

$$\begin{aligned} \frac{(z(f * \phi)'(z))'}{(h * \phi)'(z)} &= 1 + \sum_{n=2}^{\infty} [(n^2 a_n \phi_n - n b_n \phi_n) \\ &+ \sum_{r=1}^{n-2} k_r^{-1}(2b_2\phi_2, 3b_3\phi_3, \ldots, (r+1)b_{r+1}\phi_{r+1}) \\ &\times ((n-r)^2 a_{n-r}\phi_{n-r} - (n-r)b_{n-r}\phi_{n-r})] z^{n-1}, \end{aligned} \tag{3}$$

$$\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)} = 1 + \sum_{n=2}^{\infty}[(n^2 A_n \Phi_n - nB_n\Phi_n) + \sum_{r=1}^{n-2} K_r^{-1}(2B_2\Phi_2, ...(r+1)B_{r+1}\Phi_{n-r}) \times ((n-r)^2 A_{n-r}\Phi_{n-r} - (n-r)B_{n-r}\Phi_{n-r})]w^{n-1}. \tag{4}$$

On the other hand, since

$$Re\left(\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)}\right) > \alpha$$

in U, there exists $p(z) = 1 + \sum_{n=1}^{\infty} c_n\, z^n \in A$ so that

$$\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)} = \alpha + (1-\alpha)p(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n. \tag{5}$$

Also as

$$Re\left(\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)}\right) > \alpha$$

in U, there exists $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in A$ such that

$$\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)} = \alpha + (1-\alpha)q(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n. \tag{6}$$

By the Caratheodory lemma, $|c_n| \leq 2$ and $|d_n| \leq 2$. Comparing the coefficients from (3) and (5), we obtain for $n \geq 2$,

$$\sum_{n=2}^{\infty}[(n^2 a_n\phi_n - nb_n\phi_n) + \sum_{r=1}^{n-2} K_r^{-1}(2b_2\phi_2, 3b_3\phi_3, \ldots, (r+1)b_{r+1}\phi_{r+1}) \times ((n-r)^2 a_{n-r}\phi_{n-r} - (n-r)b_{n-r}\phi_{n-r})] = (1-\alpha)c_{n-1}. \tag{7}$$

Similarly, from (4) and (6), we obtain:

$$\sum_{n=2}^{\infty}[(n^2 A_n \Phi_n - nB_n\Phi_n) + \sum_{r=1}^{n-2} K_r^{-1}(2B_2\Phi_2, ...(r+1)B_{r+1}\Phi_{n-r}) \times ((n-r)^2 A_{n-r}\Phi_{n-r} - (n-r)B_{n-r}\Phi_{n-r})] = (1-\alpha)d_{n-1}. \tag{8}$$

But under the assumption $a_k = b_k = B_k = 0, 2 \leq k \leq n-1$ and $\phi_k = 0, 2 \leq k \leq n-1$, equation (7) and (8), respectively yield:

$$n^2 a_n\phi_n - nb_n\phi_n = (1-\alpha)c_{n-1}. \tag{9}$$

and

$$-n^2 A_n\Phi_n - nB_n\Phi_n = (1-\alpha)d_{n-1}. \tag{10}$$

By the definition of K_n^p, $A_n = -a_n$, Thus

$$-n^2 a_n \Phi_n - n B_n \Phi_n = (1-\alpha) d_{n-1}. \tag{11}$$

Solving either of equations (9) or (10), we obtain

$$|a_n| \leq \frac{1}{n} + \frac{2(1-\alpha)}{n^2}$$

upon noticing that $|b_n| \leq 1$ and $|B_n| \leq 1$.

Remark 2.1. For $n = 2$ equations (9) and (11) yield

$$\begin{aligned} 4a_2\phi_2 - 2b_2\phi_2 &= (1-\alpha)c_1, \\ -4a_2\Phi_2 - 2B_2\Phi_2 &= (1-\alpha)d_1. \end{aligned}$$

Solving these two equations, we have $|a_2| \leq \frac{2-\alpha}{2}$.

Remark 2.2. When $\phi(z) = \frac{1}{1-z}$, this class $Q_\Sigma^{**}(\alpha, \phi)$ will reduce to the class of all bi-quasi-convex functions of order α [16] .

Remark 2.3. When $\phi(z) = \frac{1}{1-z}$, as every convex function is starlike, immediate replacements of $(f * \phi)'$ and $(h * \phi)'$ by $(f * \phi)$ and $(h * \phi)$ respectively in a result of Libera [9], the class $Q_\Sigma^{**}(\alpha, \phi)$ will reduce to the class of bi-close-to-convex functions of order α studied by Hamidi and Jahangiri[15].

Remark 2.4. Replacing $\phi(z)$ by $\frac{1}{1-z}$ and f by h, the class $Q_\Sigma^{**}(\alpha, \phi)$ reduces to the class of bi-convex functions of order α discussed by Brannan and Taha [5]. When n=2 and $\alpha = 0$, $|a_2| \leq 1$, which is an improvement of the corresponding estimate of Brannan and Taha [5].

Remark 2.5. Replacing h by f in Remark 2.3, we notice that the class $Q_\Sigma^{**}(\alpha, \phi)$ reduces to the class of bi-starlike functions α discussed by Brannan and Taha [5]. When n=2 and $\alpha = 0$, $|a_2| \leq 1$, which is an improvement of the corresponding estimate of Brannan and Taha [5].

References

[1] H. Airault, *Symmetric sums associated to the factorization of Grunsky coefficients,* in Conference, Groups and Symmetries, Montreal, Canada, April 2007.

[2] H. Airault, *Remarks on Faber polynomials*, Int. Math. Forum, 3 (2008), 9-12.

[3] H. Airault, A. Bouali, *Differential calculus on the Faber polynomials,* Bull. Sci. Math., 130 (2006), 179-222.

[4] D.A. Brannan, J.G. Clunie, *Aspects of contemporary complex analysis*, Academic Press, London and New York, 1980.

[5] D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, Studia Univ. Babeş-Bolyai Math., 31 (1986), 70-77.

[6] A. W. Goodman, *Univalent functions*, Vol. I, II, Mariner, Tampa, FL, 1983.

[7] W. Kaplan, *Close-to-Convex schlicht functions,* Mich. Math. J. 1 (1952), 169-185.

[8] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., 18 (1967), 63-68.

[9] R.J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16 (1965), 755-758.

[10] K. I. Noor, *On Quasi-convex functions and related topics*, Internat. J. Math. and Math. Sci., 10(1987), 241-258.

[11] K. I. Noor and D.K. Thomas, *Quasi-convex univalent functions*, Int. J. Math. and Math. Sci., 3 (1980), 255-266.

[12] G. Polya and I.J. Schoenberg, *Remarks on the la Valle Poussin means and convex conformal maps of the circle*, Pacific J. Math., 8 (1958), 295-334.

[13] M.S. Robertson, *On the theory of univalent functions,* Ann. of Math., 37(1936), 374-408.

[14] St. Ruscheweyh, T. Sheil-Small, *Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture*, Comm. Math. Helv., 48(1973), 119-135.

[15] S. G. Hamidi and Jay M. Jahangiri, *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser I, 352 (2014), 17-20.

[16] G. Saravanan and K. Muthunagai, *Co-efficient estimates for the class of Bi-quasi-convex functions using Faber polynomials*, Far East J. Math. Sci. (FJMS), 102 (2017), 2267-2276.

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Heptavalent symmetric graphs of order $12p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $12p$ for each prime p. As a result, there are eleven sporadic such graphs: one for $p = 2$, one for $p = 3$ and nine for $p = 13$.

Keywords: symmetric graph, s-transitive graph, coset graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [24, 27] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G-vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be *(G, s)-arc-transitive* or *(G, s)-regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be *(G, s)-transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called *G-symmetric*. A graph X is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [20] and Potočnik [23]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [8, 30, 31]. Guo [12] determined the exact

structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs was classified in [17, 21, 22, 28, 29]. Recently, Guo [13] gave the exact structure of heptavalent case and determined heptavalent symmetric graph of order $6p$ in [10]. Thus, as an application, we classify connected heptavalent symmetric graphs of order $12p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [18, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [13, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*

(2) *For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;*

(3) *For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ or $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.*

From [9, pp.12-14], [26, Theorem 2] and [16, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.3. *Let p be a prime, and let G be a non-abelian simple group of order $|G| \,\big|\, (2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the following groups:*

Table 1: **Non-abelian simple $\{2,3,5,7,p\}-$groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	$PSL(2,8)$	$2^3{\cdot}3^2{\cdot}7$	$PSU(4,2)$	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	$PSL(2,17)$	$2^4{\cdot}3^2{\cdot}17$	$PSU(3,3)$	$2^5{\cdot}3^3{\cdot}7$
$PSL(2,7)$	$2^3{\cdot}3{\cdot}7$	$PSL(3,3)$	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	$PSL(2,27)$	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	$PSU(5,2)$	$2^{10}{\cdot}3^5{\cdot}5{\cdot}11$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	$PSL(2,31)$	$2^5{\cdot}3{\cdot}5{\cdot}31$	$PSp(4,4)$	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	$PSL(2,49)$	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	$PSp(6,2)$	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	$PSL(2,81)$	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
$PSL(2,11)$	$2^2{\cdot}3{\cdot}5{\cdot}11$	$PSL(2,127)$	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
$PSL(2,13)$	$2^2{\cdot}3{\cdot}7{\cdot}13$	$PSL(3,4)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
$PSL(2,16)$	$2^4{\cdot}3{\cdot}5{\cdot}17$	$PSU(3,4)$	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	$P\Omega^+(8,2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
$PSL(2,19)$	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	$PSU(3,5)$	$2^4{\cdot}3^2{\cdot}5^3{\cdot}7$	$Sz(8)$	$2^6{\cdot}5{\cdot}7{\cdot}13$
$PSL(2,25)$	$2^3{\cdot}3{\cdot}5^2{\cdot}13$	$PSU(3,8)$	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	$PSL(2,449)$	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	$PSp(8,2)$	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
A_{12}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$	$PSL(2,2^6)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
$PSL(2,29)$	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	$PSL(4,4)$	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$	$P\Omega^-(8,2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
$PSL(2,41)$	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	$PSL(5,2)$	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
$PSL(2,71)$	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$				

In view of [14, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let X be connected heptavalent symmetric graph of order $4p$ with p a prime. Then X is isomorphic to K_8.*

Next we construct some heptavalent symmetric graphs of order $6p$ with p a prime. In order to construct some heptavalent symmetric graphs, we need to introduce the so called coset graph (see [20, 25]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G,H,D)$ of G with respect to H and D is defined to have vertex set $[G:H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G,H,D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G,H,D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G,H,D) \cong \mathrm{Cos}(G,H^\alpha,D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. For more details regarding coset graphs, see, for example, [7, 18, 25].

By Atlas [6], S_8 has a maximal subgroup $K \cong \mathbb{Z}_2^4 \rtimes S_4$ and A_8 has a maximal subgroup $H \cong \mathbb{Z}_2^3 \rtimes PSL(3,2)$ such that $K \cap H \cong \mathbb{Z}_2^3 \rtimes S_4$. Take an element g

of order 2 in $K\backslash(K\cap H)$. Then define the coset graph:

$$\mathcal{C}_{30} = \mathrm{Cos}(\mathrm{S}_8, H, HgH).$$

Let $G \cong \mathrm{PSL}(2,13)$ as a permutation group acting naturally on 14 points, and take the following four elements:

$a = (1,13,2,12,9,5,14)(3,4,7,10,11,6,8)$,
$b = (2,5)(3,7)(6,11)(8,10)(9,12)(13,14)$,
$x = (1,4)(3,10)(6,14)(7,8)(9,12)(11,13)$,
$y = (1,4)(2,6)(5,11)(8,9)(10,12)(13,14)$.

Then $H = \langle a,b\rangle \cong D_{14}$. Define the following two coset graph:

$$\mathcal{C}_{78}^1 = \mathrm{Cos}(G, H, HxH), \quad \mathcal{C}_{78}^2 = \mathrm{Cos}(G, H, HyH).$$

We can get the classification of heptavalent symmetric graphs of order $6p$ with p a prime from [10, Theorem 3.1].

Proposition 2.5. *Let X be a connected heptavalent symmetric graph of order $6p$ with p a prime. Then X is isomorphic to $\mathcal{C}_{30}$, $\mathcal{C}_{78}^1$ or $\mathcal{C}_{78}^2$.*

Let $G \cong \mathrm{PSL}(2,8)$. Then by Atlas [6], G has a maximal subgroup $H \cong D_{14}$ and a Sylow 2-subgroup $P \cong \mathbb{Z}_2^3$ such that $P \cap H \cong \mathbb{Z}_2$. Take $g \in P\backslash H$. Then $g^2 = 1$, $H \cap H^g \cong \mathbb{Z}_2$ and $\langle H, g\rangle = G$. Define the coset graph:

$$\mathcal{C}_{36} = \mathrm{Cos}(G, H, HgH).$$

From [11, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order 36.

Proposition 2.6. *There is only one connected heptavalent symmetric graph of order 36, that is, $\mathcal{C}_{36}$.*

Now we construct some heptavalent symmetric graphs of order $12p$. If $p = 2$, then by [5], there is only one such graph of order 24, and we define this graph as follows:

Construction 2.7. *Set $a = (3,5)(4,7)(6,8)$, $b = (1,2)(4,8)(6,7)$ and $c = (1,4,8,3,5,6,7)$. Then $G = \langle a,b,c\rangle \cong \mathrm{PGL}(2,7)$. Let $H = \langle a,c\rangle$. Clearly, $H \cong D_{14}$ and b centralizes a. Define the coset graph:*

$$\mathcal{C}_{24} = \mathrm{Cos}(G, H, HbH).$$

Then $\mathcal{C}_{24}$ is a connected heptavalent symmetric graph of order 24. By Magma [3], $\mathrm{Aut}(\mathcal{C}_{24}) \cong \mathrm{PGL}(2,7)$ and any connected heptavalent symmetric graph of order 24 admitting $\mathrm{PGL}(2,7)$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{24}$.

By using the isomorphisms of coset graph and calculation of Magma [3], we have that there are nine non-isomorphic heptavalent symmetric graphs of order 156 with PSL$(2,13)$ as an automorphism group.

Construction 2.8. *Take the following six elements in* S_{16}*:*

$$\begin{aligned}
a &= (3,13,11,9,7,5)(4,14,12,10,8,6),\\
b &= (1,2,9)(3,8,10)(4,5,12)(6,13,14),\\
c &= (15,16),\\
f_1 &= (2,10)(3,13)(4,11)(5,14)(6,7)(8,12),\\
f_2 &= (2,10)(3,13)(4,11)(5,14)(6,7)(8,12)(15,16),\\
f_3 &= (1,10,8,5,14,12,2)(3,6,11,4,7,13,9).
\end{aligned}$$

Then $G = \langle a,b,c\rangle \cong \mathrm{PSL}(2,13)\times\mathbb{Z}_2$*,* $H_1 = \langle f_1, f_3\rangle \cong D_{14}$ *and* $H_2 = \langle f_2, f_3\rangle \cong D_{14}$*. Take five elements in* G*:*

$$\begin{aligned}
x_1 &= (3,9)(4,10)(5,11)(6,12)(7,13)(8,14)(15,16),\\
x_2 &= (2,6)(3,14)(4,8)(7,12)(9,11)(10,13)(15,16),\\
x_3 &= (2,9)(3,10)(4,13)(5,6)(7,11)(12,14)(15,16),\\
x_4 &= (1,9)(2,6)(3,13)(5,8)(7,10)(12,14)(15,16),\\
x_5 &= (1,9)(2,4)(3,14)(5,13)(8,12)(10,11)(15,16).
\end{aligned}$$

Define the following five coset graphs:

$$\begin{aligned}
\mathcal{SG}^i_{156} &= \mathrm{Cos}(G, H_1, H_1x_iH_1), i = 1,2.\\
\mathcal{SG}^i_{156} &= \mathrm{Cos}(G, H_2, H_2x_iH_2),\ i = 3,4,5.
\end{aligned}$$

By Magma [3], $\mathrm{Aut}(\mathcal{SG}^1_{156}) \cong \mathrm{PSL}(2,13)\times\mathbb{Z}_2$*,* $\mathrm{Aut}(\mathcal{SG}^2_{156}) \cong \mathrm{PGL}(2,13)\times\mathbb{Z}_2$ *and* $\mathrm{Aut}(\mathcal{SG}^i_{156}) \cong \mathrm{PSL}(2,13)\times\mathbb{Z}_2$ *for* $i = 3,4,5$*. In particular,* $\mathcal{SG}^1_{156}$ *and* $\mathcal{SG}^2_{156}$ *are bipartite.*

Take the following eight elements:

$$\begin{aligned}
d &= (1,2)(3,6)(4,5)(7,14)(8,13)(9,12)(10,11),\\
e &= (1,4)(2,5)(3,8)(6,13)(7,10)(11,14),\\
f &= (1,9,4,14,3,8,11)(2,10,13,6,7,5,12),\\
y_1 &= (3,9)(4,10)(5,11)(6,12)(7,13)(8,14)\\
y_2 &= (2,11)(3,6)(4,14)(5,12)(7,8)(9,13),\\
y_3 &= (2,13)(3,11)(4,6)(5,8)(7,14)(9,10),\\
y_4 &= (2,9,5,12)(3,14,7,13)(6,10,11,8),\\
y_5 &= (2,13,6,10)(3,4,14,8)(7,9,12,11),
\end{aligned}$$

Set $T = \langle a,b,d\rangle \cong \mathrm{PGL}(2,13)$*,* $K_1 = \langle d, f\rangle \cong D_{14}$ *and* $K_2 = \langle e, f\rangle \cong D_{14}$*. Define the following coset graphs:*

$$\begin{aligned}
\mathcal{CG}^i_{156} &= \mathrm{Cos}(T, K_1, K_1y_iK_1), i = 1,2,3.\\
\mathcal{CG}^i_{156} &= \mathrm{Cos}(T, K_2, K_2y_iK_2),\ i = 4,5.
\end{aligned}$$

By Magma [3], $\mathcal{CG}^5_{156} \cong \mathcal{SG}^2_{156}$*, and* $\mathrm{Aut}(\mathcal{CG}^i_{156}) \cong \mathrm{PGL}(2,13)$ *for* $i = 1,2,3,4$*. In particular,* $\mathcal{CG}^4_{156}$ *is bipartite.*

We use the same notations as above to state the following lemma.

Lemma 2.9. *Let* X *be a connected heptavalent symmetric graph of order* 156 *and* $A = \mathrm{Aut}(X)$*. Then*

(1) *If* A *has an arc-transitive subgroup isomorphic to* G*, then* $X \cong \mathcal{SG}^i_{156}$ *for* $i = 1,2,3,4,5$*;*

(2) *If* A *has an arc-transitive subgroup isomorphic to* T*, Then* $X \cong \mathcal{CG}^i_{156}$ *for* $i = 1,2,3,4,5$*;*

Proof. Assume that A has an arc-transitive subgroup G. Then $X \cong \mathrm{Cos}(G, G_v, G_v g G_v)$ with $G_v \cong D_{14}$, $\langle G_v, g\rangle = G$ and $|G_v : G_v^g \cap G_v| = 7$. In particular, g can be chosen as a 2-element. By Magma [3], G has two conjugacy classes of subgroups isomorphic to D_{14} with H_1 and H_2 as their representatives.

Suppose that $G_v = H_1$. Then by Magma [3], g has 42 choices, denoted this set by U. Note that $N_G(G_v) \cong D_{28}$ by Atlas [6]. Thus, again by Magma [3], $N_G(G_v)$ acting on U has 6 orbits. Clearly, the coset graphs formed by the elements in the same orbit are isomorphic each other. Thus, we obtain six coset graphs. By Magma [3], these six representatives of 6 orbits form two coset graphs, and $X \cong \mathcal{SG}^i_{156}$ with $i = 1,2$.

Suppose that $G_v = H_2$. Then by Magma [3], g has 84 choices, denoted this set by V. Since $N_G(G_v) \cong D_{28}$, we have that $N_G(G_v)$ acting on V has 12 orbits by Magma [3]. All the representatives of 12 orbits form three coset graphs, and $X \cong \mathcal{SG}^i_{156}$ with $i = 3,4,5$.

Assume that A has an arc-transitive subgroup T. Then $X \cong \mathrm{Cos}(T, T_v, T_v t T_v)$ with $T_v \cong D_{14}$, $\langle T_v, t\rangle = T$ and $|T_v : T_v^t \cap T_v| = 7$. In particular, t can be chosen as a 2-element. By Atlas [6], T has two conjugacy classes of subgroups isomorphic to D_{14} with K_1 and K_2 as their representatives.

Suppose that $T_v = K_1$. Then by Magma [3], t has 112 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has seven orbits. These seven orbits form seven coset graphs and by Magma [3], there are 3 non-isomorphic graphs, that is, $X \cong \mathcal{CG}^i_{156}$ with $i = 1,2,3$.

Suppose that $T_v = K_2$. Then by Magma [3], t has 98 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has five orbits. These five orbits form five coset graphs and by Magma [3], there are 2 non-isomorphic graphs, that is, $X \cong \mathcal{CG}^i_{156}$ with $i = 4,5$. □

3. Classification

This section is devoted to classify connected heptavalent symmetric graphs of order $12p$ for each prime p.

Theorem 3.1. *Any connected heptavalent symmetric graph of order* $12p$ *with* p *a prime is isomorphic to* $\mathcal{C}_{24}$, $\mathcal{C}_{36}$, $\mathcal{SG}^i_{156}$ *with* $i = 1, 2, 3, 4, 5$, *or* $\mathcal{CG}^j_{156}$ *with* $j = 1, 2, 3, 4$.

Proof. Let X be a connected heptavalent symmetric graph of order $12p$ and $A = \mathrm{Aut}(X)$. If $p = 2$, then by [5] and Construction 2.7, $X \cong \mathcal{C}_{24}$ and $A \cong \mathrm{PGL}(2, 7)$. If $p = 3$, then by Proposition 2.6, $X \cong \mathcal{C}_{36}$. Thus, in the following we assume that $p \geq 5$. Take $v \in V(X)$. Then by Proposition 2.2, $|A_v| \,\big|\, 2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$. We separate the proof into two cases: A has a solvable minimal normal subgroup; A has no solvable minimal normal subgroup.

Case 1: A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, and N is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or p and k a positive integer. By Proposition 2.1, N is semiregular and X_N is also a connected heptavalent A/N-symmetric graph. It follows that $|N| \,\big|\, 12p$ and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent regular graph of odd order. Thus, $N \ncong \mathbb{Z}_2^2$. Since there is no connected heptavalent regular graph of order 12 by [19] or [5], we have that $N \ncong \mathbb{Z}_p$. This forces that $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ and X_N is a heptavalent symmetric graph of order $6p$ or $4p$. By Propositions 2.4 and 2.5, $X_N \cong K_8$, $\mathcal{C}_{30}$, $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$. If $X_N \cong K_8$, then $N \cong \mathbb{Z}_3$ and $p = 2$, contrary to our assumption. Thus, $N \cong \mathbb{Z}_2$ and $X_N \cong \mathcal{C}_{30}$, $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$.

Assume that $X_N \cong \mathcal{C}_{30}$. Then $p = 5$. Since $\mathrm{Aut}(\mathcal{C}_{30}) \cong \mathrm{S}_8$ by Proposition 2.5, we have that $A/N \lesssim \mathrm{S}_8$. By Magma [3], $\mathrm{Aut}(\mathcal{C}_{30})$ has a minimal arc-transitive subgroup isomorphic to S_7. It follows that A/N has an arc-transitive subgroup $M/N \cong \mathrm{S}_7$. Note that $\mathcal{C}_{30}$ is bipartite. Set $H/N \cong \mathrm{A}_7$. Then H/N has two orbits on $V(X_N)$ and hence $H_v \cong \mathrm{PSL}(2, 7)$. Since $N \leq C_H(N)$ and H/N is simple, we have that $C_H(N) = H$. It forces that $H \cong \mathbb{Z}_2 \times \mathrm{A}_7$ or $\mathbb{Z}_2.\mathrm{A}_7$. If $H \cong \mathbb{Z}_2 \times \mathrm{A}_7$, then H has a characteristic subgroup $K \cong \mathrm{A}_7$. The normality of H in M implies that K is also normal in M. Thus, the block graph X_K has order 4. By Proposition 2.1, K is semiregular and $|K| \,\big|\, 12p$. This is impossible because $K \cong \mathrm{A}_7$. If $H \cong \mathbb{Z}_2.\mathrm{A}_7$, then $|H_v| = 168$. However, by Magma [3], $\mathbb{Z}_2.\mathrm{A}_7$ has no subgroups of order 168, a contradiction.

Assume that $X_N \cong \mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$. Then $p = 13$. By Atlas [6], $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$ has a minimal arc-transitive subgroup isomorphic to $\mathrm{PSL}(2, 13)$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathrm{PSL}(2, 13)$. Since $N \cong \mathbb{Z}_2$, we have that $C_M(N) = M$. By Atlas [6], $\mathrm{Mult}(\mathrm{PSL}(2, 13)) = 2$. It follows that $M \cong \mathrm{SL}(2, 13)$ or $\mathrm{PSL}(2, 13) \times \mathbb{Z}_2$. If $M \cong \mathrm{SL}(2, 13)$, then $M_v \cong \mathbb{Z}_{14}$ by Magma [3]. This is impossible by Proposition 2.2. It follows that $M \cong \mathrm{PSL}(2, 13) \times \mathbb{Z}_2$. Note that M is arc-transitive. Thus, by Construction 2.8 and Lemma 2.9, $X \cong \mathcal{SG}^i_{156}$ with $i = 1, 2, 3, 4, 5$.

Case 2: A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Since N is non-solvable and minimal normal, we have that $N = T^k$ with T a

non-abelian simple group and k an positive integer. It follows that T has at least 3-prime factors. Note that $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$. Thus, T is one of the simple groups listed in Proposition 2.3. By Proposition 2.1, N has at most two orbits on $V(X)$, and hence $|N| = 12p|N_v|$ or $6p|N_v|$.

Assume that $k \geq 2$. Since T is a non-abelian simple group, we have that $2^2 \,\big|\, |T|$ and $T_v \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that p divides the order of X_N. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p \leq 7$. Recall that $|T^k| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$ and $6{\cdot}p \,\big|\, |T^k|$.

Let $p = 5$. Then $7^2 \nmid |T^k|$ and T is a simple $\{2,3,5\}$-group. By Proposition 2.3, $T \cong \mathrm{A}_5$ or A_6. The normality of N in A implies that $N_v \trianglelefteq A_v$. If $k \geq 3$, then $5^2 \,\big|\, |N_v|$ and hence N_v has a subgroup isomorphic to A_7 by Proposition 2.2. It follows that $7 \,\big|\, |T|$, a contradiction. Thus, $k = 2$. If $T \cong \mathrm{A}_5$, then $N \cong \mathrm{A}_5^2$ and $|N_v| = 60$ or 120. By Atlas [6], for $|N_v| = 60$, we have that $N_v \cong D_{10} \times \mathrm{S}_3$, $\mathrm{A}_4 \times \mathbb{Z}_5$ or A_5, and for $|N_v| = 120$, we have that $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_2$ or $\mathrm{A}_4 \times D_{10}$. However, by Proposition 2.2, A_v has no such normal subgroups, a contradiction. If $T \cong \mathrm{A}_6$, then $N \cong \mathrm{A}_6^2$, and $|N_v| = 2^4{\cdot}3^3{\cdot}5$ or $2^5{\cdot}3^3{\cdot}5$. By Magma [3], for $|N_v| = 2^4{\cdot}3^3{\cdot}5$, we have that $N_v \cong \mathrm{A}_6 \times \mathrm{S}_3$ or $\mathrm{A}_5 \times F_{36}$, and for $|N_v| = 2^5{\cdot}3^3{\cdot}5$, we have that $N_v \cong \mathrm{A}_6 \times \mathrm{A}_4$. Similarly, A_v has no such normal subgroups, a contradiction.

Let $p = 7$. Then $|T^k| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7^2$ and $6{\cdot}7 \,\big|\, |T^k|$. It follows that $7 \,\big|\, |T|$ and $k = 2$. By Proposition 2.3, T is isomorphic to one of the following groups:

$$\mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{PSL}(3,4).$$

If $T \cong \mathrm{PSL}(2,7)$, then $|N_v| = 2^5{\cdot}3{\cdot}7$ or $2^4{\cdot}3{\cdot}7$. By Magma [3], for $|N_v| = 2^5{\cdot}3{\cdot}7$, $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_4$; for $|N_v| = 2^4{\cdot}3{\cdot}7$, $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$. The normality of N_v in A_v implies that $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ and $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$ by Proposition 2.2. It follows that N has two orbits on $V(X)$ and $|A| = 2|A_vN| = 6|N|$. Clearly, $C_A(N) \cap N = 1$. Thus, $|C_A(N)| \,\big|\, 6$ and $C_A(N)$ is solvable. Since $C_A(N) \trianglelefteq A$, we have that $C_A(N) = 1$ by our assumption. By "N/C-Theorem" (see [15, Chapter I, Theorem 4.5]), $A \cong A/C_A(N) \lesssim \mathrm{Aut}(N)$. However, by Magma [3], $|\mathrm{Aut}(N)| = 8|N|$, which is contrary to the fact that $|A| = 6|N|$. If $T \cong \mathrm{PSL}(2,8)$, then $|N_v| = 2^5{\cdot}3^3{\cdot}7$ or $2^4{\cdot}3^3{\cdot}7$. By Magma [3], the only possible is that $N_v \cong \mathrm{PSL}(2,8) \times \mathrm{S}_3$. However, A_v has no such normal subgroup by Proposition 2.2, a contradiction. If $T \cong \mathrm{A}_7$, then by Magma [3], the only possible is that $|N_v| = 2^5{\cdot}3^3{\cdot}5^2{\cdot}7$ and $N_v \cong \mathrm{A}_7 \times \mathrm{A}_5$. Similarly, A_v has no such normal subgroup, a contradiction. If $T \cong \mathrm{A}_8$ or $\mathrm{PSL}(3,4)$, then $|N_v| = |N|/6p$ or $|N|/12p$. By Magma [3], A_8^2 or $\mathrm{PSL}(3,4)^2$ has no subgroups of such orders, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. Since $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, we have that N is isomorphic to one of the simple group listed in Proposition 2.3. By Proposition 2.1, we have that N has at most two orbits on $V(X)$ and $|N_v| = |N|/6p$ or $|N|/12p$.

Subcase 2.1: Suppose that $p = 5$. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^3{\cdot}7$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l} \mathrm{A}_5,\ \mathrm{A}_6,\ \mathrm{PSU}(4,2),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10} \\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2). \end{array}$$

Note that $|N_v| = |N|/6{\cdot}5$ or $|N|/12{\cdot}5$.

Let $N \cong \mathrm{A}_5$. Then $N_v \cong \mathbb{Z}_2$ or 1. If $N_v \cong \mathbb{Z}_2$, then since $N_v \trianglelefteq A_v$, we have that $A_v \cong D_{28}$, $F_{42} \times \mathbb{Z}_2$ or $F_{42} \times \mathbb{Z}_6$ by Proposition 2.2. Note that $C_A(N) \cong C_A(N)N/N \leq A/N$ and $|A : A_vN| = 2$. Thus, it is easy to deduce that $C_A(N)$ is solvable, which is contrary to our assumption that A has no solvable minimal normal subgroup. If $N_v = 1$, then N is regular on $V(X)$. By "N/C-Theorem" (see [15, Chapter I, Theorem 4.5]), $A/C_A(N) \lesssim \mathrm{Aut}(N)$. Since $\mathrm{Aut}(\mathrm{A}_5) \cong \mathrm{S}_5$, we have that $7 \,\big|\, |C_A(N)|$ and since $C_A(N) \trianglelefteq A$, we have that $C_A(N)$ is non-solvable and has no solvable characteristic subgroup by our assumption. Clearly, $C_A(N)$ is transitive or has two orbits on $V(X)$. Thus, $|A : A_vC_A(N)| = 2$ and $N \cong C_A(N)N/C_A(N) \trianglelefteq A/C_A(N)$. Since $N \cong \mathrm{A}_5$ and $|A/C_A(N) : A_vC_A(N)/C_A(N)| = 2$, we have that $N \leq A_vC_A(N) \cong A_v/(C_A(N) \cap A_v)$. It follows that A_v has a normal subgroup isomorphic to A_5, this is impossible by Proposition 2.2.

Let $N \cong \mathrm{A}_6$. Then $|N_v| = 12$ or 6. By Atlas [6], $N_v \cong \mathrm{A}_4$, or S_3. Since $N \trianglelefteq A$, we have that $N_v \trianglelefteq A_v$. By Proposition 2.2, $N_v \cong \mathrm{A}_4$ and $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$. It follows that $|A| = 2^4{\cdot}3{\cdot}7{\cdot}|N|$. Similar as above, $C_A(N) \neq 1$ and $C_A(N)$ is no-solvable. Clearly, $C_A(N) \cap N = 1$. Thus, $|C_A(N)| \,\big|\, 2^4{\cdot}3{\cdot}7$. This implies that $C_A(N)$ acting on $V(X)$ has at least five orbits. By Proposition 2.1, $C_A(N)$ is regular and hence solvable, a contradiction.

Let $N \cong \mathrm{PSU}(4,2)$, A_7, A_8, A_9, A_{10}, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(6,2)$, J_2 or $\mathrm{P}\Omega^+(8,2)$. Then $|N_v| = |N|/30$ or $|N|/60$. However, by Magma [3], these simple groups have no subgroups of such orders, a contradiction.

Subcase 2.2: Suppose that $p = 7$. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7^2$ and $6{\cdot}7 \,\big|\, |N|$ or $12{\cdot}7 \,\big|\, |N|$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l} \mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{PSU}(3,3),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10} \\ \mathrm{PSL}(2,49),\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2). \end{array}$$

Let $N \cong \mathrm{PSL}(2,7)$. Then $N_v \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. Set $C = C_A(N)$. If $C = 1$, then by "N/C-Theorem", $A \cong A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$. However, since $p = 7$ and X is arc-transitive, we have that $7^2 \,\big|\, |A|$, a contradiction. Thus, $C \neq 1$. Since $C \trianglelefteq A$, we have that C is non-solvable by our assumption. Note that $C \cong CN/N \trianglelefteq A/N$ and $A_v/N_v \cong A_vN/N \leq A/N$ with $|A : A_vN| \leq 2$. Thus, A_v is non-solvable. Since $N_v \trianglelefteq A_v$, we have that $N_v \cong \mathbb{Z}_2^2$ and $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ by Proposition 2.2. It follows that $A_v/N_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_3$ and $C \cong \mathrm{PSL}(2,7)$ because C is non-solvable and has no solvable characteristic subgroup. Thus, $|A/C : NC/C| = 12$ and $N \lesssim A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$, this is impossible.

Let $N \cong \mathrm{PSL}(2,8)$. Then $|N_v| = 6$ or 12. By Atlas [6], $N_v \cong \mathrm{S}_3$. Since $N \trianglelefteq A$, we have that $N_v \trianglelefteq A_v$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to S_3, a contradiction.

Let $N \cong \mathrm{PSU}(3,3)$, A_8, A_{10}, $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(6,2)$, J_2 or $\mathrm{P\Omega}^+(8,2)$. Then $|N_v| = |N|/42$ or $|N|/84$. By Magma [3], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{A}_7$. Then $|N_v| = 30$ or 60. By Atlas [6], $N_v \cong \mathrm{A}_5$. The normality of N in A implies that A_v has a normal subgroup isomorphic to A_5. This is impossible by Proposition 2.2.

Let $N \cong \mathrm{A}_9$. Then by Atlas [6], $N_v \cong (\mathrm{A}_6 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. This is also impossible because A_v has no normal subgroup isomorphic to $(\mathrm{A}_6 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Subcase 2.3: Suppose that $p > 7$. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$ and $6{\cdot}p \,\big|\, |N|$ or $12{\cdot}p \,\big|\, |N|$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l}
\mathrm{PSL}(2,17),\ \mathrm{PSL}(3,3),\ \mathrm{PSL}(2,11),\ \mathrm{PSL}(2,13),\ \mathrm{PSL}(2,16)\\
\mathrm{PSL}(2,19),\ \mathrm{PSL}(2,25),\ \mathrm{PSL}(2,27),\ \mathrm{PSL}(2,31),\ \mathrm{PSL}(2,81)\\
\mathrm{PSL}(2,127),\ \mathrm{PSU}(3,4),\ \mathrm{PSU}(3,8),\ \mathrm{PSU}(5,2),\ \mathrm{PSp}(4,4),\ \mathrm{M}_{11},\ \mathrm{M}_{12}\\
{}^2F_4(2)',\ \mathrm{A}_{11},\ \mathrm{A}_{12},\ \mathrm{PSL}(2,29),\ \mathrm{PSL}(2,41),\ \mathrm{PSL}(2,71),\ \mathrm{PSL}(2,449)\\
\mathrm{PSL}(2,2^6),\ \mathrm{PSL}(4,4),\ \mathrm{PSL}(5,2),\ \mathrm{PSp}(8,2),\ \mathrm{M}_{22},\ \mathrm{P\Omega}^-(8,2),\ G_2(4).
\end{array}$$

Let $N \cong \mathrm{PSL}(2,17)$. Then by Atlas [6], $N_v \cong \mathrm{A}_4$ or S_4. Since $N_v \trianglelefteq A_v$, we have that $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ by Proposition 2.2. Similar as above, $C_A(N) \cong \mathrm{PSL}(2,7)$. Since $p \nmid |C_A(N)|$, we have that $C_A(N)$ has at least p orbits on $V(X)$, and by Proposition 2.1, $C_A(N)$ is semiregular and hence solvable, a contradiction.

Let $N \cong \mathrm{PSL}(3,3)$. Then $|N_v| = 2^2{\cdot}3^2$ or $2^3{\cdot}3^2$. By Atlas [6], N_v has a characteristic subgroup isomorphic to $\mathbb{Z}_3^2$. The normality of N_v in A_v implies that A_v has a normal subgroup isomorphic to $\mathbb{Z}_3^2$. This is impossible by Proposition 2.2.

Let $N \cong \mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,25)$, M_{11}, M_{12} or A_{12}. Then by Atlas [6], $N_v \cong \mathbb{Z}_5$ or D_{10} for $N \cong \mathrm{PSL}(2,11)$; $N_v \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$ for $N \cong \mathrm{PSL}(2,15)$; $N_v \cong \mathrm{A}_5$ or S_5 for $N \cong \mathrm{M}_{11}$; $N_v \cong \mathrm{A}_6 \rtimes \mathbb{Z}_2^2$ or $\mathrm{A}_6 \times \mathbb{Z}_2$ for $N \cong \mathrm{M}_{12}$; $N_v \cong \mathrm{A}_{10}$ or S_{10} for $N \cong \mathrm{A}_{12}$. However, By Proposition 2.2, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSL}(2,13)$. Then by Atlas [6], $N_v \cong \mathbb{Z}_7$ or D_{14}. Since $N_v \trianglelefteq A_v$, we have that A_v is solvable by Proposition 2.2. Note that $C_A(N) \cong C_A(N)N/N \trianglelefteq A/N$ and $|A/N : A_vN/N| \leq 2$. Thus, $C_A(N)$ is solvable. By our assumption, $C_A(N) = 1$ and by "N/C-Theorem", $A \cong A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,13)$. By Construction 2.8 and Lemma 2.9, $X \cong \mathcal{CG}_{156}^i$ with $i = 1,2,3,4$.

For the remaining simple groups listed in the above, with the calculation of Magma [3] or check the information of maximal subgroups in Atlas [6], we can deduce that all the groups do not have subgroups of order $|N_v| = |N|/6{\cdot}p$ or $|N|/12{\cdot}p$, a contradiction. □

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References

[1] N. Biggs, *Algebraic graph theory,* Second ed., Cambridge University Press, Cambridge, 1993.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Elsevier Science Ltd, New York, 1976.

[3] W. Bosma, C. Cannon, C. Playoust, *The MAGMA algebra system I: The user language*, J. Symbolic Comput., 24 (1997), 235-265.

[4] Y. Cheng, J. Oxley, *On the weakly symmetric graphs of order twice a prime*, J. Combin. Theory B, 42 (1987), 196-211.

[5] M.D.E. Conder, *A complete list of all connected symmetric graphs of order* 2 *to* 30*,* `https://www.math.auckland.ac.nz/conder/symmetricgraphs-orderupto30.txt`.

[6] H.J. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A Wilson, *Atlas of finite group*, Clarendon Press, Oxford, 1985.

[7] X.G. Fang, C.E. Praeger, *Finite two-arc transitive graphs admitting a Suzuki simple group,* Comm. Algebra, 27 (1999), 3727-3754.

[8] Y.Q. Feng, J.H. Kwak, K.S. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$, European J. Combin., 26 (2005), 1033-1052.

[9] D. Gorenstein, *Finite simple groups,* Plenum Press, New York, 1982.

[10] S.T. Guo, *Heptavalent symmetric graphs of order* $6p$, Ita. J. Pure Appl. Math., 38 (2017), 741-750.

[11] S.T. Guo, *Heptavalent symmetric graphs of order* $18p$, Submitted.

[12] S.T. Guo, Y.Q. Feng, *A note on pentavalent s-transitive graphs*, Discrete Math., 312 (2012), 2214-2216.

[13] S.T. Guo, Y.T. Li, X.H. Hua, (G, s)*-transitive graphs of valency* 7, Algeb. Colloq., 23 (2016), 493-500.

[14] S.T. Guo, J. Shi, Z.J. Zhang, *Heptavalent symmetric graphs of order* $4p$*,* South Asian J. Math., 1 (2011), 131-136.

[15] B. Huppert, *Eudiche Gruppen I*, Springer-Verlag, Berlin, 1967.

[16] A. Jafarzadeh, A. Iranmanesh, *On simple K_n-groups for $n = 5, 6$, (In Campbell, CM, Quick, MR, Robertson, EF, Smith, GC, eds.)* Groups St. Andrews 2005. London Mathematical Society lecture note series, vol. 2, pp. 668-680, Cambridge University Press , Cambridge, (2007).

[17] B. Ling, C.X. Wu, B. Lou, *Pentavalent symmetric graphs of order* $30p$, Bull. Aust. Math. Soc., 90 (2014), 353-362.

[18] P. Lorimer, *Vertex-transitive graphs: Symmetric graphs of prime valency,* J. Graph Theory, 8 (1984), 55-68.

[19] B.D. Mckay, *Transitive graphs with fewer than 20 vertices*, Math. Comp., 33 (1979), 1101-1121.

[20] R.C. Miller, *The trivalent symmetric graphs of girth at most six*, J. Combin. Theory B, 10 (1971), 163-182.

[21] J. Pan, B. Lou, C. Liu, *Arc-transitive pentavalent graphs of order* $4pq$, Electron. J. Comb., 20 (2013), 1215-1230.

[22] J.M. Pan, X. Yu, *Pentavalent Symmetric Graphs of Order Twice a Prime Square*, Algeb. Colloq., 22 (2015), 383-394.

[23] P. Potočnik, *A list of* 4*-valent* 2*-arc-transitive graphs and finite faithful amalgams of index* $(4, 2)$, European J. Combin., 30 (2009), 1323-1336.

[24] D.J. Robinson, *A Course in the theory of groups*, Springer-Verlag, New York, 1982.

[25] B.O. Sabidussi, *Vertex-transitive graphs*, Monatsh Math., 68 (1964), 426-438.

[26] W.J. Shi, *On simple K4-groups*, Chinese Science Bull, 36 (1991), 1281-1283 (in Chinese).

[27] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

[28] D.W. Yang, Y.Q. Feng, *Pentavalent symmetric graphs of order* $2p^3$, J. Korean Math. Soc., 18 (2016), 1-18.

[29] D.W. Yang, Y.Q. Feng, J.L. Du, *Pentavalent symmetric graphs of order* $2pqr$, Discrete Math., 339 (2016), 522-532.

[30] J.X. Zhou, *Tetravalent s-transitive graphs of order* $4p$, Discrete Math., 309 (2009), 6081-6086.

[31] J.X. Zhou, Y.Q. Feng, *Tetravalent s-transitive graphs of order twice a prime power*, J. Aust. Math. Soc., 88 (2010), 277-288.

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Existence result for a class of functional integral equations via the measure of non-compactness and applications

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Abstract. In the present paper, by expedient assumptions and using the meaning of measure of non-compactness and essential fixed point theorems such as Darbo's theorem, we present an existence of solutions for some nonlinear functional integral equations on $C[0,T]$. Our existence results subtend many key integral and functional equations that emerge in nonlinear analysis and its applications. We show applications of the obtained results for specific scenarios of known equations.

Keywords: nonlinear functional integral equations, measure of non-compactness, fixed point theorem, Darbo condition, Banach algebra

1. Introduction

Some problems investigated in the vehicular traffic theory, queuing theory, biology and etc., lead to functional integral equations, see [7, 13]. For instance, the most important frequently considered integral equations, in linear or its nonlinear counterparts cases, are the Hammerstein integral equation and its generalization the Urysohn integral equation.

In this study, we discuss on some of these equations in the following general form:

$$\varphi(t) = \psi(t, \varphi(t)) + \int_0^t \rho(t, \tau, \varphi(\tau)) d\tau, \tag{1}$$

where $t \in [0,T]$. Many important integral equations, for example; the nonlinear Volterra integral equations such as the Urysohn type integral equation, which were considered in many papers and monographs [7, 11, 13, 14], are especial cases of Eq. (1). The main tool for study on existence of solutions for functional integral equations (1) is the fixed point theorem which satisfies the Darbo condition [1, 2, 8, 9], with respect to a measure of non-compactness in the Banach algebra of continuous functions on the interval $[0,T]$. Fixed point theorems have many important applications in nonlinear analysis literature [1, 9, 12, 10, 15].

The rest of this paper is organized as follows. Section 2 is devoted to collect some definitions and auxiliary results, which will be used in this work. In Section 3, applying the technique associated with measure of non-compactness, we prove an existence theorem for Eq. (1). In the last section, we present some examples that verify the application of our reviewed results for nonlinear functional integral equations.

2. Notation and auxiliary facts

We recall some basic definitions and properties of measure of non-compactness [2, 3, 5, 6, 8], which are utilized for obtaining our main results.

Definition 1. *Let (X, d) be a complete metric space and B the family of nonempty and bounded subsets of X. A function ξ, defined on B in $\mathbb{R}_+ = [0, \infty)$, is called the measure of non-compactness on X if $\xi(\overline{co}(\mathcal{B})) = \xi(\mathcal{B})$ for all bounded subsets $\mathcal{B} \in B$, where $\overline{co}(\mathcal{B})$ denotes the convex closure of $\mathcal{B}$.*

For $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2 \in B$, the measure of non-compactness ξ is said to be:

1. Monotone: if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ implies $\xi(\mathcal{B}_1) \subseteq \xi(\mathcal{B}_2)$;
2. Regular: if $\xi(\mathcal{B}) = 0$ is equivalent to the relative compactness of $\mathcal{B}$ in X;
3. Nonsingular: if $\xi(\mathcal{B}) = 0$ for every finite set $\mathcal{B}$;
4. Semi-additive: if $\xi(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\xi(\mathcal{B}_1), \xi(\mathcal{B}_2)\}$.

As an example of a measure of non-compactness, possessing all above properties, we may consider the non-compactness measure of Hausdorff ξ_1 defined on $\mathcal{B} \in B$ as follows,

$$\xi_1(\mathcal{B}) = \inf\{\varepsilon > 0 : \mathcal{B} \text{ can be covered by a finite number of balls with radii} \leq \varepsilon\}.$$

Without confusion, the Kuratowski measure of non-compactness ξ_2 defined by

$$\xi_2(\mathcal{B}) = \inf\{\varepsilon > 0 : \mathcal{B} \text{ can be covered by finitely many sets with diameter} \leq \varepsilon\},$$

where the diameter of $\mathcal{B}_i$ is defined by $diam\mathcal{B}_i = \sup\{|b_1 - b_2| : b_1, b_2 \in \mathcal{B}_i\}, i = 1, 2, \ldots, m$.
It is well-known that the Kuratowski measure of non-compactness ξ_2, verifies properties 1-4, as well as the Hausdorff measure of non-compactness.

If X is a Banach space, then for the Kuratowski or Hausdorff measure of non-compactness ξ, we also have:

5. Semi-homogeneity: $\xi(\alpha\mathcal{B}) = |\alpha|\xi(\mathcal{B})$, where $\alpha \in \mathbb{R}$ and $\alpha\mathcal{B} = \{\alpha b : b \in \mathcal{B}\}$;
6. Algebraic semi-additivity: $\xi(\mathcal{B}_1 + \mathcal{B}_2) \leq \xi(\mathcal{B}_1) + \xi(\mathcal{B}_2)$, where $\mathcal{B}_1 + \mathcal{B}_2 = \{b_1 + b_2 : b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$;

7. Lipschitzianity: $|\xi(\mathcal{B}_1) - \xi(\mathcal{B}_2)| \leq L_\xi d_l(\mathcal{B}_1, \mathcal{B}_2)$, where $L_{\xi_1} = 1, L_{\xi_2} = 2$ and $d_l(\mathcal{B}_1, \mathcal{B}_2)$ denotes the Hausdorff metric of $\mathcal{B}_1$ and $\mathcal{B}_2$, i.e.

$$d_l(\mathcal{B}_1, \mathcal{B}_2) = \max\{\sup_{t\in\mathcal{B}_2} d(t, \mathcal{B}_1), \sup_{t\in\mathcal{B}_1} d(t, \mathcal{B}_2)\},$$

here $d(.,.)$ is the distance from an element of X to a set of X;

8. Continuity: for every $\mathcal{B} \in B$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|\xi(\mathcal{B}) - \xi(\mathcal{B}_1)| < \varepsilon$ for all $\mathcal{B}_1$ satisfying $d_l(\mathcal{B}, \mathcal{B}_1) < \delta$.

Theorem 1. *The relationship of the Hausdorff measure of non-compactness ξ_1, and the Kuratowski measure of non-compactness ξ_2, on $\mathcal{B} \in B$ is*

$$\xi_1(\mathcal{B}) \leq \xi_2(\mathcal{B}) \leq 2\xi_1(\mathcal{B}).$$

The expressed property allows us to characterize solutions of Eq. (1), and will be used in the next section. Further facts concerning measures of non-compactness and their properties may be found in [2, 5].

Theorem 2 (Darbo fixed point theorem) ([2, 9])**.** *Let $D \neq \emptyset$ be a bounded, closed and convex subset of a Banach space X, ξ_2 be the Kuratowski measure of non-compactness on X and suppose that $\Lambda : D \to D$ is a continuous operator such that there exists a constant $\eta \in [0, 1)$ with $\xi_2(\Lambda\Phi) \leq \eta\xi_2(\Phi)$ for all $\Phi \in D$. Then Λ has a fixed point in D.*

In what follows, we will work in the classical Banach space $C[0, T]$, consisting of the all real and continuous functions defined on the interval $[0, T]$. This space is equipped with the standard(uniform) norm

$$\|x\| = \sup\{|x(t)| : \ t \in [0, T]\}.$$

Obviously, the space $C[0, T]$ also has the structure of a Banach algebra.

Now, we present the definition of a special measure of non-compactness in $C[0, T]$ which will be used in the sequel, a measure that was introduced and studied in [4, 11]. To do this, let us fix a subset Φ belong to the family of all nonempty and bounded subsets of $C[0, T]$. For $\varepsilon > 0$ and $\varphi \in \Phi$ denote by $\mu(\varphi, \varepsilon)$ the *modulus of continuity* of φ defined by

$$\mu(\varphi, \varepsilon) = \sup\{|\varphi(t_1) - \varphi(t_2)| : \ |t_1 - t_2| \leq \varepsilon, \ t_1, t_2 \in [0, T]\}.$$

Further, we set

$$\mu(\Phi, \varepsilon) = \sup\{\mu(\varphi, \varepsilon) : \varphi \in \Phi\},$$

$$\mu_0(\Phi) = \lim_{\varepsilon \longrightarrow 0} \mu(\Phi, \varepsilon),$$

the function $\mu_0(\Phi)$ is a regular measure of non-compactness in the space $C[0, T]$.

3. Main result

According to the announcement given in the Introduction, we study the solvability of the nonlinear functional integral equations (1) for $\varphi \in C[0,T]$. The assumptions are formulated for Eq. (1), namely, we assume the following hypotheses.

(H_1) $\psi : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist nonnegative constants β such that $|\psi(t,0)| \leq \beta$, for $t \in [0,T]$. Also, exists the continuous function $\psi_1 : [0,T] \longrightarrow [0,T]$ such that

$$|\psi(t,\varphi_1) - \psi(t,\varphi_2)| \leq \psi_1(t)|\varphi_1 - \varphi_2|,$$

and let $\kappa = \max\{|\psi_1(t)| : t \in [0,T]\}$.

(H_2) $\rho(t,\tau,\varphi) : [0,T] \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies in sublinear condition, i.e.

$$|\rho(t,\tau,\varphi)| \leq \gamma + \delta|\varphi|,$$

for $\gamma, \delta \in \mathbb{R}_+$, $t, \tau \in [0,T]$ and $\varphi \in \mathbb{R}$.

(H_3) $\kappa < 1 - \delta T$.

Theorem 3. *Under the tacit assumptions $(H_1) - (H_3)$ above, Eq. (1) has at least one solution in the Banach algebra $C = C[0,T]$.*

Proof. To prove this result, we need to define operators Λ on the space C[0, T] in the following way

$$(\Lambda\varphi)(t) = \psi(t,\varphi(t)) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau.$$

So, Λ transforms the Banach algebra $C[0,T]$ into itself. Let us fix $\varphi \in C[0,T]$, then using our assumptions for $t \in [0,T]$, we get

$$\begin{aligned}
|(\Lambda\varphi)(t)| &= \left|\psi(t,\varphi(t)) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq |\psi(t,\varphi(t)) - \psi(t,0)| + |\psi(t,0)| + \left|\int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq \kappa|\varphi(t)| + \beta + \left|\int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq \kappa|\varphi(t)| + \beta + T(\gamma + \delta|\varphi(t)|) \\
&\leq (\kappa + \delta T)|\varphi| + (\beta + \gamma T).
\end{aligned}$$

Obviously, in view of the assumptions $(H_1) - (H_3)$, we have $\beta + \gamma T < \infty$ and $\kappa + \delta T < 1$, also the operator Λ transforms $\mathcal{B}_r$ into it self for $r = \frac{\beta+\gamma T}{1-(\kappa+\delta T)}$. Where $\mathcal{B}_r(\theta)$ denotes the closed ball with radius r centered at θ as the zero element of $C[0,T]$.

Next, we show that the operator Λ is continuous on the ball $\mathcal{B}_r(\theta)$. To do this, fix $\varepsilon > 0$ and take arbitrary $\varphi_1, \varphi_2 \in \mathcal{B}_r(\theta)$ such that $\|\varphi_1 - \varphi_2\| \leq \varepsilon$. Then for $t \in [0, T]$, we get

$$\begin{aligned} |(\Lambda\varphi_1)(t) - (\Lambda\varphi_2)(t)| &= \Big|\psi(t, \varphi_1(t)) + \int_0^t \rho(t, \tau, \varphi_1(\tau))d\tau - \psi(t, \varphi_2(t)) \\ &\quad - \int_0^t \rho(t, \tau, \varphi_2(\tau))d\tau\Big| \\ &\leq \kappa|\varphi_1 - \varphi_2| + \int_0^t |\rho(t, \tau, \varphi_1(\tau)) - \rho(t, \tau, \varphi_2(\tau))|d\tau \\ &\leq \kappa\varepsilon + T \cdot \mu(\rho, \varepsilon), \end{aligned}$$

where

$$\mu(\rho, \varepsilon) = \sup\{|\rho(t, \tau, \varphi_1) - \rho(t, \tau, \varphi_2)| : |\varphi_1 - \varphi_2| \leq \varepsilon, \varphi_1, \varphi_2 \in [-r, r], t, \tau \in [0, T]\}.$$

On the other hand, using the uniformly continuous of the function $\rho = \rho(t, \tau, \varphi)$ on the bounded subset $[0, T] \times [0, T] \times [-r, r]$, we infer that $\mu(\rho, \varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus, the above assessment shows that the operator Λ is continuous on $\mathcal{B}_r(\theta)$.

Now, we prove that the operator Λ satisfies the Darbo condition with respect to the measure μ_0, defined in Section 2, on the ball $\mathcal{B}_r(\theta)$. Take a nonempty subset Φ of $\mathcal{B}_r(\theta)$ and $\varphi \in \Phi$, then for a fixed $\varepsilon > 0$ and $t_1, t_2 \in [0, T]$ such that, without loss of generality, we may assume that $t_1 \leq t_2$ and $t_2 - t_1 \leq \varepsilon$, we obtain

$$\begin{aligned} &|(\Lambda\varphi)(t_2) - (\Lambda\varphi)(t_1)| \\ &\quad = \Big|\psi(t_2, \varphi(t_2)) + \int_0^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau - \psi(t_1, \varphi(t_1)) \\ &\qquad - \int_0^{t_1} \rho(t_1, \tau, \varphi(\tau))d\tau\Big| \\ &\quad \leq |\psi(t_2, \varphi(t_2)) - \psi(t_2, \varphi(t_1))| + |\psi(t_2, \varphi(t_1)) - \psi(t_1, \varphi(t_1))| \\ &\qquad + \left|\int_0^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau - \int_0^{t_1} \rho(t_1, \tau, \varphi(\tau))d\tau\right| \\ &\quad \leq \kappa|\varphi(t_2) - \varphi(t_1)| + \mu_\psi(\varepsilon, .) \\ &\qquad + \left|\int_0^{t_1} \Big(\rho(t_2, \tau, \varphi(\tau)) - \rho(t_1, \tau, \varphi(\tau))\Big)d\tau + \int_{t_1}^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau\right| \\ &(2) \qquad \leq \kappa|\varphi(t_2) - \varphi(t_1)| + \mu_\psi(\varepsilon, .) + T\mu_\rho(\varepsilon, ., .) + M\varepsilon. \end{aligned}$$

For simplicity, we employ of the following notations

$$\mu_\psi(\varepsilon, .) = \sup\left\{|\psi(t, \varphi) - \psi(\hat{t}, \varphi)| : \ \varphi \in [-r, r], \ |t - \hat{t}| \leq \varepsilon, \ t, \hat{t} \in [0, T]\right\},$$

$$\mu_\rho(\varepsilon, ., .) = \sup\left\{|\rho(t, \tau, \varphi) - \rho(\hat{t}, \tau, \varphi)| : \ \varphi \in [-r, r], \ |t - \hat{t}| \leq \varepsilon, \ t, \hat{t}, \tau \in [0, T]\right\},$$

$$M = \sup\left\{|\rho(t, \tau, \varphi)| : \ \varphi \in [-r, r], \ t, \tau \in [0, T]\right\}.$$

Then, using relation (2), yields

$$\mu(\Lambda\varphi,\varepsilon) \leq \kappa\mu(\varphi,\varepsilon) + \mu_{\psi}(\varepsilon,.) + T\mu_{\rho}(\varepsilon,.,.) + M\varepsilon.$$

As in because our assumptions, we conclude that the functions $\psi = \psi(t,\varphi)$, $\rho = \rho(t,\tau,\varphi)$ are uniformly continuous on the sets $[0,T]\times\mathbb{R}$ and $[0,T]\times[0,T]\times\mathbb{R}$, respectively. Hence, we figure out $\mu_{\psi}(\varepsilon,.) \longrightarrow 0$, and $\mu_{\rho}(\varepsilon,.,.) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Consequently, we get

$$\mu_0(\Lambda\Phi) \leq \kappa\mu_0(\Phi). \tag{3}$$

Finally, we understand the operator Λ satisfies the Darbo condition on $\mathcal{B}_r(\theta)$ respect to the measure μ_0 with constant κ. Also, under the assumption (H_3), we know that $\kappa < 1$ and operator Λ is a contraction on $\mathcal{B}_r(\theta)$ with respect to μ_0. Therefore, applying Theorem 2 we conclude that Λ has at least one fixed point in $\mathcal{B}_r(\theta)$. Consequently, functional integral equations (1) has at least one solution in $\mathcal{B}_r(\theta)$. This completes the proof. □

4. Applications

In this section, we present some examples of the classical integral and functional equations considered in nonlinear analysis, which are particular cases of Eq. (1) and consequently, the existence of their solutions can be established using Theorem 3.

Example 1. Setting $\psi(t,s) = f(t)$, Eq. (1) reduces to the well-known nonlinear Volterra-Urysohn integral equation

$$\varphi(t) = f(t) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau. \tag{4}$$

The equation of this type appears in many applications. For example, it can be applied in modeling the some problems in engineering, economics and physics. Also, a lot of problems considered in the theory of partial differential equations lead us to the Urysohn integral equation.

Example 2. In Eq. (4) if we set $\rho(t,\tau,\varphi(\tau)) = k(t,\tau)v(\tau,\varphi(\tau))$, a nonlinear integral equation will be appeared known as the Hammerstein type. Solution of many problems in mathematical physics, control theory, the dynamic model of chemical reactor, and studying of an elliptic partial differential equation with nonlinear boundary conditions, encounter with this equation type.

Example 3. This example shows that in general the assumptions of our existence Theorem 3, can be easily verified in objective situations. For the following

functional integral equations

$$\varphi(t) = \frac{t^2}{4}\exp(-t) + \int_0^t \frac{(t-\tau)^2}{4}\exp(\tau - t)\varphi(\tau)d\tau,$$

$$\varphi(t) = \sqrt{t} + 0.1 - t\sin t + \int_0^t \frac{\sin t\varphi^2(\tau)}{(\sqrt{\tau}+0.1)^2}d\tau,$$

$$\varphi(t) = \frac{1}{3}\sin(3\varphi(t)) + \frac{t}{t^3-1}\int_0^t \frac{\exp(-t\tau + \tau^2\varphi^5(\tau))}{1+\varphi^2(\tau)}d\tau,$$

$$\varphi(t) = \cos(t\varphi(t)) + \int_0^t \left(\ln(\tau t^2\varphi^2(\tau)) - \frac{\tan^{-1}(\tau + 2t^2 + \varphi^2(\tau))}{5 - \sqrt{|\sin(\varphi(\tau))|}}\right)d\tau,$$

the hypothesis of Theorem 3 are satisfied.

References

[1] Ravi P. Agarwal, E. Karapinar, D. O'Regan, A.F. Roldán-López-deHierro, *Fixed point theory in metric type spaces*, Springer, Country Cham, Switzerland, 2015.

[2] J.M. Ayerbe Toledano, T. Dominguez Benavides, G. Lopez Acedo, *Measures of noncompactness in metric fixed point theory*, Birkhäuser, Basel, 1997.

[3] J. Banaś, *Measures of noncompactness in the study of solutions of nonlinear differential and integral equations*, Cent. Eur. J. Math., 10 (2012), 2003-2011.

[4] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[5] J. Banaś, M. Mursaleen, *Sequences spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New Delhi, 2014.

[6] J. Banaś, R. Nalepa, *On a measure of noncompactness in the space of functions with tempered increments*, J. Math. Anal. Appl., 435 (2016), 1634-1651.

[7] J. Banaś, B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl., 284 (2003), 165-173.

[8] J. Cao, Q. Tong, X. Huang, *Nonlocal fractional functional differential equations with measure of noncompactness in Banach space*, Math. Sci., 9 (2015), 59-69.

[9] Chen, J., Tang, X., *Generalizations of Darbo's fixed point theorem via simulation functions with application to functional integral equations*, J. Comput. Appl. Math., 296 (2016), 564-575.

[10] Jagan Mohan, J., Shobanadevi, N., Deekshitulu, G.V.S.R., *Stability of nonlinear nabla fractional difference equations using fixed point theorems*, Ital. J. Pure Appl. Math., 32 (2014), 165-184.

[11] Maleknejad, K., Nouri, K., Mollapourasl, R., *Investigation on the existence of solutions for some nonlinear functional-integral equations*, Nonlinear Anal., 71 (2009), 1575-1578.

[12] K. Nouri, M. Nazari, B. Keramati, *Existence results for a coupled system of fractional integro-differential equations with time-dependent delay*, J. Fixed Point Theory Appl., 68 (2017), 2927-2943.

[13] D. O'Regan, *Existence theory for nonlinear Volterra integrodifferential and integral equations*, Nonlinear Anal., 31 (1998), 317-341.

[14] D. O'Regan, M. Meehan, *Existence theory for nonlinear integral and integrodifferential equations*, Kluwer Academic, Dordrecht, 1998.

[15] H.C. Wu, C.Z. Cheng, W.J. Han, *Strong convergence theorems for fixed point problems and equilibrium problems with applications*, Ital. J. Pure Appl. Math., 33 (2014), 161-174.

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Application of scaling group transformation on viscoelastic fluid with Cattaneo Christov heat flux model

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Abstract. In this paper Cattaneo Christov heat flux model is considered to analyze the heat transfer in radiated Maxwell fluid with velocity slip and temperature jump condition over a stretching plate and the Lie symmetry group transformations are used to convert the boundary layer equations into non-linear ordinary differential equations. The dimensionless governing equations are solved numerically using Bvp4c with MATLAB, which is a collocation method equivalent to the fourth order mono-implicit Runge-Kutta method. The effects of some physical parameters, such as elasticity number, velocity slip coefficient, the relaxation time of the heat flux, thermal slip parameter, radiation parameter and the Prandtle number on velocity and temperature fields are analyzed.

Keywords: scaling group transformation, UCM fluid, Cattaneo Christov heat flux.

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1. Introduction

The mechanism of heat transfer in different scenarios is successfully described through a famous heat flux model given by Fourier [1] in 1822. This model is an empirical formula, it gives a way to investigate transfer of heat phenomenon and provide the basis to study heat conduction in the successive two centuries. This model has a drawback that it produces a parabolic energy equation which means that initial disturbance is instantly experienced by the medium under investigation. To minimize this drawback, a modification to Fourier's Law is given by Cattaneo [2] by introducing relaxation time for heat flux and in result it produces hyperbolic energy equation that permits heat transportation through promulgation of thermal waves with fixed speed. The process of such heat transportation has useful applications that span from nanofluid flows to the modeling of skin burn injury (see Tibullo and Zampoli [3]). Christove [4] modified the Maxwell Catteneo model to maintain the material invariant formulation by using the Oldroyd's upper-convected derivative and this modification in literature is known as Cataneo Christov heat flux model. Uniqueness of the solution's for the Cattaneo Christov model is proved by Ciarletta and Straughan [5]. Cattaneo-Christov Model is discussed in the study of slip flow and transfer of heat through Maxwell fluid by Han et al [6]. Cattaneo Christov model in rotating flow with upper convected Maxwell fluid is studied by Mustafa [7]. Recently Hayat et al [8] applied the Cattaneo Christov heat flux model on magneto hydrodynamic flow of Oldroyd B fluid with homogeneous-heterogeneous reactions.

In the last few decades, researchers have given more attention to study Maxwell fluid due to its vast applications in industry. Maxwell fluid is an important class of viscoelastic fluid. An effort has been put by different researchers to investigate the flow of upper convected Maxwell fluid. Choi et al [9] investigated the effects of suction, viscoelasticity and inertia in porous medium. Sadeghy et al [10] studied Sakiadis flow and showed that wall skin friction decreases by increasing the value of Deborah number. Magnetohydrodynamic flows in porous medium were analyzed in [11, 12, 13, 14] and stagnation point flows of UCM model were discussed in [15, 16, 17, 18]. Hayat et al [19] discussed the effects of MHD and chemical reaction on boundary layer flow over a shirking sheet. Akber et al [20] discussed viscous dissipation, thermal radiation and heat transfer of an upper convected Maxwell fluid.

Lie group analysis, also called symmetry analysis, was developed by Sophus Lie to find point transformations which map a given differential equation to itself. This method has been used by many investigators to analyze various flow phenomenon arising in fluid mechanics [21, 22, 23, 24, 25, 26, 27, 28]. Moreover, a special form of Lie group transformations is the scaling group technique and it has been applied by many researchers [29, 30, 31, 32, 33, 34, 35, 36, 37] to study different flow models over different nonlinear dynamical systems, aerodynamics, and some other engineering branches. For example, Rashidi et al [38] investi-

gated free convective flow of a nanofluid over a chemically reacting horizontal plate in a porous media using scaling group of transformation. Cao et al [39] studied the MHD effects on the convectively heated stretching porous surface with the heat source/sink. Aziz et al [40] used this method to studied MHD flow over an inclined radiating plate with the temperature dependent thermal conductivity, variable reactive index and heat generation.

In this paper, we shall focus on the application of scaling group of transformation to investigate the coupled flow and heat transfer of viscoelastic fluid with Cattaneo Christov heat flux model, velocity slip and temperature jump conditions by considering the radiation effects. By implementing appropriate scaling group of transformation, the set of governing partial differential equations are converted into a set of ordinary differential equations along with boundary conditions. The solutions of the transformed system of ordinary differential equations is then solved numerically and shown the results graphically.

2. Formulation

Consider a two dimensional steady boundary layer flow of the Maxwell fluid over a plate. For present problem the governing boundary layer equations for expressing conservation of mass and momentum can be written as [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{2.1}$$

$$\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}} + \lambda_1 \left(\overline{u}^2 \frac{\partial^2 \overline{u}}{\partial \overline{x}^2} + \overline{v}^2 \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} + 2\bar{u}\bar{v}\frac{\partial^2 \overline{u}}{\partial \overline{x}\partial \overline{y}} \right) = \nu \frac{\partial^2 \overline{u}}{\partial \overline{y}^2}. \tag{2.2}$$

The boundary conditions with velocity slip on boundary are

$$\overline{u} = a\overline{x} + \lambda_0 \frac{2-\sigma_\nu}{\sigma_\nu}\frac{\partial \overline{u}}{\partial \overline{y}},\ and \quad \overline{v} = 0\ at \quad \overline{y} = 0,\ \overline{u} \to 0\ as \quad \overline{y} \to \infty, \tag{2.3}$$

where α is positive constant, σ_ν is the tangential momentum accommodation coefficient and λ_0 is the molecular mean free path. We use the Cattaneo–Christov heat flux model to study heat transfer with radiation effects of the Maxwell fluid and the heat flux model is of the following form [4]

$$\mathbf{q} + \lambda_2 \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V}.\nabla\mathbf{q} - \mathbf{q}.\nabla\mathbf{V} + (\nabla.\mathbf{V})\mathbf{q} \right) = -k\nabla\overline{T}. \tag{2.4}$$

Where $\mathbf{q}$ is the heat flux, λ_2 is the relaxation time of the heat flux, T is the temperature of the Maxwell fluid and k is the thermal conductivity. $\mathbf{V} = (\overline{u},\ \overline{v})$ is the velocity vector of the Maxwell fluid. When $\lambda_2 = 0$, Eq. (2.4) is simplified to Fourier's law. Since the fluid is incompressible, which satisfies $\nabla.\mathbf{V} = 0$. Eq. (2.4) becomes the following form [3]

$$\mathbf{q} + \lambda_2 \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V}.\nabla\mathbf{q} - \mathbf{q}.\nabla\mathbf{V} \right) = -k\nabla\overline{T}. \tag{2.5}$$

The energy balance equation for the steady boundary layer flow in the presence of thermal radiation[20] is

$$\rho C_p \mathbf{V}.\nabla\overline{T} = -\nabla.\mathbf{q} - \frac{\partial q_r}{\partial y}. \tag{2.6}$$

where C_p = specific heat at constant pressure; and q_r = radiative heat flux. Using the Rosseland approximation [41] for thermal radiation, the radiative heat flux is simplified as

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial \overline{y}} = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial \overline{T}}{\partial \overline{y}}. \tag{2.7}$$

Eliminating $\mathbf{q}$ between Eqs. (2.5) and (2.6) and on using eq. (2.7), we can obtain the temperature governing equation for the steady flow as

$$\begin{aligned} &\overline{u}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{T}}{\partial \overline{y}} + \lambda_2\left(\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}\frac{\partial \overline{T}}{\partial \overline{y}} + \overline{u}\frac{\partial \overline{v}}{\partial \overline{x}}\frac{\partial \overline{T}}{\partial \overline{y}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}\frac{\partial \overline{T}}{\partial \overline{x}}\right.\\ &\left.+2\overline{uv}\frac{\partial^2 \overline{T}}{\partial \overline{x}\partial \overline{y}} + \overline{u}^2\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \overline{v}^2\frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\right)\\ &= \alpha\frac{\partial^2 \overline{\overline{T}}}{\partial \overline{y}^2} + \frac{16\sigma^* T_\infty^3}{3kk^*}\frac{\partial^2 \overline{T}}{\partial \overline{y}^2}. \end{aligned} \tag{2.8}$$

Where $\alpha = \frac{K}{\rho C_P}$ is thermal diffusivity. The corresponding temperature jump boundary condition [42] is

$$\overline{T} = T_w + k\frac{\partial \overline{T}}{\partial \overline{y}} \quad at \quad \overline{y} = 0, \qquad \overline{T} \to T_\infty \quad as \quad \overline{y} \to \infty. \tag{2.9}$$

The following non dimensional variables can be introduced as

$$x = \frac{\overline{x}}{\sqrt{\nu/a}},\ y = \frac{\overline{y}}{\sqrt{\nu/a}},\ u = \frac{\overline{u}}{\sqrt{a\nu}},\ \ v = \frac{\overline{v}}{\sqrt{a\nu}},\ \theta = \frac{\overline{T} - T_\infty}{T_w - T_\infty}. \tag{2.10}$$

The stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y} \ \text{ and } \ v = -\frac{\partial \psi}{\partial x}$$

leads to equations (2.1), (2.2) and (2.8) in the following non dimensional form

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - \psi_{yyy} + \beta\left[(\psi_y)^2\psi_{xxy} + (\psi_x)^2\psi_{yyy} - 2\psi_x\psi_y\psi_{xyy}\right] = 0, \tag{2.11}$$

$$\begin{aligned} &\psi_y\theta_x - \psi_x\theta_y + \gamma\big[\psi_y\psi_{xy}\theta_x + \psi_x\psi_{xy}\theta_y - \psi_y\psi_{xx}\theta_y - \psi_x\psi_{yy}\theta_x - 2\psi_y\psi_x\theta_{xy}\\ &+(\psi_y)^2\theta_{xx} + (\psi_x)^2\theta_{yy}\big] - (\frac{1}{Pr} + \frac{4}{3}R)\theta_{yy} = 0. \end{aligned} \tag{2.12}$$

The boundary conditions (2.3) and (2.8) can be written as

$$\psi_y = x + b\psi_{yy},\ and\quad \psi_x = 0\ at\quad y = 0,\ \psi_y \to 0\ as\quad y \to \infty, \tag{2.13}$$
$$\theta = 1 + c\,\theta_y\quad at\ \ y = 0,\ \ \theta \to 0\ as\ \ y \to \infty,$$

where

$$b = \lambda_0 \frac{2-\sigma_\nu}{\sigma_\nu}\sqrt{\frac{a}{\nu}}, \gamma = \lambda_2 a,\ R = \frac{4\sigma^* T_\infty^3}{kk^*}\ \text{ and } c = \sqrt{\frac{a}{\nu}}k.$$

3. Lie point symmetries and similarity solutions

The symmetry groups of equations (2.11)-(2.12) are calculated using the classical Lie group approach. The one parameter infinitesimal Lie group of transformations leaving (2.11)-(2.12) invariant is defined as

$$x^* = xe^{\in a_1}, y^* = ye^{\in a_2}, \psi^* = \psi e^{\in a_3}, u^* = ue^{\in a_4}, v^* = ve^{\in a_5}, \theta^* = \theta e^{\in a_6}, \tag{3.1}$$

where the coordinates $(x,\ y,\ \psi,\ u,\ v,\ \theta) \to (x^*,\ y^*,\ \psi^*, u^*, v^*, \theta^*)$. Substituting (3.1) into (2.11) and (2.12), we get

$$\begin{aligned} &e^{\epsilon(a_1+2a_2-2a_3)}({\psi^*}_{y^*}{\psi^*}_{x^*y^*} - {\psi^*}_{x^*}{\psi^*}_{y^*y^*}) \\ &- e^{\epsilon(3a_2-a_3)}{\psi^*}_{y^*y^*y^*} + \ \beta e^{\epsilon(a_1+2a_2-2a_3)} \\ &(({\psi^*}_{y^*})^2{\psi^*}_{x^*x^*y^*} + ({\psi^*}_{x^*})^2{\psi^*}_{y^*y^*y^*} - 2{\psi^*}_{x^*}{\psi^*}_{y^*}{\psi^*}_{x^*y^*y^*}) = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &e^{\epsilon(a_1+a_2-a_3-a_6)}({\psi^*}_{y^*}{\theta^*}_{x^*} - {\psi^*}_{x^*}{\theta^*}_{y^*}) + \gamma e^{\epsilon(2a_1+2a_2-2a_3-a_6)}\big({\psi^*}_{y^*}{\psi^*}_{x^*y^*}{\theta^*}_{x^*} \\ &+ {\psi^*}_{x^*}{\psi^*}_{x^*y^*}{\theta^*}_{y^*} - {\psi^*}_{y^*}{\psi^*}_{x^*x^*}{\theta^*}_{y^*} - {\psi^*}_{x^*}{\psi^*}_{y^*y^*}{\theta^*}_{x^*} \\ &- 2{\psi^*}_{x^*}{\psi^*}_{y^*}{\theta^*}_{x^*y^*} + ({\psi^*}_{y^*})^2{\theta^*}_{x^*x^*}\big) - ({\psi^*}_{x^*})^2{\theta^*}_{y^*y^*} \\ &- (\frac{1}{Pr} + \frac{4}{3}R)e^{\epsilon(2a_2-a_6)}{\theta^*}_{y^*y^*} = 0. \end{aligned} \tag{3.3}$$

Since the system remains invariant under the group of transformations, we have$a_1 = a_3,\ a_2 = a_4 = a_6 = 0$, the characteristic equations for finding the similarity transformations would be

$$\frac{dx}{x} = \frac{dy}{0} = \frac{d\psi}{\psi} = \frac{d\theta}{0}. \tag{3.4}$$

The corresponding similarity variable and functions are

$$\eta = y,\ \psi = xF(\eta),\ \theta = \theta(\eta), \psi = \psi(\eta). \tag{3.5}$$

Substituting (3.5) into (2.11)-(2.12) obtains the following ordinary equations

$$F'''(\eta)+F(\eta)F''(\eta)-(F'(\eta))^2+\beta(2F(\eta)F'(\eta)F''(\eta)-(F(\eta))^2F'''(\eta))=0, \tag{3.6}$$

$$(\frac{1}{Pr} + \frac{4}{3}R)\theta''(\eta) + F(\eta)\theta'(\eta) - \gamma(F(\eta)F'(\eta)\theta'(\eta) + (F(\eta))^2\theta''(\eta)) = 0. \tag{3.7}$$

And the boundary conditions from Eq. (2.13) are transformed as

$$F'(\eta) = 1 + bF''(\eta),\ F(\eta) = 0\ as\ \eta = 0,\ F'(\eta) \to 0,\ \eta \to \infty,$$
$$\theta(\eta) = 1 + c\,\theta'(\eta)\quad at\ \ \eta = 0,\quad \theta(\eta) \to 0\ \ as\ \ \eta \to \infty. \tag{3.8}$$

4. Results and discussions

In this paper the nonlinear ordinary differential equations (3.6) and (3.7) along with boundary conditions (3.8) are solved numerically using BVP4C for several values of the elasticity number, slip coefficient, the relaxation time of the heat flux, Prandtle number, radiation and temperature jump parameter on velocity and temperature fields. The presence of elastic term in upper convected Maxwell model shows that it has some effects on flow and heat transfer of viscoelastic fluids. The effects of elasticity number β on the velocity and temperature distribution shown in figs. 1-3 which indicates that on increasing the value of elasticity number β increase the elastic force and for $\beta = 0$ shows that the elastic force disappears and the fluid becomes the Newtonian fluid. Figs. 1 and 2 indicates that the value of $F(\eta)$ and $F'(\eta)$ become smaller with the increase of β. This shows that the velocity boundary layer is increasingly thin with the increase of β. Fig. 3 describes that fluid temperature $\theta(\eta)$ increases with increasing the value β. This shows that the elastic force promotes heat transfer of the viscoelastic fluid. The effects of slip coefficient b on velocity distribution $F(\eta)$ and $F'(\eta)$ are shown in figs. 4 and 5. On increasing the value of slip coefficient b, the velocity distribution becomes smaller. Fig. 6 indicates the effects of γ on temperature profile and analyzed that for $\gamma = 0$ corresponding results for the Fourier's law obtained and on enhancing the value of γ temperature decreases which means that temperature in Cattaneo Christov heat flux model is smaller than the Fourier's model. Fig. 7 shows the impact of Prandtl number Pr on the temperature field $\theta(\eta)$. Prandtl number has inverse relationship with thermal diffusivity. Therefore an increase in Prandtl number reduces the temperature and enhances the thermal boundary layer thickness. Fig.8 shows that the increasing values of thermal radiation parameter R , due to this we saw enhancement in the temperature field. Similarly, the influences of temperature jump parameter c upon temperature distribution is depicted in Fig. 9; the higher values of c lead to thinner thickness of thermal boundary layer. The influence of c on the rate of heat transfer is also shown in Fig. 10. It displays that the values of $-\theta'(0)$ increase with decreasing c.

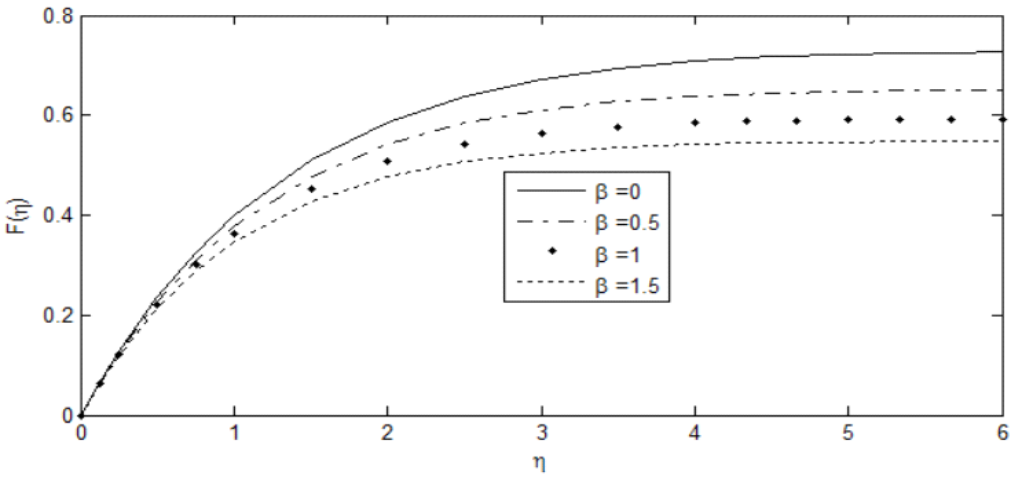


Figure 1: Profile of $F(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

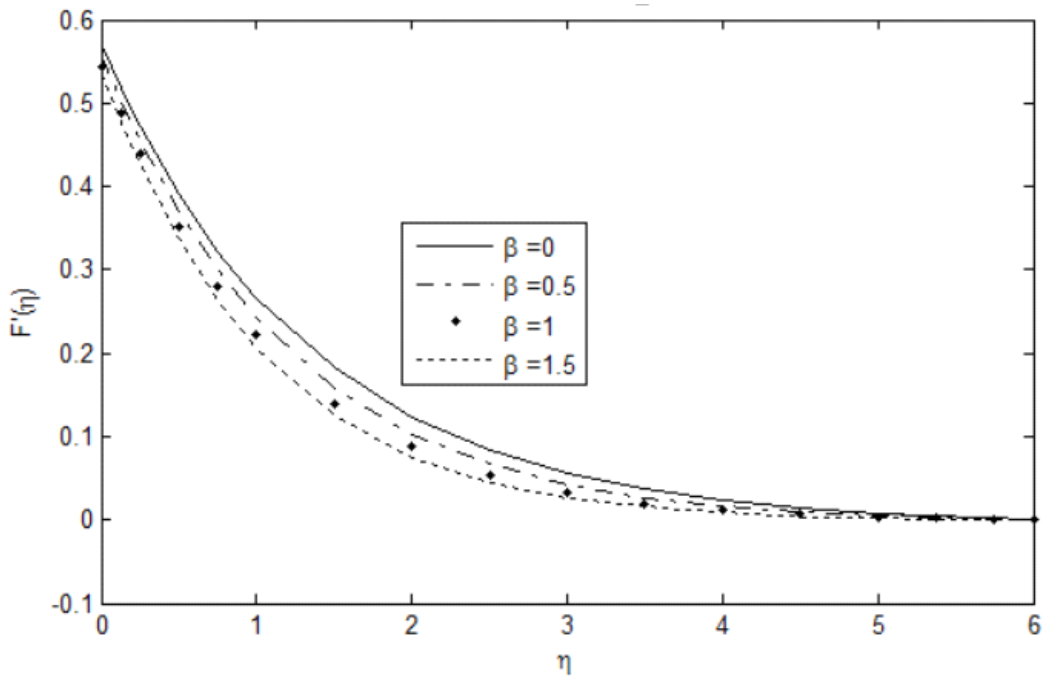


Figure 2: Profile of $F'(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

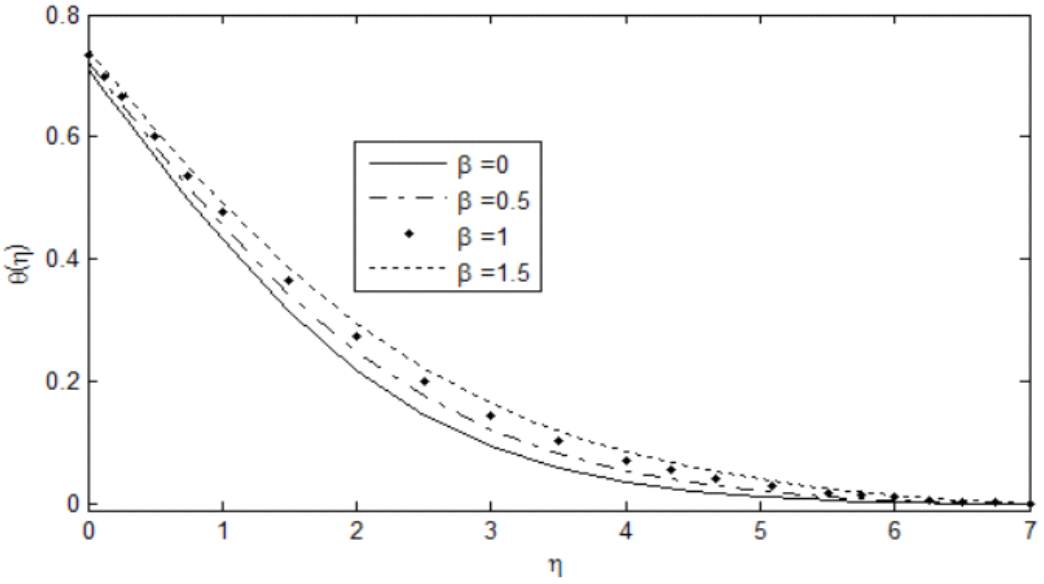


Figure 3: Profile of $\theta(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

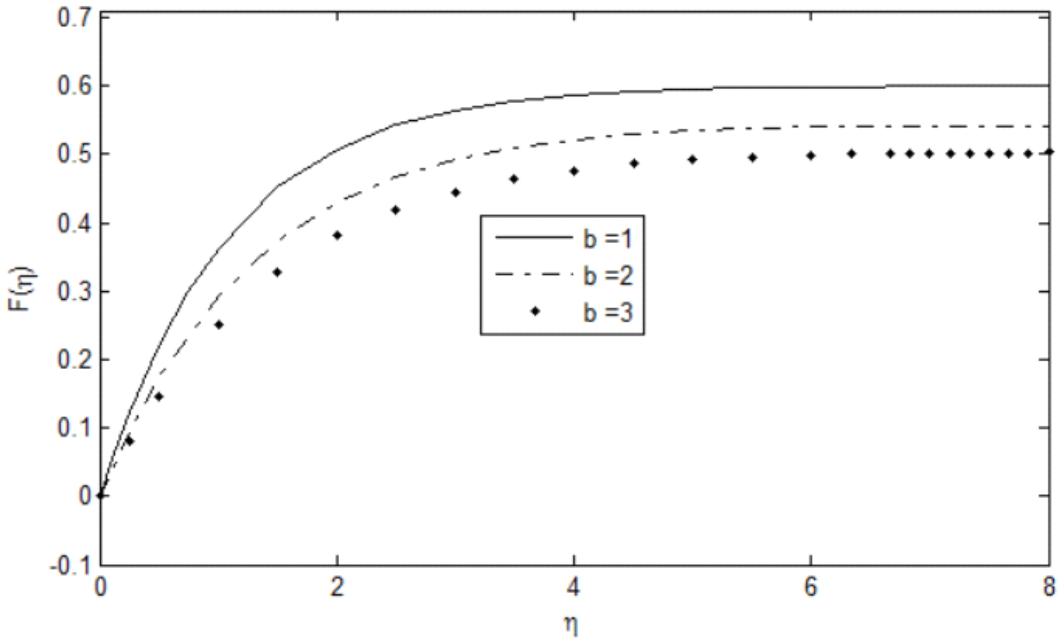


Figure 4: Profile of $F(\eta)$ for different values of b when $\beta = c = Pr = \gamma = 1, R = 0.2$.

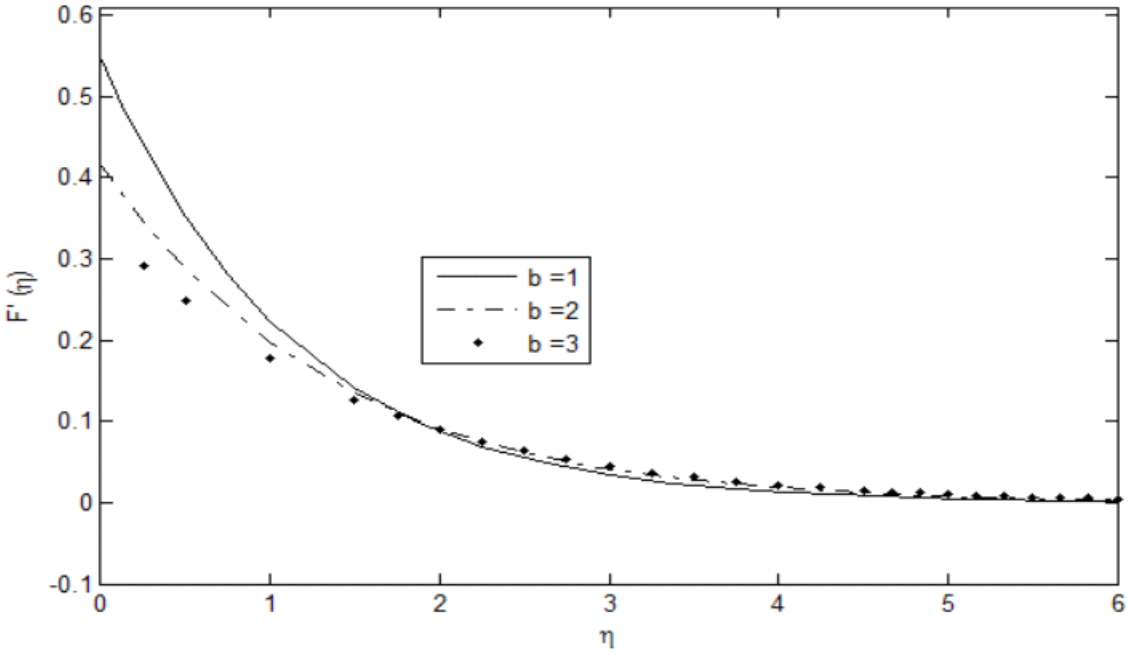


Figure 5: Profile of $F'(\eta)$ for different values of b when $\beta = c = Pr = \gamma = 1, R = 0.2$.

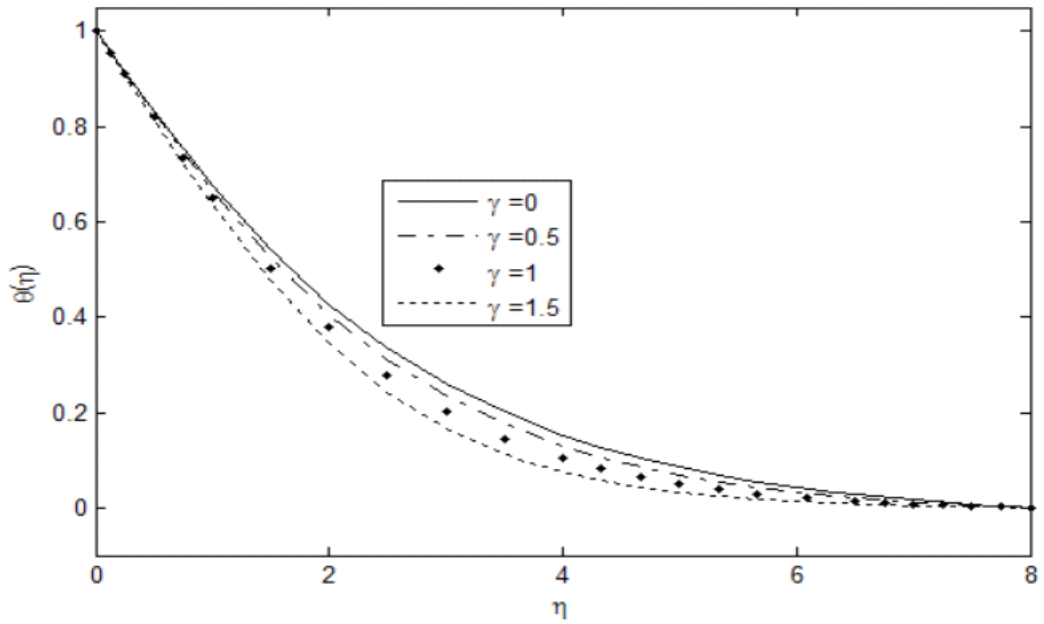


Figure 6: Profile of $\theta(\eta)$ for different values of γ when $\beta = c = Pr = b = 1, R = 0.2$.

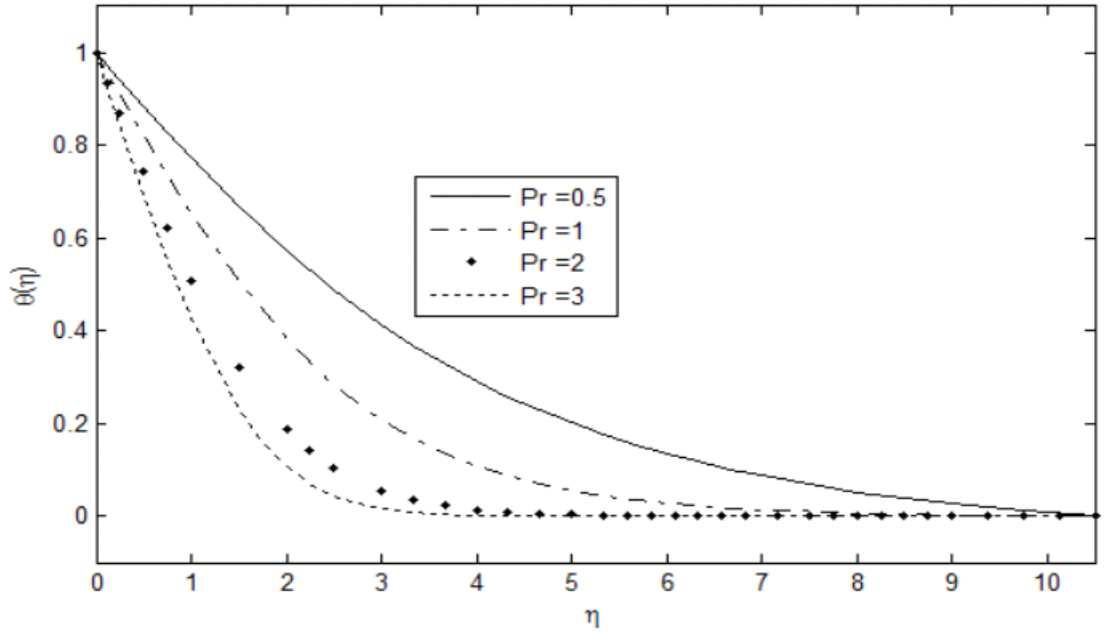


Figure 7: Profile of $\theta(\eta)$ for different values of Pr when $\beta = \gamma = c = b = 1, R = 0.2$.

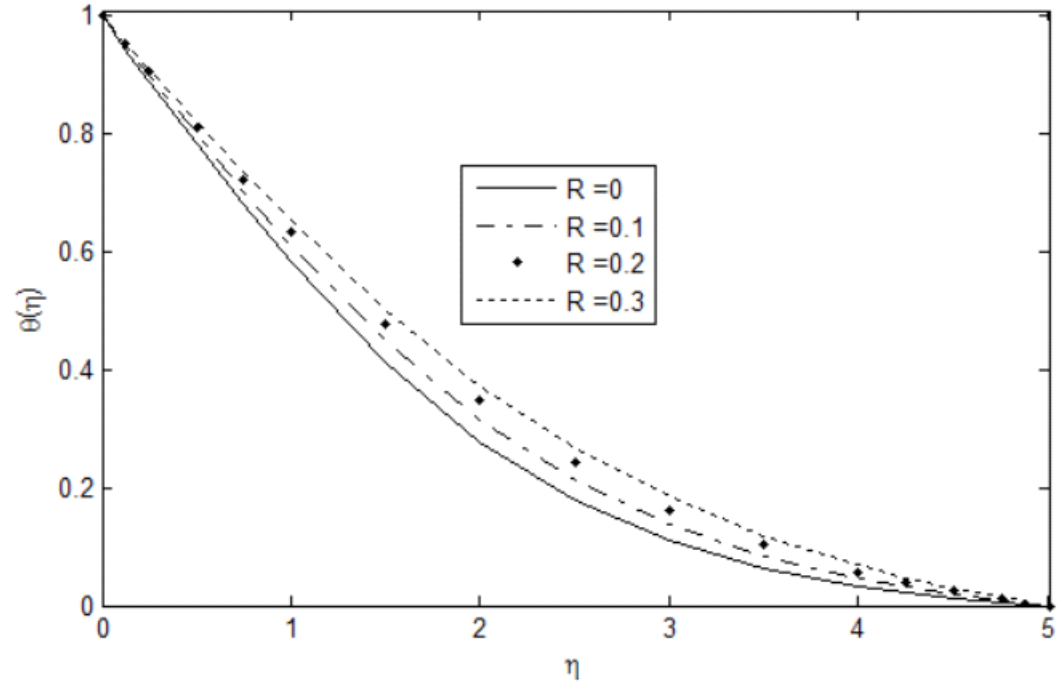


Figure 8: Profile of $\theta(\eta)$ for different values of R when $\beta = \gamma = c = Pr = b = 1$.

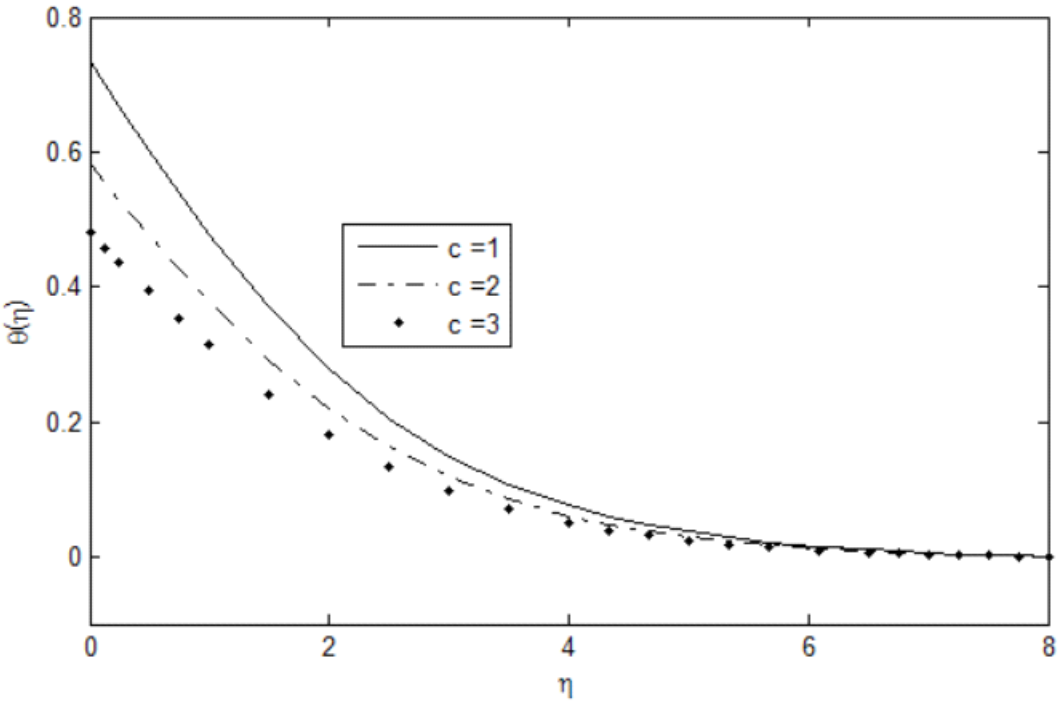


Figure 9: Profile of $\theta(\eta)$ for different values of c when $\beta = \gamma = Pr = b = 1, R = 0.2$.

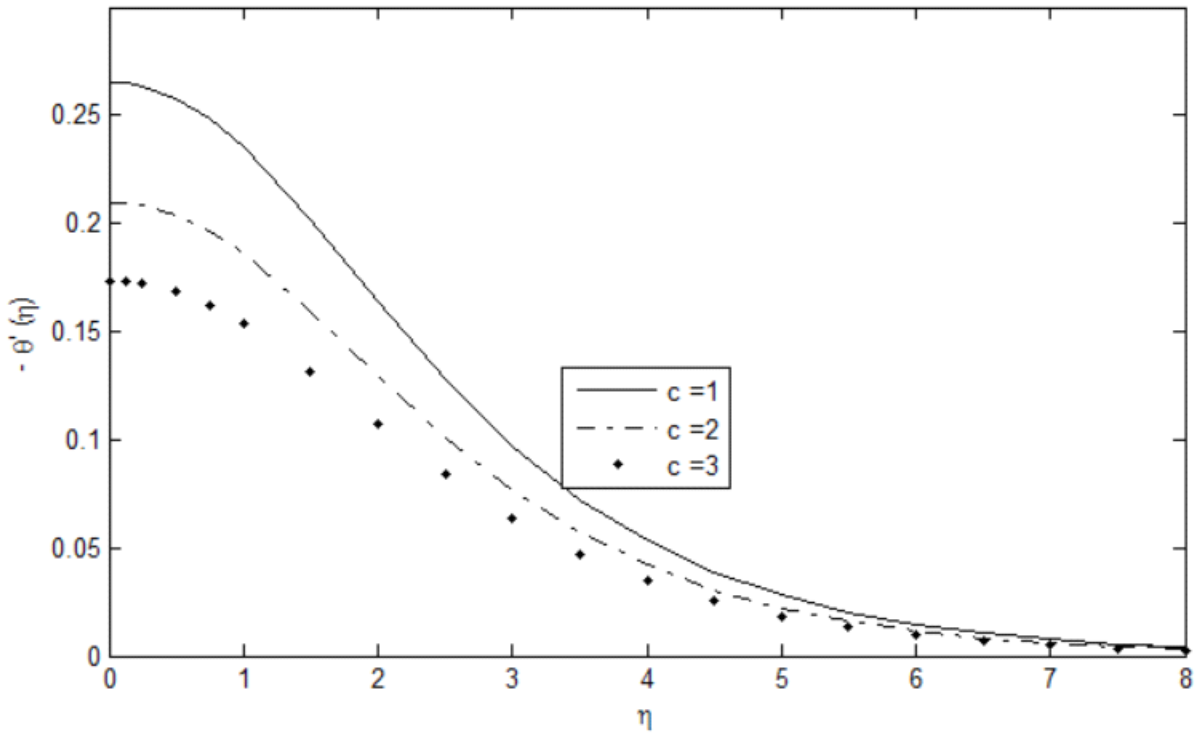


Figure 10: Profile of $-\theta(\eta)$ for different values of c when $\beta = \gamma = Pr = b = 1, R = 0.2$.

References

[1] J.B.J. Fourier, *Théorie analytique De La Chaleur*, Paris.

[2] C. Cattaneo, *Sulla conduzione del calore*, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, 3 (1948), 83-101.

[3] V. Tibullo, V. Zampoli, *A uniqueness result for the Cattaneo-Christov heat conduction model applied to incompressible fluids*, Mech. Res. Commun., 38 (2011), 77-79.

[4] C.I. Christov, *On frame indifferent formulation of the Maxwell-Cattaneo model of finite-speed heat conduction*, Mech. Res. Commun., 36 (2009), 481-486.

[5] M. Ciarletta, B. Straughan, *Uniqueness and structural stability for the Cattaneo-Christov equations*, Mech. Res. Commun., 37 (2010), 445-447.

[6] S. Han, L. Zheng, Li Chunrui, X. Zhang, Coupled flow and heat transfer in viscoelastic fluid with Cattaneo-Christov heat flux model, Appl. Math. Lett., 38 (2014), 87-93.

[7] M. Mustafa, *Cattaneo-Christov heat flux model for rotating flow and heat transfer of upper convected Maxwell fluid*, AIP Adv., 5 (2015), 047109.

[8] T. Hayat, Imtiaz Maria, *On Cattaneo-Christov heat flux in MHD flow of Oldroyd-B fluid with homogeneous-heterogeneous reactions*, Journal of Magnetism and Magnetic Materials, 401 (2016) , 296-303.

[9] J.J. Choi, Z. Rusak, J.A. Tichy, *Maxwell fluid suction flow in a channel*, J. Non-Newtonian Fluid Mech., 85 (1999), 165-187.

[10] K. Sadeghy, A.H. Najafi, M. Saffaripour, *Sakiadis flow of an upper- convected Maxwell fluid*, Int. J. Non-Linear Mech., 40 (2005), 1220-1228.

[11] Z. Abbas, M. Sajid, T. Hayat, *MHD boundary-layer flow of an upper-convected Maxwell fluid in a porous channel*, Theor. Comput. Fluid Dyn., 20 (2006), 229-238.

[12] T. Hayat, Z. Abbas, M. Sajid, *Series solution for the upper-convected Maxwell fluid over a porous stretching plate*, Phys. Lett., 358 (2006), 396-403.

[13] Pahlavan, A. A., Akbar, V. A., Farahani, F. V. and Sadeghy, K., *MHD flows of UCM fluids above porous stretching sheets using two-auxiliary-parameter homotopy analysis method*, Commun. Nonlinear Sci. Numer. Simul., 14 (2009) , 473-488.

[14] B. Raftari, A. Yildirim, *The application of homotopy perturbation method for MHD flows of UCM fluids above porous stretching sheets*, Comput. Math. Appl., 59 (2010), 3328-3337.

[15] K. Sadeghy, H. Hajibeygi, S.M. Taghavi, *Stagnation-point flow of upper-convected Maxwell fluids*, Int. J. Non-Linear Mech., 41 (2006), 1242-1247.

[16] T. Hayat, Z. Abbas, M. Sajid, *MHD stagnation-point flow of an upper-convected Maxwell fluid over a stretching surface*, Chaos Solitons Fractals, 39 (2009), 840-848.

[17] M. Kumaria, G. Nath, *Steady mixed convection stagnation-point flow of upper convected Maxwell fluids with magnetic field*, Int. J. Non-Linear Mech., 44 (2009) , 1048-1055.

[18] T. Hayat, M. Awais, M. Qasim, A.A. Hendi, *Effects of mass transfer on the stagnation point flow of an upper-convected Maxwell (UCM) fluid*, Int. J. Heat Mass Transfer., 54 (2011), 3777-3782.

[19] T. Hayat, Z. Abbas, N. Ali, *MHD flow and mass transfer of a upper-convected Maxwell fluid past a porous shrinking sheet with chemical reaction species*, Phys. Lett., A, 372 (2008), 4698-4704.

[20] V. Aliakbar, A. Alizadeh-Pahlavan, K. Sadeghy, *The influence of thermal radiation on MHD flow of Maxwellian fluids above stretching sheets*, Commun. Nonlinear Sci. Numer. Simul., 14 (2009), 779-794.

[21] L.V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York, 1982.

[22] G.W. Bluman, S. Kumei, *Symmetries and differential equations*, Springer, New York, 1989.

[23] J. Olver, *Application of Lie groups to differential equations*, Springer, New York, 1989.

[24] M.B. Abd-el-Malek, N.A. Badrana, H.S. Hassanb, *Using group theoretic method to solve multi-dimensional diffusion equation*, J. Comput. Appl. Math., 147 (2002), 385-395.

[25] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, *Lie-group method for unsteady flows in a semi-infinite expanding or contracting pipe with injection or suction through a porous wall*, J. Comput. Appl. Math., 197 (2006) , 465-494.

[26] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, *Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability*, Appl. Math. Model., 31 (2007), 1092-1108.

[27] M.B. Abd-el-Malek, A.M. Aminb, *Lie group analysis of nonlinear inviscid flows with a free surface under gravity*, J. Comput. Appl. Math., 258 (2014), 17-29.

[28] K.S. Mekheimer, M.F. El-Sabbagh, R.E. Abo-Elkhair, *Lie group analysis and similarity solutions for hydro-magnetic Maxwell fluid through a porous medium*, Bound. Value Probl., 2012.

[29] M. Pakdemirli, M. Yurusoy, *Similarity transformations for partial differential equations*, SIAM Rev., 40 (1998), 96-101.

[30] M. Yurusoy, M. Pakdemirli, O.F. Noyan, *Lie group analysis of creeping flow of a second grade fluid*, Int. J. Non-Linear Mech., 36 (2001), 955-960.

[31] S. Mukhopadhyay, G.C. Layek, S.A. Samad, *Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity*, Int. J. Heat Mass Transf., 48 (2005), 4460-4466.

[32] R. Kanadasamy, I. Muhaimin, *Lie group analysis for the effect of temperature dependent fluid viscosity and thermophoresis particle deposition on free convective heat and mass transfer under variable stream conditions* Appl. Math. Mech., 31 (2010), 317-328.

[33] J. Zhu, L. Zheng, Z. Zhang, *Effects of slip condition on MHD stagnation-point flow over a power-law stretching sheet*, Appl. Math. Mech., 31 (2010), 439-448.

[34] M. A. A. Hamad, *Analytical solution of natural convection flow of a nanofluid over a linearly stretching sheet in the presence of magnetic field*, Int. Commun. Heat Mass Transf., 38 (2011), 487-492.

[35] K. Das, *Lie group analysis for nanofluid flow past a convectively heated stretching surface*, Appl. Math. Comput., 221 (2013), 547-557.

[36] P.K. Kundu, K. Das, S. Jana, *Nanofluid flow towards a convectively heated stretching surface with heat source/sink: a lie group analysis*, Afr. Math., 25 (2014), 363-377.

[37] S. Asghar, M. Jalil, M. Hussan, M. Turkyilmazoglu, *Lie group analysis of flow and heat transfer over a stretching rotating disk*, Int. J. Heat Mass Transf., 69 (2014), 140-146.

[38] M.M. Rashidi, E. Momoniat, *Lie group solution for free convective flow of a nanofluid past a chemically reacting horizontal plate in a porous media*, Hindawi, 2014.

[39] L. Cao, X. Si, *Lie group analysis for MHD effects on the convectively heated stretching porous surface with the heat source/sink*, Boundary value problem, 2015.

[40] A. Aziz, M.J. Uddin, M.A.A. Hamad, A.I. Md.Ismail, *MHD flow over an inclined radiating plate with the temperature-dependent thermal conductivity variable reactive index and heat generation*, Heat Transfer, 41 (2012), 241-259.

[41] M.Q. Brewster, *Thermal radiative transfer properties*, John Wiley and Sons, (1972).

[42] L. Zheng, J. Niu, X. Zhang, Y. Gao, *MHD flow and heat transfer over a porous shrinking surface with velocity slip and temperature jump*, Mathematical and Computer Modelling, 56 (2012), 133-144.

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Schur convexity of generalized geometric Bonferroni mean involving three parameters

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Abstract. In this paper, we discuss the Schur convexity, Schur geometric convexity and Schur harmonic convexity of the generalized geometric Bonferroni mean. At the end of the paper, two inequalities related to the generalized geometric Bonferroni mean are established to illustrate the applications of the obtained results.

Keywords: Schur convexity, Schur geometric convexity, Schur harmonic convexity, generalized geometric Bonferroni means, inequalities.

1. Introduction

The Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians. There are numerous articles written on this topic in recent years, see [1, 2] and the references therein. As supplements to the Schur convexity of functions, the

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Schur geometrically convex functions and Schur harmonically convex functions were investigated by Zhang and Yang [3], Chu et al. [4, 5, 6], Shi et al. [7, 8], Meng et al. [9], Zheng et al. [10]. These properties of functions have been found to have an important application in discovering and proving the inequalities for special means (see [11, 12, 13, 14]).

Recently, it comes to our attention that a type of means which is symmetrical on n variables $x_1,\ x_2,\ \ldots,\ x_n$ and involves two parameters, it was initially proposed by Bonferroni [15], as follows

$$B^{p,q}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} x_i^p x_j^q\right)^{\frac{1}{p+q}}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i \geq 0, i = 1, 2, \ldots, n,\ p, q \geq 0$ and $p + q \neq 0$.

$B^{p,q}(\boldsymbol{x})$ is called the Bonferroni mean. It has important application in multi criteria decision-making (see [16, 17, 18, 19, 20, 21]).

Beliakov et al. [22] generalized the Bonferroni mean by introducing three parameters p, q, r, i.e.,

$$B^{p,q,r}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)(n-2)} \sum_{i,j.k=1,i\neq j\neq k}^{n} x_i^p x_j^q x_k^r\right)^{\frac{1}{p+q+r}}, \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i \geq 0, i = 1, 2, \ldots, n,\ p, q, r \geq 0$ and $p + q + r \neq 0$.

Motivated by the the Bonferroni mean $B^{p,q}(\boldsymbol{x})$ and the geometric mean $G(\boldsymbol{x}) = \prod_{i=1}^{n}(x_i)^{\frac{1}{n}}$, Xia et al. [23] introduced a new mean which is called the geometric Bonferroni mean, as follows

$$GB^{p,q}(\boldsymbol{x}) = \frac{1}{p+q} \prod_{i,j=1,i\neq j}^{n} (px_i + qx_j)^{\frac{1}{n(n-1)}}, \tag{3}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i > 0, i = 1, 2, \ldots, n,\ p, q \geq 0$ and $p + q \neq 0$.

An extension of the geometric Bonferroni mean was given by Park and Kim in [19], which is called generalized geometric Bonferroni mean, i.e.,

$$GB^{p,q,r}(\boldsymbol{x}) = \frac{1}{p+q+r} \prod_{i,j,k=1,i\neq j\neq k}^{n} (px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}}, \tag{4}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i > 0, i = 1, 2, \ldots, n,\ p, q, r \geq 0$ and $p + q + r \neq 0$.

Remark 1. For $r = 0$, it is easy to observe that

$$GB^{p,q,0}(\boldsymbol{x}) = \frac{1}{p+q+0} \prod_{i,j=1,i\neq j}^{n} \Big[\prod_{k=1,i\neq j\neq k}^{n} (px_i + qx_j + 0 \times x_k)\Big]^{\frac{1}{n(n-1)(n-2)}}$$

$$= \frac{1}{p+q} \prod_{i,j=1,i\neq j}^{n} \left[(px_i + qx_j)^{(n-2)}\right]^{\frac{1}{n(n-1)(n-2)}}$$

$$= \frac{1}{p+q} \prod_{i,j=1,i\neq j}^{n} (px_i + qx_j)^{\frac{1}{n(n-1)}}$$

$$= GB^{p,q}(\boldsymbol{x}).$$

Remark 2. If $q = 0, r = 0$, then the generalized geometric Bonferroni mean reduces to the geometric mean, i.e.,

$$GB^{p,0,0}(\boldsymbol{x}) = GB^{p,0}(\boldsymbol{x}) = \frac{1}{p} \prod_{i,j=1,i\neq j}^{n} (px_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}} = G(\boldsymbol{x}).$$

Remark 3. If $\boldsymbol{x} = (x, x, \ldots, x)$, then

$$GB^{p,q,r}(\boldsymbol{x}) = GB^{p,q,r}(x, x, \ldots, x) = x.$$

For convenience, throughout the paper $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0,\ i = 1, 2, \ldots, n\}.$$

In a recent paper [24], Shi and Wu investigated the Schur m-power convexity of the geometric Bonferroni mean $GB^{p,q}(\boldsymbol{x})$ and obtained the following results:

Proposition 1. *For fixed positive real numbers p, q, (i) if $m < 0$ or $m = 0$, then $GB^{p,q}(\boldsymbol{x})$ is Schur m-power convex on $\mathbb{R}^n_{++}$; (ii) if $m = 1$ or $m \geq 2$, then $GB^{p,q}(\boldsymbol{x})$ is Schur m-power concave on $\mathbb{R}^n_{++}$.*

In this paper we discuss the Schur convexity, Schur geometric convexity and Schur harmonic convexity of the generalized geometric Bonferroni mean $GB^{p,q,r}(\boldsymbol{x})$. Our main results are as follows.

Theorem 1. *For fixed non-negative real numbers p, q, r with $p + q + r \neq 0$, and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), n \geq 3$, (i) if $m \geq 1$ then $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power concave on $\mathbb{R}^n_{++}$; (ii) if $m < 0$, then $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power convex on $\mathbb{R}^n_{++}$.*

Corollary 1. *For fixed non-negative real numbers p, q, r with $p + q + r \neq 0$, if $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $n \geq 3$, then $GB^{p,q,r}(\boldsymbol{x})$ is Schur concave and Schur harmonic convex on $\mathbb{R}^n_{++}$.*

2. Preliminaries

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 1 ([1]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$,the function φ: $\Omega \to \mathbb{R}$ is said to be Schur convex on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be Schur concave function on Ω if and only if $-\varphi$ is Schur convex function on Ω.

Definition 2 ([1]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$ implies

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega.$$

Definition 3 ([1]). (*i*) A set $\Omega \subset \mathbb{R}^n$ is called symmetric, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

The following proposition is called Schur's condition. It provides an approach for testing whether a vector valued function is Schur convex or not.

Proposition 2 ([1]). *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur convex function (Schur concave function) if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left[\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - \frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0 \ (\leq 0) \tag{5}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Definition 4 ([25]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$.

(*i*) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0,1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}_+^n$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur geometrically convex function on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Definition 5 ([25]). Let $\Omega \subset \mathbb{R}_+^n$.

(*i*) A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = (x_1y_1, x_2y_2, \ldots, x_ny_n)$ and $\frac{1}{\boldsymbol{\lambda x}+\boldsymbol{(1-\lambda)y}} = \big(\frac{1}{\lambda x_1 + (1-\lambda)y_1}, \frac{1}{\lambda x_2 + (1-\lambda)y_2}, \cdots, \frac{1}{\lambda x_n + (1-\lambda)y_n}\big)$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur harmonically convex on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Proposition 3 ([25]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points, and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω^0. Then φ is Schur harmonically convex (Schur harmonically concave) on Ω if and only if*

$$(x_1 - x_2)\left[x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0) \tag{6}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Proposition 4 ([26]). *Let $\Omega \subset \mathbb{R}_{++}^n$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex (Schur m-power concave) on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0), \quad \text{if } \ m \ne 0 \tag{7}$$

and

$$(\log x_1 - \log x_2)\left[x_1 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0), \quad \text{if } \ m = 0 \tag{8}$$

for all $\boldsymbol{x} \in \Omega^\circ$.

Lemma 1 ([12]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$, $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$, $n \ge 2$ and $1 \le \lambda \le \frac{n}{n-1}$. Then*

$$\begin{aligned}&(\lambda A_n(\boldsymbol{x}) + (1-\lambda)x_1, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n)\\ &\prec (x_1, x_2, \ldots, x_n).\end{aligned} \tag{9}$$

Lemma 2 ([2]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\sum_{i=1}^n x_i = (n-2)c$, $n \ge 2$. Then*

$$\left(\frac{c+x_1}{n-1}, \frac{c+x_2}{n-1}, \ldots, \frac{c+x_n}{n-1}\right) \prec (c - x_1, c - x_2, \ldots, c - x_n). \tag{10}$$

3. Proof of main results

Proof of Theorem 1.

Note that the generalized geometric Bonferroni mean is defined by

$$GB^{p,q,r}(\boldsymbol{x}) = \frac{1}{p+q+r} \prod_{i,j,k=1,i\neq j\neq k}^{n} (px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}},$$

Taking the natural logarithm gives

$$\log GB^{p,q,r}(\boldsymbol{x}) = \log \frac{1}{p+q+r} + \frac{1}{n(n-1)(n-2)} Q,$$

where

$$\begin{aligned}
Q = & \sum_{j,k=3,j\neq k}^{n} [\log(px_1 + qx_j + rx_k) + \log(px_2 + qx_j + rx_k)] \\
& + \sum_{i,k=3,i\neq k}^{n} [\log(px_i + qx_1 + rx_k) + \log(px_i + qx_2 + rx_k)] \\
& + \sum_{i,j=3,i\neq j}^{n} [\log(px_i + qx_j + rx_1) + \log(px_i + qx_j + rx_2)] \\
& + \sum_{k=3}^{n} [\log(px_1 + qx_2 + rx_k) + \log(px_2 + qx_1 + rx_k)] \\
& + \sum_{j=3}^{n} [\log(px_1 + qx_j + rx_2) + \log(px_2 + qx_j + rx_1)] \\
& + \sum_{i=3}^{n} [\log(px_i + qx_1 + rx_2) + \log(px_i + qx_2 + rx_1)] \\
& + \sum_{i,j,k=3,\, i\neq j\neq k}^{n} \log(px_i + qx_j + rx_k).
\end{aligned}$$

Differentiating $GB^{p,q,r}(\boldsymbol{x})$ with respect to x_1 and x_2 respectively, we have

$$\begin{aligned}
\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_1} &= \frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)} \cdot \frac{\partial Q}{\partial x_1} \\
&= \frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)} \Bigg[\sum_{j,k=3,j\neq k}^{n} \frac{p}{px_1 + qx_j + rx_k} + \sum_{i,k=3,i\neq k}^{n} \frac{q}{px_i + qx_1 + rx_k} \\
&+ \sum_{i,j=3,i\neq j}^{n} \frac{r}{px_i + qx_j + rx_1} + \sum_{k=3}^{n} \Big(\frac{p}{px_1 + qx_2 + rx_k} + \frac{q}{px_2 + qx_1 + rx_k} \Big)
\end{aligned}$$

$$+\sum_{j=3}^{n}\Big(\frac{p}{px_1+qx_j+rx_2}+\frac{r}{px_2+qx_j+rx_1}\Big)$$
$$+\sum_{i=3}^{n}\Big(\frac{q}{px_i+qx_1+rx_2}+\frac{r}{px_i+qx_2+rx_1}\Big)\Big].$$

$$\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_2}=\frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)}\cdot\frac{\partial Q}{\partial x_2}$$
$$=\frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)}\Bigg[\sum_{j,k=3,j\neq k}^{n}\frac{p}{px_2+qx_j+rx_k}+\sum_{i,k=3,i\neq k}^{n}\frac{q}{px_i+qx_2+rx_k}$$
$$+\sum_{i,j=3,i\neq j}^{n}\frac{r}{px_i+qx_j+rx_2}+\sum_{k=3}^{n}\Big(\frac{q}{px_1+qx_2+rx_k}+\frac{p}{px_2+qx_1+rx_k}\Big)$$
$$+\sum_{j=3}^{n}\Big(\frac{r}{px_1+qx_j+rx_2}+\frac{p}{px_2+qx_j+rx_1}\Big)$$
$$+\sum_{i=3}^{n}\Big(\frac{r}{px_i+qx_1+rx_2}+\frac{q}{px_i+qx_2+rx_1}\Big)\Bigg].$$

It is easy to see that $GB^{p,q,r}(\boldsymbol{x})$ is symmetric on $\mathbb{R}^n_{++}$. Without loss of generality, we may assume that $x_1\geq x_2>0$. Hence, for $m\neq 0$ and $n\geq 3$, we have

$$\Delta=\frac{x_1^m-x_2^m}{m}\left[x_1^{1-m}\frac{GB^{p,q,r}(\boldsymbol{x})}{\partial x_1}-x_2^{1-m}\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_2}\right]$$
$$=\frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}$$
$$\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\Big(\frac{x_1^{1-m}}{px_1+qx_j+rx_k}-\frac{x_2^{1-m}}{px_2+qx_j+rx_k}\Big)$$
$$+q\sum_{i,k=3,i\neq k}^{n}\Big(\frac{x_1^{1-m}}{px_i+qx_1+rx_k}-\frac{x_2^{1-m}}{px_i+qx_2+rx_k}\Big)$$
$$+r\sum_{i,j=3,i\neq j}^{n}\Big(\frac{x_1^{1-m}}{px_i+qx_j+rx_1}-\frac{x_2^{1-m}}{px_i+qx_j+rx_2}\Big)$$
$$+\sum_{k=3}^{n}\Big(\frac{px_1^{1-m}-qx_2^{1-m}}{px_1+qx_2+rx_k}+\frac{qx_1^{1-m}-px_2^{1-m}}{px_2+qx_1+rx_k}\Big)$$
$$+\sum_{j=3}^{n}\Big(\frac{px_1^{1-m}-rx_2^{1-m}}{px_1+qx_j+rx_2}+\frac{rx_1^{1-m}-px_2^{1-m}}{px_2+qx_j+rx_1}\Big)$$

$$+\sum_{i=3}^{n}\Big(\frac{qx_1^{1-m}-rx_2^{1-m}}{px_i+qx_1+rx_2}+\frac{rx_1^{1-m}-qx_2^{1-m}}{px_i+qx_2+rx_1}\Big)\Bigg],$$

i.e.,

$$\begin{aligned}
\Delta &= \frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}\\
&\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\frac{px_1x_2(x_1^{-m}-x_2^{-m})+(qx_j+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)}\\
&+q\sum_{i,k=3,i\neq k}^{n}\frac{qx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)}\\
&+r\sum_{i,j=3,i\neq j}^{n}\frac{rx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)}\\
&+\sum_{k=3}^{n}\frac{(p^2+q^2)x_1x_2(x_1^{-m}-x_2^{-m})+2pq(x_1^{2-m}-x_2^{2-m})+(p+q)rx_k(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)}\\
&+\sum_{j=3}^{n}\frac{(p^2+r^2)x_1x_2(x_1^{-m}-x_2^{-m})+2pr(x_1^{2-m}-x_2^{2-m})+(p+r)qx_j(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)}\\
&+\sum_{i=3}^{n}\frac{(q^2+r^2)x_1x_2(x_1^{-m}-x_2^{-m})+2qr(x_1^{2-m}-x_2^{2-m})+(q+r)px_i(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)}\Bigg].
\end{aligned}$$

In view of $x_1^{2-m}-x_2^{2-m}=(x_1^{1-m}-x_2^{1-m})(x_1+x_2)-(x_1^{-m}-x_2^{-m})x_1x_2$, we can rewrite the expression of Δ as

$$\begin{aligned}
\Delta &= \frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}\\
&\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\frac{px_1x_2(x_1^{-m}-x_2^{-m})+(qx_j+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)}\\
&+q\sum_{i,k=3,i\neq k}^{n}\frac{qx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)}\\
&+r\sum_{i,j=3,i\neq j}^{n}\frac{rx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)}\\
&+\sum_{k=3}^{n}\frac{(p-q)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2pq(x_1+x_2)+(p+q)rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)}
\end{aligned}$$

$$+\sum_{j=3}^{n}\frac{(p-r)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2pr(x_1+x_2)+(p+r)qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)}$$
$$+\sum_{i=3}^{n}\frac{(q-r)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2qr(x_1+x_2)+(q+r)px_i)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)}\Bigg].$$

If $m \geq 1$, by the assumption $x_1 \geq x_2 > 0$, $p \geq 0, q \geq 0, r \geq 0$, one has $x_1^{-m} - x_2^{-m} \leq 0$, $x_1^{1-m} - x_2^{1-m} \leq 0$ and $x_1^m - x_2^m \geq 0$, which implies $\Delta \leq 0$. Using Proposition 4, we conclude that $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power concave for $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

If $m < 0$, by the assumption $x_1 \geq x_2 > 0$, $p \geq 0, q \geq 0, r \geq 0$, we obtain $x_1^{-m} - x_2^{-m} \geq 0$, $x_1^{1-m} - x_2^{1-m} \geq 0$ and $x_1^m - x_2^m \leq 0$. Thus, $\Delta \geq 0$. From Proposition 4, it follows that $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power convex for $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

The proof of Theorem 1 is completed.

Remark 4. As a direct consequence of Theorem 1, taking $m = 1$ and $m = -1$ in Theorem 1 respectively, together with the definitions of Schur concave functions and Schur harmonic convex functions, we arrive at the assertions of Corollary 1.

4. Applications

As an application of Theorem 1, we establish the following interesting inequalities for generalized geometric Bonferroni mean.

Theorem 2. *Let p, q, r be non-negative real numbers with $p+q+r \neq 0$, and let $1 \leq \lambda \leq \frac{n}{n-1}$. Then for $\boldsymbol{x} \in \mathbb{R}^n_{++} (n \geq 3)$ we have the inequality*

$$\begin{aligned} GB^{p,q,r}(\boldsymbol{x}) \leq GB^{p,q,r}\,&(\lambda A_n(\boldsymbol{x}) + (1-\lambda)x_1, \lambda A_n(\boldsymbol{x}) \\ &+(1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n)\,. \end{aligned} \tag{11}$$

Proof. It follows form Theorem 1 that $GB^{p,q,r}(\boldsymbol{x})$ is Schur concave on $\mathbb{R}^n_{++}$.

Using Lemma 1, one has $(\lambda A_n(\boldsymbol{x})+(1-\lambda)x_1, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_n) \prec (x_1, x_2, \ldots, x_n)$.

Thus, we deduce from Definition 1 that $GB^{p,q,r}(\lambda A_n(\boldsymbol{x})+(1-\lambda)x_1, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n) \geq GB^{p,q,r}(x_1, x_2, \ldots, x_n)$.

The Theorem 2 is proved. □

Putting $\lambda = 1$ in Theorem 2 gives

Corollary 2. *Let p, q, r be non-negative real numbers with $p+q+r \neq 0$. Then for $\boldsymbol{x} \in \mathbb{R}^n_{++}$ $(n \geq 3)$ we have the inequality*

$$GB^{p,q,r}(\boldsymbol{x}) \leq A_n(\boldsymbol{x}). \tag{12}$$

Theorem 3. *Let p, q, r be non-negative real numbers with $p+q+r \neq 0$, and let $\sum_{i=1}^{n} x_i = (n-2)c$. Then for $\boldsymbol{x} \in \mathbb{R}_{++}^{n}$ $(n \geq 3)$ we have the inequality*

$$GB^{p,q,r}(c(n-2)^2 + \boldsymbol{x}) \geq (n-1)\, GB^{p,q,r}(c(n-2) - \boldsymbol{x}). \tag{13}$$

Proof. By the relationship of majorization given in Lemma 2, we have

$$\left(\frac{c+x_1}{n-1}, \frac{c+x_2}{n-1}, \ldots, \frac{c+x_n}{n-1}\right) \prec (c-x_1, c-x_2, \ldots, c-x_n)\,.$$

Using the properties of majorization yields

$$\left(\frac{c+x_1}{n-1} + (n-3)c,\ \frac{c+x_2}{n-1} + (n-3)c,\ \ldots,\ \frac{c+x_n}{n-1} + (n-3)c\right)$$
$$\prec (c-x_1 + (n-3)c,\ c-x_2 + (n-3)c,\ \ldots,\ c-x_n + (n-3)c)\,,$$

i.e.

$$\left(\frac{c(n-2)^2 + x_1}{n-1},\ \frac{c(n-2)^2 + x_2}{n-1},\ \ldots,\ \frac{c(n-2)^2 + x_n}{n-1}\right)$$
$$\prec (c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n)\,.$$

It follows from Theorem 1 that

$$GB^{p,q,r}\left(\frac{c(n-2)^2 + x_1}{n-1},\ \frac{c(n-2)^2 + x_2}{n-1},\ \ldots,\ \frac{c(n-2)^2 + x_n}{n-1}\right)$$
$$\geq GB^{p,q,r}\,(c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n)\,.$$

Hence

$$GB^{p,q,r}\left(c(n-2)^2 + x_1,\ c(n-2)^2 + x_2,\ \ldots,\ c(n-2)^2 + x_n\right)$$
$$\geq (n-1) GB^{p,q,r}\,(c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n)\,,$$

that is

$$GB^{p,q,r}(c(n-2)^2 + \boldsymbol{x}) \geq (n-1)\, GB^{p,q,r}(c(n-2) - \boldsymbol{x}).$$

This completes the proof of Theorem 3. □

References

[1] A. W. Marsshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its application* (Second Edition), Springer, New York, 2011.

[2] H. N. Shi, *Theory of Majorization and Analytic Inequalities,* (in Chinese) Harbin Institute of Technology Press, Harbin, 2012.

[3] X. M. Zhang, Z. H. Yang, *Differential criterion of n-dimensional geometrically convex functions*, J. Appl. Anal., 13 (2007), 197-208.

[4] Y. Chu, X. M. Zhang and G.D. Wang, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal., 15 (2008), 707-718.

[5] Y. M. Chu, W. F. Xia, *Necessary and sufficient conditions for the Schur harmonic convexity of the generalized Muirhead mean*, Proceedings of a Razmadze Mathematical Institute, 152 (2010), 19-27.

[6] Y. M. Chu, G. D. Wang, X. H. Zhang, *The Schur multiplicative and harmonic convexities of the complete symmetric function*, Mathematische Nachrichten, 284 (2011), 653-663.

[7] H. N. Shi, J. Zhang, *Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions,* J. Inequal. Appl., 2013 (2013), Article ID 527, 9 pages.

[8] H. N. Shi, J. Zhang, *Schur-convexity, Schur geometric and Schur harmonic convexities of dual form of a class symmetric functions,* J. Math. Inequal., 8 (2014), 349-358.

[9] J. X. Meng, Y. M. Chu, X. M. Tang, *The Schur-harmonic-convexity of dual form of the Hamy symmetric function*, Matematicki Vesnik, 62 (2010), 37-46.

[10] N. G. Zheng, Z. H. Zhang, X. M. Zhang, *Schur-convexity of two types of one-parameter mean values in n variables*, J. Inequal. Appl., 2007 (2007), Article ID 78175, 10 pages.

[11] S. H. Wu, H. N. Shi, *A relation of weak majorization and its applications to certain inequalities for means,* Mathematica Slovaca, 61 (2011), 561-570

[12] S. H. Wu, *Generalization and sharpness of power means inequality and their applications*, J. Math. Anal. Appl., 312 (2005), 637-652.

[13] H. N. Shi, S. H. Wu, F. Qi, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl., 9 (2006), 219-224.

[14] L. L. Fu, B. Y. Xi, H. M. Srivastava, *Schur-convexity of the generalized Heronian means involving two positive numbers*, Taiwanese J. Math., 15 (2011), 2721-2731.

[15] C. Bonferroni, *Sulle medie multiple di potenze*, Bollettino dell' Unione Matematica Italiana, 5 (1950), 267-270.

[16] R. R. Yager, *On generalized Bonferroni mean operators for multi-criteria aggregation*, International Journal of Approximate Reasoning, 50 (2009), 1279-1286.

[17] J. H. Park, E. J. Park, *Generalized fuzzy Bonferroni harmonic mean operators and their applications in group decision making*, Journal of Applied Mathematics, 2013 (2013), Article ID 604029, 14 pages.

[18] M. Xia, Z. Xu, B. Zhu, *Generalized intuitionistic fuzzy Bonferroni means*, International Journal of Intelligent Systems, 27 (2012), 23-47.

[19] J. H. Park, J. Y. Kim, *Intuitionistic fuzzy optimized weighted geometric Bonferroni means and their applications in Group Decision Making*, Fundamenta Informaticae, 144 (2016), 363-381.

[20] G. Beliakov, S. James, J. Mordelová, T. Rückschlossová, R. R. Yager, *Generalized Bonferroni mean operators in multicriteria aggregation*, Fuzzy Sets and Systems, 161 (2010), 2227-2242.

[21] Z. Xu, R. R. Yager, *Intuitionistic Fuzzy Bonferroni Means*, IEEE Transactions on Systems, Man, and Cybernetics, Part B, 41 (2011), 568-578

[22] G. Beliakov, S. James, J. Mordelová, T. Rückschlossová, and R. R. Yager, *Generalized Bonferroni mean operators in multi-criteria aggregation*, Fuzzy Sets and Systems, 161 (2010), 2227-2242.

[23] M. Xia, Z. Xu, B. Zhu, *Generalized intuitionistic fuzzy Bonferroni means*, International Journal of Intelligent Systems, 27 (2012), 23-47.

[24] H. N. Shi, S. H. Wu, *Schur m-power convexity of geometric Bonferroni mean*, Italian Journal of Pure and Applied Mathematics, 38 (2017), 769-776.

[25] H. N. Shi, J. Zhang, Q. H. Ma, *Schur-convexity, Schur-geometric and Schur-harmonic convexity for a composite function of complete symmetric function,* SpringerPlus, 5 (2016), Article ID 296, 8 pages.

[26] Z. H. Yang, *Schur power convexity of Stolarsky means*, Publ. Math. Debrecen, 80 (2012), 43-66.

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On TL-fuzzy ideals and their L-fuzzy roughness in lattices

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Abstract. In this paper, we study TL-fuzzy ideals of lattices. First, using a left continuous t-norm T on a complete lattice L, we introduce the concept of TL-fuzzy ideals of lattices and study the corresponding relations between TL-fuzzy ideals of two homomorphic lattices and discuss the image and the inverse image of generated TL-fuzzy ideals. Moreover, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices. And we study some related properties of TL-fuzzy ideals with respect to T-upper and θ-lower fuzzy rough approximations.

Keywords: TL-fuzzy ideal, TL-fuzzy congruence relation, T-upper fuzzy rough approximation, θ-lower fuzzy rough approximation.

1. Introduction

As is known, an important task of the artificial intelligence is to make the computers simulate human being in dealing with certainty and uncertainty in information. Logic gives a technique for laying the foundations of this task. While information processing dealing with certain information is based on the classical two-valued logic, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, vagueness, etc. Therefore, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information

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and fuzzy information. And various logical algebras have been proposed as the semantical systems of non-classical logical systems, for instance, MV-algebras [1], BL-algebras [27], etc. Among these logical algebras, lattices are very basic algebraic structures. Filters and ideals are important tools in studying lattice algebraic systems. The sets of provable formulas in corresponding inference systems are described by filters, and from the point of view of uncertain information, by fuzzy filters of algebraic semantics. Moreover, the properties of the sets of filters/ideals have a strong influence on the structure properties of those algebras. Based on these, many authors introduced and studied fuzzy filters of logical algebras [15, 29, 23, 8]. Recently, fuzzy filters based t-norm of BL-algebras has been given in [26], TL-filters of integral residuated l-monoids have been studied in [28], and so on.

An important notion in fuzzy set theory is that of t-norms: t-norms are used to define a generalized intersection. By a fuzzy subset μ in a given universe X we understand a mapping μ: $X \to [0,1]$, the membership degrees $\mu(x)$ can in a natural way be understood as the truth value (in fuzzy logic) of the statement "x belongs to μ". In the same way, the intersection $\mu \cap \nu$ of fuzzy sets μ, ν can be viewed as having the membership degree $(\mu \cap \nu)(x)$ corresponding to the truth degree of the statement "x belongs to μ" AND "x belongs to ν". Here AND refers to a suitably defined conjunction connective, defined according to the different possibilities which one has to determine the membership degrees $(\mu \cap \nu)(x)$. For example, AND can be understood as taking the minimum or as taking the (usual, i.e. algebraic) product, or more generally, it also can be understood as a t-norm. Accordingly, the t-norms were considered as the candidates for generalized conjunction connectives of the background many-valued logic. The fuzzy logic based on t-norms, especially left continuous t-norms, was developed significantly by Hájek, Esteva et al (see [3, 5, **?**, 20]) Also, there are many applications of t-norms in several fields of mathematics and artificial intelligence [10].

Rough sets theory was developed by Pawlak [21] as a formal tool for representing and processing information in data tables-consequently, this formalism acts on partial information. It has been found practical applications in many areas such as knowledge discovery, machine learning, data analysis, approximation classification and so on, see [13, 16, 22]. The fuzzy set theory [39], on the other hand, offers a wide variety of techniques for analyzing imprecise data. It seems therefore natural to combine methods developed within both theories in order to construct hybrid structures capable to deal with both aspects of incompleteness. Such structures, called fuzzy rough sets and rough fuzzy sets, have been proposed in the literature [4]. In [24], Radzikowska investigated fuzzy rough sets taking the interval $[0,1]$ as a basic structure. However, as Goguen [6] pointed out, in some situations it may be impossible to use the linearly ordered set $[0,1]$ to represent degrees of membership. From this reason, the concept of an L-fuzzy set was then introduced.

The integration of the fuzzy set theory and rough set theory is an interesting and valuable research work, which has been studying until now. And it makes sense to study rough sets on fuzzy algebra structures. The initiation and majority of studies on fuzzy rough sets for algebraic structures such as semigroups, groups, rings, and hypermodules have been concentrated on the interval $[0, 1]$ as a basic structure, see [35, 9]. The interval $[0, 1]$, however, seems to be restrict the application of the generalized rough set model for algebraic sets. To solve this problem, Radzikowska and Kerre [25] proposed the concept of L-fuzzy rough sets. It differs from fuzzy rough sets in that it takes a complete residuated lattice L as its basic structure. This is a fairly wide constructive setting because diverse residuated pairs can be chosen and, in case $L = [0, 1]$, the fuzzy rough set theory follows. From this point of view, this paper considers the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices, see [34].

Based on the importance of ideals in studying logic algebras and extensive applications of t-norms and rough sets in fuzzy logic and artificial intelligence. In this paper, we study the relationships among L-fuzzy sets, rough sets and lattices theory. We hope that these links may invoke some new research topics for rough set theory and quantales and provide more applications in the fields such as logic, approximate reasoning and information science. Using a left continuous t-norm T on a complete lattice L, we introduce the notion of TL-fuzzy ideals in lattices and study the corresponding relations between TL-fuzzy ideals of two homomorphic lattices and discuss the image and the inverse image of generated TL-fuzzy ideals of the lattice X. Moreover, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice.

2. Preliminaries

In this section, we recall some notions and definitions that will be used in the sequel.

First, we recall some basic notions about t-norms that we shall use in the following paragraphs.

Let $(L, \wedge, \vee, \leq, 0, 1)$ denote a complete lattice with the top and bottom elements 1 and 0, respectively.

Definition 1 ([18]). *A binary operation T on L is called a t-norm if it satisfies the following conditions: for all $a, b, c \in L$,*

$(T1)$ $aT1 = a$;

$(T2)$ $aTb = bTa$;

$(T3)$ $(aTb)Tc = aT(bTc)$;

$(T4)$ *if $b \leq c$, then $aTb \leq aTc$.*

Because of associative and commutative properties, for all $a_1, a_2, \cdots, a_n \in L$ $(n \geq 1)$, $a_1Ta_2T\cdots Ta_n$ is well defined and its value is irrelevant to the order of $a_1, a_2, \cdots, a_n$. We write $T_{i=1}^{n} a_i = a_1Ta_2T\cdots Ta_n$. The subset D_T of L is

defined as $D_T = \{a \in L | aTa = a\}$. When $T = \wedge$, D_T coincides with L. If $aT(\vee_{i\in I} b_i) = \vee_{i\in I}(aTb_i)$, for all $a, b_i \in L$, where I is an arbitrary nonempty index set, then T is called a left continuous t-norm.

In what follows, let $L = [0, 1]$, there are some examples of the most popular t-norms: for all $x, y \in [0, 1]$,

(1) the Lukasiewicz t-norm: $xT_L y = max\{x + y - 1, 0\}$.

(2) the algebraic product: $xT_P y = xy$.

(3) the standard min operation: $xT_M y = min\{x, y\}$.

Given a t-norm T, the following binary operation on L, for any $a, b \in L$, $a\theta b = \vee\{c \in L | aTc \leq b\}$ is called the implication operator induced by T.

The implication operators induced by T_P, T_M and T_L, respectively, are: for any $x, y \in [0, 1]$,

(1) the Gaines implication: $x\theta_P y = \begin{cases} 1, & x \leq y, \\ \frac{y}{x}, & otherwise. \end{cases}$

(2) the Gödel implication: $x\theta_M y = \begin{cases} 1, & x \leq y, \\ y, & otherwise. \end{cases}$

(3) the Lukasiewicz implication: $x\theta_L y = min\{1, 1 - x + y\}$.

Some basic properties that will be used in the sequel of the implication operator θ are mentioned in the following. For further information about the implication operator θ please refer to [18, 5].

Proposition 1 ([18])**.** *The implication operator θ of a left continuous t-norm T satisfies the following properties: for any $a, b, c, a_i \in L$, $i \in I$,*

(θ1) $a\theta 1 = 1$ *and* $1\theta a = a$*;*

(θ2) $a \leq b \Longrightarrow c\theta a \leq c\theta b, a\theta c \geq b\theta c$*;*

(θ3) $(a\theta b)Tc \leq a\theta(bTc)$*;*

(θ4) $b\theta(\vee_{i\in I} a_i) \geq \vee_{i\in I}(b\theta a_i)$.

Throughout this paper, $(L, \wedge, \vee, \leq, 0, 1)$ always represents a given complete lattice with a left continuous t-norm T and an implication operator θ induced by T. In fact, such kind of lattice is a complete residuated lattice. Unless otherwise stated, the universe $(X, \wedge', \vee', \leq')$ indicates a given lattice.

An L-fuzzy subset of X is a mapping from X to L. The family of all L-fuzzy subsets of X is denoted by $LF[X]$ ([6]). When $L = [0, 1]$, the L-fuzzy subsets of X are known as fuzzy subsets of X ([39]). Let $\mu, \nu \in LF[X]$ be given, Then μ is said to be included in ν and written as $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$. And the union, intersection of μ and ν are defined as L-fuzzy subsets of X by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$. In general, $(\cup_{i\in I}\mu_i)(x) = \vee_{i\in I}\mu_i(x)$, $(\cap_{i\in I}\mu_i)(x) = \wedge_{i\in I}\mu_i(x)$.

In what follows, we recall some basic notions about (θ, T)-fuzzy rough approximation operators, which will be needed in the following sections.

An L-fuzzy binary relation R on X is referred to an L-fuzzy subset of $X \times X$, where $R(x, y)$ is the degree of relation between x and y, for all $(x, y) \in X \times X$.

Definition 2 ([12]). *An L-fuzzy binary relation R on X is called a TL-fuzzy equivalence relation if it satisfies the following conditions: for any $x, y, z \in X$,*

(1) $R(x, x) = 1$;
(2) $R(x, y) = R(y, x)$;
(3) $R(x, y) \geq R(x, z)TR(z, y)$.

In particular, a TL-fuzzy equivalence relation on X is simply referred to as an L-fuzzy equivalence relation on X when $T = \wedge$. Moreover, when $L = [0, 1]$, an L-fuzzy equivalence relation on X is referred to as a fuzzy equivalence relation on X.

In the sequel, unless otherwise stated, R denotes a TL-fuzzy equivalence relation on X. In this case, the pair (X, R) is called a TL-fuzzy approximation space.

Definition 3 ([33, 31]). *Let (X, R) be a TL-fuzzy approximation space. Define the following two mappings $\overline{R}^T$ and $\underline{R}_\theta$: $LF[X] \to LF[X]$, respectively called T-upper fuzzy rough approximation operator and θ-lower fuzzy rough approximation operator, as follows: for any $\mu \in LF[X]$, $x \in X$, $\overline{R}^T\mu(x) = \vee_{y\in X} R(x, y)T\mu(y)$ and $\underline{R}_\theta\mu(x) = \wedge_{y\in X} R(x, y)\theta\mu(y)$.*

$\overline{R}^T\mu$ and $\underline{R}_\theta\mu$ are called a T-upper fuzzy rough approximation of μ and θ-lower fuzzy rough approximation of μ, respectively. And the pair $R^T_\theta\mu = (\underline{R}_\theta\mu, \overline{R}^T\mu)$ is called a (θ, T)-fuzzy rough set of μ with respect to the TL-fuzzy equivalence relation R if $\underline{R}_\theta\mu \neq \overline{R}^T\mu$.

Remark. In particular, if R is a crisp equivalence relation of X, then it is easy to see that: for any $\mu \in LF[X]$, $x \in X$, $\overline{R}^T\mu(x) = \vee_{y\in[x]_R}\mu(y)$ and $\underline{R}_\theta\mu(x) = \wedge_{y\in[x]_R}\mu(y)$.

Furthermore, if μ is a crisp subset of X, then $\overline{R}^T\mu = \{x \in X | [x]_R \cap \mu \neq \phi\}$ and $\underline{R}_\theta\mu = \{x \in X | [x]_R \subseteq \mu\}$, where $[x]_R$ denotes the equivalence class generated by R.

Let us review several properties of fuzzy rough approximation operators. For further information about (θ, T)-fuzzy rough sets please refer to [33].

Proposition 2 ([33]). *The (θ, T)-fuzzy rough approximation operators $\underline{R}_\theta$, $\overline{R}^T$ satisfy the following properties: for any $\mu, \nu \in LF[X]$,*

($R1$) $\underline{R}_\theta\mu \subseteq \mu \subseteq \overline{R}^T\mu$;
($R2$) *if $\mu \subseteq \nu$, then $\underline{R}_\theta\mu \subseteq \underline{R}_\theta\nu$, $\overline{R}^T\mu \subseteq \overline{R}^T\nu$;*
($R3$) $\overline{R}^T(\underline{R}_\theta\mu) = \underline{R}_\theta\mu$, $\underline{R}_\theta(\overline{R}^T\mu) = \overline{R}^T\mu$;
($R4$) $\overline{R}^T(\overline{R}^T\mu) = \overline{R}^T\mu$, $\underline{R}_\theta(\underline{R}_\theta\mu) = \underline{R}_\theta\mu$.

3. TL-fuzzy ideals of lattices

In this section, we shall introduce the notion of TL-fuzzy ideals in lattices and investigate some related properties of them. Specifically, we give the formula

for calculating the TL-fuzzy ideal generated by L-fuzzy subsets and study the lattice structure of TL-fuzzy ideals in lattices. Also, there is an open problem in this section.

First, we shall introduce TL-fuzzy ideals in lattices.

Definition 4. *Let $(X, \wedge', \vee', \leq')$ be a lattice and μ be an L-fuzzy subset of X. Then μ is called a TL-fuzzy ideal of X if it satisfies the following conditions: for all $x, y \in X$,*

(i) $\mu(x \vee' y) \geq \mu(x) T \mu(y)$,

(ii) $\mu(x \wedge' y) \geq \mu(x) \vee \mu(y)$.

We shall denote the set of all TL-fuzzy ideals of the lattice X as $TLFI[X]$.

Now, the following result gives an equivalent version of the concept of TL-fuzzy filters in lattices.

Theorem 1. *Let $(X, \wedge', \vee', \leq')$ be a lattice and μ be an L-fuzzy subset of X. Then μ is a TL-fuzzy ideal of X if and only if it satisfies the following conditions: for all $x, y \in X$,*

(i) $\mu(x \vee' y) \geq \mu(x) T \mu(y)$,

(ii) if $y \leq' x$, *then* $\mu(y) \geq \mu(x)$.

Proof. The proof is straightforward. □

Let f be a mapping from X_1 into X_2, $\mu \in LF[X_1]$ and $\nu \in LF[X_2]$. The L-fuzzy sets $f(\mu)$ of X_2 and $f^{-1}(\nu)$ of X_1, defined by

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) | f(x) = y, x \in X_1\}, & f^{-1}(y) \neq \phi \\ 0, & f^{-1}(y) = \phi \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$, for all $x \in X_1$, $y \in X_2$, are called, respectively, the image of μ under f and the inverse image of ν under f.

Now we discuss properties of TL-fuzzy filters under lattice homomorphisms.

Proposition 3. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2 and μ be a TL-fuzzy ideal of X_1. Then $f(\mu)$ is also a TL-fuzzy ideal of X_2.*

Proof. Suppose that μ is a TL-fuzzy ideal of X_1.

Since f is surjective, then for all $y_1, y_2 \in X_2$, there are $x_1, x_2 \in X_1$ such that $f(x_1) = y_1, f(x_2) = y_2$. Then we have

$$\begin{aligned} & f(\mu)(y_1 \vee' y_2) = \vee\{\mu(x) | f(x) = y_1 \vee' y_2, x \in X_1\} \\ & \geq \vee\{\mu(x_1 \vee' x_2) | f(x_1 \vee' x_2) = y_1 \vee' y_2, x_1, x_2 \in X_1\} \\ & \geq \vee\{\mu(x_1) T \mu(x_2) | f(x_1) = y_1, f(x_2) = y_2, x_1, x_2 \in X_1\} \\ & = (\vee\{\mu(x_1) | f(x_1) = y_1, x_1 \in X_1\}) T (\vee\{\mu(x_2) | f(x_2) = y_2, x_2 \in X_1\}) \\ & = f(\mu)(y_1) T f(\mu)(y_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&f(\mu)(y_1 \wedge' y_2) = \vee\{\mu(x) | f(x) = y_1 \wedge' y_2, x \in X_1\} \\
&\geq \vee\{\mu(x_1 \wedge' x_2) | f(x_1 \wedge' x_2) = y_1 \wedge' y_2, x_1, x_2 \in X_1\} \\
&\geq \vee\{\mu(x_1) \vee \mu(x_2) | f(x_1) = y_1, f(x_2) = y_2, x_1, x_2 \in X_1\} \\
&= (\vee\{\mu(x_1) | f(x_1) = y_1, x_1 \in X_1\}) \vee (\vee\{\mu(x_2) | f(x_2) = y_2, x_2 \in X_1\}) \\
&= f(\mu)(y_1) \vee f(\mu)(y_2).
\end{aligned}$$

Combining the above arguments, we have that $f(\mu)$ is a TL-fuzzy ideal of X_2. □

Definition 5. *Let f be a mapping from X_1 into X_2 and μ be an L-fuzzy subset of X_1. Then μ is said to be $f-$invariant if for any $x_1, x_2 \in X_1$, $f(x_1) = f(x_2)$, then $\mu(x_1) = \mu(x_2)$.*

Obviously, if μ is $f-$invariant, then $f^{-1}(f(\mu)) = \mu$.

Proposition 4. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and ν be a TL-fuzzy ideal of X_2. Then $f^{-1}(\nu)$ is also a TL-fuzzy ideal of X_1 and $f-$invariant.*

Proof. Assume that ν be a TL-fuzzy ideal of X_2. For any $x_1, x_2 \in X_1$, we have that $f^{-1}(\nu)(x_1 \vee' x_2) = \nu(f(x_1 \vee' x_2)) = \nu(f(x_1) \vee' f(x_2)) \geq \nu(f(x_1)) T \nu(f(x_2)) = f^{-1}(\nu)(x_1) T f^{-1}(\nu)(x_2)$.

Since f is a homomorphism from X_1 into X_2, then f is isotonic. Therefore, if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$ and $f^{-1}(\nu)(x_1) = \nu(f(x_1)) \geq \nu(f(x_2)) = f^{-1}(\nu)(x_2)$. Thus, we get that $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.

Combining the above arguments, we have that $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. For any $x_1, x_2 \in X_1$, if $f(x_1) = f(x_2)$, then $f^{-1}(\nu)(x_1) = \nu(f(x_1)) = \nu(f(x_2)) = f^{-1}(\nu)(x_2)$. Therefore, $f^{-1}(\nu)$ is $f-$invariant. □

In order to discuss the corresponding relations between TL-fuzzy ideals of two homomorphic lattices, we recall the following lemmas.

Lemma 1 ([19])**.** *Let f be a mapping from X_1 to X_2 and μ , ν be L-fuzzy subsets of X_1 and X_2, respectively. Then:*

(1) *$f(f^{-1}(\nu)) \subseteq \nu$ and the equality holds if f is surjective,*

(2) *$f^{-1}(f(\mu)) \supseteq \mu$ and the equality holds if f is injective.*

Based on the discussion above, the following theorem shows the corresponding relations between TL-fuzzy filters of two homomorphic lattices.

Theorem 2. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2. $TLFI_f[X_1]$ is the set of all TL-fuzzy ideals which are $f-$invariant of X_1, then there is an one to one order-preserving mapping from $TLFI_f[X_1]$ to $TLFI[X_2]$.*

Proof. Let φ: $TLFI_f[X_1] \longrightarrow TLFI[X_2]$ be defined by $\varphi(\mu) = f(\mu)$, for all $\mu \in TLFI_f[X_1]$.

(i) Clearly, φ is a map from $TLFI_f[X_1]$ to $TLFI[X_2]$.

(ii) Let us check that φ is injective. For any $\mu_1, \mu_2 \in TLFI_f[X_1]$, if $\varphi(\mu_1) = \varphi(\mu_2)$, which means $f(\mu_1) = f(\mu_2)$, then $f^{-1}(f(\mu_1)) = f^{-1}(f(\mu_2))$. Since μ_1 and μ_2 are $f-$invariant, it follows that $f^{-1}(f(\mu_1)) = \mu_1$ and $f^{-1}(f(\mu_2)) = \mu_2$. Hence $\mu_1 = \mu_2$. Therefore, φ is injective.

(iii) Now, we prove that φ is surjective. For all $\nu \in TLFI[X_2]$, by Proposition 4, we have that $f^{-1}(\nu)$ is also a TL-fuzzy ideal of X_1 and $f-$invariant. This means that $f^{-1}(\nu) \in TLFI_f[X_1]$. Since f is surjective, we have that $\varphi(f^{-1}(\nu)) = f(f^{-1}(\nu)) = \nu$ by Lemma 1. It follows that φ is surjective.

(iv) We shall check that φ is order-preserving. For any $\mu_1, \mu_2 \in TLFI_f[X_1]$, if $\mu_1 \subseteq \mu_2$, then for $x \in X_1$, $\mu_1(x) \leq \mu_2(x)$. Notice this, for any $y \in X_2$, we have that $f(\mu_1)(y) = \vee\{\mu_1(x)|f(x) = y, x \in X_1\} \leq \vee\{\mu_2(x)|f(x) = y, x \in X_1\} = f(\mu_2)(y)$, which means $f(\mu_1) \subseteq f(\mu_2)$. That is, $\varphi(\mu_1) \subseteq \varphi(\mu_2)$. Therefore, φ is order-preserving.

Combining the above arguments, we have that φ is an one to one order-preserving mapping from $TLFI_f[X_1]$ to $TLFI[X_2]$. □

Definition 6. *Let μ be an L-fuzzy subset in a lattice X. A TL-fuzzy ideal ν of the lattice X is said to be generated by μ, if $\mu \subseteq \nu$ and for any TL-fuzzy ideal ω of X, $\mu \subseteq \omega$ implies $\nu \subseteq \omega$. The TL-fuzzy ideal generated by μ will be denoted by $(\mu]_{TL}$.*

It follows from Definition 6 that $(\mu]_{TL}$ is the smallest TL-fuzzy ideal of the lattice X containing μ. And we can easily get that $(\mu]_{TL} = \cap_{i\in I}\{\mu_i \in TLFI[X]|\mu_i \supseteq \mu, i \in I\}$.

In what follows, we give the formula for calculating the TL-fuzzy ideals generated by L-fuzzy subsets.

Theorem 3. *Let μ be an L-fuzzy subset in the lattice X. Then for any $x \in X$, $(\mu]_{TL}(x) = \vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)|x \leq' a_1 \vee' a_2 \vee' \cdots \vee' a_n, a_i \in X, i \in I\}$.*

Proof. The proof is straightforward. □

Now, we discuss the image and the inverse image of the generated TL-fuzzy ideals of the lattice X. In order to get this, we recall the following lemma.

Lemma 2 ([19])**.** *Let f be a mapping from X_1 to X_2 and $\mu, \mu' \in LF[X_1]$, $\nu, \nu' \in LF[X_2]$, then:*

(1) $\mu \subseteq \mu'$ implies $f(\mu) \subseteq f(\mu')$, which means f is order-preserving. (2) $\nu \subseteq \nu'$ implies $f^{-1}(\nu) \subseteq f^{-1}(\nu')$, which means f^{-1} is order-preserving, too.

Now, we arrive at one of our main theorems.

Theorem 4. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2 and μ be an L-fuzzy subset of X_1. Then $f((\mu]_T) = (f(\mu)]_T$.*

Proof. By Proposition 3, we get that $f((\mu]_T)$ is a TL-fuzzy ideal of X_2. Notice $(\mu]_T \supseteq \mu$, we have $f((\mu]_T) \supseteq f(\mu)$ by Lemma 2. Now, suppose that ν is an any TL-fuzzy ideal of X_2 and $\nu \supseteq f(\mu)$, then $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1 by Proposition 4. Since $\nu \supseteq f(\mu)$, we have that $f^{-1}(\nu) \supseteq f^{-1}(f(\mu)) \supseteq \mu$ by Lemmas 1 and 2, which implies $f^{-1}(\nu) \supseteq (\mu]_T$. It follows from Lemmas 1 and 2, we get that $\nu \supseteq f(f^{-1}(\nu)) \supseteq f((\mu]_T)$. This means that $f((\mu]_T)$ is the smallest TL-fuzzy ideal of X_2 which contains $f(\mu)$. Therefore, $f((\mu]_T) = (f(\mu)]_T$. □

Theorem 5. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and ν be a TL-fuzzy subset of X_2. Then $f^{-1}((\nu]_T) = (f^{-1}(\nu)]_T$.*

Proof. By Proposition 4, we get that $f^{-1}((\nu]_T)$ is a TL-fuzzy ideal of X_1. Notice $(\nu]_T \supseteq \nu$, we have $f^{-1}((\nu]_T) \supseteq f^{-1}(\nu)$ by Lemma 2. Now, suppose that ω is an any TL-fuzzy ideal of X_1 and $\omega \supseteq f^{-1}(\nu)$, then $f(\omega)$ is a TL-fuzzy ideal of X_2 by Proposition 3. Since $\omega \supseteq f^{-1}(\nu)$, notice that f is surjective, we have that $f(\omega) \supseteq f(f^{-1}(\nu)) = \nu$ by Lemmas 1 and 2, which implies $f(\omega) \supseteq (\nu]_T$. It follows from Lemmas 1 and 2, we get that $\omega = f^{-1}(f(\omega)) \supseteq f^{-1}((\nu]_T)$. This means that $f^{-1}((\nu]_T)$ is the smallest TL-fuzzy ideal of X_1 which contains $f^{-1}(\nu)$. Therefore, $f^{-1}((\nu]_T) = (f^{-1}(\nu)]_T$. □

4. T-upper and θ-lower fuzzy rough approximations of TL-fuzzy ideals

In this section, based on the discussion in the previous sections, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices. We will investigate some properties of (θ, T)-fuzzy rough approximations of TL-fuzzy filters in lattices. We start by introducing the concept of TL-fuzzy congruence relation on lattices.

Definition 7. *A TL-fuzzy equivalence relation R on the lattice $(X, \wedge', \vee')$ is called a TL-fuzzy congruence relation if it satisfies the following conditions: for $x_1, x_2, y_1, y_2 \in X$, $R(x_1 \wedge' x_2, y_1 \wedge' y_2) \geq R(x_1, y_1)TR(x_2, y_2)$, $R(x_1 \vee' x_2, y_1 \vee' y_2) \geq R(x_1, y_1)TR(x_2, y_2)$.*

In particular, by a fuzzy congruence relation R on the lattice $(X, \wedge', \vee')$, we mean a TL-fuzzy congruence relation on $(X, \wedge', \vee')$ when $T = T_M$ and $L = [0, 1]$.

Proposition 5. *Let R be a TL-fuzzy congruence relation on the lattice X. If μ is a TL-fuzzy ideals of X, then $\overline{R}^T\mu$ is a TL-fuzzy ideals of X.*

Proof. Suppose that μ is a TL-fuzzy ideal of X. For any $x, y \in X$, we have $\overline{R}^T\mu(x \vee' y) = \vee_{z\in X}R(x \vee' y, z)T\mu(z) \geq \vee_{s,t\in X}R(x \vee' y, s \vee' t)T\mu(s \vee' t) \geq (\vee_{s,t\in X}R(x, s)TR(y, t))T(\mu(s)T\mu(t)) = (\vee_{s\in X}R(x, s)T\mu(s))T(\vee_{t\in X}R(y, t)T\mu(t))$ $= \overline{R}^T\mu(x)T\overline{R}^T\mu(y)$. On the other hand, we have $\overline{R}^T\mu(x \wedge' y) = \vee_{z\in X}R(x \wedge' y, z)T\mu(z) \geq \vee_{s,t\in X}R(x \wedge' y, s \wedge' t)T\mu(s \wedge' t) \geq (\vee_{s,t\in X}R(x, s)TR(y, t))T(\mu(s) \vee \mu(t)) = (\vee_{s\in X}R(x, s)T\mu(s)) \vee (\vee_{t\in X}R(y, t)T\mu(t)) = \overline{R}^T\mu(x) \vee \overline{R}^T\mu(y)$.

Combining the above arguments, we have that $\overline{R}^T\mu$ is a TL-fuzzy ideal of X. □

Next, let us consider the relationships between T-upper fuzzy rough approximation operator and the generated TL-fuzzy ideal.

Proposition 6. *Let R be a TL-fuzzy congruence relation on the lattice X. If μ is an L-fuzzy subset of X, then $\overline{R}^T(\mu]_T = \overline{R}^T(\overline{R}^T\mu]_T$.*

Proof. Since $\mu \subseteq (\mu]_T$, it follows that $\overline{R}^T\mu \subseteq \overline{R}^T(\mu]_T$. By Proposition 5, we have that $\overline{R}^T(\mu]_T$ is a TL-fuzzy ideal of the lattice X. We conclude that $(\overline{R}^T\mu]_T \subseteq \overline{R}^T(\mu]_T$. Again, by $(R2)$ and $(R4)$ of Proposition 2, we have that $\overline{R}^T(\overline{R}^T\mu]_T \subseteq \overline{R}^T\overline{R}^T(\mu]_T = \overline{R}^T(\mu]_T$. On the other hand, since $\mu \subseteq \overline{R}^T\mu$, it follows that $(\mu]_T \subseteq (\overline{R}^T\mu]_T$. Hence $\overline{R}^T(\mu]_T \subseteq \overline{R}^T(\overline{R}^T\mu]_T$.

Combining the above arguments, we have $\overline{R}^T(\mu]_T = \overline{R}^T(\overline{R}^T\mu]_T$. □

Open problem 2. Whether $\underline{R}_\theta\mu$ is a TL-fuzzy ideal of X if μ is a TL-fuzzy ideal of the lattice X and R is a TL-fuzzy congruence relation on the lattice X.

In what follows, we discuss some homomorphism properties of (θ, T)-fuzzy rough approximations of TL-fuzzy filters in lattices. In order to do this, we recall the following definition [30] and give some lemmas.

Let f be a mapping from X_1 into X_2 let R_1 and R_2 be two L-fuzzy relations on X_1 and X_2, respectively. The L-fuzzy relations $f(R_1)$ on X_2, defined by $f(R_1)(y_1, y_2) = \vee\{R_1(x_1, x_2)|f(x_1) = y_1, f(x_2) = y_2\}$, for any $(y_1, y_2) \in X_2 \times X_2$, if both y_1 and y_2 have inverse elements in X_1. Otherwise, $f(R_1)(y_1, y_2) = 0$. The L-fuzzy relations $f^{-1}(R_2)$ on X_1, is defined by $f^{-1}(R_2)(x_1, x_2) = R_2(f(x_1), f(x_2))$, for any $(x_1, x_2) \in X_1 \times X_1$. The L-fuzzy relations $f(R_1)$ and $f^{-1}(R_2)$ are called, respectively, the image of R_1 under f and the inverse image of R_2 under f.

Lemma 3. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and let R_1 and R_2 be two TL-fuzzy congruence relations on X_1 and X_2, respectively.*

(1) *If f is an isomorphism, then $f(R_1)$ is a TL-fuzzy congruence relation on X_2.*

(2) *$f^{-1}(R_2)$ is a TL-fuzzy congruence relation on X_1.*

Proof. It is easy to check them. □

Lemma 4. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and let R_1 be a TL-fuzzy congruence relation on the lattice X_1. Suppose that μ is a L-fuzzy subset of X_1, then $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta\mu)$, $\overline{f(R_1)}^T f(\mu) = f(\overline{R_1}^T\mu)$.*

Proof. Since $f : X_1 \longrightarrow X_2$ is an isomorphism from a lattice X_1 to a lattice X_2 and R_1 is a TL-fuzzy congruence relation on X_1. By Lemma 3, we have that $f(R_1)$ is a TL-fuzzy congruence relation on X_2. Then, for any $y \in X_2$, we have that

$\underline{f(R_1)}_\theta f(\mu)(y) = \wedge_{y' \in X_2} f(R_1)(y, y')\theta f(\mu)(y') = \wedge_{x' \in X_1} R_1(x, x')\theta\mu(x') = \underline{R_1}_\theta\mu(x) = f(\underline{R_1}_\theta\mu)(y)$, where $f(x) = y, f(x') = y'$. Therefore, $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta\mu)$.

In a similar way, for any $y \in X_2$, we have that

$\overline{f(R_1)}^T f(\mu)(y) = \vee_{y' \in X_2} f(R_1)(y, y')Tf(\mu)(y') = \vee_{x' \in X_1} R_1(x, x')T\mu(x') = \overline{R_1}^T\mu(x) = f(\overline{R_1}^T\mu)(y)$, where $f(x) = y, f(x') = y'$. Therefore, $\overline{f(R_1)}^T f(\mu) = f(\overline{R_1}^T\mu)$. □

Lemma 5. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and let R_2 be a TL-fuzzy congruence relation on X_2. Suppose that ν is a L-fuzzy subset of X_2, then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) \supseteq f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) \subseteq f^{-1}(\overline{R_2}^T\nu)$. In particular, when f is surjective, then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) = f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) = f^{-1}(\overline{R_2}^T\nu)$.*

Proof. Since $f : X_1 \longrightarrow X_2$ is a homomorphism from a lattice X_1 to a lattice X_2 and R_2 is a TL-fuzzy congruence relation on X_2. By Lemma 3, we have that $f^{-1}(R_2)$ is a TL-fuzzy congruence relation on X_1. Then, for any $x \in X_1$, we have that $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)(x) = \wedge_{x' \in X_1} f^{-1}(R_2)(x, x')\theta f^{-1}(\nu)(x') = \wedge_{x' \in X_1} R_2(f(x), f(x'))\theta\nu(f(x')) \geq \wedge_{y \in X_2} R_2(f(x), y)\theta\nu(y) = \underline{R_2}_\theta\nu(f(x)) = f^{-1}(\underline{R_2}_\theta\nu)(x)$. Therefore, we have $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) \supseteq f^{-1}(\underline{R_2}_\theta\nu)$.

In a similar way, for any $x \in X_1$, we have that $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)(x) = \vee_{x' \in X_1} f^{-1}(R_2)(x, x')Tf^{-1}(\nu)(x') = \vee_{x' \in X_1} R_2(f(x), f(x'))T\nu(f(x'))$
$\leq \vee_{y \in X_2} R_2(f(x), y)T\nu(y) = \overline{R_2}^T\nu(f(x)) = f^{-1}(\overline{R_2}^T\nu)(x)$. Therefore, we have $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) \subseteq f^{-1}(\overline{R_2}^T\nu)$.

It is easy to see that the equality "=" in the above proof holds if f is surjective. Then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) = f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) = f^{-1}(\overline{R_2}^T\nu)$. □

Combining the above lemmas and theorems, we have the following results.

Theorem 6. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and R_1 be a TL-fuzzy congruence relation on a lattice X_1. Suppose that μ is a L-fuzzy subset of X_1. Then*

(1) *$\underline{R_1}_\theta\mu$ is a TL-fuzzy ideal of X_1 iff $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2,*

(2) *$\overline{R_1}^T\mu$ is a TL-fuzzy ideal of X_1 iff $\overline{f(R_1)}^T f(\mu)$ is a TL-fuzzy ideal of X_2.*

Proof. (1) $\Longrightarrow$ Assume that $\underline{R_1}_\theta\mu$ is a TL-fuzzy ideal of X_1. By Proposition 3, we have that $f(\underline{R_1}_\theta\mu)$ is a TL-fuzzy ideal of X_2. Since f is an isomorphism from

X_1 to X_2, we get that $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta \mu)$ by Lemma 4. Hence, $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2.

$\Longleftarrow$ If $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2. By Proposition 4, we obtain that $f^{-1}(\underline{f(R_1)}_\theta f(\mu))$ is a TL-fuzzy ideal of X_1. Since f is injective, we can get that $f^{-1}(\underline{f(R)}_\theta f(\mu)) = \underline{f^{-1}(f(R_1))}_\theta f^{-1}(f(\mu)) = \underline{R_1}_\theta \mu$ by use of Lemmas 1 and 5. Therefore, $\underline{R_1}_\theta \mu$ is a TL-fuzzy ideal of X_1.

(2) In a similar way, we can prove that $\overline{R_1}^T \mu$ is a TL-fuzzy ideal of X_1 if and only if $\overline{f(R_1)}^T f(\mu)$ is a TL-fuzzy ideal of X_2.

□

Theorem 7. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and R_2 be a TL-fuzzy congruence relation on the lattice X_2. Suppose that ν is a L-fuzzy subset of X_2. Then*

(1) *$\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2 iff $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1,*

(2) *$\overline{R_2}^T \nu$ is a TL-fuzzy ideal of X_2 iff $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.*

Proof. (1) $\Longrightarrow$ Suppose that $\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2. By Proposition 4, we have that $f^{-1}(\underline{R_2}_\theta \nu)$ is a TL-fuzzy ideal of X_1. Since f is an epimorphism from X_1 to X_2, we get $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) = f^{-1}(\underline{R_2}_\theta \nu)$ by Lemma 5. Hence, $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.

$\Longleftarrow$ If $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. By Proposition 3, we obtain that $f(\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu))$ is a TL-fuzzy ideal of X_2. Since f is surjective, we get that $f(\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)) = \underline{f(f^{-1}(R_2))}_\theta f(f^{-1}(\nu)) = \underline{R_2}_\theta \nu$ by use of Lemmas 1 and 4. Therefore, $\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2.

(2) In a similar way, we can prove that $\overline{R_2}^T \mu$ is a TL-fuzzy ideal of X_2 if and only if $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. □

5. Conclusion

Based on the importance of ideals in studying logic algebras and extensive applications of t-norms and rough sets in fuzzy logic and artificial intelligence, in this paper, using a left continuous t-norm T on a complete lattice L, we introduce the notion of TL-fuzzy ideals of lattices and investigate some properties of them. We study the relationships among L-fuzzy sets, rough sets and lattices theory. We hope that the results presented in this paper can hopefully provide more insight into and a full understanding of lattices and fuzzy rough sets theory.

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References

[1] C.C. Chang, *Algebraic analysis of many valued logics*, Trans. Am. Math. Soc., 88 (1958), 467-490.

[2] I. Chon, *Fuzzy subgroups as products*, Fuzzy Sets and Systems, 141 (2004), 505-508.

[3] P. Cintula, P. Hájek, *Triangular norm based predicate fuzzy logics*, Fuzzy Sets and Systems, 161 (2010), 311-346.

[4] D. Dubois, H. Prade, *Rough fuzzy sets and fuzzy rough sets*, International Journal of General Systems, 17 (1990), 191-209.

[5] F. Esteva, L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, 124 (2001), 271-288.

[6] J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., 18 (1967), 145-174.

[7] P. Hájek, *Observations on the monoidal t-norm logic*, Fuzzy Sets and Systems, 132 (2002), 107-112.

[8] Y.B. Jun, Y. Xu, X.H. Zhong, *Fuzzy filters of MTL-algebras*, Information Sciences, 175 (2005), 120-138.

[9] O. Kazanci, B. Davvaz, *On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings*, Information Sciences, 178 (2008), 1343-354.

[10] E.P. Klement, R. Mesiar, E. Pap, *t-norms*, vol. 8 of Trends in Logic Studia Logica Library, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[11] F. Li, Y.Q. Yin, L.X. Lu, *(ϑ, T)-fuzzy rough approximation operators and TL-fuzzy rough ideals on a ring*, Information Sciences, 177 (2007), 4711-4726.

[12] S.L. Li, Y.D. Yu, Z.D. Wang, *T-congruence L-relations on groups and rings*, Fuzzy Sets and Systems, 92 (1997), 365-381.

[13] J. Liang, J. Wang, Y. Qian, *A new measure of uncertainty based on knowledge granulation for rough sets*, Information Sciences, 179 (2009), 458-470.

[14] G.L. Liu, *Axiomatic systems for rough sets and fuzzy rough sets*, International Journal of Approximate Reasoning, 48 (2008), 857-867.

[15] L.Z. Liu, K.T. Li, *Fuzzy filters of BL-algebras*, Information Sciences, 173 (2005), 141-154.

[16] D.Q. Miao, Y. Zhao, Y.Y. Yao, H.X. Li, F.F. Xu, *Relative reducts in consistent and inconsistent decision tables of the Pawlak rough set model*, Information Sciences, 179 (2009), 4140-4150.

[17] A. Mieszkowicz-Rolka, L. Rolka, *Fuzzy rough approximations of process data*, International Journal of Approximate Reasoning, 49 (2008), 301-315.

[18] N.N. Morsi, M.M. Yakout, *Axiomatics for fuzzy rough sets*, Fuzzy Sets and Systems, 100 (1998), 327-342.

[19] J.N. Mordeson, D.S. Malik, *Fuzzy commutative algebra*, World Scientific, 1998.

[20] C. Noguera, F. Esteva, L. Godo, *Generalized continuous and left-continuous t-norms arising from algebraic semantics for fuzzy logics*, Information Sciences, 180 (2010), 1354-1372.

[21] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, 11 (1982), 341-356.

[22] Y. Qian, J. Liang, D. Li, H. Zhang, C. Dang, *Measures for evaluating the decision performance of a decision table in rough set theory*, Information Sciences, 178 (2008), 181-202.

[23] J. Rachunek, D. Salounova, *Fuzzy filters and fuzzy prime filters of bounded Rℓ-monoids and pseudo BL-algebras*, Information Sciences, 178 (2008), 3474-3481.

[24] A.M. Radzikowska, E.E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems, 126 (2001), 137-155.

[25] Radzikowska AM, Kerre EE, *Fuzzy rough sets based on residuated lattices*, Transactions on Rough Sets, Lecture Notes in Computer Sciences, 3135 (2004), 278-296.

[26] R. Tayebi Khoramia, A. Borumand Saeidb, *New representation for filters of BL-algebras*, Journal of Intelligent and Fuzzy Systems, 23 (2012), 225-235.

[27] E. Turunen, *BL-algebras and basic fuzzy logic*, Mathware and Soft Computing, 6 (1999), 49-61.

[28] Z.D. Wang, *TL-filters of integral residuated l-monoids*, Information Sciences, 177 (2007), 887-896.

[29] W. Wang, X.L. Xin, *On fuzzy filters of pseudo BL-algebras*, Fuzzy Sets and Systems, 162 (2011), 27-38.

[30] Z.D. Wang, Y.D. Yu, Y.Y. Dai, *On T-congruence L-relations on groups and rings*, Fuzzy Sets and Systems, 119 (2001), 393-407.

[31] W.Z. Wu, Y. Leung, J.S. Mi, *On characterizations of* (I,T)*-fuzzy rough approximation operators*, Fuzzy Sets and Systems, 154 (2005), 76-102.

[32] W.Z. Wu, J.S. Mi, W.X. Zhang, *Generalized fuzzy rough sets*, Information Sciences, 151 (2003), 263-284.

[33] W.Z. Wu, W.X. Zhang, *Constructive and axiomatic approaches of fuzzy approximation operators*, Information Sciences, 159 (2004), 233-254.

[34] Q.M. Xiao, Q.G. Li, X.N. Zhou, *Rough ideals in lattices*, Neural Comput Appl., 21 (2012), 245-253.

[35] Q.M. Xiao, Z.L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Information Sciences, 176 (2006), 725-733.

[36] S. Yamak, O. Kazanci, D. Bayrak, *A solution to an open problem on the* T*-product of* TL*-fuzzy subgroups*, Fuzzy Sets and Systems, 178 (2011), 102-106.

[37] Y.Q. Yin, J.M. Zhan, P. Corsini, *Fuzzy roughness of* n*-ary hypergroups based on a complete residuated lattice*, Neural Comput. Appl., 20 (2011), 41-57.

[38] Y.Q. Yin, J.M. Zhan, P. Corsini, L*-fuzzy roughness of* n*-ary polygroups*, Acta Mathematica Sinica English Series, 27 (2011), 97-118.

[39] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

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A criterion of p-nilpotency of finite groups with some weakly s-semipermutable subgroups

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Abstract. Let H be a subgroup of a group G. H is called a weakly s-semipermutable subgroup of G if there are a subnormal subgroup T of G and an s-semipermutable subgroup H_{ssG} in G contained in H such that $G = HT$ and $H \cap T \leq H_{ssG}$. In this paper, we got a criterion of p-nilpotency of G by some weakly s-semipermutable subgroups.
Keywords: finite group, p-nilpotent group, weakly s-semipermutable subgroup.

1. Introduction

Throughout this paper, all groups are finite. Let G be a group and H a subgroup of G. Recall that H is said to be s-permutable (or s-quasinormal, π-quasinormal) of G if H permutes with every Sylow subgroup of G; H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H; H is said to be s-semipermutable in G if H permutes with every Sylow p-subgroup G_p of G with $(|H|, p) = 1$. More recently, the authors introduced in [4] the following concept.

Definition 1.1. *Let H be a subgroup of a group G. H is called a weakly s-semipermutable subgroup of G if there are a subnormal subgroup T of G and an s-semipermutable subgroup H_{ssG} in G contained in H such that $G = HT$ and $H \cap T \leq H_{ssG}$.*

Li et al. in [4] get the following result:

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Theorem I. Let G be a group and let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P of order $|D|$ and every cyclic subgroup of P of order 4 (if P is non-abelian 2-subgroup and $|D| = 2$) is weakly s-semipermutable in G. Then G is p-nilpotent.

A celebrated theorem of Frobenius [9, Satz. IV.5.8] asserts that G is p-nilpotent if either $N_G(H)$ is p-nilpotent or $N_G(H)/C_G(H)$ is a p-subgroup for every non-identity p-subgroup H of G. In this paper we replace some of the conditions of Frobenius' theorem and Theorem I, namely, H is restricted to be a p-subgroup of a fixed order, the condition that p is the smallest prime of $|G|$ is changed by either $N_G(H)$ is p-nilpotent or $N_G(H)/C_G(H)$ is a p-group and we assume that H is a weakly s-semipermutable subgroup of G. Our main result is the following theorems, which can be considered as a complement to Frobenius' theorem and Theorem I with weakly s-semipermutable subgroups.

Main result

Let p be an odd prime dividing the order of a group G and P a Sylow p-subgroup of G. If there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly s-semipermutable in G and either $N_G(H)$ is p-nilpotent or $N_G(H)/C_G(H)$ is a p-group, then G is p-nilpotent.

Remark 1.2. Let H be a p-subgroup of a group G. If $N_G(H)$ is p-nilpotent, then obviously $N_G(H)/C_G(H)$ is a p-group. But the converse is not true in general. For example, let $G = A_4 \times C_2$. Evidently $N_G(C_2)/C_G(C_2)$ is a 2-group, but $G = N_G(C_2)$ is not 2-nilpotent.

All unexplained notations and terminologies are standard. The reader is referred to [9] if necessary.

2. Preliminaries

In this section, we list some lemmas which will be useful for the proofs of our results.

Lemma 2.1 ([4, Lemma 2.1])**.** (*a*) *An s-permutable subgroup of G is subnormal in G.*

(*b*) *If $H \leq K \leq G$ and H is s-permutable in G, then H is s-permutable in K.*

(*c*) *Let $K \trianglelefteq G$. If H is s-permutable in G, then HK/K is s-permutable in G/K.*

(*d*) *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.2 ([4, Lemma 2.2])**.** *Let G be a group. Suppose that H is an s-semipermutable subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is s-semipermutable in K.*

(2) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(3) *If $H \leq O_p(G)$, then H is s-permutable in G.*

(4) *Suppose that H is a p-subgroup of G for some prime $p \in \pi(G)$ and N is normal in G. Then $H \cap N$ is also an s-semipermutable subgroup of G.*

Lemma 2.3 ([4, Lemma 2.3]). *Let G be a group and U a weakly s-semipermutable subgroup of G and N a normal subgroup of G. Then:*

(1) *If $U \leq H \leq G$, then U is weakly s-semipermutable in H.*

(2) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly s-semipermutable in G/N.*

(3) *Suppose that U is a p-group for some prime p. and N is a p'-subgroup. Then UN/N is weakly s-semipermutable in G/N.*

(4) *If U is a p-group for some prime p and N a normal subgroup of G contained in $O^p(G)$, then $U \cap N$ is an s-semipermutable subgroup of G.*

Lemma 2.4 ([9, III, 5.2 and IV, 5.4]). *Suppose p is a prime and G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent. Then:*

(a) G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$;

(b) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;

(c) The exponent of P is p or 4.

Lemma 2.5. *Let G be a group and P a p-subgroup of G, where p is a prime divisor of $|G|$. Suppose that $N_G(P)/C_G(P)$ is a p-group. Let N be a normal subgroup of G. If either N is a p'-group or $N \leq P$, then $N_{G/N}(PN/N)/C_{G/N}(PN/N)$ is also a p-group.*

Proof Let $N_{G/N}(PN/N) = K/N$. Then $PN \trianglelefteq K$. By the hypothesis, P is a Sylow p-subgroup of NP. Thus by the Frattini argument $K = N_K(P)PN = N_G(P)N \leq K$, that is, $N_{G/N}(PN/N) = N_G(P)N/N$. It follows from $N_G(P)/C_G(P)$ is a p-group that

$$N_G(P)N/N/C_G(P)N/N \simeq N_G(P)N/C_G(P)N \simeq N_G(P)/C_G(P)(N_G(P) \cap N)$$

is a p-group. Evidently $C_G(P)N/N \leq C_{G/N}(PN/N)$.

Hence $N_{G/N}(PN/N)/C_{G/N}(PN/N)$ is a p-group as desired.

Lemma 2.6. *Let G be a group and H a p-subgroup of G, where p is a prime divisor of $|G|$. Let M be a subgroup of G containing H. If $N_G(H)/C_G(H)$ is a p-group, then $N_M(H)/C_M(H)$ is also a p-group.*

Proof Since $N_M(H)/C_M(H) = N_G(H) \cap M/C_G(H) \cap M \cong (N_G(H) \cap M)C_G(H)/C_G(H) \leq N_G(H)/C_G(H)$, the result is obvious.

3. Proof of main result

Let H be a p-subgroup of G. By Remark 1.2., we only need to prove that if there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly s-semipermutable in G and $N_G(H)/C_G(H)$ is a p-group, then G is p-nilpotent. Assume it is false and G is a counterexample with minimal order. Now we derive a contradiction from the following several steps.

Step 1. $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Then by Lemmas 2.3(3) and 2.5, we have $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

Step 2. Let T be a subgroup of G such that $P \leq T < G$, then T is p-nilpotent.

Let H be a subgroup of T of order $|D|$. By hypothesis and Lemma 2.6, we have that $N_T(H)/C_T(H)$ is a p-group and by Lemma 2.3(1) H is weakly s-semipermutable in T. Hence T satisfies the hypothesis of the theorem. The minimality of G forces that T is p-nilpotent.

Step 3. If $|P : D| > p$, then every subgroup H of P of order $|D|$ is s-semipermutable in G and $O^p(G) = G$.

Assume that P has a subgroup H such that $|H| = |D|$ and H is not s-semipermutable in G. By hypothesis, there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K < H$. It follows that $K < G$. Let M be a normal maximal subgroup of G containing K. Then $|G : M| = p$. By Lemmas 2.3(1) and 2.6, the hypothesis is still true for M since $|P : D| > p$. The minimal choice of G implies that M is p-nilpotent and so G is p-nilpotent, a contradiction. Hence every subgroup H of P of order $|D|$ is s-semipermutable in G. Similarly, we have that $O^p(G) = G$.

Step 4. $|D| > p$.

Suppose that $|D| = p$. Since G is not p-nilpotent, G has, by [9, IV, 5.4], a p-closed Schmidt subgroup $K = [K_p]K_q$, where $K_p \leq P$. By hypothesis and Lemma 2.4, the exponent of K_p is p. Now every subgroup H of K of order p is weakly s-semipermutable in K by Lemma 2.3 and $N_K(H)/C_K(H)$ is a p-group by Lemma 2.6. Let x be an element in K_p of order p. Then $\langle x\rangle$ is weakly s-semipermutable in K. Let T be a subnormal subgroup of K and $\langle x\rangle_{ssK}$ an s-semipermutable subgroup of K contained in $\langle x\rangle$ such that $K = \langle x\rangle T$ and $\langle x\rangle \cap T \leq \langle x\rangle_{ssK}$. Since $O^p(K) = K$, $T = K$ and so $\langle x\rangle = \langle x\rangle_{ssK}$ is s-semipermutable in K. By Lemmas 2.2(3) and 2.1(d) $\langle x\rangle$ is normal in K. It follows that $N_K(\langle x\rangle)/C_K(\langle x\rangle) = K/C_K(\langle x\rangle)$ is a p-group. Hence $\langle x\rangle \in Z(K)$, which implies that every element of K of order p were in $Z(K)$, then K would be p-nilpotent by [9, Satz IV.5.5], a contradiction. Hence $|D| > p$.

Step 5. $O_p(G) \neq 1$.

Since G is not p-nilpotent, by the Glauberman-Thompson Theorem, $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Then $P \leq N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$.

Step 6. Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then $|N| < |D|$.

If $|N| = |D|$, then by hypothesis, $G/C_G(N) = N_G(N)/C_G(N)$ is a p-group. So $O^p(G) \leq C_G(N)$. If N is a Sylow p-subgroup of $C_G(N)$, then by Schur-Zassenhaus Theorem, N has a complement K in $C_G(N)$ and so $C_G(N) = N \times K$. It follows that K is normal in G, which contradicts Step 1. Hence N is not a Sylow p-subgroup of $C_G(N)$. It is easy to see that $C_G(N)$ satisfies the hypothesis. If $C_G(N) < G$, by induction $C_G(N)$ is p-nilpotent and then G is p-nilpotent, a contradiction. Hence $C_G(N) = G$, that is, $N \leq Z(G)$ and so $|N| = |D| = p$, which contradicts step 4.

Now suppose that $|N| > |D|$. Let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Consider $C_G(N)$ and NQ. Obviously $N \leq C_G(N)$, $N \leq NQ$ and $P \cap C_G(N)$ is a Sylow p-subgroup of $C_G(N)$. It is easy to see that NQ and $C_G(N)$ satisfy the hypothesis of the theorem. If $C_G(N) = G$, then $|N| = p$ and so $|D| = 1$, contradicts the hypothesis. Hence $C_G(N) < G$ and then $C_G(N)$ is p-nilpotent by the choice of G. Hence Step 1 forces that $C_G(N) = N$. Now consider NQ. If $NQ < G$, then by induction NQ is p-nilpotent and so $Q \leq C_G(N) = N$, a contradiction. Hence $NQ = G$. It follows that $N = P$ is a minimal normal subgroup of G. Let H be a subgroup of N with order $|D|$, T a subnormal subgroup of G and H_{ssG} an s-semipermutable subgroup of G contained in H such that $G = HT$ and $H \cap T \leq H_{ssG}$. If $T < G$, then $G = NT$ and $N \cap T = 1$ for the minimality of N. Therefore $N = H$, contradicts $|N| > |D|$. Hence $H = H_{ssG}$ is s-semipermutable in G. By Lemma 2.2(3), H is s-permutable in G and $O^p(G) \leq N_G(H)$ follows from Lemma 2.1(d). Noticing that $H \trianglelefteq N$, H is normal in G, contradicts the minimality of N and the hypothesis $1 < |D|$. Hence $|N| < |D|$.

Step 7. G has a maximal subgroup M such that $G = MN$, $M \cap N = 1$. Furthermore, $C_G(N) = N = O_p(G)$ is the unique minimal normal subgroup of G.

By Step 6 and Lemma 2.3, it is easy to see that G/N satisfies the hypothesis of the theorem, so the choice of G yields that G/N is p-nilpotent. The uniqueness of N and $N \leq \Phi(G)$ are obvious. So G has a maximal subgroup M such that $G = MN$, $M \cap N = 1$ and $M \cong G/N$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, the uniqueness of N yields $O_p(G) \cap M = 1$ and so $N = O_p(G)$. Noticing that G is p-solvable, $C_G(N) \leq N$ follows from Step 1 and [7, Theorem 9.3.1]. On the other hand, N is abelian, so $C_G(N) = N$.

Step 8. $|N| = p$.

By Step 7, $P = N(P \cap M)$. If $|P : D| > p$, pick a maximal subgroup N_1 of N and pick a subgroup S_1 of $M \cap P$ such that $H = N_1S_1$ is of order $|H| = |D|$.

Then by Step 3 every subgroup H of P of order $|D|$ is s-semipermutable in G. So $N_1 = N \cap H$ is s-semipermutable in G by Lemma 2.2(4). Furthermore, N_1 is s-permutable in G by Lemma 2.2(3), Therefore, N_1 is normal in G by Lemma 2.1(4) and Step 3. Then $N_1 = 1$ by the minimality of N and so $|N| = p$.

Now assume that $|P : D| = p$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P_1 \cap N$ is s-semipermutable in G by the unique minimality of N and Lemma 2.3(4), then $P_1 \cap N$ is s-permutable in G by Lemma 2.2(3). Hence $P_1 \cap N$ is normal in G follows from Lemma 2.1 and the fact that $P_1 \cap N$ is normal in P. The minimality of N implies that $P_1 \cap N = 1$. Hence N has order p.

Step 9. The final contradiction.

By Step 8, $Aut(N)$ is a cyclic group of order $p-1$. For $G/N \cong N_G(N)/C_G(N) \lesssim Aut(N)$, we have that $P \leq C_G(N) = N$, contradicts Step 6 and the hypothesis. The final contradiction completes the proof. □

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References

[1] A. N. Skiba, *On weakly s-permutable subgroups of finite groups,* J. Algebra, 315 (2007), 192-209.

[2] Xinjian Zhang, Xianhua Li, *A criterion of p-nilpotentcy of finite groups*, Comm. Algebra, 40 (2012), 3652-3657.

[3] O. H. Kegel, *Sylow-Gruppen and Subnoramlteiler endlicher Gruppen*, Math. Z., 78 (1962), 205-221.

[4] Yangming Li, Shouhong Qiao,Ning Su, Yanming Wang, *On weakly s-semipermutable subgroups of finite groups*, Journal of Algebra, 371 (2012), 250-261.

[5] Huaquan Wei, Yanming Wang, *On c^*- normality and its properties,* J. Group Theory, 10 (2007), 211-223.

[6] Yangming Li, Yanming Wang, Huaquan Wei, *The influence of π-quasinormal of some subgroups of a finite group*, Arch. Math., 81 (2003), 245-252.

[7] D. J. S. Robinson, *A course in the theory of groups*, Springer, New York, Berlin, 1993.

[8] D. Gorenstein, *Finite groups*, New York-Evanston-London, 1968.

[9] B. Huppert, *Endliche gruppen I*, Springer, New York, Berlin, 1967.

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Co-small monoform modules

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Abstract. The concept of small monoform module was introduced by Hadi and Marhun, where a module U is called small monoform if for each non-zero submodule V of U and for every non-zero homomorphism $f \in Hom_R(V, U)$, implies that ker f is small submodule of V. In this paper the author dualizes this concept; she calls it co-small monoform module. Many fundamental properties of co-small monoform module are given. Partial characterization of co-small monoform module is established. Also, the author dualizes the concept of small quasi-Dedekind modules which given by Hadi and Ghawi. She show that co-small monoform is contained properly in the class of the dual of small quasi-Dedekind modules. Furthermore, some subclasses of co-small monoform are investigated. Other generalizations of co-small monoform are introduced.

Keywords: essential submodules, small monoform modules, Co-small monoform modules.

1. Introduction

"Throughout this paper, all rings R are commutative with identity and all modules are unitary left R-modules. A submodule V of an R-module U is called small if (denoted by $V \ll U$) if, for every proper submodule L of U, $V + L \neq U$ ([3], P.20)". "A non-zero module U is called monoform if for each non-zero homomorphism $f \in Hom_R(V, U)$ with every non-zero submodule V of U; f is monomorphism [8]". In 2014, Hadi and Marhun introduced the concept of small monoform module as a generalization of monoform module, "where an R-module U is called small monoform if for each non-zero submodule V of U and for each non-zero homomorphism $f \in Hom_R(V, U)$; ker f is small submodule of V [8]". "A submodule V of U is called essential (denoted by $V \leq_e U$ if every non-zero submodule of U has a non-zero intersection with V ([3], P.15)".

The main goal of this work is to dualize the concept of small monoform module; we call it co-small monoform module, where a non-zero R-module U is called co-small monoform if for each proper submodule V of U and for every non-zero homomorphism $f : U \to \frac{U}{V}$; $f(U)$ is essential submodule of $\frac{U}{V}$. This paper consists of five sections, in section 2; fundamental properties of co-

small monoform are investigated. We prove that under the class of co-small monoform modules, the class of small submodules coincides with the class of e-small submodules, see theorem (2.13), "where a submodule V of U is called e-small if $N + W = U$ with $W \leq_e U$, implies that W = U [15]". In section 3; the relationship of co-small monoform module with coquasi-Dedekind module is studied, see proposition (3.1) and proposition (3.7). Also, we show that every coquasi-Dedekind ring is co-small monoform, see corollary (3.4). In section 4; we show that epiform, anti-hopfian and almost finitely generated modules are subclasses of co-small monoform modules. Section 5 is devoted to introduce some generalizations of co-small monoform module such as co-small quasi-Dedekind, relatively co-small monoform and E-coprime modules. Besides, partial equivalent between co-small monoform module and each one of these generalizations are obtained, see theorems (5.3) and (5.6).

2. Co-small monoform modules

In this section we introduce the dual notion of small monoform module, we start by following.

Definition 2.1. *A non-zero module R-module U is called co-small monoform module if for each proper submodule V of U and for every non-zero homomorphism $f : U \to \frac{U}{V}$; $f(U)$ is essential submodule of $\frac{U}{V}$. A ring R is called co-small monoform if R is co-small monoform R-module.*

Remarks and Examples 2.2.

i. Every simple module is co-small monoform module. Since if U is small, then the only proper submodule of U is (0), and by Schur's Lemma, for each non-zero homomorphism $f \in (Hom_R(U, \frac{U}{(0)}))$; f is epimorphism ([12], P.73), hence $f(U) \leq_e \frac{U}{(0)}$.

ii. Semisimple module is not co-small monoform module, as we will see in remark (3.6)(b).

iii. The set of integer numbers Z as Z-module is not co-small monoform module. In fact if we consider the homomorphism $f : Z \to \frac{Z}{6Z}$ which is defined by $f(x) = 3x + 6Z$ for all $x \in Z$. Clearly $\frac{Z}{6Z} \cong Z_6$, and $f(Z) = \frac{3Z}{6Z} \cong (\bar{3})$ and $(\bar{3})$ is not essential submodule of Z_6.

iv. The Z-module Z_4 is a co-small monoform module. In fact the only non-zero homomorphism $f \in (Hom_R(Z_4, \frac{Z_4}{(2)}))$ is defined as: $f(x) = 2x \; \forall x \in Z$, so $f(Z4) = (\bar{2})$ is essential submodule of $\frac{Z_4}{(2)}$.

v. Z_{P^∞} is a co-small monoform Z-module, see Proposition 4.8.

vi Uniform module may not be co-small monoform, "where an R-module U is called uniform if all non-zero submodules of U are essential in U [2]". For example Z is a uniform Z-module, but it is not co-small monoform as we verified in (iii).

Proposition 2.3. *If all non-zero factor of an R-module U is uniform, then U is a co-small monoform module.*

Proof. Let $f : U \to \frac{U}{V}$ be a non-zero homomorphism, and V be a proper submodule of U. Note that $\frac{U}{V} \neq 0$, and since $\frac{U}{V}$ is a uniform module, then $f(U) \leq_e \frac{U}{V}$, and we are done. □

Remark 2.4. If we replace the word "all" in the proposition (2.3), by "there exists", then we cannot guarantee U is co-small monoform. In fact in example (2.2)(iii) we see that Z is not co-small monoform. Note that not all non-zero factor of Z are uniform modules.

Hadi and Marhun in [8] proved that an epimorphic image of small monoform is not necessarily small monoform. For co-small monoform modules we have the following.

Proposition 2.5. *The epimorphic image of co-small monoform module is co-small monoform.*

Proof. Assume that U is a co-small monoform module and $f : U \to U_1$ is an epimorphism, where U_1 is any R-module. Let $g : U_1 \to \frac{U_1}{V_1}$ be a non-zero homomorphism where V_1 is any proper submodule of U_1, so we have following compositions of homomorphisms:

$$U \xrightarrow{f} U_1 \xrightarrow{g} \frac{U_1}{V_1} \xrightarrow{h} \frac{U}{f^{-1}(V)}$$

Where h is defend as follows: for $u_1 \in U_1$, since f is an epimorphism, then there exists $u \in U$ such that $f(u) = u_1$. So set $h(u_1 + V_1) = u + f^{-1}(V_1)$ for each $u_1 \in U_1$. One can show that h is an isomorphism. Since U is co-small monoform module, then $((hgf)(U)) \leq_e \frac{U}{f^{-1}(V_1)}$. We claim that $g(U_1) \leq_e \frac{U_1}{V_1}$, to show that; $(hgf)(U) = (hg)(U_1) \leq_e \frac{U}{f^{-1}(V_1)}$. This implies that $h^{-1}((hg)(U_1)) \leq_e h^{-1}(\frac{U}{f^{-1}(V_1)})$ ([3], Prop.(1.1), P.16). But h is an isomorphism, thus $g(U_1) \leq_e \frac{U_1}{V_1}$, hence U_1 is co-small monoform. □

Corollary 2.6. *The quotient of co-small monoform module is co-small monoform.*

Corollary 2.7. *A direct summand of co-small monoform module is co-small monoform.*

Proof. Let $U = V_1 \oplus V_2$, where both of V_1 and V_2 are submodules of U. Consider the projection homomorphism $f : U \to V_1$. Since U is a co-small monoform module, then by corollary (2.6); $\frac{U}{V_2}$ is co-small monoform. But $V_1 \cong \frac{U}{V_2}$, thus V_1 is co small. □

Remark 2.8. The direct sum of co-small monoform module may not be co-small monoform; for example both of Z_2 and Z_3 are simple Z-modules, hence they are co-small monoform modules. But the Z-module $Z_2 \oplus Z_3$ is not, in fact $Z_2 \oplus Z_3 \cong Z_6$ which is semisimple, and by example (2.2)(ii) it is not co-small monoform module.

Proposition 2.9. *Let U be a co-small monoform module, then for each element $0 \neq r \in R$; either $rU = 0$ or $rU \leq_e U$.*

Proof. Let $0 \neq r \in R$ and $f \in End_R(U)$ defined by $f_r(u) = ru \; \forall u \in U$. Assume that $rU \neq 0$, then $f \neq 0$. But U is co-small monoform, then $f(U) \leq_e U$. Thus $rU \leq_e U$. □

"Recall that an R-module U is called divisible, if $rU = U$ for every non-zero divisor element $r \in R$ ([3], P.102)". This concept and proposition (2.9) led us to introduce the following concept.

Definition 2.10. *An R-module U is called essentially divisible if $rU \leq_e U$ for each non-zero divisor element $r \in R$.*

Remark 2.11.
i. "It is clear that every divisible R-module is essentially divisible. The converse is not true in general", for example: Z is essentially divisible Z-module, in fact $rZ \leq_e Z \; \forall \; 0 \neq r \in Z$ while Z is not divisible.

ii. If a module U is semisimple, then we can easily show that the class of divisible module coincides with the class of essentially divisible module.

Proof (ii). Let $0 \neq r \in R$, since U is essentially divisible, then $rU \leq_e U$. But U is semisimple, therefore $rU \leq_c U$, hence $rU = U$. □

Proposition 2.12. *If U is faithful co-small monoform module then U is essentially divisible.*

Proof. Let $0 \neq r \in R$, since U is co-small monoform, then by proposition (2.9) either $rU = 0$ or $rU \leq_e U$. If $rU = 0$ and since U is faithful, then $r = 0$. But this is a contradiction, therefore $rU \leq_e U$. □

In the following proposition we use co-small monoform as a useful condition under which e-small submodule can be small.

Theorem 2.13. *Let U be a co-small monoform module, then V is small submodule if and only if V is e-small.*

Proof. The necessity is clear. Conversely, Assume that V is an e-small submodule, and let $V + L = U$ where $L \leq U$. Define $\Psi : U \to \frac{U}{V \cap L}$ as follows: $\forall u \in U; u = x + y$ where $x \in V$ and $y \in L$. Set $\Psi(u) = y + V \cap L$. It is clear that Ψ is well defined and homomorphism. If $\Psi = 0$, then $y \in V$ for all $y \in L$. This implies that $L \subseteq V$, hence $V = U$. But this is a contradiction, thus $\Psi \neq 0$. Since U is co-small monoform, then $\Psi(U) \leq_e (\frac{U}{V \cap L})$, so that $(\frac{L}{V \cap L}) \leq_e (\frac{U}{V \cap L})$. Thus

$L \leq_e U$ ([12], Exc.(1.64), P.32). On the other hand, V is an e-small submodule, therefore $L = U$ that is V is a small submodule of U. □

"An R-module U is called hollow if every proper submodule of U is small, and U is called e-hollow if every proper submodule V of U e-small [4]".

Corollary 2.14. *Let U be a co-small monoform module, then U is hollow module if and only if U is e-hollow module.*

Proof. The necessity is obvious. Conversely, assume that U is an e-hollow module and let $V \lneq U$. Since U is co-small monoform, so by theorem (2.13); V is small submodule of U, hence U is a hollow module. □

"Recall that an R-module U is called couniform if every proper submodule V of U is either zero or there exists a proper submodule W of V such that $(\frac{V}{W}) \ll (\frac{U}{W})$ [5]".

Corollary 2.15. *Let U be a co-small monoform module. If U is couniform and Artinian then U is an e-hollow module.*

Proof. Since U is a couniform and Artinian module, then U is hollow [5]. By corollary (2.14), U is e-hollow. □

3. Co-small monoform and coquasi-Dedekind modules

"Recall that an R-module U is called coquasi-Dedekind if $Hom_R(U, V) = 0$ for each proper submodule V of U. Equivalently; U is coquasi-Dedekind if every $0 \neq f \in End_R(U)$ is epimorphism [14]". The two concepts co-small monoform module and coquasi-Dedekind are independent for example; The integer numbers Z-module Z is a coquasi-Dedekind module. But it is not co-small monoform, see (2.2)(iii). On the other hand, Z_4 is co-small monoform as we have be seen in example (2.2)(iv), while it is not coquasi-Dedekind [14]. In fact we will show later that in the category of rings; every coquasi-Dedekind ring is co-small monoform.

However, this section is devoted to study how can be relate between them, before that; an R-module U is called self-generator if for every submodule V of U, $V = \Sigma_f f(U)$ where $f \in Hom_R(U, V)$ [11].

Proposition 3.1. *If U is a self-generator module, then every coquasi-Dedekind module is co-small monoform.*

Proof. Let V be a proper submodule of the self-generator module U, then$V = \Sigma_f f(U)$ where $f \in Hom_R(U, V)$. Since U is coquasi-Dedekind, then $Hom_R(U, V) = 0$, hence $f = 0$. This implies that $V = \Sigma_f f(U) = 0$, therefore U is simple. By example (2.2)(i), U is co-small monoform. □

"An R-module U is called multiplication if for every submodule V of U, there exists an ideal I of R such that $V = IU$ [11]". Since every multiplication module is self-generator [11], then we have the following.

Corollary 3.2. *If U is a multiplication module, then every coquasi-Dedekind module is co-small monoform.*

Since any ring R is multiplication R-module, so by using corollary (3.2) we deduce the following implication in the category of rings.

Corollary 3.3. *Every coquasi-Dedekind ring is a co-small monoform ring.*

Corollary 3.4. *Every cyclic coquasi-Dedekind module is a co-small monoform module.*

Now, we give a useful property for co-small monoform module.

Proposition 3.5. *Every co-small monoform module satisfies the following: condition* $(*)$*: For every non-zero* $f \in End_R(U)$*;* $f(U) \leq_e U$*.*

Proof. Let U be co-small monoform, and $0 \neq f \in End_R(U)$. Let V be a proper submodule of U, so we have the following composition of homomorphisms:

$$U \xrightarrow{f} U \xrightarrow{\pi} \frac{U}{V}$$

Since U is co-small monoform, then $(\pi f)(U) \leq_e (\frac{U}{V})$. This implies that $(\frac{f(U)}{V}) \leq_e (\frac{U}{V})$, hence $f(U) \leq_e U$ ([12], Exc (1.64), P.32). □

Remark 3.6.

a. The converse of proposition (3.5) is not true in general, for example the set of rational numbers Q is a uniform Z-module so every non-zero submodule of Q is essential, in particular $f(U) \leq_e U$. On the other hand Q is not co-small monoform. In fact Q is a torsion free module and $\frac{Q}{Z}$ is torsion, therefore $Hom_R(Q, \frac{Q}{Z}) = 0$, that is $f(Q) \nleq_e \frac{Q}{Z}$. Thus Q is not co-small monoform.

b. We can use proposition (3.5) to show that the semisimple module is not co-small monoform, in fact if U is co-small monoform , then by proposition (3.5), for any proper submodule V of an R-module U and any non-zero homomorphism $f \in End_R(U)$; $f(U) \leq_e U$. On the other hand, since U is semisimple, then $f(U)$ is a direct summand of U, so we have a contradiction, thus U is not co-small monoform.

"Recall that a ring R is called (Von Newoman) regular if for each $r \in R$ there exists $x \in R$ such that $r = rxr$ ([2], P.(186))".

Proposition 3.7. *Let U be an R-module, with* $End_R(U)$ *is regular ring, then every co-small monoform module is coquasi-Dedekind.*

Proof. Let $0 \neq f \in End_R(U)$, and V be a proper submodule of U. Consider the following composition of homomorphisms:

$$U \xrightarrow{f} U \xrightarrow{\pi} \frac{U}{V}$$

It is clear that πf is a non-zero homomorphism. Since U is co-small monoform, then $(\pi f)(U) \leq_e \frac{U}{V}$. This implies that $\frac{f(U)}{V} \leq_e \frac{U}{V}$, hence $f(U) \leq_e U$ ([12], Exe.(1.64), P.32), but $End_R(U)$ is a regular ring, then $f(U)$ is a direct summand of U [3]. Thus $f(U) = U$, that is U is coquasi-Dedekind. □

Note the condition "$End_R(U)$ is regular ring" in proposition (3.7) is necessary, because if $End_R(U)$ is not regular, we cannot guarantee that every co-small monoform module is coquasi-Dedekind. For example the Z-module Z_4 is co-small monoform as we saw in (2.2)(iv), but Z_4 is not coquasi-Dedekind [14], in fact $End_R(Z_4) \cong Z_4$ which is not regular ring.

Corollary 3.8. *Every semisimple co-small monoform module is coquasi-Dedekind.*

Proof. Assume that U is co-small monoform, since U is semisimple, then $End(U)$ is regular ([12], P.91), and the result follows by proposition (3.7). □

4. Subclasses of co-small monoform modules

In this section we show that epiform, anti-hopfian and almost finitely generated are contained in the class of co-small monoform modules." A non-zero module U is called epiform if every non-zero homomorphism $f : U \to \frac{U}{V}$ with V a proper submodule of U is an epimorphism [2]. Equivalently; U is epiform if every proper submodule of U is corational, where a submodule V of U is called corational if $Hom_R(U, \frac{V}{W}) = 0$ for all submodules W of U such that $W \subseteq V \subseteq U$ ([2], P.85)".

Remark 4.1. It can be easily show that each epiform module is co-small monoform, but the reverse is not always true; for example Z_4 is co-small monoform, see example (2.2)(iv) , but not epiform, to see this; if Z_4 is epiform, then every proper submodule of U is corational, but clearly the submodule $(\bar{2})$ is not corational, so Z_4 is not epiform.

As applications of (4.1), we have the following two propositions. The first one is about the relationship between co-small monoform and hollow modules. In fact there is no direct implication between co-small monoform and hollow modules; however, we can relate them by using the class of noncosingular module, "where a module U is called noncosingular if for any non-zero R-module T and for every nonzero homomorphism $f : U \to T$; $f(U)$ is not small submodule of T [5]".

Proposition 4.2. *Let U be a noncosingular module. If U is hollow then U is a co-small monoform module.*

Proof. Suppose that U is a hollow module. Since U is noncosingular, then U is epiform [1], and by remark (4.1), U is co-small monoform. □

"Recall that an R-module U is called cosemisimple if $Rad(\frac{U}{V}) = 0$ for all submodules V of U [2]. The second application of remark (4.1) is given by the following.

Proposition 4.3. *Let U be a couniform module. If U is Artinian and cosemisimple, then U is a co-small monoform module.*

Proof. Since U is couniform and Artinian, then U is a hollow module. But U is cosemisimple, thus U is epiform [1], hence U is a co-small monoform module.□

Theorem 4.4. *Let U be a self-generator semisimple module, then the following statements are equivalent:*

1. *U is a simple module.*
2. *U is an epiform module.*
3. *U is a co-small monoform module.*
4. *U is a coquasi-Dedekind module.*

Proof. (1) ⇒ (2) [1]
(2) ⇒ (3) It is clear.
(3) ⇒ (4) Since U is co-small monoform and semisimple, then by corollary (3.8), U is coquasi-Dedekind.
(4) ⇒ (1) Since U coquasi-Dedekind and self-generator, then by proposition (3.1), U is simple. □

"Recall that an R-module U is called anti-hopfian if U is not simple and all non-zero factor modules of U are isomorphic to U; that is U is not simple and $\frac{U}{V} \cong U$ for every proper submodule V of U [9,10]. The following proposition shows that the anti-hopfian module is subclass of co-small monoform modules.

Proposition 4.5. *Every anti-hopfian module is co-small monoform.*

Proof. Since U is anti-hopfian, then U is an epiform module [6], and by remark (4.1), U is co-small monoform. □

By using the concept of anti-hopfian we have the following.

Proposition 4.6. *If U is a coquasi-Dedekind module such that every proper submodule of U is anti-hopfian, then U is a co-small monoform module.*

Proof. Assume that U is coquasi-Dedekind module. Since every proper submodule of U is anti-hopfian, then U is an epiform ([6], Prop.(3.8)), hence U is co-small monoform. □

"An R-module U is called almost finitely generated if U is not finitely generated and every proper submodule of U is finitely generated [13]". We need the following lemma which is appeared in [13].

Lemma 4.7. *Let U be an almost finitely generated R-module. If U_1 and U_2 are almost finitely generated, then for all $0 \neq f \in Hom_R(U_1, U_2)$; f is an epimorphism.*

Proposition 4.8. *Every almost finitely generated R-module is co-small monoform.*

Proof. Suppose that U is an almost finitely generated. Let V be a proper submodule of U and $0 \neq f \in Hom_R(U, \frac{U}{V})$. Note that $\frac{U}{V}$ is almost finitely generated [13], Prop.(1.1)), by lemma (4.7), f is an epimorphism, hence we are through. □

The converse of proposition (4.8) is not true in general as the following example shows.

Example 4.9. As we saw in example (2.2)(iv), that the Z-module Z_4 is a co-small monoform module, while it is not almost finitely generated, since Z_4 is finitely generated. Thus the class of almost finitely generated is contained properly in the class of co-small monoform module.

5. Some generalizations of co-small monoform modules

In this section; some generalizations of co-small monoform are introduced. We start by the first one.

Definition 5.1. *An R-module U is called co-small quasi-Dedekind if U satisfies the condition* $(*)$ *in the proposition (3.5).*

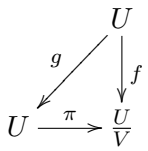
Remarks 5.2. a. It is worth mentioning to say that the class of co-small quasi-Dedekind module forms a dual of the class of small quasi-Dedekind which appeared in [7], "where an R-module U is called small quasi-Dedekind if for every $0 \neq f \in End_R(U)$; $kerf \ll U$".

b. It is clear that every uniform module is co-small quasi-Dedekind.

Under certain conditions the class of co-small monoform coincides with the class of co-small quasi-Dedekind modules as the following theorem shows.

Theorem 5.3. *Let U be a self-projective uniform module, then U is co-small monoform if and only if U is a co-small quasi-Dedekind module.*

Proof. The necessity follows by proposition (3.5). Conversely, assume that U is co-small quasi-Dedekind module, and $f : U \to \frac{U}{V}$ is a non-zero homomorphism with a proper submodule V of U. Consider the following diagram:



Where π is the natural epimorphism. Since U is a self-projective module, then there exists a homomorphism $g : U \to U$ such that $(\pi g) = f$. It is clear

that $(\pi g) \neq 0$ and homomorphism. By assumption, $g(U) \leq_e U$, since U is uniform then $V \leq_e U$. This implies that $(\pi g)(U) \leq_e \pi(U)$ ([12], Prop.(2.6), P.76), hence $f(U) \leq_e \frac{U}{V}$, thus U is co-small monoform. □

We have mentioned in remark (2.2)(vi) that uniform module is not necessary co-small monoform, as consequence of theorem (5.3) we have the following.

Corollary 5.4. *If U is a uniform and self-projective module, then U is co-small monoform.*

Proof. Since U is uniform, then by (5.2)(b), U is co-small quasi-Dedekind. But U is self-projective, so by (5.3), U is a co-small monoform module. □

Now, we introduce another type for the generalizations of co-small monoform module.

Definition 5.5. *"An R-module U is called relatively co-small monoform if for every proper closed submodule V of U and every non-zero homomorphism $f : U \to \frac{U}{V}$; $f(U) \leq_e \frac{U}{V}$".*

"It is clear that every co-small monoform is relatively co-small monoform, but the converse is not true in general, for example": in the Z-module Z; the only closed submodule is (0), so clearly Z is relatively co-small monoform, but it is not co-small monoform, see example (2.2)(iii).

Theorem 5.6. *For a semisimple module U, the following statements are equivalent:*

i. U is a co-small monoform module.

ii. U is a relatively co-small monoform module.

Proof. (i) ⇒ (ii) It is straightforward.
(ii) ⇒ (i) Let V a proper submodule of U, and $f : U \to \frac{U}{V}$ be a non-zero homomorphism. Since U is semisimple, then V is closed, and by assumption we get the result. □

Proposition 5.7. *Every uniform module is relatively co-small monoform module.*

Proof. Let U be a uniform module, so the only proper closed submodule in U is (0), and for every non-zero homomorphism $f : U \to \frac{U}{(0)}$; $f(U) \leq_e \frac{U}{(0)}$. That is U is relativly co-small monoform.. □

It is well-known that every non-zero extending and indecomposable module is uniform, so we have the following.

Corollary 5.8. *If U is a non-zero extending and indecomposable module, then U is relatively co-small monoform..*

Following [14], a non-zero module U is called coprime if for each element of R, either $rU = 0$ or $rU = U$. This motivates us to introduce the third generalizations of co-small monoform module; we call it an E-coprime module.

Definition 5.9. *A non-zero module U is called E-coprime if for each non-zero element r of R, either $rU = 0$ or $rU \leq_e U$.*

It is clear that every a coprime module is E-coprime.

Remark 5.10. i. Every co-small monoform module is E-coprime.

ii. If $ann(\frac{U}{V}) = ann(U)$ for each proper submodule V of U, then U is E-coprime.

Proof. i. The result follows directly by proposition (2.9).

ii. Let $r \in R$, and assume that $rU \nleq_e U$, then rU is a proper submodule of U. By assumption $ann(U) = ann(\frac{U}{V})$. Now, $r(\frac{U}{rU}) = 0$, this implies that $r \in ann(U)$, hence $rU = 0$. □

The following theorem gives a partial characterization of E-coprime module.

Theorem 5.11. *Let U be a semisimple module, then U is E-coprime if and only if $ann(\frac{U}{V}) = ann(U)$ for each proper submodule V of U.*

Proof. Suppose that U is an E-coprime module, and V be a proper submodule of U. Let $t \in ann(\frac{U}{V})$, then $tU \subseteq V$. But V is proper, so $tU \neq U$. Since U is semisimple, then $tU \nleq_e U$. But U is E-coprime, therefore $tU = 0$, hence $t \in ann(U)$. On the other hand, clearly $ann(U) \subseteq ann(\frac{U}{V})$, thus $ann(U) = ann(\frac{U}{V})$. The sufficiency follows by remark (5.10)(ii). □

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References

[1] B.A. Al-Hashimi, M.A. Ahmed, *Some results on epiform modules*, J. Al-Anbar, Univ. for Pure Sci., 4 (2010), 54-56.

[2] J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, *Lifting modules*, Frontiers in Mathematics, Birkhauser-Basel, 2006.

[3] K.R. Goodearl, *Ring theory, nonsingular rings and modules*, Marcel Dekker, New York and Basel, 1976.

[4] I.M. Hadi, S.H. Aidi, *e-Hollow modules*, International J. Advanced Scientific and Technical Research, 3 (2015), 435-461.

[5] I.M. Hadi, M.A. Ahmed, *Couniform modules*, Baghdad Sc. J., 10 (2013), 243-250.

[6] I.M. Hadi, M.A. Ahmed, *Epiform* modules*, International Mathematical Forum, 7 (2012), 2749-2758.

[7] I.M. Hadi, Th.Y. Ghawi, *Small quasi-Dedekind modules*, J. Al-Qadisiyah for Computer Science and Mathematics, 3 (2011), 88-98.

[8] I.M. Hadi, H.K. Marhun, *Small monoform modules*, Ibn AL-Haitham J. Pure and Applied Sci., 27 (2014), 229-240.

[9] Y. Hirano, I. Mogami, *On restricted anti-hopfian modules*, Math. J. Okayama Univ., 28 (1986), 119-131.

[10] G. Oman, A. Salminen, *modules which are isomorphic to their factor modules*, Comm in Algebra, 41 (2013), 1300-1315.

[11] N. Orhan, *Fully idempotent multiplication modules*, Palestine J. Mathematics, 3 (2014), 432- 437.

[12] A. Tercan, C. Yucel, *Module theory. extending modules and generalizations*, Springer International Publishing Switzerland, 2016.

[13] W.D. Weakly, *Modules whose proper submodules are finitely generated*, J. Algebra, 84 (1983), 189-219.

[14] S.M. Yasseen, *Coquasi-Dedekined modules*, Ph.D. Thesis, Univ. of Baghdad, 2003.

[15] D.X. Zhou, X.R. Zhang, *Small-essential submodules and morita duality*, Southeast Asian Bull. Math., 35 (2011), 1051-1062.

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$p-[a,b]$-paracompactness in bitopological spaces

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Abstract. In this paper, we introduce a new definition of paracompactness in bitopological spaces, we give a equivalent statements for this notation. Finally a product theorem is given.
Keywords: $p-[a,b]-$paracompact space, $p-$locally$-a$ family, $s-[a,b]$ compact space.

1. Introduction

In 1963, Kelly [4] introduced the concept of bitopological space. A set X with two topologies τ_1, τ_2 is a bitopological space, and denoted by $X = (X, \tau_1, \tau_2)$. A cover $\mathcal{U}$ of a bitopological space $X = (X, \tau_1, \tau_2)$ is called $\tau_1\tau_2-$ open cover (family) [4], if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$, and it is called $p-$ open cover (family) [4],if it is $\tau_1\tau_2-$ open cover and contains at least one nonempty member of τ_1 and one nonempty member of τ_1. A space $X = (X, \tau_1, \tau_2)$ is called $s-[a,b]-(p-[a,b]-)$ compact space [1], if every $\tau_1\tau_2-(p-)$ open cover of X with cardinality $\leq b$ has a subcover with cardinality $< a$.

For a $\tau_1\tau_2-$ open covers $\mathcal{U}$, $\mathcal{V}$ in a bitopological space $X = (X, \tau_1, \tau_2)$, $\mathcal{U}$ is called a parallel refinement of $\mathcal{V}$ [2], if for each $U \in \mathcal{U} \cap \tau_i$ is contained in some $V \in \mathcal{V} \cap \tau_i$ for $i = 1, 2$.

In this paper the letters a, b are infinite regular cardinals, ω_0, ω_1 stands for the cardinality of $\mathbb{N}, \mathbb{R}$, respectively. If $X = (X, \tau_1, \tau_2)$ is a bitopological space and $A \subseteq X$, $int_{\tau_i}(A)$, $\overline{A}^{\tau_i}$ denote the interior and the closure of A in τ_i, respectively for $i = 1, 2$. When $X = (X, \tau_1, \tau_2)$ has a topological property Q this means that both τ_1 and τ_2 have this property. For the concepts not defined here, see [3] and [5].

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2. Preliminaries

Definition 2.1. A family $\mathcal{A}$ of subsets of a bitopological space $X = (X, \tau_1, \tau_2)$ is called p- locally-a family, if for all $x \in X$ there exists a τ_1- open set U containing x such that U meets less than a members of $\mathcal{A} \cap \tau_2$ or there exists a τ_2- open set V containing x such that V meets less than a members of $\mathcal{A} \cap \tau_1$.

Definition 2.2. A space $X = (X, \tau_1, \tau_2)$ is called $p-[a,b]-$ paracompact if every $p-$ open cover of X with cardinality $\leq b$ has a $p-$ locally$-a$ $p-$ open parallel refinement.

Theorem 2.3. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological spaces. Then X is $p-[a,b]-$ paracompact if and only if each τ_j- open cover of a τ_i- proper closed subset of X with cardinality $\leq b$ has τ_i- locally $-a$ τ_j- open parallel refinement for $i \neq j; i, j = 1, 2$.*

Proof. ($\Rightarrow$) Let F be a τ_i-closed proper subset of X. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a τ_j-open cover of F with $|\Delta| \leq b$. Now $\{U_\alpha | \alpha \in \Delta\} \cup \{X - F\}$ is $p-$open cover of X, but X is $p-[a,b]-$paracompact, so there exists a $p-$open parallel refinement

$$\mathcal{V} = \{V_\alpha | \alpha \in \Delta^* \subseteq \Delta\} \bigcup \{W_\gamma | \gamma \in \Gamma\}.$$

Now $\{V_\alpha | \alpha \in \Delta^* \subseteq \Delta\}$ is a τ_i-locally$-a$ τ_j- open refinement, for $i \neq j; i, j = 1, 2$.

($\Leftarrow$) Let $\mathcal{U} = \{W_\alpha | \alpha \in \Delta\} \cup \{V_\gamma | \gamma \in \Gamma\}$ be a $p-$open cover of X with $|\Delta \cup \Gamma| \leq b$ where $W_\alpha \in \tau_i$ for all $\alpha \in \Delta$ and $V_\gamma \in \tau_j$ for all $\gamma \in \Gamma$, for $i \neq j; i, j = 1, 2$. So we have the following cases:

Case (i) If $\bigcup_{\gamma \in \Gamma} V_\gamma = X$, then take $\alpha_0 \in \Delta$ such that $W_{\alpha_0} \neq \phi$. Consider the set $F = X - W_{\alpha_0}$, then F is a proper τ_i- closed subset of X and $\{V_\gamma | \gamma \in \Gamma\}$ is a τ_j- open cover of F with $|\Delta| \leq b$, then F has a τ_j- open τ_i- locally $a-$ refinement

$$\mathcal{V}^* = \{V_\lambda^* | \lambda \in \Lambda\}.$$

Finally the family

$$\mathcal{U}^* = \mathcal{V}^* \cup \{W_{\alpha_0}\}$$

is the required $p-$ locally $-a$ $p-$ open parallel refinement.

Case (ii) If $\bigcup_{\gamma \in \Gamma} V_\gamma \neq X$, then

$$E_1 = X - \bigcup_{\gamma \in \Gamma} V_\gamma$$

is a τ_j- closed subset of X, and

$$E_1 \subseteq \bigcup_{\alpha \in \Delta} W_\alpha,$$

so $\{W_\alpha | \alpha \in \Delta\}$ has τ_i- open τ_j- locally $-a$ parallel refinement

$$\mathcal{W}^* = \{W_\lambda^* | \lambda \in \Lambda\},$$

if $\bigcup_{\lambda\in\Lambda} W^*_\lambda = X$, we are done. if not let

$$E_2 = X - \bigcup_{\lambda\in\Lambda} W^*_\lambda,$$

then E_2 is τ_i- closed and hence there exists a τ_j- open τ_i- locally $-a$ refinement

$$\mathcal{V}^* = \{V^*_\omega | \omega \in \Omega\}.$$

Finally the family

$$\mathcal{U}^* = \mathcal{V}^* \cup \mathcal{W}^*$$

is p- locally $-a$ $p-$ open parallel refinement, hence the result.

□

Theorem 2.4. *Every $p-[\omega_0,\infty]-$ paracompact $p-T_2-$ bitopological space $X=(X,\tau_1,\tau_2)$ is $p-T_4$.*

Proof. Let E and F be disjoint closed sets such that E is a τ_2-closed and F is a τ_1- closed . Let $e\in E$, since X is $p-T_2-$ space, for each $f\in F$ there exists a τ_1- open set U_f and a τ_2- open set V_f such that $e\in U_f$ and $f\in V_f$ with $U_f\cap V_f=\phi$.

Let

$$\mathcal{V} = \{V_f | f\in F\}\bigcup\{X-F\},$$

then $\mathcal{V}$ is a $p-$ open cover of X, and hence there exists a $p-$ locally $-\omega_0$ $p-$ open parallel refinement $\mathcal{V}^*$ such that if $V\in\mathcal{V}^*$ and $V\cap F\neq\phi$, so we have $V\in\tau_2$. Now let

$$V_e = \bigcup\{V | V\in\mathcal{V}^* \quad \text{and} \quad V\cap F\neq\phi\},$$

then V_e is τ_2- open and $F\subseteq V_e$. Let U be a τ_1- open set containing e that intersects $<\omega_0$ of $\mathcal{V}^*$ say $V_1,V_2,...,V_n$ and $V_k\subseteq V_{f_k}, 1\leq k\leq n$. Let $U_e = U\cap U_{f_1}\cap U_{f_2}\cap\ldots\cap U_{f_n}$. Now $e\in U_e$ and $F\subseteq V_e$ with $U_e\cap V_e=\phi$, hence X is $p-T_3$.

Now let

$$\mathcal{U} = \{U_e | e\in E\}\cup\{X-E\},$$

then $\mathcal{U}$ is a $p-$ open cover of X, so there exists a $p-$ locally $-\omega_0$ $p-$open parallel refinement $\mathcal{U}^*$, notice that if $U^*\in\mathcal{U}^*$ and $U^*\cap E\neq\phi$, then $U^*\in\tau_1$. Let

$$W = \bigcup\{U^* | U^*\in\mathcal{U}^* \quad \text{and} \quad U^*\cap E\neq\phi\},$$

then W is a τ_1- open and $E\subseteq W$, now for each $f\in F$ there exists τ_2- open set U^*_f that intersects $<\omega_0$ of $\mathcal{U}^*$ say $U^*_1,U^*_2,\ldots,U^*_m$. Now let $U_{e_s}\in\mathcal{U}^*$ with $U^*_s\subseteq U_{e_s}$ for $s=1,2,\ldots,m$ and let $U_f = U^*_f\cap Ve_1\cap Ve_2\cap\ldots\cap Ve_m$, let $V=\bigcup\{U_f | f\in F\}$. Then $V\in\tau_2$, $F\subseteq V$ and $W\cap F=\phi$, hence X is $p-T_4$.

□

3. $p-[\omega_0,\omega_1]$-paracompact space

Definition 3.1. A family $\mathcal{A}$ of subsets of a bitopological space $X=(X,\tau_1,\tau_2)$ is called $p-$point$-a$ family if for all $x\in X$, x meets $<a$ of $\mathcal{A}\cap\tau_i$, $i=1$ or 2.

Lemma 3.2. *If $X=(X,\tau_1,\tau_2)$ is normal, $p-$ normal space, then for each $p-$open $p-$point$-\omega_0$ cover $\mathcal{G}$ with $|\mathcal{G}|<\omega_1$ has a parallel refinement $\mathcal{V}$ such that $\overline{V_1}^{\tau_i}\cup\overline{W_1}^{\tau_j}\subseteq G$ where $V_1,W_1\in\mathcal{V}\cap\tau_i$ for some $G\in\mathcal{G}\cap\tau_i$,$i\neq j;i,j=1,2$.*

Proof. Let $\mathcal{G}=\{G_k|k\in\Delta,|\Delta|<\omega_1\}$ be a $p-$open $p-$point$-\omega_0$ of X, write $\Delta=\{1,2,\ldots\}$.

Let

$$F_1^i=X-\bigcup_{k>1}(G_k\cap\tau_i),$$

and

$$F_1^j=X-\bigcup_{k>1}(G_k\cap\tau_j),$$

then F_1^i is a τ_i- closed and F_1^j is a τ_j-closed for $i\neq j;i,j=1,2$. Now let

$$F_1=F_1^i\cup F_1^j,$$

then

$$F_1^i\subseteq F_1\subseteq G_1$$

and

$$F_1^j\subseteq F_1\subseteq G_1.$$

Without loss of generality assume that G_1 is a τ_i-open, since X is normal and by [6], there exists τ_i-open sets V_1^i,W_1^i such that

$$F_1^i\subseteq V_1^i\subseteq\overline{V_1^i}^{\tau_i}\subseteq G_1$$

and

$$F_1^j\subseteq W_1^i\subseteq\overline{W_1^i}^{\tau_j}\subseteq G_1,$$

so

$$F_1\subseteq\overline{V_1^i}^{\tau_i}\cup\overline{W_1^i}^{\tau_j}\subseteq G_1,\ \text{ for }\ i\neq j;i,j=1,2.$$

Let

$$V_1=V_1^i\cup W_1^i$$

and

$$F_\alpha^i=X-\big(\bigcup_{\beta<\alpha}V_\beta\big)\cup\big(\bigcup_{\gamma>\alpha}(G_\gamma\cap\tau_i)\big),$$

$$F_\alpha^j=X-\big(\bigcup_{\beta<\alpha}int_{\tau_j}(V_\beta)\big)\cup\big(\bigcup_{\gamma>\alpha}(G_\gamma\cap\tau_j)\big),i\neq j;i,j=1,2,$$

and $F_\alpha = F_\alpha^i \cup F_\alpha^j$. Again without loss of generality assume G_α is a τ_i-open, so

$$F_\alpha^i \cup F_\alpha^j \subseteq G_\alpha$$

and hence there exists τ_i- open sets V_α^i, W_α^i such that

$$F_\alpha^i \subseteq V_\alpha^i \subseteq \overline{V_\alpha^i}^{\tau_i} \subseteq G_\alpha,$$

$$F_\alpha^j \subseteq W_\alpha^i \subseteq \overline{W_\alpha^i}^{\tau_j} \subseteq G_\alpha,$$

and

$$F_\alpha \subseteq \overline{V_\alpha^i}^{\tau_i} \cup \overline{W_\alpha^i}^{\tau_j} \subseteq G_\alpha, i \neq j; i, j = 1, 2.$$

Let $V_\alpha = V_\alpha^i \cup W_\alpha^i$,then $\mathcal{V} = \{V_\alpha | \alpha \in \Delta\}$ is a $p-$open $p-$point$-\omega_0$ parallel refinement of X. For instance, let $x \in X$ x meets $< \omega_0$ of $\mathcal{G} \cap \tau_i$ for $i = 1, 2$ say $G_{\alpha_1}^i, G_{\alpha_2}^i, \ldots, G_{\alpha_n}^i$, let $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Now $x \notin G_\alpha$ for any $\gamma > \alpha$ and if $x \notin V_\beta$ for any $\beta < \alpha, V_\beta \in \tau_i$, $x \in F_\alpha \subseteq V_\alpha$, hence $x \in V_\beta$ for some $\beta \leq \alpha$, therefore $\mathcal{V}$ is a $p-$point$-\omega_0$ parallel refinement of X. □

Theorem 3.3. *For a normal, $p-$normal bitopological space $X = (X, \tau_1, \tau_2)$. The following are equivalent:*

(i) X is $p - [\omega_0, \omega_1]-$ paracompact,

(ii) Every $p-$open cover of X has a $p-$point$-\omega_0$ refinement,

(iii) Every $p-$open cover $\mathcal{U}$ of X with cardinality $< \omega_1$ has a parallel refinement $\mathcal{V}$ such that for $V_1, W_1 \in \mathcal{V} \cap \tau_i$, we have $\overline{V_1}^{\tau_i} \cup \overline{W_1}^{\tau_j} \subseteq U$ for some $U \in \mathcal{U} \cap \tau_i$,$i \neq j; i, j = 1, 2$,

(iv) Given a decreasing sequence of a $p-$closed family $\mathcal{F} = \{F_k | k \in \Delta\}$ with $\bigcap_{k\in\Delta} F_k = \phi$, there exists a sequence of $p-$ open family $\mathcal{G} = \{G_k | k \in \Delta\}$ with $\bigcap_{k\in\Delta} G_k = \phi$, such that $F_k^i \subseteq G_k^i$ and $F_k^i \subseteq G_k^j$, where F_k^i is a τ_i- closed in $\mathcal{F}$, $G_k^i \in \mathcal{G} \cap \tau_i$ and $G_k^j \in \mathcal{G} \cap \tau_j$, $i \neq j; i, j = 1, 2$,

(v) Given a decreasing sequence of a $p-$closed family $\mathcal{F} = \{F_k | k \in \Delta\}$ with $\bigcap_{k\in\Delta} F_k = \phi$, there exists a sequence of $p-$closed $(G_\delta)-$ family $\mathcal{A} = \{A_k | k \in \Delta\}$ with $\bigcap_{k\in\Delta} A_k = \phi$, and $F_k \subseteq A_k$.

Proof. $(i) \to (ii)$ trivial.

$(ii) \to (iii)$ Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $p-$open cover of X with $|\Delta| < \omega_1$, so by (ii), for all $x \in X$, $\mathcal{U}$ has a parallel refinement $\mathcal{S}$ such that x meets $< \omega_0$ of $\mathcal{S} \cap \tau_i, i = 1$ or 2. For each $S \in \mathcal{S}$, let $U(S)$ be the first U_α containing S.

Let

$$G_\alpha = \bigcup_{U(S)=U_\alpha} S,$$

clearly $G_\alpha \in \tau_1 \cup \tau_2, G_\alpha \subseteq U_\alpha$. Let $\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$ is a $p-$point$-\omega_0$ cover of X which is $p-$open cover (if necessary add $U \in \mathcal{U} \cap \tau_i, i = 1$ or 2), finally the result comes from Lemma 3.2 .

$(iii) \to (iv)$ Let $\mathcal{F} = \{F_k | k \in \Delta || \Delta| < \omega_1\}$ be a sequence of $p-$closed family such that $F_{k+1} \subseteq F_k$, and $\bigcap_{k\in\Delta} F_k = \phi$.

Let

$$U_k^i = X - F_k^i,$$

and

$$U_k^j = X - F_k^j,$$

where F_k^i is a τ_i-closed and F_k^j is a τ_j- closed for $i \neq j; i, j = 1, 2$. Then $\mathcal{U} = \{U_k^i | k \in \Delta\} \cup \{U_k^j | k \in \Delta\}$ is a $p-$open cover of X, so by (iii) there exists $V_k, W_k \in \mathcal{V} \cap \tau_i$, where $\mathcal{V}$ is a parallel refinement with $\overline{V_k}^{\tau_i} \cup \overline{W_k}^{\tau_j} \subseteq U_k^i$, we may assume that $U_k^i \in \mathcal{U} \cap \tau_i$. Let

$$G_k^i = X - \overline{V_k}^{\tau_i} \quad \text{and} \quad G_k^j = X - \overline{W_k}^{\tau_j},$$

then

$$\mathcal{G} = \{G_k^i | k \in \Delta\} \cup \{G_k^j | k \in \Delta\},$$

is a $p-$open family and $\bigcap_{k\in\Delta}(G_k^i \cup G_k^j) = \phi$ with $\overline{V_k}^{\tau_i} \subseteq U_k^i$ and $\overline{W_k}^{\tau_i} \subseteq U_k^i$, hence $X - U_k^i \subseteq X - \overline{V_k}^{\tau_i}$ and $X - U_k^i \subseteq X - \overline{W_k}^{\tau_j}$, so we have $F_k^i \subseteq G_k^i$ and $F_k^i \subseteq G_k^j$ for $i \neq j; i, j = 1, 2$.

$(iv) \to (v)$ Let $\mathcal{F} = \{F_k, k \in \Delta\}$ be a decreasing sequence of $p-$ closed subset of X with $\bigcap_{k\in\Delta} F_k = \phi$, by (iv) there exists a sequence $\mathcal{G} = \{G_k, k \in \Delta\}$ of $p-$open subsets of X with $\bigcap_{k\in\Delta} G_k = \phi$, such that $F_k^j \subseteq G_k^i$ and $F_k^j \subseteq G_k^j$ where $G_k^i \in \mathcal{G} \cap \tau_i$, $G_k^j \in \mathcal{G} \cap \tau_j$, and F_k^j is τ_j- closed in $\mathcal{F}$, $i \neq j; i, j = 1, 2$. So by Urysohn's lemma there exists a continuous function $f_k^j : (X, \tau_j) \to (\mathbb{R}, \tau_u)$ such that $f_k^j(F_k^j) = \{0\}$ and $f_k^j(X - G_k^j) = \{1\}$, and by [5] there exists a $p-$continuous function $g_k^i : (X, \tau_1, \tau_2) \to (\mathbb{R}, \tau_l, \tau_r)$ such that $g_k^i(F_k^j) = \{0\}$ and $g_k^i(X - G_k^i) = \{1\}$.

Let

$$M_{km}^j = \{x | f_k^j(x) < \frac{1}{m}\},$$

and

$$N_{km}^i = \{x | g_k^i(x) < \frac{1}{m}\},$$

then

$$M_k^j = \bigcap_m M_{km}^j = \{x | f_k^j(x) = 0\},$$

and

$$N_k^i = \bigcap_m N_{km}^i = \{x | g_k^i(x) = 0\}.$$

Now $\mathcal{A} = \{M_k^j | k \in \Delta\} \cup \{N_k^i | k \in \Delta\}$ is a $p-$closed $(G_\delta-)$set with $\bigcap_{k\in\Delta} A_k = \phi$ and $F_k \subseteq A_k$.

$(v) \to (i)$ Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta, |\Delta| < \omega_1\}$ be a $p-$open cover of X .

Let

$$F_\alpha^i = X - \bigcup_{\beta\leq\alpha}(U_\beta \cap \tau_i),$$

and

$$F_\alpha^j = X - \bigcup_{\beta\leq\alpha}(U_\beta \cap \tau_j),$$

then

$$F_{\alpha+1}^i \subseteq F_\alpha^i,$$

and

$$F_{\alpha+1}^j \subseteq F_\alpha^j.$$

Let

$$\mathcal{F} = \{F_\alpha^i | \alpha \in \Delta\} \cup \{F_\alpha^j | \alpha \in \Delta\},$$

then

$$F_{\alpha+1}^i \cup F_{\alpha+1}^j \subseteq F_\alpha^i \cup F_\alpha^j,$$

and

$$\bigcap_{\alpha\in\Delta} F_\alpha = \phi,$$

by (v) there exists a sequence $\mathcal{A}$ of $p-$closed $G_\delta-$set with $\bigcap_{\alpha\in\Delta} A_\alpha = \phi$, and $F_\alpha^i \subseteq A_\alpha$, $F_\alpha^j \subseteq A_\alpha$, but $X - A_\alpha$ is a $F_\alpha-$set, let $X - A_\alpha = \bigcup_\alpha B_{\alpha\gamma}$ where each $B_{\alpha\gamma}$ is a τ_i- closed and the family $\mathcal{B} = \{B_{\alpha\gamma} | \alpha \in \Delta\}$ is a $p-$ closed family, since X is normal $p-$normal space, we can assume that

$$B_{\alpha\gamma} \subseteq int_{\tau_i}(B_{\alpha,\gamma+1})$$

and

$$B_{\alpha\gamma} \subseteq int_{\tau_j}(B_{\alpha,\gamma+1}),$$

then

$$X - A_\alpha \subseteq \bigcup_\alpha int_{\tau_i}(B_{\alpha\gamma}),$$

and

$$X - A_\alpha \subseteq \bigcup_\alpha int_{\tau_j}(B_{\alpha\gamma}),$$

$$B_{\alpha\gamma} \subseteq X - A_\alpha \subseteq X - (F_\alpha^i \cup F_\alpha^j) = \bigcup_{\beta\leq\alpha} U_\beta.$$

Let $x \in X$ and U_α be the first element in $\mathcal{U}$ such that $x \in U_\alpha$, so we have two cases:

(*i*) $U_\alpha \in \tau_i$, let $V_\alpha^i = U_\alpha - \bigcup_{\alpha<\gamma} B_{\alpha\gamma}$ and each $B_{\alpha\gamma}$ is a τ_i- closed.

(*ii*) $U_\alpha \in \tau_j$, let $V_\alpha^j = U_\alpha - \bigcup_{\alpha<\gamma} B_{\alpha\gamma}$ and each $B_{\alpha\gamma}$ is a τ_j- closed for $i \neq j; i, j = 1, 2$.

Now in each case for $\alpha < \gamma$

$$B_{\alpha\gamma} \subseteq \bigcup_{k<\alpha} U_k \subseteq \bigcup_{k<\gamma} U_k,$$

$$U_\alpha - \bigcup_{k<\gamma} U_k \subseteq V_\gamma^i \cup V_\gamma^j,$$

hence

$$\mathcal{V} = \{V_\alpha^i | \alpha \in \Delta\} \cup \{V_\alpha^j | \alpha \in \Delta\}$$

is a $p-$open parallel refinement cover of X. Finally, we need to show that $\mathcal{V}$ is $p-$ locally $-\omega_0$, let $x \in X$, there exists $\alpha \in \Delta$ such that $x \notin A_\alpha$, so for some $k, x \in int_{\tau_i}(B_{\alpha k})$ and $x \in int_{\tau_j}(B_{\alpha k})$.

If $\gamma > \alpha$ and $\gamma > k$,

$$int_{\tau_i}(B_{\alpha k}) \subseteq B_{\alpha\gamma}$$

and

$$int_{\tau_j}(B_{\alpha k}) \subseteq B_{\alpha\gamma},$$

with

$$int_{\tau_i}(B_{\alpha k}) \cap V_\alpha^i = \phi,$$

and

$$int_{\tau_j}(B_{\alpha k}) \cap V_\alpha^j = \phi.$$

So $int_{\tau_i}(B_{\alpha k})$ is a τ_i- open set contains x and meets $< \omega_0$ of $\mathcal{V} \cap \tau_i$, $int_{\tau_j}(B_{\alpha k})$ is a τ_j- open set contains x and meets $< \omega_0$ of $\mathcal{V} \cap \tau_j$ for $i \neq j; i, j = 1, 2$, hence $\mathcal{V}$ is $p-$locally $-\omega_0$, therefore X is $p-[\omega_0, \omega_1]-$paracompact space.

□

4. A product theorem

Theorem 4.1. *Let $X = (X, \tau_1, \tau_2)$ be a $s-[w_0, \infty]-$ compact space and $Y = (Y, \sigma_1, \sigma_2)$ is a $p-[\omega_0, \omega_1]-$ paracompact space. Then $X \times Y = (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $p-[\omega_0, \omega_1]-$ paracompact space.*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\, |\Delta| < \omega_1\}$ be a $p-$ open cover of $X \times Y$. Let $x \in X$ and $V_\alpha = \{x \in X | x \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha\}$, for $x \in V_\alpha$ and $y \in Y$ with $(x, y) \in x \times Y$ there exists a set $O_x \times Q_x$ with $x \in O_x \in \tau_1 \cup \tau_2$ and $y \in Q_x \in \sigma_1 \cup \sigma_2$. Let $\mathcal{Q} = \{Q_x | x \in X\}$, then $\mathcal{Q}$ is a $\tau_1\tau_2-$ open cover of Y and hence $\mathcal{Q}$ has a $\tau_1\tau_2-$open subcover with cardinality $< \omega_0$, say $Q_{x_1}, Q_{x_2}, \cdots, Q_{x_n}$.

Let

$$O_1 = \bigcap_m (O_m \cap \tau_1)$$

and

$$O_2 = \bigcap_m (O_m \cap \tau_2) \quad \text{for} \quad 1 \leq m \leq n.$$

Let $O = O_1 \cup O_2$, then $x \in O \subseteq \tau_1 \cup \tau_1$ and

$$O \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha,$$

and

$$x \times Y \subseteq O \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha$$

and hence $V_\alpha \in \tau_1 \cup \tau_1$. Now $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma, |\Gamma| < \omega_1\}$,(if $\mathcal{V} \cap \tau_i = \phi$, add $U_\alpha \in \mathcal{U} \cap \tau_i$ for some $\alpha \in \Delta, i = 1, 2$). Since for $x \in X$, $x \times Y$ is $s - [\omega_0, \infty]-$ compact, so it contains in $< \omega_0$ of $\mathcal{U}$ and hence $x \in V_\alpha$, so $\mathcal{V}$ has a $p-$ locally $-\omega_0$ $p-$ open parallel refinement say $\mathcal{B}$. For $B \in \mathcal{B}$, let $V_B \in \mathcal{V}$ be the first $V_\gamma \in \mathcal{V}$ such that $B \subseteq V_B$ and $G_\gamma = \bigcup_{V_B = V_\gamma} B$, then $G_\gamma \in \tau_1 \cup \tau_2$ and $\mathcal{G} = \{G_\gamma | \gamma \in \Gamma\}$ is a $p-$locally $-\omega_0$ $p-$ open cover of X. If $\alpha \leq \gamma$, let $W_{\gamma\beta} = (G_\gamma \times Y) \cap U_\alpha$, and $\mathcal{W} = \{W_{\gamma\beta} | \alpha \leq \gamma\}$. For $(x, y) \in X \times Y$, $x \in G_\gamma$ for some $\gamma \in \Gamma$ and $(x, y) \in G_\gamma \times Y$, also $x \in G_\gamma \subseteq V_\gamma$, $(x, y) \in X \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha$, so $(x, y) \in W_{\gamma\beta}$, therefore $\mathcal{W}$ is a $p-$ open cover of $X \times Y$. Again for $(x, y) \in X \times Y$, $x \in K$ where $K \in \tau_i$ which meets $< \omega_0$ of $\mathcal{G} \cap \tau_j$ for $i \neq j; i, j = 1, 2$. Now $K \times Y$ meets $< \omega_0$ of $(\mathcal{G} \cap \tau_j) \cap Y$, hence $\mathcal{W}$ is $p-$ locally$-\omega_0$ $p-$ open parallel refinement of $\mathcal{U}$, hence $X \times Y$ is $p - [\omega_0, \omega_1]-$paracompact space.

□

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References

[1] Fuad A. Abushaheen and Hasan Z. Hdeib, *On [a,b] compactness in bitopological spaces,* International Journal of Pure and Applied Mathematics, 110 (2016), 519-535.

[2] M.C. Datta, *Projective bitopological spaces*, J. Austral Math. Soc., 13 (1972), 327-334.

[3] R. Engelking, *General topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.

[4] P. Fletcher et. al., *The compaison of topologies*, Duke Math. J., 36 (1969), 325-331.

[5] J.C. Kelly, *Bitopological spaces*, Proc. Londan Math. Soc., 13 (1963), 71-89.

[6] L. Reilly, et. al., *On bitopological compactness*, J. Londan Math. Soc., 9, 518-522.

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Hydrodynamic lubrication of a porous slider: shuns simplifying assumption of a small porous facing thickness

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Abstract. This investigation analyzes the lubricating characteristics of a ferrofluid based plane porous slider without the use of the simplifying assumption of a small porous facing thickness. The magnetic fluid flow model of Neuringer and Rosensweig has been adopted here. It is observed that the performance characteristics getting improved owing to magnetization of the lubricant. Further, significant deviations from the past results are observed regarding the ranges of some parameters for which the simplifying assumption yield that satisfactory results.

Keywords: slider bearing, porosity, magnetic fluid, infinite series solution, load carrying capacity.

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1. Introduction

Many authors have conducted investigations on the fluid film lubrication between porous plates (Prakash and Vij [1], Wu [2], Prakash and Vij [3], Bhat and Patel [4]). Sanni and Ayomidele [5] consider hydrodynamic lubrication and porous slider bearing without the use of the signifying assumption of a small porous facing thickness. Solutions were obtained in the form of infinite series. Das [6] studied the optimum load bearing capacity for slider bearings lubricated with couple stress fluids in magnetic field. It was observed that the couple stress effect improved the bearing performance with suitable inlet to outlet film thickness ratio.

Deheri et. al. [7] discussed the performance of longitudinally rough slider bearing with squeeze film formed by a magnetic field. It was noted that the bearing performance was significantly affected by all the parameters characterizing the surface roughness. Ochonski [8] investigated some new design of slider bearing lubricated with ferrofluid exploring the possibility of using them in computer and audio-visual equipment. Agrawal [9] dealt with a porous inclined slider bearing lubricated with a ferrofluid in the presence of an externally applied magnetic field. The load carrying capacity was found to be more as compared to that of a viscous porous inclined slider bearing.

Patel and Deheri [10] studied the effect of surface roughness on the performance of a magnetic fluid based parallel plate porous slider bearing with slip velocity. Here, in spite of the positive effect of a magnetization the slip velocity caused significant load reduction. Further, the friction was unaltered. Lin [11] observed that the performance characteristics of a finite porous slider bearing by using the Brinkmans equation to account for the effect of viscous shearing stress.

Singh and Gupta [12] evaluated the effect of ferrofluid on the dynamic characteristics of a curved slider bearing using Shliomis model based ferrofluid lubrication. It was observed that the effect of rotation of magnetic particle improved the stiffness and damping capacity of the bearing. Bhat and Deheri [13] extended the analysis of Agrawal [9] by considering the magnetic fluid based porous composite slider bearing with its slider consisting of an inclined pad and a flat pad. Here, it observed that magnetic fluid increased the load carrying capacity, unaltered the friction, decreased the coefficient of friction and shifted the centre of pressure toward the inlet. Deheri et. al. [14] discussed transversely rough slider bearings with squeeze film formed by a magnetic fluid. Here, it is concluded that the use of magnetic fluid as a lubricant increases the load carrying capacity, decreases the coefficient of friction and affects the centre of pressure marginally. Further, it is seen that the effect of magnetization[15] on the plane and secant shaped bearings is nominal while the effect of exponential and hyperbolic slider bearings is significant.

Here, it has been sought to study the effect of magnetic fluid lubrication on the configuration of Sanni and Ayomidele [5].

2. Analysis

Geometry and coordinates system of the problem is shown below. With usual

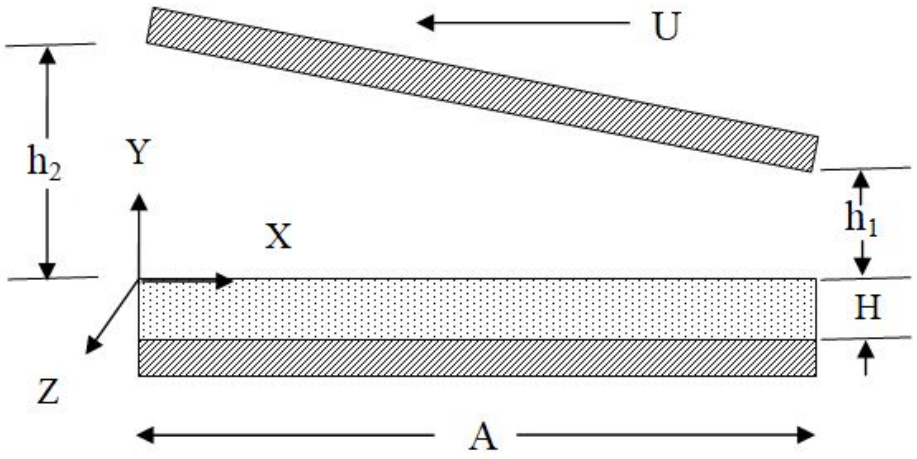


Figure 1: Geometry and coordinates of the bearing system

assumption of conventional lubrication theory, after neglecting the side leakage the modified Reynolds equation is found in the form of equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3\ \frac{\mathrm{d}p^*}{\mathrm{d}x}\right) = -6\mu U\frac{\mathrm{d}h}{\mathrm{d}x} + 12\phi\left[\frac{\partial P}{\partial y}\right]_{y=0}. \tag{1}$$

where P the pressure in the porous region, p^* is the pressure in the lubricant region , μ is the absolute viscosity of lubricant , $h = h_2 - (h_2 - h1)\frac{x}{A}$, ϕ is the permeability of the porous material, U is the tangential velocity of the slider.

In the porous region P is govern by the Laplacian equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0. \tag{2}$$

The boundary condition on p^* and P are

$$p^*(0) = p^*(A) = 0, \tag{3}$$

$$P(0, y) = P(A, y) = 0, \tag{4}$$

$$\left[\frac{\partial P}{\partial y}\right]_{y=-H} = 0. \tag{5}$$

Further, the matching condition for the continuity for the flow at the porous plate film interface is given by

$$P(x, 0) = p^*(x). \tag{6}$$

The pressure distribution in the porous matrix comes out to be

$$P = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\overline{p}(n)(e^{n\pi H^*}\sinh n\pi\overline{y} + e^{-n\pi\overline{y}}\cosh n\pi H^*)\sin n\pi\overline{x}}{\cosh n\pi H^*}. \tag{7}$$

The matching condition yields the pressure in the film region to be equation 3. Now substituting this in equation 2 in equation 1 gets

$$p^*(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\overline{p}(n)\sin n\pi\overline{x} \tag{8}$$

In the view of the magnetic fluid lubrication equation 1 is converted to

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(h^3\ \frac{\mathrm{d}}{\mathrm{d}x}\{p^* - 0.5\mu_0\overline{\mu}H^2\}\big) = -6\mu U\frac{\mathrm{d}h}{\mathrm{d}x} + 12\phi\big[\frac{\partial P}{\partial y}\big]_{y=0}. \tag{9}$$

where

$$H^2 = kx(A-x)$$

k has been suitably chosen constant to manufacture a required magnetic strength. Substituting Equations 7 and 8 into Equation 9, one obtains

$$\begin{aligned} &6nA\frac{\mathrm{d}h}{\mathrm{d}x}\overline{p}(n)h^2\cos n\pi\overline{x} \\ &- 2\pi n^2\overline{p}(n)h^3\sin n\pi\overline{x} - 0.5A^2\mu_0\overline{\mu}k\big[3h^2A\frac{\mathrm{d}h}{\mathrm{d}x} - 2kn^3 - 6kxh^2\frac{\mathrm{d}h}{\mathrm{d}x}\big] \\ &= -6A^2\mu U\frac{\mathrm{d}h}{\mathrm{d}x} + 24A\phi n\overline{p}(n)\tanh n\pi H^*\sin n\pi\overline{x}. \end{aligned} \tag{10}$$

The constants $\overline{p}(n)$ are calculated using orthogonality of eigen function $\sin n\pi\overline{x}$ as

$$\overline{p}(n) = \frac{24\mu UAH^*\{1-(-1)^n\}(1-r) + 4\pi\mu_0\overline{\mu}kA^2H^*nh_1^2D_n}{nh_1^2\{C_nH^* + 48\Psi n\pi\tanh n\pi H^*\}}. \tag{11}$$

Where

$$\begin{aligned} r &= \frac{h_2}{h_1}, \\ D_n &= \big[\frac{r^3-(-1)^n}{n\pi} + \frac{6(1-r)^2[(-1)^n - r]}{n^3\pi^3}\big] \\ &\quad - 1.5(1-r)\big[\frac{r^2+(-1)^n}{n\pi} - \frac{2(1-r)^2[(-1)^n+1] - 8(1-r)[(-1)^n+r]}{n^3\pi^3}\big] \end{aligned}$$

and

$$C_n = n^2\pi^2(1+r+r^2+r^3) + (1-r-r^2+r^3). \tag{12}$$

Substituting the value of $\overline{p}(n)$ in Equation 8 one obtains

$$p^*(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{24\mu UAH^*\{1-(-1)^n\}(1-r)+4\pi\mu_0\overline{\mu}kA^2H^*nh_1^2D_n}{nh_1^2\{C_nH^*+48\Psi n\pi\tanh n\pi H^*\}}\sin n\pi\overline{x}. \tag{13}$$

The pressure distribution in non-dimensional form is found as

$$\overline{p}=\sum_{n=1}^{\infty}\frac{[48H^*[1-(-1)^n]](r-1)+8\pi\mu^*n\frac{A}{h_1}D_nH^*}{n[C_nH^*+48\Psi n\pi\tanh n\pi H^*]}\sin n\pi\overline{x}. \tag{14}$$

Hence, the load bearing capacity in dimensionless form can be calculated as

$$\begin{aligned}\overline{W} &= \int_0^1 \overline{p}d\overline{x}\\ &= \sum_{n=1}^{\infty}\frac{[48H^*[1-(-1)^n]](r-1)+8\pi\mu^*n\frac{A}{h_1}D_nH^*}{n[C_nH^*+48\Psi n\pi\tanh n\pi H^*]}\times\frac{1-(-1)^n}{n\pi}.\end{aligned} \tag{15}$$

3. Result and discussions

It is seen from equation 15 that the load carrying capacity enhances by

$$\sum_{n=1}^{\infty}\frac{8\pi\mu^*n\frac{A}{h_1}D_nH^*}{n[C_nH^*+48\Psi n\pi\tanh n\pi H^*]}\frac{1-(-1)^n}{n\pi} \tag{16}$$

as compare to the conventional fluid based bearing system. This may be due to the effect that magnetization increases the viscosity of lubricant leading to increased pressure and hence the load. A closed observation of equation 15 suggest that the expression is linear with respect to μ^*, accordingly an increased in μ^* will lead to increased load carrying capacity. In the absence of magnetization this investigation reduces to the study of Sanni and Ayomidele [5].

The effect that load bearing capacity increases sharply with respect to magnetization is presented in Figures 2 4. The effect of H^* on the load carrying capacity with respect to μ^* remains negligible (C.f. Figure 2). Besides, the porosity effect up to the porosity value 0.001. Further, the porosity effect up to 0.001 remains either almost nominal or negligible as can be seen from Figures 6 and 7.

Indeed, some amount of fluid enters into the holes and when the surface dries up the fluids comes up to the surface. Therefore, the flow is maintained. But, the pressure decreases (and hence load carrying capacity decreases) because of the obstruction due to the presence of holes. When H^* increases more space is occupied by the fluid which results in increased pressure.

A composition of the result presented here and results of Sanni and Ayomidele [5] indicates that all the cases there appears elevated load.

4. Conclusions

It is established that the results presented here compare well with some of earlier published works. The Neuringer-Rosensweig model based magnetic fluid lubrication results.in significant enhancement of the performance of bearing characteristic. It is revealed that the porosity effect can be neutralized by the positive effect of magnetic fluid lubrication at least for smaller to nominal values of porosity. In addition, this type of bearing systems become some amount load under the absence of flow as well, a properly rarely seen in traditional lubrication.

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6. Nomenclature

A= length of bearing
h=film thickness = $h = h_2 - (h_2 - h_1)\frac{x}{A}$
h_1=minimum film thickness
h_2=maximum film thickness
r= film thickness ration =$\frac{h_2}{h_1}$
μ_0=permeability of the free space
$\overline{\mu}$=magnetic permeability
μ=absolute viscosity of the lubricant
U= tangential velocity of the slider
$H^* = \frac{H}{A}$
Ψ=permeability parameter= $\phi\frac{H}{h_1^3}$
p^*=Pressure in the lubricant region
H = Bearing wall thickness
ϕ= Permeability of porous material
P = Pressure in porous region
μ^*= Magnetization parameter
$\overline{P} = -\frac{h_1^2}{\mu UA}p^*$, Dimensionless pressure
$\overline{W} = -\frac{h_1^2}{\mu UBA^2}W$ Dimensionless load carrying capacity

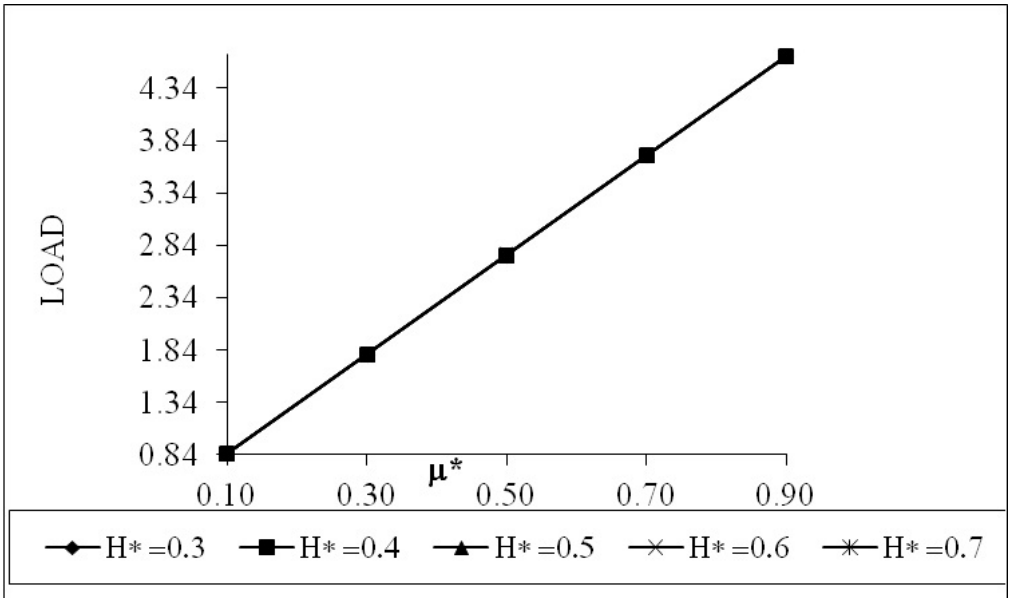


Figure 2: Variation of load caring capacity with respect to $\mu^* and H^*$

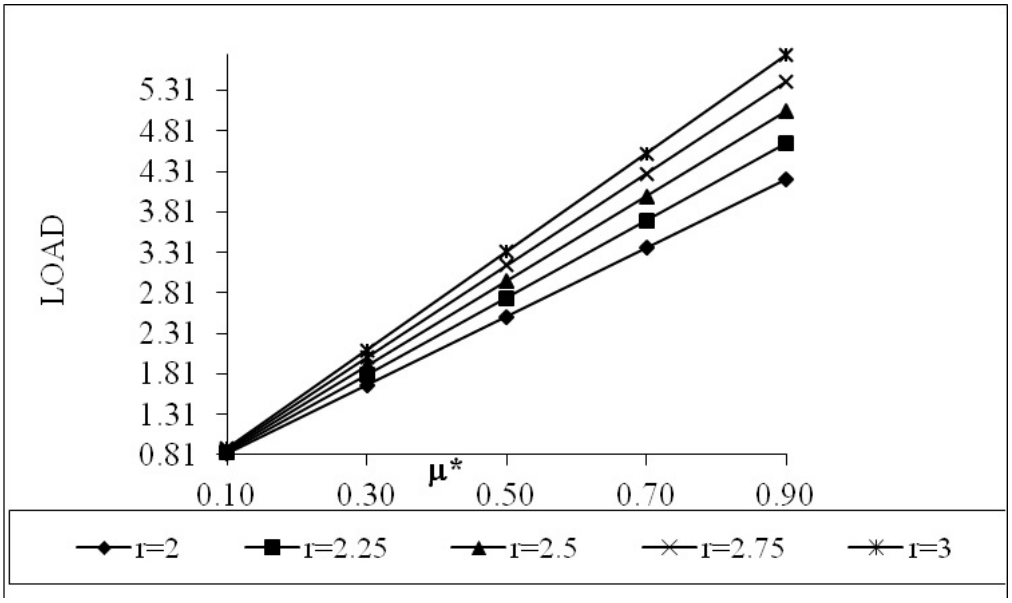


Figure 3: Variation of load caring capacity with respect to μ^* and r

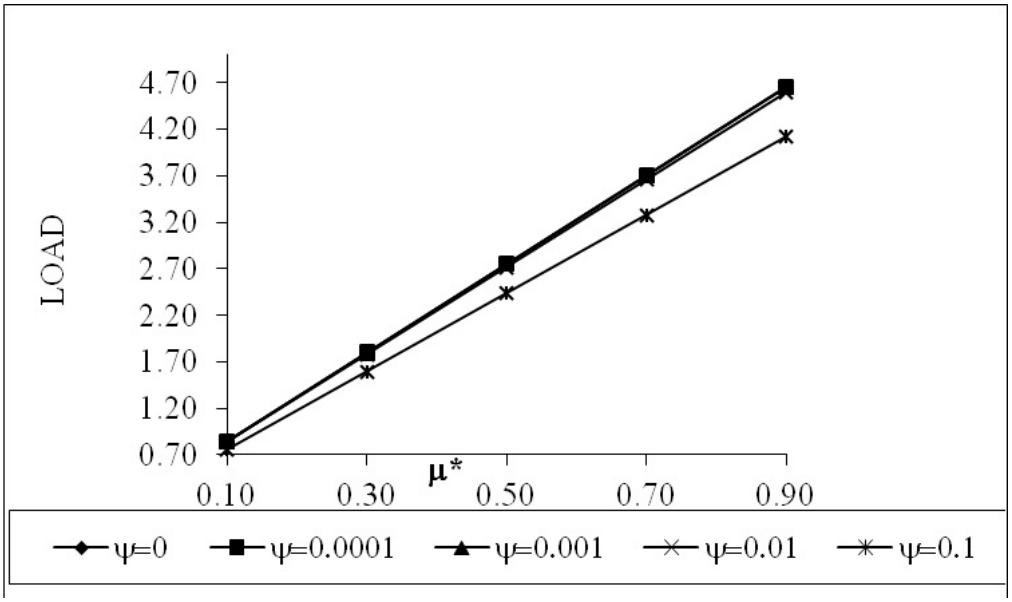


Figure 4: Variation of load caring capacity with respect to $\mu^* and \Psi$

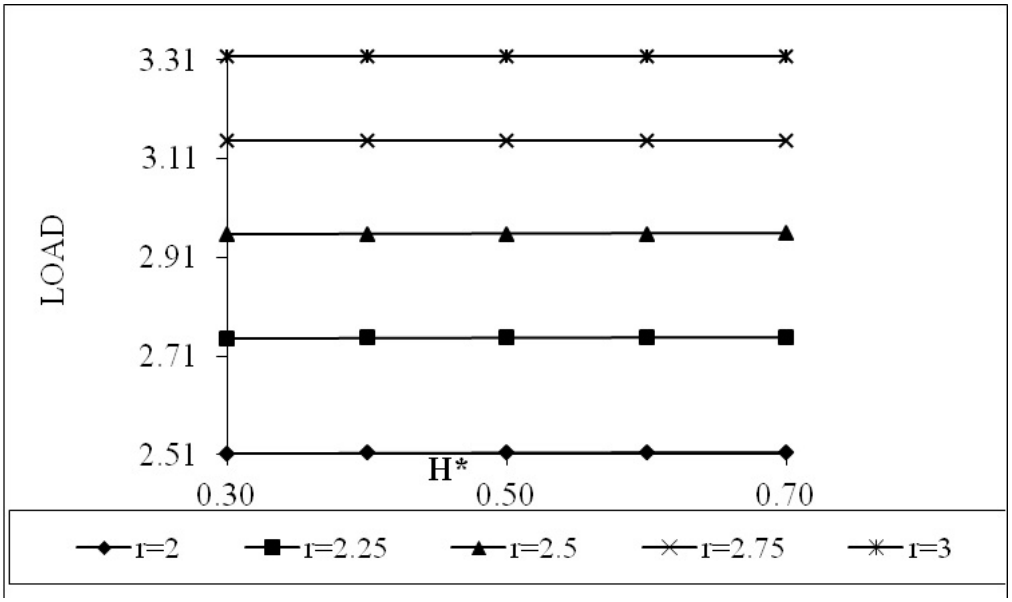


Figure 5: Variation of load caring capacity with respect to H^* and r

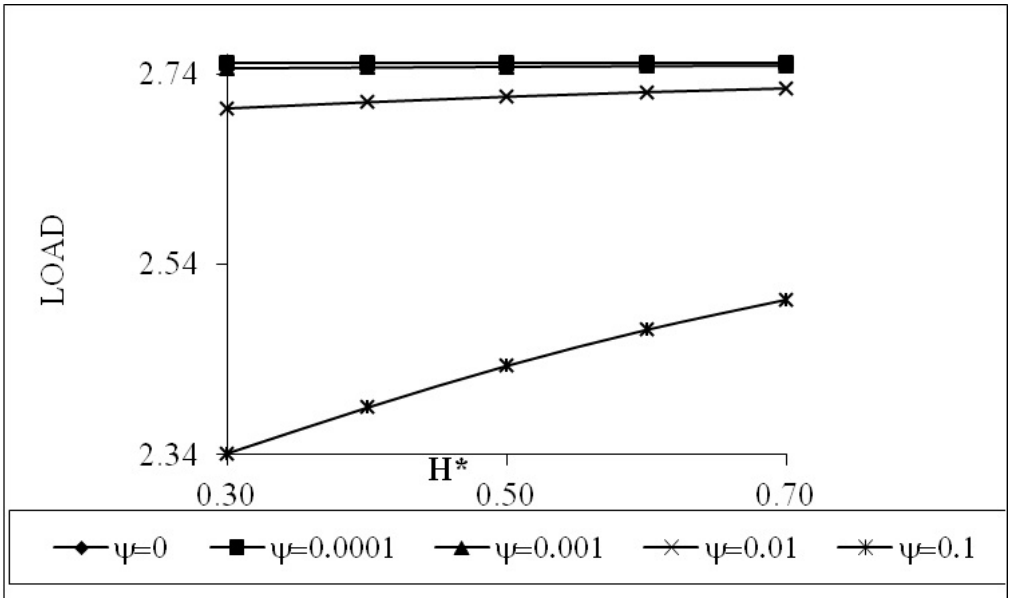


Figure 6: Variation of load caring capacity with respect to $H^{*}and\Psi$

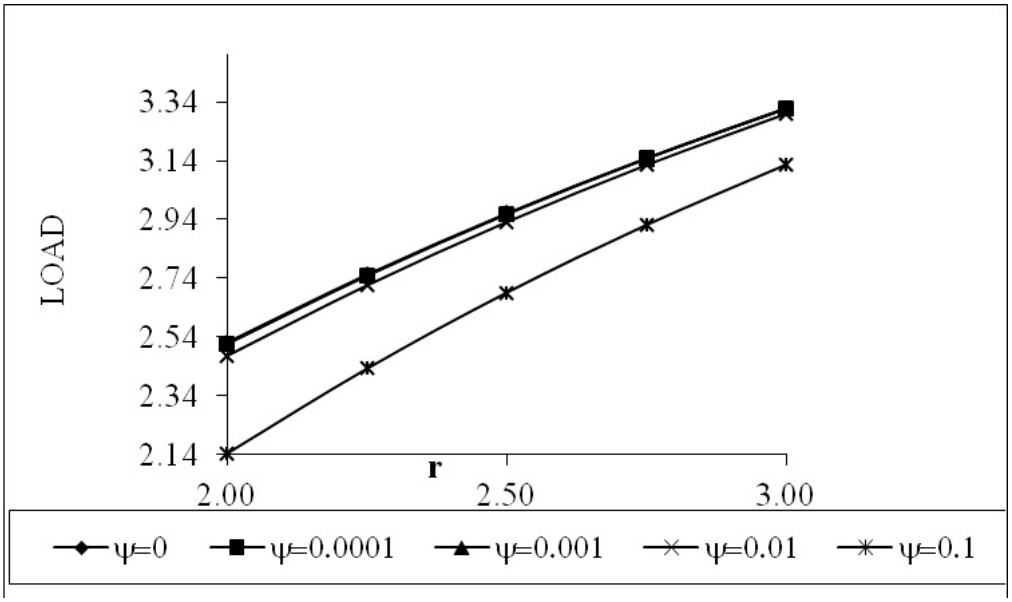


Figure 7: Variation of load caring capacity with respect to $rand\Psi$

References

[1] J. Prakash and S. Vij, *Load capacity and time-height relations for squeeze films between porous plates,* Wear, 24 (1973), 309-322.

[2] H. Wu, *An analysis of the squeeze film between porous rectangular plates,* Journal of Lubrication Technology, 94 (1972), 64-68.

[3] J. Prakash and S. Vij, *Hydrodynamic lubrication of a porous slider,* Journal of Mechanical Engineering Science, 15 (1973), 232-234.

[4] M. Bhat and C. Patel, *The squeeze film in an inclined porous slider bearing,* Wear, 66 (1981), 189-193.

[5] S. Sanni and I. Ayomidele, *Hydrodynamic lubrication of a porous slider: limitations of the simplifying assumption of a small porous facing thickness,* Wear, 147 (1991), 1-7.

[6] N. Das, *A study of optimum load-bearing capacity for slider bearings lubricated with couple stress fluids in magnetic field,* Tribology Inter- national, 31 (1998), 393-400.

[7] G. Deheri, P. Andharia, and R. Patel, *Longitudinally rough slider bearings with squeeze film formed by a magnetic fluid,* Industrial Lubrica- tion and Tribology, 56 (2004), 177-187.

[8] W. Ochoski, *Sliding bearings lubricated with magnetic fluids,* Indus- trial Lubrication and Tribology, 59 (2007), 252-265.

[9] V. Agrawal, *Magnetic-fluid-based porous inclined slider bearing,* Wear, 107 (1986), 133-139.

[10] N. Patel and G. Deheri, *Effect of surface roughness on the performance of a magnetic fluid based parallel plate porous slider bearing with slip velocity,* Journal of the Serbian society for Computational Mechanics, 5 (2011), 104-118.

[11] J.-R. Lin, *Performance characteristics of finite porous slider bearings: Brinkman model,* Tribology international, 34 (2001), 181-189.

[12] U. P. Singh and R. Gupta, *Dynamic performance characteristics of a curved slider bearing operating with ferrofluids,* Advances in Tribology, Hindawi Publication Corporation, Article ID 278723.

[13] M. Bhat and G. Deheri, *Porous composite slider bearing lubricated with magnetic fluid,* Japanese journal of applied physics, 30 (1991), 2513-2514.

[14] G. Deheri, P. Andharia, and R. Patel, *Transversely rough slider bearings with squeeze film formed by a magnetic fluid,* International Journal of Applied Mechanics and Engineering, 10 (2005), 53-76.

[15] J. L. Neuringer and R. E. Rosensweig, *Ferrohydrodynamics,* The Physics of Fluids, 7 (1964), 1927.

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An algorithm to compute the number of Rosenberg hypergroups of order less than 7

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Abstract. Using some special type of Boolean matrices, named very good matrices, we enumerate the non-isomorphic Rosenberg hypergroups of order less than 7. Moreover, the regular and reversible Rosenberg hypergroups are identified. Also the algorithm behind the method is presented.

Keywords: Rosenberg hypergroup, binary relation, regular reversible hypergroup, Boolean matrix.

1. Introduction

One of the most investigated hypergroups associated with binary relations is that introduced by Rosenberg [20] in 1998. It represents object of study of numerous papers written in order to compute the number of finite such hyperstructures. This line of research has been inspired by Migliorato's paper [19], where the structure of all non-isomorphic hypergroups of order 3 and of total regular abelian hypergroups have been determined. Later on Bayon and Lygeros [1, 2, 3] computed the number of finite abelian hypergroups and H_v-groups.

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Turning back to Rosenberg hypergroups, several algorithms written in $C\#$, Mathematica or MS Visual Basic have been presented by Spartalis and Mamaloukas [21], Massouros and Tsitouras [17, 18], Cristea et al. [9] and Jafarpour et al. [15], where particular types of such hypergroups have been computed. More exactly, in [9] the authors determined the number of Rosenberg hypergroups that are (general) mutually associative or complementary hypergroups. The paper [15], like the current one, deals with regular reversible Rosenberg hypergroups. Both papers use an algorithm based on representation of binary relations by Boolean matrices. All regular reversible Rosenberg hypergroups of order less than 5 have been computed, up to isomorphism [15], but not those of bigger order. That is why we propose here another method, consisting in representing every Boolean matrix by a non-negative integer obtained as explained in Section 4. In this way we compute the number of idempotent Rosenberg hypergroups, regular Rosenberg hypergroups and reversible Rosenberg hypergroups of order 6, up to isomorphism.

2. Preliminaries

As we have mentioned before, this paper is a continuation of a previous work [15]. Since we want the paper to be self-contained, we recall here the basic concepts regarding the regular, reversible and Rosenberg hypergroups, that are also included in [15].

Let us briefly recall some basic notions and results about hypergroups; for a comprehensive overview of this subject, the reader is refereed to [4, 8].
Let H be a non-empty set and $P^*(H)$ denote the set of all non-empty subsets of H. Let $\circ$ be a *hyperoperation* (or *join operation*) on H, that is, a function from the cartesian product $H \times H$ into $P^*(H)$. The image of the pair $(a, b) \in H \times H$ under the hyperoperation $\circ$ in $P^*(H)$ is denoted by $a \circ b$. The join operation can be extended in a natural way to subsets of H as follows: for non-empty subsets A, B of H, define $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its element a. The hyperstructure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c.$$

A semihypergroup $(H, \circ)$ is called a *hypergroup* if the reproduction law holds: $a \circ H = H \circ a = H$, for all $a \in H$. A non-empty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* if it is a hypergroup, too. An element e of H is called an *identity* if, for all $a \in H$, $a \in a \circ e \cap e \circ a$ and $a' \in H$ is called an *inverse* of a in H if $e \in a \circ a' \cap a' \circ a$.

Definition 2.1. (i) A hypergroup $(H, \circ)$ is *regular* if it has at least one identity and each element has at least one inverse.

(ii) A regular hypergroup $(H, \circ)$ is called *reversible* if, for any $(x, y) \in H^2$, it satisfies the following conditions:
(1) if $y \in a \circ x$, then there exists an inverse a' of a, such that $x \in a' \circ y$;
(2) if $y \in x \circ a$, then there exists an inverse a'' of a, such that $x \in y \circ a''$.

Definition 2.2. Let $(H, \circ)$ and $(H', \circ')$ be two hypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if it satisfies the condition: for any $x, y \in H$,

$$f(x \circ y) \subseteq f(x) \circ' f(y).$$

f is a *good homomorphism* if, for any $x, y \in H$, $f(x \circ y) = f(x) \circ' f(y)$. We say that the two hypergroups are *isomorphic* if there is a good homomorphism between them which is also a bijection.

The Rosenberg partial hypergroupoid $H_\rho = (H, \circ_\rho)$ associated with a binary relation ρ defined on a nonempty set H is constructed as follows. For any $x, y \in H$,

$$x \circ_\rho x = \{z \in H \mid (x, z) \in \rho\} \quad \text{and} \quad x \circ_\rho y = x \circ_\rho x \cup y \circ_\rho y.$$

The set $\mathbb{D}(\rho) = \{x \in H \mid \exists y \in H : (x, y) \in \rho\}$ is called the *domain* of ρ, while $\mathbb{R}(\rho) = \{y \in H \mid \exists x \in H : (x, y) \in \rho\}$ is the *range* of the relation ρ. An element $x \in H$ is called *outer element* of ρ if there exists $h \in H$ such that $(h, x) \notin \rho^2$.

The next theorem gives necessary and sufficient conditions, obtained by Rosenberg, under which the partial hypergroupoid H_ρ is a hypergroup.

Theorem 2.3 ([20])**.** *H_ρ is a hypergroup if and only if*

(i) *ρ has full domain: $\mathbb{D}(\rho) = H$;*

(ii) *ρ has full range: $\mathbb{R}(\rho) = H$;*

(iii) *$\rho \subset \rho^2$;*

(iv) *If $(a, x) \in \rho^2$ then $(a, x) \in \rho$, whenever x is an outer element of ρ.*

Adding new conditions to the previous ones, Corsini characterized the regular reversible Rosenberg hypergroups.

Theorem 2.4 ([5])**.** *The hypergroup H_ρ is regular if and only if $K = \{e \in H \mid P \subset e \circ_\rho e\} \neq \emptyset$, where e is an identity for H and $P = \{x \in H \mid x \notin x \circ_\rho x\}$.*

Moreover, if $K \neq \emptyset$ and ρ is symmetric, then H_ρ is a regular reversible hypergroup.

The rest of the section gathers properties of regular reversible Rosenberg hypergroups based on the representation of binary relations by Boolean matrices, having all elements 0 or 1 and satisfying the rules in a Boolean algebra: $0 + 1 = 1 + 0 = 1 + 1 = 1$, while $0 + 0 = 0$, and $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$ [21]. Let ρ be a binary relation defined on a finite set $H = \{a_1, ..., a_n\}$.

The associated Boolean matrix $M(\rho) = (a_{ij})$, $i, j \in \{1, 2, \ldots, n\}$, is obtained as follows: $a_{ij} = 1$, if $(a_i, a_j) \in \rho$ and $a_{ij} = 0$, if $(a_i, a_j) \notin \rho$. It is clear that $M(\rho^2) = M^2(\rho)$ and in this way, 2^{n^2} Rosenberg partial hypergroupoids can be defined on every set with n elements.

We recall now some definitions and results from [9, 15].

Definition 2.5 ([9])**.** The matrix $M(\rho)$ is called *very good* if the Rosenberg hypergroupoid H_ρ is a hypergroup.

Definition 2.6 ([15])**.** A Rosenberg hypergroup H_ρ is called *i-Rosenberg hypergroup* if $M(\rho)$ is an idempotent very good matrix (i.e. $M(\rho)^2 = M(\rho)$).

Definition 2.7 ([15])**.** (i) A very good matrix $M(\rho)$ is called *i-very good* only if H_ρ is an i-Rosenberg hypergroup.

(ii) A very good matrix $M(\rho)$ is called *regular* if H_ρ is a regular hypergroup.

(iii) A regular matrix $M(\rho)$ is called *reversible* if H_ρ is a regular reversible hypergroup.

For the transpose matrix of M we use the standard notation M^T. The next theorem characterizes the very good matrices.

Theorem 2.8 ([9])**.** *A matrix $M = M(\rho)$ is a very good matrix if and only if, for any j, with $1 \leq j \leq n$, the following assertions hold:*

(i) $M_j^T \neq (0)$*;*

(ii) $M_j \neq (0)$*;*

(iii) if $M_j^2 \neq (1)$*, then* $M_j = M_j^2$*,*

where M_j and M_j^2 are, respectively, the j-column vectors of the matrices $M(\rho)$ and $M(\rho^2)$, while (0) is the zero column vector and (1) is the column vector with all elements equal to 1.

As an immediate consequence, we obtain the following characterization of an idempotent very good matrix.

Proposition 2.9 ([15])**.** *An idempotent matrix $M = M(\rho)$ (i.e. $M^2 = M$) is a very good matrix if and only if, for any j, with $1 \leq j \leq n$, the following assertions hold:*

(i) $M_j^T \neq (0)$*;*

(ii) $M_j \neq (0)$*.*

The aim of this paper is to count all non-isomorphic Rosenberg hypergroups satisfying certain conditions. The next result gives the necessary and sufficient conditions such that two Rosenberg hypergroups, having the same support set, are isomorphic. Two matrices $M(\rho)$ and $M(\rho')$ are called *isomorphic* if the associated Rosenberg hypergroups H_ρ and $H_{\rho'}$ are isomorphic.

Theorem 2.10 ([9]). *Let $H = \{a_1, ..., a_n\}$ be a finite set, ρ and ρ' be two binary relations on H and $M(\rho) = (t_{ij}), M(\rho') = (t'_{ij})$ be their associated matrices. The hypergroups H_ρ and $H_{\rho'}$ are isomorphic if and only if $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for σ a permutation of the set $\{1, 2, \ldots, n\}$.*

Now we check when a very good matrix is regular.

Theorem 2.11 ([15]). *Let $M(\rho) = (t_{ij})_{n\times n}$ be a very good matrix. Then $M(\rho)$ is regular if and only if there exists k, $1 \leq k \leq n$, such that, for all i, $1 \leq i \leq n$, we have $t^c_{ii} \leq t_{ki}$, where $t^c_{ij} = 0$ if and only if $t_{ij} = 1$.*

Finally, we present how to combine two idempotent very good matrices in order to obtain another idempotent very good matrix.

Theorem 2.12 ([15]). *Let $M = (t_{ij})_{n\times n}$, $M' = (t'_{ij})_{m\times m}$ be two idempotent very good matrices. Then $M \oplus M' = (m_{ij})_{k\times k}$, where $k = n + m$, and*

$$m_{ij} = \begin{cases} t_{ij}, & \text{if } \ i \leq n, \quad j \leq n \\ t'_{ij}, & \text{if } \ n < i, \quad n < j \\ 0 & \text{elsewhere} \end{cases}$$

is an idempotent very good matrix.

Theorem 2.13 ([15]). *Let $M = (t_{ij})_{n\times n}$, $M' = (t'_{ij})_{m\times m}$ be two idempotent very good matrices. Then $M \boxplus M' = (m_{ij})_{k\times k}$, where $k = n + m$, and*

$$m_{ij} = \begin{cases} t_{ij}, & \text{if } \ i \leq n, \quad j \leq n \\ t'_{ij}, & \text{if } \ n < i, \quad n < j \\ 1, & \text{if } \ i \leq n, \quad j > n \\ 0 & \text{if } \ n < i, \quad j \leq n \end{cases}$$

is an idempotent very good matrix. Moreover, if M is regular then $M \boxplus M'$ is regular, too.

We conclude this section with the following result.

Theorem 2.14 ([15]). *A regular matrix $M = M(\rho)$ is reversible if $M = M^T$.*

3. The main algorithm

Comparing to previous approaches, what makes our method so efficient is that we represent every $n \times n$ Boolean matrix $M = (m_{ij}), i, j \in \{1, 2, \ldots, n\}$ by a non-negative integer as follows. Let $m_{i1}, ..., m_{in}$ be the elements on the i'th row. Then the corresponding integer is given by the formula:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \times 2^{(i-1)n+j-1}.$$

From now on, we denote the integer by $I(M)$. Let us see one example. For instance consider the following matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

then the corresponding integer is

$$\begin{aligned} I(M) &= \sum_{i=1}^{4}\sum_{j=1}^{4} m_{ij} \times 2^{4(i-1)+j-1} \\ &= \sum_{i=1}^{4}(m_{i1} \times 2^{4(i-1)} + m_{i2} \times 2^{4(i-1)+1} + m_{i3} \times 2^{4(i-1)+2} + m_{i4} \times 2^{4(i-1)+3}) \\ &= 2^0 + 2^3 + 2^4 + 2^5 + 2^9 + 2^{11} + 2^{14} + 2^{15} \\ &= 51769. \end{aligned}$$

Obviously there are 2^{n^2} Boolean matrices of order n, which we denote by $M_1, ..., M_{2^{n^2}}$ and all of them can easily be generated by computer. So computation of vg (Number of very good matrices) and vgi (Number of very good idempotent matrices) is straightforward. The tough part is when we want to enumerate all non-isomorphic matrices, and this is where our integer representation of boolean matrices comes to play. Let A and A_2 are initialized as two $1 \times m$ arrays with all entries equal to 0. Once a Boolean matrix is approved to be a very good matrix, we have to decide weather any of its isomorphic copies has already been approved. Let $I(M)$ be the integer representation of the matrix M. If the $I(M)$'th element of array A_2 is already equal to 1, then this is a copy of another matrix that is already considered and we skip it, otherwise it is new and we increment the variable *co*3 (the number of very good matrices up to isomorphism) by 1. Also we produce all its $n!$ isomorphic copies which are denoted by M_σ where σ is a permutation of $\{1, ..., n\}$ and compute $I(M_\sigma)$ (the set of corresponding integers) and set the corresponding array in A_2 equal to 1. We follow a similar approach in the idempotent case and decide if a very good idempotent matrix is new (up to isomorphism) by looking at corresponding entry in array A and increment the variable *co*2 (the number of very good idempotent matrices up to isomorphism) if the case is approved to be new (up to isomorphism). The Main Algorithm is summarized in the following:

```
co2:=0; co3:=0;vgi:=0;vg:=0; rc:=0; rv:=0;
for j=1 to 2^{n^2}
    A(j) := 0; A_2(j) := 0
end
for j=1 to 2^{n^2}
    M := M_j
    if M ≤ M^2 then
        if M satisfies in conditions (i), (ii), (iii) of Theorem 2.3 then
            if A_2(I(A)) == 0 then
                co3=co3+1;
                for every permutation σ of {1, ..., n} set A_2(I(M_σ)) = 1
                if M is regular rc=rc+1;
                if M is reversible rv=rv+1;
            end
            vg=vg+1;
            if M == M^2 then
                vgi=vgi+1;
                if A(I(M)) == 0 then
                    co2=co2+1;
                    for every permutation σ of {1, ..., n} set A(I(M_σ)) = 1
                end
            end
        end
    end
end
```

We prepared a C++ code, based on the above algorithm, which will be available upon request from authors.

4. Conclusions

Hypregroups constructed from binary/ternary/n-ary relations have been obtained and studied in [5, 6, 7, 10, 11, 12, 13, 14, 16, 20, 22], as a method to construct new hyperstructures, and their properties are identified based on the properties of the involved relations. The Rosenberg hypergroups represent the subject of several papers [9, 15, 17, 18], dedicated to the computation of their number. In this paper, the number of all non-isomorphic regular reversible Rosenberg hypergroups has been computed, applying a program written in C++. The results of this computation (for $n = 2, 3, 4, 5$ and 6) are summarized in the following table:

vg: Number of very good matrices
co3: Number of very good matrices up to isomorphism
vgi: Number of very good idempotent matrices
co2: Number of very good idempotent matrices up to isomorphism
rc: Number of regular matrices up to isomorphism
rv: Number of reversible matrices up to isomorphism

N=	2	3	4	5	6
vg	6	149	9729	2921442	4578277389
co3	4	33	501	26409	6502030
vgi	4	35	559	14962	636217
co2	3	10	44	239	1668
rc	4	32	466	22672	5000329
rv	3	9	31	147	1095

By using Theorems 2.12 and 2.13 and results presented in above table we can provide a lower bound for very good idempotent matrices of higher orders. For example for order 11 the lower bound would be 14962+636217=651179.

References

[1] R. Bayon, N. Lygeros, *Number of abelian H_v groups of order n*, in: N.J.A. Sloane (Ed.), The On-line Encyclopedia of Integer Sequences, 2005.

[2] R. Bayon, N. Lygeros, *Number of abelian hypergroups of order n*, in: N.J.A. Sloane (Ed.), The On-line Encyclopedia of Integer Sequences, 2007.

[3] R. Bayon, N. Lygeros, *Advanced results in enumeration of hyperstructures*, J. Algebra, 320 (2008), 821–835.

[4] P. Corsini, *Prolegomena of Hypergroups Theory*, Aviani Editore, 1993.

[5] P. Corsini, *On the hypergroups associated with binary relations*, Multi. Val. Logic, 5 (2000), 407-419.

[6] P. Corsini, *Binary relations and hypergroupoids*, Ital. J. Pure Appl. Math., 7 (2000), 11-18.

[7] P. Corsini, V. Leoreanu, *Hypergroups and binary relations*, Algebra Universalis, 43 (2000), 321-330.

[8] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, in: Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[9] I. Cristea , M. Jafarpour, S. Mousavi, A. Soleymani, *Enumeration of Rosenberg hypergroups*, Comput. Math. Appl., 60 (2010), 2753-2763.

[10] I. Cristea, *Several aspects on the hypergroups associated with n-ary relations*, An. Şt. Univ. Ovidius Constanta, 17 (2009), no.3, 99-110.

[11] I. Cristea, M. Stefanescu, *Binary relations and reduced hypergroups*, Discrete Math., 308 (2008), 3537-3544.

[12] I. Cristea, M. Stefanescu, *Hypergroups and n-ary relations*, European J. Combin., 31 (2010), 780-789.

[13] B. Davvaz, V.Leoreanu-Fotea, *Binary relations on ternary semihypergroups*, Comm. Algebra, 38 (2010), 3621-3636.

[14] M. De Salvo, G. Lo Faro, *Hypergroups and binary relations*, Multi. Val. Logic, 8 (2002), 645-657.

[15] M. Jafarpour, I. Cristea and A. Tavakoli, *A method to compute the number of regular reversible Rosenberg hypergroup*, Ars combinatorics, 128(2016), 309-329.

[16] M. Jafarpour, S.Sh. Mousavi, *A construction of hypersemigroups from a family of preordered semigroups*, Scientiae Mathematicae Japonicae Online, e-2009, 587-594.

[17] Ch. G. Massouros, Ch. Tsitouras, *Enumeration of hypercompositional structures defined by binary relations*, Ital. J. Pure Appl. Math, 28 (2011), 53-64.

[18] Ch. G. Massouros, Ch. Tsitouras, *On enumeration of hypergroups of order* 3, Comput. Math. Appl., 59 (2010), 519-523.

[19] R. Migliorato, *Ipergruppi di cardinalita* 3 *e isomorfismi di ipergruppoidi commutativi totalmente regolari*, in: Atti Convegno su Ipergruppi, altre Strutture Multivoche e loro applicazioni, Udine, 1985.

[20] I.G. Rosenberg, *Hypergroups and join spaces determined by relations*, Ital. J. Pure Appl. Math., 4 (1998), 93-101.

[21] S. Spartalis, C. Mamaloukas, *On hyperstructures associated with binary relations*, Comput. Math. Appl., 51 (2006), 41-50.

[22] S. Spartalis, M. Konstantinidou-Serafimidou, A. Taouktsoglou, *C-hypergroupoids obtained by special binary relations*, Comput. Math. Appl., 59 (2010), 2628-2635.

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Certain notions of single-valued neutrosophic K-algebras

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Abstract. We apply the notion of single-valued neutrosophic sets to K-algebras. We develop the concept of single-valued neutrosophic K-subalgebras, and present some of their properties. Moreover, we study the behavior of single-valued neutrosophic K-subalgebras under homomorphism. Finally, we discuss $(\in, \in \vee q)$-single-valued neutrosophic K-algebras.

Keywords: Single-valued neutrosophic sets, K-algebras, homomorphism, $(\in, \in \vee q)$-single-valued neutrosophic K-algebras.

1. Introduction

A new kind of logical algebra, known as K-algebra, was introduced by Dar and Akram [9]. A K-algebra was built on a group G by adjoining the induced binary operation on G. The group G is particularly of the type in which each non-identity element is not of order 2. This algebraic structure is, in general, non-commutative and non-associative with right identity element [5, 10, 11]. Akram et.al [2, 3, 4] introduced fuzzy K-algebras. They then developed fuzzy K-algebras with other researchers worldwide. The concepts and results of K-algebras have been broadened to the fuzzy setting frames by applying Zadeh's fuzzy set theory and its generalizations, namely, interval- valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, bipolar fuzzy sets and vague sets.

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In handling information regarding various aspects of uncertainty, non-classical logic (a great extension and development of classical logic) is considered to be a more powerful technique than the classical logic. The non-classical logic has nowadays become a useful tool in computer science. Moreover, non-classical logic deals with fuzzy information and uncertainty. In 1998, Smarandache [15] introduced neutrosophic sets as a generalization of fuzzy sets [19] and intuitionistic fuzzy sets [6]. A neutrosophic set is identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval $]^{-}0, 1^{+}[$, where $^{-}0 = 0 - \epsilon$, $1^{+} = 1 + \epsilon$, ϵ is an infinitesimal number. To apply neutrosophic set in real-life problems more conveniently, Smarandache [15] and Wang et al. [16] defined single-valued neutrosophic sets which takes the value from the subset of [0, 1]. Thus, a single-valued neutrosophic set is an instance of neutrosophic set, and can be used expediently to deal with real-world problems, especially in decision support. Algebraic structures have a vital place with vast applications in various disciplines. Neutrosophic set theory has been applied to algebraic structures [1, 8, 13]. In this research article, we introduce the notion of single-valued neutrosophic K-subalgebra and investigate some of their properties. We discuss K-subalgebra in terms of level sets using neutrosophic environment. We study the homomorphisms between the single-valued neutrosophic K-subalgebras. We discuss characteristic K-subalgebras and fully invariant K-subalgebras. Finally, we discuss $(\in, \in \vee q)$-single-valued neutrosophic K-algebras.

2. Single-valued neutrosophic K-algebras

The concept of K-algebra was first developed by Dar and Akram in [14].

Definition 2.1. Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a K- *algebra* is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = x.y^{-1}$ and satisfies the following axioms:

(i) $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(ii) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(iii) $x \odot x = e$,

(iv) $x \odot e = x$,

(v) $e \odot x = x^{-1}$,

for all x, y, $z \in G$.

Definition 2.2. [16] Let Z be a space of objects with a general element $z \in Z$. A single-valued neutrosophic set $\mathcal{A}$ in Z is characterized by three membership

functions, $\mathcal{T}_{\mathcal{A}}$-truth membership function, $\mathcal{I}_{\mathcal{A}}$-indeterminacy membership function and $\mathcal{F}_{\mathcal{A}}$-falsity membership function, where $\mathcal{T}_{\mathcal{A}}(z), \mathcal{I}_{\mathcal{A}}(z), \mathcal{F}_{\mathcal{A}}(z) \in [0,1]$, for all $z \in Z$.

$\mathcal{A}$ can also be written as $\mathcal{A} = \{< z, \mathcal{T}_{\mathcal{A}}(z), \mathcal{I}_{\mathcal{A}}(z), \mathcal{F}_{\mathcal{A}}(z) > \mid z \in Z\}$.

Definition 2.3. A single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in a K-algebra $\mathcal{K}$ is called a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

(a) $\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\}$,
(b) $\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\}$,
(c) $\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\}$, for all $s, t \in G$.

Note that $\mathcal{T}_{\mathcal{A}}(e) \geq \mathcal{T}_{\mathcal{A}}(s)$, $\mathcal{I}_{\mathcal{A}}(e) \geq \mathcal{I}_{\mathcal{A}}(s)$, $\mathcal{F}_{\mathcal{A}}(e) \leq \mathcal{F}_{\mathcal{A}}(s)$, for all $s \in G$.

Example 2.1. Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Cayley's table for $\odot$ is given as:

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
e	e	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^8	x^7	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^8	x^7	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^8	x^7	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^8	x^7	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^8	x^7	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e	x^8	x^7
x^7	x^7	x^6	x^5	x^4	x^3	x^2	x	e	x^8
x^8	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	e

We define a single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in K-algebra as follows:

$\mathcal{T}_{\mathcal{A}}(e) = 0.8, \mathcal{I}_{\mathcal{A}}(e) = 0.7, \mathcal{F}_{\mathcal{A}}(e) = 0.4$,

$\mathcal{T}_{\mathcal{A}}(s) = 0.2, \mathcal{I}_{\mathcal{A}}(s) = 0.3, \mathcal{F}_{\mathcal{A}}(s) = 0.6$, for all $s \neq e \in G$.

Clearly, $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Example 2.2. Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra on dihedral group $D4$ given as $G = \{e, a, b, c, x, y, u, v\}$, where $c = ab, x = a^2, y = a^3$, $u = a^2b, v = a^3b$ and

Cayley's table for $\odot$ is given as:

$\odot$	e	a	b	c	x	y	u	v
e	e	y	b	c	x	a	u	v
a	a	e	c	u	y	x	v	b
b	b	c	e	y	u	v	x	a
c	c	u	a	e	v	b	y	x
x	x	a	u	v	e	y	b	c
y	y	x	v	b	a	e	c	u
u	u	v	x	a	b	c	e	y
v	v	b	y	x	c	u	a	e

We define a single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ in K-algebra as follows: $\mathcal{T}_\mathcal{A}(e) = 0.9, \mathcal{I}_\mathcal{A}(e) = 0.3, \mathcal{F}_\mathcal{A}(e) = 0.3$, $\mathcal{T}_\mathcal{A}(s) = 0.6, \mathcal{I}_\mathcal{A}(s) = 0.2, \mathcal{F}_\mathcal{A}(s) = 0.4$, for all $s \neq e \in G$. By routine calculations, it can be verified that $\mathcal{A}$ is a single-valued neutrosophic K-subalgebra ok $\mathcal{K}$.

Proposition 2.1. If $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then

1. $(\forall s, t \in G), (\mathcal{T}_\mathcal{A}(s \odot t) = \mathcal{T}_\mathcal{A}(t) \Rightarrow \mathcal{T}_\mathcal{A}(s) = \mathcal{T}_\mathcal{A}(e))$.
   $(\forall s, t \in G)(\mathcal{T}_\mathcal{A}(s) = \mathcal{T}_\mathcal{A}(e) \Rightarrow \mathcal{T}_\mathcal{A}(s \odot t) \geq \mathcal{T}_\mathcal{A}(t))$.
2. $(\forall s, t \in G), (\mathcal{I}_\mathcal{A}(s \odot t) = \mathcal{I}_\mathcal{A}(t) \Rightarrow \mathcal{I}_\mathcal{A}(s) = \mathcal{I}_\mathcal{A}(e))$.
   $(\forall s, t \in G)(\mathcal{I}_\mathcal{A}(s) = \mathcal{I}_\mathcal{A}(e) \Rightarrow \mathcal{I}_\mathcal{A}(s \odot t) \geq \mathcal{I}_\mathcal{A}(t))$.
3. $(\forall s, t \in G), (\mathcal{F}_\mathcal{A}(s \odot t) = \mathcal{F}_\mathcal{A}(t) \Rightarrow \mathcal{F}_\mathcal{A}(s) = \mathcal{F}_\mathcal{A}(e))$.
   $(\forall s, t \in G)(\mathcal{F}_\mathcal{A}(s) = \mathcal{F}_\mathcal{A}(e) \Rightarrow \mathcal{F}_\mathcal{A}(s \odot t) \leq \mathcal{F}_\mathcal{A}(t))$.

Proof. 1. Assume that $\mathcal{T}_\mathcal{A}(s \odot t) = \mathcal{T}_\mathcal{A}(t)$, for all $s, t \in G$. Taking $t = e$ and using (iii) of Definition 2.1, we have $\mathcal{T}_\mathcal{A}(s) = \mathcal{T}_\mathcal{A}(s \odot e) = \mathcal{T}_\mathcal{A}(e)$. Let for $s, t \in G$ be such that $\mathcal{T}_\mathcal{A}(s) = \mathcal{T}_\mathcal{A}(e)$. Then $\mathcal{T}_\mathcal{A}(s \odot t) \geq \min\{\mathcal{T}_\mathcal{A}(s), \mathcal{T}_\mathcal{A}(t)\} = \min\{\mathcal{T}_\mathcal{A}(e), \mathcal{T}_\mathcal{A}(t)\} = \mathcal{T}_\mathcal{A}(t)$.

2. Assume that $\mathcal{I}_\mathcal{A}(s \odot t) = \mathcal{I}_\mathcal{A}(t)$, for all $s, t \in G$. Taking $t = e$ and by (iii) of Definition 2.1, we have $\mathcal{I}_\mathcal{A}(s) = \mathcal{I}_\mathcal{A}(s \odot e) = \mathcal{I}_\mathcal{A}(e)$. Also let $s, t \in G$ be such that $\mathcal{I}_\mathcal{A}(s) = \mathcal{I}_\mathcal{A}(e)$. Then $\mathcal{I}_\mathcal{A}(s \odot t) \geq \min\{\mathcal{I}_\mathcal{A}(s), I_(t)\} = \min\{\mathcal{I}_\mathcal{A}(e), \mathcal{I}_\mathcal{A}(t)\} = \mathcal{I}_\mathcal{A}(t)$.

3. Consider that $\mathcal{F}_\mathcal{A}(s \odot t) = \mathcal{F}_\mathcal{A}(t)$, for all $s, t \in G$. Taking $t = e$ and again by (iii) of Definition 2.1, we have $\mathcal{F}_\mathcal{A}(s) = \mathcal{F}_\mathcal{A}(s \odot e) = \mathcal{F}_\mathcal{A}(e)$. Let $s, t \in G$ be such that $\mathcal{F}_\mathcal{A}(s) = F_(e)$. Then $\mathcal{F}_\mathcal{A}(s \odot t) \leq \max\{\mathcal{F}_\mathcal{A}(s), \mathcal{F}_\mathcal{A}(t)\} = \max\{\mathcal{F}_\mathcal{A}(e), \mathcal{F}_\mathcal{A}(t)\} = \mathcal{F}_\mathcal{A}(t)$.

   This completes the proof.

□

Definition 2.4. Let $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ be a single-valued neutrosophic set in a K-algebra $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in [0,1] \times [0,1] \times [0,1]$ with $\alpha + \beta + \gamma \leq 3$. Then level subsets of $\mathcal{A}$ are defined as:

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \{s \in G \mid \mathcal{T}_\mathcal{A}(s) \geq \alpha, \mathcal{I}_\mathcal{A}(s) \geq \beta, \mathcal{F}_\mathcal{A}(s) \leq \gamma\},$$
$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \{s \in G \mid \mathcal{T}_\mathcal{A}(s) \geq \alpha\} \cap \{s \in G \mid \mathcal{I}_\mathcal{A}(s) \geq \beta\} \cap \{s \in G \mid \mathcal{F}_\mathcal{A}(s) \leq \gamma\},$$
$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \cup(\mathcal{T}_\mathcal{A}, \alpha) \cap \cup'(\mathcal{I}_\mathcal{A}, \beta) \cap L(\mathcal{F}_\mathcal{A}, \gamma)$$

are called (α, β, γ) -level subsets of single-valued neutrosophic set $\mathcal{A}$.

The set of all $(\alpha, \beta, \gamma) \in \mathrm{Im}(\mathcal{T}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A})$ is known as image of $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$. The set $\mathcal{A}_{(\alpha,\beta,\gamma)} = \{s \in G \mid \mathcal{T}_\mathcal{A}(s) > \alpha, \mathcal{I}_\mathcal{A}(s) > \beta, \mathcal{F}_\mathcal{A}(s) < \gamma\}$ is known as strong (α, β, γ)- level subset of $\mathcal{A}$.

Proposition 2.2. If $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then the level subsets $\cup(\mathcal{T}_\mathcal{A}, \alpha) = \{s \in G \mid \mathcal{T}_\mathcal{A}(s) \geq \alpha\}$, $\cup'(\mathcal{I}_\mathcal{A}, \beta) = \{s \in G \mid \mathcal{I}_\mathcal{A}(s) \geq \beta\}$ and $L(\mathcal{F}_\mathcal{A}, \gamma) = \{s \in G \mid \mathcal{F}_\mathcal{A}(s) \leq \gamma\}$ are k-subalgebras of $\mathcal{K}$, for every $(\alpha, \beta, \gamma) \in \mathrm{Im}(\mathcal{T}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A}) \times \mathrm{Im}(\mathcal{F}_\mathcal{A}) \subseteq [0,1]$, where $\mathrm{Im}(\mathcal{T}_\mathcal{A})$, $\mathrm{Im}(\mathcal{I}_\mathcal{A})$ and $\mathrm{Im}(\mathcal{F}_\mathcal{A})$ are sets of values of $T_(\mathcal{A})$, $\mathcal{I}_(\mathcal{A})$ and $F_(\mathcal{A})$, respectively.

Proof. Assume that $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in \mathrm{Im}(\mathcal{T}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A}) \times \mathrm{Im}(\mathcal{F}_\mathcal{A})$ be such that $\cup(\mathcal{T}_\mathcal{A}, \alpha) \neq \emptyset, \cup'(\mathcal{I}_\mathcal{A}, \beta) \neq \emptyset$ and $L(\mathcal{F}_\mathcal{A}, \gamma) \neq \emptyset$. Now to prove that $\cup, \cup'$ and L are level K-subalgebras. Let for $s, t \in \cup(\mathcal{T}_\mathcal{A}, \alpha)$, $\mathcal{T}_\mathcal{A}(s) \geq \alpha$ and $\mathcal{T}_\mathcal{A}(t) \geq \alpha$. It follows from Definition 2.3 that $\mathcal{T}_\mathcal{A}(s \odot t) \geq \min\{\mathcal{T}_\mathcal{A}(s), \mathcal{T}_\mathcal{A}(t)\} \geq \alpha$. It implies that $s \odot t \in \cup(\mathcal{T}_\mathcal{A}, \alpha)$. Hence $\cup(\mathcal{T}_\mathcal{A}, \alpha)$ is a level K-subalgebra of $\mathcal{K}$. Similar result can be proved for $\cup'(\mathcal{I}_\mathcal{A}, \beta)$ and $L(\mathcal{F}_\mathcal{A}, \gamma)$. □

Theorem 2.1. Let $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ be a single-valued neutrosophic set in K-algebra $\mathcal{K}$. Then $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if and only if $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-sublagebra of $\mathcal{K}$, for every $(\alpha, \beta, \gamma) \in \mathrm{Im}(\mathcal{T}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A}) \times \mathrm{Im}(\mathcal{F}_\mathcal{A})$ with $\alpha + \beta + \gamma \leq 3$.

Proof. Let $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ be a single-valued neutrosophic set in a K-algebra $\mathcal{K}$. Assume that $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ be a single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Let for $(\alpha, \beta, \gamma) \in \mathrm{Im}(\mathcal{T}_\mathcal{A}) \times \mathrm{Im}(\mathcal{I}_\mathcal{A}) \times \mathrm{Im}(\mathcal{F}_\mathcal{A})$ with $\alpha + \beta + \gamma \leq 3$ be such that $\mathcal{A}_{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $s, t \in \mathcal{A}_{(\alpha,\beta,\gamma)}$ be such that

$$\mathcal{T}_\mathcal{A}(s) \geq \alpha, \mathcal{T}_\mathcal{A}(t) \geq \alpha',$$
$$\mathcal{I}_\mathcal{A}(s) \geq \beta, \mathcal{I}_\mathcal{A}(t) \geq \beta',$$
$$\mathcal{F}_\mathcal{A}(s) \leq \gamma, \mathcal{F}_\mathcal{A}(t) \leq \gamma'.$$

Without loss of generality we can assume that $\alpha \leq \alpha^{'}$, $\beta \leq \beta^{'}$ and $\gamma \geq \gamma^{'}$. It follows from Definition 2.3 that

$$\begin{aligned}
\mathcal{T}_{\mathcal{A}}(s \odot t) &\geq \alpha = \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\},\\
\mathcal{I}_{\mathcal{A}}(s \odot t) &\geq \beta = \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\},\\
\mathcal{F}_{\mathcal{A}}(s \odot t) &\leq \gamma = \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\}.
\end{aligned}$$

It implies that $s \odot t \in \mathcal{A}_{(\alpha,\beta,\gamma)}$. So, $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$.

Conversely, we suppose that $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$. If the condition of the Definition 2.3 is not true, then there exist $u, v \in G$ such that

$$\begin{aligned}
\mathcal{T}_{\mathcal{A}}(u \odot v) &< \min\ \{\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)\},\\
\mathcal{I}_{\mathcal{A}}(u \odot v) &< \min\ \{\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)\},\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &> \max\{\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)\}.
\end{aligned}$$

Taking $\alpha_1 = \frac{1}{2}(\mathcal{T}_{\mathcal{A}}(u \odot v) + \min\{\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)\})$, $\beta_1 = \frac{1}{2}(\mathcal{I}_{\mathcal{A}}(u \odot v) + \min\{\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)\})$, $\gamma_1 = \frac{1}{2}(\mathcal{F}_{\mathcal{A}}(u \odot v) + \min\{\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)\})$.

We have $\mathcal{T}_{\mathcal{A}}(u \odot v) < \alpha_1 < \min\{\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)\}$, $\mathcal{I}_{\mathcal{A}}(u \odot v) < \beta_1 < \min\{\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)\}$ and $\mathcal{F}_{\mathcal{A}}(u \odot v) > \gamma_1 > \max\{\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)\}$. It implies that $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$ and $u \odot v \notin \mathcal{A}_{(\alpha,\beta,\gamma)}$, a contradiction. Therefore, the condition of Definition 2.3 is true. Hence $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic k-subalgebra of $\mathcal{K}$. □

Theorem 2.2. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic k-subalgebra and $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in \mathrm{Im}(\mathcal{T}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{I}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{F}_{\mathcal{A}})$ with $\alpha_j + \beta_j + \gamma_j \leq 3$ for $j = 1, 2$. Then $\mathcal{A}_{(\alpha_1,\beta_1,\gamma_1)} = \mathcal{A}_{(\alpha_2,\beta_2,\gamma_2)}$ if $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$.

Proof. If $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$, then clearly $\mathcal{A}_{(\alpha_1,\beta_1,\gamma_1)} = \mathcal{A}_{(\alpha_2,\beta_2,\gamma_2)}$.

Assume that $\mathcal{A}_{(\alpha_1,\beta_1,\gamma_1)} = \mathcal{A}_{(\alpha_2,\beta_2,\gamma_2)}$. Since $(\alpha_1, \beta_1, \gamma_1) \in \mathrm{Im}(\mathcal{T}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{I}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{F}_{\mathcal{A}})$, there exists $s \in G$ such that $\mathcal{T}_{\mathcal{A}}(s) = \alpha_1, \mathcal{I}_{\mathcal{A}}(s) = \beta_1$ and $\mathcal{F}_{\mathcal{A}}(s) = \gamma_1$. It follows that $s \in \mathcal{A}_{(\alpha_1,\beta_1,\gamma_1)} = \mathcal{A}_{(\alpha_2,\beta_2,\gamma_2)}$. So that $\alpha_1 = \mathcal{T}_{\mathcal{A}}(s) \geq \alpha_2, \beta_1 = \mathcal{I}_{\mathcal{A}}(s) \geq \beta_2$ and $\gamma_1 = \mathcal{F}_{\mathcal{A}}(s) \leq \gamma_2$. Also $(\alpha_2, \beta_2, \gamma_2) \in \mathrm{Im}(\mathcal{T}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{I}_{\mathcal{A}}) \times \mathrm{Im}(\mathcal{F}_{\mathcal{A}})$, there exists $t \in G$ such that $\mathcal{T}_{\mathcal{A}}(t) = \alpha_2, \mathcal{I}_{\mathcal{A}}(t) = \beta_2$ and $\mathcal{F}_{\mathcal{A}}(t) = \gamma_2$. It follows that $t \in \mathcal{A}_{(\alpha_2,\beta_2,\gamma_2)} = \mathcal{A}_{(\alpha_1,\beta_1,\gamma_1)}$. So that $\alpha_2 = \mathcal{T}_{\mathcal{A}}(t) \geq \alpha_1$, $\beta_2 = \mathcal{I}_{\mathcal{A}}(t) \geq \beta_1$ and $\gamma_2 = \mathcal{F}_{\mathcal{A}}(t) \leq \gamma_1$. Hence $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$. □

Theorem 2.3. Let H be a K-subalgebra of K-algebra $\mathcal{K}$. Then there exists a single-valued neutrosophic K-subalgebra $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ of K-algebra $\mathcal{K}$ such that $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}}) = H$, for some $\alpha, \beta \in (0, 1], \gamma \in [0, 1)$.

Proof. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic set in K-algebra $\mathcal{K}$ given by

$$\mathcal{T}_{\mathcal{A}}(s) = \begin{cases} \alpha \in (0, 1], & \text{if } s \in H,\\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{I}_{\mathcal{A}}(s)=\begin{cases}\beta\in(0,1], & \text{if } s\in H,\\ 0, & \text{otherwise,}\end{cases},\quad \mathcal{F}_{\mathcal{A}}(s)=\begin{cases}\gamma\in[0,1), & \text{if } s\in H,\\ 0, & \text{otherwise.}\end{cases}$$

Let $s,t\in G$. If $s,t\in H$, then $s\odot t\in H$ and so

$\mathcal{T}_{\mathcal{A}}(s\odot t)\geq\min\{\mathcal{T}_{\mathcal{A}}(s),\mathcal{T}_{\mathcal{A}}(t)\}$, $\mathcal{I}_{\mathcal{A}}(s\odot t)\geq\min\{\mathcal{I}_{\mathcal{A}}(s),\mathcal{I}_{\mathcal{A}}(t)\}$, $\mathcal{F}_{\mathcal{A}}(s\odot t)\leq\max\{\mathcal{F}_{\mathcal{A}}(s),\mathcal{F}_{\mathcal{A}}(t)\}$. But if $s\notin H$ or $t\notin H$, then $\mathcal{T}_{\mathcal{A}}(s)=0$ or $\mathcal{T}_{\mathcal{A}}(t),\mathcal{I}_{\mathcal{A}}(s)=0$ or $\mathcal{I}_{\mathcal{A}}(t)$ and $\mathcal{F}_{\mathcal{A}}(s)=0$ or $\mathcal{F}_{\mathcal{A}}(t)$. It follows that $\mathcal{T}_{\mathcal{A}}(s\odot t)\geq\min\{\mathcal{T}_{\mathcal{A}}(s),\mathcal{T}_{\mathcal{A}}(t)\}$, $\mathcal{I}_{\mathcal{A}}(s\odot t)\geq\min\{\mathcal{I}_{\mathcal{A}}(s),\mathcal{I}_{\mathcal{A}}(t)\}$, $\mathcal{F}_{\mathcal{A}}(s\odot t)\leq\max\{\mathcal{F}_{\mathcal{A}}(s),\mathcal{F}_{\mathcal{A}}(t)\}$. Hence $\mathcal{A}=(\mathcal{T}_{\mathcal{A}},\mathcal{I}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ is a SVN K-subalgebra of $\mathcal{K}$. Consequently $\mathcal{A}_{(\alpha,\beta,\gamma)}=H$. □

The following Theorem shows that any K-subalgebra of $\mathcal{K}$ can be perceived as a level K-subalgebra of some single-valued neutrosophic K-subalgebras of $\mathcal{K}$.

Theorem 2.4. Let $\mathcal{K}$ be a K-algebra. Given a chain of K-subalgebras: $\mathcal{A}_0\subset\mathcal{A}_1\subset\mathcal{A}_2\subset\ldots\subset A_n=G$. Then there exists a single-valued neutrosophic K-subalgebra whose level K-subalgebras are exactly the K-subalgebras in this chain.

Proof. Let $\{\alpha_k\mid k=0,1,\ldots,n\},\{\beta_k\mid k=0,1,\ldots,n\}$ be finite decreasing sequences and $\{\gamma_k\mid k=0,1,\ldots,n\}$ be finite increasing sequence in $[0,1]$ such that $\alpha_i+\beta_i+\gamma_i\leq 3$, for $i=0,1,2,\ldots,n$. Let $\mathcal{A}=(\mathcal{T}_{\mathcal{A}},\mathcal{I}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic set in $\mathcal{K}$ defined by $\mathcal{T}_{\mathcal{A}}(\mathcal{A}_0)=\alpha_0,\mathcal{I}_{\mathcal{A}}(\mathcal{A}_0)=\beta_0,\mathcal{F}_{\mathcal{A}}(\mathcal{A}_0)=\gamma_0$, $\mathcal{T}_{\mathcal{A}}(\mathcal{A}_k\setminus\mathcal{A}_{k-1})=\alpha_k,\mathcal{I}_{\mathcal{A}}(\mathcal{A}_k\setminus\mathcal{A}_{k-1})=\beta_k$ and $\mathcal{F}_{\mathcal{A}}(\mathcal{A}_k\setminus\mathcal{A}_{k-1})=\gamma_k$, for $0<k\leq n$. We claim that $\mathcal{A}=(\mathcal{T}_{\mathcal{A}},\mathcal{I}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Let $s,t\in G$. If $s,t\in\mathcal{A}_k\setminus\mathcal{A}_{k-1}$, then it implies that $\mathcal{T}_{\mathcal{A}}(s)=\alpha_k=\mathcal{T}_{\mathcal{A}}(t),\mathcal{I}_{\mathcal{A}}(s)=\beta_k=\mathcal{I}_{\mathcal{A}}(t)$ and $\mathcal{F}_{\mathcal{A}}(s)=\gamma_k=\mathcal{F}_{\mathcal{A}}(t)$. Since each $\mathcal{A}_k$ is a K-subalgebra, it follows that $s\odot t\in\mathcal{A}_k$. So that either $s\odot t\in\mathcal{A}_k\setminus\mathcal{A}_{k-1}$ or $s\odot t\in\mathcal{A}_{k-1}$. In any case, we conclude that

$$\begin{aligned}\mathcal{T}_{\mathcal{A}}(s\odot t)&\geq\alpha_k=\min\{\mathcal{T}_{\mathcal{A}}(s),\mathcal{T}_{\mathcal{A}}(t)\},\\ \mathcal{I}_{\mathcal{A}}(s\odot t)&\geq\beta_k=\min\{\mathcal{I}_{\mathcal{A}}(s),\mathcal{I}_{\mathcal{A}}(t)\},\\ \mathcal{F}_{\mathcal{A}}(s\odot t)&\leq\gamma_k=\max\{\mathcal{F}_{\mathcal{A}}(s),\mathcal{F}_{\mathcal{A}}(t)\}.\end{aligned}$$

For $i>j$, if $s\in\mathcal{A}_i\setminus\mathcal{A}_{i-1}$ and $t\in\mathcal{A}_j\setminus\mathcal{A}_{j-1}$, then $\mathcal{T}_{\mathcal{A}}(s)=\alpha_i$, $\mathcal{T}_{\mathcal{A}}(t)=\alpha_j,\mathcal{I}_{\mathcal{A}}(s)=\beta_i,\mathcal{I}_{\mathcal{A}}(t)=\beta_j$ and $\mathcal{F}_{\mathcal{A}}(s)=\gamma_i,\mathcal{F}_{\mathcal{A}}(t)=\gamma_j$ and $s\odot t\in\mathcal{A}_i$ because $\mathcal{A}_i$ is a K-subalgebra and $\mathcal{A}_j\subset\mathcal{A}_i$. It follows that

$$\begin{aligned}\mathcal{T}_{\mathcal{A}}(s\odot t)&\geq\alpha_i=\min\{\mathcal{T}_{\mathcal{A}}(s),\mathcal{T}_{\mathcal{A}}(t)\},\\ \mathcal{I}_{\mathcal{A}}(s\odot t)&\geq\beta_i=\min\{\mathcal{I}_{\mathcal{A}}(s),\mathcal{I}_{\mathcal{A}}(t)\},\\ \mathcal{F}_{\mathcal{A}}(s\odot t)&\leq\gamma_i=\max\{\mathcal{F}_{\mathcal{A}}(s),\mathcal{F}_{\mathcal{A}}(t)\}.\end{aligned}$$

Thus, $\mathcal{A}=(\mathcal{T}_{\mathcal{A}},\mathcal{I}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ and all its non empty level subsets are level K-subalgebras of $\mathcal{K}$. Since $\mathrm{Im}(\mathcal{T}_{\mathcal{A}})=\{\alpha_0,\alpha_1,\ldots,\alpha_{\mathrm{n}}\},\mathrm{Im}(\mathcal{I}_{\mathcal{A}})=\{\beta_0,\beta_1,\ldots,\beta_{\mathrm{n}}\}$, $\mathrm{Im}(\mathcal{F}_{\mathcal{A}})=\{\gamma_0,\gamma_1,\ldots,\gamma_{\mathrm{n}}\}$. Therefore,

the level K-subalgebras of $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ are given by the chain of K-subalgebras:

$$\begin{aligned}
\cup(\mathcal{T}_\mathcal{A}, \alpha_0) \subset \cup(\mathcal{T}_\mathcal{A}, \alpha_1) \subset ... \subset \cup(\mathcal{T}_\mathcal{A}, \alpha_n) = G,\\
\cup'(\mathcal{I}_\mathcal{A}, \beta_0) \subset \cup'(\mathcal{I}_\mathcal{A}, \beta_1) \subset ... \subset \cup'(\mathcal{I}_\mathcal{A}, \beta_n) = G,\\
\mathrm{L}(\mathcal{F}_\mathcal{A}, \gamma_0) \subset \mathrm{L}(\mathcal{F}_\mathcal{A}, \gamma_1) \subset ... \subset \mathrm{L}(\mathcal{F}_\mathcal{A}, \gamma_n) = G,
\end{aligned}$$

respectively. Indeed,

$$\begin{aligned}
\cup(\mathcal{T}_\mathcal{A}, \alpha_0) = \{s \in G \mid \mathcal{T}_\mathcal{A}(s) \geq \alpha_0\} = \mathcal{A}_0,\\
\cup'(\mathcal{I}_\mathcal{A}, \beta_0) = \{s \in G \mid \mathcal{I}_\mathcal{A}(s) \geq \beta_0\} = \mathcal{A}_0,\\
\mathrm{L}(\mathcal{F}_\mathcal{A}, \gamma_0) = \{s \in G \mid \mathcal{F}_\mathcal{A}(s) \leq \gamma_0\} = \mathcal{A}_0.
\end{aligned}$$

Now we prove that $\cup(\mathcal{T}_\mathcal{A}, \alpha_k) = \mathcal{A}_k, \cup'(\mathcal{I}_\mathcal{A}, \beta_k) = \mathcal{A}_k$ and $L(\mathcal{F}_\mathcal{A}, \gamma_k) = \mathcal{A}_k$, for $0 < k \leq n$. Clearly, $\mathcal{A}_k \subseteq \cup(\mathcal{T}_\mathcal{A}, \alpha_k), \mathcal{A}_k \subseteq \cup'(\mathcal{I}_\mathcal{A}, \beta_k)$ and $\mathcal{A}_k \subseteq L(\mathcal{F}_\mathcal{A}, \gamma_k)$. If $s \in \cup(\mathcal{T}_\mathcal{A}, \alpha_k)$, then $\mathcal{T}_\mathcal{A}(s) \geq \alpha_k$ and so $s \notin \mathcal{A}_i$, for $i > k$.

Hence $\mathcal{T}_\mathcal{A}(s) \in \{\alpha_0, \alpha_1, ..., \alpha_k\}$ which implies that $s \in \mathcal{A}_i$, for some $i \leq k$ since $\mathcal{A}_i \subseteq \mathcal{A}_k$. It follows that $s \in \mathcal{A}_k$. Consequently, $\cup(\mathcal{T}_\mathcal{A}, \alpha_k) = \mathcal{A}_k$ for some $0 < k \leq n$. Similar case can be proved for $\cup'(\mathcal{I}_\mathcal{A}, \beta_k) = \mathcal{A}_k$. Now if $t \in L(\mathcal{F}_\mathcal{A}, \gamma_k)$, then $\mathcal{F}_\mathcal{A}(s) \leq \gamma_k$ and so $t \notin \mathcal{A}_i$, for some $j \leq k$.Thus, $\mathcal{F}_\mathcal{A}(s) \in \{\gamma_0, \gamma_1, ..., \gamma_k\}$ which implies that $s \in \mathcal{A}_j$, for some $j \leq k$. Since $\mathcal{A}_j \subseteq \mathcal{A}_k$. It follows that $t \in \mathcal{A}_k$. Consequently, $L(\mathcal{F}_\mathcal{A}, \gamma_k) = \mathcal{A}_k$, for some $0 < k \leq n$. Hence the proof. □

2.1 Homomorphism of single-valued neutrosophic K-algebras

Definition 2.5. Let $\mathcal{K}_1 = (G_1, \cdot, \odot, e_1)$ and $\mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ be two K-algebras and let ϕ be a function from $\mathcal{K}_1$ into $\mathcal{K}_2$. If $\mathcal{B} = (\mathcal{T}_\mathcal{B}, \mathcal{I}_\mathcal{B}, \mathcal{F}_\mathcal{B})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then the *preimage* of $\mathcal{B} = (\mathcal{T}_\mathcal{B}, \mathcal{I}_\mathcal{B}, \mathcal{F}_\mathcal{B})$ under ϕ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$ defined by $\phi^{-1}(\mathcal{T}_\mathcal{B})(s) = \mathcal{T}_\mathcal{B}(\phi(s))$, $\phi^{-1}(\mathcal{I}_\mathcal{B})(s) = \mathcal{I}_\mathcal{B}(\phi(s))$ and $\phi^{-1}(\mathcal{F}_\mathcal{B})(s) = \mathcal{F}_\mathcal{B}(\phi(s))$, for all $s \in G_1$.

Theorem 2.5. Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $\mathcal{B} = (\mathcal{T}_\mathcal{B}, \mathcal{I}_\mathcal{B}, \mathcal{F}_\mathcal{B})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then $\phi^{-1}(\mathcal{B})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. It is easy to see that $\phi^{-1}(\mathcal{T}_\mathcal{B})(e) \geq \phi^{-1}(\mathcal{T}_\mathcal{B})(s)$, $\phi^{-1}(\mathcal{I}_\mathcal{B})(e) \geq \phi^{-1}(\mathcal{I}_\mathcal{B})(s)$ and $\phi^{-1}(\mathcal{F}_\mathcal{B})(e) \leq \phi^{-1}(\mathcal{F}_\mathcal{B})(s)$ for all $s \in G_1$. Let $s, t \in G_1$, then

$$\begin{aligned}
\phi^{-1}(\mathcal{T}_\mathcal{B})(s \odot t) &= \mathcal{T}_\mathcal{B}(\phi(s \odot t))\\
\phi^{-1}(\mathcal{T}_\mathcal{B})(s \odot t) &= \mathcal{T}_\mathcal{B}(\phi(s) \odot \phi(t))\\
\phi^{-1}(\mathcal{T}_\mathcal{B})(s \odot t) &\geq \min\{\mathcal{T}_\mathcal{B}(\phi(s)), \mathcal{T}_\mathcal{B}(\phi(t))\}\\
\phi^{-1}(\mathcal{T}_\mathcal{B})(s \odot t) &\geq \min\{\phi^{-1}(\mathcal{T}_\mathcal{B})(s), \phi^{-1}(\mathcal{T}_\mathcal{B})(t)\},
\end{aligned}$$

$$\begin{aligned}
&\phi^{-1}(\mathcal{I}_{\mathcal{B}})(s \odot t) = \mathcal{I}_{\mathcal{B}}(\phi(s \odot t))\\
&\phi^{-1}(\mathcal{I}_{\mathcal{B}})(s \odot t) = \mathcal{I}_{\mathcal{B}}(\phi(s) \odot \phi(t))\\
&\phi^{-1}(\mathcal{I}_{\mathcal{B}})(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{B}}(\phi(s)), \mathcal{I}_{\mathcal{B}}(\phi(t))\}\\
&\phi^{-1}(\mathcal{I}_{\mathcal{B}})(s \odot t) \geq \min\{\phi^{-1}(\mathcal{I}_{\mathcal{B}})(s), \phi^{-1}(\mathcal{I}_{\mathcal{B}})(t)\},
\end{aligned}$$

$$\begin{aligned}
&\phi^{-1}(\mathcal{F}_{\mathcal{B}})(s \odot t) = \mathcal{F}_{\mathcal{B}}(\phi(s \odot t))\\
&\phi^{-1}(\mathcal{F}_{\mathcal{B}})(s \odot t) = \mathcal{F}_{\mathcal{B}}(\phi(s) \odot \phi(t))\\
&\phi^{-1}(\mathcal{F}_{\mathcal{B}})(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{B}}(\phi(s)), \mathcal{F}_{\mathcal{B}}(\phi(t))\}\\
&\phi^{-1}(\mathcal{F}_{\mathcal{B}})(s \odot t) \leq \max\{\phi^{-1}(\mathcal{F}_{\mathcal{B}})(s), \phi^{-1}(\mathcal{F}_{\mathcal{B}})(t)\}.
\end{aligned}$$

Hence $\phi^{-1}(\mathcal{B})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$. □

Theorem 2.6. Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $\mathcal{B} = (\mathcal{T}_{\mathcal{B}}, \mathcal{I}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$ and $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is the *preimage* of $\mathcal{B}$ under ϕ. Then $\mathcal{A}$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. It is easy to see that $\mathcal{T}_{\mathcal{A}}(e) \geq \mathcal{T}_{\mathcal{A}}(s)$, $\mathcal{I}_{\mathcal{A}}(e) \geq \mathcal{I}_{\mathcal{A}}(s)$ and $\mathcal{F}_{\mathcal{A}}(e) \leq \mathcal{F}_{\mathcal{A}}(s)$, for all $s \in G_1$. Now for any $s, t \in G_1$,

$$\begin{aligned}
&\mathcal{T}_{\mathcal{A}}(s \odot t) = \mathcal{T}_{\mathcal{B}}(\phi(s \odot t))\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) = \mathcal{T}_{\mathcal{B}}(\phi(s) \odot \phi(t))\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{B}}(\phi(s)), \mathcal{T}_{\mathcal{B}}(\phi(t))\}\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_{\mathcal{A}}(s \odot t) = \mathcal{I}_{\mathcal{B}}(\phi(s \odot t))\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) = \mathcal{I}_{\mathcal{B}}(\phi(s) \odot \phi(t))\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{B}}(\phi(s)), \mathcal{I}_{\mathcal{B}}(\phi(t))\}\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}_{\mathcal{A}}(s \odot t) = \mathcal{F}_{\mathcal{B}}(\phi(s \odot t))\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) = \mathcal{F}_{\mathcal{B}}(\phi(s) \odot \phi(t))\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{B}}(\phi(s)), \mathcal{F}_{\mathcal{B}}(\phi(t))\}\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\}.
\end{aligned}$$

Hence $\mathcal{A}$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$. □

Definition 2.6. Let a mapping $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ from $\mathcal{K}_1$ into $\mathcal{K}_2$ of K-algebras and let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic set of $\mathcal{K}_2$. The map $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is called the *preimage* of $\mathcal{A}$ under ϕ, if $\mathcal{T}^{\phi}_{\mathcal{A}}(s) = \mathcal{T}_{\mathcal{A}}(\phi(s))$, $\mathcal{I}^{\phi}_{\mathcal{A}}(s) = \mathcal{I}_{\mathcal{A}}(\phi(s))$ and $\mathcal{F}^{\phi}_{\mathcal{A}}(s) = \mathcal{F}_{\mathcal{A}}(\phi(s))$ for all $s \in G_1$.

Proposition 2.3. Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then $\mathcal{A}^\phi = (\mathcal{T}^\phi_\mathcal{A}, \mathcal{I}^\phi_\mathcal{A}, \mathcal{F}^\phi_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. For any $s \in G_1$, we have

$$\begin{aligned}
\mathcal{T}^\phi_\mathcal{A}(e_1) &= \mathcal{T}_\mathcal{A}(\phi(e_1)) = \mathcal{T}_\mathcal{A}(e_2) \geq \mathcal{T}_\mathcal{A}(\phi(s)) \ = \mathcal{T}^\phi_\mathcal{A}(s),\\
\mathcal{I}^\phi_\mathcal{A}(e_1) &= \mathcal{I}_\mathcal{A}(\phi(e_1)) = \mathcal{I}_\mathcal{A}(e_2) \geq \mathcal{I}_\mathcal{A}(\phi(s)) \ = \mathcal{I}^\phi_\mathcal{A}(s),\\
\mathcal{F}^\phi_\mathcal{A}(e_1) &= \mathcal{F}_\mathcal{A}(\phi(e_1)) = \mathcal{F}_\mathcal{A}(e_2) \leq \mathcal{F}_\mathcal{A}(\phi(s)) = \mathcal{F}^\phi_\mathcal{A}(s).
\end{aligned}$$

For any $s, t \in G_1$, since $\mathcal{A}$ ia a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$

$$\begin{aligned}
&\mathcal{T}^\phi_\mathcal{A}(s \odot t) = \mathcal{T}_\mathcal{A}(\phi(s \odot t))\\
&\mathcal{T}^\phi_\mathcal{A}(s \odot t) = \mathcal{T}_\mathcal{A}(\phi(s) \odot \phi(t))\\
&\mathcal{T}^\phi_\mathcal{A}(s \odot t) \geq \min\{\mathcal{T}_\mathcal{A}(\phi(s)), \mathcal{T}_\mathcal{A}(\phi(t))\}\\
&\mathcal{T}^\phi_\mathcal{A}(s \odot t) \geq \min\{\mathcal{T}^\phi_\mathcal{A}(s), \mathcal{T}^\phi_\mathcal{A}(s)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}^\phi_\mathcal{A}(s \odot t) = \mathcal{I}_\mathcal{A}(\phi(s \odot t))\\
&\mathcal{I}^\phi_\mathcal{A}(s \odot t) = \mathcal{I}_\mathcal{A}(\phi(s) \odot \phi(t))\\
&\mathcal{I}^\phi_\mathcal{A}(s \odot t) \geq \min\{\mathcal{I}_\mathcal{A}(\phi(s)), \mathcal{I}_\mathcal{A}(\phi(t))\}\\
&\mathcal{I}^\phi_\mathcal{A}(s \odot t) \geq \min\{\mathcal{I}^\phi_\mathcal{A}(s), \mathcal{I}^\phi_\mathcal{A}(s)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}^\phi_\mathcal{A}(s \odot t) = \mathcal{F}_\mathcal{A}(\phi(s \odot t))\\
&\mathcal{F}^\phi_\mathcal{A}(s \odot t) = \mathcal{F}_\mathcal{A}(\phi(s) \odot \phi(t))\\
&\mathcal{F}^\phi_\mathcal{A}(s \odot t) \leq \max\{\mathcal{F}_\mathcal{A}(\phi(s)), \mathcal{F}_\mathcal{A}(\phi(t))\}\\
&\mathcal{F}^\phi_\mathcal{A}(s \odot t) \leq \max\{\mathcal{F}^\phi_\mathcal{A}(s), \mathcal{F}^\phi_\mathcal{A}(s)\}.
\end{aligned}$$

Hence $\mathcal{A}^\phi = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$. □

Proposition 2.4. Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $\mathcal{A}^\phi = (\mathcal{T}^\phi_\mathcal{A}, \mathcal{I}^\phi_\mathcal{A}, \mathcal{F}^\phi_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. Since there exists $s \in G_1$ such that $t = \phi(s)$, for any $t \in G_2$

$$\begin{aligned}
\mathcal{T}_\mathcal{A}(t) &= \mathcal{T}_\mathcal{A}(\phi(s)) \ = \mathcal{T}^{\phi(s)}_\mathcal{A} \leq \mathcal{T}^{\phi(e_1)}_\mathcal{A} = \mathcal{T}_\mathcal{A}(\phi(e_1)) = \mathcal{T}_\mathcal{A}(e_2),\\
\mathcal{I}_\mathcal{A}(t) &= \mathcal{I}_\mathcal{A}(\phi(s)) \ = \mathcal{I}^{\phi(s)}_\mathcal{A} \leq \mathcal{I}^{\phi(e_1)}_\mathcal{A} = \mathcal{I}_\mathcal{A}(\phi(e_1)) = \mathcal{I}_\mathcal{A}(e_2),\\
\mathcal{F}_\mathcal{A}(t) &= \mathcal{F}_\mathcal{A}(\phi(s)) = \mathcal{F}^{\phi(s)}_\mathcal{A} \geq \mathcal{F}^{\phi(e_1)}_\mathcal{A} = \mathcal{F}_\mathcal{A}(\phi(e_1)) = \mathcal{F}_\mathcal{A}(e_2).
\end{aligned}$$

for any $s,t \in G_2$, $u,v \in G_1$ such that $s = \phi(u)$ and $t = \phi(v)$. It follows that

$$\begin{aligned}
&\mathcal{T}_{\mathcal{A}}(s \odot t) = \mathcal{T}_{\mathcal{A}}(\phi(u \odot v))\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) = \mathcal{T}^{\phi}_{\mathcal{A}}(u \odot v)\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}^{\phi}_{\mathcal{A}}(u), \mathcal{T}^{\phi}_{\mathcal{A}}(v)\}\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(\phi(u)), \mathcal{T}_{\mathcal{A}}(\phi(v))\}\\
&\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_{\mathcal{A}}(s \odot t) = \mathcal{I}_{\mathcal{A}}(\phi(u \odot v))\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) = \mathcal{I}^{\phi}_{\mathcal{A}}(u \odot v)\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}^{\phi}_{\mathcal{A}}(u), \mathcal{I}^{\phi}_{\mathcal{A}}(v)\}\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(\phi(u)), \mathcal{I}_{\mathcal{A}}(\phi(v))\}\\
&\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}_{\mathcal{A}}(s \odot t) = \mathcal{F}_{\mathcal{A}}(\phi(u \odot v))\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) = \mathcal{F}^{\phi}_{\mathcal{A}}(u \odot v)\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}^{\phi}_{\mathcal{A}}(u), \mathcal{F}^{\phi}_{\mathcal{A}}(v)\}\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(\phi(u)), \mathcal{F}_{\mathcal{A}}(\phi(v))\}\\
&\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\}.
\end{aligned}$$

Hence $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$. □

From above two propositions, we obtain the following theorem.

Theorem 2.7. Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. Then $\mathcal{A}^{\phi} = (\mathcal{T}^{\phi}_{\mathcal{A}}, \mathcal{I}^{\phi}_{\mathcal{A}}, \mathcal{F}^{\phi}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$ if and only if $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$.

Definition 2.7. A single-valued neutrosophic K-subalgebra $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ of a K-algebra $\mathcal{K}$ is called *characteristic* if $\mathcal{T}_{\mathcal{A}}(\phi(s)) = \mathcal{T}_{\mathcal{A}}(s)$, $\mathcal{I}_{\mathcal{A}}(\phi(s)) = \mathcal{I}_{\mathcal{A}}(s)$ and $\mathcal{F}_{\mathcal{A}}(\phi(s)) = \mathcal{F}_{\mathcal{A}}(s)$, for all $s \in G$ and $\phi \in Aut(\mathcal{K})$.

Definition 2.8. A K-subalgebra S of a K-algebra $\mathcal{K}$ is said to be *fully invariant* if $\phi(S) \subseteq S$, for all $\phi \in End(\mathcal{K})$, where $End(\mathcal{K})$ is the set of all endomorphisms of a K-algebra $\mathcal{K}$. A single-valued neutrosophic K-subalgebra $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ of a K-algebra $\mathcal{K}$ is called *fully invariant* if $\mathcal{T}_{\mathcal{A}}(\phi(s)) \leq \mathcal{T}_{\mathcal{A}}(s)$, $\mathcal{I}_{\mathcal{A}}(\phi(s)) \leq \mathcal{I}_{\mathcal{A}}(s)$ and $\mathcal{F}_{\mathcal{A}}(\phi(s)) \leq \mathcal{F}_{\mathcal{A}}(s)$, for all $s \in G$ and $\phi \in End(\mathcal{K})$.

Definition 2.9. Let $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, \mathcal{F}_{\mathcal{A}_1})$ and $\mathcal{A}_2 = (\mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$ be single-valued neutrosophic K-subalgebras of $\mathcal{K}$. Then $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, _{\mathcal{A}_1})$ is said to be the same type of $\mathcal{A}_2 = (\mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$ if there exists $\phi \in Aut(\mathcal{K})$ such that $\mathcal{A}_1 = \mathcal{A}_2 \circ \phi$, i.e., $\mathcal{T}_{\mathcal{A}_1}(s) = \mathcal{T}_{\mathcal{A}_2}(\phi(s))$, $\mathcal{I}_{\mathcal{A}_1}(s) = \mathcal{I}_{\mathcal{A}_2}(\phi(s))$ and $\mathcal{F}_{\mathcal{A}_1}(s) = \mathcal{F}_{\mathcal{A}_2}(\phi(s))$, for all $s \in G$.

Theorem 2.8. Let $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, \mathcal{F}_{\mathcal{A}_1})$ and $\mathcal{A}_2 = (\mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$ be single-valued neutrosophic K-subalgebras of $\mathcal{K}$. Then $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, \mathcal{F}_{\mathcal{A}_1})$ is a single-valued neutrosophic K-subalgebra is of the same type of $\mathcal{A}_2 = (\mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$ if and only if $\mathcal{A}_1$ is isomorphic to $\mathcal{A}_2$.

Proof. Sufficient condition holds trivially so we only need to prove the necessary condition. Let $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, \mathcal{F}_{\mathcal{A}_1})$ be a single-valued neutrosophic K-subalgebra having same type of $\mathcal{A}_2 = (\mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$. Then there exists $\phi \in Aut(\mathcal{K})$ such that $\mathcal{T}_{\mathcal{A}_1}(s) = \mathcal{T}_{\mathcal{A}_2}(\phi(s))$, $\mathcal{I}_{\mathcal{A}_1}(s) = \mathcal{I}_{\mathcal{A}_2}(\phi(s))$ and $\mathcal{F}_{\mathcal{A}_1} = \mathcal{F}_{\mathcal{A}_2}(\phi(s))$, for all $s \in G$.

Let $f : \mathcal{A}_1(K) \to \mathcal{A}_2(K)$ be a mapping defined by $f(\mathcal{A}_1(s)) = \mathcal{A}_2(\phi(s))$, for all $s \in G$, that is, $f(\mathcal{T}_{\mathcal{A}_1}(s)) = \mathcal{T}_{\mathcal{A}_2}(\phi(s))$, $f(\mathcal{I}_{\mathcal{A}_1}(s)) = \mathcal{I}_{\mathcal{A}_2}(\phi(s))$ and $f(\mathcal{F}_{\mathcal{A}_1}(s)) = \mathcal{F}_{\mathcal{A}_2}(\phi(s))$, for all $s \in G$. Clearly, f is surjective. Also, f is injective because if $f(\mathcal{T}_{\mathcal{A}_1}(s)) = f(\mathcal{T}_{\mathcal{A}_1}(t))$, for all $s, t \in G$, then $\mathcal{T}_{\mathcal{A}_2}(\phi(s)) = \mathcal{T}_{\mathcal{A}_2}(\phi(t))$ and we have $\mathcal{T}_{\mathcal{A}_1}(s) = \mathcal{T}_{\mathcal{A}_1}(t)$. Similarly, $\mathcal{I}_{\mathcal{A}_1}(s) = \mathcal{I}_{\mathcal{A}_1}(t)$, $\mathcal{F}_{\mathcal{A}_1}(s) = \mathcal{F}_{\mathcal{A}_1}(t)$.

Therefore, f is a homomorphism, such that for $s, t \in G$, we have

$$
\begin{aligned}
f(\mathcal{T}_{\mathcal{A}_1}(s \odot t)) &= \mathcal{T}_{\mathcal{A}_2}(\phi(s \odot t)) = \mathcal{T}_{\mathcal{A}_2}(\phi(s) \odot \phi(t)), \\
f(\mathcal{I}_{\mathcal{A}_1}(s \odot t)) &= \mathcal{I}_{\mathcal{A}_2}(\phi(s \odot t)) = \mathcal{I}_{\mathcal{A}_2}(\phi(s) \odot \phi(t)), \\
f(\mathcal{F}_{\mathcal{A}_1}(s \odot t)) &= \mathcal{F}_{\mathcal{A}_2}(\phi(s \odot t)) = \mathcal{F}_{\mathcal{A}_2}(\phi(s) \odot \phi(t)).
\end{aligned}
$$

Hence $\mathcal{A}_1 = (\mathcal{T}_{\mathcal{A}_1}, \mathcal{I}_{\mathcal{A}_1}, \mathcal{F}_{\mathcal{A}_1})$ is isomorphic to $\mathcal{A}_2 = \mathcal{T}_{\mathcal{A}_2}, \mathcal{I}_{\mathcal{A}_2}, \mathcal{F}_{\mathcal{A}_2})$. Hence the proof. □

3. $(\tilde{a}, \tilde{b})$-single-valued neutrosophic K-algebras

Definition 3.1. A single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in a set G is called an $(\tilde{a}, \tilde{b})-$single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

- $u_{(\alpha_1,\beta_1,\gamma_1)}\ \tilde{a}\ \mathcal{A},\ v_{(\alpha_2,\beta_2,\gamma_2)}\ \tilde{a}\ \mathcal{A} \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))}\ \tilde{b}\ \mathcal{A}$,

for all $u, v \in G, \alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$.

Twelve different types of single-valued neutrosophic K-subalgebras can be obtained by replacing the values of $\tilde{a}(\neq \in \wedge q)$ and $\tilde{b}$ by any two values in the set $\{\in, q, \in \vee q, \in \wedge q\}$ in Definition 3.1.

Remark 3.1. Every $(\in, \in)$-single-valued neutrosophic K-subalgebra is in fact, a single-valued neutrosophic K-subalgebra.

Proposition 3.1. Every $(\in, \in)$-single-valued neutrosophic K-subalgebra is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra.

Proof. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Let $u, v \in G$ and $\alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$ be such that $u_{(\alpha_1,\beta_1,\gamma_1)} \in \mathcal{A}$, $v_{(\alpha_2,\beta_2,\gamma_2)} \in \mathcal{A}$. Then $u_{(\alpha_1,\beta_1,\gamma_1)} \in \mathcal{A}$, $v_{(\alpha_2,\beta_2,\gamma_2)} \in \mathcal{A} \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))} \in \vee q\ \mathcal{A}$. Hence $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$. □

Proposition 3.2. Every $(\in \vee q, \in \vee q)$-single-valued neutrosophic K-subalgebra is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Definition 3.2. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic set in G. The set $\underline{\mathcal{A}}$=$\{u \in G \mid \mathcal{T}_{\mathcal{A}}(u) \neq 0,\ \mathcal{I}_{\mathcal{A}}(u) \neq 0,\ \mathcal{F}_{\mathcal{A}}(u) \neq 0\}$ is called the *support* of $\mathcal{A}$.

Lemma 3.1. If $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a non-zero $(\in, \in)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{K}$.

Proof. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a non-zero $(\in, \in)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ and let $u, v \in \underline{\mathcal{A}}$. Then $\mathcal{T}_{\mathcal{A}}(u) \neq 0$ and $\mathcal{T}_{\mathcal{A}}(v) \neq 0$, $\mathcal{I}_{\mathcal{A}}(u) \neq 0$ and $\mathcal{I}_{\mathcal{A}}(v) \neq 0$ and $\mathcal{F}_{\mathcal{A}}(u) \neq 0$, $\mathcal{F}_{\mathcal{A}}(v) \neq 0$. Let $\mathcal{T}_{\mathcal{A}}(u \odot v) = 0, \mathcal{I}_{\mathcal{A}}(u \odot v) = 0$ and $\mathcal{F}_{\mathcal{A}}(u \odot v) = 0$. Since $u_{\mathcal{T}_{\mathcal{A}}}(u) \in \mathcal{A}$ and $v_{\mathcal{T}_{\mathcal{A}}}(v) \in \mathcal{A}$, $u_{\mathcal{I}_{\mathcal{A}}}(u) \in \mathcal{A}$ and $v_{\mathcal{I}_{\mathcal{A}}}(v) \in \mathcal{A}$, $u_{\mathcal{F}_{\mathcal{A}}}(u) \in \mathcal{A}$ and $v_{\mathcal{F}_{\mathcal{A}}}(v) \in \mathcal{A}$ but
$(u \odot v)_{(\min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)), \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)), \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)))} \notin \mathcal{A}$.

Since $\mathcal{T}_{\mathcal{A}}(u \odot v) = 0, \mathcal{I}_{\mathcal{A}}(u \odot v) = 0$ and $\mathcal{F}_{\mathcal{A}}(u \odot v) = 0$. A contradiction. Hence $\mathcal{T}_{\mathcal{A}}(u \odot v) \neq 0, \mathcal{I}_{\mathcal{A}}(u \odot v) \neq 0$ and $\mathcal{F}_{\mathcal{A}}(u \odot v) \neq 0$ which shows that $(u \odot v) \in \underline{\mathcal{A}}$, consequently $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{A}$. □

Lemma 3.2. If $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a non-zero $(\in, q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{K}$.

Lemma 3.3. If $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a non-zero $(q, \in)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{K}$.

Lemma 3.4. If $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a non-zero (q, q)-single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{K}$.

Theorem 3.1. If $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a non-zero $(\tilde{a}, \tilde{b})$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$, then $\underline{\mathcal{A}}$ is a K-subalgebra of $\mathcal{K}$.

Definition 3.3. A neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in a K-algebra $\mathcal{K}$ is called an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

(a) $e_{(\alpha,\beta,\gamma)} \in \mathcal{A} \Rightarrow (u)_{(\alpha,\beta,\gamma)} \in \vee q\ \mathcal{A}$,
(b) $u_{(\alpha_1,\beta_1,\gamma_1)} \in \mathcal{A}, v_{(\alpha_2,\beta_2,\gamma_2)} \in \mathcal{A} \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2), \min(\beta_1,\beta_2), \max(\gamma_1,\gamma_2)} \in \vee q\ \mathcal{A}$,

For all $u, v \in G, \alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)$.

Example 3.1. Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6\}$ is the cyclic group of order 7 and Cayley's table for $\odot$ is given as:

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6
e	e	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e

We define a single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ in $\mathcal{K}$ as follows:

$$\mathcal{T}_\mathcal{A}(u) = \begin{cases} 1, & \text{if } u = e, \\ 0.7, & \text{otherwise} \end{cases}$$

$$\mathcal{I}_\mathcal{A}(u) = \begin{cases} 1, & \text{if } u = e, \\ 0.6, & \text{otherwise} \end{cases}$$

$$\mathcal{F}_\mathcal{A}(u) = \begin{cases} 0, & \text{if } u = e, \\ 0.5, & \text{otherwise} \end{cases}$$

Now take $\alpha = 0.4, \alpha_1 = 0.5, \alpha_2 = 0.3$, $\beta = 0.5, \beta_1 = 0.6, \beta_2 = 0.3$, $\gamma = 0.6, \gamma_1 = 0.6, \gamma_2 = 0.5$, where $\alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)$.

By direct calculations, it is easy to see that $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Theorem 3.2. Let $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ be a single-valued neutrosophic set in $\mathcal{K}$. Then $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if and only if

$$\text{(i)} \quad \begin{aligned} \mathcal{T}_\mathcal{A}(u) &\geq \min(\mathcal{T}_\mathcal{A}(e), 0.5), \\ \mathcal{I}_\mathcal{A}(u) &\geq \min(\mathcal{I}_\mathcal{A}(e), 0.5), \\ \mathcal{F}_\mathcal{A}(u) &\leq \max(\mathcal{F}_\mathcal{A}(e), 0.5). \end{aligned}$$

$$\text{(ii)} \quad \begin{aligned} \mathcal{T}_\mathcal{A}(u \odot v) &\geq \min(\mathcal{T}_\mathcal{A}(u), \mathcal{T}_\mathcal{A}(v), 0.5), \\ \mathcal{I}_\mathcal{A}(u \odot v) &\geq \min(\mathcal{I}_\mathcal{A}(u), \mathcal{I}_\mathcal{A}(v), 0.5), \\ \mathcal{F}_\mathcal{A}(u \odot v) &\leq \max(\mathcal{F}_\mathcal{A}(u), \mathcal{F}_\mathcal{A}(v), 0.5), \text{ for all } u, v \in G. \end{aligned}$$

Proof. Assume that $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra.

Let for $u, v \in G$. Assume that $\mathcal{T}_\mathcal{A}(u \odot v) < \min(\mathcal{T}_\mathcal{A}(u), \mathcal{T}_\mathcal{A}(v), 0.5)$, $\mathcal{I}_\mathcal{A}(u \odot v) < \min(\mathcal{I}_\mathcal{A}(u), \mathcal{I}_\mathcal{A}(v), 0.5)$, $\mathcal{F}_\mathcal{A}(u \odot v) > \max(\mathcal{F}_\mathcal{A}(u), \mathcal{F}_\mathcal{A}(v), 0.5)$. Then $\mathcal{T}_\mathcal{A}(u \odot v) < \min(\mathcal{T}_\mathcal{A}(u), \mathcal{T}_\mathcal{A}(v))$, $\mathcal{I}_\mathcal{A}(u \odot v) < \min(\mathcal{I}_\mathcal{A}(u), \mathcal{I}_\mathcal{A}(v))$ and $\mathcal{F}_\mathcal{A}(u \odot v) > \max(\mathcal{F}_\mathcal{A}(u), \mathcal{F}_\mathcal{A}(v))$. Take α, β, γ such that $\mathcal{T}_\mathcal{A}(u \odot v) < \alpha < \min(\mathcal{T}_\mathcal{A}(u), \mathcal{T}_\mathcal{A}(v)$,

$\mathcal{I}_{\mathcal{A}}(u \odot v) < \beta < \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)$, $\mathcal{F}_{\mathcal{A}}(u \odot v) > \gamma > \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)$. Then $u_\alpha, v_\alpha \in \mathcal{T}_{\mathcal{A}}$, $u_\beta, v_\beta \in \mathcal{I}_{\mathcal{A}}$ and $u_\gamma, v_\gamma \in \mathcal{F}_{\mathcal{A}}$ but $(u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))} \overline{\in \vee q}\ \mathcal{A}$, a contradiction.

Now if $\mathcal{T}_{\mathcal{A}}(u \odot v) < 0.5$, $\mathcal{I}_{\mathcal{A}}(u \odot v) < 0.5$, $\mathcal{F}_{\mathcal{A}}(u \odot v) > 0.5$. Then $u_{(0.5,0.5,0.5)}, v_{(0.5,0.5,0.5)} \in \mathcal{A}$, but $(u \odot v)_{(0.5,0.5,0.5)} \overline{\in \vee q} \mathcal{A}$ which is also a contradiction. Hence (i) holds.

Let $u_{(\alpha_1,\beta_1,\gamma_1)}, v_{(\alpha_2,\beta_2,\gamma_2)} \in \mathcal{A}$ which means that $\mathcal{T}_{\mathcal{A}}(u) \geq \alpha_1, \mathcal{T}_{\mathcal{A}}(v) \geq \alpha_2$, $\mathcal{I}_{\mathcal{A}}(u) \geq \beta_1, \mathcal{I}_{\mathcal{A}}(v) \geq \beta_2$, $\mathcal{F}_{\mathcal{A}}(u) \leq \gamma_1, \mathcal{F}_{\mathcal{A}}(v) \leq \gamma_2$. We have $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5) \geq \min(\alpha_1, \alpha_2, 0.5)$, $\mathcal{I}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v), 0.5) \geq \min(\beta_1, \beta_2, 0.5)$, $\mathcal{F}_{\mathcal{A}}(u \odot v) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), 0.5) \leq \max(\gamma_1, \gamma_2, 0.5)$. If $\min(\alpha_1, \alpha_2) > 0.5, \min(\beta_1, \beta_2) > 0.5, \max(\gamma_1, \gamma_2) < 0.5$, then $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq 0.5 \Rightarrow \mathcal{T}_{\mathcal{A}}(u \odot v) + \min(\alpha_1, \alpha_2) > 1$, $\mathcal{I}_{\mathcal{A}}(u \odot v) \geq 0.5 \Rightarrow \mathcal{I}_{\mathcal{A}}(u \odot v) + \min(\beta_1, \beta_2) > 1$, $\mathcal{F}_{\mathcal{A}}(u \odot v) \leq 0.5 \Rightarrow \mathcal{F}_{\mathcal{A}}(u \odot v) + \max(\gamma_1, \gamma_2) < 1$.

But if $\min(\alpha_1, \alpha_2) \leq 0.5, \min(\beta_1, \beta_2) \leq 0.5, \max(\gamma_1, \gamma_2) \geq 0.5$, then $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \min(\alpha_1, \alpha_2), \mathcal{I}_{\mathcal{A}}(u \odot v) \geq \min(\beta_1, \beta_2)$, $\mathcal{F}_{\mathcal{A}}(u \odot v) \leq \max(\gamma_1, \gamma_2)$. Hence $(u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))} \in \vee q\mathcal{A}$. Which completes the proof. □

Theorem 3.3. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic set in $\mathcal{K}$. Then $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if and only if each non-empty $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$, for $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$.

Proof. Assume that $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ and let $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$. To prove that $\mathcal{A}_{(\alpha,\beta,\gamma)} = \{u \in G \mid \mathcal{T}_{\mathcal{A}}(u) \geq \alpha, \mathcal{I}_{\mathcal{A}}(u) \geq \beta, \mathcal{F}_{\mathcal{A}}(u) \leq \gamma\}$ is a K-subalgebra of $\mathcal{K}$. If $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$, then $\mathcal{T}_{\mathcal{A}}(u) \geq \alpha, \mathcal{T}_{\mathcal{A}}(v) \geq \alpha, \mathcal{I}_{\mathcal{A}}(u) \geq \beta, \mathcal{I}_{\mathcal{A}}(v) \geq \beta, \mathcal{F}_{\mathcal{A}}(u) \leq \gamma, \mathcal{F}_{\mathcal{A}}(v) \leq \gamma$. Thus, $\mathcal{T}_{\mathcal{A}}(e) \geq \min(\mathcal{T}_{\mathcal{A}}(u), 0.5) \geq \min(\alpha, 0.5) = \alpha, \mathcal{I}_{\mathcal{A}}(e) \geq \min(\mathcal{I}_{\mathcal{A}}(u), 0.5) \geq \min(\beta, 0.5) = \beta, \mathcal{F}_{\mathcal{A}}(e) \leq \max(\mathcal{F}_{\mathcal{A}}(u), 0.5) \geq \min(\gamma, 0.5) = \gamma$ and $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5) \geq \min(\alpha, 0.5) = \alpha$, $\mathcal{I}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5) \geq \min(\beta, 0.5) = \beta, \mathcal{F}_{\mathcal{A}}(u \odot v) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), 0.5) \leq \max(\gamma, 0.5) = \gamma$. Thus, $u \odot v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$. Hence $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$. Converse part is obvious. □

Theorem 3.4. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic set in $\mathcal{K}$. Then $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$ if and only if

(a) $\max(\mathcal{T}_{\mathcal{A}}(u \odot v), 0.5) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v))$,
$\max(\mathcal{I}_{\mathcal{A}}(u \odot v), 0.5) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v))$,
$\min(\mathcal{F}_{\mathcal{A}}(u \odot v), 0.5) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v))$,

(b) $\max(\mathcal{T}_{\mathcal{A}}(e), 0.5) \geq (\mathcal{T}_{\mathcal{A}}(u)$,
$\max(\mathcal{I}_{\mathcal{A}}(e), 0.5) \geq (\mathcal{I}_{\mathcal{A}}(u)$,
$\min(\mathcal{F}_{\mathcal{A}}(e), 0.5) \leq (\mathcal{F}_{\mathcal{A}}(u)$, for all $u, v \in G$.

Proof. Suppose that $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$ and let $\max(\mathcal{T}_{\mathcal{A}}(u \odot v), 0.5) < \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)) = \alpha$, $\max(\mathcal{I}_{\mathcal{A}}(u \odot v), 0.5) < \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)) = \beta$, $\min(\mathcal{F}_{\mathcal{A}}(u \odot v), 0.5) > \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)) = \gamma$. Then for $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0.5, 1)$ and $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$, $\mathcal{T}_{\mathcal{A}}(u \odot v) < \alpha, \mathcal{I}_{\mathcal{A}}(u \odot v) < \beta, \mathcal{F}_{\mathcal{A}}(u \odot v) > \gamma$. Since $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$ and $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$, so $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$ or $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \alpha, \mathcal{I}_{\mathcal{A}}(u \odot v) \geq \beta, \mathcal{F}_{\mathcal{A}}(u \odot v) \leq \gamma$, a contradiction.

Conversely, suppose that conditions (a) and (b) holds. Let for $u, v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$, we have $0.5 < \alpha \leq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v)) \leq \max(\mathcal{T}_{\mathcal{A}}(u \odot v), 0.5) \Rightarrow \mathcal{T}_{\mathcal{A}}(u \odot v) \geq \alpha$, $0.5 < \beta \leq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v)) \leq \max(\mathcal{I}_{\mathcal{A}}(u \odot v), 0.5) \Rightarrow \mathcal{I}_{\mathcal{A}}(u \odot v) \geq \beta$, $0.5 > \gamma \geq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v)) \geq \min(\mathcal{F}_{\mathcal{A}}(u \odot v), 0.5) \Rightarrow \mathcal{F}_{\mathcal{A}}(u \odot v) \leq \gamma$. $0.5 < \alpha \leq \mathcal{T}_{\mathcal{A}}(u) \leq \max(\mathcal{T}_{\mathcal{A}}(e), 0.5) \Rightarrow \mathcal{T}_{\mathcal{A}}(mu) \geq \alpha$, $0.5 < \beta \leq \mathcal{I}_{\mathcal{A}}(u) \leq \max(\mathcal{I}_{\mathcal{A}}(e), 0.5) \Rightarrow \mathcal{I}_{\mathcal{A}}(mu) \geq \beta$, $0.5 > \gamma \geq \mathcal{F}_{\mathcal{A}}(u) \geq \min(\mathcal{F}_{\mathcal{A}}(e), 0.5) \Rightarrow \mathcal{F}_{\mathcal{A}}(mu) \leq \gamma$, for some $m \in G$ $u \odot v \in \mathcal{A}_{(\alpha,\beta,\gamma)}$. Hence $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$. □

Theorem 3.5. The intersection of any family of $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Proof. Let $\{\mathcal{A}_j : j \in I\}$ be a family of $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebras of $\mathcal{K}$.

Let $\mathcal{A} = \bigcap_{j \in I} \mathcal{A}_j = (\sup_{j \in I} \mathcal{T}_{\mathcal{A}_i}, \sup_{j \in I} \mathcal{I}_{\mathcal{A}_i}, \inf_{j \in I} \mathcal{F}_{\mathcal{A}_i})$, for $u, v \in G$, we have

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5),\\
&\mathcal{I}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v), 0.5),\\
&\mathcal{F}_{\mathcal{A}}(u \odot v) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), 0.5).\\
&\mathcal{T}_{\mathcal{A}}(u \odot v) = \sup_{j \in I} \mathcal{T}_{\mathcal{A}_i}(u \odot v),\\
&\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \sup_{j \in I} \min(\mathcal{T}_{\mathcal{A}_i}(u), \mathcal{T}_{\mathcal{A}_i}(v), 0.5),\\
&\mathcal{T}_{\mathcal{A}}(u \odot v) = \min(\sup_{j \in I} \mathcal{T}_{\mathcal{A}_i}(u), \sup_{j \in I} \mathcal{T}_{\mathcal{A}_i}(v), 0.5),\\
&\mathcal{T}_{\mathcal{A}}(u \odot v) = \min(\bigcap_{j \in I} \mathcal{T}_{\mathcal{A}_i}(u), \bigcap_{j \in I} \mathcal{T}_{\mathcal{A}_i}(v), 0.5),\\
&\mathcal{T}_{\mathcal{A}}(u \odot v) = \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5),\\
&\mathcal{I}_{\mathcal{A}}(u \odot v) = \sup_{j \in I} \mathcal{I}_{\mathcal{A}_i}(u \odot v),\\
&\mathcal{I}_{\mathcal{A}}(u \odot v) \geq \sup_{j \in I} \min(\mathcal{I}_{\mathcal{A}_i}(u), \mathcal{I}_{\mathcal{A}_i}(v), 0.5),\\
&\mathcal{I}_{\mathcal{A}}(u \odot v) = \min(\sup_{j \in I} \mathcal{I}_{\mathcal{A}_i}(u), \sup_{j \in I} \mathcal{I}_{\mathcal{A}_i}(v), 0.5),\\
&\mathcal{I}_{\mathcal{A}}(u \odot v) = \min(\bigcap_{j \in I} \mathcal{I}_{\mathcal{A}_i}(u), \bigcap_{j \in I} \mathcal{I}_{\mathcal{A}_i}(v), 0.5),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{I}_{\mathcal{A}}(u \odot v) &= \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v), 0.5),\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &= \inf_{j\in I} \mathcal{F}_{\mathcal{A}_i}(u \odot v),\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &\leq \inf_{j\in I} \max(\mathcal{F}_{\mathcal{A}_i}(u), \mathcal{F}_{\mathcal{A}_i}(v), 0.5),\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &= \max(\inf_{j\in I} \mathcal{F}_{\mathcal{A}_i}(u), \inf_{j\in I} \mathcal{F}_{\mathcal{A}_i}(v), 0.5),\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &= \max(\bigcap_{j\in I} \mathcal{F}_{\mathcal{A}_i}(u), \bigcap_{j\in I} \mathcal{F}_{\mathcal{A}_i}(v), 0.5),\\
\mathcal{F}_{\mathcal{A}}(u \odot v) &= \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), 0.5).
\end{aligned}$$

It follows that $\mathcal{A}$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$. □

Definition 3.4. Let $\epsilon_1, \epsilon_2 \in [0,1]$ and $\epsilon_1 < \epsilon_2$. Let $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be a single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Then $\mathcal{A}$ is called a single-valued neutrosophic K-subalgebra with thresholds (ϵ_1, ϵ_2) of $\mathcal{K}$ if

$$\begin{aligned}
&\max(\mathcal{T}_{\mathcal{A}}(u \odot v), \epsilon_1) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), \epsilon_2),\\
&\max(\mathcal{I}_{\mathcal{A}}(u \odot v), \epsilon_1) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v), \epsilon_2),\\
&\min(\mathcal{F}_{\mathcal{A}}(u \odot v), \epsilon_1) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), \epsilon_2), \text{ for all } u, v \in G.
\end{aligned}$$

Example 3.2. Consider a K-algebra on a cyclic group of order 9 and ... table for $\odot$ is given in example 2.1. It is easy to see that $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a single-valued neutrosophic K-subalgebra with thresholds $(\epsilon_1 = 0.3, \epsilon_2 = 0.56)$ and for $(\epsilon_1 = 0.55, \epsilon_2 = 0.78)$.

Remark 3.2. Let for $\epsilon_1, \epsilon_2 \in [0,1]$ and $\epsilon_1 < \epsilon_2$ unless otherwise specified.

(i) When $\epsilon_1 = 0$ and $\epsilon_2 = 1$ in single-valued neutrosophic K-subalgebra with thresholds (ϵ_1, ϵ_2), $\mathcal{A}$ is an ordinary single-valued neutrosophic K-subalgebra.

(ii) When $\epsilon_1 = 0$ and $\epsilon_2 = 0.5$ in single-valued neutrosophic K-subalgebra with thresholds (ϵ_1, ϵ_2), $\mathcal{A}$ is an $(\in, \in \vee q)$single-valued neutrosophic K-subalgebra.

Theorem 3.6. A single-valued neutrosophic set $\mathcal{A}$ in $\mathcal{K}$ is a single-valued neutrosophic K-subalgebra with thresholds (ϵ_1, ϵ_2) if and only if $\cup(\mathcal{T}_{\mathcal{A}}, \alpha)$, $\cup'(\mathcal{I}_{\mathcal{A}}, \beta)$, $L(\mathcal{F}_{\mathcal{A}}, \gamma)(\neq \phi), \alpha, \beta, \gamma \in (\epsilon_1, \epsilon_2]$ is a K-subalgebra of $\mathcal{K}$.

Proof. Assume that $\mathcal{A}$ is a single-valued neutrosophic K-subalgebra with thresholds (ϵ_1, ϵ_2). Let for $u, v \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha)$ and $\alpha \in (\epsilon_1, \epsilon_2)$, $\mathcal{T}_{\mathcal{A}}(u) \geq \alpha$ and $\mathcal{T}_{\mathcal{A}}(v) \geq \alpha$. Since $\mathcal{A}$ is a single-valued neutrosophic K-subalgebra, it follows that $\max(\mathcal{T}_{\mathcal{A}}(u \odot v), \epsilon_1) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), \epsilon_2) = \alpha$, so that $u \odot v \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha)$. Hence $\cup(\mathcal{T}_{\mathcal{A}}, \alpha)$ is a K-subalgebra of $\mathcal{K}$. Similarly, we can proof for $\cup'(\mathcal{I}_{\mathcal{A}}, \beta)$ and $L(\mathcal{F}_{\mathcal{A}}, \gamma)$. Hence $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$.

Conversely, suppose that for $(\epsilon_1, \epsilon_2) \in [0,1]$ and $\epsilon_1 < \epsilon_2$, $\mathcal{A}$ be a ..... K-subalgebra of K such that $\max(\mathcal{T}_{\mathcal{A}}(u \odot v), \epsilon_1) < \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), \epsilon_2) = \alpha$, then $\mathcal{T}_{\mathcal{A}}(u \odot v) < \alpha$, where $u \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha), v \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha), \alpha \in (\epsilon_1, \epsilon_2]$. Since $u, v \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha)$ and $\cup(\mathcal{T}_{\mathcal{A}}, \alpha)$ is a K-subalgebra, $u \odot v \in \cup(\mathcal{T}_{\mathcal{A}}, \alpha)$, i.e., $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \alpha$, a contradiction. Similar results can be obtained for $\cup'(\mathcal{I}_{\mathcal{A}}, \beta)$ and $L(\mathcal{F}_{\mathcal{A}}, \gamma)$. □

References

[1] A. A. A. Agboola, B. Davvaz, *Introduction to neutrosophic BCI/BCK-algebras*, International Journal of Mathematics and Mathematical Sciences, Article ID 370267, (2015) 1-6.

[2] M. Akram, K. H. Dar, P. K. Shum, *Interval-valued (α, β)-fuzzy K-algebras*, Applied Soft Computing, 11 (2011), 1213-1222.

[3] M. Akram, B. Davvaz, F. Feng, *Intutionistic fuzzy soft K-algebras*, Mathematics in Computer Science, 7 (2013), 353-365.

[4] M. Akram, K. H. Dar, Y. B. Jun, E. H. Roh, *Fuzzy structures of $K(G)$-algebra*, Southeast Asian Bulletin of Mathematics, 31 (2007), 625-637.

[5] M. Akram, K. H. Dar, *Generalized fuzzy K-algebras*, VDM Verlag Dr. Miller, 2010, ISBN-13: 978-363927095.

[6] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[7] S.K. Bakhat, P. Das, $(\in, \in \vee q)$-*fuzzy subgroup*, Fuzzy Sets and Systems, 80 (1996), 359-368.

[8] R.A. Borzooei, H. Farahani, M. Moniri, *Neutrosophic deductive filters on BL-algebras*, Journal of Intelligent & Fuzzy Systems, 26(2014), 2993-3004.

[9] K.H. Dar, M. Akram, *On a K-algebra built on a group*, Southeast Asian Bulletin of Mathematics, 29 (2005), 41-49.

[10] K.H. Dar, M. Akram, *Characterization of a $K(G)$-algebra by self maps*, Southeast Asian Bulletin of Mathematics, 28 (2004), 601-610.

[11] K.H. Dar, M. Akram, *On K-homomorphisms of K-algebras*, International Mathematical Forum, 46 (2007), 2283-2293.

[12] D. Coker, M. Demirci, *On intuitionistic fuzzy points*, Notes on intuitionistic fuzzy sets, 1 (1995), 79-84.

[13] Y. B. Jun, S.-Z. Song, F. Smarandache, H. Bordbar, *Neutrosophic quadruple BCK/BCI-algebras*, Axioms, 7 (2018), 41.

[14] P. M. Pu, Y. M. Liu, *Fuzzy topology, I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, Journal of Mathematical Analysis and Applications, 76 (1980), 571-599.

[15] F. Smarandache, *Neutrosophy neutrosophic probability, set, and logic*, American Research Press, Rehoboth, USA, 1998.

[16] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, *Single valued neutrosophic sets*, Multispace and Multistruct, 4 (2010), 410-413.

[17] C. K. Wong, *Fuzzy points and local properties of fuzzy topology*, Jounal of Mathematical Analysis and Applications, 46 (1974), 316-328.

[18] X. Yuan, C. Zhang, Y. Ren, *Generalized fuzzy groups and many-valued implications*, Fuzzy sets and Systems, 138 (2003), 205-211.

[19] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

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On hyperconnected spaces via m-structures

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Abstract. In this paper, we introduce and investigate the notion of m-hyperconnectedness in a topological space (X, τ) with a minimal structure m_X on X. Several characterizations and preservation theorems of m-hyperconnectedness are obtained.

Keywords: m-structure, m-hyperconnected, semi-m_X-open, semi-m_X-interior, somewhere dense.

1. Introduction

A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure [11] if $\phi \in m_X$ and $X \in m_X$. In [2], the present authors introduced and investigated the notion of m^*-connected spaces, m-separated sets and m-connected sets in a topological space (X, τ) with a minimal structure m_X. In this paper, we introduced the notion of m-hyperconnectedness in a topological space (X, τ) with a minimal structure m_X. We obtain several characterizations and preservation theorems of m-hyperconnectedness. And also, we investigate the

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relationship between m-hyperconnectedness and hyperconnectedness. Recently papers [3, 4, 12] have introduced some new classes of sets via m-structures.

2. Minimal structures

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [11] if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed*.

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be

1. α-open [10] if $A \subset Int(Cl(Int(A)))$,
2. semi-open [7] if $A \subset Cl(Int(A))$,
3. preopen [9] if $A \subset Int(Cl(A))$,
4. b-open [6] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,
5. β-open [1] or semi-preopen [5] if $A \subset Cl(Int(Cl(A)))$.

The family of all α-open (resp. semi-open, preopen, b-open, semi-preopen) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\mathrm{SPO}(X)$).

Definition 2.3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [8] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 2.4. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\mathrm{SPO}(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{bInt}(A)$, $\mathrm{spInt}(A)$).

Lemma 2.5 (Maki et al. [8])**.** *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2.6 (Popa and Noiri [11])**.** *Let X be a nonempty set with an m-structure m_X and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 2.7. An m-structure m_X on a nonempty set X is said to have

1. *property* $\mathcal{B}$ [8] if the union of any family of sets belonging to m_X belongs to m_X.

2. *property* $\mathcal{I}$ if the intersection of any finite family of sets belonging to m_X belongs to m_X.

Remark 2.8. Let (X, τ) be a topological space. Then the families $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$ and $\mathrm{SPO}(X)$ are m-structures on X with property $\mathcal{B}$.

Lemma 2.9 (Popa and Noiri [11])**.** *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is* m_X*-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is* m_X*-closed.*

A topological space (X, τ) with an m-structure m_X on X is called a mixed space and is denoted by (X, τ, m_X).

Definition 2.10. A subset A of a mixed space (X, τ, m_X) is said to be:

1. m_X-dense if $mCl(A) = X$.

2. m_X-nowhere dense if $Int(mCl(A)) = \phi$.

3. α-m_X-open if $A \subseteq Int(mCl(Int(A)))$.

4. semi-m_X-open if $A \subseteq mCl(Int(A))$.

5. pre-m_X-open if $A \subseteq Int(mCl(A))$.

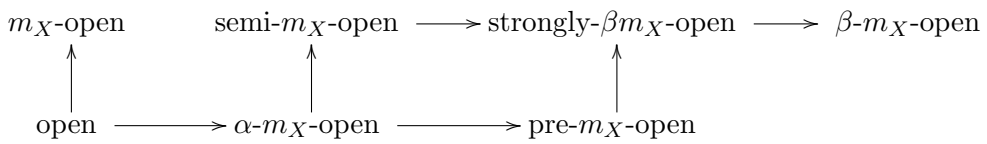
6. β-m_X-open if $A \subseteq Cl(Int(mCl(A)))$.

7. semi-m_X^*-open if $A \subseteq Cl(mInt(A))$.

8. strongly-βm_X-open if $A \subseteq mCl(Int(mCl(A)))$.

Lemma 2.11. *If $\tau \subseteq m_X$, then every semi-m_X-open set is semi-m_X^*-open.*

If $\tau \subseteq m_X$, the following diagram holds:



Lemma 2.12. *Let A be a subset of a mixed space (X, τ, m_X). Then the following properties hold:*

1. *A is semi-m_X-open if and only if there exists $B \in \tau$ such that $B \subseteq A \subseteq mCl(B)$.*
2. *If there exists $B \in m_X$ such that $B \subseteq A \subseteq Cl(B)$, then A is semi-m_X^*-open.*
3. *A is semi-m_X^*-open if and only if $Cl(A) = Cl(mInt(A))$.*

3. m-hyperconnected spaces

Definition 3.1. A mixed space. (X, τ, m_X) is said to be

1. m-hyperconnected (resp. hyperconnected [13]) if A is m_X-dense (resp. dense) for every nonempty open set A of X,
2. $m^\star$-connected [2] if X cannot be written as the disjoint union of a nonempty m_X-open set and a nonempty open set.

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $m_X = \{\phi, X, \{a\}, \{a, c\}, \{a, b\}\}$. Then the mixed space (X, τ, m_X) is m-hyperconnected.*

If $\tau \subseteq m_X$, the following diagram holds:

$$\begin{array}{ccc} (X,\tau,m_X) \text{ is } m\text{-hyperconnected} & \longrightarrow & (X,\tau) \text{ is hyperconnected} \\ \downarrow & & \downarrow \\ (X,\tau,m_X) \text{ is } m^*\text{-connected} & \longrightarrow & (X,\tau) \text{ is connected} \end{array}$$

Theorem 3.3. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. The following properties are equivalent:*

1. *X is m-hyperconnected;*
2. *A is m_X-dense or m_X-nowhere dense for every subset A of X;*
3. *$A \cap B \neq \phi$ for every nonempty open subset A and every nonempty m_X-open subset B of X;*
4. *$A \cap B \neq \phi$ for every nonempty semi-m_X-open subset A and every nonempty m_X-open subset B of X.*

Proof. (1) $\Rightarrow$ (2): Let X be m-hyperconnected and $A \subseteq X$. Suppose that A is not m_X-nowhere dense. Then $Cl[X - mCl(A)] = X - Int(mCl(A)) \neq X$. Since $Int(mCl(A)) \neq \phi$, by (1) $mCl(Int(mCl(A))) = X$. Since $mCl(Int(mCl(A))) = X \subseteq mCl(A)$, then $mCl(A) = X$. Thus A is m_X-dense.

(2) $\Rightarrow$ (3): Suppose that $A \cap B = \phi$ for some nonempty sets $A \in \tau$ and $B \in m_X$. Then $mCl(A) \cap B = \phi$ and A is not m_X-dense. Moreover, since $A \in \tau$, $\phi \neq A \subseteq Int(mCl(A))$ and A is not m_X-nowhere dense.
(3) $\Rightarrow$ (4): Suppose that $A \cap B = \phi$ for some nonempty semi-m_X-open set A and some nonempty m_X-open set B of X. Since A is nonempty, $Int(A)$ is nonempty and by (3) $\phi \neq Int(A) \cap B \subseteq A \cap B$. This is a contradiction.
(4) $\Rightarrow$ (1): Suppose that X is not m-hyperconnected. Then there exists a nonempty open set V such that $mCl(V) \neq X$. Therefore, there exists a point $x \notin mCl(V)$ and by Lemma 2.6 $U \cap V = \phi$ for some $U \in m_X$ containing x. Since V is open, V is semi-m_X-open. This is contrary to (4). □

Lemma 3.4. *Let (X, τ, m_X) be an m-hyperconnected mixed space. Then every open mixed subspace $(Y, \tau_Y, m_X(Y))$ of (X, τ, m_X) is m-hyperconnected.*

Proof. Let $Q = Y \cap O$ be a nonempty open set in Y, then $mCl_Y(Q) = Y \cap mCl(Y \cap O) = Y \cap X = Y$ and hence Q is m-dense in Y. □

Definition 3.5. Let (X, τ, m_X) be a mixed space and $A \subseteq X$. The semi-m_X-closure (resp. semi-m_X^*-closure, pre-m_X-closure, strongly-βm_X-closure) of A, denoted by $S\text{-}mCl(A)$ (resp. $S^*\text{-}mCl(A)$, $P\text{-}mCl(A)$, $s\beta\text{-}mCl(A)$), is defined by the intersection of all semi-m_X-closed (resp. semi-m_X^*-closed, pre-m_X-closed, strongly-βm_X-closed) sets of X containing A.

Lemma 3.6. *Let (X, τ, m_X) be a mixed space. The following properties hold for a subset $A \subseteq X$.*

1. *$S\text{-}mCl(A) = A \cup mInt(Cl(A))$;*
2. *$S^*\text{-}mCl(A) = A \cup Int(mCl(A))$.*

Theorem 3.7. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. The following properties are equivalent:*

1. *X is m-hyperconnected;*
2. *H is m_X-dense for every nonempty strongly-βm_X-open subset $H \subseteq X$;*
3. *$S^*\text{-}mCl(H) = X$ for every nonempty strongly-βm_X-open subset $H \subseteq X$.*

Proof. (1) $\Rightarrow$ (2): Let (X, τ, m_X) be an m-hyperconnected space. Let H be any nonempty strongly-βm_X-open subset of X. We have $Int(mCl(H)) \neq \phi$. Thus $X = mCl(Int(mCl(H))) = mCl(H)$.
(2) $\Rightarrow$ (3): Let H be any nonempty strongly-βm-open subset of X. Thus, by Lemma 3.6 $S^*\text{-}mCl(H) = H \cup Int(mCl(H)) = H \cup Int(X) = X$.
(3) $\Rightarrow$ (1): Let G be any open set. Then G is strongly-βm_X-open and by (3) we have $X = S^*\text{-}mCl(G) = G \cup Int(mCl(G)) \subset mCl(G)$. Therefore, X is m-hyperconnected. □

Lemma 3.8. *If a mixed space (X, τ, m_X) is m-hyperconnected, then every set having nonempty interior is semi-m_X-open.*

Proof. Let A be a nonempty set containing a nonempty open set U. Since X is m-hyperconnected and U is m-dense, we have $mCl(U) = X$ and so $U \subseteq A \subseteq mCl(U) = X$. Hence, by Lemma 2.12, A is semi-m_X-open. □

Theorem 3.9. *Let (X, τ, m_X) be a mixed space such that the intersection of an open set and an m_X-open set is m_X-open. Then X is m-hyperconnected if and only if the union of a non dense set and a non m_X-dense set is non m_X-dense.*

Proof. Suppose that X is m-hyperconnected. Let U be non dense and V be non m_X-dense in X. If one is empty, then there is nothing to prove. If not, there exist a nonempty open set G and an m_X-open H such that $U \cap G = \phi = V \cap H$. Now, $(G \cap H) \cap (U \cup V) = (G \cap H \cap U) \cup (G \cap H \cap V) = \phi$. Since X is m-hyperconnected, $G \cap H \neq \phi$. By assumption, $G \cap H \in m_X$ and so $U \cup V$ is non m_X-dense. Conversely, suppose that X is not m-hyperconnected. Then there exist a nonempty open set U and a nonempty m_X-open set V such that $U \cap V = \phi$. Then $(X - U) \cup (X - V) = X$. If $X - U$ is dense, then $Cl(X - U) = X$ which implies that $U = \phi$ and if $X - V$ is m_X-dense, then $mCl(X - V) = X$ which implies that $V = \phi$. This is a contradiction. Therefore, $X - U$ is non dense and $X - V$ is non m_X-dense. By assumption, $(X - U) \cup (X - V) = X$ is non m_X-dense and so X is non m_X-dense, which is a contradiction. Therefore, X is m-hyperconnected. □

Theorem 3.10. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. If A is m-hyperconnected set of X and $A \subseteq B \subseteq mCl(A)$, then B is m-hyperconnected.*

Proof. Let V be any nonempty open set of the subspace B. Then $V = U \cap B$ for some open set U of X. Since V is nonempty and $B \subseteq mCl(A)$, $U \cap A$ is a nonempty open set of A and hence $mCl_A(U \cap A) = A$. Moreover, we have $A = mCl(U \cap A) \cap A \subseteq mCl(U \cap B) \cap B = mCl_B(V)$ and hence $B = mCl(A) \cap B = mCl_B(A) \subseteq mCl_B(V)$. This shows that B is an m-hyperconnected set of X. □

The semi-m_X-interior of a subset A of a mixed space (X, τ, m_X), denoted by $S\text{-}mInt(A)$, is defined by the union of all semi-m_X-open sets of X contained in A.

Definition 3.11. A function $f : (X, \tau, m_X) \to (Y, \sigma)$ is said to

1. semi-m-continuous if for every open set A of Y, $f^{-1}(A)$ is semi-m_X-open.
2. almost semi-m-continuous if for every nonempty regular open set A of Y, $f^{-1}(A) \neq \phi$ implies $S\text{-}mInt[f^{-1}(A)] \neq \phi$.

Definition 3.12. A function $f : (X, \tau) \to (Y, \sigma, m_Y)$ is said to almost semi-m-open if for every nonempty regular open set B of X, $S\text{-}mInt[f(B)] \neq \phi$

Lemma 3.13. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$ is almost semi-m-continuous.*

Proof. Let $f : (X, \tau, m_X) \to (Y, \sigma)$ be a semi-m-continuous function. Suppose that G is a nonempty regular open subset of (Y, σ, m_Y) such that $f^{-1}(G) \neq \phi$. This implies that $f^{-1}(G)$ is a nonempty semi-m_X-open set in (X, τ, m_X).

Thus, $f^{-1}(G) \subseteq mCl(Int(f^{-1}(G)))$ and hence $\phi \neq f^{-1}(G) \subseteq f^{-1}(G) \cap mCl(Int(f^{-1}(G))) = S\text{-}mInt(f^{-1}(G)$. Consequently, $f : (X, \tau, m_X) \to (Y, \sigma)$ is almost semi-m-continuous.

□

Theorem 3.14. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. Then, the following properties are equivalent:*

1. *(X, τ, m_X) is m-hyperconnected;*
2. *Every almost semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space, is constant;*
3. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space, is constant;*
4. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is the two point discrete space, is constant.*

Proof. (1) $\Rightarrow$ (2): Let X be an m-hyperconnected space. Suppose that there exist a Hausdorff space Y and an almost semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$ such that f is not constant. There exist two distinct points x and y in X such that $f(x) \neq f(y)$. Since Y is Hausdorff, then there exist two open sets A and B in X such that $f(x) \in A$, $f(y) \in B$ and $A \cap B = \phi$. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. This implies that G and H are nonempty regular open and $G \cap H = \phi$. Since f is almost semi-m-continuous, $S\text{-}mInt[f^{-1}(G)] \neq \phi$ and $S\text{-}mInt[f^{-1}(H)] \neq \phi$. Since $S\text{-}mInt[f^{-1}(G)]$ is nonempty semi-m_X-open, $Int(S\text{-}mInt[f^{-1}(G)]) \neq \phi$. We have $Int(S\text{-}mInt[f^{-1}(G)]) \cap S\text{-}mInt[f^{-1}(H)] \subseteq S\text{-}mInt[f^{-1}(G)] \cap S\text{-}mInt[f^{-1}(H)] \subseteq f^{-1}[G \cap H] = \phi$. Since $Int(S\text{-}mInt[f^{-1}(G)])$ is m_X-open and $S\text{-}mInt[f^{-1}(H)]$ is semi-m_X-open, by Theorem 3.3 (4) X is not m-hyperconnected. This is a contradiction.

(2) $\Rightarrow$ (3): It follows from Lemma 3.13 and (2).

(3) $\Rightarrow$ (4): It follows from (3).

(4) $\Rightarrow$ (1): Suppose (X, τ, m_X) is not m-hyperconnected. Then there exists a nonempty open set U such that $mCl(U) \neq X$. Let $Y = \{a, b\}$, $\sigma = \{\phi, \{a\}, \{b\}, Y\}$ and define $f : (X, \tau, m_X) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a, & x \in mCl(U); \\ b, & x \notin mCl(U). \end{cases}$$

Then f is not a constant function. Since $mCl(U) = mCl(Int(U)) \subseteq mCl(Int(mCl(U)))$, $mCl(U)$ is semi-m_X-open. Hence σ is the two point discrete topology on $\{a, b\}$ and f is a semi-m-continuous function. Moreover, f is

not constant. This is contrary to the hypothesis. Therefore, (X, τ, m_X) is an m-hyperconnected space. □

Theorem 3.15. *Let (X, τ, m_X) be an m-hyperconnected mixed space and $\tau \subseteq m_X$. If $f : (X, \tau, m_X) \to (Y, \sigma)$ is an almost semi-m-continuous surjection, then Y is hyperconnected.*

Proof. Suppose Y is not hyperconnected. Then there exist disjoint nonempty open sets $A \subseteq Y$ and $B \subseteq Y$. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. Then G and H are nonempty regular open sets and $G \cap H = \phi$.

We have $Int(S\text{-}mInt[f^{-1}(G)]) \cap S\text{-}mInt[f^{-1}(H)] \subseteq S\text{-}mInt[f^{-1}(G)] \cap S\text{-}mInt[f^{-1}(H)] \subseteq f^{-1}(G) \cap f^{-1}(H) = \phi$. Since f is almost semi-m-continuous, $S\text{-}mInt[f^{-1}(G)] \neq \phi$ and $S\text{-}mInt[f^{-1}(H)] \neq \phi$.

Hence $Int(S\text{-}mInt[f^{-1}(G)])$ is not empty. By Theorem 3.3, X is not m-hyperconnected. This is a contradiction. □

Theorem 3.16. *Let (Y, σ, m_Y) be an m-hyperconnected mixed space and $\sigma \subseteq m_Y$. If $f : (X, \tau) \to (Y, \sigma, m_Y)$ is an almost semi-m-open injection, then X is hyperconnected.*

Proof. Let A and B be nonempty open sets of X. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. Then G and H are nonempty regular open sets. Since f is almost semi-m-open, $S\text{-}mInt[f(G)] \neq \phi$ and $S\text{-}mInt[f(H)] \neq \phi$. Hence $Int(S\text{-}mInt[f(G)]) \neq \phi$. Since Y is an m-hyperconnected space, then by Theorem 3.3, $\phi \neq Int(S\text{-}mInt[f(G)]) \cap S\text{-}mInt[f(H)] \subseteq f(G) \cap f(H)$. Since f is injective, then $G \cap H \neq \phi$. Then $A \cap B \neq \phi$ and hence X is hyperconnected. □

4. m_X-somewhere dense sets

Definition 4.1. A subset A of a mixed space (X, τ, m_X) is said to be m_X-somewhere dense if $Int(mCl(A)) \neq \phi$. In other words, a subset A of a mixed space (X, τ, m_X) is said to be m_X-somewhere dense if there exists a nonempty open set G such that $G \subseteq mCl(A)$. If A is not m_X-somewhere dense in X, then it is m_X-nowhere dense in X.

Theorem 4.2. *Let (X, τ, m_X) be a mixed space, $A \cap B \in m_X$ for any $A \in \tau$ and $B \in m_X$ and m_X have property $\mathcal{B}$. Then, for a subset N of X, the following properties are equivalent:*

1. *N is m_X-nowhere dense in X,*
2. *$X - mCl(N)$ is dense in X,*
3. *For each nonempty open set U in X, there exists a nonempty m_X-open set V in X such that $V \subseteq U$ and $V \cap N = \phi$.*

Proof. (1) $\Rightarrow$ (2): Let N be m_X-nowhere dense in X. Then $Int(mCl(N)) = \phi$ and $Cl(X - mCl(N)) = X$. Hence $X - mCl(N)$ is dense in X.
(2) $\Rightarrow$ (3): By letting $V = U \cap (X - mCl(N))$, we have the desired result.
(3) $\Rightarrow$ (1): Suppose that $Int(mCl(N)) \neq \phi$. Then, for any nonempty m_X-open set V such that $V \subseteq Int(mCl(N))$, $V \cap N \neq \phi$. Because, if $V \cap N = \phi$, then $V \cap mCl(N) = \phi$ and $V \cap Int(mCl(N)) = \phi$. Therefore, $V = \phi$. This is a contradiction. Hence $V \cap N \neq \phi$. This shows that (3) $\Rightarrow$ (1). □

Theorem 4.3. *Let (X, τ, m_X) be a mixed space. Then, every nonempty strongly-βm_X-open set of X is m_X-somewhere dense.*

Proof. Suppose that A is a nonempty strongly-βm_X-open set. Then $A \subseteq mCl(Int(mCl(A))) \subseteq mCl(mCl(A)) \subseteq mCl(A)$. Therefore the set $Int(mCl(A))$ is nonempty open and $Int(mCl(A)) \subseteq mCl(A)$. Thus A is m_X-somewhere dense. □

Theorem 4.4. *Let (X, τ, m_X) be a mixed space. The union of an arbitrary family of nonempty m_X-somewhere dense subsets of (X, τ, m_X) is m_X-somewhere dense.*

Proof. Assume that $\{A_\alpha : \alpha \in \Delta\}$ is a family of m_X-somewhere dense sets. Then, for each $\alpha \in \Delta$ there exists a nonempty open set G_α such that $G_\alpha \subseteq mCl(A_\alpha) \subseteq mCl(\cup_{\alpha\in\Delta} A_\alpha)$. Hence $\cup_{\alpha\in\Delta} A_\alpha$ is an m_X-somewhere dense set. □

Theorem 4.5. *Let (X, τ, m_X) a hyperconnected mixed space and $A \cap B \in m_X$ for any $A \in \tau$ and $B \in m_X$. If M is open and N is m_X-somewhere dense, then $M \cap N$ is m_X-somewhere dense.*

Proof. Suppose that N is an m_X-somewhere dense subset of (X, τ, m_X). Then there is a nonempty open set G which is contained in $mCl(N)$. Therefore $M \cap G \subseteq M \cap mCl(N) \subseteq mCl(M \cap N)$. Since (X, τ, m_X) is hyperconnected, then $M \cap G \neq \phi$, Hence $M \cap N$ is m_X-somewhere dense. □

A function $f : (X, m_X) \to (Y, m_Y)$ is said to be M-continuous [11] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subseteq V$.

Theorem 4.6. *If a function $f : (X, \tau, m_X) \to (Y, \sigma, m_Y)$ is open and M-continuous, then the image of each m_X-somewhere dense set is m_X-somewhere dense.*

Proof. Let A be an m_X-somewhere dense subset of (X, τ, m_X). Then there is a nonempty open set G such that $G \subseteq mCl(A)$. Now, $f(G) \subseteq f(mCl(A))$. Since f is open and M-continuous, then $f(G)$ is open and $f(mCl(A)) \subseteq mCl(f(A))$ (Theorem 3.1 of [11]). Therefore $f(A)$ is m_X-somewhere dense. □

Lemma 4.7. *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{I}$. If G is m_X-open in X, then $mCl(A) \cap G \subseteq mCl(A \cap G)$ for every $A \subseteq X$.*

Proposition 4.8. *Let (X, τ, m_X) be an m-hyperconnected mixed space such that m_X has property $\mathcal{I}$. If Y is an m_X-open set in X, then Y is m-hyperconnected.*

Proof. Let U be any nonempty open set of Y. Then there exists an open set V of X such that $U = V \cap Y$. By Lemma 4.7, we have $mCl(V) \cap Y \subseteq mCl(V \cap Y) \cap Y = mCl(U) \cap Y = mCl_Y(U)$. Since X is m-hyperconnected, we obtain that $mCl(V) = X$ and hence $Y = mCl_Y(U)$ then Y is m-hyperconnected. □

Theorem 4.9. *Let (X, τ, m_X) be a mixed space. Then $X - A$ is m_X-somewhere dense if and only if there exists a proper closed set F such that $mInt(A) \subseteq F$.*

Proof. Suppose $X - A$ is m_X-somewhere dense, then there exists a nonempty open set G such that $G \subseteq mCl(X - A)$. Thus $mInt(A) = X - mCl(X - A) \subseteq X - G$, take $F = X - G$ which is a proper closed set and hence $mInt(A) \subseteq F$. Conversely, suppose that there exists a proper closed set F such that $mInt(A) \subseteq F$. Then $X - F \subseteq X - mInt(A) = mCl(X - A)$ and $X - F$ is a nonempty open set. Therefore, $X - A$ is m_X-somewhere dense. □

Theorem 4.10. *Let (X, τ, m_X) be a mixed space and $m_X \subseteq \tau$. Then A or $X - A$ is m_X-somewhere dense.*

Proof. Suppose that A is not m_X-somewhere dense, then $Int(mCl(A)) = \phi$, so we have $Cl(A) \subseteq mCl(A) \neq X$. Then $X - Cl(A)$ is a nonempty open subset of $X - A$ and hence $X - Cl(A) \subseteq mCl(X - A)$. Therefore, $mInt(A) \subseteq Cl(A)$ and, by Theorem 4.9, $X - A$ is m_X-somewhere dense. □

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.

[2] A. Al-Omari, H. Al-Saadi and T. Noiri, *On connected spaces via m-structures,* submitted.

[3] A. Al-Omari, H. Al-Saadi and T. Noiri, *On extremally disconnected spaces via m-structures,* Commun. Korean Math. Soc., 34 (2019), 351-359..

[4] A. Al-Omari, T. Noiri and S. Modak, *Paracompact spaces with m-structures,* An. Univ. Oradea Mat., 24 (2017), 155-162.

[5] D. Andrijević, *Semi-preopen sets,* Mat. Vesnik, 38 (1986), 24-32.

[6] D. Andrijević, *On b-open sets,* Mat. Vesnik, 48 (1996), 59-64.

[7] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly, 70 (1963), 36–41.

[8] H. Maki, K. C. Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci., 49 (1999), 17-29.

[9] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings,* Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[10] O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math., 15 (1965), 961–970.

[11] V. Popa and T. Noiri, *On M-continuous functions,* An. Univ. Dunarea de Jos Galati, Ser. Mat. Fiz. Mec. Teor., 18 (2000), 31-41.

[12] A. Mahammad and H. Hassan, *Preopen sets in bitopological spaces*, J. Kufa. Math. Comput., 2 (2015) 27-43.

[13] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology,* Holt, Rinehart and Winster, New York, 1970.

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A review on empirical mode decomposition in forecasting time series

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Abstract. This study aims to survey and summarize the studies that introduced forecasting time series method based on EMD, providing references for researchers relating to this topic. We highlight results that have published during $1998 - 2017$ (since presented the EMD technique). In this survey, we also present some studies that improved EMD methodology to overcome its limitations. In this survey, we present some studies that improved EMD methodology to overcome its limitations, as well studies that have introduced an expansion of EMD methodology. There has been tremendous progress in many areas, but we find that there are a large number of topics that need to be further developed. Finally, we summary some remarks may it will help the researcher in this area.

Keywords: empirical mode decomposition, intrinsic mode function, hybrid approach, forecasting time series, review.

1. Introduction

Forecasting time series is one of the core activities in scientific research, but limited by the availability of prediction methods in the past. To accommodate the variety of data generated by nonlinear and nonstationary processes in nature,

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the forecasting method would have to be adaptive. Hilbert-Huang transform (HHT), consisting of Empirical Mode Decomposition (EMD) and Hilbert spectral analysis [1], is a newly developed adaptive data analysis method, which has been used extensively in prediction research.

A number of studies in literature review that have introduced to review of EMD, such as [2], [3], [4], [5] and [6]. While, [2] presented a short review of EMD method as data analyzing method with example of its application on Earthquake data, [3] presented the recent developments until 2008 with summarize some applications in various geophysical research areas, [4] presented the principle of EMD and some of its extensions method with their application in image fusion, [5] presented review on EMD in fault diagnosis of rotating machinery and [6] presented review on EMD in fault diagnosis of rotating machinery diagnosis.

In this review, we will display the EMD methodology steps, a highlight on some of the studies that tried to develop on EMD method and introduced an expand of EMD method will be presented, with some studies that presented a comparison of EMD method with another decomposition methods. In addition, the aim of this study, the studies introduced forecasting time series method based on EMD. Moreover, for each forecasting technique, we present its methodology, the data used and the methods that have used in a comparison. We hope this study will be an assistant reference for interested and concerned researchers in this area in the future.

1.1 Hilbert-Huang transform

The Hilbert-Huang transform (HHT) has presented by [1] as a integrate of empirical mode decomposition (EMD) and Hilbert transform analysis (HT). The strength of HHT is the ability to process non-stationary and non-linear data. Moreover, HHT does not move from the time domain into the frequency domain - Information is maintained in the time domain [7]. While the HT is applies on intrinsic mode functions (IMF) along with a residual, and obtain instantaneous frequency data. Its mean, the component decomposition of signals must be performed beforehand before applied HT. Mathematically, the HT of a time series $x(t)$ is given by formula 1 in [1].

$$\mathcal{H}[x(t)] = \frac{1}{\pi}\ \mathrm{PV} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau \tag{1}$$

in which the PV indicates the principal value of the singular integral. In the next section, we will present a different analysis method from the previous methods that have been dealt with, while this method have a different components in its results (seasonal, trend and remainder) and also does not use the transformation technique in its algorithm. In this study the HT is not discussed, only the EMD procedure with applications in time series forecasting are presented.

2. Empirical Mode Decomposition (EMD)

The Empirical mode decomposition (EMD) is a new decomposition method has been described by [1]. The main idea for EMD is to decompose a non-stationary and nonlinear time series into a nearly orthogonal combination of simple time series [8]. These components known as intrinsic mode functions (IMFs) and residual (r). The EMD methodology analyze the time series with keeping this time series in time domain. This decomposition method (EMD) is adaptive, intuitive, direct and highly efficient. After this definition and some property of EMD, in the subsection of this section, the EMD algorithm process, several of application of EMD, theoretical developments on EMD, and forecasting methods based on EMD will be presented.

2.1 Sifting Process

The algorithm process of EMD will be presented as 6 steps. This process is named the Sifting process of EMD. While the main idea of EMD is decompose of time series into IMFs and $r(t)$. So, the time series -let $x(t)$- can be constructed back as the equation (2). The sifting decomposition process is based on the local characteristic time scale of the data as presented by [1].

$$x(t) = \sum_{i=1}^{n} IMF_i(t) + r(t), \tag{2}$$

where $x(t)$ represents the original time series, $r(t)$ represents the residue of the original time series data decomposition, and IMF_i represents the i^{th} intrinsic mode function (IMF) series. In order to estimate the IMFs should initiate the steps of sifting process of time series $x(t)$ as presented by [1]. Thus, this is summarized as described below:-

Step 1. The first step begins by taking the original time series as a $x(t)$ for sifting process. Also, we assume that the value of the two repetition indicators are $i = 1$ and $j = 1$.

Step 2. Then evaluating all the local extrima values (local upper and local lower) of the time series $x(t)$. Figure 1 shows an example of step 2. Here, the black line is the original time series $x(t)$, the red circle represents the local upper, and the green circle represents the local lower.

Step 3. After that, form the local upper (local maximum) envelope function $e_u(t)$ by connecting all local maxima values using the cubic spline line. In a similar way, form the local lower (local minimum) envelope function $e_l(t)$. Should all observations in $x(t)$ cover between $e_u(t)$ and $e_l(t)$. After that, form the mean envelop function denoted by $m_j(t)$ from $e_u(t)$ and $e_l(t)$ by using formula (3). Figure 2 shows an example of step 3. Here the black line represents the original time series $x(t)$, the red line represents the upper envelope line

$e_u(t)$, the green line represents the lower envelope line $e_l(t)$, and the blue line represents the mean envelop $m_j(t)$.

$$m_j(t) = \frac{e_u(t) + e_l(t)}{2} \tag{3}$$

Step 4. Next, define a new function $h_j(t)$ using the mean envelope $m_j(t)$ and the signal $x(t)$ on formula (4).

$$h_j(t) = x(t) - m_j(t) \tag{4}$$

Check if the function $h_j(t)$ which is an IMF or not, according to IMF conditions (will be presented in section 2.2). If the function $h_j(t)$ has satisfied IMF conditions, then go to step 5. If not, renew the value of $x(t)$ such that it become $h_j(t)$. Also, the iteration index value j is renewed. Such that it becomes $j = j + 1$, and repeat the steps again from step 2 until step 4.

Step 5. This step has three processes. Firstly, save $h_j(t)$ which obtained from the last step as a IMF_i, where $\text{IMF}_i(t) = h_j(t)$. Secondly, obtain the residue function $r_i(t)$ using the $\text{IMF}_i(t)$ and the signal $x(t)$ by the formula (5). Thirdly, renew the iterations index values of i and j. Such that it become $i = i + 1$ and $j = 1$.

$$r_{i+1}(t) = x(t) - IMF_i(t) \tag{5}$$

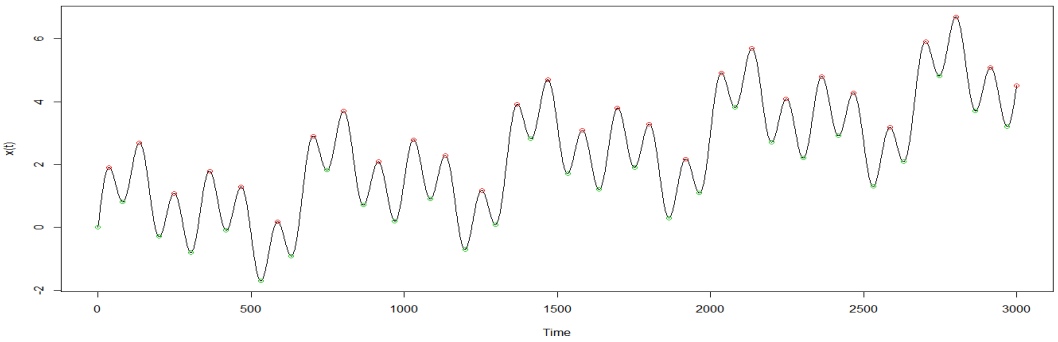


Figure 1: The extraction of local extremum values of time series

Step 6. In this step, according to the residual function's characteristics $r_i(t)$, which obtained from step 5, it will be decided whether the sifting process is over or not.

If $r_i(t)$ is monotonic or constant function from which cannot extract more IMF or the value of SD_h (standard deviation) between 0.2 and 0.3 by [1], where SD_h defined as formula (6), then the residue and all the IMF's will be saved,

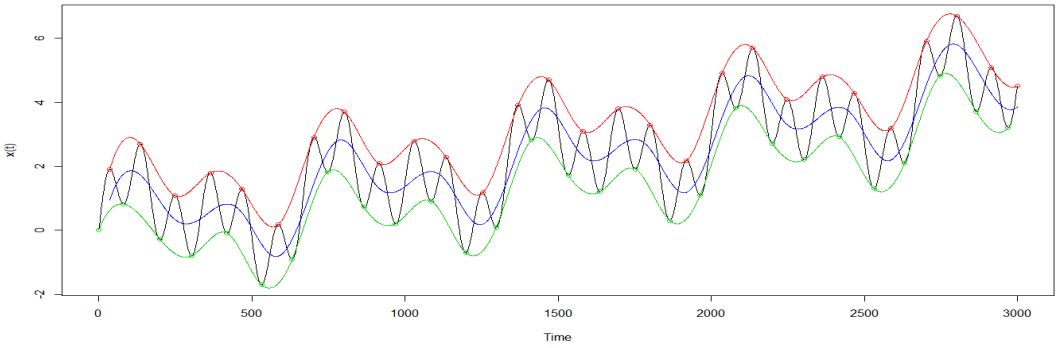


Figure 2: The evaluation of upper and lower envelopes of the time series.

and stop the sifting process. If the residue $r_i(t)$ is not, the value of $x(t)$ will be renewed, such that it becomes $r_i(t)$, and go back to step 2. This step is named "Stoppage criteria of the sifting process".

$$SD_h = \sum_{t=0}^{T} \frac{|h_{k-1}(t) - h_k(t)|^2}{h_{k-1}^2(t)}. \tag{6}$$

The steps 1 through 6 which were discussed above allow the sifting process (EMD algorithm) to separate the time-altering signal properties. Figure 3 is a flowchart summarizes all the sifting process steps.

2.2 Intrinsic Mode Function (IMF)

Based on the EMD algorithm which presented in the previous section, the IMFs' produced by the sifting process need to satisfy two conditions these are

1. $$|Num[extrima] - Num[cross - zero]| \leq 1.$$

   Where Num.extreme represents the number of local extreme points (all local maxima and all local minima), also Num[cross-zero] represent the number of cross-zero points.

2. $$|m(t)| = |\frac{e_u(t) + e_l(t)}{2}| < \varepsilon.$$

   Where $u(t)$ represents the envelope function generated by using cubic spline line on all local maxima, $l(t)$ represents the envelope function generated by using a cube spline line on all local minima, $m(t)$ represents the mean function that it was obtained by evaluating the mean of $u(t)$ and $l(t)$, and ε is a very small positive number which is close to zero. Sometimes, it is equal to zero.

The UK stock market data are taken as an example to show the original time series with its IMFs and residue. The results are displayed in Figure 4.

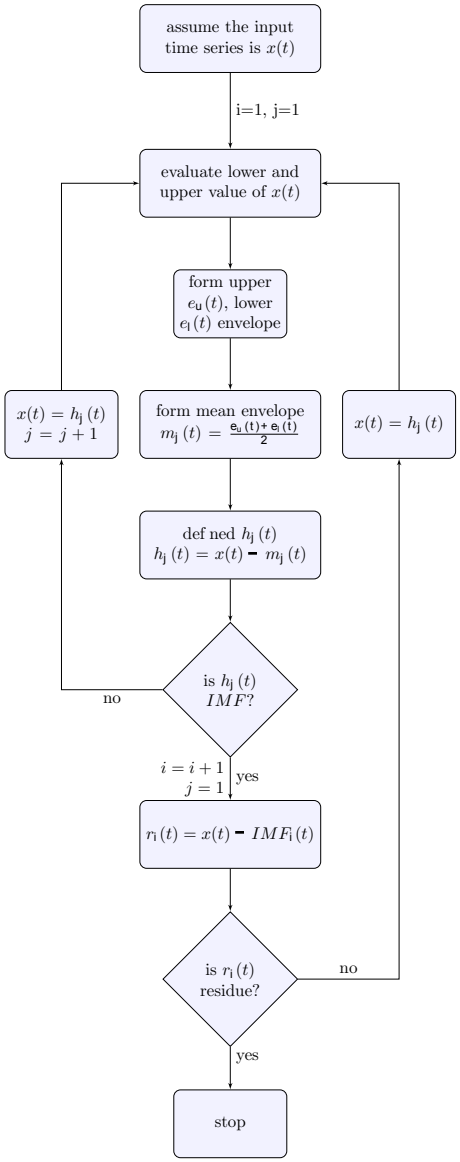


Figure 3: Flowchart of empirical mode decomposition estimation process.

2.3 Limitation, extension, comparison, and applications for EMD

After the introducing the EMD in the research field, it has been widely used in **application** for many research area. Such as, in financial time series by [**?**], [10]. In Medicine by [11]. In Mechanical Engineering by [12], the EMD method was employed in EMD-Golay de-noising algorithm to reduce the noise effectively on Lidar Signal. In electronics engineering by [13]. In sciences such as biological by [14], climate by [15], and dynamic by [16]. In civil & construction Engineering by [17]. Also, it has used in traffic by [**?**] and [19]. It is worth mentioning that

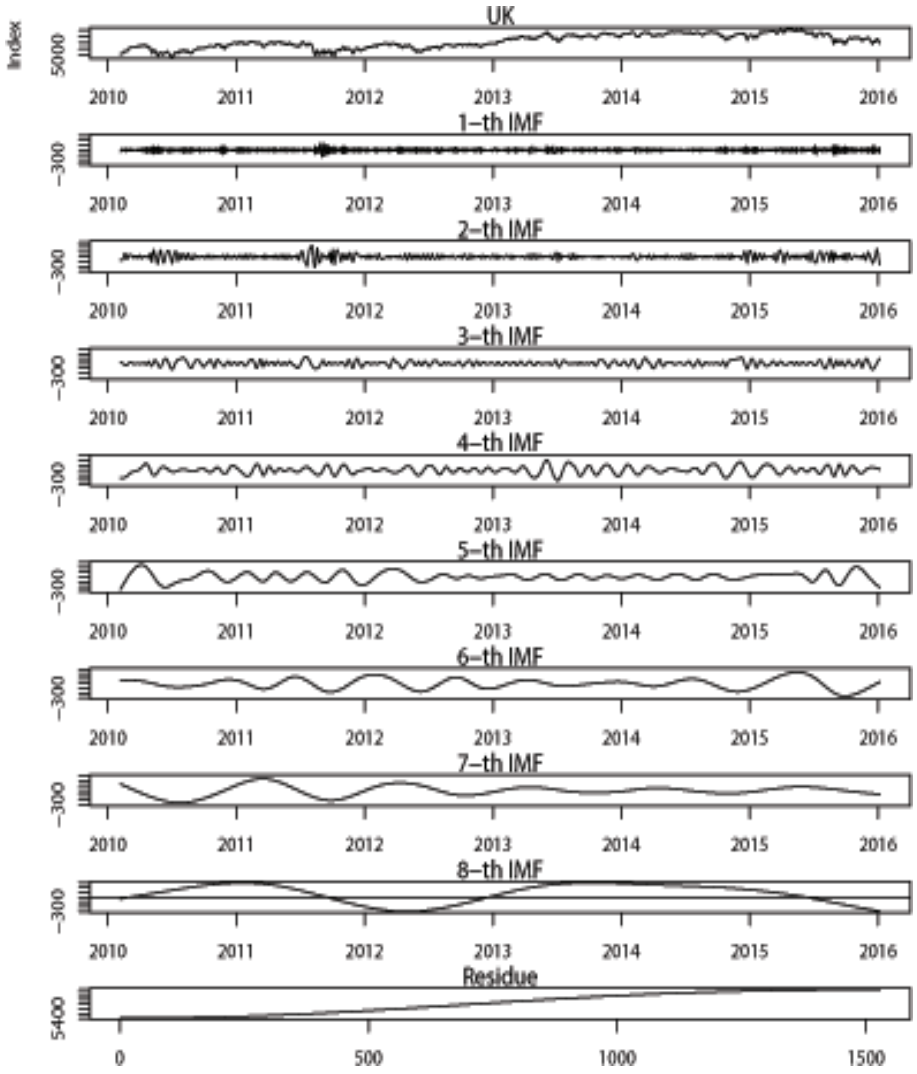


Figure 4: The UK stock market data with its IMFs and residue

the EMD technology has a number of **limitations** in its algorithm. The first is that the theoretical base has not fully established. While that the most of the EMD methodology steps without mathematical expressions. Moreover, [20] declared that there is no theory for EMD. Therefore, many studies have put some theoretical assumption for EMD, despite it still defined largely as steps of an algorithm. Such as, for a time series of size N, it usually only needs $\log_2 N$ IMFs [21]. Also, the average period of each IMF can be calculated by $\frac{2\times N}{(\#\, of\ zero\ crossings)}$ [22].

In [23], assumes that the IMF components are all normally distributed and the Fourier spectra of the IMF components are all identical and cover the same area on a semi-logarithmic period scale. [24] tried to develop the theoretical fun-

damentals of EMD algorithm by introducing three hypotheses on EMD sifting process. However, the theoretical part of the EMD is still poor [25].

The second limitation is the sensitivity to endpoint treatments (the boundary effect) when using the EMD algorithm [15]. To overcome this limitation, many studies have been developed on the EMD methodology. Such as [26] has extended both the beginning and end of the time series by the addition of typical waves. While, [27] companied local polynomial quantile regression (LLQ) with sifting process for automatic boundary correction.

The third limitation is mode mixing [28]. [29] is one of the studies that tried to solve this limitation, this by increasing amount of EMD iteration with additional mathematical operators based on differentiation and integration. Also, [30] by introducing an approach based on Partial Differential Equation as an alternative implementation to the algorithmic of the sifting process, they applied this technique on image analysis in [31]. Plus, [32] introduced a novel method based on the revised blind source separation, [33] applied the differential operation into the separation of the IMFs, and [8] introduced wavelet-bounded EMD to overcome the third limitation.

A number of studies have provided an **extension** of the EMD technology. Such as [34] generalize the EMD technique for two-dimensional. After that, [35] present multivariate extensions of EMD. Also, [36] present a fast three-dimensional EMD (TEMD) to decompose a volume into three-dimensional IMFs. On the other hand, [37] present Variational mode decomposition (VMD) as an alternative decomposition method to the EMD. In [38], the VMD has been applied to noise reduction of the diesel engine. In [39], the Complex VMD (CVMD) has been presented as a development of VMD to applied in Complex-valued signals.

In [40], the Ensemble EMD (EEMD) has been presented as an extension of EMD. The EEMD method has been applied in [41] and [42]. In [43], the complete EEMD (CEEMD) has been presented as a development of EEMD method. Moreover, the EMD algorithm has been modified by [44], [45], and [46].

Recently, a number of studies have presented a **comparison** between the EMD technique with another decomposition method. Such as, [47] present the comparison between EMD and Wavelet Decomposition (WD) in the nonlinear time series analysis. The authors deduce that the accuracy of the decomposition of the EMD is better than that Wavelet decomposition. Moreover, there is a difficult problem in the WD is the selection of the wavelet basis function and decomposition levels; while in the EMD there is none. Also, the authors have inferred that the EMD-based HT was decent for decomposing the linear and nonlinear regime. While the method of the WD-based WT was accurate for decompose the linear regime.

In [48], the EMD method was applied on Ultrasonic Signals for comparison with the Chirplet Signal Decomposition (CSD) method.Both these methods were applied to ultrasonic signal feature extraction with the aim of gaining

more detailed feature information. The EMD method has accurate parameter estimation of time series. It is dynamically method to track the changing in time series.

2.4 Forecasting methods based on EMD

The hybrid models combine the strengths of some traditional models to get a better forecasting accuracy. Recently, several studies addressed hybrid models have applied EMD in the literature for time series forecasting. That by using EMD to decompose the non-stationary and non-linear time series data into Intrinsic Mode Functions (IMFs) and residual components. Then use the forecasting model to forecast each component. Later, the forecasting results were aggregated to get the final forecasted value of the original time series.

A hybrid EMD-ARIMA model has been used in [49, 50, 51, **?**] to forecast the short-term wind speed data, the monthly prices of rice data, the exchange rates data, and the traffic speed data, respectively. This by applying the ARIMA model on EMD components. Then aggregate all results of forecasting. In per of these studies, the forecasting results of the EMD-ARIMA model are superior to the forecasting results of the selected techniques.

A hybrid EMD-LSSVR (least squares support vector regression) forecasting model has been presented and applied on foreign exchange rate in [52]. The results show that the EMD-LSSVR model outperforms EMD-ARIMA, LSSVR, and ARIMA models. The methodology of EMD-LSSVR was by applying the LSSVR to forecast each component of EMD. Then all the forecasted values were aggregated to produce the final forecasted value.

In [53], the EMD-BPN (back-propagation neural network) was presented. All EMD components were modeled and forecasted by BPN. The final forecasting value can be obtained by the sum of these forecasting results. The EMD-NPN model was applied to wind power for short-term forecasting. The results show that the EMD-NPN model outperforms the BPN and ARIMA models.

In [54], the EMD-RBFNN (Radial basis function neural networks) was presented and applied on wind farm power. The results show that the EMD-RBFNN model has better forecasting accuracy than RBFNN model. The RBFNN forecasting model was built for each EMD component according to its feature. After that, all forecasting values were aggregated to obtain the final forecasting value.

In [55], the hybrid MFES model was presented and applied to forecast a half-hour electricity demand data. The MFES combines a multi-output FFNN (feedforward neural network) with EMD-based signal filtering and seasonal adjustment. The results demonstrated that the MFES model forecasting outcome was more accurate than MFE, MFS and MFES models.

In [56], the EMD-ANFIS (Adaptive Neuro-Fuzzy Inference System) model was presented. The ANFIS models were developed for EMD components. Then, these models were applied on IMFs to estimate the forecasting value. All the

results were combined together to get the final forecasting. The EMD-ANFIS was applied to forecast Electric Peak Load data. The forecasting results of EMD-ANFIS model were more accurate than traditional Artificial Neural Network (ANN) models and the EMD-ANN.

[57] used a hybrid of EMD, Least Squares Support Vector Machine (LSSVM), and autoregressive (AR) to forecast wind speed data. The results show that the EMD-LSSVM-AR provides better forecasting compared with two hybrid model; the first model is the EMD-AR by [49] and the second model is EMD-LSSVM by [58]. The EMD components were classified into two sets (low and high correlated) according to the obtained partial autocorrelation function (PACF) factor and frequency. LS-SVM was applied on low correlated and AR model with Kalman filter model was applied on the highly correlated. The forecasting results were aggregated.

In [59], the EMD-ANN was proposed. This proposed method apply ANN on each EMD component before forecasting. The results were added together. The EMD-ANN model was applied on the stream flow data of river to compare with ANN model. The results show that the EMD-ANN model provided a superior alternative to the ANN model.

In [60], a new hybrid has been recommended and applied on uterine electromyography. The entropy ratios values of both instantaneous amplitude and instantaneous frequency of the first ten EMD components were computed. Six different classifiers were implemented in order to evaluate the forecasting performance. The results show an improvement in forecasting accuracy of compared with the existent techniques.

In [61], the integration of EMD and LSSVM model was used to forecast the water demand series data. LSSVM was built to forecast all EMD components individually, PACF was used as an input data. All of these forecasting values were then aggregated. The results show that the EMD-LSSVM model were better than the single LSSVM and ANN model without EMD and EMD-ANN model.

In [62], each EMD component represents the high and low frequencies as well as the patterns. Then meaningful signals were identified using Pearson product moment correlation coefficient. After the identification process was done, the new data set was obtained where the less meaningful signal was omitted from the signals sets. Then, a LSSVM was applied to forecast each IMF. This model was applied for river flow forecasting. The results was proven that EMD-LSSVM model outperforms a single LSSVM based on several performance criteria.

In [63], the integration of Phase space reconstruction (PSR), EMD, and ANN techniques (PSR-EMD-NNPSO) optimized by particle swarm optimization was applied to forecast stock index data.

In [64], the AR-EMD-SVR was presented. This model was employed to short-term forecast of ship motion time series. The time series was decomposed into IMFs and a residual by AR- EMD. The components was forecasted individually using SVR model. The forecasting results were aggregated. The AR-

EMD-SVR model results were better than forecasting results of AR, EMD-AR, and SVR models.

In [65], the EMD-SVR model was suggested. The attribute selection module was used to determine the parameter and compose the input vector for the SVR model of each EMD component. Then, the SVR module was trained and forecasted. Using wind speed data, the EMD-SVR model was compared with five forecasting models namely SVR, EMD-SVR by [66], EMD-SVR by [67], EMD-RBFNN by [68], and EMD-BP by [69]. The results show that the EMD-SVR model has significantly better performance than the selected method.

In [**?**], the EMD and Dynamic Regression (EMD-DR) was presented. Each EMD component was fit and forecasted with suitable DR model. Then the forecasting results were aggregated. The EMD-DR was employed for short-term load (electricity demand and reactive power) forecasting. The forecasting results from EMD-DR model was better than the forecasting results from a single Dynamic Regression model.

In [71], the hybrid EMD-AR model is presented. The AR model was applied on each EMD component to find the forecasting value. The results were aggregated to attain the final forecasting. The EMD-AR model was applied on ocean waves data. Forecasting results show that the hybrid EMD-AR model was superior to the AR model.

In [72], the DEMD-QPSO-SVR-AR model was presented and applied on two real electric load data to compare with ARIMA, BPNN, GA-ANN, PSO-BP, SVR, PSO-SVR, and, AFCM models. The differential EMD (DEMD) was applied on time series to decompose into a number of IMFs and residual. Quantum-Behaved Particle Swarm Optimization with support vector regression model (QPSO-SVR) was applied on each EMD component to find the forecasting value. Auto regression model (AR) was applied on residual to find the forecasting value. Then the final forecasting value were obtained from the IMFs and residual forecasting value.

In [73], an integrated forecasting model of EMD, ARIMA with SVR was presented. The original time series was decomposed into two part, linear and nonlinear. ARIMA was used to analyze and forecast the linear part. EMD was applied on the non-linear part, each component was forecasted by SVR model. Then, the forecasting values for all component were added to the forecasting value from the first part to get the final forecasting. This technique was applied to the stock index of four countries. The forecasting results of this model were better accuracy than ARIMA, SVR, EMD+SVR, ARIMA+SVR.

In [74], the DSF-ANN (Decomposition Selection Forecasting -ANN) and DSF-SVM forecasting models were proposed. For each EMD component, the initial features and targets were constructed. Then, a feature selection process was introduced to constitute the relevant and informative features. Then, the ANN or SVM model was built using the selected feature to evaluate the final forecasting. Three wind speed time series were used to compare these models with SVM, ANN, Decomposition Forecasting Aggregation-ANN (DFA-ANN)

and DFA-SVM. The results show that these two models have satisfactory performance for the wind speed forecasting.

[75] presented EMD with radial basis function neural network (RBFNN) to forecast the monthly groundwater depth data. The RBFNN was used to forecast and stack each EMD separation sequence. The results showed that EMD-RBFNN model better than the conventional time series mode [76].

In [**?**], the modified EMD-LSSVM (MEMD-LSSVM) model was presented and used to forecast the exchange rate data. The EMD components were clustered into several groups based on Permutation distribution clustering (PDC). After that, LSSVM was used to train and forecast each group. All forecasting values added up together. The result shows that the MEMD-LSSVM outperforms single LSSVM and hybrid model of EMD-LSSVM. Moreover, the MED-LSSVM results were better than the MEMD-ARIMA result on the same data in [78]. In [79], a hybrid EMD with Holt-Winter method (HW) was applied to forecasting stock market data. The HW method was applied to forecast each EMD component, all forecasting results ware aggregated. Based on the three forecast accuracy measures, the results indicate that EMD-HW forecasting performance was superior to traditional HW forecasting method.

The combined EMD with exponential smoothing models (EMD-EXP) model was presented and applied to forecast stock market data in [80]. The EXP model was applied for forecast each EMD component. Then, all forecasting values were aggregated. The results show that EMD-EXP outperform four selected forecasting models based on five error forecasting measures.

In [81], a new combined TOPSIS-EMD-FNN model (Technique for order preference by similarity to an ideal solution, EMD, and ANN) was presented and applied on four time series. The TOPSIS-EMD reconstruction method was used to determine the weight for each component. The FNN was used to build a forecasting model for each component, all forecasting results were aggregated. Four forecasting methods were employed for comparison based on MSE, MAE, and MAPE. The results indicated that the TOPSIS-EMD-FNN method performs better than the other four models.

[82] applied the EMD with particle swarm optimization (PSO) and LSSVR model to forecast carbon price. The PSO-LSSVR was employed in forecasting the EMD components, the forecasting values of all the components were aggregated. The model's results were superior to four selected forecasting models.

In [83], a modified EMD-ANN model (MEMD-ANN) was applied to forecast tourism arrival data. The components produced via EMD by reconstructing some components through trial and error method (decomposition). This decomposition and the remaining these components were predicted using ANN model. The forecasted results were aggregated. The results show that, the MEMD-ANN outperformed the ANN and EMD-ANN models based on two measures. The same steps were applied in [84] with Group Method of Data Handling (GMDH) instead of ANN method. This model named MEMD-GMDH and was applied on tourism arrivals. The results showed that the EMD-GMDH model

performed better than the traditional GMDH and EMD-GMDH [85] models based on RMSE and MAPE.

In [86], the integrated EMD with moving average (EMD-MA) model was presented and applied on stock market data. The MA model was applied for forecast each EMD component. Then, all forecasting values were aggregated. The results showed that EMD-MA outperform four selected forecasting models based on five error forecasting measures.

In [87], the hybrid model of EMD, phase space reconstruction, and extreme learning machine (EMD-PSR-ELM) was presented and applied to forecast exchange rates. The EMD components phase space was reconstructed to reveal its unseen dynamics according to the optimum time delay and embedding dimension. A regression forecast model was set up for each components by using ELM. All forecasting values were added up. The results show that the EMD-PSR-ELM superior than six existing method.

In [88], a combination EMD-SVM model was applied for forecast river flow data. The meaningful signals were identified for each EMD component using a statistical measure and the new dataset was obtained. After that, applied SVM to perform forecasting. The experiment results stated that the proposed EMD-SVM have outperformed selected models based on three measure.

In [89], the EMD and BPANN (back-propagation ANN) optimized by particle swarm optimization was presented. The three-layer BPANN was constructed to forecast each EMD component. Then, all forecasting results were aggregating. The outpatient visits data were used, the results showed that their method attains a better than the selected methods.

In [90], a hybrid of improved EEMD, ARIMA, extreme learning machine (ELM), and polynomial function (PF) were applied to forecast a hog price. Then, the EEMD components were composed into the high-frequency (HF), the low-frequency (LF), and trend terms. Then, the ELM, ARIMA, and PF were applied to forecast the HF, LF and trend terms, respectively. The forecasting results were aggregated. The results showed that the improved EEMD-ELM, ARIMA, PF approach outperforms the selected methods based on RMSE and SMAPE. In [92], a combination EEMD-ARIMA was applied to improving daily occupancy forecasting accuracy for hotels. The result showed that EEMD-ARIMA model improve accuracy compared to the ARIMA method.

In [**?**], a novel hybrid model of EMD with chaotic LSSVM (EMD-CLSSVM) was applied for annual runoff data. The LSSVM was applied to forecast the EMD components that possess chaotic characteristics, the rest were simulated by a polynomial method. The results were aggregated. The results reveal that the EMD-CLSSVM model better than the CLSSVM hybrid model based on RATED, RMSE, MARE, and MAE. In [93], EMD-HW bagging was used the EMD in bagging forecastingand applied on stock market data. The EMD with quantile regression (QR) were used to separate the data into regression line (RL), IMFs and residual. The IMFs were clustering into two clusters (HF and LF). Then the HF was resembling using a moving block bootstrap, then new HF series

were add to LF, RL, and R. An ensemble of a hybrid EMD with HW model was applied to estimate the new series. The resulting point forecasts were combined by using the median. Based on three error measures, the results indicate that the EMD-HW bagging method outperforms seven forecasting models.

3. Conclusion

In this paper, we have attempted to provide a review of time series forecasting methods based on EMD. While, the EMD was effective to improve the forecast accuracy in all studies that have used it. Even after all the studies have done to improve the EMD, still there are open problems such as end effects, mode mixing, and Spline problem. Moreover, there is a lot of time series data need to improve its forecasting accuracy, because it is nonlinear and nonstationary, such as exchange rates time series.

Table 1 presents the summary of these literature review where contain the cite, Year , Methods are used and data are used. As a salvation to this part, there is no study was conducted using EMD with MA, EMD with HW, EMD with RW or EMD with EXP in the content of the stock market.

References

[1] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N.-C. Yen, C. C. Tung, H. H. Liu, *The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis*, in: 420 Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 454, The Royal Society, 1998, 903995.

[2] N. E. Huang, *Review of empirical mode decomposition, in: Aerospace/Defense Sensing*, Simulation, and Controls, International Society for Optics and Photonics, 2001, 7180.

[3] N. E. Huang, Z. Wu, *A review on hilbert-huang transform: method and its applications to geophysical studies*, Reviews of Geophysics, 46 (2008).

[4] X. Xiaohong, N. Jiping, *Review of emd-based image fusion,* in: Intelligence Science and Information Engineering (ISIE), 2011 International Conference, IEEE, 2011, 282285.

[5] Y. Lei, J. Lin, Z. He, M. J. Zuo, *A review on empirical mode decomposition in fault diagnosis of rotating machinery,* Mechanical Systems and Signal Processing, 35 (2013), 108126.

[6] L. M. H. SM. Firdaus Isham, M. Salman Leong, Z. A. B. Ahmad, *Empirical mode decomposition: a review on mode selection method for rotating machinery diagnosis*, International Journal of Engineering & Technology IJET-IJENS, 8 (2017), 1626.

Table 1: Related work used EMD in hybrid forecasting method.

	Cite	Year	Method	Data Category
1	[49]	2008	EMD-ARIMA	Wind speed
2	[52]	2012	EMD-LSSVR	Exchange rate
3	[53]	2012	EMD-BPN	Tourism demand
4	[54]	2012	EMD-RBFNN	Wind power
5	[55]	2013	EMD, FFNN	Electricity demand
6	[56]	2013	EMD, ANFIS	Peak Load
7	[57]	2013	EMD, LS-SVM, AR	Wind speed
8	[50]	2014	EMD-ARIMA	Prices of rice
9	[59]	2014	EMD-ANN	River stage
10	[51]	2015	EMD-ARIMA	Exchange rates
12	[61]	2015	EMD-LSSVM	Water demand
13	[64]	2015	AR-EMD-SVR	Ship motion
14	[**?**]	2015	EMD, ARIMA	Traffic speed
15	[60]	2015	EMD, novel nonlinear	Preterm Delivery
16	[62]	2015	EMD, LSSVM	River Flow
17	[63]	2015	EMD, PSR-EMD-NNPSO	Financial data
18	[65]	2016	EMD-SVR	Wind speed
19	[**?**]	2016	EMD, DR	Electricity demand
20	[71]	2016	EMD-AR	Ocean waves
21	[72]	2016	DEMD-QPSO-SVR-AR	Electric load
22	[73]	2016	EMD, SVR, ARIMA	Stock market
23	[74]	2016	EMD, SVM,ANN	Wind speed
24	[75]	2016	EMD, RBFNN	Groundwater depth
25	[**?**]	2016	MEMD, LS-SVM	Exchange Rate
26	[81]	2017	TOPSIS-EMD-FNN	Chaotic, temperature, sunspot, & flow
27	[82]	2017	EMD, LS-SVM	Carbon Price
28	[83]	2017	EMD, ANN	Tourism
29	[87]	2017	EMD, PSR, ELM	Exchange Rates
30	[88]	2017	EMD-SVM	River flow
31	[89]	2017	EMD and BPANN	outpatient visits
32	[90]	2017	iEEMD, ARIMA, ELM, PF	Hog price
33	[92]	2017	EEMD-ARIMA	Occupancy of hotels
34	[**?**]	2017	EMD,CLSSVM	Runoff

[7] A. Y. Ayenu-Prah, S. A. Mensah, N. O. Attoh-Okine, *The Hilbert-Huang transform and the Fourier transform in the analysis of pavement profiles,* in: 1st Annual Inter-University Symposium on Infrastructure Management (AISIM), University of Waterloo, Waterloo, ON, Canada, 2005.

[8] K. J. Moore, M. Kurt, M. Eriten, D. M. McFarland, L. A. Bergman, A. F. Vakakis, *Wavelet-bounded empirical mode decomposition for measured time series analysis*, Mechanical Systems and Signal Processing, 99 (2018), 1429.

[9] H.-S. Oh, J.-H. Suh, D.-H. Kim, *A multi-resolution approach to nonstationary financial time series using the Hilbert-Huang transform*, Korean Journal of Applied Statistics, 22 (2009), 499513.

[10] A. M. Jaber, M. T. Ismail, A. M. Altaher, *Application of empirical mode decomposition with local linear quantile regression in financial time series forecasting*, The Scientific World Journal, 2014.

[11] P. Masselot, F. Chebana, D. Belanger, A. St-Hilaire, B. Abdous, P. Gosselin, T. B. Ouarda, *Emd-regression for modelling multi-scale relationships, and application to weather-related cardiovascular mortality*, Science of The Total Environment 612 (2018), 10181029.

[12] Y. Zhang, X. Ma, D. Hua, Y. Cui, L. Sui, *An emd-based denoising method for lidar signal*, in: Image and Signal Processing (CISP), 2010 3rd International Congress on, Vol. 8, IEEE, 2010, 40164019.

[13] L. Suvasini, S. Prethivika, S. S. Murugan, V. Natarajan, *Extraction of binary sequences in a frequency shift keying-modulated signal by empir460 ical mode decomposition algorithm against ambient noises in underwater acoustic channel,* in: Artificial Intelligence and Evolutionary Algorithms in Engineering Systems, Springer, 2015, 371378.

[14] W. Huang, Y.-P. Sher, K. Peck, Y. C. B. Fung, *Matching gene activity with physiological functions*, Proceedings of the National Academy of Sciences, 99 (2002), 26032608.

[15] K. Coughlin, K.-K. Tung, *11-year solar cycle in the stratosphere extracted by the empirical mode decomposition method*, Advances in Space Research, 34 (2004), 323329.

[16] R. R. Zhang, S. Ma, E. Safak, S. Hartzell, *Hilbert-huang transform anal ysis of dynamic and earthquake motion recordings*, Journal of Engineering Mechanics, 129 (2003), 861875.

[17] E. J. OBrien, A. Malekjafarian, A. Gonzalez, *Application of empirical mode decomposition to drive-by bridge damage detection*, European Journal of Mechanics-A/Solids, 61 (2017), 151163.

[18] H. Wang, L. Liu, S. Dong, Z. Qian, H. Wei, *A novel work zone short-term vehicle-type specific traffic speed prediction model through the hybrid emdarima framework*, Transportmetrica B: Transport Dynamics, 2015, 128.

[19] L. Zhang, B. Pan, *Characteristics analysis on speed time series with empirical mode decomposition as vehicle driving towards an intersection*, in: Information Technology and Intelligent Transportation Systems: Volume 1, Proceedings of the 2015 International Conference on Information Technology and Intelligent Transportation Systems ITITS 2015, held December 12-13, 2015, Xian China, Springer, 2017, 341353.

[20] P. Flandrin, G. Rilling, P. Goncalves, *Empirical mode decomposition as a 485 filter bank*, IEEE Signal Processing Letters, 11 (2004), 112114.

[21] Z. Wu, E. K. Schnieder, Z.-Z. Hu, L. Cao, *The impact of global warming on ENSO variability in climate records,* Vol. 110, Center for Ocean-Land-Atmosphere Studies, 2001.

[22] G. Amirthanathan, G. Pegram, T. Mcmahon, F. Chiew, *Issues with the application of empirical mode decomposition analysis,* in: in: MODSIM 2005 International Congress on Modelling and Simulation, Modelling and Simulation Society of Australia and New Zealand, edited by, Citeseer, 2005.

[23] Z. Wu, N. E. Huang, *A study of the characteristics of white noise using the empirical mode decomposition method,* Proceedings of the Royal Soci495 ety of London. Series A: Mathematical, Physical and Engineering Sciences, 460 (2004), 15971611.

[24] S. Kizhner, K. Blank, T. Flatley, N. E. Huang, D. Petrick, P. Hestnes, *On certain theoretical developments underlying the hilbert-huang transform,* in: Aerospace Conference, 2006 IEEE, IEEE, 2006, 14pp.

[25] G. Rilling, P. Flandrin, *On the influence of sampling on the empirical mode decomposition*, in: ICASSP (3), 2006, 444447.

[26] K. Coughlin, K. K. Tung, *Empirical mode decomposition and climate variability,* in: HilbertHuang Transform and its Applications, 2014, 223 239.

[27] A. M. Jaber, M. T. Ismail, A. M. Altaher, *Empirical mode decomposition combined with local linear quantile regression for automatic boundary correction*, in: Abstract and Applied Analysis, Vol. 2014, Hindawi Publishing Corporation, 2014.

[28] G. Rilling, P. Flandrin, P. Goncalves, et al., *On empirical mode decomposi tion and its algorithms,* in: IEEE-EURASIP Workshop on Nonlinear Signal and Image Processing, Vol. 3, NSIP-03, Grado (I), 2003, 811.

[29] S.-D.Wu, J.-C. Chiou, E. Goldman, *Solution for mode mixing phenomenon of the empirical mode decomposition,* in: Advanced Computer Theory and Engineering (ICACTE), 2010 3rd International Conference on, Vol. 2, IEEE, 2010, V2500.

[30] O. Niang, E. Delechelle, J. Lemoine, *A spectral approach for sifting process in empirical mode decomposition*, IEEE Transactions on Signal Processing, 58 (2010), 56125623.

[31] O. Niang, A. Thioune, M. C. El Gueirea, E. Delechelle, J. Lemoine, *Partial differential equation-based approach for empirical mode decomposition: application on image analysis,* IEEE Transactions on Image Processing, 21 (2012), 39914001.

[32] B. Tang, S. Dong, T. Song, *Method for eliminating mode mixing of empirical mode decomposition based on the revised blind source separation,* Signal Processing, 92 (2012), 248258.

[33] Z. Li, Y. Jiang, C. Hu, Z. Peng, *Difference equation based empirical mode decomposition with application to separation enhancement of multi-fault vibration signals,* Journal of Difference Equations and Applications, 23 (2017), 457467.

[34] Y. Xu, B. Liu, J. Liu, S. Riemenschneider, *Two-dimensional empirical mode decomposition by finite elements*, in: Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 462, The Royal Society, 2006, 30813096.

[35] N. Rehman, D. P. Mandic, *Multivariate empirical mode decomposition,* in: Proceedings of The Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 466, The Royal Society, 2010, 12911302.

[36] Z. He, J. Li, L. Liu, Y. Shen, *Three-dimensional empirical mode decomposition (temd): A fast approach motivated by separable filters*, Signal Processing, 131 (2017), 307319.

[37] K. Dragomiretskiy, D. Zosso, *Variational mode decomposition*, IEEE Transactions on Signal Processing, 62 (2014), 531544.

[38] J. Yao, Y. Xiang, S. Qian, S. Wang, S. Wu, *Noise source identification of diesel engine based on variational mode decomposition and robust independent component analysis*, Applied Acoustics, 116 (2017), 184194.

[39] Y. Wang, F. Liu, Z. Jiang, S. He, Q. Mo, *Complex variational mode decomposition for signal processing applications*, Mechanical Systems and Signal Processing, 86 (2017), 7585.

[40] Z. Wu, N. E. Huang, *Ensemble empirical mode decomposition: a noiseassisted data analysis method*, Advances in Adaptive Data Analysis, 1, 550 (2009), 141.

[41] C. Montalvo, C. Gavilan-Moreno, A. Garca-Berrocal, *Cofrentes nuclear power plant instability analysis using ensemble empirical mode decomposition (eemd)*, Annals of Nuclear Energy, 101 (2017), 390396.

[42] N. Henzel, *Qrs complex detection based on ensemble empirical mode de composition,* in: Innovations in Biomedical Engineering, Springer, 2017, 286293.

[43] M. E. Torres, M. A. Colominas, G. Schlotthauer, P. Flandrin, *A complete ensemble empirical mode decomposition with adaptive noise*, in: 2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), IEEE, 2011, 41444147.

[44] K. Zeng, M.-X. He, *A simple boundary process technique for empirical mode decomposition*, in: Geoscience and Remote Sensing Symposium, 2004. IGARSS'04. Proceedings. 2004 IEEE International, Vol. 6, IEEE, 2004, 42584261.

[45] R. Deering, J. F. Kaiser, *The use of a masking signal to improve empirical mode decomposition,* in: Acoustics, Speech, and Signal Processing, 2005. Proceedings.(ICASSP05). IEEE International Conference on, Vol. 4, IEEE, 2005, iv485.

[46] S. A. Bagherzadeh, D. Asadi, *Detection of the ice assertion on aircraft using empirical mode decomposition enhanced by multi-objective optimization*, Mechanical Systems and Signal Processing, 88 (2017), 924.

[47] X. Wang, G. Feng, A. Feng, *On performance differences of emd and wd in the nonlinear time series analysis*, in: Fuzzy Systems and Knowledge Discovery (FSKD), 2011 Eighth International Conference on, Vol. 1, IEEE, 575 (2011), 207211.

[48] Y. Lu, E. Oruklu, J. Saniie, *Chirplet signal and empirical mode decompositions of ultrasonic signals for echo detection and estimation*, Journal of Signal and Information Processing, 4 (2013), 149.

[49] R. Li, Y. Wang, *Short-term wind speed forecasting for wind farm based on empirical mode decomposition,* in: Electrical Machines and Systems, 2008. ICEMS 2008. International Conference on, IEEE, 2008, 25212525.

[50] S. Abadan, A. Shabri, *Hybrid empirical mode decomposition-arima for forecasting price of rice*, Applied Mathematical Sciences, 8 (2014), 31333143.

[51] S. S. Abadan, A. Shabri, S. Ismail, *Hybrid empirical mode decompositionarima for forecasting exchange rates*, in: AIP Conference Proceedings, Vol. 1643, AIP, 2015, 256263.

[52] C.-S. Lin, S.-H. Chiu, T.-Y. Lin, *Empirical mode decompositionbased least squares support vector regression for foreign exchange rate forecasting,* Economic Modelling, 29 (2012), 25832590.

[53] C.-F. Chen, M.-C. Lai, C.-C. Yeh, *Forecasting tourism demand based on empirical mode decomposition and neural network*, Knowledge-Based Systems, 26 (2012), 281287.

[54] Z.-W. Zheng, Y.-Y. Chen, X.-W. Zhou, M.-M. Huo, B. Zhao, M.-Y. Guo, *Short-term wind power forecasting using empirical mode decomposition and rbfnn*, International Journal of Smart Grid and Clean Energy, 2 (2012), 192199.

[55] N. An, W. Zhao, J. Wang, D. Shang, E. Zhao, *Using multi-output feedforward neural network with empirical mode decomposition based signal filtering for electricity demand forecasting*, Energy, 49 (2013), 279288.

[56] V. A. Okolobah, Z. Ismail, *New approach to peak load forecasting based on emd and anfis,* Indian Journal of Science and Technology, 6 (2013), 56005606.

[57] S. Tatinati, K. C. Veluvolu, *A hybrid approach for short-term forecasting of wind speed*, The Scientific World Journal, 2013.

[58] J. Hu, J. Wang, G. Zeng, *A hybrid forecasting approach applied to wind speed time series*, Renewable Energy, 60 (2013), 185194.

[59] O. Kisi, L. Latifoglu, F. Latifoglu, *Investigation of empirical mode decomposition in forecasting of hydrological time series*, Water Resources Management, 28 (2014), 40454057.

[60] P. Ren, S. Yao, J. Li, P. A. Valdes-Sosa, K. M. Kendrick, *Improved prediction of preterm delivery using empirical mode decomposition analysis of uterine electromyography signals*, PloS one, 10 (2015), e0132116.

[61] A. Shabri, R. Samsudin, U. Teknologi, *Empirical mode decomposition least squares support vector machine based for water demand forecasting,* Int. J. Adv. Soft Comput. Its Appl., 7 (2015).

[62] S. Ismail, A. Shabri, S. S. Abadan, *Empirical mode decomposition coupled with least square support vector machine for river flow forecasting*, in: AIP Conference Proceedings, Vol. 1643, AIP, 2015, 232241.

[63] Y. Heng-Li, L. Han-Chou, *Integrating emd, chaos-based neural network and pso for financial time series forecasting*, Economic Computation & Economic Cybernetics Studies & Research, 49 (2015).

[64] W.-Y. Duan, L.-M. Huang, Y. Han, Y.-H. Zhang, S. Huang, *A hybrid ar-emd-svr model for the short-term prediction of nonlinear and nonstationary ship motion*, Journal of Zhejiang University Science A, 16 (2015), 562576.

[65] Y. Ren, P. N. Suganthan, N. Srikanth, *A novel empirical mode decomposition with support vector regression for wind speed forecasting*, IEEE Transactions on Neural Networks and Learning Systems, 27 (2016) ,1793 1798.

[66] Y. Lin, L. Peng, *Combined model based on emd-svm for short-term wind power prediction [j]*, Proceedings of the CSEE, 31 (2011), 102108.

[67] Z.-h. Han, X.-X. Zhu, *Training set of support vector regression extracted by empirical mode decomposition*, in: Power and Energy Engineering Conference (APPEEC), 2011 Asia-Pacific, IEEE, 2011, 14.

[68] Z. Guo, W. Zhao, H. Lu, J. Wang, *Multi-step forecasting for wind speed using a modified emd-based artificial neural network model*, Renewable Energy, 37 (2012), 241249.

[69] H. Liu, C. Chen, H.-Q. Tian, Y.-F. Li, *A hybrid model for wind speed prediction using empirical mode decomposition and artificial neural networks*, Renewable Energy, 48 (2012), 545556.

[70] N. Akrom, Z. Ismail, *Emd-dr models for forecasting electricity load demand*, Contemporary Engineering Sciences, 9 (2016), 297314.

[71] W.-y. Duan, L.-M. Huang, Y. Han, D.-T. Huang, *A hybrid emd-ar model for nonlinear and non-stationary wave forecasting*, Journal of Zhejiang University Science A, 17 (2016), 115129.

[72] L.-L. Peng, G.-F. Fan, M.-L. Huang, W.-C. Hong, *Hybridizing demd and quantum pso with svr in electric load forecasting*, Energies, 9 (2016), 221.

[73] H.-L. Yang, H.-C. Lin, *An integrated model combined arima, emd with svr for stock indices forecasting,* International Journal on Artificial Intelligence Tools, 25 (2016), 1650005.

[74] C. Zhang, H. Wei, J. Zhao, T. Liu, T. Zhu, K. Zhang, *Short-term wind speed forecasting using empirical mode decomposition and feature selec tion*, Renewable Energy, 96 (2016), 727737.

[75] Q. Fu, D. Liu, T. Li, S. Cui, Y. Hu, *Emd-rbfnn coupling prediction model of complex regional groundwater depth series: a case study of the jiansanjiang administration of heilongjiang land reclamation in China,* Water, 8 (2016), 340.

[76] H. Erdogan, E. Gulal, *The application of time series analysis to describe the dynamic movements of suspension bridges*, Nonlinear Analysis: Real World Applications, 10 (2009), 910927.

[77] N. I. A. Rashid, R. Samsudin, A. Shabri, *Exchange rate forecasting using modified empirical mode decomposition and least squares support vector machine*, Int. J. Advance Soft Compu. Appl., 8 (2016).

[78] N. I. A. Rashid, A. Shabri, R. Samsudin, *Comparison between memd-lssvm and memdarima in forecasting exchange rate*, Journal of Theoretical and Applied Information Technology, 95 (2017), 328.

[79] A. M. Awajan, M. T. Ismail, *A hybrid approach emd-hw for short-term forecasting of daily stock market time series data*, in: AIP Conference Proceedings, Vol. 1870, AIP Publishing, 2017, p. 060006.

[80] M. T. Ismail, A. M. Awajan, *A new hybrid approach emd-exp for shortterm forecasting of daily stock market time series data*, Electronic Journal of Applied Statistical Analysis, 10 (2017), 307327.

[81] W. Jun, T. Lingyu, L. Yuyan, G. Peng, *A weighted emd-based prediction model based on topsis and feed forward neural network for noised time series*, Knowledge-Based Systems, 132 (2017), 167178.

[82] B. Zhu, J. Chevallier, *Forecasting carbon price with empirical mode decomposition and least squares support vector regression*, in: Pricing and Forecasting Carbon Markets, Springer, 2017, 133143.

[83] N. A. Yahya, R. Samsudin, A. Shabri, *Tourism forecasting using hybrid modified empirical mode decomposition and neural network*, International Journal of Advances in Soft Computing & Its Applications, 9 (2017).

[84] N. A. Yahya, R. Samsudin, A. Shabri, *Tourism forecasting using modified empirical mode decomposition and group method of data handling*, 890 (2017), 012140.

[85] A. Shabri, *A novel hybrid ensemble learning paradigm for tourism forecasting*, in: AIP Conference Proceedings, Vol. 1643, AIP, 2015, 192200.

[86] A. M. Awajan, M. T. Ismail, S. Al Wadi, *A hybrid emd-ma for forecasting stock market index*, Italian Journal of Pure and Applied Mathematics, 38 (2017), 313332.

[87] H.-L. Yang, H.-C. Lin, *Applying the hybrid model of emd, psr, and elm to exchange rates forecasting*, Computational Economics, 49 (2017), 99116.

[88] S. Ismail, A. Shabri, *Combination model of empirical mode decomposi- tion and svm for river flow forecasting*, in: American Institute of Physics Conference Series, Vol. 1830, 2017.

[89] D. Huang, Z. Wu, *Forecasting outpatient visits using empirical mode decomposition coupled with back-propagation artificial neural networks optimized by particle swarm optimization*, PloS one, 12 (2017), e0172539.

[90] T. Xiong, C. Li, Y. Bao, et al., *An improved eemd-based hybrid approach for the short-term forecasting of hog price in china*, Agricultural Economics (Zemedelska Ekonomika), 63 (2017), 136148.

[91] G. Zhang, J. Wu, B. Pan, J. Li, M. Ma, M. Zhang, J. Wang, *Improving daily occupancy forecasting accuracy for hotels based on eemd-arima model*, Tourism Economics, 2017, 1354816617706852.

[92] X. Zhao, X. Chen, Y. Xu, D. Xi, Y. Zhang, X. Zheng, *An emd-based chaotic least squares support vector machine hybrid model for annual runoff forecasting*, Water, 9 (2017), 153.

[93] A. M. Awajan, M. T. Ismail, S. A. Wadi, *Forecasting time series using emd-hw bagging*, International Journal of Statistics & Economics, 18 (2017), 921.

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On the Diophantine equation $X^4 + hY^3 = Z^4 + hW^3$

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Abstract. First we prove that, under a certain condition, the Diophantine equation $X^4 + hY^3 = Z^4 + hW^3$, where h is a rational number, has infinitely many nontrivial integral solutions. Using this result we deduce that, under a condition, for $k \geq 2$, the Diophantine equation $X^4+Y_1^3+Y_2^3+\cdots+Y_k^3 = Z^4+W_1^3+W_2^3+\cdots+W_k^3$ has infinitely many nontrivial integral solutions. Then, we conjecture, by some evidences, that the above conditions may be removed.

Keywords: diophantine equation, elliptic curve

1. Introduction

One of the ancient challenging problems is to find the rational solutions as well as the number of solutions of the symmetric Diophantine equations. A symmetric Diophantine equation in n variables is of the form

$$f(x_1, x_2, \ldots, x_n) = f(z_1, z_2, \ldots, z_n),$$

where f is a function with integer coefficients [1]. Actually, most of the studies are done on degrees 3 to 6 since the higher degrees are much more complicated.

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The more attractive case is the equation

$$AX^m + BY^n = AZ^m + BW^n,\ A, B \in \mathbb{Z}. \tag{1}$$

Most people concentrate on the case $m = n$ in lower degrees. For example, see [2] for the equation $X^4+hY^4 = Z^4+hW^4$, $h \in \mathbb{Q}$, and [7] for the equation $AX^3+BY^3 = AZ^3+BW^3$, $A, B \in \mathbb{Z}$. For more details on the symmetric Diophantine equations (1) in low degrees we cite the comprehensive classic references [3, 4].

The equation (1) is also a special case of the more general Diophantine equation

$$\sum_{i=1}^{m} a_i {X_i}^k = \sum_{j=1}^{n} b_j {Z_j}^l,$$

where $a_i, b_j \in \mathbb{Z}$.

Consider the elliptic curve

$$E : y^2 = x^3 + ax^2 + bx + c$$

where $a, b, c \in \mathbb{Q}$, and assume $b \neq 0$. Then the Mordell-Weil group $E(\mathbb{Q})$ of rational points on E is a finitely generated and hence satisfies

$$E(\mathbb{Q}) \equiv \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tors}},$$

where r is called the rank of E and $E(\mathbb{Q})_{\text{tors}}$ the torsion subgroup of $E(\mathbb{Q})$. For more details we cite the classical references [8, 9, 10]. It is clear that an elliptic curve with positive rank has infinitely many rational points. For this reason, one of the methods to show that a Diophantine equation has infinitely many rational solutions is to transfer it by a birational map to an elliptic curve with positive rank.

By a trivial solution of (1) we mean $X = Z$ and $Y = W$ and we consider all solutions are nontrivial. In this article we are interested on the the solutions of the equation (1) in the case $m = 4, n = 3$ and prove the following main theorem by the elliptic curve method mentioned above. The second and third authors have experienced this method in [5] to solve the equation $X^6+6Y^3 = Z^6 \pm 6W^3$.

Theorem 1.1. *Given a nonzero rational number h, if the elliptic curve*

$$y^2 = x^3 - 9x^2 - 8h^2x$$

has positive rank, then the Diophantine equation

$$X^4 + hY^3 = Z^4 + hW^3, \tag{2}$$

has infinitely many integral solutions.

In fact, putting $h = B/A$ we get exactly the equation (1). Using Theorem 1.1 we solve an extensive Diophantine equation.

Corollary 1.2. *In Theorem 1.1 if, in addition, $h = \sum_{i=1}^{k} h_i^3$ for some positive integer k, then the Diophantine equation*

$$X^4 + \sum_{i=0}^{k} Y_i^3 = Z^4 + \sum_{i=0}^{k} W_i^3, \tag{3}$$

has infinitely many integral solutions.

Appealing to Remark 2.1, as an evidence, we conjecture that the condition of Theorem 1.1 may be removed.

Conjecture 1.3. *For any rational number h, the Diophantine equation*

$$X^4 + hY^3 = Z^4 + hW^3$$

has infinitely many integral solutions.

Now, the next conjecture is clear.

Conjecture 1.4. *The Diophantine equation*

$$X^4 + \sum_{i=0}^{k} Y_i^3 = Z^4 + \sum_{i=0}^{k} W_i^3$$

has infinitely many integral solutions.

2. Proofs

Proof of Theorem 1.1. Let the nonzero rational number h be such that the elliptic curve $y^2 = x^3 - 9x^2 - 8h^2x$ has positive rank. Put

$$X = u - h,\ Y = u + v,\ Z = u + h,\ W = u - 2v.$$

Then, with some straightforward calculations, the equation (2) turns to

$$8u^3 + 8uh^2 - 9u^2v + 9uv^2 - 9v^3 = 0.$$

Next, taking $u - v = 1$ we get

$$9v^2 = u^3 - 9u^2 - 8h^2u.$$

Finally, putting $x = u,\ y = 3v$, we obtain the elliptic curve

$$E_h : y^2 = x^3 - 9x^2 - 8h^2x. \tag{4}$$

Now, let the rational number h be such that the elliptic curve E_h has positive rank. Take a non-torsion point $(x, y) = (\frac{r}{s^2}, \frac{t}{s^3})$ on E_h, where $r, s, t \in \mathbb{Z}$. Let (X_0, Y_0, Z_0, W_0) be the corresponding rational solution of (2). Then

$$\begin{aligned} X_0 &= u - h, \\ Y_0 &= u + v, \\ Z_0 &= u + h, \\ W_0 &= u - 2v. \end{aligned}$$

h	1	2	6	12	13	15	18	19	24	25	26	27	32	36	41	46	47	49
t	2	2	2	2	3	2	3	4	2	2	2	2	9	5	2	2	2	5
R	1	1	1	1	1	2	1	1	1	1	1	1	1 or 2	1	1	1	1	2

Table 1: E_h has null rank and $R = \text{rank}(E_{ht^3}) > 0$

We have $u = \dfrac{r}{s^2}$ and $v = \dfrac{t}{3s^3}$. From the fact $u - v = 1$ we get $t = 3rs - 3s^3$.

Clearly, if (X_0, Y_0, Z_0, W_0) is a rational solution of (2), then

$$(\mu^3 X_0, \mu^4 Y_0, \mu^3 Z_0, \mu^4 W_0)$$

is also a rational solution, for any $\mu \in \mathbb{Q}$. Let $h = c/d$, where $c, d \in \mathbb{Z}$, $d \neq 0$, and take $\mu = 3sd$. Since $X_0 = u - h = \dfrac{r}{s^2} - h$, we have

$$\mu^3 X_0^3 = 27d^3 rs - 27d^3 s^3 h.$$

Adding $\pm 27s^3 d^3$ to the right hand side and using the relation $t = 3rs - 3s^3$ turns the above equation to

$$\mu^3 X_0^3 = 9d^3 t + 27s^3 d^3 - 27d^2 s^3 c.$$

The same calculations with $\mu^3 Y_0, \mu^4 Z_0, \mu^3 W_0$ yields the following integral solution.

$$\begin{aligned} X = \mu^3 X_0^3 &= 9d^3 t + 27s^3 d^3 - 27d^2 s^3 c, \\ Y = \mu^4 Y_0^3 &= 81rs^2 d^4 + 27sd^4 t, \\ Z = \mu^3 Z_0^3 &= 9d^3 t + 27s^3 d^3 + 27d^2 s^3 c, \\ W = \mu^4 W_0^3 &= 81rs^2 d^4 - 54sd^4 t. \end{aligned}$$

Therefore, we have infinitely many integral solutions □

Remark 2.1. If the rank of E_h is zero, then we may find an appropriate t such that replacing h in (2) with ht^3 yields an elliptic curve with positive rank. Table 1 shows all such h up to 50. Note that for those h not in the table, E_h has positive rank. In fact, Table 1 guarantees the truth of Conjecture 1.3 for $1 \leq h \leq 50$.

Proof of Corollary 1.2. In Theorem 1.1 take $h = \sum_{i=1}^{k} h_i^3$. Then we have

$$X^4 + \sum_{i=0}^{k} h_i^3 Y^3 = Z^4 + \sum_{i=0}^{k} h_i^3 W^3.$$

Putting $Y_i = h_i Y$ and $W_i = h_i W$ now gives the result. □

Tables 2 and 3 shows some h in the cases $k = 2$ and $k = 3$, respectively. All rank computations in Tables 1, 2 and 3 are done using MWRANK software.

h	9	16	28	35	65	91	126	133	189	243
h_1, h_2	1, 2	2, 2	1, 3	2, 3	1, 4	3, 4	1, 5	2, 5	4, 5	3, 6
R	1	1	1	1	2	2	2	1	1	1

Table 2: $R = \text{rank}(E_h)$ for $h = h_1^3 + h_2^3$

h	10	17	29	43	55	73	80	129	134	160
h_1, h_2, h_3	1, 1, 2	2, 1, 2	1, 1, 3	2, 2, 3	1, 3, 3	1, 2, 4	2, 2, 4	1, 4, 4	1, 2, 5	1, 3, 5
R	1	2	1	1	2	1	1	1	1	1

Table 3: $R = \text{rank}(E_h)$ for $h = h_1^3 + h_2^3 + h_3^3$

3. A subfamily of E_h with positive rank

In this section, we attempt to construct a subfamily of E_h with positive rank. This guarantees the existence of positive-rank elliptic curves E_h in Theorem 1.1. Following the method of MacLeod [6], if in the elliptic curve (4) we impose $x = h^2$, then we will have a point of infinite order if

$$h^2 - 17 = t^2. \tag{5}$$

We try to parameterize (5). To do this, we use tangent-chord method. By a simple search we find the solution $h = 9, t = 8$. The line $t = 8 + k(h - 9)$ meets (5) at one further point giving

$$h = \frac{9k^2 - 16k + 9}{k^2 - 1}, \ k \neq \pm 1$$

Substituting h in (4) and defining

$$x = \frac{u}{(k^2 - 1)^4}, \ y = \frac{v}{(k^2 - 1)^6}$$

gives the subfamily of elliptic curves

$$v^2 = u^3 - 9(k^2 - 1)^4 u^2 - 8(k^2 - 1)^6 (9k^2 - 16k + 9)u$$

with rank at least 1.

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References

[1] A. Choudhry, *Symmetric Diophantine equations*, Rocky Mountain J. Math., 34 (2004), 1281–1298.

[2] A. Choudhry, *On the Diophantine equation* $A^4 + hB^4 = C^4 + hD^4$, Indian J. Pure Appl. Math., 26 (1995), 1057–1061.

[3] L.E. Dickson, *History of the theory of numbers, Vol. 2: Diophantine Analysis*, Dover Publication Inc., 2005.

[4] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers, 5th edition*, Clarendon Press, 1979.

[5] A.S. Janfada, A. Abbaspour, *On Diophantine equations* $X^6 + 6Z^3 = Y^6 \pm 6W^3$, Int. J. Pure Appl. Math., 105 (2015), 709–713.

[6] A.J. MacLeod, *A Simple method for high-rank families of elliptic curves with specified torsion*, arXiv:1410.1662v1, 2014.

[7] A. Oppenheim, *The rational integral solution of the equation* $a(x^3 + y^3) = b(u^3 + v^3)$ *and allied Diophantine equations*, Acta Arith., 9 (1964), 221–226.

[8] J.H. Silverman, *The arithmetic of elliptic curves*, Springer, New York, 1986.

[9] J.H. Silverman, J. Tate, *Rational points on elliptic curves,* Springer, New York, 1992.

[10] L.C. Washington, *Elliptic Curves*, *Number theory and cryptography, second edition*, CRC press, 2008.

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Fuzzy congestion in data envelopment analysis

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Abstract. Congestion is an important aspect of economic concepts and data envelopment analysis (DEA) such that in recent years, special attention has been for mathematicians. Several models and methods have been proposed for measuring congestion in non-deterministic environment which cannot show inexact congestion. More important, the congestion is less been investigated with imprecise data which a comprehensive model has not been presented yet. In this paper, our aim is to introduce extension of measuring congestion in DEA with common set of weights and fuzzy inputs and outputs such that, comparisons in models are fuzzy. Finally the proposed models for fuzzy congestion will be shown by examples.

Keywords: data envelopment analysis, fuzzy congestion, common weights.

1. Introduction

Congestion is said to occur when an increase in one or more inputs can be associated with a decrease in one or more outputs, without improving any other inputs or outputs [2]. First, the research on congestion began by Fare and Svensson [5]. Then, it was completed in 1983 and 1985 by the Fare and Grosskopf [6, 7]. They presented a model according to the concept of data envelopment analysis. Another approach was presented by Cooper et al. [3]. Brockett et al. [1] and Cooper et al. [2] developed a new DEA-based approach to measure input congestion. While a significant literature exists on the subject, the two latter methodologies are considered to be fundamental congestion consideration. Another notable method for measuring congestion is Noura et al. [13]. First

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in this paper, we introduce a new methodology for congestion by common set of weights [11, 16] then the congestion of DEA model will be considered with input and output fuzzy number in which fuzzy congestion is proposed by using fuzzy definitions and fuzzy comparisons.

The paper is arranged as follows: in section 2, measuring congestion with common set of weights (CSW) is introduced. In section 3, the background of fuzzy numbers is presented. Congestion with fuzzy numbers is contributed in section 4. Numerical example and conclusion are drawn in section 5 and 6 respectively.

2. Measuring congestion with CSW

Based on Noura et al. methodology [14], we proposed the following multi objective linear programming (MOLP) with common set of weights (CSW).

$$\begin{aligned} Min \quad & v^t x_j + v_0 - u^t y_j, && j = 1, \ldots, n \\ s.t. \quad & v^t x_j + v_0 - u^t y_j \geq 0, && j = 1, \ldots, n \\ & u^t \geq 1_s \epsilon, \qquad v^t \geq 1_m \epsilon. \end{aligned} \tag{1}$$

Equal weights method is applied to solve the above MOLP as follows:

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} (v^t x_j + v_0 - u^t y_j) \\ s.t. \quad & v^t x_j + v_0 - u^t y_j \geq 0, \qquad j = 1, \ldots, n \\ & u^t \geq 1_s \epsilon, \qquad v^t \geq 1_m \epsilon. \end{aligned} \tag{2}$$

$\Rightarrow$

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} (v^t x_j + v_0 - u^t y_j) \\ s.t. \quad & v^t x_j + v_0 - u^t y_j - \Delta_j = 0, \qquad j = 1, \ldots, n \\ & u^t \geq 1_s \epsilon, \qquad v^t \geq 1_m \epsilon \\ & \Delta_j \geq 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{3}$$

From $v^t x_j + v_0 - u^t y_j - \Delta_j = 0$ we get $v^t x_j + v_0 - u^t y_j = \Delta_j$ so

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} \Delta_j \\ s.t. \quad & v^t x_j + v_0 - u^t y_j - \Delta_j = 0, \qquad j = 1, \ldots, n \\ & u^t \geq 1_s \epsilon, \qquad v^t \geq 1_m \epsilon \\ & \Delta_j \geq 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{4}$$

Above model is similar to Liu Peng's model [12] and Noura Hoseini's model [15]. Now we suppose $(u^*, v^*)^t$ will be the optimal solution of problem (4),

which is called common set of weights (CSW) for ranking and comparing DMUs. According to achieved CSW, the efficiency score of $DMU_j (j = 1, \ldots, n)$ will be $\varphi_j^* = \frac{v^{*t} x_j + v_0^*}{u^{*t} y_j}$. If φ_j^* is equal to one then the DMU under evaluation is efficient.

Now as Noura et al. [13], the efficient set of DMUs (E) is defined as follows:

$$E = \{j : \varphi_j^* = 1\}. \tag{5}$$

The highest value in each input among DMUs of E for all components is introduced with x_i^*.

$$x_i^* = \max\{x_{ij} : j \in E\}, \quad i = 1, \ldots, m. \tag{6}$$

So the following revised definition is suggested to identifying congestion.

Definition 2.1. *Congestion in DMUo eventually occurs if for the optimal solution of DMUo* (φ_o^*)*, the following condition is satisfied:*
$\varphi_o^* > 1$*, and there is at least one* $x_{io} > x_i^*, i = 1, \ldots, m$.

The amount of congestion in the ith input of DMUo is shown with s_{io}^c as follow:

$$s_{io}^c = x_{io} - x_i^*. \tag{7}$$

The sum of all s_{io}^c is the amount of congestion in DMUo.

$$s_o^c = \sum_{i=1}^{m} s_{io}^c. \tag{8}$$

Congestion does not present in DMUo when $x_{io} \leq x_i^*$ or $s_{io}^c = 0$ for all $i = 1, \ldots, m$.

3. Fuzzy background

In this section some definition have been given for better understanding of contains of the next section. Several definitions of LR-fuzzy numbers have been published. All of them are variations on the original definition by Dubios and Parade [4]. In this paper, we will use the following definition.

LR-fuzzy number: A fuzzy number $(\tilde{A})$ is called LR-fuzzy number if and only if it has the following membership function.

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \\ R\left(\frac{x-m}{\beta}\right), & x > m. \end{cases} \tag{9}$$

Where L and R are non-increasing functions from R^+ to [0,1] such that L(0)=R(0)=1 and it is shown to $(m, \alpha, \beta)_{LR}$.

Triangular fuzzy number. LR-fuzzy number is called triangular fuzzy number if and only if $L(x) = R(x) = 1 - x$.

Addition and subtraction of triangular fuzzy numbers. Assume $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)$ then

$$\begin{aligned} \tilde{A}_1 + \tilde{A}_2 &= (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2), \\ \tilde{A}_1 - \tilde{A}_2 &= (m_1 - m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2). \end{aligned} \tag{10}$$

In what follow, we give some results for triangular fuzzy numbers that belong to the same family and have different shapes.

Comparison of fuzzy numbers by dubois and paraden [4]. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two fuzzy numbers. Then, $\tilde{A}_1 > \tilde{A}_2 \Leftrightarrow \tilde{A}_1 \cup \tilde{A}_2 = \tilde{A}_1$, where $\tilde{A}_1 \cup \tilde{A}_2$ represents the maximum of those fuzzy numbers.

Comparison of fuzzy numbers by ramik and rimanek [17]. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two triangular fuzzy numbers. Then

$$\tilde{A}_1 \leq \tilde{A}_2 \Leftrightarrow m_1 \leq m_2, m_1 - \alpha_1 \leq m_2 - \alpha_2, m_1 + \beta_1 \leq m_2 + \beta_2. \tag{11}$$

Proposition ([17]). Let $\tilde{A}_1$ and $\tilde{A}_2$ be two fuzzy numbers. Then, $\tilde{A}_1 \cup \tilde{A}_2 = \tilde{A}_1$ if and only if $inf\{s : \mu_{\tilde{A}_1}(s) \geq w\} \geq inf\{t : \mu_{\tilde{A}_2}(t) \geq w\}$ and $sup\{s : \mu_{\tilde{A}_1}(s) \geq w\} \geq sup\{t : \mu_{\tilde{A}_2}(t) \geq w\}$ hold for all grades of membership w in [0, 1].

Comparison of fuzzy numbers by tanaka et al. [8] : Let $\tilde{A}_1$ and $\tilde{A}_2$ be two fuzzy numbers and h a real number, h$\in$[0, 1], Then $\tilde{A}_1 \leq^h \tilde{A}_2$ if and only if $infs : \mu_{\tilde{A}_1}(s) \geq w \leq inf\{t : \mu_{\tilde{A}_2}(t) \geq w\}$ and $sup\{s : \mu_{\tilde{A}_1}(s) \geq w\} \leq sup\{t : \mu_{\tilde{A}_2}(t) \geq w\}$. Hence, for two LR-fuzzy numbers $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)$ at a given possibility level h; $\tilde{A}_1 \leq^h \tilde{A}_2$, then it's necessary to check $m_1 - \alpha_1 L^{-1}(w) \leq m_2 - \alpha_2 L^{-1}(w)$ and $m_1 + \beta_1 R^{-1}(w) \leq m_2 + \beta_2 R^{-1}(w), w \in [h, 1]$.

4. Fuzzy congestion

In this section, we assume that inputs and outputs are triangular fuzzy number in model 2. So suppose $\tilde{x}_{ij} = (\alpha_{ij}^L, x_{ij}, \alpha_{ij}^U), \tilde{y}_{rj} = (\beta_{rj}^L, y_{rj}, \beta_{rj}^U)$ in the possibility level of h; $0 \leq h \leq 1$ therefore

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} (v^t \tilde{x}_j + v_0 - u^t \tilde{y}_j), \\ s.t. \quad & v^t \tilde{x}_j + v_0 - u^t \tilde{y}_j \geq 0, \qquad j = 1, \ldots, n \\ & u^t \geq 1_s \epsilon, \qquad v^t \geq 1_m \epsilon. \end{aligned} \tag{12}$$

According to the Guo et al. [8, 9, 10] the following model obtains.

$$
\begin{aligned}
& Min \quad \sum_{j=1}^{n}(v^t x_j - (1-h)v^t\alpha_j^L + v_0 - u^t y_j + (1-h)u^t\beta_j^L), \\
& s.t. \quad v^t x_j - (1-h)v^t\alpha_j^L + v_0 - u^t y_j + (1-h)u^t\beta_j^L \geq 0, j = 1,\ldots,n \\
& \qquad v^t x_j + (1-h)v^t\alpha_j^U + v_0 - u^t y_j - (1-h)u^t\beta_j^U \geq 0, j = 1,\ldots,n \\
& \qquad u^t \geq 1_s\epsilon, \qquad v^t \geq 1_m\epsilon.
\end{aligned} \tag{13}
$$

The fuzzy efficiency of an evaluated DMU_j with the triangular fuzzy input vector $\tilde{x}_{ij} = (\alpha_{ij}^L, x_{ij}, \alpha_{ij}^U)$ and output vector $\tilde{y}_{rj} = (\beta_{rj}^L, y_{rj}, \beta_{rj}^U)$ is defined a triangular fuzzy number $\tilde{e}_j = (L_j, \theta_j, R_j)$ as follows:

$$\phi_j = \frac{v^{*t}x_j + v_0^*}{u^{*t}y_j} \quad \theta_j = \frac{1}{\phi_j}, \tag{14}$$

$$L_j = \theta_j - \frac{u^{*t}y_j - (1-h)u^{*t}\beta_j^L}{v^{*t}x_j + (1-h)v^{*t}\alpha_j^U + v_0^*}, \tag{15}$$

$$R_j = \frac{u^{*t}y_j + (1-h)u^{*t}\beta_j^U}{v^{*t}x_j - (1-h)v^{*t}\alpha_j^L + v_0^*} - \theta_j, \tag{16}$$

where v^* and u^* are obtained coefficient vectors from (13). The DMU_j with $\theta_j + R_j > 1$ for the h possibility level is called an h-possibilistic efficient DMU. On the contrary, the DMU with $\theta_j + R_j \leq 1$ for the h possibility level is called an h-possibilistic inefficient DMU.

Now, the fuzzy efficient set (E) is defined as follows:

$$E = \{j : \theta_j + R_j > 1\}. \tag{17}$$

Also, we obtain the highest value in each input for all components among the DMUs in set E based on fuzzy comparison as $\tilde{x}_i^*$.

Moreover, in the crisp mode Congestion occurs when an increase in one or more inputs can be associated with a decrease in one or more outputs, without improving any other inputs or outputs. So, with extended this definition of congestion in the fuzzy mode for DMUo, we have:

$$
\begin{aligned}
& \text{If} \quad \tilde{X}_o \to \tilde{X}_o + \tilde{\Delta X}; \tilde{\Delta X} \geq 0, \tilde{\Delta X} \neq 0, \\
& \text{Then} \quad \tilde{Y}_o \to \tilde{Y}_o - \tilde{\Delta Y}; \tilde{\Delta Y} \geq 0, \tilde{\Delta Y} \neq 0.
\end{aligned} \tag{18}
$$

or

$$
\begin{aligned}
& \text{If} \quad \tilde{X}_o \to \tilde{X}_o - \tilde{\Delta X}; \tilde{\Delta X} \geq 0, \tilde{\Delta X} \neq 0, \\
& \text{Then} \quad \tilde{Y}_o \to \tilde{Y}_o + \tilde{\Delta Y}; \tilde{\Delta Y} \geq 0, \tilde{\Delta Y} \neq 0.
\end{aligned} \tag{19}
$$

Therefore, with respect to fuzzy increasing and fuzzy decreasing (based on fuzzy comparison), definition of fuzzy congestion is exactly suggested as follows.

Definition 4.1. *Fuzzy congestion eventually occurs if for every optimal solution of DMUj, the following condition is satisfied:* $\theta_j + R_j \leq 1$, *and there is at least one* $\tilde{x}_{ij} > \tilde{x}_i^*, i = 1, \ldots, m$.

The amount of fuzzy congestion in the ith input of DMUj is shown by $\tilde{S}_{ij}^c$ as follow:

$$\tilde{S}_{ij}^c = \tilde{x}_{ij} - \tilde{x}_i^*. \tag{20}$$

The sum of all $\tilde{S}_{ij}^c$ is the amount of fuzzy congestion in DMUj.

$$\tilde{S}_j^c = \sum_{i=1}^{m} \tilde{S}_{ij}^c. \tag{21}$$

Fuzzy congestion does not present when $\tilde{x}_{ij} \leq \tilde{x}_i^*$ or $\tilde{S}_{ij}^c = 0$ for all $i = 1, \ldots, m$.

5. Numerical examples

Example 5.1. Table 1 shows five DMUs where input and output are symmetrical triangular fuzzy numbers. The fuzzy efficiencies of DMUs were obtained by model 13 for the different h values (table 2).

With considering table 2, we have $h = 0$, $E = \{DMU_1, DMU_2, DMU_3, DMU_4\}$, $\forall DMU \in E : \tilde{x}_1^* = \max Input1 = (1, 5, 1)$. For $DMU_5, \tilde{x}_{15} = (0.5, 5, 0.5) \leq \tilde{x}_1^*$ so there is no congestion.

$h = 0.25$, $E = \{DMU_1, DMU_2, DMU_4\}$, $\forall DMU \in E : \tilde{x}_1^* = \max Input1 = (1, 5, 1)$. For DMU_3, $\tilde{x}_{13} = (0.6, 3, 0.6) \leq \tilde{x}_1^*$ so there is no congestion. For DMU_5, $\tilde{x}_{15} = (0.5, 5, 0.5) \leq \tilde{x}_1^*$ so there is no congestion.

$h = 0.5, 0.75, 1$, $E = \{DMU_2\}$, $\forall DMU \in E : \tilde{x}_1^* = \max Input1 = (0.5, 3, 0.5)$.
For $DMU_1, \tilde{x}_{11} = (0.5, 2, 0.5) \leq \tilde{x}_1^*$ so there is no congestion.
For $DMU_3, \tilde{x}_{13} = (0.6, 3, 0.6) \leq \tilde{x}_1^*$ so there is no congestion.
For DMU_4, $\tilde{x}_{14} = (1, 5, 1) > \tilde{x}_1^* = (0.5, 3, 0.5)$, so DMU_4 has congestion and we have, $\tilde{S}_{14}^c = (0.5, 5, 0.5) - (0.5, 3, 0.5) = (1.5, 2, 1.5) = \tilde{S}_4^c$.
For DMU_5, $\tilde{x}_{15} = (0.5, 5, 0.5) > \tilde{x}_1^* = (0.5, 3, 0.5)$, so DMU_5 has congestion and we have, $\tilde{S}_{15}^c = (1, 5, 1) - (0.5, 3, 0.5) = (1, 2, 1) = \tilde{S}_5^c$.

Example 5.2. Table 3 shows eight DMUs where inputs and output are non-symmetrical triangular fuzzy numbers. The fuzzy efficiencies of DMUs were obtained by model 13 for the different h values which have listed in table 4.

With considering table 4, we have

For $h = 0, 0.25, 0.5, 0.75$, $E = \{DMU_3, DMU_4, DMU_8, \forall\ DMU \in E : \tilde{x}_1^* =$ max Input $1 = (3, 6, 1)$.

Table 1: DMUs with single fuzzy input and single fuzzy output. Source: [10]

DMUs	Input s	Outputs
DMU1	(0.5,2.0,0.5)	(0.3,1.0,0.3)
DMU2	(0.5,3.0,0.5)	(0.7,3.0,0.7)
DMU3	(0.6,3.0,0.6)	(0.6,2.0,0.4)
DMU4	(1.0,5.0,1.0)	(1.0,4.0,1.0)
DMU5	(0.5,5.0,0.5)	(0.2,2.0,0.2)

Table 2: Fuzzy efficiency ($\tilde{e}_j = (L_j, \theta_j, R_j)$) of DMUs with different h values for example 1

h	DMU1	DMU2	DMU3	DMU4	DMU5
0	(0.33, 0.63, 0.81)	(0.38, 1, 0.61)	(0.25, 0.67, 0.44)	(0.27, 0.69, 0.45)	(0.06, 0.34, 0.09)
0.25	(0.27, 0.63, 0.51)	(0.30, 1, 0.42)	(0.20, 0.67, 0.30)	(0.22, 0.69, 0.31)	(0.05, 0.34, 0.07)
0.5	(0.12, 0.48, 0.14)	(0.18, 0.97, 0.21)	(0.12, 0.65, 0.14)	(0.16, 0.78, 0.20)	(0.03, 0.39, 0.04)
0.75	(0.06, 0.49, 0.07)	(0.09, 0.98, 0.11)	(0.07, 0.66, 0.06)	(0.08, 0.79, 0.10)	(0.02, 0.40, 0.02)
1	(0.00, 0.50, 0.00)	(0.00, 1.00, 0.00)	(0.00, 0.67, 0.00)	(0.00, 0.80, 0.00)	(0.00, 0.40, 0.00)

For $DMU_1,\ \tilde{x}_{11} = (0.5, 0.5, 0.5) \leq \tilde{x}_1^*$, so there is no congestion in Input 1.

For $DMU_2,\ \tilde{x}_{12} = (2, 4, 2) \leq \tilde{x}_1^*$, so there is no congestion in Input 1.

For $DMU_5,\ \tilde{x}_{15} = (4, 8, 4) > \tilde{x}_1^* = (3, 6, 1)$, so DMU_5 has congestion in Input 1 and we have $\tilde{S}_{15}^c = (4, 8, 4) - (3, 6, 1) = (5, 2, 7)$.

For $DMU_6,\ \tilde{x}_{16} = (3, 6, 1) \leq \tilde{x}_1^*$, so there is no congestion in Input 1.

For $DMU_7,\ \tilde{x}_{17} = (4, 7, 2) > \tilde{x}_1^* = (3, 6, 1)$, so DMU_7 has congestion in Input 1 and we have $\tilde{S}_{17}^c = (4, 7, 2) - (3, 6, 1) = (5, 1, 5)$, $\forall\ DMU \in E : \tilde{x}_2^* = \max$ Input $2 = (3, 6, 1)$.

For $DMU_1,\ \tilde{x}_{21} = (2, 4, 2) \leq \tilde{x}_2^*$, so there is no congestion in Input 2.

For $DMU_2,\ \tilde{x}_{22} = (0.5, 0.5, 0.5) \leq \tilde{x}_2^*$, so there is no congestion in Input 2.

For $DMU_5,\ \tilde{x}_{25} = (3, 6, 1) \leq \tilde{x}_2^*$, so there is no congestion in Input 2.

For $DMU_6,\ \tilde{x}_{26} = (4, 8, 4) > \tilde{x}_2^*$, so DMU_6 has congestion in Input 2 and we have $\tilde{S}_{26}^c = (4, 8, 4) - (3, 6, 1) = (5, 2, 7)$.

For $DMU_7,\ \tilde{x}_{27} = (4, 7, 2) > \tilde{x}_2^*$, so DMU_7 has congestion in Input 2 and we have $\tilde{S}_{27}^c = (4, 7, 2) - (3, 6, 1) = (5, 1, 5)$

$\Rightarrow \tilde{S}_5^c = (5, 2, 7), \tilde{S}_6^c = (5, 2, 7), \tilde{S}_7^c = (5, 1, 5) + (5, 1, 5) = (10, 2, 10)$.

For $h = 1, E = \{DMU_3, DMU_8\},\ \forall DMU \in E : \tilde{x}_1^* = \max$ Input $1 = (2, 4, 2)$.

For $DMU_1,\ \tilde{x}_{11} = (0.5, 0.5, 0.5) \leq \tilde{x}_1^*$, so there is no congestion in Input 1.

For $DMU_2,\ \tilde{x}_{12} = (2, 4, 2) \leq \tilde{x}_1^*$, so there is no congestion in Input 1.

For $DMU_4,\ \tilde{x}_{14} = (3, 6, 1) > \tilde{x}_1^*$, so DMU_4 has congestion in Input 1 and we have $\tilde{S}_{14}^c = (3, 6, 1) - (2, 4, 2) = (3, 2, 5)$.

For $DMU_5,\ \tilde{x}_{15} = (4, 8, 4) > \tilde{x}_1^*$, so DMU_5 has congestion in Input 1 and we have $\tilde{S}_{15}^c = (4, 8, 4) - (2, 4, 2) = (6, 4, 6)$.

Table 3: DMUs with fuzzy inputs and single fuzzy output. Source: [18]

DMUs	Input 1	Input 2	Output
DMU1	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	(2, 4, 2)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
DMU2	(2, 4, 2)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
DMU3	(2, 4, 2)	(2, 4, 2)	(3, 5, 5)
DMU4	(3, 6, 1)	(3, 6, 1)	(4, 7, 2)
DMU5	(4, 8, 4)	(3, 6, 1)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
DMU6	(3, 6, 1)	(4, 8, 4)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
DMU7	(4, 7, 2)	(4, 7, 2)	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
DMU8	(1, 1, 1)	(1, 1, 1)	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

Table 4: Fuzzy efficiency ($\tilde{e}_j = (L_j, \theta_j, R_j)$) of DMUs with different h values for example 2

DMUs	h=0	h=0.25	h=0.5	h=0.75	h=1
DMU1	(0.13,0.13,0.47)	(0.12,0.15,0.33)	(0.10,0.17,0.21)	(0.07,0.19,0.11)	(0.00,0.22,0.00)
DMU2	(0.13,0.13,0.47)	(0.12,0.15,0.33)	(0.10,0.17,0.21)	(0.07,0.19,0.11)	(0.00,0.22,0.00)
DMU3	(0.55,0.75,2.25)	(0.48,0.78,1.51)	(0.39,0.85,0.92)	(0.24,0.92,0.43)	(0.00,1.00,0.00)
DMU4	(0.44,0.70,1.10)	(0.36,0.73,0.72)	(0.26,0.77,0.43)	(0.15,0.81,0.20)	(0.00,0.86,0.00)
DMU5	(0.04,0.04,0.13)	(0.36,0.04,0.08)	(0.03,0.05,0.05)	(0.02,0.05,0.02)	(0.00,0.05,0.00)
DMU6	(0.04,0.04,0.13)	(0.36,0.04,0.08)	(0.03,0.05,0.05)	(0.02,0.05,0.02)	(0.00,0.05,0.00)
DMU7	(0.04,0.04,0.16)	(0.36,0.04,0.10)	(0.03,0.05,0.05)	(0.02,0.05,0.02)	(0.00,0.05,0.00)
DMU8	(0.15,0.15,∞)	(0.17,0.19,6.81)	(0.19,0.26,2.74)	(0.23,0.42,1.24)	(0.00,1.00,0.00)

For DMU_6, $\tilde{x}_{16} = (3, 6, 1) > \tilde{x}_1^*$, so DMU_6 has congestion in Input 1 and we have $\tilde{S}_{16}^c = (3, 6, 1) - (2, 4, 2) = (3, 2, 5)$.

For DMU_7, $\tilde{x}_{17} = (4, 7, 2) > \tilde{x}_1^*$, so DMU_7 has congestion in Input 1 and we have $\tilde{S}_{17}^c = (4, 7, 2) - (2, 4, 2) = (4, 3, 6)$, $\forall\ DMU \in E : \tilde{x}_2^* = \max$ Input $2 = (2, 4, 2)$.

For DMU_1, $\tilde{x}_{21} = (2, 4, 2) \leq \tilde{x}_2^*$, so there is no congestion in Input 2.

For DMU_2, $\tilde{x}_{22} = (0.5, 0.5, 0.5) \leq \tilde{x}_2^*$, so there is no congestion in Input 2.

For DMU_4, $\tilde{x}_{24} = (3, 6, 1) > \tilde{x}_2^*$, so DMU_4 has congestion in Input 2 and we have $\tilde{S}_{24}^c = (3, 6, 1) - (2, 4, 2) = (3, 2, 5)$.

For DMU_5, $\tilde{x}_{25} = (3, 6, 1) > \tilde{x}_2^*$, so DMU_5 has congestion in Input 2 and we have $\tilde{S}_{25}^c = (3, 6, 1) - (2, 4, 2) = (3, 2, 5)$.

For DMU_6, $\tilde{x}_{26} = (4, 8, 4) > \tilde{x}_2^*$, so DMU_6 has congestion in Input 2 and we have $\tilde{S}_{26}^c = (4, 8, 4) - (2, 4, 2) = (6, 4, 6)$.

For DMU_7, $\tilde{x}_{27} = (4, 7, 2) > \tilde{x}_2^*$, so DMU_7 has congestion in Input 2 and we have $\tilde{S}_{27}^c = (4, 7, 2) - (2, 4, 2) = (4, 3, 6)$.

$\Rightarrow \tilde{S}_4^c = (3, 2, 5) + (3, 2, 5) = (6, 4, 10, \tilde{S}_5^c = (6, 4, 6) + (3, 2, 5) = (9, 6, 11), \tilde{S}_6^c = (3, 2, 5) + (6, 4, 6) = (9, 6, 11), \tilde{S}_7^c = (4, 3, 6) + (4, 3, 6) = (8, 6, 12)$.

Table 5: Results of example2 by Rostami et al.'s method [18]

DMUs	h'_{BCC}	Congestion
DMU1	1	No
DMU2	1	No
DMU3	1	No
DMU4	1	No
DMU5	10	Yes
DMU6	10	Yes
DMU7	10	Yes
DMU8	1	No

Table 6: Results of example2 by proposed method

DMUs	Measure of fuzzy congestion($\tilde{S}^c$	Congestion
DMU1	0	No
DMU2	0	No
DMU3	0	No
DMU4	0	No
DMU5	(5,2,7)	Yes
DMU6	(5,2,7)	Yes
DMU7	(10,2,10)	Yes
DMU8	1	No

Results of measuring congestion by proposed method and Rostami et al.'s (2011) method are shown in tables 5 and 6. As can be observed, two methods get similar results for identifying congestion but Rostami et al.'s (2011) method don't determine measure of congestion while the proposed method is suggested measure of congestion by a fuzzy number.

6. Conclusion

In this paper, we proposed a new method for calculating fuzzy congestion with applying fuzzy rectangular inputs and outputs based on Noura et al.'s method and Common weights. The main advantage of the proposed method is to solve one linear programming in order to measuring fuzzy congestion for all DMUs. Therefore, calculation of this method is very low and it provides an easy understandable knowledge of fuzzy congestion. Moreover, congestion is less been investigated with imprecise data while the proposed method suggests measure of congestion by a fuzzy number. This important was shown by examples and compared with other methods.

References

[1] P.L. Brockett, W.W. Cooper, H.C. Shin, Y. Wang, *Inefficiency and congestion in Chinese production before and after the 1978 economic reforms*, Socio-Economic Planning Sciences, 32 (1978), 1-20.

[2] W.W. Cooper, L.M. Seiford, J. Zhu, *A unified additive model approach for evaluating inefficiency and congestion whit associated measures in DEA*, Socio-Economic Planning Sciences, 34 (2000), 1-25.

[3] W.W. Cooper, R.G. Thompson, R.M. Thrall, *Extensions and new developments in DEA,* Ann. Oper. Res., 66 (1996), 3-45.

[4] D. Dubois, H. Prade, *Fundamentals of fuzzy sets*, Kluwer Academic, Dordrecht, 2000.

[5] R. Fre, L. Svensson, *Congestion of production factors*, Econometrica, 48 (1980), 1745-53.

[6] R. Fre, S. Grosskopf, *Measuring congestion in production*, Journal of Economics, 43 (1983), 257-71.

[7] R. Fre, S. Grosskopf, C. Lovell, *The measurement of efficiency of production*, Kluwer Nijhoff, Boston, 1985.

[8] P. Guo, H. Tanaka, *Fuzzy DEA: a perceptual evaluation method*, Fuzzy set and systems, 119 (2001), 149-160.

[9] P. Guo, H. Tanaka, *Decision making based on fuzzy data envelopment analysis*, to appear in Intelligent Decision and Policy Making Support Systems In D. Ruan, K. Meer, (Eds.), Springer, Berlin/Heidelberg, 2008, 3954.

[10] P. Guo, *Fuzzy data envelopment analysis and its application to location problems*, Information Sciences, 179 (2009), 820829.

[11] A.R. Hajihosseini, A.A. Noura, F. Hosseinzadeh Lotfi, *A New approach to measuring congestion in DEA with common weights*, Indian Journal of Science and Technology, 8 (2015), 574-580.

[12] F.H.F. Liu, H.H. Peng, *Ranking of units on the DEA frontier with common weights*, Computers Operations Research, 35 (2008), 1624-1637.

[13] A.A. Noura, F. Hosseinzadeh Lotfi, G.R. Jahanshahloo, S. Fanati Rashidi, B.R. Parker, *A new method for measuring congestion in data envelopment analysis*, Socio-Economic Planning Sciences, 44 (2010), 240-246.

[14] A.A. Noura, A. Payan, M. Nozohour, *Improvement of ranking method based on effectiveness of units in society by common weights approach in DEA*, Wseas Transactions on Mathematics, 11 (2012), 742-750.

[15] A.A. Noura, E. Hoseini, *Measuring congestion in data envelopment analysis with common weights*, Int. J. Data Envelopment Analysis, 3 (2013), 125-133.

[16] A.A. Noura, F. Hosseinzadeh Lotfi, A.R. Hajihosseini, *Congestion in DEA under Weight Restriction Using Common Weights*, Journal of Mathematical Extension, 10 (2016), 121-133.

[17] J. Ramik, J. Rimanek, *Inequality relation between fuzzy numbers and its use in fuzzy optimization*, Fuzzy Sets and Systems, 16 (1985), 123138.

[18] M. Rostamy-Malkhalifeh, S. Ebrahimkhani Ghazi, H. Sale, N. Ebrahimkhani Ghazi, *Congestion in DEA model with fuzzy data,* International Journal of Applied Operational Research, 1 (2011), 49-56.

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On asymptotically lacunary statistical equivalent (Wijsman sense) set sequences via ideal and modulus function

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Abstract. For any non-trivial ideal $\mathcal{I} \subseteq \wp(\mathbb{N})$, and any non-empty closed subset $A_k \subseteq X$, where (X, ρ) is a metric space. Let f be a modulus function, The objective of this paper is to introduce and study the new notation by using a modulus function, $\mathcal{I}(W_L)$, $\mathcal{I}(fW)$, $\mathcal{I}[WN]_\theta^L$, $\mathcal{I}(Wf^L)$, $\mathcal{I}(_\theta^W f)$, $\mathcal{I}(WS^L)$ and $\mathcal{I}(WS_\theta^L)$. Which are natural combinations of the definitions for asymptotically lacunary equivalent (*Wijsman sense*). In addition, some relations among these new notions are also obtained.

1. Introduction and preliminaries

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [7] (see, also [26]). as follows: if K is a subset of natural numbers $\mathbb{N}$, K_n will denote the set $\{\kappa \in \mathrm{K} : \kappa \leqslant n\}$ and $|\mathrm{K}_n|$ will denote the cardinality of K_n. Natural density of K [9] is given by

$$\delta(\mathrm{K}) := \lim_n \frac{1}{n}|\mathrm{K}_n|$$

, if $\bar{\delta}(\mathrm{K}) = \underline{\delta}(\mathrm{K})$ then we say that the natural density of K exists and it is denoted simply by $\delta(\mathrm{K})$. A sequence (x_n) of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $\mathrm{K}(\varepsilon) = n \in N : |x_n - L| \geq \varepsilon$ has natural density zero. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [26]. In [3], Borwein introduced and studied strongly summable functions. The concept of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades in 2012. Similar to this concept, the concept of Wijsman lacunary statistical convergence was presented by Ulusu and Nuray in 2012. Kostyrko et al. [11] generalized statistical convergence with the help of an admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of positive integers and called it $\mathcal{I}$-convergence. Quite recently, Savaş et al. [25] unified the notions of statistical convergence and $\mathcal{I}$-convergence to introduce new concepts of $\mathcal{I}$-statistical convergence.

In order to compare the rate of growth of two sequences, Marouf [16] defined asymptotically equivalent sequences of real numbers and studied its relations with certain matrix-transformed sequences. In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, many authors have shown their interest to solve different problems arising in this area. In this work, we define asymptotically equivalent sequences using lacunary sequences, ideals, Wijsman sense and a modulus function and obtain some revelent connections between these notions.

2. Preliminaries

Definition 2.1 ([16]). *Two nonnegative sequences $x = (x_n)$ and $y = (y_n)$ are said to be asymptotically equivalent to multiple l provided that*

$$\lim_{\kappa\to\infty} \frac{x_n}{y_n} = L,$$

(denoted by $x \sim y$) and simply asymptotically equivalent if L=1.

Definition 2.2 ([21]). *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$.*

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{\kappa \le n : \left|\frac{x_n}{y_n} - L\right| \ge \varepsilon\right\}\right| = 0.$$

(*denoted by $x \sim^{S_L} y$*)

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $K_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Let, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. Using lacunary sequences, Fridy et al. [8] defined S_θ-convergence, a generalized statistical convergence as follows.

Definition 2.3 ([22]). *Let $\theta = k_r$ be a lacunary sequence, the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$.*

$$\lim_{r} \frac{1}{h_r}\left|\left\{\kappa \in I_r : \left|\frac{x_n}{y_n} - L\right| \ge \varepsilon\right\}\right| = 0.$$

(*denoted by $x \sim^{S_\theta^L} y$*) *and simply asymptotically lacunary statistically equivalent if L=1.*

Definition 2.4 ([22]). *Let $\theta = k_r$ be a lacunary sequence, the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are strongly asymptotically lacunary equivalent of multiple L provided that.*

$$\lim_{r} \frac{1}{h_r} \sum_{\kappa\in I_r} \left|\frac{x_n}{y_n} - L\right| = 0.$$

(*denoted by* $x \sim^{N_\theta^L} y$) *and strongly simply asymptotically lacunary equivalent if* $L = 1$.

Let (X, σ) be a metric space. For any point $x \in X$ and any nonempty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \sigma(x, A).$$

Definition 2.5 ([1]). *Let* (X, σ) *be a metric space. For any nonempty closed subset* A, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman convergent to* A *if*

$$\lim_{k \to \infty} d(x, A_k) = d(x, A),$$

for each $x \in X$ *In this case we write* $W - \lim A_k = A$.

Definition 2.6 ([18]). *Let* (X, σ) *a metric space. For any nonempty closed subset* A, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman statistically convergent to* A *if* $\{d(x, A_k)\}$ *is statistically convergent to* $d(x, A)$*; that is, for* $\varepsilon > 0$ *and for each* $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{\kappa \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ *or* $A_k \to A(Ws)$.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades.

Definition 2.7 ([18]). *Let* (X, σ) *a metric space. For any nonempty closed subset,* $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is bounded if*

$$\sup_k d(x, A_k) < \infty,$$

for each $x \in X$. *In this case we write* $\{A_k\} \in L_\infty$.

Definition 2.8 ([27]). *Let* (X, σ) *a metric space and let* $\theta = \{K_r\}$ *be a lacunary sequence. For any nonempty closed subset* A, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman statistically convergent to* A *if* $\{d(x, A_k)\}$ *is statistically convergent to* $d(x, A)$*; that is, for* $\varepsilon > 0$ *and for each* $x \in X$,

$$\lim_{h_r \to \infty} \frac{1}{h_r} |\{\kappa \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim_W = A$ *or* $A_k \to A(WS_\theta)$.

Definition 2.9 ([10, 28]). *family* $I \in \wp(X)$ *is said to be an ideal in* X *if*

1. $\emptyset \in \mathcal{I}$;

2. $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$;

3. $A \in \mathcal{I}$ *and* $B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$.

Definition 2.10. *A non-empty family* $\mathcal{F} \in \wp(X)$ *is said to be a filter in* X *if*

1. $\emptyset \notin \mathcal{F}$;

2. $A \in \mathcal{F}$ *and* $A \subset B$ *implies* $B \in \mathcal{F}$;

3. $A \in \mathcal{F}$ *and* $B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if it contains all the singleton sets. Moreover, if $\mathcal{I}$ is a non-trivial ideal on X, then $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on X and conversely. The filter $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Using ideals, Kostyrko et al. [11] defined $\mathcal{I}$-convergence, a stronger convergence in a metric space, whereas Dass et al. [6] unified this idea with statistical convergence for real sequences.

Definition 2.11 ([11])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and* (X, σ) *be a metric space. A sequence* $x \in (x_k)$ *in* X *is said to be* $\mathcal{I}$*-convergence to* ξ *if for each* $\epsilon > 0$*, the set*

$$A(\epsilon) = \left\{ K \in \mathbb{N} : \rho(x_k, \xi) \geq \epsilon \right\} \in \mathcal{I}.$$

Definition 2.12 ([25])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and* (X, σ) *be a metric space. A sequence* $x \in (x_k)$ *in* X *is said to be* $\mathcal{I}$*-statistically convergence to* ξ *provided that for each* $\epsilon > 0$ *and every* $\delta > 0$*,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{\kappa \leq n : |x_k - \xi| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}S - \lim_{k \to \infty} x_k = \xi$ *or* $x_k \to \xi(\mathcal{I} - S)$

Definition 2.13 ([13])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *the two non-negative sequences* $x = (x_k)$ *and* $y = (y_k)$ *are said to be asymptotically* $\mathcal{I}$*-*$[C, 1]$*-equivalent of multiple* L *provided that for each* $\delta > 0$*,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{\kappa=1}^{n} \left| (\frac{x_n}{y_n}) - L \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $x \sim^{[C,1]^{L_{\mathcal{I}}}} y$*) and simply asymptotically* $\mathcal{I}$*-*$[C, 1]$*-equivalent if* $L = 1$*.*

Definition 2.14 ([12])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and let* $\theta = \{K_r\}$ *be a lacunary sequence. The non-negative sequences* $x = (x_k)$ *and* $y =$

(y_k) are said to be asymptotically lacunary statistical equivalent multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\delta \geq 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_n}{y_n} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $x \sim^{\mathcal{I}(S_\theta)} y$) and simply asymptotically lacunary statistical equivalent with respect to the ideal if $L = 1$.

Definition 2.15 ([13]). *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$. The non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}$-statistical equivalent multiple L provided that for each $\epsilon > 0$ and $\delta \geq 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| (\frac{x_n}{y_n}) - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $x \sim^{S^{L(\mathcal{I})}} y$) and simply asymptotically $\mathcal{I}$-statistical equivalent if $L = 1$.

Nakano [17] introduced the notion of a modulus function in 1953 as follows. By a modulus function, we mean a function f from $[0, \infty)$ to $[0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$;
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;
3. f is increasing;
4. f is continuous from the right at 0;

It follows that f must be continuous on $[0, 1)$. A modulus may be bounded or unbounded. Many authors, including Connor [5], Kolk [14], Maddox [15], Ö ztürk et al. [20], Pehlivan et al. [23, 24] and many others used a modulus f to construct some sequence spaces. Recently, Bilgin [4] used modulus function to define some notions of asymptotically equivalent sequences and studied some of their connections.

Definition 2.16 ([19]). *Let (X, σ) be a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically equivalent ($Wijsman\ sense$) if for each $x \in X$,*

$$\lim_{\kappa \to \infty} \frac{d(x, A_k)}{d(x, B_k)} = L,$$

(denoted by $A_k \sim B_k$).

Definition 2.17 ([19]). *Let (X,σ) be a metric space. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically statistically equivalent $(Wijsman\ sense)$ of multiple L if for each $x \in X$,*

$$\lim_n \frac{1}{n}\left|\left\{\kappa \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| = 0$$

$(denoted\ by\ \{A_k\} \sim^{WS_L} \{B_k\})$ and simply asymptotically statistical equivalent $(Wijsman\ sense)$ if $L = 1$.

Definition 2.18 ([19]). *Let (X,σ) be a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically lacunary equivalent $(Wijsman\ sense)$ of multiple L if for each $x \in X$,*

$$\lim_r \frac{1}{h_r}\sum_{\kappa \in I_r}\left|\frac{d(x,A_k)}{d(x,A_k)}\right| = L$$

$(denoted\ by\ \{A_k\} \sim^{WN_\theta^L} \{B_k\})$ and simply asymptotically lacunary equivalent $(Wijsman\ sense)$ if $L = 1$.

Definition 2.19 ([19]). *Let (X,σ) a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically lacunary equivalent $(Wijsman\ sense)$ of multiple L if for each $x \in X$,*

$$\lim_r \frac{1}{h_r}\sum_{\kappa \in I_r}\left|\frac{d(x,A_k)}{d(x,A_k)} - L\right| = 0$$

$(denoted\ by\ \{A_k\} \sim^{[WN]_\theta^L} \{B_k\})$ and simply strongly asymptotically lacunary equivalent $(Wijsman\ sense)$ if $L = 1$.

Definition 2.20 ([19]). *Let (X,σ) a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically lacunarily statistical equivalent $(Wijsman\ sense)$ of multiple L if for each $x \in X$,*

$$\lim_r \frac{1}{h_r}\left|\left\{\kappa \in I_r : \left|\frac{d(x,A_k)}{d(x,A_k)} - L\right| \geq \varepsilon\right\}\right| = 0$$

$(denoted\ by\ \{A_k\} \sim^{WS_\theta^L} \{B_k\})$ and simply asymptotically lacunarily statistical equivalent $(Wijsman\ sense)$ if $L = 1$.

3. Results and discussion

We now consider our main results. We begin with the following definitions.

Definition 3.1. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically equivalent $(Wijsman\ sense)$ of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$) and simply strongly asymptotically equivalent with respect to the ideal $(Wijsman\ sense)$, if $L = 1$.

Definition 3.2. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$ and (X, σ) a metric space. The nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent $(Wijsman\ sense)$ multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\delta \geq 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$) and simply asymptotically statistical equivalent $(Wijsman\ sense)$with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3.3. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space and $\theta = \{K_r\}$ be a lacunary sequence. The nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent $(Wijsman\ sense)$ multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\delta \geq 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ and simply asymptotically lacunary statistical equivalent $(Wijsman\ sense)$with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3.4. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space and $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically lacunary statistically equivalent $(Wijsman\ sense)$ of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$) and simply strongly asymptotically lacunary statistically equivalent $(Wijsman\ sense)$ with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3.5. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space and f be a modulus function. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are f-asymptotically equivalent ($Wijsman\ sense$) of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon \geq 0$,*

$$\left\{k \in \mathbb{N} : f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(f_W)} \{B_k\}$ and simply f-asymptotically equivalent (Wijsman sense) with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3.6. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space and f be a modulus function and $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly f-asymptotically lacunary equivalent ($Wijsman\ sense$) of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon \geq 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(_\theta^W f)} \{B_k\}$ and simply strongly f-asymptotically lacunary equivalent ($Wijsman\ sense$) with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3.7. *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, (X, σ) a metric space. For any nonempty closed subset A_k, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly f-asymptotically equivalent ($Wijsman\ sense$) of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \epsilon\right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$) and simply strongly f-asymptotically equivalent with respect to the ideal ($Wijsman\ sense$), if $L = 1$.

Theorem 3.8 ([2, 24])**.** *Let f be a modulus function and let $0 < \delta < 1$. Then for $y \neq 0$ and each $\left(\frac{x}{y}\right) > \delta$, we have $f\left(\frac{x}{y}\right) \leq \frac{2f(1)}{\delta}\left(\frac{x}{y}\right)$.*

Theorem 3.9. *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, (X, ρ) be a metric space and f be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ and*
2. *$\lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0$ then $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$.*

Proof. (1) Let $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$ and $\epsilon \geq 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. We can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{n}\sum_{k=1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) + \frac{1}{n}\sum_{k=2} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right),$$

where the first summation runs over $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq \delta$, and the complement of first a summation is $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| > \delta$. Moreover, using the definition of the modulus function f , we have

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) < \epsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|.$$

Thus, for any $\gamma > 0$

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\}\\ \subseteq &\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{(\gamma-\epsilon)\delta}{2f(1)}\right\}.\end{aligned}$$

Since $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$, it follows the final set, and hence, the first set in above expression belongs to $\mathcal{I}$. This prove that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$.

(2) If $\lim_{k\to\infty} \frac{f(k)}{k} = \beta > 0$, then we have $f(k) \geq k\beta$ for all $k > 0$. Know suppose that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. Since

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{1}{n}\sum_{k=1}^{n} \beta\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \beta\left(\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$

it follows that for each $\epsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \beta\epsilon\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ $\epsilon > 0$ it follows that the later set belongs to $\mathcal{I}$, from (1) and (2), the theorem is proved. □

Theorem 3.10. *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, (X, ρ) be a metric space and f be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$ and*
2. *If f is bounded, then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$.*

Proof. (1) Suppose $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, and let $\epsilon \geq 0$ be given, then we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$
$$\geq \frac{f(\varepsilon)}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right|.$$

Consequently, for $\gamma > 0$, we have

$$\left\{n \in \mathbb{N} : \left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right| \geq \frac{\gamma}{f(\epsilon)}\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, it follows by Definition 3.7 that the

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\} \in \mathcal{I},$$

and therefore $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$.

(2) Suppose f is bounded and $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$. Since f is bounded, there exists a real number M such that $\sup f(t) \leq M$. Moreover, for $\epsilon > 0$, we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{n}\Bigg[\sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$
$$+ \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \epsilon}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\Bigg]$$
$$\leq \frac{M}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + f(\frac{\epsilon}{2}).$$

Thus if we denote the sets

$$B(\epsilon) = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right\}$$

and

$$A(\epsilon) = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| \geq \frac{f(\epsilon)}{2M}\right\},$$

then we have the inclusion $B(\epsilon) \subset A(\epsilon)$. Since $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$ it follows that $A(\epsilon) \in \mathcal{I}$, and therefore $B(\epsilon) \in \mathcal{I}$, by applying the operators $\epsilon \to 0$ This shows that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. □

Theorem 3.11. *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, (X, ρ) be a metric space, f be a modulus function and $\theta = \{K_r\}$ be a lacunary sequence. If $\liminf_r q_r > 1$, then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\} \Rightarrow \{A_k\} \sim^{\mathcal{I}(W_\theta f)} \{B_k\}$.*

Proof. Suppose $\liminf_r q_r > 1$, then there exist $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$. Let $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. For a sufficiently large r, we obtain the following:

$$\begin{aligned}\frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) &\geq \frac{1}{k_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\ &\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right).\end{aligned}$$

Which gives for $\epsilon > 0$

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \epsilon\right\}\\ \subseteq&\left\{r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{\epsilon\delta}{1+\delta}\right\}.\end{aligned}$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, its follows that the set

$$\left\{r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{\epsilon\delta}{1+\delta}\right\} \in \mathcal{I},$$

and hence the set

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \epsilon\right\} \in \mathcal{I}.$$

This shows that $\{A_k\} \sim^{\mathcal{I}(W_\theta f)} \{B_k\}$. □

Theorem 3.12. *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, (X, ρ) be a metric space and f be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(W_\theta f)} \{B_k\}$ and*
2. *$\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$ then $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(W_\theta f)} \{B_k\})$.*

Proof. The proof is similar to the proof of Theorem 3.9 □

Theorem 3.13. *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $\theta = \{K_r\}$ be a lacunary sequence, (X, ρ) be a metric space and f be a modulus function. Then,*

1. *If* $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\}$ *then* $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$ *and*
2. *If* f *is bounded, then* $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$.

Proof. (1) Suppose $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)\{B_k\}}$, and let $\epsilon \geq 0$ be given, then

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$
$$\geq \frac{f(\varepsilon)}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right|.$$

For $\eta > 0$, if we denote sets

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \eta\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \eta f(\epsilon)\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\}$, so

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \eta f(\epsilon)\right\} \in \mathcal{I}.$$

By the definition of ideal

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \eta\right\} \in \mathcal{I}$$

and therefore $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$. (2) Suppose f is bounded and $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$. Since f is bounded, there exists a real number $M > 0$ such that $|f(t)| \leq M$ for all $t \geq 0$, by using this fact

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{h_r}\Bigg[\sum_{\substack{k\in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$
$$+ \sum_{\substack{k\in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\Bigg]$$
$$\leq \frac{M}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + f(\tfrac{\epsilon}{2}).$$

Thus if we denote the sets

$$B(\epsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right\}$$

and

$$A(\epsilon)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\frac{\epsilon}{2}\right\}\right|\geq\frac{f(\epsilon)}{2M}\right\},$$

then we have the inclusion $B(\epsilon)\subset A(\epsilon)$. Since $\{A_k\}\sim^{\mathcal{I}(WS_\theta^L)}\{B_k\}$ it follows that $A(\epsilon)\in\mathcal{I}$, and therefore $B(\epsilon)\in\mathcal{I}$, by applying the operators $\epsilon\to 0$ This shows that $\{A_k\}\sim^{\mathcal{I}(_\theta^W f)}\{B_k\}$. □

Theorem 3.14. *Let $\mathcal{I}\subset\wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, (X,ρ) be a metric space, $\theta=\{K_r\}$ be a lacunary sequence and let A_k, $B_k\subseteq X$ be a non-empty closed subset of X.*

1. *If $\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}$ then $\{A_k\}\sim^{\mathcal{I}(WS_\theta^L)}\{B_k\}$ and $\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}$ is proper.*
2. *If $\{A_k\},\{B_k\}\in\mathcal{L}_\infty$ such that $\{A_k\}\sim^{\mathcal{I}(WS_\theta^L)}\{B_k\}$, then $\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}$.*
3. *$\{A_k\}\sim^{\mathcal{I}(WS_\theta^L)}\{B_k\}\cap\ell_\infty=\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}\cap\ell_\infty$.*

Here ℓ_∞ denotes the class of bounded sequences.

Proof. (1) Suppose A_k, $B_k\subseteq X$ be a non-empty closed subset of X such that $\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}$. We show that $\{A_k\}\sim^{\mathcal{I}(WS_\theta^L)}\{B_k\}$. Let $\epsilon>0$. Since

$$\begin{aligned}\sum_{k\in I_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|&\geq\sum_{\substack{k=1\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\\&\geq\varepsilon.\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\right\}\right|,\end{aligned}$$

given $\delta>0$

$$\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\right\}\right|\ \textit{implies that}\ \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\delta.$$

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\right\}\right|\geq\delta\right\}\\\subseteq&\left\{n\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\delta\right\}.\end{aligned}$$

Since $\{A_k\}\sim^{\mathcal{I}[WN]_\theta^L}\{B_k\}$, it follows by the definition that the set in the right side belong to $\mathcal{I}$ which immediately implies

$$\left\{n\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

This shows that $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$.

We next give an example to show that the inclusion $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\} \subset \{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ is proper. Consider the sequences A_k, B_k defined by

$$A_k = \begin{cases} \{k\},, & \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r}\right] r = 1, 2, ... \\ \{0\}, & \text{otherwise} \end{cases}$$

$\{B_k\} = 0$ for all k. Then clearly $\{A_k\} \notin \ell_\infty$. We have for every $\epsilon > 0$ and for each $x \in X$,

$$\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \epsilon\right\}\right| = \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0 \,\text{as}\, r \to \infty.$$

That is $\{A_k\} \sim^{\mathcal{I}(WS_\theta^1)} \{B_k\}$. On the other hand,

$$\frac{1}{h_r}\sum_{k \in I_r}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \nrightarrow 0 \,\text{as}\, r \to \infty.$$

Hence $\{A_k\} \nsim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$. □

(2) *Suppose A_k, $B_k \in \ell_\infty$ such that $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$. Then we can find a number M such that*

$$\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \leq M,$$

for all $k \in \mathbb{N}$. Given $\epsilon > 0$, We have

$$\begin{aligned} \frac{1}{h_r}\sum_{k \in I_r}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| &= \frac{1}{h_r}\sum_{\substack{k \in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon}} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &\quad + \frac{1}{h_r}\sum_{\substack{k \in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \epsilon}} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &\leq \frac{M}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + \frac{\epsilon}{2}. \end{aligned}$$

This shows that $\{A_k\} \nsim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$.

(3) *This immediately follows from* (1) *and* (2).

4. Conclusions

We observe that if the modulus function f satisfies $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then the notions $\mathcal{I}(W_L)$ and $\mathcal{I}[WN]_\theta^L$ respectively coincide with the notions $\mathcal{I}(Wf^L)$ and $\mathcal{I}(^W_\theta f)$. However, if f is bounded, the notions $\mathcal{I}(Wf^L)$ and $\mathcal{I}(^W_\theta f)$ coincides respectively with the notions $\mathcal{I}(WS^L)$ and $\mathcal{I}(WS_\theta^L)$.

References

[1] M. Baronti and P. L. Papini, *Convergence of sequences of sets*, in Methods of Functional Analysis in Approximation Theory, Birkhäuser, Basel, Switzerland, 76 (1986), 135-155.

[2] M. Baśarir, S. Altundağ, *On asymptotically equivalent difference sequences with respect to a modulus function*, Ricerche Mat., 60 (2011), 299-311.

[3] D. Borwein, *Linear functionals with strong Cesaro summability*, Journal London Math.Soc., 40 (1965), 628-634.

[4] T. Bilgin, *f-Asymptotically lacunary equivalent sequences*, Acta Univ. Apulensis., 28 (2011), 271-278.

[5] J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., 32 (1989), 194-198.

[6] P. Das, E. Savas and S. Ghosal, *On generalizations of certain summability methods using ideals*, Appl. Math. Lett., 24 (2011), 1509-1514

[7] H. Fast, *Sur la convergence ststistique*, Colloq. Math., 2 (1951) 241-244.

[8] J. A. Fridy and C. Orhan, *Lacunary statistical convergent*, Pacific J. Math., 160 (1993), 43-51.

[9] H. Halberstem, K. F. Roth, *Sequence*, Springer-Verlag, New York, 1993.

[10] K. Kuratowski, *Topology*, Vol. I. NewYork: Academic Press, 1966.

[11] P. Kostyrko, T. Salat and W.Wilczynski, *$\mathcal{I}$-convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669-686.

[12] V. Kumar and A. Sharma, *Asymptotically lacunary equivalent sequences defined by ideals and modulus function*, Mathematical Sciences., 6 (2012), 1-5.

[13] V. K. Kaushik, A. Sharma, *On asymptotically generalized statistical equivalent sequences via ideals*, Tamkang Journal of Mathematics., 43 (2012), 469-478.

[14] E. Kolk, *On strong boundedness and summability with respect to a sequence of moduli*, TartuUl. Toimetised., 960 (1993), 41-50.

[15] I.J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc., 100 (1986), 161-166.

[16] M. Marouf, *Asymptotic equivalence and summability,* Internat. J. Math. Math. Sci., 16 (1993), 755-762.

[17] H. Nakano, *Concave modulars*, J. Math. Soc. Japan., 5 (1953), 29-49.

[18] F. Nuray and B. E. Rhoades, *Statistical convergence of sequences of sets*, Fasciculi Mathematici., 49 (2012), 87-99.

[19] F. Nuray, *On asymptotically lacunary statistical equivalent set sequences*, Journal of Mathematics, 2013.

[20] E. Öztürk, T. Bilgin, *Strongly summable sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., 25 (1994), 621-625.

[21] R. F. Patterson, *On asymptotically statistical equivalent sequences*, Demonstratio Mathematica., 36 (2003), 149-153.

[22] R. F. Patterson and E. Savaş, *On asymptotically lacunary statistically equivalent sequences*, Thai Journal of Mathematics, 4 (2006), 267-272.

[23] S. Pehlivan, B. Fisher, *On some sequence spaces*, Indian J. Pure Appl. Math., 25 (1994), 1067-1071.

[24] S. Pehlivan, B. Fisher, *Some sequence spaces defined by a modulus*, Math. Slovaca., 45 (1995), 275-280.

[25] E. Savaş, P. Das, *A generalized statistical convergence via ideals,* Appl.Math. Lett., 24 (2011), 826-830.

[26] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 (1959), 361-375.

[27] U. Ulusu and F. Nuray, *Lacunary statistical convergence of sequence of sets*, Progress in Applied Mathematics., 4 (2012), 99-109.

[28] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, 1960.

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New types of soft ordered mappings via soft α-open sets

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Abstract. The concept of soft topological ordered spaces is an extension of the soft topological spaces notion. The motivation of this paper is twofold: One is to generalize and extend the existing ordered maps and other is to contribute on making a general framework for studying soft topological ordered spaces. In this study, we utilize soft α-open sets to introduce new ordered maps, which generalize existing comparable notions, namely soft $x\alpha$-continuous, soft $x\alpha$-open, soft $x\alpha$-closed and soft $x\alpha$-homeomorphism maps, for $x \in \{I, D, B\}$, via soft topological ordered spaces. We show the relationships among these concepts and discuss the equivalent conditions for each one of them. Also, we derive that an extended soft topologies notion guarantees the equivalent between the soft maps initiated herein and their counterparts of maps on topological ordered spaces. For illustration and comparison, various examples are provided.

Keywords: soft $I(D, B)\alpha$-continuous map, Soft $I(D, B)\alpha$-open map and soft $I(D, B)\alpha$-homeomorphism map.

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1. Introduction and preliminaries

By combining a topological structure τ with a partial order relation $\preceq$ on a non-empty set X, Nachbin [42] named the triple $(X, \tau, \preceq)$ a topological ordered space and established some topological ordered notions about it which depend on topological structure and partial order relation. In his work, he introduced the concepts of normally ordered, regularly ordered and completely ordered spaces. McCartan [37] renamed some ordered separation axioms and characterised T_i-ordered spaces ($i = 0, 1, 2, 3, 4$). Also, he presented interesting examples to show that T_i-spaces are proper generalization of T_i-ordered spaces, for $i = 0, 1, 2$, and derived some results which connected T_i-ordered axioms and some topological notions like one-point compactification and local compactness. Some scholars investigated another types of ordered spaces by replacing a partial order relation by preorder relation or any binary relation (see, for example, [29, 38, 39, 45]). As a generalization of topological ordered spaces, Das [20] introduced and studied ordered spaces. For more investigation on ordered spaces, we refer the interested readers to [3, 7, 10, 13, 17, 22, 23, 24, 28, 29, 32, 33, 34, 49].

Molotdsov [41], in 1999, came up with an idea, namely soft sets, in order to approach uncertainties and vagues of data on some reality situations and phenomena. He discussed the strengths of soft set theory in compared with fuzzy theory and probability theory and investigated its applications on different fields. As a result of advantages of soft sets in overcoming incomplete data, many researchers began introducing some soft operations between soft sets and applying it in several situations arising in information science, decision making, mathematics and other related disciplines (see, for example, [1, 2, 19, 27, 44]).

To insert soft sets on topology studies, Shabir and Naz [48] formulated the soft topological spaces notion by analogy with the definition of topology. They fixed a parameters set to avoid anomaly arising in appearing various non-null soft sets. They gave elementary ideas on soft topological spaces and studied soft separation axioms. Min [40] observed some mistakes on Shabir and Naz's work and corrected them. In this regard, we think the existence of these mistakes was due to the lack of knowledge of shape of soft open sets in soft regular spaces. Later on, the desire of obtaining a deeper understanding of soft topology prompted interested researchers to carry out many studies on soft topological notions and their features (see, for example, [18, 30, 31, 43]). [8, 9, 12, 25] investigated and corrected some errors which appeared in some previous studies. In [4], the authors gave the definition of soft α-open sets and examined some related properties. They [5] also defined soft α-separation axioms and characterised them. In the regard of studying generalized soft open sets, Al-shami [11] originated a concept of somewhere dense sets and investigated many of its properties.

Recently, the authors of [14] introduced a concept of soft topological ordered spaces and established the notions of increasing (decreasing) soft sets and increasing (decreasing) soft maps. Also, they defined p-soft T_i-ordered spaces

($i = 0, 1, 2, 3, 4$) depending on totally non belong relations, which defined in [26], and monotone soft neighborhoods. Based on soft β-open sets, some of soft ordered maps were introduced and studied in [16]. In the end of this literature, we indicate to that the authors of [1, 2, 46] defined new types of soft subsets, soft equality, soft union and intersection by relaxing the conditions on the parameters sets.

The purpose of this study is to introduce and study the concepts of soft $x\alpha$-continuous, soft $x\alpha$-open, soft $x\alpha$-closed and soft $x\alpha$-homeomorphism maps, for $x \in \{I, D, B\}$, via soft topological ordered spaces and to obtain a deeper understanding of them. Also, we give various examples to show the relationships among these maps and illustrate that soft $x\alpha$-continuous, soft $x\alpha$-open, soft $x\alpha$-closed and soft $x\alpha$-homeomorphism maps are strictly stronger than soft α-continuous, soft α-open, soft α-closed and soft α-homeomorphism maps, respectively, for $x \in \{I, D, B\}$. Furthermore, we characterize each one of the initiated soft maps and clarify a significant role of extended soft topologies on studying the interrelations between these soft maps and their counterparts of maps in topological ordered spaces.

We allocate the rest of this section to present some fundamental definitions and findings that will be needed in the sequels.

Definition 1.1 ([41]). *An ordered pair (G, E) is said to be a soft set over X if G is a map of a set of parameters E into 2^X.*

Remark 1.2. (i) For short, we use the notation G_E instead of (G, E).

(ii) A soft set G_E can be written as a set of ordered pairs $G_E = \{(e, G(e)) : e \in E \text{ and } G(e) \in 2^X\}$.

Definition 1.3 ([36]). *A soft set G_E over X is called a null soft set, denoting by $\widetilde{\Phi}$, if $G(e) = \emptyset$, for each $e \in E$ and is called an absolute soft set, denoting by $\widetilde{X}$, if $G(e) = X$, for each $e \in E$.*

Definition 1.4 ([6]). *The relative complement of a soft set G_E is denoted by G_E^c, where $G^c : E \to 2^X$ is a mapping defined by $G^c(e) = X \setminus G(e)$, for each $e \in E$.*

In this connection, it is worth noting that $x \notin G_E$ does not imply that $x \in G_E^c$.

Definition 1.5 ([48]). *For $x \in X$ and a soft set G_E over X, we say that $x \in G_E$ if $x \in G(e)$, for each $e \in E$ and $x \notin G_E$ if $x \notin G(e)$, for some $e \in E$.*

Definition 1.6 ([48]). *A soft topology on a non-empty set X is a collection τ of soft sets over X under a parameters set E satisfying the following axioms:*

(i) $\widetilde{X}$ and $\widetilde{\emptyset}$ belong to τ.

(ii) The soft intersection of finite members in τ belongs to τ.

(iii) The soft union of any members in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space. Every member of τ is called soft open and its relative complement is called soft closed.

Proposition 1.7 ([48]). *Let (X, τ, E) be a soft topological space. Then $\tau_e = \{G(e) : G_E \in \tau\}$ defines a topology on X, for each $e \in E$.*

Definition 1.8 ([4]). *A soft subset H_E of (X, τ, E) is said to be soft α-open if $H_E \widetilde{\subseteq} int(cl(int(H_E)))$. Its complement is said to be soft α-closed.*

Definition 1.9 ([4], [48]). *For a soft subset H_E of (X, τ, E), we define the following four operators:*

(i) $int(H_E)$ is the largest soft open set contained in H_E.

(ii) $cl(H_E)$ is the smallest soft closed set containing H_E.

(iii) $int_\alpha(H_E)$ is the largest soft α-open set contained in H_E.

(iv) $cl_\alpha(H_E)$ is the smallest soft α-closed set containing H_E.

Definition 1.10 ([43]). *Consider (X, τ, E) is a soft topological space and τ_e is a topology on X as in the above proposition. Then $\tau^\star = \{G_E : G(e) \in \tau_e$, for each $e \in E\}$ is a soft topology on X finer than τ.*
In this work, we term $\tau^\star$ an extended soft topology.

Definition 1.11 ([50]). *Consider $f : X \to Y$ and $\phi : A \to B$ are two maps and let $f_\phi : S(X_A) \to S(Y_B)$ be a soft map. Let G_K and H_L be soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then*

(i) $f_\phi(G_K) = (f_\phi(G))_B$ is a soft subset of $S(Y_B)$ such that

$$f_\phi(G)(b) = \begin{cases} \bigcup_{a \in \phi^{-1}(b) \bigcap K} f(G(a)), & \phi^{-1}(b) \bigcap K \neq \emptyset \\ \emptyset, & \phi^{-1}(b) \bigcap K = \emptyset \end{cases}$$

for each $b \in B$.

(ii) $f_\phi^{-1}(H_L) = (f_\phi^{-1}(H))_A$ is a soft subset of $S(X_A)$ such that

$$f_\phi^{-1}(H)(a) = \begin{cases} f^{-1}(H(\phi(a))), & \phi(a) \in L \\ \emptyset, & \phi(a) \notin L \end{cases}$$

for each $a \in A$.

Remark 1.12. Henceforth, a soft map $f_\phi : S(X_A) \to S(Y_B)$ implies that a map f of the universe set X into the universe set Y and a map ϕ of a set of parameters A into a set of parameters B

Definition 1.13 ([50]). *A soft map $f_\phi : S(X_A) \to S(Y_B)$ is said to be injective (resp. surjective, bijective) if f and ϕ are injective (resp. surjective, bijective).*

Proposition 1.14 ([43]). *Consider $f_\phi : S(X_A) \to S(Y_B)$ is a soft map and let G_A and H_B be two soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then we have the following results:*

(i) $G_A \widetilde{\subseteq} f_\phi^{-1} f_\phi(G_A)$ and the equality relation holds if f_ϕ is injective.

(ii) $f_\phi f_\phi^{-1}(H_B) \widetilde{\subseteq} H_B$ and the equality relation holds if f_ϕ is surjective.

Definition 1.15 ([4]). *A soft map $f_\phi : (X, \tau, A) \to (Y, \theta, B)$ is said to be:*

(i) Soft α-continuous if the inverse image of each soft open subset of (Y, θ, B) is a soft α-open subset of (X, τ, A).

(ii) Soft α-open (resp. soft α-closed) if the image of each soft open (resp. soft closed) subset of (X, τ, A) is a soft α-open (resp. soft α-closed) subset of (Y, θ, B).

(iii) Soft α-homeomorphism if it is bijective, soft α-continuous and soft α-open.

Definition 1.16 ([21], [43]). *A soft subset P_E over X is called soft point if there exists $e \in E$ and there exists $x \in X$ such that $P(e) = \{x\}$ and $P(a) = \emptyset$, for each $a \in E \setminus \{e\}$. A soft point will be shortly denoted by P_e^x and we say that $P_e^x \in G_E$, if $x \in G(e)$.*

Definition 1.17 ([14]). *Let $\preceq$ be a partial order relation on a non-empty set X and let E be a set of parameters. A triple $(X, E, \preceq)$ is said to be a partially ordered soft set.*

Definition 1.18 ([14]). *We define an increasing soft operator $i : (SS(X)_E, \preceq\ ) \to (SS(X)_E, \preceq)$ and a decreasing soft operator $d : (SS(X)_E, \preceq\ ) \to (SS(X)_E, \preceq)$ as follows, for each soft subset G_E of $SS(X_E)$*

(i) $i(G_E) = (iG)_E$, where iG is a mapping of E into X given by $iG(e) = i(G(e)) = \{x \in X : y \preceq x, \text{ for some } y \in G(e)\}$.

(ii) $d(G_E) = (dG)_E$, where dG is a mapping of E into X given by $dG(e) = d(G(e)) = \{x \in X : x \preceq y, \text{ for some } y \in G(e)\}$.

Definition 1.19 ([14]). *A soft subset G_E of a partially ordered soft set $(X, E, \preceq)$ is said to be increasing (resp. decreasing) if $G_E = i(G_E) (resp.\ G_E = d(G_E))$.*

Theorem 1.20 ([14]). *If a soft map $f_\phi : (S(X_A), \preceq_1) \to (S(Y_B), \preceq_2)$ is increasing, then the inverse image of each increasing (resp. decreasing) soft subset of $\widetilde{Y}$ is an increasing (resp. a decreasing) soft subset of $\widetilde{X}$.*

Definition 1.21 ([14]). *A quadrable system $(X, \tau, E, \preceq)$ is said to be a soft topological ordered space, where (X, τ, E) is a soft topological space and $(X, E, \preceq)$ is a partially ordered soft set. Henceforth, the two notations $(X, \tau, E, \preceq_1)$ and $(Y, \theta, F, \preceq_2)$ stand for soft topological ordered spaces.*

Definition 1.22 ([47]). *A map $(X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is said to be:*

(i) *I (resp. D, B) α-continuous if the inverse image of each open set is I (resp. D, B) α-open.*

(ii) *I (resp. D, B) α-open if the image of each open set is I (resp. D, B) α-open.*

(iii) *I (resp. D, B) α-closed if the image of each open set is I (resp. D, B) α-closed.*

(iv) *I (resp. D, B) α-homeomorphism if it is bijective, I (resp. D, B) α-continuous and I (resp. D, B) α-open.*

Definition 1.23 ([15]). *The composition of two soft maps $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2\ )$ and $g_\lambda : (Y, \theta, F, \preceq_2\ ) \to (Z, \upsilon, K, \preceq_3)$ is a soft map $f_\phi \circ g_\lambda : (X, \tau, E, \preceq_1) \to (Z, \upsilon, K, \preceq_3)$ and is given by $(f_\phi \circ g_\lambda)(P_e^x) = f_\phi(g_\lambda(P_e^x))$.*

2. Soft $I(D, B)\alpha$-continuous maps

The purpose of this section is to define soft $I(D, B)\alpha$-continuity at soft point, ordinary point and on the universe set and to elucidate the relationships among them with the help of examples. The characterisations of of the given soft maps are studied and the interrelations between these soft maps and their counterparts of maps on topological ordered spaces are discussed.

Definition 2.1. *A soft subset H_E of $(X, \tau, E, \preceq_1)$ is said to be:*

(i) *Soft I (resp. Soft D, Soft B) α-open if it is soft α-open and increasing (resp. decreasing, balancing).*

(ii) *Soft I (resp. Soft D, Soft B) α-closed if it is soft α-closed and increasing (resp. decreasing, balancing).*

Definition 2.2. *A soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is called:*

(i) *Soft I (resp. Soft D, Soft B) α-continuous at $P_e^x \in \widetilde{X}$ if for each soft open set H_F containing $f_\phi(P_e^x)$, there exists a soft I (resp. soft D, soft B) α-open set G_E containing P_e^x such that $f_\phi(G_E) \widetilde{\subseteq} H_F$.*

(ii) *Soft I (resp. Soft D, Soft B) α-continuous at $x \in X$ if it is soft I (resp. soft D, soft B) α-continuous at each P_e^x.*

(*iii*) *Soft I (resp. Soft D, Soft B) α-continuous if it is soft I (resp. soft D, soft B) α-continuous at each $x \in X$.*

Theorem 2.3. *A soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) α-continuous if and only if the inverse image of each soft open subset of $\widetilde{Y}$ is a soft I (resp. soft D, soft B) α-open subset of $\widetilde{X}$.*

Proof. We only present theorem's proof in case of f_ϕ is soft Iα-continuous and the cases between parenthesis can be made similarly.
To prove the necessary part, let G_F be a soft open subset of $\widetilde{Y}$, Then we have the following two cases:

(i) Either $f_\phi^{-1}(G_F) = \widetilde{\emptyset}$.

(ii) Or $f^{-1}(G_F) \neq \widetilde{\emptyset}$. By choosing $P_e^x \in X$ such that $P_e^x \in f_\phi^{-1}(G_F)$, we obtain that $f_\phi(P_e^x) \in G_F$. So there exists a soft Iα-open set H_E containing P_e^x such that $f_\phi(H_E) \widetilde{\subseteq} G_F$. Since P_e^x is chosen arbitrary, then $f_\phi^{-1}(G_F) = \widetilde{\bigcup}_{P_e^x \in f_\phi^{-1}(G_F)} H_E$.

From the two cases above, we conclude that $f_\phi^{-1}(G_F)$ is a soft Iα-open subset of $\widetilde{X}$. To prove the sufficient part, let G_F be a soft open subset of $\widetilde{Y}$ containing $f_\phi(P_e^x)$. Then $P_e^x \in f_\phi^{-1}(G_F)$. By hypothesis, $f_\phi^{-1}(G_F)$ is a soft Iα-open set. Since $f_\phi(f_\phi^{-1}(G_F)) \widetilde{\subseteq} G_F$, then f_ϕ is a soft Iα-continuous map at $P_e^x \in X$ and since P_e^x is chosen arbitrary, then f_ϕ is a soft Iα-continuous map. □

Remark 2.4. From Definition (2.2), we can note the following:

(i) Every soft I (D, B) α-continuous map is always soft α-continuous.

(ii) Every soft Bα-continuous map is soft Iα-continuous or soft Dα-continuous.

To elucidate that the converse of the two results of the remark above need not be true, we construct the following two examples.

Example 2.5. Let the two universe sets $X = \{2, 3, 5\}$, $Y = \{7, 11, 13\}$ and the two parameters sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$. Consider a map $\phi : A \to B$ is defined as, $\phi(a_1) = \phi(a_2) = b_1$ and a map $f : X \to Y$ is defined as, $f(2) = 7$ and $f(3) = f(5) = 11$. We define a partial order relation on X as $\preceq = \triangle \bigcup \{(3, 2), (2, 5), (3, 5)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_A\}$ and $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, H_B\}$, where $F_A = \{(a_1, \{2\}), (a_2, \emptyset)\}$ and $H_B = \{(b_1, \{7\}), (b_2, \emptyset)\}$. Since $f_\phi^{-1}(H_B) = \{(a_1, \{2\}), (a_2, \{2\})\}$ is a soft α-open set, then $f_\phi : S(X_A) \to S(Y_B)$ is a soft α-continuous map. On the other hand, $f_\phi^{-1}(H_B)$ is neither a soft Dα-open nor a soft Iα-open set. Hence f_ϕ is not soft I (soft D, soft B) α-continuous.

Example 2.6. In Example above, if we replace only the partial order relation by $\preceq = \triangle \bigcup \{(2, 5)\}$(resp. $\preceq = \triangle \bigcup \{(3, 2)\}$), then the soft map f_ϕ is soft D-continuous (resp. soft I-continuous), but is not soft B-continuous.

Definition 2.7. *For a soft subset H_E of $(X, \tau, E, \preceq)$, we define the following six operators:*

(*i*) $H_E^{i\alpha o}$(*resp.* $H_E^{d\alpha o}, H_E^{b\alpha o}$) *is the largest soft I (resp. soft D, soft B) α-open set contained in H_E.*

(*ii*) $H_E^{i\alpha cl}$(*resp.* $H_E^{d\alpha cl}, H_E^{b\alpha cl}$) *is the smallest soft I (resp. soft D, soft B) α-closed set containing H_E.*

Lemma 2.8. *We have the following three properties for a soft subset H_E of $(X, \tau, E, \preceq)$:*

(*i*) $(H_E^{d\alpha cl})^c = (H_E^c)^{i\alpha o}$.

(*ii*) $(H_E^{i\alpha cl})^c = (H_E^c)^{d\alpha o}$.

(*iii*) $(H_E^{b\alpha cl})^c = (H_E^c)^{b\alpha o}$.

Proof. (i) $(H_E^{d\alpha cl})^c = \{\widetilde{\bigcap} F_E : F_E$ is a soft Dα-closed set containing $H_E\}^c$ $=\widetilde{\bigcup}\{F_E^c : F_E^c$ is a soft Iα-open set contained in $H_E^c\} = (H_E^c)^{i\alpha o}$.

By analogy with (i), one can prove (ii) and (iii). □

Theorem 2.9. *The following five properties of a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ are equivalent:*

(*i*) f_ϕ *is soft Iα-continuous;*

(*ii*) $f_\phi^{-1}(L_F)$ *is a soft Dα-closed subset of $\widetilde{X}$, for each soft closed subset L_F of $\widetilde{Y}$;*

(*iii*) $(f_\phi^{-1}(M_F))^{d\alpha cl} \widetilde{\subseteq} f_\phi^{-1}(cl(M_F))$, *for every* $M_F \widetilde{\subseteq} \widetilde{Y}$;

(*iv*) $f_\phi(N_E^{d\alpha cl}) \widetilde{\subseteq} cl(f_\phi(N_E))$, *for every* $N_E \widetilde{\subseteq} \widetilde{X}$;

(*v*) $f_\phi^{-1}(int(M_F)) \widetilde{\subseteq} (f_\phi^{-1}(M_F))^{i\alpha o}$, *for every* $M_F \widetilde{\subseteq} \widetilde{Y}$.

Proof. (*i*) $\Rightarrow$ (*ii*) : Consider L_F is a soft closed subset of $\widetilde{Y}$. By hypothesis, $f_\phi^{-1}(L_F^c)$ is a soft Iα-open subset of $\widetilde{X}$ and by the fact that $f_\phi^{-1}(L_F^c) = (f_\phi^{-1}(L_F))^c$, we obtain that $f_\phi^{-1}(L_F)$ is soft Dα-closed as required.

(*ii*) $\Rightarrow$ (*iii*) : It follows from statement (*ii*) that $f_\phi^{-1}(cl(M_E))$ is a soft Dα-closed subset of $\widetilde{X}$, for every $M_F \widetilde{\subseteq} \widetilde{Y}$. So $(f_\phi^{-1}(M_F))^{d\alpha cl} \widetilde{\subseteq} (f_\phi^{-1}(cl(M_F))^{d\alpha cl} = f_\phi^{-1}(cl(M_F))$.

(*iii*) $\Rightarrow$ (*iv*) : From the fact that $N_E^{d\alpha cl} \widetilde{\subseteq} (f_\phi^{-1}(f_\phi(N_E))^{d\alpha cl}$ and from (*iii*), we have $(f_\phi^{-1}(f_\phi(N_E))^{d\alpha cl} \widetilde{\subseteq} f_\phi^{-1}(cl(f_\phi(N_E))$.

This implies that $f_\phi(N_E^{d\alpha cl}) \widetilde{\subseteq} cl(f_\phi(N_E))$.

(*iv*) $\Rightarrow$ (*v*) : For any soft subset M_F of $\widetilde{Y}$, we obtain from Lemma (2.8) that $f_\phi(\widetilde{X} - (f_\phi^{-1}(N_E))^{i\alpha o}) = f_\phi(((f_\phi^{-1}(N_E))^c)^{d\alpha cl})$. It follows from statement

(iv), that $f_\phi(((f_\phi^{-1}(N_E))^c)^{d\alpha cl}) \widetilde{\subseteq} cl(f_\phi(f_\phi^{-1}(N_E))^c) = cl(f_\phi(f_\phi^{-1}(N_E^c))) \widetilde{\subseteq} cl(\widetilde{Y} - N_E) = \widetilde{Y} - int(N_E)$. Therefore $(\widetilde{X} - (f_\phi^{-1}(N_E))^{i\alpha o}) \widetilde{\subseteq} f_\phi^{-1}(\widetilde{Y} - int(N_E)) = \widetilde{X} - f_\phi^{-1}(int(N_E))$. Thus $f_\phi^{-1}(int(N_E)) \widetilde{\subseteq} (f_\phi^{-1}(N_E))^{i\alpha o}$.

$(v) \Rightarrow (i)$: Consider M_F is a soft open subset of $\widetilde{Y}$. Then $f_\phi^{-1}(M_F) = f_\phi^{-1}(int(M_F)) \widetilde{\subseteq} (f_\phi^{-1}(M_F))^{i\alpha o}$. So $(f_\phi^{-1}(M_F))^{i\alpha o} = f_\phi^{-1}(M_F)$ and this means that $f_\phi^{-1}(M_F)$ is a soft Iα-open subset of $\widetilde{X}$. Hence the desired result is proved. □

Theorem 2.10. *The following five properties of a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ are equivalent:*

(i) f_ϕ is soft Dα-continuous (resp. soft Bα-continuous);

(ii) $f_\phi^{-1}(L_F)$ is a soft Iα-closed (resp. soft Bα-closed) subset of $\widetilde{X}$, for each soft closed subset L_F of $\widetilde{Y}$;

(iii) $(f_\phi^{-1}(M_F))^{i\alpha cl} \widetilde{\subseteq} f_\phi^{-1}(cl(M_F))$(resp. $(f_\phi^{-1}(M_F))^{b\alpha cl} \widetilde{\subseteq} f_\phi^{-1}(cl(M_F))$, for every $M_F \widetilde{\subseteq} \widetilde{Y}$;

(iv) $f_\phi(N_E^{i\alpha cl}) \widetilde{\subseteq} cl(f_\phi(N_E))$(resp. $f_\phi(N_E^{b\alpha cl}) \widetilde{\subseteq} cl(f_\phi(N_E))$, for every $N_E \widetilde{\subseteq} \widetilde{X}$;

(v) $f_\phi^{-1}(int(M_F)) \widetilde{\subseteq} (f_\phi^{-1}(M_F))^{d\alpha o}$(resp. $f_\phi^{-1}(int(M_F)) \widetilde{\subseteq} (f_\phi^{-1}(M_F))^{b\alpha o}$, for every $M_F \widetilde{\subseteq} \widetilde{Y}$.

Proof. The proof is similar to that of Theorem (2.9). □

Theorem 2.11. *Let $\tau^\star$ be an extended soft topology on X. Then a soft map $g_\phi : (X, \tau^\star, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) α-continuous if and only if a map $g : (X, \tau_e^\star, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-continuous.*

Proof. Necessity: Let U be an open subset of $(Y, \theta_{\phi(e)}, \preceq_2)$. Then there exists a soft open subset G_F of $(Y, \theta, F, \preceq_2)$ such that $G(\phi(e)) = U$. Since g_ϕ is a soft I (resp. soft D, soft B) α-continuous map, then $g_\phi^{-1}(G_F)$ is a soft I (resp. soft D, soft B) α-open set. From Definition (1.11), it follows that a soft subset $g_\phi^{-1}(G_F) = (g_\phi^{-1}(G))_E$ of $(X, \tau, E, \preceq_1)$ is given by $g_\phi^{-1}(G)(e) = g^{-1}(G(\phi(e)))$, for each $e \in E$. By hypothesis, $\tau^\star$ is an extended soft topology on X, we obtain that a subset $g^{-1}(G(\phi(e))) = g^{-1}(U)$ of $(X, \tau_e, \preceq_1)$ is I (resp. D, B) α-open. Hence a map g is I (resp. D, B) α-continuous.
Sufficiency: Let G_F be a soft open subset of $(Y, \theta, F, \preceq_2)$. Then from Definition (1.11), it follows that a soft subset $g_\phi^{-1}(G_F) = (g_\phi^{-1}(G))_E$ of $(X, \tau^\star, E, \preceq_1)$ is given by $g_\phi^{-1}(G)(e) = g^{-1}(G(\phi(e)))$, for each $e \in E$. Since a map g is I (resp. D, B) α-continuous, then a subset $g^{-1}(G(\phi(e)))$ of $(X, \tau_e^\star, \preceq_1)$ is I (resp. D, B) α-open. By hypothesis, $\tau^\star$ is an extended soft topology on X, we obtain that $g_\phi^{-1}(G_F)$ is a soft I (resp. soft D, soft B) α-open subset of $(X, \tau^\star, E, \preceq_1)$. Hence a soft map g_ϕ is soft I (resp. soft D, soft B) α-continuous. □

Proposition 2.12. *Let $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ be a surjective soft Bα-continuous. If $\preceq_1$ is linearly ordered, then θ is the soft indiscrete topology.*

3. Soft $I(D,B)\alpha$-open and soft $I(D,B)\alpha$-closed maps

In this part, we establish the notions of soft $I(D,B)\alpha$-open and soft $I(D,B)\alpha$-closed maps and show the relationships among them. Also, we give the equivalent conditions for each one of these soft maps and investigate the interrelations between these soft maps and their counterparts of maps on topological ordered spaces.

Definition 3.1. *A soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \tau, F, \preceq_2)$ is called:*

(*i*) *Soft I (resp. Soft D, Soft B) α-open if the image of every soft open subset of $\widetilde{X}$ is a soft I (resp. soft D, soft B) α-open subset of $\widetilde{Y}$.*

(*ii*) *Soft I (resp. Soft D, Soft B) α-closed if the image of every soft closed subset of $\widetilde{X}$ is a soft I (resp. soft D, soft B) α-closed subset of $\widetilde{Y}$.*

Remark 3.2. From Definition (3.1), we can note the following:

(i) Every soft I (D, B) α-open map is soft α-open.

(ii) Every soft I (D, B) α-closed map is soft α-closed.

(iii) Every soft Bα-open (resp. soft Bα-closed) map is soft Iα-open or soft Dα-open (resp. soft Iα-closed or soft Dα-closed).

In the following two examples, we show that the converse of the three statements of remark above fails.

Example 3.3. Let the two universe sets X, Y, the two parameters sets A, B and a map $f : X \to Y$ be the same as in Example (2.5). Consider a map $\phi : A \to B$ is defined as, $\phi(a_m) = b_m$, for $m \in \{1, 2\}$. We define a partial order relation on Y as $\preceq = \triangle \bigcup \{(7, 11), (11, 13), (7, 13)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_A\}$ and $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, H_B\}$, where $F_A = \{(a_1, \{2, 3\}), (a_2, \{5\})\}$ and $H_B = \{(b_1, \{11\}), (b_2, \{11\})\}$. Since $f_\phi(F_A) = \{(b_1, \{7, 11\}), (b_2, \{11\})\}$ is a soft α-open set, then $f_\phi : S(X_A) \to S(Y_B)$ is a soft α-open map and since $f_\phi(F_A^c) = \{(b_1, \{11\}), (b_2, \{7, 11\})\}$ is a soft α-closed set, then $f_\phi : S(X_A) \to S(Y_B)$ is a soft α-closed map. On the other hand, $f_\phi(F_A)$ is neither a soft Dα-open nor a soft Iα-open set and $f_\phi(F_A^c)$ is neither a soft Dα-closed nor a soft Iα-closed set. Hence f_ϕ is not a soft I (soft D, soft B) α-open map and not a soft I (soft D, soft B) α-closed map.

Example 3.4. In Example above, if we replace only the partial order relation by $\preceq = \triangle \bigcup \{(7, 11)\}$(resp. $\preceq = \triangle \bigcup \{(11, 13)\}$), then the soft map f_ϕ is soft Iα-open and soft Iα-closed (resp. soft Dα-open and soft Dα-closed), but is not soft Bα-open and soft Bα-closed.

Theorem 3.5. *The following three properties of a soft map* $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ *are equivalent:*

(*i*) f_ϕ *is soft Iα-open;*

(*ii*) $int(f_\phi^{-1}(M_F))\widetilde{\subseteq} f_\phi^{-1}(M_F^{i\alpha o})$, *for every* $M_F\widetilde{\subseteq}\widetilde{Y}$;

(*iii*) $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{i\alpha o}$, *for every* $N_E\widetilde{\subseteq}\widetilde{X}$.

Proof. (*i*) $\Rightarrow$ (*ii*): Given a soft subset M_F of $\widetilde{Y}$, it is obvious that $int(f_\phi^{-1}(M_F))$ is a soft open subset of $\widetilde{X}$. Then, by hypothesis, it follows that $f_\phi(int(f_\phi^{-1}(M_F)))$ is a soft Iα-open subset of $\widetilde{Y}$. Since $f_\phi(int(f_\phi^{-1}(M_F)))\widetilde{\subseteq} f_\phi(f_\phi^{-1}(M_F))\widetilde{\subseteq} M_F$, then $int(f_\phi^{-1}(M_F))\widetilde{\subseteq} f_\phi^{-1}(M_F^{i\alpha o})$.

(*ii*) $\Rightarrow$ (*iii*): Given a soft subset N_E of $\widetilde{X}$, from (*ii*), we obtain that $int(f_\phi^{-1}(f_\phi(N_E)))\widetilde{\subseteq} f_\phi^{-1}((f_\phi(N_E))^{i\alpha o})$.

Since $int(N_E)\widetilde{\subseteq} f_\phi^{-1}(f_\phi(int(f_\phi^{-1}(f_\phi(N_E)))))\widetilde{\subseteq} f_\phi^{-1}((f_\phi(N_E))^{i\alpha o})$, then $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{i\alpha o}$ as required.

(*iii*) $\Rightarrow$ (*i*): Let G_E be a soft open subset of $\widetilde{X}$. Then $f_\phi(int(G_E)) = f_\phi(G_E)\widetilde{\subseteq}(f_\phi(G_E))^{i\alpha o}$. Hence f_ϕ is a soft Iα-open map. □

The following theorem can be proved in a similar manner.

Theorem 3.6. *The following three properties of a soft map* $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ *are equivalent:*

(*i*) f_ϕ *is soft Dα-open (resp. soft Bα-open);*

(*ii*) $int(f_\phi^{-1}(M_F))\widetilde{\subseteq} f_\phi^{-1}(M_F^{d\alpha o})$(*resp.* $int(f_\phi^{-1}(M_F))\widetilde{\subseteq} f_\phi^{-1}(M_F^{b\alpha o})$), *for every* $M_F\widetilde{\subseteq}\widetilde{Y}$;

(*iii*) $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{d\alpha o}$(*resp.* $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{b\alpha o}$), *for every* $N_E\widetilde{\subseteq}\widetilde{X}$.

Theorem 3.7. *The following three statements hold for a soft map* f_ϕ:$(X, \tau, E\preceq_1) \to (Y, \theta, F, \preceq_2)$:

(*i*) f_ϕ *is soft Iα-closed if and only if* $(f_\phi(G_E))^{i\alpha cl}\widetilde{\subseteq} f_\phi(cl(G_E))$, *for every* $G_E\widetilde{\subseteq}\widetilde{X}$.

(*ii*) f_ϕ *is soft Dα-closed if and only if* $(f_\phi(G_E))^{d\alpha cl}\widetilde{\subseteq} f_\phi(cl(G_E))$, *for every* $G_E\widetilde{\subseteq}\widetilde{X}$.

(*iii*) f_ϕ *is soft Bα-closed if and only if* $(f_\phi(G_E))^{b\alpha cl}\widetilde{\subseteq} f_\phi(cl(G_E))$, *for every* $G_E\widetilde{\subseteq}\widetilde{X}$.

Proof. We only give a proof for the first statement and the others follow similar lines.

Necessity: Since f_ϕ is soft Iα-closed, then $f_\phi(cl(G_E))$ is a soft Iα-closed subset of $\widetilde{Y}$ and since $f_\phi(G_E)\widetilde{\subseteq}f_\phi(cl(G_E))$, then $(f_\phi(G_E))^{i\alpha cl}\widetilde{\subseteq}f_\phi(cl(G_E))$.

Sufficiency: Consider H_E is a soft closed subset of $\widetilde{X}$.

Then $f_\phi(H_E)\widetilde{\subseteq}(f_\phi(H_E))^{i\alpha cl}\widetilde{\subseteq}f_\phi(cl(H_E)) = f_\phi(H_E)$. Therefore $f_\phi(H_E) = (f_\phi(H_E))^{i\alpha cl}$. This means that $f_\phi(H_E)$ is a soft Iα-closed set. Hence the proof is complete. □

Theorem 3.8. *The following three statements hold for a bijective soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$:*

(*i*) *f_ϕ is soft I (resp. soft D, soft B) α-open if and only if f_ϕ is soft D (resp. soft D, soft B) α-closed.*

(*ii*) *f_ϕ is soft I (resp. soft D, soft B) α-open if and only if f_ϕ^{-1} is soft I (resp. soft D, soft B) α-continuous.*

(*iii*) *f_ϕ is soft D (resp. soft I, soft B) α-closed if and only if f_ϕ^{-1} is soft I (resp. soft D, soft B) α-continuous.*

Proof. For the sake of economy, we only give proofs of cases outside the parenthesis for the three statements above and the cases between parenthesis can be made similarly.

(i) To prove the necessary condition, let H_E be a soft closed subset of $\widetilde{X}$ and consider f_ϕ is a soft Iα-open map. Then H_E^c is soft open and $f_\phi(H_E^c)$ is soft Iα-open. It follows from the bijectiveness of f_ϕ, that $f_\phi(H_E^c) = [f_\phi(H_E)]^c$. This automatically implies that $f_\phi(H_E)$ is soft Dα-closed. Thus f_ϕ is a soft Dα-closed map. In a similar manner, we can prove the sufficiency condition.

(ii) Necessity: Let G_E be a soft open subset of $\widetilde{X}$ and consider f_ϕ is a soft Iα-open map. Then $f_\phi(G_E)$ is soft Iα-open. It follows from the bijectiveness of f_ϕ, that $f_\phi(G_E) = (f_\phi^{-1})^{-1}(G_E)$. This automatically implies that $(f_\phi^{-1})^{-1}(G_E)$ is soft Iα-open. Thus f_ϕ^{-1} is a soft Iα-continuous map. In a similar manner, we can prove the sufficiency condition.

(iii) The proof of this statement comes immediately from (i) and (ii) above.

□

Theorem 3.9. *Let $\theta^\star$ be an extended soft topology on Y and ϕ is an injective map. Then a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) α-open if and only if a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-open.*

Proof. Necessity: Let U be an open subset of $(X, \tau_e, \preceq_1)$ and $\phi(e) = f$. Then there exists a soft open subset G_E of $(X, \tau, E, \preceq_1)$ such that $G(e) = U$. Since g_ϕ is a soft I (resp. soft D, soft B) α-open map, then $g_\phi(G_E)$ is a soft I (resp. soft D, soft B) α-open set. From Definition (1.11), it follows that a soft subset $g_\phi(G_E) = (g_\phi(G))_F$ of $(Y, \theta, F, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{e\in\phi^{-1}(f)} g(G(e))$, for each $f \in F$. By hypothesis, $\theta^\star$ is an extended soft topology on Y, a subset $\bigcup_{e\in\phi^{-1}(f)} g(G(e)) = g(U)$ of $(Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-open. Hence a map g is I (resp. D, B) α-open.

Sufficiency: Let G_E be a soft open subset of $(X, \tau, E, \preceq_1)$. Then from Definition (1.11), it follows that a soft subset $g_\phi(G_E) = (g_\phi(G))_F$ of $(Y, \theta^\star, F, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{e\in\phi^{-1}(f)} g(G(e))$, for each $f \in F$. Since a map g is I (resp. D, B) α-open, then a subset $\bigcup_{e\in\phi^{-1}(f)} g(G(e))$ of $(Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-open. By hypothesis, $\theta^\star$ is an extended soft topology on Y, $g_\phi(G_E)$ is a soft I (resp. soft D, soft B) α-open subset of $(Y, \theta^\star, F, \preceq_2)$. Hence a soft map g_ϕ is soft I (resp. soft D, soft B) α-open. □

The result above is restated in case of a soft I (resp. soft D, soft B) α-closed map. One can prove them similarly and so the proof will be omitted.

Theorem 3.10. *Let $\theta^\star$ be an extended soft topology on Y and ϕ is an injective map. Then a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) α-closed if and only if a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-closed.*

Proposition 3.11. *Consider τ is not the indiscrete topology on X. If an injective soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft Bα-open or soft Bα-closed, then $\preceq_2$ is not linearly ordered.*

Proposition 3.12. *Let $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ and $g_\lambda : (Y, \theta, F, \preceq_2\ ) \to (Z, \upsilon, K, \preceq_3)$ be two soft maps. Then then following properties hold, for $x \in \{I, D, B\}$.*

(*i*) *If f_ϕ is a soft xα-continuous map and g_λ is a soft continuous map, then $g_\lambda \circ f_\phi$ is a soft x-continuous map.*

(*ii*) *If f_ϕ is a soft open (resp. soft closed) map and g_λ is a soft xα-open (resp. xα-closed) map, then $g_\lambda \circ f_\phi$ is a soft x-open (resp. xα-closed) map.*

(*iii*) *If $g_\lambda \circ f_\phi$ is a soft x-open map and f_ϕ is surjective soft continuous, then g_λ is a soft x-open map.*

(*iv*) *If $g_\lambda \circ f_\phi$ is a soft closed map and g_λ is an injective soft x-continuous map, then f_ϕ is a soft y-closed map, where $(x, y) \in \{(I, D), (D, I), (B, B)\}$.*

4. Soft $I(D,B)\alpha$-homeomorphism maps

The concepts of soft $I(D,B)\alpha$-homeomorphism maps are introduced and their main properties are discussed. Some examples are constructed to illustrate the relationships among the initiated soft maps.

Definition 4.1. *A bijective soft map $g_\phi : (X,\tau,E,\preceq_1) \to (Y,\theta,F,\preceq_2)$ is called soft I (resp. soft D, soft B) α-homeomorphism if it is soft Iα-continuous and soft Iα-open (resp. soft Dα-continuous and soft Dα-open, soft Bα-continuous and soft Bα-open).*

Remark 4.2. From Definition (4.1), we can note the following:

(i) Every soft I (soft D, soft B) α-homeomorphism map is soft α-homeomorphism.

(ii) Every soft Bα-homeomorphism map is soft Iα-homeomorphism or soft Dα-homeomorphism.

To elucidate that the two items of the remark above are not conversely, we give two examples below.

Example 4.3. Let $X = \{3,5,7,11\}$ be a universe set and $A = \{a_1,a_2\}$ be a parameters set. Consider $\phi : A \to A$ and $f : X \to X$ are both identity maps. We define two partial order relations on X and Y, respectively, as $\preceq_1 = \triangle\bigcup\{(7,5)\}$ and $\preceq_2 = \triangle\bigcup\{(3,11)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau = \{\widetilde{\emptyset},\widetilde{X},F_A\}$ and $\theta = \{\widetilde{\emptyset},\widetilde{Y},F_A,H_A\}$, where $F_A = \{(a_1,\{3\}),(a_2,\{5,7\})\}$ and $H_A = \{(a_1,\{3,5\}),(a_2,\{5,7\})\}$. Then one can readily check that a soft map $f_\phi : S(X_A) \to S(Y_B)$ is soft α-homeomorphism. On the other hand, $f_\phi(F_A) = F_A$ is not a soft Iα-open set and $f_\phi^{-1}(H_A) = H_A$ is not a soft Dα-open set. Hence f_ϕ is not soft I (soft D, soft B) α-homeomorphism.

Example 4.4. In Example above, if we replace only the partial order relation $\preceq_2$ by $\preceq = \triangle\bigcup\{(5,11)\}$, then the soft map f_ϕ is soft D-homeomorphism, but is not soft B-homeomorphism. Also, if we replace only the partial order relation $\preceq_1$ by $\preceq = \triangle\bigcup\{(11,3)\}$, then the soft map f_ϕ is soft I-homeomorphism, but is not soft B-homeomorphism.

Theorem 4.5. *If a bijective soft map $f_\phi : (X,\tau,E,\preceq_1) \to (Y,\theta,F,\preceq_2)$ is soft Iα-continuous (resp. soft Dα-continuous, soft Bα-continuous), Then the following three statements are equivalent:*

(i) f_ϕ is soft Iα-homeomorphism (resp. soft Dα-homeomorphism, soft Bα-homeomorphism);

(ii) f_ϕ^{-1} is soft Iα-continuous (resp. soft Dα-continuous, soft Bα-continuous);

(iii) f_ϕ is soft Dα-closed (resp. soft Iα-closed, soft Bα-closed).

Proof. $(i) \Rightarrow (ii)$ Since f_ϕ is a soft Iα-homeomorphism (resp. soft Dα-homeomorphism, soft Bα-homeomorphism) map, then f_ϕ is soft Iα-open (resp. soft Dα-open , soft Bα-open). It follows from item (ii) of Theorem (3.8), that f_ϕ^{-1} is soft Iα-continuous (resp. soft Dα-continuous, soft Bα-continuous).

$(ii) \Rightarrow (iii)$ The proof follows from item (iii) of Theorem (3.8).

$(iii) \Rightarrow (i)$ It sufficient to prove that f_ϕ is a soft Iα-open (resp. soft Dα-open, soft Bα-open) map. This follows from item (i) of Theorem (3.8). □

Theorem 4.6. *Let $\tau^\star$ and $\theta^\star$ be extended soft topologies on X and Y, respectively. Then a soft map $g_\phi : (X, \tau^\star, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) α-homeomorphism if and only if a map $g : (X, \tau_e^\star, \preceq_1) \to (Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) α-homeomorphism.*

Proof. The proof is obtained immediately from Theorem (2.11) and Theorem (3.9) □

Proposition 4.7. *Let the two soft topologies τ and θ on X and Y, respectively, do not belong to {soft discrete topology, soft indiscrete topology}. If a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft Bα-homeomorphism, then $\preceq_1$ and $\preceq_2$ is not linearly ordered.*

Conclusion

Al-shami et al. [14] combined a soft partially ordered set with a soft topological space to constitute a soft topological ordered space concept. As a contribution of this, we have established the concepts of soft $x\alpha$-continuous, soft $x\alpha$-open, soft $x\alpha$-closed and soft $x\alpha$-homeomorphism maps, for $x \in \{I, D, B\}$. We have completely described these concepts and have deduced some results which connect the initiated soft maps with those maps via topological ordered spaces. To some extent there is a similarity between the results obtained herein and those presented in [15]. The newly ordered maps initiated herein will be an important basis for the further developments on soft topological ordered spaces. In our upcoming research, we intend to give more soft ordered topological concepts and investigate their properties.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Abbas, B. Ali and S. Romaguera, *On generalized soft equality and soft lattice structure*, Filomat, 28 (2014), 1191-1203.

[2] M. Abbas, M. I. Ali and S. Romaguera, *Generalized operations in soft set theory via relaxed conditions on parameters,* Filomat, 31 (2017), 5955-5964.

[3] M. Abo-Elhamayel and T. M. Al-shami, *Supra homeomorphism in supra topological ordered spaces*, Facta Universitatis, Series: Mathematics and Informatics, 31 (2016), 1091-1106.

[4] M. Akdag and A. Ozkan, *Soft α-open sets and soft α-continuous functions*, Abstract and Applied Analysis, Article ID 891341(2014), 7pages.

[5] M. Akdag and A. Ozkan, *On soft α-separation axioms*, Journal of Advanced Studies in Topology, 5 (2014), 16-24.

[6] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, 57 (2009), 1547-1553.

[7] T. M. Al-shami, *Supra β-bicontinuous maps via topological ordered spaces*, Mathematical Sciences Letters, 6 (2017), 239-247.

[8] T. M. Al-shami, *Corrigendum to "On soft topological space via semi-open and semi-closed soft sets, Kyungpook Mathematical Journal, 54 (2014) 221-236"*, Kyungpook Mathematical Journal, 58 (2018), 583-588.

[9] T. M. Al-shami, *Corrigendum to "Separation axioms on soft topological spaces, Ann. Fuzzy Math. Inform., 11 (2016) 511-525"*, Annals of Fuzzy Mathematics and Informatics, 15 (2018), 309-312.

[10] T. M. Al-shami, *On some maps in supra topological ordered spaces*, Journal of New Theory, 20 (2018), 76-92.

[11] T. M. Al-shami, *Soft somewhere dense sets on soft topological spaces*, Communications of the Korean Mathematical Society, 33 (2018), 1341-1356.

[12] T. M. Al-shami, *Comments on "Soft mappings spaces"*, The Scientific World Journal, Volume 2019, Article ID 6903809, 2 pages.

[13] T. M. Al-shami and M. K. Tahat, *I (D, B)-supra pre maps via supra topological ordered spaces*, Journal of Progressive Research in Mathematics, 12 (2017), 1989-2001.

[14] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, *On soft topological ordered spaces*, Journal of King Saud University-Science, https://doi.org/10.1016/j.jksus.2018.06.005.

[15] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, *On soft ordered maps*, General Letters in Mathematics, 5 (2018), 118-131.

[16] T. M. Al-shami, M. E. El-Shafei and B. A. Asaad, *Other kinds of soft β mappings via soft topological ordered spaces*, European Journal of Pure and Applied Mathematics, 12 (2019), 176-193.

[17] S. D. Arya and K. Gupta, *New separation axioms in topological ordered spaces*, Indain Journal Pure and Appllied Mathematics, 22 (1991), 461-468.

[18] A. Aygünoğlu and H. Aygün, *Some notes on soft topological spaces,* Neural Computers and Applications, 21 (2012), 113-119.

[19] K. V. Babitha and J.J. Sunil, *Soft set relations and functions*, Computers and Mathematics with Applications, 60 (2010), 1840-1849.

[20] P. Das, *Separation axioms in ordered spaces*, Soochow Journal of Mathematics, 30 (2004), 447-454.

[21] S. Das and S. K. Samanta, *Soft metric,* Annals of Fuzzy Mathematics and Informatics, 6 (2013), 77-94.

[22] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, *Strong separation axioms in supra topological ordered spaces*, Mathematical Sciences Letters, 6 (2017), 271-277.

[23] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, *Supra R-homeomorphism in supra topological ordered spaces*, International Journal of Algebra and Statistics, 6 (2017), 158-167.

[24] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, *Generating ordered maps via supra topological ordered spaces,* International Journal of Modern Mathematical Sciences, 15 (2017), 339-357.

[25] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, *Two notes on "On soft Hausdorff spaces"*, Annals of Fuzzy Mathematics and Informatics, 16 (2018), 333-336.

[26] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, *Partial soft separation axioms and soft compact spaces,* Filomat, 32 (2018), 4755-4771.

[27] F. Feng and Y. M. Li, *Soft subsets and soft product operations*, Information Science, 232 (2013), 1468-1470.

[28] M. H. Ghanim, *On some types of topologies*, Master Thesis, Al-Azher University, 1976.

[29] M. D. Green, *Locally convex topology on a preordered space*, Pascific Journal of Mathematics, 26 (1968), 487-491.

[30] T. Hida, *A comprasion of two formulations of soft compactness*, Annals of Fuzzy Mathematics and Informatics, 8 (2014), 511-524.

[31] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, *γ-operation and decompositions of some forms of soft continuity in soft topological spaces*, Annals of Fuzzy Mathematics and Informatics, 7 (2014), 181-196.

[32] M. K. R. S. V. Kumar, *Homeomorphism in topological ordered spaces*, Acta Ciencia Indian, XXVIII(M) (2012), 67-76.

[33] H. P. A. Künzi, *Completely regular ordered spaces*, Order, 7 (1990), 283-293.

[34] H. P. A. Künzi and T. A. Richmond, *Completely regularly ordered spaces versus T_2-ordered spaces which are completely regular,* Topology and its Applications, 135 (2004), 185-196.

[35] D. S. Leela and G. Balasubramanian, *New separation axioms in ordered topological spaces*, Indian Journal of Pure and Applied Mathematics, 33 (2002), 1011-1016.

[36] P. K. Maji, R. Biswas and R. Roy, *Soft set theory*, Computers and Mathematics with Applications, 45 (2003), 555-562.

[37] S. D. McCartan, *Separation axioms for topological ordered spaces*, Mathematical Proceedings of the Cambridge Philosophical Society, 64 (1986), 965-973.

[38] S. D. McCartan, *Bicontinuous preordered topological spaces*, Pacific Journal of Mathematics, 38 (1971), 523-529.

[39] O. Mendez, L. H. Popescu and E. D. Schwab, *Inner separation structures for topological spaces*, Blakan Journal of Geometry and Its Applications, 13 (2008), 59-65.

[40] W. K. Min, *A note on soft topological spaces*, Computers and Mathematics with Applications, 62 (2011), 3524-3528.

[41] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, 37 (1999), 19-31.

[42] L. Nachbin, *Topology and ordered*, D. Van Nostrand Inc. Princeton, New Jersey, 1965.

[43] S. Nazmul and S. K. Samanta, *Neigbourhood properties of soft topological spaces,* Annals of Fuzzy Mathematics and Informatics, 6 (2013), 1-15.

[44] D. Pei and D. Miao, *From soft sets to information system*, In Proceedings of the IEEE International Conference on Granular Computing, 2 (2005), 617-621.

[45] L. Popescu, *R-Separated spaces*, Blakan Journal of Geometry and Its Applications, 6 (2001), 81-88.

[46] K. Qin and Z. Hong, *On soft equality*, Journal of Computational and Applied Mathematics, 234 (2010), 1347-1355.

[47] K. K. Rao and R. Chudamani, *α-homeomorphism in topological ordered spaces*, International Journal of Mathematical Sciences, Technology and Humanities, 52 (2012), 541-560.

[48] M. Shabir and M. Naz, *On soft topological spaces*, Computers and Mathematics with Applications, 61 (2011), 1786-1799.

[49] L. E. Ward, *Partially ordered topological spaces*, Proceedings of the American Mathematical Society, 5 (1954), 144-169.

[50] I. Zorlutuna, M. Akdag, W. K. Min and S. K. Samanta, *Remarks on soft topological spaces*, Annals of Fuzzy Mathematics and Informatics, 2 (2012), 171-185.

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Forecasting earnings of firm's listed in ASE using ARIMA model

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Abstract. The study aims to estimate and forecast earnings of the firms listed in Amman Stock exchange (ASE) using a time series data of earning per share (EPS) for the period from 1978 till 2016. The data has been extracted from firms' annual reports. A wavelet Transform (WT) decomposes the data and detects the fluctuations and outlay values. The parameters p, d, and q are estimated using the ARIMA model, the results show that the ARIMA models accuracy criteria MASE and RSME have the lowest values (0.7089 and 0.0709) respectively, thus the forecasting accuracy is high. It is concluded that firms' earnings show slow increasing trend for the upcoming 38 financial years.

Keywords: forecasting, WT, ARIMA model, earning per share, Jordan.

1. Introduction

Earning refers to after-tax net income, sometimes called the bottom line, or a profit, it's a crucial aspect that affect the firm's share price, also it indicates whether the business will be profitable and successful in the long run or not, so it measured a firm performance which is formed based on the accrual basis of accounting. Earning is an important figure that it's used extensively as measure of firm performance by users of financial information like investors, management, creditors, customers and all other stakeholders to make decision for future actions, therefore the need to forecast firms' future performance is

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highly importance for them. The main goal of financial reporting is to provide useful information to make informed financial decisions. Thus, the evaluation of accounting information in forecasting future profitability has been always studied and presented in financial accounting studies (Oskouei and Zadeh, 2017; Alsinglawi and Aladwan, 2018).

The aim of this research is to forecast the future firm's earning based on past earnings' pattern by using Earning per Share (EPS) as a single figure that summarize the firm's' performance, EPS is generally considered as the most important factor to determine the share price and the firms' value (Islam et al., 2014). This study is supposed to offer a valuable understanding for the firm's expected financial performance. Therefore, a more accurate forecasting would be helpful for decision makers to make a data-driven decision.

Financial data generally are a type of time series data that often show trending and seasonality that presenting a challenge to develop an effective forecasting model. The issue of how effectively modeling financial series data and how to increase the superiority of forecasting is still unresolved in literatures. Plenty of previous studies used different types of models in forecasting like; Autoregressive Integrated Moving Average (ARIMA) model; ARIMA is considered as most widely used approaches in time series forecasting and able to describe the autocorrelations in the data. The ARIMA framework to forecasting is formerly developed by (Box et al., 2015).

The primary objective of this study is to explore whether the use of earnings' time series based on Earning per share (EPS) would help to forecast the future direction and movement of firms' earnings? Therefore, to be able to capture the future earnings of firms, we will try to achieve the following objectives that are: 1) detect the earning's trend of firms listed in ASE;2) select the best ARIMA model firms; 3) forecast earning's trend for firms and finally develop ARIMA model that is able to predict future earnings of the firms.

The rest of the paper is organized as follows. Section 2 describes the literature review. Section 3 introduces the methodology and framework. Sections 4 describe the data descriptions and analytical results. Finally, Section 5 presents our conclusions.

2. Literature review

Many investors depend on earnings to understand firms' performance to make their investment decisions; the basic measurement of earnings is earnings per share (EPS). This metric is considered as a summary indicator for performance (Schroeder et al., 2009). Most of the time the earnings forecasts are based on analysts' prospecting of firms' growth and profitability; to forecast earnings, stock analysts shape financial models that estimate prospective revenues and costs, and incorporate other factors like economic growth, currencies and other macroeconomic factors that influence firms' growth (Ramezani et al., 2002).

The analysts use market research reports to get a sense of firms' growth trends in order to understand the dynamics of the firms under consideration.

Many researchers employed mathematical modeling like ARIMA model to forecast financial series data like (Mills et al., 2011; Steel, 2014; Uko and Nkoro 2012; Devi et al., 2013; Jaya and Sundar, 2012), but still nobody used series data of EPS as a proxy to forecast earnings.

2.1 Time series forecasting

The time series is defined as data points indexed in time order, it is a series taken at consecutive similarly spaced points in time. Therefore, it is ordered and the requirement to maintain this ordering imposes certain restrictions on its processing. Furthermore, the time series data usually ordered by factors such as distance but naturally the time factor is the main ordering encounter. Thus, some groups are referred to as time series. Time series analysis has widely applications in many fields like; Sales, accounting, economic and stock market forecasting. Time series analysis is used to recognize the relationship between the attributes of current value to that of its previous or later values (Mills et al., 2011). Based on this for financial data like earnings we are interested in identifying previous values to see how current values get affected, also we are interested to forecast the future values (Steel, 2014). ARIMA model is widely used to analyze time series data and to understand the impact of past values in forecasting future values. Wrathful to say It was rarely to find in literatures studies that used ARIMA model to forecast future earning based on time data series of earning per share (EPS), most research work focused on forecasting stock market indices, like Junior et al (2014) who assessed the performance of ARIMA model for time series forecasting of IBOVESPA, they concluded that the ARIMA model can be used for financial time series data forecasting. A study by Uko and Nkoro (2012) that analytically compared the influence of ECM, VAR, and ARIMA models in forecasting inflation of Nigeria, it revealed that ARIMA is a superior forecaster of inflation at Nigeria.

Devi et al (2013) has used ARIMA model with its parameters to forecast the NSE Nifty Midcap50 companies among them top 4 companies the results were implemented using criterions like AIC & BIC. Paul et al (2013) empirically found that the best ARIMA (2, 1, 2) model for forecasting based on AIC, SIC AME, RMSE, MAPE in content of SPL data series. Jaya and Sundar (2012) used ARIMA model for 19 IT firms and analyzed the market capitalization of the firms. Authors found that firms are categorized into three trends i.e. companies on an upward, linear and downward trends. Shrimal and Prasad (2016) found the best ARIMA model for predicting market capitalization using some of mathematical criteria such as RMSE and MAPE; they applied this model on 21companies.

2.2 Why earning per share (EPS)?

The term earnings per share (EPS) summarize a firm's earnings that are a net income after taxes and preferred stock dividends. The EPS is evaluated by dividing net income earned in a specific annually reporting period, by the entire number of shares remaining during the same period, EPS is most important variable that affect the share's price, it is a key driver of share prices, and it is a main component to compute the price-earnings valuation ratio. EPS is used as an indicator to capture a firm's profitability per each unit of shareholder ownership called per share, furthermore profitability can be assessed by prior earnings, current earnings or future projected earnings, therefore earning per share is widely considered to be the most popular method of quantifying a firm's profitability, EPS is generally viewed as a more accurate measure and is more commonly cited (Besley and Brigham, 2007).

3. Methodology

This section consists of the research framework include the mathematical model that is used to achieve the purpose of this research and mathematical criteria used.

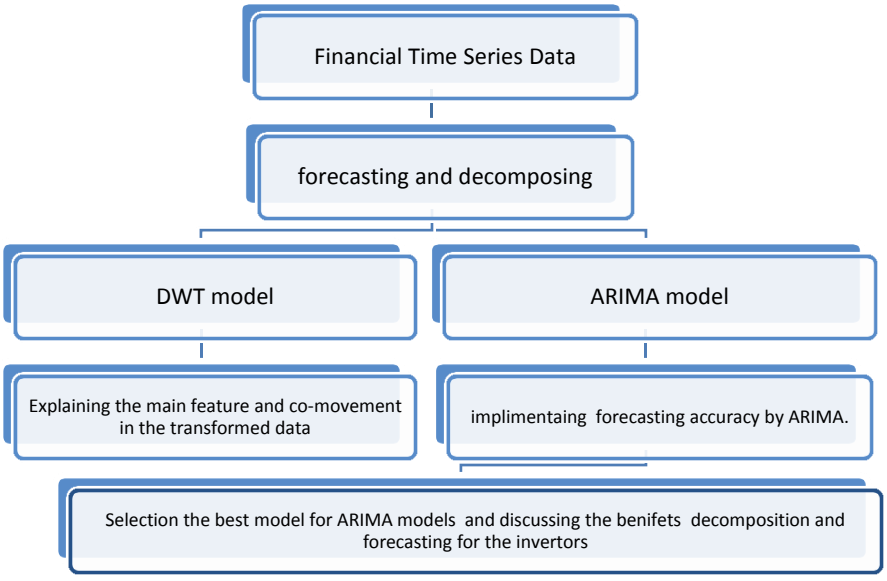


Figure 1: research framework

3.1 ARIMA model

An Auto regressive (AR) process is a series depends on its lagged values. The AR (p) model is a regression model which defined as:

$$Yt = \alpha_0 + \alpha_1 Yt - 1 + \alpha_2 Yt - 2 + ... + \alpha_p Yt - p. \tag{1}$$

Moving average (MA) model is related if the AR process is not the only mechanism that generates Y, but it contains past values with its error terms. MA (q) process is defined as:

$$\epsilon t = \beta_1 \epsilon t + \beta_2 \epsilon t - 2 + \beta_3 \epsilon t - 3 + ... + \beta q Y \epsilon t - q. \tag{2}$$

That contains the white noise errors. When Y has the equations 1 and 2 it is called by ARMA (p, q) model. Since the financial time series data are type of non-stationary, therefore differencing the series will yield a stationary time series (Gujarathi et al., 2012; Box et al., 2015).

If the financial data becomes stationary when differenced d times, we name the series as I (d). Consequently, if ARMA (p,q) is applied to a series financial data which is I(d), then the original time series is ARIMA(p, d, q). The ARIMA methodology proposed that finding the values of p and q for AR and MA respectively by referring to the correlogram.

3.2 Wavelet transform formula

WT is a mathematical model employed to convert the original observations into a time-scale domain. The model is very appropriate with the non-stationary data since most of the financial data are non-stationary. WT can be divided into Discrete Wavelet transform (DWT) and continuous wavelet transforms (CWT). DWT consists of many functions such as Haar, Daubechies, and Maximum overlapping Wavelet transform (MODWT) and others. All of these functions have the same properties with different applications. In this article, the WT will be presented with its equation for all functions. For more details please refer to (Daubechies, 1992; Chiann and Morettin, 1998; Genay et al., 2002; Al Wadi, 2011).

WT is based on Fourier analysis (FT), which represents any function as the sum of the sine and cosine functions. WT is simply a function of time t that obeys a basic rule, known as the wavelet admissibility condition (Genay et al., 2002):

$$C_\varphi = \int_0^\infty (\frac{|\varphi f|}{f}) df < \infty. \tag{3}$$

Where is the FT, it was introduced to solve problems associated with the FT with non-stationary signals and the signals that are localized Time-Domain dimensions.

Father wavelets describe low-frequency parts of a signal where the mother wavelets describe high-frequency components. Equation (4) represents the father and mother wavelets respectively, with j=1,2,3,..., J in the J-level wavelet decomposition(Gencay, et, al., 2002):

$$\Phi j,k = 2^{-\frac{j}{2}}\phi(t - \frac{2^j k}{2^j}). \tag{4}$$

Father and mother wavelets and satisfy the following function:

$$\int \phi(t)dt = 1 and \int \varphi(t)td = 0. \tag{5}$$

The WT approximation to $f(t)$ is defined by:

$$F(t) = \sum S_{J,K}\Phi_{J,K}(t) + \sum d_{J,K}\varphi_{J,K}(t) + \sum d_{J-1,K}\varphi_{J-1,K}(t) + ... + \sum d_{1,K}\varphi_{1,K}(t), \tag{6}$$

$$S_J(t) = \sum S_{J,K}\Phi_{J,K}(t), \tag{7}$$

$$D_J(t) = \sum d_{J,K}\Phi_{J,K}(t). \tag{8}$$

Where S_J (t), and D_J(t) are introducing the smooth and details coefficients respectively. The smooth coefficients dives the most important features of the data set and the details coefficients are used to detect the main features in the dataset. For more details about the WT and its functions please refer to (Wadi, 2015; Al-Khazaleh et al., 2015).

3.2.1 Accuracy criteria

In this section the researchers will present the criteria which have been used to make a fair comparison, and then present the framework comparison with more details. The researches have been adopted to compare the performance of the models within two types of accuracy criteria which are Mean absolute square error (MSE), Root mean squared error (RMSE), and Mean absolute error (MAE). For more details about the mathematical model refer to (Aggarwal et al., 2008; Al Wadi et al., 2011).

4. Results

4.1 Data description

In this research, the statistical population includes firms listed in ASE over the time period of 1978-2016, a complete 38 years of time series data for 266 firms listed firms in ASE; because of variations in the number of listed companies from a year to year, we have used average earning per share(EPS) as a proxy to the firms' earning, EPS is considered as summary indicator that explain the

firms' performance in term of profitability (Schroeder et al., 2009; Aladwan et al., 2018). The time series data were accessed from the ASE and extracted from firms' annual financial reports. The MATLAB and MINTAB software were used to analyze the data.

Figure 2 represents the model of the study.

Research hypothesis	Independent Variable	Dependent Variable	Statistics
Modeling and forecasting EPS	Time	EPS	Histogram, accumulated histogram and descriptive statistics, WT, ARIMA

Figure 2: Data analysis matrix

Histogram, accumulated histogram and descriptive statistics of the time series are shown in figure 3, which shows a linear path with positive slope; therefore, it is none stationary homogenous type, characterized by constant changes from one period to another.

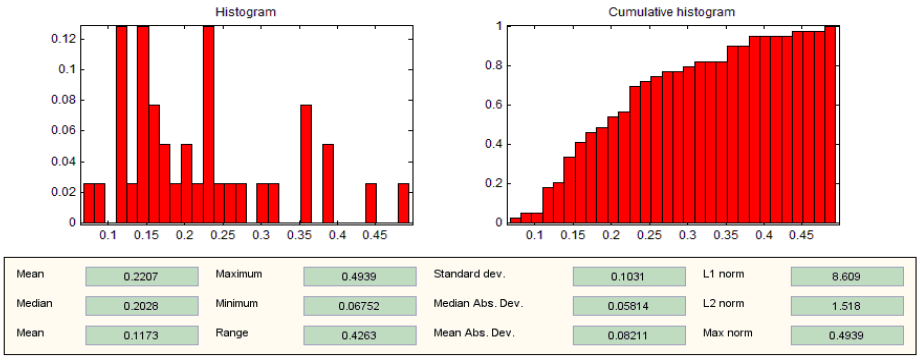


Figure 3: Statistical Analysis

4.1.1 Decomposing time series

Usually, the time series data has three components that are a trend, noise and seasonal components. Decomposition of the time series means separating original time series into these components. Therefore, Fig (4) shows the decomposition of based on WT. The decomposition consists of a1, which are the approximated coefficients used for the proper forecasting and d1 that show the fluctuations of data. Mathematically, the equation can be represented as S= a1+d1 where S is the original data.

Refer to the Fig (4) a visual inspection of the time plots shows that EPS data time series has a trend of random fluctuations this means the data are

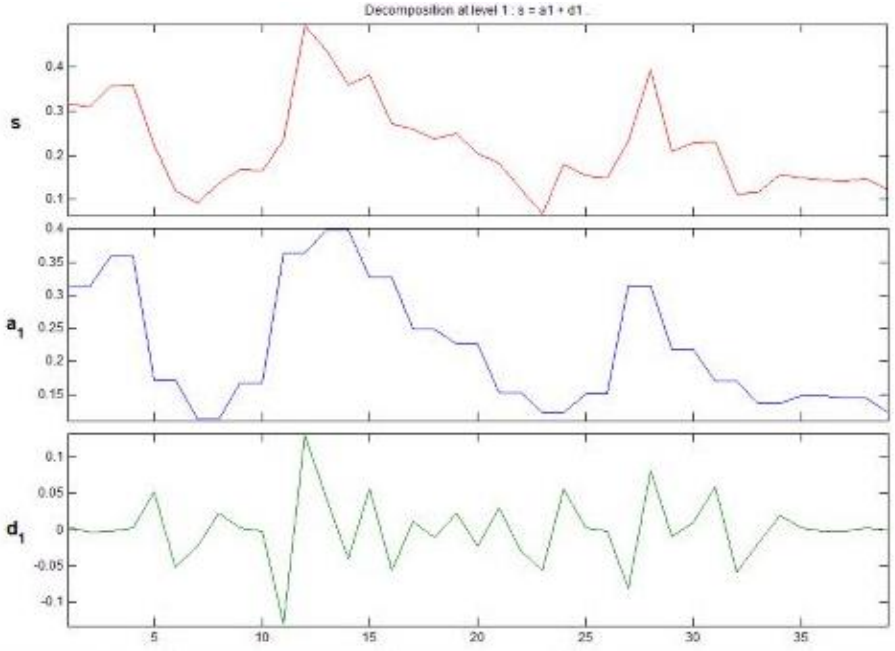


Figure 4: EPS 1978-2016 using WT

non-stationary and it is not constant around mean and variance. This type of non-stationary time series data contains a seasonal trend can be carried out by spectral Analysis function which is WT. Yield, random trend can be transformed into a linear trend. Before conducting additional analysis using ARIMA model then the data has to be discussed with its behaviors. Therefore, d1 is used to clarify the main features and fluctuations of the time series data, it has been clear that there was much fluctuation we descript that there is many events that ASE faced during the period under study which summarized in table 1

Table 1: events that effect EPS

obs	**Year**	**Event that affect the EPS**
4	1981	high earnings from exports to Iraq due to Iraq-Iranian Ware
11	1988	shrinking in liquidity because of Releasing ties with Palestine
13	1990	Iraq Kuwait conflict that enhanced demand and support earnings
17	1994	pull back in demand of products because of sanctions on Iraq
28	2005	Enhanced liquidity because of foreign investment from Gulf investors
31	2008	International financial crisis
34	2011	Syrian conflict

After we have done the decomposing process of time series data, we applied the ARIMA forecasting process.

The fitted ARIMA models were diagnosed using MASE and RMSE. Parameter estimation for the ARIMA models was done using the Gaussian MLE

Table 2: Average of EPS Earnings

	ARIMA Average EPS. Jordanian Dinar
Model	(2,2,1)
MASE	0.7089
RMSE	0.0709

criterion. The ARIMA models fitted based on the lowest value of MASE (0.7089) and RMSE (0.0709) as appear in table 2, with a fit ARIMA is ARIMA (2,2,1).

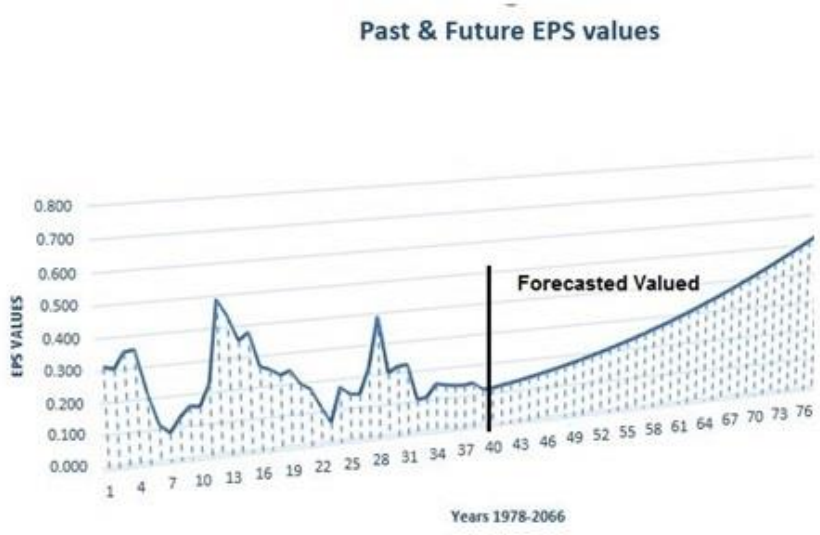


Figure 5: past and future EPS values

Figure 3 shows the EPS original time data series from (1978 till 2016) and the forecasted data of the EPS for the coming forty years (2017 till 2066), this suggests that the EPS long term trend is up ward slopping, which gives an indication that ASE firms' long-term profitability is growing. Furthermore, the EPS as experienced high fluctuation in the past 38 years, in the year of 1991, we are expecting the EPS value will reach its high past level in the year of 2044.

5. Conclusion

In this paper we deployed a WT model to decompose the data to detect the fluctuation and outlier values, then we utilized ARIMA model in forecasting firms' future earnings using earning per share (EPS) time series of firms' listed ASE of the years 1978 to 2016. It is clear that ARIMA model offers an excellent technique for forecasting any variable like EPS. It is strength lies in its fitting varieties of different types of time series with any pattern of change. In

the process of model building, the original data is found none stationary then converted to be stationary. An ARIMA (2, 2, 1) model is developed for analyzing and forecasting EPS for ASE firms among all of various tentative ARIMA models as it has lowest BIC values. From the results, it can be observed that influence R Square value is (95%) high and Mean Absolute Percentage Error is very small for the fitted model. Therefore, the forecasting accuracy is high. It is concluded that firms' earnings show slow fluctuations and increasing trend for upcoming seventy-six financial years.

References

[1] L.M. Saini Aggarwal, Kumar Ashwani, *Price forecasting using wavelet transform and LSE based mixed model in Australian electricity market*, International Journal of Energy Sector Management, 2 (2008), 521-546.

[2] O. Alsinglawi, M. Aladwan, *The effect of invisible intangibles on volatility of stock prices,* Modern Applied Science, 11 (2018),290-300.

[3] M. Aladwan, O. Alsinglawi, O. Hawatmeh, *The applicability of target costing in Jordanian hotels industry*, Academy of Accounting and Financial Studies Journal, 22 (2018), 1-13.

[4] A.M. Al-Khazaleh, S. Al Wadi, F. Ababneh, *Wavelet transform asymmetric winsorized mean in detecting outlier values*, Far East Journal of Mathematical Sciences, 96 (2015).

[5] S. Al Wadi, M.T. Ismail, S.A.A. Karim, *A comparison between the Daubechies wavelet transformation and the fast Fourier transformation in analyzing insurance time series data*, Far East J. Appl. Math., 45 (2010), 53-63.

[6] S. Alwadi, *Existing outlier values in financial data via wavelet transform*, European Scientific Journal, 11 (2015).

[7] M.T.I.S. Al Wadia, M. Ismail Tahir, *Selecting wavelet transforms model in forecasting financial time series data based on ARIMA model*, Applied Mathematical Sciences, 7 (2011), 315-326.

[8] S. Besley, E. Brigham, *Essentials of managerial finance*, Cengage Learning, 2007.

[9] G.E. Box, G.M. Jenkins, G.C. Reinsel, G.M. Ljung, *Time series analysis: forecasting and control*, John Wiley & Sons, 2015.

[10] C. Chiann, P.A. Morettin, *A wavelet analysis for time series*, Journal of Nonparametric Statistics, 10 (1998), 1-46.

[11] I. Daubechies, *Ten lectures on wavelets*, vol. 61 of CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Philadelphia, Pa, USA, 1992.

[12] B.U. Devi, D. Sundar, P. Alli, *An effective time series analysis for stock trend prediction using ARIMA model for nifty midcap-50*, International Journal of Data Mining and Knowledge Management Process, 3 (2013), 65.

[13] R. Gençay, F. Selçuk, B. Whitcher, *An introduction to wavelets and other filtering methods in finance and economics*, Waves in Random Media, 12 (2002), 399-412.

[14] Porter Gunasekar Gujarathi, *Basic econometrics,* McGraw Hill Pvt. Ltd., 2012.

[15] M. Islam, T.R. Khan, T.T. Choudhury, A.M. Adnan, *How earning per share (EPS) affects on share price and firm value*, European Journal of Business and Management, 6 (2014), 97-108.

[16] M. Jaya, K. Sundar, *Forecasting of market capitalization through arima (with special reference to Indian information technology firms)*, Asian Journal of Research in Business Economics and Management, 2 (2012), 1-21.

[17] P.R. Junior, F.L.R. Salomon, Pamplona, E. de Oliveira, *Arima: an applied time series forecasting model for the bovespa stock index*, Applied Mathematics, 5 (2014), 3383.

[18] B.C. Mills, P.R. Mecklenburg, R. Morselli, R.D. Sedgewick, U.S. Patent Application, 12/603, 154, 2011.

[19] Z.H. Oskouei, R.B.H. Zadeh, *The prediction of future profitability using life cycle theory based on cash flow pattern*, Advances in Economics and Business, 5 (2017), 167-175.

[20] J.C. Paul, M.S. Hoque, M.M. Rahman, *Selection of best ARIMA Model for forecasting average daily share price index of pharmaceutical companies in Bangladesh: a case study on square pharmaceutical Ltd.*, Global Journal of Management and Business Research, 2013.

[21] N.D. Princeton, *Model based inference in the life sciences: a primer on evidence*, 2008.

[22] C.A. Ramezani, L. Soenen, A. Jung, *Growth, corporate profitability, and value creation*, Financial Analysts Journal, 56-67, 2002.

[23] R.G. Schroeder, M.W. Clark, J.M. Cathey, *Financial accounting theory and analysis: text and cases*, John Wiley & Sons, 2009.

[24] K. Shrimal, H. Prasad, *Prediction of market capitalization trend through selection of best ARIMA model with reference to indian infrastructural companies.*

[25] A. Steel, *Predictions in financial time series data*, 2014.

[26] A.K. Uko, E. Nkoro, *Inflation forecasts with ARIMA, vector autoregressive and error correction models in Nigeria*, European Journal of Economics, Finance and Administrative Science, 50 (2012), 71-87.

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Rough set theory applied to UP-algebras

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Abstract. In this paper, the concept of roughness in UP-algebras is introduced. We study the lower and upper approximations of UP-subalgebras and UP-ideals and prove that the lower/upper approximation of UP-subalgebra (resp., UP-ideals) is a UP-subalgebra (resp., UP-ideals). A connection between rough sets and UP-Algebras with their weak and strong ideals have also been taken under consideration and some related results have been shown.
Keywords: UP-algebra, UP-ideal, congruence, lower and upper approximations, definable.

1. Introduction

The notion of rough sets and its approximations (lower/upper) spaces were introduced in early 1980s by Pawlak in his papers [26], [27] and [28] to deal with uncertain knowledge in information system, artificial intelligence and cognitive sciences in fields such as machine learning, knowledge aquisition, decision analysis etc. As an R and D (research and developement) a connection between rough set theory with algebraic systems (structures) came into existence. As a result many authors introduced lot of concepts. Kuroki [12] introduced roughness in semigroups and its ideals which can be considered to be concept of roughness in classical algebras. Then Biswas and Nanda [18] introduced the notion of rough groups and rough subgroups. Xiao and Zhang [15] studied rough prime ideals and rough fuzzy prime ideals in semigroups. The basic logical algebras have been established and investigated widely by many authors. Li. and Yin [7] defined ϑ-lower and T-upper fuzzy rough approximation operators on a semigroup whereas Qi. and Liu. [16] studied the cocepts of rough appoximations in Boolean Algebras. Davvaz [3] introduced the concept of roughness in rings.

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Hu and Li. [14] gone through BCH Algebras. Prabpayak and Leerawat [19] introduced KU-ideals which can be consider to be an interesting idea in logical algebras, further they studied homomorphisms of KU-Algebras [6] and investigated some related results and properties of KU-algebras. Yaqoob et al. [13] introduced cubic KU-ideals which further became an interesting direction for study of diffrent types of classical properties, logical properties, fuzzificational (intuitionistic) properties, Neutrosophic properties in modern modern algebra. Ameri et al. [17] , studied rough set theory applied to hyper BCK-Algebras, where Dudek et al. [21] studied rough set theory applied to hyper BCI-Algebras. Further Jun et al. [23] gone through the concepts of roughness in BCC-algebras.

Fuzzy sets was introduced by Zadeh [25] in 1965. From then till now many authors have considered and studied fuzziness in different branches of sciences technologies and engineering. Fuzzyness [10], Neutrosophic [4] and in different types of Logical Algebras [22], [20], [11] are some recent trend and interest of study for numerous researchers and authors. Including numerous authors Moin and Ali [11] have studied roughness in KU-algebras recently. Ahn [2] et. al have studied rough fuzzy ideals in BCK/BCI algebras.

Both logical and classical algebras have been the basic building tools in the study of different types and directions of applied algebras more precisely in computer applications related to artificial intelligence which simulate a human being in dealing with certainty and uncetainty in information with the help of logical techniques. Tasks related to these concepts can easily be solved by these techniques. Some imoprtant types of basic logical algebras are BCI/BCK algebras, BL-algebras [8] and many more. Torkzadeh and Ghorbani [9] studied rough filters in B-Algebras. Different types of these algebras are perfect BL-algebras and local BL-algebras, SBL-algebras etc. which have been studied by many authors. These concepts are an interesting part for roughness and softness with or without their hyper structures.

We have applied roughness concept in UP-algebras which is introduced recently by Impan [1]. Some classical results based on rough set theory is applied to UP-algebras and related results have been studied based on this concept. By means of lower and upper approximations we have shown properties of rough ideals of a UP-algebra. It is shown that a strong UP-ideal with respect to its upper and lower approximation of a UP-algebra is again a strong ideal.

2. Preliminaries

In this section we shall define some basic concepts including UP-algebras, UP-subalgebras, UP-ideals with examples based on them.

Definition 1 ([6]). *An algebra* $(X, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X$,

$(ku1) : (x * y) * [(y * z) * (x * z)] = 0,$

$(ku2) : x * 0 = 0,$

$(ku3) : 0 * x = x,$
$(ku4) : x * y = 0 = y * x$ *implies* $x = y.$

In a KU-algebra X a binary relation $' \leq'$ can be considered by: $x \leq y$ if and only if $y * x = 0.$

Proposition 1 ([6]). $(X, *, 0)$ *is a KU-algebra if and only if it satisfies:*
$(ku5) : (y * z) * (x * z) \leq (x * y),$
$(ku6) : 0 \leq x,$
$(ku7) : x \leq y,\ y \leq x$ *implies* $x = y,$
$(ku8) : x \leq y$ *if and only if* $y * x = 0.$

Proposition 2. *In a KU-algebra, the following identities are true [20]*
(1) $z * z = 0,$
(2) $z * (x * z) = 0,$
(3) $x \leq y$ *imply* $y * z \leq x * z,$
(4) $z * (y * x) = y * (z * x)$, *for all* $x, y, z \in X,$
(5) $y * [(y * x) * x] = 0.$

Definition 2 ([1]). *By a UP-algebra we mean an algebra* $(A, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X,$
$(UP - 1) : (y * z) * [(x * y) * (x * z)] = 0,$
$(UP - 2) : 0 * x = x,$
$(UP - 3) : x * 0 = 0,$
$(UP - 4) : x * y = 0 = y * x$ *implies* $x = y.$

Example 1 ([1]). Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cap A' = A' \cap B = B - A$ for all $A, B \in P(X)$. Then $(P(X); *, \emptyset)$ is a UP-algebra which is the power UP-algebra of type 1.

Example 2 ([1]). Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B\ \forall\ A, B \in P(X)$. Then $(P(X); *, X)$ is a UP-algebra which is a power UP-algebra of type 2.

Example 3. Let $A = \{0, a, b, c\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

It is easy to see that $A = \{0, a, b, c\}$ is UP-algebra.

Example 4. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	b	0	0	0
c	0	b	b	0	0
d	0	b	b	d	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Example 5. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Example 6. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	a	0	c	0
c	0	a	0	0	0
d	0	a	b	c	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Proposition 3. *In a UP-algebras A the following properties hold for any $x, y, z \in A$:*

(1) $x * x = 0$,
(2) $x * y = 0$ *and* $y * z = 0 \Rightarrow x * z = 0$,
(3) $x * y = 0 \Rightarrow (z * x) * (z * y) = 0$,
(4) $x * y = 0 \Rightarrow (y * z) * (x * z) = 0$,
(5) $x * (y * x) = 0$,
(6) $(y * x) * x = 0 \Longleftrightarrow x = y * x$, *and*
(7) $x * (y * y) = 0$

Proposition 4. *Let $A = (A, *, 0)$ be UP-algeras, then define a binary relation $\leq$ on A as follows: for all $x, y, z \in A$*

$x \leq y \Leftrightarrow x * y = 0$. *Based on this binary relation we have that in an UP-algebra A, the following properties are true for any $x, y, z \in A$:*

(1) $x \leq x$,
(2) $x \leq y$ *and* $y \leq x \Rightarrow x = y$,
(3) $x \leq y$ *and* $y \leq z \Rightarrow x \leq z$,
(4) $x \leq y \Rightarrow z * x \leq z * y$,
(5) $x \leq y \Rightarrow y * z \leq x * z$,
(6) $x \leq y * x$, *and*
(7) $x \leq y * y$.

Definition 3. *Let $A = (A, *, 0)$ be a UP-algebra. Then a subset S of A is called UP-subalgebras of A if the constant 0 of A is in S and $(S, *, 0)$ itself form a UP-algebra. Clearly, A and $\{0\}$ are UP-algebras of A.*

Definition 4. *Let A be a UP-algebra. Then a subset B of A is called a UP-ideal of A if it satisfies:*
(i) The constant 0 of A is in B and
(ii) for ant $x, y, z \in A$, $x * (y * z) \in B$ and $y \in B \Rightarrow x * z \in B$.
Clearly, A and $\{0\}$ are UP-ideals of A.

Example 7. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	0	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

We find here that $A = (A; *, 0)$ is a UP-algebras. Further $\{0, a, b\}$ and $\{0, a, c\}$ are UP-ideals of A.

Definition 5. *Let S be a nonempty subset of a UP-algebra A and $0 \in S$. Then,*
(1) *S is called a weak UP-ideal of A if $y * x \in S$ and $y \in S \Rightarrow x \in S$, for all $x, y \in A$;*
(2) *S is called a strong UP-ideal of A if $(y * x) \cap S \neq \emptyset$ and $y \in S \Rightarrow x \in S$, for all $x, y \in A$.*

3. Rough approximations in UP-algebras

Let V be a set and E an equivalence relation on V and let $P(V)$ denote the power set of V. For all $a \in V$, let $[a]_E$ denote the equivalence class of a with respect to E. Define the functions $E_-, E^- : P(V) \to P(V)$ as follows: $\forall\ S \in P(V)$,

$$E_-(S) = \{a \in V : [a]_E \subseteq S\}$$

and

$$E^-(S) = \{x \in V : [a]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an approximation space. Let S be a subset of V. Then S is said to be definable if $E_-(S) = E^-(S)$ and rough otherwise. $E_-(S)$ is called the lower approximation of S while $E^-(S)$ is called the upper approximation.

Throughout this section A will represent a UP-algebra. Let I be a UP-ideal of A. Define a relation Θ on A by $(a, b) \in \Theta$ if and only if $a * b \in I$ and $b * a \in I$. Then Θ is an equivalence relation on A related to a UP-ideal I of A. Moreover satisfies $(a, b) \in \Theta$ and $(u, v) \in \Theta$ imply $(a * u, b * v) \in \Theta$.

Hence Θ is a congruence relation on A. Let I_a denote the equivalence class of a with respect to the equivalence relation Θ related to a UP-ideal I of A, and A/I denote the collection of all equivalence classes, that is, $A/I = \{I_a : a \in A\}$. Then $I_0 = I$. If $I_a * I_b$ is defined as the class containing $a*b$, that is, $I_a * I_b = I_{a*b}$, then $(A/I, *, I_0)$ is a UP-algebra. Let Θ be an equivalence relation on A related to a UP-ideal I of A. For any nonempty subset S of A, the lower and upper approximation of S are denoted by $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$ respectively, that is,

$$\underline{\Theta}(I, S) = \{a \in A : I_a \subseteq S\}$$

and

$$\overline{\Theta}(I, S) = \{a \in a : I_a \cap S \neq \emptyset\}.$$

If $I = S$, then $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$ are denoted by $\underline{\Theta}(I)$ and $\overline{\Theta}(I)$, respectively.

Definition 6 ([28])**.** Given an approximation space (U, Θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, Θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.

Definition 7 ([28])**.** Let (U, Θ) be an approximation space and X be a nonempty subset of U.

(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.

(ii) If $\overline{Apr}(X) = \emptyset$, then X is called empty interior.

(iii) If $Apr(X) = U$, then X is called empty exterior.

Example 8. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	0	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

We find here that $A = (A; *, 0)$ is a UP-algebras. Further $\{0, a, b\}$ and $\{0, a, c\}$ are UP-ideals of A.

So $I = \{0, a\}$ is a UP-ideal of A ($I \lhd A$) and let Θ be an equivalence relation on A related to I. Then $I_0 = K_1 = K$, $K_2 = \{2\}$, $K_3 = \{3\}$, and $K_4 = \{4\}$.

Hence

$$\begin{aligned}
\underline{\Theta}(K,\{0,a\}) &= \{0,a\} \lhd A \\
\underline{\Theta}(K,\{0,b\}) &= \{b\} \\
\underline{\Theta}(K,\{0,c\}) &= \{c\} \\
\underline{\Theta}(K,\{0,a,b,c\}) &= \{0,a,b,c\} \lhd A
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Theta}(K,\{0,a\}) &= \{0,a\} \lhd A \\
\overline{\Theta}(K,\{0\}) &= \{0,a\} \\
\overline{\Theta}(K,\{b\}) &= \{0,b\} \\
\overline{\Theta}(K,\{a,b,c\}) &= \{0,a,b,c\} \lhd A \\
\overline{\Theta}(K,\{0,b,c\}) &= \{0,a,b,c\} \lhd A \\
\overline{\Theta}(K,\{a,b,c,d\}) &= \{0,a,b,c,d\} \lhd A.
\end{aligned}$$

In above example 8, we know that there exists a non-UP-ideal S of A such that their lower and upper approximation are UP-ideals of A. Also we choose some non-UP-ideals S of A such that their lower and upper approximation are UP-ideals of A.

Proposition 5. *Let Θ and Ξ be equivalence relations on A related to UP-ideals I and J of A, respectively. If S and T are nonempty subsets of A. Then*

(1) $\underline{\Theta}(I,S) \subseteq S \subseteq \overline{\Theta}(I,S)$;
(2) $\underline{\Theta}(I,\emptyset) = \emptyset = \overline{\Theta}(I,\emptyset)$
(3) $\overline{\Theta}(I,S\cup T) = \overline{\Theta}(I,S) \cup \overline{\Theta}(I,T)$;
(4) $\underline{\Theta}(I,S\cap T) = \underline{\Theta}(I,S) \cap \underline{\Theta}(I,T)$;
(5) $S \subseteq T$ *implies* $\underline{\Theta}(I,S) \subseteq \underline{\Theta}(I,T)$ *and* $\overline{\Theta}(I,S) \subseteq \overline{\Theta}(I,T)$;
(6) $\underline{\Theta}(I,S) \cup \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S\cup T)$;
(7) $\overline{\Theta}(I,S\cap T) \subseteq \overline{\Theta}(I,S) \cap \overline{\Theta}(I,T)$;
(8) $\Theta \subseteq \Xi$ *and* $I \subseteq J$ *implies* $\underline{\Xi}(J,S) \subseteq \overline{\Theta}(I,S)$ *and* $\overline{\Theta}(I,S) \subseteq \overline{\Xi}(J,S)$.

Proof. (1) If $x \in \underline{\Theta}(I,S)$, then $x \in I_x \subseteq S$. Hence $\underline{\Theta}(I,S) \subseteq$S. Next, if $x \in S$, then, since $x \in I_x$, we have $I_x \cap S \neq \phi$, and so $x \in \overline{\Theta}(I,S)$. Thus $S \subseteq \overline{\Theta}(I,S)$.

(2) is straightforward.

(3) Note that

$$\begin{aligned}
x \in \overline{\Theta}(I,S\cup T) &\iff I_x \cap (S\cup T) \neq \phi \\
&\iff (I_x \cap S) \cup (I_x \cap T) \neq \phi \\
&\iff I_x \cap S \neq \phi \ \text{ or } I_x \cap T \neq \phi \\
&\iff x \in \overline{\Theta}(I,S) \text{ or } a \in \overline{\Theta}(I,T) \\
&\iff x \in \overline{\Theta}(I,S) \cup \overline{\Theta}(I,T).
\end{aligned}$$

Thus

$$\overline{\Theta}(I,S\cup T) = \overline{\Theta}(I,S) \cup \overline{\Theta}(I,T).$$

(4) Note that

$$\begin{aligned} x \in \underline{\Theta}(I, S \cap T) &\iff I_x \subseteq S \cap T \\ &\iff I_x \subseteq S \ \text{ and } \ I_x \subseteq T \\ &\iff x \in \underline{\Theta}(I, S) \text{ and } x \in \underline{\Theta}(I, T) \\ &\iff x \in \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T). \end{aligned}$$

Thus

$$\underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T).$$

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, by (3) we have

$$\underline{\Theta}(I, S) = \underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T).$$

This implies that $\underline{\Theta}(I, S) \subseteq \underline{\Theta}(I, T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, by (2) we have

$$\overline{\Theta}(I, T) = \overline{\Theta}(I, S \cup T) = \overline{\Theta}(I, S) \cup \overline{\Theta}(I, T).$$

This implies that $\overline{\Theta}(I, S) \subseteq \overline{\Theta}(I, T)$.

(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by (4) we have

$$\underline{\Theta}(I, S) \subseteq \underline{\Theta}(I, S \cup T) \quad \text{and} \quad \underline{\Theta}(I, T) \subseteq \underline{\Theta}(I, S \cup T).$$

This implies $\underline{\Theta}(I, S) \cup \underline{\Theta}(I, T) \subseteq \underline{\Theta}(I, S \cup T)$.

(7) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by (4) we have

$$\overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, S) \quad \text{and} \quad \overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, T).$$

This implies $\overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, S) \cap \overline{\Theta}(I, T)$.

(8) Since $\Theta \subseteq \Xi$. If $x \in \underline{\Xi}(J, S)$, then $J_x \subseteq S$. But $\Theta \subseteq \Xi$, then $I_x \subseteq J_x \subseteq S$, that is, $I_x \subseteq S$. Thus $x \in \underline{\Theta}(I, S)$. Hence

$$\underline{\Xi}(J, S) \subseteq \underline{\Theta}(I, S).$$

Now let x be any element of $\overline{\Theta}(S)$. So $I_x \cap S \neq \phi$, then there exists $y \in I_y \cap S$ such that $y \in I_y$ and $y \in S$. Hence $(y, x) \in \Theta$, that is $y * x \in I$. Since $I \subseteq J$, it follows that $y * x \in J$ and $x * y \in J$ so that $(y, x) \in \Xi$, that is, $y \in J_x$. Therefore $y \in J_x \cap S$, which means that $x \in \Xi(J, S)$. This completes the proof. □

Proposition 6. *Let Θ be an equivalence relation on A related to a UP-ideal I of A. If S is a nonempty subset of A. Then*

(1) $\underline{\Theta}(I, \underline{\Theta}(I, S)) = \underline{\Theta}(I, S)$;
(2) $\overline{\Theta}(I, \overline{\Theta}(I, S)) = \overline{\Theta}(I, S)$;
(3) $\overline{\Theta}(I, \underline{\Theta}(I, S)) = \underline{\Theta}(I, S)$;
(4) $\underline{\Theta}(I, \overline{\Theta}(I, S)) = \overline{\Theta}(I, S)$;
(5) $\underline{\Theta}(I, S) = (\overline{\Theta}(I, S^c))^c$;
(6) $\overline{\Theta}(I, S) = (\underline{\Theta}(I, S^c))^c$;
(7) $\underline{\Theta}(I, I_x) = X = \overline{\Theta}(I, I_x)$, *for all* $x \in A$.

Proof. The proof is straightforward. □

Proposition 7. *Let Θ be an equivalence relation on X related to a UP-ideal I of A. If S is a nonempty subset of A. Then*
(1) $\overline{\Theta}(I,S) * \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I, S*T)$;
(2) If Θ is congruence relation, then $\underline{\Theta}(I,S) * \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I, S*T)$.

Proof. (1) Let c be any element of $\overline{\Theta}(I,S) * \overline{\Theta}(I,T)$. Then $c = p*q$ with $p \in \overline{\Theta}(I,S)$ and $q \in \overline{\Theta}(I,T)$. Thus there exist elements $x, y \in S$ such that

$$x \in I_p \cap S \quad \text{and} \quad y \in I_q \cap T.$$

Thus $x \in I_p$, $y \in I_q$, $x \in S$, and $y \in T$. Since Θ is a congruence on S, it follows that

$$x * y \in I_p * I_q \in I_{p*q}.$$

On the other hand, since $x * y \in S * T$. We have $x * y \in I_{p*q} \cap S * T$, and so $c = p * q \in \overline{\Theta}(I, S*T)$. Thus we have

$$\overline{\Theta}(I,S) * \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I, S*T).$$

(2) Assume that Θ is complete, let c be any element of $\underline{\Theta}(I,S) * \underline{\Theta}(I,T)$. Then $c = p*q$ with $p \in \underline{\Theta}(I,S)$ and $q \in \underline{\Theta}(I,T)$. It follows that $I_p \subseteq S$ and $I_q \subseteq T$. Since Θ is a congruence relation on S, we have

$$I_{p*q} = I_p * I_q \subseteq S * T.$$

So $c = p*q \in \underline{\Theta}(I, S*T)$. Thus

$$\underline{\Theta}(I,S) * \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I, S*T).$$

This completes the proof. □

Proposition 8. *Let Θ and Ξ be equivalence relations on A related to UP-ideals I and J of A, respectively. If S and T are nonempty subsets of A. Then*
(1) $\overline{\Theta \cap \Xi}(I \cap J, S) \subseteq \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S)$;
(2) $\underline{\Theta \cap \Xi}(I \cap J, S) \supseteq \underline{\Theta}(I,S) \cap \underline{\Xi}(J,S)$.

Proof. (1) Note that $\Theta \cap \Xi$ is also a congruence relation on S. Let $c \in \overline{\Theta \cap \Xi}(I \cap J, S)$, then $[I \cap J]_c \cap S \neq \phi$. Then there exists an element $x \in [I \cap J]_c \cap S$. Since $(x,c) \in \Theta \cap \Xi$, we have

$$(x,c) \in \Theta \text{ and } (x,c) \in \Xi.$$

Thus we have $x \in I_c$ and $x \in J_c$. Since $x \in S$, we have $x \in I_c$, $x \in S$ and $x \in J_c$, $x \in S$. This implies that

$$x \in I_c \cap S \quad \text{and} \quad x \in J_c \cap S$$

$$I_c \cap S \neq \phi \quad \text{and} \quad J_c \cap S \neq \phi.$$

So $c \in \overline{\Theta}(I,S)$ and $c \in \overline{\Xi}(J,S)$, hence $c \in \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S)$. Thus we obtain

$$\overline{\Theta \cap \Xi}(I \cap J, S) \subseteq \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S).$$

(2) Since $\Theta \cap \Xi \subseteq \Theta$ and $\Theta \cap \Xi \subseteq \Xi$, which implies that

$$\underline{\Theta}(I,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S) \quad \text{and} \quad \underline{\Xi}(J,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S)$$

$$\Longrightarrow \underline{\Theta}(I,S) \cap \underline{\Xi}(J,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S).$$

This completes the proof. □

Theorem 1. *Let (A,Θ) be an approximation space. Then*
(1) *for every $S \subseteq X$, $\underline{\Theta}(I,S)$ and $\overline{\Theta}(I,S)$ are definable sets,*
(2) *for every $x \in X$, I_x is definable set.*

Proof. (1) By Proposition 6 part (1) and (3), we have

$$\underline{\Theta}(I, \underline{\Theta}(I,S)) = \underline{\Theta}(I,S) = \overline{\Theta}(I, \underline{\Theta}(I,S)).$$

Hence $\underline{\Theta}(I,S)$ is definable. On the other hand by Proposition 6 (2) and (4), we have

$$\overline{\Theta}(I, \overline{\Theta}(I,S)) = \overline{\Theta}(I,S) = \underline{\Theta}(I, \overline{\Theta}(I,S)).$$

Therefore $\overline{\Theta}(I,S)$ is a definable set.

(2) By Proposition 6 (7) the proof is clear. □

Definition 8. *A nonempty subset S of A is called an upper (resp. a lower) rough UP-subalgebra of A if the upper (resp. nonempty lower) approximation of S is a UP-subalgebra of A. If S is both an upper and a lower rough UP-subalgebra of A, we say that S is a rough UP-subalgebra of A.*

Theorem 2. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a UP-subalgebra of I, then*
(1) *$\overline{\Theta}(I,S)$ is a UP-subalgebra of A.*
(2) *$\underline{\Theta}(I,S)$ is a UP-subalgebra of A.*

Proof. (1) Let $x, y \in \overline{\Theta}(I,S)$. Then

$$I_x \cap S \neq \emptyset \text{ and } I_y \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_x$ and $b \in I_y$. It follows that $(a,x) \in \Theta$ and $(b,y) \in \Theta$. Since Θ is a congruence relation on A, we have $(a*b, x*y) \in \Theta$. Hence $a*b \in I_{x*y}$. Since S is a UP-subalgebra of A, we get $a*b \in S$, and therefore $a*b \in I_{x*y} \cap S$, that is, $I_{x*y} \cap S \neq \emptyset$. This shows that $x*y \in \overline{\Theta}(I,S)$, and consequently $\overline{\Theta}(I,S)$ is a UP-subalgebra of A.

(2) Let $x, y \in \underline{\Theta}(I,S)$. Then $I_x \subseteq S$ and $I_y \subseteq S$. Since S is a UP-subalgebra of A, it follows that

$$I_{x*y} = I_x * I_y \subseteq S$$

so that $x*y \in \underline{\Theta}(I,S)$. Hence $\underline{\Theta}(I,S)$ is a UP-subalgebra of A. □

The following example shows that the converse of Theorem 2(1) may not be true.

Example 9. Let $A = \{0, a, b, c, d\}$ be a UP-algebra with the Cayley's table as follows:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	d
b	0	0	0	a	d
c	0	0	0	0	d
d	0	a	a	a	0

Let $I = \{0, a, b\}$ be a UP-ideal of A ($I \lhd A$) and let Θ be an equivalence relation on A related to I. Then $I_0 = I_a = I_b = I$, $I_c = \{c\}$, and $I_d = \{d\}$. Note that $S = \{a, c\}$ is not a UP-subalgebra of A, but $\overline{\Theta}(I, S) = \{0, a, b, c\}$ is UP-subalgebra of A.

Definition 9. *A nonempty subset S of A is called an upper (resp. a lower) rough UP-ideal of A if the upper (resp. nonempty lower) approximation of S is a UP-ideal of A. If S is both an upper and a lower rough UP-ideal of A, we say that S is a rough UP-ideal of A.*

Theorem 3. *Let Θ be a congruence relation on A related to a UP-ideal I of A. If S is a UP-ideal of A containing I, then*

(1) $\overline{\Theta}(I, S)$ *is a UP-ideal of A.*

(2) $\underline{\Theta}(I, S)$ *is a UP-ideal of A.*

Proof. (1) Let S be a UP-ideal of A containing I. Obviously $0 \in \overline{\Theta}(I, S)$. Let $x, y, z \in A$ be such that $y \in \overline{\Theta}(I, S)$ and $x * (y * z) \in \overline{\Theta}(I, S)$. Then

$$I_y \cap S \neq \emptyset \text{ and } I_{x*(y*z)} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_y$ and $b \in I_{x*(y*z)}$. Hence $(a, y) \in \Theta$ and $(b, (x*(y*z))) \in \Theta$, which implies $y*a \in A \subseteq S$ and $(x*(y*z))*b \in I \subseteq S$. Since $a, b \in S$ and S is a UP-ideal, we get

$$y \in S \text{ and } x * (y * z) \in S,$$

it follows from Definition 5 (2) that $x * z \in S$. Note that $x * z \in I_{x*z}$, thus $x*z \in I_{x*z} \cap S$, that is, $I_{x*z} \cap S \neq \emptyset$. Hence $x*z \in \overline{\Theta}(I, S)$ and therefore $\overline{\Theta}(I, S)$ is a UP-ideal of A.

(2) Let S be a UP-ideal of A containing I. Let $x \in I_0$. Then $x \in I \subseteq S$, and so $I_0 \subseteq S$. Hence $0 \in \underline{\Theta}(I, S)$. Let $x, y, z \in X$ be such that $y \in \underline{\Theta}(I, S)$ and $x * (y * z) \in \underline{\Theta}(I, S)$. Then

$$I_y \in S \text{ and } I_x * (I_y * I_z) = I_{x*(y*z)} \subseteq S.$$

Let $w \in I_{x*z} = I_x * I_z$. Then $w = I_x * I_z$ for some $a \in I_x$ and $c \in I_z$. From $a \in I_x$ and $c \in I_z$, we have $(a, x) \in \Theta$ and $(c, z) \in \Theta$. Taking $b \in I_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(a * (b * c), x * (y * z)) \in \Theta \text{ and so } a * (b * c) \in I_{x*(y*z)} \subseteq S.$$

Since S is a UP-ideal of A, it follows from Definition 5 (2) that $w \in a * c \in S$, so that $I_{x*z} \subseteq S$. Hence $x * z \in \underline{\Theta}(I, S)$ and therefore $\underline{\Theta}(I, S)$ is a UP-ideal of A. □

Theorem 4. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a weak UP-ideal of A containing I, then*
(1) *$\overline{\Theta}(I, S)$ is a weak UP-ideal of A.*
(2) *$\underline{\Theta}(I, S)$ is a weak UP-ideal of A.*

Proof. (1) Let S be a weak UP-ideal of A containing I. Obviously $0 \in \overline{\Theta}(I, S)$. Let $x, y \in A$ be such that $y \in \overline{\Theta}(I, S)$ and $y * x \in \overline{\Theta}(I, S)$. Then

$$I_y \cap S \neq \emptyset \text{ and } I_{y*x} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_y$ and $b \in I_{y*x}$. Hence $(a, y) \in \Theta$ and $(b, (y * x)) \in \Theta$, which implies

$$y * a \in I \subseteq S \text{ and } (y * x) * b \in I \subseteq S.$$

Since $a, b \in S$ and S is a weak UP-ideal, we get $y \in S$ and $y * x \in S$, it follows from definition 5 (2) that $x \in S$. Note that $x \in I_x$, thus $x \in I_x \cap S$, that is, $I_x \cap S \neq \emptyset$. Hence $x \in \overline{\Theta}(I, S)$ and therefore $\overline{\Theta}(I, S)$ is a weak UP-ideal of A.

(2) Let S be a weak UP-ideal of A containing I. Let $x \in I_0$. Then $x \in I \subseteq S$, and so $I_0 \subseteq S$. Hence $0 \in \underline{\Theta}(I, S)$. Let $x, y \in A$ be such that $y \in \underline{\Theta}(I, S)$ and $y * x \in \underline{\Theta}(I, S)$. Then

$$I_y \in S \text{ and } I_y * I_x = I_{y*x} \subseteq S.$$

Let $w \in I_x$. Then $w = I_x$ for some $a \in I_x$. From $a \in I_x$, we have $(a, x) \in \Theta$. Taking $b \in I_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(b * a, y * x) \in \Theta \text{ and } b * a \in I_{y*x} \subseteq S.$$

Since S is a weak UP-ideal of A, it follows from definition 5 (2) that $w = a \in S$, so that $I_x \subseteq S$. Hence $x \in \underline{\Theta}(I, S)$ and therefore $\underline{\Theta}(I, S)$ is a weak UP-ideal of A. □

Theorem 5. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a strong UP-ideal of A containing I, then*
(1) *$\overline{\Theta}(I, S)$ is a strong UP-ideal of A.*
(2) *$\underline{\Theta}(I, S)$ is a strong UP-ideal of A.*

Proof. (1) Let $x, y \in X$ be such that

$$(y * x) \cap \overline{\Theta}(I, S) \neq \emptyset \text{ and } y \in \overline{\Theta}(I, S).$$

Then $I_y \cap S \neq \emptyset$ and so there exist $z \in A$ such that $z = y * x$ and $z \in \overline{\Theta}(I, S)$. Hence $I_z \cap S \neq \emptyset$ and so there exist $c, d \in X$ such that

$$c \in I_z \cap S \text{ and } d \in I_y \cap S.$$

Hence $c\Theta z$ and $d\Theta y$ where $c, d \in S$. Thus we $z * c \in I \subseteq S$ and $y * d \in I \subseteq S$. Since S is a strong UP-ideal and $c, d \in S$, we have $z \in S$ and $y \in S$. Thus we have proved that $(y * x) \cap I \neq \emptyset$ and $y \in I$. Since S is a strong UP-ideal, we have $x \in S$ and so $I_x \cap S \neq \emptyset$ which means that $\overline{\Theta}(I, S)$ is a strong UP-ideal of S.

(2) Let $x, y \in X$ be such that

$$(y * x) \cap \underline{\Theta}(I, S) \neq \emptyset \text{ and } y \in \underline{\Theta}(I, S).$$

Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on A, $b * a\Theta y * x$. Since $(y * x) \cap \underline{\Theta}(I, S) \neq \emptyset$, then there exist $t \in A$ such that $t \in y * x$ and $t \in \underline{\Theta}(I, S)$. Now, $t \in b * a\Theta y * x$ implies that there exist $z \in b * a$ such that $z\Theta t$ and so $I_t = I_z \subseteq S$. Hence $z \in S$ and so $(b * a) \cap S \neq \emptyset$. On the other hand, we have $b \in I_y \subseteq S$. Since S is a strong UP-ideal of A, then we have $a \in S$ which implies $I_x \subseteq S$ that means $x \in \underline{\Theta}(I, S)$. Therefore, $\underline{\Theta}(I, S)$ is a strong UP-ideal of S. □

4. Conclusion

We have studied the connection between rough sets and UP-algebras. We have presented definitions and examples of the lower and upper approximations of a UP-algebra and UP-subalgebras with respect to UP-ideals. In the future further study is possible in the direction of roughness with different types of ideals in UP-algebras.

References

[1] A. Iampan, *A new branch of the logical algebra: UP-algebras*, Journal of Algebra and Related Topics, 5 (2017), 35-54.

[2] S.S. Ahn, J.M. Ko, *Rough fuzzy ideals in BCK/BCI-algebras*, J. Comput. Anal. Appl., 25 (2018), 75-84

[3] B. Davvaz, *Roughness in rings,* Info. Sc., 164 (2004), 147-163.

[4] B. Davvaz, S. M. Mostafa and F.F. Kareem, *Neutrosophic ideals of Neutrosophic KU-Algebras*, Gazi University J. of Science, 30 (2017), 463-472.

[5] C. Prabpayak and U. Leerawat, *On ideals and congruencess in KU-algebras*, Scientia Magna J., 5 (2009), 54-57.

[6] C. Prabpayak and U. Leerawat, *On isomorphismsm of KU-algebras*, Scientia Magna J., 5 (2009), 25-31.

[7] F. Li., Y. Q. Yin, *The ϑ-lower and T-upper fuzzy rough approximation operators on a semigroups,* Inf. Sc., 195 (2012), 241-255.

[8] Hajek Petr, *Mathematics of fuzzy logic,* Klower Academic Publishers, Dordrecht, 1998.

[9] L. Torkzadeh and S. Ghorbani, *Rough filters in B-algebras,* Int. J. Math. Math. Sci., 2011 (2011).

[10] M. Akram, N. Yaqoob and J. Kavikumar, *Interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras,* Int. J. Pure Appl. Math., 92 (2014), 335-349.

[11] Moin A. Ansari and Ali N.A. Koam, *Rough approximations in KU-algebras*, Italian Journal of Pure and Applied Mathematics, accepted.

[12] N. Kuroki, *Rough ideals in semigroups*, Info. Sc., 100 (1997), 139-163.

[13] N. Yaqoob, S. M. Mostafa and Moin A. Ansari, *On cubic KU-ideals of KU-algebras*, ISRN, Algebras, 2013 (2013), 10 Pages.

[14] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Semin. Notes, Kobe University, 11 (1983), 313-320.

[15] Q. M. Xiao and Z. L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci., 176 (2006), 725-733.

[16] Qi., W. Liu, *Rough operations on boolean algebras*, Info. Sc., 173 (2005), 49-63.

[17] R. Ameri, R. Moradian and R.A. Borzooei, *Rough set theory applied to hyper BCK-algebras*, Ratio Math., 26 (2014), 3-20.

[18] R. Biswas and S. Nanda, *Rough groups and rough subgroups,* Bull. Polish Acad. Sci. Math., 42 (1994), 251-254.

[19] S. Keawrahun and U. Leerawat, *On isomorphisms of SU-algebras*, Sci. Magna, 7 (2011), 39-44.

[20] S. M. Mostafa, M.A. Abd-Elnaby and M.M.M. Yousef, *Fuzzy ideals of KU algebras*, Int. Math. Forum, 6 (2011), 3139-3149.

[21] W. Dudek, Y. B. Jun and H.S. Kim, *Rough set theory applied to BCI-algebras*, Quasi-groups and Relt. Syst., 9 (2002), 45-54.

[22] X. Mao and H. Zhou, *The application of rough set theory in pseudo BCK-algebra*, J. Math. Res. Appl., 36 (2016), 23-35.

[23] Y. B. Jun and K.H. Kim, *Rough set theory applied to BCC-algebras*, Int. Math. Forum, 41 (2007), 2023-2029.

[24] Y.B. Jun, *Roughness of ideals in BCK-algebras*, Sci. Math. Japon., 7 (2002), 115-119.

[25] Lotfi Aliasker Zadeh, *Fuzzy sets*, Info. Control, 8 (1965), 338-353.

[26] Z. Pawlak, *Classification of objects by means of attributes, reports*, 429, Institute of Comp. Sc., Polish Academy of Sciences Warsaw, Poland, 1981.

[27] Z. Pawlak, *Rough relations*, Reports, 435, Institute of Comp. Sc. Polish Academy of Sciences, Warsaw, Poland, 1981.

[28] Z. Pawlak, *Rough sets*, Int. J. of Comp. Sc. and Inf. Sci., 11 (1982), 341-356.

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Some separation axioms using hereditary classes in generalized topological spaces

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Abstract. we introduce a new class of generalized topological spaces namely T_2 modulo $\mathcal{H}$ and discuss its various properties and characterizations. Also we define $\mathcal{H}$-regular generalized topological spaces and study some properties of the same.

Keywords: generalized topologies, hereditary classes, T_2 modulo $\mathcal{H}$, $\mathcal{H}$-regular spaces, $*$- generalized topology.

1. Introduction and preliminaries

Ideals in topological spaces play an important role in the study of general topology. Many authors like Kurakowski, Hamlett, Jankovic, Noiri, Jha [10, 7, 8, 9, 16] studied various concepts like compactness, separation axioms, continuity in topological spaces via ideals. A topological space (X, τ), together with an ideal $\mathcal{I}$ is called an ideal space and denoted by $(X, \tau, \mathcal{I})$. Kuratowski [10], in his classic text, defined a local function for each subset of X with respect to $\mathcal{I}$ and τ. Vaidyanathaswamy [17, 18] extended the study of ideals and local functions to define the star topology τ^* generated by the ideal $\mathcal{I}$ and the topology τ^*, which is finer than τ. Also the modern research in general topology and real analysis concerns the different variations of compactness, separation axioms and connectedness etc in the various setting of topological spaces. Newcomb [12] in his Ph. D. thesis introduced the concept of compactness modulo an ideal, which is further investigated by Hamlett and Jankovic [7]. Shanthi and Rajesh [15] investigated the separation axioms in topological ordered spaces. In 1994, Hamlett and Jankovic [8] introduced and studied the concept of $\mathcal{I}$- regular spaces. V. Renuka Devi [13], in her Ph. D. thesis, introduced a new class of spaces, called Hausdorff modulo $\mathcal{I}$ (T_2 modulo $\mathcal{I}$) and investigated properties and characterizations for the same. She also studied some properties of $\mathcal{I}$- regular spaces.

On the other hand, generalized open sets are very important tools in topological spaces which is discussed by many authors from time to time. Cśaszár [4] introduced the concept of generalized open sets in generalized topological spaces in 2002. Further in 2008, he [6] defined the concept of hereditary classes instead of ideals in generalized topological spaces and extended the study of heredi-

tary classes to define star generalized topology finer than the given generalized topology. Later many authors studied the various concepts like compactness, connectedness, continuity and separation axioms in the settings of generalized topological spaces. Asaad [2] defined a γ operation on generalized open sets in X and studied its applications. In 2017-2018, Ahmad and Asaad [3, 1] introduced an operation γ on semi generalized open subsets of X and discussed some types of separation axioms, functions and closed spaces with respect to γ.

In this paper we introduce a new class of generalized topological spaces namely Hausdorff modulo $\mathcal{H}$ and discuss its properties and characterizations. Also we define $\mathcal{H}$-regular spaces and investigate them to find their properties. The purpose of this paper is to check whether the properties of Hausdorff modulo $\mathcal{I}$ spaces in ideal topological spaces are satisfied by Hausdorff modulo $\mathcal{H}$ spaces in the settings of generalized topological spaces or not. Some counter examples are also given for the properties which are not satisfied.

We refer to the following concepts:

Definition 1.1 ([4])**.** Let X be a non empty set and $expX$ be the power set of X. A collection $\mu \subset expX$ is called a generalized topology on X if $\emptyset \in \mu$ and μ is closed for arbitrary unions. (X, μ) is called a generalized topological space and the members of μ are called μ-open sets and their complements are called μ-closed sets and $cl_\mu(A)$ is defined as the intersection of all μ-closed sets containing A for each $A \subset X$.

Definition 1.2 ([6])**.** An ideal $\mathcal{I}$ on X is a non empty family of subsets of X satisfying (i) $A \subset B$, $B \in \mathcal{I}$ implies $A \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. If τ is a topology on X and $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space.

Definition 1.3 ([6])**.** A non empty family $\mathcal{H}$ of subsets of X is called a hereditary class if it satisfies only condition (i) of ideals, that is $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$. We will call $(X, \mu, \mathcal{H})$ a hereditary space. $\mathcal{H}$ is called codense if $\mu \cap \mathcal{H} = \{\emptyset\}$.

Definition 1.4. [6] The operator $()^* : expX \to expX$, with respect to hereditary class $\mathcal{H}$ and generalized topology μ is defined by $A^* = \{x \in X : U \cap A \notin \mathcal{H}$ for every $U \in \mu, x \in U\}$ for $A \subset X$. This operator defines another operator $c^* : expX \to expX$, by $c^*(A) = A \cup A^*$ for $A \subset X$, which is monotonic, enlarging and idempotent. This operator c^* induces another generalized topology μ^*, called $*$-generalized topology, which is finer than μ. The members of μ^* are called $*$-open sets and their complements are called $*$-closed sets.

Definition 1.5. Let $f : (X, \mu) \to (Y, \nu)$ be a mapping between two generalized topological spaces. Then f is called (μ, ν)-continuous [4] if $f^{-1}(V)$ is μ-open set in X for each ν-open set V in Y. f is called (μ, ν)-open mapping [11] if $f(U)$ is

ν-open set in Y for each μ-open set U in X. f is called (μ,ν)-homeomorphism if it is (μ,ν)-continuous, (μ,ν)-open bijection.

Definition 1.6 ([14])**.** A generalized topological space (X,μ) is called μ-Hausdorff if for each pair of distinct points x and y in X, there exist two disjoint μ-open sets U and V such that $x \in U$ and $y \in V$.

2. Results

Definition 2.1. A hereditary space $(X,\mu,\mathcal{H})$ is said to be *Hausdorff mod* $\mathcal{H}$ or T_2 *mod* $\mathcal{H}$ if for each pair of distinct points x and y in X, there exists μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$.

Theorem 2.2. *Every μ-Hausdorff space is T_2 mod $\mathcal{H}$.*

Proof. Proof is obvious, since $\emptyset \in \mathcal{H}$.

The converse of Theorem 2.2 need not be true in general, shown as in Example 2.3. Even if $\mathcal{H}$ is codense, then also the converse need not be true, shown in Example 2.5. Although the converse is true in case of ideal topological spaces for codense ideal, that is, if $(X,\tau,\mathcal{I})$ is ideal topological space, which is T_2 *mod* $\mathcal{I}$, where $\mathcal{I}$ is codense, then $(X,\tau,\mathcal{I})$ is Hausdorff. In particular, if we take $\mathcal{H} = \{\emptyset\}$, then μ-Hausdorffness and Hausdorffness mod $\mathcal{H}$ are equivalent.

Example 2.3. Let $X = \{a,b,c\}$, $\mu = \{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},X\}$ and $\mathcal{H} = \{\emptyset,\{b\}\}$. Then X is T_2 *mod* $\mathcal{H}$, but not μ-Hausdorff.

Theorem 2.4. *A space (X,μ) is μ-Hausdorff if and only if (X,μ) is Hausdorff mod $\mathcal{H}$, where $\mathcal{H} = \{\emptyset\}$.*

Example 2.5. Let $X = \{a,b,c\}$, $\mu = \{\emptyset,\{b\},\{a,b\},\{a,c\},X\}$ and $\mathcal{H} = \{\emptyset,\{a\}\}$. Then $\mathcal{H}$ is codense and X is T_2 *mod* $\mathcal{H}$, but not μ-Hausdorff.

Theorem 2.6. *If a hereditary space $(X,\mu,\mathcal{H})$ is Hausdorff mod $\mathcal{H}$ and $\mathcal{H} \subset \mathcal{K}$, then $(X,\mu,\mathcal{K})$ is Hausdorff mod $\mathcal{K}$.*

The following theorem gives characterizations of T_2 *mod* $\mathcal{H}$ spaces.

Theorem 2.7. *Let $(X,\mu,\mathcal{H})$ be a hereditary space. Then the following are equivalent:*

1. *$(X,\mu,\mathcal{H})$ is T_2 mod $\mathcal{H}$.*
2. *For each $x,y \in X$ such that $y \neq x$, there exists a μ-open set U containing x such that $y \notin U^*$.*

3. *For each* $x \in X$, $\cap\{U_x^* : U_x \in \mu, x \in U_x\}$ *is either* $\emptyset$ *or* $\{x\}$.

Proof. $1 \Rightarrow 2$. Let $x, y \in X$ such that $y \neq x$, then by hypothesis, there exist μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$. Therefore $(U \cap V)^* = \emptyset$ and so $U^* \cap V = \emptyset$. Therefore $y \notin U^*$, which proves 2.

$2 \Rightarrow 3$. For each $x \in X$, and each $y \neq x$, there exists a μ-open set U containing x such that $y \notin U^*$. Therefore $y \notin \cap\{U_x^* : U_x \in \mu, x \in U_x\}$ and so $\cap\{U_x^* : U_x \in \mu, x \in U_x\}$ is either $\emptyset$ or $\{x\}$, which proves 3.

$3 \Rightarrow 1$. Let x and y be any two distinct points in X. Then by hypothesis, $y \notin \cap\{U_x^* : U_x \in \mu, x \in U_x\}$, which implies that $y \notin U_x^*$ for some $U_x \in \mu, x \in U_x$. Therefore there exists a μ-open neighbourhood V_y of y such that $U_x \cap V_y \in \mathcal{H}$ and therefore $(X, \mu, \mathcal{H})$ is T_2 mod $\mathcal{H}$, which proves 1.

It is proved that for an ideal space $(X, \tau, \mathcal{I})$, (X, τ) is T_2 mod $\mathcal{I}$ if and only if (X, τ^*) is T_2 mod $\mathcal{I}$. But if we take hereditary space $(X, \mu, \mathcal{H})$ instead of ideal space, where (X, μ^*) is T_2 mod $\mathcal{H}$ then (X, μ) need not be T_2 mod $\mathcal{H}$ as shown in the following example:

Example 2.8. Let $X = \{a, b\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, X, \{a\}, \{b\}\}$, which is T_2 *mod* $\mathcal{H}$, but μ is not T_2 *mod* $\mathcal{H}$.

Theorem 2.9. *Let* X *be any set.* $A \subset X$ *and* $\mathcal{H}$ *be a hereditary class on* X. *Then* $\mathcal{H}_A = \{A \cap H : H \in \mathcal{H}\} = \{H \in \mathcal{H} : H \subset A\}$ *is a hereditary class on* A.

Theorem 2.10. *Let* $(X, \mu, \mathcal{H})$ *be* T_2 *mod* $\mathcal{H}$ *space and* $A \subset X$. *Then* $(A, \mu_A, \mathcal{H}_A)$ *is* T_2 *mod* $\mathcal{H}_A$, *where* μ_A *is subspace generalized topology on* A.

Proof. Let x and y be two distinct points in A, then there exists μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$. Then $x \in U \cap A \in \mu_A$, $y \in V \cap A \in \mu_A$ and $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A \in \mathcal{H}_A$. Therefore $(A, \mu_A, \mathcal{H}_A)$ is T_2 *mod* $\mathcal{H}_A$.

Theorem 2.11. *Let* $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ *be a mapping, where* $(X, \mu, \mathcal{H})$ *is a hereditary space and* (Y, ν) *is a generalized topological space. Then* $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ *is a hereditary class on* Y. *If* f *is an injection mapping and* $\mathcal{K}$ *is a hereditary class on* Y, *then* $f^{-1}(\mathcal{K}) = \{f^{-1}(K) : K \in \mathcal{K}\}$ *is a hereditary class on* X.

The following theorem shows that the property of being T_2 *mod* $\mathcal{H}$ is preserved under open bijection mapping in generalized topological space:

Theorem 2.12. *Let* $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ *be an* (μ, ν)*-open bijection and* $(X, \mu, \mathcal{H})$ *be* T_2 *mod* $\mathcal{H}$. *Then* (Y, ν) *is* T_2 *mod* $f(\mathcal{H})$.

Proof. Let y_1 and y_2 be two distinct points in Y. Since f is bijection, there exists distinct points x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then there exists μ-open sets U and V such that $x_1 \in U$, $x_2 \in V$ and $U \cap V \in \mathcal{H}$. Then $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{H})$. Since f is (μ, ν)-open mapping, $f(U)$ and $f(V)$ are ν-open sets in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Therefore (Y, ν) is T_2 *mod* $f(\mathcal{H})$.

Theorem 2.13. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{K})$ be a (μ, ν)-continuous injection mapping and $(Y, \nu, \mathcal{K})$ be T_2 mod $\mathcal{K}$. Then (X, μ) is T_2 mod $f^{-1}(\mathcal{K})$.*

Proof. Let x_1 and x_2 be two distinct points in X. Since $(Y, \nu, \mathcal{K})$ is T_2 *mod* $\mathcal{K}$, there exist ν-open sets U and V such that $f(x_1) \in U$, $f(x_2) \in V$ and $U \cap V \in \mathcal{K}$. Since f is injection, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in f^{-1}(\mathcal{K})$. Since f is (μ, ν)-continuous mapping, $f^{-1}(U)$ and $f^{-1}(V)$ are μ-open sets in X such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore (X, μ) is T_2 *mod* $f^{-1}(\mathcal{K})$.

Definition 2.14. Let (X, μ) be a generalized topological space and $\mathcal{A}$ be a collection of subsets of X. Then the smallest hereditary class on X containing $\mathcal{A}$, denoted by $\mathcal{H}(\mathcal{A})$, is called the hereditary class generated by $\mathcal{A}$.

It can be easily seen that $\mathcal{H}(\mathcal{A}) = \{H : H \subset A \in \mathcal{A}\}$.

Theorem 2.15. *Let $(X_\alpha, \mu_\alpha, \mathcal{H}_\alpha)$ be a collection of T_2 mod $\mathcal{H}_\alpha$ hereditary spaces for each $\alpha \in \Lambda$, where Λ is indexing set. If $P_\alpha : \Pi X_\alpha \to X_\alpha$ is the projection mapping for each α, $\mathcal{A} = \{P_\alpha^{-1}(H_\alpha) : H_\alpha \in \mathcal{H}_\alpha, \alpha \in \Lambda\}$, $\mathcal{H}(\mathcal{A})$ is hereditary class generated by $\mathcal{A}$ and $\mathcal{H}$ is hereditary class finer than $\mathcal{H}(\mathcal{A})$ on ΠX_α, then ΠX_α with product generalized topology is T_2 mod $\mathcal{H}$.*

Proof. Let x and y be two distinct points in ΠX_α, then there exists some $\alpha \in \Lambda$ such that $x_\alpha \neq y_\alpha$. Since each $(X_\alpha, \mu_\alpha, \mathcal{H}_\alpha)$ is T_2 mod $\mathcal{H}_\alpha$, there exist μ_α-open sets U_α and V_α such that $x_\alpha \in U_\alpha$, $y_\alpha \in V_\alpha$ and $U_\alpha \cap V_\alpha \in \mathcal{H}_\alpha$. Then $P_\alpha^{-1}(U_\alpha) \cap P_\alpha^{-1}(V_\alpha) = P_\alpha^{-1}(U_\alpha \cap V_\alpha) \in \mathcal{H}(\mathcal{A}) \subset \mathcal{H}$. Since P_α is projection mapping, $P_\alpha^{-1}(U_\alpha)$ and $P_\alpha^{-1}(V_\alpha)$ are generalized open sets in ΠX_α containing x and y respectively. Therefore ΠX_α with product generalized topology is T_2 mod $\mathcal{H}$.

Theorem 2.16. *Let (X_α, μ_α) be a collection of generalized topological spaces for each $\alpha \in \Lambda$, where Λ is indexing set. Let $\mathcal{H}$ be hereditary class on ΠX_α. If ΠX_α is T_2 mod $\mathcal{H}$, then (X_α, μ_α) is T_2 mod $P_\alpha(\mathcal{H})$ for each $\alpha \in \Lambda$, where P_α is the projection mapping for each α.*

Proof. Each X_α is (μ, ν)-homeomorphic to a subspace Y of the space ΠX_α, so each $P_\alpha|Y : Y \to X_\alpha$ is a (μ, ν)-homeomorphism. Since ΠX_α is T_2 mod $\mathcal{H}$, Y is T_2 mod $\mathcal{H}_Y$ and so X_α is T_2 mod $(P_\alpha|Y)(\mathcal{H}_Y)$. Since $(P_\alpha|Y)(\mathcal{H}_Y) \subset (P_\alpha)(\mathcal{H}_Y) \subset P_\alpha(\mathcal{H})$, therefore each (X_α, μ_α) is T_2 mod $P_\alpha(\mathcal{H})$.

Hamlett and Jankovic introduced and studied $\mathcal{I}$-regular topological spaces. Here we introduce $\mathcal{H}$-regularity for generalized topological spaces.

Definition 2.17. A hereditary space $(X, \mu, \mathcal{H})$ is said to be $\mathcal{H}$-regular if for each μ-closed set F and $x \notin F$, there exists disjoint μ-open sets U and V such that $x \in U$ and $F - V \in \mathcal{H}$.

Theorem 2.18. *Let $(X, \mu, \mathcal{H})$ be a hereditary space. Then the following are equivalent:*

1. *X is $\mathcal{H}$-regular.*
2. *For each $x \in X$ and μ-open set U containing x, there exists a μ-open set V containing x such that $cl_\mu(V) - U \in \mathcal{H}$.*
3. *For each $x \in X$ and μ-open set U containing x, there exists a μ-closed set F containing x such that $F - U \in \mathcal{H}$.*
4. *For each $x \in X$ and μ-closed set A not containing x, there exists a μ-open set V containing x such that $cl_\mu(V) \cap A \in \mathcal{H}$.*

Proof. $1 \Rightarrow 2$. Let $x \in X$ and U be a μ-open set containing x. Then $X - U$ is a μ-closed set not containing x. Since X is $\mathcal{H}$-regular, there exist disjoint μ-open sets V and W such that $x \in V$ and $(X - U) - W \in \mathcal{H}$. If $(X - U) - W = H \in \mathcal{H}$, then $(X - U) \subset W \cup H$. Since V and W are disjoint, $V \subset X - W$ implies $cl_\mu(V) \subset X - W$ and so $cl_\mu(V) - U \subset (X - W) - U = H \in \mathcal{H}$. Therefore $cl_\mu(V) - U \in \mathcal{H}$, this proves 2.

$2 \Rightarrow 4$. Let A be a μ-closed set in X such that $x \notin A$. Then by hypothesis, there exists a μ-open set V containing x such that $cl_\mu(V) - (X - A) \in \mathcal{H}$. Therefore $cl_\mu(V) \cap A \in \mathcal{H}$, which proves 4.

$4 \Rightarrow 1$. Let A be a μ-closed set in X such that $x \notin A$. Then by hypothesis, there is a μ-open set V containing x such that $cl_\mu(V) \cap A \in \mathcal{H}$. Then $A - (X - cl_\mu(V)) \in \mathcal{H}$. V and $X - cl_\mu(V)$ are disjoint μ-open sets such that $x \in V$ and so $A - (X - cl_\mu(V)) \in \mathcal{H}$. Hence X is $\mathcal{H}$-regular, which proves 1.

The equivalence of 2 and 3 is obvious.

Corollary 2.19. *Let $(X, \mu, \mathcal{H})$ be a hereditary space, where $\mathcal{H}$ be codense. Then the following are equivalent:*

1. *X is $\mathcal{H}$-regular.*
2. *For each $x \in X$ and μ-open set U containing x, there exists a μ-open set V containing x such that $V^* - U \in \mathcal{H}$.*
3. *For each $x \in X$ and μ-closed set A not containing x, there exists a μ-open set V containing x such that $V^* \cap A \in \mathcal{H}$.*

Proof. $\mathcal{H}$ is codense if and only if $cl(V) = V^*$ for every μ-open set V.

The following theorems show that $\mathcal{H}$-regularity is hereditary property as well as it is preserved under (μ, ν)-homeomorphism:

Theorem 2.20. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and $A \subset X$. Then $(A, \mu_A, \mathcal{H}_A)$ is $\mathcal{H}_A$-regular.*

Proof. Let $F \subset A$ be μ_A-closed in A and $x \in A$ such that $x \notin F$. Then $F = A \cap K$ where K is μ-closed in X and $x \notin K$. Since $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, there exists disjoint μ-open sets U and V such that $x \in U$ and $K - V \in \mathcal{H}$. So $A \cap U$ and $A \cap V$ are μ_A-open sets in A such that $x \in A \cap U$ and $(A \cap U) \cap (A \cap V) = (U \cap V) \cap A = \emptyset$. If $K - V = H \in \mathcal{H}$, then $K \subset H \cup V$ and so $F = A \cap K \subset A \cap (H \cup V) = (A \cap H) \cup (A \cap V)$ implies $F - (A \cap V) \subset A \cap H \in \mathcal{H}_A$. Therefore $F - (A \cap V) \in \mathcal{H}_A$. Hence $(A, \mu_A, \mathcal{H}_A)$ is $\mathcal{H}_A$-regular.

Theorem 2.21. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and let $f : (X, \mu, \mathcal{H}) \to (Y, \nu, f(\mathcal{H}))$ be a (μ, ν)-homeomorphism. Then $(Y, \nu, f(\mathcal{H}))$ is $f(\mathcal{H})$-regular.*

Proof. Let F be ν-closed in Y and $y \in Y$ such that $y \notin F$. Let $x = f^{-1}(y)$. Since f is (μ, ν)-continuous, $f^{-1}(F)$ is μ-closed in X not containing x. Since $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, there exist disjoint μ-open sets U and V in X such that $x \in U$ and $f^{-1}(F) - V \in \mathcal{H}$. Let $f^{-1}(F) - V = H \in \mathcal{H}$, so $f^{-1}(F) \subset H \cup V \Rightarrow f(f^{-1}(F)) \subset f(H \cup V) \Rightarrow F \subset f(H) \cup f(V) \Rightarrow F - f(V) \subset f(H) \in f(\mathcal{H}) \Rightarrow F - f(V) \in f(\mathcal{H})$. Therefore $f(U)$ and $f(V)$ are disjoint ν-open sets in Y such that $y \in f(U)$ and $F - f(V) \in f(\mathcal{H})$. Hence $(Y, \nu, f(\mathcal{H}))$ is $f(\mathcal{H})$-regular.

Theorem 2.22. *Let (X_α, μ_α) be a collection of generalized topological spaces for each $\alpha \in \Lambda$, where Λ is indexing set. Let $\mathcal{H}$ be hereditary class on ΠX_α. If ΠX_α is $\mathcal{H}$-regular, then (X_α, μ_α) is $P_\alpha(\mathcal{H})$-regular for each $\alpha \in \Lambda$, where P_α is the projection mapping for each α.*

Theorem 2.23. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and let x and y be two distinct points in X. Then either $cl_\mu(\{x\}) = cl_\mu(\{y\})$ or $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \in \mathcal{H}$.*

Proof. If $x \in cl_\mu(\{y\})$ and $y \in cl_\mu(\{x\})$. Then $cl_\mu(\{x\}) \subset cl_\mu(cl_\mu(\{y\})) = cl_\mu(\{y\}) \subset cl_\mu(cl_\mu(\{x\})) = cl_\mu(\{x\})$ and so $cl_\mu(\{x\}) = cl_\mu(\{y\})$. If $y \notin cl_\mu(\{x\})$. Since $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, there exists a μ-open set V containing y such that $cl_\mu(V) \cap cl_\mu(\{x\}) \in \mathcal{H}$, therefore $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \subset cl_\mu(\{x\}) \cap cl_\mu(V) \in \mathcal{H}$. Hence $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \in \mathcal{H}$. This completes the proof.

Theorem 2.24. *Let $(X, \mu, \mathcal{H})$ be a hereditary space. If each point of X has a μ-closed neighbourhood A which is $\mathcal{H}_A$-regular, then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Let $x \in X$ and U be a μ-open set containing x. Then there exists a μ-closed neighbourhood A of x, which is $\mathcal{H}_A$-regular. $A \cap U$ is μ_A-open set in A containing x. Therefore there is a μ_A-closed set F containing x in A such that $F - (A \cap U) \in \mathcal{H}_A$. Since F is μ_A-closed in A and A is μ-closed in X, F is μ-closed in X containing x and $F - U \subset F - (A \cap U) \in \mathcal{H}_A \subset \mathcal{H}$. Thus F is μ-closed set containing x in X such that $F - U \in \mathcal{H}$. Hence $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.

Theorem 2.25. *Let $(X, \mu, \mathcal{H})$ be a hereditary space, which is $\mathcal{H}$-regular. Then for every nonempty set A and a μ-closed set F in X such that $F \cap A = \emptyset$, there exist disjoint μ-open sets U and V such that $A \cap U \neq \emptyset$ and $F - V \in \mathcal{H}$.*

Proof. Let $(X, \mu, \mathcal{H})$ be $\mathcal{H}$-regular. Let A be a nonempty set and F be a μ-closed set in X such that $F \cap A = \emptyset$. Then for each $x \in A$, there exist disjoint μ-open sets U and V such that $x \in U$ and $F - V \in \mathcal{H}$. Also $A \cap U \neq \emptyset$.

It is proved that the star topology with respect to a given ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular if the given ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular. That is if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular then $(X, \tau^*, \mathcal{I})$ is $\mathcal{I}$-regular. But in case of generalized topological spaces, this result need not be true. That is if the hereditary space $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular then $(X, \mu^*, \mathcal{H})$ need not be $\mathcal{H}$-regular as shown in the following example:

Example 2.26. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a, c\}, \{b, c\}, X\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, but $(X, \mu^*, \mathcal{H})$ is not $\mathcal{H}$-regular, since $\{a\}$ is a μ^*-closed set not containing c and there do not exist disjoint μ^*-open sets U and V such that $c \in U$ and $\{a\} - V \in \mathcal{H}$.

Although there exist spaces where this result is true, shown in the following example:

Example 2.27. Let $X = \{a, b\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a\}, \{b\}, X\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, as well as $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular.

The following example shows that if the star generalized topology is $\mathcal{H}$-regular, then the given generalized topology need not be $\mathcal{H}$-regular. But if the hereditary class is codense, then this will hold. Therefore it can be concluded that $\mathcal{H}$-regularity of generalized topology and $\mathcal{H}$-regularity of star generalized topology do not depend on each other.

Example 2.28. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\{a, c\}$ is a μ-closed not contain-

ing b and there do not exist disjoint μ-open sets U and V such that $b \in U$ and $\{a, c\} - V \in \mathcal{H}$. Therefore $(X, \mu, \mathcal{H})$ is not $\mathcal{H}$-regular, but $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular.

Theorem 2.29. *Let $(X, \mu, \mathcal{H})$ be a hereditary space such that $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular and $\mathcal{H}$ is codense. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Let A be μ-closed set in X such that $x \notin A$. Since $\mu \subset \mu^*$, A is μ^*-closed, therefore by Corollary 2.19, there exists a μ^*-open set V containing x such that $V^* \cap A \in \mathcal{H}$. Since V is μ^*-open set containing x, there exist $U \in \mu$ and $H \in \mathcal{H}$ such that $x \in U - H \subset V$. Therefore $U^* \subset V^*$ implies $U^* \cap A \subset V^* \cap A \in \mathcal{H}$ and so $U^* \cap A \in \mathcal{H}$. Hence $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.

The hereditary space in Example 2.28 is a T_2 mod $\mathcal{H}$ space which is not $\mathcal{H}$-regular. The following example shows that the $\mathcal{H}$-regular space need not be T_2 mod $\mathcal{H}$. Hence T_2 mod $\mathcal{H}$ and $\mathcal{H}$-regular are independent concepts:

Example 2.30. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, which is not T_2 mod $\mathcal{H}$.

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References

[1] N. Ahmad, B. A. Asaad, *More properties of an operation on semi-generalized open sets*, Italian Journal of Pure and Applied Mathematics, 39 (2018), 608-627.

[2] B. A. Asaad, *Some applications of generalized open sets via operations*, New Trends in Mathematical Sciences, 5 (2017), 145-157.

[3] B. A. Asaad, N. Ahmad, *Operation on semi generalized open sets with its separation axioms*, AIP Conference Proceedings, 1905, 020001 (2017).

[4] Á. Cśaszár, *Generalized topology, generalized continuity*, Acta Math. Hungar., 96 (2002), 351-357

[5] Á. Cśaszár, *Generalized open set in generalized topologies*, Acta Math. Hungar., 106 (2005), 53-66.

[6] Á. Cśaszár, *Modifications of generalized topologies via hereditary classes*, Acta Math. Hungar., 120 (2008), 275-279.

[7] T.R. Hamlett, D. Janković, *Compactness with respect to an Ideal*, Boll. U. M. I., 7 (1990), 849-861.

[8] T.R. Hamlett, D. Janković, *On weaker forms of paracompactness, countable compactness and lindelofness*, Annals of the New York Academy of Sciences, Papers on General Topology and Applications, 728 (1994), 41-49.

[9] D. Janković, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97 (1990), 295-310.

[10] K. Kurakowski, *Topology 1*, Academic Press, New York, 1966.

[11] W. K. Min, *Some results on generalized topological spaces and generalized systems*, Acta Math. Hungar., 108 (2005), 171-181.

[12] R. L. Newcomb, *Topologies which are compact modulo an ideal*, Ph. D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.

[13] V. Renuka Devi, *A study on ideals in topological spaces and star topological spaces*, Ph. D. Dissertation, Manonmaniam Sundaranar University, 2006.

[14] B. Roy, *On a type of generalized open sets*, Applied General Topology, 12 (2011), 163-173.

[15] S. Shanthi, N. Rajesh, *Separation axioms in topological ordered spaces*, Italian Journal of Pure and Applied Mathematics, 40 (2018), 464-473.

[16] M. Shrivastava, T. Noiri, P. Jha, *Contra weakly-θ_I-precontinuous functions in ideal topological spaces*, Italian Journal of Pure and Applied Mathematics, 40 (2018), 736-747.

[17] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci., 20 (1945), 51-61.

[18] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, New York, 1946.

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Edge maximal non-bipartite Hamiltonian graphs without theta graphs of order 7

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Abstract. For a set of graphs $\mathcal{F}$, let $\mathcal{H}(n;\mathcal{F})$ denote the class of non-bipartite Hamiltonian graphs on n vertices that does not contain any graph of $\mathcal{F}$ as a subgraph and $h(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;\mathcal{F})\}$ where $\mathcal{E}(G)$ is the number of edges in G. In this paper, we determine $h(n;\{\theta_4,\theta_5,\theta_7\})$ and we establish an upper bound of $h(n;\theta_7)$ for sufficiently even large n. Our results confirms the conjecture made in [1] for $k = 3$.

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Keywords: Tuŕan number, theta graph, extremal graph.

1. Introduction and preliminaries

We consider undirected graphs without loops and multiple edges. Let $V(G)$ be the vertex set and $E(G)$ be the edge set of a graph G. The order of a graph G is the number of vertices of G and is denoted by $\mathcal{V}(G)$. The size of G, denoted by $\mathcal{E}(G)$, is the number of edges of G. A complete k-partite graph is a graph whose vertices can be partitioned into k disjoint sets, such that two vertices are adjacent if and only if they belong to different sets. We often denote by $x_1x_2\dots x_nx_1$ the cycle C_n having n vertices $x_1, x_2, \dots, x_n$ and the edges $x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ and x_nx_1. A theta graph is a cycle C_n with a new edge (a chord) joining two non-adjacent vertices of C_n. The set of all theta graphs of order n will be denoted by θ_n. It is easy to check that the set θ_n contains $\left\lfloor \frac{n}{2} \right\rfloor - 1$ (non-isomorphic) graphs.

If F is a subgraph of G, then $G - F$ is the graph that contains all vertices of G which are not in F and all edges of G connecting two vertices of $G - F$. If P and Q are two subgraphs of G, then $E(P, Q)$ is the set containing all edges of G, which connect a vertex in P and a vertex in Q and $\mathcal{E}(P, Q) = |E(P, Q)|$. An induced subgraph $G[V(Q)]$ of a graph G consists of the vertices in Q and all edges of G connecting two vertices in Q. The join $G = G_1 \vee G_2$ of graphs $G_1$and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph G_1 union G_2 together with all the edges joining $V(G_1)$ and $V(G_2)$.

For a set of graphs S, the Turán number $ex(n, S)$ is defined as the maximum number of edges in a graph of order n having no member of S as a subgraph. If S contains only one graph G, we write simply $ex(n, G)$. The problem was formulated by Turán [14], who showed that $ex(n, K_r) = \lfloor \frac{rn^2}{2(r+1)} \rfloor$, where K_r is the complete graph having r vertices.

We now introduce some additional notation. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ (and $\mathcal{H}(n; \mathcal{F})$) denote the class of non-bipartite $\mathcal{F}$-free graphs (the subclass of $\mathcal{G}(n; \mathcal{F})$ which consists of all the Hamiltonian members in $\mathcal{G}(n; \mathcal{F})$) on n vertices, and

$$\begin{aligned} f(n; \mathcal{F}) &= \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}, \\ h(n; \mathcal{F}) &= \max\{\mathcal{E}(G) : G \in \mathcal{H}(n; \mathcal{F})\}. \end{aligned}$$

Hendry and Brandt [10] proved that $h(n; C_5) \leq \frac{(n-3)^2}{4} + 5$ for odd $n \geq 7, n \neq 9$, and $h(9; C_5) = 15$. However, they did not characterize the extremal graphs. Caccetta and Jia [7] characterized the extremal graphs and proved that $f(n; C_5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 9$. Also, they proved $h(n; C_5) \leq \frac{(n-4)^2}{4} + 7$ for even $n \geq 12$. Further, the extremal graphs were characterized. Jia [13]

conjectured that $f(n; C_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Bataineh [1] settled the above conjecture for $n \geq 36k$. Further, he showed that equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. Furthermore he proved the following result:

Theorem 1 (Bataineh [1]). *For positive integers $k \geq 3$ and $n > (4k+2)(4k^2 + 10k)$,*

$$h(n; C_{2k+1}) = \begin{cases} \dfrac{(n-2k+1)^2}{4} + 4k - 3, & \text{if } n \text{ is odd} \\ \dfrac{(n-2k)^2}{4} + 4k + 1, & \text{if } n \text{ is even.} \end{cases}$$

For θ_5-graph, Bataineh et al. [2] proved that for $n \geq 5$

$$f(n; \theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Later on, Bataineh et al. [3], [4] and Jaradat et al. [11] proved the following results

Theorem 2 (Jaradat et al. [11]). *For positive integers n and k, let G be a graph on $n \geq 6k+3$ vertices which contains no θ_{2k+1} as a subgraph, then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 3 (Jaradat et al. [11] and Bataineh et al. [3] and [4]). *For sufficiently large integer n and for $k \geq 3$,*

$$f(n; \theta_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Caccetta and Jia [7] proved the following results:

Theorem 4 (Caccetta and Jia [7]). *Let $G \in \mathcal{G}(n; C_3, C_5, \ldots, C_{2k+1})$. Then*

$$\mathcal{E}(G) \leq \lfloor \frac{1}{4}(n-2k+1)^2 \rfloor + 2k - 1.$$

Theorem 5 (Caccetta and Jia [7]). *Let $\mathcal{F}_k = \{C_3, C_5, C_7, \ldots, C_{2k+1}\}$. For even $n \geq 4k+4, k \geq 2$, we have*

$$h(n; \mathcal{F}_k) = \frac{(n-4k-4)^2}{4} + 8k - 11.$$

Analogously, In [1], Bataineh proved the following result concerning theta graphs:

Theorem 6 (Bataineh [1]). *Let $\Theta_k = \{\theta_4\} \cup \{\theta_5, \theta_7, \ldots, \theta_{2k+1}\}$, then for $k \geq 5$ and large odd n, we have*

$$h(n; \Theta_k) = \frac{(n-2k+3)^2}{4} + 2k - 3.$$

Jaradat et al. [12] proved the following result.

Theorem 7 (Jaradat et al. [12]). *For sufficiently large odd n, let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 7$. Then*

$$h(n; \theta_7) \leq \frac{(n-3)^2}{4} + 3.$$

Furthermore, the bound is best possible.

Bataineh [1] made the following conjecture

Conjecture 1. *Let $k \geq 3$ be a positive integer. For even $n \geq 4k+4$, $h(n; \theta_{2k+1}) \leq \frac{(n-2k+2)^2}{4} + 2k$.*

In this paper, we investigates the values of $h(n; \mathcal{F})$, for sufficiently large even n where $\mathcal{F} = \{\theta_4, \theta_5, \theta_7\}$ and $\mathcal{F} = \{\theta_7\}$. In fact, we settle the above conjecture for $k = 3$ under a constrain on the minimum degree.

2. Main results

For the sake of completeness, we start this section, by listing the following three results of Jaradat et al. [12] which will be used in the sequel.

Lemma 1 ([12]). *Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. If $u \in V(H-C)$, then $\mathcal{E}(u, C) \leq 3$. Also, if $B = \{u \in V(H-C) : \mathcal{E}(u, C) = 3\}$, then $|B| \leq 1$. Further, if $C = x_1x_2x_3x_4x_5x_6x_7x_1$ and $u \in B$, then $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$ for some $i = 1, 2, \ldots, 7$ ($x_j = x_{j-7}$ for $j > 7$).*

Lemma 2 ([12]). *Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$ such that H contains a cycle C of length 7. If $|B| = 1$ and uv is an edge in the subgraph $H - C - B$, then $\mathcal{E}(uv, C) \leq 3$ where B is as defined in Lemma 1.*

The following remark follows from the fact that if $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$, C is a cycle of length 7 in H and $\mathcal{E}(u, C) = 3$, then $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$.

Remark 1 ([12]). Let $H \in \mathcal{H}(n, \{C_3, \theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. Then $B = \varnothing$ where B is as defined in Lemma 1.

To investigate $h(n; \{\theta_4, \theta_5, \theta_7\})$ and $h(n; \theta_7)$ for even n, we prove the following lemmas.

Lemma 3. *For any $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$, if H contains cycles of lengths 3 and 7, then*

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n.

Proof. Let $C_7 = x_1x_2 \dots x_7x_1$ and $C_3 = y_1y_2y_3y_1$ be cycles of length 7 and 3 in H, respectively. Let $A = H[x_1, x_2, \dots, x_7]$ and $R_1 = H - A$. We distinguish two cases:

Case 1. $V(C_3) \subseteq V(R_1)$. Let $R_2 = R_1 - C_3$. By Lemma 2 we have $\mathcal{E}(R_2, A) \leq 2(n-10)$. Notice that if $u \in V(H - C_3)$, then $\mathcal{E}(u, C_3) \leq 1$, otherwise θ_4 is produced as a subgraph of H. Thus, $\mathcal{E}(R_2, C_3) \leq n - 10$. Observe that for $i = 1, 2, \dots, 7$ and $j = 1, 2, 3$, if x_i is adjacent to y_j, then neither x_{i+1} nor x_{i-1} can be adjacent to y_s for some $s = 1, 2, 3$, and $s \neq j$, otherwise θ_5 is produced as a subgraph. Now, if x_{i-1}, x_i and x_{i+1} are all adjacent to the same y_j, then θ_4 is produced as a subgraph, hence, $\mathcal{E}(C_3, A) \leq 4$. By Theorem 2 we have

$$\mathcal{E}(R_2) \leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(R_2) + \mathcal{E}(R_2, A) + \mathcal{E}(R_2, C_3) + \mathcal{E}(A) + \mathcal{E}(A, C_3) + \mathcal{E}(C_3) \\ &\leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor + 2(n-10) + n - 10 + 7 + 4 + 3 \\ &\leq \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor \\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

Case 2. $V(C_3) \nsubseteq V(R_1)$. Then $|V(C_3) \cap V(A)| = 2$ or 1, accordingly, we split this case into two subcases:

Subcase 2.1. $|V(C_3) \cap V(A)| = 2$. Withoutloss of generality assume $x_1, x_2 \in N_A(y_1)$ and let $A_1 = H[y_1, A]$ and $R_3 = H - A_1$, then by Lemma 1, we get $\mathcal{E}(y_1, A) \leq 3$, hence $\mathcal{E}(A_1) \leq 10$. Also, by Theorem 2 we have

$$\mathcal{E}(R_3) \leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor.$$

Now, we consider the case $\mathcal{E}(y_1, A) = 3$, then $\mathcal{E}(A_1) = 10$. By Lemma 1 $\mathcal{E}(x, A) \leq 2$ for each $x \in V(R_3)$. On the other hand, one can notice that if there is an $x \in V(R_3)$ such that $y_1x \in E(H)$, then $\mathcal{E}(x, A) = 0$ as otherwise a θ_4 or θ_5 or θ_7 is produced as a subgraph of H, which implies that $\mathcal{E}(x, A_1) \leq 1$ and so $\mathcal{E}(R_3, A_1) \leq 2(n-8) - 1$. If $y_1x \notin E(H)$ for each $x \in V(R_3)$, then $\mathcal{E}(x, A_1) = \mathcal{E}(x, A)$, but by Lemma 2 we get $\mathcal{E}(R_3, A_1) = \mathcal{E}(R_3, A) \leq 2(n-8)-1$. Therefore,

$$\mathcal{E}(R_3, A_1) \leq 2(n-8) - 1.$$

Consequently, we have

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_3) + \mathcal{E}(R_3, A_1) + \mathcal{E}(A_1)\\ &\leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor + 2(n-8) - 1 + 10\\ &= \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor\\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.\end{aligned}$$

We now consider the case $\mathcal{E}(y_1, A) = 2$, then $\mathcal{E}(A_1) \leq 9$. Now, for $x \in V(R_3)$ if $xy_1 \in E(H)$, then $x_i x \notin E(H)$ for each $i = 1, 2, 3, 5, 7$ as otherwise a θ_4 or θ_5 or θ_7 is produced as a subgraph of H. Further, y_1 can not be adjacent to both x_4 and x_1 as otherwise $x_4x_3x_2y_1xx_6x_5x_4x$ is a θ_7-graph of H. Thus, $\mathcal{E}(x, A) \leq 1$, which implies that $\mathcal{E}(x, A_1) \leq 2$. Also, if $uv \in E(H - A_1)$ and $y_1u \in E(H)$, then as above $N_{C_7}(u) \subseteq \{x_4\}$ or $N_{C_7}(u) \subseteq \{x_6\}$; and $vx_i \notin E(H)$ for each $i = 1, 2, 4, 6$ as otherwise θ_5 or θ_7 is produced as a subgraph of H. Further, v is adjacent to at most one of x_3, x_5 and x_7, to see this, note that: (1) If v is adjacent to both x_3 and x_5, then $vx_5x_4x_3x_2y_1uvx_3$ is a θ_7-graph in H; (2) if v is adjacent to both of x_7 and x_5, then by symmetry we get a θ_7-graph in H; (3) if v is adjacent to both of x_3 and x_7, then $vuy_1x_1x_7x_6x_5vx_5$ is a θ_7-graph in H. In addition, if $vy_1 \in E(H)$, then $ux_4, ux_6 \notin E(H)$, to see that let $ux_4 \in E(H)$, then $ux_4x_3x_2x_1y_1vy_1$ is a θ_7-graph in H. Thus, $\mathcal{E}(uv, A_1) \leq 3$. Therefore, from the above and using Lemma 2, we conclude that

$$\mathcal{E}(R_3, A_1) \leq 2(n-8).$$

And so,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_3) + \mathcal{E}(R_3, A_1) + \mathcal{E}(A_1)\\ &\leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor + 2(n-8) + 9\\ &= \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor\\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.\end{aligned}$$

Subcase 2.2. $|V(C_3) \cap V(A)| = 1$. Without loss of generality assume y_1, y_2 are adjacent to x_1. Let $A_2 = H[y_1y_2, A]$ and $R_4 = H - A_2$. One can easily see that $\mathcal{E}(y_1y_2, A) = 2$, because otherwise θ_4 or θ_5 or θ_7 is produced, hence $\mathcal{E}(A_2) = 10$. Now, if $x \in V(R_4)$, then x cannot be adjacent to both y_1 and y_2, as otherwise θ_4 is produced. Moreover, if x is adjacent to either y_1 or y_2, then $N_A(x) \subseteq \{x_3, x_6\}$ as otherwise θ_4 or θ_5 or θ_7 is produced.
Now, let $x, x^* \in R_4$ be adjacent to y_1 or y_2 and assume $N_A(x) = \{x_3, x_6\}$, then x^* is adjacent to at most one of x_3 and x_6. To see this, assume $N_A(x^*) = \{x_3, x_6\}$.

Then, If $xy_1, x^*y_1 \in E(H)$, then $xx_6x_7x_1y_1x^*x_3xy_1$ is θ_7. A similar argument holds if $xy_2, x^*y_2 \in E(H)$. If $xy_1, x^*y_2 \in E(H)$, then $x^*x_3xx_6x_7x_1y_2x^*x_6$ is a θ_7.

Let

$$S = \{x \in R_4 : xy_1 \text{ or } xy_2 \in E(H)\},$$

and

$$S^* = \{x \in R_4 : xy_1, xy_2 \notin E(H)\}.$$

Then from the above argument

$$\mathcal{E}(S, A_2) \leq 2|S| + 1, \tag{1}$$

and by Lemma 2

$$\mathcal{E}(S^*, A_2) \leq 2|S^*| + 1. \tag{2}$$

Hence, combining 1 and 2, we get

$$\begin{aligned} \mathcal{E}(R_4, A_2) &\leq 2|S| + 1 + 2|S^*| + 1 \\ &\leq 2(n-9) + 2. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(R_4) + \mathcal{E}(R_4, A_2) + \mathcal{E}(A_2) \\ &\leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor + 2(n-9) + 2 + 9 \\ &= \left\lfloor \frac{n^2 - 10n + 49}{4} \right\rfloor \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

□

Lemma 4. *For any $H \in \mathcal{H}(n, \{C_3, \theta_4, \theta_5, \theta_7\})$, if H contains a cycle of length 5 and a cycle of length 7, then*

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n.

Proof. Let $C_5 = y_1y_2y_3y_4y_5y_1$ and $C_7 = x_1x_2x_3 \dots x_7x_1$ be cycles of length 5 and 7 in H, respectively. As in Lemma 3, we let $R_1 = H - A$ where $A = H[x_1, x_2, \dots, x_7]$. Now we consider two cases:

Case 1. $V(C_5) \subseteq V(R_1)$. Let $R_5 = R_1 - C_5$. Notice that $A = C_7$ and $H[C_5] = C_5$, otherwise θ_7 or θ_5 is produced as a subgraph, and so $\mathcal{E}(A) = 7$ and

$\mathcal{E}(H[C_5]) = \mathcal{E}(C_5) = 5$. By Lemma 2 we have $\mathcal{E}(R_5, A) \leq 2(n-12)$. Now, if $x \in V(R_5)$, then $\mathcal{E}(x, C_5) \leq 2$, otherwise θ_4 or θ_5 is produced as a subgraph.

Claim 1. Let $xy \in E(R_5)$, then $\mathcal{E}(xy, C_5) \leq 2$.

Proof of the claim. Suppose that $\mathcal{E}(x, C_5) = 2$. Then, by taking into account the symmetry, we have $N_{C_5}(x) = \{y_i, y_{i+2}\}$, otherwise C_3 is produced. Without loss of generality we may assume that $N_{C_5}(x) = \{y_1, y_3\}$, then we have the following possibilities:

1. y is adjacent to y_1. Then the trail $xyy_1y_2y_3xy_1$ is a θ_5-graph.
2. y is adjacent to y_2. Then the trail $xyy_2y_1y_5y_4y_3xy_1$ is a θ_7-graph.
3. y is adjacent to y_3. Then the trail $xyy_3y_2y_1xy_3$ is a θ_5-graph.
4. y is adjacent to y_4. Then the trail $xyy_4y_5y_1y_2y_3xy_1$ is a θ_7-graph.
5. y is adjacent to y_5. Then the trail $xyy_5y_4y_3y_2y_1xy_3$ is a θ_7-graph.

Thus $\mathcal{E}(y, C_5) = 0$, and so $\mathcal{E}(xy, C_5) \leq 2$. This completes the proof of the claim. Since H is a Hamiltonian graph, then there is an edge e in R_5. Thus, by the Claim 1, $\mathcal{E}(e, C_5) \leq 2$, and so $\mathcal{E}(R_5, C_5) \leq 2(n-12) - 2$. Also, by Claim 1, one can see that $\mathcal{E}(C_5, A) \leq 6$. Further, by Theorem 2, we have

$$\mathcal{E}(R_5) \leq \left\lfloor \frac{(n-12)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(R_5) + \mathcal{E}(R_5, A) + \mathcal{E}(R_5, C_5) + \mathcal{E}(A) + \mathcal{E}(A, C_5) + \mathcal{E}(C_5) \\
&\leq \left\lfloor \frac{(n-12)^2}{4} \right\rfloor + 2(n-12) + 2(n-12) - 2 + 7 + 6 + 5 \\
&\leq \left\lfloor \frac{n^2 - 8n + 10}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor - 1 \\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

Case 2. $V(C_5) \nsubseteq V(R_1)$. Then $|V(C_5) \cap V(A)| = 1$ or 2 or 3 or 4. Thus, we split this case into 4 subcases:

Subcase 2.1. $|V(C_5) \cap V(A)| = 1$. Without loss of generality, assume $C_5 = x_1y_1y_2y_3y_4x_1$ is in H, then let $T_1 = H[y_1, y_2, y_3, y_4, A]$ and $D_1 = H - T_1$. From Remark 1, $\mathcal{E}(x, A) \leq 2$ for any $x \in H - A$. Since $\mathcal{E}(H[C_5]) = \mathcal{E}(C_5) = 5, \mathcal{E}(A) = 7$ and $\mathcal{E}(y_j, A) \leq 2$ for $j = 1, 2, 3, 4$, then $\mathcal{E}(T_1) \leq 18$. Also by Theorem 2 we have

$$\mathcal{E}(D_1) \leq \left\lfloor \frac{(n-11)^2}{4} \right\rfloor.$$

Claim 2. For each $x \in V(D_1)$, $\mathcal{E}(x, T_1) \leq 3$.

Proof of the claim. Let $x \in V(D_1)$, then as above both $\mathcal{E}(x, A), \mathcal{E}(x, C_5) \leq 2$. If $\mathcal{E}(x, C_5) \leq 1$, then $\mathcal{E}(x, T_1) \leq 3$. Also, if $xx_1 \in E(H)$, then $\mathcal{E}(x, T_1) \leq 3$ because x_1 is a common vertex of both A and C_5. To this end, we consider the case where $\mathcal{E}(x, C_5) = 2$ and $xx_1 \notin E(H)$. Then, x is either adjacent to both y_1 and y_3 or adjacent to both y_2 and y_4 or to both y_1 and y_4. If $xy_1, xy_4 \in E(H)$, then $C_5^* = xy_1y_2y_3y_4x$ is a cycle of length 5 such that $V(C_5^*) \subseteq V(R_5)$ and so we get Case 1. If $xy_1, xy_3 \in E(H)$, then

1- $xx_2 \notin E(H)$ as otherwise $xy_1y_2y_3y_4x_1x_2xy_3$ is a θ_7.
2- $xx_3 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_2x_3xy_1$ is a θ_7.
3- $xx_7 \notin E(H)$ as otherwise $xy_1y_2y_3y_4x_1x_7xy_3$ is a θ_7.
4- $xx_6 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_7x_6xy_1$ is a θ_7.

Thus, $N_{C_7}(x) \subseteq \{x_4, x_5\}$. Also, if $xx_4, xx_5 \in E(H)$, then C_3 is produced. Hence x is adjacent to either x_4 or x_5 but not to both, and so, $\mathcal{E}(x, T_1) \leq 3$. Similarly, by using the symmetry, one can show that if $xy_2, xy_4 \in E(H)$, then $\mathcal{E}(x, T_1) \leq 3$. This completes the proof of the claim.
Therefore, by Claim 2, $\mathcal{E}(D_1, T_1) \leq 3(n-11)$. Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(D_1) + \mathcal{E}(D_1, T_1) + \mathcal{E}(T_1) \\
&\leq \left\lfloor \frac{(n-11)^2}{4} \right\rfloor + 3(n-11) + 18 \\
&= \left\lfloor \frac{n^2 - 10n + 61}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 9 \\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

Subcase 2.2. $|V(C_5) \cap V(A)| = 2$. Without loss of generality assume that $C_5 = x_1y_1y_2y_3x_2x_1$ is in H. Let $T_2 = H[y_1, y_2, y_3, A]$ and $D_2 = H - T_2$. As above, $\mathcal{E}(x, C_5) \leq 2$ and $\mathcal{E}(x, A) \leq 2$.

Claim 3. For each $x \in V(D_2)$, $\mathcal{E}(x, T_2) \leq 3$.

Proof of the claim. Suppose that $\mathcal{E}(x, T_2) = 4$. Then $\mathcal{E}(x, C_5) = 2$. Note that if $xx_1 \in E(H)$, then x_1 is a common vertex of both C_5 and A and so $\mathcal{E}(x, T_2) \leq 3$, similarly if $xx_2 \in E(H)$, then x_2 is a common vertex of both C_5 and A and so $\mathcal{E}(x, T_2) \leq 3$. Thus, $N_{C_5}(x) = \{y_1, y_3\}$, and

1- $xx_3 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_2x_3xy_1$ is a θ_7.
2- $xx_4 \notin E(H)$ as otherwise $xy_1y_2y_3x_2x_3x_4xy_3$ is a θ_7.
3- $xx_6 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_7x_6xy_1$ is a θ_7.
4- $xx_7 \notin E(H)$ as otherwise $xy_1y_2y_3x_2x_1x_7xy_3$ is a θ_7.

Thus, x is adjacent to at most x_5, and so $\mathcal{E}(x, T_2) \leq 3$, as claimed.
Hence, $\mathcal{E}(D_2, T_2) \leq 3(n-10)$. Recall that for $j = 1, 2, 3, \mathcal{E}(y_j, A) \leq 2$. Observe that y_2 cannot be adjacent to x_1 or x_2, as otherwise C_3 is produced as a subgraph of H. Thus, $N_A(y_2) = \{x_3\}$ or $\{x_4\}$ or $\{x_5\}$ or $\{x_6\}$ or $\{x_7\}$ or $\{x_3, x_7\}$ as

otherwise C_3, θ_4, θ_5 or θ_7 is produced as a subgraph of H. If $N_A(y_2) = \{x_3, x_7\}$, then $N_A(y_1) = \{x_1\}$ and $N_A(y_3) = \{x_2\}$ as otherwise C_3, θ_4, θ_5 or θ_7 is produced as a subgraph of H. Thus, $\mathcal{E}(T_2) \leq 14$. By Theorem 2 we have

$$\mathcal{E}(D_2) \leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(D_2) + \mathcal{E}(D_2, T_2) + \mathcal{E}(T_2) \\ &\leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor + 3(n-10) + 14 \\ &= \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor \\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

Subcase 2.3. $|V(C_5) \cap V(A)| = 3$. Without loss of generality, assume that $C_5 = x_1y_1y_2x_3x_2x_1$ is in H, then let $T_3 = H[y_1, y_2, A]$ and $D_3 = H - T_3$. Now, $\mathcal{E}(A) = 7$ and by Lemma 2 $\mathcal{E}(y_1y_2, A) \leq 3$, thus $\mathcal{E}(T_3) \leq 11$. Now, let $x \in V(D_3)$, then x is adjacent to at most one of y_1 and y_2 as otherwise C_3 is produced. Further, by Remark 1, $\mathcal{E}(x, A) \leq 2$. Thus, $\mathcal{E}(x, T_3) \leq 3$. Let $B_1 = \{x \in V(D_3) : \mathcal{E}(x, T_3) = 3\}$.

Claim 4. $|B_1| = 0$.

Proof of the claim. Let $x \in B_1$, then $N_{T_3}(x) = \{y_2, x_2, x_4\}$ or $\{y_2, x_2, x_6\}$ or $\{y_1, x_2, x_5\}$ or $\{y_1, x_2, x_7\}$.

If $N_{T_3}(x) = \{y_2, x_2, x_4\}$, then the trail $x_2x_1y_1y_2x_3x_4xx_2x_3$ is a θ_7-graph. If $N_{T_3}(x) = \{y_2, x_2, x_6\}$, then the trail $xx_6x_7x_1x_2x_3y_2x_2$ is a θ_7-graph. By symmetry we get similar trails if $N_{T_3}(x) = \{y_1, x_2, x_5\}$ or $\{y_1, x_2, x_7\}$. The proof of the claim is complete.

Thus, $\mathcal{E}(x, T_3) \leq 2$ for any $x \in V(D_3)$, which implies that

$$\mathcal{E}(D_3, T_3) \leq 2(n-9).$$

Also, by Theorem 2 we have

$$\mathcal{E}(D_3) \leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor.$$

Therefore,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(D_3) + \mathcal{E}(D_3, T_3) + \mathcal{E}(T_3)\\ &\leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor + 2(n-9) + 11\\ &= \left\lfloor \frac{n^2 - 10n + 53}{4} \right\rfloor\\ &= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 7\\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.\end{aligned}$$

Subcase 2.4. $|V(C_5) \cap V(A)| = 4$. Without loss of generality, assume that $C_5 = x_1y_1x_4x_3x_2x_1$ is in H, then let $T_4 = H[y_1, A]$ and $D_4 = H - T_4$. By Remark 1 $\mathcal{E}(x, A) \leq 2$ for any $x \in H - A$. Therefore, $\mathcal{E}(y_1, A) = 2$, and so $\mathcal{E}(T_4) = 9$. Now, let $x \in V(D_4)$, if x is not adjacent to y_1, then $\mathcal{E}(x, T_4) \leq 2$; if x is adjacent to y_1, then

1. $xx_1 \notin E(H)$ as otherwise the trail xy_1x_1x is a C_3.
2. $xx_4 \notin E(H)$ as otherwise the trail xy_1x_4x is a C_3.
3. $xx_5 \notin E(H)$ as otherwise the trail $y_1xx_5x_4x_3x_2x_1y_1x_4$ is a θ_7-graph.
4. $xx_7 \notin E(H)$ as otherwise the trail $y_1xx_7x_1x_2x_3x_4y_1x_1$ is a θ_7-graph.

Thus, $N_{C_7}(x) \subseteq \{x_2, x_3, x_6\}$. Now, If x is adjacent to x_2, then it is neither adjacent to x_3 (as otherwise $C_3 = xx_2x_3x$ is produced) nor to x_6 (as otherwise $\theta_7 = xx_6x_5x_4y_1x_1x_2xy_1$ is produced). Similarly if x is adjacent to x_3, then it can not be adjacent to x_6 (as otherwise $\theta_7 = xx_6x_7x_1y_1x_4x_3xy_1$ is produced). Thus, $\mathcal{E}(x, T_4) \leq 2$, and so $\mathcal{E}(D_4, T_4) \leq 2(n-8)$. Also, by Theorem 2 we have

$$\mathcal{E}(D_4) \leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor.$$

Consequently we have

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(D_4) + \mathcal{E}(D_4, T_4) + \mathcal{E}(T_4)\\ &\leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor + 2(n-8) + 9\\ &= \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor\\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.\end{aligned}$$

□

Now, we give the following construction: Let $\mathcal{H}_1$ be the class of graphs that obtained from $\overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$ by replacing one edge, say $u_1u_2 \in \overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$,

by the path $u_1w_2w_3w_4w_5u_2$ in addition to one of the two edges u_1w_3 and w_2w_4. Note that if $H \in \mathcal{H}_1$, then H is a non-bipartite Hamiltonian graph which has none of $\{\theta_4, \theta_5, \theta_7\}$ as a subgraph of H and $\mathcal{E}(H) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5$. Thus, we establish that

$$h(n; \{\theta_4, \theta_5, \theta_7\}) \geq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5 \text{ for even } n. \tag{3}$$

Theorem 8. *Let $H \in \mathcal{H}(n; \{\theta_4, \theta_5, \theta_7\})$, then*

$$h(n; \{\theta_4, \theta_5, \theta_7\}) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n. Furthermore, the bound is best possible.

Proof. By 3, it is suffices to prove the upper bound of $h(n; \{\theta_4, \theta_5, \theta_7\})$. Let $H \in \mathcal{H}(n; \{\theta_4, \theta_5, \theta_7\})$. If H has no cycles of length 7, then by Theorem 1 we have

$$\begin{aligned} \mathcal{E}(H) &\leq \frac{(n-6)^2}{4} + 13 \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

Now, we assume that H has cycles of length 7. If H contains neither cycles of length 3 nor cycles of length 5, then by Theorem 5 we have

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

To this end, If H contains cycles of length 5, then the results follows from Lemma 4. Finally, if H contains no cycles of length 5 but it contains cycles of length 3, then the results follows from Lemma 3. □

In the following theorem we give an upper bound of $h(n; \theta_7)$ for sufficiently large even n under a constrain of the minimum degree.

Theorem 9. *For sufficiently large even n, let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 22$. Then*

$$h(n; \theta_7) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

Proof. Let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 22$. Suppose that H has θ_5-graph, say $\theta_5 = x_1x_2x_3x_4x_5x_1x_4$. For $i = 1, 2, 3$, let A_i be a set that consist of 6 neighbors of x_i in $H - \theta_5$ selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[x_1, x_2, x_3, x_4, x_5, A_1, A_2, A_3]$ and $B = H - T$. Let $u \in V(B)$, if u is adjacent to a vertex in one of the sets A_1, A_2 and A_3, then u cannot be adjacent to any vertex in the other two sets as otherwise H would have a θ_7-graph. Also, if u

is adjacent to a vertex in A_i for some $i = 1, 2, 3$, then u cannot be adjacent to any of x_{i+1} and x_{i-1}, otherwise, H would have a θ_7-graph. Thus, $\mathcal{E}(u, T) \leq 9$, which implies $\mathcal{E}(B, T) \leq 9(n-23)$. Also, by Theorem 2 we have

$$\mathcal{E}(B) \leq \left\lfloor \frac{(n-23)^2}{4} \right\rfloor \quad \text{and} \quad \mathcal{E}(T) \leq \left\lfloor \frac{(23)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(B) + \mathcal{E}(B,T) + \mathcal{E}(T) \\ &\leq \left\lfloor \frac{(n-23)^2}{4} \right\rfloor + 9(n-23) + \left\lfloor \frac{(23)^2}{4} \right\rfloor \end{aligned}$$

$$\begin{aligned} &\leq \left\lfloor \frac{n^2 - 10n + 230}{4} \right\rfloor \\ &= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 51 \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

So, we consider that H contains no θ_5-graph. If H contains no θ_4-graph as a subgraph, then by Theorem 8 we have

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

If H contains θ_4-graph as a subgraph, then let $\theta_4 = x_1x_2x_3x_4x_1x_3$. For $i = 2, 3, 4$, let A_i be a set that consist of 5 neighbors of x_i in H selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[\, x_1, x_2, x_3, x_4, A_2, A_3, A_4]$ and $B = H - T$. Also, let $u \in V(B)$. If u is adjacent to a vertex in one of the sets A_2, A_3 and A_4, then u cannot be adjacent to a vertex in the other two sets as otherwise H would have a θ_7-graph. Also, if u is adjacent to a vertex in A_i for some $i = 2, 3, 4$, then u cannot be adjacent to x_{i+1} and x_{i-1}, otherwise H would have a θ_5-graph. Thus, $\mathcal{E}(u, T) \leq 7$. Therefore, $\mathcal{E}(B, T) \leq 7(n-19)$. By Theorem 2 we have

$$\mathcal{E}(B) \leq \left\lfloor \frac{(n-19)^2}{4} \right\rfloor \quad \text{and} \quad \mathcal{E}(T) \leq \left\lfloor \frac{(19)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(B) + \mathcal{E}(B,T) + \mathcal{E}(T) \\
&\leq \left\lfloor \frac{(n-19)^2}{4} \right\rfloor + 7(n-19) + \left\lfloor \frac{(19)^2}{4} \right\rfloor \\
&\leq \left\lfloor \frac{n^2 - 10n + 190}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 41 \\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

□

In the above theorem, we have proved that if G is a θ_7-free graph with n vertices and minimum degree greater than or equal to 22, then $\mathcal{E}(G) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5 \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 6$ which confirm Conjecture 1 in the case $k = 3$. Now consider H is the graph obtained from $\overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$ by replacing one edge, say $u_1u_2 \in \overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$, by the path $u_1w_2w_3w_4w_5u_2$ in addition to the two edges u_1w_3 and w_2w_4. Note that H is a non-bipartite Hamiltonian graph which has no θ_7 as a subgraph of H. Furthermore, $\mathcal{E}(H) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 6$. Thus, we establish the upper bound of the Conjecture 1 in the case $k = 3$.

References

[1] M. Bataineh, *Some extremal problems in graph theory*, Ph.D. thesis, Curtin University of Technology, Australia, 2007.

[2] M. Bataineh, M.M.M. Jaradat, E. Al-Shboul, *Edge-maximal graphs without θ_5-graphs*, Ars Combinatoria, 124 (2016), 193-207.

[3] M. Bataineh, M.M.M. Jaradat, E. Al-Shboul, *Edge-maximal graphs without θ_7-graphs*, SUT Journal of Mathematics, 47 (2011), 91-103.

[4] M.S.A. Bataineh, M.M.M. Jaradat, I.Y. Al-Shboul, *Edge-maximal graphs with-out theta graphs of order seven: Part II*, Proceeding of the Annual International Conference on Computational Mathematics, Computational Geometry& Statistics. DOI#10.5176/2251-1911_CMCGS66.

[5] J.A. Bondy, *Pancyclic graphs*, J. Combinatorial Theory Ser B, 11 (1971), 80-84.

[6] J.A. Bondy, *Large cycle in graphs*, Discrete Mathematics 1 (1971), 121-132.

[7] L. Caccetta, R. Jia, *Edge maximal non-bipartite graphs without odd cycles of prescribed length*, Graphs and Combinatorics, 18 (2002), 75-92.

[8] L. Caccetta and K. Vijayan, *Maximal cycles in graphs*, Discrete Mathematics 98 (1991), 1-7.

[9] R. Häggkvist, R.J. Faudree and R.H. Schelp, *Pancyclic graphs-connected Ramsey number*, Ars Combinatoria 11 (1981), 37-49.

[10] G.R.T. Hendry and S. Brandt, *An extremal problem for cycles in Hamiltonian graphs*, Graphs Comb., 11 (1995), 255-262.

[11] M.M.M. Jaradat, M.S. Bataineh and E. Al-Shboul, *Edge-maximal graphs without θ_{2k+1}-graphs,* Akce International Journal of Graphs and Combinatorics, 11 (2014), 57-65.

[12] M.M.M. Jaradat, M. Bataineh, A. Al-Rhayyel and Z. Mustafa, *Extremal number of theta graphs of order* 7*,* Bol. Soc. Paran. Mat. (Accepted).

[13] R. Jia, *Some extermal problems in graph theory*, Ph.D. thesis, Curtin University of Technology, Australia, 1998.

[14] P. Turán, *On a problem in graph theory*, Mat. Fiz. Lapok, 48 (1941), 436-452.

[15] D. Woodall, *Maximal circuits of graphs I*, Acta Math. Acad. Sci. Hungar., 28 (1976), 77-80.

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Some Drazin invertible elements in Banach algebras and applications to operator equations solutions

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Abstract. In this paper, some conditions of Drazin invertibility of operators has been established as well as some results for operations on Drazin invertible operators leading to the solvability of some oparator equations in Banch algebras.
Keywords: generalized Drazin inverse, idempotent operator, consistent equation.

1. Introduction

Troughout this paper A denotes a Banach algebra. Recall that an element $a \in A$ is said to be Drazin invertible if there exists $x \in A$ such that

$$ax = xa,\ xax = x \text{ and } a^k = a^{k+1}x \tag{1}$$

for some positive integer k. The element x above is unique if it exists and is denoted by a^D. The least such k is called Drazin index of a, denoted by $\text{ind}(a)$. If $\text{ind}(a) = 1$, then a^D is called group inverse of a and is denoted by $a^{\pm}$. By A^D we mean the set of all Drazin invertible elements of A. For two commutative generalized Drazin invertible elements a, b in a Banach algebra, Kolina gave the expression of $(ab)^D$. Meanwhile, the representation of $(a+b)^D$ was obtained under the condition $ab = ba = 0$ in a Banach algebra. Later, Dordjevic and Wei [10] gave the expression of $(a+b)^D$ under the assumption $ab = 0$ in the context of the Banach algbra of all bounded linear operators on an arbitrary

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complex Banach space. In [2], Castro-Gonzàlez and Kolina obtained a formula for $(a+b)^D$ under the conditions

$$a^\pi b = b, \quad ab^\pi = a, \quad b^\pi a b a^\pi = 0, \tag{2}$$

which are weaker than $ab = 0$ in Banach algebra. In [8], Deng and Wei derived necessary and sufficient conditions for the existence of $(P+Q)^D$ under the condition $FQ = QP$, where P, Q are bounded linear operators. Moreover, the expression of $(P+Q)^D$ was given. In [4], Cvetkovic-Illè, Liu and Wei extended the result of [8] to Banach algebra. More results on generalized Drazin inverse can be found in [2, 4, 7, 8, 12, 14].

In [15], Liu, Wu and Yu deduced the explicit expression for the Drazin inverses of product ab and sum $a+b$ under the conditions $a^2b = aba$ and $b^2a = bab$, where a and b are complex matrices. In [18] the corresponding results of [15] have been studied for the pseudo Drazin inverse which is a special case of generalized Drazin inverse [19].

In this paper, we prove some interesting condition of Drazin invertibility of operators, some results for operations on Drazin invertible operators are given, some conditions of solvability of oparator equations and other important results are also given. Our work will extend the above results.

2. Preliminaries

The following lemmas will be useful later.

Lemma 2.1 ([9]). *For $a, b \in A^D$, the following assertions hold.*

1. *Whenever $ab = ba$, we have $a^D b = b a^D$.*
2. *Whenever $ab = ba = 0$, we have $(a+b)^D = a^D + b^D$.*

Lemma 2.2 ([21]). *Let $a, b \in A^D$ such that $ab = ba$. Then $ab \in A^D$ and*

$$(ab)^D = b^D a^D = a^D b^D.$$

Lemma 2.3 ([2]). *Let $a, b \in A$ such that $ab \in A^D$. Then $ba \in A^D$ and*

$$(ba)^D = b\left((ab)^D\right)^2 a.$$

Lemma 2.4 ([5]). *1. Let $a, b \in A$. Then $1-ab \in A^D$ if and only if $1-ba \in A^D$.*

2. Let $a, b \in A^D$ and $p^2 = p \in A$, if $ap = pa$ and $bp = pb$, then $ap + b(1-p) \in A^D$ and $(ap + b(1-p))^D = a^D p + b^D (1-p)$.

Proposition 2.5. *Let $a, b, c \in A$ such that $ca = bc$. Then*

$$ca^n = b^n c, \ \forall\, n \in \mathbb{N}.$$

Proof. We proceed by recurrence on n. For $n = 1$, we get $ca = bc$ which is true by hypothesis. So, assume that $ca^n = b^n c$. We have next

$$ca^{n+1} = caa^n = bca^n = bb^n c = b^{n+1}c.$$

The second equality follows from the hypothesis on a, b, c. The third one follows from the recurrence hypothesis.

Proposition 2.6. *Let $a, b, c \in A$ be such that $p^2 = p$, $cp = pc$ and $c\,(pa) = (bp)\,c$. Then*

$$c\,(pa^n) = (b^n p)\,c,\ \forall\, n \in \mathbb{N}.$$

Proof. For $n = 0$, the result follows from the hypothesis $cp = pc$. For $n = 1$, it follows from the hypothesis $c(pa) = (bp)c$. So, assume that $c(pa^n) = (b^n p)c$. We have

$$\begin{aligned} c(pa^{n+1}) &= c(paa^n) \\ &= c(pa)a^n \\ &= (b^n p)ca \text{ (This is from the recurrence hypothesis)} \\ &= b^n(pc)a \\ &= b^n(cp)a \text{ (From the hypothesis cp=pc)} \\ &= b^n(bp)c \text{ (From the hypothesis c(pa)=(bp)c)} \\ &= (b^{n+1}p)c. \end{aligned}$$

Definition 2.7. One say that the operator p is idempotent if $p^2 = p$.

Corollary 2.8. *If p is idempotent then $p^D = p$.*

Theorem 2.9. *Let $A, B, C \in B\,(H)$. Then the equation*

$$AXB = C \tag{1}$$

is consistent if and only if for some $A^{(1)}, B^{(1)}$

$$AA^{(1)}CB^{(1)}B = C \tag{2}$$

in which case the general solution is

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AXBB^{(1)} \tag{3}$$

for arbitrary $Y \in B\,(H)$.

Proof. Let $AA^{(1)}CB^{(1)}B = C$. Then $X_0 = A^{(1)}CB^{(1)}$ is a particular solution of $AXB = C$. Conversely assume that X_0 satisfies $AXB = C$. Then $AA^{(1)}CB^{(1)}B = AA^{(1)}AXBB^{(1)}B = AXB = C$. Further, $A(X - X_0)B = 0$.

3. Main results

In what follows, $a, b \in A$, p and q always mean two arbitrary idempotent elements in the Banach algebra A. We aim to study the Drazin invertibility of the operators $ap(1-q)$, $bq(1-p)$, $ap(1-q)p$ and $bq(1-p)q$.

Theorem 3.1. *Let $a, b \in A^D$ and assume that $ap = pa$, $aq = qa$, $bp = pb$, $bq = qb$ and $pq = qp$. Then*

1. *$ap(1-q) \in A^D$ and $bq(1-p) \in A^D$.*
2. *$ap(1-q) + bq(1-p) \in A^D$ and*

$$(ap(1-q) + bq(1-p))^D = (ap(1-q))^D + (bq(1-p))^D.$$

Proof. We will prove the two assertions in one block. Since $p^2 = p$, we get $p^D = p$. Thus, $p, q \in A^D$. Recall next that $ap = pa$ and $bp = pb$. So, we obtain by Lemma 2.2,

$$(ap)^D = a^D p \text{ and } (bq)^D = b^D q.$$

Next, notice that $aq = qa$, $bp = pb$ and $ap(1-q) = (1-q)ap$. We obtain

$$(ap(1-q))^D = a^D p(1-q) \text{ and } bq(1-p) = (1-p)bq.$$

Consequently,

$$(bq(1-p))^D = b^D q(1-p).$$

Now, observe that

$$ap(1-q)bq(1-p) = bq(1-p)ap(1-q) = 0.$$

We obtain

$$(ap(1-q) + bq(1-p))^D = a^D p(1-q) + b^D q(1-p).$$

Corollary 3.2. *For $a = I = b$ and $pq = qp$, we have*

$$(p(1-q) + q(1-p))^D = p(1-q) + q(1-p).$$

Theorem 3.3. *Let $a, b \in A^D$ and assume that $p^2 = p$, $q^2 = q$ and $pq = qp$. The following assertions hold.*

1. *$ap(1-q)p \in A^D$ and $bq(1-p)q \in A^D$*
2. *$ap(1-q)p + bq(1-p)q \in A^D$ and*

$$(ap(1-q)p + bq(1-p)q)^D = a^D p(1-q)p + b^D q(1-p)q.$$

Proof. The first part is similar to Theorem 3.1. So, let as prove the second. We have

$$ap(1-q)p = ap - apqp = ap - aqp^2 = ap - aqp = ap - apq = ap(1-q) \in A^D.$$

Theorem 3.4. *Let $a, b \in A^D$ and $c \in A$, p idempotent such that $c(pa) = (bp)c$ and $cp = pc$. Then $c\left(pa^D\right) = \left(b^D p\right)c$.*

Proof. It is straightforward that $\forall\, n \in \mathbb{N}$, we have

$$\begin{aligned} bb^D pc - bb^D bcpa^D &= bb^D pc - bb^D cpaa^D \\ &= bb^D pc\left(1 - aa^D\right) \\ &= \left(bb^D\right)^n pc\left(1 - aa^D\right) \\ &= \left(b^D\right)^n (b)^n pc\left(1 - aa^D\right) \\ &= \left(b^D\right)^n cpa^n\left(1 - aa^D\right), \end{aligned}$$

which yields that

$$\begin{aligned} \|bb^D pc - bb^D bcpa^D\| &= \|\left(b^D\right)^n cpa^n\left(\left(1 - aa^D\right)\right)\| \\ &\leq \|b^D\|\|cp\|^{\frac{1}{n}}\|a^n\left(1 - aa^D\right)\|^{\frac{1}{n}} \longrightarrow 0 \end{aligned}$$

whenever $n \longrightarrow \infty$. Thus,

$$bb^D pc = bb^D bcpa^D,$$

or equivalently,

$$b^D pc = b^D bcpa^D.$$

On the other hand, we have

$$\begin{aligned} cpa^D a - b^D pcaa^D a &= cpa^D a - b^D bpca^D a \\ &= \left(1 - b^D b\right) cpa^D a \\ &= \left(1 - b^D b\right)\, cp\left(a^D a\right)^n \\ &= \left(1 - b^D b\right) cpa^n\left(a^D\right)^n \\ &= \left(1 - b^D b\right) b^n pc\left(a^D\right)^n. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|cpa^D a - b^D pcaa^D a\|^{\frac{1}{n}} &= \|\left(1 - b^D b\right) b^n pc\left(a^D\right)^n\|^{\frac{1}{n}} \\ &\leq \|\left(1 - b^D b\right) b^n\|^{\frac{1}{n}}\|pc\|^{\frac{1}{n}}\|a^D\| \longrightarrow 0 \end{aligned}$$

whenever $n \longrightarrow \infty$. So,

$$cpa^D a = b^D pcaa^D a,$$

which means that

$$cpa^D = b^D pcaa^D.$$

Therefore, we deduce that

$$cpa^D = b^D cp.$$

Corollary 3.5. *Let $a, b \in A^D$ and $c \in A$. If $ca = bc$, then $ca^D = b^D c$.*
We obtain the result in [3] as a particular case $p = I$.

Corollary 3.6. *Let $a \in A^D$ and $ca = ac$, then $ca^D = a^D c$.*
We obtain the result in [10] as a particular case $a = b$.

Theorem 3.7. *Let $A, B, C \in B(H)$. The equation*

$$AXB = C \tag{1}$$

is consistent if and only if for some A^D, B^D,

$$AA^DCB^DB = C \tag{2}$$

in which case the general solution is

$$X = A^DCB^D + Y - A^DAYBB^D \tag{3}$$

for arbitrary $Y \in B(H)$.

Proof. If (2) holds then $X = A^DCB^D$ is solution of (1). Conversly, if X is any solution of (1) then

$$C = AXB = AA^DAXBB^DB = AA^DCBB^D.$$

Moreover, it follows from (2) and the definition of A^D and B^D that every X of the from (3) satisfies (1). On the other hand, let X be any solution of (1). Then clearly

$$\begin{aligned} C &= AXB \\ &= AA^DCB^DB + AYB - AA^DAYBB^DB \\ &= AA^DCB^DB + AYB - AYB \\ &= AA^DCB^DB. \end{aligned}$$

Then $X = A^DCB^D$.

Theorem 3.8. *Let $A, B, C \in B(H)$ such that $CA = BC$. Then the equation*

$$AXB = C \tag{4}$$

is consistent if and only if for some A^D, B^D

$$A^DCB = C \tag{5}$$

in which case the general solution is

$$X = A^DCB + Y - A^DAYBB^D \tag{6}$$

for arbitrary $Y \in B(H)$.

Proof. If $CA = BC$ then $CA^D = B^DC$ (see Corrollary 3.2). By Theorem 3.7 we obtain

$$\begin{aligned} AXB = C &\iff AA^DCB^DB = C \\ &\iff AA^DA^DCB = C \\ &\iff A^DCB = C. \end{aligned}$$

Now, whenever $X = A^DCB^D + Y - A^DAYBB^D$ we obtain

$$AXB = AA^DCB^DB + AYB - AA^DAYBB^DB.$$

Consequently,

$$AXB = A^DCB + AYB - AYB.$$

Which means that

$$AXB = A^DCB.$$

Corollary 3.9. *Let $A, C, \in B(H)$ such that $AC = CA$. Then the equation*

$$AXA = C \tag{7}$$

is consistent if and only if for some A^D we have $AA^DCA = C$.

Proof. We have the following implications.

$$AXA = C \implies AA^DCA^DA = C \implies AA^DA^DCA = C \implies AA^DCA = C.$$

References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized inverses. Theory and applications,* Second ed. Springer, New Yook, 2003.

[2] N. Castro-Gonzalez, JJ. Koliha, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburgh, 134 (2004), 1085-1097.

[3] J. Chen, H. Zhu, *Drazin invertibility of product and difference of idempotents in a ring*, Department of Mathematics, Southeast University, Nanjing 210096, China.

[4] DS. Cvetkovic-Ilic, *The generalized Drazin inverse with commutativity up to a factor in a Banach algebra*, Linear Algebra Appl., 431 (2009), 783-791.

[5] DS. Cvetkovic-Ilic, Liu XJ, YM. Wei, *Some additive results for the generalized Drazin inverse in a Banach algebra*, Elect. J. Linear Algebra, 22 (2011), 1049-1058.

[6] CY. Deng, *Generalized Drazin inverses of anti-triangular block matrices*, J. Math. Anal. Appl., 368 (2010), 1-8.

[7] CY. Deng, Cvetkovic-Ilic DS, YM. Wei, *Some results on the generalized Drazin inverse of operator matrices*, Linear Multilinear Algebra, 58 (2010), 503-521.

[8] CY. Deng, YM. Wei, *New additive results for the generalized Drazin inverse*, J. Math. Anal. Appl., 370 (2010),313-321.

[9] CY. Deng, YM. Wei, *Perturbation of the generalized Drazin inverse*, Electron J. Linear Algebra, 21 (2010), 85-97.

[10] DS. Djordjevic, YM. Wei, *Additive results for the generalized Drazin inverse*, J. Aust. Math. Soc., 73 (2002), 115-125.

[11] E. Kaniuth, *A course in commutative banach algebras*, New York, NY, USA: Springer-Verlag, 2009.

[12] JJ. Koliha, *A generalized Drazin inverse*, Glasgow Math. J., 38 (1996), 367-381.

[13] MZ. Kolundzija, D. Mosic, DS. Djordjevic, *Further results on the generalized Drazin inverse of block matrices in Banach algebras*, Bull. Malays. Math. Sci. Soc., 38 (2015), 483-498.

[14] YH. Liao, JL. Chen, J. Cui, *Cline's formula for the generalized Drazin inverse*, Bull. Malays. Math. Sci. Soc., 37 (2014), 37-42.

[15] XJ. Liu, SX. Wu, YM. Yu, *On the Drazin inverse of the sum of two matrices*, J. Appl. Math., 2011, doi:10.1155/2011/831892.

[16] D. Mosic, *A note on Cline's formula for the generalized Drazin inverse*, Linear Multilinear Algebra, 63 (2015), 1106-1110.

[17] D. Mosic, DS. Djordjevic, *Formulae for the generalized Drazin inverse of a block matrix in terms of Banachiewic-Schur forms*, J. Math. Anal. Appl., 413 (2014), 114-120.

[18] D. Mosic, DS. Djordjevic, *Several expressions for the generalized Drazin inverse of a block matrix in a Banach algebra*, Appl. Math. Comput., 220 (2013), 374-381.

[19] Z. Wang, JL. Chen, *Pseudo Drazin inverses in associative rings and Banach algebras*, Linear Algebra Appl., 437 (2012), 1332-1345.

[20] HH. Zhu, JL. Chen, P. Patrcio, *Representations for the pseudo Drazin inverse of elements in a Banach algebra*, Taiwanese J. Math., 19 (2015), 349-362

[21] G.F. Zhuang, J.L. Chen, D.S. Cvetković-Ilić and Y.M. Wei, *Additive property of Drazin invertibility of elements in a ring*, Linear Multilinear Algebra, 60 (2012), 901-910.

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Recognizable iso-triangular picture languages

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Abstract. In modern civilization, the art of tiling has become a Prerogative pattern. Tiling patterns have been used to decorate and cover floors and walls. Motivated by the problem of tiling, H. Wang introduced a tile in 1961, called by his name, Wang tile and in which the edges are coloured. A finite set of Wang tiles admits a valid tiling of the plane if copies of tiles can be arranged one by one without rotation or reflection to fill the plane so that gluable tiles which have the same colour. Wang tiling patterns motivate Kalyani et.al to introduce an iso-triangular labelled Wang tiles. A finite automaton is one of the machines that scan the pictures and recognizes the pictures. In this paper, we define iso-triangular Wang automaton and iso-triangular Wang P system. It is showed that an iso-triangular Wang automaton stimulates the computational power of the P system. The iso-triangular Wang P system recognizes the iso-triangular pictures. The computational complexity of iso-triangular Wang automaton and iso-triangular Wang P system is examined with the computational powers of iso-triangular Wang system, iso-triangular tiling system and hexagonal tiling system.

Keywords: iso-triangular tiles, iso-triangular Wang tiles, finite automata, Wang automata, P system.

1. Introduction

Tiling patterns have been introduced in modern civilization. The interest on pattern recognition and image processing motivates the researcher to introduce more types of tiling patterns on two-dimensional picture languages. A two-dimensional pattern generating model called tile pasting P system that generates tiling patterns by pasting square tiles at the edges has been introduced and studied in [9, 10]. Iso-Triangular tiles and iso-triangular tile pasting system are defined to generate the iso-triangular picture languages [6], in which the labelled edges of iso-triangular tiles are pasted. H. Wang [11] generated a tiling of the plane using the set of square tiles with marked edges with a condition that the

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edges of the tiles should be of the same colour. An aperiodic tile set containing 13 tiles over five tiles is defined in [7] by Karel Culik II et. al. Automata theory is a scanning strategy to recognize the strings. The finite symbols or words are taken as input and the symbols are read by the transition rules [5, 12]. The working rule in automaton is, the input string or alphabet in the initial state can be read by the machine and the read alphabet can be transmitted to the next state. Likewise the transition is processed. The automaton stop reading the symbol after the output string is collected in the final state of the automata.

In [4] P automaton is introduced for the multisets of objects. In traditional automata the order or the position of the words or symbols is considered. But for multisets of objects concern the order is not important to recognize the picture.

Membrane computing is an area of theory of computation and it focusing on computing models from the structure and functioning of biological living cells. P system is one among the areas of membrane computing models introduced in [8]. It contains a finite number of membranes and finite regions. The multisets of objects are placed in the regions of the membranes. In each region the pasting rules are applied to the existing iso-arrays. The pasting rules governing the modification of the objects in time and transfers the pictures to another membrane or retained in the same membrane with the presence of target symbol. Using the concepts of iso-triangular tile pasting, a membrane computing model called Triangular tile pasting P system is defined and examined its generative powers in [2]. In membrane computing, p automaton is a special branch which has not been considered so for the iso-triangular wang tiles. This survey is motivated to define the iso-triangular Wang automaton for scanning the iso-triangular pictures. Generating tessellations and tiling patterns by tile pasting system is studied in [6]. Computation of iso picture languages by rules of context-free iso-array grammar by re-writing iso-arrays has been explained in [1]. In [3] triangular picture languages are generated by triangular tile pasting P system and triangular array token Petri net are introduced. It is noted that triangular tile pasting p system is not comparable with triangular array token Petri net.

In this paper, a theoretical model called an iso-triangular Wang P System (ITWPS) is introduced to recognize or generate the iso triangular pictures. The computational complexity of the system is examined with the computational powers of HTS and ITWS. A non-deterministic machine model called iso-triangular Wang automaton is introduced to recognize the two-dimensional picture languages. It simulates the computational power of P system. This paper introduces Wang P system to check the acceptability of the iso-picture languages. We noticed that the iso-triangular Wang automata stimulate the computational power of iso-triangular Wang P system.

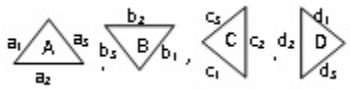
This paper is organized as follows. Section two reveals the preliminary definitions and section three introduces the iso-triangular Wang p system, in this section the generating power of iso-triangular Wang P system is compared with the generating power of hexagonal, iso-triangular tiling systems and iso-triangular

Wang system. In section four, the iso-triangular Wang automaton is introduced with suitable example and noticed that Wang automaton stimulates the Wang P automata when generating the iso-triangular picture languages. Section five gives the conclusion part of the paper.

2. Preliminaries

The art of tiling is a well-known theory in the applications of pattern generation. In this section, we recollect the notion of triangular tiles and tile pasting system. A tile is a topological disc with closed boundary in the XOY plane, whose edges are gluable. A tiling is a family of countable and gluable tiles with no gaps or overlaps that cover the Euclidean plane.

Definition 2.1. *Consider the labelled triangular tiles*

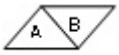


whose horizontal (Vertical) and other side edges are of length 1 unit and $1/\sqrt{2}$ *unit respectively.*

Definition 2.2. *A pasting rule is a pair* (x, y) *of labelled tiles with distinct edges.*

For example, the triangular tile 

and tile 

are joined by the edges (z, t)*, which means that the edge* z *of tile* A *is glued with the edge* t *of tile* B*, we get the pattern* 

Note that the edges are of the same length.

The set of all edge labels is called an edge set denoted by E*. Tile pasting rules of the tiles* A, B, C, D *are given below:*

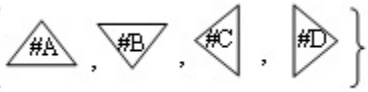
1. *Tile* A *can be glued with tile* B *by the pasting rules* $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ *with tile* C *by the rule* $\{(a_3, c_1)\}$ *and with tile* D *by the rule* $\{(a_1, d_3)\}$*.*
2. *Tile* B *can be glued with tile* A *by the pasting rules* $\{(b_1, a_1), (b_2, a_2), (b_3, a_3)\}$ *with tile* C *by the rule* $\{(b_1, c_3)\}$ *and with tile* D *by the rule* $\{(b_3, d_1)\}$*.*
3. *Tile* C *can be glued with tile* A *by the pasting rule* $\{(c_1, a_3)\}$ *with tile* B *by* $\{(c_3, b_1)\}$ *and with the tile* D *by the pasting rules* $\{(c_1, d_1), (c_2, d_2), (c_3, d_3)\}$*.*
4. *Tile* D *can be glued with tile* A *by the pasting rule* $\{(d_3, a_1)\}$ *with tile* B *by* $\{(d_1, b_3)\}$ *and with the tile* C *by the pasting rules* $\{(d_1, c_1), (d_2, c_2), (d_3, c_3)\}$*.*

Definition 2.3. $S = (\Sigma, P, t_0)$ *is a construct of triangular tile pasting system (TTPS), where* Σ *is a finite set of labelled iso-triangular tiles,* P *is a finite set of pasting rules and* t_0 *is the an axiom of the pattern. A pattern* p_2 *is generated from a pattern* p_1 *by applying the pasting rules in a parallel manner to the edges of the pattern* p_1*, where pasting is possible. Note that the labels of pasted edges in*

a pattern are ignored once the tiles are pasted. The set of all patterns generated from the axiom t_0 constitutes the triangular picture language $T(S)$ of S.

Definition 2.4. *Let $p \in \Sigma_I^{**}$ be an iso-triangular picture. Let Γ and Σ be two finite sets of iso triangular tiles and $\pi : \Gamma^T \to \Sigma^T$ is a mapping which we call, a projection. The projection by mapping π of the picture p is the picture $p' \in \Sigma_I^{**}$ such that $p'(i,j,k) = \pi(p(i,j,k))$ for all $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq k \leq 2j-1$, where (n,m) is the size of the iso-triangular picture p. We denote $p' = \pi(p)$. By extension, we note $\pi(L)$ is in Σ_I^{**}. The projection by mapping π of L over Σ^T is the language $L' = \{p'/p' = \pi(p),\ \forall\ p \in L\} \subseteq \Sigma_I^{**}$.*

Definition 2.5. *An iso-triangular tiling system ITTS is a 4-tuple $(\Sigma, \Gamma, \pi, \theta)$ where Σ and Γ are two finite sets of symbols, $\pi : \Gamma \to \Sigma$ is a projection and θ is a set of iso-arrays of size 2 over the alphabet*

$\Gamma \cup \{$  $\}$

Definition 2.6. *A labelled iso-triangular Wang tile is a 4-tuple consisting of 3 colours, chosen in a finite set of colours. There is particular colour b (black) used only to delimit the boundary of picture. Two iso-triangular Wang tiles are adjacent if the edges of the tiles are of the same colour, whereas, two iso-triangular labelled Wang tiles are adjacent if the gluable edges are of the same colour.*

Definition 2.7. *An iso-triangular Wang system is a four tuple $ITWS = (\Sigma, T_w, C, P_r)$, where Σ is a finite set of alphabets, T_w is a finite set of iso-triangular Wang tiles, C is a finite set of colours and P_r is the set of pasting rules. We denote by $L(IW)$, the language of iso-pictures generated by an iso-triangular Wang system. An iso-picture language $L \subseteq \Sigma_I^{**}$ is iso-triangular Wang recognizable if there exists an iso-triangular Wang system ITWS such that $L = L(ITWS)$.*

Now we recall the notions of Hexagonal picture languages.

Definition 2.8. Hexagonal Tiles *Let I be an alphabet, a finite non-empty set of symbols. The set of all hexagonal tiles over I (including the empty array #) is denoted by I^{**H} and $I^{++H} = I^{**H} - \{\#\}$.*

The size of the hexagonal tile is defined by parameters LU (Left up), LD (Left down), RU (Right up), RD (Right down), U (Up), D (Down) as shown in Figure 1 For $X \in V^{**H}$ the length of left upper side of X is denoted by $|X|_{LU}$; similarly we define $|X|_{LD}$, $|X|_{RU}$, $|X|_{RD}$, $|X|_U$ and $|X|_L$. Here it is enlighten that $|X|_{LD} = |X|_{RD} = |X|_{LU} = |X|$, $|X|_L = |X|_U$.

Definition 2.9. *The projection of the hexagonal picture is defined as $\pi : \Gamma^H \to \Sigma^H$, where Γ^H and Σ^H be two finite sets of hexagonal tiles. Let $L' \subset \Gamma^{++H}$ be a hexagonal picture language. The projection by mapping π of the hexagonal picture language L is $L^H = \{p'/p' = \pi(p),\ \forall\ p \in L\} \subseteq \Sigma^{**H}$.*

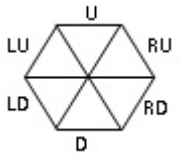


Figure 1: hexagonal picture

Definition 2.10. *A hexagonal tiling system T is a 4-tuple $(\Sigma', \Gamma', \pi, \theta')$ where Σ' and Γ' are two finite sets of symbols, $\pi : \Gamma' \to \Sigma'$ is a projection and θ', is a set of hexagonal tiles over the alpha $\Gamma \cup \{\#\}$.*

The hexagonal picture language $L' \subset \Gamma^{++H}$, if there exists a tiling system $T = (\Sigma', \Gamma', \pi, \theta')$ such that $L' = \pi(\theta')$.

3. Iso-triangular Wang P system

In this section we introduce an iso-triangular Wang tile pasting P system with suitable example and computational complexity of the system is examined by compared with other introduced systems.

Definition 3.1. *Iso-Triangular Wang tiling P system is a construct $\Pi = (\Sigma, \mu, C, T_w, F_1, F_2, \ldots, F_m, R_1, R_2, \ldots, R_m, i_0)$, where*

Σ - is a finite set of labels,

C - Finite set of distinct colours,

μ - Membrane structure, it contains m membranes labelled in a one-to-one manner with labels $1, 2, 3 \ldots m$.

F_i - initial pictures inside the regions of the membranes $(1 \leq i \leq m)$.

T_w - The finite set of iso-triangular Wang tiles.

R_i - Finite set of pasting rules associated with m regions of the membranes.

i_0 - Out put membrane.

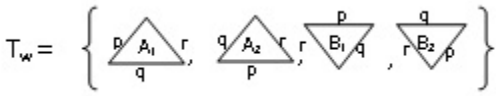
The pasting rules in R_i, is defined by $f(P_i, (T_w \times T_w)/(C \times C))_{Tar} \to P_j$.

The computation starts with axiom iso-triangular Wang tile. In every computational steps, the rules in R_i $(1 \leq i \leq m)$ is applied to all possible edges and the target symbol associated with the pasting rules decides that whether the resultant picture will be communicated to the next immediate membrane or retained in the same membrane. It depends the target symbol from $Tar = \{here, in_j, out\}$ associated with the rules in the region. It is assumed that the tiles defined in T_w can have more number of copies during the computation. The selection of the pasting rules in R_i will be considered non-deterministically. The process of evaluation continues in this way and finally it stops, if no rule can be further applicable. It means the computation stops and then the resultant picture pattern is collected in the output membrane. The set of all languages collected in this manner by Π is denoted by $L(\Pi) = \{p/p \in i_0\}$. The family of all language by Π is denoted by $L(ITWPS)_m$

Example 3.1. A class of hexagonal picture language L_H can be generated by an iso-triangular Wang system consists of 4 iso-triangular Wang tiles.

Consider the iso-triangular Wang system $ITWS = (\Sigma, T_w, C, P_r)$, where Σ is a set of all labels, $C = \{p, q, r\}$ and

$p_r = \{((A_1, B_1)/(r, r)), ((B_2, A_1)/(p, p)), ((A_2, B_1)/(p, p)), ((A_2, B_1)/(r, r)),$
$(B_2, A_1)/(r, r)), ((B_1, A_2)/(p, p)), ((A_2, B_1)/(q, q)), ((A_1, B_2)/(q, q)),$
$((B_1, A_2)/(q, q)), ((A_1, B_2)/(p, p)), ((A_2, B_2)/(r, r))\}.$

$T_w =$ 

Here the rules are considered in the non-deterministic order. The edges are glued with the condition that if the edges of iso-triangular Wang tiles have the same colour. The language L' is generated and which is recognized by the iso-triangular Wang system.

The hexagonal picture language is shown in figure 2

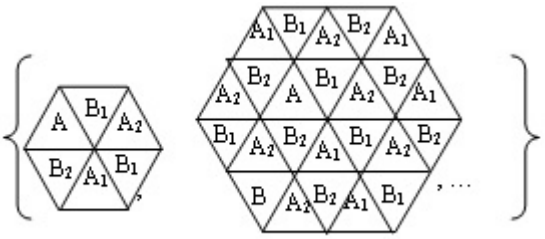
$L_H =$ 

Figure 2: A class of hexagonal picture language

Example 3.2. An iso-triangular Wang P system with two membrane structure generates a class of hexagonal picture language.

Consider the $ITWPS\ \Pi_1 = (\Sigma, [_1[_2]_2]_1, \{p, q, r\}, \{A_1, A_2, B_1, B_2\}, F_1, F_2, R_1, R_2, 1)$,

where $F_1 =$  , $F_2 = \phi$ and A_1, A_2, B_1, B_2 are the iso-triangular Wang tiles which are defined in Example 3.1 and the rules are

$R_1 = \{((A_1, B_1)/(r, r))_{in2}, ((B_2, A_1)/(p, p))_{in2}, ((A_2, B_1)/(p, p))_{here},$
$((A_2, B_1)/(r, r))_{in2}, ((B_2, A_1)/(r, r))_{in2}, ((B_1, A_2)/(p, p))_{in2},$
$((A_2, B_1)/(q, q))_{here}\},$
$R_2 = \{((A_1, B_2)/(q, q))_{out}, ((B_1, A_2)/(q, q))_{out}, ((A_1, B_2)/(p, p))_{here},$
$((B_2, A_1)/(q, q))_{out}, (((B_2, A_1)/(p, p))_{out}, ((A_2, B_2)/(r, r))_{out}\}.$

In region one, the axiom of pattern, tile A_1 is considered and the rule of R_1 is applied, the picture is grown in right up direction and then the existing pattern is entered into the membrane two. In region two, the rule of R_2 is applied. Wang tile B_2 is pasted with the resultant picture that is the coloured edge of the resultant picture is pasted with the same coloured edge of tile B_2, the pattern is

developed its shape in left down direction and it is communicated to the region one of the membrane one by the target symbol associated with the rule. Again in membrane one, the rule (p, p) of the tile A_1 is pasted with the tile B_2. The picture is sent to the region two, in which the tile B_1 is pasted with the tile A_1 and the pattern halts in the region one. Thus the computation gives the first member of the language, is collected in output membrane. The computational process starts again to generate the second member of the language and it is collected finally. During the computation the copies of iso-triangular Wang tiles are considered and the pasting rules are applied non-deterministically.

The members of hexagonal picture language are derived in the derivation steps. The first and second members are shown in fig. 3 and 4.

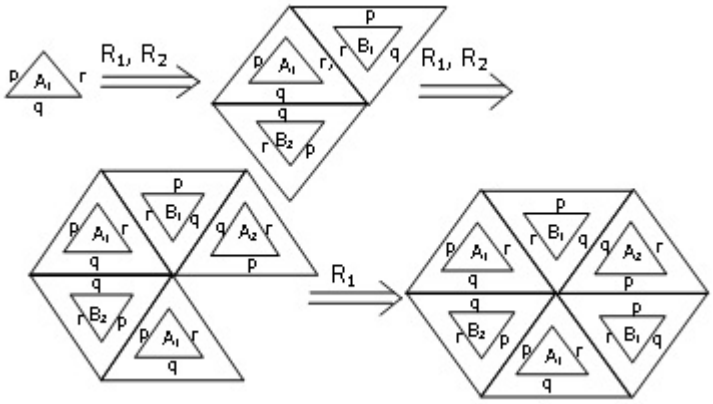
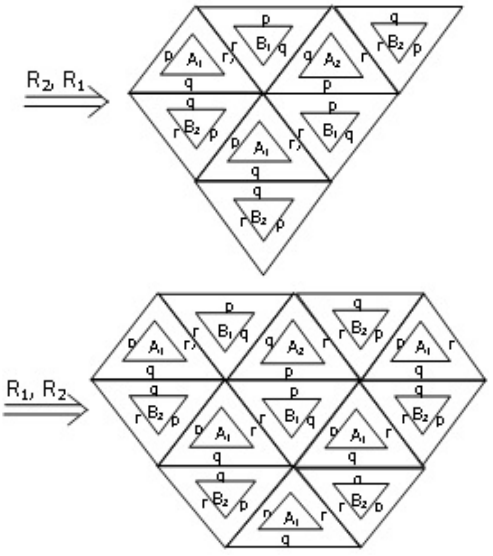


Figure 3: 1^{st} member of hexagonal picture language



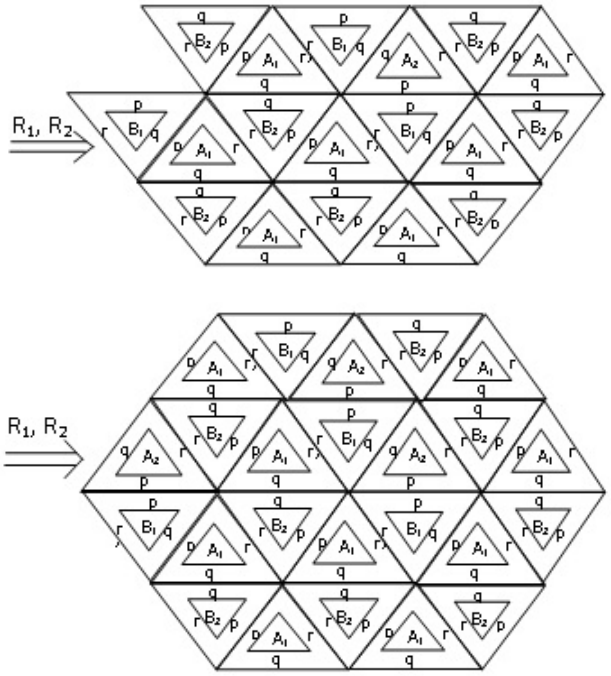


Figure 4: 2^{nd} member of hexagonal picture language

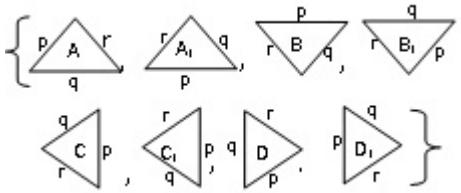
Theorem 3.1. *ITWPS generating a sub class of hexagonal picture languages*

Proof. It is clear from the example 3.2.

Theorem 3.2. $L(ITWPS) - L(HTS) \neq \phi$

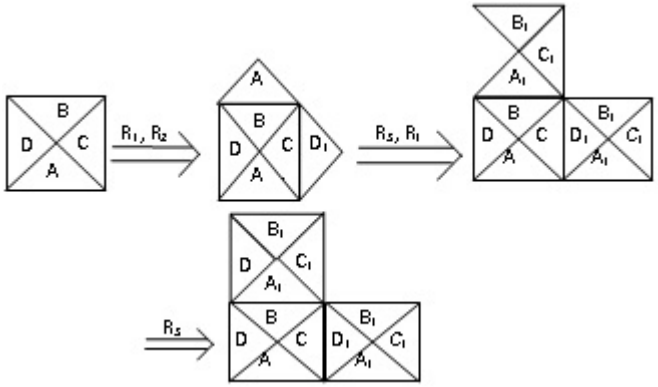
Proof. In example 3.2, a class of hexagonal picture language is generated by iso-triangular Wang tilie Pasting p system. This hexagonal language is recogonized by the HTS. It shows that HTS can generate the hexagonal picture language, but not all the triangular picture languages which are generated by IWTPPS. In order To explain that we considered the triangular picture language of stair case model, which can be generated by ITWTPPS and it cannot be generated by any HTS.

Consider the ITWPS $\Pi_2 = (\Sigma, [_1[_2]_2[_3]_3]_1, \{p, q, r\}, T'_w, F_1, F_2, F_3, R_1, R_2, R_3, 1)$, where Σ is the finite set of all labels



$F_1 =$  , $F_2 = \phi$, $F_3 = \phi$,

$R_1 = \{((A, C)/(r, r))_{in2}, ((B, A_1)/(p, p))_{in2}, ((A_1, C_1)/(q, q))_{in2},$
$((B_1, A)/(q, q))_{in2}, ((B_1, D)/(r, r))_{here}\},$
$R_2 = \{((C, B)/q, q))_{in3}, ((C, D_1)/(p, p))_{in3}, ((C_1, B_1)/(p, p))_{out},$
$((C_1, D_1)/(p, p))_{in3}\},$
$R_3 = \{((D_1, A_1)/(r, r))_{out}, ((D_1, A)/(p, p))_{out}, ((B, D)/(r, r))_{out}\}.$
The derivation step of one member of the picture language is shown below.



□

Theorem 3.3. $L(ITWS) \subseteq L(ITWPS)$

Proof. It is clear from the definitions 2.7 and 3.1. □

Theorem 3.4. $L(ITTS) \subseteq L(ITWPS)$

Proof. It is true from definitions 2.5 and 3.1. □

4. Iso-triangular Wang automaton and Wang P system

As every non-deterministic automaton is converted to deterministic automaton and we do not concentrate on explicit form. In this section iso-triangular, Wang automaton is defined non-deterministically to recognize the iso-triangular pictures.

A non-deterministic iso-triangular Wang automata is a six tuple (ITWA) $A_w = (T_w, Q, C, q_0, F, \Delta)$ where T_w-The finite set of iso-triangular wang tiles, Q-The finite set of states, C-The finite set of colors, q_0-The initial states, F-The set of final states, Δ-The set of transition rules, $\Delta \subseteq Q \times T_w \times C$, the relation in the transition consisting of transition rules, which is of the form $(q, T_w, p) \in \Delta$. The working rule is as follows;

1. A tile of T_w can be considered as an initial symbol in the initial state.
2. A Wang tile is glued with one another, only if the edges have the same colour.
3. After the glueing is occurred by the transition rules of Δ, the resultant picture transmit to the next state.

4. The process continued in this way until the iso-triangular picture is read by the scanner tape presented in the machine and the picture is transmitted to the final state.

5. The computation halts if the reading head of the automata reach the rightmost end and further no more rule can be applicable.

The automata A_w accepts the picture $p \in \Sigma_I^{**}$, if the picture p is collected in the set of final states F. Suppose, if the picture p does not in the state of final state set F, then the picture p is rejected by A_w.

The set of all iso-triangular pictures is a two dimensional picture language which is recognized by the automaton A_w and it is denoted as $L(A_w) = \{p \in \Sigma_I^{**}/A_w \text{ accepts } p\}$.

The strings are considered in automaton to stimulate the P systems. We now define iso-triangular Wang P system with iso-triangular Wang automaton to recognize the iso pictures. Iso-triangular Wang P system is defined as a structure with iso-triangular Wang automaton $A_w = (T_w, Q, C, q_0, F, \Delta)$. The P system is $\Pi_{ITW} = (V, \mu, F_i, \alpha_i, R_i, i_0)$ where $V = \Sigma \cup (T_w \times C) \cup Q$, μ-the membrane structure consists of m membranes which are labelled. F_i - initial pictures inside the membranes, $\alpha_i = (rep, rot)$, R_i - the set of transition rules Δ associated with the target indications and of the form $(p, T_w, \alpha_i, q)_{Tar}$, $Tar = (here, in_j, out\}$ $(1 \leq i \leq m, 2 \leq j \leq m)$.

The rules in R_i are applied in parallel manner. If a Wang tile goes to a point on running head of the automaton in the P system, the suitable transition rule is applied in that particular state and the edges of the tiles are glued when the edges have the same colour. In string case the automata can read the first symbol. But as for the iso-triangular Wang tiles only the suitable tile can be read and it is pasted with a iso-triangular tile, which has the same coloured edge. The resultant picture is scanned and transmitted from the present state to the next state. This mechanism takes place until the resultant picture goes to final state of F. Note that the final state is presented in the region of the output membrane. The computation is success if the picture halts in the output membrane.

The picture language is denoted by $L = \{p_i/p_i \in F_{Aw} \in i_0\}$.

Proposition 4.1. *The iso-triangular Wang automata recognize the iso-triangular picture language of size* (n, m). *The Wang automata* $A_w = (T_w, Q, C, q_0, F, \Delta)$, *with the three iso-triangular Wang tiles generates the iso-picture language, whose members are the iso-triangles an odd number of iso-triangular Wang tiles in the base* $(n \geq 3)$ *with* m *length of tiles. Here the initial state is* q_0 *and the final state is* F, *the three colours are labelled as* p, q *and* r *which are present in the colour set* C. Q*-the set of all states and* Δ *consists of the following transition rules,* $\{(q_0, A_1, q_1)$, (q_0, A_2, q_2), (q_0, B_1, q_0), (q_1, B_1, q_2), (q_2, A_2, q_3), (q_3, B_1, q_0), $(q_3, A_2, q_3)\}$. *In the transition rules all* $A's$ *and* $B's$ *are the labelled iso-triangular wang tiles. For space and time concern the labels of Wang tiles are considered*

in the rules. The transition graph of the language of iso-triangles is given in figure 5. Initial state is defined by enclosed circle and an arrow, the final state is represented by a concentric circle. The iso-triangular Wang tile is pasted with other Wang tile, only if the edges have the same colour. Initial state is q_0 and the initial iso-Wang tile is A_1. The transition rules are applied and the member of the language is collected in the final state of the automata. The language is recognizable since the language is accepted by the automata.

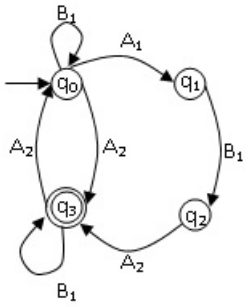


Figure 5: Transition graph of the iso-triangular picture language

Lemma 4.1. *Two pictures patterns p_1 and p_2 shown in figure 6 are equal in shape but not in size. Because the picture pattern p_1 is generated by the iso-triangular wang tiles with no geometrical operation the second one is generated by the equilateral-triangular wang tiles with rotation.*

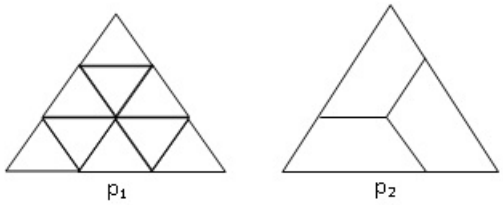
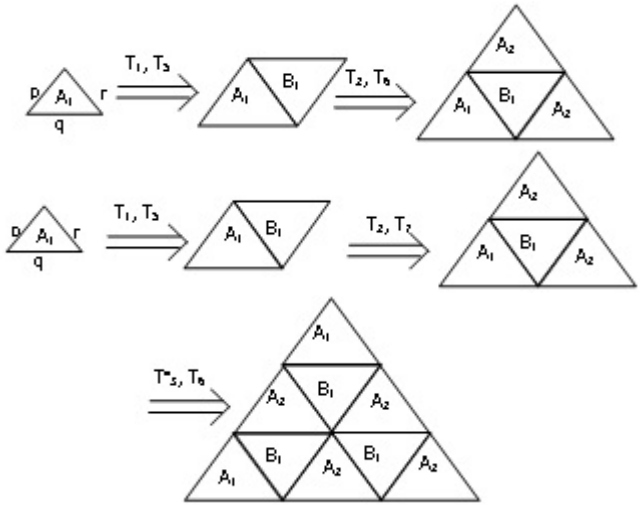


Figure 6: Triangular pictures

Theorem 4.1. *The iso-triangular Wang automata A_w stimulate the iso-triangular Wang P system.*

Consider The iso-triangular Wang automata $A_w = (T_w, Q, C, q_0, q_3, \Delta)$ and the Wang p system $(V, [_1[_2]_2[_3]_3]_1, F_1, F_2, F_3, \alpha_i, R_1, R_2, R_3, 3)$, where $v = T_w \cup Q$, $R_1 = \{T_1, T_2, T_3, T_4\}$, $R_2 = \{T_5\}$, $R_3 = \{T_6, T_7\}$, $F_2 = \phi$, $F_3 = \phi$ and $\alpha_i = \phi$. The states in Q are q_0, q_1, q_2, q_3 and the Wang tiles are $T_w = \{A_1, A_2, B_1\}$. The transition rules are $T_1 = (q_0, A_1, q_1)_{in2}$, $T_2 = (q_2, A_2, q_3)_{in3}$, $T_3 = (q_0, B_1, q_0)_{here}$, $T_4 = (q_0, A_2, q_3)_{in3}$, $T_5 = (q_1, B_1, q_2)_{out}$, $T_6 = (q_3, A_2, q_3)_{here}$, $T_7 = (q_3, A_2, q_0)_{out}$. The P system generates the iso-triangular picture language and recognized by

the automata. The graph is shown fig. 5 and the two members are given below with two derivation steps.



5. Conclusion

This paper, introduced an iso-triangular Wang P system and iso-triangular Wang automata with suitable examples, which are generating iso triangular arrays. Also, the computational complexity of ITWPS is examined by comparing the generative powers of ITWS, ITTS. It is noticed that the ITWA stimulates the ITWPS to generate iso-picture languages.

References

[1] K. Bhuvaneswari, T. Kalyani, D. Gnanaraj Thomas, A.K. Nagar and T. Robinson, *Iso-array rewriting P systems with context-free rules,* Journal of Application for Mathematics, 3 (2014), 1-16.

[2] K. Bhuvaneswar and T. Kalyani, *Triangular tile pasting P system for Pattern generation,* Proceeding of ICAMTCS-2013, 26-30.

[3] K. Bhuvaneswari, T. Kalyani and D. Lalitha, *Triangular tile pasting P system and array generating Petri nets,* International Journal of Pure and Applied Mathematics, 107 (2016), 111-128.

[4] E. Csuhaj-varju and G. Vaszil, *P automata or purely communicating accepting P systems*, In A. Salomaa, Gh. Paun, G. Rozenberg and C. Zandron (eds.), Membrane Computing, International Workshop WMC 2002, LNCS vol. 2597, 219-233, 2002, Springer-Verlag, Berlin.

[5] J. Hopcroft, R. Motwani and J. Ullman, *Introduction to automata theory*, Languages and Computation (second edition). Addison Wesley, 2001.

[6] T. Kalyani, K. Sasikala, V.R. Dare and T. Robinson, *Triangular pasting system*, In K.G. Subramanian, K. Rangarajan, M. Mukund (Eds.), Formal Models, Languages and Applications, Series in Machine Perception and Artificial Intelligence, 66 (2006), 195-211.

[7] II Karel Culik, *An aperiodic set of 13 Wang tiles*, Discrete Mathematics, Elsevier Science, 160 (1996), 245-251.

[8] G.H. Păun, *Membrane computing: an introduction*, Springer-Verlag, Berlin, Heidelberg, 2010.

[9] K.G. Subramanian, T. Robinson and Atulya K. Nagar, *Tile pasting P system model for pattern generation*, in Third Asia International Conference on Modelling and Simulation, Indonesia, 2009.

[10] T. Robinson, V.R. Dare and K.G. Subramanian, *A parallel pattern generating system*, Proc. of the International Workshop on Parallel Image Processing and Analysis, Madras, 23-32, 1999.

[11] H. Wang, *Dominoes and the AEA case of decision problems*, In Mathematical Theory of Automata, Symposium on Mathematical Theory of Automata, 1963, vol. 12 of Brooklyn Polytechnic Institute. Microwave Research Institute, Symposia series, pages 23-55, Polytechnic Press, 1963.

[12] Xian Xu, *P systems and finite automata*, Proceedings of the First International Conference on Complex, Intelligent and Software Intensive Systems (CISIS07).

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Γ-convergence: an application to eigenvalue problems

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Abstract. We apply an elementry result of $\Gamma-$ convergence to show the dependance on $p(\cdot)$ of the first eigenvalue $\lambda_{p(x)}$ of the $p(x)-$Laplacian problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda |u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Keywords: $p(x)-$Laplacian, eigenvalue problem, stability, $\Gamma-$convergence, equicoercivenes.

1. Introduction and main result

The eigenvalue problem associated with the $p(x)-$Laplacian is defined by

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u, \\ u \in W_0^{1,p(x)}(\Omega), \end{cases} \tag{1.1}$$

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where Ω is a bounded set of $\mathbb{R}^N$ with Lipschitzien boundary, $N \geq 2$ and $p(\cdot) \in \mathcal{P}^{log}(\Omega)$ which is defined by

$$\mathcal{P}^{log}(\Omega) = \{p(\cdot) \in \mathcal{P}(\Omega)/\ p(\cdot)\ is\ \log-\text{Hölder continuous}\}$$

$$\mathcal{P}(\Omega) = \{p(\cdot) \in C(\overline{\Omega})/\ 1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \sup_{x\in\overline{\Omega}} p(x) =: p^+ < N\ \forall x \in \overline{\Omega}\}.$$

Recall that $p(\cdot)$ is $\log-$Hölder continuous if

$$|p(x) - p(y)| \leq\ -\frac{L}{\log(|x-y|)}$$

for some $L > 0$ and for all $x, y \in \Omega$, with $0 < |x-y| \leq \frac{1}{2}$.

A pair $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ is a weak solution of (1.1) provided that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\ dx = \lambda \int_\Omega |u|^{p(x)-2} uv\ dx\ \forall v \in W_0^{1,p(x)}(\Omega) \tag{1.2}$$

u is called an eigenfunction and λ is called the associated eigenvalue.

(1.2) is the Euler-Lagrange equation associated to the minimization of the Rayleigh ratio $R(u) := \frac{\int_\Omega |\nabla u|^{p(x)}\ dx}{\int_\Omega\ |u|^{p(x)}\ dx}$.

In [7], the authors showed that problem (1.1) has an infinite sequence of eigenvalues defined by

$$\lambda^{m,p(x)}(u) = \inf_{K\in\mathcal{W}_{m,p(\cdot)}(\Omega)} \sup_{u\in K} \int_\Omega |\nabla u|^{p(x)}\ dx, \tag{1.3}$$

where $\mathcal{W}_{m,p(\cdot)}(\Omega)$ is the set of symmetric and compact subsets of $\{u \in W_0^{1,p(x)}(\Omega) : \int_\Omega |u|^{p(x)}\ dx = 1\}$ such that $i(K) \geq m$, and i denotes the Krasnoselskii's genus.[1]

When $m = 1$ we use the notation $\lambda^{p(\cdot)} = \lambda^{1,p(\cdot)}$.

The normalization condition $\int_\Omega |u|^{p(x)}\ dx = 1$ taken on u is necessary to avoid the fact that $\lambda^{m,p(\cdot)}(u) \equiv 0$. Indeed, the example introduced in [8] shows that this may happen even if $u \not\equiv 0$.

The first eigenvalue (i.e. $m = 1$) is characterized to be

$$\lambda^{p(\cdot)} = \inf_{u\in\mathcal{M}} \int_\Omega |\nabla u|^{p(x)}\ dx\ ;\ \mathcal{M} = \{u \in W_0^{1,p(x)}(\Omega)/\ \int_\Omega |u|^{p(x)}\ dx = 1\}$$

is a closed manifold.

In the constant exponent case E. Parini showed in [11] the continuity with respect to p of the m$^{\text{th}}$ variational eigenvalue in a sufficiently regular domain.

1. Let E be a real Banach space and $\Sigma(E)$ the set of all closed subsets of E which do not contain 0 and are symmetric with respect to 0. The Krasnoselkii's genus $i(A)$ of a non empty set $A \in \Sigma(A)$ is defined to be the smallest integer $m \in \mathbb{N}$ for which there exists an odd mapping $g \in C(A, \mathbb{R}^m \setminus \{0\})$. If no such mapping exists we write $i(A) = +\infty$.

More precisely the authors require the domain to be bounded with Lipschitz boundary.

A few years later, M. Degiovanni and M. Marzocchi showed in [5] the continuity of the m$^{\text{th}}$ variational eigenvalue but requiring the domain to be only bounded open and connected subset of $\mathbb{R}^N$.

For other articles treating the stability see for example [6],[2],[10] and the references therein.

As far as we know, the stability of eigenvalues associated to the $p(x)-$Laplacian problem have not been treated before. Our main goal in this work is to show that the first eigenvalue associated to the $p(x)-$Laplacian problem $\lambda^{p(\cdot)}$ is continuous with respect to $p(\cdot)$, i.e. for all $h \in \mathbb{N}$

$$\lambda^{p_h(\cdot)} \longrightarrow \lambda^{p(\cdot)} \; as \; p_h(\cdot) \underset{h\to\infty}{\longrightarrow} p(\cdot) \; uniformly.$$

To achieve this goal we utilized an elementary result found in [3] concerning convergence of minimum values of a family of functionals F_h to the minimum value of a limit F.

Since it has been introduced in 1985 by De Giorgi, De Giorgi's $\Gamma-$convergence covers almost all other variational convergences in term of applications as it plays a central role for its compactness properties and for the large number of results concerning $\Gamma-$limits of integral functionals.

The most difficult issue in the study of the Rayleigh ratio with integrals is the non homogeneity of the integrals in marked contrast to the ratio with norms.

Plan of the paper. In Section 2, we collect various preliminaries necessary to apprehend the following section. In section 3 we will prove the main result, that is, the first eigenvalue is continuous with respect to uniform variations of $p(\cdot)$.

2. Preliminaries

2.1 Lebesgue and Sobolev spaces with variable exponents

We refer to the monograph [4] for basics of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, but remind their definitions and some important properties.

Let Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitzien boundary and $p(\cdot) \in \mathcal{P}(\Omega)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx$$

is finite. We define the Luxembourg norm on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \; \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the space of $L^{p(\cdot)}(\Omega)$ functions u whose distributional gradient exists and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Under the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \ \forall u \in W^{1,p(x)}(\Omega)$$

$W^{1,p(\cdot)}(\Omega)$ is a Banach space.

Note that the variable exponent Sobolev spaces resembles classical Sobolev spaces in many aspects: they are Banach spaces and they are reflexive and separable if and only if $1 < p^- \leq p^+ < \infty$.

We require $p(\cdot)$ to be in addition in $\mathcal{P}^{\log}(\Omega)$ to define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ (the space of indefinitely derivable functions with compact support) in $W^{1,p(x)}(\Omega)$ (see [4], Corollary 11.2.4).

Proposition 2.1. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^N)$ be bounded, let $q \in \mathcal{P}(\mathbb{R}^N)$ and suppose that*

$$q(x) \leq p^*(x) - \frac{\omega(|x|)}{log(e + \frac{1}{|x|})}$$

where $\omega : [0, \infty) \to [0, \infty)$ is increasing and continuous with $\omega(0) = 0$.

Then the embedding $W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is compact and continuous. with $p^*(x) := \frac{Np(x)}{N-p(x)}$.

1. *(**Poincaré inequality.**) There is a positive constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \ \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

2.2 $\Gamma-$convergence

We recall from [3] basic definitions concerning $\Gamma-$convergence.

Definition 2.2. *Let X be a metric space. A sequence F_h of functionals $(F_h) : X \to \overline{\mathbb{R}}$ is said to $\Gamma(X)$ converges to $F : X \to \overline{\mathbb{R}}$ and we write $\Gamma(X) - \lim_{h\to\infty} F_h = F$ if the following hold*

i. (liminf inequality) For every $u \in X$ and $(u_h) \subset X$ such that $u_h \to u$ in X we have

$$F(u) \leq \liminf_{h\to\infty} F_h(u_h).$$

ii. (limsup inequality) For every $u \in X$ there exists a sequence $(u_h) \subset X$ (called recovery sequence) such that $u_h \to u$ in X and

$$F(u) \geq \limsup_{h\to\infty} F_h(u_h).$$

The condition ii. is characterized by the following (see [9]).

Proposition 2.3. *Let X be a topological space that satisfies the 1st axiom of countability*[2] *and assume that (u_h) is a sequence such that $u_h \to u$ in X as $h \to \infty$, $\limsup_{h\to\infty} F(u_h) \leq F(u)$ and such that for every $m \in \mathbb{N}$ there exists a sequence $\{u_{m,h}\}_h$, $u_{m,h} \to u_m$ as $h \to \infty$ with $\limsup_{h\to\infty} F_h(u_{m,h}) \leq F(u_m)$, then there exists a recovering sequence of u in the sense of ii. of the preceding definition.*

Definition 2.4. *We say that a sequence (F_h) is equicoercive (on X) if for every $t \in \mathbb{R}$ there exists a closed countably compact subset K_t of X such that $\{F_h \leq t\} \subseteq K_t$ for all $h \in \mathbb{N}$.*

Proposition 2.5 ([3], Proposition 7.7)**.** *The sequence (F_h) is equicoercive if and only if there exists a lower semi-continuous coercive function $\psi : X \to \overline{\mathbb{R}}$ such that $F_h \geq \psi$ on X for all $h \in \mathbb{N}$.*

Theorem 2.6 ([3], Theorem 7.8)**.** *Suppose that (F_h) is equicoercive on X, then the $\Gamma - \underbrace{\limsup_{h\to\infty} F_h}_{F'}$ and $\Gamma - \underbrace{\liminf_{h\to\infty} F_h}_{F''}$ are coercive and*

$$\min_{x\in X} F'(x) = \liminf_{h\to\infty} \inf_{x\in X} F_h(x).$$

If in addition (F_h) $\Gamma-$converges to a function F in X then F is coercive and

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x).$$

3. Main result

$F : M \to [0, +\infty]$ defined by:

$$F(u) = \int_\Omega |\nabla u|^{p(x)}\, dx \tag{3.1}$$

and $F_h : M_h \to [0, +\infty]$ defined by:

$$\int_\Omega |\nabla u|^{p_h(x)}\, dx. \tag{3.2}$$

To show the continuity of the first eigenvalue we have to prove the following convergence

$$\inf_{u\in M_h} \int_\Omega |\nabla u|^{p_h(x)}\, dx \longrightarrow \inf_{u\in M} \int_\Omega |\nabla u|^{p(x)}\, dx \tag{3.3}$$

as $p_h(\cdot) \to p(\cdot)$ uniformly, where $M_h = \{u \in W_0^{1,p_h(\cdot)}(\Omega) / \int_\Omega |u|^{p_h(x)}\, dx = 1\}$ and $M = \{u \in W_0^{1,p(\cdot)}(\Omega) / \int_\Omega |u|^{p(x)}\, dx = 1\}$.

2. A topological space satisfies the first axiom of countability if the defining system of neighbourhoods of every point has a countable base.

In view of Theorem 2.6, the convergence (3.3) will be attained if we show that the family F_h $\Gamma-$converges to F as $h \to \infty$ and F_h is equicoercive. Our main concern in this section is to prove the following theorem:

Theorem 3.1. *1. The family of functions (F_h) is equicoercive.*

2. Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set with Lipschitzien boundary and let $(p_h) \subset \mathcal{P}^{\log}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega)$ such that $p_h \to p$ uniformly in Ω. Then

$$\Gamma(L^{p(\cdot)})(\Omega) - \lim_{h\to\infty} F_h = F.$$

Proof. 1. For all $u \in X$ such that $\|u\|_X \geq 1$ we have $\frac{F(u)}{\|u\|_X} \geq \|u\|_X^{p_- -1}$. This implies

$$\Rightarrow \lim_{\|u\|_X\to\infty} \frac{F(u)}{\|u\|_X} = \infty.$$

So F is coercive and clearly lower semi-continuous so we deduce the equicoerciveness by simply taking $\Psi = F$ in Proposition 2.5.

2. `liminf inequality`.

Let $u_h \to u$ in $L^{p(\cdot)}(\Omega)$. If $\liminf_{h\to\infty} F_h(u_h) = +\infty$ there is nothing to prove. Thus, we may assume, without loss of generality, that $u_h \in W^{1,p_h(\cdot)}(\Omega)$ and, up to a subsequence

$$\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h) < +\infty.$$

Since $p_h \to p$ uniformly, $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ $\forall h \geq N_\epsilon$ $|p_h - p| < \epsilon$

$$\Rightarrow \ p(x) - \epsilon < p_h(x) \ \forall x \in \Omega \ \forall h \geq N_\epsilon \tag{3.4}$$

Thus, for all $h \geq N_\epsilon$ we have $W_0^{1,p_h(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and

$$\|\nabla u_h\|_{p(\cdot)-\epsilon} \leq (1 + |\Omega|)\|\nabla u_h\|_{p_h(\cdot)} \leq D, \tag{3.5}$$

where D is a constant not depending on ϵ.

Then(u_h) is a bounded sequence in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ which is reflexive, hence there exists a subsequence (u_h) such that $u_h \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ with $u \in W_0^{1,p(\cdot)-\epsilon}(\Omega)$ for all $\epsilon > 0$ sufficiently small.

We shall next show that the limit $u \in W_0^{1,p(\cdot)}(\Omega)$ for ϵ sufficiently small.

By the lower semicontinuity of the modular and inequality (3.5) we have

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)-\epsilon}\, dx &\leq \liminf_{h\to\infty} \int_\Omega |\nabla u_h|^{p(x)-\epsilon}\, dx \\
&\leq \liminf_{h\to\infty} \max\{\|\nabla u_h\|_{p(\cdot)-\epsilon}^{p^+-\epsilon}, \|\nabla u_h\|_{p(\cdot)-\epsilon}^{p^--\epsilon}\} \\
&\leq \liminf_{h\to\infty} \max\ \{\left[(1 + |\Omega|)|\nabla u_h|_{p_h(\cdot)}\right]^{p^+-\epsilon}, \\
&\quad \left[(1 + |\Omega|)|\nabla u_h|_{p_h(\cdot)}\right]^{p^--\epsilon}\} \\
&\leq \max\left\{D^{p^+-\epsilon}, D^{p^--\epsilon}\right\} < +\infty.
\end{aligned}$$

Thus

$$\sup_{\epsilon>0}\int_\Omega |\nabla u|^{p(x)-\epsilon}\,dx < +\infty$$

and by Fatou's lemma

$$|\nabla u|^{p(x)} = \liminf_{\epsilon\to 0}|\nabla u|^{p(x)-\epsilon} \in L^1(\Omega).$$

It follows that

$$u \in W_0^{1,p(\cdot)}(\Omega).$$

By Young's inequality and (3.4) we have

$$\int_\Omega |\nabla u_h|^{p(x)-\epsilon}\,dx \le \int_\Omega \frac{p(x)-\epsilon}{p_h(x)}|\nabla u_h|^{p_h(x)}\,dx + \int_\Omega \frac{p_h(x)-p(x)+\epsilon}{p_h(x)}\,dx$$
$$\le \int_\Omega |\nabla u_h|^{p_h(x)}\,dx + \epsilon\int_\Omega \frac{1}{p_h(x)}\,dx + \frac{1}{p_h^-}\|p_h - p\|_\infty.$$

In view of the weak convergence to u in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and the uniform convergence of p_h to p we have

$$\int_\Omega |\nabla u|^{p(x)-\epsilon}\,dx \le \liminf_{h\to\infty}\int_\Omega |\nabla u_h|^{p_h(x)}\,dx + \epsilon\frac{|\Omega|}{p^-}.$$

Since the inequality holds for each $\epsilon > 0$, letting $\epsilon \searrow 0$ and applying Fatou's lemma we obtain

$$\int_\Omega |\nabla u|^{p(x)}\,dx \le \liminf_{h\to\infty}\int_\Omega |\nabla u_h|^{p_h(x)}\,dx.$$

`limsup inequality`

Suppose that $u \in W_0^{1,p(\cdot)}(\Omega)$ (if $u \notin W_0^{1,p(\cdot)}(\Omega)$ $F_h(u) = +\infty$ and there is nothing to prove). Since $p(\cdot)$ is $\log-$Hölder continuous $\overline{C_0^\infty(\Omega)} = W_0^{1,p(\cdot)}(\Omega)$ and then it exists a sequence $(u_n) \subset C_0^\infty(\Omega)$ such that

$$u_n \to u \in W_0^{1,p(\cdot)}(\Omega)$$

$$\text{i.e}\int_\Omega |\nabla u|^{p(x)}\,dx = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)}\,dx. \tag{3.6}$$

On the other hand, for each $m \in \mathbb{N}$

$$|u_m|^{p_h(x)} \longrightarrow |u_m|^{p(x)} \text{ as } h \to \infty \text{ for a.e } x \in \Omega$$

and by the uniform convergence of p_h to p $\forall\epsilon > 0$ we can find $N_\epsilon \in \mathbb{N}$ such that $p_h < p + \epsilon\ \forall\ h \ge N_\epsilon$ and then

$$|u_m|^{p_h(x)} \le |u_m|^{p(x)+\epsilon} + 1 \in L^1(\Omega)\ \forall\ x \in \Omega\ \forall\ h \ge N_\epsilon,$$

whence by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_\Omega |u_m|^{p_h(x)}\,dx = \int_\Omega |u_m|^{p(x)}\,dx\ \forall m \in \mathbb{N}.$$

The proof is completed by Proposition 2.3 . □

Theorem 3.2. *The first eigenvalue associated to the $p(x)-$Laplacian $\lambda^{p(\cdot)}$ is continuous with respect to the exponent $p(\cdot)$.*

Proof. By Theorem 3.1 the functional F defined in (3.2) is the $\Gamma-$limit of F_h and since the functionals F_h are coercive $\forall\ h \in \mathbb{N}$ by Theorem 2.6

$$\inf_{u\in M_h} F_h \longrightarrow \inf_{u\in M} F \text{ as } h \to \infty.$$

Define $\lambda^{p_h(\cdot)} := \inf_{u\in M_h} F_h(u)$ and $\lambda^{p(\cdot)} := \inf_{u\in M} F(u)$ we conclude that

$$\lambda^{p_h(\cdot)} \longrightarrow \lambda^{p(\cdot)}$$

when $p_h(\cdot) \to p(\cdot)$ uniformly. □

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References

[1] T. Champion, L. De Pascale, *Asymptotic behaviour of nonlinear eigenvalue problems involving $p-$Laplacian-type operators*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 137 (2007), 1179-1195.

[2] F. Colasuonno, M. Squassina, *Stability of eigenvalues for variable exponent problems.* Nonlinear Analysis: Theory, Methods & Applications, 123 (2015), 56-67.

[3] G. Dal Maso, *An introduction to $\Gamma-$converegnce*, 8 (2012), Springer Science & Business Media.

[4] L. Diening, P. Harjulehto, P. Hästö, M. Rúžička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, 2017 (2011), Springer-Verlag, Heidelberg.

[5] M. Degiovanni, M. Marzocchi, *On the dependence on p of the variational eigenvalues of the $p-$Laplace operator*, arXiv preprint arXiv:1504.07403, 2015.

[6] A. El Khalil, P. Lindqvist, A. Touzani, *On the stability of the first eigenvalue of $A_p u + \lambda g(x)|u|^{p-2}u = 0$ with varying p*, Rend. Mat. Appl., 24 (2004), 321-336.

[7] X. Fan, Q. Zhang, D. Zhao, *Eigenvalues of $p(x)-$Laplacian Dirichlet problem.* Journal of Mathematical Analysis and Applications, 302 (2005), 306-317.

[8] X. Fan, D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$ *Dirichlet problem*, J. Math. Anal. Appl., 263 (2001), 424-446.

[9] J. Jost, X. Li-Jost, *Calculus of variations*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 64 (1998).

[10] P. Lindqvist, *Stability for the solutions of* $\operatorname{div}(\nabla u|^{p-2}\nabla u) = f$ *with varying* p, Journal of Mathematical Analysis and Applications, 127 (1987), 93-10.

[11] E. Parini, *Continuity of the variational eigenvalues of the* $p-$*Laplacian with respect to* p, Bulletin of the Australian Mathematical Society, 83 (2011), 376-381.

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On connectedness of the Hausdorff fuzzy metric spaces

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Abstract. Several properties of the Hausdorff fuzzy metric, as completeness, precompactness and completion were discussed by Rodríguez-López and Romaguera [The Hausdorff fuzzy metric on compact sets, Fuzzy Sets and Systems 147 (2004) 273–283]. It is necessary to seek other more properties of the Hausdorff fuzzy metric. In the paper, we show that a fuzzy metric space is connected if and only if the corresponding Hausdorff fuzzy metric space on compact (finite) sets is connected.
Keywords: fuzzy metric, the Hausdorff fuzzy metric, compact subset, finite subset, connected.

1. Introduction

The notion of fuzzy metric space has been defined by many authors in different ways [4, 6, 17, 18]. In particular, by extending the notion of Menger space to the fuzzy setting, Kramosil and Michalek [18] obtained the notion of fuzzy metric space with the help of continuous t-norms. In order to make the topology generated by a fuzzy metric to be Hausdorff, George and Veeramani [6] modified in a slight but appealing way the notion given by Kramosil and Michalek. In [15], Gregori and Romaguera proved that the topological space generated by a modified fuzzy metric is metrizable and then, some classical theorems on metric properties are adapted to the realm of the modified version of fuzzy metric. In view of them, some authors became interested in the new version of fuzzy metric. Dinarvand [2] gave Some fixed point results for admissible Geraghty contraction type mappings in fuzzy metric spaces. Romaguera and Sanchis [25] introduced a notion of fuzzy metric group and investigated properties of the quotient subgroups of a fuzzy metric group. In [1], an arclength notion

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of continuous curves in fuzzy metric spaces was proposed, and some arclength properties , including invariance, additive, continuity and boundedness were studied by Chen et al. In recent years, Gregori et al [9, 10, 11, 12, 13, 14, 15, 16] gave much progress to the study of fuzzy metric spaces. We can also find other more studies of fuzzy metric spaces in [7, 8, 19, 20, 21, 23, 26, 27, 28].

To explore hyperspaces in given fuzzy metric spaces, Rodríguez-López and Romaguera [24] introduced the Hausdorff fuzzy metric on the collection of nonempty compact sets and studied completeness, precompactness and compactness of the Hausdorff fuzzy metric spaces. In [5], an identification theorem for the completion of the Hausdorff fuzzy metric was explored. Recently, we gave several equivalent conditions for the Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be complete in [22]. It is nature to seek other more properties of the Hausdorff fuzzy metric. In the paper we do it. Here, we obtain that connectedness of a fuzzy metric space and connectedness of the corresponding Hausdorff fuzzy metric space on compact (finite) sets coincide.

2. Preliminaries

Throughout the paper the letter $\mathbb{N}$ shall denote the set of all positive integer numbers. Our basic reference for general topology is [3].

Definition 2.1 ([6]). *A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:*

(i) $*$ *is associative and commutative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a$ *for all* $a \in [0,1]$*;*

(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, and* $a, b, c, d \in [0,1]$*.*

Observe that $a * b = \min\{a, b\}$*,* $a * b = a \cdot b$ *and* $a * b = \max\{a + b - 1, 0\}$ *are three common examples of continuous t-norms.*

Clearly, Definition 2.1 shows that, if $1 \geq r > s \geq 0$, then there exists a $\delta \in (s, 1)$ such that $r * \delta \geq s$.

Definition 2.2 ([6]). *An ordered triple $(X, M, *)$ is said to be a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:*

(i) $M(x, y, t) > 0$*;*

(ii) $M(x, y, t) = 1$ *if and only if* $x = y$*;*

(iii) $M(x, y, t) = M(y, x, t)$*;*

(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(v) the function $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

If $(X,M,*)$ *is a fuzzy metric space, then we shall call* $(M,*)$ *a fuzzy metric on* X.

It is well known that $M(x,y,\cdot)$ *is a non-decreasing function on* $(0,\infty)$ *for all* $x,y \in X$.

Definition 2.3 ([6]). *Let* $(X,M,*)$ *be a fuzzy metric space and let* $r \in (0,1), t > 0$ *and* $x \in X$. *The set*

$$B_M(x,r,t) = \{y \in X | M(x,y,t) > 1-r\}$$

is called the open ball with center x *and radius* r *with respect to* t.

In [6], George and Veeramani proved that $\{B_M(x,r,t) | x \in X, t > 0, r \in (0,1)\}$ forms a base of a topology τ_M in X and $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M for every $x \in X$.

Definition 2.4 ([15]). *A fuzzy metric space* $(X,M,*)$ *is said to be compact if* (X,τ_M) *is compact.*

3. Connectedness of the Hausdorff fuzzy metric spaces

Given a fuzzy metric space $(X,M,*)$, we shall denote by $\mathcal{P}_0(X)$, $\mathcal{K}_0(X)$ and $\mathcal{F}_0(X)$, the collection of nonempty subsets, the collection of nonempty compact subsets and the collection of nonempty finite subsets of (X,τ_M), respectively. Put $M(x,A,t) := \sup_{a\in A} M(x,a,t)$, $M(A,x,t) := \sup_{a\in A} M(a,x,t)$ for all $x \in X$, $A \in \mathcal{P}_0(X)$ and $t > 0$ (see Definition 2.4 of [28]). It is obvious that $M(x,A,t) = M(A,x,t)$. In the following, for any $B \in \mathcal{P}_0(X)$, the cardinality of B will be denote by $|B|$.

Definition 3.1 ([24]). *Let* $(X,M,*)$ *be a fuzzy metric space. For every* $A,B \in \mathcal{K}_0(X)$ *and* $t > 0$, *define* $H_M : \mathcal{K}_0(X) \times \mathcal{K}_0(X) \times (0,\infty) \to [0,1]$ *by*

$$H_M(A,B,t) = \min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\}.$$

Then $(\mathcal{K}_0(X), H_M, *)$ *is a fuzzy metric space. We call* $(H_M,*)$ *the Hausdorff fuzzy metric on* $\mathcal{K}_0(X)$.

Lemma 3.1 ([19]). *Let* $(X,M,*)$ *be a fuzzy metric space. Then* $H_M(A,B,t) = 1 - \inf\{r | A \subseteq B_M(B,r,t), B \subseteq B_M(A,r,t)\}$ *for all* $A,B \in \mathcal{K}_0(X)$ *and* $t > 0$, *where* $B_M(A,r,t) = \bigcup_{a\in A} B_M(a,r,t)$.

Lemma 3.2 ([24]). *Let* $(X,M,*)$ *be a fuzzy metric space. Then* M *is a continuous function on* $X \times X \times (0,\infty)$.

Definition 3.2. *Let $(X, M, *)$ be a fuzzy metric space. For every $A \in \mathcal{K}_0(X)$ and $t > 0$, define $diam_t(A)$ by*

$$\operatorname{diam}_t(A) = \inf\{M(a, b, t) | a, b \in A\}$$

and call it the fuzzy diameter of A with t.

Proposition 3.1. *Let $(X, M, *)$ be a fuzzy metric space, $A \in \mathcal{K}_0(X)$ and $t > 0$. Then $diam_t(A) > 0$.*

Proof. By Lemma 3.2, we conclude that $\{M(a, b, t) | a, b \in A\}$ is a closed subset of [0,1]. Hence

$$\inf\{M(a, b, t) | a, b \in A\} \in \{M(a, b, t) | a, b \in A\}.$$

Thus, there exist $a_0, b_0 \in A$ such that

$$\inf\{M(a, b, t) | a, b \in A\} = M(a_0, b_0, t) > 0.$$

We complete the proof. □

Lemma 3.3. *Let $(Y, M, *)$ be a compact subspace of a fuzzy metric space $(X, M, *)$, and let $\mathcal{U}$ be a family of open subsets of X which covers Y. Then there exist $r \in (0, 1)$ and $t > 0$ with the property that every $A \in \mathcal{K}_0(X)$ with $diam_t(A) > 1 - r$ and $A \cap Y \neq \emptyset$ is contained in an element $U \in \mathcal{U}$.*

Proof. Assume that such r and t do not exist. Then, for every $n \in \mathbb{N}$, we can choose an $A_n \in \mathcal{K}_0(X)$ such that

(1) $\operatorname{diam}_{\frac{1}{n}}(A_n) > 1 - \frac{1}{n}$,

(2) $A_n \cap Y \neq \emptyset$,

(3) A_n is not contained in any element of $\mathcal{U}$.

Take $y_n \in A_n \cap Y$ for every $n \in \mathbb{N}$. Since Y is compact, there is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ such that the subsequence converges to a point $y \in Y$. Without loss of generality, we can assume that

$$y = \lim_{n\to\infty} y_n.$$

Since $\mathcal{U}$ covers Y, we can find a $U \in \mathcal{U}$ such that $y \in U$. Note that U is open, there exists $l \in \mathbb{N}$ such that $B_M(y, \frac{1}{l}, \frac{1}{l}) \subseteq U$. In addition, there exists $N \in \mathbb{N}$ such that $y_m \in B_M(y, \frac{1}{l+1}, \frac{1}{l+1})$ for all $m \geq N$. Now choose $m \geq N$ so large that $\frac{1}{m} \leq \frac{1}{l(l+1)}$ and $(1 - \frac{1}{m}) * (1 - \frac{1}{l+1}) \geq [1 - \frac{1}{2}(\frac{1}{l} + \frac{1}{l+1})] > 1 - \frac{1}{l}$. Since

$$\operatorname{diam}_{\frac{1}{m}}(A_m) = \inf\{M(a, b, \frac{1}{m}) | a, b \in A_m\} > 1 - \frac{1}{m},$$

we get that $M(a, b, \frac{1}{m}) > 1 - \frac{1}{m}$ whenever $a, b \in A_m$. Let $m \geq N$ so large. Then, for every $x \in A_m$, we have that $M(x, y, \frac{1}{l}) \geq M(x, y, \frac{1}{m} + \frac{1}{l+1}) \geq M(x, y_m, \frac{1}{m}) * M(y_m, y, \frac{1}{l+1}) \geq (1 - \frac{1}{m}) * (1 - \frac{1}{l+1}) > 1 - \frac{1}{l}$. Hence $x \in B_M(y, \frac{1}{l}, \frac{1}{l})$. It follows that $A_m \subseteq B_M(y, \frac{1}{l}, \frac{1}{l}) \subseteq U$, which contradicts (3). This concludes the proof.□

Lemma 3.4. *Let $(Y, M, *)$ be a compact subspace of a fuzzy metric space $(X, M, *)$, and let U be an open neighborhood of Y in X. Then there exist $r \in (0,1)$ and $t > 0$ such that $B_M(Y, r, t) \subseteq U$.*

Proof. Observe that $\{U\}$ covers Y. According to Lemma 3.3, we can find $r \in (0,1)$ and $t > 0$ with the property that every $A \in \mathcal{K}_0(X)$ with $\mathrm{diam}_t(A) > 1 - r$ and $A \cap Y \neq \emptyset$ is contained in $U \in \{U\}$. Let $x \in B_M(Y, r, t) = \bigcup_{y \in Y} B_M(y, r, t)$. Then there exists a $y_0 \in Y$ such that $x \in B_M(y_0, r, t)$, which means that $M(x, y_0, t) > 1 - r$. It follows that $\mathrm{diam}_t(\{x, y_0\}) = M(x, y_0, t) > 1 - r$. Note that $\{x, y_0\} \in \mathcal{K}_0(X)$ and $\{x, y_0\} \cap Y \neq \emptyset$. Hence $\{x, y_0\} \subseteq U$, which implies that $x \in U$. Thus $B_M(Y, r, t) \subseteq U$. □

The proof of the next lemma is straightforward.

Lemma 3.5. *Let $(X, M, *)$ be a fuzzy metric space and $A \in \mathcal{P}_0(X)$. If $0 < r_1 < r_2 < 1$ and $0 < t_1 < t_2$, then $B_M(A, r_1, t_1) \subseteq B_M(A, r_2, t_2)$.*

Let $(X, M, *)$ be a fuzzy metric space. For every finite family $\mathcal{V}$ of subsets of X, put

$$\langle \mathcal{V} \rangle = \{A \in \mathcal{K}_0(X) | A \subseteq \bigcup \mathcal{V} \text{ and for each } V \in \mathcal{V}, V \cap A \neq \emptyset\}.$$

Lemma 3.6. *Let $(X, M, *)$ be a fuzzy metric space, and let $\mathcal{V}$ be a finite family of open subsets of X. Then $\langle \mathcal{V} \rangle$ is an open subset of $\mathcal{K}_0(X)$.*

Proof. Let $A \in \langle \mathcal{V} \rangle$. For each $V \in \mathcal{V}$ we choose a point $x_V \in V \cap A$. Notice that $\langle \mathcal{V} \rangle$ is finite, V is open and $A \in \mathcal{K}_0(X)$ with $A \subseteq \bigcup \mathcal{V}$. Then, according to Lemma 3.4 and Lemma 3.5, we can find $r \in (0,1)$ and $t > 0$ such that

(1) $B_M(x_V, r, t) \subset V$ for every $V \in \mathcal{V}$,

(2) $B_M(A, r, t) \subseteq \bigcup \mathcal{V}$.

Let $B \in B_{H_M}(A, r, t)$. Then, due to Lemma 3.1, we obtain that $B \subseteq B_M(A, r, t)$. It follows from (2) that $B \subseteq \bigcup \mathcal{V}$. Since $A \subseteq B_M(B, r, t)$, we can find a $y_V \in B$ such that $M(x_V, y_V, t) > 1 - r$ for every $V \in \mathcal{V}$. According to (1), we conclude that $y_V \in V$ for every $V \in \mathcal{V}$. Thus, for each $V \in \mathcal{V}$, $y_V \in V \cap B$, i.e., $V \cap B \neq \emptyset$. Hence $B \in \langle \mathcal{V} \rangle$. We immediately deduce that $B_{H_M}(A, r, t) \subseteq \langle \mathcal{V} \rangle$. The proof is finished. □

Definition 3.3. *A fuzzy metric space $(X, M, *)$ is said to be connected if (X, τ_M) is connected.*

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space. If $(\mathcal{K}_0(X), H_M, *)$ is connected, then so is $(X, M, *)$.*

Proof. Let $(\mathcal{K}_0(X), H_M, *)$ be a connected fuzzy metric space. Suppose that $(X, M, *)$ is not connected. Then we can find two disjoint nonempty open subsets U and V of X such that $X = U \cup V$. Notice that

$$\mathcal{K}_0(X) = \langle \{U\} \rangle \cup \langle \{V\} \rangle \cup \langle \{U, V\} \rangle.$$

Obviously, by Lemma 3.6, $\langle\{U\}\rangle$, $\langle\{V\}\rangle$ and $\langle\{U,V\}\rangle$ are pairwise disjoint, nonempty and open in $\mathcal{K}_0(X)$. It follows that $(\mathcal{K}_0(X), H_M, *)$ is not connected, which is a contradiction. We are done. □

Lemma 3.7 ([24]). *Let Y be a dense subset of a fuzzy metric space $(X, M, *)$. Then $\mathcal{F}_0(Y)$ is dense in $(\mathcal{K}_0(X), H_M, *)$.*

Lemma 3.8 ([3]). *Let A be a connected subspace of a topological space (X, τ_X). Then the closure $\overline{A}$ of A is also connected.*

Theorem 3.2. *Let $(X, M, *)$ be a fuzzy metric space. If $(\mathcal{F}_0(X), H_M, *)$ is connected, then so is $(X, M, *)$.*

Proof. Suppose that $(\mathcal{F}_0(X), H_M, *)$ is connected. According to Lemma 3.7 and Lemma 3.8, we get that $(\mathcal{K}_0(X), H_M, *)$ is connected. It follows from Theorem 3.1 that $(X, M, *)$ is connected. □

Lemma 3.9 ([3]). *A finite cartesian product of connected spaces is connected.*

Lemma 3.10 ([3]). *The image of a connected space under a continuous mapping is connected.*

Lemma 3.11 ([3]). *The union of collection of connected subspaces of a topological space (X, τ_X) that have a point in common is connected.*

Let $(X, M, *)$ be a fuzzy metric space. For each $n \in \mathbb{N}$, put $\mathcal{F}_0^n(X) = \{A \subseteq X | 1 \leq |A| \leq n\}$, which we regard as a subspace of $\mathcal{K}_0(X)$.

Lemma 3.12. *Let $(X, M, *)$ be a fuzzy metric space. For each $n \in \mathbb{N}$, define the function $g_n : X^n \to \mathcal{F}_0^n(X)$ by*

$$g_n(x_1, x_2, \cdots, x_n) = \{x_1, x_2, \cdots, x_n\}.$$

Then g_n is a continuous surjection for every $n \in \mathbb{N}$.

Proof. Obviously, g_n is a surjection for every $n \in \mathbb{N}$. We will now prove that g_n is continuous for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let $(a_1, a_2, \cdots, a_n) \in X^n$. Then $A = g_n(a_1, a_2, \cdots, a_n) = \{a_1, a_2, \cdots, a_n\} \in \mathcal{F}_0^n(X)$. To complete our proof, it suffices to show that, for $r \in (0, 1)$ and $t > 0$,

$$g_n(B_M(a_1, r, t) \times B_M(a_2, r, t) \times \cdots \times B_M(a_1, r, t)) \subseteq B_{H_M}(A, r, t) \cap \mathcal{F}_0^n(X).$$

Let $(b_1, b_2, \cdots, b_n) \in B_M(a_1, r, t) \times B_M(a_2, r, t) \times \cdots \times B_M(a_n, r, t)$. Then $b_i \in B_M(a_i, r, t)$ $(1 \leq i \leq n)$, which means that $M(a_i, b_i, t) > 1 - r$ $(1 \leq i \leq n)$. It follows that $A \subseteq B_M(B, r, t)$ and $B \subseteq B_M(A, r, t)$, where $B = \{b_1, b_2, \cdots, b_n\}$. Due to Lemma 3.1, we obtain that

$$B \in B_{H_M}(A, r, t) \cap \mathcal{F}_0^n(X),$$

i.e., $g_n(b_1, b_2, \cdots, b_n) \in B_{H_M}(A, r, t) \cap \mathcal{F}_0^n(X)$. This concludes the proof. □

Theorem 3.3. *Let $(X, M, *)$ be a connected fuzzy metric space. Then $(\mathcal{F}_0(X), H_M, *)$, and hence $(\mathcal{K}_0(X), H_M, *)$, is connected.*

Proof. Assume that (X, τ_M) is connected. According to Lemma 3.9, we get that X^n is connected. For each $n \in \mathbb{N}$, define the function $g_n : X^n \to \mathcal{F}_0^n(X)$ by

$$g_n(x_1, x_2, \cdots, x_n) = \{x_1, x_2, \cdots, x_n\}.$$

Then, by Lemma 3.10 and Lemma 3.12, we obtain that $(\mathcal{F}_0^n(X), H_M, *)$ is connected for all $n \in \mathbb{N}$. Observe that $\bigcup_{n=1}^{\infty} \mathcal{F}_0^n(X) = \mathcal{F}_0(X)$ and $\bigcap_{n=1}^{\infty} \mathcal{F}_0^n(X) = \mathcal{F}_0^1(X) \neq \emptyset$. It follows from Lemma 3.11 that $\mathcal{F}_0(X)$ is connected. This concludes the proof. □

As a consequence of Theorem 3.1, Theorem 3.2 and Theorem 3.3, we immediately deduce the next corollary.

Corollary 3.1. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent.*

(i) $(X, M, *)$ *is connected.*

(ii) $(\mathcal{F}_0(X), H_M, *)$ *is connected.*

(iii) $(\mathcal{K}_0(X), H_M, *)$ *is connected.*

4. Conclusion

In this work, we have proven that a fuzzy metric space is connected if and only if the corresponding Hausdorff fuzzy metric space on compact (finite) sets is connected.

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References

[1] S. Chen, P. Shi, C. C. Lim, *Curve arclength in fuzzy metric spaces*, Fuzzy Sets and Systems, 313 (2017), 105-113.

[2] M. Dinarvand, *Some fixed point results for admissible Geraghty contraction type mappings in fuzzy metric spaces,* Iranian Journal of Fuzzy Systems, 14 (2017), 161-177.

[3] R. Engelking, *General topology*, PWN-Polish Science Publishers, warsaw, 1977.

[4] M. A. Erceg, *Metric spaces in fuzzy set theory,* Journal of Mathematical Analysis and Applications, 69 (1979), 205-230.

[5] J. G. García, S. Romaguera, M. Sanchis, *An identification theorem for the completion of the Hausdorff fuzzy metric,* Fuzzy Sets and Systems, 227 (2013), 96-106.

[6] A. George, P. Veeramani, *On some resules in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395-399.

[7] A. George, P. Veeramani, *Some theorems in fuzzy metric spaces*, The Journal of Fuzzy Mathematics, 3 (1995), 933-940.

[8] A. George, P. Veeramani, *On some resules of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems, 90 (1997), 365-368.

[9] V. Gregori, J. A. Mascarell, A. Sapena, *On completion of fuzzy quasi-metric spaces,* Topology and its Application, 153 (2005), 886-899.

[10] V. Gregori, J. J. Miñana, S. Morillas, *A note on convergence in fuzzy metric spaces*, Iranian Journal of Fuzzy Systems, 11 (2014), 75-85.

[11] V. Gregori, J. J. Miñana, S. Morillas, A. Sapena, *Characterizing a class of completable fuzzy metric spaces*, Topology and its Application, 203 (2016), 3-11.

[12] V. Gregori, J. J. Miñana, A. Sapena, *Completable fuzzy metric spaces*, Topology and its Application, 225 (2017), 103-111.

[13] V. Gregori, S. Morillas, B. Roig, *Fuzzy quasi-metrics for the Sorgenfrey line*, Fuzzy Sets and Systems, 222 (2013), 98-107.

[14] V. Gregori, S. Morillas, A. Sapena, *On a class of completable fuzzy metric spaces*, Fuzzy Sets and Systems, 161 (2011), 95-111.

[15] V. Gregori, S. Romaguera, *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, 115 (2000), 485-489.

[16] V. Gregori, S. Romaguera, *On completion of fuzzy metric spaces,* Fuzzy Sets and Systems, 130 (2002), 399-404.

[17] O. Kaleva, S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), 215-229.

[18] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), 326-334.

[19] C. Q. Li, *On some results of metrics induced by a fuzzy ultrametric*, Filomat, 27 (2013), 1133-1140.

[20] C. Q. Li, L. Yang, *Some properties of the Hausdorff fuzzy metric on finite sets*, Journal of Computational Analysis and Applications, 19 (2015), 359-364.

[21] C. Q. Li, Z. Q. Yang, *Fuzzy ultrametrics based on idempotent probability measures*, The Journal of Fuzzy Mathematics, 22 (2014) 463-476.

[22] C. Q. Li, Y. L. Zhang, K. X. Zhang, *A note on completeness of the Hausdorff fuzzy metric spaces*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 49-58.

[23] D. Qiu, R. Dong, H. Li, *On metric spaces induced by fuzzy metric spaces*, Iranian Journal of Fuzzy Systems, 13 (2016), 145-160.

[24] J. Rodríguez-López, S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, 147 (2004), 273-283.

[25] S. Romaguera, M. Sanchis, *On fuzzy metric group*, Fuzzy Sets and Systems, 124 (2001), 109-115.

[26] A. Savchenko, M. Zarichnyi, *Fuzzy ultrametrics on the set of probability measures,* Topology, 48 (2009), 130-136.

[27] P. Tirado, *On compactness and G-completeness in fuzzy metrics spaces,* Iranian Journal of Fuzzy Systems, 9 (2012), 151-158.

[28] P. Veeramani, *Best approximation in fuzzy metric spaces,* The Journal of Fuzzy Mathematics, 9 (2001), 75-80.

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A new criterion of optimization of the cross multipole coefficients in a modified surface stress operator for the elastic two-dimensional case

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Abstract. The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified' Green's function technique. In this work, a new criterion of optimality based on the minimization of the norm of the surface stress operator (modified traction operator) is established.

Keywords: cross multipole coefficients, modified traction operator, Green's function, integral equations, linear elasticity.

1. Introduction

The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified' Green's function technique, where the simple and cross multipole coefficients must satisfy some suitable and mild conditions (2.5) [3]. Some criteria to determine an optimal choice for these multipole coefficients are developed recently; the first criterion is based on the minimization of the norm of the modified integral operator [2, 8], and motivated by enlarging the radius of convergence of the numerical method used (successive approximations). The second criterion is based on the minimization of the norm of the modified Green's function [9], and motivated by the minimization of the norm of the difference between the modified and exact Green's function. In [1], Argyropoulos et al.

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have presented another criterion based on the minimization of the condition number of the boundary integral equations describing the problem. In [6], we have developed a new criterion based on the minimization of the norm of the surface stress operator, or the norm of the modified traction operator using simple multipole coefficients, motivated this time by the minimization of the norm of the difference between the modified and exact kernel of the integral operator. In this work we generalize the same criterion but for the case of cross multipole coefficients.

2. Formulation of the problem

An exterior Neumann boundary value problem in two-dimensional elastic case can be described through a boundary integral equation of the form [3]:

$$\left(\frac{1}{2}I+\bar{K}_0^*\right)(\varphi)(p)=f(p),\ p\in\partial D, \tag{2.1}$$

where f is a Holder continuous density, and the integral operator K_0 is defined as :

$$(K_0\varphi)(p)=\frac{1}{2\pi}\int_{\partial D}T_pG_0(p,\ q)\ \varphi(q)\ ds_q,\ p\in\partial D, \tag{2.2}$$

G_0 is the Green's function (fundamental solution), and T is the surface stress operator.

Using the modified Green's function technique, by introduce a regular solution [3], the modified Green's function is written as:

$$\begin{aligned} &G_1(p,\ Q)\\ &=\frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2}\left[\begin{array}{c} F_m^{\sigma l}(P)\otimes\hat{F}_m^{\sigma l}(Q)+a_m^{\sigma l}\ F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q)\\ +(-1)^{\sigma+l}\,b_m\ F_m^{\sigma l}(P)\otimes F_m^{(3-\sigma)(3-l)}(Q)\end{array}\right] \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} F_m^{\sigma 1}(P)&=\operatorname{grad}\left(H_m^1(kr_p)\ E_m^{\sigma}(\theta_p)\right),\\ F_m^{\sigma 2}(P)&=rot\left(H_m^1(Kr_p)\ E_m^{\sigma}(\theta_p)\ \hat{e}_3\right), \end{aligned} \tag{2.4}$$

$\hat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [3], and

$$E_m^{\sigma}(\theta_p)=\sqrt{\varepsilon_m}\begin{cases}\cos(m\theta_p),\ \sigma=1\\ \sin(m\theta_p),\ \sigma=2\end{cases},\ \text{with } \varepsilon_m=\begin{cases}1,\ m=0,\\ 2,\ m>0,\end{cases}$$

$a_m^{\sigma l}$ and b_m are the simple and cross multipole coefficients, which must satisfy the following conditions:

$$\bar{b}_m\left(a_m^{\sigma 1}+\frac{1}{2}\right)+b_m\left(\bar{a}_m^{\sigma 2}+\frac{1}{2}\right)=0, \tag{2.5}$$

and

$$\left|a_m^{\sigma l}+\frac{1}{2}\right|^2+|b_m|^2-\frac{1}{4}<0,\ \forall m=0:\infty \text{ and } \forall\sigma,\ l=1:2.$$

3. Main results

3.1 General case

We now consider the question of how to choose the simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m in the modification (2.3) so to minimize $\|T_pG_1\|_{L_2(\partial D)}$. The question is answered by the following theorem.

Theorem 1. *There are uniquely defined simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m which minimize the quantity:*

$$\int\limits_{r_p=A}\|T_pG_1\|^2_{L_2(\partial D)}\ ds_p,\ \forall A\geq\max r_q,\ q\in\partial D. \tag{3.1}$$

These simple and cross multipole coefficients are given by the relations:

$$a_m^{\sigma l}=\frac{\bar B_m^{\sigma l}M_{m,1}^{\sigma l}+\bar\beta_m^{\sigma l}M_{m,2}^{\sigma l}-A_m^{(3-\sigma)(3-l)}N_{m,1}^{\sigma l}-\alpha_m^{(3-\sigma)(3-l)}N_{m,2}^{\sigma l}}{\Delta^\sigma_{m,\partial D}}, \tag{3.2}$$

and

$$(-1)^{\sigma+l}\,b_m=\frac{B_m^{\sigma l}N_{m,1}^{\sigma l}+\beta_m^{\sigma l}N_{m,2}^{\sigma l}-A_m^{\sigma l}M_{m,1}^{\sigma l}-\alpha_m^{\sigma l}M_{m,2}^{\sigma l}}{\Delta^\sigma_{m,\partial D}}, \tag{3.3}$$

with

$$M_{m,1}^{\sigma l}=\Delta^\sigma_{m,A}\left[B_m^{(3-\sigma)(3-l)}g_m^{(3-\sigma)(3-l)}-A_m^{(3-\sigma)(3-l)}h_m^{\sigma l}\right], \tag{3.4}$$

$$M_{m,2}^{\sigma l}=\Delta^\sigma_{m,\partial D}\left[\beta_m^{(3-\sigma)(3-l)}g_m^{(3-\sigma)(3-l)}-\alpha_m^{(3-\sigma)(3-l)}h_m^{\sigma l}\right], \tag{3.5}$$

$$N_{m,1}^{\sigma l}=\Delta^\sigma_{m,A}\left[B_m^{(3-\sigma)(3-l)}h_m^{(3-\sigma)(3-l)}-A_m^{(3-\sigma)(3-l)}g_m^{\sigma l}\right], \tag{3.6}$$

$$N_{m,2}^{\sigma l}=\Delta^\sigma_{m,\partial D}\left[\beta_m^{(3-\sigma)(3-l)}h_m^{(3-\sigma)(3-l)}-\alpha_m^{(3-\sigma)(3-l)}g_m^{\sigma l}\right], \tag{3.7}$$

$$g_m^{\sigma l}=-\left\langle\bar\beta_m^{\sigma l}T\hat F_m^{(3-\sigma)(3-l)}+\alpha_m^{\sigma l}T\ \hat F_m^{\sigma l},\ TF_m^{\sigma l}\right\rangle, \tag{3.8}$$

$$h_m^{\sigma l}=-\left\langle\bar\beta_m^{\sigma l}T\hat F_m^{(3-\sigma)(3-l)}+\alpha_m^{\sigma l}T\ \hat F_m^{\sigma l},\ TF_m^{(3-\sigma)(3-l)}\right\rangle, \tag{3.9}$$

$$\alpha_m^{\sigma l}=\left\|TF_m^{\sigma l}\right\|^2_A,\ \beta_m^{\sigma l}=-\left\langle TF_m^{\sigma l},\ TF_m^{(3-\sigma)(3-l)}\right\rangle_A, \tag{3.10}$$

$$A_m^{\sigma l}=\left\|TF_m^{\sigma l}\right\|^2_{\partial D},\ B_m^{\sigma l}=-\left\langle TF_m^{\sigma l},\ TF_m^{(3-\sigma)(3-l)}\right\rangle_{\partial D}, \tag{3.11}$$

$$\Delta^\sigma_{m,A}=\left(\alpha_m^{\sigma 1}A_m^{\sigma 1}\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}-\beta_m^{\sigma 1}B_m^{\sigma 1}\bar\beta_m^{\sigma l}\bar B_m^{\sigma l}\right)_A, \tag{3.12}$$

$$\Delta^\sigma_{m,\partial D}=\left(\alpha_m^{\sigma 1}A_m^{\sigma 1}\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}-\beta_m^{\sigma 1}B_m^{\sigma 1}\bar\beta_m^{\sigma l}\bar B_m^{\sigma l}\right)_{\partial D}, \tag{3.13}$$

where $\langle .,\ .\rangle_{\partial D}$ *and* $\langle .,\ .\rangle_A$ *are the inner product on the boundary* ∂D *and on a circle of radius* A *respectively.*

Proof. Step 1:

We have

$$\int\limits_{r_p=A} \|T_pG_1\|^2_{L_2(\partial D)}\, ds_p = \int\limits_{r_p=A}\int\limits_{\partial D} T_pG_1\,(P,\ q) : \overline{T_pG_1}\,(q,\ P)\, ds_p ds_q \tag{3.14}$$

$$\begin{aligned}
&= \sum_{m=0}^{+\infty}\sum_{\sigma+1}^{2}\alpha_m^{\sigma 1}\begin{bmatrix} \|\hat{F}_m^{\sigma 1}\|^2_{\partial D} + \bar{a}_m^{\sigma 1}\langle \hat{F}_m^{\sigma 1},\ F_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{\sigma 1},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ +a_m^{\sigma 1}\langle F_m^{\sigma 1},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{\sigma 1}\bar{a}_m^{\sigma 1}A_m^{\sigma 1} \\ -(-1)^{\sigma}a_m^{\sigma 1}\bar{b}_m B_m^{\sigma 1} - (-1)^{\sigma}b_m\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{\sigma 1}b_m\bar{B}_m^{\sigma 1} + b_m\bar{b}_m A_m^{(3-\sigma)2} \end{bmatrix} \\
&+ \beta_m^{\sigma 1}\begin{bmatrix} \langle \hat{F}_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} + \bar{a}_m^{(3-\sigma)2}\langle \hat{F}_m^{\sigma 1},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{\sigma 1},\ F_m^{\sigma 1}\rangle_{\partial D} + a_m^{\sigma 1}\langle F_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} \\ +a_m^{\sigma 1}\bar{a}_m^{(3-\sigma)2}B_m^{\sigma 1} - (-1)^{\sigma}a_m^{\sigma 1}\bar{b}_m A_m^{\sigma 1} \\ -(-1)^{\sigma}b_m\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} - (-1)^{\sigma}\bar{a}_m^{(3-\sigma)2}b_m A_m^{(3-\sigma)2} \\ +b_m\bar{b}_m\bar{B}_m^{\sigma 1} \end{bmatrix} \\
&+ \overline{\beta}_m^{\sigma 1}\begin{bmatrix} \langle\ \hat{F}_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\bar{a}_m^{\sigma 1}\bar{B}_m^{\sigma 1} \\ -(-1)^{\sigma}a_m^{(3-\sigma)2}\bar{b}_m A_m^{(3-\sigma)2} - (-1)^{\sigma}b_m\langle F_m^{\sigma 1},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{\sigma 1}b_m A_m^{\sigma 1} + b_m\bar{b}_m B_m^{\sigma 1} \end{bmatrix} \\
&= \alpha_m^{(3-\sigma)2}\begin{bmatrix} \|\hat{F}_m^{(3-\sigma)2}\|^2_{\partial D} + \bar{a}_m^{(3-\sigma)2}\langle \hat{F}_m^{(3-\sigma)2},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{(3-\sigma)2},\ F_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\bar{a}_m^{(3-\sigma)2}A_m^{(3-\sigma)2} \\ -(-1)^{\sigma}a_m^{(3-\sigma)2}\bar{b}_m B_m^{\sigma 1} - (-1)^{\sigma}b_m\langle F_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{(3-\sigma)2}b_m B_m^{\sigma 1} + b_m\bar{b}_m A_m^{\sigma 1} \end{bmatrix}.
\end{aligned}$$

A necessary condition for the existence of a minimum of (3.14) is the vanishing of the gradient with respect to the simple and cross multipole coefficients $\bar{a}_m^{\sigma 1},\ \bar{a}_m^{(3-\sigma)2}$ and b_m. So, we obtain the following relations:

$$\begin{aligned}
&\alpha_m^{\sigma 1}A_m^{\sigma 1}a_m^{\sigma 1} + \bar{\beta}_m^{\sigma 1}\bar{B}_m^{\sigma 1}a_m^{(3-\sigma)2} - (-1)^{\sigma}(\alpha_m^{\sigma 1}\bar{B}_m^{\sigma 1} + A_m^{\sigma 1}\bar{\beta}_m^{\sigma 1})b_m = g_m^{\sigma 1}, \qquad (3.15)\\
&\beta_m^{\sigma 1}B_m^{\sigma 1}a_m^{\sigma 1} + \alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}a_m^{(3-\sigma)2} \\
&- (-1)^{\sigma}(\alpha_m^{(3-\sigma)2}B_m^{\sigma 1} + A_m^{(3-\sigma)2}\beta_m^{\sigma 1})b_m = g_m^{(3-\sigma)2}, \qquad (3.16)\\
&- (-1)^{\sigma}(B_m^{\sigma 1}\alpha_m^{\sigma 1} + A_m^{\sigma 1}\beta_m^{\sigma 1})a_m^{\sigma 1} \\
&- (-1)^{\sigma}(A_m^{(3-\sigma)2}\bar{\beta}_m^{\sigma 1} + \bar{B}_m^{\sigma 1}\alpha_m^{(3-\sigma)2})a_m^{(3-\sigma)2} \\
&+ (A_m^{(3-\sigma)2}\alpha_m^{\sigma 1} + \bar{B}_m^{\sigma 1}\beta_m^{\sigma 1} + B_m^{\sigma 1}\bar{\beta}_m^{\sigma 1} + A_m^{\sigma 1}\alpha_m^{(3-\sigma)2})b_m \qquad (3.17)\\
&= -(-1)^{\sigma}(h_m^{\sigma 1} + h_m^{(3-\sigma)2}).
\end{aligned}$$

Exploiting the Schwartz inequality and linear independence of the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ we can show that the determinant of the system (3.15 – 3.17) is greater than zero:

$$\begin{aligned}\Delta_m^{\sigma} = &\left[\begin{array}{c} A_m^{\sigma 1}\alpha_m^{(3-\sigma)2} + \alpha_m^{\sigma 1}A_m^{(3-\sigma)2} \\ -\left(B_m^{\sigma l}\bar{\beta}_m^{\sigma l} + \beta_m^{\sigma l}\bar{B}_m^{\sigma 1}\right)\end{array}\right] \\ &+\left(A_m^{\sigma 1}A_m^{(3-\sigma)2} - \left|B_m^{\sigma l}\right|^2\right)\left(\alpha_m^{\sigma 1}\alpha_m^{(3-\sigma)2} - \left|\beta_m^{\sigma l}\right|^2\right) > 0. \end{aligned} \tag{3.18}$$

A fact that ensures this system has a unique solution. Its solution is given by the relations (3.2) and (3.3).

Step 2:

In order to prove that these simple and cross multipole coefficients really minimize the quantity given by (3.1), we have to show that the sufficient conditions for a minimum are also satisfied. So, we assume that the integrand in (3.1) is a function of the variables $x_m^{\sigma 1},\ y_m^{\sigma 1},\ x_m,\ x_m^{(3-\sigma)2},\ y_m^{(3-\sigma)2}$ and y_m , where:

$$\alpha_m^{\sigma 1} = x_m^{\sigma 1} + iy_m^{\sigma 1},\ a_m^{(3-\sigma)2} = x_m^{(3-\sigma)2} + iy_m^{(3-\sigma)2} \text{ and } b_m = x_m + iy_m. \tag{3.19}$$

Then we see that the optimal choice of the simple and cross multipole coefficients defined in (3.2) and (3.3), and which ensure that first derivatives vanish, also make the determinants of the second derivatives greater than zero:

$$\begin{aligned}&\left|Hessien\left[\int\limits_{r_p=A}\left\|T_pG_1\right\|_{L_2(\partial D)}^2 ds_p\right]\right| = \\ &\left(16\alpha_m^{\sigma 1}A_m^{\sigma 1}\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^{\sigma} + 16\left(\mathrm{Re}^4\left(\beta_m^{\sigma 1}B_m^{\sigma 1}\right) + \mathrm{Im}^4\left(\beta_m^{\sigma 1}B_m^{\sigma 1}\right)\right),\end{aligned} \tag{3.20}$$

$$|H_{11}| = |H_{44}| = \left(16\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^{\sigma} > 0, \tag{3.21}$$

$$|H_{22}| = |H_{55}| = \left(8\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^{\sigma} > 0, \tag{3.22}$$

$$|H_{33}| = |H_{66}| = \left(16\alpha_m^{\sigma 1}A_m^{\sigma 1}\right)\Delta_m^{\sigma} > 0. \tag{3.23}$$

So the required sufficient conditions can be satisfied. □

3.2 Circular case

As shown in [4, 5], the lack of orthogonality between the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ does not permit a tractable analytical approach to show that the choice of the simple and cross multipole coefficients given in (3.2) and (3.3) satisfy the conditions (2.5), so ensuring the unique solvability of the modified integral equation given by (2.1). so we treat the problem by considering only the special circular case as an indication of the general case.

Lemma 1. *If the boundary ∂D is a circle of radius , then the simple and cross multipole coefficients of theorem 3.1.1 are given by the relations:*

$$a_m^{11} = a_m^{21} = -\frac{1}{2}\left[\frac{\hat{\alpha}_m^1 \alpha_m^2 - \bar{\beta}_m \hat{\beta}_m}{\Delta_m'}\right], \tag{3.24}$$

$$a_m^{12} = a_m^{22} = -\frac{1}{2}\left[\frac{\alpha_m^1 \hat{\alpha}_m^2 - \beta_m \hat{\varphi}_m}{\Delta_m'}\right], \tag{3.25}$$

$$b_m = \frac{1}{2}\left[\frac{\hat{\alpha}_m^1 \beta_m - \alpha_m^1 \hat{\beta}_m}{\bar{\Delta}_m}\right] = \frac{1}{2}\left[\frac{\bar{\beta}_m \hat{\alpha}_m^2 - \alpha_m^2 \hat{\varphi}_m}{\Delta_m'}\right], \tag{3.26}$$

where

$$a_m^1 = 2\pi a \begin{bmatrix} \left|k^2\left(2\mu H_m''(ka) - \lambda H_m(ka)\right)\right|^2 \\ \\ + \left|\frac{2\mu m}{a}\left(kH_m'(ka) - \frac{H_m(ka)}{a}\right)\right|^2 \end{bmatrix}, \tag{3.27}$$

$$\hat{a}_m^1 = 2\pi a \begin{bmatrix} k^4\left(2\mu J_m''(ka) - \lambda J_m(ka)\right)\left(2\mu \bar{H}_m''(ka) - \lambda \bar{H}_m(ka)\right) \\ \\ + \left(\frac{2\mu m}{a}\right)^2 \left(k\bar{H}_m'(ka) - \frac{\bar{H}_m(ka)}{a}\right)\left(kJ_m'(ka) - \frac{J_m(ka)}{a}\right) \end{bmatrix}, \tag{3.28}$$

$$\alpha_m^2 = 2\pi a \begin{bmatrix} \left|\mu k^2\left(2\mu H_m''(Ka) + H_m(Ka)\right)\right|^2 \\ \\ + \left|\frac{2\mu m}{a}\left(KH_m'(ka) - \frac{H_m(Ka)}{a}\right)\right| \end{bmatrix}, \tag{3.29}$$

$$\hat{\alpha}_m^2 = 2\pi a \begin{bmatrix} \left(\mu K^2\right)^2\left(2J_m''(Ka) + J_m(Ka)\right)\left(2\mu \bar{H}_m''(ka) + \bar{H}_m(ka)\right) \\ \\ + \left(\frac{2\mu m}{a}\right)^2 \left(KJ_m'(Ka) - \frac{J_m(Ka)}{a}\right)\left(K\ \bar{H}_m'(ka) - \frac{\bar{H}_m(Ka)}{a}\right) \end{bmatrix}, \tag{3.30}$$

$$\beta_m = 2\pi\mu m \begin{bmatrix} k^2\left(2\mu H_m''(ka) - \lambda H_m(ka)\right)\left(K\bar{H}_m'(Ka) - \frac{\bar{H}_m(Ka)}{a}\right) \\ \\ + \mu K^2\left(kH_m'(ka) - \frac{H_m(ka)}{a}\right)\left(2\bar{H}_m''(Ka) + \bar{H}_m(Ka)\right) \end{bmatrix}, \tag{3.31}$$

(3.32)
$$\hat{\beta}_m = 4\pi\mu m \left[\begin{array}{l} k^2 \left(2\mu J_m^{''}(ka) - \lambda J_m(ka)\right) \left(K\bar{H}_m^{'}(Ka) - \dfrac{\bar{H}_m(Ka)}{a}\right) \\ \\ +\mu K^2 \left(kJ_m^{'}(ka) - \dfrac{J_m(ka)}{a}\right) \left(2\bar{H}_m^{''}(Ka) + \bar{H}_m(Ka)\right) \end{array} \right],$$

(3.33)
$$\hat{\varphi}_m = 4\pi\mu m \left[\begin{array}{l} k^2 \left(2\mu \bar{H}_m^{''}(ka) - \lambda \bar{H}_m(ka)\right) \left(KJ_m^{'}(Ka) - \dfrac{J_m(Ka)}{a}\right) \\ \\ +\mu K^2 \left(kJ_m^{'}(ka) - \dfrac{\bar{H}_m(ka)}{a}\right) \left(2J_m^{''}(Ka) + J_m(Ka)\right) \end{array} \right],$$

(3.34)
$$\Delta_m = (2\pi\mu)^2 \left[\begin{array}{c} \mu k^2 K^2 \left(2\mu H_m^{''}(ka) - \lambda H_m(ka)\right) \left(2H_m^{''}(Ka) + H_m(Ka)\right) \\ \\ -\left(\dfrac{2\mu m}{a}\right)^2 \left(kH_m^{'}(ka) - \dfrac{H_m(ka)}{a}\right) \left(KH_m^{'}(Ka) - \dfrac{H_m(Ka)}{a}\right) \end{array} \right].$$

Proof. Using the well-known relations which hold for the inner product of the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius a we can obtain, after easy calculations [3, 7, 10]:

(3.35)
$$\begin{aligned} g_m^{\sigma 1} &= -\bar{\beta}_m^{\sigma 1} \left\langle T\hat{F}_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l} \right\rangle_a - \alpha_m^{\sigma 1} \left\langle T\hat{F}_m^{\sigma l},\ TF_m^{\sigma l} \right\rangle_a \\ &= -\left\langle TF_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l} \right\rangle_a \left\langle T\hat{F}_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l} \right\rangle_a \\ &\quad - \left\| TF_m^{\sigma l} \right\|_a^2 \left\langle T\hat{F}_m^{\sigma l},\ TF_m^{\sigma l} \right\rangle_a . \end{aligned}$$

So

(3.36)
$$g_m^{\sigma 1} = -\left(-(-1)^{\sigma}\,\bar{\beta}_m\right)\left(-(-1)^{\sigma}\,\hat{\varphi}_m\right) - \alpha_m^1 \hat{\alpha}_m^1,$$

(3.37)
$$g_m^{(3-\sigma)2} = -\left(-(-1)^{\sigma}\,\beta_m\right)\left(-(-1)^{\sigma}\,\hat{\beta}_m\right) - \alpha_m^2 \hat{\alpha}_m^2,$$

and

(3.38)
$$\begin{aligned} h_m^{\sigma 1} &= -\bar{\beta}_m^{\sigma 1} \langle T\hat{F}_m^{(3-\sigma)(3-l)}, TF_m^{(3-\sigma)(3-l)} \rangle_a - \alpha_m^{\sigma 1} \langle T\hat{F}_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)} \rangle_a \\ &= -\langle TF_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l} \rangle_a \langle T\hat{F}_m^{(3-\sigma)(3-l)},\ T_m^{(3-\sigma)(3-l)} F \rangle_a \\ &\quad - \|TF_m^{\sigma l}\|_a^2 \langle T\hat{F}_m^{\sigma l},\ TF_m^{(3-\sigma)(3-l)} \rangle_a . \end{aligned}$$

So

(3.39)
$$h_m^{\sigma 1} = -\left(-(-1)^{\sigma}\,\bar{\beta}_m \hat{\alpha}_m^2\right) - \alpha_m^1 \left(-(-1)^{\sigma}\,\hat{\beta}_m\right),$$

(3.40)
$$h_m^{(3-\sigma)2} = -\left(-(-1)^{\sigma}\,\beta_m \hat{\alpha}_m^1\right) - \alpha_m^2 \left(-(-1)^{\sigma}\,\hat{\varphi}_m\right).$$

From (3.38) to (3.40) we obtain (3.24) to (3.26).

Note here, that we have $a_m^{11} = a_m^{21}$ and $a_m^{12} = a_m^{22}$, and we can show [10] that the two ewpressions found for b_m (3.26) which appear to be different, are equal. □

Theorem 2. *If the boundary ∂D is a circle of radius , then the optimal choice of the simple and cross multipole coefficients given by (3.24) to (3.26) yield the exact Green's function for the Neumann problem, i.e.:*

$$G_1^N(p,\ q) = G_{exact}^N(p,\ q). \tag{3.41}$$

Proof. As it was proved in [10], the exact Green's function for the exterior Neumann problem for the circle of radius is given by the relation:

$$\begin{aligned} G_{exact}^N(p,\ q) \;=\; & G_0(p,\ q) + \\ & \frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\left[\begin{array}{l} \left(\dfrac{\hat{\alpha}_m^1\alpha_m^2 - \bar{\beta}_m\hat{\beta}_m}{\Delta_m'}\right)F_m^{11}(p)\otimes F_m^{11}(q) + \\ \left(\dfrac{\hat{\alpha}_m^1\beta_m - \alpha_m^1\hat{\beta}_m}{\Delta_m'}\right)F_m^{11}(p)\otimes F_m^{22}(q) \\ + \left(\dfrac{\alpha_m^1\hat{\alpha}_m^2 - \beta_m\hat{\varphi}_m}{\Delta_m'}\right)F_m^{12}(p)\otimes F_m^{12}(q) + \\ \left(\dfrac{\bar{\beta}_m\hat{\alpha}_m^2 - \alpha_m^2\hat{\varphi}_m}{\Delta_m'}\right)F_m^{12}(p)\otimes F_m^{21}(q) \\ + \left(\dfrac{\hat{\alpha}_m^1\alpha_m^2 - \bar{\beta}_m\hat{\beta}_m}{\Delta_m'}\right)F_m^{21}(p)\otimes F_m^{21}(q) + \\ \left(\dfrac{\hat{\alpha}_m^1\beta_m - \alpha_m^1\hat{\beta}_m}{\Delta_m'}\right)F_m^{21}(p)\otimes F_m^{12}(q) \\ + \left(\dfrac{\alpha_m^1\hat{\alpha}_m^2 - \beta_m\hat{\varphi}_m}{\Delta_m'}\right)F_m^{22}(p)\otimes F_m^{22}(q) \\ + \left(\dfrac{\bar{\beta}_m\hat{\alpha}_m^2 - \alpha_m^2\hat{\varphi}_m}{\Delta_m'}\right)F_m^{22}(p)\otimes F_m^{11}(q) \end{array}\right] \end{aligned} \tag{3.42}$$

.

then from (3.24) to (3.26) we conclude that when ∂D is a circle of radius a, the simple and cross multipole coefficients yield the exact Green's function for the Neumann problem. This result ensures the unique solvability of the modified integral equation defined by (2.1). □

References

[1] E. Argyropoulos, D. Gintides and K. Kiriaki, *On the condition number of integral equations in linear elasticity using the modified Green's function*, Australian Mathematical Society, 2002, 1-16.

[2] E. Argyropoulos and K. Kiriaki, *A criterion of optimization of the modified Green's function in linear elasticity*, Internat. J. Engrg. Sci., 37 (1999), 1441-1460.

[3] L. Bencheikh, *Scattering of elastic waves by cylindrical cavities: integral-equation methods and low frequency asymptotic expansions*, Ph.D. Thesis, Department of Mathematics, University of Manchester UK, 1986.

[4] R. E. Kleinman and G. F. Roach, *Operators of minimal norm via modified Green's functions*, Proc. Roy. Soc. Edinburgh, 94 A. (1983), 163-178.

[5] R. E. Kleinman and G. F. Roach, *On modified Green functions in exterior problems for the Helmholtz equation*, Proc. Roy. Soc. London, A 383 (1982), 313-332.

[6] B. Sahli, *A new criterion of optimization of the simple multipole coefficients in a modified Green's function for the elastic two-dimensional case*, Applied Mathematics Letters (Elsevier), 25 (2012), 77-80.

[7] B. Sahli, *Optimisation des coefficients des multipôles de la fonction de Green modifiée par minimisation de la norme du noyau de l'opérateur intégral en élasticité*, Thèse de Doctorat en Sciences, Département de mathématiques, Université de Sétif, Algérie, 2010.

[8] B. Sahli, *Operators of minimal norm via modified Green's function in two-dimensional elastic waves*, Int. J. Open Problems Comput. Sci. Math., 3 (2010), 278-294.

[9] B. Sahli and L. Bencheikh, *A criterion of optimization of a modified fundamental solution for two dimensional elastic waves*, Int. J. Open Problems Comput. Sci. Math., 2 (2009), 113-137.

[10] B. Sahli, *Optimisation des coefficients des multipôles de la fonction de Green modifiée, par minimisation de la norme de l'opérateur intégral en élasticité*, Thèse de Magister, Département de mathématiques, Université de Batna, Algérie, 1999.

[11] S. Boulaaras and R. Guefaifa, *Existence of positive weak solutions for a class of Kirrchoø elliptic systems with multiple parameters*, Math. Meth. Appl. Sci., 41 (2018), 5203-5210.

[12] R. Guefaifa and S. Boulaaras, *Existence of positive radial solutions for* $(p(x), q(x))$*-Laplacian systems,* Appl. Math. E-Notes, 18 (2018), 209-218.

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On certain properties of hyperbolically convex functions

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Abstract. The aim of this paper is to prove that the envelope of hyperbolically convex functions is hyperbolically convex function. Furthermore, we study the standard functional operations of hyperbolically convex functions and introduce a class $BH[a, b]$ of functions representable as the difference of two hyperbolically convex functions.

Keywords: generalized convex functions, hyperbolically convex functions, supporting functions, majorization.

1. Introduction

One of the powerful properties of functions, which play a very important role in many areas of mathematics both pure and applied, is convexity. An arbitrary function f defined on an interval I is said to be convex if each point on the chord between $(u, f(u))$ and $(v, f(v))$ is above the graph of f for any $u, v \in I$. In fact, there are families of real functions $\{F(x)\}$ which are not topologically equivalent to the family $\{L(x)\}$ of all non vertical line segments terminating on $x = u$ and $x = v$. So that, there are many activities concerning to generalize the notion of a convex function to other classes of functions. But many properties of convex functions are satisfied for these general functions. Generalized convex

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functions were first defined and systematically investigated by Beckenbach [2] and studied furthermore by Beckenbach and Bing [3]. More generally, let $\{F(x)\}$ be a family of real functions $F(x)$ defined in an interval I, such that for given points $p_1 : (u_1, v_1)$ and $p_2 : (u_2, v_2)$, $u_1, u_2 \in I$ with $u_1 < u_2$, there is a unique member of $\{F(x)\}$ through p_1 and p_2. Functions $f(x)$ dominated by $\{F(x)\}$ are said to be convex relative to $\{F(x)\}$. In this work, we concern with one of these generalizations in the sense of Beckenbach by replacing the family of linear functions with a family of hyperbolic functions,

$$H(x) = A\cosh px + B\sinh px,$$

where A, B arbitrary constants and p is a fixed positive constant.

The class of hyperbolically convex functions is closed under addition but it is not closed under subtraction and scalar multiplication . We may consider the class of functions representable as the difference of two hyperbolically convex functions. This larger class is closed under addition, substraction and scalar multiplication, thus forming a linear space.

2. Definitions and preliminary results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [1], [4], [5], [6].

Definition 2.1. *A function $f : I \to \mathbb{R}$ is said to be sub H-function on I or hyperbolically convex function, if for any arbitrary closed subinterval $[u, v]$ of I the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the function, determined by the equation $H(x) = H(x, u, v, f) = A\cosh px + B\sinh px$; $p > 0$ where A and B are chosen such that $H(u) = f(u)$, and $H(v) = f(v)$.*

Equivalently, for all $x \in [u, v]$

$$f(x) \le H(x) = \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)}. \tag{2.1}$$

Remark 2.1. The hyperbolically convex functions possess a number of properties analogous to those of convex functions. For example: If $f : I \to \mathbb{R}$ is hyperbolically convex function, then for any $u, v \in I$, the inequality $f(x) \ge H(x)$ holds outside the interval $[u, v]$.

Note that, from the above definitions $H(x)$ majorizes the function f(x) and $H_{ij}(x)$ denote the member satisfying $H_{ij}(x_i) = f(x_i), H_{ij}(x_j) = f(x_j), (a < x_i < x_j < b)$.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function. Then f is hyperbolically convex function on I if and only if $f''(x) - p^2 f(x) \ge o$ for all x in I.*

Definition 2.2. *Let a function $f : I \to \mathbb{R}$ be hyperbolically convex function*

$$S_u(x) = A\cosh px + B\sinh px$$

is said to be supporting function for $f(x)$ *at the point* $u \in (a,b)$ *if*

1. $S_u(u) = f(u)$
2. $S_u(x) \leq f(x), \forall x \in I.$

That is, if $f(x)$ *and* $S_u(x)$ *agree at* $x = u$ *the graph of* $f(x)$ *does not lie under the support curve.*

Proposition 2.1. *If* $f : I \to \mathbb{R}$ *is a differentiable hyperbolically convex function, then the supporting function for* $f(x)$ *at the point* $u \in I$ *has the form*

$$S_u(x) = f(u)\cosh p(x-u) + \frac{f'(u)}{p}\sinh p(x-u).$$

Theorem 2.2. *If* $f : I \to \mathbb{R}$ *is differentiable function. Then* f *is convex if and only if* f' *is increasing.*

Theorem 2.3. *If* $f : (a,b) \to \mathbb{R}$ *and* $g : (a,b) \to \mathbb{R}$ *are both non-negative, increasing (decreasing), and convex, then* $h(x) = f(x)g(x)$ *also preserve these three properties.*

Definition 2.3. *Let* $f : I \to \mathbb{R}$, f *is said to satisfy a Lipschitz condition if there exists a constant* $K > 0$ *such that for every* $x, y \in I$ *we have*

$$|f(x) - f(y)| \leq K|x - y|.$$

Theorem 2.4. *A function* f *that is* F ρ*-convex satisfies a Lipschitz condition in every compact subinterval* J *of* (a,b)*, and thus is absolutely continuous and has a derivative almost everywhere that is bounded in* J*.*

Definition 2.4. *Let* $f : [a,b] \to \mathbb{R}$ *be a function and* $\Pi = \{x_0, x_1, x_2, ..., x_n\}$ *a partition of* $[a,b]$*. We denote* $V_\Pi(f) = \Sigma_{k=0}^{n-1}|f(x_{k+1}) - f(x_k)|$ *and set*

$$V_a^b(f) = \sup_\Pi V_\Pi(f),$$

where the supremum is taken over all partitions of $[a,b]$*. A function* f *is said to be of bounded variation on* $[a,b]$ *if* $V_a^b(f)$ *is finite. If* f *is of bounded variation on* $[a,b]$ *we write* $f \in V[a,b]$*.*

Theorem 2.5. *If* $f : [a,b] \to \mathbb{R}$ *satisfies a Lipschitz condition on* $[a,b]$ *with constant* K*, then* $f \in V[a,b]$ *and* $V_a^b(f) \leq K(b-a)$*.*

3. Results

Definition 3.1. *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of hyperbolically convex functions majorized by f, i.e $f_\alpha \leq f$, $g(x)$ is called* ***envelope of*** *$f(x)$ if*

$$g(x) = envelopef(x) = \sup_\alpha f_\alpha(x).$$

Lemma 3.1. *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary bounded family of hyperbolically convex functions and $f(x) = \sup_\alpha f_\alpha(x)$ then f is hyperbolically convex function.*

Proof. Let $x \in [a, b] \subseteq I$, since $f_\alpha(x)$ is hyperbolically convex function. Then,

$$\begin{aligned} f(x) &= \sup_\alpha f_\alpha(x) \\ &\leq \sup_\alpha [\frac{f_\alpha(a)\sinh p(b-x) + f_\alpha(b)\sinh p(x-a)}{\sinh p(b-a)}] \\ &\leq [\frac{\sup_\alpha f_\alpha(a)\sinh p(b-x) + \sup_\alpha f_\alpha(b)\sinh p(x-a)}{\sinh p(b-a)}]. \end{aligned}$$

Hence,

$$f(x) \leq [\frac{f(a)\sinh p(b-x) + f(b)\sinh p(x-a)}{\sinh p(b-a)}].$$

Then, f is hyperbolically convex function.

Theorem 3.1. *Let $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary bounded family of hyperbolically convex functions majorized by f if $f_\alpha(x) \leq f(x)$ and $g(x) = \sup_\alpha f_\alpha(x)$, where the supremum is taken over all hyperbolically convex functions then*

1. *g is hyperbolically convex function.*
2. *If f is hyperbolically convex function, then $f = g$.*
3. *g is continuous function.*

Proof.

1. g is hyperbolically convex function.
It is obvious from Lemma 3.1.
2. If f is hyperbolically convex function, then $f = g$.
Since, $f(x)$ is hyperbolically convex function $f(x) \leq \sup_\alpha f_\alpha(x) = g(x)$, then

$$f(x) \leq g(x). \tag{3.1}$$

Since $f_\alpha(x) \leq f(x)$, then $f(x)$ is an upper bound.
Since $f_\alpha(x) \leq \sup_\alpha f_\alpha(x)$, then $\sup f_\alpha(x) \leq f(x)$. Thus,

$$g(x) \leq f(x). \tag{3.2}$$

Hence, from 3.1 and 3.2 we get $f(x) = g(x)$.

3. g is continuous function.

We shall show that $g(x)$ is continuous at an arbitrary x_o, with $a < x_o < b$.

Let $a < x_1 < x_o < x_2 < b$. Since $g(x)$ is hyperbolically convex function, for $x_1 < x_o - h < x_o + h < x_2$. Then,

$$H_{o2}(x_o - h) \leq g(x_o - h) \leq H_{1o}(x_o - h),$$

$$H_{1o}(x_o + h) \leq g(x_o + h) \leq H_{o2}(x_o + h), \tag{3.3}$$

$$-H_{1o}(x_o - h) \leq -g(x_o - h) \leq -H_{o2}(x_o - h), \tag{3.4}$$

Hence, from 3.3 and 3.4 we get $H_{1o}(x_o+h)-H_{1o}(x_o-h) \leq g(x_o+h)-g(x_o-h) \leq H_{o2}(x_o+h)-H_{o2}(x_o-h)$, $-|H_{1o}(x_o+h)-H_{1o}(x_o-h)| \leq H_{1o}(x_o+h)-H_{1o}(x_o-h) \leq g(x_o+h)-g(x_o-h) \leq H_{o2}(x_o+h)-H_{o2}(x_o-h) \leq |H_{o2}(x_o+h)-H_{o2}(x_o-h)|$, $-|H_{1o}(x_o+h)-H_{1o}(x_o-h)| \leq g(x_o+h)-g(x_o-h) \leq |H_{o2}(x_o+h)-H_{o2}(x_o-h)|$, $|g(x_o+h)-g(x_o-h)| \leq \max[|H_{1o}(x_o+h)-H_{1o}(x_o-h)|, |H_{o2}(x_o+h)-H_{o2}(x_o-h)|] < \epsilon$, $\forall \epsilon > o \exists \delta > o$ s.t $|h| < \delta \rightarrow |g(x_o + h) - g(x_o - h)| < \epsilon$. Hence, $g(x)$ is continuous.

Theorem 3.2. *If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are hyperbolically convex functions and $\alpha \geq 0$ then $f + g$ and αf are hyperbolically convex functions.*

Proof. Since f, g are hyperbolically convex function, then $\forall\ x \in [u, v] \subseteq I$,

$$\begin{aligned} f(x) &\leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)}, \\ g(x) &\leq \frac{g(u)\sinh p(v-x) + g(v)\sinh p(x-u)}{\sinh p(v-u)}. \end{aligned}$$

Then,

$$\begin{aligned} f(x)+g(x) &\leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)} \\ &+ \frac{g(u)\sinh p(v-x) + g(v)\sinh p(x-u)}{\sinh p(v-u)}, \\ f(x)+g(x) &\leq \frac{(f(u)+g(u))\sinh p(v-x) + (f(v)+g(v))\sinh p(x-u)}{\sinh p(v-u)}. \end{aligned}$$

Thus,

$$(f+g)(x) \leq \frac{(f+g)(u))\sinh p(v-x) + (f+g)(v)\sinh p(x-u)}{\sinh p(v-u)}.$$

Hence, $f + g$ is hyperbolically convex function.

$$\text{Since, } f(x) \leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)} \qquad \textit{and} \qquad \alpha \geq o$$

then

$$\alpha f(x) \leq \frac{\alpha f(u)\sinh p(v-x) + \alpha f(v)\sinh p(x-u)}{\sinh p(v-u)}.$$

Hence, αf is hyperbolically convex function.

Remark 3.1. If $\alpha < 0$ then αf is not hyperbolically convex function. For example if $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2$ is hyperbolically convex function. As $2 - x^2 \geq 0$ from theorem 2.1, but $-x^2$ is not hyperbolically convex function. While αf is hyperbolically convex function either for $f(x) = \cosh px$ or $f(x) = \sinh px$ for any $\alpha \in \mathbb{R}$, as shown in the following lemma.

Lemma 3.2. *For any real number m, $m\sinh px$ and $m\cosh px$ are hyperbolically convex functions.*

Proof. Let $x \in [u, v]$, Since

$$\sinh px = \frac{\sinh pu\sinh p(v-x) + \sinh pv\sinh p(x-u)}{\sinh p(v-u)}.$$

Then, $m\sinh px$ is hyperbolically convex function. Similarly, $m\cosh px$.

Theorem 3.3. *Assume that $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ where $range(f) \subseteq J$, let f and g are both non-negative, hyperbolically convex function, two times continuously differentiable functions and g is increasing. If $f'(x) \geq 1$, then the composite function gof is hyperbolically convex function on I.*

Proof. Since g is increasing, then

$$g'(x) \geq o, \qquad \forall x \in J. \tag{3.5}$$

Since f and g are hyperbolically convex functions, then using theorem 2.1,

$$\text{we have} \quad f''(x) - p^2 f(x) \geq 0, \qquad \forall x \in I, \tag{3.6}$$

$$\text{and} \quad g''(x) - p^2 g(x) \geq 0, \qquad \forall x \in J. \tag{3.7}$$

Since,

$$\begin{aligned} h(x) &= g(f(x)) \\ h'(x) &= g'(f(x)).f'(x) \\ h''(x) &= g''(f(x)).(f'(x))^2 + g'(f(x)).f''(x). \end{aligned}$$

Then,

$$\begin{aligned} h''(x) - p^2 h(x) &= g''(f(x)).(f'(x))^2 + g'(f(x)).f''(x) - p^2 g(f(x)) \\ &= (f'(x))^2[g''(f(x) - \frac{p^2}{(f'(x))^2}g(f(x))] + g'(f(x)).f''(x). \end{aligned}$$

Now using 3.5, 3.6, 3.7 and $f'(x) \geq 1$ we conclude that

$$h''(x) - p^2 h(x) \geq 0.$$

Hence, $h(x)$ is hyperbolically convex function.

Theorem 3.4. *If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are both non-negative, increasing, two times continuously differentiable functions and hyperbolically convex functions then $k(x) = f(x)g(x)$ also exhibits theses three properties.*

Proof. Since f and g are non-negative and increasing functions, then

$$f(x) \geq 0,\ g(x) \geq 0, \tag{3.8}$$

and

$$f'(x) \geq 0,\ g'(x) \geq 0. \tag{3.9}$$

Since f and g are hyperbolically convex functions, then using theorem 2.1,

$$\text{we have} \quad f''(x) - p^2 f(x) \geq 0, \quad \forall x \in I, \tag{3.10}$$

$$\text{and} \quad g''(x) - p^2 g(x) \geq 0, \quad \forall x \in I. \tag{3.11}$$

Since,

$$\begin{aligned} k(x) &= f(x).g(x) \\ k'(x) &= f'(x).g(x) + f(x).g'(x) \\ k''(x) &= f''(x).g(x) + 2f'(x).g'(x) + f(x).g''(x). \end{aligned}$$

Then,

$$\begin{aligned} k''(x) - p^2 k(x) &= f''(x).g(x) + 2f'(x).g'(x) + f(x).g''(x) - p^2 f(x).g(x) \\ &= (f''(x) - p^2 f(x)).g(x) + 2f'(x).g'(x) + f(x).g''(x). \end{aligned}$$

Now using 3.8, 3.9, 3.10 and 3.11, we conclude that

$$k''(x) - p^2 k(x) \geq 0.$$

Hence, $k(x)$ is hyperbolically convex function.

Theorem 3.5. *If (f_n) is a real valued sequence of hyperbolically convex functions converging to a finite limit function f on I, then f is hyperbolically convex function.*

Proof. Let $x \in [a, b] \subseteq I$

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &\leq \lim_{n\to\infty} \frac{f_n(a)\sinh p(b-x) + f_n(b)\sinh p(x-a)}{\sinh p(b-a)} \\ &= \frac{\lim_{n\to\infty} f_n(a)\sinh p(b-x) + \lim_{n\to\infty} f_n(b)\sinh p(x-a)}{\sinh p(b-a)} \\ &= \frac{f(a)\sinh p(b-x) + f(b)\sinh p(x-a)}{\sinh p(b-a)}. \end{aligned}$$

It follows that f is hyperbolically convex function.

Definition 3.2. *Let $BH[a,b]$ be the class of functions $f : [a,b] \to R$ representable in the form $f = g - h$ where g and h are hyperbolically convex functions on $[a,b]$ and $g'_+(a), g'_-(b), h'_+(a), h'_-(b)$ are all finite.*

Theorem 3.6. *$BH[a,b]$ is closed under addition, subtraction and scalar multiplication.*

Proof. Let $f, g \in BH[a,b]$, $f = f_1 - f_2$, $g = g_1 - g_2$. For addition: $f + g = f_1 - f_2 + g_1 - g_2 = (f_1 + g_1) - (f_2 + g_2)$. Since $f_1 + g_1, f_2 + g_2$ are hyperbolically convex functions, by theorem 3.2, then $f + g \in BH[a,b]$. For subtraction: $f - g = f_1 - f_2 - g_1 + g_2 = (f_1 + g_2) - (f_2 + g_1)$. Since $f_1 + g_2, f_2 + g_1$ are hyperbolically convex functions, by theorem3.2, then $f - g \in BH[a,b]$.

For scalar multirlication:

Case(1): let $\alpha \geq o$, then $\alpha f = \alpha f_1 - \alpha f_2$.

Since $\alpha f_1, \alpha f_2$ are hyperbolically convex functions, by theorem3.2, then $\alpha f \in BH[a,b]$. Case(2): let $\alpha < o$, then $\alpha f = \alpha f_1 - \alpha f_2 = -\alpha f_2 - (-\alpha) f_1$.

Since $-\alpha f_2, -\alpha f_1$ are hyperbolically convex functions, by theorem3.2, then $\alpha f \in BH[a,b]$.

Finally, it is clear that all the previous functions have finite endpoint derivatives.

Corollary 3.1. *BH[a,b] is a linear space.*

Theorem 3.7. *If $f \in BH[a,b]$, then f satisfies lipschitz condition and consequently absolutely continuous on $[a,b]$.*

Proof. Let $f \in BH[a,b], f = f_1 - f_2$. Since f_1 and f_2 are hyperbolically convex functions. Then, from theorem 2.4, f_1, f_2 satisfies lipschitz condition,

$$|f_1(x) - f_1(y)| \leq k|x - y| \qquad \text{and} \qquad |f_2(x) - f_2(y)| \leq m|x - y|.$$

$$\begin{aligned}
f(x) - f(y) &= f_1(x) - f_2(x) - (f_1(y) - f_2(y)) \\
&= f_1(x) - f_1(y) - (f_2(x) - f_2(y)). \\
|f(x) - f(y)| &= |f_1(x) - f_1(y) - (f_2(x) - f_2(y))| \\
&\leq |f_1(x) - f_1(y)| + |-(f_2(x) - f_2(y)| \\
&\leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \\
&\leq k|x - y| + m|x - y| \\
&= (k + m)|x - y| \\
&= h|x - y|.
\end{aligned}$$

Then

$$|f(x) - f(y)| \leq h|x - y|, \forall x, y \in [a,b].$$

Hence, f satisfies lipschitz condition.

Let $\epsilon > o$, choose $\delta = \frac{\epsilon}{h}$ such that for any collection $\{(x_i, y_i) : i = 1, 2, ..., n\}$ of disjoint open subinterval of $[a, b]$ with $\sum_1^n |x_i - y_i| < \delta$,

$$then \sum_1^n |f(x_i) - f(y_i)| < \sum_1^n h|x_i - y_i| = h \sum_1^n |x_i - y_i| < h\frac{\epsilon}{h} = \epsilon.$$

Hence, f is absolutely continuous on $[a, b]$.

Corollary 3.2. *If $f \in BH[a, b]$, then $V_a^b(f) < \infty$.*

Proof. Let $f \in BH[a, b]$ then, from theorem 3.7, f satisfies lipschitz condition. then, from theorem 2.5, $f \in V[a, b]$. Hence, $V_a^b(f) < \infty$.

References

[1] Mohamed S. S. Ali, *On certain properties for two classes of generalized convex functions,* Abstract and Applied Analysis, 2016, Article ID 4652038, 7 pages, 2016.

[2] E. F. Beckenbach, *Generalized convex functions*, Bulletin of the American Mathematical Society, 43 (1937), 363371.

[3] E. F. Beckenbach and R. H. Bing, *On generalized convex functions*, Transactions of the American Mathematical Society, 58 (1945), 220230.

[4] R. Kannan and C.K. Kreuger, *Advanced analysis on the real line*, Springer-Verlag, 1996.

[5] M. M. Peixoto, *Generalized convex functions and second order differential inequlities,* Bulletin of the American Mathematical Society, 55 (1949), 563-572.

[6] A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press, New York, 1973.

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Fuzzy order relative to fuzzy B-algebras

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Abstract. Let $(X; *, 0)$ be a B-algebra and μ a fuzzy B-algebra defined on X. For each $x \in X$, the fuzzy order of x, denoted by $FO_\mu^B(x)$, is the least positive integer n such that $\mu(x^n) = \mu(0)$. Also, the order of an element $x \in X$, denoted by $|x|_B$, is the least positive integer n such that $x^n = 0$. In this paper, some properties of $FO_\mu^B(x)$ are established. Also, the relationship between $FO_\mu^B(x)$ and the order $|x|_B$ of x is also explored.

Keywords: B-algebra, fuzzy B-algebras, order of an element, fuzzy order.

1. Introduction

Neggers and Kim [8] first introduced the notion of *B-algebras* in 2002. It is an algebra $(X; *, 0)$ of type $(2, 0)$ such that the following axioms are satisfied for all $x, y, z \in X : (I)x * x = 0, (II)x * 0 = x$, and $(III)(x * y) * z = x * (z * (0 * y))$. Meanwhile, the concept of *subalgebra* of a B-algebra was introduced in [7]. It is a nonempty subset A of X such that $x * y \in A$ whenever $x, y \in A$. It was proven (see [2,6]) that the class of B-algebras and the class of groups coincide; that is, one can view a B-algebra as a group and vice versa. Because of this, new researches were published to further explore the properties of a B-algebra when viewed as a group. One such research is the concept of *cyclic B-algebras* which was introduced by Gonzaga and Vilela [4] in 2015. Endam and Teves [3] meanwhile provided more properties of cyclic B-algebras and went to introduce the concept of "order of an element" of a B-algebra similar to that in groups. On the other hand, Zadeh [9] initiated the notion of *fuzzy sets*. It is a generalization of an ordinary set in which each element is associated with a *membership value* within the interval $[0, 1]$. That is, it is a function $\mu : A \to [0, 1]$ where A is any

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nonempty (crisp) set. Since it generalizes an ordinary set, various researches in fuzzy sets were established in an attempt to generalize some concepts in modern algebra. An example of this is the notion of *fuzzy B-algebra* which was introduced by Jun, Roh, and Kim [5]. It is a fuzzy set μ defined on a B-algebra $(X; *, 0)$ such that the inequality $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ holds for all $x, y \in X$. Ahn and Bang [1] meanwhile explored the properties of fuzzy B-algebras with respect to the subalgebras of the underlying B-algebra. Continuing this research trend, this paper will introduce the concept of fuzzy order of an element of a B-algebra as a generalization of the concept introduced in [3]. Some of its properties are extensively investigated, and its relationship to the ordinary"order" is also explored.

2. Preliminaries

Throughout this paper, X will denote the B-algebra $(X; *, 0)$ unless otherwise specified. Recall the following property from [8]:

Lemma 2.1. *If X is a B-algebra, then $x = 0 * (0 * x)$ for all $x \in X$.*

Let $n \in \mathbb{Z}^+$ and $x \in X$. In [4], the exponential expression x^n is defined as follows: $x^n = x * (0* \prod^{n-1} x)$ where the expression $0* \prod^{n-1} x$ is equal to $[\ldots[(0 * x) * x] * \ldots] * x$ such that x occurs $n-1$ times. By convention, $x^1 = x$, while $x^0 = 0$ is taken from [8]. Also, the notations $-x$ and x^{-n} are equal to $0 * x$ and $(-x)^n$, respectively. Using these observations, the following results are obtained:

Corollary 2.2. *For all $x \in X$ and for all $m, n \in \mathbb{Z}$, $x^m * x^n = x^{m-n}$.*

Corollary 2.3. *For all $x \in X$ and for all $n \in \mathbb{Z}$,*

i. $x^{-n} = (-x)^n = -(x^n) = 0 * x^n$;

ii. $(x^{-n})^{-1} = (x^{-1})^{-n} = x^n$.

Proposition 2.4. *Let $x \in X$. Then for all $m, n \in \mathbb{Z}$, $(x^m)^n = x^{mn}$.*

For a subset $A \subseteq X$, the *cyclic subalgebra* $\langle A \rangle_B$ of X genereted by A is the intersection of all subalgebras of X containing A [4]. If A is a singleton set, that is, $A = a$, then $\langle A \rangle_B = \langle a \rangle_B$, the cyclic subalgebra of X containing a. If there is an $x \in X$ such that $\langle x \rangle_B = X$, then X is a *cyclic B-algebra*.

The following propositions hold for all fuzzy B-algebra defined on a B-algebra, see [5]:

Proposition 2.5. *Every fuzzy B-algebra μ defined on X satisfies the inequality $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proposition 2.6. *If a fuzzy set μ defined on X is a fuzzy B-algebra, then $\mu(-x) \geq \mu(x)$ for all $x \in X$.*

Proposition 2.6 is strengthen in the following lemma:

Lemma 2.7. *If a fuzzy set μ defined on X is a fuzzy B-algebra, then $\mu(-x) = \mu(x)$ for all $x \in X$.*

Proof. It remains to show that $\mu(-x) \leq \mu(x)$ for all $x \in X$. Indeed, $\mu(x) = \mu(0 * (0 * x)) \geq \min\{\mu(0), \mu(0 * x)\} = \mu(0 * x) = \mu(-x)$ by Lemma 2.1 and Proposition 2.5. □

Let μ be a fuzzy B-algebra defined on a B-algebra X. For all $t \in [0, 1]$, the subset $\mu_t = \{x \in X | \mu(x) \geq t\}$ of X is called the *t-level subsets* of μ[1]. For the construction of examples in this paper, consider the following theorem [1]:

Theorem 2.8. *Let X be a B-algebra and μ be a fuzzy set on X such that every t-level subsets of X is a subalgebra of X, where $0 \leq t \leq \mu(0)$. Then μ is a fuzzy B-algebra defined on X.*

The following lemma will be used extensively in this study.

Lemma 2.9. *Let X be a B-algebra and μ a fuzzy B-algebra defined on X. Then for all $x \in X$ and for all $n \in \mathbb{Z}$, $\mu(x^n) \geq \mu(x)$.*

Proof. Let $x \in X$ and $n \in \mathbb{Z}$. If $n = 0$, then $\mu(x^0) = \mu(0) \geq \mu(x)$ by Proposition 2.5. Suppose $n > 0$. Applying (1) repeatedly and Lemma 2.7,

$$
\begin{aligned}
\mu(x^n) &= \mu(x * (0 * \prod^{n-1} x)) \\
&\geq \min\{\mu(x), \mu(0 * \prod^{n-1} x)\} \\
&= \min\{\mu(x), \mu([\ldots[(0 * \underbrace{x) * x] * \ldots] * x]}_{n-1}\} \\
&\geq \min\{\mu(x), \min\{\mu([\ldots[(0 * \underbrace{x) * x] * \ldots] * x}_{n-2}], \mu(x)\}\} \\
&\geq \min\{\mu(x), \min\{\min\{\mu([\ldots[(0 * \underbrace{x) * x] * \ldots] * x}_{n-3}], \mu(x)\}, \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu([\ldots[(0 * \underbrace{x) * x] * \ldots] * x}_{n-3}], \mu(x)\}\} \\
&\vdots \\
&= \min\{\mu(x), \min\{\mu(0 * x), \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu(-x), \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu(x), \mu(x)\}\} \\
&= \min\{\mu(x), \mu(x)\} \\
&= \mu(x).
\end{aligned}
$$

If $n < 0$, then $-n > 0$. Thus, $\mu(x^n) = \mu(-(x^{-n})) = \mu(x^{-n} \geq \mu(x)$ by Corollary 2.3 and Lemma 2.7. □

3. Main results

In [3], the order of an element $x \in X$, denoted by $|x|_B$, is the least positive integer n such that $x^n = 0$.

Definition 3.1. Let μ be a fuzzy B-algebra defined on a B-algebra X and $x \in X$. The *fuzzy order* of x with respect to μ, denoted by $FO^B_\mu(x)$, is the least positive integer n such that $\mu(x^n) = \mu(0)$. If there is such least n satisfying the equality, then x is of μ-finite order; otherwise, x is of μ-infinite order.

By Proposition 2.5, for all $x \in X$, $\mu(0) \geq \mu(x^n)$ for any fuzzy B-algebra μ defined on X. Thus, the following remark is valid.

Remark 3.2. Let $x \in X$ where X is a B-algebra and $n \in \mathbb{Z}^+$. Then $n = FO^B_\mu(x)$ if and only if n is the least positive integer such that $\mu(x^n) \geq \mu(0)$.

The following example shows that the fuzzy order of an element is not equal to its order and fuzzy orders of two elements are equal but their respective orders are not.

Example 3.3. Consider the B-algebra $X = \{0, 1, 2, 3\}$ with the following table of operation [5], and μ defined on X:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

$$\mu(x) = \begin{cases} 0.9, & \text{if } x \in \{0, 2\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy B-algebra, $FO^B_\mu(0) = FO^B_\mu(2) = 1$ and $FO^B_\mu(1) = FO^B_\mu(3) = 2$. But $|2|_B = 2 \neq 1 = FO^B_\mu(2)$ and $|0|_B = 1 \neq 2 = |2|_B$.

On the other hand, the following example shows that the orders of two elements are equal but their fuzzy orders are not.

Example 3.4. Consider the Klein-4 group $V = \{e, a, b, ab\}$ with binary operation "$\cdot$" defined in the Cayley table below:

$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

Then by routine calculations, $\mathbf{V} = (V; \cdot, e)$ is a B-algebra (this is a special case where the operation "$\cdot$" of a group and the operation "$*$" of its derived B-algebra coincide). Define the fuzzy set μ for all $x \in V$, where $t \in (0, 1]$ by

$$\mu(x) = \begin{cases} t, & \text{if } x \in \{e, ab\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the only level subsets of μ are $\{e, ab\}$ and V. Since $\{e, ab\} = \langle ab \rangle$ in view of the group V, $\{e, ab\} = \langle ab \rangle_B$ is a subalgebra of $\mathbf{V}$ which is cyclic. Therefore by Theorem 2.8, μ is a fuzzy B-algebra defined on $\mathbf{V}$. Now, $|a|_B = |ab|_B = 2$ since $\langle a \rangle_B = \{e, a\}$. But then, $FO^B_\mu(a) = 2 \neq 1 = FO^B_\mu(ab)$.

In view of Examples 3.3 and 3.4, the following remarks hold.

Remark 3.5. Let μ be a fuzzy B-algebra defined on X and $x, y \in X$.

a) The equality $|x|_B = FO^B_\mu(x)$ may not hold in general.

b) The equality $|x|_B = |y|_B$ does not imply that $FO^B_\mu(x) = FO^B_\mu(y)$, and vice versa.

Let μ be a fuzzy B-algebra defined on X. If $x \in X$ is of finite order (that is, $|x|_B < +\infty$), then x is also of μ-finite order. To see this, note that if $|x|_B = n$, then $\mu(x^n) = \mu(0)$. Thus, $FO^B_\mu(x) \leq n$. Observe that the inequality is still possible because the definition of μ may vary along the interval $[0, 1]$. That is, there might be an $r \in \mathbb{Z}^+$ with $r < n$ such that $\mu(x^r) = \mu(0)$. If there is no such r, however, then it is guaranteed that $FO^B_\mu(x) = n$.

The following proposition gives the idea that any element of a B-algebra that is of infinite order may still be of μ-finite order.

Proposition 3.6. *Let $x \in X$ be of infinite order. Then there is a fuzzy B-algebra defined on X such that x is of μ-finite order.*

Proof. Consider the fuzzy set $\mu : X \to [0, 1]$ defined for all $y \in X$, where $r_1, r_2 \in [0, 1]$ and $n \in \mathbb{Z}^+$ with $r_1 \geq r_2$ as follows:

$$\mu(y) = \begin{cases} r_1, & \text{if } y \in \langle x^n \rangle, \\ r_2, & \text{otherwise.} \end{cases}$$

Then we see that $\langle x^n \rangle$ and X are the only level subsets of X, which are all subalgebras of X. By Theorem2.8, μ is a fuzzy B-algebra defined on X. Now, observe that $\mu(x^n) = r_1 = \mu(0)$ since $0 \in \langle x^n \rangle$. Hence, $FO^B_\mu(x) = n$. □

A natural question after observing Examples 3.3 and 3.4 is the following: what then is the relationship between $FO^B_\mu(x)$ and $|x|_B$ for any element x of a B-algebra X and for any fuzzy B-algebra μ defined on X. In Example 3.3, $FO^B_\mu(1)$ divides $|1|_B$. This observation is generalized in the next result.

Theorem 3.7. *Let μ be a fuzzy B-algebra defined on X, $x \in X$ such that x is of μ-finite order and $m \in \mathbb{Z}^+$. Then $\mu(x^m) = \mu(0)$ if and only if $FO^B_\mu(x)$ divides m. In particular, $FO^B_\mu(x)$ divides $|x|_B$.*

Proof. Set $FO^B_\mu(x) = n$. Suppose $m \in \mathbb{Z}^+$ with $n|m$. Then $m = nq$ for some $q \in \mathbb{Z}^+$. By Proposition 2.4, Lemma 2.9 and Proposition 2.5, $\mu(x^m) = \mu(x^{nq}) = \mu((x^n)^q) \geq \mu(x^n) = \mu(0) \geq (x^m)$. Hence, $\mu(x^m) = \mu(0)$. Conversely, let $m \in \mathbb{Z}^+$ such that $\mu(x^m) = \mu(0)$. By the Division Algorithm (applied to m and n), there are unique $q, r \in \mathbb{Z}$ with $0 \leq r < n$ such that $m = nq + r$. By Corollary 2.2, Proposition 2.4, Lemma 2.9 and Proposition 2.5,

$$\begin{aligned}
\mu(x^r) = \mu(x^{m-nq}) = \mu(x^m * x^{nq}) &\geq \min\{\mu(x^m), \mu(x^{nq})\} \\
&\geq \min\{\mu(0), \mu((x^n)^q)\} \\
&= \mu((x^n)^q) \\
&\geq \mu(x^n) \\
&= \mu(0) \\
&\geq \mu(x^r).
\end{aligned}$$

Thus, $\mu(x^r) = \mu(0)$. But this contradicts to the minimality of n. Hence, $r = 0$ and so $m = nq$. This shows that $n|m$. The particular statement holds since $\mu(x^m) = \mu(0)$ whenever $|x|_B = m < +\infty$ and, in the extended real number system, $+\infty$ is divisible by both $+\infty$ and a positive real number. □

The next results further characterize the concept of fuzzy order of an element of a B-algebra with respect to a well-defined fuzzy B-algebra.

Proposition 3.8. *Let μ be a fuzzy B-algebra defined on X. Then $FO^B_\mu(x) = FO^B_\mu(-x)$ for all $x \in X$.*

Proof. If $FO^B_\mu(x) = +\infty$, then $\mu(x^m) \neq \mu(0)$ for all $m \in \mathbb{Z}^+$. By Corollary 2.3(i) and Lemma 2.7, $\mu((-x)^m) = \mu(-(x^m)) = \mu(x^m) \neq \mu(0)$ for all $m \in \mathbb{Z}^+$. Hence, $FO^B_\mu(-x) = +\infty$. Suppose now that $FO^B_\mu(x) = n$ where $n \in \mathbb{Z}^+$. Then n is the least positive integer such that $\mu(x^n) = \mu(0)$. Again by Corollary 2.3(i) and Lemma 2.7, $\mu((-x)^n) = \mu(-(x^n)) = \mu(x^n) = \mu(0)$. Let $r \in \mathbb{Z}^+$ such that $\mu((-x)^r) = \mu(0)$ but $r < n$. Then by Corollary 2.3,

$$\mu(x^r) = \mu(-(x^{-r})) = \mu(x^{-r}) = \mu((-x)^r) = \mu(0),$$

a contradiction to the minimality of n. Therefore, $FO^B_\mu(-x) = n$. □

Theorem 3.9. *Let μ be a fuzzy B-algebra defined on X. Let $x \in X$ with $FO^B_\mu(x) = n < +\infty$. Then $FO^B_\mu(x^m) = \frac{n}{gcd(m,n)}$ for all $m \in \mathbb{Z}^+$.*

Proof. Set $FO^B_\mu(x^m) = t$ and $d = gcd(m, n)$. Then $d|m$ and so $m = dk$ for some $k \in \mathbb{Z}^+$. By Proposition 2.5, Corrolary 2.3, and Lemma 2.9, $\mu(0) \geq \mu((x^m)^{\frac{n}{d}}) =$

$\mu((x^{dk})^{\frac{n}{d}}) = \mu(x^{nk}) = \mu((x^n)^k) \geq (x^n) = \mu(0)$. Thus, $\mu((x^m)^{\frac{n}{d}}) = \mu(0)$. By Theorem 3.7, $t|\frac{n}{d}$. Also, $d = gcd(m,n)$ implies there exist $p, q \in \mathbb{Z}$ such that $d = mp + nq$. Now, by Proposition 2.5, Corollary 2.2, Proposition 2.4, and Lemmas 2.7 and 2.9,

$$\begin{aligned}
\mu(0) \geq \mu(x^{td}) = \mu(x^{t(mp+nq)}) &= \mu(x^{tmp+tnq}) \\
&= \mu(x^{mtp} * x^{-tnq}) \\
&\geq \min\{\mu(x^{mtp}), \mu(x^{-ntq})\} \\
&= \mu\{(((x^m)^t)^p), \mu(-(x^n)^{tq})\} \\
&\geq \min\{\mu(((x^m)^t)), \mu((x^n)^{tq})\} \\
&\geq \min\{\mu(0), \mu(x^n)\} \\
&= \min\{\mu(0), \mu(0)\} \\
&= \mu(0).
\end{aligned}$$

So, $\mu(x^t d) = \mu(0)$. This means that $n|td$ by Theorem 3.7. Hence, $ns = td$ for some $s \in \mathbb{Z}^+$, that is, $(\frac{n}{d})s = t$. This implies that $\frac{n}{d}|t$. Consequently, $t = \frac{n}{d}$. □

Proposition 3.10. *Let μ be a fuzzy B-algebra defined on X and $x \in X$ which is of μ-finite order. If m is an integer such that m and $FO^B_\mu(x)$ are relatively prime, then $\mu(x^m) = \mu(x)$.*

Proof. Set $FO^B_\mu(x) = n$ and suppose $m \in \mathbb{Z}$ such that $gcd(m,n) = 1$. Then there exist $r, s \in \mathbb{Z}$ such that $nr + ms = 1$. By Corollary 2.2, Proposition 2.4, and Lemma 2.7, Proposition 2.5, and Lemma 2.9,

$$\begin{aligned}
\mu(x) = \mu(x^{nr+ms}) = \mu(x^{nr} * x^{-ms}) &\geq \min\{\mu(x^{nr}), \mu(x^{-ms})\} \\
&\geq \min\{\mu((x^n)^r), \mu(-(x^m)^s\} \\
&\geq \min\{\mu(x^n), \mu((x^m)^s)\} \\
&\geq \min\{\mu(0), \mu(x^m)\} \\
&= \mu(x^m) \\
&= \mu(x).
\end{aligned}$$

Therefore, $\mu(x^m) = \mu(x)$. □

Theorem 3.11. *Let μ be a fuzzy B-algebra defined on X. Suppose $FO^B_\mu(x) = n < +\infty$ where $x \in X$ and $i, j \in \mathbb{Z}$. If $i \equiv j(modn)$, then $FO^B_\mu(x^i) = FO^B_\mu(x^j)$.*

Proof. Set $FO^B_\mu(x^i) = s$ and $FO^B_\mu(x^j) = t$. Now, $i \cong j(modn)$ implies $nk = i - j$ for some $k \in Z$. By Proposition 2.5, Corollary 2.2, Proposition 2.4, and

Lemmas 2.7 and 2.9,

$$\begin{aligned}\mu(0) \geq \mu((x^i)^t) = \mu((x^{j+nk})^t) = \mu(x^{jt+nkt}) &= \mu(x^{jt}) * x^{-nkt}) \\ &\geq \min\{\mu(x^{jt}), \mu(x^{-nkt}\} \\ &= \min\{\mu((x^j)^t), \mu(-(x^{nkt}))\} \\ &= \min\{\mu(0), \mu(x^{nkt})\} \\ &= \mu((x^n)^{kt}) \\ &\geq \mu(x^n) \\ &= \mu(0).\end{aligned}$$

Thus, $\mu((x^i)^t) = \mu(0)$. By Theorem 3.7, $s|t$. Also by Proposition 2.5, Corollary 2.2, Proposition 2.4, and Lemma 2.9,

$$\begin{aligned}\mu(0) \geq \mu((x^j)^s) = \mu((x^{i-nk})^s) = \mu(x^{it-nks}) &= \mu(x^{is} * x^{nks}) \\ &\geq \min\{\mu(x^{is}), \mu(x^{nks}\} \\ &= \min\{\mu((x^i)^s), \mu((x^n)^{ks})\} \\ &\geq \min\{\mu(0), \mu(x^n)\} \\ &= \min\{\mu(0), \mu(0)\} \\ &= \mu(0).\end{aligned}$$

So, $\mu((x^j)^s) = \mu(0)$. By Theorem 3.7, $t|s$. Consequently, $s = t$. □

References

[1] S. S. Ahn and K. Bang, *On Fuzzy subalgebras in B-algebras,* Commun. Korean Math. Soc., 18 (2003), 429-437.

[2] P. J. Allen, J. Neggers, and H. S. Kim, *B-algebras and groups*, Scientiae Mathematicae Japonicae Online, 9 (2003), 159-165.

[3] J. C. Endam and R. C. Teves, *Some properties of cyclic B-algebras*, International Mathematical Forum, 11 (2016), 387-394.

[4] N. C. Gonzaga and J. P. Vilela, *On cyclic B-algebras*, Applied Mathematical Sciences, 9 (2015), 5507-5522.

[5] Y. B. Jun, E. H. Roh, and H. S. Kim, *On fuzzy B-algebras*, Czechoslovak Mathematical Journal, 52 (2002), 375-384.

[6] M. Kondo and Y. B. Jun, *The class of B-algebras Coincides with the Class of Groups*, Scientiae Mathematicae Japonicae Online, 7 (2002), 175-177.

[7] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras,* Int. Math. Journal, 2 (2002), 207-214.

[8] J. Neggers and H. S. Kim, *On B-algebras*, Math. Vesnik, 54 (2002), 21-29.

[9] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 87-96.

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On counting by a pair of Fibonacci generated countable sets of irrationals

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Abstract. A Fibonacci sequence representation of any rational number of the real line R is used to construct a pair of phi-nary (in relation to the Golden ratio φ)unique countable sets of irrational numbers. These sets, which have a potential for novel applications, are further employed in the proof that R should be populated "much more with irrational numbers than with rational numbers".

Keywords: Fibonacci numbers, irrational numbers, rational numbers, multiplicative decomposition.

1. Introduction

It is well known that the set of rationals Q in the set of real numbers R is infinite and countable, while the set of irrationals $Q^c = R\backslash Q$ is infinite but uncountable, and are equally dense, [1, p. 36], as can be deduced from the following famous results, in which $k \in \mathbb{N} = \{1, 2, 3, ...\}$ and $\gamma \in \Gamma \subset Q^c$,where Γ is an uncountable set.

Theorem 1 ([2, p. 14], Density of rational numbers)**.** *There is a countable set* $A = \{r_k\}_{k=1}^{\infty} \subset [Q \cap (a,\, b)]$ *between any two distinct real numbers a,b* $\in R$.

Corollary 1 (*Density of irrational numbers*)**.** *There is a countable family of uncountable sets,*$\{\{\mathcal{S}_{\gamma k}\}_{\gamma\in\Gamma}\}_{k=1}^{\infty} \triangleq \{\mathcal{S}_{\gamma k}\} \subset [Q^c \cap (a, b)]$, *between any two distinct real numbers* $a, b \in R$.

The proofs of these results are given for the interested reader in the Appendix. The corollary and its proof indicate, moreover, that "Q^c is however more populated than Q". And our major aim in this note is to construct Fibonacci-generated, see e.g. [3, p. 143], countable phinary sets of irrationals for a related multiplicative decomposition principle of rationals. These phi-nary sets, which have a potential for novel applications, are demonstrated, in section 2, to be asymptotically rationally distinct (ARD), then used to assert Corollary 1, without invoking Theorem 1.

2. Main result

The double subscript in the set family $\{\{\mathcal{S}_{\gamma k}\}_{\gamma\in\Gamma}\}_{k=1}^{\infty}$ emphasizes the fact that its existence is tied to the existence of $\mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a,b)$.This being a fact, and not a restriction, should by no means affect the reality of the next remark.

Remark 1. The ratio $\frac{\#\,\mathfrak{A}}{\#\{\{S_{\gamma k}\}_{\gamma}\}_{k=1}^{\infty}} = \frac{\aleph_0}{\aleph_0\,\aleph_1} = \frac{\aleph_0}{\aleph_1}$ does not have a quantitative meaning. However, in terms of the Lebesgue measure μ, the previous ratio reduces to $\frac{\mu(\mathfrak{A})}{\mu(\{\{S_{\gamma k}\}_{\gamma}\}_{k=1}^{\infty})} = \frac{0}{\infty} = 0$, for any $(a,b) \subset R$, and this can imply that Q^c should be more populated than Q.

Let $p, q \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, ...\}$, and $q \neq 0$, $r = \frac{p}{q} \in Q$, and $s, z \in Q^c$. Rather trivially $rs, \frac{s}{r}, \frac{r}{s} \in Q^c$, while $r \pm s$, $sz \in R$, facts that can easily be proved by contradiction in assuming the opposite to be true. Equally simple, is the proof of

$$r = s\ z. \tag{1}$$

Incidentally Dedekind cuts of the Q set can be used, see e.g. [4, p. 38], as a basis for the construction of R. Hence the multiplicative decomposability of r into two irrational numbers, (1), happens to represent a sort of reversed Dedekind construction.

As for $R^+ = [0,\infty)$ and $r = \frac{p}{q} \in [R^+ \cap Q]$, $p \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$ happen to generate a sequence $< r_k >$, arranged viz $0 \leq r_1 \leq r_2 \leq r_3 \leq .....$, in a countable set $\{r_k\}_{k=1}^{\infty} \subset [R^+ \cap Q]$, for which $\triangle r_k = r_{k+1} - r_k$ is not constant with varying k, but has a fractal structure, as indicated by the Minkowski lattice [5]. Being essential for the completeness of the R^+ set, the irrational numbers are in fact defined by Dedekind, [4, p. 38], in 1901 as the cut which is created in order to fill the gaps produced by the r_k members of this structure.

Indeed, since s, in (1), can be varied within a certain set $\{s_\gamma\} = \mathcal{F} \subset Q^c$, which could be countable, $\boldsymbol{G}$, or uncountable , $\boldsymbol{U}$. Then z varies, for a fixed r, as $z_\gamma = \frac{p}{q\ s_\gamma} \in \{z_\gamma\} = \hbar \subset Q^c$,with $\hbar$ similarly countable as $\mathcal{F}$, i.e. with corresponding countable $\boldsymbol{H}$ and uncountable $\boldsymbol{V}$.

As an example of this, if the s irrationals are chosen, not from rule-generated sequences, to be, e.g. $s_1 = 1 - \sqrt{2}$, $s_2 = e^3$, $s_3 = \pi^2$, ...etc, then the corresponding, via (1), z irrationals will also be not rule-generated, namely : $z_1 = \frac{p}{q(1-\sqrt{2})}$, $z_2 = \frac{p}{qe^3}$, $z_3 = \frac{p}{q\pi^2}$, ... Needless to say here that π and e are transcendental irrational numbers, while φ is an algebraic irrational number. Also π^n , e^n , and φ^n ,$\forall n \in \mathbb{N}$, are all irrationals. Therefore, possible examples of rule-generated $\boldsymbol{G}$ for s_n , over $(0,\infty) \cap Q^c$could be

$$\begin{aligned} &\{n\pi\}_{n=1}^{\infty}, \{ne\}_{n=1}^{\infty}, \left\{n\frac{e}{\pi}\right\}_{n=1}^{\infty}, \left\{n\frac{\pi}{e}\right\}_{n=1}^{\infty}, \{n\varphi\}_{n=1}^{\infty}, \quad \text{or;} \\ &\{\pi^n\}_{n=1}^{\infty}, \{e^n\}_{n=1}^{\infty}, \left\{\left(\frac{\pi}{e}\right)^n\right\}_{n=1}^{\infty}, \{\varphi^n\}_{n=1}^{\infty}, \left\{\frac{\pi}{e}\varphi^n\right\}_{n=1}^{\infty}, \left\{\frac{e}{\pi}\varphi^n\right\}_{n=1}^{\infty} . \end{aligned} \tag{2}$$

While over $(0,1)\cap Q^c$, a rule-generated $\boldsymbol{G}$ could be

$$\begin{aligned} &\left\{\pi^{-n}\right\}_{n=1}^{\infty}, \left\{e^{-n}\right\}_{n=1}^{\infty}, \left\{\left(\frac{\pi}{e}\right)^{-n}\right\}_{n=1}^{\infty}, \\ &\left\{\varphi^{-n}\right\}_{n=1}^{\infty}, \left\{\frac{\pi}{e}\varphi^{-n}\right\}_{n=1}^{\infty}, \left\{\frac{e}{\pi}\varphi^{-n}\right\}_{n=1}^{\infty}. \end{aligned} \tag{3}$$

In a similar fashion the associated $\boldsymbol{H}=\{z_n=\frac{p}{qs_n}\}_{n=1}^{\infty}$ could be defined.

A basic inconvenience in the suggested $\boldsymbol{G}'$s, of (2-3), is their restricted applicability to certain domains and the possibility for their asymptotic behavior to be uncertain, [5-7]. Hence the construction of a universal and computationally robust $\boldsymbol{G}$, that could be blended with the (p,q) pair, in a $\frac{p}{q}$ representation, should be of some value for several applications.

Unlike $\boldsymbol{G}$ and $\boldsymbol{H}$, the uncountable $\boldsymbol{U}=\{s_\gamma\}$ and $\boldsymbol{V}=\{z_\gamma=\frac{p}{qs_\gamma}\}$, with $\#\ \boldsymbol{U}=\#\ \boldsymbol{V}=\aleph_1$, can exist in a variety of rather chaotic fashions that may, or may not, follow any specific rule.

So, in order to develop a nonqualitative (or precise) approach to the analysis of the relative population of Q and Q^c, we shall resort, when generating the $\boldsymbol{G}$ and $\boldsymbol{H}$ sets, to the sequence $\langle F_n\rangle$ of Fibonacci numbers F_n,

Table 1: Some Fibonacci numbers

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$< F_n >$	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
F_n	-3	2	-1	1	0	1	1	2	3	5	8	13	21

which is an example of a complete sequence in the sense of Zeckendorf [8], and defined recursively viz $F_n=F_{n-1}+F_{n-2}$, with $F_{-n}=(-1)^{n+1}F_n$ when $n\in\mathbb{Z}$.

This sequence, though aperiodic [9], is structurally periodic (or semi-periodic) as every third number in it is even. Also, any three consecutive Fibonacci numbers are pairwise coprime, [9, 10, p. 33], and every F_n that is prime must have a prime index n; with the exception of $F_4=3$. Furthermore, a characteristic feature of the spiral, defined by $\langle F_n\rangle$, is that $\lim_{n\to\infty}\frac{F_n}{F_{n-1}}=\varphi$. Defined via $\varphi=[1;1,1,1,1,....]=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+....}}}}=\sum_{k=1}^{\infty}\varphi^{-k}$,
is the Golden ratio, which is a famous irrational number 1.618033988..., satisfying the remarkably unique properties

$$\varphi-1=\varphi^{-1},\quad \varphi+1=\varphi^2, \tag{4}$$

properties that enter in the structure of Binet's formula

$$F_n=\frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}. \tag{5}$$

Definition 1. An ordered countable set of irrational numbers $\boldsymbol{B} = \{z_\gamma\}_{\gamma\in\Gamma}$ is asymptotically rationally distinct (ARD) if $\lim_{\gamma\to\infty} z_\gamma = \rho \in Q$.

Now if $f(n)$ is a function of integers, then $\lfloor f(n)\rfloor$ is the floor function for $f(n)$, i.e. the largest integer less than or equal to n. Obviously $n - \lfloor n\rfloor = 0$, and $N^n - \lfloor N^n\rfloor = 0, \forall n, N \in \mathbb{N}$, are rational distinctivity properties for the sequences $< n >$ and $< N^n >$, respectively.

φ is also special in several other ways. The powers $\varphi^n = \sqrt{5}F_n + (1-\varphi)^n$, which satisfy $\lim_{n\to\infty} \varphi^n = \sqrt{5}F_n$, though are all irrationals, they however lie unexpectedly [6-7] close to integers: for instance $\varphi^{11} = 199.005$ is unusually close to 199. Moreover, it happens, [6], rather puzzlingly, that

$$\lim_{n\to\infty} (\varphi^n - \lfloor\varphi^n\rfloor) = 0, \tag{6}$$

despite the fact that $\varphi^n - \lfloor\varphi^n\rfloor \neq 0$, for all finite n.

An explanation of the reason for the puzzle (6), representing an asymptotic rational distinctivity for the sequence $< \varphi^n >$, is provided by the lemma to follow.

Remark 2. Throughout this paper we need to distinguish between asymptotic computational infinity ∞, as a very large (but finite as a must) rational number and mathematical infinity $\bowtie$,which is neither rational or irrational, but irrational (as a must) when conceived as $\bowtie = \frac{P}{0}$, $p \in \mathbb{N}$.

Accordingly, $\varphi^{\bowtie} = \bowtie \in Q^c$, despite the computational fact (6), which stipulates that

$$\varphi^{\infty} = \infty \in Q. \tag{7}$$

Lemma 1. *An ordered countable set of irrational numbers* $B = \{z_n\}_{n=1}^{\infty}$ *is asymptotically rationally distinct (ARD), i.e.* $\lim_{n\to\infty} z_n = \rho \in Q$, *if*

(*i*) $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = s \in Q^c$,

(*ii*) $\lim_{n\to\infty} (z_n - z_{n-1}) = s^* \in Q^c, or = 0$,

(*iii*) *such that* $\frac{s^*}{s-1} \in Q$.

Proof. Satisfaction of (i) means that

$$\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = s \in Q^c, \tag{8}$$

while satisfaction of (ii) can be expressed as

$$\lim_{n\to\infty} (z_n - z_{n-1}) = \lim_{n\to\infty} z_{n-1}\left(\frac{z_n}{z_{n\ -1}} - 1\right) = s^* \in Q^c. \tag{9}$$

Now in view of (8), relation (9) is the same as

$$\lim_{n\to\infty} (z_n - z_{n-1}) = (s-1)\lim_{n\to\infty} z_{n-1} = s^*. \tag{10}$$

Furthermore, as $(s-1)=s^o\in Q^c$, then (10), by (iii) becomes $\lim_{n\to\infty} z_{n-1}=\frac{s^*}{s^o}=\rho\in Q$, which is the same as the required result $\lim_{n\to\infty} z_n=\rho\in Q$; and here the proof ends.

Corollary 2. *An ordered countable set of irrational numbers* $B=\{z_n\}_{n=1}^{\infty}$ *is asymptotically rationally distinct (ARD), when*

(*a*) $s^*=0$, *if* (*i*), *of Lemma 1, is violated.*

(*b*) *if both* (*i*), *and* (*ii*), *of Lemma 1, are simultaneously violated, provided that* $s\neq 1$.

Proof. By direct substitution of the stated conditions in Lemma 1.

Example 1. Test the ARD for The phi-nary set $\{\varphi^n\}_{n=1}^{\infty}$ whose sequence $\langle\varphi^n\ ,\ n\in\mathbb{N}\rangle$ diverges as an expanding Golden spiral.

Clearly ,

(i) $\lim_{n\to\infty}\frac{z_n}{z_{n-1}}=\lim_{n\to\infty}\frac{\varphi^n}{\varphi^{n-1}}=\varphi=s\in Q^c$.

(ii) $\lim_{n\to\infty}(z_n-z_{n-1})=(\varphi-1)\ \ \lim_{n\to\infty}\varphi^{n-1}$.

Additionally, since $\lim_{n\to\infty}\varphi^{n-1}=\ \lim_{n\to\infty}\left\lfloor\varphi^{n-1}\right\rfloor$ is rational, then $\lim_{n\to\infty}(z_n-z_{n-1})=s^*\in Q^c$. Furthermore $\frac{s^*}{s-1}=\frac{(\varphi-1)}{(\varphi-1)}\lim_{n\to\infty}\left\lfloor\varphi^{n-1}\right\rfloor=\rho\in Q$; and (iii) is satisfied. Hence this set is ARD. Interestingly, this turns out to to be consistent with (6).

Example 2. Test the ARD for The phi-nary set $\{\varphi^{-n}\}_{n=1}^{\infty}$ whose sequence $\langle\varphi^{-n}\ ,\ n\in\mathbb{N}\rangle$ converges to 0 as a contracting Golden spiral.

Here correspondingly,

(i) $\lim_{n\to\infty}\frac{z_n}{z_{n-1}}=\lim_{n\to\infty}\frac{\varphi^{-n}}{\varphi^{-n+1}}=\varphi^{-1}=s\in Q^c$.

(ii) $\lim_{n\to\infty}(z_n-z_{n\ -1})=(1-\varphi)\ \ \lim_{n\to\infty}\varphi^{-n}=0=s^*\in Q$.

(iii) $\frac{s^*}{s-1}=\frac{(1-\varphi)}{(\varphi^{-1}-1)}\lim_{n\to\infty}\varphi^{-n}=-\varphi\lim_{n\to\infty}\varphi^{-n}=0=\rho\in Q$.

Therefore this set is also ARD.

Example 3. Test the ARD for $\boldsymbol{B}=\left\{z_n=(n+\sqrt{2})\right\}_{n=1}^{\infty}$ of algebraic irrationals.

Clearly $\lim_{n\to\infty}\frac{z_n}{z_{n-1}}=\lim_{n\to\infty}\frac{n+\sqrt{2}}{n-1+\sqrt{2}}=1\notin Q^c$; violation of (i). Here Corollary 2 can be applied. Moreover, $\lim_{n\to\infty}(z_n-z_{n-1})=\ \lim_{n\to\infty}[n+\sqrt{2}-(n-1+\sqrt{2})]=1\notin Q^c$ or 0; violation of (ii). Then $\frac{s^*}{s-1}=\frac{1}{0}=\bowtie\notin Q$; violation of (iii). Therefore this $\boldsymbol{B}$ in not ARD.

In addition to the demonstrated ARD behavior of $\varphi^n$and φ^{-n}, it is rather straightforward, [3, p. 214], to prove that

$$\varphi^n=\varphi^{n-1}+\varphi^{n-2}=\varphi F_n+F_{n-1}, \tag{11}$$

$$\varphi^{-n} = \varphi^{2-n} - \varphi^{1-n} = (-1)^n[F_{n+1} - \varphi F_n]. \tag{12}$$

Remark 3. There are two canonical countable sets $\boldsymbol{C}$ and $\boldsymbol{D}$ in $[\ R^+\cap\ Q^c]$, and removing them from the set $[R^+\cap\ Q^c]$ does not alter the cardinality of this set.

Proof. According to the decomposition principle (1), each $r = \frac{p}{q} = sz \in [\ R^+\cap\ Q]$ defines countable $\boldsymbol{C}$ and $\boldsymbol{D} \subset [R^+\cap\ Q^c\ ]$. Moreover as $\boldsymbol{C} \sim \boldsymbol{D} \sim [R^+\cap\ Q]$, then $\#\ \boldsymbol{C} = \#\ \boldsymbol{D} = \aleph_0$. Also $\boldsymbol{C} \precsim [R^+\cap\ Q^c]$ and $\boldsymbol{D} \precsim [R^+\cap\ Q^c\ ]$, then since $R^+\cap\ Q^c$ is not denumerable, then $\boldsymbol{C} \prec [R^+\cap\ Q^c]$ and $\boldsymbol{D} \prec [R^+\cap\ Q^c]$.

Now when $\boldsymbol{C}\cup\ \boldsymbol{D}$ is taken out from $R^+\cap\ Q^c$ to form $R^+\cap\ Q$, the set

$$R^+ \cap Q^c \setminus\ (\boldsymbol{C} \cup \boldsymbol{D}), \tag{13}$$

remains inside $R^+\cap\ Q^c$ and

$$\#[\ R^+ \cap\ \ Q^c] \setminus\ (\boldsymbol{C} \cup \boldsymbol{D}) = \#[R^+ \cap Q^c] = \aleph_1. \tag{14}$$

Therefore in the least sense $\#\ [\ R^+\cap\ Q^c] = 2\ \#\ [R^+\cap\ Q] = 2\aleph_0$, but the previous process can be repeated an infinite number of times with the same result. However $\aleph_0 = \infty$, and if the inequality $2\dot{\infty} > \infty$ has any meaning (actually it does not), then we may claim that $\#\ [R^+\cap\ Q^c] > \infty(2\aleph_0) = \infty\aleph_0$.

This holds regardless of the fact that $\frac{\#\ [R^+\cap\ Q^c]}{\#\ [R^+\cap\ Q]} = \infty\frac{\aleph_0}{\aleph_0} = \frac{\aleph_1}{\aleph_0}$, does not have a quantitative meaning. Hence since removal of $\boldsymbol{C}$ and $\boldsymbol{D}$ can indefinitely be repeated from $R^+\cap\ Q^c$, without affecting the cardinality of the remaining set, we may say that "$R^+\cap\ Q^c$ should be much more populated than $R^+\cap\ Q$". Finally, since $R^+ = [R^+\cap\ Q]\cup[R^+\cap\ Q^c]$, then there exists much more irrational numbers in R^+ than rational numbers. Here the proof ends.

The only weakness in the previous proof of Remark 3, is that $\boldsymbol{C}$ and $\boldsymbol{D}$ are only qualitative or symbolic in nature, i.e. without a rule of generation, and with rather uncertain asymptotic behavior. Hence in a move towards stregthening this proof, it is quite desirable to redefine these sets in a rule-generated and in a computationally ARD fashion.

Remark 4. The arguments used in Remark 3 can also be applied on $R^-\cap\ Q^c$, then for $R\cap\ Q^c = Q^c$.

Theorem 3 (Fibonacci generated countable sets of irrational factors of r)**.** *The two irrational factors of $r \in [R^+ \cap\ Q]$ form a pair of phi-nary, Fibonacci-generated for r, countable ARD sets $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$.*

Proof. Any two positive integers that stand in $r = \frac{p}{q}$, $p \in \mathbb{Z}^+$, $q \in \mathbb{N}$ can be represented in terms of Fibonacci numbers viz $p = p(m,i)\ = [F_m - i] \in (F_{m-1}, F_m]$, $0 \le i \le\ F_{m-2} - 1, m \ge 0, m \ne 2$, yielding

$$\mathbb{Z}^+ = \left\{ \{F_m - i\}_{i=0}^{F_{m-2}-1} \right\}_{m=0,m\neq 2}^{\infty}$$

and $q = p(n,j) = [F_n - j] \in (F_{n-1}, F_n],\ 0 \le j \le F_{n-2} - 1, n \ge 1, n \ne 2$, yielding $\mathbb{N} = \left\{\{F_n - j\}_{j=0}^{F_{n-2}-1}\right\}_{n=1,n\ne 2}^{\infty}$.

It should be noted that $m \ne 2$ and $n \ne 2$ in the previous relations happen to be reflection of the remarkable fact that $F_1 = F_2 = 1$.

Therefore $r_k = r(m,n,i,j) = \frac{p(m,i)}{q(n,j)} = \frac{[Fm-i];m\in\mathbb{Z}^+\setminus\{2\},0\le i\le F_{m-2}-1}{[F_n-j];n\in\mathbb{N}\setminus\{2\},0\le j\le F_{n-2}-1} \ge 0$.

Apparently, different p's may have the same m (that defines the closest F_m), but different i's. Also different p's may have different m's but the same i. Regardless of that, $p(m,i) = [F_m - i]$ should be unique, i.e. exists one and only one p that can be defined by the (m,i) pair. This follows directly from the uniqueness of $F_m, \forall m \ne 2$.

Make then use of (5) in this r_k to rewrite it as $r_k = \frac{\varphi^m - (1-\varphi)^{-m} - i\sqrt{5}}{\varphi^n - (1-\varphi)^{-n} - j\sqrt{5}}$, which in view of (4) decomposes to

$$r_k = \frac{\pi n}{em(i+j)} \frac{[1-(-1)^m(1-\varphi^{-1})^m]/i - \varphi^{-m}\sqrt{5}}{[1-(-1)^n(1-\varphi^{-1})^n]/j - \varphi^{-n}\sqrt{5}} \ \frac{e}{\pi}\ \frac{m}{n}(i+j)\frac{i}{j}\varphi^{m-n}. \tag{15}$$

By virtue of (1) , it is possible to write

$$\begin{aligned} \wp_{m-n,i,j}\ (\varphi) &= \frac{e}{\pi}\frac{m}{n}(i+j)\frac{i}{j}\ \varphi^{m-n} \\ &= \frac{e}{\pi}\frac{m}{n}(i+j)\frac{i}{j}\begin{cases} \varphi F_{m-n} + F_{m-n-1}, & m-n>0 \\ (-1)^{n-m}[F_{n-m+1} - \varphi F_{n-m}], & m-n<0. \end{cases} \end{aligned} \tag{16}$$

Since distinct $r = \frac{p}{q}'$s may have the same $\frac{i}{j}\ \varphi^{m-n}$, to entail a virtual contradiction. However, the multiplication of $\frac{i}{j}\ \varphi^{n-m}$ by $\frac{n}{m}(i+j)$, in (16), is meant to guarantee uniqueness of $\wp_k(\varphi) = \wp_{m-n,i,j}\ (\varphi)$.A uniqueness that follows from the fact that the system of equations:

$\varphi^{m-n} = \alpha, \frac{m}{n} = \beta, \frac{i}{j} = \sigma,\ i+j = \theta$, with $\begin{cases} \alpha > 1, & \beta > 1, m-n > 0, \\ \alpha < 1, & \beta < 1, m-n < 0 \end{cases}$, has always the unique solution vector $(\frac{\beta ln\alpha}{(\beta-1)ln\varphi}, \frac{ln\alpha}{(\beta-1)ln\varphi}, \frac{\gamma\theta}{1+\sigma}, \frac{\sigma}{1+\sigma}) = (m,n,i,j) \triangleq r_k$.

Now as this $\wp_{m-n,i,j}\ (\varphi) \in \boldsymbol{C}^*$ is an irrational factor of r_k , and $\boldsymbol{C}^* \sim [R^+ \cap\ Q]$, then

$$\frac{\pi n}{em(i+j)} \frac{[1-(-1)^m(1-\varphi^{-1})^m]/i\ - \varphi^{-m}\sqrt{5}}{[1-(-1)^n(1-\varphi^{-1})^n]/j - \varphi^{-n}\sqrt{5}} = \Im_{m,n,i,j}(\varphi), \tag{17}$$

can only be an irrational belonging in some other $\boldsymbol{D}^* \sim [R^+ \cap\ Q]$, and distinct from $\boldsymbol{C}^*$.

Hence, the multiplicative decomposition principle (1) writes in phi-nary factors as

$$\begin{aligned} r_k &= \ \wp_{m-n,i,j}\ (\varphi)\ \Im_{m,n,i,j}\ (\varphi), \\ (m,n,i,j)\ &\triangleq \frac{p}{q} = r_k \in \{r_k\}_{k=1}^{\infty} \subset [R^+ \cap\ Q]. \end{aligned} \tag{18}$$

The proof that $\boldsymbol{C}^*$ is ARD is trivial, and unlike the the proof of the ARD of $\boldsymbol{D}^*$,which is indirect. Moreover, $\wp_k = \wp_{m-n,i,j}(\varphi) \in \{\wp_k\}_{k=1}^{\infty} = \boldsymbol{C}^*$ and $\Im_k = \Im_{m,n,i,j}\ (\varphi) \in \{\Im_k\}_{k=1}^{\infty} = \boldsymbol{D}^*$, establishing that both $\boldsymbol{C}^*$and $\boldsymbol{D}^*$ are Fibonacci-generated for $\{r_k\}_{k=1}^{\infty}$, countable irrational phinary ARD sets. In particular, $\wp_k = \wp_{m-n,i,j}(\varphi) \in \boldsymbol{C}^*$ is Fibonacci-generated uniquely for every r_k via the vector $(\frac{n}{m}, (n-m), \frac{i}{j}, (i+j))$. Also $\Im_k = \Im_{m,n,i,j}\ (\varphi) \in \boldsymbol{D}^*$ corresponds, to the same r_k , via its m, n, i, j. Hence the pair of irrational numbers $(\wp_k\ ,\ \Im_k\ ) = (\wp_{m-n,i,j}(\varphi)\ ,\ \Im_{m,n,i,j}\ (\varphi))$ uniquely represents any rational number r_k via the Fibonacci-based rules (16)-(17).

Here the proof ends.

Corollary 3. *The set* $[\boldsymbol{C}^* U \boldsymbol{D}^*]$ *is a is a unique countable set of irrationals on* R^+.

Proof. Each element $\wp_k = \wp_{m-n,i,j}(\varphi) \in \boldsymbol{C}^*$and $\Im_k = \Im_{m,n,i,j}(\varphi)\ \in \boldsymbol{D}^*$ is , by virtue of (16) and (17), respectively unique. So are the pertaining $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ sets.

Result 1. Remark 3 holds, when each of the $\boldsymbol{C}$ and $\boldsymbol{D}$ sets, in its proof, is respectively replaced by $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ of Theorem 3.

As $\boldsymbol{C}^* \cup \boldsymbol{D}^*$ may preserve some of the aperiodicity (or semi-periodicity) features, [10, p.33], of the Fibonacci sequence, then Result 1 should not be the only possible application for this set. One can envisage, e.g., some role to be played by $\boldsymbol{C}^* \cup \boldsymbol{D}^*$, in some modifications to Fibonacci search algorithms [11-12], to be investigated in the future. Specifically, in certain optimization problems, a stable (resonance free) optimal frequency, that may be sought by such algorithms, may not be admissible in the Q set, [5]. Such situations invoke a modification of the Fibonacci to "irrational-Fibonacci" search algorithm, to yield a stable optimal frequency in $\boldsymbol{C}^* \cup \boldsymbol{D}^*$.

3. Conclusion

It is worthwhile to remark that this work has nothing to do with Fibonacci coding [7-8], or with base φ representations [13] of whole numbers. The reported result of this note illustrates, nevertheless, that the Fibonacci generated countable irrational $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ sets can improve arguments of the proofs of some related real analysis results and may have potential application in some irrational-Fibonacci search algorithms. It also poses a question on its possible relation to the subject of completeness of the $\langle F_n \rangle$ sequence. Zeckendorf completeness means that every positive integer, like p or q, can be written in a unique way as a sum of one or more distinct F_n's, in such a way that the sum does not include any two consecutive F_n's. Our Theorem 3 demonstrates that by using $\langle F_n \rangle$ sequence representations, any r_k rational is multiplicatively decomposable to two unique irrationals, $\wp_k \in \boldsymbol{C}^*$ and $\Im_k \in \boldsymbol{D}^*$. The tenuous re-

lationship between this result and completeness is perhaps one between additive and multiplicative decomposability, but not of the same objects though.

4. Appendix

Q and Q^c are both equally dense, as this word has a precise mathematical meaning that they share. Density here is a topological concept, and has nothing to do with cardinality #; see e.g. [1]. As the word "more dense"sounds meaningless in topology, one way to measure how relatively populated Q and Q^c are, is the fact that Q has a Lebesgue measure 0, whereas Q^c has full measure in any interval $(a, b) \subset R$.Hence if one picks at random a number x from $(0, 1)$, the probability of $x = r \in A = \{r_k\}_{k=1}^{\infty} \subset Q$ is zero, while the probability of $x = s \in B = \{s_k\} \subset Q^c$ is one. Remarkably, this happens to take place while the countable A and uncountable B are both infinite sets and both are equally dense.

This may sound physically paradoxical, if density is conceived conventionally as a "number of points per unit length". The paradox is resolvable however, if one simulates rational numbers by a tiny amount of a solute of small molecules, dissolved in a pond of a solvent of large molecules. Topologically, the species density should be some ratio like

$$\frac{\text{Molecular size}}{\text{Average distance between any two molecules}},$$

for each of the two species. Such a density can be the same for the solvent (B) and solute (A), despite the infinitely larger relative population of the solvent molecules in B.

Indeed B appears to be more populated that A, as can be deduced from Theorem 1 and its corollary.

Proof of Theorem 1. Reapply the same classical proof, see e.g. [2], of this theorem on the existence of a rational number between any two real numbers $a,b \in R$. This invokes the Archimedian property of R to start with $0 < \frac{1}{n_k} < b - a$, $n_k \in N = \{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$.

Take then $m_* = \frac{\min m}{n_k} \geq b$, $m \in \mathbb{N}$, to arrive at $a < r_k = \frac{m_*-1}{n_k} < b$, $r_k \in A = \{r_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$. Here the proof ends.

Proof of Corollary 1. Reconsider the open interval (a, b) of Theorem 1 as $(\frac{a}{z_\gamma}, \frac{b}{z_\gamma})$, $\gamma \in \Gamma$, with an arbitrary $z_\gamma \in \mathfrak{B} \subset Q^c \cap (a, b)$, not necessarily countable, with $\#\mathfrak{B} = \aleph_1$. This theorem stipulates the existence of $\rho_k \in \mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$ satisfying $\frac{a}{z\gamma} < \rho_k < \frac{b}{z_\gamma}$, $\rho_k \in \mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$, with $\#\mathfrak{A} = \aleph_0$, i.e. $a < s_{\gamma k} = z_\gamma \, \rho_k < b$, and $s_{\gamma k} \triangleq (z_\gamma \, , \, \rho_k) \in \mathfrak{B} \times \mathfrak{A}$.Now since $z_\gamma \, \rho_k$ is irrational, then $s_{jk} \in \{\{s_{\gamma k}\}_{\gamma \in \Gamma}\}_{k=1}^{\infty} \subset Q^c \cap (a, b)$, which is the required result.

References

[1] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill, Auckland, 1964.

[2] M. R. Spiegel, *Real variables*, Schaum's Outline Series, McGraw-Hill, New York, 1969.

[3] R. A. Dunlap, *The golden ratio and Fibonacci numbers*, World Scientific, New York, 1997.

[4] P. S. Alexandrov, A. N. Kolmogorov, *An introduction to the theory of functions of a real variable* (in Russian), GTTI, Moscow, 1938.

[5] K. Dombrowski, *Rational numbers distribution and resonance*, Progress in Physics, 1 (2005), 65-67.

[6] T. Tao, The Works of Yves Meyer, *The Abel prize laureate 2017 international page*, https://en.wikipedia.org/wiki/Abel_Prize.

[7] C. Rousseau, *The Phi number system revisited*, Mathematics Magazine, 68 (1995), 283-284.

[8] E. Zeckendorf, *Représentation de nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. R. Sci. Liège, 41 (1972), 179-182.

[9] T. D. Noe, J. Vos Post, *Primes in Fibonacci n-step and Lucus n-step sequences*, *Mathematics of Computation,* 68 (1999), 417-427.

[10] R. P. Grimaldi, *Fibonacci and Catalan numbers: an introduction*, Wiley, New York, 2011.

[11] J. Kiefer, *Sequential minimax search for a maximum*, Proceedings of the American Mathematical Society, 4 (1953), 502-506.

[12] M. Avriel, D. J. Wilde, *Optimality proof for the symmetric Fibonacci search technique*, Fibonacci Quarterly, 4 (1966), 265-269.

[13] G. Bergman, *A number system with an irrational base*, Mathematics Magazine, 31 (1957), 98-110.

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A modified quasi-Newton methods for unconstrained optimization

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Abstract. In this paper, we present a new modified SR1 method based on the new quasi-Newton equation. Under some suitable condition, we prove the global convergence of method with Wolfe condition. The Numerical results show that the proposed method effective for the given some test problems.
Keywords: quasi-Newton equations, symmetric rank-one update, convergence.

1. Introduction

In this paper, the proposed method is designed to solve the following problem of unconstrained optimization:

$$Minf(x), x \in \mathbb{R}^n, \tag{1}$$

where is a continuously differentiable objective function. [7] This problem can be solved by using several numerical methods including quasi-Newton methods, which are well-known to be the effective ones. The iterative formula of the Quasi-Newton methods is given by

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

where α_k is the parameter determined by exact line-search as:

$$\alpha_k = \frac{\nabla f(x_x)^T d_k}{d_k^T Q d_k} \tag{3}$$

and calculate the search direction by:

$$B_k d_k + \nabla f(x_k) = 0. \tag{4}$$

The updated matrix B_k is generated by:

$$B_{k+1} s_k = y_k, \tag{5}$$

where $s_k = x_{k+1} - x_k$ and $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$, (5) is very famous and named quasi-Newton equation. Based on the above equation (5), the very famous update B_k is the SR1 formula:

$$B_{k+1}^{SRI} = B_k + \frac{(y_k - B_k s_k)(y_k - B_k s_k)^T}{(y_k - B_k s_k)^T s_k}. \tag{6}$$

For more details can be found in [11]. The SR1 method of updates"that approximates the inverse Hessian can be obtained from(5):

$$H_{k+1}^{SRI} = H_k + \frac{(s_k - H_k y_k)(s_k - H_k y_k)^T}{(s_k - H_k y_k)^T y_k}. \tag{7}$$

For standard SR1 method, the steplength α_k in (3) is determined by Wolfe-Powell line search:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k \nabla f(x_k)^T d_k, \tag{8}$$

$$d_k^T g(x_k + \alpha_k d_k) \geq \sigma f(x_k)^T d_k, \tag{9}$$

where $0 \prec \delta \prec \sigma$. For more details can be found in (Wolfe, [8,9]). These methods can be modified using other alternatives of QN algorithms for upgrading their efficiency whilst maintaining important properties (see [2]). Furthermore, the idea of modified quasi-Newton equation had been studied by many researchers for example, see (Powell, [6]); (Li and Fukushima, [3]); (Wei, Li, and Qi, [10]) ; and (Zhang, Deng, and Chen, [14]).

Moreover, among all QN updating formulas, the SR1 formula has very effective properties of correction; therefore, it is currently regarded as the most effective one (see [5]).

Now we will derive new quasi-Newton equations and analyze its convergence.

2. Motivation based on quasi-Newton equation

In 1999, Zhang et al. [14] presented a modified secant condition exploits not only the gradients but also the function values. Using the Taylor expansion to third-order terms, f and $\nabla f(x_k)^T s_k$ can be written as respectively:

$$f_k = f_{k+1} - \nabla f(x_{k+1})^T + \frac{1}{2} s_k^T G_{k+1} + \frac{1}{6} s_k^T T_{k+1} s_k + O(\|s_k\|^4), \tag{10}$$

$$\nabla f(x_k)^T s_k = \nabla f(x_{k+1}^T s_k - s_k^T G_{k+1} s_k + \frac{1}{2} s_k^T (T_{k+1} s_k) s_k + O(\|s_k\|^4), \tag{11}$$

where $T_{k+1} \in R^{n \times n \times n}$ is the tensor of at x_{k+1} . Canceling the terms which include the tensor yields:

$$s_k^T G_{k+1} s_k = y_k^T s_k = 6(f_k - f_{k+1}) + 3(\nabla f(x_{k+1} + \nabla f(x_k))^T s_k. \tag{12}$$

For more details can be found in [12,14].

Now, the motivation quasi-Newton equation based on the above relation:

$$\frac{1}{6}s_k^T G_{k+1}s_k = (f_k - f_{k+1}) + \frac{2}{3}s_{k+1}^T + \frac{1}{3}s_k^T \nabla f(x_k). \tag{13}$$

It follows from the meaning of α_k:

$$s_k^T G_{k+1}s_k = (f_k - f_{k+1}) + \frac{2}{3}s_k^T \nabla f(x_{k+1}) - \frac{1}{2}s_k^T \nabla f(x_k). \tag{14}$$

It follows from the meaning ofy_k we get:

$$y_k^T s_k = \nabla f(x_{k+1})^T s_k - \nabla f(x_k)^T s_k. \tag{15}$$

By putting (15)in (14) we get:

$$s_k^T G_{k+1}s_k = \frac{1}{2}s_k^T y_k + (f_k - f_{k+1}) + \frac{1}{6}s_k^T \nabla f(x_{k+1}). \tag{16}$$

To obtain a higher truth in estimating the Hessian matrix by B_{k+1} it is reasonable to let B_{k+1} satisfy:

$$s_k^T B_{k+1}s_k = \frac{1}{2}s_k^T y_k + (f_k - f_{k+1}) + \frac{1}{6}s_k^T \nabla f(x_{k+1}). \tag{17}$$

good choice to estimate $B_{k+1}s_k$ is known by:

$$B_{k+1}s_k = y_k^*, y_k^* = \frac{1}{2}y_k + \frac{(f_k - f_{k+1}) + 1/6(\nabla f(x_{k+1})^T s_k)}{s_k^T u_k}u_k, \tag{18}$$

where u_k is any vector such that $s_k^T u_k \neq 0$.

Here we will give our two choices of vector u_k in above equation.

1. Putting $u_k = \nabla f(x_{k+1})$ in (18) we have:

$$y_k^* = \frac{1}{2}y_k + \frac{(f_k - f_{k+1}) + 1/6(\nabla f(x_{k+1})^T s_k)}{s_k^T \nabla f(x_{k+1})}\nabla f(x_{k+1}). \tag{19}$$

2. Putting $u_k = y_k$ in (18) we have:

$$y_k^* = \frac{1}{2}y_k + \frac{(f_k - f_{k+1}) + 1/6(\nabla f(x_{k+1})^T s_k)}{s_k^T y_k}y_k. \tag{20}$$

Applying any new quasi-Newton equation to the SR1 formula, this yields the new formula.

The new algorithm method of Modified SR1-type is expressed formally as follows.

2.1 New algorithm

1. Given an initial pointx_0 an initial positive definite matrix $H_0 = I$.
2. If the convergence criterion is achieved, the stop.
3. Compute a quasi-Newton direction by $d_k = -H_k \nabla f(x_k)$.
4. Find an acceptable steplength such that the Wolfe condition.
5. Set $x_{k+1} = x_k + \alpha_k d_k$.
6. Calculate y_k^* by using equation.
7. Compute the next inverse Hessian approximation H_{k+1} by (7).
8. Set $k = k + 1$ and go to Step 1.

The following theorem shows good property of the new secant equation.

Theorem 2.1. *Let* $(\alpha_k,\ x_{k+1},\ g_{k+1},\ d_{k+1})$ *be generated through the new algorithm. Then* H_{k+1} *is non negative definite for* $\forall k$ *provided that* $s_k^T y_k^* \succ 0$.

Proof of Theorem 2.1. And by evaluating the amount, we have by the Wolfe conditions (4) that:

$$s_k^T y_k^* = \frac{1}{2} s_k^T y_k + (f_k - f_{k+1}) + \frac{1}{6} s_k^T \nabla f(x_{k+1}),$$

$$s_k^T y_k^* = (f_k - f_{k+1}) + \frac{2}{3} s_k^T \nabla f(x_{k+1}) - \frac{1}{2} s_k^T \nabla f(x_k). \tag{21}$$

By Wolfe condition (5), we get:

$$\begin{aligned} s_k^T y_k^* &\geq -\delta \nabla f(x_k)^T s_k + \frac{2}{3}\sigma \nabla f(x_k)^T s_k - \frac{1}{2} s_k^T \nabla f(x_k) \\ &\geq s_k^T \nabla f(x_k)(\delta + \frac{1}{2} - \frac{2}{3}\sigma) \geq s_k^T \nabla f(x_k)(\delta + \frac{1}{2} - \frac{2}{3}\delta) \geq s_k^T \nabla f(x_k)(\frac{1}{2} + \frac{1}{3}\delta), \end{aligned}$$

$$s_k^T y_k^* \geq s_k^T \nabla f(x_k)(\frac{1}{2} + \frac{1}{3}\delta). \tag{22}$$

Noting the $s_k^T \nabla f(x_k) = \alpha_k d_k^T \nabla f(x_k) \prec 0$, we know that there exists a constant $m \succ 0$ such that:

$$s_k^T y_k \geq -m d_k^T \nabla f(x_k) \succ 0 \tag{23}$$

The proof is complete.

3. Global convergence

Throughout this section, the following assumptions will be considered about the objective function f.

Assumption:

Let G be the matrix of second derivative of f.

i. The objective function f is continuously differentiable in a neighbor-hood Q of the level set $D = \{x \in R^n : f(x) \leq f(x_0)\}$ and bounded below R^nin.

ii. The "gradient is Lipschitz continuous, i.e., there exists a constant $L \succ 0$ such that:

$$||\nabla f(x_2) - \nabla f(x_1)|| \leq ||x_2 - x_1||, \forall x_1, x_2 \in Q. \tag{24}$$

Let us define:

$$\cos(\theta_k) = \frac{\nabla f(x_k)^T d_k}{||\nabla f(x_k)||||d_k||}. \tag{25}$$

For more details can be found in [1,11].

Theorem 3.1. *Let x_0 be a starting point for which f satisfies Assumption. Consider $\{x_k\}$ the sequence of points generated by the updating scheme $x_{k+1} = x_k + \alpha_k d_k$ where the $\{B_k\}$ sequence is generated by BFGS and α_k satisfies the Wolfe conditions. Then:*

$$\sum \cos^2(\theta_k) = ||\nabla f(x_k)||^2 \prec \infty. \tag{26}$$

Proof of Theorem 3.1. *From Wolfe conditions we have that:*

$$(\nabla f(x_{k+1}) - \nabla f(x_k))^T d_k \geq (\sigma - 1)\nabla f(x_k)^T d_k. \tag{27}$$

Moreover, from Lipschitz condition we obtain:

$$(\nabla f(x_{k+1}) - \nabla f(x_k))^T d_k \leq \alpha_k L||d_k||^2. \tag{28}$$

Combining the above inequalities gives:

$$(\nabla f(x_k)^T d_k)/||d_k||^2 \frac{\sigma - 1}{L} \leq \alpha_k. \tag{29}$$

Therefore, by using the first Wolfe condition and above inequality, we have:

$$f_{k+1} \leq f_k + \delta(\frac{\sigma - 1}{L})(\nabla f(x_k)^T d_k)^2/||d_k||^2. \tag{30}$$

We now use definition $\cos(\theta_k)$ to write this relation as:

$$f_{k+1} \leq f_k + c\cos^2\theta_k||\nabla f(x_k)||^2, \tag{31}$$

where $c = \delta(\sigma - 1)/LSumming$ this expression and recalling that f is bounded below we obtain:

$$\sum \cos^2\theta_k||\nabla f(x_k)||^2 \prec \infty, \tag{32}$$

which concludes the proof.

Table 1: Comparison of different SR1-algorithms with different test functions and different dimensions

$P.No.$	n	SR1 algorithm		BSR1 with $u_k = y_k$		BSR1 with $u_k = \nabla f(x_{k+1})$	
		NI	NF	NI	NF	NI	NF
1	2	4	34	10	28	4	11
2	2	13	53	12	49	3	30
3	2	15	73	13	68	8	44
4	2	2	27	2	27	2	27
5	3	34	162	15	82	7	20
6	3	16	51	17	75	5	15
7	3	2	4	2	-4	2	4
8	3	2	27	2	27	2	27
9	3	2	27	2	27	2	27
10	4	16	52	17	99	6	18
11	4	19	86	19	65	4	13
12	4	19	101	10	74	5	12
13	4	15	70	14	44	5	16
14	5	2	27	2	27	2	27
15	6	6	21	6	21	4	11
16	11	3	31	3	31	3	31
17	20	22	118	13	88	4	12
18	400	17	76	9	53	6	18
19	400	2	27	2	27	2	27
20	200	2	5	2	5	2	5
21	100	2	27	2	27	2	27
22	500	8	49	11	41	6	43
23	500	2	4	2	4	2	4
24	500	11	82	11	82	6	15
25	500	10	69	8	52	6	41
26	500	2	4	2	4	2	4
27	500	3	7	3	7	3	7
28	500	3	7	3	7	3	7
Total		235	1321	229	1145	103	543

4. Numerical result

Table 1 lists numerical results. In Table 1, “problem” and “n” stand for the test problem name and the“dimension”of the test problem, respectively. NI , NF are the total number of the“iterations, the function evaluations, respectively.

In this section, we report the detailed numerical result of a number of problems by Algorithm 2.1. Furthermore, the original SRI methods are given. All

cods were written in Matlab code. we used unconstrained test problems form More Garbow and Hillstrom (see 4). We test the following three BEGS methods: $\sigma_1 = 0.1$, $\sigma_2 = 0.9$, $\epsilon = 10^{-5}$. We stop the iteration: if $|f(x_k)| \succ 10^{-5}$ let $stop1 = \frac{|f(x_k)-f(x_{k+1}|}{|f(x_k)|}$ Otherwise, let $stop1 = |f(x_k) - f(x_{k+1}|$. For every problem, if $||\nabla f(x_k)|| \leq \epsilon$ $stop1 \prec 10^{-5}$ or is satisfied, the program will be stopped. Himmeblau (see 13).

Problems numbers indicant for: "1. is the Froth, 2. is the Badscb, 3. is the Beale, 4. is the Jensam, 5. is the Helix, 6. is Bard, 7. is the Gauss, 8. is the Gulf, 9. is the Box, 10. is the Sing, 11. is the Wood, 12. is the Kowosb, 13. is the Bd. 14. is the Osb1, 15. is the Biggs, 16. is the Osb2, 17. is the Watson, 18. is the Singx, 19. is the Pen1, 20. is the Pen2, 21. is Vardim, 22. is the Trig, 23. is the Bv, 24. is the Trid, 25. is the Band, 26. is the Lin, 27. is the Lin1, 28. is the Lino".

5. Conclusions

We have presented the BSR1 method with a new quasi-Newton equation. Under suitable conditions, we proved that the"proposed method" is globally convergent. Some limited"numerical results" are also reported, which show that the BSR1 method is more efficient than the $SR1$ method. Commonly, we can compute the percentage performance of the new proposed algorithms $BSR1$ compared in opposition to the standard SR1 algorithm for the general tools NI and NF as follows:

Table 2: Relative efficiency of the new Algorithms

	SR1 algorithm	BSR1 with $u_k = y_k$	BSRI with $u_k = \nabla f(x_{k+1})$
NI	100	97.44	43.82
NF	100	86.67	41.10

References

[1] M. Farzin, H.M. Abu and J.L. Wah, *Improved Hessian approximation with modified secant equations for symmetric rank-one method*, Journal of Computational and Applied Mathematics, 235 (2011) 2423-2431.

[2] B. Fahimeh, A.H. Malik and J.L. Wah, *New quasi-Newton methods via higher order tensor models*, Journal of Computational and Applied Mathematics, 235 (2011), 2412-2422.

[3] D. Li, M. Fukushima, *A modified BFGS method and its global convergence in non convex minimization*, J. Comput. Appl. Math., 129 (2001), 15-35.

[4] J. More, B. Garbow, K. Hillstrome, *Testing unconstrained optimization software*, ACM Trans. Math. Software, 7 (1981), 17-41.

[5] J. Nocedal, *Theory of algorithms for unconstrained optimization*, Acta Numerica, 1 (1992), 199242.

[6] D. Powell, *Algorithms for Nonlinear Constraints that Use Lagrange Functions,*, Math. Programming., 14 (1978), 224-248.

[7] Y. Shengwei, N. Liangshuo, *An adaptive three-term conjugate gradient method based on self-scaling memoryless BFGS matrix*, Journal of Computational and Applied Mathematics, 2017, 1-27.

[8] P. Wolfe, *Convergence conditions for ascent methods*, SIAM Rev., 11 (1969), 226-235.

[9] P. Wolfe, P. Wolfe, *Convergence conditions for ascent methods*, II:Some corrections SIAM Review, 13 (1970), 185-188.

[10] Z. Wei, G. Li, L. Qi, *New quasi-Newton methods for unconstrained optimization problems*, Math. Program. Applied Mathematics and Computation., 175 (2006): 1156-1188.

[11] J.L. Wah, A.H. Malik, *Scaled memoryless symmetric rank one method for large-scale optimization*, Applied Mathematics and Computation, 218 (2011), 413-418.

[12] X. Yunhai, W. Zengxin, *A modified BFGS method without line searches for nonconvex unconstrained optimization*, Advances in theoretical and applied mathematics, 2006, 149-162.

[13] Y. Yuan and W. Sun, *Theory and Methods of Optimization*, Science Press of China, 1999.

[14] J.Z. Zhang, N.Y. Deng, L.H. Chen, *New quasi-Newton equation and related methods for unconstrained optimization,* Journal of Optimization Theory and Applications, 102 (1999), 147-167.

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On some weak structures in intuitionistic fuzzy soft topological spaces

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Abstract. Intuitionistic fuzzy set is fruitful and more realistic model of uncertainty than the fuzzy sets. Extending this approach further, intuitionistic fuzzy soft sets were introduced, as a generalization of fuzzy soft sets and standard soft sets. Intuitionistic fuzzy soft sets used to handle the problem of muti-criteria decision making. In this paper, We initiate and establish the topological structures of intuitionistic fuzzy soft semi-open sets and intuitionistic fuzzy soft semi-closed sets. We also define and explore the properties of intuitionistic fuzzy soft semi-interior, intuitionistic fuzzy soft semi-closure and discuss the relationship between them. Moreover, we establish and discuss intuitionistic fuzzy soft semi-open neighborhood systems. We believe that the findings in this paper will enhance the further study proceed towards muti-criteria decision making problems.

Keywords: intuitionistic fuzzy soft sets, intuitionistic fuzzy soft topology, intuitionistic fuzzy soft semi-open(closed), intuitionistic fuzzy soft semi-interior(closure), intuitionistic fuzzy soft semi-open nbd systems.

1. Introduction

Molodtsov [15] introduced soft sets theory as a new mathematical tool to deal with uncertainties, imprecise and unclear defined objects and to overcome incompatibility with the parametrization tools, which were not solved by existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. This theory proved to be useful in different fields not limited to decision making [6][18], data analysis [4][22], forecasting [19] and so on. The topological structures of soft sets are studied and discussed in [7], [9-10]. In [17], Molodtsov et. al mentioned several directions for the applications of soft sets such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability and theory of measurement for modelling the problems in engineering, computer science, economics, social sciences

and medical sciences. In soft systems a very general frame work has been provided with the involvement of parameters.

Zadeh [21] initiated the concept of fuzzy sets. intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were initiated and explored by Atanassov [1-3] to generalize the concept of fuzzy set. Maji et al. introduced the concept of intuitionistic fuzzy soft sets[13-15], which is a generalization of fuzzy soft sets[12] and standard soft sets[15]. It is to be noted that the parameters may not always be crisp, rather they may be intuitionistic fuzzy in nature. Different algebraic structures of intuitionistic fuzzy soft sets are studied and explored in [20]. Coker [5] introduced and studied the concept of intuitionistic fuzzy topological spaces. Li et al. [11] initiated intuitionistic fuzzy topological structures of intuitionistic fuzzy soft sets. They explored the notions of intuitionistic fuzzy soft open(closed) sets, intuitionistic fuzzy soft interior(closure) and intuitionistic fuzzy soft base in intuitionistic fuzzy soft topological spaces. Recently, Hussain [8] initiate the concept of intuitionistic fuzzy soft boundary and explored the characterizations and properties of intuitionistic fuzzy soft boundary in general as well as in terms of intuitionistic fuzzy soft interior and intuitionistic fuzzy soft closure.

In this paper, we initiate and establish the topological structures of intuitionistic fuzzy soft semi-open sets and intuitionistic fuzzy soft semi-closed sets. We also define and explore the properties of intuitionistic fuzzy soft semi-interior, intuitionistic fuzzy soft semi-closure and discuss the relationship between them. Moreover, we establish and discuss intuitionistic fuzzy soft semi-open neighborhood systems.

2. Preliminaries

We recall the following:

Definition 2.1 ([21]). A fuzzy set f on X is a mapping $f : X \to I = [0, 1]$. The value of $f(x)$ represents the degree of membership of $x \in X$ in the fuzzy set f, for $x \in X$.

Definition 2.2 ([16]). Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Definition 2.3 ([12]). Let I^X denotes the set of all fuzzy sets on X and $A \subseteq X$. A pair (f, A) is called a fuzzy soft set over X, where $f : X \to I^X$ is a function. That is, for each $a \in A$, $f(a) = f_a : X \to$ I is a fuzzy set on X .

Definition 2.4 ([2]). An intuitionistic fuzzy set A over the universe X is defined as: $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ with the property that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$. The values $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and nonmembership of x to A respectively.

Definition 2.5 ([2]). Let $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)); x \in X\}$ are intuitionistic fuzzy set over the universe X. Then

(1) $A^c = \{(x, \nu_A(x), \mu_A(x)); x \in X\}$.
(2) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$.
(3)$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(4) $A \cap B = \{(x, min\{\mu_A(x), \mu_B(x)\}, max\{\nu_A(x), \nu_B(x)\} : x \in X\}$.
(5) $A \cup B = \{(x, max\{\mu_A(x), \mu_B(x)\}, min\{\nu_A(x), \nu_B(x)\} : x \in X\}$.

Definition 2.6 ([2]). An intuitionistic fuzzy set A over the universe X is said to be intuitionistic fuzzy null set denoted as $\tilde{0}$, and is defined as: $A = \{(x, 0, 1) : x \in X\}$.

Definition 2.7 ([2]). An intuitionistic fuzzy set A over the universe X is said to be intuitionistic fuzzy absolute set denoted as $\tilde{1}$, and is defined as: $A = \{(x, 1, 0) : x \in X\}$.

Definition 2.8 ([14]). Let X be the initial universal set and E be the set of parameters. Let IF^X denotes the set of all intuitionistic fuzzy soft sets on X and $A \subseteq X$. A pair (IF, A) is called a IF soft set over X, where $f : A \to IF^X$ is a function. That is, for each $a \in A$, $f(a) = f_a : A \to IF^X$, is an intuitionistic fuzzy set on X and is defined as: $f(a) = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$.

From now on, for our convenience, we will represent the intuitionistic fuzzy soft set (IF, A) as IF soft set f_A.

Now we give the example of IF soft sets as:

Example 2.9. Let $(IF, A) = f_A$ describe the character of the employees with respect to the given parameters, for finding the best employee of the financial year. Let the set of employees under consideration be $X = \{x_1, x_2, x_3, x_4\}$. Let $E = \{$ regular workload (r), conduct (c), field performances (g), sincerity(s), pleasing personality (p)$\}$ be the set of parameters framed to choose the best employee. Suppose the administrator Mr. X has the parameter set $A = \{r, c, p\} \subseteq E$ to choose the best employee. Then f_A be the an IF soft set over X, defined as follows:
$f(r)(x_1) = (0.8, 0.1), f(r)(x_2) = (0.7, 0.5), f(r)(x_3) = (0.9, 0.1), f(r)(x_4) = (0.7, 0.2)$
$f(c)(x_1) = (0.6, 0.2),\ f(c)(x_2) = (0.7, 0.1),\ f(c)(x_3) = (0.5, 0.3),\ f(c)(x_4) = (0.3, 0.6)$
$f(p)(x_1) = (0.6, 0.2),\ f(p)(x_2) = (0.7, 0.1),\ f(p)(x_3) = (0.5, 0.3),\ f(p)(x_4) =$

$(0.3, 0.6)$
That is

$$f_A = (IF, A)\tilde{\tilde{=}}\{F(r) = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.05), (x_3, 0.9, 0.1), (x_4, 0.7, 0.2)\},$$
$$F(c) = \{(x_1, 0.6, 0.2), (x_2, 0.7, 0.1), (x_3, 0.5, 0.3), (x_4, 0.3, 0.6)\},$$
$$F(p) = \{(x_1, 0.6, 0.2), (x_2, 0.7, 0.1), (x_3, 0.5, 0.3), (x_4, 0.3, 0.6)\}\}.$$

The tabular representation of IF soft sets f_A is:

	x_1	x_2	x_3	x_4
r	(0.8, 0.1)	(0.7, 0.5)	(0.9, 0.1)	(0.7, 0.2)
c	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.3)	(0.3, 0.6)
p	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.3)	(0.3, 0.6)

In short, we will represent f_A as:
$f_A\tilde{\tilde{=}}\{\{x_{(0.8,0.1)}, x_{(0.7,0.05)}, x_{(0.9,0.1)}, x_{(0.7,0.2)}\}, \{x_{(0.6,0.2)}, x_{(0.7,0.1)}, x_{(0.5,0.3)}, x_{(0.3,0.6)}\},$
$\{x_{(0.6,0.2)}, x_{(0.7,0.1)}, x_{(0.5,0.3)}, x_{(,0.3,0.6)}\}\}.$

Definition 2.10 ([14])**.** Two IF soft sets f_A and g_B over a common universe X, we say that f_A is a IF soft subset of g_B if

(1) $A \subseteq B$ and

(2) for all $a \in A$, $f_a \subseteq g_a$; implies f_a is a IF subset of g_a.

We denote it by $f_A\tilde{\tilde{\subseteq}}g_B$. f_A is said to be a IF soft super set of g_B, if g_B is a IF soft subset of f_A. We denote it by $f_A\tilde{\tilde{\supseteq}}g_B$.

Note that two IF soft sets f_A and g_B over a common universe X are said to be IF soft equal, if f_A is an IF soft subset of g_B and g_B is an If soft subset of f_A.

Definition 2.11 ([14])**.** The union of two IF soft sets of f_A and g_B over the common universe X is the IF soft set h_C, where $C = A \cup B$ and for all $c \in C$,

$$h(c) = \begin{cases} f(c), & \text{if } c \in A - B \\ g(c), & \text{if } c \in B - A \\ f(c) \cup g(c), & \text{if } c \in A \cap B \end{cases}$$

We write $f_A\tilde{\tilde{\cup}}g_B = h_C$.

Definition 2.12 ([14])**.** The intersection h_C of two IF soft sets f_A and g_B over a common universe X , denoted $f_A\tilde{\tilde{\cap}}g_B$, is defined as $C = A \cap B$, and $h(c) = f(c) \cap g(c)$, for all $c \in C$.

Definition 2.13 ([14])**.** The relative complement of an IF soft set f_A is an IF soft set f_A^c, which is denoted by $(f_A)^c$ and where $f^c : A \to IF(X)$ is a IF set-valued function that is, for each $x \in A$, $f^c(A)$ is an IF set in X, whose

membership function $f_a^c(x) = (f_a(x))^c$, for all $x \in A$. Here f_a^c is the membership function of $f^c(a)$.

Definition 2.14 ([11]). Let τ be the collection of IF soft sets over X, then τ is said to be an IF soft topology on X, if

(1) $\widetilde{\widetilde{\Phi}}_A$, $\widetilde{\widetilde{X}}_A$ belong to τ.

(2) If $(f_A)_i \in \tau$, for all $i \in I$, then $\widetilde{\bigcup}_{i\in I}(f_A)_i \in \tau$.

(3) For $f_a, g_b \in \tau$ implies that $f_a \widetilde{\widetilde{\cap}} g_b \in \tau$.

The triplet (X, τ, A) is called an IF soft topological space over X. Every member of τ is called IF soft open set. An IF soft set is called IF soft closed if and only if its complement is IF soft open.

Example 2.15. Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and $\tau = \{\widetilde{\widetilde{\Phi}}_A, \widetilde{\widetilde{X}}_A, f_A, g_A, h_A, k_A\}$, where f_A, g_A, h_A, k_A are IF soft sets over X, defined as follows

$$\begin{aligned}
&f(e_1)(x_1) = (0.2, 0.8), \quad f(e_1)(x_2) = (0.6, 0.3),\\
&f(e_2)(x_1) = (0.2, 0.5), \quad f(e_2)(x_2) = (0.9, 0.1),\\
&g(e_1)(x_1) = (0.1, 0.8), \quad g(e_1)(x_2) = (0.6, 0.1),\\
&g(e_2)(x_1) = (0.2, 0.8), \quad g(e_2)(x_2) = (0.8, 0.1),\\
&h(e_1)(x_1) = (0.2, 0.8), \quad h(e_1)(x_2) = (0.6, 0.1),\\
&h(e_2)(x_1) = (0.2, 0.5), \quad h(e_2)(x_2) = (0.9, 0.1),\\
&k(e_1)(x_1) = (0.1, 0.8), \quad k(e_1)(x_2) = (0.6, 0.3),\\
&k(e_2)(x_1) = (0.2, 0.8), \quad k(e_2)(x_2) = (0.8, 0.1).
\end{aligned}$$

Then

$$\begin{aligned}
\tau = &\{\widetilde{\widetilde{\Phi}}_A, \widetilde{\widetilde{X}}_A, (\{x_{(0.2,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\}), (\{x_{(0.1,0.8)}, x_{(0.6,0.1)}\},\\
&\{x_{(0.2,0.8)}, x_{(0.8,0.1)}\}), (\{x_{(0.2,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\}), (\{x_{(0.1,0.8)},\\
&x_{(0.6,0.3)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\})\}
\end{aligned}$$

is an IF soft topology on X and hence (X, τ, A) is an IF soft topological space over X.

Theorem 2.16 ([11]). *Let τ be the collection of IF soft sets over X. Then*

(1) $\widetilde{\widetilde{\Phi}}_A$, $\widetilde{\widetilde{X}}_A$ *are IF soft closed sets over X.*

(2) *The intersection of any number of IF soft closed sets is an IF soft closed set over X.*

(3) *The union of any two IF soft closed sets is an IF soft closed set over X.*

Definition 2.17 ([11]). Let (X, τ, A) be an IF soft topological space over X and f_A be an IF soft set over X. Then IF soft interior of IF soft set f_A over X is

denoted by $int(f_A)$ and is defined as the union of all IF soft open sets contained in f_A. Thus $int(f_A)$ is the largest IF soft open set contained in f_A.

Definition 2.18 ([11])**.** Let (X, τ, A) be an IF soft topological space over X and f_A be an IF soft set over X. Then the IF soft closure of f_A, denoted by $cl(f_A)$ is the intersection of all IF soft closed super sets of f_A. Clearly $cl(f_A)$ is the smallest IF soft closed set over X which contains f_A.

3. Properties of intuitionistic fuzzy (IF) soft semi-open(closed) sets

We define:

Definition 3.1. Let (X, τ, A) be an IF soft topological space over X, where X is a nonempty set and τ is a family of IF soft sets. An IF soft set f_A in an IF topological space (X, τ) is called IF soft semi-open, if there exists an IF soft open set g_A such that $g_A \tilde{\tilde{\subseteq}} f_A \tilde{\tilde{\subseteq}} cl(g_A)$. The class of all IF soft semi-open sets in X is denoted by $\widetilde{\widetilde{\widetilde{SSO}}}(X)$.

Definition 3.2. An IF soft set f_A in an IF soft topological space (X, τ, A) is IF soft semi-closed if and only if its complement $(f_A)^c$ is IF soft semi-open. The class of an IF soft semi-closed sets is denoted by $\widetilde{\widetilde{\widetilde{SSC}}}(X)$.

Remark 3.3. From the definition, it is obvious that an IF soft open set is IF soft semi-open in an IF soft topological space (X, τ, A). The following example shows that the converse is not true in general.

Example 3.4. Let us consider an IF soft topological space (X, τ, A) of Example 2.15. The IF soft closed sets are

$$\widetilde{\widetilde{\Phi_A}} \tilde{\tilde{=}} (\{x_{(0,1)}, x_{(0,1)}\}, \{x_{(0,1)}, x_{(0,1)}\}), \widetilde{\widetilde{X_A}} \tilde{\tilde{=}} (\{x_{(1,0)}, x_{(1,0)}\}, \{x_{(1,0)}, x_{(1,0)}\}),$$
$$(\{x_{(0.8,0.2)}, x_{(0.3,0.6)}\}, \{x_{(0.5,0.2)}, x_{(0.1,0.9)}\}), (\{x_{(0.8,0.1)}, x_{(0.1,0.6)}\},$$
$$\{x_{(0.8,0.2)}, x_{(0.1,0.8)}\}), (\{x_{(0.8,0.2)}, x_{(0.1,0.6)}\}, \{x_{(0.5,0.2)}, x_{(0.1,0.9)}\}),$$
$$(\{x_{(0.8,0.1)}, x_{(0.3,0.6)}\}, \{x_{(0.8,0.2)}, x_{(0.1,0.8)}\}).$$

Now we consider an IF soft set l_A over X defined by
$l(e_1)(x_1) = (0.2, 0.7)$, $l(e_1)(x_2) = (0.6, 0.2)$,
$l(e_2)(x_1) = (0.2, 0.4)$, $l(e_2)(x_2) = (0.9, 0.1)$,
That is, $l_A \tilde{\tilde{=}} (\{x_{(0.2,0.7)}, x_{(0.6,0.2)}\}, \{x_{(0.2,0.4)}, x_{(0.9,0.1)}\})$. Then there exists an IF soft open set $f_A \tilde{\tilde{=}} (\{x_{(0.2,0.8)}, x_{(0.6,0.3)}\}, \{x_{(0.2,0.5)}, x_{(0.9,0.1)}\})$ such that $f_A \tilde{\subseteq} l_A \tilde{\subseteq} cl(f_A) \tilde{\tilde{=}} (\{x_{(1,0)}, x_{(1,0)}\}, \{x_{(1,0)}, x_{(1,0)}\}) \tilde{\tilde{=}} \widetilde{\widetilde{X_A}}$.

Hence l_A is an IF soft semi-open set, but l_A is not an IF soft open set.

Proposition 3.5. *Let f_A be an IF soft set in an IF soft topological space (X, τ, A). Then f_A is IF soft semi-open if and only if $f_A \tilde{\subseteq} cl(int(f_A))$.*

Proof. ($\Rightarrow$) Suppose that f_A is IF soft semi-open, then the there exists an IF soft open set g_A such that $g_A \tilde{\subseteq} f_A \tilde{\subseteq} cl(g_A)$. Now $g_A \tilde{\subseteq} int(f_A)$ implies that $cl(g_A) \tilde{\subseteq} cl(int(f_A))$. Therefore $f_A \tilde{\subseteq} cl(g_A) \tilde{\subseteq} cl(int(f_A))$.

($\Leftarrow$) Suppose that $f_A \tilde{\subseteq} cl(int(f_A))$. Take $g_A \tilde{=} int(f_A)$, we have

$$g_A \tilde{\subseteq} f_A \tilde{\subseteq} cl(g_A).$$

This completes the proof.

Theorem 3.6. *Let (X, τ, A) be an IF soft topological space. Then an arbitrary union of IF soft semi-open sets is an IF soft semi-open set.*

Proof. Let $\{g_{A_\alpha} : \alpha \in I\}$ be a collection of IF soft semi-open sets and $h_A \tilde{=} \tilde{\cup}_{\alpha \in I}(g_A)_\alpha$. Since each g_{A_α} is IF soft semi-open, then there exist an IF soft open set f_{A_α} such that $f_{A_\alpha} \tilde{\subseteq} g_{A_\alpha} \tilde{\subseteq} cl(f_{A_\alpha})$ and so $\tilde{\cup}_{\alpha \in I} f_{A_\alpha} \tilde{\subseteq} \tilde{\cup}_{\alpha \in I} g_{A_\alpha}$ $\tilde{\subseteq} \tilde{\cup}_{\alpha \in I} cl(f_{A_\alpha}) \tilde{\subseteq} cl(\tilde{\cup}_{\alpha \in I} f_{A_\alpha})$. Let $f_A \tilde{=} \tilde{\cup}_{\alpha \in I} f_{A_\alpha}$. Then f_A is IF soft open and $f_A \tilde{\subseteq} \tilde{\cup}_{\alpha \in I} g_{A_\alpha} \tilde{\subseteq} cl(f_A)$. Therefore, $\tilde{\cup}_{\alpha \in I} g_{A_\alpha}$ is an IF soft semi-open set. Hence the proof.

Proposition 3.7. *Let f_A be an IF soft semi-open set and h_A be an IF soft set in an IF soft topological space (X, τ, A). Suppose $f_A \tilde{\subseteq} h_A \tilde{\subseteq} cl(f_A)$. Then h_A is an IF soft semi-open set in X.*

Proof. f_A be an IF soft semi-open set implies that there exists an IF soft open set g_A such that $g_A \tilde{\subseteq} f_A \tilde{\subseteq} cl(g_A)$. Now $g_A \tilde{\subseteq} h_A$ and $cl(f_A) \tilde{\subseteq} cl(g_A)$ implies that $h_A \tilde{\subseteq} cl(g_A)$. Therefore $g_A \tilde{\subseteq} h_A \tilde{\subseteq} cl(g_A)$. Hence h_A is an IF soft semi-open set in X. This completes the proof.

Proposition 3.8. *Let f_A be an IF soft set in an IF soft topological space (X, τ, A). Then f_A is IF soft semi-closed if and only if there exists an IF soft closed set h_A such that $int(h_A) \tilde{\subseteq} f_A \tilde{\subseteq} h_A$.*

Proof. This follows directly from the definition of an IF soft semi-closed set.

The following proposition is obvious.

Proposition 3.9. *Every IF soft closed set is IF soft semi-closed in an IF soft topological space (X, τ, A).*

The following example shows that the converse of the above proposition is not true in general.

Example 3.10. Consider an IF soft topological space (X, τ, A) over X of Example 3.4. Take an IF soft set $m_A \widetilde{\widetilde{=}} (l_A)^c$ over X as:

$m_A \widetilde{\widetilde{=}} (\{x_{(0.7,0.2)}, x_{(0.2,0.6)}\}, \{x_{(0.4,0.2)}, x_{(0.1,0.9)}\})$.

We observe that m_A is an IF soft semi-closed, but m_A is not an IF soft closed sets.

Next we give the characterization of an IF soft semi-closed set.

Theorem 3.11. *Let f_A be an IF soft set in an IF soft topological space (X, τ, A). Then f_A is IF soft semi-closed if and only if $int(cl(f_A)) \widetilde{\widetilde{\subseteq}} f_A$.*

Proof. ($\Rightarrow$) Suppose that f_A is IF soft semi-closed, then by Proposition 3.8, there exists an IF soft closed set h_A such that $int(h_A) \widetilde{\widetilde{\subseteq}} f_A \widetilde{\widetilde{\subseteq}} h_A$. This follows that $cl(f_A) \widetilde{\widetilde{\subseteq}} cl(h_A) \widetilde{\widetilde{=}} h_A$. Thus $int(cl(f_A)) \widetilde{\widetilde{\subseteq}} int(h_A)$.

Therefore, $int(cl(f_A)) \widetilde{\widetilde{\subseteq}} int(h_A) \widetilde{\widetilde{\subseteq}} f_A$.

($\Leftarrow$) Suppose that f_A be an IF soft set in (X, τ, A) such that $int(cl(f_A)) \widetilde{\widetilde{\subseteq}} f_A$. Take $cl(f_A) \widetilde{\widetilde{=}} h_A$. Then $int(h_A) \widetilde{\widetilde{\subseteq}} f_A \widetilde{\widetilde{\subseteq}} h_A$. This implies that f_A is an IF soft semi-closed set. Hence the proof.

Theorem 3.12. *Let $\{f_{A_\alpha} : \alpha \in I\}$ be a collection of an IF soft semi-closed sets in an IF soft topological space (X, τ, A). Then the intersection $\widetilde{\widetilde{\cap}}_{\alpha \in I}(f_{A_\alpha})$ is an IF soft semi-closed set in (X, τ, A).*

Proof. Since each $\alpha \in I$, f_{A_α} is an IF soft semi-closed set, then by Proposition 3.8, there exists an IF soft closed set h_{A_α} such that $int(h_{A_\alpha}) \widetilde{\widetilde{\subseteq}} (f_{A_\alpha}) \widetilde{\widetilde{\subseteq}} (h_{A_\alpha})$. This follows that $\widetilde{\widetilde{\cap}}_{\alpha \in I}(int(h_{A_\alpha})) \widetilde{\widetilde{\subseteq}} \widetilde{\widetilde{\cap}}_{\alpha \in I}(int(h_{A_\alpha})) \widetilde{\widetilde{\subseteq}} \widetilde{\widetilde{\cap}}_{\alpha \in I}(f_{A_\alpha}) \widetilde{\subseteq} \widetilde{\widetilde{\cap}}_{\alpha \in I}(h_{A_\alpha})$. Take $\widetilde{\widetilde{\cap}}_{\alpha \in I}(h_{A_\alpha}) \widetilde{\widetilde{=}} (h_A)$. Then by Theorem 2.16, h_A is an IF soft closed and hence $\widetilde{\widetilde{\cap}}_{\alpha \in I}(f_{A_\alpha})$ is an IF soft semi-closed. This completes the proof.

Theorem 3.13. *Let f_A be an IF soft semi-closed set and g_A be an IF soft set in an IF soft topological space (X, τ, A). If $int(f_A) \widetilde{\widetilde{\subseteq}} g_A \widetilde{\widetilde{\subseteq}} f_A$, then g_A is an IF soft semi-closed set.*

Proof. Since f_A is an IF soft semi-closed, then by Proposition 3.8, there exists an IF soft closed set h_A such that $int(h_A) \widetilde{\widetilde{\subseteq}} f_A \widetilde{\widetilde{\subseteq}} h_A$. Then $g_A \widetilde{\widetilde{\subseteq}} h_A$. Also $int(int(h_A)) \widetilde{\widetilde{=}} int(h_A) \widetilde{\widetilde{\subseteq}} int(f_A)$. This implies that $int(h_A) \widetilde{\widetilde{\subseteq}} g_A$.

Therefore, $int(h_A) \widetilde{\widetilde{\subseteq}} g_A \widetilde{\widetilde{\subseteq}} h_A$. Hence g_A is an IF soft semi-closed set. Hence the proof.

Definition 3.14. Let f_A be an IF soft sets in an IF soft topological spaces (X, τ, A). Then

(1) the IF soft semi-interior of f_A is denoted by $\widetilde{\widetilde{sint}}(f_A)$ and is defined as the union of all IF soft semi-open subsets of f_A.

Clearly, $\widetilde{\widetilde{sint}}(f_A)$ is the largest IF soft semi-open set over X contained in f_A.

(2) the IF soft semi-closure of f_A is denoted by $\widetilde{\widetilde{scl}}(f_A)$ and is defined as the intersection of all IF soft semi-closed supersets of f_A.

Note that $\widetilde{\widetilde{scl}}(f_A)$ is the smallest IF soft semi-closed set over X which contains f_A.

Example 3.15. Let us consider an IF soft topological space (X, τ, A) over X and an IF soft closed sets of Example 3.10. It is clear that for any IF soft closed set k_A, $\widetilde{\widetilde{scl}}(k_A)\tilde{\tilde{=}}k_A$.

Example 3.16. Let us consider an IF soft topological space (X, τ, A) over X and an IF soft open sets of Example 3.4. It is clear that for any IF sot open set f_A, $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{=}}(f_A)$.

From Remark 3.3, Proposition 3.9 and Definition 3.14, we have:

Remark 3.17. Clearly, if f_A be an IF soft set in an IF soft topological space (X, τ, A), then $int(f_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\subseteq}}(f_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\subseteq}}cl(f_A)$.

Theorem 3.18. *Let f_A and g_A are two IF soft sets in an IF soft topological space (X, τ, A) over X. Then*

(1) $\widetilde{\widetilde{sint}}(\tilde{\Phi}_A)\tilde{\tilde{=}}\widetilde{\widetilde{scl}}(\tilde{\Phi}_A)\tilde{\tilde{=}}\tilde{\Phi}_A$ *and* $\widetilde{\widetilde{sint}}(\widetilde{X_A}) \tilde{\tilde{=}}\widetilde{\widetilde{scl}}(\widetilde{X_A})\tilde{\tilde{=}}\widetilde{X_A}$.

(2) f_A *is an IF soft semi-open (respt. IF soft semi-closed) set if and only if* $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{=}}f_A$ *(respt.* $\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{=}}f_A$*).*

(3) $\widetilde{\widetilde{sint}}(\widetilde{\widetilde{sint}}(f_A))\tilde{\tilde{=}}f_A$.

(4) $f_A\tilde{\tilde{\subseteq}}(g_A)$ *implies* $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(g_A)$ *and* $\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(g_A)$.

(5) $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A)\tilde{\tilde{=}}\widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)$ *and* $\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{scl}}(g_A)\tilde{\tilde{\supseteq}}\widetilde{\widetilde{scl}}(f_A\tilde{\tilde{\cap}}g_A)$.

(6) $\widetilde{\widetilde{sint}}(f_A)\tilde{\cup}\widetilde{\widetilde{sint}}(g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(f_A\tilde{\cup}g_A)$ *and* $\widetilde{\widetilde{scl}}(f_A)\tilde{\cup}\widetilde{\widetilde{scl}}(g_A)\tilde{\tilde{=}}\widetilde{\widetilde{scl}}(f_A\tilde{\cup}g_A)$.

Proof.(1)–(4) follow directly from the definitions of IF soft semi-interior and If soft semi-closure and are therefore omitted.

(5) (i) Using (4), we have $(f_A\tilde{\tilde{\cap}}g_A)\tilde{\subset}f_A$, $(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}g_A$ implies

$$\widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(f_A), \widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(g_A),$$

so that $\widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A)$. Also, since $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\subseteq}}f_A$ and $\widetilde{\widetilde{sint}}(g_A)\tilde{\tilde{\subseteq}}g_A$ implies $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A)\tilde{\tilde{\subseteq}}(f_A\tilde{\tilde{\cap}}g_A)$.

Thus $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A)$ is a IF soft semi-open subset of $(f_A\tilde{\tilde{\cap}}g_A)$. Hence $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)$. Thus $\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{sint}}(g_A) = \widetilde{\widetilde{sint}}(f_A\tilde{\tilde{\cap}}g_A)$.

(ii) Using (4), we have $(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}f_A$, $(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}g_A$ implies $\widetilde{\widetilde{scl}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(f_A)$, $\widetilde{\widetilde{scl}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(g_A)$, so that $\widetilde{\widetilde{scl}}(f_A\tilde{\tilde{\cap}}g_A)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\cap}}\widetilde{\widetilde{scl}}(g_A)$.

(6) The proof is similar to (5) by using the property that $f_A\tilde{\subseteq}(f_A\tilde{\cup}g_A)$, $g_A\tilde{\subseteq}(f_A\tilde{\cup}g_A)$. Hence the proof.

Theorem 3.19. *Let f_A be an IF soft set in an IF soft topological space (X,τ,A) over X. Then*

(1) $(\widetilde{\widetilde{sint}}(f_A))^c\tilde{\tilde{=}}\widetilde{\widetilde{scl}}((f_A)^c)$.

(2) $(\widetilde{\widetilde{scl}}(f_A))^c\tilde{\tilde{=}}\widetilde{\widetilde{sint}}((f_A)^c)$.

(3) $\widetilde{\widetilde{sint}}(int(f_A))\tilde{\tilde{=}}int(\widetilde{\widetilde{sint}}(f_A))\tilde{\tilde{=}}int(f_A)$.

(4) $\widetilde{\widetilde{scl}}(cl(f_A))\tilde{\tilde{=}}cl((\widetilde{\widetilde{scl}}(f_A)))\tilde{\tilde{=}}cl(f_A)$.

Proof. (1) $\widetilde{\widetilde{sint}}(f_A)\tilde{\subseteq}f_A$ implies that $(f_A)^c\tilde{\subseteq}(\widetilde{\widetilde{sint}}(f_A))^c$.

Now by Theorem 3.18(2) and since $(\widetilde{\widetilde{sint}}(f_A))^c$ is a IF soft semi-closed set, we have $\widetilde{\widetilde{scl}}((f_A)^c)\tilde{\subseteq}\widetilde{\widetilde{scl}}((\widetilde{\widetilde{sint}}(f_A))^c)\tilde{\tilde{=}}(\widetilde{\widetilde{sint}}(f_A))^c$. For the reverse inclusion, $(f_A)^c\tilde{\subseteq}\widetilde{\widetilde{scl}}((f_A)^c)$ implies that $(\widetilde{\widetilde{scl}}((f_A)^c))^c\tilde{\subseteq}((f_A)^c)^c\tilde{\tilde{=}}f_A$. $\widetilde{\widetilde{scl}}((f_A)^c)$ being IF soft semi-closed implies that $(\widetilde{\widetilde{scl}}((f_A)^c))^c$ is IF soft semi-open. Thus $(\widetilde{\widetilde{scl}}((f_A)^c))^c\tilde{\subseteq}\widetilde{\widetilde{sint}}(f_A)$ and hence $(\widetilde{\widetilde{sint}}(f_A))^c\tilde{\subseteq}((\widetilde{\widetilde{scl}}((f_A)^c))^c)^c\tilde{\tilde{=}}\widetilde{\widetilde{scl}}((f_A)^c)$.

(2) The proof is same as (1).

(3) By Remark 3.3, $int(f_A)$ is an IF soft open implies that it is IF soft semi-open. Therefore, by Theorem 3.18(2), $\widetilde{\widetilde{sint}}(int(f_A))\tilde{\tilde{=}}int(f_A)$. Now by Remark 3.17, $int(f_A)\tilde{\subseteq}\widetilde{\widetilde{sint}}(f_A)\tilde{\tilde{=}}f_A$. This implies that $\widetilde{\widetilde{sint}}(int(f_A))\tilde{\tilde{=}}int(f_A)$.

(4) By Proposition 3.9, $cl(f_A)$ being an IF soft closed implies that it is IF soft semi-closed. Therefore, by Theorem 3.18(2), $\widetilde{\widetilde{scl}}(cl(f_A))\tilde{\tilde{=}}cl(f_A)$. Now by Remark 3.17, $f_A\tilde{\subseteq}\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\subseteq}}cl(f_A)$. Hence $\widetilde{\widetilde{scl}}(f_A)\tilde{\tilde{\subseteq}}cl(\widetilde{\widetilde{scl}}(f_A))\tilde{\tilde{\subseteq}}\widetilde{\widetilde{scl}}(f_A)$. This implies that $cl((\widetilde{\widetilde{scl}}(f_A)))\tilde{\tilde{=}}cl(f_A)$. This completes the proof.

4. Intuitionistic fuzzy (IF) soft semi-neighborhood systems

Definition 4.1. An IF soft set f_A is called an IF soft point over X, denoted by e_f, if for the element $e\in A$, $f(e)\neq(\Phi,\Phi)$ and $f(e')=(\Phi,\Phi)$, for all $e'\in A-\{e\}$.

Definition 4.2. An IF soft point e_f is said to be in an IF soft set g_A, denoted by $e_f\ \tilde{\tilde{\in}}\ g_A$, if for the element $e\in A$, $f(e)\subseteq g(e)$.

Example 4.3. Let $X=\{x_1,x_2\}$, $A=\{e_1,e_2\}$ and let us consider an IF soft point as: $e_{1f}\tilde{\tilde{=}}(e_1,\{x_{(0.1,0.8)},x_{(0.6,0.3)}\})$ and an IF soft set g_A defined by:

$g(e_1)(x_1) = (0.1, 0.7)$, $g(e_1)(x_2) = (0.6, 0.1)$,
$g(e_2)(x_1) = (0.1, 0.8)$, $g(e_2)(x_2) = (0.9, 0.1)$.

Then $g_A \widetilde{\widetilde{=}} (\{x_{(0.1,0.7)}, x_{(0.6,0.1)}\}, \{x_{(0.1,0.8)}, x_{(0.9,0.1)}\})$. It is clear that $e_{1f} \widetilde{\widetilde{\in}} g_A$, since $f(e_1) \subseteq g(e_1)$. Obviously, $f(e_2) = \Phi = \{x_{(0,1)}, x_{(0,1)}\} \subseteq g(e_2)$.

Proposition 4.4. *Let e_F be an IF soft point over X and g_A be an IF soft set over X. If $e_f \widetilde{\widetilde{\in}} g_A$, then $e_f \widetilde{\widetilde{\notin}} (g_A)^c$.*

Proof. If $e_f \widetilde{\widetilde{\in}} g_A$, then for $e \in A$, $f(e) \subseteq g(e)$. This implies $f(e) \not\subseteq g^c(e)$. Therefore, we have $e_f \widetilde{\widetilde{\notin}} g_A^c \widetilde{\widetilde{=}} (g_A)^c$. This completes the proof.

Remark 4.5. The following example shows that the converse of the above proposition is not true.

Example 4.6. Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and let us consider the IF soft point as: $e_{1f} \widetilde{\widetilde{=}} (e_1, \{x_{(0.2,0.8)}, x_{(0.6,0.3)}\})$ and an IF soft set g_A defined by:

$g(e_1)(x_1) = (0.1, 0.8)$, $g(e_1)(x_2) = (0.6, 0.1)$,
$g(e_2)(x_1) = (0.2, 0.8)$, $g(e_2)(x_2) = (0.8, 0.1)$.

Then $g_A \widetilde{\widetilde{=}} (\{x_{(0.1,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\})$. It is clear that $e_{1f} \widetilde{\widetilde{\notin}} g_A$. Moreover, $e_{1f} \widetilde{\widetilde{\notin}} (g_A)^c \widetilde{\widetilde{=}} (\{x_{(0.1,0.8)}, x_{(0.6,0.1)}\}, \{x_{(0.2,0.8)}, x_{(0.8,0.1)}\})$.

Definition 4.7. Let e_h be an IF soft point in an IF soft topological space (X, τ, A) over X. If there is an IF soft semi-open set g_A such that $e_h \widetilde{\widetilde{\in}} g_A$, then g_A is called an IF soft semi-open neighborhood (in short: nbd) of e_h. The set of all IF soft semi-open nbds of e_h, denoted $\widetilde{\widetilde{\widetilde{snbd}}}(e_h)$, is called an IF soft semi-open nbd systems of e_h. That is, $\widetilde{\widetilde{\widetilde{snbd}}}(e_h) \widetilde{\widetilde{=}} \{g_A : g_A$ is an IF soft semi-open set and $e_h \widetilde{\widetilde{\in}} g_A\}$.

The following theorem gives us important properties of an IF soft open nbd system:

Proposition 4.8. *Let f_A, g_A and h_A are IF soft sets in an IF soft topological spaces (X, τ, A) over X such that $g_A, h_A \tilde{\subseteq} f_A$. Then the collection of an IF soft semi-open nbd $\widetilde{\widetilde{\widetilde{snbd}}}(e_k)$ at e_k in (X, τ, A) has the following properties*:

(1) *If $g_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$, then $e_k \widetilde{\widetilde{\in}} g_A$.*

(2) *If $g_A, h_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$, then $g_A \widetilde{\cap} h_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$.*

(3) *If $g_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$ and $g_A \tilde{\subseteq} h_A$, then $h_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$.*

(4) *If $g_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$, then there is an $h_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_k)$ such that $g_A \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{snbd}}}(e_l)$, for each $e_l \widetilde{\widetilde{\in}} h_A$.*

(5) *$g_A \tilde{\subseteq} f_A$ is an IF soft semi-open if and only if g_A contains an IF soft semi-open nbd of each of its IF soft points.*

Proof. (1) is obvious, since g_A is an IF soft semi-open nbd of $e_k\tilde{\tilde{\in}}f_A$. So g_A is an IF soft semi-open set such that $e_k\tilde{\tilde{\in}}g_A$.

(2) If $g_A, h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$, then there exist an IF soft semi-open sets m_A and n_A such that $e_k\tilde{\tilde{\in}}m_A\tilde{\tilde{\subseteq}}g_A$ and $e_k\tilde{\tilde{\in}}n_A\tilde{\tilde{\subseteq}}h_A$. Therefore $e_k\tilde{\tilde{\in}}m_A\tilde{\tilde{\cap}}n_A\tilde{\tilde{\subseteq}}g_A\tilde{\tilde{\cap}}h_A$ and hence $g_A\tilde{\tilde{\cap}}h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$.

(3) Since $g_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$, then there exists an IF soft semi-open set m_A such that $e_k\tilde{\tilde{\in}}m_A\tilde{\tilde{\subseteq}}g_A$. Then $e_k\tilde{\tilde{\in}}m_A\tilde{\tilde{\subseteq}}g_A\tilde{\tilde{\subseteq}}h_A$ or $e_k\tilde{\tilde{\in}}m_A\tilde{\tilde{\subseteq}}h_A$. Hence $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$.

(4) Since $g_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$, then $e_k\tilde{\tilde{\in}}h_A\tilde{\tilde{\subseteq}}g_A$, for h_A an IF soft semi-open set in f_A. Since $e_k\tilde{\tilde{\in}}h_A\tilde{\tilde{\subseteq}}h_A$ then $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$. If $e_k\tilde{\tilde{\in}}h_A$, then by (3) $h_A\tilde{\tilde{\subseteq}}g_A$ implies $g_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$, for each $e_k\tilde{\tilde{\in}}h_A$.

(5) (i) Suppose g_A is an IF soft semi-open in f_A, then $e_k\tilde{\tilde{\in}}g_A\tilde{\tilde{\subseteq}}f_A$ implies g_A is an IF soft semi-open nbd of each $e_k\tilde{\tilde{\in}}g_A$.

(ii) If each $e_k\tilde{\tilde{\in}}g_A$ has an IF soft semi-open nbd $(h_A)_{e_k}\tilde{\tilde{\subseteq}}g_A$, then $g_A = \{e_k : e_k\tilde{\tilde{\in}}g_A\}\tilde{\tilde{\subseteq}}\tilde{\tilde{\bigcup}}_{e_k\tilde{\tilde{\in}}g_A}(h_A)_{e_k}\tilde{\tilde{\subseteq}}g_A$ or $g_A = \tilde{\tilde{\bigcup}}_{e_k\tilde{\tilde{\in}}g_A}(h_A)_{e_k}$. This gives g_A is an IF soft semi-open in f_A. Hence the proof.

Definition 4.9. Let (X, τ, A) be an IF soft topological space over X. An IF soft semi-nbd base at $e_k\tilde{\tilde{\in}}f_A$ is a subcollection $\widetilde{\widetilde{snbd}}^s(e_k)$ of IF soft semi-nbd $\widetilde{\widetilde{snbd}}(e_k)$ having the property that each $g_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}(e_k)$ contains some $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$. That is, $\widetilde{\widetilde{snbd}}(e_k)$ must be determined by $\widetilde{\widetilde{snbd}}^s(e_k)$ as follows:

$\widetilde{\widetilde{snbd}}(e_k) = \{g_A\tilde{\tilde{\subseteq}}f_A : h_A\tilde{\tilde{\subseteq}}g_A$, for some $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)\}$.

Each $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$ is called a basic IF soft semi-open neighborhood of e_k.

For an IF soft basic semi-nbd system, the following properties directly follows by the corresponding properties of Proposition 4.8.:

Proposition 4.10. *Let f_A be an IF soft set in an IF soft topological space (X, τ, A) over X and for each $e_k\tilde{\tilde{\in}}f_A$, let $\widetilde{\widetilde{snbd}}^s(e_k)$ be an IF soft semi-nbd base at e_k. Then we have the following properties.*

(1) *If $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$, then $e_k\tilde{\tilde{\in}}h_A$.*

(2) *If g_A, $m_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$, then there is some $n_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$ such that $n_A\tilde{\tilde{\subseteq}}g_A\tilde{\cap}m_A$.*

(3) *If $h_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$, then there is some $l_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_k)$ such that if $e_p\tilde{\tilde{\in}}l_A$, then there is some $m_A\tilde{\tilde{\in}}\widetilde{\widetilde{snbd}}^s(e_p)$ with $m_A\tilde{\tilde{\subseteq}}h_A$.*

(4) *$g_A\tilde{\tilde{\subseteq}}f_A$ is an IF soft semi-open if and only if g_A contains an IF soft semi-basic nbd of each of its IF soft points.*

Conclusion. The parameters may not always be crisp, rather they may be intuitionistic fuzzy in nature in real life situations. The problems of object recognition have received paramount importance in recent years. The recognition problem may be viewed as multiobserver decision making problem, where the final identification of the object is based on the set of inputs from different observers who provide the overall object characterization in terms of diverse set of parameters. Intuitionistic fuzzy set is fruitful and more realistic model of uncertainty than the fuzzy sets. Extending this approach further, intuitionistic fuzzy soft sets were introduced, as a generalization of fuzzy soft sets and standard soft sets. Intuitionistic fuzzy soft sets used to handle the problem of muti-criteria decision making. Here, we defined and established the topological structures of intuitionistic fuzzy soft semi-open sets and intuitionistic fuzzy soft semi-closed sets. We also defined and explored the properties of intuitionistic fuzzy soft semi-interior, intuitionistic fuzzy soft semi-closure and discussed the relationship between them. Moreover, we established and discussed intuitionistic fuzzy soft semi-open neighborhood systems. We hope that the findings in this paper will enhance and promote the further study proceed towards practical life application.

References

[1] K. Atanassov, *lntuitionistic fuzzy sets,* in: V. Sgurev, Ed., VII ITKR's Session, Sofia (June 1983 Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[2] K. Atanassov, *lntuitionistic fuzzy sets*, Fuzzy Sets and Systems,20 (1986), 87-96.

[3] K. Atanassov and S. Stoeva, *Intuitionistic fuzzy sets,* in: Polish Syrup. on Interval and Fuzzy Mathematics, Poznan, August (1983), 23-26.

[4] D. Chen, E. C. C. Tsang, D. S. Yeung, X. Wang, *The parameterization reduction of soft sets and its applications,* Comput. Math. Appl., 49 (2005), 757-763.

[5] D. Coker: *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems, 88 (1997), 81-89.

[6] F. Feng, Y.B. Jun, X. Liu, L. Li, *An adjustable approach to fuzzy soft set based decision making,* J. Comput. Appl. Math., 234 (2010), 10-20.

[7] S. Hussain, *A note on soft connectedness,* Journal of Egyptian Mathematical Society, 23 (2015), 6-11.

[8] S. Hussain, *On some properties of Intuitionistic fuzzy soft boundary,* submitted.

[9] S. Hussain, B. Ahmad, *Some properties of soft topological spaces,* Computers and Mathematics with Applications, 62 (2011), 4058-4067.

[10] S. Hussain, B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacettepe Journal of Mathematics and Statistics, 44 (2015), 559-568.

[11] Z. Li, R. Cui, *On the topological structurs of intuitionistic fuzzy soft sets,* Annals of fuzzy Mathematics and Informatics, 5 (2013), 229-239.

[12] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy soft sets,* J. Fuzzy Math., 9(2001), 589-602.

[13] P. K. Maji, R. Biswas and A. R. Roy, *Intuitionistic fuzzy soft sets,* The Journal of Fuzzy Mathematics, 9 (2001), 677-691.

[14] P. K. Maji, R. Biswas and A. R. Roy, *On Intuitionistic fuzzy soft sets,* The Journal of Fuzzy Mathematics, 12 (2004), 669-683.

[15] P. K. Maji, *More on intuitionistic fuzzy soft sets,* LNCS 5908(2006), 231-240.

[16] D. Molodtsov, *Soft set theory: first results*, Computers and Mathematics with Applications, 37 (1999), 1931.

[17] D. Molodtsov, V.Y. Leonov, D.V. Kovkov, *Soft sets technique and its application*, Nechetkie Sistemy i Myagkie Vychisleniya, 9 (2006), 8-39.

[18] A. R. Roy, P.K. Maji, *A fuzzy soft set theoretic approach to decision making problems*, J. Comput. Appl. Math., 203 (2007), 412418.

[19] Z. Xiao, K. Gong, Y. Zou, *A combined forecasting approach based on fuzzy soft sets,* J. Comput. Appl. Math., 228 (2009), 326333.

[20] Y. Yin, H. Li, Y. B. Jun, *On algebraic structure of intuitionistic fuzzy soft sets,* Computers and Mathematics with Applications, 64 (2012), 28962911.

[21] L. A. Zadeh, *Fuzzy sets,* Information and Control, 8 (1996), 338-353.

[22] Y. Zou, Z. Xiao, *Data analysis approaches of soft sets under incomplete information,* Knowledge-Based Syst., 21 (2008), 941-945.

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On fuzzy soft compact and fuzzy soft locally compact spaces

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Abstract. In this paper, we define and explore the properties and characterization of fuzzy soft compact spaces at fuzzy soft point. We also initiate and investigate the concept of fuzzy soft locally compact spaces at fuzzy soft point through its properties and characterization. Examples of defined notions are also presented to validate their existence. We hope that the obtained results will proceed towards more natural findings and applications in real life problems with the issues of uncertainties and ambiguous environment.

Keywords: fuzzy soft sets, fuzzy soft topology, fuzzy soft open(closed), fuzzy soft classes, fuzzy soft compact, fuzzy soft locally compact.

1. Introduction

Fuzzy set theory initiated and studied by Zadeh [38] proved to be an important mathematical tool to solve different types of complicated problems having uncertainties in real life problems with ambiguous environment such as sociology, economics, engineering, computer and medical sciences etc. The real life problems associated with uncertainties are handled with probability theory, rough set theory, vague set theory, interval mathematics theory and fuzzy set theory. It is observed that these theories have their limitations and difficulties which require the pre-specification of some initial parameters to deal with them.

To overcome this situation of inadequacy of parametrization tools, Molodtsov [29] initiated soft sets as a modern and sufficient approach for modelling the complicated problems with uncertainties. In recent days, the application of soft set theory have been observed and studied by many researchers in the fields of information science, computer science, demand analysis, forecasting, engineering, medical science, social science and decision making [3], [7-8], [11], [13-14], [24], [26], [30-32].

Shabbir and Naz[34] defined and explored soft topological spaces. Later on, different algebraic structures, soft mappings and generalizations of soft sets have been discussed in [2], [5], [15-16], [21-23].

Maji and Biswas [27] introduced fuzzy soft sets as a generalization of soft sets. Z. Kong et. al [25] presented the application of fuzzy soft set as new theoretic approach to decision making problems. Chang [4] initiated and discussed the notion of fuzzy topological spaces. In [35-36], authors initiated fuzzy soft topology and discussed the algebraic structures of fuzzy soft topology. The applications and structural properties of fuzzy soft sets and fuzzy soft topology are discussed and explored in [1], [6], [9-10], [12], [17], [28], [33], [37].

Recently, S. Hussain [18-20] defined and investigated the properties and characterizations of generalized structures, generalized fuzzy soft mappings and weak and strong forms of fuzzy soft open sets in fuzzy soft topological spaces.

2. Preliminaries

Definition 2.1 ([38]). A fuzzy set f on X is a mapping $f : X \to I = [0, 1]$. The value $f(x)$ represents the degree of membership of $x \in X$ in the fuzzy set f, for $x \in X$.

Definition 2.2 ([29]). Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Definition 2.3 ([27]). Let I^X denotes the set of all fuzzy sets on X and $A \subseteq X$. A pair (f, A) is called a fuzzy soft set over X, where $f : X \to I^X$ is a function. That is, for each $a \in A$, $f(a) = f_a : X \to I$, is a fuzzy set on X .

Definition 2.4 ([27]). For two fuzzy soft sets (f, A) and (g, B) over a common universe X, we say that (f, A) is a fuzzy soft subset of (g, B) if

(1) $A \subseteq B$ and

(2) for all $a \in A$, $f_a \leq g_a$; implies f_a is a fuzzy subset of g_a.

We denote it by $(f, A)\tilde{\leq}(g, B)$. (f, A) is said to be a fuzzy soft super set of (g, B), if (g, B) is a fuzzy soft subset of (f, A). We denote it by $(f, A)\tilde{\geq}(g, B)$.

Definition 2.5 ([27]). Two fuzzy soft sets (f, A) and (g, B) over a common universe X are said to be fuzzy soft equal, if (f, A) is a fuzzy soft subset of (g, B) and (g, B) is a fuzzy soft subset of (f, A).

Definition 2.6 ([27]). The union of two fuzzy soft sets of (f, A) and (g, B) over the common universe X is the fuzzy soft set (h, C), where $C = A \cup B$ and for

all $c \in C$,

$$h_c = \begin{cases} f_c, & \text{if } c \in A - B, \\ g_c, & \text{if } c \in B - A, \\ f_c \vee g_c, & \text{if } c \in A \cap B. \end{cases}$$

We write $(f, A)\tilde{\vee}(g, B) = (h, C)$.

Definition 2.7 ([27]). The intersection (h, C) of two fuzzy soft sets (f, A) and (g, B) over a common universe X , denoted $(f, A)\tilde{\wedge}(g, B)$, is defined as $C = A \cap B$, and $h_c = f_c \wedge g_c$, for all $c \in C$.

Definition 2.8 ([27]). The difference (h, C) of two fuzzy soft sets (f, A) and (g, B) over X, denoted by $(f, A)\tilde{\setminus}(g, B)$, is defined as $(f, A)\tilde{\setminus}(g, B) = (f, A)\tilde{\wedge}(f, B)^c$.

For our convenience, we will use the notation f_A for fuzzy soft set instead of (f, A).

Definition 2.9 ([35]). Let τ be the collection of fuzzy soft sets over X, then τ is said to be a fuzzy soft topology on X if

(1) $\tilde{0}_A$, $\tilde{1_A}$ belong to τ.

(2) If $(f_A)_i \in \tau$, for all $i \in I$, then $\tilde{\bigvee}_{i\in I}(f_A)_i \in \tau$.

(3) $f_a, g_b \in \tau$ implies that $f_a\tilde{\wedge}g_b\tilde{\in}\tau$.

The triplet (X, τ, A) is called a fuzzy soft topological space over X. Every member of τ is called fuzzy soft open set. A fuzzy soft set is called fuzzy soft closed if and only if its complement is fuzzy soft open.

Definition 2.10 ([36]). Let (X, τ, A) be a fuzzy soft topological space over X and f_A be a fuzzy soft set over X. Then fuzzy soft closure of f_A , denoted by $\overline{f_A}$ is the intersection of all fuzzy soft closed super sets of f_A. Clearly $\overline{f_A}$ is the smallest fuzzy soft closed set over X which contains f_A.

Definition 2.11 ([1]). Let $F(X, A)$ and $F(Y, B)$ be families of fuzzy soft sets. $u : X \to Y$ and $p : A \to B$ be mappings. Then a function $f_{pu} : F(X, A) \to F(Y, B)$ defined as:

(1) Let f_A be a fuzzy soft set in $F(X, A)$. The image of f_A under f_{pu} , written as $f_{pu}(f_A)$, is a fuzzy soft set in $F(Y, B)$ such that for $\beta \in p(A) \subseteq B$ and $y \in Y$,

$$f_{pu}(f_A)(\beta)(y) = \begin{cases} \bigvee_{x\in u^{-1}(y)}(\bigvee_{\alpha\in p^{-1}(\beta)\cap A}(f_A(\alpha)), & u^{-1}(y) \neq \phi, p^{-1}(\beta) \cap A \neq \phi \\ 0, & \text{otherwise} \end{cases},$$

for all $y \in B$. $f_{pu}(f_A)$ is known as a fuzzy soft image of a fuzzy soft set f_A.

(2) Let g_B be a fuzzy soft set in $F(Y, B)$. Then the inverse image of g_B under f_{pu} , written as $f^{-1}_{pu}(g_B)$, is a fuzzy soft set in $F(X, A)$ such that

$$f^{-1}_{pu}(g_B)(\alpha)(x) = \begin{cases} g(p(\alpha))(u(x)), & p(\alpha) \in B \\ 0, & \text{otherwise} \end{cases},$$

for all $x \in A$. $f_{pu}^{-1}(g_B)$ is known as a fuzzy soft inverse image of a fuzzy soft set g_B.

The fuzzy soft function f_{pu} is called fuzzy soft surjective, if p and u are surjective. The fuzzy soft function f_{pu} is called fuzzy soft injective, if p and u are injective.

Theorem 2.12 ([1]). *Let $f_A, g_A \in F(X, A)$. Then for a function $f_{pu} : F(X, A) \to F(Y, B)$, the following statements are true:*

(1) $f_{pu}(\tilde{0}_A) = \tilde{0}_B$.
(2) $f_{pu}(\tilde{1}_A) = \tilde{1}_B$.
(3) $f_{pu}(f_A \tilde{\vee} g_A) = f_{pu}(f_A) \tilde{\vee} f_{pu}(g_A)$. *In general* $f_{pu}(\tilde{\bigvee}_i((f_A)_i) = \tilde{\bigvee}_i(f_{pu}(f_A)_i)$.
(4) $f_{pu}(f_A \tilde{\wedge} g_A) \tilde{\leq} f_{pu}(f_A) \tilde{\wedge} f_{pu}(g_B)$. *In general* $f_{pu}(\tilde{\bigwedge}_i(f_A)_i) \tilde{\leq} \tilde{\bigwedge}_i(f_{pu}(f_A)_i)$.
(5) *If* $f_A \tilde{\leq} g_A$, *then* $f_{pu}(f_A) \tilde{\leq} f_{pu}(g_A)$.

Definition 2.13 ([36]). Let (X, τ_1, A) and (Y, τ_2, B) are fuzzy soft topological spaces and $f_{pu} : F(X, A) \to F(Y, B)$ be a fuzzy soft mapping. Then fuzzy soft function $f_{pu} : F(X, A) \to F(Y, B)$ is fuzzy soft pu-continuous, if for any fuzzy soft open set g_A in (Y, τ_2, B), $f_{pu}^{-1}(g_A)$ is fuzzy soft open in (X, τ_1, A).

Definition 2.14 ([36]). Let (X, τ_1, A) and (Y, τ_2, B) are fuzzy soft topological spaces and $f_{pu} : F(X, A) \to F(Y, B)$ be a fuzzy soft function. Then f_{pu} is said to be fuzzy soft pu-open(resp. fuzzy soft pu-closed), if for any fuzzy soft open (resp. fuzzy soft closed) set h_A in (X, τ_1, A), $f_{pu}(h_A)$ is fuzzy soft open(resp. fuzzy soft closed) in (Y, τ_2, B).

Definition 2.15 ([20]). A fuzzy soft set f_A is said to be a fuzzy soft point in (X, τ, A) denoted by $e(f_A)$, if for the element $e \tilde{\in} A$, $f(e) \tilde{\neq} \tilde{0}$ and $f(e^c) \tilde{=} \tilde{0}$, for all $e^c \in A \backslash \{e\}$.

Definition 2.16 ([20]). Let (X, τ, A) be a fuzzy soft topological space over X and f_A be a fuzzy soft set over X. Then f_A is called a fuzzy soft neighborhood (briefly: nbd) of the fuzzy soft point $e(f_A)$, if there exists a fuzzy soft open set h_A such that $e(f_A) \tilde{\in} h_A \tilde{\leq} g_A$.

3. Some properties of fuzzy soft compact spaces

Definition 3.1. Let (X, τ, A) be a fuzzy soft topological space over X and f_A be nonempty fuzzy soft subset set over X. Then

$$\tilde{\tau}_{f_A} \tilde{=} \{v_A \tilde{\wedge} f_A, \text{ where } v_A \text{ is fuzzy soft open set in } \tilde{X}\}$$

is said to be the fuzzy soft relative topology on f_A and $(f_A, \tilde{\tau}_{f_A}, A)$ is called a fuzzy soft subspace of (X, τ, A).

Definition 3.2. Let (X, τ, A) be a fuzzy soft topological space and f_A be fuzzy soft set over X. A collection $\tilde{\Omega} \tilde{=}\{g_{A_i}, i \in I\}$ of fuzzy soft sets is a fuzzy soft cover of a fuzzy soft set f_A, if $f_A \tilde{\leq} \tilde{\bigvee}_{i\in I}(g_{A_i})$. It is fuzzy soft open cover, if each member of $\tilde{\Omega}$ is a fuzzy soft open set.

If we take $\tilde{1}_A$ instead of f_A, then family $\tilde{\Omega}$ of fuzzy soft sets is called a fuzzy soft cover of (X, τ, A).

A finite subcollection $\tilde{\Gamma} \tilde{=}\{g_{A_i}, i = 1, 2, ..., n\}$ of $\tilde{\Omega}$ is called a finite fuzzy soft subcover of f_A, if $f_A \tilde{\leq} \tilde{\vee}\{g_{A_i} : i = 1, 2, ..., n\}$.

Definition 3.3. Let (X, τ, A) be a fuzzy soft topological space and f_A be a fuzzy soft set over X. Then f_A is called fuzzy soft compact, if every fuzzy soft open cover of f_A has a finite fuzzy soft subcover. In particular, (X, τ, A) is called fuzzy soft compact, if every fuzzy soft open cover of (X, τ, A) has a finite fuzzy soft subcover.

Example 3.4. Let (X, τ, A) be a fuzzy soft topological space. Then any finite fuzzy soft subset f_A of (X, τ, A) is fuzzy soft compact. If we take $f_A \tilde{=} \tilde{1}_A$, then (X, τ, A) is fuzzy soft compact.

Example 3.5. Any infinite fuzzy soft set f_A with the fuzzy soft discrete topology is not fuzzy soft compact.

Since fuzzy soft topology is fuzzy soft discrete, then $\{e(g_A)\}$ is fuzzy soft open, for each fuzzy soft point $e(g_A)$. Now $f_A \tilde{=} \tilde{\bigvee}_{e(g_A)\tilde{\in} f_A}\{e(g_A)\}$ and thus $\tilde{\Omega} \tilde{=}\{\{e(g_A)\} : e(g_A) \tilde{\in} f_A\}$ is fuzzy soft open cover of f_A. Since f_A is infinite, this fuzzy soft open cover has no finite fuzzy soft subcover of f_A. Thus f_A is not fuzzy soft compact. If we take $f_A \tilde{=} \tilde{1}_A$, then (X, τ, A) is fuzzy soft compact.

Example 3.6. Let f_A be any fuzzy soft set with the cofinite fuzzy soft topology in fuzzy soft topological space (X, τ, A). Then f_A is fuzzy soft compact.

Let $\tilde{\Omega} \tilde{=}\{g_{A_i}, i \in I\}$ of fuzzy soft sets is a fuzzy soft open cover of a fuzzy soft set f_A. Choose any $g_{A_{i_0}} \tilde{\in} \tilde{\Omega}$. Since fuzzy soft topology is cofinite, then $(g_{A_{i_0}})^c$ is finite. Let $(g_{A_{i_0}})^c \tilde{=}\{e(h_{A_1}), e(h_{A_2}), ..., e(h_{A_n})\}$. Since $f_A \tilde{=} \tilde{\bigvee}_{i\in I}\ (g_{A_i})$, then for each $i = 1, 2, ..., n$, there exists some $g_{A_i} \tilde{\in} \tilde{\Omega}$ such that $e(h_{A_i}) \tilde{\in} g_{A_i}$. Then

$$\begin{aligned} f_A &\tilde{=} g_{A_{i_0}} \tilde{\vee} (g_{A_{i_0}})^c \tilde{=} g_{A_{i_0}} \tilde{\vee}\{e(h_{A_1})\}\tilde{\vee}\{e(h_{A_2})\}, ..., \tilde{\vee}\{e(h_{A_n})\} \\ &\tilde{\leq} g_{A_{i_0}} \tilde{\vee} g_{A_{i_1}} \tilde{\vee} g_{A_{i_2}} \tilde{\vee} .... \tilde{\vee} g_{A_{i_n}}. \end{aligned}$$

Thus $\{g_{A_{i_0}}, g_{A_{i_1}}, g_{A_{i_2}}, ..., g_{A_{i_n}}\}$ is finite fuzzy soft subcover of f_A. Hence f_A is fuzzy soft compact. If we take $f_A \tilde{=} \tilde{1}_A$, then (X, τ, A) is fuzzy soft compact.

Theorem 3.7. *Let (X, τ, A) be a fuzzy soft compact fuzzy soft spaces and f_A be a fuzzy soft set over X. If f_A is fuzzy soft closed then f_A is fuzzy soft compact.*

Proof. Let $\{f_{A_\alpha} : \alpha \in I\}$ be fuzzy soft open cover of f_A. Since f_A is fuzzy soft closed then f_A^c is fuzzy soft open. Now $\tilde{1}_A \tilde{=} f_A \tilde{\vee} f_A^c \tilde{=} (\tilde{\bigvee}_{\alpha\in I} f_{A_\alpha}) \tilde{\vee} f_A^c$. This

follows that $\{f_A^c, f_{A_\alpha}; \alpha \in I\}$ is fuzzy soft open cover of (X, τ, A). (X, τ, A) is fuzzy soft compact implies that this fuzzy soft open cover has finite fuzzy soft subcover $\{f_A^c, f_{A_{\alpha_1}}, f_{A_{\alpha_2}}, ..., f_{A_{\alpha_n}}\}$ of (X, τ, A). Now

$$\begin{aligned} f_A &\tilde{=} f_A \tilde{\wedge} \tilde{1}_A \tilde{=} f_A \tilde{\wedge} [f_A^c \tilde{\vee} f_{A_{\alpha_1}} \tilde{\vee} f_{A_{\alpha_2}} \tilde{\vee} ..... \tilde{\vee} f_{A_{\alpha_n}}] \\ &\tilde{=} (f_A \tilde{\wedge} f_A^c) \tilde{\vee} (f_A \tilde{\wedge} f_{A_{\alpha_1}}) \tilde{\vee} (f_A \tilde{\wedge} f_{A_{\alpha_2}}) \tilde{\vee} ..... \tilde{\vee} (f_A \wedge f_{A_{\alpha_n}}) \\ &\tilde{=} \tilde{0}_A \vee (f_A \tilde{\wedge} f_{A_{\alpha_1}}) \tilde{\vee} (f_A \tilde{\wedge} f_{A_{\alpha_2}}) \tilde{\vee} ..... \tilde{\vee} (f_A \wedge f_{A_{\alpha_n}}) \\ &\tilde{\leq} f_{A_{\alpha_1}} \tilde{\vee} f_{A_{\alpha_2}} \tilde{\vee} ..... \tilde{\vee} f_{A_{\alpha_n}}. \end{aligned}$$

Thus $\{f_{A_{\alpha_1}}, f_{A_{\alpha_2}}, ..., f_{A_{\alpha_n}}\}$ is finite fuzzy soft subcover of f_A. Therefore, f_A is fuzzy soft compact. Hence the proof.

The following example shows that the converse of above theorem is not true in general.

Example 3.8. Let $X = \{h_1, h_2, h_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{\tilde{0}, \tilde{1}, (f_A)_1\}$, where $(f_A)_1$ is fuzzy soft sets over X, defined as follows

$f_1(e_1)(h_1) = 0.3$, $f_1(e_1)(h_2) = 0.4$, $f_1(e_1)(h_3) = 0.2$,
$f_1(e_2)(h_1) = 0.4$, $f_1(e_2)(h_2) = 0.2$, $f_1(e_2)(h_3) = 0.6$,
Then τ is a fuzzy soft topology on X and hence (X, τ, A) is a fuzzy soft topological space over X. Let us take $f_A \tilde{=} \{\{h_{0.3}, h_{0.4}, h_{0.5}\}, \{h_{0.4}, h_{0.2}, h_{0.6}\}\}$. Then f_A being finite is fuzzy soft compact. But f_A is not fuzzy soft closed, since $f_A^c \tilde{=} \{\{h_{0.7}, h_{0.6}, h_{0.5}\}, \{h_{0.6}, h_{0.8}, h_{0.4}\}\}$ is not fuzzy soft open.

Definition 3.9 ([17])**.** Let (X, τ, A) be a fuzzy soft topological space over X. Then (X, τ, A) is said to be fuzzy soft T_2-space, if for any two fuzzy soft points $e(g_A)$, $e(k_A)$ of fuzzy soft set f_A in (X, τ, A) with $e(g_A) \tilde{\neq} e(k_A)$, there exist fuzzy soft open sets h_A and s_A such that $e(g_A) \tilde{\in} h_A$, $e(k_A) \tilde{\in} s_A$ and $h_A \tilde{\wedge} s_A \tilde{=} \tilde{0}_A$.

The converse of the above Theorem 3.7 is true, if (X, τ, A) is fuzzy soft T_2-space. To prove this, we need the following theorem:

Theorem 3.10. *Let (X, τ, A) be a fuzzy soft T_2-space, f_A be a fuzzy soft compact subset of (X, τ, A) and fuzzy soft point $e(g_A) \tilde{\in} f_A$. Then there exist two fuzzy soft open sets h_A and s_A such that $e(g_A) \tilde{\in} h_A$, $f_A \tilde{\leq} s_A$ and $h_A \tilde{\wedge} s_A \tilde{=} 0_A$.*

Proof. Let us take fuzzy soft point $e(k_A) \tilde{\in} f_A$ with $e(g_A) \tilde{\neq} e(k_A)$. Since (X, τ, A) be a fuzzy soft T_2-space, then there exists fuzzy soft open sets $(h_{e(k_A)})_A$ and $(s_{e(k_A)})_A$ such that

$$(*) \qquad e(g_A) \tilde{\in} (h_{e(k_A)})_A, e(k_A) \tilde{\in} (s_{e(k_A)})_A \text{ and } (h_{e(k_A)})_A \tilde{\wedge} (s_{e(k_A)})_A \tilde{=} 0_A ...$$

Now $f_A \tilde{=} \tilde{\bigvee}_{e(k_A) \tilde{\in} f_A} \{e(k_A)\} \tilde{\leq} \tilde{\bigvee}_{e(k_A) \tilde{\in} f_A} (s_{e(k_A)})_A$, and so $\{(s_{e(k_A)})_A : e(k_A) \tilde{\in} f_A\}$ is fuzzy soft open cover of f_A. Since f_A is fuzzy soft compact, then this fuzzy soft open cover has a finite fuzzy soft subcover $\{(s_{e(k_A)})_{A_1}, (s_{e(k_A)})_{A_2}, ..., (s_{e(k_A)})_{A_n}\}$ (say) of f_A. Let $(h_{e(k_A)})_{A_1}, (h_{e(k_A)})_{A_2}, ..., (h_{e(k_A)})_{A_n}$ be the fuzzy soft sets corresponding to $(s_{e(k_A)})_{A_1}, (s_{e(k_A)})_{A_2}, ..., (s_{e(k_A)})_{A_n}$ which satisfy (*).

Put $h_A \tilde{=} \tilde{\wedge}\{(h_{e(k_A)})_{A_i} : i = 1, 2, ..., n\}$, $s_A \tilde{=} \tilde{\vee}\{(s_{e(k_A)})_{A_i} : i = 1, 2, ..., n\}$. Clearly, h_A and s_A are fuzzy soft open. Further, $e(g_A) \tilde{\in} h_A$, since $e(g_A) \tilde{\in} (h_{e(k_A)})_{A_i}$, for each $i = 1, 2, ..., n$.

Moreover, $f_A \tilde{\leq} s_A$, since $\{(s_{e(k_A)})_{A_1}, (s_{e(k_A)})_{A_2}, ..., (s_{e(k_A)})_{A_n}\}$ is finite fuzzy soft subcover of f_A. We claim that $h_A \tilde{\wedge} s_A \tilde{=} 0_A$. To verify the claim, we contrarily suppose that $h_A \tilde{\wedge} s_A \tilde{\neq} 0_A$ and let $e(l_A) \tilde{\in} h_A \tilde{\wedge} s_A$.

Then, we have $e(l_A) \tilde{\in} \tilde{\wedge}\{(h_{e(k_A)})_{A_i} : i = 1, 2, ..., n\}$ and $e(l_A) \tilde{\in} \tilde{\vee}\{(s_{e(k_A)})_{A_i} : i = 1, 2, ..., n\}$. This implies that $e(l_A) \tilde{\in} (h_{e(k_A)})_{A_i}$, for each $i = 1, 2, ..., n$ and $e(l_A) \tilde{\in} (s_{e(k_A)})_{A_i}$, for some $i = 1, 2, ..., n$.

This follows that $(h_{e(k_A)})_{A_i} \tilde{\wedge} (s_{e(k_A)})_{A_i} \tilde{\neq} 0_A$. This contradicts (*). Hence $h_A \tilde{\wedge} s_A \tilde{=} 0_A$. This completes the proof.

Using above Theorem 3.10, we have the following.

Theorem 3.11. *Let (X, τ, A) be a fuzzy soft T_2-space, f_A be a fuzzy soft subset of (X, τ, A). If f_A is fuzzy soft compact then f_A is fuzzy soft closed.*

Proof. To show that f_A is fuzzy soft closed, we prove that $(f_A)^c$ is fuzzy soft open. Let $e(g_A) \tilde{\in} (f_A)^c$. Then $e(g_A) \tilde{\notin} f_A$ and f_A is fuzzy soft compact. So, by Theorem 3.10, there exist two fuzzy soft open sets h_A and s_A such that $e(g_A) \tilde{\in} h_A$, $f_A \tilde{\leq} s_A$ and $h_A \tilde{\wedge} s_A \tilde{=} 0_A$. This follows that $h_A \tilde{\leq} (s_A)^c \tilde{\leq} (f_A)^c$. Therefore, $e(g_A) \tilde{\in} h_A \tilde{\leq} (f_A)^c$. This implies that $(f_A)^c$ is fuzzy soft open and therefore f_A is fuzzy soft closed. Hence the proof.

Definition 3.12. Let (X, τ, A) be a fuzzy soft topological space and f_A be a fuzzy soft set over X. A collection $\tilde{\Omega} \tilde{=} \{g_{A_i}, i \in I\}$ of fuzzy soft subsets of X is said to have the finite intersection property, if for any finite fuzzy soft subcollection $\{g_{A_i}, i = 1, 2, ..., n\}$ of $\tilde{\Omega}$, we have $\tilde{\wedge}\{g_{A_i} : i = 1, 2, ..., n\} \tilde{\neq} \tilde{0}_A$.

The following theorem gives the useful characterization of fuzzy soft compact spaces.

Theorem 3.13. *Let (X, τ, A) be a fuzzy soft topological space. Then the following statements are equivalent:*

(1) *(X, τ, A) is fuzzy soft compact.*

(2) *For any collection $\{g_{A_i}, i \in I\}$ of fuzzy soft closed subsets of (X, τ, A) which satisfy the finite intersection property implies $\tilde{\bigwedge}_{i \in I}(g_{A_i}) \tilde{\neq} 0_A$.*

Proof. (1) $\Rightarrow$ (2) Let (X, τ, A) is fuzzy soft compact and $\{g_{A_i}, i \in I\}$ be any collection of fuzzy soft closed subsets of (X, τ, A) with the finite intersection property. We show that $\tilde{\bigwedge}_{i \in I}(g_{A_i}) \tilde{\neq} 0_A$. For this contrarily suppose that $\tilde{\bigwedge}_{i \in I}(g_{A_i}) \tilde{=} 0_A$. Then $(\tilde{\bigwedge}_{i \in I}(g_{A_i}))^c \tilde{=} (\tilde{0}_A)^c$ or $\tilde{\bigvee}_{i \in I}(g_{A_i})^c \tilde{=} \tilde{1}_A$. Thus $\{(g_{A_i})^c : i \in I\}$ is a fuzzy soft open cover for (X, τ, A).

Since (X, τ, A) is fuzzy soft compact, then this fuzzy soft open cover has a finite fuzzy soft subcover, $\{(g_{A_{i_1}})^c, (g_{A_{i_2}})^c, ..., (g_{A_{i_n}})^c\}$. Then $\tilde{\bigvee}_{k=1}^{n}(g_{A_{i_k}})^c \tilde{=} \tilde{1}_A$

or $(\tilde{\bigwedge}_{k=1}^{n}(g_{A_{i_k}}))^c \tilde{=} \tilde{1}_A$ or $\tilde{\bigwedge}_{k=1}^{n}(g_{A_{i_k}}) \tilde{=} \tilde{0}_A$. This contradicts the fact that $\{g_{A_i}, i \in I\}$ satisfy the finite intersection property. Thus $\tilde{\bigwedge}_{i\in I}(g_{A_i}) \tilde{\neq} 0_A$.

(2) $\Rightarrow$ (1) Suppose (2) holds and let $\{h_{A_i}, i \in I\}$ be any fuzzy soft open cover of (X, τ, A). Contrarily suppose that $\{h_{A_i}, i \in I\}$ has no finite fuzzy soft subcover of (X, τ, A). Then for every finite subcollection $\{h_{A_{i_1}}, h_{A_{i_2}}, ..., h_{A_{i_n}}\}$ of $\{h_{A_i}, i \in I\}$, $\tilde{\bigvee}_{k=1}^{n}(h_{A_{i_k}}) \tilde{\neq} \tilde{1}_A$ or $(\tilde{\bigvee}_{k=1}^{n}(h_{A_{i_k}}))^c \tilde{\neq} \tilde{0}_A$ or $\tilde{\bigwedge}_{k=1}^{n}(h_{A_{i_k}})^c \tilde{\neq} \tilde{0}_A$. Thus $\{(h_{A_i})^c, i \in I\}$ is a collection of fuzzy soft closed sets with the finite intersection property. Hence by (2), $\tilde{\bigwedge}_{i\in I}(h_{A_i})^c \tilde{\neq} \tilde{0}_A$ or $(\tilde{\bigvee}_{i\in I}(h_{A_i}))^c \tilde{\neq} \tilde{0}_A$ or $\tilde{\bigvee}_{i\in I}(h_{A_i}) \tilde{\neq} \tilde{1}_A$. This contradicts the fact that $\{h_{A_i}, i \in I\}$ is fuzzy soft open cover of (X, τ, A). Thus $\{h_{A_i}, i \in I\}$ has finite fuzzy soft subcover of (X, τ, A). Therefore, (X, τ, A) is fuzzy soft compact. This completes the proof.

Theorem 3.14. *Let (X, τ_1, A) be a fuzzy soft compact fuzzy soft topological space and (Y, τ_2, B) be a fuzzy soft topological spaces. Suppose fuzzy soft function $f_{pu} : F(X, A) \to F(Y, B)$ be fuzzy soft pu-continuous, fuzzy soft surjective function form fuzzy soft classes $F(X, A)$ of (X, τ, A) to the fuzzy soft classes $F(Y, B)$ of (Y, τ_2, B). Then (Y, τ_2, B) is fuzzy soft compact.*

Proof. Let $\tilde{\Psi} \tilde{=} \{g_{A_i}, i \in I\}$ be a fuzzy soft open cover of (Y, τ_2, B). Since f_{pu} is fuzzy soft pu-continuous, the family of all fuzzy soft sets of the form $\{h_{A_i} : h_{A_i} \tilde{=} f_{pu}^{-1}(g_{A_i}), i \in I\}$, is a fuzzy soft open cover of (X, τ_1, A). Since (X, τ_1, A) is fuzzy soft compact, then there are indices $i_1, i_2, ..., i_n$ such that $\tilde{1}_A \tilde{=} \tilde{\bigvee}_{k=1}^{n} h_{A_{i_k}}$. Now $f_{pu}(\tilde{1}_A) \tilde{=} f_{pu}(\tilde{\bigvee}_{k=1}^{n} h_{A_{i_k}}) \tilde{=} \tilde{\bigvee}_{k=1}^{n} f_{pu}(h_{A_{i_k}}) \tilde{=} \tilde{\bigvee}_{k=1}^{n}(g_{A_{i_k}}) \tilde{\leq} f_{pu}(\tilde{1}_A)$. This follows that $f_{pu}(\tilde{1}_A) \tilde{=} \tilde{\bigvee}_{k=1}^{n}(g_{A_i})$. Thus (Y, τ_2, B) is fuzzy soft compact. Hence the proof.

Definition 3.15 ([28])**.** A fuzzy soft topological space (X, τ, A) is said to be fuzzy soft normal, if for any two fuzzy soft closed set f_A and g_A in (X, τ, A) with $f_A \tilde{\wedge} g_A \tilde{=} \tilde{0}_A$, there are fuzzy soft open sets h_A and k_A such that $f_A \tilde{\leq} h_A$, $g_A \tilde{\leq} k_A$ and $f_A \tilde{\wedge} g_A \tilde{=} \tilde{0}_A$.

Theorem 3.16. *Let (X, τ, A) be a fuzzy soft topological space. If (X, τ, A) is a fuzzy soft compact and fuzzy soft T_2 space then it is fuzzy soft normal.*

Proof. Let (X, τ, A) be a fuzzy soft compact and fuzzy soft T_2-space. Let f_A and k_A are any two fuzzy soft closed and fuzzy soft disjoint subsets of (X, τ, A). Then by Theorem 3.10, for any $e(g_A) \tilde{\in} f_A$ there exist two fuzzy soft open sets h_A and s_A such that $e(g_A) \tilde{\in} h_A$, $k_A \tilde{\leq} s_A$ and $h_A \tilde{\wedge} s_A \tilde{=} 0_A$. The collection $\{(h_{e(g_A)})_A : e(g_A) \tilde{\in} f_A\}$ forms fuzzy soft open cover for f_A. Since f_A is fuzzy soft closed, then by Theorem 3.7, f_A is fuzzy soft compact. So there exists finite fuzzy soft subcover of f_A. That is, $f_A \tilde{\leq} \tilde{\bigvee}_{i=1}^{n}(h_{e(g_A)})_{A_i}$. Let $l_A \tilde{=} \tilde{\bigvee}_{i=1}^{n}(h_{e(g_A)})_{A_i}$, $m_A \tilde{=} \tilde{\bigwedge}_{i=1}^{n}(s_{e(g_A)})_{A_i}$. Then $f_A \tilde{\leq} l_A$, $k_A \tilde{\leq} m_A$ and $l_A \tilde{\wedge} m_A \tilde{=} \tilde{0}_A$. Hence (X, τ, A) is fuzzy soft normal. This completes the proof.

4. Fuzzy soft locally compact spaces

Definition 4.1. Let (X, τ, A) be a fuzzy soft topological space and f_A be any fuzzy soft set in (X, τ, A). Then f_A is said to be fuzzy soft locally compact, if for any fuzzy soft point $e(g_A)\tilde{\in}f_A$, there exist a fuzzy soft compact set h_A and a fuzzy soft open set k_A such that $e(g_A)\tilde{\in}k_A\tilde{\leq}h_A$. That is, each fuzzy soft point $e(g_A)\tilde{\in}f_A$ has a fuzzy soft compact nbd.

If we take $\tilde{1}_A$ instead of f_A, then (X, τ, A) is called fuzzy soft locally compact.

Example 4.2. Let X be any set, A be the set of any parameters and τ_D is a fuzzy soft discrete topology on X, then (X, τ_D, A) is a fuzzy soft discrete topological space over X. Clearly (X, τ_D, A) is fuzzy soft locally compact space. Because, if we take any fuzzy soft point $e(f_A)$ in (X, τ_D, A), $k_A\tilde{=}\{e(f_A)\}$ and $g_A\tilde{=}\{e(f_A)\}$. Then k_A is fuzzy soft compact, g_A is fuzzy soft open and $e(f_A)\tilde{\in}g_A\tilde{\leq}k_A$. Hence (X, τ_D, A) is fuzzy soft locally compact.

Example 4.3. Let X be any set, A be the set of any parameters and τ_I is a fuzzy soft indiscrete topology on X, then (X, τ_I, A) is a fuzzy soft indiscrete topological space over X. Clearly (X, τ_I, A) is fuzzy soft locally connected space. Because, if we take any fuzzy soft point $e(f_A)$ in (X, τ_I, A), $k_A\tilde{=}\tilde{1}_A$ and $g_A\tilde{=}\tilde{1}_A$. Then k_A is fuzzy soft compact, g_A is fuzzy soft open and $e(f_A)\tilde{\in}g_A\tilde{\leq}k_A$. Hence (X, τ_I, A) is fuzzy soft locally compact.

Theorem 4.4. *Let (X, τ, A) be a fuzzy soft locally compact fuzzy soft space. If f_A be a fuzzy soft closed subset of (X, τ, A). Then f_A is fuzzy soft locally compact.*

Proof. Let f_A be a fuzzy soft closed set in a fuzzy soft locally compact space (X, τ, A). We show that f_A is fuzzy soft locally compact. Let us take fuzzy soft point $e(g_A)\tilde{\in}f_A$. Then $e(g_A)$ is also fuzzy soft point in (X, τ, A). Since (X, τ, A) is fuzzy soft locally compact, then there exists a fuzzy soft compact set k_A and fuzzy soft open set s_A such that $e(g_A)\tilde{\in}s_A\tilde{\leq}k_A$. Take $h_A\tilde{=}f_A\tilde{\wedge}k_A$ and $v_A\tilde{=}f_A\tilde{\wedge}s_A$. Then $e(g_A)\tilde{\in}v_A\tilde{\leq}h_A$. Since f_A is fuzzy soft closed, $f_A\tilde{\wedge}k_A$ is fuzzy soft closed subset of fuzzy soft compact set k_A and thus $f_A\tilde{\wedge}k_A$ is fuzzy soft compact in k_A. Hence $h_A\tilde{=}f_A\tilde{\wedge}k_A$ is also fuzzy soft compact in f_A. Clearly, v_A is fuzzy soft open in f_A, since s_A is fuzzy soft open in (X, τ, A). Thus f_A is fuzzy soft locally compact. This completes the proof.

Theorem 4.5. *Let (X, τ_1, A) and (Y, τ_2, B) are two fuzzy soft topological spaces and $u : X \to Y$ and $p : A \to B$ be mappings such that (X, τ_1, A) be a fuzzy soft locally compact space. If $f_{pu} : F(X, A) \to F(Y, B)$ be fuzzy soft pu-continuous, fuzzy soft pu-open and fuzzy soft surjective then (Y, τ_2, B) is fuzzy soft locally compact.*

Proof. To show that (Y, τ_2, B) is fuzzy soft locally compact, let us take fuzzy soft point $e(g_B)$ in (Y, τ_2, B). Since f_{pu} is fuzzy soft surjective, then there

exist a fuzzy soft point $e(h_A)$ in (X, τ_1, A) such that $e(g_B) \tilde{=} f_{pu}(e(h_A))$. Also (X, τ_1, A) is fuzzy soft locally compact implies that there exists a fuzzy soft compact set k_A in (X, τ_1, A) and fuzzy soft open set v_A in (X, τ_1, A) such that $e(h_A) \tilde{\in} v_A \tilde{\leq} k_A$. This implies that $e(g_B) \tilde{=} f_{pu}(e(h_A)) \tilde{\in} f_{pu}(v_A) \tilde{\leq} f_{pu}(k_A)$. Further, $f_{pu}(v_A)$ is fuzzy soft open, since v_A is fuzzy soft open in (X, τ_1, A) and f_{pu} is fuzzy soft pu-open. Moreover, $f_{pu}(k_A)$ is fuzzy soft compact in (Y, τ_2, B), since k_A is fuzzy soft compact in (X, τ_1, A) and f_{pu} is fuzzy soft pu-continuous. Thus (Y, τ_2, B) is fuzzy soft locally compact. Hence the proof.

Theorem 4.6. *Let (X, τ, A) be a fuzzy soft locally compact and fuzzy soft T_2 fuzzy soft space. Then for any fuzzy soft point $e(f_A)$ in (X, τ, A), there exists a fuzzy soft open set g_A such that $e(f_A) \tilde{\in} g_A$ and $\overline{g_A}$ is fuzzy soft compact.*

Proof. Let $e(f_A)$ be a fuzzy soft point in (X, τ, A). Since (X, τ, A) is fuzzy soft locally compact, then there exists a fuzzy soft compact set k_A and a fuzzy soft open set g_A such that $e(f_A) \tilde{\in} g_A \tilde{\leq} k_A$. Now we prove that $\overline{g_A}$ is fuzzy soft compact. Since k_A is a fuzzy soft compact subset of fuzzy soft T_2 space (X, τ, A), then by Theorem 3.11, k_A is fuzzy soft closed. Thus $\overline{g_A} \tilde{\leq} \overline{k_A} \tilde{=} k_A$. Hence by Theorem 3.7, $\overline{g_A}$ being a fuzzy closed subset of a fuzzy soft compact set k_A is fuzzy soft compact. This completes the proof.

Theorem 4.7. *Let (X, τ, A) be a fuzzy soft locally compact, fuzzy soft T_2 space and h_A be a fuzzy soft subset in (X, τ, A). If h_A is fuzzy soft open, then h_A is fuzzy soft locally compact.*

Proof. Let $e(f_A)$ be a fuzzy soft point in h_A. By above Theorem 4.6, there exists a fuzzy soft open set g_A such that $e(f_A) \tilde{\in} g_A$ and $\overline{g_A}$ is fuzzy soft compact. Take $v_A \tilde{=} g_A \tilde{\wedge} h_A$ and $k_A \tilde{=} \overline{g_A} \tilde{\wedge} h_A$. Then $e(f_A) \tilde{\in} v_A \tilde{\leq} k_A$. Clearly, v_A is fuzzy soft open in h_A and k_A is fuzzy soft compact in h_A. Therefore, h_A is fuzzy soft locally compact. Hence the proof.

Now we give the characterization of fuzzy soft locally compact spaces.

Theorem 4.8. *Let (X, τ, A) be a fuzzy soft T_2 fuzzy soft space. Then the following statements are equivalent:*

(1) *(X, τ, A) is fuzzy soft locally compact.*

(2) *For any fuzzy soft compact set k_A in (X, τ, A), there exists a fuzzy soft open set v_A such that $k_A \tilde{\leq} v_A$ and $\overline{v_A}$ is fuzzy soft compact.*

Proof. (1) $\Rightarrow$ (2) Given (X, τ, A) is fuzzy soft locally compact and fuzzy soft T_2 fuzzy soft space. Then by Theorem 4.6, for any fuzzy soft point $e(f_A)$ in (X, τ, A), there exists a fuzzy soft open set $(v_{e(f_A)})_A$ such that $e(f_A) \tilde{\in} (v_{e(f_A)})_A$ and $\overline{(v_{e(f_A)})_A}$ is fuzzy soft compact. Since k_A is fuzzy soft compact, then the fuzzy soft open cover $\{(v_{e(f_A)})_A : e(f_A) \tilde{\in} k_A\}$ has a finite fuzzy soft subcover $\{(v_{e(f_A)})_{A_1}, (v_{e(f_A)})_{A_2}, ..., (v_{e(f_A)})_{A_n}\}$ (say). Let $v_A \tilde{=} \tilde{\bigvee}_{i=1}^{n} (v_{e(f_A)})_{A_i}$. Clearly, v_A is fuzzy soft open, $k_A \tilde{\leq} v_A$ and $\overline{v_A}$ is fuzzy soft compact.

(2) $\Rightarrow$ (1) For any fuzzy soft point $e(f_A)$ in (X, τ, A), take $k_A \tilde{=} \{e(f_A)\}$. Then there exists a fuzzy soft open set v_A such that $\{e(f_A)\} \tilde{\leq} v_A$ with $\overline{v_A}$ is fuzzy soft compact. Hence (X, τ, A) is fuzzy soft locally compact. This completes the proof.

References

[1] B. Ahmad, A. Kharal, *Mappings on fuzzy soft classes,* Advances in Fuzzy Systems, (2009), Article ID 407890, 6 Pages.

[2] B. Ahmad, S. Hussain, *On some structures of soft topology*, Mathematical Sciences, 6 (2012), 7 Pages.

[3] H. Aktas, N. Cagman, *Soft sets and soft groups,* Information Science, 177 (2007), 2726-2735.

[4] C. L. Chang, *Fuzzy topological spaces,* Journal of Mathematical Analysis and Applications, 24(1968), 182-190.

[5] B. Chen, *Soft semi-open sets and related properties in soft topological spaces,* Applied Mathematics Information Sciences, 7 (2013), 287-294.

[6] S. Du, Q. Qin, Q. Wang, and B. Li, *Fuzzy description of topological relations I: a unified fuzzy 9-intersection model,* Proceedings of the 1st International Conference on Advances in Natural Computation (ICNC –05), vol. 3612 of Lecture Notes in Computer Science, 1261-1273, Changsha, China, August 2005.

[7] F. Feng, Y. M. Li, N. Cagman, *Generalized uni-int decision making scheme based on choice value soft sets*, Europeon Journal of Operaion Research, 220 (2012), 162-170.

[8] F. Feng, M. Akram, B. Davvaz, V. Leoreanu-Fotea, *Attribute analysis of information systems based on elementry soft implications*, Knowledge Based Systems, 70 (2014), 281-292.

[9] F. Feng, W. Pedrycz, *On scalar products and decomposition theorems of fuzzy soft sets*, Journal of Multi-valued Logic and Soft Computing, 25 (2015), 45-80.

[10] F. Feng, Y. B. Jun, X. Y. Liu, L. F. Li, *An adjustable approach to fuzzy soft set based decision making*, J. Comput. Appl. Math., 234 (2009), 10-20.

[11] F. Feng, X. Y. Liu, *Soft rough sets with applications to demand analysis*, Int. Workshop Intell. Syst. Appl. (ISA 2009), (2009), 1-4.

[12] C. Gundaz, S. Bayramov, *Some results on fuzzy soft topological spaces*, Mathematical Problems in Engineering, (2013), Article ID 935308, 10 Pages.

[13] T. Herawan, M. M. Deris, *On multi-soft sets construction in information systems,* In: Emerging Intelligent Computing Technology and Applications with aspects of Artificial Intelligence, 101-110. Springer, Berlin, 2009.

[14] T. Herawan, A. N. M. Rose, M. M. Deris, *Soft set theoretic approach for dimensionality reduction.*, In: Database Theory and Application, 171-178, Springer, Berlin, 2009.

[15] S. Hussain, *On some soft functions*, Mathematical Science Letters, 4 (2015), 55-61.

[16] S. Hussain, *Properties of soft semi-open and soft semi-closed sets*, Pensee Journal, 76 (2014), 133-143.

[17] S. Hussain, *On properties of fuzzy soft locally connected spaces*, Hacettepe Journal of Mathematics and Statistics, 47 (2018), 589-599.

[18] S. Hussain, *On some generalized structures in fuzzy soft topological spaces*, Information Science Letters, 4 (2015), 107-115.

[19] S. Hussain, *On some generalized soft mappings*, Hacettepe Journal of Mathematics and Statistics, 45 (2016), 743-754.

[20] S. Hussain, *On weak and strong forms of fuzzy soft open sets,* Fuzzy Information and Engineering, 8 (2016), 451-463.

[21] S. Hussain: *A note on soft connectedness,* Journal of Egyptian Mathematical Society, 23 (2015), 6-11.

[22] S. Hussain, B. Ahmad, *Some properties of soft topological spaces,* Computers and Mathematics with Applications, 62 (2011), 4058-4067.

[23] S. Hussain, B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacettepe Journal of Mathematics and Statistics, 44 (2015), 559-568.

[24] Y. K. Kim, W. K. Min, *Full soft sets and full soft decision systems*, J. Intell. Fuzzy Syst., 26 (2014), 925-933.

[25] Z. Kong, L. Q. Gao, L. F. Wong, *Comment on a fuzzy soft set theoretic approach to decision making problems*, J. Comp. Appl. Math., 223 (2009), 540-542.

[26] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Computers and Mathematics with Applications, 45 (2003), 555-562.

[27] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy soft sets*, J. Fuzzy Maths., 9 (2001), 589-602.

[28] J. Mahanta, P. K. Das, *Results on fuzzy soft topological spaces,* Arxiv: 1203.0634v1 [math.GM] 3 Mar. 2012.

[29] D. Molodtsov, *Soft set theory first results,* Computers and Mathematics with Applications, 37 (4-5) (1999), 19-31.

[30] D. Molodtsov, V.Y. Leonov, D.V. Kovkov, *Soft sets technique and its application*, Nechetkie Sistemy i Myagkie Vychisleniya, 9 (2006), 28-39.

[31] M. Mushrif, S. Sengupta, A. K. Ray, *Texture classification using a novel, soft set theory based classification algorithm,* Springer Berlin, Heidelberg , 254-264, 2006.

[32] D. Pie, D. Miao, *From soft sets to information systems,* X. Hu, Q. Liu, A. Skowron, Y. Y. Lin, R. R. Yager, B. Zhang (Eds.) Proceedings of Granular Computing, Vol. 2, IEEE (2005), 2005, 617-621.

[33] A. R. Roy, P. K. Maji, *A fuzzy soft set theoretic approach to decision making problems*, J. Comput. Appl. Math., 203 (2007), 412-418.

[34] M. Shabir, M. Naz, *On soft topological spaces*, Computers and Mathematics with Applications, 61 (2011), 1786-1799.

[35] B. Tanay and M. B. Kandemir, *Topological structure of fuzzy soft sets*, Computers and Mathematics with Applications, 61 (2011), 2952-2957.

[36] B. P. Varol and H. Aygun, *Fuzzy soft topology,* Hacettepe Journal of Mathematics and Statistics, 41 (2012), 407-419.

[37] Z. Xiao, K. Gong, Y.Zou, *A combined forecasting approach based on fuzzy soft sets*, J. Comput. Appl. Math., 228 (2009), 326-333.

[38] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1996), 338-353.

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Fuzzy sets in fully UP-semigroups

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Abstract. In this paper, we introduce several types of subsets and of fuzzy sets of fully UP-semigroups, and investigate the algebraic properties of fuzzy sets under the operations of intersection and union. Further, we discuss the relation between t-characteristic fuzzy sets and UP_s-subalgebras (resp., UP_i-subalgebras, UP_s-filters, UP_i-filters, UP_s-ideals, UP_i-ideals, strongly UP_s-ideals and strongly UP_i-ideals).
Keywords: fully UP-semigroup, fuzzy set, t-characteristic fuzzy set.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [7], BCI-algebras [8], B-algebras [17], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iseki [8] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iseki [7, 8] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Several researches introduced a new class of algebras related to logical algebras and semigroups such as: In 1993, Jun et al. [10] introduced the notion of BCI-semigroups. In 1998, Jun et al. [14] renamed the BCI-semigroup as the IS-algebra. In 2006, Kim [15] introduced the notion of KS-semigroups. In 2015, Endam and Vilela [2] introduced the notion of JB-semigroups. In 2018, Iampan [5] introduced the notion of fully UP-semigroups.

The concept of a fuzzy subset of a set was first considered by Zadeh [23] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [23], several researches were conducted on

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the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 1998, Jun et al. [11] applied the notion of fuzzy sets to BCI-semigroups (it was renamed as an IS-algebra for the convenience of study), and introduced the concept of fuzzy I-ideals. In 2000, Roh et al. [18] considered the fuzzification of an associative I-ideal of an IS-algebra. They proved that every fuzzy associative I-ideal is a fuzzy I-ideal. By giving an appropriate example, they verified that a fuzzy I-ideal may not be a fuzzy associative I-ideal. They gave a condition for a fuzzy I-ideal to be a fuzzy associative I-ideal, and they investigated some related properties. In 2003, Jun and Kondo [12] proved that some concepts of BCK/BCI-algebras expressed by a certain formula can be naturally extended to the fuzzy setting and that many results are obtained immediately with the use of our method. Moreover They proved that these results can be extended to fuzzy IS-algebras. In 2003, Jianming and Dajing [9] introduced the concept of intuitionistic fuzzy associative I-ideals of IS-algebras and they investigated some related properties. In 2007, Prince Williams and Husain [22] studied fuzzy KS-semigroups. In 2016, Endam and Manahon [1] introduced the notion of fuzzy JB-semigroups and they investigated some of its properties.

In this paper, we introduce several types of subsets and of fuzzy sets of fully UP-semigroups, and investigate the algebraic properties of fuzzy sets under the operations of intersection and union. Further, we discuss the relation between t-characteristic fuzzy sets and $\mathrm{UP_s}$-subalgebras (resp., $\mathrm{UP_i}$-subalgebras, $\mathrm{UP_s}$-filters, $\mathrm{UP_i}$-filters, $\mathrm{UP_s}$-ideals, $\mathrm{UP_i}$-ideals, strongly $\mathrm{UP_s}$-ideals and strongly $\mathrm{UP_i}$-ideals).

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1 ([4])**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

Example 1.2 ([20])**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω.

Example 1.3 ([20])**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω.

In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is the power UP-algebra of type 1 and $(\mathcal{P}(X), *, X)$ is the power UP-algebra of type 2.

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [4, 6]).

(1.1) $(\forall x \in A)(x \cdot x = 0)$,

(1.2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(1.3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(1.4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(1.5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(1.6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(1.7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(1.8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(1.9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(1.10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(1.11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(1.12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(1.13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

Definition 1.4 ([3, 4, 21])**.** A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if for any $x, y \in S$, $x \cdot y \in S$.

(2) a *UP-filter* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y \in A$, $x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(3) a *UP-ideal* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y, z \in A$, $x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(4) a *strongly UP-ideal* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and

(ii) for any $x, y, z \in A$, $(z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [3] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

Definition 1.5. A nonempty subset S of a semigroup $(A, *)$ is called

(1) a *subsemigroup* of A if for any $x, y \in S$, $x * y \in S$.

(2) an *ideal* of A if for any $x \in A$ and $s \in S$, $x * s, s * x \in S$.

Clearly, an ideal is a subsemigroup.

Definition 1.6 ([5])**.** Let A be a nonempty set, $\cdot$ and $*$ are binary operations on A, and 0 is a fixed element of A (i.e., a nullary operation). An algebra $A = (A, \cdot, *, 0)$ of type $(2, 2, 0)$ in which $(A, \cdot, 0)$ is a UP-algebra and $(A, *)$ is a semigroup is called a *fully UP-semigroup* (in short, an *f-UP-semigroup*) if the operation "$*$" is distributive (on both sides) over the operation "$\cdot$".

Definition 1.7 ([23])**.** A *fuzzy set* F in a nonempty set U (or a *fuzzy subset* of U) is described by its membership function $\mathrm{f_F}$. To every point $x \in U$, this function associates a real number $\mathrm{f_F}(x)$ in the interval $[0, 1]$. The number $\mathrm{f_F}(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F, that is, $\mathrm{F} := \{(u, \mathrm{f_F}(u)) \mid u \in U\}$. If $A \subseteq U$ and $t \in (0, 1]$, the *t-characteristic function* [13] χ_A^t of U is a function of U into $\{0, t\}$ defined as follows:

$$\chi_A^t(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

By the definition of t-characteristic function, χ_A^t is a function of U into $\{0, t\} \subset [0, 1]$. We denote the fuzzy set F_A^t in U is described by its membership function χ_A^t, is called the *t-characteristic fuzzy set* of A in U. We say that a fuzzy set F in U is *constant* if its membership function $\mathrm{f_F}$ is constant.

Definition 1.8 ([16])**.** Let $\{\mathrm{F}_i\}_{i \in I}$ be a nonempty family of fuzzy sets in a nonempty set U where I is an arbitrary index set. The *intersection* of F_i, denoted by $\bigwedge_{i \in I} \mathrm{F}_i$, is described by its membership function $\mathrm{f}_{\bigwedge_{i \in I} \mathrm{F}_i}$ which defined as follows:

$$\mathrm{f}_{\bigwedge_{i \in I} \mathrm{F}_i}(x) = \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i \in I} \text{ for all } x \in U.$$

The *union* of F_i, denoted by $\bigvee_{i \in I} \mathrm{F}_i$, is described by its membership function $\mathrm{f}_{\bigvee_{i \in I} \mathrm{F}_i}$ which defined as follows:

$$\mathrm{f}_{\bigvee_{i\in I}\mathrm{F}_i}(x) = \sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \text{ for all } x \in U.$$

Lemma 1.9. *Let S be a nonempty subset of a UP-algebra $(A, \cdot, 0)$ and $t \in (0, 1]$. Then the constant 0 of A is in S if and only if $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$.*

Proof. Assume that $0 \in S$. Then for all $x \in A$, $\chi_S^t(0) = t \geq \chi_S^t(x)$.

Conversely, assume that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Since S is a nonempty subset of A, we have an element a in S, that is, $\chi_S^t(a) = t$. Thus $t \geq \chi_S^t(0) \geq \chi_S^t(a) = t$. So $\chi_S^t(0) = t$, that is, $0 \in S$. □

Rosenfeld [19] introduced the notion of fuzzy subsemigroups (resp., fuzzy ideals) of semigroups as follows:

Definition 1.10. A fuzzy set F in a semigroup $A = (A, *)$ is called

(1) a *fuzzy subsemigroup* of A if for any $x, y \in A$,

$$\mathrm{f}_{\mathrm{F}}(x * y) \geq \min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\}.$$

(2) a *fuzzy ideal* of A if for any $x, y \in A$,

$$\mathrm{f}_{\mathrm{F}}(x * y) \geq \max\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\}.$$

Clearly, a fuzzy ideal is a fuzzy subsemigroup.

Theorem 1.11. *Let S be a nonempty subset of a semigroup $A = (A, *)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a subsemigroup of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy subsemigroup of A, and

(2) S is an ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy ideal of A.

Proof. (1) Assume that S is a subsemigroup of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is a subsemigroup of A, we have $x * y \in S$ and so $\chi_S^t(x * y) = t$. Therefore, $\chi_S^t(x * y) = t \geq t = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S^t(x) = 0$ or $\chi_S^t(y) = 0$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = 0$. Therefore, $\chi_S^t(x * y) \geq 0 = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy subsemigroup of A.

Conversely, assume that F_S^t is a fuzzy subsemigroup of A. Let $x, y \in S$. Then $\chi_S^t(y) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy subsemigroup of A, we have $t \geq \chi_S^t(x * y) \geq \min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x * y) = t$, that is, $x * y \in S$. Hence, S is a subsemigroup of A.

(2) Assume that S is an ideal of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\max\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is an ideal of A, we have $x * y \in S$ and so $\chi_S^t(x * y) = t$. Therefore, $\chi_S^t(x * y) = t \geq t = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x*y \in S$, then $\chi_S^t(x*y) = t$. Therefore, $\chi_S^t(x*y) = t \geq \max\{\chi_S^t(x), \chi_S^t(y)\}$. If $x * y \notin S$, then $x, y \notin S$. Thus $\chi_S^t(x * y) = 0$ and $\chi_S^t(x) = 0 = \chi_S^t(y)$. Therefore, $\chi_S^t(x * y) = 0 \geq 0 = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy ideal of A.

Conversely, assume that F_S^t is a fuzzy ideal of A. Let $s \in S$ and $x \in A$. Then $\chi_S^t(s) = t$, so $\max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Since F_S^t is a fuzzy ideal of A, we have $t \geq \chi_S^t(s*x), \chi_S^t(x*s) \geq \max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Thus $\chi_S^t(s*x) = t = \chi_S^t(x*s)$, that is $s * x, x * s \in S$. Hence, S is an ideal of A. □

Somjanta et al. [21] and Guntasow et al. [3] introduced the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) of UP-algebras as follows:

Definition 1.12. A fuzzy set F in a UP-algebra $A = (A, \cdot, 0)$ is called

(1) a *fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}.$$

(2) a *fuzzy UP-filter* of A if for any $x, y \in A$,

(i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and

(ii) $\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\}$.

(3) a *fuzzy UP-ideal* of A if for any $x, y, z \in A$,

(i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and

(ii) $\mathrm{f_F}(x \cdot z) \geq \min\{\mathrm{f_F}(x \cdot (y \cdot z)), \mathrm{f_F}(y)\}$.

(4) a *fuzzy strongly UP-ideal* of A if for any $x, y, z \in A$,

(i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and

(ii) $\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((z \cdot y) \cdot (z \cdot x)), \mathrm{f_F}(y)\}$.

Guntasow et al. [3] also proved that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Theorem 1.13 ([3])**.** *Fuzzy strongly UP-ideals and constant fuzzy sets coincide in UP-algebras.*

Lemma 1.14. *Let* F *be a fuzzy UP-filter of a UP-algebra* $A = (A, \cdot, 0)$*. Then for any* $x, y \in A$*,*

$$x \leq y \text{ implies } \mathrm{f_F}(x) \leq \mathrm{f_F}(y) \leq \mathrm{f_F}(x \cdot y).$$

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$, so

$$\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(x)\} = \mathrm{f_F}(x).$$

By (1.5), we have $y \leq x \cdot y$ and thus $\mathrm{f_F}(y) \leq \mathrm{f_F}(x \cdot y)$. □

Theorem 1.15. *Let S be a nonempty subset of a UP-algebra $A = (A, \cdot, 0)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a UP-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-subalgebra of A,

(2) S is a UP-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-filter of A,

(3) S is a UP-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-ideal of A, and

(4) S is a strongly UP-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly UP-ideal of A.

Proof. (1) Assume that S is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\chi_S^t(x \cdot y) = t$. Therefore, $\chi_S^t(x \cdot y) = t \geq t = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S^t(x) = 0$ or $\chi_S^t(y) = 0$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = 0$. Therefore, $\chi_S^t(x \cdot y) \geq 0 = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy UP-subalgebra of A.

Conversely, assume that F_S^t is a fuzzy UP-subalgebra of A. Let $x, y \in S$. Then $\chi_S^t(y) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy UP-subalgebra of A, we have $t \geq \chi_S^t(x \cdot y) \geq \min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x \cdot y) = t$, that is, $x \cdot y \in S$. Hence, S is a UP-subalgebra of A.

(2) Assume that S is a UP-filter of A. Since $0 \in S$, it follows from Lemma 1.9 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$. Thus $\chi_S^t(y) = t \geq \chi_S^t(x \cdot y) = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x \notin S$, then $\chi_S^t(x) = 0$. Thus $\chi_S^t(y) \geq 0 = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$. If $y \notin S$, then $\chi_S^t(y) = 0$. Since S is a UP-filter of A, we have $x \cdot y \notin S$ or $x \notin S$ and so $\chi_S^t(x \cdot y) = 0$ or $\chi_S^t(x) = 0$. Thus $\chi_S^t(y) = 0 \geq 0 = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$.

Hence, F_S^t is a fuzzy UP-filter of A.

Conversely, assume that F_S^t is a fuzzy UP-filter of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 1.9 that $0 \in S$. Next, let $x, y \in A$ be such that $x \cdot y \in S$ and $x \in S$. Then $\chi_S^t(x \cdot y) = t = \chi_S^t(x)$, so $\min\{\chi_S^t(x \cdot y), \chi_S^t(x)\} = t$.

Since F_S^t is a fuzzy UP-filter of A, we have $t \geq \chi_S^t(y) \geq \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\} = t$. Thus $\chi_S^t(y) = t$, that is, $y \in S$. Hence, S is a UP-filter of A.

(3) Assume that S is a UP-ideal of A. Since $0 \in S$, it follows from Lemma 1.9 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z), y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Since S is a UP-ideal of A, we have $x \cdot z \in S$ and so $\chi_S^t(x \cdot z) = t$. Thus $\chi_S^t(x \cdot z) = t \geq t = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Case 2: $x \cdot (y \cdot z) \notin S$ or $y \notin S$. Then $\chi_S^t(x \cdot (y \cdot z)) = 0$ or $\chi_S^t(y) = 0$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = 0$. Thus $\chi_S^t(x \cdot z) \geq 0 = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy UP-ideal of A.

Conversely, assume that F_S^t is a fuzzy UP-ideal of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 1.9 that $0 \in S$. Next, let $x, y, z \in A$ such that $x \cdot (y \cdot z) \in S$ and $y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy UP-ideal of A, we have $t \geq \chi_S^t(x \cdot z) \geq \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x \cdot z) = t$, that is, $x \cdot z \in S$. Hence, S is a UP-ideal of A.

(4) It is straightforward by Theorem 1.13, and A is the only one strongly UP-ideal of itself. □

2. Special subsets of fully UP-semigroups

In this section, we introduce the notions of $\mathrm{UP_s}$-subalgebras, $\mathrm{UP_i}$-subalgebras, $\mathrm{UP_s}$-filters, $\mathrm{UP_i}$-filters, $\mathrm{UP_s}$-ideals, $\mathrm{UP_i}$-ideals, strongly $\mathrm{UP_s}$-ideals, and strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples and prove its generalizations.

From now on, we shall let A be an f-UP-semigroup $A = (A, \cdot, *, 0)$ unless otherwise specified.

Definition 2.1. A subset S of an f-UP-semigroup A is called

(1) a *UP_s-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP_i-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 2.2, 2.8, and 2.13 directly from Definition 1.5.

Theorem 2.2. *Every UP_i-subalgebra of A is a UP_s-subalgebra of A.*

Example 2.3. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	3	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a $\mathrm{UP_s}$-subalgebra of A. Since $1 \in S$ and $3 \in A$ but $3 * 1 = 3 \notin S$, we have S is not an ideal of $(A, *)$. Thus S is not a $\mathrm{UP_i}$-subalgebra of A.

Definition 2.4. A subset S of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is called

(1) a *UP_s-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP_i-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 2.5, 2.7, 2.10, 2.12, 2.15, and 2.17 directly from a result quoted in Definition 1.4.

Theorem 2.5. *Every UP_s-filter of A is a UP_s-subalgebra of A.*

Example 2.6. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 2\}$. Then S is a $\mathrm{UP_s}$-subalgebra of A. Since $2 \cdot 1 = 0 \in S$ and $2 \in S$ but $1 \notin S$, we have S is not a UP-filter of $(A, \cdot, 0)$. Thus S is not a $\mathrm{UP_s}$-filter of A.

Theorem 2.7. *Every UP_i-filter of A is a UP_i-subalgebra of A.*

In Example 2.6, we have S is a $\mathrm{UP_i}$-subalgebra of A. Since S is not a UP-filter of $(A, \cdot, 0)$, we have S is not a $\mathrm{UP_i}$-filter of A.

Theorem 2.8. *Every UP_i-filter of A is a UP_s-filter of A.*

In Example 2.3, we have S is a $\mathrm{UP_s}$-filter of A. Since S is not an ideal of $(A, *)$, we have S is not a $\mathrm{UP_i}$-filter of A.

Definition 2.9. A subset S of an f-UP-semigroup A is called

(1) a *UP*$_{\mathrm{s}}$*-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP*$_{\mathrm{i}}$*-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 2.10. *Every UP*$_{\mathrm{s}}$*-ideal of A is a UP*$_{\mathrm{s}}$*-filter of A.*

Example 2.11. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1\}$. Then S is a UP$_{\mathrm{s}}$-filter of A. Since $2 \cdot (1 \cdot 3) = 0 \in S$ and $1 \in S$ but $2 \cdot 3 = 2 \notin S$, we have S is not a UP-ideal of $(A, \cdot, 0)$. Thus S is not a UP$_{\mathrm{s}}$-ideal of A.

Theorem 2.12. *Every UP*$_{\mathrm{i}}$*-ideal of A is a UP*$_{\mathrm{i}}$*-filter of A.*

In Example 2.11, we have S is a UP$_{\mathrm{i}}$-filter of A. Since S is not a UP-ideal of $(A, \cdot, 0)$, we have S is not a UP$_{\mathrm{i}}$-ideal of A.

Theorem 2.13. *Every UP*$_{\mathrm{i}}$*-ideal of A is a UP*$_{\mathrm{s}}$*-ideal of A.*

In Example 2.3, we have S is a UP$_{\mathrm{s}}$-ideal of A. Since S is not an ideal of $(A, *)$, we have S is not a UP$_{\mathrm{i}}$-ideal of A.

Definition 2.14. A subset S of an f-UP-semigroup A is called

(1) a *strongly UP*$_{\mathrm{s}}$*-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *strongly UP*$_{\mathrm{i}}$*-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 2.15. *Every strongly UP*$_{\mathrm{s}}$*-ideal of A is a UP*$_{\mathrm{s}}$*-ideal of A.*

Example 2.16. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a UP$_{\mathrm{s}}$-ideal of A. Since $S \neq A$, we have S is not a strongly UP-ideal of $(A, \cdot, 0)$. Thus S is not a strongly UP$_{\mathrm{s}}$-ideal of A.

Theorem 2.17. *Every strongly UP_{i}-ideal of A is a UP_{i}-ideal of A.*

In Example 2.16, we have S is a $\mathrm{UP_i}$-ideal of A. Since S is not a strongly UP-ideal of $(A, \cdot, 0)$, we have S is not a strongly $\mathrm{UP_i}$-ideal of A.

Theorem 2.18. *Strongly UP_{s}-ideals and strongly UP_{i}-ideals coincide in A and it is only A.*

Proof. It is straightforward by A is the only one strongly UP-ideal of itself. □

3. Fuzzy sets in fully UP-semigroups

In this section, we introduce the notions of fuzzy $\mathrm{UP_s}$-subalgebras, fuzzy $\mathrm{UP_i}$-subalgebras, fuzzy $\mathrm{UP_s}$-filters, fuzzy $\mathrm{UP_i}$-filters, fuzzy $\mathrm{UP_s}$-ideals, fuzzy $\mathrm{UP_i}$-ideals, fuzzy strongly $\mathrm{UP_s}$-ideals, and fuzzy strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples, prove its generalizations and investigate the algebraic properties of fuzzy sets under the operations of intersection and union.

Definition 3.1. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy UP_{s}-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy UP_{i}-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy $\mathrm{UP_i}$-subalgebra is a fuzzy $\mathrm{UP_s}$-subalgebra.

In Example 2.16, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and } \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-subalgebra of A. Since $\mathrm{f_F}(2 * 3) = \mathrm{f_F}(1) = 0.4 \not\geq 0.5 = \max\{0.5, 0.2\} = \max\{\mathrm{f_F}(2), \mathrm{f_F}(3)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-subalgebra of A.

Theorem 3.2. *The intersection of any nonempty family of fuzzy UP_{s}-subalgebra of A is also a fuzzy UP_{s}-subalgebra of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-subalgebra of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x \cdot y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x \cdot y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\} \text{ and} \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A. □

In Example 2.16, we define two membership functions $\mathrm{f_{F1}}$ and $\mathrm{f_{F2}}$ as follow:

A	0	1	2	3
$\mathrm{f_{F1}}$	0.7	0.5	0.7	0.3
$\mathrm{f_{F2}}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3{*}2) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1) = 0.5 \not\geq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Theorem 3.3. *A nonempty subset S of A is a UP_{s}-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{s}-subalgebra of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (1). □

Theorem 3.4. *The intersection of any nonempty family of fuzzy UP_{i}-subalgebra of A is also a fuzzy UP_{i}-subalgebra of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-subalgebra of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\} \text{ and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x*y)\}_{i\in I}\\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \max\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{F}_i$ is a fuzzy $\mathrm{UP_i}$-subalgebra of A. □

In Example 2.11, we define two membership functions $\mathrm{f_{F1}}$ and $\mathrm{f_{F2}}$ as follow:

A	0	1	2	3
$\mathrm{f_{F_1}}$	0.9	0.7	0.1	0.1
$\mathrm{f_{F_2}}$	0.8	0.4	0.5	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_i}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1{\cdot}3) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2) = 0.5 \not\geq 0.6 = \min\{0.7, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_i}$-subalgebra of A.

Theorem 3.5. *A nonempty subset S of A is a UP_{i}-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{i}-subalgebra of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (1). □

Definition 3.6. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy UP_s-filter* of A if F is a fuzzy UP-filter of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy UP_i-filter* of A if F is a fuzzy UP-filter of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy UP_i-filter is a fuzzy UP_s-filter.

In Example 2.16, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and } \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy UP_s-filter of A. Since$\mathrm{f_F}(2 * 3) = \mathrm{f_F}(1) = 0.4 \not\geq 0.5 = \max\{0.5, 0.2\} = \max\{\mathrm{f_F}(2), \mathrm{f_F}(3)\}$, we have F is not a fuzzy UP_i-filter of A.

Theorem 3.7. *The intersection of any nonempty family of fuzzy UP_s-filter of A is also a fuzzy UP_s-filter of A.*

Proof. Let F_i be a fuzzy UP_s-filter of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I}\\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x),\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot y), \mathrm{f}_{\mathrm{F}i}(x)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x)\},\ \text{and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy UP_s-filter of A. □

In Example 2.16, we define two membership functions $\mathrm{f_{F1}}$ and $\mathrm{f_{F2}}$ as follow:

A	0	1	2	3
$\mathrm{f_{F_1}}$	0.7	0.5	0.7	0.3
$\mathrm{f_{F_2}}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy UP_s-filters of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2 * 3) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1) = 0.5 \not\geq 0.6 = \min\{0.7, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy UP_s-filter of A.

Theorem 3.8. *A nonempty subset S of A is a UP_{s}-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{s}-filter of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (2). □

Theorem 3.9. *The intersection of any nonempty family of fuzzy UP_{i}-filter of A is also a fuzzy UP_{i}-filter of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-filter of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I}\\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x),\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot y), \mathrm{f}_{\mathrm{F}i}(x)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x)\}, \text{ and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x*y)\}_{i\in I}\\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \max\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_i}$-filter of A. □

Example 3.10. Let $A = \{0,1,2,3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

We define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.9	0.9	0.5	0.5
$\mathrm{f}_{\mathrm{F}_2}$	1	0.5	0.6	0.5

Then F_1 and F_2 are fuzzy $\mathrm{UP_i}$-filters of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3) = 0.5 \ngeq 0.6 = \min\{0.9, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2\cdot 3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_i}$-filter of A.

Theorem 3.11. *A nonempty subset S of A is a UP_{i}-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{i}-filter of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (2). □

We have Theorem 3.12, 3.13, 3.19, 3.20, and 3.27 directly from a result quoted in Definition 1.12.

Theorem 3.12. *Every fuzzy UP_s-filter of A is a fuzzy UP_s-subalgebra of A.*

In Example 2.6, we define a membership function f_F as follows:

$$f_F(0) = 1,\ f_F(1) = 0.2,\ f_F(2) = 0.9,\ \text{and } f_F(3) = 0.1.$$

Then F is a fuzzy UP_s-subalgebra of A. Since $f_F(1) = 0.2 \ngeq 0.9 = \min\{1, 0.9\} = \min\{f_F(0), f_F(2)\} = \min\{f_F(2 \cdot 1), f_F(2)\}$, we have F is not a fuzzy UP_s-filter of A.

Theorem 3.13. *Every fuzzy UP_i-filter of A is a fuzzy UP_i-subalgebra of A.*

In Example 2.6, we define a membership function f_F as follows:

$$f_F(0) = 1,\ f_F(1) = 0.2,\ f_F(2) = 0.9,\ \text{and } f_F(3) = 0.1.$$

Then F is a fuzzy UP_i-subalgebra of A. Since $f_F(1) = 0.2 \ngeq 0.9 = \min\{1, 0.9\} = \min\{f_F(0), f_F(2)\} = \min\{f_F(2 \cdot 1), f_F(2)\}$, we have F is not a fuzzy UP_i-filter of A.

Definition 3.14. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy UP_s-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy UP_i-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy UP_i-ideal is a fuzzy UP_s-ideal.

In Example 2.16, we define a membership function f_F as follows:

$$f_F(0) = 1,\ f_F(1) = 0.4,\ f_F(2) = 0.5,\ \text{and } f_F(3) = 0.2.$$

Then F is a fuzzy UP_s-ideal of A. Since $f_F(3 * 2) = f_F(1) = 0.4 \ngeq 0.5 = \max\{0.2, 0.5\} = \max\{f_F(3), f_F(2)\}$, we have F is not a fuzzy UP_i-ideal of A.

Theorem 3.15. *The intersection of any nonempty family of fuzzy UP_s-ideal of A is also a fuzzy UP_s-ideal of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-ideal of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot z)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z)), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z))\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot(y\cdot z)), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}, \text{ and} \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x*y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-ideal of A. □

In Example 2.16, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.7	0.5	0.7	0.3
$\mathrm{f}_{\mathrm{F}_2}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-ideals of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3*2) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-ideal of A.

Theorem 3.16. *A nonempty subset S of A is a $\mathrm{UP_s}$-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_s}$-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (3). □

Theorem 3.17. *The intersection of any nonempty family of fuzzy $\mathrm{UP_i}$-ideal of A is also a fuzzy $\mathrm{UP_i}$-ideal of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-ideal of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot z)\}_{i\in I}
\end{aligned}$$

$$\begin{aligned}
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z)),\mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z))\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot(y\cdot z)),\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\},\ \text{and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x*y)\}_{i\in I}\\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}i}(x),\mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \max\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x),\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_i}$-ideal of A. □

In Example 3.10, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.7	0.3	0.4	0.3
$\mathrm{f}_{\mathrm{F}_2}$	0.8	0.5	0.2	0.2

Then F_1 and F_2 are fuzzy $\mathrm{UP_i}$-ideals of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(0\cdot 3)=\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3)=0.3\ngeq 0.4=\min\{0.4,0.5\}=\min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2),\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1)\}=\min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(0\cdot(1\cdot 3)),\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1)\}$, we have $\mathrm{F}_1\vee\mathrm{F}_2$ is not a fuzzy $\mathrm{UP_i}$-ideal of A.

Theorem 3.18. *A nonempty subset S of A is a UP_{i}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{i}-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (3). □

Theorem 3.19. *Every fuzzy UP_{s}-ideal of A is a fuzzy UP_{s}-filter of A.*

In Example 2.11, we define a membership function f_{F} as follows:

$$\mathrm{f}_{\mathrm{F}}(0)=0.8,\ \mathrm{f}_{\mathrm{F}}(1)=0.6,\ \mathrm{f}_{\mathrm{F}}(2)=0.3,\ \text{and}\ \mathrm{f}_{\mathrm{F}}(3)=0.3.$$

Then F is a fuzzy $\mathrm{UP_s}$-filter of A. Since $\mathrm{f}_{\mathrm{F}}(2\cdot 3)=\mathrm{f}_{\mathrm{F}}(2)=0.3\ngeq 0.6=\min\{0.8,0.6\}=\min\{\mathrm{f}_{\mathrm{F}}(0),\mathrm{f}_{\mathrm{F}}(1)\}=\min\{\mathrm{f}_{\mathrm{F}}(2\cdot(1\cdot 3)),\mathrm{f}_{\mathrm{F}}(1)\}$, we have F is not a fuzzy $\mathrm{UP_s}$-ideal of A.

Theorem 3.20. *Every fuzzy UP_{i}-ideal of A is a fuzzy UP_{i}-filter of A.*

In Example 2.11, we define a membership function f_{F} as follows:

$$\mathrm{f}_{\mathrm{F}}(0)=0.8,\ \mathrm{f}_{\mathrm{F}}(1)=0.6,\ \mathrm{f}_{\mathrm{F}}(2)=0.3,\ \text{and}\ \mathrm{f}_{\mathrm{F}}(3)=0.3.$$

Then F is a fuzzy $\mathrm{UP_i}$-filter of A. Since $\mathrm{f}_{\mathrm{F}}(2\cdot 3)=\mathrm{f}_{\mathrm{F}}(2)=0.3\ngeq 0.6=\max\{0.8,0.6\}=\max\{\mathrm{f}_{\mathrm{F}}(0),\mathrm{f}_{\mathrm{F}}(1)\}=\max\{\mathrm{f}_{\mathrm{F}}(2\cdot(1\cdot 3)),\mathrm{f}_{\mathrm{F}}(1)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-ideal of A.

Definition 3.21. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy strongly UP_{s}-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy strongly UP_{i}-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Theorem 3.22. *Fuzzy strongly UP_{s}-ideals, fuzzy strongly UP_{i}-ideals, and constant fuzzy sets coincide in A.*

Proof. It is straightforward by Theorem 1.13. □

If a fuzzy set F_i is constant for all $i \in I$, then we see that the fuzzy sets $\bigwedge_{i\in I} \mathrm{F}_i$ and $\bigvee_{i\in I} \mathrm{F}_i$ are constant. From this, we have Theorem 3.23 and 3.24.

Theorem 3.23. *The intersection and union of any nonempty family of fuzzy strongly UP_{s}-ideal of A are also a fuzzy strongly UP_{s}-ideal of A.*

Theorem 3.24. *The intersection and union of any nonempty family of fuzzy strongly UP_{i}-ideal of A are also a fuzzy strongly UP_{i}-ideal of A.*

Theorem 3.25. *A nonempty subset S of A is a strongly UP_{s}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly UP_{s}-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (4). □

Theorem 3.26. *A nonempty subset S of A is a strongly UP_{i}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly UP_{i}-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (4). □

Theorem 3.27. *Every fuzzy strongly UP_{s}-ideal (fuzzy strongly UP_{i}-ideal) of A is a fuzzy UP_{s}-ideal and a fuzzy UP_{i}-ideal of A.*

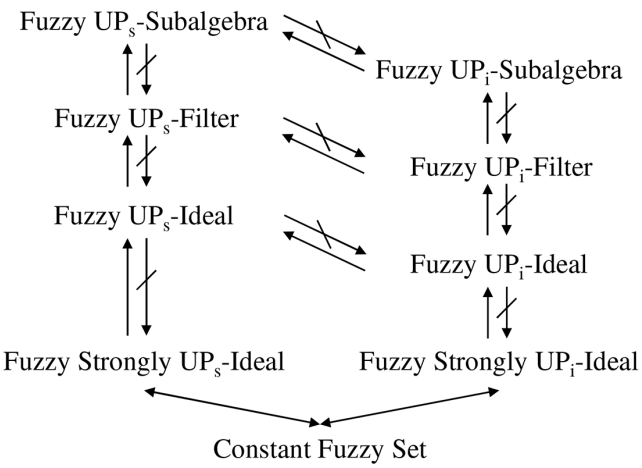
In Example 2.3, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 0.7,\ \mathrm{f_F}(1) = 0.5,\ \mathrm{f_F}(2) = 0.3,\ \text{and } \mathrm{f_F}(3) = 0.6.$$

Then F is a fuzzy $\mathrm{UP_i}$-ideal of A. Since F is not constant, we have F is not a fuzzy strongly $\mathrm{UP_s}$-ideal and a fuzzy strongly $\mathrm{UP_i}$-ideal of A.

4. Conclusions and future works

In this paper, we have introduced the notions of fuzzy $\mathrm{UP_s}$-subalgebras, fuzzy $\mathrm{UP_i}$-subalgebras, fuzzy $\mathrm{UP_s}$-filters, fuzzy $\mathrm{UP_i}$-filters, fuzzy $\mathrm{UP_s}$-ideals, fuzzy $\mathrm{UP_i}$-ideals, fuzzy strongly $\mathrm{UP_s}$-ideals, and fuzzy strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, proved its generalizations and investigated some of its important properties. Then we have the generalization diagram of fuzzy sets in fully UP-semigroups below.



In our future study of fully UP-semigroups, may be the following topics should be considered:

- To get more results in fuzzy sets in fully UP-semigroups.
- To define fuzzy translations in fully UP-semigroups.
- To define fuzzy soft sets over fully UP-semigroups.

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References

[1] J. C. Endam and M. D. Manahon, *On fuzzy JB-semigroups*, Int. Math. Forum, 11 (2016), 379–386.

[2] J. C. Endam and J. P. Vilela, *On JB-semigroups*, Appl. Math. Sci., 9 (2015), 2901–2911.

[3] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, *Fuzzy translations of a fuzzy set in UP-algebras*, J. Indones. Math. Soc., 23 (2017), 1–19.

[4] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Top., 5 (2017), 35–54.

[5] A. Iampan, *Introducing fully UP-semigroups*, Discuss. Math., Gen. Algebra Appl., 38 (2018), 297–306.

[6] A. Iampan, *UP-algebras: the beginning*, Copy House and Printing, Thailand, 2018.

[7] Y. Imai and K. Iseki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy, 42 (1966), 19–22.

[8] K. Iseki, *An algebra related with a propositional calculus*, Proc. Japan Acad., 42 (1966), 26–29.

[9] Z. Jianming and X. Dajing, *Intuitionistic fuzzy associative I-ideals of IS-algebras*, Sci. Math. Jpn. Online, 10 (2004), 93–98.

[10] J. B. Jun, S. M. Hong, and E. H. Roh, *BCI-semigroups*, Honam Math. J., 15 (1993), 59–64.

[11] Y. B. Jun, S. S. Ahn, J. Y. Kim, and H. S. Kim, *Fuzzy I-ideals in BCI-semigroups*, Southeast Asian Bull. Math., 2 (1998), 147–153.

[12] Y. B. Jun and M. Kondo, *On transfer principle of fuzzy BCK/BCI-algebras*, Sci. Math. Jpn. Online, 9 (2003), 95–100.

[13] Y. B. Jun, M. A. Öztürk, and G. Muhiuddin, *A generalization of* $(\in, \in \vee q)$*-fuzzy subgroups*, Int. J. Math. Stat., 5 (2016), 7–18.

[14] Y. B. Jun, X. L. Xin, and E. H. Roh, *A class of algebras related to BCI-algebras and semigroups*, Soochow J. Math., 24 (1998), 309–321.

[15] K. H. Kim, *On structure of KS-semigroups*, Int. Math. Forum, 1 (2006), 67–76.

[16] K. H. Lee, *First course on fuzzy theory and applications*, Springer-Verlag Berlin Heidelberg, 2018.

[17] J. Neggers and H. S. Kim, *On B-algebras*, Mat. Vesnik, 54 (2002), 21–29.

[18] E. H. Roh, Y. B. Jun, and W. H. Shim, *Fuzzy associative I-ideals of IS-algebras*, Int. J. Math. Math. Sci., 24 (2000), 729–735.

[19] A. Rosenfeld, *Fuzzy groups*, J. Math, Anal. Appl., 35 (1971), 512–517.

[20] A. Satirad, P. Mosrijai, and A. Iampan, *Generalized power UP-algebras*, Int. J. Math. Comput. Sci., 14 (2019), 17–25.

[21] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform., 12 (2016), 739–756.

[22] D. R. Prince Williams and S. Husain, *On fuzzy KS-semigroups*, Int. Math. Forum, 2 (2007), 1577–1586.

[23] L. A. Zadeh, *Fuzzy sets*, Inf. Cont., 8 (1965), 338–353.

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Effect of thermal diffusion and chemical reaction on MHD free convective flow past an infinite isothermal vertical plate with heat source

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Abstract. An attempt has been made to study the heat and mass transfer effect on an MHD free convection flow of an electrically conducting viscous fluid past over a moving infinite isothermal vertical plate subject to transverse magnetic field in presence of Soret effect, chemical reaction, heat source and thermal radiation. Exact solutions for velocity field, temperature profile and concentration distribution are obtained using Laplace transformation technique in closed form. Expressions for skin friction, rate of heat transfer and rate of mass transfer are also derived. The effects of different physical parameters on the various fields are studied graphically.

Keywords: heat and mass transfer, thermal radiation, chemical reaction, thermal diffusion and heat source.

Introduction

The phenomenon concerning heat and mass transfer with MHD flow is important due to its numerous applications in science and technology, industrial processes and devices namely MHD accelerator, casting and levitation atmospheric winds etc. Study of thermal radiation effects on MHD flow in fluid mechanics is very important to the engineers showing meaning to this field in our economic growth. Thermal radiation and its effects on MHD flow are major input of researchers in recent past. Prasad et al. (2012) examined the thermal radiation effects on magneto hydrodynamic free convection heat and mass transfer from a sphere in a variable porosity regime. Raju et al. (2014) carried out a analytical study

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of MHD free convection, dissipative boundary layer flow past a porous vertical surface in presence of thermal radiation, chemical reaction and constant suction. Radiation and mass transfer effects on MHD oscillatory flow in a channel filled with porous medium in presence of chemical reaction was investigated by Ibrahim et al. (2015). Bhuvaneswari et al. (2010) reported the exact analysis of radiation convection flow with heat and mass transfer over an inclined plate in a porous medium. Several researchers investigated MHD flow through a fluid saturated porous and non-porous medium near vertical plate considering different aspects of the problem. Relevant studies are due to Raptis et al. (2004), Raptis and Kafoussias (1982), Geindrean and Auriault (2002), Chen (2009), Seth et al.(2011, 2014, 2015, 2017) etc.

It is well known that, the driving potential for the phenomena of transport of mass is not alone concentration gradients, but high temperature gradient also. Mass transfer created by temperature gradient is called as thermal diffusion effect i.e. Soret effect. The Soret effect has been utilized for isotope separation and in mixture between gases of very light molecular weight and medium molecular weight. The importance of this effect in convection transport has been reported by Eckert and Darke (1972). The study of heat source effects on heat transfer is very important because of effects are crucial in controlling the heat transfer and in cooling processes. Turkyilmazoglu and Pop (2012) studied Soret and heat source effects on the unsteady radiative MHD free convection flow from an impulsively started infinite vertical plate. Ahmed (2012b) investigated the effects of heat and mass transfer in Hartmann flow with Soret effect in presence of a constant heat source. Singh (2001) discussed MHD free convection and mass transfer flow with heat source and thermal diffusion. Ibrahim and Suneetha (2015) analyzed chemical reaction and Soret effects on unsteady MHD flow of a viscoelastic fluid past an impulsively started infinite vertical plate with heat Source. Ahmed (2012a) discussed the Soret and radiation effects on transient MHD free convection from an impulsively started infinite vertical plate. Zimmerman et al. (1992) studied convection in two component system with Soret effect. Raptis (1982) considered free convection and mass transfer effects on the oscillatory flow past an infinite moving vertical isothermal plate with constant suction and heat source. Krishnendu Bhattacharyya (2011) investigated the effects of radiation and heat source/sink on unsteady MHD boundary layer flow and heat transfer over a stretching sheet with suction/injection. Chamkha (2000) studied thermal radiation and buoyancy effects on hydromagnetic flow over an accelerating permeable surface with heat source or sink.

The study of heat and mass transfer with chemical reaction is of great practical importance for engineers and scientists because of its almost universal occurrence in many branches of science and engineering. In most of chemical engineering processes chemical reaction occurs between a foreign mass on the fluid. It can be classified as either heterogeneous or homogeneous processes. Possible applications of this type of flow can be found in many industries like power and chemical process industries. Astarita (1967) discussed the mass trans-

fer with chemical reaction. Effects of chemical reaction on a moving isothermal surface with suction were studied by Muthucumaraswamy(2002). Raju et al. (2013) presented unsteady MHD free convection and chemically reactive flow past an infinite vertical porous plate. Chemically reacting MHD boundary layer flow of heat and mass transfer over a moving vertical plate with suction was investigated by Ibrahim and Makinde (2010).

The objective of the present problem is to examine the effect of thermal diffusion and chemical reaction on MHD free convection flow past an infinite isothermal vertical plate with heat source. This present work is a generalization of the work done by Muthucumaraswamy and Sivkumar(2016) to consider the effect of heat source on the flow characteristics.

Mathematical analysis

Consider two dimensional unsteady free convection flow of a viscous incompressible fluid past an infinite isothermal moving vertical plate in presence of thermal diffusion, thermal radiation, chemical reaction and heat source. We take x' axis along the plate in the direction of fluid flow and y' -axis normal to it. A uniform transverse magnetic field of strength B_0 is assumed to be applied in the positive y' direction namely normal to the plate and chemical reaction is also taking place in the flow.

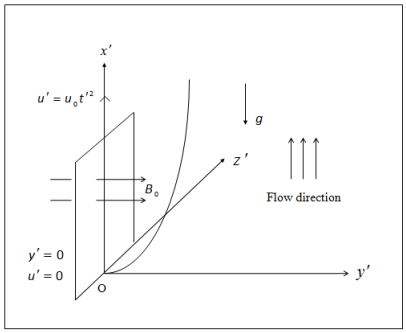


Figure 1: Physical configuration of the problem.

At $t' \leq 0$, the plate and fluid are at rest and at the uniform temperature T'_∞ and uniform species concentration C'_∞. At $t' > 0$, the plate begins to move with a velocity $u' = u_0 t'^2$ against the gravitational force g .In this analysis, the magnetic Reynolds number is taken to be small so that induced magnetic field can be neglected. The fluid is gray, absorbing-emitting radiant but a non scattering medium. Chemical reaction is assumed to be homogeneous and of first order. Under the Boussinesq approximation Equatoins governing the conservation of mass, momentum, energy and concentration can be written

as

$$\frac{\partial v'}{\partial y'} = 0, \tag{1}$$

$$\frac{\partial u'}{\partial t'} = g\beta(T' - T'_\infty) + g\beta_c(C' - C'_\infty) + \nu\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_o^2}{\rho}u', \tag{2}$$

$$\rho C_p\frac{\partial T'}{\partial t'} = \kappa\frac{\partial^2 T'}{\partial y'^2} - \frac{\partial q_r}{\partial y'} - Q'(T' - T'_\infty), \tag{3}$$

$$\frac{\partial C'}{\partial t'} = D_M\frac{\partial^2 C'}{\partial y'^2} - K_1(C' - C'_\infty) + D_T\frac{\partial^2 T'}{\partial y'^2} \tag{4}$$

Initial and boundary conditions are

$$\begin{aligned} &u' = 0, \quad T' = T'_\infty, \quad C' = C'_\infty \quad \text{for all } y',\, t' \le 0 \\ &t' > 0, \quad u' = u_0 t'^2, \quad T' = T'_w, \quad C' = C'_w \text{ at } y' = 0 \\ &u' \to 0, \quad T' \to T'_\infty, \quad C' \to C'_\infty \quad as\ y' \to \infty \end{aligned} \tag{5}$$

The local radiant for the case of an optically thick gray gas is expressed by

$$\frac{\partial q_r}{\partial y'} = -4a^*\sigma^*\left(T'^4_\infty - T'^4\right) \tag{6}$$

We assume that the temperature differences within the flow are sufficiently small and on using Taylors Series to expand T'^4 about T'_∞ and neglecting higher order terms, we derive

$$T'^4 = 4T'^3_\infty T' - 3T'^4_\infty \tag{7}$$

By using (6) and (7); equation (3) reduces to

$$\rho C_p\frac{\partial T'}{\partial t'} = \kappa\frac{\partial^2 T'}{\partial y'^2} + 16a^*\sigma^*T'^3_\infty\left(T'_\infty - T'\right) - Q'(T' - T'_\infty) \tag{8}$$

We introduce the following non-dimensional quantities:

$$\begin{aligned} &u = u'\left(\frac{1}{u_0\nu^2}\right)^{\frac{1}{5}},\ t = \left(\frac{u_0^2}{\nu}\right)^{\frac{1}{5}}t',\ y = y'\left(\frac{u_0}{\nu^3}\right)^{\frac{1}{5}}, \\ &Pr = \frac{\mu C_p}{\kappa},\ Sc = \frac{\nu}{D_M},\ K = K_1\left(\frac{\nu}{u_0^2}\right)^{\frac{1}{5}} \\ &Gr = \frac{g\beta\,(T'_w - T'_\infty)}{\left(\nu u_0^3\right)^{\frac{1}{5}}},\quad Gm = \frac{g\beta_c\,(C'_w - C'_\infty)}{\left(\nu u_0^3\right)^{\frac{1}{5}}}, \end{aligned} \tag{9}$$

$$M = \frac{\sigma B_0^2}{\rho}\left(\frac{\nu}{u_0^2}\right)^{\frac{1}{5}}, \quad \theta = \frac{T' - T'_\infty}{T'_w - T'_\infty}$$
$$\phi = \frac{C' - C'_\infty}{C'_w - C'_\infty}, \quad N = \left(\frac{\nu^3}{u_0}\right)^{\frac{2}{3}} \frac{16a^*\sigma^* T'^3_\infty}{\kappa},$$
$$Sr = \frac{D_T\,(T'_w - T'_\infty)}{\nu\,(C'_w - C'_\infty)}, \quad Q = \frac{Q'\nu^{\frac{1}{5}}}{\rho C_p u_0^{2/5}}.$$

The non-dimensional form of the governing equations (2), (3) and (4) are as follows:

$$\frac{\partial u}{\partial t} = Gr\theta + Gm\,\phi + \frac{\partial^2 u}{\partial y^2} - Mu, \tag{10}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} - \frac{1}{Pr}N\theta - Q\theta, \tag{11}$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{Sc}\frac{\partial^2 \phi}{\partial y^2} - K\phi + Sr\frac{\partial^2 \theta}{\partial y^2}. \tag{12}$$

Corresponding initial and boundary conditions in dimensionless forms become

$$\begin{aligned} &u = 0, \quad \theta = 0, \quad \phi = 0 \qquad \text{for all } y,\ t \le 0,\\ &t > 0, \quad u = t^2, \ \theta = 1, \ \phi = 1 \quad \text{at } y = 0,\\ &u \to 0, \ \theta \to 0, \ \phi \to 0 \qquad \text{as } y \to 0. \end{aligned} \tag{13}$$

Taking Laplace transform of equations (10), (11), (12) and (13) the equations are transformed into 2nd order ordinary differential equation. These equations are further inverted and solved subject to the transformed boundary condition. So that the following solutions for ϕ, $\theta$and u are obtained

$$\theta = f_1\,(Pr, y, A_2, t)\,, \tag{14}$$

$$\begin{aligned} \phi = {} & (1 + A_4)\,f_1\,(Sc, y, K, t) + A_5\,e^{A_3 t} f_1\,(Sc, y, A_3, t)\\ & - A_4\,f_1\,(Pr, y, A_2, t) - A_5\,e^{A_3 t}\,f_1\,(Pr, y, A_{31}, t)\,, \end{aligned} \tag{15}$$

$$\begin{aligned} u = {} & \left\{\left(\frac{y^2 + 4Mt^2}{8M}\right) f_2\,(M, y, t) + \left(\frac{y\,(1 - 4Mt)}{16M^{3/2}}\right)\, f_3\,(M, y, t)\right.\\ & \left. - \left(\frac{y\sqrt{t}}{2M\sqrt{\pi}}\right)\, e^{-\left(\frac{y^2}{4t} + Mt\right)}\right\}\\ & + A_{19}\,e^{A_{11}t}\,f_1\,(1, y, A_{25}, t) + A_{20} f_1\,(1, y, M, t)\\ & + A_{21}\,e^{A_{12}t} f_1\,(1, y, A_{26}, t) + A_{22} e^{A_3 t} f_1\,(1, y, A_{27}, t)\\ & + A_{23} e^{A_3 t} f_1\,(Pr, y, A_{28}, t) - A_{14} f_1\,(Pr, y, A_2, t)\\ & + A_{15} f_1\,(Sc, y, K, t) + A_{24} e^{A_{11} t} f_1\,(Sc, y, A_{29}, t)\\ & - A_{16} e^{A_3 t} f_1\,(Sc, y, A_{30}, t)\\ & + A_{18} e^{A_3 t} f_1\,(Pr, y, A_{31}, t)\,, \end{aligned} \tag{16}$$

where

$$f_1(x,y,z,t) = \frac{1}{2}\left[e^{y\sqrt{xz}}erfc\left(\frac{y}{2}\sqrt{\frac{x}{t}}+\sqrt{zt}\right)+e^{-y\sqrt{xz}}erfc\left(\frac{y}{2}\sqrt{\frac{x}{t}}-\sqrt{zt}\right)\right],$$

$$f_2(z,y,t) = \left[e^{y\sqrt{z}}erfc\left(\frac{y}{2\sqrt{t}}+\sqrt{zt}\right)+e^{-y\sqrt{z}}erfc\left(\frac{y}{2\sqrt{t}}-\sqrt{zt}\right)\right],$$

$$f_3(z,y,t) = \left[e^{-y\sqrt{z}}erfc\left(\frac{y}{2\sqrt{t}}-\sqrt{zt}\right)-e^{-y\sqrt{z}}erfc\left(\frac{y}{2\sqrt{t}}+\sqrt{zt}\right)\right].$$

Skin friction

The expression for the skin friction or shear stress at the plate is given by

$$\begin{aligned}\tau = & \left.\frac{\partial u}{\partial y}\right]_{y=0} \\ & = t^2 f_6(M,t) + f_4(M,t) - f_5(M,t) + A_{19}e^{A_{11}t} f_6(A_{25},t) + A_{20}f_6(M,t) \\ & + A_{21}e^{-A_{12}t} f_6(A_{26},t) + A_{22}e^{A_3 t} f_6(A_{27},t) + A_{23}e^{A_{11}t} f_6(Pr,A_{28},t) \\ & - A_{14}f_6(Pr,A_2,t) + A_{15}f_6(Sc,K,t) + A_{24}e^{A_{12}t} f_6(Sc,A_{29},t) \\ & - A_{16}e^{A_3 t} f_6(Sc,A_{30},t) + A_{18}e^{A_3 t} f_6(Pr,A_{31},t), \end{aligned} \tag{17}$$

where

$$f_4(M,t) = \left(\frac{1-4Mt}{16M^{3/2}}\right)\left[erfc\left(-\sqrt{Mt}\right)-erfc\left(\sqrt{Mt}\right)\right],$$

$$f_5(M,t) = \frac{e^{-Mt}}{2M}\left(\sqrt{\frac{t}{\pi}}\right),$$

$$f_6(x,z,t) = \frac{1}{2}\left[\sqrt{xz}\left\{erfc\left(\sqrt{zt}\right)-erfc\left(-\sqrt{zt}\right)\right\}-\frac{2\sqrt{x}e^{-zt}}{\sqrt{\pi t}}\right].$$

Nusselt number

Expression for the rate of heat transfer at the plate in terms of Nu is given by

$$\begin{aligned} Nu = & -\left.\frac{\partial\theta}{\partial y}\right]_{y=0} \\ & = -f_6(Pr,A_2,t) \end{aligned} \tag{18}$$

Sherwood number

Expression for the rate of mass transfer at the plate in terms of Sh is given by

$$\begin{aligned} Sh = & \left.\frac{\partial\phi}{\partial y}\right]_{y=0} \\ & = (1+A_4) f_6(Sc,K,t) + A_5 e^{A_3 t} f_6(Sc,A_{30},t) \\ & - A_4 f_6(Pr,A_2,t) - A_5 e^{A_3 t} f_6(Pr,A_{31},t) \end{aligned} \tag{19}$$

Results and discussion

In order to get a clear insight of the physical problem, numerical computations have been carried out for various values of different parameters such as Prandtl number (Pr), thermal Grashof number (Gr), mass Grashof number (Gm), magnetic parameter (M), Soret number (Sr), thermal radiation (N), heat source (Q) and chemical reaction (K). For illustrations of the results, numerical values are plotted in the figures ($2-18$).

Figures ($2-9$) demonstrate the pattern of the velocity, temperature and concentration distributions obtained from the analytical solutions of the problem.The effects of Soret number on the velocity field and concentration profile are shown in Figures 2 and 3. These figures exhibit that the velocity and concentration of the fluid increase with increase in Soret number. Soret number is associated with the ratio of temperature difference and concentration gradient. Hence the greater Soret number stands for larger temperature difference and precipitous gradient. Thus the fluid velocity rises due to greater thermal diffusion factor.

Figures $4, 5$ and 6 illustrate the dimensionless velocity, temperature and concentration distribution for different values of heat source parameter. It is observed that the velocity and temperature profiles decrease as heat source parameter increases in contrast the concentration profile raise for increasing values of heat Source. Figure 7 reveals the effects of the magnetic field parameter (M) on velocity field. From this figure it is observed that the velocity of fluid decreases for increasing of magnetic parameter (M). It is due to the fact that the presence of transverse magnetic field produces a resistive force called Lorentz force, which leads to slow down the motion of fluid.Figure 8 and 9 exhibit the velocity field and concentration distribution against y for different values of chemical reaction parameter. From these two figures we observe that as chemical reaction parameter increases velocity and concentration decrease. It is noticed that the fluid motion and concentration increases quickly up to some thin layer of the fluid adjacent to the plate and then decreases asymptotically as we move far away from the plate.

The influence of radiation parameter, Soret number and heat Source effects on skin friction with respect to time t is shown graphically in Figures $10, 11$ and 12 respectively. It is observed that the viscous drag increases due to an increase in either of the Soret number, or heat source or the thermal radiation effect. Thus the effects thermal-diffusion, thermal radiation and heat source are effective mechanism in controlling the viscous drag at the plate to a considerable extent. The magnitude of the heat transfer coefficient for various values of thermal radiation (N) and heat source (Q) are given in Figures 13 and 14. It is observed that the rate of heat transfer coefficient in case of air decreases whenever there is an increase in either thermal radiation (N) or heat source (Q). That is to say that the effect of radiation or heat source causes the rate of heat transfer from the plate to fall to some extent.The rate of mass transfer coefficient

for various values of thermal-diffusion and heat source is given in Figures 15 and 16. It is noticed that the rate of mass transfer at the plate increases when either there is an increase in Soret number (Sr) or there is an increase in the heat source parameter (Q).In other words the rate of mass transfer from the plate to the fluid gets enhance due to Soret number as well as heat source effect.

In the absence of thermal diffusion and heat source (i.e. $Sr = 0$, $Q = 0$), the results obtained in the present work are in good agreement with that obtained by Muthucumaraswamy and Sivkumar (2016), Figure 17 shows the effect of magnetic field parameter M (when $Sr = 0$ and $Q = 0$) on the velocity profile and the outcome is compared with Figure 18 (Figure 3 of the work Muthucumaraswamy and Sivkumar (2016)). It is seen that both the figure exactly coincide, depicting similar result that the velocity decreases with the application of transverse magnetic field.

Conclusion

Our theoretical investigation can be summarized to the following conclusions:

- The velocity of fluid decreases with increasing values of heat Source but reverese trend is shown in case of concentration profile.
- Increase in Soret number results and enhancement in velocity and concentration of the fluid.
- With increasing values of heat Source parameter the temperature profile falls decrease.
- Application of transverse magnetic field causes the velocity of the fluid to reduce comprehensively.
- The viscous drag at the plate is enhanced in magnitude with increase in thermal radiation, Soret number and heat Source.
- The rate of heat transfer decreases with the increasing values of thermal radiation and heat source parameter.
- The rate of mass transfer contributes to increase with the increasing values of thermal diffusion and heat Source.

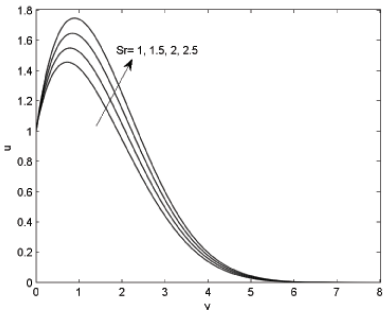


Fig. 2 Velocity versus y for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, N = 1, M = 1, Q = .2, K = .50, t = 1$

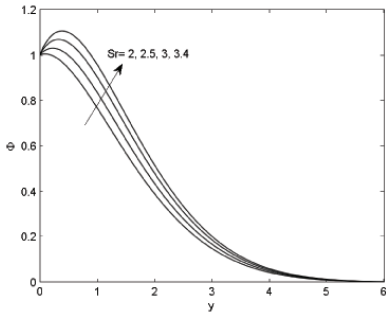


Fig. 3 Concentration versus y for $Pr = .71, Sc = .60, N = 1, Q = .2, K = .50, t = 1$

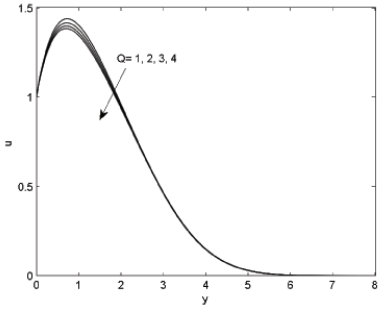


Fig. 4 Velocity versus y for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, Sr = 1, N = 1, M = 1, K = .50, t = 1$

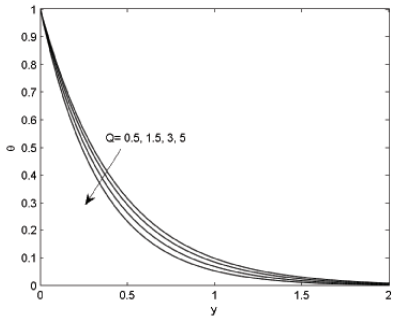


Fig. 5 Temperature versus y for $Pr = .71, N = 5, t = .7$

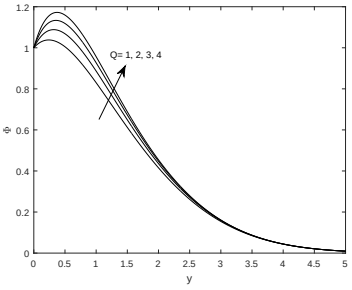


Fig. 6 Concentration versus y for $Pr = .71, Sc = .60, Sr = 2, N = 1, K = .50, t = 1$

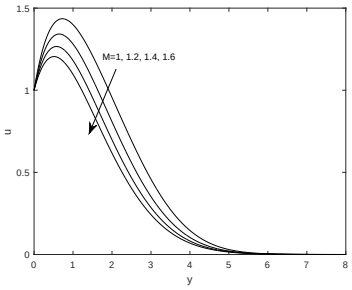


Fig. 7 Velocity versus y for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, Sr = 1, N = 1, Q = 1, K = .50, t = 1$

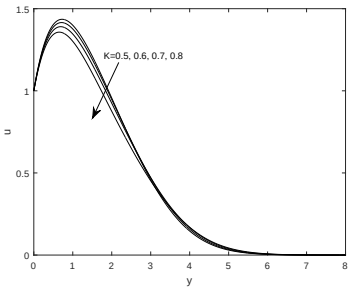


Fig. 8 Velocity versus y for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, Sr = 1, N = 1, M = 1, Q = 1, t = 1$

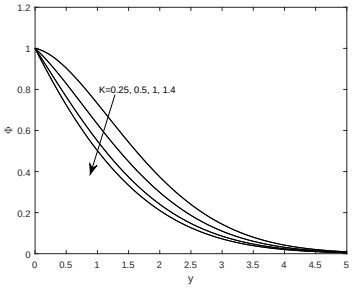


Fig. 9 Concentration versus y for $Pr = .71, Sc = .60, Sr = 1, N = 1, Q = .2, t = 1$

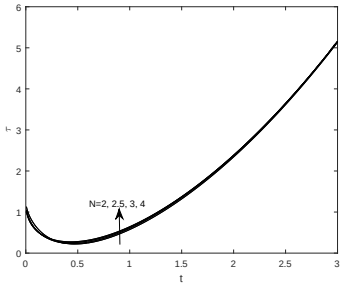


Fig. 10 Skin friction versus t for $Pr = .71, Gr = 2, Gm = 2, Sc = .60, Sr = 2, Q = .5, K = 1, M = 1$

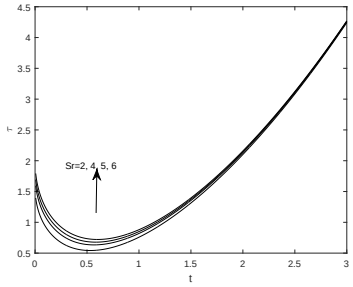


Fig. 11 Skin friction versus t for $Pr = .71, Gr = 2, Gm = 2, Sc = .60, N = .25, Q = .5, K = 1, M = 1$

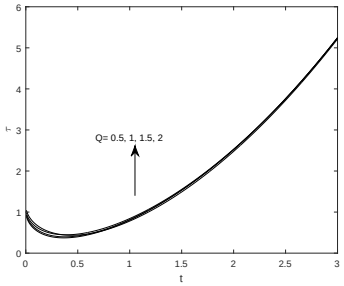


Fig. 12 Skin friction versus t for $Pr = .71, Gr = 2, Gm = 2, Sc = .60, N = .25, Q = .5, K = 1, M = 1$

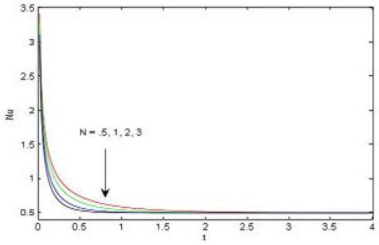


Fig. 13 Nusselt number versus t for $Pr = .71, Q = .5$

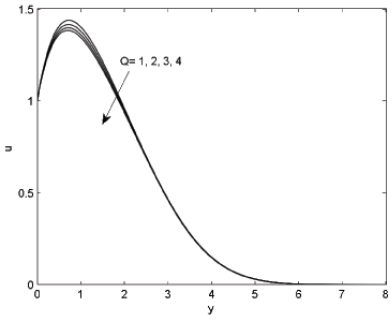


Fig. 14 Nusselt number versus t for $Pr = .71, N = .5$

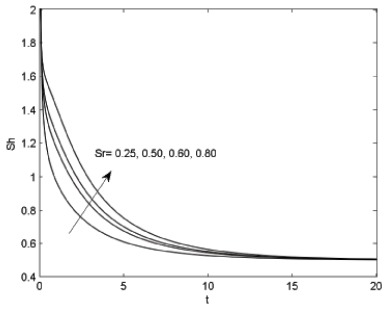


Fig. 15 Sherwood number versus t for $Pr = .71, Sc = .60, N = .50, Q = .5, K = .2$

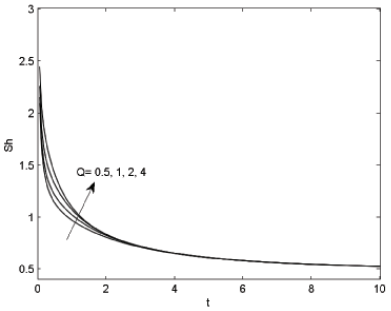


Fig. 16 Sherwood number versus t for $Pr = .71, Sc = .60, N = .50, Sr = .5, K = .2$

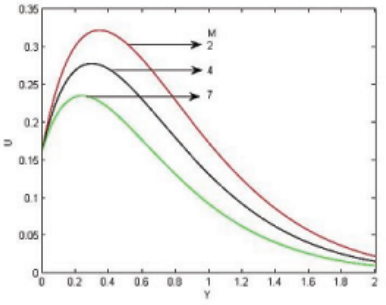


Fig. 17 Velocity versus y for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, Sr = 0, N = 10, Q = 1, K = 10, t = 0.4$

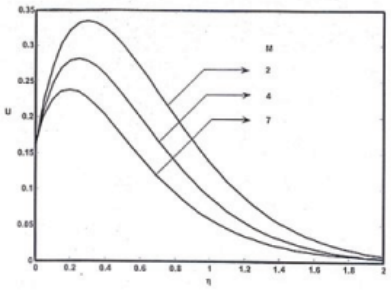


Fig. 18 Velocity versus η for $Pr = .71, Gr = 3, Gm = 3, Sc = .60, Sr = 0, N = 10, Q = 1, K = 10, t = 0.4$ (Figure 3 of the work Muthucumaraswamy and Sivkumar(2016)

Nomenclature

a^*	mean absorption coefficient
A	positive real constant
B_0	strength of applied magnetic field
$\vec{B}$	magnetic induction vector
C'	species concentration
C_p	specific heat at constant pressure
C'_w	dimensional species concentration at the plate
C'_∞	dimensional species concentration in the free stream
D_M	mass diffusion coefficient
D_T	thermal diffusion coefficient
erf	error function
$erfc$	complementary error function
g	acceleration due to gravity
Gm	solutal Grashof number
Gr	thermal Grashof number
$\vec{J}$	current density vector
K	chemical reaction parameter
K_1	chemical reaction rate
K_T	thermal diffusion ratio
M	Hartmann number
N	radiation parameter
Nu	Nusselt number
p	fluid pressure
Pr	Prandtl number
$\vec{q}$	fluid velocity vector
q_r	magnitude of radiative heat flux
Q	non-dimensional heat source parameter
Q'	dimensional heat source
Sc	Schmidt number
Sr	Soret number
t	dimensionless time
t'	dimensional time
T'	fluid temperature
T'_m	mean fluid temperature
T'_∞	fluid temperature at free stream
T'_w	fluid temperature at the plate

u non-dimensional velocity component along x direction
u' x' component of fluid velocity
β coefficient of volume expansion for heat transfer
β_c coefficient of volume expansion for mass transfer
κ thermal conductivity
μ coefficient of viscosity
ν kinematic viscosity
ϕ dimensionless concentration
ρ fluid density
σ electrical conductivity
σ^* Stefan Boltzmann constant
τ skin friction coefficient
θ dimensionless temperature
and the other symbols have their usual meanings.

Appendix

$$A_1 = \frac{Sr\,Sc\,Pr}{Pr - Sc}, A_2 = \frac{Q\,Pr + R}{Pr}, A_3 = \frac{Sc\,K - (Q\,Pr + R)}{Pr - Sc},$$
$$A_4 = -\frac{A_1 A_2}{A_3}, A_5 = \frac{A_1\,(A_2 + A_3)}{A_3}, A_6 = \frac{Gr}{1 - Pr},$$
$$A_7 = \frac{Gc\,(1 + A_4)}{Sc - 1},\ A_8 = \frac{Gc\,A_5}{Sc - 1},\ A_9 = \frac{Gc\,A_4}{Pr - 1},$$
$$A_{10} = \frac{Gc\,A_5}{Pr - 1}, A_{11} = \frac{M - Pr\,A_2}{Pr - 1}, A_{12} = \frac{M - Sc\,K}{Sc - 1},$$
$$A_{13} = A_6 + A_9,\ A_{14} = \frac{A_{13}}{A_{11}}, A_{15} = \frac{A_7}{A_{12}},$$
$$A_{16} = \frac{A_8}{A_3 - A_{12}},\ A_{18} = \frac{A_{10}}{A_3 - A_{11}}, A_{19} = A_{18} - A_{14},$$
$$A_{20} = A_{14} - A_{15}, A_{21} = A_{15} - A_{16},\ A_{22} = A_{16} - A_{18},\ A_{23} = A_{14} - A_{18},$$
$$A_{24} = A_{16} - A_{15}, A_{25} = M + A_{11},\ A_{26} = M + A_{12},\ A_{27} = M + A_3,$$
$$A_{28} = A_2 + A_{11},, A_{29} = K + A_{12},\ A_{30} = K + A_3,\ A_{31} = A_2 + A_3.$$

References

[1] N. Ahmed, *Soret and radiation effects on transient MHD free convection from an impulsively started infinite vertical plate*, Journal of Heat Transfer (ASME), 134 (2012a), 1-9.

[2] N. Ahmed, *Heat and mass transfer in Hartmann flow with Soret effect in presence of a constant heat source,* Turkish J. of Phy., 33 (2012b), 446-460.

[3] G. Astarita, *Mass transfer with chemical reaction*, Elsevier Publishing Company, Amsterdam, 1967.

[4] K. Bhattacharyya, *Effect of radiation and heat source/sink on unsteady MHD boundary layer flow and heat transfer over a stretching sheet with suction/injection*, Frontiers of Chemical Science and Engineering, 5 (2011), 376-384.

[5] M. Bhuvaneswari, S. Sivasankaram, Y.J. Kim, *Exact analysis of radiation convective flow heat and mass transfer over an inclined plate in a porous medium*, World Appl. Sci. J., 10 (2010), 774-778.

[6] A.J. Chamkha, *Thermal radiation and buoyancy effects on hydromagnetic flow over an accelerating permeable surface with heat source or sink*, Int. J. Eng. Sci., 38 (2000), 1699-1712.

[7] C.H. Chen, *Magnetohydrodynamic mixed convection of a power law fluid past a stretching surface in the presence of thermal radiation and internal heat generation/absorption*, Int. J. Nonlinear Mech., 44 (2009), 596-603.

[8] E.R.G. Eckert, R.M. Drake, *Analysis of heat and mass transfer*, McGraw Hill, New York, 1972.

[9] C. Geindreau, J.L. Auriaut, *Magnetohydrodynamic flow in porous media*, J. Fluid. Mech., 466 (2002), 343-363.

[10] S.M. Ibrahim, K. Gangadhar, N. Bhaskar Reddy, *Radiation and mass transfer effects on MHD oscillatory flow in a channel filled with porous medium in presence of chemical reaction*, Journal of Applied Fluid Mechanics, 8 (2015), 529-537.

[11] S.M. Ibrahim, K. Suneetha, *Chemical reaction and Soret effects on unsteady MHD flow of a viscoelastic fluid past an impulsively started infinite vertical plate with heat source/sink*, International Journal of Mathematics and computational Science, 1 (2015), 5-14.

[12] S.Y. Ibrahim, O.D. Makinde, *Chemically reacting MHD boundary layer flow of heat and mass transfer over a moving vertical plate with suction*, Scientific Research and Essays, 19 (2010), 2875-2882.

[13] R. Muthucumaraswamy, *Effect of chemical reaction on a moving isothermal surface with suction*, Acta Mechanica, 155 (2002), 65-72.

[14] R. Muthucumaraswamy, P. Sivkumar, *MHD flow past a parabolic flow past an infinite isothermal vertical plate in the presence of thermal radiation and chemical reaction*, Int. J. of Applied Mechanics and Engineering, 21 (2016), 95-105.

[15] V.R. Prasad, B. Vasu, O.A. Beg, R.D. Prasad, *Thermal radiation effects on magnetohydrodynamic free convection heat and mass transfer from a sphere in a variable porosity regime*, Commun Nonlinear Sci. Numer. Simul., 17 (2012), 654-671.

[16] M.C. Raju, N.A. Reddy, S.V.K. Varma, *Analytical study of MHD free convective, dissipative boundary layer flow past a porous vertical surface in presence of thermal radiation*, Chemical Reaction and Constant Suction, Ain Shams Engineering Journal, 5 (2014),1361-1369.

[17] M.C. Raju, S.V.K. Varma, R.R.K. Rao, *Unsteady MHD free convection and chemically reactive flow past an infinite vertical porous plate*, Journal of Future Engineering and Technology, 8 (2013), 35-40.

[18] A. Raptis, *Free convection and mass transfer effects on the oscillatory flow past an infinite moving vertical isothermal plate with constant suction and heat source*, Astrophy. Space Sci., 86 (1982), 43-53.

[19] A. Raptis, N.G. Kafoussias, *Magnetohydrodynamics free convection flow and mass transfer through porous medium bounded by an infinite vertical porous plate with constant heat flux*, Can. J. Phy., 60 (1982), 1725-1729.

[20] A. Raptis, C. Perdikis, H.S. Takhar, *Effect of thermal radiation on MHD flow*, Applied Mathematics and Computation, 153 (2004), 645-649.

[21] G.S. Seth, M.S. Ansari, R. Nandkeolyar, *MHD natural convection flow with radiative heat transfer past an impulsively moving plate with ramped wall temperature*, Heat Mass Transfer, 47 (2011), 551-561.

[22] G.S. Seth, A. Bhattacharyya, R. Tripathi, *Effect of Hall current on MHD natural convection heat and mass transfer flow of rotating fluid past a vertical plate with ramped wall temperature*, Frontiers in Heat and Mass Transfer, 9-21, 2017.

[23] G.S. Seth, S.N. Hussain, S. Sarkar, *Effects of Hall current and rotation on unsteady MHD natural convection flow with heat and mass transfer past an impulsively moving vertical plate in the presence of radiation and chemical reaction*, Bulg. Chem. Comm., 46 (2014), 704-718.

[24] G.S. Seth, B. Kumbhakar, S. Sarkar, *Soret and Hall effects on unsteady MHD free convection flow of radiating and chemically reactive fluid past a moving vertical plate with ramped temperature in rotating system*, Int. J. Eng. Sci. and Tech., 7 (2015), 94-108.

[25] A.K. Singh, *MHD free convection and mass transfer flow with heat source and thermal diffusion*, J. of Energy, Heat and Mass Transfer, 23 (2001), 227-249.

[26] M. Turkyilmazoglu, I. Pop, *Soret and heat source effects on the unsteady radiative MHD free convection flow from an impulsively started infinite vertical plate*, International Journal of Heat and Mass Transfer, 55 (2012), 7635-7644.

[27] G. Zimmerman, U. Muller, C. Benard, *Convection in a two component system with Soret effect*, Int. J. Heat and Mass Transfer, 35 (1992), 2245-2256.

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Use of data envelopment analysis for clustering of decision making units

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Abstract. In this paper, clustering based on data envelopment analysis (DEA) of eight educational groups, with each group having four inputs and outputs, as well as DMU_k ID, is specified. Minitab software and optimal weights for the formation of clusters of DMUs were used for the analysis. The statistical descriptions of DMUs showed, that the elementary educational group (DMU_2) with the best educational performance ranked first. The correlation between the variables of different DMUs was also investigated. In an analysis of the main variables, only three items of eigenvalue were greater than one. In clustering based on DMUs as highlighted in green, with the four clusters, more than 97% of variables could be retained. Based on observations as shown in red, considering the three clusters, more than 78% of information could be maintained and in this stage the abnormal variable of O_3 (scientific products) was removed from the process. Regarding the analysis, the results of path coefficients showed that DMU_2 had seen the best economy performance.

Keywords: data envelopment analysis, clustering, decision making units, cluster analysis, optimal weights.

1. Introduction

Data envelopment analysis (DEA) is a useful method for the clustering of variables of decision-making units (DMUs). DEA has been widely studied and applied in various areas for 38 years since [1] first proposed the method with the CCR model. One of the main forms of DEA models and their extensions includes those of the BCC model [2]. DEA is being used extensively by various researchers in different fields of interests including DMU clustering.

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Cluster analysis is a branch of statistical multivariate analysis and unsupervised learning in pattern recognition [3],[4],[5]. It is a method for classifying groups of variables set into the same cluster but not groups set into different clusters. The method of clustering is as old as science itself. Clustering is the task of grouping similar objects into clusters. The process involves classifying existing variables in such a way that the variation in the same-group variables is as low as possible while it is very high between groups.

More accurately, clustering can be defined as partitioning the variables set into subgroups called clusters so that variables in the same cluster are more similar to each other than variables in other clusters. Cluster analysis enables the analyst to find the underlying structure of variables and come up with hypotheses that would answer many questions regarding the variables. This study considers optimal weights corresponding to DMUs. It is thus obvious that every DMU will be represented by a weight vector. This provides us with a multidimensional representation of DMUs in the given problem. These weight vectors can then be used to form clusters of the DMUs. The use of these clusters will be used to treat all the DMUs separately. This paper aims to investigate clustering based on the DEA of educational groups from Islamic Azad University, Khash Branch, Iran. Based on the obtained results of the group's performance, it is necessary and compulsory to revise some of the input variables that affect the output results of each DMU of the university's Khash Branch in the future. It is important to find the best DMUs for managerial decision-making where decision-makers are interested in knowing the required changes in input resources to get the desired result.

The rest of this paper is organized as follows: Section 2 provides a survey of the relevant literature and introduces the DEA-related works that have been used for clustering of DMUs. Section 3 describes the related methodological framework, presents the data used in our study, the empirical results of analysis, and discussion of the clustering based on DEA. Finally, the conclusion is stated in Section 4.

2. Related work

In the clustering literature, several authors have discussed the robustness for clustering [6], [7]. The robustness of the DEA-based clustering algorithm is another interesting research issue. Some notable works include: Kaufman and Rousseeuw's [4] "finding groups in data: an introduction to cluster analysi" and Dave and Krishnapuram's [7] "Robust clustering methods: a unified view".

Other works include: McLachlan and Basford's [8] "Mixture models: inference and applications to clustering" and Wu and Yang's [9] "Alternative c-means clustering algorithms". The clustering is done by assigning different efficiency ratio grades [10],[11]. Generally, clustering methods can be divided into the following categories: hierarchical clustering [4] is a category of cluster analysis. K-means clustering algorithm [3],[9],[12] include clustering procedures that min-

imize dissimilarities and can also be considered as a feature analysis technique. In port cluster resources based on DEA model [13] and in a new clustering approach using DEA [14], using mixed DEA cluster model [15]. Besides, there are data envelopment-based clustering approach for public sugar factories in the privatizing process [16], clustering of DMUs using full-dimensional efficient facets (FDEFs) of PPS with BCC technology [17], and weight-based clustering in DEA [18], efficiency evaluation using DEA game, and cluster analysis [19]. In this study, in contrast to the previous literature on DEA, we present a new approach.

3. Research methodology

Several methods have been proposed to obtain the meaningful classification of variables.The DEA method can be used to cluster the variables with input and output items of the DMUs. In this study, we propose a clustering approach based on the input and output variables of educational groups from Islamic Azad University, Khash Branch, Iran, using machine learning, which involves DEA. The proposed DEA-based clustering approach employs machine learning techniques derived from the DEA method for clustering the observed variables with input and output items of all DMUs.

3.1 Data envelopment analysis

3.1.1 Theoretical foundation

Let there be n DMUs, $DMU_i, i = 1, 2, \cdots, n$. Every DMU has m inputs and s outputs. The m inputs are denoted by $X_1, X_2, \ldots, X_m$ and the vector $\underline{X} = (X_1, X_2, \ldots, X_m)$ is called the input vector. The s outputs are denoted by $Y_1, Y_2, \ldots, Y_s$ and the vector $\underline{Y} = (Y_1, Y_2, \ldots, Y_s)$ is the output vector. These together give a vector of dimension $m+s$. Denoted by $\underline{T} = (\underline{X}, \underline{Y}), X \geq 0, Y \geq 0$, this vector is called the production vector.

3.1.2 Model M

Suppose there are n DMUs, each having m inputs, and are denoted by vector$\underline{X} \geq 0$ and the s outputs are denoted by vector $\underline{Y} \geq 0$. Therefore, the m coefficients $\underline{V} = (V_1, V_2, \ldots, V_m)$, $\underline{V} \geq 0$ for inputs and s coefficients $\underline{U} = (U_1, U_2, \ldots, U_s)$, $\underline{U} \geq 0$ for outputs are to be determined by solving the following constrained optimization problem.

Model M:

$$\max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

Subject to the constraints:

$$(1)\qquad \begin{aligned} \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} &\leq 1 \qquad \text{for } j = 1, 2, \ldots, n \\ u_r &\geq 0 \qquad \text{for } r = 1, 2, \ldots, s \\ v_i &\geq 0 \qquad \text{for } i = 1, 2, \ldots, m \end{aligned}$$

where u_r is the weight of the output y_{ro}, and r $(r = 1, 2, \ldots, s)$ is the number of the generated products, y_{ro} - the amount of the product r is generated by DMU_o $(o = 1, 2, \ldots, n)$, x_{i0} - the amount of the resource i is used by DMU_o, v_i - the weight of the input x_{io}, i $(i = 1, 2, \ldots, m)$ is the number of the resources used, y_{rj} - the amount of the product r is generated by DMU_j, x_{ij} - the amount of the resource i is used by DMU_j, and j $(j = 1, 2, \cdots, n)$ is the number of DMUs.

To provide the optimal solution, model M produces vectors of optimal coefficients for all the n DMUs in the data. The optimal coefficient matrix is formed by treating these vectors as its rows. This matrix is denoted by T and can be written as follows:

$$(2)\qquad T = \begin{bmatrix} v_{11}, & v_{12}, & \cdots & v_{1m}, & u_{11}, & u_{12}, & \cdots & u_{1s} \\ v_{21}, & v_{22}, & \cdots & v_{2m}, & u_{21}, & u_{22}, & \cdots & u_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{j1}, & v_{j2}, & \cdots & v_{jm}, & u_{j1}, & u_{j2}, & \cdots & u_{js} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1}, & v_{n2}, & \cdots & v_{nm}, & u_{n1}, & u_{n2}, & \ldots & u_{ns} \end{bmatrix}$$

The n rows of T correspond to n DMUs, the first m column of T corresponds to m inputs, and the last s column of T corresponds to s outputs. There are many distance functions that can be used for this purpose.

3.2 Clustering of decision making units

The first step toward the formation of clusters is to construct the distance matrix. It is a square symmetric matrix of order $n \times n$, where the $(i, j)^{\text{th}}$ element denotes the distance between the i^{th} row and the j^{th} row of matrix T. The distance function is a non-negative, symmetric, reflexive, and transitive function that may desirably satisfy triangle inequality.

There are some s standard distance functions that are commonly used because they are meaningful. Euclidean distance, Manhattan distance, binary distance, Minkowski distance, and Canberra distance are some of the commonly used distance functions. Matrix D can be written as $D_{nn} = ((d_{ij}))$, where

$$(3)\qquad d_{ij} = d(\underline{T_i}, \underline{T_j}) \qquad \text{for } i, j = 1, 2, \cdots, n.$$

Here d is the distance function and $d(\underline{T_i}, \underline{T_j})$ denotes the distance between the i^{th} and j^{th} rows of the matrix T.

3.2.1 Data analysis

Clustering is one of the most important topics, and in peer groups its classification has many applications. In this section, using the data obtained from the research titled "Assess the relative efficiency of educational groups of Islamic Azad University Khash Branch" in academic year 2011-12, eight educational groups are specified, with each academic department having four inputs (I_i) and four outputs (O_j), and $\mathrm{DMU_k}$ ID ($i, j = 1, \ldots, 4; k = 1, \ldots, 8$). In order to evaluate DMUs regarding all the inputs and outputs of the eight DMUs using the clustering method, Minitab 17 statistical software was used. As mentioned above, the inputs and outputs of the DMUs and their values in this study are presented in Tables 1 and 2.

Table 1: Inputs and outputs for each DMU & field of study.

Inputs: I_i		**Outputs:** O_j		$\mathrm{DMU_k}$	**Field of study**
I_1:	Head of department	O_1:	Number of students	$\mathrm{DMU_1}$	Persian literature
I_2:	Specialty books	O_2:	Number of graduates	$\mathrm{DMU_2}$	Elementary Educational
I_3:	Faculty members	O_3:	Scientific products	$\mathrm{DMU_3}$	Management
I_4:	Area of department	O_4:	Accepting in higher academic level	$\mathrm{DMU_4}$	Geography
				$\mathrm{DMU_5}$	Civil engineering
				$\mathrm{DMU_6}$	Computer
				$\mathrm{DMU_7}$	Electro-technique
				$\mathrm{DMU_8}$	Physical education

Table 2: The values of inputs and outputs for each of educational groups.

$\mathrm{DMU_k}$	Score of HOD	Number of library books	Faculty members	Area of department	Number of students	Number of graduates	Scientific product	Accepting in higher academic level
$\mathrm{DMU_1}$	2.973	2000	12	20	298	54	0	65
$\mathrm{DMU_2}$	2.671	3300	6	40	250	126	0	65
$\mathrm{DMU_3}$	2.590	3000	5	20	431	31	0	65
$\mathrm{DMU_4}$	2.806	2000	5	20	125	10	3	20
$\mathrm{DMU_5}$	2.500	2000	2	20	29	4	0	75
$\mathrm{DMU_6}$	2.806	1500	4	20	77	9	0	50
$\mathrm{DMU_7}$	2.447	1000	3	30	35	8	0	35
$\mathrm{DMU_8}$	2.198	700	5	20	33	10	1	35

3.2.2 Description

By applying the described process on the eight educational groups of the DMUs, the average allocation to each of these groups was used. These descriptions are provided in Table 3 for the different DMUs.

Table 3: Descriptive statistics of DMUs.

$\mathbf{DMU_k}$	$\mathbf{DMU_1}$	$\mathbf{DMU_2}$	$\mathbf{DMU_3}$	$\mathbf{DMU_4}$	$\mathbf{DMU_5}$	$\mathbf{DMU_6}$	$\mathbf{DMU_7}$	$\mathbf{DMU_8}$
Mean	20.91	31.70	17.21	9.15	14.31	12.18	11.12	10.40
St. Dev	24.75	44.40	21.88	7.10	25.31	16.43	13.50	11.62

The mean of the obtained values in Table 3 show that the maximum amount related to DMU_2 is allocated to the elementary education group. This shows that the DMU_2 has the best academic performance due to input variables compared to other first-ranked DMUs. DMU_1 is ranked second after DMU_2 shows. However DMU_4 did not give good performance in comparison with other DMUs.

3.2.3 Correlation

To find out the relationship and the correlation between the input and output of each DMU, variables related to the analysis and a covariance matrix are used as measurement variables.

This matrix is symmetric, hence a bottom triangle. Since the diagonal matrix is always 1, the same variables have a value of 1 and have a full relationship. Thus, there is complete solidarity and a significant amount. Therefore, it is automatically eliminated from the analysis of the matrix. Thus, it seems that the obtained correlation matrix of Table 4 shows a matrix of 7×7.

Table 4: Correlation and significance.

C \ R	I_1	I_2	I_3	I_4	O_1	O_2	O_3
I_2	0.422 0.298						
I_3	0.582 0.130	0.208 0.621					
I_4	-0.051 0.905	0.384 0.348	-0.048 0.910				
O_1	0.435 0.282	0.766 0.027	0.563 0.146	0.071 0.867			
O_2	0.288 0.489	0.709 0.049	0.459 0.252	0.762 0.028	0.533 0.173		
O_3	0.058 0.891	-0.157 0.711	-0.044 0.917	-0.269 0.519	-0.206 0.625	-0.276 0.509	
O_4	0.165 0.696	0.573 0.138	0.177 0.675	0.111 0.793	0.444 0.271	0.420 0.300	-0.757 0.030

In each cell of this matrix, two values are visible showing the higher correlation criterion, and the lower amount shows the significance level of the test. In addition, that the calculated P value of the test is less than or equal to 0.05 shows a high correlation between the respective variables. According to the above matrix, the values marked in purple represent a high correlation between variables.

3.2.4 Analysis of the main variables

Based on the correlation matrix analysis, the eigenvalues in Table 5 are marked as numbers for eight variables. It should be noted that the eigenvalue should be more than 1 to have enough credibility. According to Table 5, there are only three eigenvalue items that are greater than one.

Table 5: Correlation and significance.

Principal Component Analysis	Eigen analysis of the Correlation Matrix							
Eigenvalue	3.5834	1.6777	1.2549	0.7377	0.4981	0.2277	0.0204	-0.000
Proportion	0.448	0.210	0.157	0.092	0.062	0.028	0.003	-0.000
Cumulative	0.448	0.658	0.814	0.907	0.969	0.997	1.000	1.000

According to Figure 1, a part from the first three clusters, the rest of the clusters have gradually become worthless.

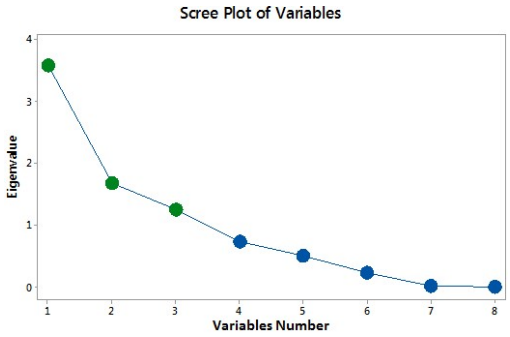


Figure 1: Scree plot for eigenvalue of main variables.

3.2.5 Clustering

Cluster analysis is a process in which observations in a sample are divided into groups so that observations in each group are similar or close to each other. This division is based on values of the observed variables, and these variables can include input and output variables in a DEA problem. Clustering is achieved by calculating the distance matrix of order $n \times n$ or correlation matrix of order

$n \times d$. Each row describes an index. The algorithm output can be grouped as indicators in distinct sets or a tree hierarchical clustering to divide variables.

3.2.6 Clustering based on educational groups

A description of this clustering and the steps using the Minitab software are presented in Table 6.

Table 6: Cluster analysis of groups.

Correlation Coefficient Distance		Complete Linkage		Amalgamation Steps			
							Number of obs. in new cluster
Steps	Number of clusters	Similarity level	Distance level	Clusters joined		New cluster	
1	7	99.4184	0.011632	5	6	5	2
2	6	99.0167	0.019666	4	7	4	2
3	5	97.4019	0.051962	5	8	5	3
4	4	97.2872	0.054257	1	3	1	2
5	3	89.1056	0.217888	4	5	4	5
6	2	84.5944	0.308111	1	2	1	3
7	1	66.8580	0.662841	1	4	1	8

As highlighted with green in above Table 4, with the adoption of the four clusters, more than 97% of variables can be retained. So, with respect to Figure 2, on the basis of similarities that can be determined for the variables, DMUs, DMU_1 and DMU_3 are located in a cluster. DMU_4 and DMU_7 adopted a common feature and together constitute a cluster.

DMU_5, DMU_6, and DMU_8 have formed a cluster with the highest subscription. Owing to the unique nature of DMU_2, this group is in a separate cluster. The clustering of the groups based on the criteria of input and output is shown in Figure 2.

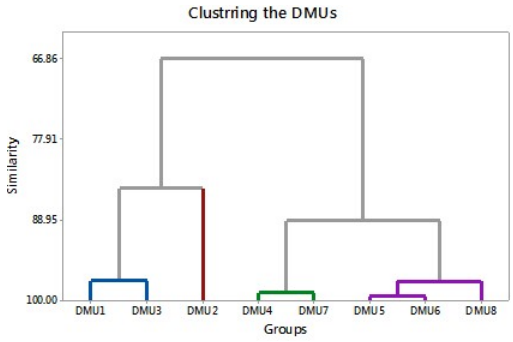


Figure 2: Clustering main variables based on educational groups.

3.2.7 Clustering based on observations

As evident in Table 7, this output provides a structure of cluster analysis variables.

Table 7: Cluster analysis of observations.

Euclidean Distance			Complete Linkage			Amalgamation Steps	
							Number of obs. in new cluster
Steps	Number of clusters	Similarity level	Distance level	Clusters joined		New cluster	
1	7	95.7546	110.88	4	5	4	2
2	6	89.5105	273.97	1	4	1	3
3	5	88.5067	300.19	7	8	7	2
4	4	86.0799	363.58	2	3	2	2
5	3	78.9891	548.78	1	6	1	4
6	2	49.1619	1327.82	1	7	1	6
7	1	0.0000	2611.87	1	2	1	8

According to Table 7, the criterion of similarity level of this process at level 110 arrives from 95.75 in the first stage of the process to the amount of 78.99 at level 548.78 in the fifth stage and then this with a huge disparity in the sixth stage reached almost 49. Therefore, as shown in Table 7 in red, considering the three clusters, more than 78% of information can be maintained. This amount can be a desirable value for selecting the three clusters from this clustering. The results of the clustering are shown in Figure 3.

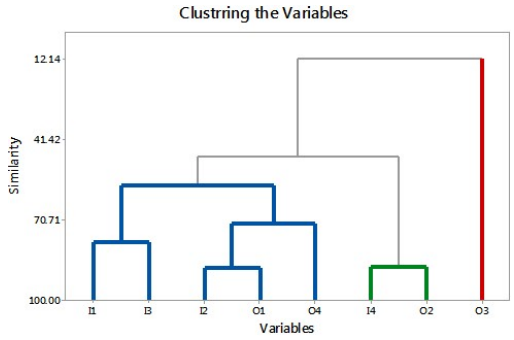


Figure 3: Clustering of main variables based on observations.

According to Figure 3, variables I_1 , I_2, I_3, O_1,and O_4 are in the same blue cluster, the variables I_4 and O_2 are in a green cluster, and O_3 alone is in a red cluster. Of the five variables that have been identified in distinct blue clusters, I_1 and I_3 have the same structure and characteristics of variables I_2 , O_1, and O_4. They are also similar in terms of features and structure to fit together in a group. Moreover, according to the clustering of variables plot shown in Figure

3, variable O_3 is clearly not compatible with other variables, which may be due to a lack or shortage of variables. Figure 4 shows a lack of consistency of O_3 variable. Therefore, to optimize the model and prevent skewness, O_3 is deleted from all educational groups to arrive at actual results.

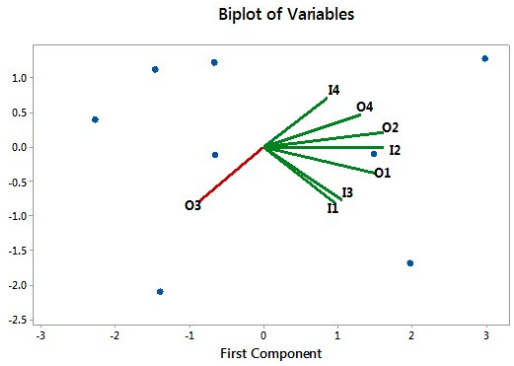


Figure 4: Clustering of main variables based on observations.

3.2.8 Structural equation modeling

The results of the various paths to connect the variables are shown in Figure 5.

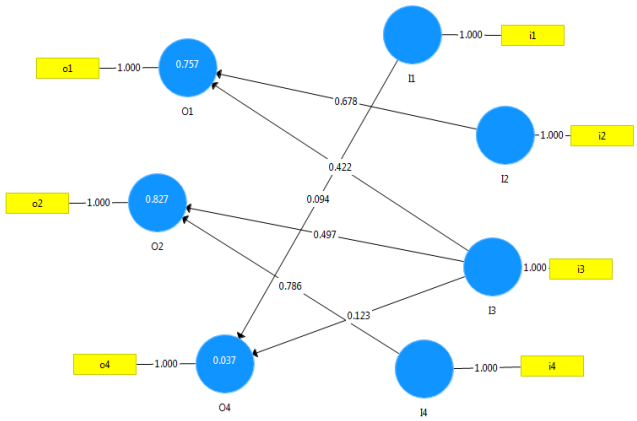


Figure 5: Different possible paths for the relationships between different variables.

In the above figure , different paths may be specified for different variables' relation ships and it has been shown how and with what relationship and amount each of the various inputs (I_i) is connected with the different outputs (O_j).

3.2.9 Analysis of path coefficients

According to the plot, the values of the path coefficients are highly desirable in most cases. This is because the route from I_2 to O_1 has the coefficient of 0.678, the route from I_3 to O_1 has the coefficient of 0.422, the variable I_3 to O_2 has the coefficient of 0.497, and the route from I_4 to O_2 has the coefficient of 0.786. Thus, the relationship between I_4 to O_2 shows the highest correlation.

Instead, the correlation between I_1 and O_4 has the minimum value of 0.094. The results of the variables and interpretation show a good fit of the coefficients.

3.3 Results and discussion

In order to investigate the performance of DMUs from Islamic Azad University, Khash Branch, using data analysis, and to compare the values described in Table 3, the main focus is on DMU_2. Table 4 shows the state of the correlation coefficient and the significance level of the different DMUs. In an analysis of the main variables, based on Table 5, only three items of eigenvalue are greater than one, therefore, we can choose only three clusters, as Figure 1 shows.

In a cluster analysis of DMUs , based on the green-highlighted part in Table 6 and with the adoption of the four clusters, more than 97% of data can be retained, but in this stage with respect to Figure 2, the DMU_2 is in a separate cluster. Therefore, in clustering based on observations regarding the data marked in red in Table 7, more than 78% of information can be maintained by considering the three clusters, because in this stage, regarding Figure 3 and Biplot 4, the variable O_3 is eliminated from all DMUs.

Finally, Figure 5 shows that the path values of I_4 to O_2 with the coefficient of 0.786 is the maximum. Instead, the relationship between I_1 and O_4 has the minimum value of 0.094. According to the results of data interpretation, the model in Figure 5 shows good utility.

4. Conclusions

According to the results of this study, DMU_2 and DMU_1 have been are ranked as first and second, respectively, and have shown the best performance. The other DMUs need to be revised and changed to accommodate some effective input variables, including faculty members, for not showing a good performance. In Figure 3, O_3 was eliminated from the cluster. Figure 5 shows that the path coefficients of I_4 to O_2 with a value of 0.786 has the highest correlation. In contrast, I_1 to O_4 is associated with the minimum value (0.094) and has the lowest correlation. The model in Figure 5 shows good utility.

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References

[1] A. Charnes, W.W. Cooper, and E. Rhodes, *Measuring the efficiency of decision making units*, Eur. J. Oper. Res., 2 (1978), 429-444.

[2] R.D. Banker, A. Charnes, and W.W. Cooper, *Some models for estimating technical and scale inefficiency in data envelopment analysis*, Manage Sci., 3 (1984), 1078-1092.

[3] R.O. Duda, and P.E. Hart, *Pattern classification and scene analysis*, NY, USA, Wiley- Intersci. Publ., 1973, 271-273.

[4] L. Kaufman, and P.J. Rousseeuw, *Finding groups in data: an introduction to cluster analysis*, NY, USA, Wiley, 1990.

[5] A.K. Jain, R.P.W. Duin, and J. Mao, *Statistical pattern recognition: a review*, Ieee T. Pattern Anal., 22 (2000), 4-37.

[6] J.M. Jolion, P. Meer, and S. Bataouche, *Robust clustering with applications in computer vision*, Ieee T. Pattern Anal., 13 (1991), 791-802.

[7] N. Dave, and R. Krishnapuram, *Robust clustering methods: A unified view*, Ieee T. Fuzzy Syst., 5 (1997), 270-293.

[8] G.J. McLachlan, and K.E. Basford, *Mixture models: inference and applications to clustering*, Marcel Dekker, New York, 1988.

[9] W. Kuo-Lung, and Y. Miin-Shen, *Alternative c-means clustering algorithms*, Pattern Recogn., 35 (2002), 2267-2278.

[10] G.R. Jahanshahloo, F. Hosseinzadeh Lotfi, N. Shoja, G. Tohidi, and S. Razavyan, *A one-model approach to classification and sensitivity analysis in DEA*, Appl. Math. Comput., 169 (2005), 887-896.

[11] W.D. Cook, and K. Bala, *Performance measurement and classication data in DEA: Input oriented model*, Omega-Int. J. Manage S., 35 (2007), 39-52.

[12] X. Wang, Y. Wang, and L. Wang, *Improving fuzzy c-means clustering based on feature weight learning*, Pattern Recogn. Lett., 25 (2004), 1123-1132.

[13] Z. Li, X. Zhao, W. Zhang, and B. Wang, *Analysis on integration efficiency of port cluster resources based on DEA model*. In: IEEE International Conference on service Operations and Logistics, and Informatics, 2 (2008), 2604-2608.

[14] R.W. Po, Y.Y. Guh, and M.S. Yang, *A new clustering approach using data envelopment analysis*, Eur. J. Oper. Res., 199 (2009), 276-284.

[15] W. Yawei, and L. Baosong, *Efficiency evaluation of city circular economy based on the super-efficient mixed dea cluster model*, In: International Conference on Management and Service Science, (2010), 1-4.

[16] E.A. Demirtas, *A data envelopment-based clustering approach for public sugar factories in privatizing process*, Math. Probl. Eng., (2011), 11.

[17] G.M.R. Moazami, and A.M.R. Jaber, *Clustering decision making units (DMUs) using full dimensional efficient facets (FDEFs) of PPS with BCC technology*, Appl. Mathl. Ski., 6 (2012), 1431-1452.

[18] L.A. Alves, and J.C.C.B.S. Mello, *Weights based clustering in data envelopment analysis using kohonen neural network: an application in brazilian electrical sector*, Ieee Lat. Am. T., 13 (2015), 188-194.

[19] L.G. Machado, J.C.C.B.S. de Mello and M. C. Roboredo, *Efficiency evaluation of brazililian electrical distributors using DEA game and cluster analysis*, Ieee Lat. Am. T., 14 (2016), 4499-4505.

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Numerical solution of cubic free undamped Duffing oscillator equation using continuous implicit hybrid method

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Abstract. In this paper, we investigate the solution of the cubic free undamped Duffing oscillator equation based on the implicit hybrid method. We investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of the proposed method. In addition, we study the zero stability, the order, and the error constant of the block method which is generated from the proposed method. Numerical results are presented to show the efficiency of the proposed method.

Keywords: cubic free undamped Duffing oscillator equation, implicit hybrid method, consistency, zero stable, convergence.

1. Introduction

In this paper, we study the cubic free undamped Duffing oscillator [1] of the form

$$y''(t) + ay(t) + by^3(t) = 0, 0 < t < T$$

∗. Corresponding author

subject to

$$y(0) = \alpha_0, y'(0) = 0$$

where α_0, a, b are constants. Several methods were used to solve this problem numerically such as homtopy perturbation method [2], the modified homtopy perturbation method [3], variational iteration method [4-8], Adomian decomposition method [9], artificial parameter decomposition [10], He's parameter expanding method [11]. More methods can be found in [16-30].

In addition, many analytical solutions for Duffing problem were developed using many methods such as method of multiple scales [12], the Krylov-Bogolubov method [13], the straightforward expansion [14], and Linstedt-Poincare method [13].

We can rewrite the above system in the form

$$y''(t) = g(y), 0 \leq t \leq T \tag{1}$$

subject to

$$y(0) = \alpha_0, y'(0) = 0. \tag{2}$$

In this paper, we use a numerical method based on the implicit hybrid method. We used collocation and interpolation techniques on the power series approximate solution. The proposed method has high order method which is very accurate comparing with other method such as the one-step methods. This method is cheaper than the multi-step methods. We investigate the consistency, zero stable, convergence, the order, error constant, and region of absolute stability of the proposed method. In addition, we study the zero stability, the order, and the error constant of the block method which is generated from the proposed method. Numerical results are presented to show the efficiency of the proposed method. More details about this method can be found in [23]. In the next section, we describe the proposed method. In section 3, we analyze the proposed method. Finally, we draw some conclusions.

2. Method of solution

In this section, we derive the proposed method. Approximate the solution of Eqn. (1) by

$$y(t) = \sum_{i=0}^{5} a_i t^i. \tag{3}$$

Then, the first derivative of the solution of Eqn. (3) is given by

$$y'(t) = \sum_{i=1}^{5} i a_i t^{i-1}. \tag{4}$$

Let$\{t_0 = 0, t_1 = \Delta, \ldots, t_M = M\Delta = T\}$ be a uniform partition of $[0, T]$ where $t_i = i\Delta, i = 0 : M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (3) at $t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}$ and collocate Eqn. (4) at $t_{n+\frac{j}{3}}, j = 0, 1, 2, 3$, to get the following linear system

$$
(5) \qquad \begin{pmatrix} 1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & t^3_{n+\frac{1}{3}} & t^4_{n+\frac{1}{3}} & t^5_{n+\frac{1}{3}} \\ 1 & t_{n+\frac{2}{3}} & t^2_{n+\frac{2}{3}} & t^3_{n+\frac{2}{3}} & t^4_{n+\frac{2}{3}} & t^5_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_n & 12t^2_n & 20t^3_n \\ 0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}
$$

Let

$$t_{n+\frac{2}{3}} = t - \Delta s, t_{n+1} = t - \Delta s + \frac{\Delta}{3},$$

$$t_{n+\frac{1}{3}} = t - \Delta s - \frac{\Delta}{3}, t_n = t - \Delta s - \frac{2\Delta}{3}.$$

Then, using the above change of variables and solving System (5), we get

$$
\begin{aligned}
a_0(s) &= -3s, \\
a_1(s) &= 1 + 3s, \\
a_2(s) &= -\frac{\Delta^2 s(7 - 90s^2 + 243s^4)}{1080}, \\
a_3(s) &= \frac{1}{360}\Delta^2 s(22 - 180s^2 + 135s^3 + 243s^4), \\
a_4(s) &= \frac{1}{360}\Delta^2 s(43 + 180s + 90s^2 - 270s^3 - 243s^4), \\
a_5(s) &= \frac{\Delta^2 s(-8 + 180s^2 + 405s^3 + 243s^4)}{\Delta^2 s(-8 + 180s^2 + 405s^3 + 243s^4)}.
\end{aligned}
$$

When $t = t_{n+1}, t_{n+\frac{2}{3}} = t_{n+1} - \Delta s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{2}{3}}}{\Delta} = \frac{\Delta}{3\Delta} = \frac{1}{3}.$$

Similarly, when $t = t_{n+\frac{2}{3}}, t_{n+\frac{1}{3}}, t_n, s = 0, -1/3, -2/3$, respectively. Thus, at $s = 1/3, -2/3$, Eqn. (3) becomes

$$
(6) \qquad \begin{aligned}
y_{n+1} &= -y_{n+\frac{1}{3}} + 2t_{n+\frac{2}{3}} + \Delta^2 \left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108} \right), \\
y_n &= 2y_{n+\frac{1}{3}} - y_{n+\frac{2}{3}} + \Delta^2 \left(\frac{f_n}{108} + \frac{5t_{n+\frac{1}{3}}}{54} + \frac{f_{n+\frac{2}{3}}}{108} \right).
\end{aligned}
$$

Using the change of variable $t_{n+\frac{2}{3}} = t - \Delta s$, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{\Delta}\frac{dy}{ds}.$$

Hence,

$$\frac{dy}{ds} = \frac{1}{\Delta}\left(\sum_{i=1}^{2} a'_{i-1}(s) y_{n+\frac{1}{3}} + \sum_{i=3}^{6} a'_{i-1}(s) f_{n+\frac{i-3}{3}}\right). \tag{7}$$

At $s = 1/3, 0, -1/3, -2/3$, Eqn. (7) implies that

$$\begin{aligned}
y'_{n+1} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{f_{n+\frac{1}{3}}}{120} + \frac{23f_{n+\frac{2}{3}}}{60} + \frac{127f_{n+1}}{1080}\right),\\
y'_{n+\frac{2}{3}} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{-7f_n}{1080} + \frac{11f_{n+\frac{1}{3}}}{180} + \frac{43f_{n+\frac{2}{3}}}{360} - \frac{f_{n+1}}{135}\right),\\
y'_{n+\frac{1}{3}} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{43f_{n+\frac{1}{3}}}{360} - \frac{11f_{n+\frac{2}{3}}}{180} + \frac{7f_{n+1}}{1080}\right),\\
y'_n &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{-127f_n}{1080} - \frac{23f_{n+\frac{1}{3}}}{60} + \frac{f_{n+\frac{2}{3}}}{120} - \frac{f_{n+1}}{135}\right).
\end{aligned} \tag{8}$$

Let

$$A_1 = \begin{pmatrix}
1 & -2 & 1 & 0 & 0 & 0\\
-2 & 1 & 0 & 0 & 0 & 0\\
\frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 0 & 1\\
\frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 1 & 0\\
\frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 1 & 0 & 0\\
\frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 0 & 0
\end{pmatrix}, Y_{1,n} = \begin{pmatrix}
y_{n+\frac{1}{3}}\\
y_{n+\frac{1}{3}}\\
y_{n+1}\\
y'_{n+\frac{1}{3}}\\
y'_{n+\frac{2}{3}}\\
y'_{n+1}
\end{pmatrix},$$

$$A_2 = \begin{pmatrix}
0 & 0\\
-1 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & -1
\end{pmatrix}, Y_{2,n} = \begin{pmatrix} y_n\\ y'_n \end{pmatrix}, F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix}
f_{n+\frac{1}{3}}\\
f_{n+\frac{2}{3}}\\
f_{n+1}
\end{pmatrix},$$

$$A_3 = \begin{pmatrix}
0\\
\frac{\Delta^2}{108}\\
\frac{\Delta}{135}\\
\frac{-7\Delta}{1080}\\
\frac{\Delta}{135}\\
\frac{-127\Delta}{1080}
\end{pmatrix}, A_4 = \begin{pmatrix}
\frac{\Delta^2}{108} & \frac{5\Delta^2}{54} & \frac{\Delta^2}{108}\\
\frac{5\Delta^2}{54} & \frac{\Delta^2}{108} & 0\\
\frac{-\Delta}{120} & \frac{23\Delta}{60} & \frac{127\Delta}{1080}\\
\frac{11\Delta}{180} & \frac{43\Delta}{360} & \frac{-\Delta}{135}\\
\frac{-43\Delta}{360} & \frac{-11\Delta}{180} & \frac{7\Delta}{1080}\\
\frac{-23\Delta}{60} & \frac{\Delta}{120} & \frac{-\Delta}{135}
\end{pmatrix}.$$

Then, Systems (6) and (8) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \tag{9}$$

Multiply both sides of Eqn. (9) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_2 F_{2,n} \tag{10}$$

where $B_1 = I_{15}$,

$$B_2 = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & \frac{2}{3} \\ 1 & 1 \\ 0 & \frac{1}{\Delta} \\ 0 & \frac{1}{\Delta} \\ 0 & \frac{1}{\Delta} \end{pmatrix}, B_3 = \begin{pmatrix} \frac{97\Delta^2}{3240} \\ \frac{28\Delta^2}{405} \\ \frac{13\Delta^2}{120} \\ 0 \\ 0 \\ 0 \end{pmatrix}, B_4 = \begin{pmatrix} \frac{19\Delta^2}{540} & \frac{-13\Delta^2}{1080} & \frac{\Delta^2}{405} \\ \frac{22\Delta^2}{135} & \frac{-2\Delta^2}{135} & \frac{2\Delta^2}{405} \\ \frac{3\Delta^2}{10} & \frac{3\Delta^2}{40} & \frac{\Delta^2}{60} \\ \frac{19\Delta}{72} & \frac{-5\Delta}{72} & \frac{\Delta}{72} \\ \frac{4\Delta}{9} & \frac{\Delta}{9} & 0 \\ \frac{3\Delta}{8} & \frac{3\Delta}{8} & \frac{\Delta}{8} \end{pmatrix}.$$

Then, we solve System (10) iteratively.

3. Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1} = -y_{n+\frac{1}{3}} + 2t_{n+\frac{2}{3}} + \Delta^2 \left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108} \right). \tag{11}$$

In addition, we study the zero stability, the order, and the error constant of the block method (10). The first and second characteristic functions are given by

$$\tau_1(z) = z^{1/3} - 2z^{2/3} + z$$

and

$$\tau_2(z) = \frac{z^{1/3}}{108} + \frac{5z^{2/3}}{54} + \frac{z}{108}.$$

Then,

$$\tau_1(1) = 0,$$
$$\tau_1'(1) = 0,$$
$$\tau_1''(1) - 2!\tau_2(1) = 0.$$

The roots of $\tau_1(z)$ for which $|z| = 1$ and $z \neq 1$ are simple. Thus, Eqn. (11) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\theta}, \theta \in [0, 2\pi].$$

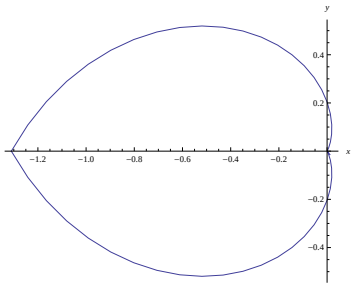


Figure 1: Region of absolute stability

Then, the interval of absolute stability is $(-1.31029, 0)$ and the region of absolute stability is given in Figure 1.

$$\bar{B_2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \bar{B_2}\right) = (s-1)s^5.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \to 0$. Using the Taylor series, Eqn. (11) becomes

$$y_{n+1} + y_{n+\frac{1}{3}} - 2t_{n+\frac{2}{3}} - \Delta^2\left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right) = \frac{-\Delta^6 y_n^{(6)}}{174960} + \cdots.$$

Thus, the order of Eqn. (11) is 4 and the error constant is $-0.000005715\Delta^6 y_n^{(6)}$. Similarly, the Taylor expansion of System (10) is give as

$$B_1Y_{1,n} - B_2Y_{2,n} - B_3F_{1,n} - B_2F_{2,n} = \begin{pmatrix} -0.0000200047\Delta^6 y_n^{(6)} \\ -0.00004572473\Delta^6 y_n^{(6)} \\ -0.000077160493\Delta^6 y_n^{(6)} \\ -0.0001085962505\Delta^6 y_n^{(6)} \\ -0.0000457247370\Delta^6 y_n^{(6)} \\ -0.0001543209876\Delta^6 y_n^{(6)} \end{pmatrix} + \ldots.$$

Thus, the block method (10) has the following order $(4,4,4,4,4,4)^t$ with error constant

$$\begin{pmatrix} -0.0000200047\Delta^6 y_n^{(6)} \\ -0.00004572473\Delta^6 y_n^{(6)} \\ -0.000077160493\Delta^6 y_n^{(6)} \\ -0.0001085962505\Delta^6 y_n^{(6)} \\ -0.0000457247370\Delta^6 y_n^{(6)} \\ -0.0001543209876\Delta^6 y_n^{(6)} \end{pmatrix}.$$

4. Numerical results

In this section, we present some of our numerical results to show the efficiency of the proposed method which is described in Section 2. Consider the following cubic free undamped Duffing oscillator equation of the form

$$y''(t) + ay(t) + by^3(t) = 0, 0 < t < T \tag{12}$$

subject to

$$y(0) = \alpha_0, y'(0) = 0. \tag{13}$$

The exact solution of Problem (12)-(13) is not Known. Therefore, the numerical solutions have determined by built in file of MATHEMATICA based on the fully explicit Runge-Kutta method and this solution is used as the standard or reference for comparison. In Tables 1 and 2, we compare our results with HPM [24] , MHPM [25], SHPM [24], and numerical solution for $\alpha_0 = 1, a = 1$, and $b = 1$ and for $\alpha_0 = 0.75, a = 1.5$, and $b = 1.5$, respectively

t	HPM	MHPM	SHPM	Present results	Numerical results
0.5	0.762476	0.768902	0.768766	0.768802	0.768802
1.0	0.176929	0.233741	0.233680	0.233692	0.233692
2.0	−1.055110	−0.891260	−0.859323	−0.859349	−0.859349
3.5	−0.461650	−0.079433	−0.093034	−0.093013	−0.093013
5.0	2.049041	−0.079433	0.947107	0.947130	0.947130

Table 1: The approximate solution for $\alpha_0 = 1, a = 1$, and $b = 1$.

t	HPM	MHPM	SHPM	Present results	Numerical results
1	0.056288	0.080176	0.080519	0.080527	0.080527
2	−0.808192	−0.739174	−0.729000	−0.729018	−0.729018
3	−0.339208	−0.239413	−0.238620	−0.238626	−0.238626
4	0.891267	0.706827	0.667953	0.668022	0.668022
5	0.893003	0.395315	0.387550	0.387551	0.387551

Table 2: The approximate solution for $\alpha_0 = 0.75, a = 1.5$, and $b = 1.5$.

5. Conclusion

We compare our results with HPM [26] , MHPM [27], SHPM [26], and numerical solution for $\alpha_0 = 1, a = 1$,and $b = 1$ and for $\alpha_0 = 1, a = 1$,and $b = 1$. From Tables 5.1 and 5.2, we see that our results agree exceptionally well with the numerical results and give more accurate results than HPM, MHPM, and SHPM.

The results in this paper confirm that the proposed method is a powerful and efficient method for solving nonlinear differential equations in different fields of sciences and engineering. In addition, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of the proposed method. Also, we study the zero stability, order, and the error constant of the block method which is generated from the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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References

[1] D. D. Ganji, A.R. Sahouli, M. Famouri, *A new modification of He's homotopy perturbation method for rapid convergence of nonlinear undamped oscillators*, J. Appl Math Comput, 30 (2009), 181-192.

[2] J. H. He, *Homotopy perturbation method: A new nonlinear analytical technique*, Appl. Math. Comput., 135 (2003), 73-79.

[3] A. Belendez, T. Belendez, A. Marquez, C. Neipp, *Application of He's homotopy perturbation method to conservative truly nonlinear oscillators*, Chaos, Solitons & Fractals, 37 (2008), 770 - 780.

[4] J. I. Ramos, *On the variational iteration method and other iterative techniques for nonlinear differential equations*, Appl. Math. Comput., 199 (2008), 39 - 69.

[5] M. I. Syam, H. I. Siyyam, *Numerical differentiation of implicitly defined curves*, J. Comput. Appl. Math., 108 (1999), 131-144.

[6] M. syam, *The modified Broyden-variational method for solving nonlinear elliptic differential equations*, Chaos, Solitions & Fractals, 32 (2007), 392-404.

[7] A.M. Wazwaz, *The variational iteration method for exact solutions of Laplace equation*, Phys. Lett. A, 363 (2007), 260–262.

[8] A.M. Wazwaz, R. Rach and J.S. Duan, *Solving the two-dimensional Lane-Emden type equations by the Adomian decomposition method*, J. Applied Mathematics and Statistics, 3 (2016), 15–26.

[9] S. Ghosh, A. Roy, D. Roy, *An adaptation of Adomian decomposition for numeric analytic integration of strongly nonlinear and chaotic oscillators*, Comput. Methods Appl. Mech. Eng., 196 (2007), 1133 -1153.

[10] J. I. Ramos, *An artificial parameter decomposition method for nonlinear oscillators: Applications to oscillators with odd nonlinearities,* J. Sound Vib., 307 (2007), 312-329.

[11] L. Xu, *Determination of limit cycle by He's parameter-expanding method for strongly nonlinear oscillators*, J. Sound Vib., 302 (2007), 178-184.

[12] A.H. Nayfeh, D.T Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.

[13] N.N. Moiseev, *Asymptotic heskie metodi nelinejnoj mehaniki*, Nauka, Moscow, 1981.

[14] J. Kevorkian, J. D. Cole, *Perturbation methods in applied mathematics*, in Applied Mathematical Sciences, 34, Springer Verlag, New York, 1981.

[15] R.E. Mickens, *Nonlinear oscillations*, Cambridge University Press, New York, 1981.

[16] M.F. El-Sayed, M.I. Syam, *Numerical study for the electrified instability of viscoelastic cylindrical dielectric fluid film surrounded by a conducting gas*, Physica A: Statistical Mechanics and its Applications 377 (2007), 381-400.

[17] M. Syam, *Cubic spline interpolation predictors over implicitly defined curves*, Journal of Computational and Applied Mathematics, 157 (2003), 283-295.

[18] M.F. El-Sayed, M.I. Syam, *Electrohydrodynamic instability of a dielectric compressible liquid sheet streaming into an ambient stationary compressible gas*, Archive of Applied Mechanics, 77 (2007), 613-626.

[19] M.I. Syam, H.I. Siyyam, *An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems*, Chaos, Solitons and Fractals, 39 (2009), 659-665.

[20] M.I. Syam, H.I. Siyyam, *Numerical differentiation of implicitly defined curves*, Journal of Computational and Applied Mathematics, 108 (1999), 131-144.

[21] M. Syam, *The modified Broyden-variational method for solving nonlinear elliptic differential equations*, Chaos, Solitons and Fractals, 32 (2009), 392-404.

[22] M. Syam, B.S. Attili, *Numerical solution of singularly perturbed fifth order two point boundary value problem*, Applied Mathematics and Computation, 170 (2005), 1085-1094.

[23] J.D. Lambert, *Numerical methods for ordinary differential systems*, New York: John Wiley, 1991.

[24] Prcious Sibanda, A. Khidir, *A new modification of the HPM for the Duffing equation with cubic nonlinearity,* Recent Researches in Applied and Computational Mathematics, ISBN: 978-1-61804-002-2, 139-145.

[25] D. D. Ganji, A.R. Sahouli, M. Famouri, *A new modification of He's homotopy perturbation method for rapid convergence of nonlinear undamped oscillators*, J. Appl. Math. Comput., 30 (2009), 181-192.

[26] M. Alquran, H.M. Jaradat, M.I. Syam, *Analytical solution of the time-fractional Phi-4 equation by using modified residual power series method*, Nonlinear Dynamics, 90 (2017), 2525-2529.

[27] M.A.Z. Raja, F.H. Shah, E.S. Alaidarous, M.I. Syam, *Design of bio-inspired heuristic technique integrated with interior-point algorithm to analyze the dynamics of heartbeat model*, Applied Soft Computing Journal, 52 (2017), 605-629.

[28] M.I. Syam, *Cubic spline interpolation predictors over implicitly defined curves*, Journal of Computational and Applied Mathematics, 157 (2003), 283-295.

[29] M.I. Syam, H.I. Siyyam, *Numerical differentiation of implicitly defined curves*, Journal of Computational and Applied Mathematics, 108 (1999), 131-144.

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Mixed monotone property and tripled fixed point theorems in partially ordered bipolar-metric spaces

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Abstract. In this paper, we obtain the existence and unique tripled fixed point theorems by using weak contractivity type and mixed monotone type mapping in a Bipolar metric space endowed with partial order. Also, an example which supports our main result is given.
Keywords: bipolar metric spaces, partially ordered set, mixed monotone property, tripled fixed point.

1. Introduction

In 1922, S. Banach [1] introduced the concept of Banach contraction principle. It is the most celebrated fixed point result in nonlinear analysis. The concept of mixed monotone mapping has been introduced by Bhaskar and Lakshmikantham [2] and established some coupled fixed point results for mixed monotone

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mappings. Subsequently to improve many authors have established coupled fixed point results for mixed monotone see ([3]-[7]). In 2011, V. Berinde and M. Borcut [8] initiated concept of triple fixed point and established some fixed point results for contractive mappings in partially ordered metric spaces. Afterward many investigators have established triples fixed point results for different spaces see ([9]-[11]).

Very recently, in 2016 Mutlu and Gürdal [12] introduced the notion of Bipolar metric spaces, which is one of the generalizations metric spaces. Also they investigated some fixed point and coupled fixed point results on this space, see ([12], [13]). Later, we proved some fixed point theorems in our earlier papers ([14]-[17]).

In this paper, we prove tripled fixed point results in bipolar metric spaces. Also, we gave an example which supports our main result.

2. Preliminaries

First we recall some basic definitions and examples.

Definition 2.1 ([12]). *Let A and B be two non-empty sets. Suppose that $d : A \times B \to [0, \infty)$ is a mapping satisfying the following properties :*

(B_0) *If $d(a, b) = 0$ then $a = b$ for all $(a, b) \in A \times B$,*

(B_1) *If $a = b$ then $d(a, b) = 0$, for all $(a, b) \in A \times B$,*

(B_2) *If $d(a, b) = d(b, a)$, for all $a, b \in A \cap B$.*

(B_3) *If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$.*

Then the mapping d is called a bipolar-metric on the pair (A, B) and the triple (A, B, d) is called a bipolar-metric space.

Example 2.2 ([12]). Let $A = (1, \infty)$ and $B = [-1, 1]$. Define $d : A \times B \to [0, +\infty)$ as $d(a, b) = |a^2 - b^2|$ for all $(a, b) \in (A, B)$, then the triple (A, B, d) is a bipolar-metric space.

Definition 2.3 ([12]). *Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets.*

The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be covariant map if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and denote $F : (A_1, B_1) \rightrightarrows (A_2, B_2)$.

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be contravariant map if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and denote $F : (A_1, B_1) \leftrightharpoons (A_2, B_2)$.

In particular, if d_1 and d_2 are bipolar metrics in (A_1, B_1) and (A_2, B_2) respectively, then some times we use the notations $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F : (A_1, B_1, d_1) \leftrightharpoons (A_2, B_2, d_2)$.

Definition 2.4 ([12]). *Let (A, B, d) be a bipolar metric space. A point $v \in A \cup B$ is said to be left point if $v \in A$, a right point if $v \in B$ and a central point if both.*

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence respectively.

In a bipolar metric space, sequence is the simple term for left or right sequence.

A sequence $\{v_n\}$ is convergent to a point v if and only if $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is sequence on the set $A \times B$. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence , if $\lim_{n,m\to\infty} d(a_n, b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.5 ([12])**.** *Let (A_1, B_1, d_1) and (A_2, B_2, d_2) be bipolar metric spaces.*

(*i*) *A map $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called left-continuous at a point $a_0 \in A_1$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_1(a_0, b) < \delta$ implies that $d_2(F(a_0), F(b)) < \epsilon$ for all $b \in B_1$.*

(*ii*) *A map $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called right-continuous at a point $b_0 \in B_1$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_1(a, b_0) < \delta$ implies that $d_2(F(a), F(b_0)) < \epsilon$ for all $a \in A_1$.*

(*iii*) *A map F is called continuous, if it left continuous at each point $a \in A_1$ and right continuous at each point $b \in B_1$.*

(*iv*) *A contravariant map $F : (A_1, B_1, d_1) \leftrightharpoons (A_2, B_2, d_2)$ is continuous if and only if it is continuous as a covariant map $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$.*

It can be seen from the definition (2.4) that a covariant or a contravariant map F from (A_1, B_1, d_1) to (A_2, B_2, d_2) is continuous if and only if $(u_n) \to v$ on (A_1, B_1, d_1) implies $F((u_n)) \to F(v)$ on (A_2, B_2, d_2).

3. Main results

Definition 3.1. *Let $(A, B, \leq)$ be a partial ordered set and $F : (A, B) \rightrightarrows (A, B)$ be a covariant mapping, we say that F is non-decreasing with respect to $\leq$ if $a, b \in A \cup B$, $a \leq b$ implies $F(a) \leq F(b)$, and similarly, a non-increasing mapping is defined.*

Definition 3.2. *Let $(A, B, \leq)$ be a partially ordered set and $F : (A^3, B^3) \rightrightarrows (A, B)$ be a covariant map. The map F has the mixed monotone property, if $F(a, b, c)$ is monotone non-decreasing in a and c and is monotone non-increasing in b, that is, for any $a, b, c \in A \cup B$.*

$$a_1, a_2 \in A \cup B, \quad a_1 \leq a_2 \Rightarrow F(a_1, b, c) \leq F(a_2, b, c)$$

$$b_1, b_2 \in A \cup B, \quad b_1 \leq b_2 \Rightarrow F(a, b_1, c) \geq F(a, b_2, c)$$

and

$$c_1, c_2 \in A \cup B, \quad c_1 \leq c_2 \Rightarrow F(a, b, c_1) \leq F(a, b, c_2)$$

Definition 3.3. *Let $F : (A^3, B^3) \rightrightarrows (A, B)$ be a covariant map, an element $(a, b, c) \in A^3 \cup B^3$ is called tripled fixed point of F if*

$$F(a, b, c) = a, \quad F(b, a, b) = b \quad \textit{and} \quad F(c, b, a) = c.$$

Let $(A, B, \leq)$ be a partially ordered set and d be a bipolar metric on (A, B) such that (A, B, d) is complete bipolar metric space. Moreover, we endow the product space (A^3, B^3) with the following partial order: For

$$(a, b, c), (p, q, r) \in A^3 \cup B^3, \quad (p, q, r) \leq (a, b, c) \Leftrightarrow a \geq p, \quad b \leq q, \quad c \geq r.$$

We begin with the following theorem that achieves the existence of a fixed point results for a mapping F on the product space (A^3, B^3).

Theorem 3.4. *Let $(A, B, \leq)$ be a partially ordered set and suppose that d is bipolar metric on (A, B) such that (A, B, d) is a complete bipolar metric spaces. Let the covariant map $F : (A^3, B^3) \rightrightarrows (A, B)$ is a continuous mapping having the mixed monotone property on (A, B) and μ , λ, κ be a non-negative constants with the condition*

$$d\left(F(l, m, n), F(r, s, t)\right) \leq \mu d\left(l, r\right) + \lambda d\left(m, s\right) + \kappa d\left(n, t\right) \tag{3.1}$$

$\forall \;\; l, m, n \in A \;\; r, s, t \in B$ *with* $l \geq r, \quad m \leq s, n \geq t$ *and* $\mu + \lambda + \kappa < 1$.
If there exist $(l_0, m_0, n_0) \in A^3 \cup B^3$ such that

$$l_0 \leq F(l_0, m_0, n_0), \quad m_0 \geq F(m_o, l_0, m_0) \textit{ and } n_0 \leq F(n_0, m_0, l_0).$$

Then there exist $(l, m, n) \in A^3 \cup B^3$ such that The mapping $F : A^3 \cup B^3 \to A \cup B$ has

$$F(l, m, n) = l, \qquad F(m, l, m) = m \quad \textit{and} \quad F(n, m, l) = n$$

Proof. Let $l_0, m_0, n_0 \in A$ and $r_0, s_0, t_0 \in B$, we choose an elements $l_1, m_1, n_1 \in A$ and $r_1, s_1, t_1 \in B$ such that

$$l_0 \leq F(l_0, m_0, n_0)) = l_1, \quad m_0 \geq F(m_o, l_0, m_0) = m_1, \text{ and } n_0 \leq F(n_0, m_0, l_0) = n_1,$$

and also

$$r_0 \leq F(r_0, s_0, t_0) = r_1, \quad s_0 \geq F(s_o, r_0, s_0) = s_1, \text{ and } t_0 \leq F(t_0, s_0, r_0) = t_1$$

similarly, we take

$$l_2 = F(l_1, m_1, n_1), \quad m_2 = F(m_1, l_1, m_1), \text{ and } n_2 = F(n_1, m_1, l_1)$$

$$r_2 = F(r_1, s_1, t_1), \quad s_2 = F(s_1, r_1, s_1) \text{ and } t_2 = F(t_1, s_1, r_1)$$

we denote

$$\begin{aligned}
F^2(l_0, m_0, n_0) &= F\left(F(l_0, m_0, n_0), F(m_0, l_0, m_0), F(n_0, m_0, l_0)\right) = F(l_1, m_1, n_1) = l_2,\\
F^2(m_0, l_0, m_0) &= F\left(F(m_0, l_0, m_0), F(l_0, m_0, n_0), F(m_0, l_0, m_0)\right) = F(m_1, l_1, m_1) = m_2,\\
F^2(n_0, m_0, l_0) &= F\left(F(n_0, m_0, l_0), F(m_0, l_0, m_0), F(l_0, m_0, n_0)\right) = F(n_1, m_1, l_1) = n_2,\\
F^2(r_0, s_0, t_0) &= F\left(F(r_0, s_0, t_0), F(s_0, r_0, s_0), F(t_0, s_0, r_0)\right) = F(r_1, s_1, t_1) = r_2,\\
F^2(s_0, r_0, s_0) &= F\left(F(s_0, r_0, s_0), F(r_0, s_0, t_0), F(s_0, r_0, s_0)\right) = F(s_1, r_1, s_1) = s_2,\\
F^2(t_0, s_0, r_0) &= F\left(F(t_0, s_0, r_0), F(s_0, r_0, s_0), F(r_0, s_0, t_0)\right) = F(t_1, s_1, r_1) = t_2,
\end{aligned}$$

for $n \geq 1$, we denote

$$\begin{aligned}
l_n &= F(l_{n-1}, m_{n-1}, n_{n-1}) \quad & r_n &= F(r_{n-1}, s_{n-1}, t_{n-1})\\
m_n &= F(m_{n-1}, l_{n-1}, m_{n-1}) \quad & s_n &= F(s_{n-1}, r_{n-1}, s_{n-1})\\
n_n &= F(n_{n-1}, m_{n-1}, l_{n-1}) \quad & t_n &= F(t_{n-1}, s_{n-1}, r_{n-1}).
\end{aligned} \tag{3.2}$$

Due to the mixed monotone property, We can obviously verify that

$$\begin{aligned}
&l_0 \leq F(l_0, m_0, n_0) = l_1 \leq F(l_1, m_1, n_1) = l_2 \leq \cdots \leq F(l_{n-1}, m_{n-1}, n_{n-1}) \leq \ldots\\
&m_0 \geq F(m_o, l_0, m_0) = m_1 \geq F(m_1, l_1, m_1) = m_2 \geq \cdots \geq F(m_{n-1}, l_{n-1}, m_{n-1}) \geq \ldots\\
&n_0 \leq F(n_0, m_0, l_0) = n_1 \leq F(n_1, m_1, l_1) = n_2 \leq \cdots \leq F(n_{n-1}, m_{n-1}, l_{n-1}) \leq \ldots\\
&r_0 \leq F(r_0, s_0, t_o) = r_1 \leq F(r_1, s_1, t_1) = r_2 \leq \cdots \leq F(r_{n-1}, s_{n-1}, t_{n-1}) \leq \ldots\\
&s_0 \geq F(s_o, r_0, s_o) = s_1 \geq F(s_1, r_1, s_1) = s_2 \geq \cdots \geq F(s_{n-1}, r_{n-1}, s_{n-1}) \geq \ldots\\
&t_0 \leq F(t_0, s_0, r_o) = t_1 \leq F(t_1, s_1, r_1) = t_2 \leq \cdots \leq F(t_{n-1}, s_{n-1}, r_{n-1}) \leq \ldots
\end{aligned}$$

thus, we obtain three bisequences satisfying the following conditions

$$l_0 \leq l_1 \leq l_2 \leq \cdots \leq l_n \leq \ldots, \quad r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq \ldots$$

$$m_0 \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq \ldots, \quad s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq \ldots$$

$$n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_n \leq \ldots, \quad t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \ldots$$

to simplify the writing, denote

$$D_n^{X_1} = d(l_{n-1}, r_n) \quad D_n^{Y_1} = d(m_{n-1}, s_n) \quad D_n^{Z_1} = d(n_{n-1}, t_n)$$

$$D_n^{X_2} = d(l_n, r_{n-1}) \quad D_n^{Y_2} = d(m_n, s_{n-1}) \quad D_n^{Z_2} = d(n_n, t_{n-1})$$

$$D_{n-1}^{X_3} = d(l_{n-1}, r_{n-1}) \quad D_{n-1}^{Y_3} = d(m_{n-1}, s_{n-1}) \quad D_{n-1}^{Z_3} = d(n_{n-1}, t_{n-1}).$$

Then by (3.1), we get

$$\begin{aligned}
D_2^{X_1} = d(l_1, r_2) &= d\left(F(l_0, m_0, n_0), F(r_1, s_1, t_1)\right)\\
&\leq \mu d\left(l_0, r_1\right) + \lambda d\left(m_0, s_1\right) + \kappa d\left(n_0, t_1\right)\\
&\leq \mu D_1^{X_1} + \lambda D_1^{Y_1} + \kappa D_1^{Z_1}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned} D_2^{Y_1} = d(m_1, s_2) &= d\left(F(m_0, l_0, m_0), F(s_1, r_1, s_1)\right) \\ &\leq \mu d\left(m_0, s_1\right) + \lambda d\left(l_0, r_1\right) + \kappa d\left(m_0, s_1\right) \\ &\leq \mu D_1^{Y_1} + \lambda D_1^{X_1} + \kappa D_1^{Y_1} \\ &= \lambda D_1^{X_1} + (\mu + \kappa) D_1^{Y_1} + 0 D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_2^{Z_1} = d(n_1, t_2) &= d\left(F(n_0, m_0, l_0), F(t_1, s_1, r_1)\right) \\ &\leq \mu d\left(n_0, t_1\right) + \lambda d\left(m_0, s_1\right) + \kappa d\left(l_0, r_1\right) \\ &\leq \mu D_1^{Z_1} + \lambda D_1^{Y_1} + \kappa D_1^{X_1} \end{aligned}$$

and

$$\begin{aligned} D_3^{X_1} = d(l_2, r_3) &= d\left(F(l_1, m_1, n_1), F(r_2, s_2, t_2)\right) \\ &\leq \mu d\left(l_1, r_2\right) + \lambda d\left(m_1, s_2\right) + \kappa d\left(n_1, t_2\right) \\ &\leq \mu D_2^{X_1} + \lambda D_2^{Y_1} + \kappa D_2^{Z_1} \\ &\leq (\mu^2 + \lambda^2 + \kappa^2) D_1^{X_1} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_1} + 2\mu\kappa D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_3^{Y_1} = d(m_2, s_3) &= d\left(F(m_1, l_1, m_1), F(s_2, r_2, s_2)\right) \\ &\leq \mu d\left(m_1, s_2\right) + \lambda d\left(l_1, r_2\right) + \kappa d\left(m_1, s_2\right) \\ &\leq \mu D_2^{Y_1} + \lambda D_2^{X_1} + \kappa D_2^{Y_1} \\ &\leq \lambda D_2^{X_1} + (\mu + \kappa) D_2^{Y_1} \\ &\leq \lambda\mu D_1^{X_1} + \lambda^2 D_1^{Y_1} + \lambda\kappa D_1^{Z_1}) + (\lambda\mu + \lambda\kappa) D_1^{X_1} \\ &\quad + (\mu + \kappa)^2 D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\leq (2\lambda\mu + \lambda\kappa) D_1^{X_1} + (\lambda^2 + (\mu + \kappa)^2) D_1^{Y_1} + \lambda\kappa D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_3^{Z_1} = d(n_2, t_3) &= d\left(F(n_1, m_1, l_1), F(t_2, s_2, r_2)\right) \\ &\leq \mu d\left(n_1, t_2\right) + \lambda d\left(m_1, s_2\right) + \kappa d\left(l_1, r_2\right) \\ &\leq \mu D_2^{Z_1} + \lambda D_2^{Y_1} + \kappa D_2^{X_1} \\ &\leq \mu(\mu D_1^{Z_1} + \lambda D_1^{Y_1} + \kappa D_1^{X_1}) + \lambda(\lambda D_1^{X_1} + (\mu + \kappa) D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\quad + \kappa(\mu D_1^{X_1} + \lambda D_1^{Y_1} + \kappa D_1^{Z_1}) \\ &\leq \mu^2 D_1^{Z_1} + \mu\lambda D_1^{Y_1} + \mu\kappa D_1^{X_1} + \lambda^2 D_1^{X_1} + (\lambda\mu + \lambda\kappa) D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\quad + \kappa\mu D_1^{X_1} + \kappa\lambda D_1^{Y_1} + \kappa^2 D_1^{Z_1} \\ &\leq (2\mu\kappa + \lambda^2) D_1^{X_1} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_1} + (\mu^2 + \kappa^2) D_1^{Z_1}. \end{aligned}$$

In order to simplify writing we also consider the matrix

$$X = \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu + \kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} = \begin{pmatrix} i_1 & j_1 & k_1 \\ l_1 & m_1 & n_1 \\ p_1 & j_1 & q_1 \end{pmatrix}$$

and further denote

$$X^2 = \begin{pmatrix} \mu^2 + \lambda^2 + \kappa^2 & 2\mu\lambda + 2\lambda\kappa & 2\mu\kappa \\ 2\lambda\mu & \lambda^2 + (\mu + \kappa)^2 & \lambda\kappa \\ 2\mu\kappa + \lambda^2 & 2\mu\lambda + 2\lambda\kappa & \mu^2 + \kappa^2 \end{pmatrix} = \begin{pmatrix} i_2 & j_2 & k_2 \\ l_2 & m_2 & n_2 \\ p_2 & j_2 & q_2 \end{pmatrix}$$

where $i_2+j_2+k_2 = l_2+m_2+n_2 = p_2+j_2+q_2 = (\mu+\lambda+\kappa)^2 < \mu+\lambda+\kappa < 1$. Now we prove by induction

$$X^n = \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix}$$

where

(3.3) $i_n+j_n+k_n = l_n+m_n+n_n = p_n+j_n+q_n = (\mu+\lambda+\kappa)^n < \mu+\lambda+\kappa < 1.$

Indeed, if we assume that (3.3) is true for n, then since

$$\begin{aligned} X^{n+1} = X^n.X &= \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix} . \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} \\ &= \begin{pmatrix} i_n\mu + j_n\lambda + k_n\kappa & i_n\lambda + j_n(\mu+\kappa) + k_n\lambda & i_n\kappa + k_n\mu \\ l_n\mu + m_n\lambda + n_n\kappa & l_n\lambda + m_n(\mu+\kappa) + n_n\lambda & l_n\kappa + n_n\mu \\ p_n\mu + j_n\lambda + q_n\kappa & p_n\lambda + j_n(\mu+\kappa) + q_n\lambda & p_n\kappa + q_n\mu \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} i_{n+1}+j_{n+1}+k_{n+1} &= i_n\mu + j_n\lambda + k_n\kappa + i_n\lambda + j_n(\mu+\kappa) + k_n\lambda + i_n\kappa + k_n\mu \\ &= (\mu+\lambda+\kappa)i_n + (\mu+\lambda+\kappa)j_n + (\mu+\lambda+\kappa)k_n \\ &= (i_n+j_n+k_n)(\mu+\lambda+\kappa) = (\mu+\lambda+\kappa)^n(\mu+\lambda+\kappa) \\ &= (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1. \end{aligned}$$

Similarly, we can prove that $l_{n+1}+m_{n+1}+n_{n+1} = p_{n+1}+j_{n+1}+q_{n+1} = (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1$. Therefore, we get

$$\begin{pmatrix} D_{n+1}^{X_1} \\ D_{n+1}^{Y_1} \\ D_{n+1}^{Z_1} \end{pmatrix} \le \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} . \begin{pmatrix} D_1^{X_1} \\ D_1^{Y_1} \\ D_1^{Z_1} \end{pmatrix}$$

that is

$$\begin{aligned} D_{n+1}^{X_1} &\le i_n D_1^{X_1} + j_n D_1^{Y_1} + k_n D_1^{Z_1} \\ D_{n+1}^{Y_1} &\le l_n D_1^{X_1} + m_n D_1^{Y_1} + n_n D_1^{Z_1} \\ D_{n+1}^{Z_1} &\le p_n D_1^{X_1} + j_n D_1^{Y_1} + q_n D_1^{Z_1}. \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned} D_2^{X_2} = d(l_2, r_1) &= d\left(F(l_1, m_1, n_1), F(r_0, s_0, t_0)\right) \\ &\le \mu d\left(l_1, r_0\right) + \lambda d\left(m_1, s_0\right) + \kappa d\left(n_1, t_0\right) \\ &\le \mu D_1^{X_2} + \lambda D_1^{Y_2} + \kappa D_1^{Z_2}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} D_2^{Y_2} = d(m_2, s_1) &= d\left(F(m_1, l_1, m_1), F(s_0, r_0, s_0)\right) \\ &\leq \mu d\left(m_1, s_0\right) + \lambda d\left(l_1, r_0\right) + \kappa d\left(m_1, s_0\right) \\ &\leq \mu D_1^{Y_2} + \lambda D_1^{X_2} + \kappa D_1^{Y_2} \\ &= \lambda D_1^{X_2} + (\mu + \kappa) D_1^{Y_2} + 0 D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_2^{Z_2} = d(n_2, t_1) &= d\left(F(n_1, m_1, l_1), F(t_0, s_0, r_0)\right) \\ &\leq \mu d\left(n_1, t_0\right) + \lambda d\left(m_1, s_0\right) + \kappa d\left(l_1, r_0\right) \\ &\leq \mu D_1^{Z_2} + \lambda D_1^{Y_2} + \kappa D_1^{X_2} \end{aligned}$$

and

$$\begin{aligned} D_3^{X_2} = d(l_3, r_2) &= d\left(F(l_2, m_2, n_2), F(r_1, s_1, t_1)\right) \\ &\leq \mu d\left(l_2, r_1\right) + \lambda d\left(m_2, s_1\right) + \kappa d\left(n_2, t_1\right) \\ &\leq \mu D_2^{X_2} + \lambda D_2^{Y_2} + \kappa D_2^{Z_2} \\ &\leq (\mu^2 + \lambda^2 + \kappa^2) D_1^{X_2} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_2} + 2\mu\kappa D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_3^{Y_2} = d(m_3, s_2) &= d\left(F(m_2, l_2, m_2), F(s_1, r_1, s_1)\right) \\ &\leq \mu d\left(m_2, s_1\right) + \lambda d\left(l_2, r_1\right) + \kappa d\left(m_2, s_1\right) \\ &\leq \mu D_2^{Y_2} + \lambda D_2^{X_2} + \kappa D_2^{Y_2} \\ &\leq \lambda D_2^{X_2} + (\mu + \kappa) D_2^{Y_2} \\ &\leq \lambda\mu D_1^{X_2} + \lambda^2 D_1^{Y_2} + \lambda\kappa D_1^{Z_2}) + (\lambda\mu + \lambda\kappa) D_1^{X_2} \\ &\quad + (\mu + \kappa)^2 D_1^{Y_2} + 0 D_1^{Z_2}) \\ &\leq (2\lambda\mu + \lambda\kappa) D_1^{X_2} + (\lambda^2 + (\mu + \kappa)^2) D_1^{Y_2} + \lambda\kappa D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_3^{Z_2} = d(n_3, t_2) &= d\left(F(n_2, m_2, l_2), F(t_1, s_1, r_1)\right) \\ &\leq \mu d\left(n_2, t_1\right) + \lambda d\left(m_2, s_1\right) + \kappa d\left(l_2, r_1\right) \\ &\leq \mu D_2^{Z_2} + \lambda D_2^{Y_2} + \kappa D_2^{X_2} \\ &\leq \mu(\mu D_1^{Z_2} + \lambda D_1^{Y_2} + \kappa D_1^{X_2}) + \lambda(\lambda D_1^{X_2} + (\mu + \kappa) D_1^{Y_2} + 0 D_1^{Z_1}) \\ &\quad + \kappa(\mu D_1^{X_2} + \lambda D_1^{Y_2} + \kappa D_1^{Z_2}) \\ &\leq \mu^2 D_1^{Z_2} + \mu\lambda D_1^{Y_2} + \mu\kappa D_1^{X_2} + \lambda^2 D_1^{X_2} + (\lambda\mu + \lambda\kappa) D_1^{Y_2} + 0 D_1^{Z_2}) \\ &\quad + \kappa\mu D_1^{X_2} + \kappa\lambda D_1^{Y_2} + \kappa^2 D_1^{Z_2} \\ &\leq (2\mu\kappa + \lambda^2) D_1^{X_2} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_2} + (\mu^2 + \kappa^2) D_1^{Z_2}. \end{aligned}$$

In order to simplify writing we also consider the matrix

$$Y = \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu + \kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} = \begin{pmatrix} i_1 & j_1 & k_1 \\ l_1 & m_1 & n_1 \\ p_1 & j_1 & q_1 \end{pmatrix}$$

and further denote

$$Y^2 = \begin{pmatrix} \mu^2 + \lambda^2 + \kappa^2 & 2\mu\lambda + 2\lambda\kappa & 2\mu\kappa \\ 2\lambda\mu & \lambda^2 + (\mu + \kappa)^2 & \lambda\kappa \\ 2\mu\kappa + \lambda^2 & 2\mu\lambda + 2\lambda\kappa & \mu^2 + \kappa^2 \end{pmatrix} = \begin{pmatrix} i_2 & j_2 & k_2 \\ l_2 & m_2 & n_2 \\ p_2 & j_2 & q_2 \end{pmatrix}$$

where $i_2+j_2+k_2 = l_2+m_2+n_2 = p_2+j_2+q_2 = (\mu+\lambda+\kappa)^2 < \mu+\lambda+\kappa < 1$. Now we prove by induction

$$Y^n = \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix}$$

where

(3.5) $i_n+j_n+k_n = l_n+m_n+n_n = p_n+j_n+q_n = (\mu+\lambda+\kappa)^n < \mu+\lambda+\kappa < 1.$

Indeed, if we assume that (3.5) is true for n, then since

$$\begin{aligned} Y^{n+1} = Y^n.Y &= \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix} . \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} \\ &= \begin{pmatrix} i_n\mu + j_n\lambda + k_n\kappa & i_n\lambda + j_n(\mu+\kappa) + k_n\lambda & i_n\kappa + k_n\mu \\ l_n\mu + m_n\lambda + n_n\kappa & l_n\lambda + m_n(\mu+\kappa) + n_n\lambda & l_n\kappa + n_n\mu \\ p_n\mu + j_n\lambda + q_n\kappa & p_n\lambda + j_n(\mu+\kappa) + q_n\lambda & p_n\kappa + q_n\mu \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} i_{n+1}+j_{n+1}+k_{n+1} &= i_n\mu + j_n\lambda + k_n\kappa + i_n\lambda + j_n(\mu+\kappa) + k_n\lambda + i_n\kappa + k_n\mu \\ &= (\mu+\lambda+\kappa)i_n + (\mu+\lambda+\kappa)j_n + (\mu+\lambda+\kappa)k_n \\ &= (i_n+j_n+k_n)(\mu+\lambda+\kappa) = (\mu+\lambda+\kappa)^n(\mu+\lambda+\kappa) \\ &= (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1. \end{aligned}$$

Similarly, we can prove that $l_{n+1}+m_{n+1}+n_{n+1} = p_{n+1}+j_{n+1}+q_{n+1} = (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1$. Therefore, we get

$$\begin{pmatrix} D_{n+1}^{X_2} \\ D_{n+1}^{Y_2} \\ D_{n+1}^{Z_2} \end{pmatrix} \le \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} . \begin{pmatrix} D_1^{X_2} \\ D_1^{Y_2} \\ D_1^{Z_2} \end{pmatrix}$$

that is

$$\begin{aligned} D_{n+1}^{X_2} &\le i_n D_1^{X_2} + j_n D_1^{Y_2} + k_n D_1^{Z_2} \\ D_{n+1}^{Y_2} &\le l_n D_1^{X_2} + m_n D_1^{Y_2} + n_n D_1^{Z_2} \\ D_{n+1}^{Z_2} &\le p_n D_1^{X_2} + j_n D_1^{Y_2} + q_n D_1^{Z_2}. \end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned} D_{n+1}^{X_3} &\le i_n D_1^{X_3} + j_n D_1^{Y_3} + k_n D_1^{Z_3} \\ D_{n+1}^{Y_3} &\le l_n D_1^{X_3} + m_n D_1^{Y_3} + n_n D_1^{Z_3} \\ D_{n+1}^{Z_3} &\le p_n D_1^{X_3} + j_n D_1^{Y_3} + q_n D_1^{Z_3}. \end{aligned} \tag{3.7}$$

Using the property (B_3), we obtain

$$\begin{aligned} &d(l_n, r_m) \leq d(l_n, r_{n+1}) + d(l_{n+1}, r_{n+1}) + \cdots d(l_{m-1}, r_m) \\ &d(m_n, s_m) \leq d(m_n, s_{n+1}) + d(m_{n+1}, s_{n+1}) + \cdots d(m_{m-1}, s_m) \\ &d(n_n, t_m) \leq d(n_n, t_{n+1}) + d(n_{n+1}, t_{n+1}) + \cdots d(n_{m-1}, t_m) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &d(l_m, r_n) \leq d(l_m, r_{m-1}) + d(l_{m-1}, r_{m-1}) + \cdots d(l_{n+1}, r_n) \\ &d(m_m, s_n) \leq d(m_m, s_{m-1}) + d(m_{m-1}, s_{m-1}) + \cdots d(m_{n+1}, s_n) \\ &d(n_m, t_n) \leq d(n_m, t_{m-1}) + d(n_{m-1}, t_{m-1}) + \cdots d(n_{n+1}, t_n) \end{aligned} \tag{3.9}$$

for each $n, m \in N$ with $n < m$. Then from (3.4), (3.6), (3.7), (3.8)and (3.9), we have

$$\begin{aligned} &d(l_n, r_m) + d(m_n, s_m) + d(n_n, t_m) \\ &\leq (d(l_n, r_{n+1}) + d(m_n, s_{n+1}) + d(n_n, t_{n+1})) \\ &+ (d(l_{n+1}, r_{n+1}) + d(m_{n+1}, s_{n+1}) + d(n_{n+1}, t_{n+1})) + \cdots \\ &+ (d(l_{m-1}, r_{m-1}) + d(m_{m-1}, s_{m-1}) + d(n_{m-1}, t_{m-1})) \\ &+ (d(l_{m-1}, r_m) + d(m_{m-1}, s_m) + d(n_{m-1}, t_m)) \\ &\leq D_{n+1}^{X_1} + D_{n+1}^{Y_1} + D_{n+1}^{Z_1} + D_{n+1}^{X_3} + D_{n+1}^{Y_3} + D_{n+1}^{Z_3} + \cdots \\ &+ D_{m-1}^{X_3} + D_{m-1}^{Y_3} + D_{m-1}^{Z_3} + D_m^{X_1} + D_m^{Y_1} + D_m^{Z_1} \\ &\leq ((i_n + i_{n+1} + \cdots + i_{m-1}) + (l_n + l_{n+1} + \cdots + l_{m-1}) \\ &+ (p_n + p_{n+1} + \cdots + p_{m-1}))D_1^{X_1} \\ &+ ((j_n + j_{n+1} + \cdots + j_{m-1}) + (m_n + m_{n+1} + \cdots + m_{m-1}))2D_1^{Y_1} \\ &+ ((k_n + k_{n+1} + \cdots + k_{m-1}) + (n_n + n_{n+1} + \cdots + n_{m-1}) \\ &+ (q_n + q_{n+1} + \cdots + q_{m-1}))D_1^{Z_1} \\ &+ ((i_n + i_{n+1} + \cdots + i_{m-2}) + (l_n + l_{n+1} + \cdots + l_{m-2}) \\ &+ (p_n + p_{n+1} + \cdots + p_{m-2}))D_1^{X_3} \\ &+ ((j_n + j_{n+1} + \cdots + j_{m-2}) + (m_n + m_{n+1} + \cdots + m_{m-2}))2D_1^{Y_3} \end{aligned}$$

$$\begin{aligned} &+ ((k_n + k_{n+1} + \cdots + k_{m-2}) + (n_n + n_{n+1} + \cdots + n_{m-2}) \\ &+ (q_n + q_{n+1} + \cdots + q_{m-2}))D_1^{Z_3} \\ &\leq (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1})(3D_1^{X_1} + 4D_1^{Y_1} + 3D_1^{Z_1}) \\ &+ (\beta^n + \beta^{n+1} + \cdots + \beta^{m-2})(3D_1^{X_3} + 4D_1^{Y_3} + 3D_1^{Z_3}) \\ &\leq \beta^n . \frac{1 - \beta^{m-n}}{1 - \beta}(3D_1^{X_1} + 4D_1^{Y_1} + 3D_1^{Z_1}) + \beta^n . \frac{1 - \beta^{m-n+1}}{1 - \beta}(3D_1^{X_3} + 4D_1^{Y_3} + 3D_1^{Z_3}) \\ &\to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Similarly, we can prove $d(l_m, r_n) + d(m_m, s_n) + d(n_m, t_n) \to 0$ as $n, m \to \infty$ where $\beta = \mu + \lambda + \kappa < 1$. Which shows (l_n, r_n), (m_n, s_n) and (n_n, t_n) are three Cauchy bisequences. Since (A, B, d) is complete bipolar metric spaces, there exist $l, m, n \in A$ and $r, s, t \in B$ with

$$\begin{aligned} &\lim_{n\to\infty} l_n = r \quad \lim_{n\to\infty} m_n = s \quad \lim_{n\to\infty} n_n = t \\ &\lim_{n\to\infty} r_n = l \quad \lim_{n\to\infty} s_n = m \quad \lim_{n\to\infty} t_n = n. \end{aligned} \tag{3.10}$$

Finally, we will prove

$$\begin{aligned} &r = F(l, m, n), \quad s = F(m, n, m) \quad \text{and} \quad t = F(n, m, l). \\ &l = F(r, s, t), \quad m = F(s, r, s) \quad \text{and} \quad n = F(t, s, r). \end{aligned}$$

Let $\epsilon > 0$. Since F is continuous at (l, m, n), or (r, s, t) for a given $\frac{\epsilon}{3} > 0$, there exists a $\delta > 0$ such that $d(l, r)+d(m, s)+d(n, t) < \delta \Rightarrow d(F(l, m, n), F(r, s, t)) < \frac{\epsilon}{3}$. Then by (3.10) it follows that $\xi = \min\left\{\frac{\delta}{3}, \frac{\epsilon}{3}\right\}$, there exist n_0, m_0, p_0 such that, for $n \geq n_0, m \geq m_0, p \geq p_0$, we have

$$\begin{aligned} &d(l_n, r) < \xi \quad d(m_n, s) < \xi \quad d(n_n, t) < \xi \\ &d(l, r_n) < \xi \quad d(m, s_n) < \xi \quad d(n, t_n) < \xi. \end{aligned}$$

Now, put $k_0 = max\{n_0, m_0, p_0\}$. Then, for any integer $n \geq k_0$, we have

$$\begin{aligned} d(F(l, m, n), r) &\leq d(F(l, m, n), r_{n+1}) + d(l_{n+1}, r_{n+1}) + d(l_{n+1}, r) \\ &\leq d(F(l, m, n), F(r_n, s_n, t_n)) + d(F(l_n, m_n, n_n), F(r_n, s_n, t_n)) \\ &\quad + d(l_{n+1}, r) \\ &\leq \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \xi < \epsilon \end{aligned}$$

which implies that $d(F(l, m, n), r) = 0$ that is $F(l, m, n) = r$. Similarly, we can prove that

$$\begin{aligned} &s = F(m, n, m) \quad \text{and} \quad t = F(n, m, l). \\ &l = F(r, s, t), \quad m = F(s, r, s) \quad \text{and} \quad n = F(t, s, r). \end{aligned}$$

On the other hand (3.10), we obtain

$$\begin{aligned} d(l, r) &= d(\lim_{n\to\infty} r_n, \lim_{n\to\infty} l_n) = \lim_{n\to\infty} d(l_n, r_n) = 0 \\ d(m, s) &= d(\lim_{n\to\infty} s_n, \lim_{n\to\infty} m_n) = \lim_{n\to\infty} d(m_n, s_n) = 0 \\ d(n, t) &= d(\lim_{n\to\infty} t_n, \lim_{n\to\infty} n_n) = \lim_{n\to\infty} d(n_n, t_n) = 0. \end{aligned}$$

So $l = r, m = s$ and $n = t$. Therefore, $(l, m, n) \in A^3 \cap B^2$ is a fixed point of F. □

The achieved theorem is still valid for the covariant map F is not necessarily continuous. Instead, we require that underlying bipolar metric space (A, B, d) has an additional postulate. We discuss this in the following result.

Theorem 3.5. *Let $(A, B, \leq)$ be a partially ordered set and suppose that (A, B, d) is complete bipolar metric spaces on (A, B) such that (A, B) has the following postulate:*

(*i*) *If a non-decreasing bisequence $(\{l_n,\}, \{r_n\}) \to (l, l)$ then $(l_n, l_n) \leq (l, l)$, $\forall\ n$;*

(*ii*) *If a non-increasing bisequence $(\{m_n,\}, \{s_n\}) \to (m, m)$ then $m \leq (m_n, s_n) \leq (m, m)$, $\forall\ n$;*

(*iii*) *If a non-decreasing bisequence $(\{n_n,\}, \{t_n\}) \to (n, n)$ then $(n_n, t_n) \leq (n, n)$, $\forall\ n$.*

Let $F : \left(A^3, B^3\right) \rightrightarrows (A, B)$ be covariant mapping having the mixed monotone property on (A, B) and μ , λ, κ be a non-negative constants with the condition (3.1) satisfied for each $l \geq r, m \leq s$ and $n \geq t$ and $\mu + \lambda + \kappa < 1$. If there exist $(l_0, m_0, n_0) \in A^3 \cup B^3$ such that

$$l_0 \leq F(l_0, m_0, n_0), \quad m_0 \geq F(m_o, l_0, m_0) \text{ and } n_0 \leq F(n_0, m_0, l_0)$$

then there exist $(l, m, n) \in A^3 \cup B^3$ such that the mapping $F : A^3 \cup B^3 \to A \cup B$ has

$$F(l, m, n) = l, \quad F(m, l, m) = m \quad \text{and } F(n, m, l) = n$$

Proof. Following the proof of previous Theorem 3.4, we only have to prove $F(l, m, n) = l$, $F(m, l, m) = m$ and $F(n, m, l) = n$. Let $\epsilon > 0$.

Since $\{F^n(l_0, m_0, n_0)\} \to r$, $\{F^n(m_0, l_0, m_0)\} \to s$, $\{F^n(n_0, m_0, l_0)\} \to t$ and $\{F^n(r_0, s_0, t_0)\} \to l$, $\{F^n(s_0, r_0, s_0)\} \to m$, $\{F^n(t_0, s_0, r_0)\} \to n$. Then there exist $n_1, n_2, n_3, n_4, n_5, n_6 \in N$ such that for all $n_1 \geq n, n_2 \geq m, n_3 \geq p$ and $n_4 \geq k, , n_5 \geq q, n_6 \geq w$ every $\epsilon > 0$, we have

$$d\left(F^n(l_0, m_0, n_0), r\right) < \tfrac{\epsilon}{4}, \quad d\left(F^m(m_0, l_0, m_0), s\right) < \tfrac{\epsilon}{4}, \ d\left(F^p(n_0, m_0, l_0), t\right) < \tfrac{\epsilon}{4}$$

$$d\left(l, F^k(r_0, s_0, t_0)\right) < \tfrac{\epsilon}{4}, \quad d\left(m, F^q(s_0, r_0, s_0)\right) < \tfrac{\epsilon}{4}, \ d\left(n, F^w(t_0, s_0, r_0)\right) < \tfrac{\epsilon}{4}$$

for each $n \geq n_1$ and every $\epsilon > 0$, since $(\{F^n(l_0, m_0, n_0)\}, \{F^n(r_0, s_o, t_0)\})$, $(\{F^n(m_0, l_0, m_0)\}, \{F^n(s_0, r_o, s_0)\})$ and $(\{F^n(n_0, m_0, l_0)\}, \{F^n(t_0, s_o, r_0)\})$ are Cauchy bisequences, we get

$$d\left(F^n(l_0, m_0, n_0), F^n(r_0, s_o, t_0)\right) < \tfrac{\epsilon}{4}, \quad d\left(F^n(m_0, l_0, m_0), F^n(s_0, r_o, s_0)\right) < \tfrac{\epsilon}{4}$$
$$\text{and } d\left(F^n(n_0, m_0, l_0), F^n(t_0, s_o, r_0)\right) < \tfrac{\epsilon}{4}$$

taking $n \in N$, $n \geq \{n_1, n_2, n_3, n_4, n_5, n_6\}$ and using

$$F^n(l_0, m_0, n_0) \leq r, \quad F^n(m_0, l_0, m_0) \geq s \text{ and } F^n(n_0, m_0, l_0) \leq t$$
$$F^n(r_0, s_0, t_0) \leq l, \quad F^n(s_0, r_0, m_0) \geq m \text{ and } F^n(t_0, s_0, r_0) \leq n$$

we obtain

$$\begin{aligned}
&d\left(F\left(l,m,n\right),r\right) \leq d\left(F\left(l,m,n\right),F^{n+1}\left(r_0,s_0,t_0\right)\right)\\
&\quad + d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right) + d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&\leq d\left(F\left(l,m,n\right),F\left(F^n\left(r_0,s_0,t_0\right),F^n\left(s_0,r_0,s_0\right),F^n\left(t_0,s_0,r_0\right)\right)\right)\\
&+ d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right) + d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&\leq \mu d\left(l,F^n\left(r_0,s_0,t_0\right)\right) + \lambda d\left(m,F^n\left(s_0,r_0,s_0\right)\right) + \kappa d\left(n,F^n\left(t_0,s_0,r_0\right)\right)\\
&+ d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right) + d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&< (\mu+\lambda+\kappa)\frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.
\end{aligned}$$

This implies $d\left(F\left(l,m,n\right),r\right) = 0$, hence $F\left(l,m,n\right) = r$. Similarly, we obtain $F\left(m,l,n\right) = s$, $F\left(n,m,l\right) = t$, $F\left(r,s,t\right) = l$, $F\left(s,r,s\right) = m$ and $F\left(t,s,r\right) = n$.

On the other hand,

$$\begin{aligned}
d\left(l,r\right) &= d\left(\lim_{n\to\infty} F^n(l_0,m_0,n_0), \lim_{n\to\infty} F^n(r_0,s_0,t_0)\right)\\
&= \lim_{n\to\infty} d\left(F^n(l_0,m_0,n_0), F^n(r_0,s_0,t_0)\right) = 0.
\end{aligned}$$

Similarly, we show that $d(m,s) = 0$ and $d(n,t) = 0$. Therefore, $l = r$, $m = s$ and $n = t$, hence $F\left(l,m,n\right) = l$, $F\left(m,l,m\right) = m$ and $F\left(n,m,l\right) = n$ □

Theorem 3.6. *Adding conditions of Theorem 3.4 to the hypothesis of theorem 3.5, then the mapping $F : A^3 \cup B^3 \to A \cup B$ has unique tripled fixed point.*

Proof. Let $(l^*,m^*,n^*) \in A^3 \cup B^3$ be a another fixed point of F. Then we prove that $d\left(l,l^*\right) + d\left(m,m^*\right) + d\left(n,n^*\right) = 0$ where

$$\lim_{n\to\infty} F^n\left(l_0,m_0,n_0\right) = l \quad \lim_{n\to\infty} F^n\left(m_0,l_0,m_0\right) = m, \ \lim_{n\to\infty} F^n\left(n_0,m_0,l_0\right) = n.$$

If $(l^*,m^*,n^*) \in A^3$ and (l,m,n) is comparable to (l^*,m^*,n^*) with respect to the partial ordering in $\left(A^3,B^3\right)$, then, for every $n \in N$, we have

$$\begin{aligned}
&\left(F^n\left(l,m,n\right),F^n\left(m,l,m\right),F^n\left(n,m,l\right)\right) = \left(l,m,n\right) \text{ is comparable to}\\
&\qquad \left(F^n\left(l^*,m^*,n^*\right),F^n\left(m^*,l^*,m^*\right),F^n\left(n^*,m^*,l^*\right)\right) = \left(l^*,m^*,n^*\right).
\end{aligned}$$

Now we claim that $d(l,l^*) + d(m,m^*) + d(n,n^*) = 0$

$$\begin{aligned}
d\left(l^*,l\right) &= d\left(F^n\left(l^*,m^*,n^*\right),F^n\left(l,m,n\right)\right)\\
&= d\left(\begin{array}{c} F\left(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(n^*,m^*,l^*\right)\right),\\ F\left(F^{n-1}\left(l,m,n\right),F^{n-1}\left(m,l,m\right),F^{n-1}\left(n,m,l\right)\right)\end{array}\right)\\
&\leq \mu d(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(l,m,n\right))\\
&+ \lambda d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right))\\
&+ \kappa d(F^{n-1}\left(n^*,m^*,l^*\right),F^{n-1}\left(n,m,l\right)) \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned} d\left(m^*,m\right) &= d\left(F^n\left(m^*,l^*,m^*\right),F^n\left(m,l,m\right)\right) \\ &= d\begin{pmatrix} F\left(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(m^*,l^*,m^*\right)\right), \\ F\left(F^{n-1}\left(m,l,m\right),F^{n-1}\left(l,m,n\right)\right),F^{n-1}\left(m,l,m\right) \end{pmatrix} \\ &\leq \mu d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right)) \\ &+ \lambda d(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(l,m,n\right)) \\ &+ \kappa d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right)) \end{aligned} \tag{3.12}$$

and also

$$\begin{aligned} d\left(n^*,n\right) &= d\left(F^n\left(n^*,m^*,l^*\right),F^n\left(n,m,l\right)\right) \\ &= d\begin{pmatrix} F\left(F^{n-1}\left(n^*,m^*,l^*\right),F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(l^*,m^*,n^*\right)\right), \\ F\left(F^{n-1}\left(n,m,l\right),F^{n-1}\left(m,l,m\right),F^{n-1}\left(l,m,n\right)\right) \end{pmatrix} \\ &\leq \mu d(F^{n-1}\left(n^*,m^*,l^*\right),F^{n-1}\left(n,m,l\right)) \\ &+ \lambda d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right)) \\ &+ \kappa d(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(l,m,n\right)) \end{aligned} \tag{3.13}$$

for all $n \in N$, combining (3.11), (3.12) and (3.13)

$$\begin{aligned} \begin{pmatrix} d\left(l^*,l\right) \\ +d\left(m^*,m\right) \\ +d\left(m^*,m\right) \end{pmatrix} &\leq (\mu+\lambda+\kappa)\begin{Bmatrix} d(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(l,m,n\right)) \\ +d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right)) \\ +d(F^{n-1}\left(n^*,m^*,l^*\right),F^{n-1}\left(n,m,l\right)) \end{Bmatrix} \\ &\leq (\mu+\lambda+\kappa)^2\begin{Bmatrix} d(F^{n-2}\left(l^*,m^*,n^*\right),F^{n-2}\left(l,m,n\right)) \\ +d(F^{n-2}\left(m^*,l^*,m^*\right),F^{n-2}\left(m,l,m\right)) \\ +d(F^{n-2}\left(n^*,m^*,l^*\right),F^{n-2}\left(n,m,l\right)) \end{Bmatrix} \\ &\vdots \\ &\leq (\mu+\lambda+\kappa)^n\begin{Bmatrix} d(F\left(l^*,m^*,n^*\right),F\left(l,m,n\right)) \\ +d(F\left(m^*,l^*,m^*\right),F\left(m,l,m\right)) \\ +d(F\left(n^*,m^*,l^*\right),F\left(n,m,l\right)) \end{Bmatrix} \\ &\leq (\mu+\lambda+\kappa)^n(d(l,l^*)+d(m,m^*)+d(n,n^*)). \end{aligned}$$

Since $\mu+\lambda+\kappa<1$ which implies $d\left(l^*,l\right)+d\left(m^*,m\right)+d\left(n^*,n\right)=0$. Hence we obtain $l^*=l$, $m^*=m$ and $n=n^*$.

Similarly, If $(l^*,m^*,n^*)\in B^3$ and (l,m,n) is comparable to (l^*,m^*,n^*) with respect to the partial ordering in $\left(A^3,B^3\right)$, then we have $l^*=l$, $m^*=m$ and $n=n^*$.

If $(l^*,m^*,n^*)\in A^3$ and (l,m,n) is not comparable to (l^*,m^*,n^*), then there exist two comparable lower or upper bounds (a,b,c) , $(a^*,b^*,c^*)\in A^3\cup B^3$ of (l,m,n) and (l^*,m^*,n^*). Then, for all $n\in N$, $(F^n(a,b,c),F^n(b,a,b),F^n(c,b,b))=(a,b,c)$ and $(F^n(a^*,b^*,c^*),F^n(b^*,a^*,b^*),F^n(c^*,b^*,a^*))=(a^*,b^*,c^*)$ is comparable to $(F^n(l,m,n),F^n(m,l,m),F^n(n,m,l),)=(l,m,n)$ and $(F^n(l^*,m^*,n^*),$

$F^n(m^*,l^*,m^*), F^n(n^*,m^*,l^*)) = (l^*,m^*,n^*)$. Then we have

$$\begin{aligned}
d\left(\begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad \begin{pmatrix} l^* \\ m^* \\ n^* \end{pmatrix}\right) &= d\left(\begin{pmatrix} F^n(l,m,n) \\ F^(m,n,m) \\ F^n(n,m,l) \end{pmatrix} \quad \begin{pmatrix} F^n(l^*,m^*,n^*) \\ F^(m^*,n^*,m^*) \\ F^n(n^*,m^*,l^*) \end{pmatrix}\right) \\
&\leq d\left(\begin{pmatrix} F^n(l,m,n) \\ F^(m,n,m) \\ F^n(n,m,l) \end{pmatrix} \quad \begin{pmatrix} F^n(a,b,c) \\ F^(b,a,b) \\ F^n(c,b,a) \end{pmatrix}\right) \\
&+ d\left(\begin{pmatrix} F^n(a^*,b^*,c^*) \\ F^(b^*,a^*,b^*) \\ F^n(c^*,b^*,a^*) \end{pmatrix} \quad \begin{pmatrix} F^n(a,b,c) \\ F^(b,a,b) \\ F^n(c,b,a) \end{pmatrix}\right) \\
&+ d\left(\begin{pmatrix} F^n(a^*,b^*,c^*) \\ F^(b^*,a^*,b^*) \\ F^n(c^*,b^*,a^*) \end{pmatrix} \quad \begin{pmatrix} F^n(l^*,m^*,n^*) \\ F^(m^*,l^*,m^*) \\ F^n(n^*,m^*,l^*) \end{pmatrix}\right) \\
&\leq (\mu+\lambda+\kappa)^n(d(l,a)+d(m,b)+d(n,c)) \\
&+ (\mu+\lambda+\kappa)^n(d(a^*,a)+d(b^*,b)+d(c^*,c)) \\
&+ (\mu+\lambda+\kappa)^n(d(a^*,l^*)+d(b^*,m^*)+d(c^*,n^*)) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

so that

$$d\left(\begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad \begin{pmatrix} l^* \\ m^* \\ n^* \end{pmatrix}\right) = 0$$

which implies $l^* = l$, $m^* = m$ and $n^* = n$. Similarly, If $(l^*,m^*,n^*) \in B^3$ and (l,m,n) is incomparable to (l^*,m^*,n^*) with respect to the partial ordering in $\left(A^3,B^3\right)$, then we have $l^* = l$, $m^* = m$ and $n^* = n$. Therefore $(l,m,n) \in A^3 \cap B^3$. Hence (l,m,n) is unique tripled fixed point of F. □

Example 3.7. Let $A = \{U_m(R)/U_m(R)$ is upper triangular matrices over R$\}$ and $B = \{L_m(R)/L_m(R)$ is lower triangular matrices over R $\}$ with the bipolar metric

$$d\,(P,Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|,$$

for all $P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. On the set (A,B), we consider the following relation :

$$P,Q \in A \cup B, P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij},$$

where $\leq$ is usual ordering. Then clearly, (A,B,d) is a complete bipolar metric space and $(A,B,\preceq)$ is a partially ordered set, and (A,B) has the property as in Theorem (3.4). Let $F : \left(A^3,B^3\right) \rightrightarrows (A,B)$ be defined as

$$\begin{aligned}
F\,(P,Q,R) &= \left(\tfrac{3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij}}{12}\right)_{m\times m}, \\
&\forall \quad (P=(p_{ij})_{m\times m}, Q=(q_{ij})_{m\times m}, R=(r_{ij})_{m\times m}) \in A^3 \cup B^3.
\end{aligned}$$

Then obviously, F has the mixed monotone property, taking

$$\left\{ \begin{array}{l} (P=(p_{ij})_{m\times m}, Q=(q_{ij})_{m\times m}, R=(r_{ij})_{m\times m}) \\ (S=(s_{ij})_{m\times m}, T=(t_{ij})_{m\times m}, U=(u_{ij})_{m\times m}) \end{array} \in A^3 \cup B^3 \right\}$$

with $P \succeq S$, $Q \preceq T$ and $R \succeq U$ that is $p_{ij} \geq s_{ij}$, $q_{ij} \leq t_{ij}$ and $r_{ij} \geq u_{ij}$, we have

$$d(F(P,Q,R),F(S,T,U)) = d(\frac{3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij}}{12}, \frac{3s_{ij}+6t_{ij}-2u_{ij}+9i_{ij}}{12})$$
$$= \frac{1}{12}\sum_{i,j=1}^{m} |(3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij}) - (3s_{ij}+6t_{ij}-2u_{ij}+9i_{ij})|$$
$$\leq \frac{1}{12}\left(\sum_{i,j=1}^{m}|3p_{ij}-3s_{ij}| + \sum_{i,j=1}^{m}|6q_{ij}-6t_{ij}| + \sum_{i,j=1}^{m}|2r_{ij}-2u_{ij}|\right)$$
$$\leq \frac{1}{12}\left(3\sum_{i,j=1}^{m}|p_{ij}-s_{ij}| + 6\sum_{i,j=1}^{m}|q_{ij}-t_{ij}| + 2\sum_{i,j=1}^{m}|r_{ij}-u_{ij}|\right)$$
$$\leq \frac{1}{12}\left(3d(P,S)+6d(Q,T)+2d(R,U)\right)$$
$$\leq \frac{3}{12}d(P,S) + \frac{6}{12}d(Q,T) + \frac{2}{12}d(R,U).$$

It is easy to verified that F satisfies Theorem (3.4) and Theorem (3.6) with $\mu = \frac{1}{4}$, $\lambda = \frac{1}{2}$ and $\kappa = \frac{1}{6}$ and $(\frac{9}{5}I, \frac{9}{5}I, \frac{9}{5}I)$ is the unique tripled fixed point of F. Note that in this case we have $P = Q = R$.

Definition 3.8. *Let $F : (A \times B \times A, B \times A \times B) \rightrightarrows (A, B)$ be a covariant map, an element $(a, p, b) \in A \times B \times A$ is called tripled fixed point of F if*

$$F(a,p,b) = a, F(p,a,p) = p \text{ and } F(b,p,a) = b.$$

Theorem 3.9. *Let $F : (A \times B \times A, B \times A \times B) \rightrightarrows (A, B)$ be a covariant map. If F is a continuous mapping having the mixed monotone property on (A, B) and μ , λ, κ be a non-negative constants with the condition*

$$d\left(F(l,r,m),F(s,n,t)\right) \leq \mu d\left(l,s\right) + \lambda d\left(n,r\right) + \kappa d\left(m,t\right), \tag{3.14}$$

$\forall l, m, n \in Ar, s, t \in B$ with $l \geq s$, $n \leq r$, and $m \geq t$ with $\mu + \lambda + \kappa < 1$. If there exist $(l_0, r_0, m_0) \in (A \times B \times A) \cup (B \times A \times B)$ such that

$$l_0 \leq F(l_0, r_0, m_0), r_0 \geq F(r_o, l_0, r_0) \text{ and } m_0 \leq F(m_0, r_0, l_0),$$

then there exists $(l, r, m) \in (A \times B \times A) \cup (B \times A \times B)$ such that the mapping $F : (A \times B \times A) \cup (B \times A \times B) \to A \cup B$ has

$$F(l,r,m) = l, F(r,l,r) = r \text{ and } F(m,r,l) = l.$$

4. Conclusions

In the present research, we introduced and proved tripled fixed point theorems by using weak contractive type and mixed monotone type mappings, defined on a partially ordered bipolar metric space and gave suitable example that support our main result.

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References

[1] S. Banach, *Sur les operations dans les ensembles abstraits etleur applications aux equations integrales*, Fund. Math., 3 (1922), 133-181.

[2] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. TMA, 65 (2006), 1379-1393.

[3] Nguyen Van Luong, Nguyen Xuan Thuan, Trinh Thi Hai, *Coupled fixed point theorems in partially ordered metric spaces depended on another function*, Bulletin of Mathematical Analysis and Applications, ISSN:1821-1291, 3 (2011), 129-140.

[4] V. Lakshmikantam, Ljubomir Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially oredered metric spaces*, Nonlinear Analysis, 70 (2009), 4341-4349.

[5] N. V. Luong, N.X. Thuan, *Coupled fixed point in partially ordered metric spaces and application*, Nonlinaer Anal., 74 (2011), 983-992.

[6] W. Sintunavarat, Y. J. Ch and P. Kumam, *Coupled fixed point theorems for Nonlinear contractions without mixed monotone property*, Fixed point theory and Applications, 2012, 170.

[7] Nguyen V. Can, Vasile Berinde, Nguyen V. Luong, Nguyen X. Thuan, *A Coupled coincidence point theorem in partially ordered metric spaces*, Kragujevac Journal of Mathematics, 37 (2013), 103-119.

[8] Vasile Berinde, Marin Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Analysis, 74 (2011), 4889-4897.

[9] K.P.R.Rao, G.N.V. Kishore and K. Tas,*A unique common tripled fixed point theorem for hybrid pair of mappings*, Abstract and Applied Analysis, Volum 2012, Article ID 750403, 9 pages.

[10] K.P.R.Rao, G.N.V. Kishore, P.R. Sobhana Babu, *Triple coincidence point theorems for multivalued maps in partially ordered metric spaces*, Universal Journal of Computational Mathematics, 1 (2013), 19-23.

[11] K.P.R.Rao, G.N.V. Kishore, *A unique common triple fixed point theorem in partially ordered cone metric spaces*, Bulletin of Mathematical Analysis and Applications, ISSN: 1821-1291, Volume 3 Issue 4 (2011), pages 213-222.

[12] Ali Mutlu, Utku Gürdal, *Bipolar metric spaces and some fixed point theorems*, J. Nonlinear Sci. Appl., 9 (2016), 5362-5373.

[13] Ali Mutlu, Kübra Özkan, Utku Gürdal, *Coupled fixed point theorems on bipolar metric spaces*, European Journal of Pure and Applied Mathematics, 10 (2017), 655-667.

[14] B.Srinuvasa Rao, G.N.V.Kishore, G.Kiran Kumar, *Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy*, International Journal of Mathematics Trends and Technology (IJMTT), 63 (2018).

[15] G.N.V. Kishore, Ravi P. Agarwal, B. Srinuvasa Rao, and R.V.N. Srinivasa Rao, *Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications*, Fixed Point Theory and Applications(2018) 2018:21, https://doi.org/10.1186/s13663-018-0646-z.

[16] B. Srinuvasa Rao, G.N.V. Kishore,*Coupled fixed point theorems in partially ordered bipolar metric spaces*, International Journal of Innovative Research and Practices, Vol.6, Issue 4, April 2018.

[17] B. Srinuvasa Rao, G.N.V. Kishore, S. Ramalingeswara Rao, *Fixed point theorems under new caristi type contraction in bipolar metric space with applications*, International Journal of Engineering and Technology, 7 (2017), 106-110.

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An automorphism theorem on certain type B semigroups

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Abstract. Motivated by studying translational hulls in semigroups and rings, and also motivated M. Petrich and J. E. Ault's works in inverse semigroups in terms of translational hulls, we attempt in this article to study the idempotents of the translational hull of a type B semigroup. Our main result is to prove that the idempotent set of the translational hull of an arbitrary type B semigroup is isomorphic to the translational hull of the idempotent set of such a semigroup.
Keywords: translational hulls, type B semigroups, reductive.

1. Introduction

Recall that a mapping λ [resp., ρ] which maps a semigroup S into itself is a left [resp., right] translation of S if $\lambda(ab) = (\lambda a)b$ [resp., $(ab)\rho = a(b\rho)$], for

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all $a, b \in S$; if also $a(\lambda b) = (a\rho)b$ for all $a, b \in S$, then λ and ρ are *linked* and the pair (λ, ρ) is a *bitranslation* of S. As usual, $\Lambda(S)$ and $P(S)$ denote the set of all left and right translations of S, respectively. $\Omega(S)$ denotes the set of all bitranslation of S. If we define a multiplication on $\Omega(S)$ by the rule $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$ for all $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$, then it is easy to check that $\Omega(S)$ forms a subsemigroup of $\Lambda(S) \times P(S)$. We call the semigroup $\Omega(S)$ *the translational hull* of S. Further, we denote

$$\widetilde{\Lambda}(S) = \{\lambda \in \Lambda(S) |\ (\lambda, \rho) \in \Omega(S) \quad \text{for some} \quad \rho \in P(S)\},$$

$$\widetilde{P}(S) = \{\rho \in P(S) |\ (\lambda, \rho) \in \Omega(S) \quad \text{for some} \quad \lambda \in \Lambda(S)\} \quad \text{and}$$

$$\Pi(S) = \{(\lambda_a, \rho_a) |\ a \in S,\ \lambda_a x = ax,\ x\rho_a = xa \quad \text{for all} \quad x \in S\}.$$

The translational hull is an important concept in the algebraic theory of semigroups (see, [11-12]). Many papers studying the translational hulls of various classes of semigroups can be found in [1, 4, 5, 7, 10, 14, 15].

Following [2], the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S are generalizations of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on S, respectively. Two elements a, b of S are $\mathcal{L}^*-$ [resp., $\mathcal{R}^*-$] related on S if and only if they are $\mathcal{L}-$ [resp., $\mathcal{R}-$] related on a semigroup T such that S is a subsemigroup of T. A semigroup S is called *right abundant* [resp., *left abundant*] if each $\mathcal{L}^*$ class [resp., $\mathcal{R}^*$ class] of S contains at least one idempotent. A semigroup S is called *abundant* if it is both left and right abundant. A right [resp., left] abundant semigroup in which the idempotents commute is *right adequate* [resp., *left adequate*]. A semigroup S is called *adequate* if and only if it is both right and left adequate. Usually, we denote by a^+ [a^*] a typical idempotent $\mathcal{R}^*$-related [$\mathcal{L}^*$-related] to a; $E(S)$ denotes the set of idempotents of S. If S is a semigroup and $\mathcal{K}$ is one of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on S, then $\mathcal{K}^S$ and $\mathcal{K}^{\Omega(S)}$ denote the $\mathcal{K}$ relation on S and $\Omega(S)$, respectively.

A right adequate semigroup S is called *right type B*, if it satisfies the following conditions:

(B1) $(\forall e, f \in E(S^1), a \in S)\ (efa)^* = (ea)^*(fa)^*$, where $S^1 = S$ if S has an identity element 1, otherwise $S^1 = S \cup \{1\}$;

(B2) $(\forall a \in S, e \in E(S))\ e \leq a^* \Longrightarrow (\exists f \in E(S^1))\ \ e = (fa)^*$, where " $\leq$ " is a natural partial order on $E(S)$ (i.e., $(\forall x, y \in E(S))\quad x \leq y \Leftrightarrow x = xy = yx$ $(see, [6])$).

Dually, we can define a *left type B* semigroup. A semigroup which is both right and left type B is called *type B* (see, [3]). Recently, Li and others study some classes of type B semigroups (refer to [7, 8, 9]). In [4], Fountain proved that the translational hull of an adequate semigroup is adequate. For a type B semigroup S, the translational hull of S, $\Omega(S)$, is again a type B semigroup (refer to [7]), and thus the idempotents of $\Omega(S)$ form a semilattice (i.e., the

idempotents are commute). How the structure of this semilattice, $E(\Omega(S))$, is influenced by the structure of the semilattice of idempotents of S, $E(S)$, is seen in our main results: $E(\Omega(S)) \cong \Omega(E(S))$. It is well known that an arbitrary inverse semigroup is type B, but the converse is not true. Therefore, our results generalize the corresponding results for inverse semigroups.

2. Preliminaries

We follow the notions in [2, 3, 4, 13]. First, we recall some known results and notations which will be frequently used in this paper.

Lemma 2.1 ([2]). *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$;

(2) *for all* $x, y \in S^1$, $ax = ay$ $[xa = ya]$ *if and only if* $bx = by$ $[xb = yb]$.

Corollary 2.2 ([2]). *Let S be a semigroup and $a \in S, e \in E(S)$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*e$ $[\ a\mathcal{R}^*e]$;

(2) $ae = a$ $[ea = a]$ *and for all* $x, y \in S^1$, $ax = ay$ $[xa = ya]$ *implies* $ex = ey$ $[xe = ye]$.

It is easy to check that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. In an arbitrary semigroup, we can show $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. If S is a semigroup and a, b, are regular elements of S, then $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ if and only if $a\mathcal{L}b$ $[a\mathcal{R}b]$. In particular, for an arbitrary adequate semigroup S, $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$ for all $a, b \in S$, and S is $\mathcal{L}^*-$unipotent and $\mathcal{R}^*-$ unipotent (i.e., each $\mathcal{L}^*-$ class and each $\mathcal{R}^*-$class of S contains exactly one idempotent).

Let S be an adequate semigroup and $a \in S$, $(\lambda, \rho) \in \Omega(S)$. The authors in [4] define the mappings $\lambda^*, \lambda^+, \rho^*$ and ρ^+ which map S into itself by the rule:

$$\lambda^*a = (\lambda a^+)^*a, \quad \lambda^+a = (a^+\rho)^+a, \quad a\rho^* = a(\lambda a^*)^*, \quad a\rho^+ = a(a^*\rho)^+.$$

It is clear that $(\lambda^*, \rho^*), (\lambda^+, \rho^+) \in \Omega(S)$ and furthermore, those elements are the idempotents of $\Omega(S)$.

Lemma 2.3 ([4]). *Let S be an abundant semigroup and λ_1, λ_2 $[\ \rho_1, \rho_2]$ be left [right] translations of S. Then*

(1) $\lambda_1 = \lambda_2 \Longleftrightarrow (\forall e \in E(S)) \quad \lambda_1 e = \lambda_2 e$;

(2) $\rho_1 = \rho_2 \Longleftrightarrow (\forall e \in E(S)) \quad e\rho_1 = e\rho_2$.

Lemma 2.4 ([4]). *Let S be an adequate semigroup. Then the following statements are true:*

(1) $e\rho^* = \lambda^* e = (\lambda e)^* \in E(S)$, *for all* $e \in E(S)$;

(2) $e\rho^+ = \lambda^+ e = (e\rho)^+ \in E(S)$, *for all* $e \in E(S)$;

(3) $(\lambda^*, \rho^*)\mathcal{L}^{*\Omega(S)}(\lambda, \rho)\mathcal{R}^{*\Omega(S)}(\lambda^+, \rho^+)$;

(4) $E(\Omega(S)) = \{\ (\lambda, \rho) \in \Omega(S) \mid \lambda E(S) \bigcup E(S)\rho \subseteq E(S)\ \}$.

Lemma 2.5 ([7]). *Let S be a type B semigroup such that $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$. Then the following statements hold:*

(1) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2) \Longleftrightarrow \lambda_1 = \lambda_2 \Longleftrightarrow \rho_1 = \rho_2$;

(2) $\Omega(S)$ *is a type B semigroup;*

(3) $(\lambda_1, \rho_1)\mathcal{L}^{*\Omega(S)}(\lambda_2, \rho_2) \Longleftrightarrow \lambda_1 e \mathcal{L}^{*S} \lambda_2 e$ *for all* $e \in E(S)$;

(4) $(\lambda_1, \rho_1)\mathcal{R}^{*\Omega(S)}(\lambda_2, \rho_2) \Longleftrightarrow e\rho_1 \mathcal{R}^{*S} e\rho_2$ *for all* $e \in E(S)$.

As in [13], a semigroup S is *left reductive [resp., right reductive]* if for any $a, b \in S$, $xa = xb$ [resp., $ax = bx$] for all $x \in S$ implies that $a = b$ and S is *reductive* if it is both left and right reductive.

Lemma 2.6 ([12]). *Let S be a reductive semigroup. Then $S \cong \Pi(S)$ and $\Omega(S) \cong \widetilde{\Lambda}(S) \cong \widetilde{P}(S)$.*

3. Properties

In this section we will consider some properties of the translational hull of a type B semigroup.

Theorem 3.1. *Let S be a type B semigroup. Then*

(1) $S \cong \Pi(S)$ *and* $\Omega(S) \cong \widetilde{\Lambda}(S) \cong \widetilde{P}(S)$;

(2) $\Pi(E(S)) = E(\Pi(S))$.

Proof. (1) By Lemma 2.6, it only remains to show that S is reductive. To see it, let $a, b \in S$ be such that $ax = bx$ for all $x \in S$. Then $(ax)^* = (bx)^*$. Hence, $(a^*x)^* = (b^*x)^*$ since S is type B. Choose an idempotent a^* of S to replace the element x of the above formula, we get $a^* = b^*a^*$. Similarly, $b^* = a^*b^*$. Note that $E(S)$ is a semilattice, we have $a^* = b^*$. Thus, $a = aa^* = ba^* = bb^* = b$. Therefore, S is right reductive. Dually, S is left reductive, as required.

(2) Straightforward. □

Example 3.1 ([7]). Let $\mathbb{N}$ be the set of all non-negative integers and put $S = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \geq n\ \}$. Define a multiplication " $\bullet$ " on S by

$$(m, n) \bullet (p, q) = (m - n + t, q - p + t),$$

where $t = \max\{n, p\}$. Then, it is easy to check that $(S, \bullet)$ is a semigroup and $E(S) = \{(m, m) \in \mathbb{N} \times \mathbb{N}\}$. As in [7], Li and Wang have proved that S is a type B semigroup such that $(n, n)\mathcal{L}^*(m, n)\ \mathcal{R}^*(m, m)$ for all $(m, n) \in S$. Hence, $(m, n)^* = (n, n)$ and $(m, n)^+ = (m, m)$.

Next, we show that S is reductive. Let $(m, n), (h, k) \in S$ be such that $(m, n) \bullet (p, q) = (h, k) \bullet (p, q)$ for all $(p, q) \in S$. Then

$$(m - n + t,\ q - p + t) = (h - k + s,\ q - p + s), \tag{1.1}$$

where $t = \max\{n, p\}, s = \max\{k, p\}$. Hence, $s = t$ and $m - n = h - k$. Choose the idempotent $(0, 0)$ of S to replace the element (p, q) of the above formula (1.1), we have $n = t = s = k$, and thus $m = h$. Therefore, $(m, n) = (h, k)$, this gives that S is right reductive. Dually, S is left reductive.

Theorem 3.2. *Let S be a type B semigroup. Then the following statements are true:*

(1) *if λ is a mapping of S into itself, then $\lambda \in \Lambda(S)$ if and only if $(\lambda e)f = \lambda f$ for all $e, f \in E(S)$ with $f \leq e$, and $\lambda x = (\lambda x^+)x$ for all $x \in S$;*

(2) *if ρ is a mapping of S into itself, then $\rho \in P(S)$ if and only if $f(e\rho) = f\rho$ for all $e, f \in E(S)$ with $f \leq e$, and $x\rho = x(x^*\rho)$ for all $x \in S$;*

(3) *for $\lambda \in \Lambda(S), \rho \in P(S)$, $(\lambda, \rho) \in \Omega(S)$ if and only if $e(\lambda f) = (e\rho)f$ for all $e, f \in E(S)$.*

Proof. (1) Let $\lambda x = (\lambda x^+)x$ for all $x \in S$ and $(\lambda e)f = \lambda f$ for all $e, f \in E(S)$ with $f \leq e$. Then, for all $x, y \in S$, we have

$$\begin{aligned} \lambda(xy) &= (\lambda(xy)^+)xy && (\text{ by hypothesis }) \\ &= (\lambda x^+)(xy)^+xy && (\text{ from } (xy)^+ \leq x^+ \text{and hypothesis }) \\ &= (\lambda x^+)xy \\ &= (\lambda x)y && (\text{ by hypothesis }) \end{aligned}$$

This gives $\lambda \in \Lambda(S)$. The converse is clear.

(2) It is the dual of (1).

(3) Let $\lambda \in \Lambda(S), \rho \in P(S)$ be such that $e(\lambda f) = (e\rho)f$ for all $e, f \in E(S)$. Then, for all $x, y \in S$, we have

$$\begin{aligned} x(\lambda y) &= xx^*(\lambda y^+)y && (\text{ by (1) }) \\ &= x(x^*\rho)y^+y && (\text{ by hypothesis }) \\ &= (x\rho)y && (\text{ by (2) }) \end{aligned}$$

Therefore, $(\lambda, \rho) \in \Omega(S)$. The converse is clear. □

4. Main results

In this section, we shall give characterizations of the semilattice of idempotents of $\Omega(S)$ for an arbitrary type B semigroup S. Furthermore, we shall give the proof of our main result: $E(\Omega(S)) \cong \Omega(E(S))$.

Theorem 4.1. *Let S be a type B semigroup. Then*

$$E(\Omega(S)) \cong \{\ \lambda \in \Lambda(S) \mid \lambda E(S) \subseteq E(S)\ \}.$$

Proof. By Lemma 2.5(2), $E(\Omega(S))$ is a semilattice. Consider the mapping

$$\psi : E(\Omega(S)) \longrightarrow \{\ \lambda \in \Lambda(S) \mid \lambda E(S) \subseteq E(S)\ \},\ (\lambda, \rho) \longmapsto \lambda.$$

By Lemma 2.4(4), it is easy to see that ψ is well-defined. Clearly ψ is a homomorphism.

Let $\lambda \in \Lambda(S)$ with $\lambda E(S) \subseteq E(S)$. Then define a mapping ρ on S by

$$x\rho = x\lambda(x^*) \quad \text{for all} \quad x \in S.$$

Hence, for any $e \in E(S)$, $e\rho = e(\lambda e) = (\lambda e)e = \lambda e$ since $\lambda e \in E(S)$ and $E(S)$ is a semilattice. That is, $e\rho \in E(S)$ for all $e \in E(S)$. Note that $E(S)$ is a semilattice. For all $e, f \in E(S)$, we have

$$(fe)\rho = \lambda(fe) = \lambda(ef) = (\lambda e)f = f(\lambda e) = f(e\rho).$$

In particular, $f\rho = f(e\rho)$, for all $e, f \in E(S)$ with $f \leq e$; also, for $x \in S$, $x(x^*\rho) = x(\lambda x^*) = x\rho$. By Theorem 3.2(2), $\rho \in P(S)$. Hence, for all $e, f \in E(S)$,

$$e(\lambda f) = e(f\rho) = (ef)\rho = (fe)\rho = f(e\rho) = (e\rho)f,$$

and so $(\lambda, \rho) \in \Omega(S)$ from Theorem 3.2(3). This together with the fact $\lambda e = e\rho \in E(S)$ for all $e \in E(S)$ and with Lemma 2.4(4), yields that $(\lambda, \rho) \in E(\Omega(S))$. Therefore, ψ is surjective.

On the other hand, let $\psi[(\lambda_1, \rho_1)] = \psi[(\lambda_2, \rho_2)]$ for some $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in E(\Omega(S))$, that is, $\lambda_1 = \lambda_2$. By Lemma 2.5(1), $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. This gives that ψ is injective. Therefore, $E(\Omega(S)) \cong \{\ \lambda \in \Lambda(S) \mid \lambda E(S) \subseteq E(S)\ \}$. □

Proposition 4.2. *Let S be a type B semigroup and $\lambda \in \Lambda(S), \rho \in P(S)$. Then $(\lambda, \rho) \in E(\Omega(S))$ if and only if $\lambda^2 = \lambda, \rho^2 = \rho$ and $\lambda e = e\rho$ for all $e \in E(S)$.*

Proof. By Lemma 2.5(2), $\Omega(S)$ is a type B semigroup and $\Omega(S)$ is $\mathcal{L}^*-$unipotent. Let $(\lambda, \rho) \in E(\Omega(S))$. Then $\lambda^2 = \lambda, \rho^2 = \rho$. By Lemma 2.4(3), $(\lambda^*, \rho^*)\mathcal{L}^*(\lambda, \rho)$. Hence, $(\lambda, \rho) = (\lambda^*, \rho^*)$ since $\Omega(S)$ is $\mathcal{L}^*-$unipotent. Thus, by Lemma 2.4(1), $\lambda e = \lambda^* e = e\rho^* = e\rho$, as required.

Conversely, suppose that $\lambda \in \Lambda(S), \rho \in P(S)$ with $\lambda^2 = \lambda, \rho^2 = \rho$, and $\lambda e = e\rho$ for all $e \in E(S)$. Then, for all $e, f \in E(S)$, we have

$$(\lambda e)^2 = (\lambda e)(\lambda e) = (\lambda e)(e\rho) = \lambda(e(e\rho)) = \lambda(e\rho) = \lambda(\lambda e) = \lambda^2 e = \lambda e.$$

Hence, $\lambda e = e\rho \in E(S)$, and so $(e\rho)f = f(e\rho) = (fe)\rho = (ef)\rho = e(f\rho) = e(\lambda f)$ since $E(S)$ is a semilattice. By Theorem 3.2(3), $(\lambda, \rho) \in \Omega(S)$. This together with the facts $\lambda^2 = \lambda, \rho^2 = \rho$, yields that $(\lambda, \rho) \in E(\Omega(S))$. □

Theorem 4.3. *Let S be a type B semigroup. Then $E(\Omega(S)) \cong \Omega(E(S))$.*

Proof. By Lemma 2.4(4), it is easy to see that $\Omega((E(S))$ is indeed a semilattice. Consider a mapping θ as follows:

$$\theta : E(\Omega(S)) \longrightarrow \Omega(E(S)), \qquad (\lambda, \rho)\theta = (\lambda|_{E(S)}, \rho|_{E(S)}),$$

where $\lambda|_{E(S)}$ [resp., $\rho|_{E(S)}$] is the restriction of λ [resp., ρ] to the set $E(S)$. By Lemma 2.4(4), $\lambda|_{E(S)} \in \Lambda(E(S)), \rho|_{E(S)} \in P(E(S))$ and $\lambda|_{E(S)}$ is linked to $\rho|_{E(S)}$. Hence θ maps into $\Omega(E(S))$.

Next, we prove that θ is injective. Let $(\lambda|_{E(S)}, \rho|_{E(S)}) = (\lambda'|_{E(S)}, \rho'|_{E(S)})$. Then, by Lemma 2.3, $(\lambda, \rho) = (\lambda', \rho')$, this gives that θ is injective.

To see that θ is surjective, let $(\lambda^\ddagger, \rho^\ddagger) \in \Omega(E(S))$. Define λ and ρ on S as follows:

$$\lambda x = (\lambda^\ddagger x^+)x, \qquad x\rho = x(x^*\rho^\ddagger).$$

Then

$$\lambda x = (\lambda^\ddagger x^+)x = (\lambda^\ddagger x^+)x^+x = (\lambda x^+)x,$$

and

$$(\lambda e)f = ((\lambda^\ddagger e)e)f = \lambda^\ddagger(ef) = \lambda^\ddagger f = (\lambda^\ddagger f)f = \lambda f$$

for all $e, f \in E(S)$ with $f \leq e$. By Theorem 3.2(1), $\lambda \in \Lambda(S)$. Dually, $\rho \in P(S)$. Furthermore,

$$e(\lambda f) = e(\lambda^\ddagger f)f = e(\lambda^\ddagger f) = (e\rho^\ddagger)f = e(e\rho^\ddagger)f = (e\rho)f,$$

for all $e, f \in E(S)$. By Theorem 3.2(3), $(\lambda, \rho) \in \Omega(S)$. Note that $\lambda|_{E(S)} = \lambda^\ddagger$ and $\rho|_{E(S)} = \rho^\ddagger$. We have

$$\lambda^2 e = \lambda(\lambda e) = \lambda(\lambda^\ddagger e) = \lambda^\ddagger(\lambda^\ddagger e) = {\lambda^\ddagger}^2 e = \lambda^\ddagger e = \lambda e,$$

and similarly $e\rho^2 = e\rho$ and so $\lambda^2 = \lambda, \rho^2 = \rho$. Hence,

$$\lambda e = (\lambda e)e = e(\lambda e) = (e\rho)e = e(e\rho) = e\rho.$$

By Proposition 4.2, $(\lambda, \rho) \in E(\Omega(S))$. Thus, θ is surjective. That θ is a homomorphism is clear. Therefore, $E(\Omega(S)) \cong \Omega(E(S))$. □

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References

[1] J.E. Ault, *The translational hull of an inverse semigroup*, Glasgow Math. J., 14 (1973), 56-64.

[2] A. El-Qallali, J.B. Fountain, *Idempotent-connected abundant semigroups*, Proc. Roy. Soc. Edinburgh Sect, A, 91 (1981), 79-90.

[3] J.B. ountain, *Adequate semigroups*, Proc. Edinburgh Math. Soc., 22 (1979), 113-125.

[4] J.B. Fountain, M. Lawson, *The translational hull of an adequate semigroup*, Semigroup Forum, 32 (1985), 79-86.

[5] X.J. Guo, K.P. Shum, *On translational hulls of type-A semigroups*, J. Algebra, 269 (2003), 240-249.

[6] M.V. Lawson, *The natural partial order on an abundant semigroup*, Proc. Edinburgh Math. Soc., 30 (1987), 169-186.

[7] C.H. Li, L.M. Wang, *On the translational hull of a type B semigroup*, Semigroup Forum, 82 (2011), 516-529 (2011).

[8] C.H. Li, L.M. Wang, B.G. Xu, H.W. Huang, *On $\sim$ bisimple right type B $\omega-$ semigroups as model of some fractal structure*, Chaos, Solitions and Fractals, 89 (2016), 169-174.

[9] C.H. Li, *On fuzzy congruences of a type B semigroup I*, International Mathematical Forum., 9 (2014), 367-375.

[10] F.J. Pastijn, L.A. Oliveira, *Ress matrix covers and the translational hull of a locally inverse semigroup*, Comm. Algebra, 36 (2008), 3230-3249.

[11] Petrich, M., *Introduction to semigroups*, Merrill, Columbus, Ohio, 1973.

[12] M. Petrich, *The translational hull in semigroups and rings*, Semigroup Forum, 1 (1970), 283-360.

[13] M. Petrich, N. Reilly, *Completely regular semigroups*, Newyork: Thon Wiley and Sons, 1999.

[14] X.M. Ren, K.P. Shum, *The translational hull of strongly right or left adequate semigroup*, Vietnam Journal of Mathematics, 34 (2006), 1-7.

[15] Y. Wang, X.M. Ren, S.Y. Ma, *The translational hull of superabundant semigroups with semilattice of idempotents*, Scientia Magna, 2 (2006), 75-80.

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On hyper Q-fuzzy normal HX subgroup, conjugate and its normal level

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Abstract. In this research, we introduce a hyper Q-fuzzy HX subgroup of a HX group ν notion also we discuss a few of its properties. Also we define a hyper Q-fuzzy normal HX subgroup, normal level subgroup HX of hyper Q-fuzzy normal HX subgroup of a HX group also self conjugate and we discussed some of their important properties.

Keywords: HX group, hyper fuzzy set, fuzzy $HX-$Subgroup, Q-fuzzy set, Q-fuzzy $HX-$subgroup, hyper Q-fuzzy $HX-$subgroup, normal level sub HX group.

1. Introduction

In 1965, Zadeh [20] introduced the notion of fuzzy sets and opened a new path of thinking to engineers, computer science, mathematicians, physicists and many others due to its diverse applications in various fields. In 1975, Zadeh [19] discussed the concepts of interval valued fuzzy sets. Where the values of membership functions are intervals of real numbers instead of the real points. The concept of fuzzy subgroups was introduced by Rosenfield [17] in 1971, which was the first fuzzification of any algebraic structure, after that Biswas [1] in 1994

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define interval valued fuzzy subgroups. Piergiulio Corsini and Li Honyxing [2, 3, 5] studied HX group concept and there after Chengzhong., Mitlonghai., and Hongxing [6] discussed fuzzy HX group concept. Manikandan et al [14] introduce anti Q-fuzzy normal HX group and its lower level sub HX groups concept. A new algebraic structure of Q-fuzzy group are introduced and defined by Solairaju et al [18]. Hereafter, several results about the HX group with respect to Q-fuzzy and anti Q-fuzzy, bipolar fuzzy, intuitionistic fuzzy have been obtained see [7, 9, 10, 11, 12, 13, 15]. In 2012, Jayanta et al [4] introduced the concepts of hyper fuzzy sets, where the values of the membership functions are assumed to be a subsets of $[0, 1]$, which is a generalization of fuzzy sets and interval valued fuzzy sets. In this research we give a new algebraic structure of hyper Q-fuzzy HX subgroups also we study some of its properties. Also we define hyper Q-fuzzy normal HX subgroup and some of its properties are discussed.

2. Preliminaries

Definition 2.1 ([5])**.** An HX group on G is non empty set $\upsilon \subset 2^G - \{\phi\}$ such that υ is a group with respect to the algebraic operation defined by $WV = \{wv, w \in W \text{ and } v \in V\}$, which its unit element is denoted by U.

Definition 2.2 ([20])**.** If A is any non empty set. A fuzzy subset λ of A is a function $\lambda : A \to [0, 1]$.

Definition 2.3 ([16])**.** A fuzzy set μ is said to be fuzzy HX subgroup of υ if for $W, V \in \upsilon$, that is,

1. $\mu(WV) \geq \min\{\mu(W), \mu(V)\}$
2. $\mu(W^{-1}) = \mu(W)$.

Definition 2.4 ([16, 18])**.** Let Q and υ be any two sets. A mapping $\mu : \upsilon \times Q \to [0, 1]$ is called Q-fuzzy set in υ.

Definition 2.5 ([16, 18])**.** A Q-fuzzy set μ is called Q-fuzzy HX subgroup of υ if for all $W, V \in \upsilon$ and $q \in Q$:

1. $\mu(WV, q) \geq \min\{\mu(W, q), \mu(V, q)\}$
2. $\mu(W^{-1}, q) = \mu(W, q)$.

Definition 2.6 ([8])**.** A Q-fuzzy HX subgroup μ of υ is said to be normal if for all $W, V \in \upsilon$ and $q \in Q$. That is, $\mu(WV, q) = \mu(VW, q)$ or $\mu(WVW^{-1}, q) = \mu(V, q)$.

Definition 2.7 ([4])**.** Let υ, Q be any two sets. An interval valued Q-fuzzy set μ defined in υ is given by $\mu = \{((V, q), \bar{\lambda}_\mu \in (V, q)); V \in \upsilon \text{ and } q \in Q\}$ where $\bar{\lambda}_\mu : \upsilon \times Q \to F[0, 1]$ and $F[0, 1]$ denoted the family of all closed subinterval of $[0, 1]$ and $\bar{\lambda}_\mu(V, q) = [\lambda^L_\mu(V, q), \lambda^R_\mu(V, q)]$ such that $\lambda^L_\mu, \lambda^R_\mu$ are two Q-fuzzy subsets of υ such that $\lambda^L_\mu(V, q) \leq \lambda^R_\mu(V, q)$ for all $V \in \upsilon$ and $q \in Q$.

Definition 2.8 ([4]). Let $F_1, F_2 \in F[0,1]$. If $F_1 = [\alpha_1, \beta_1]$ and $F_2 = [\alpha_2, \beta_2]$, then $\max(F_1, F_2) = [\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)]$ and $\min(F_1, F_2) = [\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2)]$. Thus if $F_i = [\alpha_i, \beta_i] \in F[0,1]$ for $i = 1, 2, 3, \cdots$, then we define $r \sup_i F_i = [\sup_i \alpha_i, \sup_i \beta_i]$ and $r \inf_i F_i = [\inf_i \alpha_i, \inf_i \beta_i]$. Now we call $F_1 \geq F_2$ iff $\alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$. Similarly we may have $F_1 \leq F_2$ and $F_1 = F_2$.

Definition 2.9 ([4]). An interval valued Q-fuzzy subset $\bar{\lambda}_\mu$ of a HX group υ is called an interval valued Q-fuzzy subgroup if :

1. $\bar{\lambda}_\mu(WV, q) \geq r \min\{\bar{\lambda}_\mu(W, q), \bar{\lambda}_\mu(V, q)\}$
2. $\bar{\lambda}_\mu(W^{-1}, q) = \bar{\lambda}_\mu(W, q)\}$,

   Where $W, V \in \upsilon$ and $q \in Q$.

3. Hyper Q-fuzzy HX subgroups

Definition 3.1. Let υ and Q be two sets. A mapping $\mu^* : \upsilon \times Q \to \wp([0,1])$ is said to be a hyper Q-fuzzy of υ, where $\wp([0,1])$ denotes the set of all non empty subset of $[0,1]$.

Definition 3.2. Let υ and Q be two sets and μ^*, λ^* be two hyper Q-fuzzy subset of υ. Then the intersection of μ^* and λ^* is denoted by $\mu^* \cap \lambda^*$ and defined by: $\mu^* \cap \lambda^*(V, q) = \{\min\{v_1, v_2\}; v_1 \in \mu^*(V, q), v_2 \in \lambda^*(V, q)\}$ for all $V \in \upsilon$ and $q \in Q$. The union of μ^* and λ^* is denoted by $\mu^* \cup \lambda^*$ and defined by: $(\mu^* \cup \lambda^*)(V, q) = \{\max\{v_1, v_2\}; v_1 \in \mu^*(V, q), v_2 \in \lambda^*(V, q)\}$ for all $V \in \upsilon$ and $q \in Q$.

Definition 3.3. Let υ be a HX group. A mapping $\mu^* : \upsilon \times Q \to \wp([0,1])$ is called a hyper Q-fuzzy HX subgroup of υ if $\forall W, V \in \upsilon$ and $q \in Q$ the following axioms hold:

1. $\inf \mu^*(WV, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$
2. $\sup \mu^*(WV, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}$
3. $\inf \mu^*(W^{-1}, q) \geq \inf \mu^*(W, q)$
4. $\sup \mu^*(W^{-1}, q) \geq \sup \mu^*(W, q)$.

Lemma 3.4. If μ^* is a hyper Q-fuzzy HX subgroup of υ having the identity U, then for all $W \in \upsilon$:

1. $\inf \mu^*(W^{-1}, q) = \inf \mu^*(W, q)$ and $\sup \mu^*(W^{-1}, q) = \sup \mu^*(W, q)$.
2. $\inf \mu^*(U, q) \geq \inf \mu^*(W, q)$ and $\sup \mu^*(U, q) \geq \sup \mu^*(W, q)$.

Proof. 1. As μ^* is a hyper Q-fuzzy HX subgroup of υ, then $\inf \mu^*(W^{-1}, q) \geq \inf \mu^*(W, q) \forall W \in \upsilon$ and $q \in Q$, also $\inf \mu^*(W, q) = \inf \mu^*((W^{-1})^{-1}, q) \geq \inf \mu^*(W^{-1}, q)$. Then $\inf \mu^*(W, q) \geq \inf \mu^*(W^{-1}, q)$, thus $\inf \mu^*(W, q) = \inf \mu^*(W^{-1}, q)$. Similarly, we can prove that $\sup \mu^*(W^{-1}, q) = \sup \mu^*(W, q)$.

2. $\inf \mu^*(U, q) = \inf \mu^*(WW^{-1}, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(W^{-1}, q)\} = \inf \mu^*(W, q)$. Then $\inf \mu^*(U, q) \geq \inf \mu^*(W, q)$, also $\sup \mu^*(U, q) = \sup \mu^*(WW^{-1}, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(W^{-1}, q)\} = \sup \mu^*(W, q)$, then $\sup \mu^*(U, q) \geq \sup \mu^*(W, q)$. □

Proposition 3.5. A hyper Q-fuzzy subset μ^* of a HX group υ is a hyper Q-fuzzy HX subgroup iff for all $W, V \in \upsilon, q \in Q$ followings are hold:

1. $\inf \mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$.

2. $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}$.

Proof. $\Rightarrow$ Let μ^* be a hyper Q-fuzzy HX subgroup of υ and $W, V \in \upsilon, q \in Q$. Then

$$\begin{aligned} \inf \mu^*(WV^{-1}, q) &\geq \min\{\inf \mu^*(W, q), \inf \mu^*(V^{-1}, q)\} \\ &= \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}. \end{aligned}$$

And

$$\begin{aligned} \sup \mu^*(WV^{-1}, q) &\geq \min\{\sup \mu^*(W, q), \sup \mu^*(V^{-1}, q)\} \\ &= \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}. \end{aligned}$$

$\Leftarrow$ Let μ^* be a hyper Q-fuzzy subset of υ and given conditions hold. Then for all $W, V \in \upsilon, q \in Q$, we have $\inf \mu^*(U, q) = \inf \mu^*(VV^{-1}, q) \geq \min\{\inf \mu^*(V, q), \inf \mu^*(V^{-1}, q)\} = \inf \mu^*(V, q)$, $\sup \mu^*(U, q) = \sup \mu^*(VV^{-1}, q) \geq \min\{\sup \mu^*(V, q), \sup \mu^*(V^{-1}, q)\} = \sup \mu^*(V, q)$.

So, $\inf \mu^*(V^{-1}, q) = \inf \mu^*(UV^{-1}, q) \geq \min\{\inf \mu^*(U, q), \inf \mu^*(V, q)\} = \inf \mu^*(V, q)$. And $\sup \mu^*(V^{-1}, q) = \sup \mu^*(UV^{-1}, q) \geq \min\{\sup \mu^*(U, q), \sup \mu^*(V, q)\} = \sup \mu^*(V, q)$.

Again, $\inf \mu^*(WV, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V^{-1}, q)\} \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$. Therefore μ^* be a hyper Q-fuzzy HX subgroup of υ. □

Theorem 3.6. If λ^*, μ^* are two hyper Q-fuzzy $HX-$subgroups of υ, then $\lambda^* \cap \mu^*$ is hyper Q-fuzzy $HX-$subgroup of υ.

Proof. Since λ^*, μ^* are two hyper Q-fuzzy $HX-$subgroups of υ, and $W, V \in \upsilon, q \in Q$. Thus

$$\begin{aligned} &\inf(\lambda^* \cap \mu^*)(WV^{-1}, q) = \min\{\inf \lambda^*(WV^{-1}, q), \inf \mu^*(WV^{-1}, q)\} \\ &\geq \min\{\min\{\inf \lambda^*(W, q), \inf \lambda^*(V, q)\}, \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}\} \\ &= \min\{\min\{\inf \lambda^*(W, q), \inf \mu^*(W, q)\}, \min\{\inf \lambda^*(V, q), \inf \mu^*(V, q)\}\} \quad (1) \\ &= \min\{\inf(\lambda^* \cap \mu^*)(W, q), \inf(\lambda^* \cap \mu^*)(V, q)\}. \end{aligned}$$

Also

$$\begin{aligned}
&\sup(\lambda^* \cap \mu^*)(WV^{-1}, q) = \min\{\sup \lambda^*(WV^{-1}, q), \sup \mu^*(WV^{-1}, q)\} \\
&\geq \min\{\min\{\sup \lambda^*(W, q), \sup \lambda^*(V, q)\}, \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}\} \\
(2) \quad &= \min\{\min\{\sup \lambda^*(W, q), \sup \mu^*(W, q)\}, \min\{\sup \lambda^*(V, q), \sup \mu^*(V, q)\}\} \\
&= \min\{\sup(\lambda^* \cap \mu^*)(W, q), \sup(\lambda^* \cap \mu^*)(V, q)\}.
\end{aligned}$$

Therefore by (1), (2) and Proposition 3.5, we say that $\lambda^* \cap \mu^*$ is a hyper Q-fuzzy HX subgroup of υ. □

Corollary 3.7. The intersection of any collection hyper Q-fuzzy normal HX subgroup is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Straightforward. □

Definition 3.8. Let μ^* be a hyper Q-fuzzy HX subgroup of υ. Then μ^* is called hyper Q-fuzzy normal HX subgroup of υ if $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \lambda^*(WV, q) = \sup \lambda^*(VW, q)$ for all $W, V \in \upsilon, q \in Q$.

Theorem 3.9. Let μ^* be a hyper Q-fuzzy HX subgroup of υ, then the following are equivalent.

1. μ^* be a hyper Q-fuzzy HX normal subgroup of υ.
2. $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$.
3. $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \mu^*(WV, q) = \sup \mu^*(VW, q)$, for all $W, V \in \upsilon, q \in Q$.

Proof. Straightforward. □

Theorem 3.10. If λ^*, μ^* are two hyper Q-fuzzy normal $HX-$subgroups of υ, then $\lambda^* \cap \mu^*$ is hyper Q-fuzzy normal $HX-$ subgroup of υ.

Proof. Since λ^*, μ^* are two hyper Q-fuzzy HX subgroup of υ, by Theorem 3.6 $\lambda^* \cap \mu^*$ is a hyper Q-fuzzy HX subgroup of υ. Let $W, V \in \upsilon, q \in Q$, then

$$\begin{aligned}
\inf(\lambda^* \cap \mu^*)(WV, q) &= \min\{\inf \lambda^*(WV, q), \inf \mu^*(WV, q)\} \\
&= \min\{\inf \lambda^*(VW, q), \inf \mu^*(VW, q)\} \\
&= \inf(\lambda^* \cap \mu^*)(VW, q).
\end{aligned}$$

And

$$\begin{aligned}
\sup(\lambda^* \cap \mu^*)(WV, q) &= \min\{\sup \lambda^*(WV, q), \sup \mu^*(WV, q)\} \\
&= \min\{\sup \lambda^*(VW, q), \sup \mu^*(VW, q)\} \\
&= \sup(\lambda^* \cap \mu^*)(VW, q).
\end{aligned}$$

Therefore $(\lambda^* \cap \mu^*)$ is a hyper Q-fuzzy normal HX subgroup of υ. □

Corollary 3.11. The intersection of any collection hyper Q-fuzzy normal HX subgroup is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Straightforward. □

4. Level subset of hyper Q-fuzzy normal HX subgroup and self conjecate hyper Q-fuzzy HX subgroup

Definition 4.1. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Such that for any $\alpha \in [0,1]$ we define the set $L(\mu^*, \alpha) = \{V \in \upsilon, q \in Q; \inf \mu^*(V,q) \geq \alpha$ and $\sup \lambda^*(V,q) \geq \alpha\}$ is called the level subset of μ^*.

Theorem 4.2. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Then for $\alpha \in [0,1]$ such that $\inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha, L(\mu^*, \alpha)$ is normal sub HX group of υ.

Proof. For all $W, V \in L(\mu^*, \alpha)$ we have $\inf \mu^*(W,q) \geq \alpha$ and $\inf \lambda^*(V,q) \geq \alpha$. Now $\inf \mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W,q), \inf \mu^*(V,q)\} \geq \min\{\alpha, \alpha\} = \alpha$. Thus $\inf \mu^*(WV^{-1}, q) \geq \alpha$ and $\sup \mu^*(W,q) \geq \alpha$ and $\sup \lambda^*(V,q) \geq \alpha$. Now $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W,q),$
$\sup \mu^*(V,q)\} \geq \min\{\alpha, \alpha\} = \alpha$. Thus $\sup \mu^*(WV^{-1}, q) \geq \alpha$ and we get $WV^{-1} \in L(\mu^*, \alpha)$. Hence $L(\mu^*, \alpha)$ is a sub HX group of υ.

For all $V \in L(\mu^*, \alpha), W \in \upsilon, q \in Q$ and $\alpha \in [0,1]$ such that $\inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha$ we have $\inf \mu^*(V,q) \geq \alpha$ and $\sup \mu^*(V,q) \geq \alpha$. Since μ^* be a hyper Q-fuzzy normal HX subgroup of υ, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q) \geq \alpha$ and $\sup \lambda^*(WVW^{-1}, q) = \sup \lambda^*(V,q) \geq \alpha$ for all $W, V \in \upsilon, q \in Q$, therefore $WVW^{-1} \in L(\mu^*, \alpha)$. Hence $L(\mu^*, \alpha)$ is a normal sub HX group of υ. □

Theorem 4.3. Let υ be a HX group and μ^* be a hyper Q-fuzzy subset of υ such that $L(\mu^*, \alpha)$ is a normal sub HX group of υ. For $\alpha \in [0,1] \inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha$, then μ^* is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Let $W, V \in \upsilon$ and $\mu^*(W,q) = \alpha_1$ this means that $\inf \mu^*(W,q) = \sup \mu^*(W,q) = \alpha_1, \mu^*(V,q) = \alpha_2$ this means that $\inf \mu^*(V,q) = \sup \mu^*(V,q) = \alpha_2$.

Suppose that $\alpha_1 > \alpha_2$, then $W, V \in L(\mu^*, \alpha)$ as $L(\mu^*, \alpha)$ is a subgroup of $\upsilon, WV^{-1} \in L(\mu^*, \alpha)$ hence $\inf \mu^*(WV^{-1}, q) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} \geq \min\{\inf \mu^*(W,q),$ $\inf \mu^*(V,q)\}$. Thus $Inf\mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W,q), \inf \mu^*(V,q)\}$ and $\sup \mu^*(WV^{-1}, q) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} \geq \min\{\sup \mu^*(W,q), \sup \mu^*(V,q)\}$ then $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W,q), \sup \mu^*(V,q)\}$. Therefore μ^* is a hyper Q-fuzzy HX subgroup of υ. Now for any $\alpha \in [0,1] L(\mu^*, \alpha) \neq \phi$ and $L(\mu^*, \alpha)$ is normal sub HX group. Then, we have $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q), \sup \mu^*(WVW^{-1},$ $q) = \sup \mu^*(V,q)$ for all $W, V \in \upsilon, q \in Q$. Otherwise, if there exists W_0 or $V_0 \in \upsilon$ and $q \in Q$ such that $\inf \mu^*(W_0V_0W_0^{-1}, q) = \inf \mu^*(V_0, q), \sup \lambda^*(W_0V_0W_0^{-1}, q) = \sup \lambda^*(V_0, q)$, take $\alpha_0 = 0.5[\inf \mu^*(V_0,q) + \inf \mu^*(W_0V_0W_0^{-1}, q)] = 0.5[\sup \mu^*(V_0, q)$ $+ Sup\mu^*(W_0V_0W_0^{-1}, q)]$. Evidently $\alpha_0 \in [0,1]$, we can infer that $\inf \mu^*(V_0,q) > \alpha_0, \sup \mu^*(V_0,q) > \alpha_0$ and $\inf \mu^*(W_0V_0W_0^{-1}, q) < \alpha_0$ and $\sup \mu^*(W_0V_0W_0^{-1}, q) < \alpha_0$. Consequently, we have $V_0 \in L(\mu^*, \alpha_0)$ and $W_0V_0W_0^{-1} \notin L(\mu^*, \alpha)$, this contradicts that $L(\mu^*, \alpha_0)$ is normal sub HX group. Hence, we get $\inf \mu^*(V_0,q) = \inf \mu^*(W_0V_0W_0^{-1}, q), \sup \mu^*(V_0,q) = \sup \mu^*(W_0V_0W_0^{-1}, q)$ for all $W, V \in \upsilon, q \in Q$. Thus, μ^* is a hyper Q-fuzzy normal HX subgroup of υ. □

Theorem 4.4. If μ^* is a hyper Q-fuzzy normal HX subgroup of υ. If two level normal sub HX groups $L(\mu^*, \alpha_1), L(\mu^*, \alpha_2)$ for $\alpha_1, \alpha_2 \in [0,1]$ and that $\inf \mu^*(U,q) \geq \alpha_1, \alpha_2, \sup \mu^*(U,q) \geq \alpha_1, \alpha_2$ with $\alpha_1 > \alpha_2$ of μ^* are equal if and only if there is no $A \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$.

Proof. ($\Rightarrow$) Let $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$, suppose there exists $W \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$ then $L(\mu^*, \alpha_1) \subseteq L(\mu^*, \alpha_2)$. Thus $W \in L(\mu^*, \alpha_1)$ but $W \notin L(\mu^*, \alpha_2)$, which is contradicts the assumption that $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$. Hence there is no μ^* in υ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$.

($\Leftarrow$) Suppose that there is no $W \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$. Thus $L(\mu^*, \alpha_2) \subseteq L(\mu^*, \alpha_1)$. Let $W \in L(\mu^*, \alpha_2)$ and there is no W in υ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$. Hence $W \in L(\mu^*, \alpha_2)$ and $L(\mu^*, \alpha_1) \subseteq L\mu^*, \alpha_2)$ thus $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$. □

Corollary 4.5. A level subset $L(\mu^*, \alpha); \alpha \in [0,1]$ is normal sub HX group of υ, if μ^* be a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Since μ^* is a hyper Q-fuzzy normal HX subgroup of υ and the Level subsets $L(\mu^*, \alpha); \alpha \in [0,1]$ is normal sub HX group of υ, let $W \in \upsilon$ and $B \in L(\mu^*, \alpha)$, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q) \geq \alpha$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q) \geq \alpha$ since μ^* is a hyper Q-fuzzy normal HX subgroup of υ. That is, $\inf \mu^*(WVW^{-1}, q) \geq \alpha, \sup \mu^*(WVW^{-1}, q) \geq \alpha$ therefore $WVW^{-1} \in L(\mu^*, \alpha)$, hence $L(\mu^*, \alpha)$ is normal HX subgroup of υ. □

Definition 4.6. Let υ be a HX group and μ^* is a hyper Q-fuzzy normal HX subgroup of υ, let $N(\mu^*) = \{W \in \upsilon, \inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q)$ for all $V \in \upsilon\}$. Then $N(\mu^*)$ is called the hyper Q-fuzzy normalizer of μ^*.

Proposition 4.7. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Then

1. $N(\mu^*)$ is a sub HX group of υ.
2. μ^* is a hyper Q-fuzzy normal HX subgroup of υ iff $N(\mu^*) = \upsilon$.
3. μ^* is a hyper Q-fuzzy normal HX subgroup of $N(\mu^*)$.

Proof. 1. Let $W, V \in N(\mu^*)$ then $\inf \mu^*(WZW^{-1}, q) = \inf \mu^*(Z,q)$ and $\sup \mu^*(WZW^{-1}, q) = \sup \mu^*(Z,q)$ for all $Z \in \upsilon$.

Now $\inf \mu^*((WV)Z(WV)^{-1}, q) = \inf \mu^*(WVZV^{-1}W^{-1}, q) = \inf \mu^*(VZV^{-1}, q) = \inf \mu^*(Z,q)$ and $\sup \mu^*((WV)Z(WV)^{-1}, q) = \sup \mu^*(WVZV^{-1}W^{-1}, q) = \sup \mu^*(VZV^{-1}, q) = Sup\mu^*(Z,q)$ therefore, we get $\mu^*((WV)Z(WV)^{-1}, q) = \mu^*(Z,q)$ and thus $WVN(\mu^*)$. Hence $N(\mu^*)$ is a sub HX group of υ.

2. ($\Rightarrow$) Let μ^* is a hyper Q-fuzzy normal HX subgroup of υ, clearly $N(\mu^*) \subseteq \upsilon$. Let $W \in \upsilon$, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q)$ then $W \in N(\mu^*)$ and we get $\upsilon \subseteq N(\mu^*)$, hence $N(\mu^*) = \upsilon$.

($\Leftarrow$) Let $N(\mu^*) = \upsilon$. Clearly, $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon$ and hence μ^* is a hyper Q-fuzzy normal HX subgroup of υ.

3. Clear from 2.

□

Theorem 4.8. Let μ^* is a hyper Q-fuzzy HX subgroup of υ and $W \in \upsilon$. Then the hyper Q-fuzzy subset $\lambda^* : \upsilon \times Q \to \wp([0,1])$ defined by $\lambda^*(V, q) = \lambda^*(W^{-1}VW, q)$ for all $V \in \upsilon$ is a hyper Q-fuzzy $HX-$subgroup of υ.

Proof. Let $W, Z \in \upsilon$, then for all $V \in \upsilon$

$$\begin{aligned}
\inf \lambda^*(VZ^{-1}, q) &= \inf \mu^*(W^{-1}VZ^{-1}W, q) \\
&= \inf \mu^*(W^{-1}VWW^{-1}Z^{-1}W, q) \\
&= \inf \mu^*((W^{-1}VW)(W^{-1}ZW)^{-1}, q) \\
&\geq \min\{\inf \mu^*(W^{-1}VW, q), \inf \mu^*(W^{-1}ZW, q)\} \\
&= \min\{\inf \lambda^*(V, q), \inf \lambda^*(Z, q)\}.
\end{aligned}$$

Also

$$\begin{aligned}
\sup \lambda^*(VZ^{-1}, q) &= \sup \mu^*(W^{-1}VZ^{-1}W, q) \\
&= \sup \mu^*(W^{-1}VWW^{-1}Z^{-1}W, q) \\
&= \sup \mu^*((W^{-1}VW)(W^{-1}ZW)^{-1}, q) \\
&\geq \min\{\sup \mu^*(W^{-1}VW, q), \sup \mu^*(W^{-1}ZW, q)\} \\
&= \min\{\sup \lambda^*(V, q), \inf \lambda^*(Z, q)\}.
\end{aligned}$$

Therefore μ^* is a hyper Q-fuzzy $HX-$subgroup of υ. □

Definition 4.9. Let μ^* and δ^* be two hyper Q-fuzzy HX subgroup of υ. We say that δ^* is conjugate to μ^* if for some $W \in \upsilon$ we have that $\inf \delta^*(V, q) = \inf \mu^*(W^{-1}VW, q)$ and $\sup \delta^*(V, q) = \sup \mu^*(W^{-1}VW, q)$ for all $V \in \upsilon, q \in Q$.

Definition 4.10. A hyper Q-fuzzy HX subgroup μ^* of υ is said to be self conjugate hyper Q-fuzzy HX subgroup if for all $W, Z \in \upsilon$, we have $\inf \mu^*(Z, q) = \inf \mu^*(W^{-1}ZW, q)$ and $\sup \mu^*(Z, q) = \sup \mu^*(W^{-1}ZW, q)$.

Theorem 4.11. A hyper Q-fuzzy HX subgroup μ^* of υ is normal if and only if μ^* is self conjugate hyper Q-fuzzy HX subgroup.

Proof. ($\Rightarrow$) Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ, then $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \mu^*(WV, q) = \sup \mu^*(VW, q)$, thus by Theorem 3.9 we have $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon, q \in Q$. So, μ^* is a self conjugate hyper Q-fuzzy HX subgroup.

($\Leftarrow$) Let μ^* be a self conjugate hyper Q-fuzzy HX subgroup.

Then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon, q \in Q$. So, μ^* is a hyper Q-fuzzy HX subgroup of υ. □

Theorem 4.12. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ and $\lambda^* : N(\mu^*) \to \wp([0,1])$ is defined by $\lambda^*(W, q) = \mu^*(W, q)$ for all $W \in N(\mu^*)$ and $q \in Q$. Then λ^* is a hyper Q-fuzzy normal HX subgroup of $N(\mu^*)$.

Proof. Since μ^* be a hyper Q-fuzzy normal HX subgroup of υ and we have proved that $N(\mu^*)$ is a HX subgroup of υ. Then μ^* be a hyper Q-fuzzy HX subgroup of $N(\mu^*)$. Hence λ^* is a hyper Q-fuzzy HX subgroup of $N(\mu^*)$. We have to prove λ^* is normal, since $N(\mu^*)$ is a HX subgroup of υ, then $W, V \in N(\mu^*)$ thus $W^{-1}VW \in N(\mu^*)$ also,

$$\begin{aligned}\inf \lambda^*(W^{-1}VW, q) &= \inf \mu^*(W^{-1}VW, q)(\text{ since } W^{-1}VW \in N(\mu^*))\\ &= \inf \mu^*(V, q)(\text{ since } V \in N(\mu^*))\\ &= \inf \lambda^*(V, q)(\text{ since } V \in N(\mu^*)).\end{aligned}$$

Also

$$\begin{aligned}\sup \lambda^*(W^{-1}VW, q) &= \sup \mu^*(W^{-1}VW, q)(\text{ since } W^{-1}VW \in N(\mu^*))\\ &= \sup \mu^*(V, q)(\text{ since } V \in N(\mu^*))\\ &= \sup \lambda^*(V, q)(\text{ since } V \in N(\mu^*)).\end{aligned}$$

□

Conclusion

In this research,the authors presented hyper Q-fuzzy HX subgroup of a HX group, hyper Q-fuzzy normal HX subgroup of a HX group, level sub HX group and conjugate are well defined and the authors discussed some of their conceptes. In the future, we can extend this concept to cosets and its types, homomorphism and in more areas of HX. On the other side, we will use the topics in [21, 22, 23, 24] to generalize on the hyper Q-fuzzy set concept.

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References

[1] Biswas, and Rosenfelds, *Fuzzy subgroups with interval-valued membership functions*, Fuzzy Sets and System, 63 (1994), 87-90.

[2] P. Corsini, *HX-groups and hypergroups,* Analele Univ. "Ovidius", Math Series., 24 (2016), 101-121.

[3] P. Corsini, *Hypergroups associated with HX-groups*, An. St. Univ. Ovidius Constanta, 25 (2017), 49-64.

[4] Ghosh Jayanta, T.K. Samanta, *Hyper fuzzy set and hyper fuzzy group*, International Journal of Advanced Science and Technology, 41 (2012), 27-38.

[5] Li Hongxing, *HX group*, Busefal, 33 (1987), 31-37.

[6] Chengzhong Mitlonghai Luo, Hongxing Li, *Fuzzy HX group*, Busefal, 39 (1991), 323-328.

[7] K.H. Manikandan, *Pseudo fuzzy cosets of a HX group*, Applied Mathematical Sciences, 7 (2013), 4259-4271.

[8] M.O. Massa'deh, *Some properties for M-homomorphism and M-anti homomorphism over Q-fuzzy M-HX Subgroup and its level*, Journal of Informatics and Mathematical Sciences, 9 (2017), 73-78.

[9] M.O. Massa'deh, *On Q-fuzzy M-HX groups*, International Journal of Fuzzy Mathematics and Systems, 6 (2016), 7-12.

[10] M.O. Massa'deh, A.A. Fora, *Bipolar-valued Q-fuzzy HX subgroup on an HX group*, Journal of Applied Computer Science and Mathematics, 11 (2017), 20-24.

[11] R. Muthuraj, M. Sridharan, K.H. Manikandan, *Some properties of bipolar fuzzy normal HX Subgroup and its normal level sub HX groups*, Global Research and Development Journal for Engineering, 2 (2016), pp. 57-65.

[12] R. Muthuraj, M. Sridharan, *Some properties of bipolar fuzzy cosets of a bipolar fuzzy and bipolar anti fuzzy HX subgroup*, IOSR Journal of Mathematics, 9 (2014), pp. 128-137.

[13] R. Muthuraj, M. Sridharan, *Homomorphism and anti homomorphism on bipolar fuzzy sub HX groups*, Gen. Math. Notes, 17 (2013), 53-65.

[14] R. Muthuraj, K.H. Manikandan, Selvam P.M. Sithar, *On anti Q-fuzzy normal HX groups*, Elixin Discrete Mathematics, 44 (2012), 7477-7479.

[15] R. Muthuraj, K.H. Manikandan, Selvam P.M. Sithar, *Intuitionistic Q-fuzzy normal HX groups*, Journal of Physical Sciences, 15 (2011), 95-102.

[16] R. Muthuraj, K.H. Manikandan, M.S. Muthuraman, Selvam P.M. Sithar, *Anti Q-fuzzy HX group and its Lower Level Sub HX groups*, International Journal of Computer Applications, 6 (2010), 16-20.

[17] A. Rosenfeld, *Fuzzy groups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512-517.

[18] A. Solairju, R. Nagarajan, *Q-fuzzy left R-subgroups of near rings w.r.t T-norms*, Antaratica Journal of Mathematics, 5 (2008), 59-63.

[19] L.A. Zadeh, *The concepts of a linguistic variable and its applications to approximate reasoning*, Information and Control, 8 (1975), 199-249.

[20] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[21] P. Corsini, *Partial semigroups and fuzzy sets*, Journal of discrete mathemathical sciences and cryptography, 21 (2016), 1479-1494.

[22] P. Corsini, *Some remarks on hyper structures their connections with fuzzy sets and extensions to weak structures*, Ratio mathemathica, 33 (2017), 61-76.

[23] P. Corsini, *Hyper structures and some of the most recent applications*, Journal of Hyper Structures, 6 (2017), 1-13.

[24] P. Corsini, *HX-hypergroups associated with the direct products of some* $\frac{Z}{nZ}$, Algebraic Structures and their Applications, 3 (2016), 1-15.

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The Crank-Nicolson-Galerkin FEM for a nonlocal parabolic system

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Abstract. A theoretical analysis of a linearized Crank-Nicolson Galerkin finite element method for the nonlocal nonlinear coupled system of the reaction-diffusion problem is presented here. For sufficiently smooth solutions, the maximal error in the L^2-norm possesses the optimal rate of convergence $O(\delta^2 + h^{r+1})$ (where h is the mesh size and δ is the time step size with $r \geq 1$) without any time step restriction. Some important results on the energy decay and vanishing of the solutions in finite time are also presented. To confirm our theoretical analysis, some numerical experiments are performed using Matlab.

Keywords: Optimal error, Crank-Nicolson schemes, finite element method, nonlocal diffusion.

1. Introduction

The following system of reaction-diffusion equations with a specific type nonlocal non-linearity is the focus of this research article.

$$(1.1)\qquad \begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \alpha_1 |u|^{p-2}u = f_1(x,t) \quad \text{in} \quad \Omega \times (0,T) \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \alpha_2 |v|^{p-2}v = f_2(x,t) \quad \text{in} \quad \Omega \times (0,T) \\ u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

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Where $\Omega \subset \mathbb{R}^d$ is a domain with a smooth boundary $\partial\Omega$.
The reaction-diffusion equation-model is one of the most successful models for describing physical, chemical, biological and ecological systems. Amongst them, the most successful and crucial one is the following model of parabolic partial differential equations, called the reaction-diffusion problem

$$u_t - A\Delta u - f(u) = 0, \tag{1.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear function. For $n \geq 2$, A is an $n \times n$ real matrix of diffusion (see [1] and references therein) and for $n = 1$, A is a real function called diffusion coefficient (see [2, 3]). However, cross-diffusion phenomena are not uncommon (see [4] and references therein) (1.2) can be treated as a system of equations in which A is not even diagonalizable. In 2008, Raposo et al. [5] studied the reaction-diffusion coupled system in a parallel way via a parameter $\alpha = const > 0$, of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u-v) = \alpha(u-v) & \text{in} \quad \Omega \times (0,T) \\ v_t - a(l(v))\Delta v - f(u-v) = \alpha(v-u) & \text{in} \quad \Omega \times (0,T), \end{cases} \tag{1.3}$$

and they proved the existence, uniqueness and exponential decay of solutions using the energy method. They also presented some numerical simulations using an implicit finite difference scheme in one dimension and a finite volume method without an analysis in two dimension.

Recently, Duque et al.[6] considered problem (1.1). Here u and v described the densities of two populations that interact through the functions a_1 and a_2. The death in species u is assumed to be proportional to $|u|^{p-2}u$ by the factor $\alpha_1 \geq 0$ and in species v to be proportional to $|v|^{p-2}v$ by the factor $\alpha_2 \geq 0$. The external sources are denoted by f_1 and f_2. Existence and uniqueness of weak and strong solutions were proved, globally in time, and conditions on the data so that these solutions have the waiting time and stable localization properties were given. Important results on polynomial and exponential decay and vanishing of the solutions in finite time were also presented.

There is a vast literature on finite element methods for nonlinear elliptic and parabolic problems, for instance [7, 8, 9] and references therein. But there are few studies on numerical approximations of nonlocal problems. Mbehou et al. [10] proposed a linearized Cranck-Nicolson finite element methods for a nonlocal nonlinear reaction-diffusion equation. Duque et al. [11] proposed the $P1$ or linear approximation finite element method in space variables and the Euler method for the time variable for the reaction-diffusion system (1.1) and derived convergence of order $O(\delta + h^2)$ in the L^2 norm. Almeida et al. [7] presented the convergence analysis of the fully discretized in the finite element method in space variables and the Crank-Nicolson method in time variables for a nonlocal parabolic equation with moving boundaries.

The main objective of this research article is to prove the convergence of a total discrete solution using the Crank-Nicolson-Galerkin finite element method.

To the best of our knowledge, these results are new for the nonlocal reaction-diffusion coupled system (1.1). This paper is organized as follows. Section 2 is concerned with the weak variational formulation of the problem and the hypotheses on the data. In Section 3, we present the semidiscrete solution and its convergence while in Section 4, we derive the optimal L^2 error estimate of the fully discrete Galerkin method unconditionnally. Finally numerical results are presented in Section 5 to demonstrate our theoretical analysis.

2. Preliminaries and weak formulation

$H^k(\Omega)$ is the usual Sobolev space of order $k \in \mathbb{N}$ with norm $\|\cdot\|_k$. $H_0^k(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^k(\Omega)$. $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. The Lebesgue space is denoted as $L^r(\Omega)$, $1 < r \leq \infty$, with norm $\|\cdot\|_{L^r}$ (the $L^2(\Omega)$-norm is denoted by $\|\cdot\|$). Moreover, we employ the standard notation of Bochner spaces such as $L^q(0,T,X)$, with norm $\|\cdot\|_{L^q(X)}$, where X is a Hilbert space.

Following Duque et al. [6],the existence and uniqueness of strong solutions of the reaction–diffusion system (1.1) in which α_1, α_2 are nonnegative constants and $p > 1$ were proven under the following:

Hyp 1: u_0, $v_0 \in L^2(\Omega)$.

Hyp 2: $a_i : \mathbb{R}^2 \to \mathbb{R}$ is bounded with $0 < m \leq a_i(p,q) \leq M$, $p, q \in \mathbb{R}$, $i = 1, 2$.

Hyp 3: $a_i : \mathbb{R}^2 \to \mathbb{R}$ and $|a_i(p_1,q_1) - a_i(p_2,q_2)| \leq M_i|p_1 - p_2| + K_i|q_1 - q_2|$, $p_i, q_i \in \mathbb{R}$, $i = 1, 2$, a_i Lipschitz–continuous.

Hyp 4: $l_i : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form, that is, there exists a positive function $g_i \in L^2(\Omega)$ such that $l_i(w) = l_{g_i}(w) = \int_\Omega g_i(x)w(x)dx$, for all $w \in L^2(\Omega)$, $i = 1, 2$.

Hyp 5: f_1, $f_2 \in L^2(0,T,L^2(\Omega))$.

The proof of the following results (Theorem 2.1, Theorem 2.2 and Theorem 2.3) can be found in [6].

Theorem 2.1. *Let $p > 1$ and $0 < T < \infty$. If* (**Hyp 1**) – (**Hyp 5**) *hold, then system* (1.1) *admits a unique solution, that is, there exists a unique couple (u, v) such that*

$$u,\, v \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H_0^1(\Omega)) \cap L^p(0,T,L^p(\Omega))$$

$$\cap\, \mathcal{C}([0,T];L^2(\Omega)), \tag{2.1}$$

$$u_t,\, v_t \in L^2(0,T,H^{-1}(\Omega)), \tag{2.2}$$

$$\frac{d}{dt}\int_\Omega u\phi\,dx + a_1(l_1(u),l_2(v))\int_\Omega \nabla u\cdot\nabla\phi\,dx + \alpha_1\int_\Omega |u|^{p-2}u\phi\,dx$$

$$= \int_\Omega f_1\phi\,dx, \tag{2.3}$$

$$\frac{d}{dt}\int_\Omega v\psi\,dx + a_2(l_1(u), l_2(v))\int_\Omega \nabla v\cdot\nabla\psi\,dx + \alpha_2\int_\Omega |v|^{p-2}v\psi\,dx$$
$$= \int_\Omega f_2\psi\,dx, \tag{2.4}$$

for all $\phi, \psi \in H_0^1(\Omega)$, *where* (2.3) *and* (2.4) *must be understood as an equality in* $\mathcal{D}'(0,T)$,

$$u(x,t) = v(x,t) = 0, \quad \text{on} \quad \partial\Omega\times]0,T], \tag{2.5}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \tag{2.6}$$

If more regularity is imposed on u_0 and v_0, then

$$u_t,\, v_t, \Delta u,\, \Delta v \in L^2(0,T,L^2(\Omega)). \tag{2.7}$$

Denote the global energy function by

$$E(t) = \frac{1}{2}(\|u(t)\|^2 + \|v(t)\|^2).$$

Theorem 2.2. *If* $f_1 \equiv f_2 \equiv 0$ *and* (u,v) *the solution of* (1.1) *then*

$$E(t) \le E(0)\exp(-\beta t),$$

where β *is a positive constant which depends on* m *and* Ω.

Moreover if $p > 2$, *then*

$$E(t)^{\mu-1} \le \frac{E(0)^{\mu-1}}{1 + tC(\mu-1)E(0)^{\mu-1}},$$

where $\mu = p/2$ *and* C *a positive constant which depends only on* N, p *and* Ω.

Theorem 2.3. *Suppose that* $1 < p < 2$. *If* $f_1 \equiv f_2 \equiv 0$ *and* (u,v) *the solution of* (1.1) *then* (u,v) *vanishes in a finite time, that is, there exists* t^* *which depends only on* $\|u_0\|$, $\|v_0\|$, p *and* Ω *such that*

$$u(x,t) \equiv 0 \quad \text{and} \quad v(x,t) \equiv 0 \quad \text{in} \quad \Omega \quad \text{for} \quad t > t^*.$$

3. Semi discrete approximation

Following classical finite element method theory [12], we define $\mathcal{T}_h$ as a regular triangulation of Ω into elements K. Let $h = \max_{K\in\mathcal{T}_h}\{h_K\}$ denote the mesh size, where $h_K = diam(K) = \max\{\|\boldsymbol{x} - \boldsymbol{y}\|,\, \boldsymbol{x},\, \boldsymbol{y} \in K\}$. Let V_h be the finite dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree $r \ge 1$ on $\mathcal{T}_h$.

Let Π_h be an interpolation operator and let $R_h : H_0^1(\Omega) \to V_h$ be a Ritz projection operator defined by

$$\int_\Omega \nabla(u - R_h u)\cdot\nabla w dx = 0 \quad \forall w \in H_0^1(\Omega). \tag{3.1}$$

Lemma 3.1 (cf. Ref. [12]). *If $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|u - \Pi_h u\| + h\|\nabla(u - \Pi_h u)\| \le Ch^{r+1}\|u\|_{H^{r+1}} \tag{3.2}$$

$$\|u - R_h u\| + h\|\nabla(u - R_h u)\| \le Ch^{r+1}\|u\|_{H^{r+1}}, \tag{3.3}$$

where C is a positive constant which does not depend on h and r.

Lemma 3.2 (cf. Ref. [13]). *For all $p \in (1, \infty)$ and $\tau \ge 0$, there exist a generic constant $C = C(p, d)$ such that for all ξ, $\eta \in \mathbb{R}^d$ with $d \ge 1$*

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \le C|\xi - \eta|^{1-\tau}(|\xi| + |\eta|)^{p-2+\tau} \tag{3.4}$$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \ge C|\xi - \eta|^{2+\tau}(|\xi| + |\eta|)^{p-2-\tau}. \tag{3.5}$$

The semidiscrete problem, based on Theorem 2.1 consists of finding u_h, $v_h \in V_h$, for $t \ge 0$, such that

$$\begin{aligned}
&\int_\Omega (u_h)_t \phi_h \, dx + a_1(l_1(u_h), l_2(v_h)) \int_\Omega \nabla u_h \cdot \nabla \phi_h \, dx \\
&+ \alpha_1 \int_\Omega |u_h|^{p-2} u_h \phi_h \, dx = \int_\Omega f_1 \phi_h \, dx, \\
&\int_\Omega (v_h)_t \psi_h \, dx + a_2(l_1(u_h), l_2(v_h)) \int_\Omega \nabla v_h \cdot \nabla \psi_h \, dx \\
&+ \alpha_2 \int_\Omega |v_h|^{p-2} v_h \psi_h \, dx = \int_\Omega f_2 \psi_h \, dx, \\
&u_h(x, 0) = \Pi_h u_0, \quad v_h(x, 0) = \Pi_h v_0,
\end{aligned} \tag{3.6}$$

for all $\phi_h, \psi_h \in V_h$.

We have the following result.

Theorem 3.1. *The semidiscrete problem* (3.6) *admits a unique solution (u_h, v_h) and if (u, v) is the solution of* (1.1)*, then*

$$\|u - u_h\| + \|v - v_h\| \le Ch^{r+1}, \quad \textit{for all} \quad t \in (0, T), \tag{3.7}$$

where C does not depend on h.

Proof. Note that (3.6) is equivalent to a system of ordinary differential equations of first order, so the existence of the solution follows from Caratheodory's theorem. The proof of uniqueness is similar to that of Theorem 6 in Ref. [6]. To prove the convergence, let us write $u_h - u = (u_h - R_h u) + (R_h u - u) = \Theta_h^u + \eta^u$ and $v_h - v = (v_h - R_h v) + (R_h v - v) = \Theta_h^v + \eta^v$ From Lemma 3.1, we have

$$\|\eta^u\| + \|\eta^v\| \le Ch^{r+1}\|u\|_{H^{r+1}} \, .$$

Now to estimate Θ_h^u and Θ_h^v, we have

$$\begin{aligned}\int_\Omega (\Theta_h^u)_t\phi_h\,dx + a_1(l_1(u_h),l_2(v_h))\int_\Omega \nabla\Theta_h^u\cdot\nabla\phi_h\,dx \\ &= \int_\Omega (u_h)_t\phi_h dx + a_1(l_1(u_h),l_2(v_h))\int_\Omega \nabla u_h\cdot\nabla\phi_h\,dx \\ &- \int_\Omega (R_hu)_t\phi_h\,dx - a_1(l_1(u_h),l_2(v_h))\int_\Omega \nabla(R_hu)\cdot\nabla\phi_h\,dx \\ &= \int_\Omega (u-R_hu)_t\phi_h\,dx + \alpha_1\int_\Omega (|u|^{p-2}u - |u_h|^{p-2}u_h)\phi_h\,dx \\ &+ (a_1(l_1(u),l_2(v)) - a_1(l_1(u_h),l_2(v_h)))\int_\Omega \nabla u\cdot\nabla\phi_h\,dx.\end{aligned}$$

If we take $\phi_h = \Theta_h^u$, on using the hypotheses (**Hyp 2**), (**Hyp 3**), (**Hyp 4**) and Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + m\|\nabla\Theta_h^u\|^2 &\le \|\eta_t^u\|\|\Theta_h^u\| + C_1(u,u_h)\|u-u_h\|\|\Theta_h^u\| \\ &+ C_2(M_1,K_1,u)(\|u-u_h\| + \|v-v_h\|)\|\nabla\Theta_h^u\|\,,\end{aligned} \tag{3.8}$$

where $C_1(u,u) = C(\|u\|_{L^\infty(L^2(\Omega))}, \|u_h\|_{L^\infty(L^2(\Omega))})$ and $C_2(M_1,K_1,u) = C(M_1,K_1,$ $\|\nabla u\|_{L^\infty(L^2(\Omega))})$ are constants. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + m\|\nabla\Theta_h^u\|^2 &\le C\left\{\|\eta_t^u\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\} \\ &+ m\|\nabla\Theta_h^u\|^2\,.\end{aligned} \tag{3.9}$$

If we use the same argument for Θ_h^v, we will have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Theta_h^v\|^2 + m\|\nabla\Theta_h^v\|^2 \\ &\le C\left\{\|\eta_t^v\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\} + m\|\nabla\Theta_h^v\|^2\,.\end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + \frac{1}{2}\frac{d}{dt}\|\Theta_h^v\|^2 \le C\left\{\|\eta_t^v\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\}$$

and integrating with respect to time on the interval $[0,t]$ we obtain

$$\begin{aligned}\|\Theta_h^u\|^2 + \|\Theta_h^v\|^2 \le \|\Theta_h^u(0)\|^2 + \|\Theta_h^v(0)\|^2 + C\int_0^T (\|\eta_t^u(s)\|^2 \\ + \|\eta_t^v(s)\|^2 + \|\eta^u(s)\|^2 + \|\eta^v(s)\|^2)\,ds\,,\end{aligned}$$

so the inequality (3.7) follows from Lemma 3.1. □

4. Fully discrete problem

Let $\{t_n | \, t_n = n\delta;\, 0 \le n \le N\}$ be a uniform partition of $[0,T]$ with the time step $\delta = T/N$, and $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$. We denote w^m by $w^m = w(x,t_m)$. For a sequence of functions $\{w^m\}_{n=0}^N$, we define

$$\bar\partial w^n = \frac{w^n - w^{n-1}}{\delta}, \quad \bar w^n = \frac{1}{2}(w^n + w^{n-1}),\ n = 1,2...,N,$$
$$\hat w^n = \frac{1}{2}(3w^{n-1} - w^{n-2}), \quad n = 2,...,N.$$

With above notations, a linearized Crank-Nicolson FEM is as follows: Find $u_h^n,\, v_h^n \in V_h$ such that

$$\begin{aligned}
&\int_\Omega \bar\partial u_h^n \phi_h\,dx + a_1(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar u_h^n \cdot \nabla \phi_h\,dx \\
&+ \alpha_1 \int_\Omega |\hat u_h^n|^{p-2} \bar u_h^n \phi_h\,dx = \int_\Omega f_1^{n-1/2} \phi_h\,dx, \\
&\int_\Omega \bar\partial v_h^n \psi_h\,dx + a_2(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar v_h^n \cdot \nabla \psi_h\,dx \\
&+ \alpha_2 \int_\Omega |\hat v_h^n|^{p-2} \bar v_h^n \psi_h\,dx = \int_\Omega f_2^{n-1/2} \psi_h\,dx, \\
&u_h^0 = \Pi_h u_0, \quad v_h^0 = \Pi_h v_0, \quad n = 1,...,N, \quad \text{for all} \quad \phi_h, \psi_h \in V_h,
\end{aligned} \tag{4.1}$$

where $(\hat u_h^1,\, \hat v_h^1)$ is the solution of the following system of equations.

$$\begin{aligned}
&\int_\Omega \frac{\hat u_h^1 - u_h^0}{\delta/2} \phi_h\,dx + a_1(l_1(u_h^0), l_2(v_h^0)) \int_\Omega \nabla \hat u_h^1 \cdot \nabla \phi_h\,dx \\
&+ \alpha_1 \int_\Omega |u_h^0|^{p-2} \hat u_h^1 \phi_h\,dx = \int_\Omega f_1^{1/2} \phi_h\,dx, \\
&\int_\Omega \frac{\hat v_h^1 - v_h^0}{\delta/2} \psi_h\,dx + a_2(l_1(u_h^0), l_2(v_h^0)) \int_\Omega \nabla \hat v_h^1 \cdot \nabla \psi_h\,dx \\
&+ \alpha_2 \int_\Omega |v_h^0|^{p-2} \hat v_h^1 \psi_h\,dx = \int_\Omega f_2^{1/2} \psi_h\,dx.
\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $p \ge 2$. If (u,v) is the solution of system* (1.1) *and (u_h^n, v_h^n) is the solution of* (4.1) – (**??**) *then*

$$\|u^n - u_h^n\| + \|v^n - v_h^n\| \le C(h^{r+1} + \delta^2), \tag{4.3}$$

where C is a generic constant which does not depend on h and δ.

Proof. We write

$$u_h^n - u^n = (u_h^n - R_h u^n) + (R_h u^n - u^n) = \Theta_n^u + \Psi_n^u$$

and

$$v_h^n - v^n = (v_h^n - R_h v^n) + (R_h v^n - v^n) = \Theta_n^v + \Psi_n^v.$$

The estimations of Ψ_n^u and Ψ_n^v are given by Lemma 3.1. To estimate Θ_n^u and Θ_n^v, we have

$$\begin{aligned}
&\int_\Omega \bar{\partial}\Theta_n^u \phi_h dx + a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{\Theta}_n^u \cdot \nabla \phi_h dx + \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2} \bar{\Theta}_n^u \phi_h dx \\
&= \int_\Omega \bar{\partial} u_h^n \phi_h dx + a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{u}_h^n \cdot \nabla \phi_h dx + \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2} \bar{u}_h^n \phi_h dx \\
&- \int_\Omega \bar{\partial} R_h u^n \phi_h dx - a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{R}_h u^n \cdot \nabla \phi_h dx \\
&- \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2} \bar{R}_h u^n \phi_h dx = \int_\Omega f_1^{n-1/2} \phi_h dx - \int_\Omega (u_t)^{n-1/2} \phi_h dx \\
&- a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla u^{n-1/2} \cdot \nabla \phi_h dx \\
&- \alpha_1 \int_\Omega |u^{n-1/2}|^{p-2} u^{n-1/2} \phi_h dx + \int_\Omega ((u_t)^{n-1/2} - \bar{\partial} R_h u^n) \phi_h dx \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla (u^{n-1/2} - \bar{R}_h u^n) \cdot \nabla \phi_h dx \\
&+ (a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n))) \int_\Omega \nabla \bar{R}_h u^n \cdot \nabla \phi_h dx \\
&+ \alpha_1 \int_\Omega (|u^{n-1/2}|^{p-2} u^{n-1/2} - |\bar{R}_h u^n|^{p-2} \bar{R}_h u^n) \phi_h dx \\
&+ \alpha_1 \int_\Omega (|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2}) \bar{R}_h u^n \phi_h dx.
\end{aligned}$$

By applying the definition of Ritz projection operator (3.1), we obtain

$$\begin{aligned}
&\int_\Omega \bar{\partial}\Theta_n^u \phi_h dx + a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{\Theta}_n^u \cdot \nabla \phi_h dx \\
&+ \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2} \bar{\Theta}_n^u \phi_h dx = \int_\Omega ((u_t)^{n-1/2} - \bar{\partial} R_h u^n) \phi_h dx \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla (u^{n-1/2} - \bar{u}^n) \cdot \nabla \phi_h dx \\
&+ (a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n))) \int_\Omega \nabla \bar{u}^n \cdot \nabla \phi_h dx \qquad (4.4) \\
&+ \alpha_1 \int_\Omega (|u^{n-1/2}|^{p-2} u^{n-1/2} - |\bar{R}_h u^n|^{p-2} \bar{R}_h u^n) \phi_h dx \\
&+ \alpha_1 \int_\Omega (|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2}) \bar{R}_h u^n \phi_h dx.
\end{aligned}$$

Choosing $\phi_h = \bar{\Theta}_n^u$ and applying Holder's inequality, we arrive at

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 + a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2}))\|\nabla\bar{\Theta}_n^u\|^2 + \alpha_1\int_\Omega |\hat{u}_h^n|^{p-2}|\bar{\Theta}_n^u|^2 dx \\
&\le \|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\|\|\bar{\Theta}_n^u\| \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2}))\|\nabla(u^{n-1/2} - \bar{u}^n)\|\|\nabla\bar{\Theta}_n^u\| \\
&+ |(a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)))|\|\nabla\bar{u}^n\|\|\nabla\bar{\Theta}_n^u\| \\
&+ \alpha_1\int_\Omega |(|u^{n-1/2}|^{p-2}u^{n-1/2} - |\bar{R}_h u^n|^{p-2}\bar{R}_h u^n)||\bar{\Theta}_n^u| dx \\
&+ \alpha_1\int_\Omega |(|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2})||\bar{R}_h u^n||\bar{\Theta}_n^u| dx.
\end{aligned}$$

We apply Poincare's inequality, $(Hyp\,2) - (Hyp\,3)$ and Lemma 3.2, to obtain

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 + m\|\nabla\bar{\Theta}_n^u\|^2 \le \Big\{C\|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\| + M\|\nabla(u^{n-1/2} - \bar{u}^n)\| \\
&+ C_1(u)(M_1\|u^{n-1/2} - \hat{u}_h^n\| + K_1\|v^{n-1/2} - \hat{v}_h^n\|) \\
&+ C_2(u)\|u^{n-1/2} - \bar{R}_h u^n\| + C_2(u, u_h)\|\bar{R}_h u^n - \hat{u}_h^n\|\Big\}\|\nabla\bar{\Theta}_n^u\|,
\end{aligned}$$

where $C_1(u) = C(\|\nabla u\|_{L^\infty(0,T,L^2(\Omega))})$ and $C_2(w) = C(\|w\|_{L^\infty(0,T,L^\infty(\Omega))})$. Therefore,

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 \le C\Big\{&\|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\| + \|\nabla(u^{n-1/2} - \bar{u}^n)\| + \|u^{n-1/2} - \hat{u}_h^n\| \\
&+ \|v^{n-1/2} - \hat{v}_h^n\| + \|u^{n-1/2} - \bar{R}_h u^n\| + \|\bar{R}_h u^n - \hat{u}_h^n\|\Big\}^2. \qquad (4.5)
\end{aligned}$$

In the same way we can prove that

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_n^v\|^2 \le C\Big\{&\|(v_t)^{n-1/2} - \bar{\partial}R_h v^n\| + \|\nabla(v^{n-1/2} - \bar{v}^n)\| + \|u^{n-1/2} - \hat{u}_h^n\| \\
&+ \|v^{n-1/2} - \hat{v}_h^n\| + \|v^{n-1/2} - \bar{R}_h v^n\| + \|\bar{R}_h v^n - \hat{v}_h^n\|\Big\}^2. \qquad (4.6)
\end{aligned}$$

By numerical differentiation, interpolation theories and Lemma 3.1, we can prove that for $w \in \{u, v\}$,

$$\begin{aligned}
\|(w_t)^{n-1/2} - \bar{\partial}R_h w^n\| &\le \|(w_t)^{n-1/2} - \bar{\partial}w^n\| + \|\bar{\partial}\Psi_n^w\| \\
&\le C\delta^2\|w_{ttt}\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w_t\|_{L^\infty(0,T,H^{r+1}(\Omega))} \qquad (4.7)
\end{aligned}$$

$$\|\nabla(w^{n-1/2} - \bar{w}^n)\| \le C\delta^2\|\nabla w_{tt}\|_{L^\infty(0,T,L^2(\Omega))} \qquad (4.8)$$

$$\begin{aligned}
\|w^{n-1/2} - \bar{R}_h w^n\| &\le \|w^{n-1/2} - \bar{w}^n\| + \|\bar{w}^n - \bar{R}_h w^n\| \\
&\le C\delta^2\|w_{tt}\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w\|_{L^\infty(0,T,H^{r+1}(\Omega))} \qquad (4.9)
\end{aligned}$$

$$\|w^{n-1/2} - \hat{w}_h^n\| \leq \|w^{n-1/2} - \hat{w}^n\| + \|\hat{\Psi}_n^w\| + \|\hat{\Theta}_n^w\| \leq C\delta^2 \|w_{tt}\|_{L^\infty(0,T,L^2(\Omega))}$$
$$+ Ch^{r+1}\|w\|_{L^\infty(0,T,H^{r+1}(\Omega))} + C(\|\Theta_{n-1}^w\| + \|\Theta_{n-2}^w\|) \quad \text{for } n \geq 2 \tag{4.10}$$

$$\|\bar{R}_h w^n - \hat{w}_h^n\| \leq \|\bar{R}_h w^n - w^{n-1/2}\| + \|w^{n-1/2} - \hat{w}_h^n\|. \tag{4.11}$$

Hence from above inequalities (4.7)-(4.11), we obtain

$$\|\Theta_n^u\|^2 + \|\Theta_n^v\|^2 \leq C\delta(\|\Theta_{n-1}^u\|^2 + \|\Theta_{n-1}^v\|^2 + \|\Theta_{n-2}^u\|^2 + \|\Theta_{n-2}^v\|^2)$$
$$+ C\delta(\delta^2 + h^{r+1})^2 \tag{4.12}$$

By iterating and noting that for all $1 \leq n \leq N$, $\sum_{k=1}^n \delta \leq T$, we obtain

$$\|\Theta_n^u\|^2 + \|\Theta_n^v\|^2 \leq C(\|\Theta_0^u\|^2 + \|\Theta_0^v\|^2 + \|\Theta_1^u\|^2 + \|\Theta_1^v\|^2) + C(\delta^2 + h^{r+1})^2. \tag{4.13}$$

Note that by the definition of w_h^0 with $w_h^0 \in \{u_h^0, v_h^0\}$ and Lemma 3.1, we have

$$\|\Theta_0^w\| \leq Ch^{r+1}\|w_0\|_{L^\infty(0,T,H^{r+1}(\Omega))}.$$

To conclude the proof, we have to estimate $\|\Theta_1^w\|$ for $w \in \{u, v\}$.

From (4.5), we have

$$\frac{1}{2}\bar{\partial}\|\Theta_1^u\|^2 \leq C\Big\{\|(u_t)^{1/2} - \bar{\partial}R_h u^1\| + \|\nabla(u^{1/2} - \bar{u}^1)\| + \|u^{1/2} - \hat{u}_h^1\|$$
$$+ \|v^{1/2} - \hat{v}_h^1\| + \|u^{1/2} - \bar{R}_h u^1\| + \|\bar{R}_h u^1 - \hat{u}_h^1\|\Big\}^2, \tag{4.14}$$

where $(\hat{u}_h^1, \hat{v}_h^1)$ is the solution of (4.2)–(**??**). From inequalities (4.7)-(4.11) and the triangle inequality, we can write

$$\frac{1}{2}\bar{\partial}\|\Theta_1^u\|^2 \leq C\{\delta^2 + h^{r+1}\}^2 + C\Big\{\|R_h v^{1/2} - \hat{v}_h^1\| + \|R_h u^{1/2} - \hat{u}_h^1\|\Big\}^2. \tag{4.15}$$

Let us denote $\hat{\Theta}_0^u$ and $\hat{\Theta}_1^u$ by $\hat{\Theta}_0^u = \Theta_0^u$ and $\hat{\Theta}_1^u = \hat{u}_h^1 - R_h\hat{u}^1$ with $\hat{u}^1 = u^{1/2} = u(t_{1/2})$. Then from (4.2), using (3.1), we have

$$\begin{aligned}
&2\int_\Omega \bar{\partial}\hat{\Theta}_1^u \phi_h dx + a_1(l_1(u_h^0), l_2(v_h^0)) \int_\Omega \nabla\hat{\Theta}_1^u \cdot \nabla\phi_h dx \\
&+ \alpha_1 \int_\Omega |u_h^0|^{p-2}\hat{\Theta}_1^u \phi_h dx \\
&= \int_\Omega ((u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2})\phi_h dx + (a_1(l_1(u^{1/2}), l_2(v^{1/2})) \\
&- a_1(l_1(u_h^0), l_2(v_h^0))) \int_\Omega \nabla u^{1/2} \cdot \nabla\phi_h dx \\
&+ \alpha_1 \int_\Omega (|u^{1/2}|^{p-2}u^{1/2} - |R_h u^{1/2}|^{p-2}R_h u^{1/2})\phi_h dx \\
&+ \alpha_1 \int_\Omega (|u_h^0|^{p-2} - |R_h u^{1/2}|^{p-2})R_h u^{1/2}\phi_h dx.
\end{aligned} \tag{4.16}$$

By choosing $\phi_h = \hat{\Theta}_1^u$ and applying the properties of $a_1(.,.)$, Poincare's inequality, Holder's inequality and Lemma 3.2, we arrive at

$$\begin{aligned}
&\frac{2}{\delta}\|\hat{\Theta}_1^u\|^2 - \frac{2}{\delta}\int_\Omega \Theta_0^u \hat{\Theta}_1^u dx + m\|\nabla\hat{\Theta}_1^u\|^2 \\
&\le C\Big\{\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| + \|u^{1/2} - u_h^0\| + \|v^{1/2} - v_h^0\| \\
&+ \|u^{1/2} - R_h u^{1/2}\| + \|R_h u^{1/2} - u_h^0\|\Big\}\,\|\nabla\hat{\Theta}_1^u\|,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{1}{\delta}\|\hat{\Theta}_1^u\|^2 \le \frac{1}{\delta}\|\Theta_0^u\|^2 + C\Big\{&\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| + \|u^{1/2} - u_h^0\| + \|v^{1/2} - v_h^0\| \\
&+ \|u^{1/2} - R_h u^{1/2}\| + \|R_h u^{1/2} - u_h^0\|\Big\}^2. \qquad (4.17)
\end{aligned}$$

Again applying numerical differentiation and interpolation theories and Lemma 3.1, we can prove that

$$\begin{aligned}
\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| &\le \|(u_t)^{1/2} - 2\bar{\partial}u^{1/2}\| + 2\|\bar{\partial}u^{1/2} - \bar{\partial}R_h u^{1/2}\| \\
&\le C\delta\|u_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|u_t\|_{L^\infty(0,T,H^{r+1}(\Omega))} \\
\|w^{1/2} - w_h^0\| &\le \|w^{1/2} - w_0\| + \|w_0 - w_h^0\| \\
&\le C\delta\|w_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w_0\|_{L^\infty(0,T,H^{r+1}(\Omega))}.
\end{aligned}$$

Finally, we have

$$\|\hat{\Theta}_1^u\|^2 \le C\{\delta^3 + h^{2(r+1)}\}. \qquad (4.18)$$

By substituting (4.18) into (4.15), we conclude that

$$\begin{aligned}
\|\Theta_1^u\|^2 &\le C\|\Theta_0^u\|^2 + C\delta\{\delta^3 + h^{2(r+1)}\} + C\delta\{\delta^2 + h^{r+1}\}^2 \\
&\le C\{\delta^2 + h^{r+1}\}^2. \qquad (4.19)
\end{aligned}$$

In the same way, wex can prove that

$$\|\Theta_1^v\|^2 \le C\{\delta^2 + h^{r+1}\}^2. \qquad (4.20)$$

5. Numerical simulations

5.1 Example 1: Energy decay and the vanishing in finite time

In order to verify the properties of the energy decay exponentially and the finite time extinction when $1 < p < 2$, we consider the system (1.1) in $\Omega = (0,1)^2$ with $\alpha_1 = \alpha_2 = 1$ and functions satisfying (**Hyp 1**) – (**Hyp 5**).

$$\begin{aligned}
&a_1(s,z) = 3 + \sin(s) + \cos(z), \\
&a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx \\
&u_0(x,y) = 2xy(1-x)(1-y), \quad v_0(x,y) = 0.5xy(1-x)(1-y), \\
&f_1(x,y,t) = 0.01ln(x+y+1)(0.1-t)^6, \quad f_2(x,y,t) = 0.01\sin(xy)(0.1-t)^4.
\end{aligned}$$

Figure 1 and Figure 2 show that, for $p = 1.2$, the solutions $u_h^n \equiv 0$ and $v_h^n \equiv 0$ for $t \geq 0.046$ while in Figure 3 and Figure 4 ($p = 3.5$), the solutions u_h^n and v_h^n are very small but not zeros for $t \geq 0.046$. For the asymptotic behavior, the energy function $E_n = 1/2(\|u_h^n\|^2 + \|v_h^n\|^2)$ decays and is extinguished at $t \approx 0.045$ for $p = 1.2$ (See Figure 5).

Figure 1: Evolution of the solution u_h^n for some values of t and $p = 1.2$

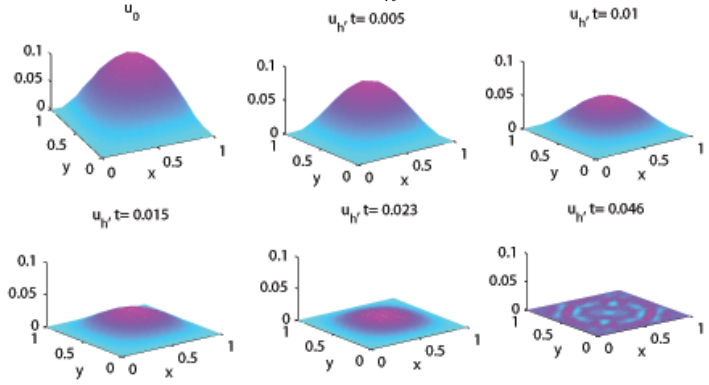


Figure 2: Evolution of the solution v_h^n for some values of t and $p = 1.2$

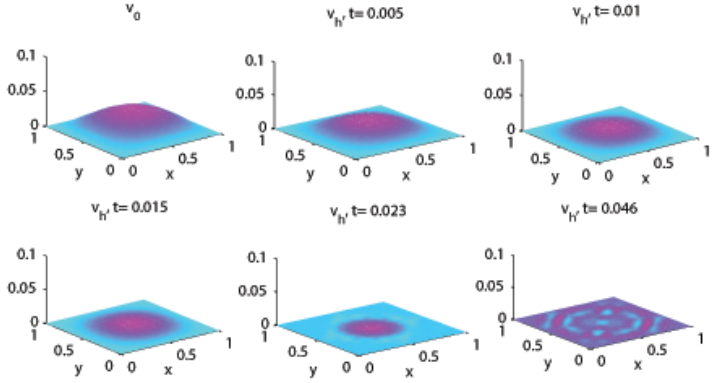


Figure 3: Evolution of the solution u_h^n for some values of t and $p = 3.5$

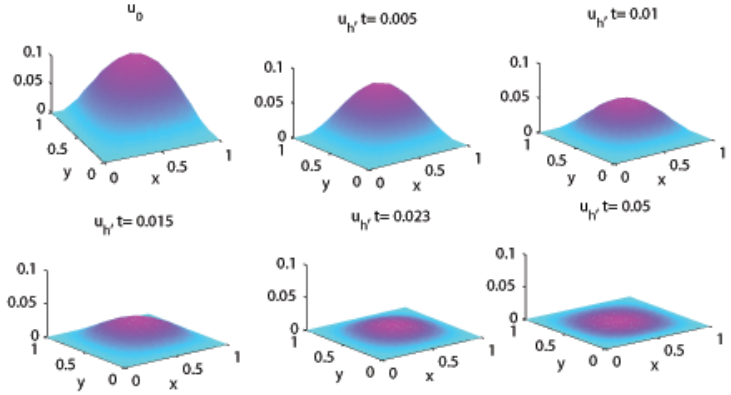
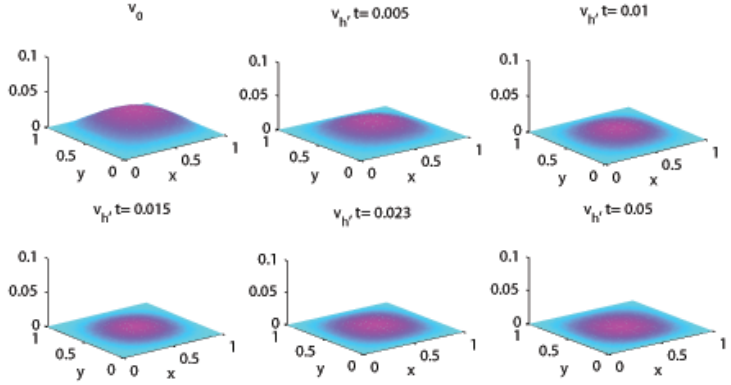


Figure 4: Evolution of the solution v_h^n for some values of t and $p = 3.5$



5.2 Example 2: Illustration of the diffusion property

We consider system (1.1) in $\Omega = (0,1)^2$ with $p = 3.5$, $\alpha_1 = \alpha_2 = 1$ and

$$a_1(s,z) = 3 + \sin(s) + \cos(z),$$

$$a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx,$$

$$u_0(x,y) = \begin{cases} 20xy(0.2 - (x-0.2)^2 - (y-0.2)^2), & \text{if} \quad 0 \le x, y \le 0.5 \\ 0 \quad \text{elsewhere} \end{cases}$$

$$v_0(x,y) = \begin{cases} 15(1-x)(1-y)(0.2 - (x-0.8)^2 - (y-0.8)^2), & \text{if} \quad 0.5 \le x, y \le 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

$$f_1(x,y,t) = 0.01ln(x+y+1)(0.1-t)^6, \quad f_2(x,y,t) = 0.01\sin(xy)(0.1-t)^4.$$

Figure 5: Evolution of the energy function E_n for $p = 1.2$ and $p = 3.5$ respectively

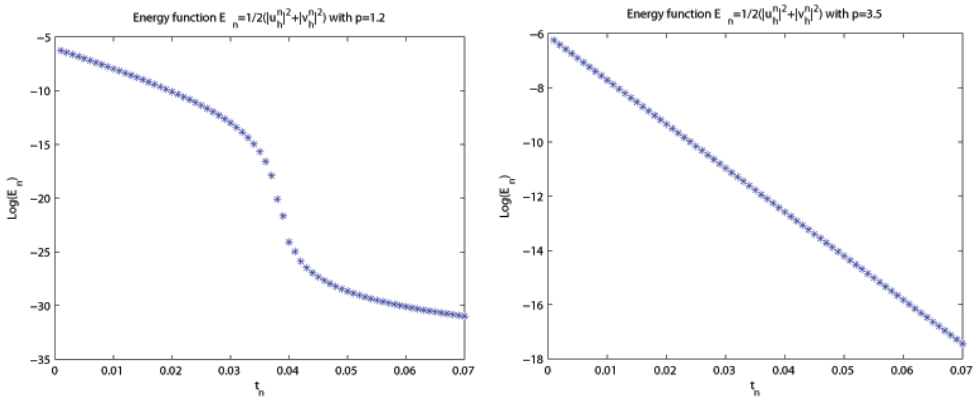


Figure 6 and Figure 7 show that, initially the solutions u_h^n and v_h^n are concentrated in two different small regions, and as time increases these solutions spread over the domain and decrease.

Figure 6: Evolution of the solution u_h^n for some values of t and $p = 3.5$

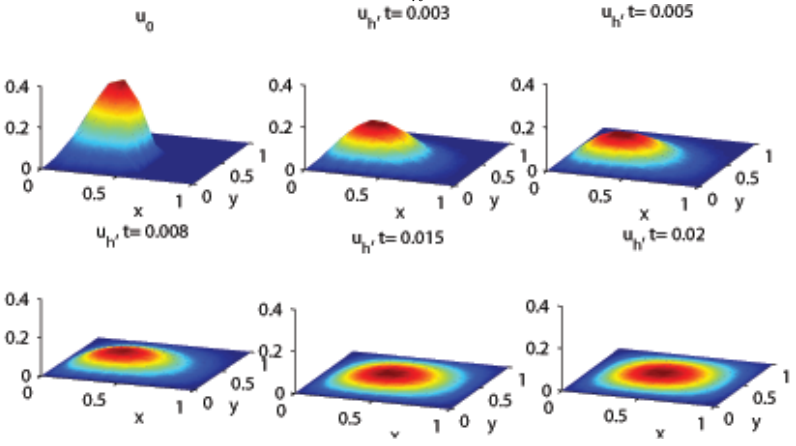
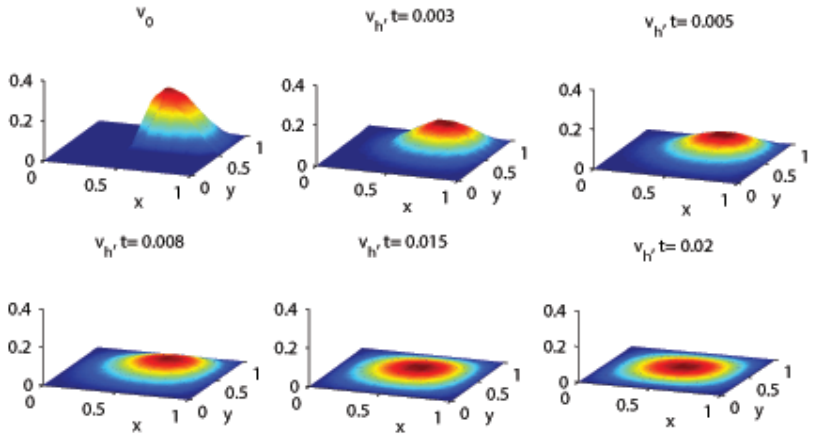


Figure 7: Evolution of the solution v_h^n for some values of t and $p = 3.5$



5.3 Example 3: Convergence rate

We consider the system (1.1) in $\Omega = (0,1)^2$ with $p = 3.5$, $\alpha_1 = \alpha_2 = 1$ and

$$a_1(s,z) = 3 + \sin(s) + \cos(z),$$
$$a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx$$
$$u(x,y,t) = \sin(\pi x)\sin(\pi y)\cos(\pi t), \quad v(x,y,t) = 0.5\sin(\pi x)\sin(\pi y)\cos(\pi t).$$

We solve the system (1.1) by the linearized Crank-Nicolson Galerkin finite element method (4.1)-(4.2) with a linear and quadratic finite element approximation. A uniform triangulation is used. For the convergence with respect to the mesh size h, $\delta = h^2$ and for the time step, $h = 0.01$. From Figure 8, we see that errors estimate in L^2-norm are in the form of $O(h^{r+1} + \delta^2)$, with $r = 1, 2$ which confirm our theoretical analysis.

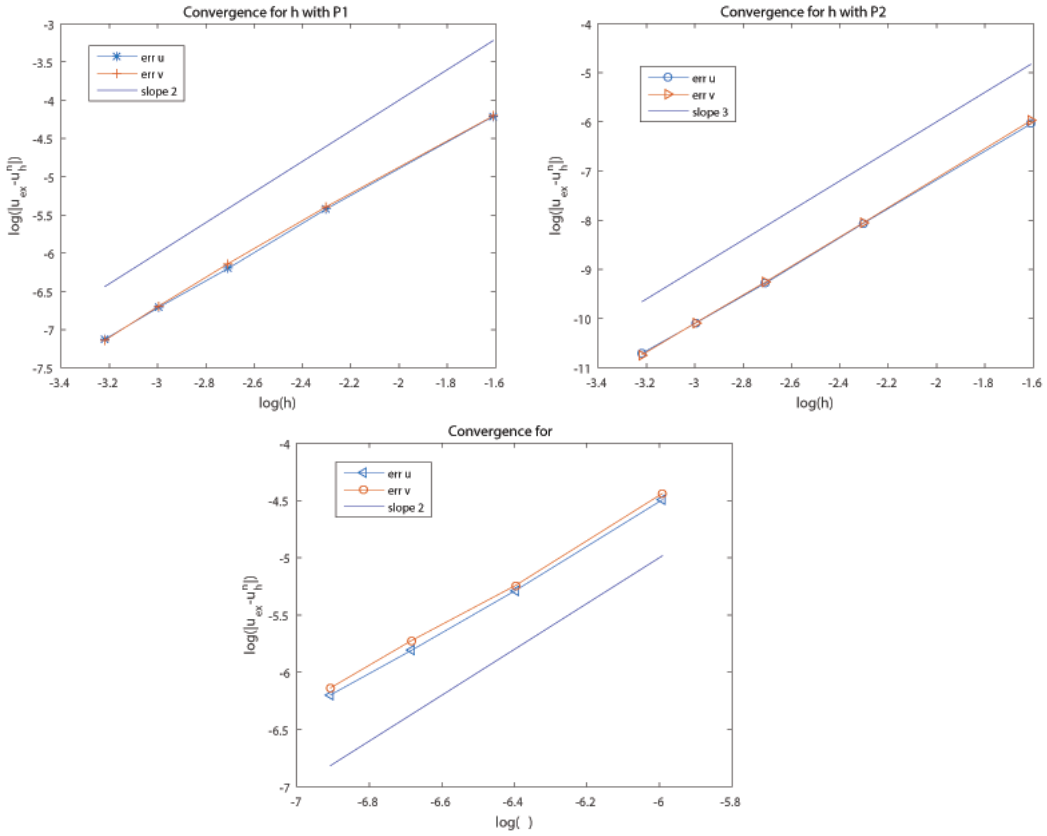
6. Conclusion

We have presented and analyzed a linearized Crank-Nicolson Galerkin finite element method for a nonlocal nonlinear parabolic system. The optimal L^2 error estimate has been obtained using sufficient conditions on the exact solution. Some numerical experiments on Matlab's environment have been carried out and our numerical results confirmed the theoretical analysis.

References

[1] J. K. Hale, *Large diffusivity and asymptotic behavior in parabolic systems,* Journal of mathematical analysis and applications, 118 (1986), 455-466.

Figure 8: Convergence rate



[2] M. Chipot and B. Lovat, *Some remarks on non local elliptic and parabolic problems,* Nonlinear Analysis: Theory, Methods & Applications, 30 (1997), 4619-4627.

[3] J. Djoko, J. Lubuma, and M. Mbehou, *On the numerical solution of the stationary power-law Stokes equations: A penalty finite element approach*, Journal of Scientific Computing, 69 (2016), 1058-1082.

[4] F. J. S. Corrêa, S. D. Menezes, and J. Ferreira, *On a class of problems involving a nonlocal operator*, Applied Mathematics and Computation, 147 (2004), 475-489.

[5] C. A. Raposo, M. Sepúlveda, O. V. Villagrán, D. C. Pereira, and M. L. Santos, *Solution and asymptotic behaviour for a nonlocal coupled system of reaction-diffusion*, Acta Applicandae Mathematicae, 102 (2008), 37-56.

[6] J. C. Duque, R. M. Almeida, S. N. Antontsev, and J. Ferreira, *A reaction-diffusion model for the nonlinear coupled system: existence, uniqueness,*

long time behavior and localization properties of solutions, IMA Journal of Applied Mathematics, 1-21, 2016.

[7] R. M. Almeida, J. Duque, J. Ferreira, and R. J. Robalo, *The Crank–Nicolson–Galerkin finite element method for a nonlocal parabolic equation with moving boundaries*, Numerical Methods for Partial Differential Equations, 31 (2015), 1515-1533.

[8] J. K. Djoko and M. Mbehou, *Finite element analysis of the stationary power-law Stokes equations driven by friction boundary conditions*, *Journal of Numerical Mathematics*, 23 (2015), 21-40.

[9] M. Mbehou, *The theta-Galerkin finite element method for coupled systems resulting from microsensor thermistor problems*, Mathematical Methods in the Applied Sciences, 41 (2018), 1480-1491.

[10] M. Mbehou, R. Maritz, and P. Tchepmo, *Numerical Analysis for a Nonlocal Parabolic Problem*, East Asian Journal on Applied Mathematics, 6 (2016), 434-447.

[11] J. C. Duque, R. M. Almeida, S. N. Antontsev, and J. Ferreira, *The Euler–Galerkin finite element method for a nonlocal coupled system of reaction–diffusion type*, Journal of Computational and Applied Mathematics, 296 (2016), 116-126.

[12] V. Thomée, *Galerkin finite element methods for parabolic problems*, 1054, Springer, 1984.

[13] J. W. Barrett and W. Liu, *Finite element approximation of the p-laplacian,* Mathematics of computation, 61 (1993), 523-537.

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A multi criteria decision making method for cubic hesitant fuzzy sets based on Einstein operational laws

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Abstract. In this paper, Einstein operations for cubic hesitant fuzzy sets have been introduced and also proved its various results. Aggregation operators play an important role to aggregate the fuzzy information, in view of this fact cubic hesitant fuzzy Einstein weighted averaging operators, cubic hesitant fuzzy Einstein weighted geometric

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operators have been introduced. Finally, by using these aggregation operators a multi criteria decision making problem of real life has been solved.

Keywords: Cubic hesitant fuzzy sets (CHFSs), aggregation operators (AOs), cubic hesitant fuzzy Einstein weighted averaging (CHFEWA) operators, cubic hesitant fuzzy Einstein weighted geometric (CHFEWG) operator, multi criteria decision making (MCDM).

1. Introduction

Fuzzy set theory which was developed by L. A. Zadeh [33] in 1965 is a useful tool to describe the situation when the information is indefinite or indeterminate. Since its inception, fuzzy set theory has been developed and used in many directions of engineering and mathematics. Fuzzy set is a necessary mathematical tool to handle a compilation of facts whose division line is not clear. These facts are given by the membership grade which belongs to $[0, 1]$. If the membership grade is 0 it means that the object does not belong to defined criteria and if this value is 1 it shows that it is fully agreed by the given condition. When defining the membership criteria someone has its own approach to determine the fact. For example when defining the term intelligent someone feels that a person is intelligent if he got 90 percent marks in his academic career while the other has the concept that he should have a better IQ level. In such types of generally fuzzy sets does not help to point being different between immaterial elements and opposite parts. In view of these facts several types of generalization of fuzzy sets have been made which includes interval valued fuzzy sets (IVFSs) [34], hesitant fuzzy sets (HFSs) [15, 16], interval valued hesitant fuzzy sets (IVHFSs) [3], intuitionistic fuzzy sets [1], interval valued intuitionistic fuzzy sets [2], cubic sets (CSs) [6], cubic hesitant fuzzy sets (CHFSs) [11, 12], etc. V. Torra [15, 16] introduced HFS that gives a set of possible values in which each value belongs to $[0, 1]$. Actually the concept of HFS was defined due to the phenomena when a group of people have the hesitancy to reach the ultimate decision. N. Chen et al [3] defined IVHFSs that generalize the concept of HFSs and allow membership value to be the closed subintervals of $[0, 1]$. T. Mahmood et al [11, 12] introduced CHFSs which is a very vital generalization and the only tool to solve such type of uncertain information when there is hesitancy in both IVFSs and fuzzy sets.

There are lots of problems in the world that needs choice of appropriate alternative amongst possible options. Such types of problems are called decision making problems and have applications in almost every field of life like economics, medical sciences, engineering and information technology etc. When decision makers suggest a collection of alternatives against a set of criteria then these sorts of arguments are called multi criteria decision making (MCDM) arguments [4, 5, 7, 13, 14, 17, 20, 22, 27, 32]. The MCDM method is acceptable when an instinctive method is not appropriate, for example considering the decision makers finger the visualization is too large and ramified to handle

instinctiveness, because it has a number of inconsistent goals, or has several collaborators with contrary views. When reaching the ultimate result in MCDM process, information given by the decision makers need to be aggregated. For this purpose many aggregation operators (AOs) have been introduced. M. Xia and Z. S. Xu [23] introduced AOs for HFSs. N. Chen et al [3] gave AOs for IVHFSs and used them in group decision making. It is noted that these AOs [24, 25, 26] are established of algebraic operational laws. Many AOs [8, 9, 10, 18, 19, 28, 29, 31] for fuzzy information have been introduced by using Einstein operations. D. Yu [28] defined AOs in order to aggregate hesitant fuzzy facts through Einstein operational laws. G. Wei and X. Zhao [21] explored AOs for IVHFSs by using Einstein operational laws. W. Wang and X. Liu [19] used Einstein operations to aggregate intuitionistic fuzzy information. H. Zhao et al [30] provided aggregation for dual hesitant fuzzy information by using Einstein operations.

The structure of this paper is as follows: Section 2 consists of primary definitions which are utilized to form the paper. Section 3 has Einstein operations for CHFSs and the proof of its various results. Section 4 gives CHFEWA and CHFEWG operators. Section 5 describes the algortim for solving MCDM problem with an example of real life. Section 6 concludes the paper.

2. Notations and preliminaries

In this section some fundamental notions and results have been studied which are used to from the paper.

Definition 2.1 ([33])**.** Assume T be a non-empty set. A fuzzy set is defined by the function $f : T \to P$ where $P = [0, 1]$. Denote P^T as the collection of all fuzzy sets (FSs) in T.

Definition 2.2 ([34])**.** Assume T be a non-empty set. An interval valued fuzzy set (IVFS) is defined by the mapping s from T to the set of closed intervals in $[0, 1]$.

Definition 2.3 ([6])**.** Let T be a non-empty set. The cubic set on T is defined by

$$C = \{< t, s(t), f(t) > /t \in T\}.$$

Where $s(t)$ represents interval valued fuzzy set (IVFS) in T and $f(t)$ is a fuzzy set in T.

Definition 2.4 ([15,16])**.** Let T be a non-empty set. A hesitant fuzzy set (HFS) is a function that when applied on T returns a finite subset of $[0, 1]$ which is denoted and defined by

$$R = \{< t, g(t) > /t \in T\}.$$

Where $g(t)$ is a set of different values in $[0,1]$, which denotes the possible membership degrees of the element $t \in T$ to the set R.

Definition 2.5 ([3]). Assume T be a non-empty set and $C[0,1]$ denotes the set of all closed subintervals of $[0,1]$. An interval valued hesitant fuzzy set (IVHFS) on T is denoted and defined by

$$T = \{< t_r, u(t_r) > /t_r \in T, r = 1, 2, ..., n\}.$$

Where $u(t_r) : T \to C[0,1]$ denotes all possible interval valued membership degrees of the element $t_r \in T$ to the set T.

Definition 2.6 ([11,12]). Let T be a non-empty set. A cubic hesitant fuzzy set (for short, CHFS) is defined by

$$Z = \{< t, u(t), g(t) > /t \in T\}.$$

Where $u(t)$ is an interval valued hesitant fuzzy element (briefly IVHFE) and $g(t)$ is hesitant fuzzy element (briefly, HFE). A CHFS $Z = \{< t, u(t), g(t) > /t \in T\}$ is simply denoted by $Z =< u, g >$.

Definition 2.7 ([11,12]). A CHFS $< u, g >$ is called an internal cubic hesitant fuzzy set (for short, ICHFS) if $\forall i \in \wedge$ (where $\wedge$ is an index) and $\forall t \in T$ we have $\aleph_i^-(t) \leq \hbar_i \leq \aleph_i^+(t)$, where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ and $\hbar_i \in g(t)$.

Definition 2.8 ([11,12]). A CHFS $< u, g >$ for a non-empty set T is said to be an external cubic hesitant fuzzy set (briefly, ECHFS) $\forall i \in \wedge$ (where $\wedge$ is an index) and $\forall t \in T$ we have $\hbar_i \notin (\aleph_i^-(t), \aleph_i^+(t))$, where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ and $\hbar_i \in g(t)$.

Definition 2.9 ([11,12]). Let $Z = \{< t, u(t), g(t) > /t \in T\}$ be a CHFS. The cubic hesitant fuzzy element (briefly, CHFE) on a non-empty set T is defined by

$$ch = \{< \aleph_i = [\aleph_i^-, \aleph_i^+] \in u(t), \hbar_i \in g(t)/\{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}.$$

Where $u(t)$ represents IVHFE and $g(t)$ represents HFE.

Definition 2.10 ([11,12]). Let ch=$\{< \aleph_i$=$[\aleph_i^-, \aleph_i^+] \in u(t), \hbar_i \in g(t)/\{[\aleph_i^-, \aleph_i^+]\}$, $\{\hbar_i\} >\}$ be the CHFE on a non-empty set T, the score of ch is defined by

$$S(ch) = \frac{1}{\circledast(ch)} \left(\aleph_i^- + \aleph_i^+ - \frac{\circledast(ch)}{2} + \hbar_i \right).$$

Where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ (an IVHFE), $\hbar_i \in g(t)$ (HFE) $\forall t \in T, \circledast(ch)$ is the number of elements in ch.

Definition 2.11 ([28]). Let g, g_1, g_2 be the three HFEs, then the Einstein operational laws are defined as follows,

1. $g_1 \oplus g_2 = \frac{\hbar_1+\hbar_2}{1+\hbar_1\hbar_2}$
2. $g_1 \otimes g_2 = \frac{\hbar_1\hbar_2}{1+(1-\hbar_1)(1-\hbar_2)}$
3. $\sigma g = \frac{(1+\hbar)^\sigma-(1-\hbar)^\sigma}{(1+\hbar)^\sigma+(1-\hbar)^\sigma}$
4. $g^\sigma = \frac{2\hbar^\sigma}{(2-\hbar)^\sigma+\hbar^\sigma}$

3. Einstein operations for CHFSs

In this section, based on Einstein operational laws Einstein operations for CHFSs are proposed.

Definition 3.1. Let $ch = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}, ch_1 = \{< \{[\aleph_{i_1}^-, \aleph_{i_1}^+]\}, \{\hbar_{i_1}\} >\}, ch_2 = \{< \{[\aleph_{i_2}^-, \aleph_{i_2}^+]\}, \{\hbar_{i_2}\} >\}$ be CHFEs. Then we define the following Einstein operational laws as follows:

1. $ch_1 \oplus_e ch_2 = \{< \{[\frac{\aleph_{i_1}^- + \aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-}, \frac{\aleph_{i_1}^+ + \aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\}, \{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)}\} >\};$
2. $ch_1 \otimes_e ch_2 = \{< \{[\frac{\aleph_{i_1}^-\aleph_{i_2}^-}{1+\left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right)}, \frac{\aleph_{i_1}^+\aleph_{i_2}^+}{1+\left(1-\aleph_{i_1}^+\right)\left(1-\aleph_{i_2}^+\right)}]\}, \{\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\} >\};$
3. $\sigma ch = \{< \{[\frac{\left(1+\aleph_i^-\right)^\sigma-\left(1-\aleph_i^-\right)^\sigma}{\left(1+\aleph_i^-\right)^\sigma+\left(1-\aleph_i^-\right)^\sigma}, \frac{\left(1+\aleph_i^+\right)^\sigma-\left(1-\aleph_i^+\right)^\sigma}{\left(1+\aleph_i^+\right)^\sigma+\left(1-\aleph_i^+\right)^\sigma}]\}, \{\frac{2(\hbar_i)^\sigma}{(2-\hbar_i)^\sigma+(\hbar_i)^\sigma}\} >\};$
4. $ch^\sigma = \{< \{[\frac{2(\aleph_i^-)^\sigma}{(2-\aleph_i^-)^\sigma+(\aleph_i^-)^\sigma}, \frac{2(\aleph_i^+)^\sigma}{(2-\aleph_i^+)^\sigma+(\aleph_i^+)^\sigma}]\}, \{\frac{(1+\hbar_i)^\sigma-(1-\hbar_i)^\sigma}{(1+\hbar_i)^\sigma+(1-\hbar_i)^\sigma}\} >\}.$

Theorem 3.2. *Assume ch, ch_1, ch_2 be the three CHFEs and $\sigma \geq 0$. Then $ch_1 \oplus_e ch_2, ch_1 \otimes_e ch_2, \sigma ch, ch^\sigma$ are also CHFEs.*

Proof. We have $ch = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}, ch_1 = \{< \{[\aleph_{i_1}^-, \aleph_{i_1}^+]\}, \{\hbar_{i_1}\} >\}, ch_2 = \{< \{[\aleph_{i_2}^-, \aleph_{i_2}^+]\}, \{\hbar_{i_2}\} >\}$ are CHFEs.

$$
\begin{aligned}
&0 \leq \aleph_i^-, \aleph_i^+, \aleph_{i_1}^-, \aleph_{i_1}^+, \aleph_{i_2}^-, \aleph_{i_2}^+, \hbar_i, \hbar_{i_1}, \hbar_{i_2} \leq 1,\\
&0 \leq \left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right),\\
&0 \leq 1-\aleph_{i_1}^- - \aleph_{i_2}^- + \aleph_{i_1}^-\aleph_{i_2}^-,\\
&\aleph_{i_1}^- + \aleph_{i_2}^- \leq 1 + \aleph_{i_1}^-\aleph_{i_2}^-,\\
&\frac{\aleph_{i_1}^- + \aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-} \leq 1.
\end{aligned}
$$

Similarly $\frac{\aleph_{i_1}^+ + \aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+} \leq 1$, also $\frac{\aleph_{i_1}^- + \aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-} \geq 0$ and $\frac{\aleph_{i_1}^+ + \aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+} \geq 0$,

$$
\begin{aligned}
&0 \leq \hbar_{i_1} \leq 1, 0 \leq \hbar_{i_2} \leq 1,\\
&0 \leq \hbar_{i_1} + \hbar_{i_2} \leq 2,\\
&0 \leq 2 - \hbar_{i_1} - \hbar_{i_2},\\
&\hbar_{i_1}\hbar_{i_2} \leq 2 - \hbar_{i_1} - \hbar_{i_2} + \hbar_{i_1}\hbar_{i_2},
\end{aligned}
$$

$$\hbar_{i_1}\hbar_{i_2} \leq 1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right),$$
$$\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)} \leq 1,$$
$$\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)} \geq 0.$$

Thus $0 \leq \frac{\aleph_{i_1}^{+}+\aleph_{i_2}^{+}}{1+\aleph_{i_1}^{+}\aleph_{i_2}^{+}} \leq 1, 0 \leq \frac{\aleph_{i_1}^{-}+\aleph_{i_2}^{-}}{1+\aleph_{i_1}^{-}\aleph_{i_2}^{-}} \leq 1, 0 \leq \frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)} \leq 1.$

So $ch_1 \oplus_e ch_2$ is a CHFE.

$$0 \leq \aleph_{i_1}^{-} \leq 1, 0 \leq \aleph_{i_1}^{+} \leq 1, 0 \leq \aleph_{i_2}^{-} \leq 1, 0 \leq \aleph_{i_2}^{+} \leq 1,$$
$$0 \leq \aleph_{i_1}^{-}+\aleph_{i_2}^{-} \leq 2,$$
$$0 \leq 2-\aleph_{i_1}^{-}-\aleph_{i_2}^{-},$$
$$\aleph_{i_1}^{-}\aleph_{i_2}^{-} \leq 2-\aleph_{i_1}^{-}-\aleph_{i_2}^{-}+\aleph_{i_1}^{-}\aleph_{i_2}^{-},$$
$$\aleph_{i_1}^{-}\aleph_{i_2}^{-} \leq 1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right),$$
$$\frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right)} \leq 1,$$
$$\frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right)} \geq 0.$$

Similarly

$$0 \leq \frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+\left(1-\aleph_{i_1}^{+}\right)\left(1-\aleph_{i_2}^{+}\right)} \leq 1,$$
$$0 \leq \left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right),$$
$$0 \leq 1-\hbar_{i_1}-\hbar_{i_2}+\hbar_{i_1}\hbar_{i_2},$$
$$\hbar_{i_1}+\hbar_{i_2} \leq 1+\hbar_{i_1}\hbar_{i_2},$$
$$\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \leq 1,$$
$$\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \geq 0,$$

Thus $0 \leq \frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right)} \leq 1, 0 \leq \frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+\left(1-\aleph_{i_1}^{+}\right)\left(1-\aleph_{i_2}^{+}\right)} \leq 1$, and $0 \leq \frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \leq 1$.

So $ch_1 \otimes_e ch_2$ is a CHFE.

$$\left(1+\aleph_i^{-}\right)^{\sigma}-\left(1-\aleph_i^{-}\right)^{\sigma} \leq \left(1+\aleph_i^{-}\right)^{\sigma}+\left(1-\aleph_i^{-}\right)^{\sigma},$$
$$\frac{\left(1+\aleph_i^{-}\right)^{\sigma}-\left(1-\aleph_i^{-}\right)^{\sigma}}{\left(1+\aleph_i^{-}\right)^{\sigma}+\left(1-\aleph_i^{-}\right)^{\sigma}} \leq 1.$$

Similarly

$$\frac{\left(1+\aleph_i^{+}\right)^{\sigma}-\left(1-\aleph_i^{+}\right)^{\sigma}}{\left(1+\aleph_i^{+}\right)^{\sigma}+\left(1-\aleph_i^{+}\right)^{\sigma}} \leq 1.$$

Also $\frac{\left(1+\aleph_i^-\right)^\sigma-\left(1-\aleph_i^-\right)^\sigma}{\left(1+\aleph_i^-\right)^\sigma+\left(1-\aleph_i^-\right)^\sigma} \geq 0, \frac{\left(1+\aleph_i^+\right)^\sigma-\left(1-\aleph_i^+\right)^\sigma}{\left(1+\aleph_i^+\right)^\sigma+\left(1-\aleph_i^+\right)^\sigma} \geq 0,$

$$
\begin{aligned}
&0 \leq \hbar_i \leq 1,\\
&2\left(\hbar_i\right)^\sigma \leq \left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma,\\
&\frac{2\left(\hbar_i\right)^\sigma}{\left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma} \leq 1,\\
&\frac{2\left(\hbar_i\right)^\sigma}{\left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma} \geq 0.
\end{aligned}
$$

Thus $0 \leq \frac{(1+\aleph_i^-)^\sigma-(1-\aleph_i^-)^\sigma}{(1+\aleph_i^-)^\sigma+(1-\aleph_i^-)^\sigma} \leq 1, 0 \leq \frac{(1+\aleph_i^+)^\sigma-(1-\aleph_i^+)^\sigma}{(1+\aleph_i^+)^\sigma+(1-\aleph_i^+)^\sigma} \leq 1$ and $0 \leq \frac{2(\hbar_i)^\sigma}{(2-\hbar_i)^\sigma+(\hbar_i)^\sigma} \leq 1$.

Hence σch is a CHFE. In the same way we can show that ch^σ is a CHFE.

Proposition 3.3. *Suppose ch, ch_1, ch_2 be the three CHFEs and $\sigma, \sigma_1, \sigma_2 > 0$ then,*

1. $ch_1 \oplus_e ch_2 = ch_2 \oplus_e ch_1$
2. $ch_1 \otimes_e ch_2 = ch_2 \otimes_e ch_1$
3. $(\sigma_1 \oplus_e \sigma_2)ch = \sigma_1 ch \oplus_e \sigma_2 ch$
4. $\sigma(ch_1 \oplus_e ch_2) = \sigma ch_1 \oplus_e \sigma ch_2$
5. $(ch_1 \otimes_e ch_2)^\sigma = ch_1^\sigma \otimes_e ch_2^\sigma$

Proof.

1. $ch_1 \oplus_e ch_2 = ch_2 \oplus_e ch_1$

$ch_1 \oplus_e ch_2 = \{< \{[\frac{\aleph_{i_1}^- + \aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-}, \frac{\aleph_{i_1}^+ + \aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\}, \{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)}\} >\},$

$= \{< \{[\frac{\aleph_{i_2}^- + \aleph_{i_1}^-}{1+\aleph_{i_2}^-\aleph_{i_1}^-}, \frac{\aleph_{i_2}^+ + \aleph_{i_1}^+}{1+\aleph_{i_2}^+\aleph_{i_1}^+}]\}, \{\frac{\hbar_{i_2}\hbar_{i_1}}{1+\left(1-\hbar_{i_2}\right)\left(1-\hbar_{i_1}\right)}\} >\} = ch_2 \oplus_e ch_1.$

2. $ch_1 \otimes_e ch_2 = ch_2 \otimes_e ch_1$

$ch_1 \otimes_e ch_2 = \{< \{[\frac{\aleph_{i_1}^-\aleph_{i_2}^-}{1+\left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right)}, \frac{\aleph_{i_1}^+\aleph_{i_2}^+}{1+\left(1-\aleph_{i_1}^+\right)\left(1-\aleph_{i_2}^+\right)}]\}, \{\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\} >\},$

$= \{< \{[\frac{\aleph_{i_2}^-\aleph_{i_1}^-}{1+\left(1-\aleph_{i_2}^-\right)\left(1-\aleph_{i_1}^-\right)}, \frac{\aleph_{i_2}^+\aleph_{i_1}^+}{1+\left(1-\aleph_{i_2}^+\right)\left(1-\aleph_{i_1}^+\right)}]\}, \{\frac{\hbar_{i_2}+\hbar_{i_1}}{1+\hbar_{i_2}\hbar_{i_1}}\} >\} = ch_2 \otimes_e ch_1.$

3. $(\sigma_1 \oplus_e \sigma_2)ch = \sigma_1 ch \oplus_e \sigma_2 ch$

$(\sigma_1 \oplus_e \sigma_2)ch = \{< \{[\frac{(1+\aleph_i^-)^{\sigma_1+\sigma_2}-(1-\aleph_i^-)^{\sigma_1+\sigma_2}}{(1+\aleph_i^-)^{\sigma_1+\sigma_2}+(1-\aleph_i^-)^{\sigma_1+\sigma_2}}, \frac{(1+\aleph_i^+)^{\sigma_1+\sigma_2}-(1-\aleph_i^+)^{\sigma_1+\sigma_2}}{(1+\aleph_i^+)^{\sigma_1+\sigma_2}+(1-\aleph_i^+)^{\sigma_1+\sigma_2}}]\},$

$\{\frac{2(\hbar_i)^{\sigma_1+\sigma_2}}{(2-\hbar_i)^{\sigma_1+\sigma_2}+(\hbar_i)^{\sigma_1+\sigma_2}}\} >\},$

$\sigma_1 ch \oplus_e \sigma_2 ch = \{< \{[\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}}, \frac{(1+\aleph_i^+)^{\sigma_1}-(1-\aleph_i^+)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}]\}, \{\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}}\} >$

$\} \oplus_e \{< \{[\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}}, \frac{(1+\aleph_i^+)^{\sigma_2}-(1-\aleph_i^+)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}]\}, \{\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}}\} >\},$

$= \{< \{[\frac{\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}}+\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}}}{1+\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}}\times\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}}},$

$$\frac{\frac{(1+\aleph_i^+)^{\sigma_1}-(1-\aleph_i^+)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}+\frac{(1+\aleph_i^+)^{\sigma_2}-(1-\aleph_i^+)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}]\},$$
$$1+\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}\times\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}$$

$$\{\frac{\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}}\times\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}}}{1+(1-\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}})(1-\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}})}\}>\},$$

$$=\{<\{[\frac{((C^-)^{\sigma_1}-(D^-)^{\sigma_1})((C^-)^{\sigma_2}+(D^-)^{\sigma_2})+((C^-)^{\sigma_1}+(D^-)^{\sigma_1})((C^-)^{\sigma_2}-(D^-)^{\sigma_2})}{((C^-)^{\sigma_1}+(D^-)^{\sigma_1})((C^-)^{\sigma_2}+(D^-)^{\sigma_2})+((C^-)^{\sigma_1}-(D^-)^{\sigma_1})((C^-)^{\sigma_2}-(D^-)^{\sigma_2})},$$
$$\frac{((C^+)^{\sigma_1}-(D^+)^{\sigma_1})((C^+)^{\sigma_2}+(D^+)^{\sigma_2})+((C^+)^{\sigma_1}+(D^+)^{\sigma_1})((C^+)^{\sigma_2}-(D^+)^{\sigma_2})}{((C^+)^{\sigma_1}+(D^+)^{\sigma_1})((C^+)^{\sigma_2}+(D^+)^{\sigma_2})+((C^+)^{\sigma_1}-(D^+)^{\sigma_1})((C^+)^{\sigma_2}-(D^+)^{\sigma_2})}]\},$$
$$\{\frac{2(E)^{\sigma_1}\times 2(E)^{\sigma_2}}{((2-E)^{\sigma_1}+(E)^{\sigma_1})((2-E)^{\sigma_2}+(E)^{\sigma_2})+((2-E)^{\sigma_1}-(E)^{\sigma_1})((2-E)^{\sigma_2}-(E)^{\sigma_2})}\}>\}.$$

Where $C^-=1+\aleph_i^-, C^+=1+\aleph_i^+, D^-=1-\aleph_i^-, D^+=1-\aleph_i^+$ and $E=\hbar_i$,

$$=\{<\{[\frac{((C^-)^{\sigma_1+\sigma_2}-(D^-)^{\sigma_1+\sigma_2})}{((C^-)^{\sigma_1+\sigma_2}+(D^-)^{\sigma_1+\sigma_2})},\frac{((C^+)^{\sigma_1+\sigma_2}-(D^+)^{\sigma_1+\sigma_2})}{((C^+)^{\sigma_1+\sigma_2}+(D^+)^{\sigma_1+\sigma_2})}]\},$$
$$\{\frac{2(E)^{\sigma_1+\sigma_2}}{((2-E)^{\sigma_1+\sigma_2}+(E)^{\sigma_1+\sigma_2})}\}>\},$$
$$=\{<\{[\frac{(1+\aleph_i^-)^{\sigma_1+\sigma_2}-(1-\aleph_i^-)^{\sigma_1+\sigma_2}}{(1+\aleph_i^-)^{\sigma_1+\sigma_2}+(1-\aleph_i^-)^{\sigma_1+\sigma_2}},\frac{(1+\aleph_i^+)^{\sigma_1+\sigma_2}-(1-\aleph_i^+)^{\sigma_1+\sigma_2}}{(1+\aleph_i^+)^{\sigma_1+\sigma_2}+(1-\aleph_i^+)^{\sigma_1+\sigma_2}}]\},$$
$$\{\frac{2(\hbar_i)^{\sigma_1+\sigma_2}}{((2-\hbar_i)^{\sigma_1+\sigma_2}+(\hbar_i)^{\sigma_1+\sigma_2})}\}>\},$$

Hence $(\sigma_1\oplus_e\sigma_2)ch=\sigma_1 ch\oplus_e\sigma_2 ch$.

4. $\sigma(ch_1\oplus_e ch_2)=\sigma ch_1\oplus_e\sigma ch_2$

$$(ch_1\oplus_e ch_2)=\{<\{[\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-},\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\},\{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)}\}>\},$$

$$\sigma(ch_1\oplus_e ch_2)=\{<\{[\frac{(1+\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma-(1-\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma}{(1+\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma+(1-\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma},\frac{(1+\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma-(1-\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma}{(1+\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma+(1-\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma}]\},$$
$$\{\frac{2(\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)})^\sigma}{(2-\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)})^\sigma+(\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)})^\sigma}\}>\},$$
$$=\{<\{[\frac{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma-(1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma+(1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma},\frac{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma-(1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma+(1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}]\},$$
$$\{\frac{2\hbar_{i_1}^\sigma\hbar_{i_2}^\sigma}{(2-\hbar_{i_1})^\sigma(2-\hbar_{i_2})^\sigma+\hbar_{i_1}^\sigma\hbar_{i_2}^\sigma}\}>\},$$
$$\sigma ch_1=\{<\{[\frac{(1+\aleph_{i_1}^-)^\sigma-(1-\aleph_{i_1}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma+(1-\aleph_{i_1}^-)^\sigma},\frac{(1+\aleph_{i_1}^+)^\sigma-(1-\aleph_{i_1}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma+(1-\aleph_{i_1}^+)^\sigma}]\},\{\frac{2(\hbar_{i_1})^\sigma}{(2-\hbar_{i_1})^\sigma+(\hbar_{i_1})^\sigma}\}>\},$$
$$\sigma ch_2=\{<\{[\frac{(1+\aleph_{i_2}^-)^\sigma-(1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_2}^-)^\sigma+(1-\aleph_{i_2}^-)^\sigma},\frac{(1+\aleph_{i_2}^+)^\sigma-(1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_2}^+)^\sigma+(1-\aleph_{i_2}^+)^\sigma}]\},\{\frac{2(\hbar_{i_2})^\sigma}{(2-\hbar_{i_2})^\sigma+(\hbar_{i_2})^\sigma}\}>\},$$
$$\sigma ch_1=\{<\{[\frac{K_1^--L_1^-}{K_1^-+L_1^-},\frac{K_1^+-L_1^+}{K_1^++L_1^+}]\},\{\frac{2M_1}{N_1+M_1}\}>\},$$
$$\sigma ch_2=\{<\{[\frac{K_2^--L_2^-}{K_2^-+L_2^-},\frac{K_2^+-L_2^+}{K_2^++L_2^+}]\},\{\frac{2M_2}{N_2+M_2}\}>\}.$$

Where $K_1^-=(1+\aleph_{i_1}^-)^\sigma, K_1^+=(1+\aleph_{i_1}^+)^\sigma, L_1^-=(1-\aleph_{i_1}^-)^\sigma, L_1^+=(1-\aleph_{i_1}^+)^\sigma, M_1=(\hbar_{i_1})^\sigma, N_1=(2-\hbar_{i_1})^\sigma, K_2^-=(1+\aleph_{i_2}^-)^\sigma, K_2^+=(1+\aleph_{i_2}^+)^\sigma, L_2^-=(1-\aleph_{i_2}^-)^\sigma, L_2^+=(1-\aleph_{i_2}^+)^\sigma, M_2=(\hbar_{i_2})^\sigma, N_2=(2-\hbar_{i_2})^\sigma$,

$$\sigma ch_1 \oplus_e \sigma ch_2 = \{< \{[\frac{\frac{K_1^- - L_1^-}{K_1^- + L_1^-} + \frac{K_2^- - L_2^-}{K_2^- + L_2^-}}{1+(\frac{K_1^- - L_1^-}{K_1^- + L_1^-})(\frac{K_2^- - L_2^-}{K_2^- + L_2^-})}, \frac{\frac{K_1^+ - L_1^+}{K_1^+ + L_1^+} + \frac{K_2^+ - L_2^+}{K_2^+ + L_2^+}}{1+(\frac{K_1^+ - L_1^+}{K_1^+ + L_1^+})(\frac{K_2^+ - L_2^+}{K_2^+ + L_2^+})}]\},$$

$$\{\frac{\frac{2M_1}{N_1+M_1} \times \frac{2M_2}{N_2+M_2}}{1+(1-\frac{2M_1}{N_1+M_1})(1-\frac{2M_2}{N_2+M_2})}\} >\},$$

$$= \{< \{[\frac{K_1^- K_2^- - L_1^- L_2^-}{K_1^- K_2^- + L_1^- L_2^-}, \frac{K_1^+ K_2^+ - L_1^+ L_2^+}{K_1^+ K_2^+ + L_1^+ L_2^+}]\}, \{\frac{2M_1 M_2}{N_1 N_2 + M_1 M_2}\} >\},$$

$$= \{< \{[\frac{(1+\aleph_{i_1}^-)^\sigma (1+\aleph_{i_2}^-)^\sigma - (1-\aleph_{i_1}^-)^\sigma (1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma (1+\aleph_{i_2}^-)^\sigma + (1-\aleph_{i_1}^-)^\sigma (1-\aleph_{i_2}^-)^\sigma}, \frac{(1+\aleph_{i_1}^+)^\sigma (1+\aleph_{i_2}^+)^\sigma - (1-\aleph_{i_1}^+)^\sigma (1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma (1+\aleph_{i_2}^+)^\sigma + (1-\aleph_{i_1}^+)^\sigma (1-\aleph_{i_2}^+)^\sigma}]\},$$

$$\{\frac{2\hbar_{i_1}^\sigma \hbar_{i_2}^\sigma}{(2-\hbar_{i_1})^\sigma (2-\hbar_{i_2})^\sigma + \hbar_{i_1}^\sigma \hbar_{i_2}^\sigma}\} >\},$$

Thus $\sigma(ch_1 \oplus_e ch_2) = \sigma ch_1 \oplus_e \sigma ch_2$.

5.$(ch_1 \otimes_e ch_2)^\sigma = ch_1^\sigma \otimes_e ch_2^\sigma$

$$ch_1^\sigma = \{< \{[\frac{2(\aleph_{i_1}^-)^\sigma}{(2-\aleph_{i_1}^-)^\sigma + (\aleph_{i_1}^-)^\sigma}, \frac{2(\aleph_{i_1}^+)^\sigma}{(2-\aleph_{i_1}^+)^\sigma + (\aleph_{i_1}^+)^\sigma}]\}, \{\frac{(1+\hbar_{i_1})^\sigma - (1-\hbar_{i_1})^\sigma}{(1+\hbar_{i_1})^\sigma + (1-\hbar_{i_1})^\sigma}\} >\},$$

$$ch_2^\sigma = \{< \{[\frac{2(\aleph_{i_2}^-)^\sigma}{(2-\aleph_{i_2}^-)^\sigma + (\aleph_{i_2}^-)^\sigma}, \frac{2(\aleph_{i_2}^+)^\sigma}{(2-\aleph_{i_2}^+)^\sigma + (\aleph_{i_2}^+)^\sigma}]\}, \{\frac{(1+\hbar_{i_2})^\sigma - (1-\hbar_{i_2})^\sigma}{(1+\hbar_{i_2})^\sigma + (1-\hbar_{i_2})^\sigma}\} >\},$$

$$ch_1^\sigma = \{< \{[\frac{2P_1^-}{Q_1^- + P_1^-}, \frac{2P_1^+}{Q_1^+ + P_1^+}]\}, \{\frac{R_1 - S_1}{R_1 + S_1}\} >\},$$

$$ch_2^\sigma = \{< \{[\frac{2P_2^-}{Q_2^- + P_2^-}, \frac{2P_2^+}{Q_2^+ + P_2^+}]\}, \{\frac{R_2 - S_2}{R_2 + S_2}\} >\}.$$

Where $P_1^- = (\aleph_{i_1}^-)^\sigma, P_1^+ = (\aleph_{i_1}^+)^\sigma, Q_1^- = (2-\aleph_{i_1}^-)^\sigma, Q_1^+ = (2-\aleph_{i_1}^+)^\sigma, R_1 = (1+\hbar_{i_1})^\sigma, S_1 = (1-\hbar_{i_1})^\sigma, P_2^- = (\aleph_{i_2}^-)^\sigma, P_2^+ = (\aleph_{i_2}^+)^\sigma, Q_2^- = (2-\aleph_{i_2}^-)^\sigma, Q_2^+ = (2-\aleph_{i_2}^+)^\sigma, R_2 = (1+\hbar_{i_2})^\sigma, S_2 = (1-\hbar_{i_2})^\sigma,$

$$ch_1^\sigma \otimes_e ch_2^\sigma = \{< \{[\frac{\frac{2P_1^-}{Q_1^- + P_1^-} \times \frac{2P_2^-}{Q_2^- + P_2^-}}{1+(1-\frac{2P_1^-}{Q_1^- + P_1^-})(1-\frac{2P_2^-}{Q_2^- + P_2^-})}, \frac{\frac{2P_1^+}{Q_1^+ + P_1^+} \times \frac{2P_2^+}{Q_2^+ + P_2^+}}{1+(1-\frac{2P_1^+}{Q_1^+ + P_1^+})(1-\frac{2P_2^+}{Q_2^+ + P_2^+})}]\},$$

$$\{\frac{\frac{R_1 - S_1}{R_1 + S_1} + \frac{R_2 - S_2}{R_2 + S_2}}{1+(\frac{R_1 - S_1}{R_1 + S_1})(\frac{R_2 - S_2}{R_2 + S_2})}\} >\},$$

$$= \{< \{[\frac{4P_1^- P_2^-}{(Q_1^- + P_1^-)(Q_2^- + P_2^-) + (Q_1^- - P_1^-)(Q_2^- - P_2^-)}, \frac{4P_1^+ P_2^+}{(Q_1^+ + P_1^+)(Q_2^+ + P_2^+) + (Q_1^+ - P_1^+)(Q_2^+ - P_2^+)}]\},$$

$$\{\frac{R_1 R_2 - S_1 S_2}{R_1 R_2 + S_1 S_2}\} >\},$$

$$= \{< \{[\frac{2P_1^- P_2^-}{Q_1^- Q_2^- + P_1^- P_2^-}, \frac{2P_1^+ P_2^+}{Q_1^+ Q_2^+ + P_1^+ P_2^+}]\}, \{\frac{R_1 R_2 - S_1 S_2}{R_1 R_2 + S_1 S_2}\} >\},$$

$$= \{< \{[\frac{2(\aleph_{i_1}^-)^\sigma (\aleph_{i_2}^-)^\sigma}{(2-\aleph_{i_1}^-)^\sigma (2-\aleph_{i_2}^-)^\sigma + (\aleph_{i_1}^-)^\sigma (\aleph_{i_2}^-)^\sigma}, \frac{2(\aleph_{i_1}^+)^\sigma (\aleph_{i_2}^+)^\sigma}{(2-\aleph_{i_1}^+)^\sigma (2-\aleph_{i_2}^+)^\sigma + (\aleph_{i_1}^+)^\sigma (\aleph_{i_2}^+)^\sigma}]\},$$

$$\{\frac{(1+\hbar_{i_1})^\sigma (1+\hbar_{i_2})^\sigma - (1-\hbar_{i_1})^\sigma (1-\hbar_{i_2})^\sigma}{(1+\hbar_{i_1})^\sigma (1+\hbar_{i_2})^\sigma + (1-\hbar_{i_1})^\sigma (1-\hbar_{i_2})^\sigma}\} >\}.$$

Now $ch_1 \otimes_e ch_2 = \{< \{[\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)}, \frac{\aleph_{i_1}^+ \aleph_{i_2}^+}{1+(1-\aleph_{i_1}^+)(1-\aleph_{i_2}^+)}]\}, \{\frac{\hbar_{i_1} + \hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\} >\},$

$$(ch_1 \otimes_e ch_2)^\sigma = \{< \{[\frac{2(\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)})^\sigma}{(2-(\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)}))^\sigma + (\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)})^\sigma},$$

$$\frac{2(\frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+(1-\aleph_{i_1}^{+})(1-\aleph_{i_2}^{+})})^{\sigma}}{(2-(\frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+(1-\aleph_{i_1}^{+})(1-\aleph_{i_2}^{+})}))^{\sigma}+(\frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+(1-\aleph_{i_1}^{+})(1-\aleph_{i_2}^{+})})^{\sigma}}]\},\ \{\frac{(1+\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^{\sigma}-(1-\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^{\sigma}}{(1+\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^{\sigma}+(1-\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^{\sigma}}\}>\},$$

$$=\{<\{[\frac{2(\aleph_{i_1}^{-})^{\sigma}(\aleph_{i_2}^{-})^{\sigma}}{(2-\aleph_{i_1}^{-})^{\sigma}(2-\aleph_{i_2}^{-})^{\sigma}+(\aleph_{i_1}^{-})^{\sigma}(\aleph_{i_2}^{-})^{\sigma}},\frac{2(\aleph_{i_1}^{+})^{\sigma}(\aleph_{i_2}^{+})^{\sigma}}{(2-\aleph_{i_1}^{+})^{\sigma}(2-\aleph_{i_2}^{+})^{\sigma}+(\aleph_{i_1}^{+})^{\sigma}(\aleph_{i_2}^{+})^{\sigma}}]\},$$

$$\{\frac{(1+\hbar_{i_1})^{\sigma}(1+\hbar_{i_2})^{\sigma}-(1-\hbar_{i_1})^{\sigma}(1-\hbar_{i_2})^{\sigma}}{(1+\hbar_{i_1})^{\sigma}(1+\hbar_{i_2})^{\sigma}+(1-\hbar_{i_1})^{\sigma}(1-\hbar_{i_2})^{\sigma}}\}>\},$$

So $(ch_1 \otimes_e ch_2)^{\sigma} = ch_1^{\sigma} \otimes_e ch_2^{\sigma}$.

4. Cubic hesitant fuzzy Einstein weighted averaging and geometric operators

In this section Einstein weighted averaging and weighted geometric operators for CHFSs are defined which helps to aggregate the given cubic hesitant fuzzy information.

Definition 4.1. Let $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. The cubic hesitant fuzzy Einstein weighted averaging (briefly, CHFEWA) operator is defined by the mapping $G^n \to G$ such that

$$CHFEWA(ch_1, ch_2, ..., ch_n) = (\oplus_e)_{s=1}^{n}(w_s ch_s).$$

Here $w = (w_1, w_2, ..., w_n)^T$ be a weighting vector of CHFEs where $w_s \in [0, 1]$ and $\sum_{s=1}^{n} w_s = 1$. As a special case when $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ then CHFEWA operator becomes CHFEA operator as

$$CHFEA(ch_1, ch_2, ..., ch_n) = (\oplus_e)_{s=1}^{n}(\frac{1}{n} ch_s).$$

Theorem 4.2. *Suppose $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. Their aggregated result obtained by using CHFEWA operator is a CHFE,*

$$CHFEWA(ch_1, ch_2, ..., ch_n) = (\oplus_e)_{s=1}^{n}(w_s ch_s)$$

$$=\{<\{[\frac{\Pi_{s=1}^{n}(1+\aleph_{i_s}^{-})^{w_s}-\Pi_{s=1}^{n}(1-\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{n}(1+\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{n}(1-\aleph_{i_s}^{-})^{w_s}},\frac{\Pi_{s=1}^{n}(1+\aleph_{i_s}^{+})^{w_s}-\Pi_{s=1}^{n}(1-\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{n}(1+\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{n}(1-\aleph_{i_s}^{+})^{w_s}}]\},$$

$$\{\frac{2\Pi_{s=1}^{n}(\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{n}(2-\hbar_{i_s})^{w_s}+\Pi_{s=1}^{n}(\hbar_{i_s})^{w_s}}\}>\}.$$

Proof. We show the result by using mathematical induction.

When $n = 2$ and by applying operational law 3 of Definition 3.1 it results,

$$w_1 ch_1 = \{<\{[\frac{(1+\aleph_{i_1}^{-})^{w_1}-(1-\aleph_{i_1}^{-})^{w_1}}{(1+\aleph_{i_1}^{-})^{w_1}+(1-\aleph_{i_1}^{-})^{w_1}},\frac{(1+\aleph_{i_1}^{+})^{w_1}-(1-\aleph_{i_1}^{+})^{w_1}}{(1+\aleph_{i_1}^{+})^{w_1}+(1-\aleph_{i_1}^{+})^{w_1}}]\},$$

$$\{\frac{2(\hbar_{i_1})^{w_1}}{(2-\hbar_{i_1})^{w_1}+(\hbar_{i_1})^{w_1}}\}>\},$$

$$w_2 ch_2 = \{<\{[\frac{(1+\aleph_{i_2}^{-})^{w_2}-(1-\aleph_{i_2}^{-})^{w_2}}{(1+\aleph_{i_2}^{-})^{w_2}+(1-\aleph_{i_2}^{-})^{w_2}},\frac{(1+\aleph_{i_2}^{+})^{w_2}-(1-\aleph_{i_2}^{+})^{w_2}}{(1+\aleph_{i_2}^{+})^{w_2}+(1-\aleph_{i_2}^{+})^{w_2}}]\},$$

$$\{\frac{2(\hbar_{i_2})^{w_2}}{(2-\hbar_{i_2})^{w_2}+(\hbar_{i_2})^{w_2}}\}>\}.$$

By applying operational law 1 of Definition 3.1 we have,

$$w_1ch_1 \oplus_e w_2ch_2 = \{< \{[\frac{\frac{(1+\aleph_{i_1}^-)^{w_1}-(1-\aleph_{i_1}^-)^{w_1}}{(1+\aleph_{i_1}^-)^{w_1}+(1-\aleph_{i_1}^-)^{w_1}}+\frac{(1+\aleph_{i_2}^-)^{w_2}-(1-\aleph_{i_2}^-)^{w_2}}{(1+\aleph_{i_2}^-)^{w_2}+(1-\aleph_{i_2}^-)^{w_2}}}{1+(\frac{(1+\aleph_{i_1}^-)^{w_1}-(1-\aleph_{i_1}^-)^{w_1}}{(1+\aleph_{i_1}^-)^{w_1}+(1-\aleph_{i_1}^-)^{w_1}})(\frac{(1+\aleph_{i_2}^-)^{w_2}-(1-\aleph_{i_2}^-)^{w_2}}{(1+\aleph_{i_2}^-)^{w_2}+(1-\aleph_{i_2}^-)^{w_2}})},$$
$$\frac{\frac{(1+\aleph_{i_1}^+)^{w_1}-(1-\aleph_{i_1}^+)^{w_1}}{(1+\aleph_{i_1}^+)^{w_1}+(1-\aleph_{i_1}^+)^{w_1}}+\frac{(1+\aleph_{i_2}^+)^{w_2}-(1-\aleph_{i_2}^+)^{w_2}}{(1+\aleph_{i_2}^+)^{w_2}+(1-\aleph_{i_2}^+)^{w_2}}}{1+(\frac{(1+\aleph_{i_1}^+)^{w_1}-(1-\aleph_{i_1}^+)^{w_1}}{(1+\aleph_{i_1}^+)^{w_1}+(1-\aleph_{i_1}^+)^{w_1}})(\frac{(1+\aleph_{i_2}^+)^{w_2}-(1-\aleph_{i_2}^+)^{w_2}}{(1+\aleph_{i_2}^+)^{w_2}+(1-\aleph_{i_2}^+)^{w_2}})}]\},$$
$$\{\frac{(\frac{2(\hbar_{i_1})^{w_1}}{(2-\hbar_{i_1})^{w_1}+(\hbar_{i_1})^{w_1}})(\frac{2(\hbar_{i_2})^{w_2}}{(2-\hbar_{i_2})^{w_2}+(\hbar_{i_2})^{w_2}})}{1+(1-\frac{2(\hbar_{i_1})^{w_1}}{(2-\hbar_{i_1})^{w_1}+(\hbar_{i_1})^{w_1}})(1-\frac{2(\hbar_{i_2})^{w_2}}{(2-\hbar_{i_2})^{w_2}+(\hbar_{i_2})^{w_2}})}\} >\},$$
$$= \{< \{[\frac{\Pi_{s=1}^2(1+\aleph_{i_s}^-)^{w_s}-\Pi_{s=1}^2(1-\aleph_{i_s}^-)^{w_s}}{\Pi_{s=1}^2(1+\aleph_{i_s}^-)^{w_s}+\Pi_{s=1}^2(1-\aleph_{i_s}^-)^{w_s}}, \frac{\Pi_{s=1}^2(1+\aleph_{i_s}^+)^{w_s}-\Pi_{s=1}^2(1-\aleph_{i_s}^+)^{w_s}}{\Pi_{s=1}^2(1+\aleph_{i_s}^+)^{w_s}+\Pi_{s=1}^2(1-\aleph_{i_s}^+)^{w_s}}]\},$$
$$\{\frac{2\Pi_{s=1}^2(\hbar_{i_s})^{w_s}}{\Pi_{s=1}^2(2-\hbar_{i_s})^{w_s}+\Pi_{s=1}^2(\hbar_{i_s})^{w_s}}\} >\}.$$

Suppose the result is true for $n = t$,

$$CHFEWA(ch_1, ch_2, ..., ch_t) = (\oplus_e)_{s=1}^t(w_sch_s)$$
$$== \{< \{[\frac{\Pi_{s=1}^t(1+\aleph_{i_s}^-)^{w_s}-\Pi_{s=1}^t(1-\aleph_{i_s}^-)^{w_s}}{\Pi_{s=1}^t(1+\aleph_{i_s}^-)^{w_s}+\Pi_{s=1}^t(1-\aleph_{i_s}^-)^{w_s}}, \frac{\Pi_{s=1}^t(1+\aleph_{i_s}^+)^{w_s}-\Pi_{s=1}^t(1-\aleph_{i_s}^+)^{w_s}}{\Pi_{s=1}^t(1+\aleph_{i_s}^+)^{w_s}+\Pi_{s=1}^t(1-\aleph_{i_s}^+)^{w_s}}]\},$$
$$\{\frac{2\Pi_{s=1}^t(\hbar_{i_s})^{w_s}}{\Pi_{s=1}^t(2-\hbar_{i_s})^{w_s}+\Pi_{s=1}^t(\hbar_{i_s})^{w_s}}\} >\}.$$

Now we have to show that the result holds for $n = t + 1$,

$$CHFEWA(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\oplus_e)_{s=1}^{t+1}(w_sch_s) = (\oplus_e)_{s=1}^t(w_sch_s) \oplus_e (w_{t+1}ch_{t+1})$$
$$= \{< \{[\frac{\Pi_{s=1}^t(1+\aleph_{i_s}^-)^{w_s}-\Pi_{s=1}^t(1-\aleph_{i_s}^-)^{w_s}}{\Pi_{s=1}^t(1+\aleph_{i_s}^-)^{w_s}+\Pi_{s=1}^t(1-\aleph_{i_s}^-)^{w_s}}, \frac{\Pi_{s=1}^t(1+\aleph_{i_s}^+)^{w_s}-\Pi_{s=1}^t(1-\aleph_{i_s}^+)^{w_s}}{\Pi_{s=1}^t(1+\aleph_{i_s}^+)^{w_s}+\Pi_{s=1}^t(1-\aleph_{i_s}^+)^{w_s}}]\},$$
$$\{\frac{2\Pi_{s=1}^t(\hbar_{i_s})^{w_s}}{\Pi_{s=1}^t(2-\hbar_{i_s})^{w_s}+\Pi_{s=1}^t(\hbar_{i_s})^{w_s}}\} >\} \oplus_e \{< \{[\frac{(1+\aleph_{i_{t+1}}^-)^{w_{t+1}}-(1-\aleph_{i_{t+1}}^-)^{w_{t+1}}}{(1+\aleph_{i_{t+1}}^-)^{w_{t+1}}+(1-\aleph_{i_{t+1}}^-)^{w_{t+1}}},$$
$$\frac{(1+\aleph_{i_{t+1}}^+)^{w_{t+1}}-(1-\aleph_{i_{t+1}}^+)^{w_{t+1}}}{(1+\aleph_{i_{t+1}}^+)^{w_{t+1}}+(1-\aleph_{i_{t+1}}^+)^{w_{t+1}}}]\}, \{\frac{2(\hbar_{i_{t+1}})^{w_{t+1}}}{(2-\hbar_{i_{t+1}})^{w_{t+1}}+(\hbar_{i_{t+1}})^{w_{t+1}}}\} >\}.$$

Now again applying operational law 1 of Definition 3.1 we obtain,

$$CHFEWA(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\oplus_e)_{s=1}^{t+1}(w_sch_s)$$
$$= \{< \{[\frac{\Pi_{s=1}^{t+1}(1+\aleph_{i_s}^-)^{w_s}-\Pi_{s=1}^{t+1}(1-\aleph_{i_s}^-)^{w_s}}{\Pi_{s=1}^{t+1}(1+\aleph_{i_s}^-)^{w_s}+\Pi_{s=1}^{t+1}(1-\aleph_{i_s}^-)^{w_s}}, \frac{\Pi_{s=1}^{t+1}(1+\aleph_{i_s}^+)^{w_s}-\Pi_{s=1}^{t+1}(1-\aleph_{i_s}^+)^{w_s}}{\Pi_{s=1}^{t+1}(1+\aleph_{i_s}^+)^{w_s}+\Pi_{s=1}^{t+1}(1-\aleph_{i_s}^+)^{w_s}}]\},$$
$$\{\frac{2\Pi_{s=1}^{t+1}(\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{t+1}(2-\hbar_{i_s})^{w_s}+\Pi_{s=1}^{t+1}(\hbar_{i_s})^{w_s}}\} >\}.$$

Hence the result holds for all n.

Definition 4.3. Let $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. The cubic hesitant fuzzy Einstein weighted geometric (briefly, CHFEWG) operator is defined by the mapping $G^n \to G$ such that

$$CHFEWG(ch_1, ch_2, ..., ch_n) = (\otimes_e)_{s=1}^n(ch_s^{w_s}).$$

Here $w = (w_1, w_2, ..., w_n)^T$ is a weighting vector of CHFEs where $w_s \in [0, 1]$ and $\sum_{s=1}^n w_s = 1$. As a special case when $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ then CHFEWG operator becomes CHFEG operator as,

$$CHFEG(ch_1, ch_2, ..., ch_n) = (\otimes_e)_{s=1}^n(ch_s)^{\frac{1}{n}}.$$

Theorem 4.4. *Suppose $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. Their aggregated result obtained by using CHFEWG operator is a CHFE,*

$CHFEWG(ch_1, ch_2, ..., ch_n) = (\otimes_e)_{s=1}^{n}(ch_s^{w_s})$

$= \{< \{[\frac{2\Pi_{s=1}^{n}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{n}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{n}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{n}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{n}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{n}(\aleph_{i_s}^{+})^{w_s}}]\},$

$\{\frac{\Pi_{s=1}^{n}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{n}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{n}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{n}(1-\hbar_{i_s})^{w_s}}\} >\}.$

Proof. We show the result by using mathematical induction.

When $n = 2$ and by applying operational law 4 of Definition 3.1 we have,

$ch_1^{w_1} = \{< \{[\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}}, \frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}}]\}, \{\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}}\} >\},$

$ch_2^{w_2} = \{< \{[\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}}, \frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}}]\}, \{\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}}\} >\}.$

By applying operational law 2 of Definition 3.1 we have,

$ch_1^{w_1} \otimes_e ch_2^{w_2} = \{< \{[\frac{(\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}})(\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}})}{1+(1-\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}})(1-\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}})},$

$\frac{(\frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}})(\frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}})}{1+(1-\frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}})(1-\frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}})}]\},$

$\{\frac{\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}}+\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}}}{1+(\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}})(\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}})}\} >\},$

$ch_1^{w_1} \otimes_e ch_2^{w_2} = \{< \{[\frac{2\Pi_{s=1}^{2}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{2}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{2}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{2}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{2}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{2}(\aleph_{i_s}^{+})^{w_s}}]\},$

$\{\frac{\Pi_{s=1}^{2}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{2}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{2}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{2}(1-\hbar_{i_s})^{w_s}}\} >\}.$

Suppose the result is true for $n = t$,

$CHFEWG(ch_1, ch_2, ..., ch_t) = (\otimes_e)_{s=1}^{t}(ch_s^{w_s})$

$= \{< \{[\frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}]\}.$

Now we have to show that the result holds for $n = t + 1$,

$CHFEWG(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\otimes_e)_{s=1}^{t+1}(ch_s^{w_s}) = (\otimes_e)_{s=1}^{t}(ch_s^{w_s}) \otimes_e ch_{t+1}^{w_{t+1}}$

$= \{< \{[\frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}]\},$

$\{\frac{\Pi_{s=1}^{t}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{t}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{t}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{t}(1-\hbar_{i_s})^{w_s}}\} >\} \otimes_e \{< \{[\frac{2(\aleph_{i_{t+1}}^{-})^{w_{t+1}}}{(2-\aleph_{i_{t+1}}^{-})^{w_{t+1}}+(\aleph_{i_{t+1}}^{-})^{w_{t+1}}},$

$\frac{2(\aleph_{i_{t+1}}^{+})^{w_{t+1}}}{(2-\aleph_{i_{t+1}}^{+})^{w_{t+1}}+(\aleph_{i_{t+1}}^{+})^{w_{t+1}}}]\}, \{\frac{(1+\hbar_{i_{t+1}})^{w_{t+1}}-(1-\hbar_{i_{t+1}})^{w_{t+1}}}{(1+\hbar_{i_{t+1}})^{w_{t+1}}+(1-\hbar_{i_{t+1}})^{w_{t+1}}}\} >\}.$

Again by applying operational law 2 of Definition 3.1 we get,

$CHFEWG(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\otimes_e)_{s=1}^{t+1}(ch_s^{w_s})$

$= \{< \{[\frac{2\Pi_{s=1}^{t+1}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t+1}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t+1}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t+1}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t+1}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t+1}(\aleph_{i_s}^{+})^{w_s}}]\},$

$\{\frac{\Pi_{s=1}^{t+1}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{t+1}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{t+1}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{t+1}(1-\hbar_{i_s})^{w_s}}\} >\}.$

Hence the result holds for all n.

5. Multi criteria decision making (MCDM) using einstein weighted averaging and geometric aggregation operators for CHFSs

In this section, CHFEWA and CHFEWG operators are applied to MCDM with cubic hesitant fuzzy information. Let $\breve{D} = \{\breve{D}_1, \breve{D}_2, \breve{D}_3, ...\breve{D}_q\}$ be a finite set of q alternatives. Let $\breve{U} = \{\breve{U}_1, \breve{U}_2, \breve{U}_3, ..., \breve{U}_p\}$ be a finite set of p criteria. Let $w = (w_1, w_2, ..., w_p)^T$ be the weight vector of criteria's $\breve{U} = \{\breve{U}_1, \breve{U}_2, \breve{U}_3, ..., \breve{U}_p\}$ with $w_l, 1 \leq l \leq p$ and $\sum_{l=1}^{p} w_l = 1$. Suppose $\breve{E} = (ch_{js})_{m\times n}$ is cubic hesitant fuzzy matrix, where $ch_{js} = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}$ is the CHFE given for alternative $\breve{D}_j (j = 1, 2, ..., q)$ with respect to criteria $\breve{U}_s (s = 1, 2, ...p)$. The CHFEWA and CHFEWG operators are utilized to develop a MCDM method with cubic hesitant fuzzy information by the following steps.

Algorithm 1;

Algorithm 1

Input: A set of criteria associated with alternatives.
Output: Best alternative.

1: Obtian a set of criteria associated with a set of alternatives.
2: Obtain CHFEWA and CHFEWG result through the weight vector associated with the mapping $G^n \to G$ and by applying Definition 4.1 and Definition 4.3 respectively.
3: Calculate the scores of CHFEWA and CHFEWG results by using Definition 2.10.
4: Rank given alternatives $\breve{D}_m, 1 \leq m \leq q$ by applying accuracy procedure through score values.
5: Obtain a best choice on a maximum score.

Next, an example is presented to select best alternative from a set of alternatives utilizing Algorithm-1.

5.1 Illustrative example

Let an organization interested to build a sugar industry in Guangzhou, China. The three districts namely, Conghua ($\breve{D}_1$), Nansha ($\breve{D}_2$) and Huangpu ($\breve{D}_3$) are alternative districts to build a sugar industry. In each district a place is selected to build a sugar industry. Since different factors need to manage before selecting a district for industry. Then the organization mainly consider three main criterias "Waste management ($\breve{U}_1$)", "Water resources planning ($\breve{U}_2$)" and "Population of area ($\breve{U}_3$)". A team of experts are selected to collect data from three districts and experts give their data on each criteria as a CHF number, that is, $< t, u(t), g(t) >$. In Table 1 CHFS values are given.

A set of weights on CHFEs is given as $w = (w_1, w_2, w_3)^T = (0.2, 0.3, 0.5)^T$. Now to calculate CHFEWG and CHFEWA, first ch_{mn}, $n, m = 1, 2, 3$ cumulated as follows:

Table 1: Matrix containing ICHFS values

	$\check{U}_1$	$\check{U}_2$	$\check{U}_3$
$\check{D}_1$	$\left\{ \begin{array}{c} < [0.1,0.2],[0.3,0.5],[0.7,0.9], \\ \{0.15,0.4,0.8\} > \end{array} \right\}$	$\{< [0.4,0.6],[0.7,0.9],\{0.5,0.8\} > \}$	$\{< [0.6,0.8],\{0.7\} >\}$,
$\check{D}_2$	$\{< [0.4,0.7],\{0.6\} >\}$	$\left\{ \begin{array}{c} < [0.2,0.4],[0.5,0.7],[0.8,0.9], \\ \{0.3,0.6,0.85\} > \end{array} \right\}$	$\{< [0.3,0.5],[0.4,0.7],\{0.4,0.6\} > \}$,
$\check{D}_3$	$\{< [0.2,0.5],[0.6,0.9],\{0.4,0.7\} > \}$	$\{< [0.5,0.9],\{0.7\} >\}$	$\left\{ \begin{array}{c} < [0.3,0.4],[0.5,0.6],[0.7,0.9], \\ \{0.35,0.55,0.8\} > \end{array} \right\}$,

$ch_{11} = \{< \{[0.1,0.2],[0.3,0.5],[0.7,0.9]\},\{0.15,0.4,0.8\} >\}$,
$ch_{12} = \{< \{[0.4,0.6],[0.7,0.9]\},\{0.5,0.8\} >\}$,
$ch_{13} = \{< \{[0.6,0.8]\},\{0.7\} >\}$.
Next, CHFEWG are calculated as follows;
$CHFEWG(ch_{11},ch_{12},ch_{13}) = \phi_1 = ((\otimes_e)_{s=1}^3)_{j=1}(ch_{js}^{w_s})$
$= \{< \{[\frac{2\Pi_{s=1}^3(\aleph_{i_{1s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{1s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{1s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{1s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{1s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{1s}}^+)^{w_s}}]\},$
$\{\frac{\Pi_{s=1}^3(1+\hbar_{i_{1s}})^{w_s}-\Pi_{s=1}^3(1-\hbar_{i_{1s}})^{w_s}}{\Pi_{s=1}^3(1+\hbar_{i_{1s}})^{w_s}+\Pi_{s=1}^3(1-\hbar_{i_{1s}})^{w_s}}\} >\}$,
$CHFEWG(ch_{11},ch_{12},ch_{13}) = \phi_1 = \{< \{[0.3867,0.5796],[0.4678,0.6740],$
$[0.5524,0.7565],[0.4636,0.6626],[0.5552,0.7632],[0.6490,0.8497]\},$
$\{0.5571,0.5936,0.6741,0.6604,0.6900,0.7543\} >\}$,
$ch_{21} = \{< \{[0.4,0.7]\},\{0.6\} >\}$,
$ch_{22} = \{< \{[0.2,0.4],[0.5,0.7],[0.8,0.9]\},\{0.3,0.6,0.85\} >\}$,
$ch_{23} = \{< \{[0.3,0.5],[0.4,0.7]\},\{0.4,0.6\} >\}$,
$CHFEWG(ch_{21},ch_{22},ch_{23}) = \phi_2 = ((\otimes_e)_{s=1}^3)_{j=2}(ch_{js}^{w_s})$
$= \{< \{[\frac{2\Pi_{s=1}^3(\aleph_{i_{2s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{2s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{2s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{2s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{2s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{2s}}^+)^{w_s}}]\},$
$\{\frac{\Pi_{s=1}^3(1+\hbar_{i_{2s}})^{w_s}-\Pi_{s=1}^3(1-\hbar_{i_{2s}})^{w_s}}{\Pi_{s=1}^3(1+\hbar_{i_{2s}})^{w_s}+\Pi_{s=1}^3(1-\hbar_{i_{2s}})^{w_s}}\} >\}$,
$CHFEWG(ch_{21},ch_{22},ch_{23}) = \phi_2 = \{< \{[0.2828,0.5036],[0.3278,0.5992],$
$[0.3726,0.5952],[0.4283,0.7000],[0.4398,0.6489],[0.5024,0.7581]\},$
$\{0.4164,0.5213,0.5068,0.6000,0.6214,0.6973\} >\}$,
$ch_{31} = \{< \{[0.2,0.5],[0.6,0.9]\},\{0.4,0.7\} >\}$,
$ch_{32} = \{< \{[0.5,0.9]\},\{0.7\} >\}$,
$ch_{33} = \{< \{[0.3,0.4],[0.5,0.6],[0.7,0.9]\},\{0.35,0.55,0.8\} >\}$,
$CHFEWG(ch_{31},ch_{32},ch_{33}) = \phi_3 = ((\otimes_e)_{s=1}^3)_{j=3}(ch_{js}^{w_s})$
$= \{< \{[\frac{2\Pi_{s=1}^3(\aleph_{i_{3s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{3s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{3s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{3s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{3s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{3s}}^+)^{w_s}}]\},$
$CHFEWG(ch_{31},ch_{32},ch_{33}) = \phi_3 = \{< \{[0.3259,0.5485],[0.4222,0.6621],$
$[0.5076,0.8121],[0.4064,0.6229],[0.5191,0.7439],[0.6164,0.9000]\},$
$\{0.4836,0.5744,0.7135,0.5486,0.6308,0.7543\} >\}$.
The operators CHFEWA are calculated as in following:
$CHFEWA(ch_{11},ch_{12},ch_{13}) = \delta_1 = \left((\oplus_e)_{s=1}^3\right)_{j=1}(w_s ch_{js})$
$= \{< \{[\frac{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^-)^{w_s}-\Pi_{s=1}^3(1-\aleph_{i_{1s}}^-)^{w_s}}{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^-)^{w_s}+\Pi_{s=1}^3(1-\aleph_{i_{1s}}^-)^{w_s}}, \frac{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^+)^{w_s}-\Pi_{s=1}^3(1-\aleph_{i_{1s}}^+)^{w_s}}{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^+)^{w_s}+\Pi_{s=1}^3(1-\aleph_{i_{1s}}^+)^{w_s}}]\},$

$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{1s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{1s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{1s}})^{w_s}}\} >\},$

$CHFEWA(ch_{11}, ch_{12}, ch_{13}) = \delta_1 = \{< \{[0.4572, 0.6628], [0.5559, 0.7745],$
$[0.4896, 0.6999], [0.5841, 0.8008], [0.5697, 0.7824], [0.6528, 0.8579]\},$
$\{0.4841, 0.5644, 0.5714, 0.6599, 0.6547, 0.7494\} >\},$

$CHFEWA(ch_{21}, ch_{22}, ch_{23}) = \delta_2 = \left((\oplus_e)_{s=1}^{3}\right)_{j=2}(w_s ch_{js})$

$= \{< \{[\frac{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{-})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{-})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{-})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{-})^{w_s}}, \frac{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{+})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{+})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{+})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{+})^{w_s}}]\},$
$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{2s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{2s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{2s}})^{w_s}}\} >\},$

$CHFEWA(ch_{21}, ch_{22}, ch_{23}) = \delta_2 = \{< \{[0.2916, 0.5192], [0.3429, 0.6256],$
$[0.3836, 0.6096], [0.4312, 0.7000], [0.5147, 0.7113], [0.5554, 0.7813]\},$
$\{0.4010, 0.4944, 0.4932, 0.6000, 0.5564, 0.6708\} >\},$

$CHFEWA(ch_{31}, ch_{32}, ch_{33}) = \delta_3 = \left((\oplus_e)_{s=1}^{3}\right)_{j=3}(w_s ch_{js})$

$= \{< \{[\frac{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{-})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{-})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{-})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{-})^{w_s}}, \frac{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{+})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{+})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{+})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{+})^{w_s}}]\},$
$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{3s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{3s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{3s}})^{w_s}}\} >\},$

$CHFEWA(ch_{31}, ch_{32}, ch_{33}) = \delta_3 = \{< \{[0.3453, 0.6431], [0.4462, 0.7154],$
$[0.5642, 0.8585], [0.4286, 0.7388], [0.5212, 0.7942], [0.6274, 0.9000]\},$
$\{0.4495, 0.5587, 0.6789, 0.5052, 0.6226, 0.7494\} >\}.$

Step 2

$S(\phi_1) = 0.4408, S(\phi_2) = 0.2935, S(\phi_3) = 0.3994,$
$S(\delta_1) = 0.4643, S(\delta_2) = 0.3069, S(\delta_3) = 0.4289.$

Step 3

Since $S(\phi_1) > S(\phi_3) > S(\phi_2)$ so by accuracy procedure $\phi_1 > \phi_3 > \phi_2$.

Also $S(\delta_1) > S(\delta_3) > S(\delta_2)$ hence by accuracy procedure we have $\delta_1 > \delta_3 > \delta_2$.

So we conclude that by applying both averaging and geometric operators we have the same result.

Hence $\check{D}_1$ is the best choice.

6. Conclusion

In our work we defined Einstein operations for CHFSs and also proved their important results. We defined CHFEWA and CHFEWG operators which are necessary and important to handle and aggregate the data given by decision makers. Cubic hesitant fuzzy sets provide such type of information which is so complicated because of its hesitancy factor. T. Mahmood et al [12] gave some generalized aggregation operators for CHFSs which worked very effectively to solve these problems. But there was a need to provide an easy and effective approach which provide more accuracy then the previous methods. So Einstein operations are the one which gave the solution to our problem.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 31 (1986), 87–96.

[2] K. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 31 (1989), 343-349.

[3] N. Chen, Z. S. Xu and M. M. Xia, *Interval valued hesitant preference relations and their applications to group decision making*, Knowledge Based Systems, 37 (2013), 528–540.

[4] D. H. Hong and C. H. Choi, *Multi criteria fuzzy decision making problems based on vague set theory*, Fuzzy Sets and Systems, 21 (2000), 1-17.

[5] C. L. Hwang and K. Yoon, *Multiple attribute decision making methods and application*, Springer New York, 1981.

[6] Y. B. Jun, C. S. Kim and K. O. Yang, *Cubic sets*, Annals of Fuzzy Math Inform., 4 (2012), 83-98.

[7] D. Li , *Multi attribute decision making models and methods using intuitionistic fuzzy sets*, Journal of Computer and System Sciences, 70 (2005), 73-85.

[8] B. Li, J. Wang, L. Yang and X. Li, *A novel generalized simplified neutrosophic number einstein aggregation operator*, International Journal of Applied Mathematics, 48(1) (2018), 67-72.

[9] P. Liu and L. Shi, *Intuitionistic uncertain linguistic powered einstein aggregation operators and their application to multi-attribute group decision making*, Journal of Applied Analysis and Computation, 5(4) (2015), 534-561.

[10] P. Liu, Y. Li and Y. Chen, *Some Generalized einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making*, Scientia Iranica, 22 (2015), 2684-2701.

[11] T. Mahmood, F. Mehmood and Q. Khan, *Cubic hesitant fuzzy sets and their applications to multi criteria decision making*, International Journal of Algebra and Statistics, 5 (2016), 19-51.

[12] T. Mahmood, F. Mehmood and Q. Khan, *Some generalized aggregation operators for cubic hesitant fuzzy sets and their applications to multi criteria decision making*, Punjab University Journal of Mathematics, 49 (2017), 31-49.

[13] V. L. G. Nayagam, S. Muralikrish and G. Sivaraman, *Multi criteria decision making method based on interval valued intuitionistic fuzzy sets*, Expert Systems with Applications, 38 (2011), 1464-1467.

[14] X. Peng, *Hesitant trapezoidal fuzzy aggregation operators based on archimedean t-norm and t-conorm and their application in madm with completely unknown weight information*, International Journal of Uncertainty Quantification, 7 (2017), 475-510.

[15] V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, in the 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 2009, 1378–1382.

[16] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems, 25 (2010), 529–539.

[17] H. E. Viedma, E. Alonso, S. Chiclana and F. Herrera, *A consensus model for group decision making with incomplete fuzzy preference relations*, IEEE Transactions on Fuzzy Systems, 15 (2007), 863–877.

[18] W. Wang and X. Liu, *Intuitionistic fuzzy geometric aggregation operators based on Einstein operations*, International Journal of Intelligent Systems, 26 (2011), 1049-1075.

[19] W. Wang and X. Liu, *Intuitionistic fuzzy information aggregation using Einstein operations*, Fuzzy Systems, IEEE Transactions, 20 (2012), 923-938.

[20] Q. Wang and Haining Sun, *Interval-valued intuitionistic fuzzy Einstein geometric choquet integral operator and its application to multiattribute group decision-making*, Mathematical Problems in Engineering, Volume 2018, Article ID 9364987, 11 Pages.

[21] G. Wei and X. Zhao, *Induced hesitant interval valued fuzzy Einstein aggregation operators and their application to multiple attribute decision making*, Journal of Intelligent and Fuzzy Systems, 24 (2013), 789-803.

[22] G. Wei, X. Zhao and R. Lin, *Some hesitant interval-valued fuzzy aggregation operators and their applications to multi attribute decision making*, Knowledge Based Systems, 2013.

[23] M. M. Xia and Z. S. Xu, *Hesitant fuzzy information aggregation in decision making*, International Journal of Approximate Reasoning, 52 (2011), 395-407.

[24] M. M. Xia, Z. S. Xu and N. Chen, *Some hesitant fuzzy aggregation operators with their application in group decision making*, Group Decision and Negotiation, 22 (2013), 259–279.

[25] Z. S. Xu and R. R. Yager, *Some geometric aggregation operators based on intuitionistic fuzzy sets*, International Journal of General Systems, 35 (2006), 417–433.

[26] Z. S. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Transactions on Fuzzy Systems, 15 (2007), 1179-1187.

[27] J. Ye, *Multi criteria fuzzy decision-making method based on a novel accuracy function under interval valued intuitionistic fuzzy environment*, Expert Systems with Applications, 36 (2009), 6899-6902.

[28] D. Yu, *Some hesitant fuzzy information aggregation operators based on Einstein operational laws*, International Journal of Intelligent Systems, 29 (2014), 320–340.

[29] W. Zhang, X. Li and Y. Ju, *Some aggregation operators based on Einstein operations under interval valued dual hesitant fuzzy setting and their application*, Mathematical Problems in Engineering. Article ID 958927, 2014.

[30] H. Zhao, Z. Xu, S. Liu, *Dual hesitant fuzzy information aggregation with Einstein T-conorm and T-norm*, Journal of Systems Science and Systems Engineering, 26(2017), 240-264.

[31] X. Zhou, Q. Li, *Generalized hesitant fuzzy prioritized Einstein weighted averaging operator and its application in group decision making*, In Fuzzy Theory and Its Applications (ifuzzy) IEEE International Conference, 2013, 435-440.

[32] X. Zhou and Q. Li, *Multiple attribute decision making based on hesitant fuzzy Einstein geometric aggregation operators*, Journal of Applied Mathematics, Article ID 745617 (2014).

[33] L. A. Zadeh, *Fuzzy sets*, Information and Control., 8 (1965), 338.353.

[34] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning-I*, Information Sciences, 8 (1975), 199-249.

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Some classes of pure sub-acts over semi-groups

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Abstract. In this paper the concepts of approximately pure subacts , approximately pure intersection subacts, and pure sub-acts relative to sub-acts have been studied. Some properties and some characterization of these notions are established and study approximately pure sub-acts with intersection property and pure sub-acts relative to sub-acts with intersection property.

Keywords: pure sub-acts, approximately pure sub-acts, approximately pure intersection sub-acts, pure sub-acts relative to sub-acts.

1. Introduction

"Let S be a monoid. A right S-act M_S is a nonempty set M together with a map (written multiplicatively) from $M \times S$ into M satisfying $m.1 = m$ and $m(st) = (ms)t$, for all $m \in M$ and $s, t \in S$. A nonempty subset N of an S-act M_S is S-subact if $ns \in N$ for all $s \in S$ and $n \in N$. We say that M_S is a cyclic S-act if $M_S = uS$ for some $u \in M_S$. An element $z \in M_S$ is called a fixed element of M_S if $zs = z$ for all $s \in S$. The set of all fixed elements of M_S will be denoted by $F(M)$. If M_S has a unique fixed element z, then z is called zero element of M_S. We will denote the zero element of M_S by O. Every S-act M_S can be extended to an S-act with fixed element z by taking the disjoint union: $M_S \cup \{za\}s$. A nonempty subset $K \subseteq S$ is called left ideal of a monoid S if $SK \subseteq K$; a right ideal of S if $KS \subseteq K$; an ideal of S if $KS \subseteq K$ and $SK \subseteq K$. Recall that, for two S-acts A_S, B_S a mapping $\theta : A_S \longrightarrow B_S$ is called a homomorphism of S-acts or just an S-homomorphism if $\theta(as) = \theta(a)s$ for all $a \in A_S, s \in S$. The set of all S-homomorphisms from A_S into B_S will be denoted by $Hom(A_S, B_S)$ or sometimes by $Hom_S(A, B)$. Note that if $\theta : A_S \longrightarrow B_S$ is an S-homomorphism then $Im\theta = \theta(A_S)$ is a subact of B_S, and the S-homomorphism $f : M_S \longrightarrow M_S$ is called an endomorphism of M_S. The set $Hom_S(M, M)$ which forms a monoid under composition of mappings is denoted by $End_S(M)$ and is called the endomorphism monoid of M_S. An equivalence relation ρ on an S-act M_S is called an S-act congruence or a congruence on M_S, if $(m, n) \in \rho$ implies $(ms, ns) \in \rho$ for $m, n \in M_S, s \in S$" [1]. "For a congruence ρ on an S-act M, the quotient set $M/\rho = \{[m] : m \in M\}$ becomes an S-act by defining $[m]s = [ms]$. Given an S-act A and a subact B of A, we define the

Rees congruence ρ_B on A by $a\rho_B b$ if and only if $a = b$ or $a, b \in Bforalla, b \in A$. We denote the quotient act A/ρ_B by A/B and call it the Rees quotient of A by B. We usually identify the ρ_B -class $\{a\} \in A/B$ with a for each $a \in A/B$, and denote the ρ-class $B \in A/B$ by 0"[2]. Recall that "let S be a semigroup An S-subsystem N of a right S-system (S-act) M_S is called R-pure (pure) in S if $N \cap Ma = Na$ for all $a \in S$ or N is R-pure (pure) if and only if $N \cap MI = NI$ for every ideal I of S"[3]. Recall "that a subact B of a right S-act A is called essential (or large) if for every right S-act C, every homomorphism $f : A \longrightarrow C$ with $f|_B$ a monomorphism, is itself a monomorphism. Also, a subact B of a right S-act A is called intersection large if $B \cap C \neq 0$, for each subact C of A. If S contains a zero, then B is intersection large in A if $B \cap C \neq 0_A$, for each non-zero subact C of A. It is easily checked that B is intersection large in A if and only if for each $a \in A$ (resp.$a \neq 0_A$) there exists $s \in S$ (resp. $s \neq 0_S$) such that $as \in B$ (resp. $0_A \neq as \in B$). Also, it is easy to see that every large subact B of a right S-act A with a zero 0 is intersection large"[4].

2. Approximately pure subacts and properties of S-acts which has approximately-pure intersection property

In this section the concept of approximately- pure subacts have been introduced.

Definition 2.1. Let A be S-act . A subact N of A is said to be approximately -pure if for each ideal I of S,$N \cap IA = IN \cup J(S)A \cap (N \cap IA)) = (IN \cup J(S)A) \cap (N \cap IA)$. where $J(S)$ is the Jacobson radical of S. It is clear that each pure subact is approximately pure subact.

Remark 2.2. 1. Let A be an S-act and N be approximately - pure subact of A. If H is approximately -pure subact of N, then H is approximately - pure submodule of A, where $J(S)$ is the Jacobson radical of S.

2. Let A be an S-act and N be approximately -pure subact of A. if K is a subact of A containing N, then N is a approximately -pure subact of K

Proof. 1. Let I be an ideal of S, since N is approximately - pure in A and H is approximately -pure in N, then $N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$ and $H \cap IN = (IN \cup J(S)A) \cap (N \cap IA)$ but H subact of N, therefore $H \cap IA \subseteq N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$ and hence $H \cap IA \subseteq H \cap [(IN \cap J(S)A) \cup (N \cap IA)] = [H \cap (IN \cup J(S)A)] \cap (N \cap IA) = H \cap [(IN \cup J(S)A) \cap (N \cap IA)] = [(H \cap IN) \cap (H \cup J(S)A)] \cap (N \cap IA) = [(H \cap IN) \cap (N \cap IA)] \cup [(H \cap J(S)A) \cap (N \cap IA)] = [(H \cap (IN \cap N) \cap IA)] \cup [(H \cap J(S)A) \cap (N \cap IA)] = [H \cap (IN \cap IA)] \cup [J(S)A \cap (H \cap IA)] = (H \cap IN) \cup [J(S)A \cap (H \cap IA)] = (H \cap IN) \cup [(H \cap IA) \cap J(S)A] = [(H \cap IN) \cup (H \cap IA)] \cap [(H \cap IN) \cup J(S)A] \subseteq [(H \cap IN) \cup (H \cap IA)] = [(IH \cup J(S)A) \cap (H \cap IN)] \cup (H \cap IA) = [[(IH \cup F) \cap (H \cap IN)] \cup H] \cap [[(IH \cup J(S)A) \cap (H \cap IN)] \cup IH] \subseteq [(IH \cup J(S)A) \cup IH] \cap [(H \cap IN) \cup IH] = (IH \cup J(S)A) \cap [(H \cap IN) \cup IH] = [(IH \cup J(S)A) \cap (H \cap IN)] \cup [(IH \cup J(S)A) \cap IH] = (IH \cup J(S)A) \cap [(H \cap IN) \cup IH] \subseteq (IH \cap J(S)A) \cap [(H \cup IH) \cap (IN \cup IH)] = (IH \cup J(S)A) \cap (H \cap IN) \subseteq$

$[(IH \cup J(S)A) \cap (H \cap IA)]$. Since $(IH \cup J(S)A) \cap (H \cap IA) \subseteq H \cap IA$, then $H \cap IA = (IH \cup J(S)A) \cap (H \cap IA)$.

2. Let I be an ideal of S, since N is approximately pure in A, then $N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$. But $K \subseteq A$, therefore, $N \cap IK \subseteq N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$, and hence $N \cap IK \subseteq [(IN \cup J(S)A) \cap (N \cap IA)] \cap IK = (IN \cup J(S)A) \cap [(N \cap IA) \cap IK]$. Since $(IN \cup J(S)A) \cap (N \cap IK) \subseteq N \cap IK$, then $N \cap IK = (IN \cup J(S)A) \cap (N \cap IK)$. □

Definition 2.3. S-act A is said to have the approximately -pure intersection property if the intersection of any two approximately - pure subacts is again approximately -pure.

Proposition 2.4. *1. If S-act A has the approximately - PIP, then every approximately - pure subact of A has the approximately - PIP.*

2. Let N be approximately - pure subact of S-act A . A has approximately - PIP if and only if A/N has approximately -PIP.

Proof. 1. Clear

2. Let L/N, K/N be two approximately pure subacts of A/N and let $J(S)$ be an ideal in S. we have to show that $(L/N \cap K/N) \cap F(A/N) = [F(L/N \cap K/N) \cup J(S)A/N] \cap [(L/N \cap K/N) \cap F(A/N)]$, where F is an ideal in S. We claim that each of L and K is approximately pure in A. To show this, let I be an ideal in S and let $x \in L \cap IA$. Since L/N is approximately pure in A/N, then $L/N \cap I(A/N) = [I(L/N) \cup J(S)A/N] \cap [(L/N \cap I(A/N)]$, thus $L/N \cap (IA \cup N)/N = [(IL \cup N)/N \cup J(S)A/N] \cap [(L/N \cap (IA \cup N)/N]$, and this implies that $[L \cap (IA \cup N)]/N = [(IL \cup N)/N \cup J(S)A/N] \cap [(L/N \cap (IA \cup N)/N] = [[(IL \cup N) \cup (J(S)A \cup N)] \cap [(L \cap (IA \cup N)]]/N$, therefore $L \cap (IA \cup N) = [(IL \cup J(S)A) \cup N] \cap [(L \cap IA) \cup (L \cap N)] = [(IL \cup J(S)A) \cup N] \cap [(L \cap IA) \cup N] = [(IL \cup J(S)A) \cap [(L \cap IA)] \cup N$, and hence $(L \cap IA) \cup N = [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$. Since $x \in L \cap IA \subseteq (L \cap IA) \cup N$, then $x \in [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$, thus $x \in (IL \cup J(S)A) \cap (L \cap IA)$ or $x \in N$, if $x \in (IL \cup J(S)A) \cap (L \cap IA)$, then $L \cap IA = (IL \cup J(S)A) \cap (L \cap IA)$, therefore L is approximately pure, if $x \in N$ thus $x \in (L \cap IA) \cup N = [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$, hence L is approximately-pure in A. Since A has the approximately -PIP, then $L \cap K$ is approximately - pure in A. Thus $(L \cap K) \cap FA = [F(L \cap K) \cup J(S)A] \cap [(L \cap K) \cap FA]$ where $F$is an ideal in S. Now, let Rees congruence $\rho_N \in [L/N \cap K/N] \cap J(S)(A/N)$, then Rees congruence $\rho_N \in J(S)(A/N)$ (Rees congruence on $J(S)A$) and $\rho_N \in L/N$ (Rees congruence on L) and $\rho_N \in K/N$ (Rees congruence on K), thus Rees congruence ρ_N on L and K therefore $\rho_N \in L \cap K$but $(L \cap K) \cap FA = [F(L \cap K) \cup J(S)A] \cap [(L \cap K) \cap FA]$ therefore $(L/N \cap K/N) \cap F(A/N) = [F(L/N \cap K/N) \cup J(S)A/N] \cap [(L/N \cap K/N) \cap F(A/N)]$. ($\Longleftarrow$) Conversely let$M$ and N be F- pure subacts of A, let L be a subact of M and L be a subact of N then M/L and N/L is approximately-pure subacts of A/L. Since A/L has approximately -PIP, then $M/L \cap N/L = (M \cap N)/L$ is approximately - pure subact of A/L. Therefore $M \cap N$ is approximately - pure subact of A. □

Theorem 2.5. *Let A be S-act, then A has the approximately -PIP if and only if $[(IM \cap IN) \cup J(S)A] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup J(S)A] \cap [(M \cap N) \cap IA]$ for every ideal I of S and for every approximately - pure subacts M and N of A.*

Proof. Suppose A has the approximately -PIP then for each approximately -pure subacts M and N, $M \cap N$ is approximately -pure. Let I be an ideal in S, then $(M \cap N) \cap IA = [I(M \cap N) \cup J(S)A] \cap [(M \cap N) \cap IA]$. It is clear that $[I(M \cap N) \cup J(S)A] \cap [(MN) \cap IA] \subseteq ([(IM \cap IN) \cup J(S)A] \cap [(M \cap N) \cap IA]$. But $[(IM \cap IN) \cup J(S)A] \cap [(M \cap N) \cap IA] \subseteq M \cap (N \cap IA) = (M \cap N) \cap IA = [I(M \cap N) \cup J(S)A] \cap [(M \cap N) \cap IA]$. Thus $[(IM \cap IN) \cup J(S)A] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup J(S)A] \cap [(M \cap N) \cap IA]$ Conversely, let L and B be F-pure subact of A and J an ideal in S. then $(L \cap B) \cap JA = L \cap (B \cap JA) = L \cap [(JB \cup F) \cap (B \cap JA)]$. Similarly $(L \cap B) \cap JA = B \cap (L \cap JA) = B \cap (JA \cup F) \cap (B \cap JA)$. But L, B are F- pure in A. Thus $(L \cap B) \cap JA \subseteq [(JL \cap JB) \cup J(S)A] \cap [(L \cap B) \cap JA] = [J(L \cap B) \cup J(S)A] \cap [(L \cap B) \cap JA]$. □

3. Approximately pure intersection large subacts

In this section the concept of approximately- pure intersection large subacts have been introduced.

Definition 3.1. A subact K of an S-act A is called approximately pure intersection large in A, if for every approximately pure subact L of A with $K \cap L \subseteq J(S)A$ implies $L \subseteq J(S)A$. A is called approximately pure intersection large of K.

Theorem 3.2. *Let K, N be subacts of A such that K is subact of N then:*

1. *If K is approximately pure intersection large in A, then K is approximately pure intersection large in N and N is approximately pure intersection large in A.*

2. *If N is approximately pure intersection large in A and $J(S)A \cap N = J(S)N$ and K is approximately pure intersection large in A then K is approximately pure intersection large in A.*

3. *If A has approximately pure intersection large finite intersection property and if N is approximately pure intersection large in A and $J(S)A \cap N = J(S)N$ then K is approximately pure intersection large in A if and only if K is approximately pure intersection large in N and N is approximately pure intersection large in M.*

Proof. 1. We have to show that N is approximately pure intersection large in A. Let L be approximately pure subact of A with $N \cap L \subseteq J(S)A$, since K is subact of N, then $K \cap L \subseteq N \cap L$ and $N \cap L \subseteq J(S)A$ thus $K \cap L \subseteq J(S)A$.

Since K is approximately pure intersection large in A, then $L \subseteq J(S)A$. Hence N is approximately pure intersection large in A.

2. Let L be an approximately pure of N with $K \cap L$ is subact of $J(S)N$, thus $K \cap L \subseteq J(S)A$, since N is subact of A and K is approximately pure intersection large in A therefore $L \subseteq J(S)A$, therefore $L = L \cap N \subseteq J(S)A \cap N$, thus $L \subseteq J(S)N$, hence K is approximately pure intersection large in N. Now we have to show that N is approximately pure intersection large in A. Let L be an approximately pure subact of A with $N \cap L \subseteq J(S)A$, thus $K \cap L \subseteq N \cap L \subseteq J(S)A$ therefore $L \subseteq J(S)A$. Hence N is approximately pure intersection large in A.

3. Suppose that K is approximately pure intersection large in N and N is approximately pure intersection large in A, we have to show that K is approximately pure intersection large in A. Let L be approximately pure subact of A with $K \cap L \subseteq J(S)A$. By assumption $N \cap L$ is approximately pure in A, thus $N \cap L$ is approximately pure in N by Remark 2.2. Since N is approximately pure intersection large in A, thus $N \cap L \subseteq J(S)N \subseteq J(S)A$, hence $L \subseteq J(S)A$. Hence K is approximately pure intersection large in A. Conversely It is clear. □

Corollary 3.3. *Let A be an S-act that has the pure finite intersection property. If H is approximately pure intersection large of N and $J(S)N \cap H = J(S)H$. Then $K \cap H$ is approximately pure intersection large in A if and only if H is approximately pure intersection large in A and K is approximately pure intersection large in A, for any subact K of A.*

Proof. $\Longrightarrow$ the proof follows by Theorem 3.2.

$\Longleftarrow$ Let L be an approximately pure subact of A with $(K \cap H) \cap L \subseteq J(S)A$, by assumption $H \cap K$ is approximately pure in A. Since K is approximately pure intersection large in A, then $H \cap L \subseteq J(S)A$. So since H is approximately pure intersection large in A, hence $L \subseteq J(S)A$, therefore $K \cap H$ is approximately pure intersection large in A. □

Recall that an S-act M_s is called decomposable if there exist two subacts A , B of M_s such that $M_s = A \cup B and A \cap B = 0$[5]. We introduce the concept of approximately pure -direct summed subact.

Definition 3.4. A subact N of S-act A is approximately pure -direct summed if there exists approximately pure subact L of A such that $A = N \cup L$ and $N \cap L = 0$.

Remark 3.5. 1. Every approximately pure-direct summed of an S-act A is approximately pure

2. Let A be an S-act. If P is approximately pure subact of A and Q is any subact of A then $P \cap Q$ is approximately pure in Q.

Definition 3.6. A subact N of an S-act A is called approximately pure closed subact of A if A does not contain a proper approximately pure intersection large extension of N.

Proposition 3.7. *Let $J(S)A \cap K = J(S)K$ for every approximately pure subact K of A, then every approximately pure -direct summed is approximately pure -closed.*

Proof. Let $A = N \oplus L$, we want to show that A is approximately pure -closed. Suppose that N is approximately pure intersection large in K we have to show that $N = K$. L is approximately pure in A and K is any subact of A. then $L \cap K$ is approximately pure in K by Remark3.5 since $N \cap L \subseteq J(S)A$. Thus $N \cap (L \cap K) \subseteq J(S)K \subseteq J(S)A$ since N is approximately pure intersection large in A therefore $L \cap K \subseteq J(S)A$. Then $K = N$, hence N is approximately pure -closed subact of A. □

Theorem 3.8. *Let M, L and N be subacts of S-act A then there exists an approximately -pure closed subact H in A which is approximately pure and $J(S)A \cap H = J(S)H$ such that N is approximately pure closed in H.*

Proof. Let $V = \{K : K$ is an approximately-pure subact of H and $J(S)A \cap K = J(S)K$ such that N is approximately pure intersection large in $K\}$. It is clear that V is S-act .By Zorn's lemma, V has a maximal element say H. To show that H is approximately pure -closed in A. Let D is a subact of A such that H is approximately pure intersection large in D. Since N is approximately pure intersection large in H and $H$is approximately pure intersection large in D. Then by Theorem3.2, N is approximately pure intersection large in D, thus $H = D$. □

Definition 3.9. Let N and K be subacts of S-act A with K is pure in A, K is called pure relative complements of N in A if K is maximal with the property $K \cap N = 0$.

Definition 3.10. Let N and K be subacts of S-act A with K is approximately pure in A, K is called relative approximately pure complement of N in A if K is maximal with $K \cap N \subseteq J(S)A$.

Proposition 3.11. *Every subact of A has a relative approximately pure complement in A.*

Proof. Let N be a given subact of A and consider the set $B = \{K \cap A, K$ is approximately pure in A with $N \cap KJ(S)A\}$. It is clear that and every chain of B has an upper bound. By Zorn's Lemma B has maximal element, which means that N has relative approximately pure complement in A. □

The following propositions give the relation between approximately pure closed subact and relative approximately pure complement subact.

Proposition 3.12. *Let N be a subact of an S-act A and $J(S)A \cap F = J(R)F$ for every approximately pure subact F of A. If N is relative approximately pure complement for some subact K of A, then N is approximately pure closed subact in A.*

Proof. Let L be an approximately pure intersection large subact of A with N is approximately pure intersection large in L. We have $N \cap K \subseteq J(S)A$, $(N \cap K) \cap L \subseteq J(S)A \cap L$. since L is approximately pure in A, then $K \cap L$ is approximately pure in L -by Remark 3.5, thus $N \cap (K \cap L) \subseteq J(S) \subseteq J(S)A$, hence $L = N$, hence N is approximately pure closed in M. □

4. Pure subacts relative to subact

Definition 4.1. Let A be S-act and F be a subact of A. A subact N of A is said to be F-pure if for each ideal I of S, $N \cap IA = IN \cup (F \cap (N \cap IA)) = (IN \cup F) \cap (N \cap IA)$.

Remark 4.2.

1. Let A be an S-act and N be F- pure subact of A. If H is F-pure subact of N, then H is F- pure submodule of A, where F is subact of A.

2. Let A be an S-act and N be F-pure subact of A. if K is a subact of A containing N, then N is a F-pure subact of K, where F is subact of A.

3. Let A be an S-act and N be F -pure subact of M. If H is a subact of N and H is subact of F, then N/H is F/H -pure subact of A/H , where F is subact of A.

4. Let A be an S-act. Let N and H be subacts of A, If H is F-pure subact of A andN/H is F/H -pure subact of A/H, then N is F-pure subact of A, where F is subact of A.

Proof. 1. Let I be an ideal of S, since N is F- pure in A and H is F-pure in N, then $N \cap IA = (IN \cap F) \cap (N \cap IA)$ and $H \cap IN = (IN \cup F) \cap (N \cap IA)$ but H subact of N, therefore $H \cap IA \subseteq N \cap IA = (IN \cup F) \cap (N \cap IA)$ and hence $H \cap IA \subseteq H \cap [(IN \cup F) \cap (N \cap IA)] = [H \cap (IN \cup F)] \cap (N \cup IA) = H \cap [(IN \cup F) \cap (N \cap IA)] = [(H \cap IN) \cup (H \cap F)] \cap (N \cap IA) = [(H \cap IN) \cap (N \cap IA)] \cup [(H \cap F) \cap (N \cap IA)] = [(H \cap (IN \cap N) \cap IA)] \cup [(H \cap F)(N \cap IA)] = [H \cap (IN \cap IA)] \cup [F \cap (H \cap IA)] = (H \cap IN) \cup [F \cap (H \cap IA)] = (H \cap IN) \cup [(H \cap IA) \cap F] = [(H \cap IN) \cup (H \cap IA)] \cap [(H \cap IN) \cup F] \subseteq [(H \cap IN) \cup (H \cap IA)] = [(IH \cup F) \cap (H \cap IN)] \cup (H \cap IA) = [[(IH \cup F) \cap (H \cap IN)] \cup H] \cap [[(IH \cup F) \cap (H \cap IN)] \cup IH] \subseteq [(IH \cup F) \cup IH] \cap [(H \cap IN) \cup IH] = (IH \cup F) \cap [(H \cap IN) \cup IH] = [(IH \cup F) \cap (H \cap IN)] \cup [(IH \cup F) \cap IH] = (IH \cup F) \cap [(H \cap IN) \cup IH] \subseteq (IH \cup F) \cap [(H \cup IH) \cap (IN \cup IH)] = (IH \cup F) \cap (H \cap IN) \subseteq [(IH \cup F) \cap (H \cap IA)]$. Since $(IH \cup F) \cap (H \cap IA) \subseteq H \cap IA$, then $H \cap IA = (IH \cup F) \cap (H \cap IA)$.

2. Let I be an ideal of S, since N is F-pure in A, then $N \cap IA = (IN \cup F) \cap (N \cap IA)$. But K subact of A, therefore, $N \cap IK \subseteq N \cap IA = (IN \cup F) \cap (N \cap IA)$, and hence $N \cap IK \subseteq [(IN \cup F) \cap (N \cap IA)] \cap IK = (IN \cup F) \cap [(N \cap IA) \cap IK]$. Since $(IN \cup F) \cap (N \cap IK) \subseteq N \cap IK$, then$N \cap IK = (IN \cup F) \cap [(N \cap IK)$.

3. Let I be an ideal of S, since N is F-pure subact of A, then $N \cap IA = (IN \cup F) \cap (N \cap IA)$. So $N/H \cap I(A/H) = N/H \cap (IA \cup H)/H = [N \cap (IA \cup H)]/H = [(N \cap IA) \cup (N \cap H)]/H = [(IN \cup F) \cap (N \cap IA)] \cup H/H = ([(IN \cup F) \cup H] \cap [(N \cap IA) \cup H])/H = [(IN \cup F) \cup H]/H \cap [(N \cap IA) \cup H])/H = [(IN \cup F)(H \cup H)]/H \cap [(N \cap IA) \cup H])/H = [(IN \cup H) \cup (F \cup H)]/H \cap [(N \cap IA) \cup H])/H = I(N/H) \cup F/H \cap [(N \cup H) \cap (IA \cup H)]/H = [I(N/H) \cup F/H] \cap (N \cup H)/H \cap (IA \cup H)/H = [I(N/H) \cup F/H] \cap [N/H \cap I(A/H)]$.

4. Clear .

□

Definition 4.3. S-act A is said to have the pure relative to subact intersection property (for short $F - PIP$) if the intersection of any two F- pure subacts is again F- pure.

Proposition 4.4. *1. If S-act A has the $F - PIP$, then every F- pure subact of A has the $F - PIP$.*

2. Let N be F- pure submodule of S-act A and N subact of F. A has $F - PIP$ if and only if A/N has $F/N - PIP$.

Proof. 1. Clear

2. Let L/N, K/N be two F/N- pure subacts of A/N and let P be an ideal in S. we have to show that $(L/N \cap K/N) \cap P(A/N) = [P(L/N \cap K/N) \cup F/N] \cap [(L/N \cap K/N) \cap P(A/N)]$. We claim that each of L and K is F-pure in A. To show this, let I be an ideal in S and let $x \in L \cap IA$. Since L/N is F/N -pure in A/N, then $L/N \cap I(A/N) = [I(L/N) \cup F/N] \cap [(L/N \cap I(A/N)]$, thus $L/N \cap (IA \cup N)/N = [(IL \cup N)/N \cup F/N] \cap [(L/N \cap (IA \cup N)/N]$, and this implies that $[L \cap (IA \cup N)]/N = [(IL \cup N)/N \cup F/N] \cap [(L/N \cap (IA \cup N)/N] = [[(IL \cup N) \cup (F \cup N)] \cap [(L \cap (IA \cup N)]]/N$, therefore $L \cap (IA \cup N) = [(IL \cup F) \cup N] \cap [(L \cap IA) \cup (L \cap N)] = [(IL \cup F) \cup N] \cap [(L \cap IA) \cup N] = [(IL \cup F) \cap [(L \cap IA)] \cup N$, and hence $(L \cap IA) \cup N = [(IL \cup F) \cap [(L \cap IA)] \cup N$. Since $x \in L \cap IA \subseteq (L \cap IA) \cup N$, then $x \in [(IL \cup F) \cap [(L \cap IA)] \cup N$, thus $x \in (IL \cup F) \cap [(L \cap IA)$ or $x \in N$, if$x \in (IL \cup F) \cap (L \cap IA)$, then $L \cap IA = (IL \cup F) \cap (L \cap IA)$, therefore L is F-pure, if $x \in N$ thus $x \in (L \cap IA) \cup N = [(IL \cup F) \cap (L \cap IA)] \cup N$, hence L is F-pure in A. Since A has the $F - PIP$, then $L \cap K$ is F- pure in A. Thus $(L \cap K) \cap PA = [P(L \cap K) \cup F] \cap [(L \cap K) \cap IA]$. Now, let Rees congruence $\rho_N \in [L/N \cap K/N] \cap P(A/N)$, then Rees congruence $\rho_N \in P(A/N)$ (Rees congruence on PA) and $\rho_N \in L/N$ (Rees congruence on L) and $\rho_N \in$

K/N (Rees congruence on K), thus Rees congruence ρ_N on L and K therefore $\rho_N \in LK$, but $(L \cap K) \cap PA = [P(L \cap K) \cup F] \cap [(L \cap K) \cap IA]$, therefore $(L/N \cap K/N) \cap P(A/N) = [P(L/N \cap K/N) \cup F/N] \cap [(L/N \cap K/N)$. Conversely let M and N be F- pure subacts of A, let L be a subact of M and L be a subact of N then M/L and N/L is F/L-pure subacts of A/L. Since A/L has $F/L - PIP$, then $M/L \cap N/L = (M \cap N)/L$ is F/L- pure subact of A/L. Therefore $M \cap N$ is F- pure subact of A. □

Theorem 4.5. *Let A be an S-act, then A has the $F - PIP$ if and only if $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$ for every ideal I of S and for every F- pure subact M and N of A.*

Proof. Suppose A has the F-PIP then for each F-pure subact M and N, $M \cap N$ is F-pure. Let I be an ideal in S, then $(M \cap N) \cap IA = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$. It is clear that $[I(M \cap N) \cup F] \cap [(M \cap N) \cap IA] \subseteq ([(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA]$. But $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] \subseteq M \cap (N \cap IA) = (M \cap N) \cap IA = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$. Thus $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$ Conversely, let L and B be F-pure subact of A and J an ideal in S. Then $(L \cap B) \cap JA = L \cap (B \cap JA) = L \cap [(JB \cup F) \cap (B \cap JA)]$. Similarly $(L \cap B) \cap JA = B \cap (L \cap JA) = B \cap (JA \cup F) \cap (B \cap JA)$. But L, B are F- pure in A. Thus $(L \cap B) \cap JA \subseteq [(JL \cap JB) \cup F] \cap [(L \cap B) \cap JA] = [J(L \cap B) \cup F] \cap [(L \cap B) \cap JA]$. □

References

[1] M. Kilp, U. Knauer and A. Mikhalev, *Monoids, acts and categories*, Walter de Gruyter, 2000.

[2] C. Miller and N. Skuc, *An introduction to presentations of monoid acts: quotients and subacts,* arXiv:1709.08916v [math.GR],26Spt2017.

[3] N. Kurokt, *Note on pure subsystems*, Proc. Japan Acad., 50 (1974), 683-687.

[4] M. Ali Naghipoor, *Retractable and coretractable acts over semigroups,* Southeast Asian Bulletin of Mathematics, 40 (2016), 887-903.

[5] S. Amer, *Extending and P-extending S-act over monoids*, International Journal of Advanced Scientific and Technical Research, Issue 7, volume 2.

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On discrete time schemes for the problem of fibre suspension flows

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Abstract. We present a time discrete stability results presented in [9], we use the tools presented in [3,6] to show that the results obtained in [9] can be generalized to obtain estimates not only for the velocity but also for the orientation tensor and a non zero body force, and without decomposing the orientation tensor into a sum of diagonal and a traceless tensors. As for the regular solutions, we prove the solutions of a discrete scheme are continuously dependent on the initial data and the same body force. Also, the restrictions imposed on the data con rm a well known result proven in [4, 8, 9], that for the rest state to be stable, the particle number Np must be less than 35/2.

Keywords: fibre suspension flows, continuously dependent, existence, stability energy. estimate.

1. Introduction

Fibre suspension flows are modelled by a coupled system of nonlinear partial differential equations, with the velocity, the pressure and orientation tensorial as primary variables variable (for more details, see [1, 2, 4, 7,8, 9]). Both the velocity and orientation tensor constitutive equations contain a fourth–order tensor, which in this paper, is approximated by the linear closure.

The constitutive equations investigated in the present work were the subject of theoretical studies by Galdi and Reddy [4], Munganga, Reddy and Diatezua [9], Munganga and Reddy [8]. Their work have helped to better understand the properties of the equations governing the flow of fibre suspensions. Galdi and Reddy, [4] showed that for the linear closure, there is a connection between stability and the particle number: in particular the rest state is unstable, in the sense of Liapounov, when the particle number exceeds 35/2.

In recent years, stability of fibre suspension flows, from thermodynamic and energetic perceptive were investigated by Munganga et al. [8,9] using linear closure and quadratic closure approximations. Under certain conditions, and when the rotary diffusivity is constant, with the use of the linear closure, Munganga et al. [8,9] proved locally and globally existence of a unique classical solution, in

time. For the quadratic closure, the same authors also established the existence of a local solutions in time.

Galdi and Reddy [4] established a unique local solution in time and instability for the rest state flow for the constitutive equations for fibre suspension flows for the linear closure, if the particle number $N_p \geq 35/2$ and the rotary diffusivity D_r is proportional to $|\mathbf{D}|$,.

Munganga and Reddy [9] extended the stability analysis started in [4] by considering others closure rules like quadratic, orthotropic, hybrid. The question of consistency of the equations with respect to the second law of thermodynamic was also investigated in [9]. For D_r constant, the local in time solution constructed in [4], was shown by Munganga and Reddy [8] to be global and stable for the linear ($N_p < 35/2$) and quadratic closure in the absence of body force. Munganga [7] established existence, uniqueness, convergence and stability of solutions to the equations of steady flows of fibre suspension flow. It is important to point out that the stability results obtained in [9] only involve the velocity.

In this paper, we present a time discrete schemes similarly to the results obtained in [9] by estimating not only the velocity but also the orientation tensor where the external force is considered to be non zero. We also prove that the regular solutions of the equations of fibre suspension flows depend continuously on the initial data with the same non zero body force. The solution is proven to be bounded for all time in the absence of body force. The solutions of the discrete problem also satisfy the same properties as the solutions of continuous problem.

The rest of the paper is organised as follows. Section 2 reviews the mathematical model and carries out simplifications to the constitutive equations. Section 3 derives a priori estimates for the continuous problem, and shows the dependence of the solution with respect to the initial data. Section 4 establishes the continuous dependence of the solution with respect to the initial conditions of a time discrete scheme, results similar to continuous model are replicated in this section. Section 5 proves the instability of the rest state for a time discrete problem in the sense of Lyapounov if $N_p \geq 35/2$. We draw our conclusions in Section 6.

1.1 Notation and function spaces

Coordinate-free notation will be used wherever convenient, and vectors and tensors will be denoted by bold-face letters.

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors, their scalar product will be denoted by $\mathbf{u} \cdot \mathbf{v}$ or by $(\mathbf{u}, \mathbf{v})$, while the scalar product *of* of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ will be denoted by $\mathbf{A} : \mathbf{B}$. In index form these expressions read $u_i v_i$ and $A_{ij} B_{ij}$, the summation convention on repeated indices being applied at all times. The magnitude of a vector $\mathbf{u}$ and a tensor $\mathbf{A}$ are then naturally defined by $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ and $|\mathbf{A}| = (\mathbf{A} : \mathbf{A})^{1/2}$.

We summarise here some notations that will occur throughout the paper:

- $\mathbf{x}$: location of fluid particle
- $\mathbf{u}$: velocity field of the flow
- $\mathbf{b}$: body force
- p: pressure
- μ: dynamic viscosity
- ρ: fluid mass density
- $\mathbf{A}$: second-order orientation tensor.
- $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ deformation tensor
- $\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$ spin tensor
- $\mathbf{T}$: symmetric Cauchy stress tensor
- $\mathcal{A}$: fourth-order orientation tensor

We will also need the following function spaces:

Here Ω denotes a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3), with regular boundary Γ. We will assume that Ω is locally located on one side of Γ, and that Γ is at least locally Lipschitz, with more regularity to be specified whenever it is needed.

$\Omega_T = \Omega \times (0, T)$, for $T > 0$.

$L^p(\Omega), 1 \leq p \leq \infty$, the Lebesgue spaces with norms defined in the usual way and their norms denoted by $\| \cdot \|_{L^p}$.

For a multi-index $\alpha = (\alpha_1, ..., \alpha_d)$ where α_i are nonnegative integers, we define

$$\partial^\alpha (\cdot) := \frac{\partial^{\alpha_1 + \cdots + \alpha_d} (\cdot)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}.$$

The following classical function spaces will be used throughout

$$\begin{aligned} L^2(\Omega, \mathbb{R}) &= \left\{ v : \Omega \longrightarrow \mathbb{R}, \int_\Omega |v|^2 < \infty \right\} \\ H^1(\Omega) &= \left\{ v \in L^2(\Omega, \mathbb{R}) : \partial^\beta v \in L^2(\Omega, \mathbb{R}), |\beta| \leq 1 \right\}. \end{aligned}$$

We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the L^2 -inner product and norm respectively. The notation will be used for scalar-, vector- and matrix-valued functions, with the norms for the latter two types of functions being defined in the usual, component-wise way. As usual, $\mathbf{u}(t)$ stands the function $\mathbf{x} \in \mathbf{\Omega} \longrightarrow \mathbf{u}(\mathbf{x}, t)$.

The double contraction $\tau, \sigma \in \mathbb{R}^{d \times d}$, two rank two tensors is denoted by $\tau : \sigma$, and given by

$$\tau : \sigma = tr\left[\tau\sigma^T\right] = tr\left[\tau^T\sigma\right] = \tau_{ij}\sigma_{ij}.$$

One should note that if τ is antisymmetric and σ is symmetric, then $\tau : \sigma = \mathbf{0}$.

The logarithm of a positive definite diagonal matrix is a diagonal matrix with, on its diagonal, the logarithm of each entry. If $\mathbf{B}$ is a symmetric matrix, the logarithm of $\mathbf{B}, ln\mathbf{B}$ is given by

$$\ln \mathbf{B} = \mathbf{Q}^T \ln \mathbf{C}\mathbf{Q}, \text{ where } \mathbf{B} = \mathbf{Q}^T\mathbf{C}\mathbf{Q},$$

with $\mathbf{Q}$ and $\mathbf{C}$ orthogonal and positive definite diagonal matrices respectively. We have the following two lemmas from [3]:

Lemma 1. *Let $\mathbf{N}, \mathbf{M}$ be two symmetric and positive definite matrices. Then*

(a) $tr \ln \mathbf{N} = \ln \det \mathbf{N}$,

(b) $\mathbf{N} - \ln \mathbf{N} - \mathbf{I}$ *is symmetric,* $tr\,(\mathbf{N} - \ln \mathbf{N} - \mathbf{I}) \geq 0$.

(c) $\mathbf{N}+\mathbf{N}^{-1}-2\mathbf{I}$ *is symmetric positive semi-definite, and* $tr\left(\mathbf{M} + \mathbf{M}^{-1} - 2\mathbf{I}\right) \geq 0$

(d) $tr\left[(\mathbf{M} - \mathbf{N})\,\mathbf{N}^{-1}\right] = tr\left[\mathbf{M}\mathbf{N}^{-1} - \mathbf{I}\right] \geq \ln \det\left[\mathbf{M}\mathbf{N}^{-1}\right] = tr\left[\ln \mathbf{M} - \ln \mathbf{N}\right]$.

(e) $tr\,(\mathbf{N}\mathbf{M}) = tr\,(\mathbf{M}\mathbf{N}) \geq 0$

Lemma 2. *If* $\mathbf{A}\,(t) \in \left(C^1\left([0,T]\right)\right)^{\frac{d(d+1)}{2}}$, *then* $\forall t \in [0,T)$:

$$\left(\frac{d}{dt}\mathbf{A}\right) \cdot \mathbf{A}^{-1} = tr\left(\mathbf{A}^{-1}\frac{d}{dt}\mathbf{A}\right) = \frac{d}{dt} tr\,(\ln \mathbf{A}) \tag{1}$$

$$\left(\frac{d}{dt}\ln \mathbf{A}\right) \cdot \mathbf{A} = tr\left(\mathbf{A}\frac{d}{dt}\ln \mathbf{A}\right) = \frac{d}{dt} tr\mathbf{A} \tag{2}$$

$H^k(\Omega)$, $k = 0, 1, ...$ the Sobolev spaces endowed with the inner product $(\cdot,\cdot)_{H^k}$ defined by

$$(\mathbf{u}, \mathbf{v})_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha \mathbf{u} D^\alpha \mathbf{v} dx, \qquad \alpha = (\alpha_1, ..., \alpha_n).$$

$W^{k,\infty}(\Omega) = \{f \in L^\infty(\Omega) : D^\alpha f \in L^\infty(\Omega), 0 \leq \alpha \leq k\}, k = 1, 2, 3, ...$ will denotes the Sobolev spaces in $L^\infty(\Omega)$.

$H_0^k(\Omega) = \{f \in H^k(\Omega) : f_{/\Gamma} = 0$ and $D^\alpha f = 0$ on Γ for $|\alpha| < k\}$

$H^{-k}(\Omega)$ the topological dual space of $H_0^k(\Omega)$.

We denote the space of vector- or tensor valued functions with components in one of the spaces introduced above as follows:

$\mathbb{L}^p(\Omega)$ will denote the space of vector-or tensor valued functions with components in $L^p(\Omega)$. The same will apply to $\mathbb{H}^k(\Omega)$,...

We will also require the following spaces:

$\mathbb{H} = \{\mathbf{u} : v_i \in L^2(\Omega), div\ \mathbf{u} = 0$ in Ω, $\mathbf{u} \cdot \mathbf{n} = 0$ on $\Gamma\}$.

$\mathbb{V} = \{\mathbf{u} : u_i \in H_0^1(\Omega), div\ \mathbf{u} = 0 \ \text{ in } \ \Omega\}$.
$\mathbb{A} = \{\mathbf{A} : A_{ij} \in L^2(\Omega), A_{ij} = A_{ji},\ A_{ii} = 0 \quad \text{a.e in } \Omega\}$.
These spaces are equipped respectively with the norms

$$\begin{aligned} &\| \cdot \|_{\mathbb{H}} \equiv \| \cdot \|_{\mathbb{L}^2} \equiv | \cdot |; \\ &\| \cdot \|_{\mathbb{V}} \equiv \| \cdot \|_{\mathbb{H}^1} \qquad and \\ &\| \cdot \|_{\mathbb{A}} \equiv \| \cdot \|_{\mathbb{L}^2}. \end{aligned}$$

$\mathbb{A}_k = \mathbb{A} \cap \mathbb{H}^k(\Omega)$.

We denotes by P the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$, and the operator $\mathcal{L}$ by

$$\mathcal{L}(\mathbf{u}) = -P\Delta\mathbf{u};$$

where Δ is the Laplacian operator. The domain of $\mathcal{L}$ is given by $D(\mathcal{L}) = \mathbb{V} \cap \mathbb{H}^2(\Omega)$, and the norm $\|\mathbf{u}\|_{D(\mathcal{L})} = \|\mathcal{L}\mathbf{u}\|_{L^2}$ is equivalent to the natural $\mathbb{H}^2$-norm.

$\|\mathbf{u}\| = |\nabla\mathbf{u}|$ is the Dirichlet norm.
$\mathbb{L}_{sym} = \left\{\mathbf{A} : A_{ij} \in L^2(\Omega), A_{ij} = A_{ji}\right\}$.
$\mathbb{H}^k_{sym} = \left\{\mathbf{A} : A_{ij} \in H^k(\Omega), A_{ij} = A_{ji}\right\}$.
$\mathcal{H} = \mathbb{H} \times \mathbb{L}_{sym}$.
$\mathbb{X}_k = \mathbb{H}^k(\Omega) \cap D(\mathcal{L})$, $\mathbb{X}_1 = \mathbb{H}^1(\Omega) \cap \mathbb{H}$ and $\mathbb{H}^0 = \mathbb{H}$.
We set

$$\|\mathbf{u}\|_{\mathbb{L}^p(0,T,\mathbb{H}^k)} = \left(\int_0^T \|\mathbf{u}(t)\|^p_{\mathbb{H}^k} dt\right)^{\frac{1}{p}}.$$

$\mathbb{L}^p(0,T;\mathbb{H}^k) = \{\mathbf{u} : [0,T] \longrightarrow \mathbb{H}^k(\Omega) : \|\mathbf{u}\|_{\mathbb{L}^p(0,T;\mathbb{H}^k)} < \infty\}$.

2. Constitutive equations

We consider a fluid that occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a regular boundary Γ, and a body force $\mathbf{b}$ per unit mass acting on the fluid. We will assume without loss of generality, that the fluid has mass density equal to unity.

The problem is then to find the velocity $\mathbf{u}(\mathbf{x},t)$, the pressure $p(\mathbf{x},t)$, and the orientation tensor $\mathbf{A}(\mathbf{x},t)$ which is solution of the set of constitutive equations (3)-(8), (for the linear closure, see [1,2,4,8,9] for more details).

$$\rho(\mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u}) - 2\mu_I\left(1 - \frac{2N_p}{35}\right)\Delta\mathbf{u} + \nabla p = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \ \text{ in } \Omega_T; \tag{3}$$

$$\mathbf{T} = \frac{2\mu_I}{7}\left[N(\mathbf{AD} + \mathbf{DA}) + N_p(\mathbf{A}:\mathbf{D})\mathbf{I}\right]; \tag{4}$$

$$\text{div}\mathbf{u} = 0 \ , \text{ in } \Omega_T; \tag{5}$$

$$\mathbf{A}' + (\mathbf{u}\cdot\nabla)\,\mathbf{A} = \mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W} + \frac{3\lambda}{7}\,(\mathbf{A}\mathbf{D} + \mathbf{D}\mathbf{A})$$

$$+\frac{4\lambda}{7}\mathbf{D} - \frac{2\lambda}{7}\,(\mathbf{A}:\mathbf{D})\,\mathbf{I} + D_r\,(\mathbf{I} - d\mathbf{A})\,, \text{ in } \Omega_T; \tag{6}$$

$$\text{tr}\mathbf{A} = 1, \text{ and } \ \mathbf{A}^T = \mathbf{A}, \quad \text{in } \Omega_T; \tag{7}$$

$$\mathbf{u}|_\Gamma = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \mathbf{A}(t=0) = \mathbf{A}_0, \mathbf{A}_0^T = \mathbf{A}_0, \text{ in } \Omega; \tag{8}$$

where for $T > 0$, $\Omega_T = \Omega \times [0,T]$, $p = p(\mathbf{x},t)$, is the hydrodynamic pressure, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the fluid velocity, $\mathbf{A} = \mathbf{A}(\mathbf{x},t)$ is the orientation tensor, $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ and $\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$, N_s and N_p are particle and shear number, respectively, $N = \frac{1}{7}(4N_p + 14N_s)$. $\lambda = \frac{r^2-1}{r^2+1}$, $r = \frac{l}{d}$, l and d, $(l \geq d)$ are the length and diameter of the particle respectively, λ, $0 \leq \lambda \leq 1$, describes the slenderness of the particle, if $\lambda = 0, (l = d)$, particle has the shape of a sphere, and if $\lambda = 1, (rarrow\infty)$, particle is an infinitely long cylinder. D_r is the rotary diffusivity due to Brownian motion. And $\mu_I = \mu(1 + hH)$ where μ is the solvent viscosity, h is the particle volume fraction, and H is a positive constant.

In lemma 3, we show that the constitutive equation (3)-(8) can be simplified further. It worth noting that lemma 3 is a generalisation of lemma 2.1 in [7].

Lemma 3. *If* $(\mathbf{u}, \mathbf{A})$ *satisfies* (3)-(8) *then* $\mathbf{A} : \mathbf{D} = 0$. *Moreover, if* $d = 2$, *then* $\mathbf{A}^{-1} : \mathbf{D} = 0$, *and if* $d = 3$, $\mathbf{A}^{-1}$: $\mathbf{D}$ *does not have a determined sign. For* $d = 2$, *if* $\mathbf{A}_0$ *is symmetric, and positive definite, so is* $\mathbf{A}(t)$.

Proof. We take the trace on both side of the evolution equation (6), to get

$$\begin{aligned} & \frac{d}{dt}tr\mathbf{A} + [\mathbf{u}\cdot\nabla]tr\mathbf{A} \\ = \ & tr(\mathbf{W}\mathbf{A}) - tr(\mathbf{A}\mathbf{W}) + \frac{3\lambda}{7}\,(tr(\mathbf{D}\mathbf{A}) + tr(\mathbf{A}\mathbf{D}) \\ & + \frac{2\lambda}{7}tr\mathbf{D} - \frac{2\lambda}{7}\,(\mathbf{A}:\mathbf{D}) + D_r d(1 - tr\mathbf{A}) \\ = \ & 2\mathbf{W}:\mathbf{A} + \frac{2\lambda}{7}tr\mathbf{D} + \frac{3\lambda}{7}\mathbf{A}:\mathbf{D} + D_r d(1 - tr\mathbf{A}). \end{aligned} \tag{9}$$

Since $\mathbf{A}$ is symmetric and $\mathbf{W}$ is antisymmetric so $\mathbf{W} : \mathbf{A} = 0$, and $tr\mathbf{D} = div\mathbf{u} = 0$. Hence returning to (9), one gets

$$\frac{d}{dt}tr\mathbf{A} + [\mathbf{u}\cdot\nabla]tr\mathbf{A} = \frac{3\lambda}{7}\mathbf{A}:\mathbf{D} + D_r d(1 - tr\mathbf{A}). \tag{10}$$

If $tr\mathbf{A} = 1$, then from (10) it is clear that $\mathbf{A} : \mathbf{D} = 0$.

Reciprocally, assume that $\mathbf{A} : \mathbf{D} = 0$, (10) implies that

$$\frac{d}{dt}tr\mathbf{A} + [\mathbf{u}\cdot\nabla]tr\mathbf{A} = dD_r(1 - tr\mathbf{A}).$$

From the initial condition on $\mathbf{A}$ (8), we obtain

$$tr\mathbf{A}(0) = tr\mathbf{A}_0 = 1\ . \tag{11}$$

Thus, we need to solve for $z = tr\mathbf{A}$ that satisfies

$$z_t + [\mathbf{u}\cdot\nabla]z + dD_r(z-1) = 0, \tag{12}$$

$$z\,(0) = 1. \tag{13}$$

One can easily check that $z(t) = tr\mathbf{A}(t) = 1$ solves (12)-(13).

If z, y are two solutions of (12)-(13) with $z(0) = y(0) = 1$, then $w = z - y$ satisfies

$$w_t + [\mathbf{u}\cdot\nabla]w + dD_r w = 0 \tag{14}$$

$$w(0) = 0. \tag{15}$$

Applying the Green's formula, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + d\int_\Omega D_r|w|^2 &= -\int_\Omega(\mathbf{u}\cdot\nabla)w\cdot w\\ &= -\frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla|w|^2\\ &= \frac{1}{2}\int_\Omega \mathrm{div}\mathbf{u}|w|^2 + \frac{1}{2}\int_\Gamma \mathbf{u}\cdot\mathbf{n}|w|^2 = 0.\end{aligned}$$

Integration of the above equation gives

$$\|w(t)\|^2 + 2d\int_0^t ds\int_\Omega D_r|w(s)|^2 = \|w(0)\|^2 = 0,$$

thus $w(t) = z(t) - y(t) = 0$. Hence, (12)-(13) has a unique solution.

We then conclude that $z(t) = tr\mathbf{A}(t) = 1$ is the unique solution of (12)-(13).

Finally, we need to show that if $d = 2$, $\mathbf{A} : \mathbf{D} = 0$, and if $\mathbf{A}$ is reversible then $\mathbf{A}^{-1} : \mathbf{D} = 0$.

1. Assume that $d = 2$, we express $\mathbf{A}$ and $\mathbf{D}$ in component form relative to the principal basis of $\mathbf{A}$. Given the definition and properties of $\mathbf{A}$, and $\mathbf{D}$, we have

$$\mathbf{A} : \mathbf{D} = (A_1 + A_2 - 1)\,D_1 = 0, \quad \text{since } A_1 + A_2 = 1. \tag{16}$$

And $\mathbf{A}^{-1} : \mathbf{D} = \frac{D_1}{A_1} - \frac{D_1}{1-A_1} = \frac{1}{A_1(1-A_1)}\,(1 - A_1 - A_1)\,D_1 = 0$.

2. Assume that $d = 3$, once again we express $\mathbf{A}$ and $\mathbf{D}$ in component form relative to the principal basis and we suppose that $A_1 > A_2 > A_3 > 0$.

$$\mathbf{A} : \mathbf{D} = 0 \quad \text{is equivalent to} \quad (A_1 - A_3)\,D_1 + (A_2 - A_3)\,D_2 = 0. \tag{17}$$

Since $A_1 - A_3 > 0, A_2 - A_3 > 0$, thus D_1 and D_2 are of opposite signs. And

$$\begin{aligned} \mathbf{A}^{-1} : \mathbf{D} &= \frac{1}{A_1} D_1 + \frac{1}{A_2} D_2 + \frac{1}{A_3} (-D_1 - D_2) \\ &= -K \left[A_2 D_1 (A_1 - A_3) + A_1 D_2 (A_2 - A_3) \right] \end{aligned} \tag{18}$$

where $K = \frac{1}{A_1 A_2 A_3} > 0$. From (17) we have $D_2 = -\frac{A_1 - A_3}{A_2 - A_3} D_1$, substituting into equation (18) yields

$$\mathbf{A}^{-1} : \mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) \tag{19}$$

If we assume that $D_1 > 0$ and $D_2 < 0$, we have $\mathbf{A}^{-1}$: $\mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) > 0$.

And if $D_1 < 0$ and $D_2 > 0$, from (19), $\mathbf{A}^{-1}$: $\mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) < 0$.

Finally, we have to show that $\mathbf{A}(t)$ is symmetric and positive definite if $\mathbf{A}_0$ is symmetric, and positive definite, which can be done following Lemma 2.1 in [3], or lemma 1 in [6]. □

Remark 4. (1) From lemma 3, and the statement (b) in lemma 1, it is obvious that the quantity $\int_\Omega tr[\mathbf{A} - \ln \mathbf{A} - \mathbf{I}]$ is well defined and always positive if $\mathbf{A}(t = 0)$ is symmetric and positive definite.

(2) The mapping $L^2_{sym}(\Omega)^{d \times d} \cap \{\text{positive semi definite}\} \ni \mathbf{A} \longrightarrow \int_\Omega tr[\mathbf{A} - \ln \mathbf{A} - \mathbf{I}] \in \mathbb{R}_+$ is not a norm.

(3) In the rest of this work, unless specified otherwise, we will assume that $n = 2$, i.e. we will work in two space dimensions.

With help of lemma 3 and Remark 4 (3), the problem (3)-(8) can be rewritten as

$$\rho(\mathbf{u}' + (\mathbf{u} \cdot \nabla)\mathbf{u}) - 2\mu_I \left(1 - \frac{2N_p}{35}\right) \Delta \mathbf{u} + \nabla p = \text{div}\mathbf{T} + \rho \mathbf{b}, \text{ in } \Omega_T; \tag{20}$$

$$\mathbf{T} = \frac{2\mu_I}{7} [N(\mathbf{AD} + \mathbf{DA})]; \tag{21}$$

$$\text{div}\mathbf{u} = 0 \text{ , in } \Omega_T; \tag{22}$$

$$\begin{aligned} &\mathbf{A}' + (\mathbf{u} \cdot \nabla)\mathbf{A} = \mathbf{WA} - \mathbf{AW} + \frac{3\lambda}{7}(\mathbf{AD} + \mathbf{DA}) \\ &+ \frac{4\lambda}{7}\mathbf{D} + D_r(\mathbf{I} - 2\mathbf{A}), \text{ in } \Omega_T; \end{aligned} \tag{23}$$

$$\text{tr}\mathbf{A} = 1, \text{ and } \mathbf{A}^T = \mathbf{A}, \quad \text{in } \Omega_T; \tag{24}$$

$$\mathbf{u}|_\Gamma = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \mathbf{A}(t = 0) = \mathbf{A}_0, \mathbf{A}_0^T = \mathbf{A}_0, \text{ in } \Omega. \tag{25}$$

3. Continuous dependence of solutions

In this section, a priori estimates for the continuous problem (3)-(8) are derived to prove that the solutions (3)-(8) are bounded for all time $t \in \mathbb{R}_+$. We also studied the dependence of the solution $(\mathbf{u}, \mathbf{A})$ with respect to the initial data $(\mathbf{u}_0, \mathbf{A}_0)$.

3.1 Boundedness of solution in time

We state first an important result which will help us to show the boundedness of the solution.

Lemma 5. *Let f be a non negative, absolutely continuous function satisfying the inequality*

$$f' + kf \leq \alpha\left(f^2 + f^3 + f^4\right) + \beta \tag{26}$$

where $k > 0$, $\alpha > 0$ and $\beta \geq 0$ are some constants. Let $M_0 > 0$, be the unique real solution of $f^5 + f^3 + f^2 + f - \frac{k}{2\alpha} = 0$, and $0 < M < M_0$. If $f(0) \leq M$, and $\beta \leq \frac{kM}{2}$, then $f(t)$ is bounded by M for all $t > 0$.

Proof. See [5,9]. □

Theorem 6. *If $N_p \leq \frac{35}{2}$, then the solution $(\mathbf{u}, \mathbf{A})$ of the problem (20)-(25) remains bounded for all time $t \in \mathbb{R}_+$, provided that the data $\mathbf{b} \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$, $(\mathbf{u}_0, \mathbf{A}_0) \in L^2(\Omega)^2 \times L^2_{sym}(\Omega)^{2\times 2}$, D_r, N_p and N_s are sufficiently small and we have*

$$E(t) \leq M, \quad \text{for all } t > 0, \tag{27}$$

where $Y = E(t) = \|\mathbf{u}\|^2 + \|\mathbf{A}\|^2$, and $0 < M < M_0$, and M_0 is the unique real solution of the equation

$$Y^5 + Y^3 + Y^2 + Y - \frac{\min\left\{\frac{1}{4N}\left(1 - \frac{2N_p}{35}\right), dD_r\right\}}{\frac{120\lambda N}{35 - 2N_p}} = 0 \tag{28}$$

Proof. First, we take the L^2 scalar product of the equation of (20) with $\mathbf{u}$, we obtain

$$\begin{aligned} \frac{7}{\mu_I N}\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{7}{N}\left(1 - \frac{2N_p}{35}\right)\|\mathbf{D}\|^2 &= -\left(\mathbf{AD} + \mathbf{DA}, \mathbf{D}\right) + \frac{7\rho}{2\mu_I N}\left(\mathbf{b}, \mathbf{u}\right) \\ &= -2\left(\mathbf{A}, \mathbf{D}_2\right) + \frac{7\rho}{2\mu_I N}\left(\mathbf{b}, \mathbf{u}\right), \end{aligned}$$

Since $(\mathbf{A}, \mathbf{D}_2) \geq 0$, we get

$$\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{2}{N}\left(1 - \frac{2N_p}{35}\right)\|\mathbf{D}\|^2 \leq 2\rho\left(\mathbf{b}, \mathbf{u}\right). \tag{29}$$

Now, we take the L^2 scalar pruduct of the equation of (23) with $\mathbf{A}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{A}\|^2 + 2D_r\|\mathbf{A}\|^2 &= \frac{6\lambda}{7}\left(\mathbf{A}_2 : \mathbf{D}\right) + \frac{4\lambda}{35}\left(\mathbf{A} : \mathbf{D}\right) + D_r\left(\mathbf{A} : \mathbf{I}\right) \\ &= \frac{6\lambda}{7}\left(\mathbf{A}_2 : \mathbf{D}\right) + \frac{4\lambda}{35}\left(\mathbf{A} : \mathbf{D}\right) + D_r \end{aligned} \tag{30}$$

We make use of Cauchy-Schwarz and Young's inequalities and we add (29) and (30), if $N_p \leq \frac{35}{2}$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)+k\left(\|\mathbf{A}\|^2+\|\mathbf{u}\|^2\right)\\&\leq 2\rho\left(\mathbf{b},\mathbf{u}\right)+\frac{6\lambda}{7}\left(\mathbf{A}_2:\mathbf{D}\right)+\frac{4\lambda}{35}\left(\mathbf{A}:\mathbf{D}\right)+D_r\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)+k\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)\\&\leq\left(\frac{60\lambda N}{35-2N_p}+1\right)\|\mathbf{A}\|^4+\frac{35N\|\mathbf{b}\|^2}{35-2N_p}+\left(\frac{16\lambda N}{35-2N_p}\right)^2+D_r\end{aligned}\tag{31}$$

where $k=\min\{\frac{7}{4N}(1-\frac{2N_p}{35}),dD_r\}$. We see from (31) that $E(t)=\|\mathbf{u}\|^2+\|\mathbf{A}\|^2$ satisfies (26) with $Y=E(t)=\|\mathbf{u}\|^2+\|\mathbf{A}\|^2$, $k=\min\{\frac{7}{4N}(1-\frac{2N_p}{35}),dD_r\}$, $\alpha=\frac{60\lambda N}{35-2N_p}+1$ and $\beta=\frac{35N\|\mathbf{b}\|^2}{35-2N_p}+(\frac{16\lambda N}{35-2N_p})^2+D_r$. The result follows from (5), if the data are small enough, i.e. if $E(t)\leq M$, and $D_r\leq\frac{M}{2}\min\{\frac{1}{4N}(1-\frac{2N_p}{35}),dD_r\}-\frac{35N\|\mathbf{b}\|^2}{35-2N_p}-(\frac{16\lambda N}{35-2N_p})^2$. □

3.2 Continuous dependence of solutions on initial data $\mathbf{u}_0,\mathbf{A}_0$

In this section, we prove the regular solutions of (20)-(25) are continuous dependent with respect initial data $(\mathbf{u}_{1,0},\mathbf{A}_{1,0})$, $(\mathbf{u}_{2,0},\mathbf{A}_{2,0})$, and the same body force $\mathbf{b}$.

Theorem 7. *For $d=2,3$, let $\mathbf{b}\in L^\infty(\mathbb{R}_+;L^2(\Omega)^d)$. For $i=1,2$, let $(\mathbf{u}^i,\mathbf{A}^i,p^i)$ be the regular solution of (20)-(25), associated with initial conditions $(\mathbf{u}_0^i,\mathbf{A}_0^i)$ respectively, as constructed in [4].*

If D_r is constant, and $N_p\leq 35/2$, then the difference $(\mathbf{u},\mathbf{A})=(\mathbf{u}_2-\mathbf{u}_1,\mathbf{A}_2-\mathbf{A}_1)$, depends continuously in the L^2-norm on the initial data $(\mathbf{u}_{2,0}-\mathbf{u}_{1,0},\mathbf{A}_{2,0}-\mathbf{A}_{1,0})$ in the following sense

$$E\left(t\right)=\|\mathbf{u}(t)\|^2+\|\mathbf{A}(t)\|^2\leq\left(\|\mathbf{u}_0\|^2+\|\mathbf{A}_0\|^2\right)\exp\left(-Kt\right),\tag{32}$$

where $K>0$ is a positive constant independent of the differences $\mathbf{u}_1-\mathbf{u}_2$, $\mathbf{A}_1-\mathbf{A}_2$, but depending on the parameters of the problem, and the domain Ω.

Proof. The difference $(\mathbf{u},\mathbf{A},p)=(\mathbf{u}_2-\mathbf{u}_1,\mathbf{A}_2-\mathbf{A}_1,p_2-p_1)$ satisfies the problem

$$\begin{cases}\mathbf{u}_t+[\mathbf{u}_2\cdot\nabla]\mathbf{u}+[\mathbf{u}\cdot\nabla]\mathbf{u}_1-2\mu_I\left(1-\frac{2N_p}{35}\right)\mathbf{D}+p\mathbf{I}-\operatorname{div}\left(\mathbf{T}_2-\mathbf{T}_1\right)=\mathbf{0},\\ \mathbf{A}_t+[\mathbf{u}_1\cdot\nabla]\mathbf{A}+[\mathbf{u}\cdot\nabla]\mathbf{A}_2=\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}+\mathbf{W}^2\mathbf{A}-\mathbf{A}\mathbf{W}^2\\ +\frac{4\lambda}{7}\mathbf{D}+\frac{3\lambda}{7}\left(\mathbf{A}\mathbf{D}_2+\mathbf{D}_2\mathbf{A}+\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1\right)-2D_r\mathbf{A},\\ \mathbf{T}_2-\mathbf{T}_1=\frac{2\mu_I N}{7}\left(\mathbf{A}\mathbf{D}_2+\mathbf{D}_2\mathbf{A}-\mathbf{A}\mathbf{D}_1-\mathbf{D}_1\mathbf{A}\right),\\ \operatorname{div}\mathbf{u}=0,\\ \mathbf{u}\left(\mathbf{x},0\right)=\mathbf{u}_0=\mathbf{u}_{2,0}-\mathbf{u}_{1,0},\qquad\mathbf{A}\left(\mathbf{x},0\right)=\mathbf{A}_0=\mathbf{A}_2-\mathbf{A}_{1,0},\ \mathbf{u}|_\Gamma=\mathbf{0}.\end{cases}\tag{33}$$

We take the $\mathbb{L}^2$–inner product of the $\mathbf{u}$–equation and $\mathbf{A}$–equation of (33) with $\mathbf{u}$ and $\mathbf{A}$ respectively, and integrate over Ω, after using Green's formula's, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + 2\mu_I\left(1-\frac{2N_p}{35}\right)\|\mathbf{D}\|^2 \\ &= -\frac{2\mu_I N}{7}\int_\Omega \mathbf{A}:(\mathbf{D}\mathbf{D}^2-\mathbf{D}_1\mathbf{D}) - \int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{u}_1\cdot\mathbf{u}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{A}\|^2+2D_r\|\mathbf{A}\|^2 &= -\int_\Omega ((\mathbf{u}\cdot\nabla)\,\mathbf{A}_2:\mathbf{A}) + \int_\Omega ((\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}):\mathbf{A}) \\ &\quad + \int_\Omega \left((\mathbf{W}^2\mathbf{A}-\mathbf{A}\mathbf{W}^2):\mathbf{A}\ \right) + \frac{4\lambda}{7}\int_\Omega \mathbf{D}:\mathbf{A} \\ &\quad + \frac{6\lambda}{7}\int_\Omega \mathbf{D}_2:[\mathbf{A}]^2 + \frac{3\lambda}{7}\int_\Omega [\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1]:\mathbf{A}. \end{aligned} \tag{35}$$

Since

$$\int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{A}_2:\mathbf{A} \le \left|\int_\Omega u_i\frac{\partial A^2_{ij}}{\partial x_k}A_{jk}\right| \le 2\|\nabla\mathbf{A}_2\|\,(\|\mathbf{A}_1\|_{L^\infty}+\|\mathbf{A}_2\|_{L^\infty})\,\|\mathbf{u}\|, \tag{36}$$

$$\begin{aligned} \int_\Omega [\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}]:\mathbf{A} &\le \|\mathbf{W}\mathbf{A}_1+\mathbf{A}_1\mathbf{W}\|\|\mathbf{A}\| \\ &\le C_0(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)\|\mathbf{A}_1\|_2\|\mathbf{A}\|, \end{aligned} \tag{37}$$

$$\int_\Omega \mathbf{D}:\mathbf{A} \le (\|\mathbf{u}_2\|_2+\|\mathbf{u}_1\|_2)\|\mathbf{A}\|, \tag{38}$$

$$\int_\Omega \mathbf{D}_2:[\mathbf{A}]^2 \le \|\nabla\mathbf{u}_2\|_{L^\infty}\|\mathbf{A}\|^2, \tag{39}$$

$$\begin{aligned} \int_\Omega [\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1]:\mathbf{A} &\le \|\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1\|\|\mathbf{A}\| \le C_0(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)\|\mathbf{A}_1\|_2\|\mathbf{A}\|, \end{aligned} \tag{40}$$

$$\int_\Omega \mathbf{A}:(\mathbf{D}\mathbf{D}^2-\mathbf{D}_1\mathbf{D}) \le \|\mathbf{A}\|\|\mathbf{D}\mathbf{D}^2+\mathbf{D}_1\mathbf{D}\| \le C_0\|\mathbf{A}\|(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)^2, \tag{41}$$

$$\begin{aligned} \int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{u}_1\cdot\mathbf{u} &\le \left|\int_\Omega u_iu_j\frac{\partial u^1_i}{\partial x_j}dx\right| \le \|\nabla\mathbf{u}_1\|_{L^\infty}\sum_{i,j=1}^{d}\int_\Omega |u_iu_j|dx \\ &\le 2\|\nabla\mathbf{u}_1\|_{L^\infty}\|\mathbf{u}\|^2. \end{aligned} \tag{42}$$

We use the estimates (36)-(42) in (34) and (35) to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right) + \left(2D_r - \frac{6C_0\lambda}{7}\|\nabla\mathbf{u}_2\|_{L^\infty}\right)\|\mathbf{A}\|^2$$

$$
\begin{aligned}
&+ 2\mu_I \left(1 - \frac{2N_p}{35}\right) \|\mathbf{D}\|^2 - 2\|\nabla \mathbf{u}_1\|_{L^\infty} \|\mathbf{u}\|^2 \\
&\leq \left(\frac{2\mu_I N}{7} C_0 \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)^2 + \frac{3\lambda C_0}{7} \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right) \|\mathbf{A}_1\|_2\right) \|\mathbf{A}\| \\
&+ \left(\frac{4\lambda}{7} \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\right) \|\mathbf{A}\| + 2 \|\nabla \mathbf{A}_2\| \left(\|\mathbf{A}_1\|_{L^\infty} + \|\mathbf{A}_2\|_{L^\infty}\right) \|\mathbf{u}\|. \qquad (43)
\end{aligned}
$$

Now, applying young inequality and Korn inequalities and assuming $N_p \leq 35/2$, (43) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) + \left(2D_r - \frac{6C_0\lambda}{7}\|\nabla \mathbf{u}_2\|_{L^\infty} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right)\|\mathbf{A}\|^2 \\
&+ \left(2\mu_I\left(1 - \frac{2N_p}{35}\right) + 2\|\nabla \mathbf{u}_1\|_{L^\infty} - \frac{\varepsilon_1}{2}\right)\|\mathbf{u}\|^2 \\
\leq\ & \frac{1}{2\varepsilon_2}\left(\frac{2\mu_I N}{7} C_0 \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)^2 + \frac{3\lambda C_0}{7}\left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\|\mathbf{A}_1\|_2\right)^2 \\
&+ \frac{1}{2\varepsilon_3}\left(\frac{4\lambda}{7}\left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\right)^2 + \frac{1}{\varepsilon_1}\|\nabla \mathbf{A}_2\|^2 \left(\|\mathbf{A}_1\|_{L^\infty} + \|\mathbf{A}_2\|_{L^\infty}\right)^2. \qquad (44)
\end{aligned}
$$

So from Agmon inequality [10], one has

$$
\|\nabla \mathbf{u}^i(t)\|_{L^\infty} \leq C\|\mathbf{u}^i(t)\|_3 \leq K_0 \quad , \quad \|\mathbf{A}^i(t)\|_{L^\infty} \leq C\|\mathbf{A}^i(t)\|_2 \leq K_0. \qquad (45)
$$

Next, we take $\epsilon_i > 0$ such that

$$
\begin{aligned}
&2D_r - \frac{6C_0\lambda}{7}\|\nabla \mathbf{u}_2\|_{L^\infty} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} \geq 0, \\
&2\mu_I\left(1 - \frac{2N_p}{35}\right) + 2\|\nabla \mathbf{u}_1\|_{L^\infty} - \frac{\varepsilon_1}{2} \geq 0, \qquad (46)
\end{aligned}
$$

Using (45) and (46), equation (44) becomes

$$
\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) - K_1\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) \leq K_2. \qquad (47)
$$

Finally, from Gronwall's lemma, (47) implies (32) thus the desired result. □

4. Time discrete scheme solutions

In this section, we present a time discrete scheme for the problem (20)-(25) that replicates the energy laws obtained in Theorem 6, and the continuous L^2–dependence of the solution with respect to the initial condition proven in theorem 7. It is worth to note that not all scheme can replicate these results.

In section 3, it was shown that solving (3)-(8) is equivalent to solve (20)-(25). We would like to know whether this equivalence holds for the discrete in time formulation we introduce next.

4.1 Preliminaries results

For some $k > 0$ and $n \in \mathbb{N}$, we let $t_n = nk$, and $u_k^n \approx u(t_n)$. Suppose that at a step $n \in \mathbb{N}$, $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ is known, where $\mathbf{A}_k^n$ is symmetric, positive semi-definite with $\mathrm{tr}\mathbf{A}_k^n = 1$. Then we seek the triplet $(\mathbf{u}_k^{n+1}, \mathbf{A}_k^{n+1}, p_k^{n+1})$ such that

$$\rho\frac{\mathbf{u}_k^{n+1}-\mathbf{u}_k^n}{k} + [\mathbf{u}_k^{n+1}\cdot\nabla]\mathbf{u}_k^{n+1} - 2\mu_I\left(1-\frac{2N_p}{35}\right)\Delta\mathbf{u}^{n+1} + p_k^{n+1}I +$$

$$= \mathrm{div}\,\widetilde{\mathbf{T}} + \rho\mathbf{b}_k^{n+1} \tag{48}$$

$$\mathrm{div}\,\mathbf{u}_k^{n+1} = 0 \text{ in } \Omega_T, \tag{49}$$

$$\widetilde{\mathbf{T}} = \frac{2\mu_I}{7}\left(N\left(\mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1}\right) + N_p\left(\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1}\right)\mathbf{I}\right), \tag{50}$$

$$\frac{\mathbf{A}_k^{n+1}-\mathbf{A}_k^n}{k} + (\mathbf{u}_k^{n+1}\cdot\nabla)\mathbf{A}_k^{n+1} = \mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}$$

$$+\frac{3\lambda}{7}\left(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}\right) + \frac{3\lambda}{7}\mathbf{D}_k^{n+1}$$

$$-\frac{2\lambda}{7}\left(\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1}\right)\mathbf{I} + D_r(I - d\mathbf{A}_k^{n+1}), \text{ in } \Omega_T \tag{51}$$

$$[\mathbf{A}_k^{n+1}]^T = \mathbf{A}_k^{n+1}, \text{ and } \mathrm{tr}\mathbf{A}_k^{n+1} = 1 \tag{52}$$

$$\mathbf{u}_k^{n+1}|_\Gamma = 0\ , \quad \mathbf{u}^0 = \mathbf{u}_0, \quad \mathbf{A}^0 = \mathbf{A}_0\ , \text{ in } \Omega; \tag{53}$$

where $\mathbf{b}_k^{n+1} = \frac{1}{k}\int_{nk}^{(n+1)k}\mathbf{b}(t)\mathrm{d}t$.

As for the continuous model, one has

Lemma 8. *Let $(\mathbf{A}_k^{n+1}, \mathbf{u}_k^{n+1})$ be the solution of (48)-(53), with D_r constant. Then*

$$\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1} = 0 \ \textit{ for all } n \quad \textit{if and only if} \quad tr\mathbf{A}_k^{n+1} = 1.$$

Moreover for $d = 2$, $[\mathbf{A}_k^{n+1}]^{-1} : \mathbf{D}_k^{n+1} = 0$ if $\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1} = 0$.

Proof. First, we take the trace on both sides of the equation (51), we have

$$\frac{\mathrm{tr}\mathbf{A}_k^{n+1} - \mathrm{tr}\mathbf{A}_k^n}{k} + \mathbf{u}_k^{n+1}\cdot\nabla\mathrm{tr}\mathbf{A}_k^{n+1} = \frac{3\lambda}{7}\mathbf{D}^{n+1} : \mathbf{A}_k^{n+1} + D_r d(1 - \mathrm{tr}\mathbf{A}_k^{n+1})$$

Since $\mathrm{tr}\mathbf{A}_k^n = 1$, if $\mathrm{tr}\mathbf{A}_k^{n+1} = 1$, we obtain $\mathbf{D}_k^{n+1} : \mathbf{A}_k^{n+1} = 0$.

Conversely, if $\mathbf{D}_k^{n+1} : \mathbf{A}_k^{n+1} = 0$, and $\mathrm{tr}\mathbf{A}_k^m = 1$ for $m \le n$, then

$$\frac{\mathrm{tr}\mathbf{A}_k^{n+1} - 1}{k} + \mathbf{u}_k^{n+1}\cdot\nabla\mathrm{tr}\mathbf{A}_k^{n+1} = dD_r(1 - \mathrm{tr}\mathbf{A}_k^{n+1}), \tag{54}$$

$$\mathrm{tr}\mathbf{A}^0 = \mathrm{tr}\mathbf{A}_0 = 1. \tag{55}$$

First, one can easily check that $\mathrm{tr}\mathbf{A}_k^{n+1} = 1$ is a solution of (54)-(55).

Next, with Lax-Milgram Theorem, one can observe that (54) has a unique solution.

Thus (54)-(55) admits a unique solution, that is $\mathrm{tr}\mathbf{A}^{n+1} = 1$.

As in Lemma 3, one can show that if $\mathbf{A}_k^n : \mathbf{D}_k^n = 0$, then $[\mathbf{A}_k^{n+1}]^{-1} : \mathbf{D}_k^{n+1} = 0$ for $d = 2$.

Thus we can reformulate the problem (48)-(53) as follows: suppose $d = 2$, and at step n, $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ is known and $\mathbf{A}_k^n$ is symmetric, positive semi–definite with $\mathrm{tr}\mathbf{A}_k^n = 1$. We are looking for the triplet $(\mathbf{u}^{n+1}, \mathbf{A}^{n+1}, p^{n+1})$ such that

$$\frac{\mathbf{u}_k^{n+1} - \mathbf{u}_k^n}{k} + [\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{u}_k^{n+1} - 2\mu_I \left(1 - \frac{2N_p}{35}\right) \Delta \mathbf{u}_k^{n+1} + \nabla p_k^{n+1}$$

$$= \mathrm{div}\tilde{\mathbf{T}} + \mathbf{b}_k^{n+1} \tag{56}$$

$$\mathrm{div}\, \mathbf{u}_k^{n+1} = 0 \tag{57}$$

$$\widetilde{\mathbf{T}} = \frac{2\mu_I}{7} N \left(\mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} \right); \tag{58}$$

$$\frac{\mathbf{A}_k^{n+1} - \mathbf{A}_k^n}{k} + (\mathbf{u}_k^{n+1} \cdot \nabla)\mathbf{A}_k^{n+1} = \mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}$$

$$+ \frac{3\lambda}{7} \left(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1} \right) + \frac{4\lambda}{7}\mathbf{D}_k^{k+1} + D_r(I - d\mathbf{A}_k^{n+1}), \tag{59}$$

$$\left(\mathbf{A}_k^{n+1}\right)^T = \mathbf{A}_k^{n+1}, \quad \text{and} \quad \mathrm{tr}\mathbf{A}_k^{n+1} = 1; \tag{60}$$

$$\mathbf{u}_k^{n+1}|_\Gamma = 0 \text{ , and } \mathbf{u}^0 = \mathbf{u}_0, \quad \mathbf{A}^0 = \mathbf{A}_0 \text{ .} \tag{61}$$

4.2 Discrete in time a priori estimates

The existence of solutions of (56)-(61)(respectively (48)-(53)) follows from [8, see Section 5], the reader can also consult [5]. However, it should be mentioned that the uniqueness of solutions is achieved only for the time step k very small.

The main result can be stated as follows

Theorem 9. *Let $\mathbf{b} \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$, $(\mathbf{u}_0, \mathbf{A}_0) \in L^2(\Omega)^2 \times L^2_{sym}(\Omega)^{2\times 2}$, with $tr\mathbf{A}_0 = 1$, positive semi-definite. If moreover, $n = 2, D_r$ is constant, and $N_p \leq 35/2$. Then, the sequence of solutions $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ of the system (56)-(61) remains bounded for all time in the following sense*

$$E_k^n \leq (1 + k\min(C_2, D_r))^{-n} E^0 + C_4 \|\mathbf{b}\|_\infty^2 \left[1 - (1 + k\min(C_2, D_r))^{-n}\right]$$

$$+ D_r C_5 \left[1 - (1 + k\min(C_2, D_r))^{-n}\right], \tag{62}$$

where C_2, C_4, C_5 are positive constant, and

$$E_k^n = \|\mathbf{u}_k^n\|^2 + \int_\Omega tr[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] \tag{63}$$

is the discrete counterpart of $E(t)$ defined in (32).

Proof. The existence of the solution $(\mathbf{u}^{n+1}, \mathbf{A}^{n+1}, p^{n+1})$ to the the problem (56)-(61) can be proven by following steps similar to [7, Theorem 3]. We take the L^2-scalar product of the equation of (56) with $2k\mathbf{u}_k^{n+1}$, and using the relation

$$2(\mathbf{u} - \mathbf{v}, \mathbf{u}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2, \quad \text{for all} \quad \mathbf{v}, \mathbf{u} \in L^2(\Omega) \tag{64}$$

the skew property of the convective term, we obtain

$$\begin{aligned}&\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla \mathbf{u}_k^{n+1}\|^2 \\ &+\frac{8\mu_I N}{7} k\underbrace{(\mathbf{A}_k^{n+1}, [\mathbf{D}_k^{n+1}]^2)}_{\geq 0} = 2k(\mathbf{b}_k^{n+1}, \mathbf{u}_k^{n+1}).\end{aligned} \tag{65}$$

It follow that

$$\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla \mathbf{u}_k^{n+1}\|^2 \leq 2k(\mathbf{b}_k^{n+1}, \mathbf{u}_k^{n+1}).$$

Next, we take the L^2-scalar product of the equation of (59) with $k(\mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1})$, gives

$$\begin{aligned}&\left(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right) + k\left([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right) \\ =\ & k\left(\mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right) \\ &+\frac{3k\lambda}{7}\left(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right) \\ &-\frac{4\lambda k}{7}\left(\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right) + D_r k\left(I - d\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\right).\end{aligned} \tag{66}$$

We now need to analyze the terms appearing in (66). First,

$$\begin{aligned}(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) &= \int_\Omega \left[\mathrm{tr}\mathbf{A}_k^{n+1} - \mathrm{tr}\mathbf{A}_k^n\right] \\ &\quad - \int_\Omega (\mathbf{A}_k^{n+1} - \mathbf{A}_k^n) : [\mathbf{A}_k^{n+1}]^{-1}.\end{aligned} \tag{67}$$

The second term in the right hand side of (67) can be treated with Lemma 1, as follows

$$\begin{aligned}-\left(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, [\mathbf{A}_k^{n+1}]^{-1}\right) &= \int_\Omega \mathrm{tr}\left((\mathbf{A}_k^n - \mathbf{A}_k^{n+1})[\mathbf{A}_k^{n+1}]^{-1}\right) \\ &= \int_\Omega \mathrm{tr}\left(\mathbf{A}_k^n[\mathbf{A}_k^{n+1}]^{-1} - \mathbf{I}\right) \\ &\geq \int_\Omega \ln\det\left(\mathbf{A}_k^n\left[\mathbf{A}_k^{n+1}\right]^{-1}\right) \\ &= \int_\Omega \mathrm{tr}\ln[\mathbf{A}_k^n[\mathbf{A}_k^{n+1}]^{-1}] \\ &= \int_\Omega \mathrm{tr}\ln\mathbf{A}_k^n + \mathrm{tr}\ln[\mathbf{A}_k^{n+1}]^{-1} \\ &= \int_\Omega \left[\mathrm{tr}\ln\mathbf{A}_k^n - \mathrm{tr}\ln\mathbf{A}_k^{n+1}\right].\end{aligned} \tag{68}$$

Thus combining (67) and (68), we obtain

$$(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) \geq \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1}] - \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^n].$$

From the divergence theorem and incompressibility condition, we get

$$\begin{aligned}
&\Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= \int_\Omega \mathbf{u}_k^{n+1} \cdot \nabla \mathrm{tr}\mathbf{A}_k^{n+1} - \Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= -\Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= -\int_\Omega \big(\mathbf{u}_k^{n+1} \cdot \nabla\big) \mathrm{tr} \ln \mathbf{A}_k^{n+1} \\
&= \int_\Omega \mathrm{div}\mathbf{u}_k^{n+1} \mathrm{tr} \ln \mathbf{A}_k^{n+1} - \int_\Gamma \mathbf{u}_k^{n+1} \cdot \mathbf{n} \mathrm{tr} \ln \mathbf{A}_k^{n+1} \\
&= 0. \qquad (69)
\end{aligned}$$

Next, a direct computation gives

$$(\mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0,$$

and from Lemma 8, and incompressibility condition

$$(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0, \qquad (70)$$

$$(\mathcal{A}\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0. \qquad (71)$$

Now, returning to (66) with (67)—(71), and adding and subtracting $\mathbf{I}$, we obtain

$$\begin{aligned}
&\int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] - \int_\Omega \mathrm{tr}[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] \\
\leq\ & D_r k(I - 2\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) \\
=\ & -D_r k \int_\Omega \mathrm{tr}[[\mathbf{A}_k^{n+1}]^{-1} + 2\mathbf{A}_k^{n+1} - 3\mathbf{I}] \\
=\ & -D_r k \int_\Omega \mathrm{tr}[[\mathbf{A}_k^{n+1}]^{-1} + \mathbf{A}_k^{n+1} - 2\mathbf{I}] - D_r k \int_\Omega \mathrm{tr}\mathbf{A}_k^{n+1} - D_r k \int_\Omega \mathrm{tr}\mathbf{I} \\
=\ & -D_r k \int_\Omega \mathrm{tr}\Big([\mathbf{A}_k^{n+1}]^{-1} + \mathbf{A}_k^{n+1} - 2\mathbf{I}\Big) - 3D_r k,
\end{aligned}$$

which with Lemma 1 part (c) gives

$$\begin{aligned}
&\int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] + D_r k \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] \\
\leq\ & \int_\Omega \mathrm{tr}[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] - 3D_r k. \qquad (72)
\end{aligned}$$

Putting together (72) and (63), we obtain

$$E_k^{n+1} + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla \mathbf{u}_k^{n+1}\|^2 + D_r k\int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}]$$
$$\leq E_k^n + 2k(\mathbf{f}_k^{n+1}, \mathbf{u}_k^{n+1}) - 3D_r k, \tag{73}$$

which with Cauchy-Schwarz inequality, and assuming that $N_p \leq 35/2$ yields

$$E_k^{n+1} \leq \frac{1}{\alpha}E_k^n + \frac{kC_3}{\alpha}\|\mathbf{f}_k^{n+1}\|^2 + \frac{3D_r k}{\alpha}, \quad \text{with} \quad \alpha = 1 + k\min(C_2, D_r). \tag{74}$$

Using (74) recursively, we find

$$\begin{aligned} E_k^n &\leq \frac{1}{\alpha^n}E^0 + kC_3\sum_{i=1}^{n}\frac{1}{\alpha^i}\|\mathbf{f}_k^{n+1-i}\|^2 + 3D_r k\sum_{i=0}^{n}\frac{1}{\alpha^i} \\ &\leq (1 + k\min(C_2, D_r))^{-n}E^0 + C_4\|\mathbf{f}\|_\infty^2\left[1 - (1 + k\min(C_2, D_r))^{-n}\right] \\ &\quad + D_r C_5\left[1 - (1 + k\min(C_2, D_r))^{-n}\right], \end{aligned} \tag{75}$$

which is the desired result. □

Remark 10. (a) Theorem 9 is the discrete counterpart of theorem 6, and no additional conditions are imposed on the data.

(b) From the estimate obtained in theorem 9,

$$\limsup_{n\to\infty} E_k^n \leq C_4\|\mathbf{f}\|_\infty^2 + D_r C_5.$$

Corollary 11. *Under the assumptions of theorem 9, and* $0 < k < \frac{1}{\min(C_2, D_r)}$, *one one obtains*

$$E_k^n \leq \exp\left(-\frac{n}{2}\,k\min(C_2, D_r)\right)E^0 + C_4\|\mathbf{f}\|_\infty^2 + D_r C_5. \tag{76}$$

The inequality (76) is deduce from (75) by observing $\exp(x/2) \leq 1 + x$ for $0 < x < 1$.

4.3 Continuous dependence

Let $(\mathbf{u}_k^{n+1}, \mathbf{A}_k^{n+1})$ be the regular solution of (56)-(61), with external force $\mathbf{f}_k^{n+1}$, and initial condition $(\mathbf{u}_0, \mathbf{A}_0)$.

Let $(\mathbf{v}_k^{n+1}, \mathbf{B}_k^{n+1})$ be the regular solution of (56)-(61), with external force $\mathbf{f}_k^{n+1}$, and initial condition $(\mathbf{v}_0, \mathbf{B}_0)$.

Theorem 12. *For* $d = 2, 3$. *If* $N_P \leq 35/2$ *and* D_r *constant. Then one can find a positive constant* $K > 1$ *such that for all* $n \geq 0$

$$\|\mathbf{u}_k^{n+1} - \mathbf{v}_k^{n+1}\|^2 + \|\mathbf{A}_k^{n+1} - \mathbf{B}_k^{n+1}\|^2 \leq K\left[\|\mathbf{u}_k^n - \mathbf{v}_k^n\|^2 + \|\mathbf{A}_k^n - \mathbf{B}_k^n\|^2\right]. \tag{77}$$

For the proof of theorem 12, it suffice to:

(a) write the equations satisfied by the difference $(\mathbf{u}_k^{n+1} - \mathbf{v}_k^{n+1}, \mathbf{A}_k^{n+1} - \mathbf{B}_k^{n+1})$.

(b) proceed as in the proof of theorem 7.

Remark 13. Using (77) recursively, we find

$$\|\mathbf{u}_k^n - \mathbf{v}_k^n\|^2 + \|\mathbf{A}_k^n - \mathbf{B}_k^n\|^2 \leq K^n \left[\|\mathbf{u}_0 - \mathbf{v}_0\|^2 + \|\mathbf{A}_0 - \mathbf{B}_0\|^2\right].$$

5. Instability of the rest state

We assume that $\mathbf{f} = \mathbf{0}$. In this section, we want to study the instability in the sense of Lyapounov of the rest state for the time discrete problem (56)-(53). We recall that in [4], the rest state $(\mathbf{0}, \overline{\mathbf{A}})$ has been shown to be unstable for the problem (20)-(25) provided that $N_p \geq 35/2$ and $\overline{\mathbf{A}} = e_1 \otimes e_1$, where e_i is the usual orthonormal basis. We want to show that the problem (56)-(61) inherited the later property.

Theorem 14. *Let $\mathbf{f} = \mathbf{0}, D_r = 0$, $\mathbf{u}_0 \in L^2(\Omega)^d$. If $N_p > 35/2$, with the time step k sufficiently small, then the rest state $\overline{\mathbf{A}} = \mathbf{e}_1 \otimes \mathbf{e}_1$ is unstable.*

Proof. We let $\mathbf{u} = u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, and take $\mathbf{A} = \mathbf{a} + \overline{\mathbf{A}}$, where $\mathbf{a}^T = \mathbf{a}$, tr$\mathbf{a} = 0$, and satisfy (48)-(53). It is easy to see that $\overline{\mathbf{A}}\mathbf{D} = 0$.

Taking the scalar product of the system (56) with $2k\mathbf{u}_k^{n+1}$ and using (64), we obtain

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 4\mu_I k\|\mathbf{D}_k^{n+1}\|^2 + 4\mu_I k N_p(\mathcal{A}\mathbf{D}_k^{n+1}, \mathbf{D}_k^{n+1}) \\
&+8\mu_I N_s k(\mathbf{a}^{n+1}, [\mathbf{D}_k^{n+1}]^2) = \mathbf{0} \\
&\mathcal{A}\mathbf{D}_k^{n+1} = -\frac{2}{35}\mathbf{D}_k^{n+1} + \frac{2}{7}\left[\mathbf{a}^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{a}^{n+1}\right].
\end{aligned}$$

together with (56) gives

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla\mathbf{u}_k^{n+1}\|^2 \\
= \quad &-8\mu_I k\left(\frac{N_p}{7} + N_s\right)(\mathbf{a}^{n+1}, [\mathbf{D}_k^{n+1}]^2) + \|\mathbf{u}_k^n\|^2.
\end{aligned}$$

For each entry, $a_{i,j}^n$ we let

$$a_{i,j}^n = \frac{1}{k}\int_{(n-1)k}^{nk} a_{i,j}(t)dt.$$

Thus from Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla\mathbf{u}_k^{n+1}\|^2 \\
&+4\mu_I k\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\|\nabla\mathbf{u}_k^{n+1}\|^2 \geq \|\mathbf{u}_k^n\|^2
\end{aligned}$$

which is re-written as

$$\begin{aligned}&\|\mathbf{u}_k^{n+1}\|^2+\|\mathbf{u}_k^{n+1}-\mathbf{u}_k^n\|^2\\&+2\mu_I k\left[\left(1-\frac{2N_p}{35}\right)+2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]\|\nabla\mathbf{u}_k^{n+1}\|^2\geq\|\mathbf{u}_k^n\|^2.\end{aligned}\tag{78}$$

Clearly, we want $(1-\frac{2N_p}{35})+2(\frac{N_p}{7}+N_s)\|\mathbf{a}^{n+1}\|_\infty\leq 0$, which possible if we take $\mathbf{a}$ such that

$$2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\leq-\left(1-\frac{2N_p}{35}\right)$$

because $\|\mathbf{a}\|_\infty$ is always positive, we should take $1-\frac{2N_p}{35}<0$, that is $N_p>35/2$. With the above restrictions, and dropping some positive terms, one obtains

$$\left[1+2\mu_I k\left[\left(1-\frac{2N_p}{35}\right)+2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]C_p\right]\|\mathbf{u}_k^{n+1}\|^2\geq\|\mathbf{u}_k^n\|^2.\tag{79}$$

We now take the step size k such[1] that

$$1+2\mu_I k\left[\left(1-\frac{2N_p}{35}\right)+2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]C_p\geq 0.$$

Combining (78) and (79), one gets

$$\|\mathbf{u}_k^{n+1}\|^2\geq\left[1+2\mu_I k\left[\left(1-\frac{2N_p}{35}\right)+2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]C_p\right]^{-1}\|\mathbf{u}_k^n\|^2.\tag{80}$$

Using (80) recursively, one obtains

$$\begin{aligned}\|\mathbf{u}_k^n\|^2\geq&\left[1+2\mu_I k\left[\left(1-\frac{2N_p}{35}\right)\right.\right.\\&\left.\left.+2\left(\frac{N_p}{7}+N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]C_p\right]^{-n}\|\mathbf{u}^0\|^2, n=1,2,3...\end{aligned}\tag{81}$$

From (81), one sees that $[1+2\mu_I k[(1-\frac{2N_p}{35})+2(\frac{N_p}{7}+N_s)\|\mathbf{a}^{n+1}\|_\infty]C_p]<1$, therefore (81) implies that

$$\limsup_{n\longrightarrow\infty}\|\mathbf{u}_k^n\|^2=\infty,$$

which ends the proof. □

1. this choice is not restrictive, since for the existence and uniqueness, we need k to be small enough

6. Concluding remarks

This study generalises the results obtained in [4,8,9], by proving instability results for the rest state not only for the velocity but also for the orientation tensor and assuming no zero body forces. It was proved that both the regular solutions and a time discrete solution dependent continuously on the initial data, furthermore, restrictions inposed on the data confirm the results found elsewhere: the rest state is unstable if particle number $N_p > 35/2$. It is important to mention that not all schemes can replicate the results obtained here For $N_p > 35/2$, the system is not mechanically acceptable, thus it is not surprising that instability results are obtained for N_p in that range. The instability of the rest state is shown irrespective of dimension which was not the case for the stability for the moment.

References

[1] S. G. Advani, *Closure approximations for three-dimensional structure tensors*, Journal of Rheology, 34 (1990), 1990.

[2] S. G. Advani, C. L. Tucker, *The use of tensors to describe and predict fiber orientation in short fiber composites*, Journal of Rheology, 31 (1987), 751-784.

[3] S. Boyaval, T. Lelièvre, C. Mangoubi, *Free-energy-dissipative schemes for the Oldroyd-B model*, ESAIM: Mathematical Modelling and Numerical Analysis, 43 (2009), 523-561.

[4] G. P. Galdi, B. Reddy, *Well-posedness of the problem of fiber suspension ows*, Journal of Non-Newtonian Fluid Mechanics, 83 (1999), 205-230.

[5] C. Guillopé, J. Saut, *Existence results for the ow of viscoelastic uids with a differential constitutive law,* Nonlinear Analysis: Theory, Methods & Applications, 15 (1990), 849-869.

[6] D. Hu, T. Lelièvre, *New entropy estimates for Oldroyd-B and related models*, Communications in Mathematical Sciences, 5 (2007), 909-916.

[7] J. M. W. Munganga, *Existence and stability of solutions for steady flows of fibre suspension flows,* Journal of Mathematical Fluid Mechanics, 15 (2013), 197-214.

[8] J. M. W. Munganga, B. D. Reddy, *Local and global existence of solutions to equations for flows of fibre suspensions*, Mathematical Models and Methods in Applied Sciences, 12 (2002), 1177-1203.

[9] J. M. W. Munganga, B. D. Reddy, K. J. Diatezua, *Aspects of the thermodynamic stability of fibre suspension ows*, Journal of Non-Newtonian Fluid Mechanics, 92 (2000), 135-150.

[10] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 2nd edition, 1998.

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A note on a finite group with all non-nilpotent maximal subgroups being normal

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Abstract. In this paper we give an elementary proof to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable.
Keywords: non-nilpotent maximal subgroup, normal, solvable.

1. Introduction

It is known that one of the important characterizations of a nilpotent group is that a finite group G is nilpotent if and only if all maximal subgroups of G are normal. As a generalization of this result, it is interesting to characterize the finite group with all non-nilpotent maximal subgroups being normal. The second author, C. Zhang and S. Guo [4, Lemma 4] used a theorem of Ballester-Bolinshe and Shemetkov [1, Theorem 2] about the p-nilpotent group to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable. In [5] the second author used a theorem of Rose [3, Theorem 1] about the non-solvable group with a nilpotent maximal subgroup of even order to give two distinct proofs of the solvability of such a group.

In this paper, without using neither [1, Theorem 2] nor [3, Theorem 1], we give an elementary proof of the solvability of such a group.

Theorem 1.1. *A finite group G with all non-nilpotent maximal subgroups being normal is solvable.*

The following lemma is necessary in the proof of Theorem 1.1 which is given in Section 2.

Lemma 1.2 ([2, Theorem 9.1.10])**.** *Let the finite group G possess a nilpotent Hall $\pi-$subgroup H. Then all Hall $\pi-$subgroups of G are conjugate.*

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2. Proof of Theorem 1.1

Proof. Suppose that the theorem is not true. Let G be a counterexample of minimal order.

If all maximal subgroups of G are non-nilpotent. By the hypothesis of the theorem, one has that all maximal subgroups of G are normal. Then G is nilpotent, a contradiction. Thus G has nilpotent maximal subgroups. It is clear that G also must have non-nilpotent maximal subgroups. It follows that G is not a simple group.

Let N be a minimal normal subgroup of G. Since the hypothesis of the theorem holds for G/N and $|G/N| < |G|$, one has that G/N is solvable. It follows that N is non-solvable. In particular, G has no solvable non-trivial normal subgroup. Let L be a nilpotent maximal subgroup of G and P be any Sylow subgroup of L. If P is not a Sylow subgroup of G, then $L < N_G(P) \leq G$. One has $N_G(P) = G$ as L is maximal in G. It follows that G possesses a solvable non-trivial normal subgroup P, a contradiction. Then any Sylow subgroup of L is also a Sylow subgroup of G. Thus we have that every nilpotent maximal subgroup of G must be a nilpotent Hall subgroup of G.

Let L_1 be a nilpotent maximal subgroup of G which is a Hall π_1-subgroup of G for the set of prime divisors π_1 of $|G|$ and L_2 be a nilpotent maximal subgroup of G which is a Hall π_2-subgroup of G for the set of prime divisors π_2 of $|G|$. We will show that $\pi_1 = \pi_2$. Otherwise, assume $\pi_1 \neq \pi_2$. By [2, Theorem 10.4.2], one has $2 \in \pi_1$ and $2 \in \pi_2$. Let $R_1 \in \mathrm{Syl}_2(L_1)$ and $R_2 \in \mathrm{Syl}_2(L_2)$. Suppose $q \in \pi_1$ but $q \notin \pi_2$. Let $Q \in \mathrm{Syl}_q(L_1)$. Then $Q \leq N_G(R_1)$. One has $q \mid |N_G(R_1)|$. Since $L_2 \leq N_G(R_2)$ and G has no solvable non-trivial normal subgroup, one has $N_G(R_2) = L_2$. Note that $|N_G(R_1)| = |N_G(R_2)|$ since R_1 and R_2 are conjugate. It follows that $q \mid |L_2|$, a contradiction. Thus we have $\pi_1 = \pi_2$. That is, all nilpotent maximal subgroups of G are nilpotent Hall $\pi-$subgroups of G for a fixed π. By Lemma 1.2, we have that all nilpotent maximal subgroups of G are conjugate.

Let $\pi_e(N) = \{p_1, p_2, \cdots, p_s\}$ be the set of prime divisors of $|N|$. For every prime divisor p_i of $|N|$ with $1 \leq i \leq s$, let $P_i \in \mathrm{Syl}_{p_i}(N)$. By Frattini argument, one has $G = N_G(P_i)N$. Since G has no solvable non-trivial normal subgroup, one has $N_G(P_i) < G$. Note that every non-nilpotent maximal subgroup of G is normal in G and N is non-solvable. It follows that every non-nilpotent maximal subgroup of G must contain N. Thus $N_G(P_i)$ is contained in some nilpotent maximal subgroup of G.

Let M_i be a nilpotent maximal subgroup of G such that $N_G(P_i) \leq M_i$. Then $P_i \leq M_i$. Since all M_i are conjugate for $1 \leq i \leq s$, there exists a $g_i \in G$ such that ${M_i}^{g_i} = M_1$ for every $2 \leq i \leq s$. Thus ${P_i}^{g_i} \leq {M_i}^{g_i} = M_1$. Note that ${P_i}^{g_i} \leq N^{g_i} = N$ and N can be generated by all its Sylow subgroups, that is, $N =< P_1, {P_2}^{g_2}, \cdots, {P_s}^{g_s} >$. Then we have $N \leq M_1$. It follows that N is nilpotent, a contradiction.

So the counterexample of minimal order does not exist and then G is solvable. □

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References

[1] A. Ballester-Bolinshe and L.A. Shemetkov, *On normalizers of Sylow subgroups in finite groups,* Sib. Math. J., 40 (1999), 1-2.

[2] D.J.S. Robinson, *A Course in the theory of groups* (Second Edition), Springer-Verlag, New York, 1996.

[3] J.S. Rose, *On finite insoluble groups with nilpotent maximal subgroups,* J. Algebra, 48 (1977), 182-l96.

[4] J. Shi, C. Zhang and S. Guo, *A note on theorem of Shlyk* (Chinese), J. Guangxi Norm. Univ., Nat. Sci., 30 (2012), 22-24.

[5] J. Shi, *A note on a finite group in which all non-nilpotent maximal subgroups are normal* (Chinese), J. Yunnan Minzu Univ., Nat. Sci., 26 (2017), 290-291.

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Existence of tube solution for a fractional initial value problem

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Abstract. In this paper, we show existence of tube solution (generalization of lower and upper solution) for a new initial value problem of a fractional nonlinear differential equation of order $0 < \alpha < 1$ with the conformable fractional derivative. By applying a variety of tools including the Schauder fixed-point theorem, some new existence results are achieved.

Keywords: fractional differential equations, existence of solutions, conformable fractional derivative, initial value problem.

1. Introduction

Fractional calculus is one of the most interesting that attracts many investigators, specially mathematics and engineering sciences. Many of natural phenomena can be present by fractional differential equations that implies importance and location of the fractional calculus in the sciences and engineering. Also, factional differential equation is applicable in various fields such as chemical physics, fluid flows, electrical networks, visco-elasticity (see [1, 2, 3, 4]).

In 2014, Khalil *et al.* [7] presented a new well-behaved definition of fractional derivative called the conformable fractional derivative. For $0 < \alpha \leq 1$, this new definition coincides by the classical definitions on polynomials and if $\alpha = 1$, the definition coincides by the classical definition of first order derivative. In 2015, Abdeljawad [8] proceed on the conformable fractional derivative and some properties its such as chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace transforms are proposed.

Recently, some authors searched existence and uniqueness of solutions for fractional differential equation with conformable fractional derivative. For example, in 2015, Prasad and Krushna [12] discussed, by utilizing fixed point theorems on cone and under suitable conditions, the existence of multiple positive solutions for a coupled system of iterative type boundary value problems,

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involving new conformable fractional order derivative. Moreover, existence and uniqueness theorem for sequential linear conformable fractional differential equations is proved in [11]. Asawasamrit *et al.* [13] showed the existence of solutions for periodic boundary value problems for impulsive fractional integro-differential equations by concept of conformable fractional derivative and the existence of solutions verified by using the method of lower and upper solutions. In 2015, Batarfi *et al.* [10] studied the existence and uniqueness of solutions for a class of differential equations with three-point boundary conditions as follow:

$$\begin{cases} D^{\alpha}\left(D+\lambda\right)x(t) = f(t,x(t)), & t \in [0,1], \\ x(0) = x'(0) = 0,\ x(1) = \beta x(\eta), \end{cases}$$

where D^{α} is the conformable fractional derivative, D is the ordinary derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a known continuous function, λ and β are real numbers, $\lambda > 0$ and $\eta \in (0,1)$. In 2017, Bayour and Torres in [6], by using the notion of tube solution and Schauder's fixed- point theorem, established existence of tube solution the following problem:

$$\begin{cases} x^{(\alpha)}(t) = f(t,x(t)), & t \in [a,b],\ a > 0, \\ x(a) = x_0, \end{cases}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order $\alpha \in (0,1)$. Notice that the tube solution method generalizes the method of lower and upper solution.

In this article, we verified existence of tube solution for a new nonlocal Cauchy problem of the form:

$$\begin{cases} z^{(\alpha)}(s) = h(s,z(s)), & s \in [0,K],\ K \in \mathbb{R}, \\ z(0) + g(z) = z_0, & \alpha \in (0,1), \end{cases} \tag{1.1}$$

where $z^{(\alpha)}$ is conformable fractional derivative of z of order $\alpha \in (0,1)$, $h \in C\left([0,K] \times \mathbb{R}, \mathbb{R}\right)$ and $z : [0,K] \to \mathbb{R}$ is a function, also suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. By utilizing the tube solution method and Schauder's fixed-point theorem, existence results are obtained.

The paper is organized as follows: In section 2, we give the initial definitions and preliminary theorems of conformable fractional calculus. Our main results involving existence of tube solution problem (1.1) is presented in section 3. Finally, some conclusions are added.

2. Preliminaries

In this section, we present some basic definitions, lemmas and theorems that will be used to prove our new results.

Definition 2.1 ([7]). *Given a function, $h : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of h of order α is defined by*

$$T_\alpha(h)(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon t^{1-\alpha}) - h(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If h is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} h^{(\alpha)}(t)$ exists, then define

$$h^{(\alpha)}(0) = \lim_{t \to 0^+} h^{(\alpha)}(t).$$

We will, sometimes, write $h^{(\alpha)}(t)$ for $T_\alpha(h)(t)$, to denote the conformable fractional derivatives of h of order α.

Definition 2.2 ([8]). *The(left) conformable fractional derivative from a of a function $h : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$\left(T_\alpha^a h\right)(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon(t-a)^{1-\alpha}) - h(t)}{\varepsilon}.$$

When $a = 0$, we write T_α. If $(T_\alpha h)(t)$ exists on (a, b) then $(T_\alpha^a h)(a) = \lim_{t \to a^+}(T_\alpha^a h)(t)$. The(right) conformable fractional derivative of order $0 < \alpha \leq 1$ at b of h is defined by

$$({}_\alpha^b T h)(t) = -\lim_{\varepsilon \to 0} \frac{h(t + \varepsilon(b-t)^{1-\alpha}) - h(t)}{\varepsilon}.$$

If $({}^b T_\alpha h)(t)$ exists on (a, b) then $({}^b T_\alpha h)(b) = \lim_{t \to b^-}({}^b T_\alpha h)(t)$.

Note that if h is differentiable then $(T_\alpha^a h)(t) = (t-a)^{1-\alpha} h'(t)$ and $({}_\alpha^b T h)(t) = -(b-t)^{1-\alpha} h'(t)$.

Notation $(I_\alpha^a h)(t) = \int_a^t h(x) d\alpha(x, a) = \int_a^t (x-a)^{\alpha-1} h(x) dx$. When $a = 0$, we write $d\alpha(x)$. Similarly, in the right case we have $({}^b I_\alpha h)(t) = \int_t^b h(x) d\alpha(b, x) = \int_a^t (b-x)^{\alpha-1} h(x) dx$. The operators I_α^a and ${}^b I_\alpha$ are called conformable left and right fractional integrals of order $0 < \alpha \leq 1$.

Theorem 2.3 ([7]). *Let $\alpha \in (0, 1)$ and f, g be α-differentiable at a point $t > 0$. Then*

1. *$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.*
2. *$T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.*
3. *$T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.*
4. *$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.*
5. *$T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.*

6. *If, in addition, f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Remark 2.4 ([6])**.** From theorem 2.3 it follows that if $f \in C^1$ (continuously differentiable functions set), then

1. $\lim_{\alpha\to 1} T_\alpha(f)(t) = f'(t)$.
2. $\lim_{\alpha\to 0} T_\alpha(f)(t) = tf'(t)$.

Example 2.5. Suppose $0 < \alpha \leq 1$. Functions $f(t) = \sin\frac{1}{\alpha}t^\alpha$, $g(t) = \cos\frac{1}{\alpha}t^\alpha$, $p(t) = (ct+d)^n$, $c, d \in \mathbb{R}$, $n \in \mathbf{N}$, $h(t) = e^{\frac{1}{\alpha}t^\alpha}$, $m(t) = \tan(at)$, $a \in \mathbb{R}$, are α-differentiable, hence their conformable fractional derivatives of order α are equal with:

1. $T_\alpha(f)(t) = \cos\frac{1}{\alpha}t^\alpha$.
2. $T_\alpha(g)(t) = -\sin\frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(p)(t) = nct^{1-\alpha}(ct+d)^{n-1}$, $c, d \in \mathbb{R}$, $n \in \mathbf{N}$.
4. $T_\alpha(m)(t) = at^{1-\alpha}(1+\tan^2(at))$, $a \in \mathbb{R}$.
5. $T_\alpha(h)(t) = e^{\frac{1}{\alpha}t^\alpha}$.

Remark 2.6 ([6])**.** Differentiability implies $\alpha-$differentiability but the contrary is not true: a non-differentiable function can be $\alpha-$differentiable.

Theorem 2.7 ([7])**.** *If a function $f : [0,\infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Definition 2.8 ([7])**.** *(conformable fractional integral) Let $\alpha \in (0,1)$ and $h : [a,\infty) \to \mathbb{R}$. The conformable fractional integral of h of order α from a to t, denoted by $I^a_\alpha(h)(t)$, is defined by*

$$I^a_\alpha(h)(t) = \int_a^t \frac{h(s)}{s^{1-\alpha}}ds,$$

where the above integral is the usual improper Riemann integral.

Lemma 2.9 ([7])**.** *Assume that $f : [a,\infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$T^a_\alpha I^a_\alpha(f)(t) = f(t).$$

Lemma 2.10 ([8])**.** *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I^a_\alpha T^a_\alpha(f)(t) = f(t) - f(a).$$

Proposition 2.11 ([8]). *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times differentiable for $t > a$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Note 2.12 ([6]). *Let $0 < a < b$ and ${}_\alpha\Im_a^b[f]$ be a value of the integral $\int_a^b \frac{f(s)}{s^{1-\alpha}} ds$, then*

$${}_\alpha\Im_a^b[f] = I_\alpha^a(f)(b).$$

Proposition 2.13 ([6]). *Assume $f \in L^1([a, b], \mathbb{R}), 0 < a < b$. Then $|{}_\alpha\Im_a^b[f]| \leq ({}_\alpha\Im_a^b[|f|])$.*

Note 2.14. We define by $C^{(\alpha)}([0, K], \mathbb{R}), K \in \mathbb{R}$, $\alpha > 0$, the set of all real-valued functions $h : [0, K] \to \mathbb{R}$ that are α-differentiable such that the α-derivative is continuous. We mostly abbreviate $C^{(\alpha)}([0, K], \mathbb{R})$ by $C^{(\alpha)}([0, K])$.

Lemma 2.15 ([6]). *Let $r \in C^{(\alpha)}([a, b]), 0 < a < b$, such that $r^{(\alpha)}(t) < 0$ on $\{t \in [a, b] : r(t) > 0\}$. If $r(a) \leq 0$, then $r(t) \leq 0$ for every $t \in [a, b]$.*

Theorem 2.16. *If $f \in L^1([0, K])$, then function $z : [0, K] \to \mathbb{R}$ as follows:*

$$z(s) = e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \tag{2.1}$$

is a solution of problem

$$\begin{cases} z^{(\alpha)}(s) + z(s) = f(s), s \in [0, K], K \in \mathbb{R}, \\ z(0) + g(z) = z_0, \alpha \in (0, 1). \end{cases} \tag{2.2}$$

Proof. Assume $z : [0, K] \to \mathbb{R}$, be a function that is as follow:

$$z(s) = e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right).$$

By Theorem 2.3 and Lemma 2.9, we can write

$$\begin{aligned} z^{(\alpha)}(s) &= s^{1-\alpha} \left(-\frac{1}{\alpha} \alpha s^{\alpha-1} \right) e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \\ &+ e^{-\frac{1}{\alpha}s^\alpha} \left[\frac{f(s)}{e^{-\frac{1}{\alpha}s^\alpha}} \right] \\ &= -e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{f(t)}{e^{\frac{-1}{\alpha}t^\alpha}} \right] \right) + f(s) \\ &= -z(s) + f(s). \end{aligned}$$

Also, we have

$$\begin{aligned} z(0) &= e^{-\frac{1}{\alpha}0^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^0\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right) \\ &= e^0(z_0 - g(z) + 0) \\ &= z_0 - g(z). \end{aligned}$$

Multiply both side (2.2) at integral factor $e^{\frac{1}{\alpha}s^\alpha}$

$$z^{(\alpha)}(s)e^{\frac{1}{\alpha}s^\alpha} + z(s)e^{\frac{1}{\alpha}s^\alpha} = f(s)e^{\frac{1}{\alpha}s^\alpha}.$$

Then,

$$z^{(\alpha)}(s)e^{\frac{1}{\alpha}s^\alpha} + z(s)\left(e^{\frac{1}{\alpha}s^\alpha}\right)^{(\alpha)} = f(s)e^{\frac{1}{\alpha}s^\alpha}.$$

Now by theorem 2.3, we have

$$T_\alpha^0\left(z(s)e^{\frac{1}{\alpha}s^\alpha}\right) = f(s)e^{\frac{1}{\alpha}s^\alpha}.$$

By lemma 2.10, we have

$$\begin{aligned} I_\alpha^0 T_\alpha^0\left(z(s)e^{\frac{1}{\alpha}s^\alpha}\right) &= I_\alpha^0 f(s)e^{\frac{1}{\alpha}s^\alpha} \\ z(s)e^{\frac{1}{\alpha}s^\alpha} - z(0) &= I_\alpha^0 f(s)e^{\frac{1}{\alpha}s^\alpha} \\ z(s) &= e^{-\frac{1}{\alpha}s^\alpha}\left(\int_0^s \frac{f(t)e^{\frac{1}{\alpha}t^\alpha}}{t^{1-\alpha}}dt + z(0)\right) \\ z(s) &= e^{-\frac{1}{\alpha}z^\alpha}\left({}_\alpha\Im_0^s\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right] + z_0 - g(z)\right). \end{aligned}$$

Thus, relation (2.1) obtains. Therefore the proof of theorem is complete. □

Proposition 2.17 ([6]). *If $z : (0,\infty) \to \mathbb{R}$ is $\alpha-$differentiable at $s \in [a,b]$, then*

$$|z(s)|^{(\alpha)} = \frac{z(s)z^{(\alpha)}(s)}{|z(s)|}.$$

Definition 2.18. *Suppose $(\mu, N) \in C^{(\alpha)}([0,K],\mathbb{R})) \times C^{(\alpha)}([0,K],[0,\infty))$. Then (μ, N) is a tube solution of problem (1.1) if*

1. $(x - \mu(s))(f(s,x) - \mu^{(\alpha)}) \leq N(s)N^{(\alpha)}(s)$ *for each* $s \in [0,K]$ *and each* $x \in \mathbb{R}$ *such that* $|x - \mu(s)| = N(s)$,
2. $\mu^{(\alpha)}(s) = f(s,\mu(s))$, $N^{(\alpha)}(s) = 0$ *for each* $s \in [0,K]$ *such that* $N(s) = 0$,
3. $|z_0 - g(z) - \mu(0)| \leq N(0)$.

Now, present the following note that, next, we utilize in the proof of theorem 3.2.

Note 2.19. We introduce the following notation:

$$T(\mu, N) = \{z \in C^{(\alpha)}([0, K], \mathbb{R}) : |z(s) - \mu(s)| \leq N(s), s \in [0, K]\}.$$

Consider the following problem:

$$\begin{cases} z^{(\alpha)}(s) + z(s) = h(s, \tilde{z}(s)) + \tilde{z}(s), \quad s \in [0, K], \ K \in \mathbb{R} \\ z(0) + g(z) = z_0, \end{cases} \tag{2.3}$$

where

$$\tilde{z}(s) = \begin{cases} \frac{N(s)}{|z(s)-\mu(s)|}(z(s) - \mu(s)) + \mu(s), & |z(s) - \mu(s)| > N(s), \\ z(s), & otherwise \end{cases} \tag{2.4}$$

Let us define the operator $M : C([0, K]) \to C([0, K])$ is defined by

$$M(z(s)) = e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}(t)) + \tilde{z}(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right). \tag{2.5}$$

3. Main result

In this section, we show existence of tube solution by Schauder fixed point theorem.

Proposition 3.1. *If $(\mu, N) \in C^{(\alpha)}([0, K], \mathbb{R}) \times C^{(\alpha)}([0, K], [0, \infty))$ is a tube solution of problem (1.1), then the operator $M : C([0, K]) \to C([0, K])$, is compact.*

Proof. Assume $\{z_m\}_{m \in \mathbf{N}}$, is a sequence of $C([0, K], \mathbb{R})$ converging to $z \in C([0, K], \mathbb{R})$. By proposition 2.13 and relation (2.5), we have

$$\begin{aligned} &|M(z_m)(s) - M(z(s))| \\ &= \left| e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) - \right. \\ &\left. e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}(t)) + \tilde{z}(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \right| \\ &\leq \frac{A}{B} \left({}_\alpha\Im_0^s \left[|(h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)) - (h(t, \tilde{z}(t)) + \tilde{z}(t))| \right] \right) \\ &\leq \frac{A}{B} \left({}_\alpha\Im_0^s \left[|h(t, \tilde{z}_m(t)) - h(t, \tilde{z}(t))| \right] +_\alpha \Im_0^s \left[|\tilde{z}_m(t) - \tilde{z}(t)| \right] \right), \end{aligned}$$

where $A = \max_{0 \leq t \leq K}\{e^{-\frac{1}{\alpha}t^\alpha}\}$ and $B = \min_{0 \leq t \leq K}\{e^{-\frac{1}{\alpha}t^\alpha}\}$. We need to show that the sequence $\{f_m\}_{m \in \mathbf{N}}$ presented by $f_m(t) = h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)$ converges in $C^{(\alpha)}([0, K])$ to function $f(t) = h(t, \tilde{z}(t)) + \tilde{z}(t)$. Since there is a constant $D > 0$

such that $\|\tilde{z}\|_{C([0,K],\mathbb{R})} < D$, there is an index M such that $\|\tilde{z}_m\|_{C([0,K],\mathbb{R})} \leq D$ for every $m > M$. Therefore, h is uniformly continuous on $[0,K] \times E_D(0)$. Hence, for $\varepsilon > 0$ given, there is a $\delta > 0$ such that

$$|y - z| < \delta < \frac{B\varepsilon\alpha}{2AK^\alpha}$$

for every $z, y \in \mathbb{R}$;

$$|h(t,y) - h(t,z)| < \frac{B\varepsilon\alpha}{2AK^\alpha}$$

for each $t \in [0,K]$. By hypothesis, there is an index $\tilde{M} > M$ such that $\|\tilde{z}_m - \tilde{z}\|_{C([0,K],\mathbb{R})} < \delta$ for $m > \tilde{M}$. Hence

$$\begin{aligned} |M(z_m)(s) - M(z)(s)| &< \frac{A}{B}\left({}_\alpha\Im_0^K\left[\frac{B\varepsilon\alpha}{2AK^\alpha}\right] +_\alpha \Im_0^K\left[\frac{B\varepsilon\alpha}{2AK^\alpha}\right]\right) \\ &= \frac{2AB\varepsilon\alpha}{2ABK^\alpha}({}_\alpha\Im_0^K[1]) \\ &= \frac{\varepsilon\alpha}{K^\alpha} \times \frac{K^\alpha}{\alpha} \\ &= \varepsilon. \end{aligned}$$

This proves continuity of M. Also, we prove that the set $M(C[0,K])$ is relatively compact. Present a sequence $\{x_m\}_{m\in\mathbf{N}}$ of $M(C[0,K])$ for every $m \in \mathbf{N}$. There is $z_m \in C([0,K])$ such that $x_m = M(z_m)$. Now by proposition 2.13 we have

$$\begin{aligned} |M(z_m)(s)| &= \left|e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right)\right| \\ &\leq A\left(|z_0 - g(z)| + \frac{1}{B}\left({}_\alpha\Im_0^K\left[|h(s,\tilde{z}_m(t)) + \tilde{z}_m(t)|\right]\right)\right) \\ &\leq A\left(|z_0 - g(z)| + \frac{1}{B}\left({}_\alpha\Im_0^K\left[|h(s,\tilde{z}_m(t))|\right]\right) + \frac{1}{B}\left({}_\alpha\Im_0^K\left[|\tilde{z}_m(t)|\right]\right)\right). \end{aligned}$$

By definition, there is a $D > 0$ such that $|\tilde{z}_m(t)| \leq D$ for every $t \in [0,K]$ and every $m \in \mathbf{N}$. The function h is compact on $[0,K] \times E_D(0)$ and we can understand the existence of a constant $R > 0$ such that

$$|h(t,\tilde{z}_m(t))| \leq R$$

for each $t \in [0,K]$. The sequence $\{x_m\}_{m\in\mathbf{N}}$ is uniformly bounded for all $m \in \mathbf{N}$. Also see that for $s_1, s_2 \in [0,K]$ we can write

$$
\begin{aligned}
&|M(z_m)(s_2) - M(z_m)(s_1)| \\
&= \left| e^{-\frac{1}{\alpha}s_2^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^{s_2} \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \right. \\
&\left. -e^{-\frac{1}{\alpha}s_1^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^{s_1} \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \right| \\
&= \left| \left(e^{-\frac{1}{\alpha}s_2^\alpha} - e^{-\frac{1}{\alpha}s_1^\alpha} \right) (z_0 - g(z)) + e^{-\frac{1}{\alpha}s_2^\alpha} \left({}_\alpha\Im_0^{s_2} \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \right. \\
&\left. -e^{-\frac{1}{\alpha}s_1^\alpha} \left({}_\alpha\Im_0^{s_1} \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \right| \\
&\leq C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{A}{B} \left| {}_\alpha\Im_0^{s_1} [h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)] - ({}_\alpha\Im_0^{s_2} [h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)]) \right| \\
&= C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{A}{B} \left| {}_\alpha\Im_{s_1}^{s_2} [h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)] \right| \\
&\leq C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{A}{B} \left| {}_\alpha\Im_{s_1}^{s_2} [h(t, \tilde{z}_m(t))] \right| + \frac{A}{B} \left| {}_\alpha\Im_{s_1}^{s_2} [\tilde{z}_m(t)] \right| \\
&\leq C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{AR}{B} \left| {}_\alpha\Im_{s_1}^{s_2} [1] \right| + \frac{AD}{B} \left| {}_\alpha\Im_{s_1}^{s_2} [1] \right| \\
&= C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{A}{B} (R + D) \left| {}_\alpha\Im_{s_1}^{s_2} [1] \right| \\
&< C \left| e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha} \right| + \frac{A(R + D)}{B} \frac{|s_1^\alpha - s_2^\alpha|}{\alpha}.
\end{aligned}
$$

Where, $C = z_0 - g(z)$. This proves that the sequence $\{x_m\}_{m \in \mathbf{N}}$ is equicontinuous. By applying the Arzela-Ascoli theorem, $M(C([0, K]))$ is relatively compact and thus M is compact.□

Theorem 3.2. *If* $(\mu, N) \in C^{(\alpha)}([0, K], \mathbb{R}) \times C^{(\alpha)}([0, K], [0, \infty))$*, is a tube solution of the problem* (1.1)*. Then the problem* (1.1) *has a solution* $z \in C^{(\alpha)}([0, K], \mathbb{R}) \cap T(\mu, N)$.

Proof. According proposition 3.1, the operator M is compact. It has a fixed point by the Schauder fixed point theorem ([5],[9]). By theorem 2.16, a such fixed point is a solution of problem (2.3)-(2.4). Therefore, it is enough, we prove that for each solution z of problem (2.3)-(2.4) , $z \in T(\mu, N)$. By (2.3) we have :

$$z^{(\alpha)}(s) = h(s, \tilde{z}(s)) + \tilde{z}(s) - z(s). \tag{3.1}$$

By (2.4), we have :

$$\tilde{z}(s) - \mu(s) = \frac{N(s)}{|z(s) - \mu(s)|}(z(s) - \mu(s)) \to \frac{\tilde{z}(s) - \mu(s)}{N(s)} = \frac{z(s) - \mu(s)}{|z(s) - \mu(s)|}. \tag{3.2}$$

Set $E = \{s \in [0, K] : |z(s) - \mu(s)| > N(s)\}$. If $s \in E$, then by proposition 2.17, (3.1), (3.2) and definition 2.18, we give

$$\begin{aligned}
& (|z(s) - \mu(s)| - N(s))^{(\alpha)} \\
= \; & (|z(s) - \mu(s)|)^{(\alpha)} - N^{(\alpha)}(s) \\
= \; & \frac{(z(s) - \mu(s))(z(s) - \mu(s))^{(\alpha)}}{|z(s) - \mu(s)|} - N^{(\alpha)}(s) \\
= \; & \frac{(z(s) - \mu(s))(z^{(\alpha)}(s) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} - N^{(\alpha)}(s).
\end{aligned}$$

Since (μ, N) is a tube solution of (1.1), then on $\{s \in E : N(s) > 0\}$

$$\begin{aligned}
& (|z(s) - \mu(s)| - N(s))^{(\alpha)} \\
& = \frac{(z(s) - \mu(s))(z^{(\alpha)}(s) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} - N^{(\alpha)}(s) \\
& = \frac{(z(s) - \mu(s))(h(s, \tilde{z}(s)) + \tilde{z}(s) - z(s) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} - N^{(\alpha)}(s) \\
& = \frac{(z(s) - \mu(s))(h(s, \tilde{z}(s)) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} + \frac{(z(s) - \mu(s))(\tilde{z}(s) - z(s))}{|z(s) - \mu(s)|} - N^{(\alpha)}(s) \\
& = \frac{(\tilde{z}(s) - \mu(s))(h(s, \tilde{z}(s)) - \mu^{(\alpha)}(s))}{N(s)} + \frac{(\tilde{z}(s) - \mu(s))(\tilde{z}(s) - z(s))}{N(s)} - N^{(\alpha)}(s) \\
& = \frac{(\tilde{z}(s) - \mu(s))(h(s, \tilde{z}(s)) - \mu^{(\alpha)}(s))}{N(s)} \\
& + \frac{(\frac{N(s)}{|z(s) - \mu(s)|}(z(s) - \mu(s)) + \mu(s) - \mu(s))(\frac{N(s)}{|z(s) - \mu(s)|}(z(s) - \mu(s)) + \mu(s) - z(s))}{N(s)} \\
& - N^{(\alpha)}(s) \\
& = \frac{(\tilde{z}(s) - \mu(s))(h(s, \tilde{z}(s)) - \mu^{(\alpha)}(s))}{N(s)} + [N(s) - |z(s) - \mu(s)|] - N^{(\alpha)}(s) \\
& \leq \frac{N(s)N^{(\alpha)}(s)}{N(s)} + [N(s) - |z(s) - \mu(s)|] - N^{(\alpha)}(s) < 0.
\end{aligned}$$

Since $s \in \{t \in E, N(t) = 0\}$, then $N(s) = 0$

$$\begin{aligned}
& (|z(s) - \mu(s)| - N(s))^{(\alpha)} \\
= \; & \frac{(z(s) - \mu(s))(h(s, \tilde{z}(s)) + \tilde{z}(s) - z(s) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} - N^{(\alpha)}(s) \\
= \; & \frac{(z(s) - \mu(s))(h(s, \tilde{z}(s)) - \mu^{(\alpha)}(s))}{|z(s) - \mu(s)|} + [N(s) - |z(s) - \mu(s)|] - N^{(\alpha)}(s) \\
\leq \; & \frac{N(s)N^{(\alpha)}(s)}{N(s)} - |z(s) - \mu(s)| - N^{(\alpha)}(s) \\
< \; & -N^{(\alpha)}(s) = 0.
\end{aligned}$$

If $\gamma(s) = |z(s) - \mu(s)| - N(s)$, then $\gamma^{(\alpha)} < 0$ on $E = \{s \in [0, K] : \gamma(s) > 0\}$. On the other hand, since (μ, N) is a tube solution of problem (1.1) and z satisfies $|z_0 - g(z) - \mu(0)| \leq N(0)$, we have, $\gamma(0) \leq 0$, also lemma 2.15 implies that $E = \emptyset$. Hence, $z \in T(\mu, N)$, therefore, the proof of theorem is complete. □

Example 3.3. Consider the fractional initial value problem

$$\begin{cases} z^{(\frac{1}{2})} = \sqrt{s}\left(-\sin(s) - \frac{1}{8}\sin(2s) - \frac{s}{4} + \frac{1}{2}\int_0^s z^2(t)dt\right), \quad s \in [0, 2], \\ z(0) = 1, \end{cases} \tag{3.3}$$

where, $g(z) = 0$. By definition 2.18, $(\mu, N) \equiv (0, 1)$ is a tube solution. Therefore, by theorem 3.2, the problem (3.3) has a solution z such that $|z(s)| \leq 1$ for all $s \in [0, 2]$.

4. Conclusions

In this paper, we have considered a new fractional initial value problem with the conformable fractional derivative. By the Schauder fixed point theorem and some new tools for fractional differential equations, existence of tube solution is proved. Also, we have presented an example to expression the results.

References

[1] K.B. Oldham, J. Spanier, *The fractional calculus*, Academic press, New York and London, 1974.

[2] B. Ross (Ed.), *The fractional calculus and its application*, in: Lecture notes in mathematics, vol.475, Springer-Verlag, Berlin, 1975.

[3] F.B. Tatom, *The relationship between fractional calculus and fractals*, Fractals, 3 (1995), 217-229.

[4] T.F. Nonnenmacher, R. Metzler, *On the Riemann-Liouville fractional calculus and some recent applications*, Fractals, 3 (1995), 557-566.

[5] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer, New York, 2003.

[6] B. Bayour, D.F.M. Torres, *Existence of solution to a local fractional nonlinear differential equation,* Journal of Computational and Applied Mathematics, 312 (2017), 127-133.

[7] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative,* J. Comput. Appl. Math. 264 (2014), 65-70

[8] T. Abdeljawad, *On conformable fractional calculus,* J. Comput. Appl. Math., 279 (2015), 57-66.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations, Elsevier B.V, Netherlands*, 2006.

[10] H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, Three-point boundary value problems for conformable fractional differential equations, Journal of Function Spaces, (2015), 1-7.

[11] A. Gökdoğan, E. Ünal, E. Çelik, *Existence and uniqueness theorems for sequential linear conformable fractional differential equations*, Miskolc Mathematical Notes, (2016), 267-279.

[12] K.R. Prasad, B.M.B. Krushna, *Existence of multiple positive solutions for a coupled system of iterative type fractional order boundary value problems*, J. Nonlinear Funct. Anal., (2015), 1-15.

[13] S. Asawasamrit, S.K. Ntouyas, P. Thiramanus, J. Tariboon, *Periodic boundary value problems for impulsive conformable fractional integro-differential equations*, Boundary Value Problems, 2016, 1-18.

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Z-quasi prime submodules and the Z-radicals of submodules

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Abstract. Let R be a commutative ring with identity and N be a proper submodule N of R-moduLe M is called prime if whenever $rx \in N$; $r \in R$, $x \in M$, implies either $x \in N$ or $r \in [N : M]$, where $[N : M] = \{r \in R : rM \subseteq N\}$. In this paper we say that N is Z-qusi prime submodule of R-moduLe M, if $[N : (x)]$ is a Z-prime ideal of R for each $x \in M$. We prove some result of this type of submodules.

Keywords: prime submodule, Z-prime submodule, qusi prime submodule, Z-regular module.

1. Introduction

A proper submodule N of module M over a ring R is said to be prime (or P-prime) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $r \in [N : M]$ or $x \in N$, where $[N : M] = \{r \in R : rM \subseteq N\}$, [1]. And M is called prime module if the zero submodule of M is a prime submodule of M, [2]. In the last year many studies and researches are published about prime submodule by many people who care with the subject of commutative algebra and some of them are J. Dauns, R.L. Mcsland, C.P. Lu, P.F. Smith, M.E. Moore. M is called faithful module if $[0 : M] = 0$,[3]. Where $[0 : M] = \{r \in R : rM = 0\}$ and $[0 : N]$ is define as $[0 : N] = \{r \in R : rN = 0\}$. There are several generalization of the notion of a prime submodules as like S-prime, [4]. N. S. Al-Mothafar and A. T. Husain [5] defined the Z-prime submodule of an R-module Mas follows: A proper submodule N of module M over a ring R is said to be Z-prime if $f(x).x \in N$ for $f \in M^* = Hom(M, R)$, $x \in M$ implies that either $f(x) \in [N : M]$ or $x \in N$ and show that every prime submodule is Z-prime submodule but the converse is not true in general. However they give condition under which a proper Z-prime submoduleis a prime submodule. In this paper we study the concept of a Z-quasi prime submodule of an R-module M, we give some examples and properties and give the relation between Z-prime submodules and Z-quasi prime submodules. Also, we define the notation of $Z - rad(N)$ using the concept of Z-prime submodules and we study some of there basic properties.

2. Z-quasi prime submodule

Definition 2.1. A proper submodule N of an R-module M is said to be Z-quasi prime submodule of M if $[N : (x)]$ is a Z-prime ideal of R for each $x \in M$.

Remarks and Examples 2.2.

1. Every quasi-prime submodule is Z-quasi prime submodule but the converse is not true, for example the submodule Z of the Z-module Q is Z-quasi prime submodule, since Hom $(Q, Z) = 0$, but not quasi-prime submodule.

2. If N is a Z-prime submodule of an R-module M such that $ann(x) = 0$ for each $x \in M$, then $[N : (x)]$ is a Z-prime ideal of R

3. If N is a Z-prime submodule of an R-module M, then N is a Z-quasi prime submodule of M and $ann(x) = 0$; for each $x \in M$.

4. Consider the Z-module $M = Z \oplus Z_p$, where P is a prime number and the Z-submodule $N = qZ \oplus Z_p$, where q is any prime number. Then, for any $x \in M$ implies that $[N : (x)] = qZ$, which is a Z-prime ideal of Z. Therefore N is a Z-quasi prime submodule of M.

Proof. (2) Let $f(a).a \in [N : (x)]$; $a \in R$, $f \in Hom(M, R) = R^*$, $x \in M$ and suppose $f(a) \notin [N : (x)]$, which implies that $f(a).\ a(x) \subseteq N$, $f(a).a(x) \in N$. Define $h : M \to R$ by $h(rx) = r$; for each $r \in R$, then $f(h(ax)).\ ax \in N$, since N is a Z-prime submodule of M and $f \circ h : M \to R$, then either $ax \in N$, which implies that $a \in [N : (x)]$ or $f(h(ax)) = f(a) \in [N : (x)]$ which is a contradiction. □

The following theorem gives some characterizations for Z-quasi prime submodule.

Theorem 2.3. *Let N be a proper submodule of an R-module M, then the following are equivalent:*

1. *N is a Z-quasi prime submodule of M.*
2. *$[N : K]$ is a Z-prime ideal of R for each submodule K of M*
3. *$[N : (rm)] = [N : (m)]$ for each $m \in M$, $r \in R$ and $r \notin [N : (m)]$.*

Proof. (1) ⇒ (2) Let N be a Z-quasi prime submodule of M. Then $[N : (x)]$ is a Z-prime ideal of R, for each $x \in M$. So $[N : (x)]$ is a Z-prime ideal for each $x \in K$ and by lemma (2.1.23) in [5], $[N : K]$ is a Z-prime ideal of R.

(2) ⇒ (3) It is clear that $[N : (m)] \subseteq [N : (rm)]$. Let $a \in [N : (rm)]$ for each $r \notin [N : (m)]$ and $m \in M$. Hence, $a(rm) \subseteq N$. It follows that $ar \in [N : (m)]$. Define $h : R \to R$ by $h(r) = r$, it is clear that $h \in R^* = Hom(R, R)$, $h(a).r \in [N : (m)]$ which is a Z-prime ideal by (2). But $r \notin [N : (m)]$ so $a \in [N : (m)]$. Thus $[N : (rm)] \subseteq [N : (m)]$. Therefore $[N : (m)] = [N : (rm)]$.

(3) $\Rightarrow$ (1) Let $x \in M$ and $f \in Hom(R, R)$, $a \in R$ such that $f(a).a \in [N : (x)]$, suppose $a \notin [N : (x)]$, hence by (3), $[N : (ax)] = [N : (x)]$. But $f(a) \in [N : (ax)]$, so $f(a) \in [N : (x)]$ and hence N is a Z-quasi prime submodule. □

The following is an immediate consequence of above theorem.

Corollary 2.4. *Let N be submodule of an R-module M. If N is a Z-quasi prime submodule of M, then $[N : M]$ is Z-prime ideal of R.*

Recall that an R-module M is said to be Z-regular R-module if for each $x \in M$ there exists $f \in M^* = Hom(M, R)$ such that $f(x).x = x$, [6].

Proposition 2.5. *Let M be a Z-regular R-module, then every Z-quasi prime submodule of M is Z-prime submodule of M.*

Proof. Let $x \in M$, $f \in M^* = Hom(M, R)$ such that $f(x).x \in N$ and $f(x) \notin [N : M]$ we have to show $x \in N$, since M is Z-regular, then for each $x \in M$ there exist $g \in M^*$, such that $g(x).x = x$.

Case 1: If $f = g$.Thus $x = f(x).x \in N$.

Case 2: If $f \neq g$. Thus $f(x).x = f(x)g(x).x$, since N is quasi-Z- prime submodule of M, then either $f(x).x \in N$ or $g(x).x \in N$, if $f(x).x \in N$ we are done, if $g(x).x \in N$, which implies that $x \in N$. Thus N is Z-prime submodule of M. □

The following proposition gives another characterization of Z-quasi prime submodule in case M is Z-regular and multiplication modules.

Proposition 2.6. *Let N be a proper submodule of multiplication, Z-regular R-module M. Then the following statements are equivalent:*

1. *N is a Z-quasi prime submodule.*
2. *$[N : M]$ is a Z-prime ideal .*
3. *N is Z-prime submodule.*

Proof. (1) $\Rightarrow$ (2) , by corollary 2.4.

(2) $\Rightarrow$ (3), by proposition 2.5.

(3) $\Rightarrow$ (1), by remark (2.2) (1). □

Proposition 2.7. *Let M and $\acute{M}$ be two modules and let $f : M \to \acute{M}$ be an epimorphism. If N is a Z-quasi prime submodule of M such that $kerf \subseteq N$, then $f(N)$ is a Z-quasi prime submodule of $\acute{M}$.*

Proof. To prove $f(N)$ is a Z-quasi prime submodule of $\acute{M}$, we must prove $[f(N) : \acute{K}]$ is Z-prime ideal of R, $\forall \acute{K} \supseteq f(N)$. Since f is an epimorphism,then $\acute{K} = f(f^{(-1)}(\acute{K}))$. Let $K = f^{(-1)}(\acute{K})$, so $f(K) = \acute{K}$. It follows that $f(K) \supseteq f(N)$. So to prove $[f(N) : f(K)]$ is a Z-prime ideal of R. Let $h \in Hom(R.R)$, $a \in R$ such that $h(a).a \in [f(N) : f(K)]$ so $h(a)af(K) \subseteq f(N)$, then for each

$x \in K$, $h(a)af(x) \in f(N)$, so that $f(h(a)ax) = f(n)$ for some $n \in N$. Which implies that $h(a)ax - n \in kerf \subseteq N$ and so $h(a)ax \in N$ for each $x \in K$. Hence $h(a).a \in [N : K]$ is Z-prime ideal since N is a Z-quasi prime submodule, so either $h(a) \in [N : K]$ or $a \in [N : K]$. Thus either $h(a)k \subseteq N$ or $aK \subseteq N$ and so either $h(a)f(K) \subseteq f(N)$ or $af(K) \subseteq f(N)$. Therefore either $h(a) \in [f(N) : f(K)]$ or $a \in [f(N) : f(K)]$, which implies that $[f(N) : f(K)]$ is a Z-prime ideal of R and $f(N)$ is a Z-quasi prime submodule of $\acute{M}$. □

Proposition 2.8. *Let M and $\acute{M}$ be two modules and let $f : M \to \acute{M}$ be an epimorphism. If N is a Z-quasi prime submodule of $\acute{M}$, then $f^{(-1)}(N)$ is a Z-quasi prime submodule of M.*

Proof. To prove $f^{(-1)}(N)$ is a Z-quasi prime submodule of M, we must prove $[f^{(-1)}(N) : K]$ is a Z-prime ideal of R, $\forall K \supseteq f^{(-1)}(N)$. Let $h \in Hom(R.R)$, $a \in R$ such that $h(a).a \in [f^{(-1)}(N) : K]$ so $h(a)aK \subseteq f^{(-1)}(N)$. Hence $f(h(a)aK) \subseteq f(f^{(-1)}(N))$, it follows that $h(a)a(f(K)) \subseteq f(f^{(-1)}(N)) = N$. Thus $h(a)a \in [N : f(K)]$. But $K \supseteq f^{(-1)}(N)$ so that $f(K) \supseteq f(f^{(-1)}(N)) = N$ (since f is epimorphism) and hence $[N : f(K)]$ is Z-prime ideal. Hence either $h(a) \in [N : f(K)]$ or $a \in [N : f(K)]$ and so either $h(a)f(K) \subseteq N$ or $a.f(K) \subseteq N$. Therefore either $f(h(a)K) \subseteq Norf(aK) \subseteq N$ that is either $h(a)K \subseteq f^{(-1)}(N)$ or $aK \subseteq f^{(-1)}(N)$. It follows that $h(a) \in [f^{(-1)}(N) : K]$ or $a \in [f^{(-1)}(N) : K]$. Thus $[f^{(-1)}(N) : K]$ is a Z-prime ideal, $\forall K \supseteq f^{(-1)}(N)$ and therefore $f^{(-1)}(N)$ is a Z-quasi prime submodule of $\acute{M}$. □

Corollary 2.9. *Let K and N be two submodules of an R-module and $K \subseteq N$. Then N/K is a Z-quasi prime submodule of M/K if and only if N is a Z-quasi prime submodule of M.*

Proof. Let $\varphi : M \to M/K$ be natural mapping, then the result follows from. proposition 2.7 and proposition 2.8. □

3. The Z-radicals of submodules

The concept of prime radical of a submodules N of of an R-module M, denoted by $rad(N)$, was defined by as the intersection of all prime submodules of M containing N, [7], if N is not contained in any prime submodule, then $rad(N) = M$. In this section, we introduce the notation of Z-rad(N) and we give some of its properties.

Definition 3.1. Let N be a submodule of an R-module. The intersection of all Z-prime submodules of M containing N is called the Z-radical of N and it is denoted by Z-rad(N). If no Z-prime submodule of M containing N, then we put $Z - rad(N) = M$.

Remark 3.2. Let N be a proper submodule of faithful, cyclic R-module M:

1. $N \subseteq Z - rad(N)$

2. $rad(N) \subseteq Z - rad(N)$

Proof. 1. It is clear.

2. Let P be a Z-prime submodule of an R-module M containing N, since M is cyclic, faithful, then P is a prime submodule of M containing N,[5] and hence $rad(N) \subseteq P$. Also, $rad(N) \subseteq \cap P$, for all Z-prime submodule P containing N. Therefore, $rad(N) \subseteq Z - rad(N)$.

□

Proposition 3.3. *Let N be a submodule of an R-module M, then $Z-rad(N) = Z - rad(Z - rad(N))$*

Proof. By remark 3.2(1) we have $Z - rad(N) \subseteq Z - rad(Z - rad(N))$. Let P be a Z-prime submodule of M containing N, then $Z - rad(N) \subseteq P$ and thus $Z - rad(Z - rad(N) \subseteq P$. Therefore $Z - rad(Z - rad(N) \subseteq \cap P$, for all Z-prime submodules P containing N. This implies that $Z - rad(Z - rad(N) \subseteq Z - rad(N)$. □

Proposition 3.4. *Let N and L be submodule of an R-module M, then $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$.*

Proof. Clearly $N \cap L \subseteq N$ and $N \cap L \subseteq L$. Thus $Z - rad(N \cap L) \subseteq Z - rad(N)$. By the same way $Z - rad(N \cap L) \subseteq Z - rad(L)$. It follows $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$ □

Proposition 3.5. *Let N and L be a submodules of an R-module M, then $Z - rad(N + L) = Z - rad(Z - rad(N) + Z + rad(L))$*

Proof. Clearly, $N \subseteq N+L$ and $L \subseteq N+L$. Hence $Z-rad(N) \subseteq Z-rad(N+L)$ and $Z-rad(L) \subseteq Z-rad(N+L)$, which implies that $Z-rad(N)-Z-rad(L) \subseteq Z-rad(N+L)$ and so $Z-rad(Z-rad(N)+Z-rad(L)) \subseteq Z-rad(Z-rad(N+L)$, but $Z - rad(Z - rad(N + L)) = Z - rad(N + L)$ by proposition 3.3, thus $Z - rad(Z - rad(N) + Z - rad(L)) \subseteq Z - rad(N + L)$.

Now, we must prove that $Z-rad(N+L) \subseteq Z-rad(Z-rad(N)+Z-rad(L))$, let P be a Z-prime submodule of M containing $Z - rad(N) + Z - rad(L)$, then P is a Z-prime submodule of M containing $N + L$ and hence $Z - rad(N + L) \subseteq P$. Also, $Z - rad(N + L) \subseteq \cap P$, for all Z-prime submodules of P containing $Z - rad(N) + Z - rad(L)$. Therefore $Z - rad(N + L) \subseteq Z - rad(Z - rad(N) + Z - rad(L))$. □

Proposition 3.6. *Let N be a submodule of cylic,faithful R-module M, then $\sqrt{([N : M])}M \subseteq Z - rad(N)$ and if M is a multiplication module, then $\sqrt{([N : M)]}M = Z - rad(N)$.*

Proof. Let P be a Z-prime submodule of M containing N, then $[N : M] \subseteq [P : M]$ and hence $\sqrt{([N : M)]} \subseteq [P : M]$, since M is cylic, faithful R-module,

hence $[P : M]$ is a prime ideal and hence $\sqrt{([N : M)]}M \subseteq [P : M]M \subseteq P$, thus $\sqrt{([N : M)]}M \subseteq Z - rad(N)$.

Now, if M is a multiplication module, then $\sqrt{([N : M)]}M = rad(N)$, [8] and thus $\sqrt{([N : M)]}M = Z - rad(N)$ □

In the following proposition, we give condition under which the other inclusion of proposition 3.4 holds.

Proposition 3.7. *Let N and L be submodule of an R-module M, such that whenever $N \cap L \subseteq P$, where P is a Z-prime submodule of M, we have $N \subseteq P$ or $L \subseteq P$, then $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$.*

Proof. If $Z - rad(N \cap L) = M$, then clearly $Z - rad(N) = Z - rad(L) = M$ and so $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L) = M$. If $Z - rad(N \cap L) \neq M$, then there exist a Z-prime submodule P, such that $N \cap L \subseteq P$. By hypothesis, $N \subseteq P$ or $L \subseteq P$ so that $Z - rad(N) \subseteq P$ or $Z - rad(L) \subseteq P$. Since this is true for all Z-prime submodule P containing $N \cap L$, thus $Z - rad(N) \cap Z - rad(L) \subseteq Z - rad(N \cap L)$ and therefore $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$. □

Corollary 3.8. *Let M be a cyclic, faithful R-module. Then $Z - rad(IM \cap N) = Z - rad(IM) \cap Z - rad(N)$, for every ideal I of R and every submodule N of M.*

Proof. Let N be a submodule of an R-module M and I be an ideal of R. We only have to show that if $IM \cap N \subseteq P$, where P is a Z-prime submodule of M, then either $N \subseteq P$ or $IM \subseteq P$. Now, suppose that $N \nsubseteq P$, then there exist $x \in N$ and $x \notin P$. $Ix \subseteq IM \cap N \subseteq P$, since M cyclic, faithful, then P is a prime submodule of M and $x \notin P$ thus $I \subseteq [P : M]$, therefore $IM \subseteq P$. □

Proposition 3.9. *If N and L are submodules of a cyclic, faithful R-module M, such that $\sqrt{([N : M])} + \sqrt{([L : M])} = R$, then $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$.*

Proof. Clearly $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$, by proposition 3.4. If P is a Z-prime submodule of M containing $N \cap L$, then $[N \cap L : M] \subseteq [P : M]$, then $[N : M] \cap [L : M] \subseteq [P : M]$. Since M is cyclic, faithful,then $[P : M]$ is a prime ideal, so either $[L : M] \subseteq [P : M]$ or $[N : M] \subseteq [P : M]$. If $[N : M] \subseteq [P : M]$, then $[L : M] \nsubseteq [P : M]$ for otherwise $\sqrt{([L : M])} \subseteq [P : M]$ which is a contradiction, therefore $N \subseteq P$. This implies that $Z - rad(N) \subseteq P$ also $Z - rad(L) \subseteq P$, then $Z - rad(N) \cap Z - rad(L) \subseteq P$, thus $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$. □

References

[1] C. P. Lu, *Prime submodules of modules*, Comment Mathematics, University Spatula, 33 (1984), 61-69.

[2] R. J. Nezhad and M. H. Naderil, *On prime and semiprime submodules of multiplication modules*, Faculty of science universityof kashan, Iran international mathematical Form, 26 (2009), 1261-1266.

[3] C. P. Lu, *M-Radicals of submodules in modules*, Math., Japonica, 34 (1989), 211-219.

[4] S. O. Dakheel, *S-prime submodules and some related concepts* M.Sc. Thesis, Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq, 2010.

[5] N. S. Al-Mothafar. and A. T. Husain, *Z-prime submodules*, Int. J. Adv. Res., 4 (2016), 355-361.

[6] J. Zelmanowitz, *Regular modules*, Trans. Amer. Math. Sec., 163 (1973), 341-355.

[7] R. L. Mc Casland, *Some commutative ring results to unitary modules* Ph.D. Thesis, University of Taxas at Arlington, 1983.

[8] Z. A. El-Bast and P. F.Smith, *Multiplication modules*, Comm. In. Algebra, 16 (1973), 755-779.

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A result on the coupled fixed point theorems in C^*-algebra valued b-metric spaces

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Abstract. The aim of this paper is to establish a new Coupled Fixed Point Theorems for C^*-algebra valued b-metric spaces. As an application of our result, we discuss the existence and uniqueness results for Couple Fixed Point Theorem in C^*-algebra valued b-metric spaces. We also give conclusion to demonstrate our result.

Keywords: fixed point theory, C^*-algebras, b- metric spaces.

1. Introduction

In Blackadar's book [1], he gave many important results and properties of the theory of C^*-algebras and Von Neumann Algebras. Huang and Zhang [7] introduced the concept of cone metric space which generalized the concept of the real valued metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions.Ma and co-authors obtained important results for fixed point theorems on both C^*-algebra valued metric space and C^*-algebra valued b-metric space in [9],[10] respectively.In [12] and [13], we considered C^*- algebras valued metric spaces and proved certain fixed-point theorem for a single valued

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mapping in such spaces. In [3], authors introduced the theory of common fixed point theorems on metric spaces.

Subsequently, some other authors studied the existence of fixed points of self mappings satisfying a contractive type condition. The author and et al introduced in [17] the notation of the Quaternion valued metric spaces as well as contraction mapping.For getting more information about the general theory, you can look valuable books [4], [11], [14] and [15] as well as significant papers [16],[8],[2],[5] and [6].

Here, we introduce a new coupled fixed point theorems for C^*-algebra b-metric spaces with some examples.

2. Preliminaries

Basic notions and facts which play a central role in the present paper for C^*-algebra valued b-metric space will be given in this section.

2.1 C^*-algebra and positive cone

Definition 2.1. A C^*-algebra $\mathcal{A}$ is a complex Banach algebra with a conjugate-linear involution $* : \mathcal{A} \longrightarrow \mathcal{A}$, such that

$$(x^*)^* = x,\ (xy)^* = y^*x^*,\ ,(x+y)^* = x^* + y^*\ , \|x^*x\| = \|x\|^2,$$

for all x, y in $\mathcal{A}$. The C^*-condition $\|x^*x\| = \|x\|^2$ implies that the involution is an isometry in the sense that $\|x^*\| = \|x\|$ for all x in $\mathcal{A}$.

A C^*-algebra is called unital if it possesses a unit. It follows easily that $\|1\| = 1$. In general C^*-algebra is non-commutative. We assume the unital C^*-algebras in our work.

Definition 2.2. A *-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between two C^*-algebras is a homomorphism such that $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital, then φ is called unital if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. An isomorphism between two C^*-algebras is a bijective *-homomorphism.

It is an important basic fact that every $*$-homomorphism between C^*-algebras is continuous. If two C^*-algebras are isomorphic, then they are automatically isometric.

Example 2.3. Let $\mathcal{H}$ be an arbitrary Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ with the operator norm $\|x\| = sup\{\|x\xi\|,\ \xi \in \mathcal{H}, \|\xi\| = 1\}$ for each $x \in \mathcal{B}(\mathcal{H})$ and the involution given by the map which assigns $x \in \mathcal{B}(\mathcal{H})$ its adjoint is a C^*-algebra.

Sub-C^*-algebra of a C^*-algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ which is a C^*-algebra with respect to the operations given on $\mathcal{A}$. Let S be a subset of a C^*-algebra $\mathcal{A}$. Then the sub-C^*-algebra generated by S, denoted by $C^*(S)$, is the smallest sub-C^*-algebra of $\mathcal{A}$ that contains S.

The following fundamental results were proved by Gelfand, Naimark and Segal [**?**]:

Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact space X. Furthermore, every C^*-algebra is isomorphic to a sub-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

2.2 Positivity in C^*-algebra

Definition 2.4. A self adjoint element in C^*-algebra is called positive if $\sigma(a) \in \mathcal{R}^+$. If a is positive, we write $a \geq 0$ and positive cone $\mathcal{A}_+$ is represented by $\mathcal{A}_+ = \{a \in \mathcal{A},\ a \geq 0\}$.

Lemma 2.5. Suppose that $\mathcal{A}$ is a C^*-algebra:

(a) If $a \in \mathcal{A}$ is normal, then $a^*a \geq 0$;

(b) If $a \in \mathcal{A}$ is self-adjoint and $\|a\| \leq 1$, then $a \geq 0$;

(c) If $a, b \in \mathcal{A}_+$ then $a + b \in \mathcal{A}_+$;

(d) $\mathcal{A}_+$ is closed in $\mathcal{A}$;

(e) If $b, c \in \mathcal{A}'_+$, then $b \leq c$ implies that $a^*ba \leq a^*ca$;

(f) If $b_n \subset \mathcal{A}$ and $\lim_{n\to\infty} b_n = 0_{\mathcal{A}}$, then $\lim_{n\to\infty} a^*b_na = 0_{\mathcal{A}}$ for any $a \in \mathcal{A}$.

For a given $a, b \in \mathcal{A}_+$, we denote $a \leq b$ if $a - b \geq 0$, $\mathcal{A}_+$ becomes a partially ordered vector space. Furthermore in a unital C^*-algebra, for all $a, b \in \mathcal{A}$, $0 \leq a \leq b$ implies that $\|a\| \leq \|b\|$, so the positive cone in a C^*-algebra is automatically normal.

Lemma 2.6. Suppose that $\mathcal{A}$ is unital C^*-algebra with a unit I:

(1) If $a \in \mathcal{A}$ with $\|a\| \leq \frac{1}{2}$ then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$;

(2) Suppose that $a, b \in \mathcal{A}$ with $a, b \geq \theta$ and $ab = ba$, then $ab \geq \theta$.

2.3 C^*-algebras valued b-metric space

C^*-algebras valued b-metric space introduced by [**?**] and studied self mapping fixed point theorem, here we will introduce some other examples and the relation between C^*-algebras valued b-metric space and Fixed Point Theory. Moreover, we will introduce the common fixed point theory in this setting.

Definition 2.7. Let X be a non-empty set and $\mathcal{A}$ is a C^*-algebra, $\mathcal{A}_+$ be its positive cone. A C^*-algebra valued b-metric space has a function $d : X \times X \longrightarrow \mathcal{A}_+$ defined on X such that for any $x, y, z \in X$ and $a \in \mathcal{A}'_+$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq a(d(x, y) + d(y, z))$.

From the the definition its automatically that $d(x, y) \geq 0,\ d(x, x) \geq 0$.

Definition 2.8. Let $(\mathrm{X}, \mathcal{A}, d)$ be a C^*-algebra valued b-metric space and $x_n \subset X$ is a sequence in X. If $x \in X$ and $\varepsilon > 0$ there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$ then (x_n) is called a convergent sequence in X to x and denote it by $\lim_{n\to\infty} x_n = x$

Moreover, if for any $\varepsilon > 0$ there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$ then (x_n) is called a cauchy sequence in X.

Definition 2.9. The tripled $(\mathrm{X}, \mathcal{A}, d)$ is a complete C^*-algebra valued b-metric space if every cauchy sequence is convergent.

Definition 2.10. Let $(\mathrm{X}, \mathcal{A}, d)$ be a C^*-algebra valued b-metric space. The mapping $T : X \longrightarrow X$, is called contractive mapping on x if there is an $a \in \mathcal{A}$ such that $\|a\| < 1$ and satisfy

$$d(T(x), T(y)) \leq a^* d(x, y)a$$

for $x, y \in X$.

Example 2.11. Let $X = [-1, 1]$ and $\mathcal{A} = M_{2\times 2}(R)$ with $\|A\| = \max(|a_1|, |a_2|, |a_3|, |a_4|)$ where a_i's are the entries of the matrice $A \in M_{2\times 2}(R)$. Then $(\mathrm{X}, \mathcal{A}, d)$ is a C^*-algebra valued b-metric space, where

$$d(x, y) = \begin{pmatrix} |x-y|^p & 0 \\ 0 & |x-y|^p \end{pmatrix}$$

for $p > 0$ and

$$|x - z|^p \leq 2^p I(|x - y|^p + |y - z|^p)$$

as well as partial ordering on $\mathcal{A}$ is given as

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \geq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$$

for $i = 1, 2, 3, 4$.

So the condition (3) is in the form satisfies

$$d(x, y) \leq 2^p I(\begin{pmatrix} |x-z|^p & 0 \\ 0 & |x-z|^p \end{pmatrix} + \begin{pmatrix} |z-y|^p & 0 \\ 0 & |z-y|^p \end{pmatrix}),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Main theorem and results

Theorem 3.1. Let $(X, \mathcal{A}, d)$ be a complete C^*-algebra valued b-metric space.We suppose that the mapping $T : X \times X \longrightarrow X$ satisfies

$$d(T(x,y),T(u,v)) \leq a[Ad(x,u)A^* + Bd(y,v)B^*], \tag{1}$$

for $x, y, u, v \in X$ and $A, B \in \mathcal{A}$, such that $\|A\| \leq \frac{1}{2}$ and $\|B\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ such that

$$x_1 = T(x_0, y_0), y_1 = T(y_0, x_0) \tag{2}$$
$$x_2 = T(x_1, y_1), y_1 = T(y_1, x_1) \tag{3}$$
$$. \tag{4}$$
$$. \tag{5}$$
$$. \tag{6}$$
$$x_m = T(x_{m-1}, y_{m-1}), y_1 = T(y_{m-1}, x_{m-1}) \tag{7}$$
$$x_{m+1} = T(x_m, y_m), y_1 = T(y_m, x_m) \tag{8}$$

From (1) and (2) we have

$$d(x_{n+1}, x_n) = d(T(x_n, y_n), T(x_{n-1}, y_{n-1}) \leq a(Ad(x_n, x_{n-1})A^* + Bd(y_n, y_{n-1})B^*)$$

and similarly

$$d(y_{n+1}, y_n) = d(T(y_n, x_n), T(y_{n-1}, x_{n-1}) \leq a(Ad(y_n, y_{n-1})A^* + Bd(x_n, x_{n-1})B^*).$$

So, we get

$$\begin{aligned} d_{nn} &\leq a(Ad(x_n, x_{n-1})A^* + Bd(y_n, y_{n-1})B^* + Ad(y_n, y_{n-1})A^* + Bd(x_n, x_{n-1})B^*) \\ &= a(A+B)d(x_n, x_{n-1})(A^* + B^*) + a(A+B)d(y_n, y_{n-1})(A^* + B^*) \\ &= a(A+B)[d(x_n, x_{n-1}) + d(y_n, y_{n-1})](A^* + B^*) \\ &= a(A+B)d_{(n-1)(n-1)}(A^* + B^*) \end{aligned}$$

by taking $d_{nn} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ for all $n \geq 0$.

In general, we have

$$\begin{aligned} d_{nn} &\leq a(A+B)d_{(n-1)(n-1)}(A^* + B^*) \\ &\leq a^2(A+B)^2 d_{(n-2)(n-2)}(A^* + B^*)^2 \\ &\vdots \\ &\leq a^n(A+B)^n d_{00}(A^* + B^*)^n \end{aligned}$$

where $d_{00} = d(x_1, x_0) + d(y_1, y_0) = C$. Put $P = (A + B)$ and $P^* = (A + B)^*$, so we obtain

$$\begin{aligned} d_n n &\leq a^n P^n C (P^*)^n = a^n P^n C^{\frac{1}{2}} C^{\frac{1}{2}} (P^*)^n \\ &= a^n (P^n C^{\frac{1}{2}})(C^{\frac{1}{2}} (P^*)^n) = a^n (P^n C^{\frac{1}{2}})(P^n C^{\frac{1}{2}})^*. \end{aligned}$$

Since C positive element in $\mathcal{A}$, so $C^* = C$ and $(C^{\frac{1}{2}})^* = C^{\frac{1}{2}}$ hold.

For $n + 1 > m$ we get

$$\begin{aligned} d_{nm} &= d(x_{n+1}, x_m) + d(y_{n+1}, y_m) \\ &\leq a^n (P^n C^{\frac{1}{2}})(P^n C^{\frac{1}{2}})^* + a^{n-1} (P^{n-1} C^{\frac{1}{2}})(P^{n-1} C^{\frac{1}{2}})^* \\ &+ ... + a^m (P^m C^{\frac{1}{2}})(P^m C^{\frac{1}{2}})^* \\ &= \sum_{k=m}^{n} a^k (P^k C^{\frac{1}{2}})(P^k C^{\frac{1}{2}})^* \\ &= \sum_{k=m}^{n} |(P^k a^{\frac{k}{2}} C^{\frac{1}{2}})|^2. \end{aligned}$$

So, we obtain

$$\begin{aligned} \|d_{nm}\| &\leq \| \sum_{k=m}^{n} |(P^k a^{\frac{k}{2}} C^{\frac{1}{2}})|^2 \| \\ &\leq \sum_{k=m}^{n} \|P^k\|^2 \|a^k C^{\frac{1}{2}}\|^2 I \\ &= \|C^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P^k\|^2 \|a\|^{\frac{k}{2}} \\ &\leq \|C^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P a^{\frac{1}{4}}\|^{2k} I, (\|P^2\| \leq \|P\|^2. \end{aligned}$$

Put $P a^{\frac{1}{4}} = Q$, then

$$\|d_{nm}\| \leq \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I.$$

So, we have

$$\|d(x_{n+1}, x_m) + d(y_{n+1}, y_m)\| \leq \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I$$

and

$$\lim_{m \to \infty} \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I = 0_{\mathcal{A}}.$$

Therefore, (x_n) and (y_n) are cauchy sequences in X with respect to $\mathcal{A}$. So, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$$

since $(\mathrm{X}, \mathcal{A}, d)$ is a completed. By using inequalities (1), we get

$$\begin{aligned} 0 &\leq d(T(x,y),x) \leq a(d(T(x,y),x_{n+1}) + d(x_{n+1},x)) \\ &= a(d(T(x,y),T(x_n,y_n)) + d(x_{n+1},x)) \\ &\leq a(Ad(x,x_n)A^* + Bd(y,y_n)B^* + d(x_{n+1},x)). \end{aligned}$$

We obtain

$$\lim_{n\to\infty} d(x_n,x) = 0, \lim_{n\to\infty} d(y_n,y) = 0, \lim_{n\to\infty} d(x_{n+1},x) = 0.$$

Thus,

$$\lim_{n\to\infty} T(x,y) = x, \lim_{n\to\infty} T(y,x) = y$$

in a similar way. So, (x, y) is coupled fixed point.

Now, we assume that (x', y') is another coupled fixed point of T then we obtain some of equalities as follows:

$$\begin{aligned} d(x',x) &= d(T(x',y'),T(x,y)) \leq a(Ad(x',x)A^* + Bd(y',y)B^*) \\ d(y',y) &\leq a(Ad(y',y)A^* + Bd(x',x)B^*). \end{aligned}$$

So,we get

$$d(x',x)+d(y',y) = a(A+B)[d(x',x)+d(y',y)](A^*+B^*) = aP[d(x',x)+d(y',y)]P^*.$$

Hence,we obtain following equality by using $\|P^*\| \leq \|P\|$ since $P \in \mathcal{A}$

$$\|d(x',x) + d(y',y)\| \leq a\|P\|\|d(x',x) + d(y',y)\|\|P^*\| \leq \|d(x',x) + d(y',y)\|.$$

This is a contradiction, so we get

$$d(x',x) + d(y',y) = 0_{\mathcal{A}}.$$

Moreover, we obtain

$$d(x',x) = 0_{\mathcal{A}}, d(y',y) = 0_{\mathcal{A}}$$

since $d(x', x)$ and $d(y', y)$ are positive elements. So, we can easily seee that $x' = x$ and $y' = y$ satisfy. Therefore, T has a unique coupled fixed point. □

Corollary 3.2. if we choose $a = I$ in the Theorem 3.1, then T has a Unique Coupled Fixed Point in the C^*-algebra valued metric space.

Corollary 3.3. Let $(\mathrm{X}, \mathcal{A}, d)$ be a complete C^*-algebra valued b- metric space.We suppose that the mapping $T : X \times X \longrightarrow X$ satisfies

$$d(T(x,y),T(u,v)) \leq A[d(x,u) + d(y,v)]A^*,$$

for $x, y, u, v \in X$ and $A \in \mathcal{A}$, where $\|A\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

References

[1] B. Blackadar, *Theory of operator C^*-algebras and Von Neumann algebras*, Springer, 2006.

[2] L.J.B. Ciric, *A generalization of Banach's contraction Principle*, Proc. Amer. Math. Soc., 45 (1974), 267-273.

[3] K.M. Das, K.V. Naik, *Common fixed theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc., 77 (1979), 369-373.

[4] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, Amer. Math. Soc., Providence, 1996.

[5] L. Gajic, V. Rakocevic, *Pair of non-self mappings and common fixed points*, App. Math. Comput., 187 (2007), 999-1006.

[6] I. Gelfand, M. Naimark, *On the embedding of normed rings into the ring of operators in Hilbert space*, Math. Sb., 12 (1943), 197-213.

[7] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332 (2007), 1468-1476.

[8] G. Jungck, *Commuting maps and fixed points*, Amer Math. Monthly, 83 (1976), 261-263.

[9] Zhenhua Ma, Lining Jiang, Hongkai Sun, *C^*-algebra valued metric spaces and realated fixed point theorems*, Fixed Point Theory and thier applications, 206 (2014), 1-11.

[10] Zhenhua Ma, Lining Jiang, *C^*-algebra valued b-metric spaces and realated fixed point theorems*, Fixed Point Theory and Thier Applications, 222, 2015.

[11] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.

[12] Ö. Özer, S. Omran, *Common fixed point in C^*-algebra b-valued metric space*, In the Application of mathematics n technical and natural scences: 8th International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'16), 1773 (1), 050005, AIP Publishing, 2016.

[13] Ö. Özer, S. Omran, *On the generalized C^*-valued metric spaces related with Banach fixed point theory*, International Journal of Advanced and Applied Sciences, 4 (2017), 35-37.

[14] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[15] V. Rakocevic, *Functional analysis*, Naucna Knjiga, Beograd, 1994.

[16] Sh. Rezapour, R. Hamlbarani, *Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl., 345 (2008), 719-724.

[17] A.E. Sayed, S. Omran, A. Asad, *Fixed point theorems in quaternion-valued metric spaces*, Abstract and Applied Analysis, 2014.

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Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring

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Abstract. In this paper, we introduce the notion of tripolar fuzzy soft semiring, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideal over semiring and study some of their properties and the relations between them.

Keywords: tripolar fuzzy set, soft set, fuzzy soft set, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideal.

1. Introduction

The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty was first introduced by Zadeh [21]. Mandal [15] studied fuzzy ideals and fuzzy interior ideals in ordered semirings. Molodtsov [16] introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties, only partially resolves the problem is that objects in universal set often does not precisely satisfy the parameters associated to each of the elements in the set. Acar et al. [1] gave the basic concept of soft ring. Aktas and Cagman [3] introduced the concept of fuzzy subgroup which was extended by Aygunoglu

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and Aygun [5]. Majumdar and Samantha [14] extended soft sets to fuzzy soft set. Ghosh et al. [7] initiated the study of fuzzy soft rings and fuzzy soft ideals. Murali Krishna Rao [17] introduced and studied fuzzy soft ideals and fuzzy soft $k-$ideals over a $\Gamma-$semiring. There are many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interior valued fuzzy sets, vague sets, bipolar fuzzy sets, cubic sets etc. In 2001, Maji et al. [13] combined the concept of fuzzy set theory which was introduced by Zadeh in 1965 and the notion of soft set theory which was introduced by Molodstov [16] in 1999. Soft semirings was introduced by Feng [6] .

In 2000, Lee [10, 11, 12] used the term bipolar valued fuzzy sets and applied it to algebraic structure. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In 1994, Zhang [22] initiated the concept bipolar fuzzy sets as a generalization of fuzzy sets. The concepts of bipolar fuzzy soft sets introduced by Akram [2] introduced the notion of bipolar fuzzy soft lie subalgebras and studied some of their properties. Jun et al. [8, 9] introduced the notion of bipolar fuzzy ideals and bipolar fuzzy filters in BCI-algebras.

The ideals of "intuitionistic fuzzy set" was first introduced by Atanassov [4] as a generalization of notion of fuzzy set. Murali Krishna Rao [18] introduced the notion of tripolar fuzzy interior ideal of $\Gamma-$semigroup. Murali Krishna Rao and Venkateswarlu [19] introduced the notion of tripolar fuzzy set to be able to deal with tripoalr information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set and introduced the notion of tripolar fuzzy ideal and tripolar fuzzy interior ideal of $\Gamma-$semiring. In this paper,we introduce the notion of tripolar fuzzy soft ideals and interior ideals over semiring. We study some of their algebraic properties and the relations between them.

2. Preliminaries

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1. A universal algebra $(M, +, \cdot)$ is called a semiring if and only if $(M, +), (M, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c) = ab + ac$, $(a + b)c = ac + bc$, for all $a, b, c \in M$.

A semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$. A semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0x = x0 = 0$, for all $x \in M$. An element $1 \in M$ is said to be unity if for each $x \in M$ such that $x1 = 1x = x$. In a semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exists $b \in M$ such that $ba = 1(ab = 1)$. In a semiring M with unity 1, an element $a \in M$ is said to be invertible if there exists $b \in M$ such that $ab = ba = 1$. A semiring M with unity 1 is said to be division semiring if every non zero element of M is invertible. An element $a \in M$ is said to be regular element of M if there exists $x \in M$ such that $a = axa$. If every element of semiring M is a

regular then M is said to be regular semiring. An element $a \in M$ is said to be idempotent of M if $a = a^2$. Every element of M is an idempotent of M then M is said to be idempotent semiring M.

Definition 2.2. A non-empty subset A of semiring M is called

(i) a subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$;

(ii) a quasi ideal of M if A is a subsemiring of M and $AM \cap MA \subseteq A$;

(iii) a bi-ideal of M if A is a subsemiring of M and $AMA \subseteq A$;

(iv) an interior ideal of M if A is a subsemiring of M and $MAM \subseteq A$;

(v) a left (right) ideal of M if A is a subsemiring of M and $MA \subseteq A(AM \subseteq A)$;

(vi) an ideal if A is a subsemiring of $M, AM \subseteq A$ and $MA \subseteq A$;

(vii) a $k-$ideal if A is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$;

(viii) a left (right) bi- quasi ideal of M if A is a subsemiring of M and $MA \cap AMA$ $(AM \cap AMA) \subseteq A$;

(ix) a bi-quasi ideal of M if A is a left bi- quasi ideal and a right bi- quasi ideal of M.

A semiring M is a left (right) simple semiring if M has no proper left (right) ideal of M. A semiring M is said to be simple semiring if M has no proper ideals.

Let M be a non-empty set. Then a mapping $f : M \to [0, 1]$ is called a fuzzy subset of M. A fuzzy subset $\mu : M \to [0, 1]$ is non-empty if μ is not the constant function. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.3. An intuitionistic fuzzy set f of a non-empty set X is an object having the form $f = (\mu_f, \lambda_f) = \{(x, \mu_f(x), \lambda_f(x)) \mid x \in X\}$, where $\mu_f : X \to [0, 1]; \lambda_f : X \to [0, 1]$ are membership functions, $\mu_f(x)$ is a degree of membership, $\lambda_f(x)$ is a degree of non membership and $0 \leq \mu_f(x) + \lambda_f(x) \leq 1$, for all $x \in X$.

Definition 2.4. A bipolar fuzzy set A of a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \delta_A(x)) \mid x \in X\}$, where $\mu_A : X \to [0, 1]; \delta_A : X \to [-1, 0]$. $\mu_A(x)$ represents degree of satisfaction of an element x to the property corresponding to fuzzy set A and $\delta_A(x)$ represents degree of satisfaction of an element x to the implicit counter property of fuzzy set A.

Definition 2.5. A tripolar fuzzy set A in a universe set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$, where $\mu_A : X \to [0, 1]; \lambda_A : X \to [0, 1]; \delta_A : X \to [-1, 0]$ such that $0 \leq \mu_A(x) + \lambda_A(x) \leq$

1. The membership degree $\mu_A(x)$ characterizes the extent that the element x satisfies to the property corresponding to tripolar fuzzy set A, $\lambda_A(x)$ characterizes the extent that the element x satisfies to the not property (irrelevant) corresponding to tripolar fuzzy set A and $\delta_A(x)$ characterizes the extent that the element x satisfies to the implicit counter property of tripolar fuzzy set A. For simplicity $A = (\mu_A, \lambda_A, \delta_A)$ has been used for $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X$ and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$.

Remark 2.6. A tripolar fuzzy set A is a generalization of a bipolar fuzzy set and an intuitionistic fuzzy set. A tripolar fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X\}$ represents the taste of food stuffs. Assuming the sweet taste of food stuff as a positive membership value $\mu_A(x)$ i.e. the element x is satisfying the sweet property. Then bitter taste of food stuff as a negative membership value $\delta_A(x)$ i.e. the element x is satisfying the bitter property and the remaining tastes of food stuffs like acidic, chilly etc., as a non memberships value $\lambda_A(x)$. i.e., the element is satisfying irrelevant to the sweet property.

Definition 2.7. A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ of a semiring M is called a tripolar fuzzy subsemiring of M if A satisfies the following conditions.

(i) $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(x)\}$;

(ii) $\lambda_A(x + y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$;

(iii) $\delta_A(x + y) \leq \max\{\delta_A(x), \delta_A(y)\}$;

(iv) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(x)\}$;

(v) $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$;

(vi) $\delta_A(xy) \leq \max\{\delta_A(x), \delta_A(y)\}$, for all $x, y \in M$.

Definition 2.8. A tripolar fuzzy subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of a semiring M is called a tripolar fuzzy ideal of M if A satisfies the following conditions

(i) $\mu_A(xy) \geq \max\{\mu_A(x), \mu_{A)}(y)\}$;

(ii) $\lambda_A(xy) \leq \min\{\lambda_A(x), \lambda_A(y)\}$;

(iii) $\delta_A(xy) \leq \min\{\delta_A(x), \delta_A(y)\}$, for all $x, y \in M$.

Definition 2.9. A tripolar fuzzy subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of a semiring M is called a tripolar fuzzy interior ideal of M if A satisfies the following conditions

(i) $\mu_A(xzy) \geq \mu_A(z)$;

(ii) $\lambda_A(xzy) \leq \lambda_A(z)$;

(iii) $\delta_A(xzy) \leq \delta_A(z)$, for all $x, y, z \in M$.

Definition 2.10. Let $(f, A), (g, B)$ be fuzzy soft sets over a semiring M. The intersection of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cap (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \text{ for all } c \in A \cup B. \\ f_c \cap g_c, & \text{if } c \in A \cap B \ , \end{cases}$$

Definition 2.11. Let $(f, A), (g, B)$ be fuzzy soft sets over a semiring M. The union of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \text{for all } c \in A \cup B. \\ f_c \cup g_c, & \text{if } c \in A \cap B, \end{cases}$$

Definition 2.12. Let $(f, A), (g, B)$ be fuzzy soft sets over a semiring M. Then (f, A) and (g, B) is denoted by $(f, A) \wedge (g, B)$ is defined by $(f, A) \wedge (g, B) = (f \cap g, C) = (h, C)$, where $C = A \times B$, $h_c(x) = \min\{f_a(x), g_b(x)\}$, for all $c = (a, b) \in A \times B$ and $x \in M$.

Definition 2.13. Let $(f, A), (g, B)$ be fuzzy soft sets over an ordered semiring M. Then (f, A) or (g, B) is denoted by $(f, A) \vee (g, B)$ is defined by $(f, A) \vee (g, B) = (h, C)$, where $C = A \times B$ and $h_c(x) = \max\{f_a(x), g_b(x)\}$, for all $c = (a, b) \in A \times B, x \in M$.

3. Tripolar fuzzy soft ideals over semiring

In this section, we introduce the notion of tripolar fuzzy sets to be able to deal with tripolar information as a generalization of fuzzy sets, bipolar fuzzy set and intuitionistic fuzzy sets. We introduce the notion of tripolar fuzzy soft ideals and interior ideals over semiring.

Definition 3.1. A tripolar fuzzy soft set (f, A) over semiring M is called a tripolar fuzzy soft semiring over M if $f(a) = \{(\mu_{f(a)}(x), \lambda_{f(a)}(x), \delta_{f(a)}(x)) \mid x \in M, a \in A\}$, where $\mu_{f(a)}(x) : M \to [0, 1]; \lambda_{f(a)}(x) : M \to [0, 1]; \delta_{f(a)}(x) : M \to [-1, 0]$ such that $0 \leq \mu_{f(a)}(x) + \lambda_{f(a)}(x) \leq 1$ and for all $x \in M$ satisfying the following conditions

(i) $\mu_{f(a)}(x + y) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(x)\}$;

(ii) $\lambda_{f(a)}(x + y) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$;

(iii) $\delta_{f(a)}(x + y) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$;

(iv) $\mu_{f(a)}(xy) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(x)\}$;

(v) $\lambda_{f(a)}(xy) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$;

(vi) $\delta_{f(a)}(xy) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$, for all $x, y \in M$ and $a \in A$.

Definition 3.2. A tripolar fuzzy soft set (f, A) over a semiring M is called a tripolar fuzzy soft ideal over M if

(i) $\mu_{f(a)}(x + y) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$;

(ii) $\lambda_{f(a)}(x + y) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$;

(iii) $\delta_{f(a)}(x + y) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$;

(iv) $\mu_{f(a)}(xy) \geq \max\{\mu_{f(a)}(x), \mu_{f(a)}(x)\}$;

(v) $\lambda_{f(a)}(xy) \leq \min\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$;

(vi) $\delta_{f(a)}(xy) \leq \min\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$, for all $x, y \in M$ and $a \in A$.

Remark 3.3. Every tripolar fuzzy soft ideal (f, A) over a semiring M is a tripolar fuzzy soft semiring (f, A) over M but the converse is not true.

Example 3.4. Let $M = \{x_1, x_2, x_3\}$. Then we define operations with the following tables:

$+$	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_2	x_3
x_3	x_3	x_3	x_2

;

$.$	x_1	x_2	x_3
x_1	x_1	x_3	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

Let $E = \{a, b, c\}$ and $B = \{a, b\}$. Then (ϕ, B) is tripolar fuzzy soft set defined as $(\phi, B) = \{\phi(a), \phi(b)\}$, where
$\phi(a) = \{(x_1, 0.2, 0.7, -0.2), (x_2, 0.3, 0.6, -0.3), (x_3, 0.6, 0.3, -0.3)\}$
$\phi(b) = \{(x_1, 0.4, 0.5, -0.3), (x_2, 0.6, 0.3, -0.5), (x_3, 0.5, 0.4, -0.2)\}$.
Then (ϕ, B) is a tripolar fuzzy soft semiring over M but not a tripolar fuzzy soft ideal over M. And (ϕ, B) is tripolar fuzzy soft interior ideal over M.

Definition 3.5. A tripolar fuzzy soft (f, A) over semiring M is called a tripolar fuzzy soft interior ideal of M if

(i) $\mu_{f(a)}(xzy) \geq \mu_{f(a)}(z)$;

(ii) $\lambda_{f(a)}(xzy) \leq \lambda_{f(a)}(z)$;

(iii) $\delta_{f(a)}(xzy) \leq \delta_{f(a)}(z)$, for all $x, y, z \in M$ and $a \in A$.

Theorem 3.6. *Every tripolar fuzzy soft ideal over a semiring M is a tripolar fuzzy soft interior ideal over a semiring M.*

Proof. Let (f, A) be tripolar fuzzy soft ideal over a semiring M. Then $f(a) = \{\mu(a), \lambda(a), f(a)\}$ is a tripolar fuzzy ideal of $M, a \in A$. Then

(i) $\mu_{f(a)}(xzy) \geq \mu_{f(a)}(xz) \geq \mu_{f(a)}(z)$;

(ii) $\lambda_{f(a)}(xzy) \leq \lambda_{f(a)}(xz) \leq \lambda_{f(a)}(z)$;

(iii) $\delta_{f(a)}(xzy) \leq \delta_{f(a)}(xz) \leq \delta_{f(a)}(z)$, for all $x, y, z \in M$ and $a \in A$.

Hence (f, A) is a tripolar fuzzy soft interior ideal over M. □

Theorem 3.7. *Every tripolar fuzzy soft interior ideal over a regular semiring M is a tripolar fuzzy soft ideal over M.*

Proof. Let (f, A) be tripolar fuzzy soft interior ideal over a regular semiring M. Then $f(a) = \{\mu_{f(a)}, \lambda_{f(a)}, \delta_{f(a)}\}$ is a tripolar fuzzy ideal of $M, a \in A$.

Suppose $x, y \in M$. Then $xy \in M$. Then there exists $z \in M$ such that $xy = xyzxy$

$$\mu_{f(a)}(xy) = \mu_{f(a)}(xyzxy) = \mu_{f(a)}\big(xy(zxy)\big) \geq \mu_{f(a)}(y),$$
$$\mu_{f(a)}(xy) = \mu_{f(a)}\big((xyz)xy\big) \geq \mu_{f(a)}(x).$$

Hence $\mu_{f(a)}$ is a fuzzy ideal of M.

$$\lambda_{f(a)}(xy) = \lambda_{f(a)}(xyzxy) \leq \lambda_{f(a)}(y),$$
$$\lambda_{f(a)}(xy) = \lambda_{f(a)}\big((xyz)xy\big) \leq \lambda_{f(a)}(x).$$

Hence $\lambda_{f(a)}$ is a fuzzy ideal of M.

$$\delta_{f(a)}(xy) = \delta_{f(a)}(xyzxy) \leq \delta_{f(a)}(y),$$
$$\delta_{f(a)}(xy) \leq \delta_{f(a)}(x).$$

Hence $\delta_{f(a)}$ is a fuzzy ideal of M.

Therefore $f(a)$ is a tripolar fuzzy ideal of the semiring M.

Thus (f, A) is a tripolar fuzzy soft ideal over semiring M. □

Theorem 3.8. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over a semiring M then $(f, A) \wedge (g, B)$ is a tripolar fuzzy soft ideal over a semiring M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft ideals over a semiring M. Then by Definition 2.12, $(f, A) \wedge (g, B) = (f \cap g, C)$, where $C = A \times B$ and $(f \wedge g)(a, b) = f(a) \cap g(b)$, for all $(a, b) \in C$. Then

$$\begin{aligned}
\mu_{f(a)\cap g(b)}(x + y) &= \min\{\mu_{f(a)}(x + y), \mu_{g(b)}(x + y)\} \\
&\geq \min\big\{\min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}, \min\{\mu_{g(b)}(x), \mu_{g(b)}(y)\}\big\} \\
&= \min\big\{\min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}, \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\}\big\} \\
&= \min\{\mu_{f(a)\cap g(b)}(x), \mu_{f(a)\cap g(b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\lambda_{f(a)\cap g(b)}(x+y) &= \min\{\lambda_{f(a)}(x+y), \lambda_{g(b)}(x+y)\} \\
&\geq \min\big\{\max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}, \max\{\lambda_{g(b)}(x), \lambda_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\lambda_{f(a)}(x), \lambda_{g(b)}(x)\}, \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\}\big\} \\
&= \max\{\lambda_{f(a)\cap g(b)}(x), \lambda_{f(a)\cap g(b)}(y)\}. \\
\delta_{f(a)\cap g(b)}(x+y) &= \min\{\delta_{f(a)}(x+y), \delta_{g(b)}(x+y)\} \\
&\geq \min\big\{\max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}, \max\{\delta_{g(b)}(x), \delta_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\delta_{f(a)}(x), \delta_{g(b)}(x)\}, \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\}\big\} \\
&= \max\{\delta_{f(a)\cap g(b)}(x), \delta_{f(a)\cap g(b)}(y)\}, \\
\mu_{f(a)\cap g(b)}(xy) &= \min\{\mu_{f(a)}(xy), \mu_{g(b)}(xy)\} \\
&\geq \min\big\{\min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}, \min\{\mu_{g(b)}(x), \mu_{g(b)}(y)\}\big\} \\
&= \min\big\{\min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}, \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\}\big\} \\
&= \min\{\mu_{f(a)\cap g(b)}(x), \mu_{f(a)\cap g(b)}(y)\}, \\
\lambda_{f(a)\cap g(b)}(xy) &= \min\{\lambda_{f(a)}(xy), \lambda_{g(b)}(xy)\} \\
&\leq \min\big\{\max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}, \max\{\lambda_{g(b)}(x), \lambda_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\lambda_{f(a)}(x), \lambda_{g(b)}(x)\}, \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\}\big\} \\
&= \max\{\lambda_{f(a)\cap g(b)}(x), \lambda_{f(a)\cap g(b)}(y)\}, \\
\delta_{f(a)\cap g(b)}(xy) &= \min\{\delta_{f(a)}(xy), \delta_{g(b)}(xy)\} \\
&\leq \min\big\{\max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}, \max\{\delta_{g(b)}(x), \delta_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\delta_{f(a)}(x), \delta_{g(b)}(x)\}, \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\}\big\} \\
&= \max\{\delta_{f(a)\cap g(b)}(x), \delta_{f(a)\cap g(b)}(y)\}.
\end{aligned}$$

Hence $(f, A) \wedge (g, B)$ is a tripolar fuzzy soft ideal over a semiring M. □

Theorem 3.9. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a semiring M then $(f, A) \wedge (g, B)$ is a tripolar fuzzy interior ideal over semiring M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a semiring M. Obviously $(f, A) \wedge (g, B)$ is a soft tripolar fuzzy subsemiring of M.

By Definition 2.12, $(f, A) \wedge (g, B) = (f \cap g, C)$, where $C = A \times B$.

Suppose $(a, b) \in C, x, y \in M$. Then

$$\begin{aligned}
\mu_{f\wedge g(a,b)}(xyz) &= \mu_{f(a)\cap g(b)}(xyz) \geq \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\} = \mu_{f(a)\cap g(b)}(y) \\
&= \mu_{f\wedge g(a,b)}(y), \\
\lambda_{f\wedge g(a,b)}(xyz) &= \lambda_{f(a)\cap g(b)}(xyz) = \min\{\lambda_{f(a)}(xyz), \lambda_{g(b)}(xyz)\} \\
&\leq \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\} = \lambda_{f(a)\cap g(b)}(y) = \lambda_{f\wedge g(a,b)}(y),
\end{aligned}$$

$$\begin{aligned}\delta_{f\wedge g(a,b)}(xyz) &= \delta_{f(a)\cap g(b)}(xyz) = \min\{\delta_{f(a)}(xyz), \delta_{g(b)}(xyz)\} \\ &\leq \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\} = \delta_{f(a)\cap g(b)}(y) = \delta_{f\wedge g(a,b)}(y).\end{aligned}$$

Hence $(f, A) \wedge (g, B)$ is a soft tripolar fuzzy interior ideal of the semiring M. □

The proofs of following theorems are similar to Theorem 3.9.

Theorem 3.10. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a semiring M then $(f, A) \cup (g, B)$ is a tripolar fuzzy interior ideal of a semiring M.*

Theorem 3.11. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a semiring M then $(f, A) \cap (g, B)$ is a tripolar fuzzy interior ideal of a semiring M.*

Theorem 3.12. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over semiring M then $(f, A) \cup (g, B)$ is a tripolar fuzzy soft ideal over M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft ideals over semiring M. Then by Definition 2.11, we have $(f, A) \cup (g, B) = (h, C)$, where $C = A \cup B$, and

$$h(c) = f \cup g(c) = \begin{cases} f(c), & \text{if } c \in A \setminus B; \\ g(c), & \text{if } c \in B \setminus A; \\ f(c), \cup g(c) & \text{if } c \in A \cap B, \quad \text{for all} \quad c \in A \cup B. \end{cases}$$

Case (i). If $c \in A \setminus B$ then $f \cup g(c) = f(c)$. Thus we have

$$\begin{aligned}\mu_{f\cup g(c)}(x+y) &= \mu_{f(c)}(x+y) \geq \min\{\mu_{f(c)}(x), \mu_{f(c)}(y)\} \\ &= \min\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}, \\ \lambda_{f\cup g(c)}(x+y) &= \lambda_{f(c)}(x+y) \leq \max\{\lambda_{f(c)}(x), \lambda_{f(c)}(y)\} \\ &= \max\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}, \\ \delta_{f\cup g(c)}(x+y) &= \delta_{f(c)}(x+y) \leq \max\{\delta_{f(c)}(x), \delta_{f(c)}(y)\} \\ &= \max\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}, \\ \mu_{f\cup g(c)}(xy) &= \mu_{f(c)}(xy) \geq \max\{\mu_{f(c)}(xy), \mu_{f(c)}(xy)\} \\ &= \max\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}, \\ \lambda_{f\cup g(c)}(xy) &= \lambda_{f(c)}(xy) \leq \min\{\lambda_{f(c)}(x), \lambda_{f(c)}(y)\} \\ &= \min\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}, \\ \delta_{f\cup g(c)}(xy) &= \delta_{f(c)}(xy) \leq \min\{\delta_{f(c)}(x), \delta_{f(c)}(y)\} \\ &= \min\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.\end{aligned}$$

Case (ii). If $c \in B \setminus A$ then $f \cup g(c) = g(c)$.

Since $g(c)$ is a tripolar fuzzy ideal of semiring M, $f \cup g(c)$ is a tripolar fuzzy ideal of semiring M.

Case (iii). If $c \in A \cap B$ then $f \cup g(c) = f(c) \cup g(c)$.

$$\begin{aligned}
\mu_{f\cup g(c)}(x+y) &= \max\{\mu_{f(c)}(x+y), \mu_{g(c)}(x+y)\} \\
&\geq \max\{\min\{\mu_{f(c)}(x), \mu_{f(c)}(y)\}, \min\{\mu_{g(c)}(x), \mu_{g(c)}(y)\}\} \\
&= \min\{\max\{\mu_{f(c)}(x), \mu_{g(c)}(x)\}, \max\{\mu_{f(c)}(y), \mu_{g(c)}(y)\}\} \\
&= \min\{\mu_{f(c)\cup g(c)}(x), \mu_{f(c)\cup g(c)}(y)\} \\
&= \min\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}
\end{aligned}$$

Similarly we can prove

$$\begin{aligned}
\lambda_{f\cup g(c)}(x+y) &\leq \max\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\} \\
\delta_{f\cup g(c)}(x+y) &\leq \max\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.
\end{aligned}$$

$$\begin{aligned}
\mu_{f\cup g(c)}(xy) &= \mu_{f(c)\cup g(c)}(xy) \\
&= \max\{\mu_{f(c)}(xy), \mu_{g(c)}(xy)\} \\
&\geq \max\{\max\{\mu_{f(c)}(x), \mu_{f(c)}(y)\}, \max\{\mu_{g(c)}(x), \mu_{g(c)}(y)\}\} \\
&= \max\{\max\{\mu_{f(c)}(x), \mu_{g(c)}(x)\}, \max\{\mu_{f(c)}(y), \mu_{g(c)}(y)\}\} \\
&= \max\{\mu_{f(c)\cup g(c)}(x), \mu_{f(c)\cup g(c)}(y)\} \\
&= \max\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}
\end{aligned}$$

Similarly we can prove

$$\begin{aligned}
\lambda_{f\cup g(c)}(xy) &\leq \min\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\} \\
\delta_{f\cup g(c)}(xy) &\leq \min\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.
\end{aligned}$$

Therefore $f \cup g(c)$ is a tripolar fuzzy ideal of M.

Hence $(f, A) \cup (g, B)$ is a tripolar fuzzy soft ideal over M. □

Corollary 3.13. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over semiring M then $(f, A) \cap (g, B)$ is a tripolar fuzzy soft ideal over M.*

Definition 3.14. Let (f, A) and (g, B) be two tripolar fuzzy soft sets over a semiring M. The the product (f, A) and (g, B) is defined as $(f, A) \circ (g, B) = (f \circ g, C)$, where $C = A \cup B$. Then by Definition 2.13, we have $(f, A) \cup (g, B) = (h, C)$, where $C = A \cup B$; and

$$\mu_{(f\cup g)(c)}(x) = \begin{cases} \mu_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \mu_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \sup_{x=yz} \{\min\{\mu_{f(c)}(y), \mu_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

$$\lambda_{(f\cup g)(c)}(x) = \begin{cases} \lambda_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \lambda_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \inf_{x=yz} \{\max\{\lambda_{f(c)}(y), \lambda_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

$$\delta_{(f\cup g)(c)}(x) = \begin{cases} \delta_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \delta_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \inf_{x=yz} \{ \max\{\delta_{f(c)}(y), \delta_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

Theorem 3.15. *If (f, A) and (g, B) are tripolar fuzzy soft interior ideals over semiring M then $(f, A)\circ(g, B)$ is a tripolar fuzzy soft interior ideal over semiring M.*

Proof. Obviously $(f, A)\circ(g, B)$ is a tripolar fuzzy soft subsemiring over M. Let $x, y, z \in M$. By Definition 3.14, $(f, A) \circ (g, B) = (f \circ g, C)$, where $C = A \cup B$ and $c \in C, x \in M$.

Case (i). If $c \in A \setminus B$ then

$$\mu_{f\circ g(c)} = \mu_{f(c)};\ \lambda_{f\circ g(c)} = \lambda_{f(c)};\ \delta_{f\circ g(c)} = \delta_{f(c)}.$$

Since (f, A) is a tripolar fuzzy soft interior ideal over M, $f \circ g(c)$ is a tripolar fuzzy soft interior ideal of M.

Case (ii). If $c \in B \setminus A$ then

$$\mu_{f\circ g(c)} = \mu_{g(c)}; \lambda_{f\circ g(c)} = \lambda_{g(c)}; \delta_{f\circ g(c)} = \delta_{g(c)}.$$

Since (g, B) is a tripolar fuzzy soft interior ideal over M, $f \circ g(c)$ is a tripolar fuzzy soft interior ideal of M.

Case (iii). If $c \in A \cap B$ then

$$\begin{aligned} \mu_{f\circ g(c)}(x) &= \sup_{x=ab} \{ \min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}\}. \\ \mu_{f\circ g(c)}(xyz) &= \sup_{x=ab} \{ \min\{\mu_{f(a)}(xyz), \mu_{g(b)}(xyz)\}\} \\ &\geq \sup_{x=ab} \{ \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\}\} = \mu_{f\circ g(c)}(y). \\ \lambda_{f\circ g(c)}(xyz) &= \inf_{x=ab} \{ \max\{\lambda_{f(a)}(xyz), \lambda_{g(b)}(xyz)\}\} \\ &\leq \sup_{x=ab} \{ \max\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\}\} = \lambda_{f\circ g(c)}(y). \\ \delta_{f\circ g(c)}(xyz) &= \inf_{x=ab} \{ \max\{\delta_{f(a)}(xyz), \delta_{g(b)}(xyz)\} \\ &\leq \inf_{x=ab} \{ \max\{\delta_{f(a)}(y), \delta_{g(b)}(y)\}\} = \delta_{f\circ g(c)}(y). \end{aligned}$$

Hence $f \circ g(c)$ is a tripolar fuzzy interior ideal of M.

Therefore $(f, A) \circ (g, B)$ is a tripolar fuzzy interior ideal over M. □

Theorem 3.16. *Let E be a parameter set and $\sum_E(M)$ be the set of all tripolar fuzzy soft interior ideals over semiring M. Then $(\sum_E(M), \cup, \cap)$ forms complete distributive lattices along with the relation $\subseteq$.*

Proof. Suppose (f, A) and (g, B) be soft interior ideals over M such that $A \subseteq E, B \subseteq E$.

By Theorems 3.10 and 3.11, $(f, A) \cap (g, B)$ and $(f, A) \cup (g, B)$ are tripolar fuzzy soft interior ideals over M.

Obviously $(f, A) \cap (g, B)$ is lub of $\{(f, A), (g, B)\}$ and $(f, A) \cup (g, B)$ is glb of $\{(f, A), (g, B)\}$.

Every sub collection of $\sum_E(M)$ has lub and glb.

Hence $\sum_E(M)$ is a complete lattice.

We can prove $(f, A) \cap \big((g, B) \cup (h, C)\big) = \big((f, A) \cap (g, B)\big) \cup \big((f, A) \cap (h, C)\big)$.

Therefore $(\sum_E(M), \cup, \cap)$ forms a complete distributive lattice. □

4. Conclusion

In this paper,we introduced the notion of tripolar fuzzy soft subsemiring, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideals over semiring and studied some of their algebraic properties and relations between them.

References

[1] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings*, Comput. Math. Appli., 59 (2010), 3458-3463.

[2] M. Akram, *Bipolar fuzzy soft lie algebras*, Quasigroups and related Systems, 21 (2013), 1-10

[3] H. Aktas and N. Cagman, *Soft sets and soft groups*, Information Science, 177 (2007), 2726-2735.

[4] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy sets and systems, 20 (1986), 87-96.

[5] A. Aygunoglu and H. Aygun, *Introduction to fuzzy soft group*, Comput. Math. Appli., 58 (2009), 1279-1286.

[6] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Comput. Math. Appli., 56 (2008), 2621-2628.

[7] J. Ghosh, B. Dinda and T. K. Samanta, *Fuzzy soft rings and fuzzy soft ideals,* Int. J. P. App. Sci. Tech., 2 (2011), 66-74.

[8] Y. B. Jun, H.S. Kim and K. J. Lee, *Bipolar fuzzy translation in BCK/BCI algebra*, J. of the Chungcheong Math. Soc., 22 (2009), 399-408.

[9] K. H. Kim and Y. B. Jun, *Intuitionistic fuzzy interior ideals of semigroups*, Int. J. of Math. and Math. Sci., 27 (2001), 261-267.

[10] K. J. Lee, *Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI algebras,* Bull. Malays. Math. Soc., 32 (2009), 361-373.

[11] K. M. Lee, *Bipolar valued fuzzy sets and their applications*, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 2000, 307-312.

[12] K. M. Lee, *Comparison of interval valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets,* J. Fuzzy Logic Intelligent Systems, 14 (2004), 125-129.

[13] P. K. Maji, R. Biswas and A. R. Roy, *Fuzzy soft sets*, The J. of Fuzzy Math., 9 (2001), 589-602.

[14] P. Majumdar and S. Samanta, *Generalized fuzzy soft set*, Comput. Math. Appl., 58 (2010), 1279-1286.

[15] D. Mandal, *Fuzzy ideals and fuzzy interior ideals in ordered semirings,* Fuzzy Inf. Engg., 6 (2014), 101-114

[16] D. A. Molodtsov, *Soft set theory-first results*, Computers and Math. with Appli., 37 (1999), 19-31.

[17] M. Murali Krishna Rao, *Fuzzy soft ideal, Fuzzy soft bi-ideal, Fuzzy soft quasi-ideal and fuzzy soft interior ideal over ordered* $\Gamma-$*Semiring,* Asia Pac. J. Math., 5 (2018), 60-84.

[18] M. Murali Krishna Rao, *Tripolar fuzzy interior ideal of* Γ*-semigroup*, Anl. of Fuzzy Math. and Info., 15 (2018), 199206.

[19] M. Murali Krishna Rao and B. Venkateswarlu, *Tripolar fuzzy interior ideal of* Γ*-semiring*, Asia Pac. J. Math., 5 (2018), 192-207.

[20] A. Rosenfeld, *Fuzzy groups,* J. Math. Anal. Appl., 35 (1971), 512-517.

[21] L. A. Zadeh, *Fuzzy sets*, Information and control, 8 (1965), 338-353.

[22] W. R. Zhang, *Bipolar fuzzy sets*, Proc. of FUZZ-IEEE, 1998, 835-840.

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Urysohn lemma in semi-linear uniform spaces

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Abstract. In topology Urysohn Lemma is widely applicable, where it is commonly used to construct continuous functions with various properties on normal space. In this paper we shall present Urysohn Lemma in semi linear uniform spaces, besides we shall give a characterization of the closure in semi-linear uniform space, then we shall use this characterization to answer the question which given in [12], by A.Tallafha and R. Khalil namely (If $\rho(x, A) = \Delta$, must $x \in A^l$).

Keywords: uniform spaces, Semi-linear uniform spaces, topological spaces, metric spaces, types of metric spaces.

1. Introduction

One of the most important generalizations of a metric spaces is uniform space. Uniform spaces is a concept lies between metric spaces and topological structure, that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. The uniform spaces have been studied extensively through years. The notion of uniformity has been investigated by several mathematician such as Weil [18] ,[19] , and [20] . L.W. Cohen [4] , and [5] . Graves [7]. The theory of uniform spaces was given by Burbaki in [3] . Also Wiel's in his booklet [20] , defined the notion of uniformly continuous mapping.

In 2009, the notion of a uniform space led A. Tallafha and R. Khalil to define a beautiful space which is a mixture of analysis and topology, namely semi-linear uniform space [12], also they studied some cases of best approximation in such spaces, besides they defined a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on semi-linear uniform space.

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Semi-linear uniform space is weaker than metric space and stronger than topological space since in [12], [13] , [14] and [15] , A. Tallafha answered the question"Is there a semi-linear uniform space which is not metrizable?". Also he defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Finally he studied the relation between ρ and δ, he showed that $\rho(x, y) = \rho(s, t)$ if and only if $\delta(x, y) = \delta(s, t)$, then he defined Lipschitz condition and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions. In [16] and [17], A. Tallafha and S. Alhihi established another properties of semi-linear uniform spaces.

In [11], A. Rawshdeh and A. Tallafha answered the question (If f is a contraction from a complete semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point) negatively, they gave an example of a complete semi-linear uniform space (X, Γ) and a contraction $f : (X, \Gamma) \to (X, \Gamma)$ which has infinitely many fixed points.

Other nice result was given in [2], by S. Alhihi and M. Fayyad, where they showed that every semi-linear uniform space induced a Tychonoff space (X, T_Γ).

2. Semi-linear uniform spaces

Let X be a none empty set and D_X be a collection of all relations on X such that each element V of D_X is reflexive and symmetric. D_X is called the family of all entourages of the diagonal.

Now the above discussion allow us to define the uniform space.

Definition 2.1 ([6])**.** *Let X be a set. A uniform space is the pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a subfamily of D_X which satisfies the following conditions:*

1) *If $U \in \mathcal{F}$ and $U \subseteq W \in D_X$, then $W \in \mathcal{F}$.*
2) *If $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$.*
3) *There exists a $W \in \mathcal{F}$ such that $W \circ W \subseteq U$, for every $U \in \mathcal{F}$,*
4) $\bigcap_{U \in F} U = \Delta$.

The notion of a uniform space led A. Tallafha and R. Khalil in 2009 to define the semi-linear uniform space:

Definition 2.2 ([12])**.** *Let Γ be a subcollection of D_X such that:*

(i) If V_1 and V_2 are in Γ, then $V \cap V_2 \in \Gamma$.
(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.
(iii) $\bigcap_{V \in \Gamma} V = \Delta$.
(vi) $\bigcup_{V \in \Gamma} V = X \times X$.
(v) Γ is a chain (a chain in $X \times X$ we mean a totally or linearly) ordered collection of subsets of $X \times X$, ordered by set inclusion.

Then the pair (X, Γ) is called a semi-linear uniform space.

In [12] and [13] , the authors defined the set valued map ρ and δ which played an important rule in the theory of fixed point on semi-linear uniform spaces.

Definition 2.3 ([12]). *Let (X,Γ) be a semi-linear uniform space. For $(x,y)\in X\times X$, let $\Gamma_{(x,y)}=\{V\in\Gamma:(x,y)\in V\}$. Then, the set valued map ρ on $X\times X$, is defined by $\rho(x,y)=\bigcap\{V:V\in\Gamma_{(x,y)}\}$.*

Clearly for all $(x,y)\in X\times X$, we have $\rho(x,y)=\rho(y,x)$, and $\Delta\subseteq\rho(x,y)$.

The following definition given in [13] , the authors defined $\delta(x,x)=\phi$, now if we define $\delta(x,x)=\rho(x,x)=\Delta$, then all the results in the literature still valid and the new definition of $\delta(x,x)$ seems to be more convenient.

Definition 2.4 ([13]). *Let (X,Γ) be a semi-linear uniform space. For $(x,y)\in X\times X$, define $\Gamma^c_{(x,y)}=\{V\in\Gamma:(x,y)\notin V\}$. Then the set valued map δ on $X\times X$, is defined by*

$$\delta(x,y)=\left\{\begin{array}{c}\cup\{V:V\in\Gamma^C_{(x,y)}\}:\ x\neq y\\ \Delta:\ x=y\end{array}\right\}.$$

By using the set valued map ρ and δ, A.Tallafha gave some important properties of semi-linear uniform spaces, some of these properties are given in the following proposition.

Proposition 2.5 ([13]). *Let (X,Γ) be a semi-linear uniform space. Then:*
i) If $V\in\Gamma^c_{(x,y)}$, then $V\subsetneqq\rho(x,y)$.
ii) $\delta(x,y)\subseteq\rho(x,y)$ for all $(x,y)\in X\times X$.
iii) If $V\in\Gamma_{(x,y)}$, then $\delta(x,y)\subseteq V$.
iv) If $(x,y)\in\rho(s,t)$, then $\rho(x,y)\subseteq\rho(s,t)$.
v) If $(x,y)\in\delta(s,t)$, then $\delta(x,y)\subseteq\delta(s,t)$.

In [1] Alhihi gave more properties of semi-linear uniform spaces as:

Theorem 2.6. *Let $A\in\mathbf{\Lambda}$, and σ a sub collection of $\mathbf{\Lambda}$. For $n\in\mathbb{N}$, we have:*
(i) $n\left(\frac{1}{n}A\right)\subseteq A$.
(ii) If $B\in\Gamma$ satisfies $nB\subseteq A$, then $B\subseteq\frac{1}{n}A$.
(iii) $\frac{1}{n+1}A\subseteq\frac{1}{n}A$.
(iv) $\frac{1}{n}A\subseteq A$.
(v) $\frac{1}{n}\bigcap_{A\in\sigma}A=\bigcap_{A\in\sigma}\frac{1}{n}A$.
(vi) $\bigcup_{A\in\sigma}\frac{1}{n}A\subseteq\frac{1}{n}\bigcup_{A\in\sigma}A$.

It is known that every metric space endues a semi-linear uniform space, the following define the semi-linear uniform space which induced by a metric space (X,d).

Definition 2.7. [13]. *Let (X,d) be a metric space. Define $V_\epsilon=\{(x,y):d(x,y)<\epsilon\}$. Then (X,Γ) is a semi-linear uniform space induced by (X,d) where $\Gamma=\{V_\epsilon:\epsilon>0\}$. This semi-linear uniform space will be denoted by (X,Γ_d).*

In [13], A. Tallafha gave the following example, which is a semi-linear uniform space but not metrizable.

Example 2.8. $(\mathbb{R}, \Gamma)$ is a semi-linear uniform space, where $\Gamma = \{V_\epsilon, \epsilon > 0\}, V_\epsilon = \{(x, y) : x^2 + y^2 < \epsilon\} \cup \{\Delta\}$.

In [11], A. Rawshdeh and A. Tallafha, give the following lemma:

Lemma 2.9. *Let (X, Γ_d) be a semi-linear uniform space induced by the metric space (X, d). Then:*
(1) $\rho(x, y) = \{(s, t) \in X \times X : d(s, t) \leq d(x, y)\}$.
(2) $\delta(x, y) = \{(s, t) \in X \times X : d(s, t) < d(x, y)\}$.

Definition 2.10 ([9])**.** *Let (X, d) be a metric space. For $x \in X, r > 0$, let $B[x, r] = \{t : d(x, t) \leq r\}$. A metric space (X, d) is convex, if for all $x, y \in X$, $B[x, r_1] \cap B[y, r_2] \neq \phi$ whenever $r_1 + r_2 \geq d(x, y)$.*

In [10] and [11], A. Rawshdeh and A. Tallafha gave the following results.

Lemma 2.11. *Let (X, Γ_d) be a semi-linear uniform space induced by convex metric space (X, d), then:*
1) $n\rho(x, y) = \{(s, t) \in X \times X : d(s, t) \leq nd(x, y)\}$.
2) $n\delta(x, y) \subseteq \{(s, t) \in X \times X : d(s, t) < nd(x, y)\}$.

Also in [12], A. Tallafha and R. Khalil, give the definition of continuos function, uniformly continuous function, converges of sequences in semi-linear uniform spaces and complete semi-linear space.

Definition 2.12. [12]. *Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$, then:*
1- f is continuous at $x_\circ$ if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x, x_\circ) \in V$, then $(f(x), f(x_\circ)) \in U$.
2- f is uniformly continuous if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x, y) \in V$, then $(f(x), f(y)) \in U$.

Definition 2.13. [12]**.** *Let (X, Γ) be a semi-linear uniform space and (x_n) be a sequence in X. Then:*
1- (x_n) converges to x in X and denoted by $x_n \to x$, if for every $V \in \Gamma$ there exists k such that $(x_n, x) \in V$ for every $n \geq k$.
2- (x_n) is called Cauchy if for every $V \in \Gamma$ there exists k such that $(x_n, x_m) \in V$ for every $n, m \geq k$.

Definition 2.14. [12]**.** *Let (X, Γ) be a semi-linear uniform space. Then (X, Γ) is called complete, if every Cauchy sequence is convergent.*

3. Some topological properties of semi-linear uniform spaces

In this section, we will shed more light on the topology induced by a semi-linear uniform space,we shall state and discuss some important topological properties of semi-linear uniform spaces.

After A. Tallafha and R. Khalil in [12], gave the definition of semi-linear uniform space, they gave the the following definition:

Definition 3.1 ([12]). *Let (X,Γ) be a semi-linear uniform space. For $x \in X$ and $V \in \Gamma$, the open ball of center x and radius V to be $B(x,V) = \{y : (x,y) \in V\}$. Equivalently $B(x,V) = \{y : \rho(x,y) \subseteq V\}$.*

Also, they showed the following:

Theorem 3.2 ([12]). *Let (X,Γ)be a semi-linear uniform space. The family $\tau = \{G \subseteq X$: for every $x \in G$ there exists a $U \in \Gamma$ such that $B(x,U) \subseteq G\}$ is topology on the set X.The topology which induced by (X,Γ) is denoted by τ_Γ.*

In [2], S. Alhihi and M. Fayyad, discussed the interior of a set and a neighborhood of $x \in X$ with respect to the topology induced by a semi-linear uniform space on the set X, they gave the following proposition and corollaries:

Proposition 3.3 ([2]). *Let (X,Γ)be a semi-linear uniform space and $A \subseteq X$. Then $int(A) = \{x \in X$: there exists a $U \in \Gamma$ such that $B(x,U) \subseteq A\}$ with respect to τ_Γ.*

Corollary 3.4 ([2]). *If the topology of a space X is induced by a semi-linear uniform Γ, then for every $x \in X$ and any $U \in \Gamma$ the set int $B(x,U)$ is a neighborhood of x.*

Corollary 3.5 ([2]). *If the topology of a space X is induced by a semi-linear uniform Γ, then for every $x \in X$ and any $A \subseteq X$ we have $x \in \bar{A}$ if and only if $A \cap B(x,U) \neq \phi$ for every $U \in \Gamma$.*

Also, they gave the following definition:

Definition 3.6 ([2]). *A family $\mathcal{B} \subset \Gamma$ is called a base for the semi-linear uniform Γ if for every $U \in \Gamma$ there exists a $W \in \mathcal{B}$ such that $W \subset U$.*

Now we can conclude from above definitions the following remark and Theorem.

Remark 3.1. Any base $\mathcal{B}$ for a semi-linear uniform space on a set X has the following properties:

1) If $U_1, U_2 \in \mathcal{B}$ there exists a $U \in \mathcal{B}$ such that $U \subseteq U_1 \cap U_2$.
2) For every $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $2W \subseteq U$.
3) $\bigcap \mathcal{B} = \Delta$.

Theorem 3.7. *Let (X,Γ)be a semi-linear uniform space. If $\mathcal{B} \subset \Gamma$ is a base for Γ then $\beta_x = \{B(x,W) : W \in \beta\}$ is a neighborhood base at x.*

Proof. Let (X,Γ)be a semi-linear uniform space and $\mathcal{B} \subset \Gamma$ is a base for Γ. To show $\beta_x = \{B(x,W) : W \in \beta\}$ is a neighborhood base at x we want to show the following:

1) $x \in B(x,W)$ for each $W \in \beta$ as $(x,x) \in \Delta \subseteq W$.

2) If $B(x,U), B(x,V) \in \beta_x$, then there exists $W \in \beta$ such that $W \subseteq U \cap V$. Hence $B(x,W) \subseteq B(x,U) \cap B(x,V)$.

3) If $B(x,U) \in \beta_x$, then $U \in \beta$.So there exists $V \in \beta$ such that $V \circ V \subseteq U$. Thus if $y \in B(x,V)$ and $z \in B(y,V)$ then $(x,z) \in V \circ V \subseteq U$, so $z \in B(x,U)$. Hence $B(y,V) \subseteq B(x,U)$,that is there exists $B(y,V) \in \beta_y$ and $B(y,V) \subseteq B(x,U)$. □

Proposition 3.8 ([2]). *If the topology of a space X is induced by a semi-linear uniform Γ and $\mathcal{B} \subset \Gamma$ is a base for Γ, then for every $x \in X$ and any $A \subseteq X$ we have $x \in \bar{A}$ if and only if $A \cap B(x,U) \neq \phi$ for every $U \in \beta$.*

In [12], A. Tallafha and R. Khalil, proved that the topology induced by semi-linear uniform space is T_2-space.Therefore the topology $(\mathbb{R}, \tau_{lef})$ where $\tau_{lef} = \{(-\infty, \alpha) : \alpha \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$ is not induced by semi-linear uniform space. Moreover in [2], S. Alhihi and M. Fayyad, showed that every semi-linear uniform space induced a Tychonoff space (X, τ_Γ), where τ_Γ is the topology on X induced by local base B_x. For more topological properties of τ_Γ we refer the reader to [2].

Definition 3.9. *Let Γ_1,Γ_2 be two semi-linear uniform spaces on a set X. We say Γ_1 is weaker than Γ_2 if and only if for all $U \in \Gamma_1$ there exists $V \in \Gamma_2$ such that $V \subseteq U$.*

Definition3.9, lead us to give the following definition:

Definition 3.10. *Let Γ_1,Γ_2 be two semi-linear uniform spaces on a set X. We say Γ_1 is equivalent to Γ_2 if Γ_1 is weaker than Γ_2 and Γ_2 is weaker than Γ_1.*

In metric space any two equivalent metric spaces induce two equivalent topologies, but we can find two equivalent metric spaces give two different semi-linear uniform spaces, for example if $d_1$is Euclidean metric on $\mathbb{R}$ and define $d_2(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$ for $x, y \in \mathbb{R}$.Then $\Gamma_{d_1} = \{U_\epsilon : \epsilon > 0\}$, where $U_\epsilon = \{(s,t) : d_1(s,t) < \epsilon\}$ is the induced semi-linear uniform space by d_1. Also $\Gamma_{d_2} = \{V_\epsilon : \epsilon > 0\}$, where $V_\epsilon = \{(s,t) : d_2(s,t) < \epsilon\}$ is the induced semi-linear uniform space by d_2. To show that Γ_{d_1} and Γ_{d_2} are different, if $\epsilon \geq 2$ then$V_\epsilon = \mathbb{R}^2$ and $\mathbb{R}^2 \notin \Gamma_{d_1}$.Thus $\Gamma_{d_1} \neq \Gamma_{d_2}$. Now the topology which induce by d_1 is the same as the topology which induce by d_2, since $f : \mathbb{R} \to (-1,1)$ which defined by $f(x) = \frac{x}{1+|x|}$ is homeomorphism and $d_1(f(x), f(y)) = d_2(x,y)$. But if we have two equivalent semi-linear uniform spaces we can get two equivalent topologies. For more details we have the following proposition:

Proposition 3.11. *Two equivalent semi-linear uniform spaces induce two equivalent topologies.*

Proof. Let Γ_1,Γ_2 be two equivalent semi-linear uniform spaces on a set X induce two topologies $\tau_{\Gamma_1}, \tau_{\Gamma_2}$ respectively. To show $\tau_{\Gamma_1}$and $\tau_{\Gamma_2}$are equivalent, we want to show $\tau_{\Gamma_1} \subseteq \tau_{\Gamma_2}$ and $\tau_{\Gamma_2} \subseteq \tau_{\Gamma_1}$. Now let $x \in G \in \tau_{\Gamma_1}$, then there exists $U \in \Gamma_1$ such that $B(x,U) \subseteq G$, as $\Gamma_1$is weaker than Γ_2, so there exists $V \in \Gamma_2$ such that $V \subseteq U$, which means $B(x,V) \subseteq B(x,U) \subseteq G$. Hence $G \in \tau_{\Gamma_2}$. The same proof for $\tau_{\Gamma_2} \subseteq \tau_{\Gamma_1}$. □

The following example is an example of two equivalent semi-linear uniform spaces:

Example 3.12. Let $X = \mathbb{R}$. Define $\Gamma_1 = \{V_t : 0 < t < \infty\}$ where $V_t = \{(x, y) : y - t < x < y + t$, and $-\infty < y < \infty\}$ and define $\Gamma_2 = \{V_t : 0 < t < \infty\}$ where $V_t = \{(x, y) : y - t \leq x \leq y + t$, and $-\infty < y < \infty\}$.Then $\Gamma_1 \neq \Gamma_2$, but $\Gamma_1$and Γ_2 are two equivalent semi-linear uniform spaces induce two topologies $\tau_{\Gamma_1}, \tau_{\Gamma_2}$ respectively. Clearly $\tau_{\Gamma_1}$and $\tau_{\Gamma_2}$are the usual topology $\mathbb{R}$.

Now we shall give some interesting facts about semi-linear uniform space:

Lemma 3.13. *Let (X, Γ)be a semi-linear uniform space. For $V \in \Gamma, U \subseteq X \times X$ we have $V \circ U \circ V = \bigcup\{B(x, V) \times B(y, V) : (x, y) \in U\}$.*

Proof. Let (X, Γ)be a semi-linear uniform space. If $(s, t) \in V \circ U \circ V$, then there exists $x, y \in X$ such that $(s, x) \in V, (x, y) \in U$ and $(y, t) \in V$, which means $s \in B(x, V)$ and $t \in B(y, V)$. Hence $(s, t) \in B(x, V) \times B(y, V)$ for $(x, y) \in U$. Now if $(s, t) \in B(x, V) \times B(y, V)$ for $(x, y) \in U$, then $(s, x), (t, y) \in V$. Thus $(s, t) \in V \circ U \circ V$. □

The proof of the following proposition is clear from above discussion:

Proposition 3.14. *If the topology of a space X is induced by a semi-linear uniform Γ, and $M \subseteq X \times X$, then for $(x, y) \in X \times X$we have $(x, y) \in \overline{M}$ if and only if $M \cap B(x, U) \times B(y, U) \neq \phi$ for every $U \in \Gamma$.*

The closure of a subset $M \subseteq X \times X$ is defined as the smallest closed subset of $X \times X$ that is contain M. The following theorem gives a useful and a new characterization of closure in term of member of semi-linear uniform space:

Theorem 3.15. *Let (X, Γ)be a semi-linear uniform space. If $M \subseteq X \times X$ then $\overline{M} = \cap\{V \circ M \circ V : V \in \Gamma\}$.*

Proof. Let (X, Γ)be a semi-linear uniform space. Let $(x, y) \in \overline{M}$ and $V \in \Gamma$,then $B(x, V) \times B(y, V) \cap M \neq \phi$, so there exists $(s, t) \in B(x, V) \times B(y, V) \cap M$, which means $(x, y) \in B(s, V) \times B(t, V)$ for $(s, t) \in M$, by Lemma 3.13, $(x, y) \in V \circ M \circ V$. Hence $\overline{M} \subseteq \cap\{V \circ M \circ V : V \in \Gamma\}$. The converse is clear by Proposition 3.14. □

Corollary 3.16. *Let (X, Γ)be a semi-linear uniform space. $\{\overline{V} : V \in \Gamma\}$ is a base for Γ.*

Proof. Let $U \in \Gamma$. Then there exists $V \in \Gamma$ such that $V \circ V \circ V \subseteq U$, by Theorem 3.15, $\overline{V} \subseteq V \circ V \circ V \subseteq U$. □

Now we will give the following definition:

Definition 3.17. *Let (X, Γ)be a semi-linear uniform space and $A \subseteq X$. For $U \in \Gamma$ define $B(A, U) = \bigcup_{x \in A} B(x, U)$.*

Proposition 3.18. *Let (X,Γ)be a semi-linear uniform space with a base β and $A \subseteq X$. Then $\bar{A} = \bigcap\{B(A,V) : V \in \beta\}$.*

Proof. Let (X,Γ)be a semi-linear uniform space and $A \subseteq X$. Let $x \in \bar{A}$. For $V \in \beta$ there exists $W \in \Gamma$such that $W \circ W \subseteq V$, so by Corollary 3.5, there exists $y \in A \cap B(x,W) \neq \phi$. As $y \in A$ and $y \in B(x,W)$, so $x \in B(y,W) \subseteq B(y,V) \subseteq B(A,V)$. Hence $x \in \bigcap\{B(A,V) : V \in \beta\}$. Now let $x \in \bigcap\{B(A,V) : V \in \beta\}$ then $x \in B(A,V)$ for all $V \in \beta$. Let $V \in \beta$, since $x \in B(A,V)$, then there exists $t \in A$ such that $x \in B(t,V)$, so $t \in B(x,V) \cap A$ which implies $x \in \bar{A}$. □

Corollary 3.19. *Let (X,Γ)be a semi-linear uniform and $A \subseteq X$. If $B(A,U) = A$ for some $U \in \Gamma$ then A is both open and closed.*

Proof. Let (X,Γ)be a semi-linear uniform and $A \subseteq X$. If $x \in \bar{A}$, then $x \in \bigcap\{B(A,V) : V \in \Gamma\}$, so $x \in B(A,U) = A$. To show that A is open, suppose $x \in A$, then $B(x,U) \subseteq A$. Hence A is open. □

In [12], A. Tallafha and R. Khalil, proved that if $x \in A^l$ then $\rho(x,A) = \Delta$. Then they asked the following question: If $\rho(x,A) = \Delta$, must $x \in A^l$. Now we shall answer this question:

Proposition 3.20. *If $\rho(x,A) = \Delta$ then $x \in \bar{A}$.*

Proof. To show that $x \in \bar{A}$, we want to show $A \bigcap B(x,U) \neq \phi$ for every $U \in \Gamma$. Suppose not, i.e., there exists $U \in \Gamma$ such that $A \bigcap B(x,U) = \phi$, which means $(x,\alpha) \notin U$ for all $\alpha \in A$, by Proposition 2.5, $U \subsetneq \rho(x,\alpha)$ for all $\alpha \in A$. Thus $U \subsetneq \bigcap_{\alpha \in A} \rho(x,\alpha) \subseteq \Delta$, which is contradiction. Hence $A \bigcap B(x,U) \neq \phi$ for every $U \in \Gamma$, and so $x \in \bar{A}$. □

So we have the following corollary:

Corollary 3.21. *$x \in \bar{A}$ if and only if $\rho(x,A) = \Delta$.*

Now we shall discuss some concept related to continuous function:

Theorem 3.22. *Let $(X,\tau_\Gamma),(Y,\Gamma_{\tau_Y})$be two topologies which induced by the semi-linear uniform spaces (X,Γ_X) and (Y,Γ_Y) respectively. Then $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$ is continuous if and only if $f : (X,\tau_{\Gamma_X}) \longrightarrow (Y,\tau_{\Gamma_Y})$ is continuous.*

Proof. Let$f : (X,\Gamma_X) \to (Y,\Gamma_Y)$be continuous at $x_\circ$. For $V \in \Gamma_Y$ take $B(f(x_\circ),V)$. Since $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$is continuous at $x_\circ$ then there exists $U \in \Gamma_X$ such that $(x,x_\circ) \in U$ then $(f(x),f(x_\circ)) \in V$. If $z \in f(B(x_\circ,U))$ then there exists $x \in B(x_\circ,U)$ such that $z = f(x)$, which means $(x,x_\circ) \in U$, and so $(f(x),f(x_\circ)) \in V$. Hence $f(x) \in B(f(x_\circ),V)$.To proof the converse, suppose $f : (X,\tau_{\Gamma_X}) \longrightarrow (Y,\tau_{\Gamma_Y})$ is continuous at $x_\circ$. For $V \in \Gamma_Y$ take $B(f(x_\circ),V)$. As $f : (X,\tau_{\Gamma_X}) \longrightarrow (Y,\tau_{\Gamma_Y})$ is continuous at $x_\circ$, so there exists open set $O \in \tau_{\Gamma_X}$ such that $x_\circ \in O$ and $f(O) \subseteq B(f(x_\circ),V)$, but as $x_\circ \in O \in \tau_{\Gamma_X}$, there exists $W \in \Gamma_X$ such that $B(x_\circ,W) \subseteq O$. So if $(x,x_\circ) \in W$ then $x \in B(x_\circ,W) \subseteq O$, which means $f(x) \in f(O) \subseteq B(f(x_\circ),V)$. Hence $(f(x),f(x_\circ)) \in V$. □

Let (X, Γ) be a semi-linear uniform space. For $A \subseteq X$, we can define a semi-linear uniform space on A by $\Gamma_A = \{(A \times A) \cap U : U \in \Gamma\}$.

Theorem 3.23. *Let (X, Γ) be a semi-linear uniform space and $A \subseteq X$. Then (A, Γ_A) can be defined by $\Gamma_A = \{(A \times A) \cap U : U \in \Gamma\}$ is a semi-linear uniform space on A and (A, Γ_A) is called the subspace of the semi-linear uniform space (X, Γ).*

Proof. Let (X, Γ)be a semi-linear uniform and $A \subseteq X$. For $U \in \Gamma$ define $A_U = (A \times A) \cap U$. From the definition of Γ_A, we note that for all $A_U \in \Gamma_A$ we have $\{(\alpha, \alpha) : \alpha \in A\} \subseteq A_U$ and $A_U = A_U^{-1}$. Now to complete the proof we need to prove the following:

(i) If A_{V_1} and A_{V_2} are in Γ_A, then $A_{V_1} \cap A_{V_2} = (A \times A) \cap (V_1 \cap V_2) \in \Gamma_A$ as $V_1 \cap V_2 \in \Gamma$. (ii) For every $A_V \in \Gamma_A$, there exists $U \in \Gamma$ such that $U \circ U \subset V$. Let $(s, t) \in [(A \times A) \cap U] \circ [(A \times A) \cap U]$. Then there exists $y \in X$ such that $(s, y) \in [(A \times A) \cap U]$ and $(y, t) \in [(A \times A) \cap U]$, so (s, y),$(y, t) \in U$ which means $(s, t) \in U \circ U \subset V$. Thus $[(A \times A) \cap U] \circ [(A \times A) \cap U] \subseteq (A \times A) \cap V = A_V$. (iii) $\bigcap_{V \in \Gamma} A_V \subseteq (\bigcap_{V \in \Gamma} V) \cap A \times A = \Delta_A$.(vi) Since (X, Γ) is a semi-linear uniform, so $\bigcup_{V \in \Gamma} V = X \times X$. Thus $\bigcup_{V \in \Gamma} A_V = A \times A$.(v) Since Γ is a chain, so Γ_A is a chain. □

From the definition of (A, Γ_A) we can note, if τ_Γ is the topology on X induced by Γ, then (A, τ_A) is the subspace topology on A.

Theorem 3.24. *If $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ is continuous, then $f : (A, \Gamma_A) \to (Y, \Gamma_Y)$ is continuous.*

Proof. Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be continuous. Let $V \in \Gamma_Y$. Then there exists $U \in \Gamma_X$, such that $(x, x_\circ) \in U$, implies $(f(x), f(x_\circ)) \in V$. So $A_U = (A \times A) \cap U$ is the set required. □

Theorem 3.25. *Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be uniform continuous. Then $f : (A, \Gamma_A) \to (Y, \Gamma_Y)$ is uniformly continuous.*

Proof. Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be uniform continuous. For $V \in \Gamma_Y$, there exists $U \in \Gamma_X$, such that $(x, y) \in U$, implies $(f(x), f(y)) \in V$. So $A_U = (A \times A) \cap U$ is the set required. □

Now we will present a new definition:

Definition 3.26. *Let (X, Γ)be a semi-linear uniform space and $A, B \subseteq X$, $U \in \Gamma$. If $(A \times B) \cap U = \phi$, then we say A and B are $U-$apart and A, B are apart if there exists $U \in \Gamma$such that A, B are $U-$apart.*

Direct application on the previous definition is the following proposition:

Proposition 3.27. *Let (X, Γ)be a semi-linear uniform space and A, B be $U-$apart. If $V \circ V \circ V \subseteq U$ for $V \in \Gamma$, then $B(A, V)$ and $B(B, V)$are $V-$apart.*

Proof. Let (X,Γ)be a semi-linear uniform space and $A,B\subseteq X$ be $U-$apart. if $(s,t)\in(B(A,V)\times B(B,V))\cap V$, then $s\in B(A,V)$ and $t\in B(B,V)$, which means $(a,s)\in V$ and $(b,t)\in V$ for $a\in A$ and $b\in B$, but $(s,t)\in V$ so we have $(a,b)\in V\circ V\circ V\subseteq U$. Which contradicts $A,B\subseteq X$ are $U-$apart. □

In topology Urysohn Lemma is widely applicable, where it is commonly used to construct continuous functions with various properties on normal space. So we shall present one of the most important facts in semi−linear uniform spaces which is called by Urysohn Lemma in semi-linear uniform spaces. Urysohn Lemma in semi-linear uniform spaces is the surprising fact in this work, the proof of this lemma will increase the strength and beauty of the previous definitions that have been defined in this paper or in the other research related to semi−linear uniform spaces. Now we will present the lemma:

Lemma 3.28 (Urysohn Lemma in semi-linear uniform spaces.)**.** *Let $A,C\subseteq X$ be apart.Then there exists a uniformly continuos function $f:(X,\Gamma)\to[0,1]$, such that $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in C$.*

Proof. Let $A,C\subseteq X$ be apart. There exists $U\in\Gamma$ such that $(A\times C)\cap U=\phi$, and as (X,Γ) is semi-linear uniform space, so for all $U\in\Gamma$ there exists $V\in\Gamma$ such that $V\circ V\subseteq U$, so we can define the following sequence $(U_0,U_1,U_2.....)$ as the following: $U_0=U$, $U_n\circ U_n\subseteq U_{n-1}$, for every $n=1,2,3....$ For $n=1,2,....$and $k=0,1,...,2^n$, define $A_r,C_r\subseteq X$ for every $r=\frac{k}{2^n}$ as the following: $A_\circ=A,A_1=X,C_\circ=X$, $C_1=C$, where $A_{\frac{2k-1}{2^n}}=X-B(C_{\frac{k}{2^{n-1}}},U_n)$ and $C_{\frac{2k-1}{2^n}}=X-B(A_{\frac{k-1}{2^{n-1}}},U_n)$. It is clear that $A_{\frac{m-1}{2^n}}$ and $C_{\frac{m}{2^n}}$ are U_n-apart $n=0,1,2,....$ and $m=1,...,2^n$. Note that $X=A_r\cup C_r$, now define the following increasing assignment: $f:r\to A_r$ by $f(x)=inf\{r:x\in A_r\}$. It is clear $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in C$. For every $\epsilon>0$ then there is $n>0$ such that $\epsilon>\frac{1}{2^{n-1}}$. Now for $x,y\in X$ with $f(x)+\epsilon\leq f(y)$ there exists $m\in\{1,2,3,...2^n\}$ such that $f(x)<\frac{m-1}{2^n}<\frac{m}{2^n}<f(y)$. Hence if $x\in A_{\frac{m-1}{2^n}}$ and $y\notin A_{\frac{m}{2^n}}$ we have $y\in C_{\frac{m}{2^n}}$, so $U_n\subsetneq\rho(x,y)$. Thus for every $(x,y)\in U_n$, we have $(x,y)\notin A_{\frac{m-1}{2^n}}\times C_{\frac{m}{2^n}}$, which means $|(f(x)-f(y)|<\epsilon$. □

References

[1] S.A. Alhihi, *More properties of semi linear uniform spaces*, Journal of applied mathematics, 6, online, 2015.

[2] S.A. Alhihi, M.A. Al-Fayyad, *Topological properties of semi-linear uniform spaces*, Global Journal of Pure and Applied Mathematics, 6 (2016), 5329-5341.

[3] Bourbaki, *Topologie générale (General topology)*, Paris 1940. ISBN 0-387-19374-X

[4] L.W. Cohen, *Uniformity properties in a topological space satisfying the first denumerability postulate*, Duke Math. J., 3 (1937), 610-615.

[5] L.W. Cohen, *On imbedding a space in a complete space*, Duke Math. J., 5 (1939), 174-183.

[6] R. Engelking, *General topology*, North-Holand, Amsterdam, 1968.

[7] L.M. Graves, *On the completing of a Housdroff space*, Ann. Math., 38 (1937), 61-64.

[8] I.M. James, *Topological and uniform spaces*, Undergraduate Texts in Mathematics, Springer-Verlag, 1987.

[9] K. Menger, *Untersuchungen ūber allegemeine Metrik,* Math. Ann., 100 (1928), 75-163.

[10] A. Rawashdeh, A. Tallafha, *Fixed point in modified semi-linear uniform spaces*, Global Journal of Pure and Applied Mathematics, 9 (2017), 4461-4476.

[11] A. Rawashdeh, A. Tallafha, *Fixed point in semi-linear uniform spaces and convex metric spaces*, submitted.

[12] A. Tallafha, R. Khalil, *Best approximation in uniformity type spaces*, European Journal of Pure and Applied Mathematics, 2 (2009), 231-238.

[13] A. Tallafha, *Some properties of semi-linear uniform spaces*, Boletin da sociedade paranaense dematematica, 2 (2011), 9-14.

[14] A. Tallafha, *Open problems in semi-linear uniform spaces*, J. Applied Functional Analysis, 2, 223-228.

[15] A. Tallafha, *Fixed point in semi-linear uniform spaces*, European Journal of Mathematical Sciences, 3 (2013), 334-340.

[16] A. Tallafha, S. Alhihi, *Fixed point in a non-metrizable spaces*, Accepted for publication as a part of the proceedings volume "Computational Analysis: Contributions from AMAT2015" in Springer, New York.

[17] A. Tallafha, S. Alhihi, *Contraction on semi-linear uniform space*, Submitted.

[18] A. Weil, *Les recouvrements des espaces topologiques: espaces complete, espaces bicompact*, C. R. Acad. Paris, 202, 1936, 1002-1005.

[19] A. Weil, *Sur les espaces a structure uniforme et sur la topologie generale*, Act. Sci. Ind., 551, Paris, 1937.

[20] A. Weil, *Sur les espaces à structure uniforme et sur la topologic générale*, Paris, 1938.

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Upper and lower nearly (I, J)-continuous multifunctions

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Abstract. In this paper the authors introduce and study upper and lower nearly (I, J)-continuous multifunctions. Some characterizations and several properties concerning upper (lower) nearly (I, J)-continuous multifunctions are obtained. The results improves many results in Literature.

Keywords: nearly (I, J)-continuous multifunctions, I-open set, I-closed set, lower nearly (I, J)-continuous multifunctions, upper almost nearly (I, J)-continuous multifunctions.

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1. Introduction

It is well known today, that the notion of multifunction is playing a very important role in general topology, upper and lower continuity have been extensively studied on multifunctions $F : (X, \tau) \to (Y, \sigma)$. Currently using the notion of topological ideal, different types of upper and lower continuity in multifunction $F : (X, \tau, I) \to (Y, \sigma)$ have been studied and characterized [2], [7], [8], [9], [14], [17]. The concept of ideal topological spaces has been introduced and studied by Kuratowski [12] and the local function of a subset A of a topological space (X, τ) was introduced by Vaidyanathaswamy [16] as follows: Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called the local function of A with respect to τ and I, is defined as follows: for $A \subseteq X$, $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I$ for every $U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(,)$ for a topology $\tau^*(\tau, I)$ called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by A^*. In 1990, Jankovic and Hamlett [10], introduced the notion of I-open set in a topological space (X, τ) with an ideal I on X. In 1992, Abd El-Monsef et al. [1] further investigated I-open sets and I-continuous functions. In 2007, Akdag [2], introduce the concept of I-continuous multifunctions in a topological space with and ideal on it. Given a multifunction $F : (X, \tau) \to (Y, \sigma)$, and two ideals I, J on X and Y respectively. Now with the topological spaces (X, τ, I) and (Y, σ, J), consider the multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$. We want to study some type of upper and lower continuity of F. In this paper, we introduce and study a new class of multifunction called a nearly (I, J)-continuous multifunctions in topological spaces. Investigate its relation with another class of continuous multifunctions given in the Literature.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if I is an ideal on X, (X, τ, I) mean an ideal topological space. For a subset A of (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular open [15] (resp. semiopen [11], preopen [13] , semi preopen [4]) if $A = \mathrm{int}(\mathrm{cl}(A))(resp. A \subseteq \mathrm{cl}(\mathrm{int}(A)), A \subseteq \mathrm{int}(\mathrm{cl}(A)), A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))))$. The complement of regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. A subset S of (X, τ, I) is an I-open [10], if $S \subseteq \mathrm{int}(S^*)$. The complement of an I-open set is called I-closed set. The I-closure and the I-interior, can be defined in the same way as $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively, will be denoted by $I\,\mathrm{cl}(A)$ and $I\mathrm{int}(A)$, respectively. The family of all I-open (resp. I-closed, regular open, regular closed, semiopen, semi closed, preopen, semi-preclosed) subsets of a (X, τ, I), denoted by $IO(X)$ (resp.

$IC(X), RO(X), RC(X), SO(X), SC(X), PO(X), SPO(X), SPC(X))$. We set $IO(X,x) = \{A : A \in IO(X) \text{ and } x \in A\}$. It is well known that in a topological space (X, τ, I), $X^* \subseteq X$ but if the ideal is codense, that is $\tau \cap I = \emptyset$, then $X \subseteq X^*$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y, denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 2.1 ([2]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be*

1. *upper I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an I-open set U containing x such that $U \subseteq F^+(V)$.*

2. *lower I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists an I-open set U containing x such that $U \subseteq F^-(V)$.*

3. *I-continuous if it is both upper I-continuous and lower I-continuous.*

Definition 2.2 ([6]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be*

1. *upper semi continuous at a point $x \in X$ if for each open set V of Y with $F(x) \in V$, there exists an open set U containing x such that $F(U) \subseteq V$.*

2. *lower semi continuous at a point $x \in X$ if for each open set V of Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(a) \cap V \neq \emptyset$ for all $a \in U$.*

Definition 2.3. *A subset A of a topological space (X, τ) is said to be N-closed [6] if every cover of A by regular open sets of X has a finite subcover.*

Definition 2.4 ([8]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper nearly continuous at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U containing x such that $F(U) \subset V$.*

2. *lower nearly continuous at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) nearly continuous on X if it has this pro-perty at every point of X.*

Definition 2.5 ([9]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper almost nearly continuous at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U containing x such that $F(U) \subset int(\mathrm{cl}(V))$.*

2. *lower almost nearly continuous at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap int(\mathrm{cl}(V)) \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) almost nearly continuous on X if it has this property at every point of X.*

Definition 2.6 ([5]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper nearly I-continuous at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an I-open set U containing x such that $F(U) \subset V$.*

2. *lower nearly I-continuous at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having N-closed complement, there exists an I-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) nearly I-continuous on X if it has this property at every point of X.*

Definition 2.7 ([7]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper almost nearly I-continuous at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an I-open set U containing x such that $F(U) \subset int(\mathrm{cl}(V))$.*

2. *lower almost nearly I-continuous at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having N-closed complement, there exists an I-open set U of X containing x such that $F(u) \cap int(\mathrm{cl}(V)) \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) almost nearly I-continuous on X if it has this property at every point of X.*

3. Upper and lower nearly (I, J)-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper nearly (I, J)-continuous at a point $x \in X$ if for each J-open set V containing $F(x)$ and having N-closed complement, there exists an I-open set U containing x such that $F(U) \subset V$.*

2. *lower nearly (I, J)-continuous at a point $x \in X$ if for each J-open set V of Y meeting $F(x)$ and having N-closed complement, there exists an I-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. upper (resp. lower) nearly (I, J)-continuous on X if it has this property at every point of X.

Example 3.2. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a multifunction $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = \{a\}$, $f(b) = \{c\}$ and $f(c) = \{b\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, \{a, b\}, \{a, c\}\}$.

In consequence, f is upper nearly (I, J)-continuous on X.

Example 3.3. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b, c\}\} = \sigma$ and two ideals $I = \{\emptyset, \{b\}\} = J$. Define a multifunction $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = \{a\}$, $f(b) = \{c\}$ and $f(c) = \{b\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X\{c\}, \{a, c\}, \{b, c\}\}$.

In consequence, f is not upper nearly (I, J)-continuous.

Recall that if (X, τ, I) is an ideal topological space and I is the empty ideal, then for each $A \subseteq X$, $A^* = cl(A)$, that is to said, every I-open set is a pre-open set, in consequence, if $f : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is upper nearly $(I, \{\emptyset\})$-continuous, then f is upper nearly I-continuous.

Example 3.4. $f : (X, \tau, I) \to (Y, \sigma)$ upper nearly I-continuous but $f : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is not upper nearly $(I, \{\emptyset\})$-continuous.

Now consider (X, τ, I) and (Y, σ, J) two ideals topological spaces, If $J \neq \{\emptyset\}$, then the concepts of upper nearly (I, J)-continuous and upper nearly I-continuous are independent, as we can see in the following examples.

Example 3.5. In the Example 3.2, the multifunction f is upper nearly (I, J)-continuous on X but is not upper nearly I-continuous on X.

Example 3.6. In the Example 3.3, the multifunction f is upper nearly I-continuous on X but is not upper nearly (I, J)-continuous on X.

Example 3.7. Let $\mathbb{R}$ be the set of real numbers with the discrete topology τ_d and $I = \{\emptyset\}$=J. Consider the multifunction $F : (\mathbb{R}, \tau_d, I) \to (\mathbb{R}, \tau_d, J)$ defined as follows: $F(x) = \{x\}$ for all $x \in \mathbb{R}$. It is easy to see that: F is upper (resp. lower) nearly (I, J)-continuous on X.

Remark 3.8. It is easy to see that if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$. If F is upper (lower) nearly I-continuous, then F is upper (lower) nearly (I, J)-continuous. Even more, if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$, we can find upper (resp. lower) nearly (I, J)-continuous on X that are not upper (lower) nearly I-continuous.

The following theorem characterize the upper nearly (I, J) continuous multifunctions.

Theorem 3.9. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is upper nearly (I, J)-continuous.*
2. *$F^+(V)$ is I-open for each J-open set V of Y having N-closed complement.*
3. *$F^-(K)$ is I-closed for every N-closed and J-closed subset K of Y.*
4. *$I\,\mathrm{cl}(F^-(B)) \subset F^-(J\,\mathrm{cl}(B))$ for every subset B of Y having N-closed J-closure.*
5. *$F^+(J\mathit{int}(B)) \subset I\mathit{int}(F^+(B))$ for every subset B of Y such that $Y\backslash J\mathit{int}(B)$ is N-closed.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be any J- open set of Y having N-closed complement. From (1), there exists an I-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since any union of I-open sets is I-open, $F^+(V)$ is I-open in (X, τ).

(2)⇒(3): Let K be any N-closed and J- closed set of Y. Then by (2), $F^+(Y\backslash K) = X\backslash F^-(K)$ is an I-open set. Then it is obtained that $F^-(K)$ is an I-closed set.

(3)⇒(4): Let B be any subset of Y having N-closed J- closure. By (3), we have $F^-(B) \subset F^-(J\,\mathrm{cl}(B)) = I\,\mathrm{cl}(F^-(J\,\mathrm{cl}(B)))$. Hence $I\,\mathrm{cl}(F^-(B)) \subset I\,\mathrm{cl}(F^-(J\,\mathrm{cl}(B))) = F^-(J\,\mathrm{cl}(B))$.

(4)⇒(5): Let B be a subset of Y such that $Y\backslash J\mathrm{int}(B)$ is N-closed.

Then by (4), we have $X\backslash I\mathrm{int}(F^+(B)) = I\,\mathrm{cl}(X\backslash F^+(B)) = I\,\mathrm{cl}(F^-(Y\backslash B)) \subset F^-(J\,\mathrm{cl}(Y\backslash B)) = F^-(Y\backslash J\mathrm{int}(B)) = X\backslash F^+(J\mathrm{int}(B))$. Therefore, we obtain $F^+(J\mathrm{int}(B)) \subset I\mathrm{int}(F^+(B))$.

(5)⇒(1): Let $x \in X$ and V be any J-open set of Y containing $F(x)$ and having N-closed complement. Then by (5), $x \in F^+(V) = F^+(J\mathrm{int}(V)) \subset I\mathrm{int}(F^+(V))$. In consequence, there exists an I-open set U containing x such that $U \subset F^+(V)$; hence $F(U) \subset V$. This shows that F is upper nearly I-continuous. □

Theorem 3.10. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is lower nearly (I, J)-continuous.*
2. *$F^-(V)$ is I-open for each J-open set V of Y having N-closed complement.*
3. *$F^+(K)$ is I-closed for every N-closed and J-closed set K of Y.*
4. *$I\,\mathrm{cl}(F^+(B)) \subset F^+(J\mathit{cl}(B))$ for every subset B of Y having N-closed closure.*

5. *$F^-(Jint(B)) \subset Iint(F^-(B))$ for every subset B of Y such that $Y\backslash Jint(B)$ is N-closed.*

Proof. The proof is similar to that of Theorem 3.9. □

Corollary 3.11. *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is upper nearly (I, J)-continuous (resp. lower nearly (I, J)-continuous) if $F^-(K)$ is I-closed (resp. $F^+(K)$ is I-closed) for every N-closed set K of Y.*

Proof. Let G be any J-open set of Y having N-closed complement. Then $Y\backslash G$ is N-closed. By the hypothesis, $X\backslash F^+(G) = F^-(Y\backslash G) = I\text{int}(F^-(Y\backslash G)) = I\,\text{cl}(X\backslash F^+(G)) = X\backslash I\text{int}(F^+(G))$ and hence, $F^+(G) = I\text{int}(F^+(G))$. It follows from Theorem 3.9, that F is upper nearly (I, J)-continuous. The proof of lower nearly (I, J)-continuous is entirely similar. □

Definition 3.12. *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper (I, J)-continuous at a point $x \in X$ if for each J-open set V containing $F(x)$, there exists an I-open set U containing x such that $F(U) \subset V$.*

2. *lower (I, J)-continuous at a point $x \in X$ if for each J-open set V of Y meeting $F(x)$, there exists an I-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) (I, J)-continuous on X if it has this property at every point of X.*

Example 3.13. The Multifunction defined in Example 3.7 is upper nearly (I, J)-continuous on X but is not upper (I, J)-continuous on X.

Remark 3.14. Every upper (resp. lower) (I, J)-continuous multifunction on X is upper (resp. lower) nearly (I, J)-continuous multifunction on X, but the converse is not necessarily true, as we can see in the following example.

Example 3.15. The Multifunction defined in Example 3.2 is upper nearly (I, J)-continuous on X but is not upper (I, J)-continuous.

Theorem 3.16. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is upper (I, J)-continuous.*

2. *$F^+(V)$ is I-open for each J-open set V of Y.*

3. *$F^-(K)$ is I-closed for every J-closed subset K of Y.*

4. *$I\,\text{cl}(F^-(B)) \subset F^-(J\,\text{cl}(B))$ for every subset B of Y.*

5. *For each point $x \in X$ and each J-open set V containing $F(x)$, $F^+(V)$ is an I-open containing x.*

6. *For each point $x \in X$ and each J-open set containing $F(x)$, there exist an I-open set U containing x such that $F(U) \subseteq V$.*

Proof. The proof is similar to that of Theorem 3.9. □

Theorem 3.17. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ and $F : (Y, \sigma, J) \to (Z, \beta, K)$ be multifunctions. If F is upper nearly (I, J)-continuous (upper (I, J)-continuous) and G upper (I, J)-continuous (upper nearly (I, J)-continuous), then $F \circ G : (X, \tau, I) \to (Z, \beta, K)$ is upper nearly (I, J)-continuous.*

Definition 3.18. *An ideal topological space (X, τ, I) is said to be I-compact [3] if every cover of X by I-open sets have a finite subcover.*

Definition 3.19. *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper (I, J)-irresolute at a point $x \in X$ if for each I-open set U containing x, there exists an I-open set V containing $F(x)$ such that $F(U) \subset V$.*

2. *lower (I, J)-irresolute at a point $x \in X$ if for each j-open set V of Y meeting $F(x)$, there exists an I-open set U containing x such that $U \subseteq F^{-}(V)$.*

3. *upper (resp. lower) (I, J)-irresolute on X if it has this property at every point of X.*

Theorem 3.20. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a surjective (I, J)-irresolute multifunction such that $F(x)$ is J-compact for each $x \in X$. If (X, τ, I) is I-compact, then (Y, σ, J) is J-compact.*

Proof. Let $\{V_i : i \in \triangle\}$ be a J-open cover of Y. For each $x \in X$, there exists a finite subset $\triangle(x)$ of $\triangle$ such that $F(x) \subseteq \bigcup\{V_i : i \in \triangle(x)\}$. Consider $V(x) = \bigcup\{V_i : i \in \triangle(x)\}$. Then $F(x) \subseteq V(x) \in JO(Y)$. using the fact that F is (I, J)-irresolute, then there exist an $U(x) \in IO(X)$ such that $F(U(x)) \subset V(x)$. Now using the that F is surjective, then the collection $\{U(x) : x \in X\}$ is an I-open cover of X. In consequence, there exists a finite number of points of X, say, $x_1, x_2, ...x_n$ such that $X = \bigcup_{i=1}^{n}\{U(x_i)\}$. It follows that $F(X) = F(\bigcup_{i=1}^{n}\{U(x_i)\} \subseteq \bigcup_{i=1}^{n}\{F(U(x_i))\}) \subseteq \bigcup_{i=1}^{n}\{V(x_i)\} \subseteq \bigcup_{i=1}^{n}\bigcup_{i\in\triangle(x_i)} U(x_i)$. It follows that Y is J-compact. □

Definition 3.21. *A multifunction $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper almost nearly (I, J)-continuous at a point $x \in X$ if for each J-open set V containing $F(x)$ and having N-closed complement, there exists an I-open set U containing x such that $F(U) \subset int(J\,\mathrm{cl}(V)))$.*

2. *lower almost nearly (I, J)-continuous at a point $x \in X$ if for each J-open set V of Y meeting $F(x)$ and having N-closed complement, there exists an I-open set U of X containing x such that $F(u) \cap int(J\,\mathrm{cl}(V)) \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) almost nearly (I, J)-continuous on X if it has this property at every point of X.*

Example 3.22. Let $X = \mathbb{R}$ the set of real numbers with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $Y = \mathbb{R}$ with the topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = \{\emptyset\}$=J. Define $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(x) = \mathbb{Q}$ if $x \in \mathbb{Q}$ and $F(x) = \mathbb{R} \setminus \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. It is easy to see that F is upper (resp. lower) almost nearly (I, J)-continuous on X.

It is clear that every upper (resp. lower) (I, J)-continuous multifunction is upper (resp. lower) nearly (I, J)-continuous multifunction and every upper (resp. lower) nearly (I, J)-continuous multifunction is upper (resp. lower) almost nearly (I, J)-continuous multifunction but the converse in both cases is not true in general as shown in the following examples.

Example 3.23. Let $\mathbb{R}$ with the finite complement topology τ_c and with the discrete topology, take $I = \{\emptyset\} = J$. Consider the multifunction $F : (\mathbb{R}, \tau_c, I) \to (\mathbb{R}, \tau_d, J)$ defined as follows: $F(x) = \{x\}$ for all $x \in \mathbb{R}$. It is easy to see that: F is upper (resp. lower) nearly (I, J)-continuous on X but is not upper (resp. lower) (I, J)-continuous on X.

Example 3.24. The multifunction F defined in Example 3.22 is upper (resp. lower) nearly almost (I, J)-continuous on X but is not upper (resp. lower) nearly (I, J)-continuous on X.

At this point, there are a question. Given a multifunction $F : (X, \tau, I) \to (Y, \sigma)$. It is possible to write a characterization for upper (resp. lower) nearly almost (I, J)-continuous on X.

References

[1] M.E. Abd El-Monsef, E.F. Lashien, A.A. Nasef, *On I-open sets and I-continuous functions,* Kyungpook Math. J., 32 (1992), 21-30.

[2] M. Akdag, *On upper and lower I-continuous multifunctions*, Far East J. Math. Sci., 25 (2007), 49-57.

[3] A. Al-Omari and M. S. M. Noorani, *Contra-I-continuous and almost I-continuous functions*, Int. J. Math. Math. Sci., (9) (2007), 169-179.

[4] D. Andrijevic, *Semi-preopen sets*, Mat. Vesnik, 38 (1986), 24-32.

[5] C. Arivazhagi and N. Rajesh, *Nearly $\mathcal{J}$-continuous multifunctions,* Bol. Soc. Paran. Mat., 37 (2019), 33-38.

[6] D. Carnahan, *Locally nearly compact spaces,* Boll. Un. mat. Ital., 6 (1972), 143-153.

[7] C. Carpintero, E. Rosas and J. Moreno, *More on upper and lower almost nearly I-continuous multifunctions*, Int. J. Pure and Appl. Math., 117 (2017), 521-537.

[8] E. Ekici, *Nearly continuous multifunctions*, Acta Math. Univ. Comenianae, 72 (2003), 229-235.

[9] E. Ekici, *Almost nearly continuous multifunctions,* Acta Math. Univ. Comenianae, 73 (2004), 175-186.

[10] D. S. Jankovic and T. R. Hamlett, *New topologies from old via ideals,* Amer. Math. Montly, 97 (1990), 295-310.

[11] N. Levine, *Semi open sets and semi-continuity in topological spaces*, Amer. Math. Montly, 70 (1963), 36-41.

[12] K. Kuratowski, *Topology*, Academic Press, New York, 1966.

[13] A.S. Mashhour, M. E. ABD El-Monsef, *El-Deep on precontinuous and weak precontinuous mappings*, Proced. Phys. Soc. Egyp, 53 (1982), 47-53.

[14] R. E. Simithson, *Almost and weak continuity for multifunctions*, Bull. Calcutta Math. Soc., 70 (1978), 383-390.

[15] M. Stone, *Applications of the theory of boolean rings to general topology*, Trans. Amer. Math. Soc., 41 (1937), 374-381.

[16] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci., 20 (1945), 51-61.

[17] I. Zorlutuna, *I-continuous multifunctions,* Filomat, 27 (2013), 155-162.

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Generalization of T-small submodules

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Abstract. Let R be associative ring with identity and let M be unitary left R - module. A submodule N of M is called, T - small in M denoted by $N \ll_T M$, in case for any submodule $X \subseteq M$, $T \subseteq N + X$ implies that $T \subseteq X$ .In this paper ,we introduce the concept of GT - small submodule in M. A submodule N of an R-module M is called GT-small submoduleo in M, denoted by $N \ll_{GT} M$, in case for every essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$. We introduce and study the concepts GT-hollow module, GT-lifiting modules and GT-supplement submodules as a generalization of T-hollow module, T-lifiting moules and T-supplement submodules respectively we supply some examples and properties of these modules.

Keywords: GT-hollow module, GT-lifting module, T-small submodule, GT-supplement submodules.

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M and denoted by $N \ll M$ [1]. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, N is called a direct summand of M and it denoted by $M = N \oplus K$. A submodule N of an R-module M is called an essential submodule and denoted by $N \leq_e M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R-module and K be a submodule of M. K is called a G-small submodule of $M (K \leq_G M)$ if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$. There are some important properties of G-small submodules in [6], [8]. The concept of small submodule has been generalized by some researchers, for this see [7, 2, 8].

In [3] the authors introduced the concept of small submodule with respect to an arbitrary submodule. Recall that a submodule N of M is called, T-small in M denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N + X$ implies that $T \subseteq X$.

In this paper, we introduce the concept of GT-small submodule in M as generalization of T-small submodule. A submodule N of an R-module M is called GT-small submodule in M, denoted by $N \ll_{GT} M$, in case for essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$. It is clear that every T-small submodule is GT-small. We show by example that GT-small submodule of M need not be T-small submodule see(1.2). Let M be a non-zero module and T be a submodule of M. M is a T-hollow module if every submodule K of M such that $T \not\subseteq K$ is a T-small submodule of M [3]. We introduce and study the concept of GT-hollow module as a generalization of T-hollow module. M is called; T-lifting module if for; any submodule N of M there exists a direct summand D of M and $H \ll_T M$ such that $N = D + H$. In section three we introduce the notion of GT-lifting modules and discus some properties of this kind of modules. In section four we introduce the notion of GT-supplement submodules, we supply some examples and properties of these submodules.

2. GT-small submodule

In this section we introduce the concept of Generalized T-small submodules (GT-small submodule) and discuss some of basic properties.

Definition 2.1. Let T be a submodule of an R-module M. A submodule N of an R-module M is called GT-small submodule in M, denoted by $N \ll_{GT} M$, in case for essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$.

Examples and remarks 2.2.

1. If $T = 0$, then every submodule of M is GT-small in M. and If $T = M$, then $N \ll_{GM} M$ if and only if $N \ll_G M$.

2. It is clear that if N is T-small submodule of M then N is GT-small submodule in M, but the converse is not true in general. For example, in the Z-module Z_{24}, let $T = \{\bar{0}, \bar{8}, \bar{16}\}$ and the only essential submodules in Z_{24} are Z_{24}, $2Z_{24}$ and $4Z_{24}$, let $N = 6Z_{24}$ then $T \subseteq 6Z_{24} + 2Z_{24}$ and $T \subseteq 2Z_{24}$ also $T \subseteq 6Z_{24} + 4Z_{24}$ and $T \subseteq 4Z_{24}$. Then the submodule $6Z_{24}$ is GT-small submodule. But is not T- small, since if $X = 3Z_{24}$, $T \subseteq 6Z_{24} + 3Z_{24}$ but T is not submodule of $3Z_{24}$.

3. Let Z be the ring of integers. It is easy to see that (0) is the only small submodule of Z and also for any nonzero integer m, the submodule (0) is the only GmZ-small submodule of Z.

Proposition 2.3. *Let M be an R-module, $K \leq L \leq M$ and $L \leq_e M$ if $K \ll_{GT} M$, then $K \ll_{GT} L$.*

Proof. Let $T \subseteq K + X$, $X \leq_e L$ and $L \leq_e M$ then $X \leq_e M$ [9], $K \ll_{GT} M$, then $T \subseteq X$ so $K \ll_{GT} L$. □

Proposition 2.4. *Let M be an R-module with submodules $N \leq K \leq M$ and $T \leq K$. If $N \ll_{GT} K$, then $N \ll_{GT} M$.*

Proof. Suppose that $T \subseteq N+X$, for some $X \leq_e M$. Then $T \subseteq (N+X) \cap K = N + (X \cap K)$. Since $N \ll_{GT} K$, $X \leq_e M$ and $K \leq_e K$ then $(X \cap K) \leq_e K$, we have $T \subseteq X \cap K \subseteq X$ so $N \ll_{GT} M$. □

Proposition 2.5. *Let M be an R-module with submodules N_1 ,N_2 and T. Then $N_1 \ll_{GT} M$ and $N_2 \ll_{GT} M$ if and only if $N_1 + N_2 \ll_{GT} M$.*

Proof. Clear. □

Proposition 2.6. *Let M be an R-module with submodules $K \leq N \leq M$ and $K \subseteq T$. If $N \ll_{GT} M$, then $K \ll_{GT} M$and $\frac{N}{K}$ is $G\frac{T}{K}$ - small in $\frac{M}{K}$.*

Proof. Suppose that $N \ll_{GT} M$ and $T \subseteq K + X$ for some $X \leq_e M$. Then $T \subseteq N + X$ and by our assumption $T \subseteq X$. Thus $K \ll_{GT} M$. Now assume that $\frac{T}{K} \subseteq \frac{N}{K} + \frac{X}{K} = \frac{(N+X)}{K}$ for some $K \subseteq X \subseteq M$ and $\frac{X}{K} \leq_e \frac{M}{K}$. Then $T \subseteq N + X$ and $X \leq_e M$ [9], so $T \subseteq X$ and $\frac{T}{K} \subseteq \frac{X}{K}$. □

Proposition 2.7. *Let M be an R=module with $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$ such that $T \subseteq M_1 \cap M_2$. Then $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$ if and only if $K_1 + K_2 \ll_{GT} M_1 + M_2$.*

Proof. Assume that $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$. Then By Proposition 2.4 $K_1 \ll_{GT} M_1+M_2$ and $K_2 \ll_{GT} M_1+M_2$. And by Proposition 2.5, $K_1+K_2 \ll_{GT} M_1 + M_2$. The other direction is clear. □

Proposition 2.8. *Let M and N be an R-modules and $f : M \longrightarrow N$ be an R-homomorphism. If K and T are submodules of M such that, $K \ll_{GT} M$, then $f(K) \ll_{Gf(T)} N$. In particular, if $K \ll_{GT} M$, $M \subseteq N$, then $K \ll_{GT} N$.*

Proof. Let $f(T) \neq 0$ and $f(T) \subseteq f(K) + X$, for some $X \leq_e N$. It is clear that $T \subseteq K + f^{-1}(X)$ and $f^{-1}(X) \leq_e M$. But Since $K \ll_{GT} M$, then $T \subseteq f^{-1}(X)$ and hence $f(T) \subseteq X$. □

Proposition 2.9. *Let T_1 and T_2 be submodules of an R-module M and K be a submodule of M. If $K \ll_{GT_1} M$, and $K \ll_{GT_2} M$, then $K \ll_{G(T_1+T_2)} M$.*

Proof. Since $K \ll_{GT_1} M$, then if $T_1 \subseteq N + X$ for some $X \leq_e M$,then $T_1 \subseteq X$ and $K \ll_{GT_2} M$, then if $T_2 \subseteq N + X$ for some $X \leq_e M$, then $T_2 \subseteq X$. Thus $T_1 + T_2 \subseteq N + X$ and $T_1 + T_2 \subseteq X$ So $K \ll_{G(T_1+T_2)} M$. □

Proposition 2.10. *Let $M = H_1 \oplus H_2$ be a module with $R = ann(H_1)+ann(H_2)$. If $H_1 \ll_{GT_1} M$, and $H_2 \ll_{GT_2} M$, then $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$.*

Proof. Let $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X$, for some $X \leq_e M$ Since $R = ann(H_1) + ann(H_2)$ then $X = X_1 \oplus X_2$. By [10] $X_1 \leq_e H_1$ and $X_2 \leq_e H_2$ and $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X_1 \oplus X_2 = (H_1 + X_1) \oplus (H_2 + X_2)$ it is clear that $T_1 \subseteq H_1 + X_1$ and $T_2 \subseteq H_2 + X_2$. Since $H_1 \ll_{GT_1} M$ and $H_2 \ll_{GT_2} M$, then $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$. Thus $T_1 \oplus T_2 \subseteq X_1 \oplus X_2 \subseteq X$ and $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$. □

Proposition 2.11. *Let M be finitely generated, faithful and multiplication module, and let I, J be ideals in R. Then $I \ll_{GJ} R$ if and only if $IM \ll_{GJM} M$.*

Proof. Assume; that $I \ll_{GJ} R$. Let I be an ideal of R. Then IM; is a submodule of M, Let $JM \subseteq IM + X$ for some essential submodule X of M, M is multiplication module then $X = KM$ for some ideal K of R by. Then $JM \subseteq IM + KM = (I + K)M$. Since M is finitely generated, faithful and multiplication module then by [4], $J \subseteq (I + K)$, since $KM \leq_e M$ then by [4, th.2.13] $K \leq_e R$. Since $I \ll_{GJ} R$ then $J \subseteq K$ thus $JM \subseteq KM = X$. Then $IM \ll_{GJM} M$.

Conversely, assume; that $IM \ll_{GJM} M$. Let J be an ideal of R such that $J \subseteq I + K$, $K \leq_e R$, M is multiplication module then $JM \subseteq IM + KM$ and by [4, th.2.13] $KM \leq_e M$, $IM \ll_{GJM} M$ thus $JM \subseteq KM$ so $J \subseteq K$. Then $I \ll_{GJ} R$. □

3. The GT-hollow module

Let M be a non-zero module and T be a submodule of M. M is a T-hollow module if every submodule K of M such that $T \not\subseteq K$ is a T-small submodule of M. And that M is a G-hollow module if every submodule of M a G-small submodule of M.

Definition 3.1. Let M be a non-zero module and T be a submodule of M. We say that M is a GT-hollow module if every submodule K of M such that $T \not\subseteq K$ is a GT-small submodule of M.

Remark 3.2. (a) Let M be a non-zero module. Then M is GM-hollow module if and only if M is G-hollow module. Z as Z-module is not Z-hollow module and not GZ-hollow module.

(b) A GT-hollow module need not to be hollow module as the following example shows : Consider the module Z_6 as Z-module. If $T = \{\bar{0}, \bar{3}\}$, then one can easily show Z_6 is GT-hollow module. But Z_6 is not hollow module.

(c) If M is uniform R-module. Then M is GM-hollow module if and only if M is hollow module.

(d) Every T-hollow module is GT-hollow module.

(e) The Z-module Z_{24} is not GT-hollow module.

Proposition 3.3. *Let M be a GT-hollow module then every essential submodule N of M such that $T \subseteq N$ is a GT-hollow module.*

Proof. Let M be a GT-hollow module and N any essential submodule of M, $T \subseteq N$. To show that N is GT-hollow module, let L be a proper Submodule of N such that $T \not\subseteq L$. Since M be a GT-hollow module, then $L \ll_{GT} M$. By proposition 2.3, then $L \ll_{GT} N$. Thus N is GT-hollow module. □

Proposition 3.4. *Let M be a GT-hollow module and let $f : M \longrightarrow N$ be an epimorphism, where N is a non-zero module. Then N is $Gf(T)$-hollow module.*

Proof. Suppose that M is a GT-hollow module and let $f : M \longrightarrow N$ be an epimorphism. To show that N is $Gf(T)$-hollow. Let $K \leq N$ such that $f(T) \not\subseteq K$. To show that $K \ll f(T)N$. Let $f(T) \subseteq K + X$, for some $X \leq_e N$. Then $f^{-1}(f(T)) \subseteq f^{-1}(K+X)$. Therefore $T + \ker f \subseteq f^{-1}(K) + f^{-1}(X)$. Thus $T \subseteq f^{-1}(K) + f^{-1}(X)$. To show that $T \not\subseteq f^{-1}(K)$. Assume $T \subseteq f^{-1}(K)$. Then $f(T) \subseteq K$ which is a contradiction. Thus $T \not\subseteq f^{-1}(K)$. Since M is GT-hollow module, then $f^{-1}(K) \ll_{GT} M$ and hence $T \subseteq f^{-1}(X)$ Therefore $f(T) \subseteq X$. Thus N is $f(T)$-hollow module. □

Proposition 3.5. *Let T and K be submodules of a module M such that $K \subseteq T$. If K is GT-small submodule of M and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module, then M is GT-hollow.*

Proof. Assume that $K \ll_{GT} M$ and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module. Let $N \leq M$ such that $T \not\subseteq N$ and let $T \subseteq N + X$ for some $X \leq_e M$. Then $\frac{T}{K} \subseteq \frac{(N+X)}{K}$ and hence $\frac{T}{K} \subseteq \frac{(N+K)}{K} + \frac{(X+K)}{K}$. To show that $\frac{T}{K} \not\subseteq \frac{(N+K)}{K}$. Assume that $T/K = (N+K)/K$. Then $T = N + K$ and hence $T \subseteq N + K$. Since $K \ll_{GT} M$, then $T \subseteq N$ which is a contradiction. Thus $T/K \not\subseteq (N+K)/K$. Since M/K is a GT/K-hollow module, then $(N+K)/K \ll_{GT/K} M/K$. Therefore $T/K \subseteq (X+K)/K$. Thus $T \subseteq X + K$. Since $K \ll_{GT} M$, then $T \subseteq X$. Thus M is GT-hollow module. □

Proposition 3.6. *Let T be a non-zero submodule of a module M. If M is GT-hollow module. Then T is indecomposable.*

Proof. Suppose that there are proper submodules K and L of T such that $T = K \oplus L$. Therefore $T \not\subseteq K$. Since M is GT-hollow module, then $K \ll_{GT} M$. But $T \subseteq K \oplus L$, therefore $T \subseteq L$ and hence $T = L$. This is a contradiction. Thus T is indecomposable. □

4. *GT*-lifting module

M is G-lifting; module if for any submodule N of M there exist; submodules L, K of M such that $N = L \oplus K$ with $L \leq N$ where L is direct summand of M; and $K \ll_G N$ [5]. M is called; T-lifting module if for; any submodule N of M

there exists a direct summand D of M and $H \ll_T M$ such that $N = D + H$. In this section we introduce the notion of GT-lifting modules and discus some properties of this kind of modules.

Definition 4.1. Let T be a submodule of a module M. M is called; GT-lifting module if for; any submodule N of M there exists a direct summand D of M and $H \ll_{GT} M$ such that $N = D + H$.

Examples and remarks 4.2.

1. Let M be a module. M is GM-lifting module if and only if M is G-lifting module.

   Proof. let M be GM-lifting module. Let N submodule of a module M. Then there exists a direct summand D of M and $H \ll_{GT} M$ such that $N = D + H$. Thus $N = N \cap (D \oplus L) = D \oplus (N \cap L)$. Let $H = N \cap L$ then $H \ll_G M$ by (2.2) thus M is G-lifting module. Other direction is clear. □

2. Let M be a module. If M is T-lifting module then M is GT-lifting module.

3. Let Z_8 as Z-module, $T = \{\bar{0}, \bar{4}\}$ and, $N = \{\bar{0}, \bar{4}\}$ then Z_8 is not GT-lifting module.

4. If M is indecomposable module. then M is not GT-lifting module for every non trivial submodule T of M.

   Proof. Let T be non trivial submodule of M. If M is GT-lifting module then $T = D + H$ where D is direct summand D of M and $H \ll_{GT} M$ but M is indecomposable module, then $D = 0$. Thus $T = H \ll_{GT} M$ which is a contradiction then M is not GT-lifting module. □

5. Le M be a GT-lifting module then every essential submodule N of M such that $T \subseteq N$ is also GT-lifting.

   Proof. Let M be GT-lifting module and N a essential submodule of M such that $T \subseteq N$ and $X \subseteq N$ then $X = D+H$ where D is direct summand D of M and $H \ll_{GT} M$. It is clear that D is direct summand D of N, $T \subseteq N$ and $N \leq_e M$ then $H \ll_{GT} N$ by (prop 2.3). Thus N is GT-lifting. □

Let H_1 be GT_1-lifting and H_2 is GT_2-lifting modules, then $M = H_1 \oplus H_2$ need not be $GT_1 \oplus GT_2$-lifting module as the following example:

Let $H_1 = Z_8$, $H_2 = Z_2$, each of H_1, H_2 is GH_i-lifting module but $M = Z_8 \oplus Z_2$ as Z- module, M is not GM-lifting module by (Ex.4.2 (1)).

Now we give a sufficient condition under which $M = H_1 \oplus H_2$ is $GT_1 \oplus GT_2$-lifting module.

Proposition 4.3. *Let $M = H_1 \oplus H_2$ be a module with $R = ann(H_1) + ann(H_2)$. If H_1 is GT_1-lifting and H_2 is GT_2-lifting modules, then M is $GT_1 \oplus GT_2$-lifting module.*

Proof. Let N submodule of M. Since $R = ann(H_1) + ann(H_2)$. then $N = N_1 \oplus N_2$ where $N_1 \subseteq H_1$ and $N_2 \subseteq H_2$. H_1 is GT_1- lifting and H_2 is GT_2-lifting modules, then for each $i \in \{1, 2\}$,there exists a direct summund D_i of H_i, such that $N_i = D_i \oplus L_i$ with $D_i \leq N_i$ and $L_i \ll_{GT} H_i$ then , $N = N_1 \oplus N_2 = (D_1 \oplus L_1) \oplus (D_2 \oplus L_2) = (D_1 \oplus D_2) \oplus (L_1 \oplus L_2)$, we have$(D_1 \oplus D_2) \leq N$, then $(D_1 \oplus D_2)$ is direct summund of M by(Prop:2.10) then $(L_1 \oplus L_2) \ll_{G(T1+T2)} M$. Thus M is $GT_1 \oplus GT_2$-lifting module. □

Proposition 4.4. *Let M be finitely generated, faithful and multiplication module. Then M is GT-lifting module if and only if R is $[GT : M]$-lifting.*

Proof. Assume that M is GT-lifting module. Let I be an ideal of R. M is GT-lifting hence there exist submodules $D \leq \oplus M$ and $H \ll_{GT} M$ such that $N = D + H$. But M is a multiplication R-module, so there are ideals J and K of R such that $D = JM$ and $H = KM$. Then $IM = JM + KM = (J + K)M$. But M is finitely generated, faithful and multiplication module then by [4] $I = J + K$, Let $M = D + L$ and $L = J'M$ for some J' of R. Then $RM = M = JM \oplus J'M = (J + J')M$ Then $R = J + J'$. Since M is finitely generated, faithful and multiplication module then $0 = JM \cap J'M = (J \cap J')M$ thus $JJ' = 0$,and $J \leq \oplus R$ by (prop. 2.11) $K \ll_{G[T:M]} R$. Thus R is $[GT : M]$-lifting. Conversely, let R be $[GT : M]$-lifting and N submodule of M. Since M is finitely generated, faithful and multiplication module then there exist I an ideal of R such that $N = IM$ and exist $J \leq \oplus R$ and $K \ll_{G[T:M]} R$ with $I = J + K$. Then $IM = JM + KM = (J + K)M$. Thus $N = JM + KM$, let $R = J \oplus J'$ for some J' of R then $M = RM = (J + J')M = JM \oplus J'M$. Since M is finitely generated, faithful and multiplication module then $JM \cap J'M = (J \cap J')M = 0M = 0$. Then $JM \leq \oplus M$ by (prop.2.11), $k \ll_{GT} M$. Then M is GT-lifting module. □

5. GT-supplemente submodule

Definition 5.1. Let M be an R-module and T, X, $Y \leq M$. Y is called a GT-supplement of X in M, if $T \subseteq X + Y$ and $X \cap Y \ll_{GT} Y$. If every submodule of M has a GT-supplement in M, then M is called a GT-supplemented module.

Examples and remarks 5.2.

1. If $T = 0$, then every submodule of M is GT-supplement in M.
2. and If $T = M$, then M is GM-supplement in M if and only if M is G-supplement in M.
3. Let Z be the ring of integers. It is easy to see that (0) is the only GmZ-small submodule of Z. Now let $T = 0$, $X = 2Z$ and $Y = 3Z$ then

$T \subseteq 2Z + 3Z$ and $2Z \cap 3Z = 6Z \ll_{GT} 2Z$. Then Y is GT-supplement in M.

4. Let Z_6 as Z-module, $T = \{\bar{0}, \bar{3}\}$, $X = \{\bar{0}, \bar{2}, \bar{4}\}$, and $Y = \{\bar{0}, \bar{3}\}$. It is clear that $T \subseteq X + Y$ and $X \cap Y = 0 \ll_{GT} Z_6$, so Y is GT-supplement in Z_6.

Proposition 5.3. *Let M be an R-module, T, X and $Y \leq M$ such that Y is GT-supplement of X in M if $T \subseteq K + Y$, for some submodule K of M. Then Y is a GT-supplement of K in M.*

Proof. Let Y be is GT-supplement of X in M, K submodule of M such that $T \subseteq K + Y$. Since $K \cap Y \subseteq X \cap Y \ll_{GT} Y$ by(Prop:2.8). Then Y is a GT-supplement of K in M. □

Proposition 5.4. *Let M be an R-module, T, X and $Y \leq M$ and Y be a GT-supplement of X in M, $L \leq Y$ and $L \ll_{GT} Y$. Then Y is a GT-supplement of $X + L$ in M.*

Proof. Let Y be a GT-supplement of X in M and $L \leq Y$ and $L \ll_{GT} Y$. Then $T \subseteq X+Y \subseteq X+Y+L$, $X \cap Y \ll_{GT} Y$. Then Y. To show that $Y \cap (X+L) \ll_{GT} Y$. Let K be essential submodule in M such that $T \subseteq Y \cap (X + L) + K$. Then $T \subseteq (X \cap Y) + L + K$, $K \subseteq L + K$ is essential submodule in M hence $T \subseteq L + K$. Since $L \ll_{GT} Y$ thus $T \subseteq K$. Then Y is a GT-supplement of $X + L$ in M. □

Proposition 5.5. *Let M and N be R-modules, and let $f : M \longrightarrow N$ be an epimorphism. If M is GT-supplemented module. Then N is $Gf(T)$-supplemented module.*

Proof. Suppose that M is a GT-supplemented module and let $f : M \longrightarrow N$ be an epimorphism. Let K be submodule of N, M is a GT-supplemented module then $T \subseteq L + f^{-1}(K)$ and $f^{-1}(K) \cap L \ll_{GT} Y$. Then $f(T) \subseteq f(L + f^{-1}(K))$. Then $f(T) \subseteq f(L) + K$. Since $f^{-1}(K) \cap L \ll_{GT} Y$ then $K \cap f(L) = f(f^{-1}(K)) \cap L \ll_{Gf(T)} f(Y)$. Therefore by(Prop:2.8) $f(L)$ is $Gf(T)$-supplement submodule of K in M. □

Proposition 5.6. *Let M be GT-lifting module and Y be a GT-supplement of X in M. Then Y contains a GT-supplement of X which is direct summand of M.*

Proof. Suppose that M is GT-lifting module and Y be a GT-supplement of X in M, Then $T \subseteq X + Y$, $X \cap Y \ll_{GT} Y$. M is GT-lifting then $Y = D + H$, where $D \leq \oplus M$ and $H \ll_{GT} M$. Since $T \subseteq X + Y$, then $T \subseteq X + D + H$ thus $T \subseteq X + D$, now $X \cap D \subseteq X \cap Y \ll_{GT} Y$ by(Prop: 2.6) $X \cap D \ll_{GT} Y$ then D is a GT-supplemente of X in M. □

References

[1] F.W. Anderson, K.R. Fuller, *Rings and categories of modules*, New York: Springer-Verlag, 1974.

[2] T. Amouzegar-Kalati, D. Keskin-Tutuncu, *Annihilators small submodules*, Bull. Iran. Math. Soc., 39 (2013), 1053-1063.

[3] R. Beyranvand, F. Moradi, *Small submodules with respect to an arbitrary submodule*, Journal of Algebra and Related Topics, 3 (2015), 43-51.

[4] Z.A. El-Bast, P.F. Smith, *Multiplication modules*, Comm. Algebra, 16 (1988), 755-774.

[5] I.M.A. Hadi, S.H. Aidi, *e-Hollow modules*, International Journal of Advanced Scientific and Technical Research May-June, 5 (2015).

[6] B. Koşar, C. Nebiyev, N. Sökmez, *G-supplemented modules*, Ukrainian Mathematical Journal, 67 (2015), 861-864.

[7] T.C. Quynh, P.H. Tin, *Some properties of e-supplemented and e-lifting modules*, Vietnam Journal of Mathematics, 41 (2013), 303-312.

[8] Y. Talebi, M. Hosseinpour, *Generalizations of ....-lifting modules*, J. Algebraic Systems, 1 (2013), 67-77.

[9] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Philadelphia, 1991.

[10] D.X. Zhou, X.R. Zhang, *Small-essential submodules and morita duality*, Southeast Asian Bulletin of Mathematics, 35 (2011), 1051-1062.

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The elasticity of determinants life insurance demand in improving the efficiency of Jordanian life insurance companies

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Abstract. The starting turbulent at any new century has brought numerous difficulties for firms and nations. During this period, success and survival analysis more and more depends on the efficiency which can be described by many different researching ways. The main target of this paper is to extend the earlier works on the elasticity of credit rate, the price of life insurance and product differentiation for improving efficiency in life insurer and increase the demand of life insurance policies. The paper strength is that the efficiency of life insurance companies is estimated from point of view of the insured. The result shows the inelastic life demand insurance, there are critical effects of credit rate, price, product differentiation on competition and efficiency in the targeted life insurance company. Moreover, there is a difficulty in understanding life insurance in Jordanian firms because they don't have the employees who are specialist in insurance. The ANOVA test and OLS are basically used in this study.

Keywords: credit rate, life insurance policy, efficiency, and elasticity.

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1. Introduction

1.1 Introduction of the paper

The economy is well enhanced by the insurance industry. The insurance stimulates investments and helps to decrease the poverty rate for several reasons. Firstly, it transfers pure risk from insured to insurer such as premature death. Furthermore, the pension is well improved by the life insurance contract. However, the ability to get the lion share from policyholders will be contributed to improving the efficiency of the insurer.

Numerous definitions of efficiency have been presented with no general agreement on any given one. The point of this study is to extend prior research on some selected elements of the used the price of life insurance, credit rate and product differentiation on efficiency in life insurer is attracting the big amount from policyholders. This study investigates the effective elasticity of some selected elements on the demand for life insurance at Jordanian insurance companies.

Previously, the life insurance and credit business expanded hugely. Consistently demonstrates the re-codes of historical lending and transaction payments to estimate the creditworthy of individuals and companies[7]. Credit rate is essentially a method for perceiving if the insurer has the capacity to remunerate at the season of loss or not. [14] said that credit rate can be used as a successful instrument in distinguish between policyholders in personal lines of life insurance.[13] investigate Iowa client's viewpoint to determine life insurance qualification. He asked 29 questions for a cross-sectional sample of Iowans, and find that is unfair, but lacks the necessary justification for the claim.

In Jordan, Several life insurance products have been offered. The pension funds have the lion share of life insurance policies in the Jordanian insurance market. In the life insurance policy, the insurer gave the promises to compensate if the insured retired or die during the time of coverage. On the other side, the insured gave the promises to pay a premium at maturity time. The efficiency of an insurance company to collect pool from insured related with the elasticity of life insurance demand. [12] found that elasticity of non-life insurance demand is smaller than unity for high wealth countries. And, he found the elasticity of insurance demand is greater than unity for low wealth countries. Furthermore, policy differentiation is another approach to enhance the efficiency of the insurer. Differentiation policies make a policy more attractive than other competing policies. Effective differentiate policy creates a competitive favorable position for an insurer, as insured see these policies as superior or unique. [3]researches the connection between the firm performance and product diversification from 1994 to 2002 in the US PC life insurance industry. The findings suggest that the product diversification has been associated with performance by the degree of geographic diversification. And, the relationship between product diversification and performance have been complex and nonlinear.

1.2 Significance of the study

This investigation differ from the prior referenced examinations for several reasons. 1. It presents the Jordanian life insurance industry. 2. It depends on the insured viewpoint to estimate the efficiency of life insurance companies. 3. It finds the elasticity of demand life insurance policy in Jordan.

1.3 Literature review

The starting turbulent at any new century has brought numerous difficulties for firms and society. During this time, sustainable development for firms increasingly depends on efficiency. The concept of efficiency changes periodically. Efficiency definition of institutions depends on their capacity to distinguish the services and product at a reasonable price. The individual risks can be estimated by the credit rating system. This system has an operational advantage of decreasing premium because it may be used by insurers to distinguish between different levels of insured risk then led to enhance the efficiency of insurance companies [8]. On the other hand, [10] found the prices and quality have a positive effect on competitive advantage in Jordanian companies. [4] investigates the role of contract performance including the quality and basis risk, financial literacy, trust, and informal risk-sharing mechanisms. This paper select 12 key variables affecting the micro-insurance demand. [2] study the non-life and life insurance demand for emerging countries, Europe and Asia. Founding that the Urbanization, the population degree of education, incomes and their distributions affected for the sustainability of the insurance sector. However, [15] discovered the direct evidence about the significance of the health insurance in Nicaragua and the transaction costs. They found about a day work need to enrollment procedure for the health insurance in Nicaragua to complete. And, the market vendors contribute about 30 percentage in hance the sign up directly at their market stall. [1]found that economic condition essentially influences the efficiency of an insurer in Nigeria. The ANOVA and OLS regression used to test the hypothesis. [5]found there is a positive significant effect between Communication Technology(ICT) on insurance companies' efficiency in Nigeria.[9] used a survey to find the effect of insurance product to create competitive advantage in life insurance firms in Kenya. The result of this study that government regulations have a significant role in creating a competitive advantage. It shows the effectiveness of the insurance product in meeting the needs of clients. [11] found the variables that determined the profitability in general insurance in Kenya. They suggest that the company can enhance the efficiency by increasing leverage, equity capital and quality of staff.

2. Hypotheses

The primary hypotheses are divided by 3 groups in our study as following:

Group A	H0: There is no critical effect of independent factors on the dependent factor
	H01: There is no critical effect of price for life insurance policies on efficiency.
	H02: There is no critical effect of product differentiation on efficiency in life insurance companies.
	H03: There is no critical effect of credit rate on efficiency in life insurance companies.
GroupB	H0: There is no critical difference in arithmetic mean of the effect of independent variables on efficiency in life insurance companies based on qualification.
	H04: There is no critical difference in arithmetic mean of the effect credit rate on efficiency in life insurance companies based on Majors.
	H05: There is no critical difference in arithmetic mean of the effect of price on life insurance policies on efficiency in life insurance companies based on Majors.
	H06: There is no critical difference in arithmetic mean of the effective product differentiation efficiency in life insurance companies based on Majors.
GroupC	H0: There is no critical difference in arithmetic mean of the effect of independent variables on efficiency in life insurance companies based on the type of policies.
	H07: There is no critical difference in arithmetic mean of the effect credit rate on efficiency in life insurance companies based on the type of policies.
	H08: There is no critical difference in arithmetic mean of the effect of price on life insurance policies on efficiency in life insurance companies based on the type of policies.
	H09: There is no critical difference in arithmetic mean of the effective product differentiation on efficiency in life insurance companies based on the type of policies.

3. Data sources

3.1 Descriptive of variables

The random sample from 250 employees who work in the department of insurance in different companies in Jordan. We select 182 questionnaires to analysis in statistical techniques in order to interpret the results by using the SPSS-22 program.

3.2 Measures and covariates

Cronbach's alpha coefficient uses to calculate the reliability of the study. The Cronbach's coefficient alpha for all factors is 65% that is more than 60%. It means there is high consistency between questions in the survey (Sekaran, 2010). Also, KMO test is utilized to test the sample size adequacy, where its value reached to 0.72 that is more than 0.5. So, the sample size is suitable. [6] Fiedel (2005).

4. Statistics and data analysis

Table 1. The descriptive statistics of demographic Factors in our data

		Frequency	Percent
Sex	Male	142	0.78
	Female	36	0.20
	Missing value	4	0.02
	Total	182	1.00
Age	Less than 30	66	0.36
	30-40	68	0.37
	More than 40	46	0.25
	Missing value	2	0.01
	Total	182	1.00
Education	Graduate	105	0.58
	postgraduate	17	0.09
	Others	58	0.32
	Missing value	2	0.01
	Total	182	1.00
Industry	industry	36	0.20
	services	86	0.47
	others	58	0.32
	Missing value	2	0.01
	Total	182	1.00
Occupation	General Manager	84	0.46
	Chair of department	59	0.32
	Employee	37	0.20
	Missing value	2	0.01
	Total	182	1.00
Experience	Less than 5 years	28	0.15
	5-10 years	64	0.35
	More than 10 years	88	0.48
	Missing value	2	0.01
	Total	182	1.00
Qualification	Life insurance	6	0.03
	Business management	38	0.21
	Others	134	0.74
	Missing value	4	0.02
	Total	182	1.00
The types of life insurance policies	Whole life insurance	80	0.44
	Temporary life insurance	36	0.20
	Umbrella life insurance	62	0.34
	Missing value	4	0.02
	Total	182	1.00

Table 1 shows the percentages and frequency for the selected employees who work at the department of insurance in different companies in Jordan. We found the percentage of postgraduate, graduate and others are 9%, 58% and 32% on succession. Furthermore, we found the percentage of females is 20%, but the percentage of the male is 78%. In addition, the percentage of employees who work in service is 47%, but the percentage of employees who work in industrial is 20%. In addition, the percentage of general chairman, manager, and em-

ployee are 32%,46% and 20% on succession. And, the percentage of employees who have a certificate in business management is 21%. Also, the percentage of employees who have a certificate in insurance is 3%. On other hands, the percentage of employees who dont have a certificate in insurance or business management is 74%. Finally, the percentage of companies that have whole life insurance policy is 44%, but the percentage of companies that have temporary life insurance is 20%.

Table 2. The arithmetic mean and standard deviation for questionner items

		Mean	S.D	Level
Credit Rate	Qa	4.40	0.81	H
	Qb	3.40	1.08	M
	Qc	4.18	0.84	H
	Qd	4.19	0.84	H
	Qe	4.10	0.90	H
	Qf	4.09	0.85	H
	Average	4.06	0.50	H
Price of life insurance premiums	Qa	3.90	1.13	H
	Qb	2.67	1.26	M
	Qc	3.84	1.01	H
	Qd	3.66	1.06	H
	Qe	3.46	1.11	M
	Average	3.55	0.65	M
Differentiation of life insurance policies	Qa	4.21	0.78	H
	Qb	3.67	1.17	M
	Qc	4.46	0.91	H
	Qd	4.30	0.73	H
	Average	4.15	0.50	H
Efficiency	Qa	3.57	1.20	M
	Qb	4.53	0.69	H
	Qc	3.95	0.87	H
	Qd	3.90	1.04	H
	Qe	4.48	0.72	H
	Qf	4.33	0.90	H
	Average	4.13	0.46	H

Five scales from Likert has been used in this survey to determine the level of agreement of each item. Table 2 shows the standard deviation and arithmetic mean for each item in the survey. We found in the table that the buying life insurance policy is highly affected by credit rate. The arithmetic mean of employees who think that the credit rate of a life insurer is highly significant for buying the life insurance policy is 4.1 and the standard deviation is 0.5. Moreover, the select of the life insurance policy is a medium affected by the price of

life insurance policy. The standard deviation of employees who think that the buying life insurance policy is affected by price is 0.65 and the arithmetic mean is 3.55. In addition, the choose of the life insurance policy is highly affected by the differentiation of life insurance policy. The standard deviation of employees who think that the buying life insurance policy is affected by differentiation is 0.5 and the arithmetic mean is 4.15.

Table 3. The correlation between survey items

	Efficiency	Price of Life insurance premium	Differentiation	Credit Rate
Efficiency	1	.318**	.443**	.496**
Price of Life insurance premium		1	.220**	.322**
Differentiation			1	.449**
Credit Rate				1

**. Correlation is significant at the 1% level (2-tailed).

The correlation between independent factors and the dependent factor presented in table 3. We found no- multicollinearity between independent factors. We found that the correlation between differentiation is of policies and credit rate of a life insurer is 0.45 that means there is a weak correlation because its less than 0.5. Moreover, the weak correlation between differentiation is of life policy and prices equal to 0.44. Finally, There is also a weak correlation between the credit rate of life insurer and price of life insurance.

Table 4. Full Regression models between dependent factor and independent factors.

Model	OLS Regression		
	B	S.E	t
Constant	1.56	0.28	5.42**
Credit Rate	0.31	0.06	4.83**
Price of Life insurance	0.11	0.05	2.26*
Differentiation	0.25	0.06	3.67**
R Square	0.42		
Durbin-Watson	2.19		
F-test	28.34		
Sig.	0.000		

Note. ** significant at 1%, *significant at 5%.

Five scales from Likert have been used. The OLS regression for determinants life insurance demand in improving the efficiency of Jordanian life insurance companies can be seen in Table4. In the table, we found there is a positive critical relationship between the efficiency of life insurer and the prices of life policy. So, we don't accept H01 (H01: there is no critical effect of price for the life insurance policy on efficiency in life insurer.) at 5% degree of significance.

We found the elasticity of prices of life policy less than one. That means there is a limited effect of demand life insurance result from a change in the price of life insurance policy. On the other hand, the inelastic demand encourages insurance companies to increase the prices. Also, the employees prefer compulsory government insurance "social security pension fund" than private life insurance. This result agrees with [15] and [12]. In another meaning, the increase in the price life insurance affects about 11 percent in hancing the efficiency of a life insurer for attracting the big share from an insured. Furthermore, we noted from the table that the relationship between the efficiency of life insurer and differentiation of life insurance policies is positively significant. So, we don't accept H02 (H02: there is no critical effect of product differentiation on efficiency in life insurer.) at 1% degree of significance. This concludes agree with [5]. In another meaning, the efficiency of life insurer affected about 25 percent by the differentiation of life insurance policy. Moreover, the elasticity of differentiation of life insurance products less than one. That means there is a limited effect of demand life insurance result from change differentiation of life insurance products. However, the table shows that the relationship between the efficiency of a life insurer for the demand for life insurance and credit rate of a life insurer is positively significant. So, we don't accept H03 (H03: there is no critical effect of credit rate on efficiency in life insurer) at 1% degree of significance. This concludes agree with[14]. In the other meaning, the creditworthy and solvency of insurer to pay the indemnity in the future are an indicator of a credit rating system. So, the efficiency of life insurer affected about 31 percent by the credit rating. The elasticity of credit rate less than one. That means there is a limited effect of demand life insurance result from the change in credit rate of life insurance policy.

In addition, we found the independent factors can interpret about 0.42 from the efficiency in life insurer. Furthermore, the Durbin-Watson test is 2.19. The OLS regression is significant at 1%.

The differences of the arithmetic mean between independent factors on dependent factor based on qualification can be seen in the ANOVA test in Table 5. In credit rating, we found that the sum of squares within a group is 43.133 and between groups is 1.033 at F-test equal 2.1 with a significant level less than 15%. So, we don't accept H04 (H04: there is no critical difference in arithmetic mean of the effect credit rating on efficiency in life insurer based on qualification). Moreover, we noted that the sum of the square within groups equal to 69.84 and between groups equal 6.97 in the price of life policy at F-test equal 8.83 that is significant at 20%. So that, we don't accept H05 (H05: there is no critical difference in arithmetic mean of the effect of price on life insurance policies on efficiency in life insurer depend on qualification). In addition, we found that the sum of the square within groups equals 41.5 and between groups equal to 2.1 in the differentiation of life policy at F-test equal 4.5 that is significant at 5%. So that, we don't accept H06 (H06: there is no critical difference in arithmetic mean of the product differentiation on efficiency in life insurer depend

on qualification). In our survey, the percentage of selected employees who have a business degree, insurance degree, and other qualification is around 21%,3%, and 74% respectively. This means there a misunderstood of life insurance policy in Jordanian companies because there are no-specialist in this area.

The differences of the arithmetic mean between independent factors on the dependent factor depend on the type of life insurance policies can be seen in the ANOVA test in Table 6. In credit rating, we found that the sum of squares within a group is 41.01 and between groups is 3.02 at F-test equal 6.4 with a significant level less than 5%. So, we don't accept H07 (H07: there is no critical difference in arithmetic mean of the effect credit rating on efficiency in life insurer based on the type of life insurance policies). Moreover, we noted that the sum of the square within groups equal to 74.2 and between groups equal to 1.8 in the price of life policy at F-test equal to 2.1 that is significant at 15%. So that, we don't accept H08 (H08: there is no critical difference in arithmetic mean of the effect of price on life insurance policies on efficiency in life insurer depend on the types of life insurance policies). In addition, we found that the sum of the square within groups equals 39.92 and between groups equal to 1.00 in the differentiation of life policy at F-test equal to 2.20 that is significant at 15%. So that, we don't accept H09 (H09: there is no critical difference in arithmetic mean of the product differentiation on efficiency in life insurer depend on the types of life insurance policies).

5. Conclusion

This paper expands prior works. It determined the elasticity of life insurance demand in enhancing the efficiency of the Jordanian life insurer. The ANOVA test and OLS regression used for this purpose. The study found that there is a critical significant effect of price, product differentiation, and credit rating on the efficiency of the Jordanian life insurer. Furthermore, there is a misunderstood of the concept of life insurance for employees who work in Jordanian companies because they are not specialist in this area. In addition, the elasticity of determined of life insurance demand less than one.

Table 5. ANOVA-test between independent factors on efficiency in life insurer based on qualification

Variables		Mean	ANOVA-test					
				Sum of Squares	df	Mean Square	F	Sig.
Credit Score	Life insurance	3.86	Between Groups	1.033	2	0.517	2.1	0.123**
	Business management	3.84	Within Groups	43.133	177	0.244		
	others	4.15	Total	44.166	179			
Price of Life insurance premium	Life insurance	3.13	Between Groups	6.967	2	3.483	8.8	0.178*
	Business management	3.48	Within Groups	69.843	177	0.395		
	others	3.59	Total	76.809	179			
Total 76.809 179 Product Differentiation	Life insurance	3.43	Between Groups	2.117	2	1.063	4.5	0.012***
	Business management	4.04	Within Groups	41.431	177	0.235		
	others	4.25	Total	43.548	179			

Note. ***significant at 5%, **significant at 15%, *significant at 20%.

Table 6. ANOVA-test for dependent Factors on independent Factors depend on the type of policies

Variables		Mean	ANOVA-test					
				Sum of Squares	df	Mean Square	F	Sig.
Credit Score	Life insurance	4.20	Between Groups	3.02	2	1.51	6.44	0.002***
	Business management	3.82	Within Groups	41.01	175	0.23		
	others	4.01	Total	44.03	177			
Price of Life insurance premium	Life insurance	3.60	Between Groups	1.76	2	0.88	2.08	0.128**
	Business management	3.34	Within Groups	74.19	175	0.42		
	others	3.61	Total	75.95	177			
Product Discrimination	Life insurance	4.23	Between Groups	1.00	2	0.50	2.20	0.114**
	Business management	4.24	Within Groups	39.92	175 0.23			
	others	4.07	Total	40.92	177			

Note. ***significant at 5%, **significant at 15%, *significant at 20%.

References

[1] M.O. Adamu, *The impact of economic environment on insurance companies in Nigeria*, Journal of Scientific Research and Development (JSRD), 13 (2011).

[2] Simona Laura Dragos, *Life and non-life insurance demand: the different effects of in uence factors in emerging countries from Europe and Asia*, Economic research-Ekonomska Istraživanja, 27(2014), 169-180.

[3] B. Elango, Yu-Luen Ma and Nat Pope, *An investigation into the diversification performance relationship in the us property liability insurance industry*, Journal of Risk and Insurance, 75 (2008), 567-591.

[4] Martin Eling, Shailee Pradhan, and Joan T. Schmit, *The determinants of microinsurance demand*, The Geneva Papers on Risk and Insurance-Issues and Practice, 39 (2014), 224-263.

[5] Olajide Solomon Fadun, *Information and communication technology (ict) and insurance companies profitability in Nigeria*, Journal Accounting Business and Management-International, 20 (2013).

[6] Babin Anderson Hair, Black and Tatham, *Multivariate data analysis*, Pearson Education, 7th ed., 2010.

[7] David J. Hand and William E. Henley, *Statistical classification methods in consumer credit scoring: a review*, Journal of the Royal Statistical Society: Series A (Statistics in Society), 160 (1997), 523-541.

[8] Noriszura Ismail and Abdul Aziz Jemain, *Construction of insurance scoring system using regression models*, Journal of Modern Applied Statistical Methods, 7 (2008).

[9] S.M. Kiragu, *Assessment of challenges facing insurance companies in building competitive advantage in kenya: A survey of insurance firms*, International Journal of Social Sciences and Entrepreneurship, 1(2014), 467-490.

[10] Momani and Mqatif, *The effect of substitution on building a strategic advantage competitiveness - a field study on companies insurance in Jordan*, Quds Open University for Research and Studies Magazine, 27, 2012.

[11] Mirie Mwangi and Cyrus Iraya, *Determinants of financial performance of general insurance underwriters in Kenya*, International Journal of Business and Social Science, 5 (2014).

[12] Hiroyuki Nakata, Yasuyuki Sawada, et al., *Demand for non-life insurance: a cross-country analysis,* CIRJE Working Paper, F-461, 2007.

[13] Y. L. Richards, Thomas J. Quinlan, et al., *Use of credit scores by the insurance industry: Iowa consumers perspective*, 2009.

[14] Gregory V. Serio, *Testimony of the national association of insurance commissioners before the sub- committee on financial institutions and consumer credit committee on financial services, United States house of representatives regarding: the fair credit reporting act: How it functions for consumers and the economy (transcript of testimony)*, Albany, NY: Superintendent of Insurance, New York, 2003.

[15] Rebecca L. Thornton, Laurel E. Hatt, Erica M. Field, Mursaleena Islam, Freddy Soliś Diaz, and Martha Azucena González, *Social security health insurance for the informal sector in Nicaragua: a randomized evaluation*, Health Economics, 19 (2010), 181-206.

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Blow up of solution for the Kelvin-Voigt type wave equation with Balakrishnan-Taylor damping and acoustic boundary

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Abstract. The purpose of this work is to study the blow up of solutions in finite time, for a nonlinear equation of the kelving voigt type with balakrishnan taylor damping and acoustic boundary in a bounded domain in $\mathbb{R}^n$.

Keywords: Kelvin-Voigt type, energy decay, Balakrishnan-Taylor damping, acoustic boundary, blow up.

1. Introduction

Let Ω be a bounded, connected set in $\mathbb{R}^n (n \geq 1)$ having a smooth boundary $\Gamma = \partial\Omega$ consisting of two parts Γ_0 and Γ_1 such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma$. We consider the following problem:

$$
\begin{aligned}
&\left|u'\right|^m u'' - \left(a^2 + b\int_\Omega |\nabla u|^2\, dx + \sigma \int_\Omega \nabla u.\nabla u' dx\right)\Delta u - 2\lambda\Delta u' \\
&= |u|^{p-2}\, u \ \text{ in } \ \Omega \times \mathbb{R}^+, \\
&u = 0 \ \text{ in } \Gamma_0 \times \mathbb{R}^+, \\
&\left(a^2 + b\int_\Omega |\nabla u|^2\, dx + \sigma \int_\Omega \nabla u.\nabla u' dx\right)\frac{\partial u}{\partial v} + 2\lambda\frac{\partial u'}{\partial v} = y' \ \text{ in } \Gamma_1 \times \mathbb{R}^+,
\end{aligned}
$$

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$$
\begin{aligned}
&u' + p(x)y' + q(x)y = 0 \ \text{ in } \Gamma_1 \times \mathbb{R}^+, \\
&u(0) = v_0, \ u'(0) = u_1 \ \text{ in } \Omega, \\
&y(0) = y_0 \ \text{ in } \Gamma_1.
\end{aligned} \tag{1.1}
$$

where primes denote the time derivative, Δ the Laplacian in $\mathbb{R}^n$ taken in space variables, v the unit normal of Γ pointing towards exterior of Ω and $\mathbb{R}^+ = (0, \infty)$ The parameters $\lambda > 0$ is a small internal material damping coefficient, $a > 0$, $b > 0$, $\sigma > 0$. are constant real numbers p and q are functions satisfying

$$
p(x) > 0, \quad q(x) > 0 \quad for\ all\ x \in \Gamma_1. \tag{1.2}
$$

Physically, the first integrodifferential equations in (1.1) occurs in the study of vibrations of damped flexible space structures in bounded domain in $\mathbb{R}^n$. The nonlinear term $|u^{'}|^m u^{''}$, where $m > 0$ is expressed materials whose density depends on the velocity $u^{'}$, The term $2\lambda\Delta u^{'}$ is the internal material damping of Kelvin-Voigt type of the structure. The model in hand, with Balakrishnan-Taylor damping ($\sigma > 0$) and $\rho = 1$, was initially proposed by Balakrishnan and Taylor [1] in 1989 and Bass and Zes [2], this problem was studied by Zaraï and Tatar [17,18,19,20] and results concerning global existence, general decay and blow up of solution have been accomplished. The boundary conditions considered here are of mixed Dirichlet and Neumann type and acoustic boundary. The analytical studies in the area of stabilization of distributed parameter system is currently of interest in view of application to vibration control of various structural elements. The phenomenon was first observed by Hunton as reported by Harrison [8]. In the absence of the Kelvin-Voigt damping ($\lambda = 0$) and $m = \sigma = 0$, we have the well known Kirchhoff [11] equation

$$
u'' = \left(a^2 + b \int_\Omega |\nabla u|^2 \, dx \right) \Delta u,
$$

it was extensively studied by several authors cited in Zaraï and Tatar [20]. Beale and Rosencrans [3] introduced acoustic boundary conditions of the general form

$$
\begin{aligned}
&\frac{\partial u}{\partial \nu} = y' && on\ \Gamma_1 \times \mathbb{R}^+ \\
&\gamma u' + m(x)y'' + p(x)y' + q(x)y = 0 && on\ \Gamma_1 \times \mathbb{R}^+
\end{aligned}
$$

Recently, wave equations with acoustic boundary conditions have been studied by many authors [4,5,9,14,15,16].

The above equation, without the nonlinear source term, has been investigated in Kang [10]. Using the multiplier technique, the author was prove an exponential decay result for the solution.

In this work we prove a finite time blow-up result of solutions, we will see that the direct method introduced and developed by Georgiev and Todorova [6], in 1994 and Salim A. Messaoudi [12,13] is efficient in our case. Combining

this method with some necessary modifications due the nature of the problem treated here.

Our paper is organized as follows: in the next section we prepare some material needed in our proofs. Section 3 is devoted to the statement and proof of the finite time blow-up result.

2. Preliminaire

In this section, we present some notations and some material which will be needed in the proof of our results. Let C be the smallest positive constant independent of t (depends only on Ω) satisfying the Poincare inequality

$$\int_\Omega u^2 dx \leq C \int_\Omega |\nabla u|^2 \, dx. \tag{2.1}$$

And also let $\overline{k}$ be the smallest positive constant independent of t (depends only on Γ_1) satisfying for $u \in H^1(\Omega)$ the embedding inequality

$$\int_{\Gamma_1} u^2 d\Gamma \leq \overline{k} \int_\Omega |\nabla u|^2 \, dx. \tag{2.2}$$

Lemma 1. *Suppose u, $v : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. Then*

$$\|u.v\|_1 < \|u\|_r \|v\|_s \tag{2.3}$$

Lemma 2. *For u, $v > 0$, we have*

$$uv \leq C\left(u^r + v^s\right), \tag{2.4}$$

for all $C > 0$ and $\frac{1}{r} + \frac{1}{s} = 1$.

For the functions p and q, we assume that $p, q \in C(\Gamma_1)$, $p(x) > 0$ and $q(x) > 0$, for all $x \in \Gamma_1$. This assumption implies that there exist positive constants p_i, q_i $(i = 0; 1)$ such that

$$p_0 \leq p(x) \leq p_1, \quad q_0 \leq q(x) \leq q_1, \quad for\ all\ x \in \Gamma_1 \tag{2.5}$$

By using Galerkin's approximation and the methods of Gorain [7] and Park [16], we can obtain existence result for the solution of (1.1) under the conditions on p and q as above.

3. Blowing up property

In this section we consider the blowing up property of the solution to problem (1.1). To this end, we use the method in [6]. We define the energy function of the solution to (1.1) by

$$\begin{aligned} E(t) &= \frac{1}{m+2}\left\|u'\right\|^{m+2} + \frac{1}{2}\left(a^2 + \frac{b}{2}\|\nabla u\|^2\right)\|\nabla u\|^2 \\ &\quad -\frac{1}{p}\|u\|^p + \frac{1}{2}\int_{\Gamma_1} q(x)(y(t))^2 d\Gamma. \end{aligned} \tag{3.1}$$

Lemma 3. *Let the pair of functions (u, y) be the solution of problem (1.1), then $E(t)$ is a nonincreasing function on $[0, t)$ and*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 - 2\lambda\left\|\nabla u^{'}\right\|^2 + \int_{\Gamma_1}\left(u^{'} + q(x)y\right)y^{'}d\Gamma \\ &= -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 - 2\lambda\left\|\nabla u^{'}\right\|^2 - \int_{\Gamma_1} p(x)\left(y^{'}\right)^2 d\Gamma \leq 0. \end{aligned} \tag{3.2}$$

Proof. By multiplying the first equation in (1.1) by u', and integrating over Ω, we get

$$\begin{aligned} \int_\Omega \left|u^{'}\right|^m u'u''dx &= \int_\Omega\left(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma\int_\Omega \nabla u.\nabla u^{'}dx\right)\Delta u u_t dx \\ &\quad +2\lambda\int_\Omega \Delta u^{'}u'dx + \int_\Omega |u|^{p-2}uu'dx \end{aligned}$$

Therefore, using integration by parts, the conditions in the borders and some manipulations, we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{m+2}\left\|u^{'}\right\|_{m+2}^{m+2} + \frac{a^2}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{b}{4}\|\nabla u\|^4 - \frac{1}{p}\|u\|_p^p + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma\right] \\ = \; &-\frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u\|^2\right)^2 - 2\lambda\left\|\nabla u^{'}\right\|^2 - \int_{\Gamma_1} p(x)\left(y^{'}\right)^2 d\Gamma. \end{aligned}$$

□

Theorem 1. *Suppose that $m > 1$, $p > 3\beta(m+1)$, assume that*

$$\begin{aligned} E_0 &= \frac{1}{m+2}\|u_1\|_{m+2}^{m+2} + \frac{1}{2}\left(a^2 + \frac{b}{2}\|\nabla u_0\|^2\right)\|\nabla u_0\|^2 \\ &\quad - \frac{1}{p}\|u_0\|^p + \frac{1}{2}\int_{\Gamma_1} q(x)(y(0))^2 d\Gamma < 0. \end{aligned} \tag{3.3}$$

Then the solution of problem (1.1) blow up in finite time

$$T^* \leq \frac{2(1-\alpha)}{(2\alpha-1)\,\gamma\,[L(0)]^{\frac{2\alpha-1}{2(1-\alpha)}}},$$

where

$$\beta = \max\left(\frac{1}{2} + \frac{\overline{k}q_1}{(p_0 a)^2}, 1\right).$$

Lemma 4. *There exists a positive constant $C > 1$ depending on Ω only such that*

$$\|u\|_p^s \leq C\left(\|\nabla u\|^2 + \|u\|_p^p\right), \tag{3.4}$$

for any $2 \leq s \leq p$ and $u \in H^1(\Omega)$.

Proof. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Soboleve embeding theorem.

If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.4) follows. □

We set

$$H(t) = -E(t).$$

That's mean

$$H'(t) = \frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u\|^2\right)^2 + 2\lambda\left\|\nabla u'\right\|^2 + \int_{\Gamma_1} p(x)\left(y'\right)^2 d\Gamma \geq 0. \tag{3.5}$$

Consequently we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p, \tag{3.6}$$

As a result of (3.1) and (3.4), we have

Corollary 1. *Let the assumptions of the lemma 4 hold. Then we have the following*

$$\begin{aligned} \|u\|_p^s \leq\ & C\left(-\frac{1}{m+2}\left\|u'\right\|_{m+2}^{m+2} - \|\nabla u\|^4\right. \\ & \left. - \int_{\Gamma_1} p(x)\left(y'\right)^2 d\Gamma - H(t) + \|u\|_p^p\right). \end{aligned} \tag{3.7}$$

for all $t \in (0,T)$, *for any* $u(.,t) \in H^1(\Omega)$ *and* $2 \leq s \leq p$.

By virtue of (3.1) and (3.3). We then define

$$L(t) = H^{2(1-\alpha)}(t) + \varepsilon \int_\Omega \left|u'\right|^m u' u dx. \tag{3.8}$$

for ε small positive to be chosen later and

$$\frac{1}{2} < \alpha < \frac{m(p-1) + 3p - 2}{2p(m+2)}. \tag{3.9}$$

By taking a derivative of (3.8) we get

$$\begin{aligned} L'(t) = {} & 2(1-\alpha)H^{(1-2\alpha)}(t)H'(t) + \varepsilon\int_\Omega \left|u'\right|^{m+2} dx \\ & + \varepsilon(m+1)\int_\Omega \left|u'\right|^m u'' u dx. \end{aligned} \tag{3.10}$$

By using the first equation in (1.1), we have

$$\begin{aligned}
&\varepsilon(m+1)\int_\Omega \left|u^{'}\right|^m u^{''}udx \\
&= -\varepsilon(m+1)a^2\left\|\nabla u\right\|^2 - \varepsilon(m+1)\frac{\sigma}{2}\left(\frac{d}{dt}\left\|\nabla u\right\|^2\right)\left\|\nabla u\right\|^2 \\
&- \varepsilon(m+1)b\left\|\nabla u\right\|^4 - \varepsilon(m+1) - 2\lambda\varepsilon(m+1)\int_\Omega \nabla u\nabla u^{'}dx \\
&+ \varepsilon(m+1)\int_\Omega |u|^p\,dx + \varepsilon(m+1)\int_{\Gamma_1} uy^{'}dx. \qquad (3.11)
\end{aligned}$$

By using the fourth equation in (1.1) we have

$$\varepsilon(m+1)\int_{\Gamma_1} uy^{'}d\Gamma = -\varepsilon(m+1)\int_{\Gamma_1}\frac{1}{p(x)}u^{'}ud\Gamma - \varepsilon(m+1)\int_{\Gamma_1}\frac{q(x)}{p(x)}yud\Gamma. \qquad (3.12)$$

By replacing (3.11) and (3.12) in (3.10) we get

$$\begin{aligned}
L^{'}(t) &= 2(1-\alpha)H^{(1-2\alpha)}(t)H^{'}(t) + \varepsilon\left\|u^{'}\right\|_{m+2}^{m+2} + \varepsilon(m+1)\left\|u\right\|_p^p \\
&-\varepsilon(m+1)a^2\left\|\nabla u\right\|^2 - \varepsilon(m+1)\frac{\sigma}{2}\left(\frac{d}{dt}\left\|\nabla u\right\|^2\right)\left\|\nabla u\right\|^2 \\
&+\varepsilon(m+1)\int_\Omega |u|^p\,dx - 2\lambda\varepsilon(m+1)\int_\Omega \nabla u\nabla u^{'}dx \\
&-\varepsilon(m+1)b\left\|\nabla u\right\|^4 - \varepsilon(m+1)\int_{\Gamma_1}\frac{1}{p(x)}u^{'}ud\Gamma \\
&-\varepsilon(m+1)\int_{\Gamma_1}\frac{q(x)}{p(x)}yud\Gamma. \qquad (3.13)
\end{aligned}$$

from (3.1) we have

$$\begin{aligned}
2\varepsilon\left\|u\right\|_p^p &= 2\varepsilon pH(t) + \frac{2\varepsilon p}{m+2}\left\|u^{'}\right\|_{m+2}^{m+2} + \varepsilon pa^2\left\|\nabla u\right\|^2 \\
&+\varepsilon\frac{pb}{2}\left\|\nabla u\right\|^4 + \varepsilon p\int_{\Gamma_1} q(x)y^2d\Gamma,
\end{aligned}$$

so, we can write

$$\begin{aligned}
\varepsilon(m+1)\left\|u\right\|_p^p &= \varepsilon\left(m-1\right)\left\|u\right\|_p^p + 2\varepsilon pH(t) + \frac{2\varepsilon p}{m+2}\left\|u^{'}\right\|_{m+2}^{m+2} + \varepsilon pa^2\left\|\nabla u\right\|^2 \\
&+\varepsilon\frac{pb}{2}\left\|\nabla u\right\|^4 + \varepsilon p\int_{\Gamma_1} q(x)y^2d\Gamma. \qquad (3.14)
\end{aligned}$$

By replacing (3.14) in (3.13) we get

$$\begin{aligned} L^{'}(t) &= 2(1-\alpha)H^{(1-2\alpha)}(t)H^{'}(t)+\varepsilon\left(1+\frac{2p}{m+2}\right)\left\|u^{'}\right\|_{m+2}^{m+2} \\ &\quad +\varepsilon a^{2}\left[p-(m+1)\right]\|\nabla u\|^{2}+\varepsilon b\left[\frac{p}{2}-(m+1)\right]\|\nabla u\|^{4} \\ &\quad +2\varepsilon pH(t)+\varepsilon\left(m-1\right)\|u\|_{p}^{p}-\varepsilon(m+1)\frac{\sigma}{2}\left(\frac{d}{dt}\|\nabla u\|^{2}\right)\|\nabla u\|^{2} \\ &\quad -2\lambda\varepsilon(m+1)\int_{\Omega}\nabla u\nabla u^{'}dx-\varepsilon(m+1)\int_{\Gamma_{1}}\frac{q(x)}{p(x)}yud\Gamma \\ &\quad -\varepsilon(m+1)\int_{\Gamma_{1}}\frac{1}{p(x)}u^{'}ud\Gamma+\varepsilon p\int_{\Gamma_{1}}q(x)y^{2}dx. \end{aligned} \tag{3.15}$$

On the other hand by using (2.1) and (2.2), we find

$$\begin{gathered} -\varepsilon(m+1)\textstyle\int_{\Gamma_{1}}\frac{1}{p(x)}u^{'}ud\Gamma\geq \\ -\varepsilon(m+1)\frac{\varepsilon_{1}\overline{k}}{2p_{0}}\left\|\nabla u^{'}\right\|^{2}-\varepsilon(m+1)\frac{\overline{k}}{2\varepsilon_{1}p_{0}}\|\nabla u\|^{2}, \end{gathered} \tag{3.16}$$

$$\begin{gathered} -\varepsilon(m+1)\textstyle\int_{\Gamma_{1}}\frac{q(x)}{p(x)}uyd\Gamma\geq \\ -\varepsilon(m+1)\frac{q_{1}\varepsilon_{2}}{2p_{0}}\overline{k}\|\nabla u\|^{2}-\varepsilon(m+1)\frac{1}{2p_{0}\varepsilon_{2}}\textstyle\int_{\Gamma_{1}}q(x)y^{2}dx, \end{gathered} \tag{3.17}$$

$$\begin{gathered} -\varepsilon\left(m+1\right)\frac{\sigma}{2}\left(\frac{d}{dt}\|\nabla u\|^{2}\right)\|\nabla u\|^{2}\geq \\ -\varepsilon\left(m+1\right)\frac{\varepsilon_{3}}{4}\sigma\|\nabla u\|^{4}-\varepsilon\left(m+1\right)\frac{\sigma}{4\varepsilon_{3}}\left(\frac{d}{dt}\|\nabla u\|^{2}\right)^{2}, \end{gathered} \tag{3.18}$$

and

$$\begin{gathered} -2\varepsilon\left(m+1\right)\lambda\textstyle\int_{\Omega}\nabla u\nabla u^{'}dx\geq \\ -\varepsilon\left(m+1\right)\lambda\varepsilon_{4}\|\nabla u\|^{2}-\varepsilon\left(m+1\right)\lambda\frac{1}{\varepsilon_{4}}\left\|\nabla u^{'}\right\|^{2}, \end{gathered} \tag{3.19}$$

by putting the last inequalities in (3.15), we get

$$\begin{aligned} L^{'}(t) &\geq \left[2\left(1-\alpha\right)H^{1-2\alpha}(0)-\varepsilon\left(m+1\right)\frac{1}{\varepsilon_{3}}\right]\left(\frac{d}{dt}\|\nabla u\|^{2}\right)^{2}+\varepsilon\|u\|_{p}^{p} \\ &\quad +\varepsilon\left\|u^{'}\right\|_{m+2}^{m+2}+\varepsilon pH(t)+(m+1)\,b\left[\frac{p}{2\left(m+1\right)}-\left(1+\frac{\varepsilon_{3}}{4b}\sigma\right)\right]\|\nabla u\|^{4} \\ &\quad +\lambda\left[2\left(1-\alpha\right)H^{1-2\alpha}(0)-\varepsilon\left(m+1\right)\left(\frac{\varepsilon_{1}\overline{k}}{2p_{0}}+\lambda\frac{1}{\varepsilon_{4}}\right)\right]\left\|\nabla u^{'}\right\|^{2} \\ &\quad +\varepsilon\left(m+1\right)\left[a^{2}\left(\frac{p}{(m+1)}-1\right)-\frac{\overline{k}}{2p_{0}}\left(\frac{1}{\varepsilon_{1}}+q_{1}\varepsilon_{2}+\frac{2p_{0}}{\overline{k}}\lambda\varepsilon_{4}\right)\right]\|\nabla u\|^{2} \\ &\quad +\varepsilon\left[p-(m+1)\frac{1}{2p_{0}\varepsilon_{2}}\right]\int_{\Gamma_{1}}q(x)y^{2}d\Gamma. \end{aligned} \tag{3.20}$$

We choose $L(0) = H^{2(1-\alpha)}(0) + \varepsilon \int_\Omega u_0 u_1 > 0$, and

$$\begin{aligned}
a_1 &= 2(1-\alpha) H^{1-2\alpha}(0) - \varepsilon (m+1) \frac{1}{\varepsilon_3}, \\
a_2 &= 2(1-\alpha) H^{1-2\alpha}(0) - \varepsilon (m+1) \left(\frac{\varepsilon_1 \overline{k}}{2p_0} + \lambda \frac{1}{\varepsilon_4} \right), \\
a_3 &= a^2 \left(\frac{p}{(m+1)} - 1 \right) - \frac{\overline{k}}{2p_0} \left(\frac{1}{\varepsilon_1} + q_1 \varepsilon_2 + \frac{2p_0}{\overline{k}} \lambda \varepsilon_4 \right), \\
a_4 &= \frac{p}{2(m+1)} - \left(1 + \frac{\varepsilon_3}{4b} \sigma \right), \\
a_5 &= p - (m+1) \frac{1}{2p_0 \varepsilon_2},
\end{aligned}$$

We choose also $\varepsilon_1 = \frac{2\overline{k}}{a^2 p_0}, \varepsilon_2 = \frac{2}{p_0}, \varepsilon_3 = \frac{2b}{\sigma}, \varepsilon_4 = \frac{a^2}{4\lambda}$, and

$$\varepsilon < \frac{2(1-\alpha) H^{1-2\alpha}(0)}{(m+1)} \min \left(\frac{2b}{\sigma}, \frac{(p_0 a)^2}{\overline{k}^2 + (2\lambda p_0 a)^2} \right),$$

so a_i $(i = 1, 2, 3, 4, 5)$ are positive.

Finely, we get

$$\begin{aligned}
L'(t) &\geq C \left(\left(\frac{d}{dt} \|\nabla u\|^2 \right)^2 + \|\nabla u\|^4 + \left\| \nabla u' \right\|^2 \right. \\
&\quad \left. + \|\nabla u\|^2 + \left\| u' \right\|_{m+2}^{m+2} + H(t) + \|u\|_p^p \right). \qquad (3.21)
\end{aligned}$$

This implies that

$$L(t) \geq L(0) > 0, \quad \text{for all } t \geq 0. \qquad (3.22)$$

Now, we estimate $\left| \int_\Omega |u'|^m u_t u dx \right|^{\frac{1}{2(1-\alpha)}}$, by using the Cauchy-Schwarz's inequality, Holder inequality and Young's inequality, we get

$$\left| \int_\Omega \left| u' \right|^m u_t u dx \right|^{\frac{1}{2(1-\alpha)}} \leq C_2 \left[\left\| u' \right\|_{m+2}^{m+2} + \|u\|_p^{\left(\frac{m+2}{[m(1-2\alpha)-(4\alpha-3)]} \right)} \right]. \qquad (3.23)$$

For $m > 1$ we have $2 \leq s = \frac{m+2}{[m(1-2\alpha)-(4\alpha-3)]} \leq p$, so by using of Corollary 1, we have

$$\left| \int_\Omega \left| u' \right|^{m+1} u dx \right|^{\frac{1}{2(1-\alpha)}} \leq C_2 \left[\left\| u' \right\|_{m+2}^{m+2} + \|u\|_p^p \right]. \qquad (3.24)$$

Therefore, we conclude

$$\begin{aligned} L^{\frac{1}{2(1-\alpha)}}(t) &= \left(H^{2(1-\alpha)}(t)+\varepsilon\int_\Omega uu_t dx\right)^{\frac{1}{2(1-\alpha)}} \\ &\leq \delta\left(H(t)+\left\|u'\right\|_{m+2}^{m+2}+\|u\|_p^p\right), \quad \text{for all } t\geq 0. \end{aligned} \tag{3.25}$$

By combining (3.21) and (3.25) we arrive for some positive cnstant $\gamma > C > \delta$

$$L'(t) \geq \gamma L^{\frac{1}{2(1-\alpha)}}(t), \quad \text{for all } t\geq 0. \tag{3.26}$$

A simple integration of (3.26) over $(0,t)$ then yields

$$L^{\frac{2\alpha-1}{2(1-\alpha)}}(t) \geq \frac{1}{-\frac{2\alpha-1}{2(1-\alpha)}\gamma t + L^{-\frac{2\alpha-1}{2(1-\alpha)}}(0)}. \tag{3.27}$$

Shows that $L(t)$ blows up in a finite time. The proof of Theorem 1 is now completed.

References

[1] A.V. Balakishnan, L.W. Taylor, *Distributed parameter nonlinear damping models for flight structures*, Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, 1989.

[2] R.W. Bass, D. Zes, *Spillover, nonlinearity and flexible structures,* The Fourth NASA Workship on Computational Control of Flexible Aerospace Systems, NASA Conference Publication 10065 (L.W. Taylor, ed.), 1991, 1-14.

[3] J.T. Beal, S.I. Rosencrans, *Acoustic boundary conditions,* Bull. Amer. Math. Soc., 80 (1974), 1276-1278.

[4] C.L. Frota, J.A. Goldstein, *Some nonlinear wave equations with acoustic boundary conditions*, J. Differ. Equ., 164 (2000), 92-109.

[5] C.L. Frota, N.A. Larkin, *Uniform stabilization for a hyperbolic equation with acoustic boundary conditions in simple connected domains*, Progr. Nonlinear Differential Equations Appl., 66 (2005), 297-312.

[6] V, Georgiev, G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations, 109 (1994), 295-308.

[7] G.C. Gorain, *Exponential energy decay estimates for the solutions of n-dimensional Kirchhoff type wave equation,* Applied Mathematics and Computation, 117 (2006), 235-242.

[8] H. Harrison, *Plane and circular motion of a string,* J. Acoust. Soc. Am., 20 (1948), 874-875.

[9] Y.H. Kang, *Energy decay rates for the Kelvin-Voigt type wave equation with acoustic boundary condition*, J. KSIAM., 16 (2012), 85-91.

[10] Y.H. Kang, *Energy decay rate for the kelving-voigt type wave equation with balakrishnan-Taylor damping and acoustic boundary*, East Asian Math. J., 32 (2016), 355-364.

[11] G. Kirchhoff, *Vorlesungen übear Mathematische Physik,* Mechanik (Teubner), 1977.

[12] S. A. Messaoudi, *Blow up in a nonlinearly damped wave equation,* Math. Nachr., 231 (2001), 1-7.

[13] S. A. Messaoudi, *Blow up and global existence in a nonlinear viscoelastic wave equation*, Math. Nachr., 260 (2003), 58-6.

[14] J.Y. Park, J.A. Kim, *Some nonlinear wave equations with nonlinear memory source term and acoustic boundary conditions*, Numer. Funct. Anal. Optim., 27 (2006), 889-903.

[15] J.Y. Park, S.H. Park, *Decay rate estimates for wave equations of memory type with acoustic boundary conditions,* Nonlinear Analysis: Theory, methods and Applications, 74 (2011), 993-998.

[16] J.Y. Park, T.G. Ha, *Well-posedness and uniform decay rates for the Klein-Gordon equation with damping term and acoustic boundary conditions,* J. Math. Phys., 50 (2009) Article No. 013506; doi:10.1063/1.3040185.

[17] N.-E. Tatar, A. Zaraï, *Exponential stability and blow up for a problem with Balakrishnan-Taylor damping,* Demonstratio Math., 44 (2011), 67-90.

[18] N.-E. Tatar, A. Zaraï, *On a Kirchhoff equation with Balakrishnan-Taylor damping and source term*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 18 (2011), 615-627.

[19] A. Zaraï, N.-E. Tatar, *Global existence and polynomial decay for a problem with Balakrishnan-Taylor damping*, Arch. Math. (Brno), 46 (2010), 157-176.

[20] A. Zaraï, N.-E. Tatar, A, Salem, *Elastic membrane equation with memory term and nonlinear boundary damping: global existence, decay and blowup of the solution*, Acta Math. Sci. Ser. B Engl. Ed., 33 (2013), 84-106

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Unique fixed point results for pairs of mappings on complete metric spaces

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Abstract. In this article, we prove a coincidence point results for four pairs of mappings on four complete metric spaces. The obtained result generalize some well known fixed point results of literature such as Namdeo et al.[Thai Journal of Mathematics, 7(1) (2009), 129 – 135] and Gupta et al. [International Journal of Applied Physics and Mathematics, 2 (3), (2012), 169 – 171].
Keywords: complete metric spaces, fixed point, Cauchy sequence.

1. Introduction

Banach contraction mapping principle [1] is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed point of certain self maps of metric spaces and provides a constructive method to find fixed points. This well-known classical principle has been studied and generalized in many different directions such as generalizing the used metric spaces, for different type of contractions, and for number of mappings. For some more related results on the theory of fixed point, authors refer to see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Some authors also generalized Banach results for two and three metric spaces such as Fisher [12], Jain [13], Namdeo *et al.* [14, 15], Nung [16], Kikina *et al.* ([17, 18]), Mishra *et al* [19] and Gupta *et al.* ([20, 21, 22]).

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This manuscript is devoted to the investigation of the existence of fixed points for four pair of mapping in four metric spaces. We generalize the result of three metric spaces to obtain a new fixed point theorem for four pairs of mappings on four metric spaces.

2. Preliminaries

Namdeo *et al* [15] proved following related fixed result for three pairs of mappings on three complete metric spaces.

Theorem 2.1. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. Let $A,B:X\mapsto Y$, $C,D:Y\mapsto Z$ and $E,F:Z\mapsto X$ satisfying the inequalities,*

$$d\left(ECAx,FDBx'\right)\leqslant c\frac{f_1\left(y,y',z,z'\right)}{g_1\left(x,x'\right)} \tag{2.1}$$

$$\rho\left(BECy,AFDy'\right)\leqslant c\frac{f_2\left(z,z',x,x'\right)}{g_2\left(y,y'\right)} \tag{2.2}$$

$$\sigma\left(DBEz,CAFz'\right)\leqslant c\frac{f_3\left(x,x',y,y'\right)}{g_3\left(z,z'\right)}, \tag{2.3}$$

$\forall\ x,x'\in X$, $y,y'\in Y$ and $z,z'\in Z$ for which $g_1(x,x')\neq 0$, $g_2(y,y')\neq 0$, $g_3(z,z')\neq 0$, where $0\leqslant c<1$ and

$$\begin{aligned}
f_1\left(y,y',z,z'\right) &= \max\left\{\rho\left(y,y'\right)d\left(Ez,Fz'\right),\sigma\left(Cy,Dy'\right)\rho\left(BEz,AFz'\right),\right.\\
&\quad \left.d\left(ECy,FDy'\right)\sigma\left(DBEz,CAFz'\right)\right\}\\
f_2\left(z,z',x,x'\right) &= \max\left\{\sigma\left(z,z'\right)\rho\left(Ax,Bx'\right),d\left(Ez,Fz'\right)\sigma\left(CAx,DBx'\right),\right.\\
&\quad \left.\rho\left(BEz,AFz'\right)d\left(ECAx,FDBx'\right)\right\}\\
f_3\left(x,x',y,y'\right) &= \max\left\{d\left(x,x'\right)\sigma\left(Cy,Dy'\right),\rho\left(Ax,Bx'\right)d\left(ECy,FDy'\right),\right.\\
&\quad \left.\sigma\left(CAx,DBx'\right)\rho\left(BECy,AFDy'\right)\right\}
\end{aligned}$$

$$\begin{aligned}
g_1\left(x,x'\right) &= \max\left\{d\left(x,x'\right),\rho\left(Ax,Bx'\right),\ \sigma\left(CAx,DBx'\right),d\left(ECAx,FDBx'\right)\right\}\\
g_2\left(y,y'\right) &= \max\left\{\rho\left(y,y'\right),\sigma\left(Cy,Dy'\right),\ d\left(ECy,FDy'\right),\rho\left(BECy,AFDy'\right)\right\}\\
g_3\left(z,z'\right) &= \max\left\{\sigma\left(z,z'\right),d\left(Ez,Fz'\right),\ \rho\left(BEz,AFz'\right),\sigma\left(DBEz,CAFz'\right)\right\}.
\end{aligned}$$

If A and C or B and D are continuous, then ECA and FDB have a unique common fixed point $u\in X$, BEC and AFD have a unique fixed point $v\in Y$, and DBE and CAF have a unique common fixed point $w\in Z$. Further, $Au=Bu=v$, $Cv=Dv=w$ and $Ew=Fw=u$.

We prove our main result for four pairs of mappings on four complete metric spaces.

3. Main result

Theorem 3.1. *Let (X, d_1), (Y, d_2), (Z, d_3) and (Q, d_4) be complete metric spaces. Let $A, B : X \mapsto Y$, $C, D : Y \mapsto Z$, $E, F : Z \mapsto Q$ and $G, H : Q \mapsto X$, satisfying the inequalities,*

$$d_1\left(GECAx, HFDBx'\right) \leqslant c\frac{f_1\left(z, z', q, q'\right)}{g_1\left(x, x'\right)}, \tag{3.1}$$

$$d_2\left(BGECy, AHFDy'\right) \leqslant c\frac{f_2\left(q, q', x, x'\right)}{g_2\left(y, y'\right)}, \tag{3.2}$$

$$d_3\left(DBGEz, CAHFz'\right) \leqslant c\frac{f_3\left(x, x', y, y'\right)}{g_3\left(z, z'\right)}, \tag{3.3}$$

$$d_4\left(FDBGq, ECAHq'\right) \leqslant c\frac{f_4\left(y, y', z, z'\right)}{g_4\left(q, q'\right)}, \tag{3.4}$$

$\forall\, x, x' \in X$; $y, y' \in Y$; $z, z' \in Z$ and $q, q' \in Q$ for which $g_1\left(x, x'\right) \neq 0$; $g_2\left(y, y'\right) \neq 0$; $g_3\left(z, z'\right) \neq 0$ and $g_4\left(q, q'\right) \neq 0$ where $0 \leqslant c < 1$ and

$$\begin{aligned}
f_1\left(z, z', q, q'\right) &= \max\left\{d_3\left(z, z'\right) d_1\left(Gq, Hq'\right), d_4\left(Fz, Fz'\right) d_2\left(BGq, AHq'\right),\right. \\
&\quad \left. d_1\left(GEz, HFz'\right) d_4\left(FDBGq, ECAHq'\right)\right\} \\
f_2\left(q, q', x, x'\right) &= \max\left\{d_4\left(q, q'\right) d_2\left(Ax, Bx'\right), d_1\left(Gq, Hq'\right) d_3\left(CAx, DBx'\right)\right. \\
&\quad \left., d_2\left(BGq, AHq'\right) d_1\left(GECAx, HFDBx'\right)\right\} \\
f_3\left(x, x', y, y'\right) &= \max\left\{d_1\left(x, x'\right) d_3\left(Cy, Dy'\right), d_2\left(Ax, Bx'\right) d_4\left(ECy, FDy'\right),\right. \\
&\quad \left. d_3\left(CAx, DBx'\right) d_2\left(BGECy, AHFDy'\right)\right\} \\
f_4\left(y, y', z, z'\right) &= \max\left\{d_2\left(y, y'\right) d_4\left(Ez, Fz'\right), d_3\left(Cy, Dy'\right) d_1\left(GEz, HFz'\right),\right. \\
&\quad \left. d_4\left(ECy, FDy'\right) d_3\left(DBGEz, CAHFz'\right)\right\} \\
g_1\left(x, x'\right) &= \max\left\{d_1\left(x, x'\right), d_2\left(Ax, Bx'\right), d_3\left(CAx, DBx'\right),\right. \\
&\quad \left. d_4\left(ECAx, FDBx'\right), d_1\left(GECAx, HFDBx'\right)\right\} \\
g_2\left(y, y'\right) &= \max\left\{d_2\left(y, y'\right), d_3\left(Cy, Dy'\right), d_4\left(ECy, FDy'\right),\right. \\
&\quad \left. d_1\left(GECy, HFDy'\right), d_2\left(BGECy, AHFDy'\right)\right\} \\
g_3\left(z, z'\right) &= \max\left\{d_3\left(z, z'\right), d_4\left(Ez, Fz'\right), d_1\left(GEz, HFz'\right),\right. \\
&\quad \left. d_2\left(BGEz, AHFz'\right), d_3\left(DBGEz, CAHFz'\right)\right\} \\
g_4\left(q, q'\right) &= \max\left\{d_4\left(q, q'\right), d_1\left(Gq, Hq'\right), d_2\left(BGq, AHq'\right),\right. \\
&\quad \left. d_3\left(DBGq, CAHq'\right), d_4\left(FDBGq, ECAHq'\right)\right\}.
\end{aligned}$$

If A, C and E or B, D and F are continuous, then $GECA$ and $HFDB$ have a unique common fixed point $u \in X$. $BGEC$ and $AHFD$ have a unique common fixed point $v \in Y$. $DBGE$ and $CAHF$ have a unique common fixed point $w \in Z$ and $FDBG$ and $ECAH$ have a unique common fixed point $t \in Q$. Further, $Au = Bu = v$, $Cv = Dv = w$, $Ew = Fw = t$, $Gt = Ht = u$.

Proof. Let $x = x_0$ be an arbitrary point in X. We define the sequence $\{x_n\}$ in X,$\{y_n\}$ in Y, $\{z_n\}$ in Z and $\{q_n\}$ in Q inductively for $n = 1, 2, \ldots$ by,

$$Ax_{2n-2} = y_{2n-1}, \quad Cy_{2n-1} = z_{2n-1}, \quad Ez_{2n-1} = q_{2n-1}, \quad Gq_{2n-1} = x_{2n-1};$$
$$Bx_{2n-1} = y_{2n}, \quad Dy_{2n} = z_{2n}, \quad Fz_{2n} = q_{2n}, \quad Hq_{2n} = x_{2n};$$

First we suppose that for some n, $g(x_{2n}, x_{2n-1})$ is zero then we obtain fixed point of mapping directly. For this suppose that,

$$\begin{aligned} g_1(x_{2n}, x_{2n-1}) &= \max\{d_1(x_{2n}, x_{2n-1}), d_2(Ax_{2n}, Bx_{2n-1}), d_3(CAx_{2n}, DBx_{2n-1}), \\ &\quad d_4(ECAx_{2n}, FDBx_{2n-1}), d_1(GECAx_{2n}, HFDBx_{2n-1})\} \\ &= \max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n+1}, y_{2n}), d_3(z_{2n+1}, z_{2n}), \\ &\quad d_4(q_{2n+1}, q_{2n}), d_1(x_{2n+1}, x_{2n})\} \\ &= 0 \qquad\qquad \text{for some } n. \end{aligned}$$

Then on putting,

$$x_{2n-1} = x_{2n} = x_{2n+1} = u, y_{2n+1} = y_{2n} = v, z_{2n+1} = z_{2n} = w, q_{2n+1} = q_{2n} = t,$$

we get that,

$$GECAu = HFDBu = u = Gt = Ht, \qquad AHFDv = v = Au = Bu,$$
$$CAHFw = w = Cv = Dv, \qquad ECAHt = t = Ew = Fw.$$

From which it follows that,

$$BGECv = v, \quad DBGEw = w, \quad FDBGt = t.$$

Similarly, $g_1(x_{2n}, x_{2n+1}) = 0$ for some n implies that there exist point $u \in X$, $v \in Y$, $w \in Z$ and $t \in Q$ such that,

$$\begin{aligned} GECAu &= HFDBu = u = Gt = Ht, \\ AHFDv &= BGECv = v = Au = Bu, \\ CAHFw &= DBGEw = w = Cv = Dv, \\ ECAHt &= FDBGt = t = Ew = Fw. \end{aligned} \tag{3.5}$$

Similarly, if one of $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$, $g_3(z_{2n+1}, z_{2n})$, $g_4(q_{2n-1}, q_{2n})$, $g_4(q_{2n+1}, q_{2n})$ is equal to zero for some n, then inequality (3.4) follows.

Next we suppose that $g_1(x_{2n-1}, x_{2n})$, $g_1(x_{2n}, x_{2n+1})$, $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$, $g_3(z_{2n+1}, z_{2n})$, $g_4(q_{2n-1}, q_{2n})$, $g_4(q_{2n+1}, q_{2n})$ are all non-zero for all n, then we also get fixed point of mappings.

Thus,

$$\begin{aligned} f_1(z, z', q, q') = \max\{&d_3(z, z')d_1(Gq, Hq'), d_4(Ez, Fz')d_2(BGq, AHq'), \\ &d_1(GEz, HFz')d_4(FDBGq, ECAHq')\} \end{aligned}$$

$$f_1(z_{2n-1}, z_{2n}, q_{2n-1}, q_{2n}) = \max\{d_3(z_{n-1}, z_{2n})d_1(x_{2n-1}, x_{2n}), d_4(q_{2n-1}, q_{2n})d_2(y_{2n}, y_{2n+1}), d_1(x_{2n-1}, x_{2n})d_4(q_{2n}, q_{2n+1})\} \tag{3.6}$$

$$f_2(q_{2n-1}, q_{2n}, x_{2n}, x_{2n-1}) = \max\{d_4(q_{2n-1}, q_{2n})d_2(y_{2n+1}, y_{2n}), d_1(x_{2n-1}, x_{2n})d_3(z_{2n+1}, z_{2n}), d_2(y_{2n}, y_{2n+1})d_1(x_{2n+1}, x_{2n})\} \tag{3.7}$$

$$f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \max\{d_1(x_{2n}, x_{2n-1})d_3(z_{2n-1}, z_{2n}), d_2(y_{2n+1}, y_{2n})d_4(q_{2n-1}, q_{2n}), d_3(z_{2n+1}, z_{2n})d_2(y_{2n}, y_{2n+1})\} \tag{3.8}$$

$$f_4(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) = \max\{d_2(y_{2n-1}, y_{2n})d_4(q_{2n-1}, q_{2n}), d_3(z_{2n-1}, z_{2n})d_1(x_{2n-1}, x_{2n}), d_4(q_{2n-1}, q_{2n})d_3(z_{2n}, z_{2n+1})\} \tag{3.9}$$

$$\begin{aligned} g_1(x_{2n}, x_{2n-1}) &= \max\{d_1(x, x'), d_2(Ax, Bx'), d_3(CAx, DBx'), \\ &\quad d_4(ECAx, FDBx')d_1(GECAx, HFDBx')\} \\ &= \max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n+1}, y_{2n}), d_3(z_{2n+1}, z_{2n}), \\ &\quad d_4(q_{2n+1}, q_{2n}), d_1(x_{2n+1}, x_{2n})\} \end{aligned} \tag{3.10}$$

$$g_2(y_{2n-1}, y_{2n}) = \max\{d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1})\} \tag{3.11}$$

$$g_3(z_{2n-1}, z_{2n}) = \max\{d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1}), d_3(z_{2n}, z_{2n+1})\} \tag{3.12}$$

$$g_4(q_{2n-1}, q_{2n}) = \max\{d_4(q_{2n-1}, q_{2n}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1}), d_3(z_{2n}, z_{2n+1}), d_4(q_{2n}, q_{2n+1})\}. \tag{3.13}$$

Applying (3.1), we obtain

$$\begin{aligned} d_1(x_{2n+1}, x_{2n}) &= d_1(GECAx_{2n}, HFDBx_{2n-1}) \\ &\leqslant \frac{cf_1(z_{2n-1}, z_{2n}, q_{2n-1}, q_{2n})}{g_1(x_{2n}, x_{2n-1})}. \end{aligned} \tag{3.14}$$

It follows from (3.6), (3.10) and (3.14),

$$d_1(x_{2n}, x_{2n+1}) \leqslant c\max\{d_1(x_{2n-1}, x_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n})\}. \tag{3.15}$$

Applying inequality (3.2),

$$\begin{aligned} d_2(y_{2n}, y_{2n+1}) &= d_2(BGECy_{2n-1}, AHFDy_{2n}) \\ &\leqslant \frac{cF_2(q_{2n-1}, q_{2n}, x_{2n}, x_{2n-1})}{g_2(y_{2n-1}, y_{2n})}. \end{aligned} \tag{3.16}$$

It follows from (3.7), (3.11) and (3.16), that

$$d_2\left(y_{2n}, y_{2n+1}\right) \leqslant c \max\left\{d_1\left(x_{2n+1}, x_{2n}\right), d_3\left(z_{2n+1}, z_{2n}\right)\right\}. \tag{3.17}$$

From (3.3), we obtain

$$\begin{aligned} d_3\left(z_{2n}, z_{2n+1}\right) = & \ d_3\left(DBGEz_{2n-1}, CAHFz_{2n}\right) \\ \leqslant & \ c\frac{f_3\left(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}\right)}{g_3\left(z_{2n-1}, z_{2n}\right)}. \end{aligned} \tag{3.18}$$

On Using (3.8), (3.12) and (3.18),

$$d_3\left(z_{2n}, z_{2n+1}\right) \leqslant c \max\left\{d_1\left(x_{2n-1}, x_{2n}\right), d_4\left(q_{2n-1}, q_{2n}\right)\right\}. \tag{3.19}$$

Using (3.4), we get

$$\begin{aligned} d_4\left(q_{2n}, q_{2n+1}\right) = & \ d_4\left(FDBGq_{2n}, ECAHq_{2n}\right) \\ \leqslant & \ c\frac{f_4\left(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}\right)}{g_4\left(q_{2n-1}, q_{2n}\right)}. \end{aligned} \tag{3.20}$$

Using (3.9), (3.13) and (3.20),

$$\begin{aligned} & d_4(q_{2n}, q_{2n+1}) \leqslant c \max\{d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n})\} \\ & d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{cd_1(x_{2n-1}, x_{2n}), cd_3(z_{2n-1}, z_{2n}), cd_4(q_{2n-1}, q_{2n})\}. \end{aligned} \tag{3.21}$$

On making use of (3.18) and (3.20), we get

$$d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n})\}. \tag{3.22}$$

Applying inequality (3.1), again we get,

$$\begin{aligned} d_1\left(x_{2n-1}, x_{2n}\right) = & \ d_1\left(GECAx_{2n-2}, HFDBx_{2n-1}\right) \\ \leqslant & \ c\frac{f_1\left(z_{2n-1}, z_{2n-2}, q_{2n-1}, q_{2n-2}\right)}{g_1\left(x_{2n-2}, x_{2n-1}\right)}. \end{aligned}$$

For which it follows that

$$\begin{aligned} d_1(x_{2n-1}, x_{2n}) \leqslant & c \max\{d_1(x_{2n-2}, x_{2n-1}), d_3(z_{n-2}, z_{2n-1}), \\ & d_4(q_{2n-2}, q_{2n-1})\} \end{aligned} \tag{3.23}$$

and similarly using inequality (3.2), (3.3) and (3.4),

$$d_2\left(y_{2n-1}, y_{2n}\right) \leqslant c \max\left\{d_1\left(x_{2n-1}, x_{2n}\right), d_3\left(z_{2n-1}, z_{2n}\right)\right\} \tag{3.24}$$

$$d_3\left(z_{2n-1}, z_{2n}\right) \leqslant c \max\left\{d_1\left(x_{2n-2}, x_{2n-1}\right), d_4\left(q_{2n-2}, q_{2n-1}\right)\right\} \tag{3.25}$$

$$d_4\left(q_{2n-1}, q_{2n}\right) \leqslant c \max\left\{d_2\left(y_{2n-2}, y_{2n-1}\right), d_3\left(z_{2n-2}, z_{2n-1}\right)\right\}. \tag{3.26}$$

Using inequalities (3.23), (3.24), (3.25) and (3.26),

$$d_2(y_{2n-1}, y_{2n}) \leqslant c \max\{d_1(x_{2n-2}, x_{2n-1}), d_3(z_{2n-2}, z_{2n-1}), d_4(q_{2n-2}, q_{2n-1})\}. \tag{3.27}$$

It follows from inequalities (3.15) and (3.23) that

$$\begin{aligned} d_1(x_n, x_{n+1}) &\leqslant c \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(z_{n-1}, z_n)\} \\ &\leqslant c^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\}. \end{aligned} \tag{3.28}$$

Similarly using inequalities (3.22), (3.27), (3.19), (3.25), (3.21) and (3.26),

$$d_2(y_n, y_{n+1}) \leqslant c^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\} \tag{3.29}$$

$$d_3(z_n, z_{n+1}) \leqslant c^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\} \tag{3.30}$$

$$d_4(q_n, q_{n+1}) \leqslant c^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\}. \tag{3.31}$$

Since $c < 1$, then it is clear that, $\{x_n\}$ is a Cauchy sequence in X with limit u, $\{y_n\}$ is a Cauchy sequence in Y with limit v, $\{z_n\}$ is a Cauchy sequence in Z with limit w, $\{q_n\}$ is a Cauchy sequence in Q with limit t.

Also, maps A, C and E are continuous then,

$$\begin{aligned} \lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty} Ax_{2n} = Au = v. \\ \lim_{n\to\infty} z_{2n-1} &= \lim_{n\to\infty} Cy_{2n-1} = Cv = w. \\ \lim_{n\to\infty} q_{2n-1} &= \lim_{n\to\infty} Ez_{2n-1} = Ew = t. \end{aligned} \tag{3.32}$$

and hence,

$$\lim_{n\to\infty} f_1(w, Z_{2n}, t, q_{2n}) = d_1(Gt, u)\, d_4(FDBGt, t). \tag{3.33}$$

$$\lim_{n\to\infty} f_2(t, q_{2n}, u, x_{2n-1}) = d_2(BGt, v)\, d_1(Gt, u). \tag{3.34}$$

$$\lim_{n\to\infty} f_3(u, x_{2n-1}, v, y_{2n}) = 0. \tag{3.35}$$

$$\lim_{n\to\infty} f_4(v, y_{2n}, w, z_{2n}) = 0. \tag{3.36}$$

$$\lim_{n\to\infty} g_1(u, x_{2n-1}) = d_1(Gt, u) \tag{3.37}$$

$$\lim_{n\to\infty} g_2(v, y_n) = \max\{d_1(Gt, u), d_2(BGt, v)\} \tag{3.38}$$

$$\lim_{n\to\infty} g_3(w, z_n) = \max\{d_1(Gt, u), d_2(BGt, v), d_3(DBGt, w)\} \tag{3.39}$$

$$\begin{aligned} \lim_{n\to\infty} g_4(t, q_{2n}) = \max\{&d_1(Gt, u), d_2(BGt, v), \\ &d_3(DBGt, w), d_4(FDBGt, t)\}. \end{aligned} \tag{3.40}$$

If $\lim_{n\to\infty} g_1(u, x_{2n}) = 0$ then, $Gt = u, GEw = u, GECv = u, GECAu = u$. And if, $\lim_{n\to\infty} g_1(u, x_{2n}) = d_1(Gt, u) \neq 0$ then applying inequalities (3.1), (3.32), (3.33) and (3.37),

$$d_1(Gt, u) = \lim_{n\to\infty} d_1(GECAu, HFDBx_{2n-1}) \leqslant cd_4(FDBGt, t) \tag{3.41}$$

Using (3.4),(3.36) and (3.40),

$$d_4(FDBGt, t) = \lim_{n\to\infty} d_4(FDBGt, ECAHq_{2n}) = 0$$

Thus by using (3.2), (3.32), (3.34) and (3.38)

$$d_2(BGt, v) = \lim_{n\to\infty} d_2(BGECv, AHFDy_{2n}) \leqslant cd_1(Gt, u) = 0$$

This implies that, $Bu = v, DBu = w, Dv = w$. Therefore from (3.3),

$$d_3(DBGt, w) = \lim_{n\to\infty} d_3(DBGEw, CAHFz_{2n}) = 0.$$

and $DBGEw = w, Bu = v, FDBGt = t, Dv = w, GECAu = u, GT = u$. Suppose that $Ht \neq u$. Now Applying inequality (3.1),

$$\begin{aligned} d_1(u, Ht) &= \lim_{n\to\infty} d_1(GECAx_{2n}, HFDBu) \\ &\leqslant c\frac{\lim_{n\to\infty} f_1(z_{2n-1}, w, q_{2n-1}, t)}{\lim_{n\to\infty} g_1(x_{2n}, u)} = cd_4(t, ECAHt). \end{aligned} \tag{3.42}$$

Applying inequality (3.4),

$$\begin{aligned} d_4(t, ECAHt) &= \lim_{n\to\infty} d_4(FDBGq_{2n}, ECAHt) \\ &\leqslant c\frac{\lim_{n\to\infty} f_4(y_{2n-1}, v, z_{2n-1}, w)}{\lim_{n\to\infty} g_4(q_{2n-1}, t)} = 0. \end{aligned}$$

This implies that $ECAHt = t$ and hence from inequality (3.42) we must have $Ht = u$. Equation (3.5) follows similarly if B, D, F are continuous.

Uniqueness

Let $GECA$ and $HFDB$ have another fixed point u', then

$$\begin{aligned} d_1(u, u') = d_1(GECAu, HFDBu') \leqslant\ & \frac{cF_1(CAu, DBu', ECAu, FDBu')}{g_1(u, u')} \\ & c\max\{d_3(CAu, DBu')\, d_1(GECAu, HFDBu'), \\ & d_4(ECAu, FDBu')\, d_2(BGECAu, AHFDBu'), \\ & d_1(GECAu, HFDBu') \\ \leqslant\ & \frac{d_4(FDBGECAu, ECAHFDBu')\}}{g_1(u, u')} \end{aligned}$$

$$\leqslant \frac{\begin{array}{l} c\max\left\{d_3\left(CAu,DBu'\right)d_1\left(u,u'\right),\right. \\ d_4\left(ECAu,FDBu'\right)d_2\left(Bu,Au'\right),d_1\left(u,u'\right) \\ \left.d_4\left(FDBu,ECAu'\right)\right\} \end{array}}{\begin{array}{l} \max\left\{d_1\left(u,u'\right),d_2\left(Au,Bu'\right),d_3\left(CAu,DBu'\right),\right. \\ \left.d_4\left(ECAu,FDBu'\right),d_1\left(GECAu,HFDBu'\right)\right\} \end{array}}$$

$$\leqslant \frac{\begin{array}{l} c\max\left\{d_3\left(w,DBu'\right)d_1\left(u,u'\right),d_4\left(t,FDBu'\right)\right. \\ \left.d_2\left(v,Au'\right),d_1\left(u,u'\right)d_4\left(t,ECAu'\right)\right\} \end{array}}{\begin{array}{l} \max\left\{d_1\left(u,u'\right),d_2\left(v,Bu'\right),d_3\left(w,DBu'\right),\right. \\ \left.d_4\left(t,FDBu'\right),d_1\left(u,u'\right)\right\}. \end{array}}$$

This implies that,

$$d_1\left(u,u'\right)\leqslant c\max\left\{d_3\left(w,DBu'\right),d_2\left(v,Au'\right),d_4\left(t,ECAu'\right)\right\}. \tag{3.43}$$

Using inequality (3.2), we get

$$d_2\left(v,Au'\right)=d_2\left(BGECAu,AHFDBu'\right)\leqslant c\frac{F_2\left(ECAu,FDBu',u,u'\right)}{g_2\left(Au,Bu'\right)}$$

$$\leqslant c\max\left\{d_2\left(v,Bu'\right),d_1\left(u,u'\right)\right\}. \tag{3.44}$$

Inequality (3.4) implies that

$$d_4\left(t,ECAu'\right)=d_4\left(FDBGECAu,ECAHFDBu'\right)\leqslant c\frac{f_4\left(Au,Bu',CAu,DBu'\right)}{g_4\left(ECAu,FDBu'\right)}$$

$$d_4\left(t,ECAu'\right)\leqslant c\max\left\{d_2\left(v,Bu'\right),d_3\left(w,DBu'\right),\ d_3\left(w,CAu'\right)\right\}. \tag{3.45}$$

On using inequality (3.2) again,

$$d_2\left(Bu',v\right)=d_2\left(BGECAu',AHFDBu\right)\leqslant c\frac{f_2\left(ECAu',FDBu,u,u'\right)}{g_2\left(Bu,Au'\right)}$$

$$\leqslant c\max\left\{d_2\left(v,Au'\right),d_1\left(u,u'\right)\right\}. \tag{3.46}$$

From inequality (3.3),

$$d_3\left(w,DBu'\right)=d_3\left(DBu',w\right)=d_3\left(DBGECAu',CAHFDBu\right) \tag{3.47}$$

$$\leqslant c\frac{f_3\left(u,u',Au',Bu\right)}{g_3\left(CAu',DBu\right)}$$

$$\leqslant c\max\left\{d_1\left(u,u'\right),d_2\left(v,Au'\right),d_3\left(w,CAu'\right)\right\}. \tag{3.48}$$

Using inequality (3.3) again, we obtain

$$d_3\left(w,CAu'\right)=d_3\left(DBGECAu,CAHFDBu'\right)\leqslant c\frac{f_3\left(u,u',Au,Bu'\right)}{g_3\left(CAu,DBu'\right)}$$

$$\leqslant d_1\left(u,u'\right),d_2\left(v,Bu'\right),d_2\left(v,Au'\right). \tag{3.49}$$

Now from inequalities (3.43)–(3.49)

$$d_1\left(u,u'\right) \leqslant c\max\left\{d_2\left(v,Au'\right),d_3\left(v,Bu'\right)\right\} \tag{3.50}$$

and then inequality (3.44), (3.46) and (3.50) implies that $u = u'$, this proves the uniqueness of u. Similarly we can prove that v is the unique common fixed point of $BGEC$, $AHFD$ and w is the unique common fixed point of $DBGE$, $CAHF$ and t is the unique common fixed point of $FDBG$, $ECAH$. □

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals,* Fund. Math., 3 (1922), 133-181.

[2] N. Mani, *Existence of fixed points and their applications in certain spaces [Ph.D. Thesis],* M. M. University, Mullana, Ambala, India, (2017).

[3] N. Mani, *Generalized C_β^ψ -rational contraction and fixed point theorem with application to second order differential equation,* Mathematica Moravica, 22 (2018), 43-54.

[4] V. Gupta, N. Mani, A. Kanwar, *A fixed point theorem on four complete metric spaces,* International Journal of Applied Physics and Mathematics, 2 (2012), 169-171.

[5] V. Gupta, A.K. Tripathi, A. Kanwar, *Related fixed points on four metric spaces,* International Journal of Mathematical Sciences, 11 (2012), 365-371.

[6] V. Gupta, A. Kanwar, R. Sharma, *Theorem on fixed point in n-metric spaces*, International Journal of Mathematical Archive, 3 (2012), 2393-2399.

[7] P. N. Dutta, B.S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory and Applications, 2008, Article ID 406368.

[8] V. Gupta, *Fixed point theorems for pair of mappings on three metric spaces,* Advances in Applied Science Research, Pelagia Research Library, 3 (2012), 2733-2738.

[9] G. Jungck, *Commuting mappings and fixed points,* Amer. Math. Monthly, 83 (1976), 261-263.

[10] M. Geraghty, *On contractive mappings,* Proc. Amer. Math. Soc., 40 (1973), 604-608.

[11] V. Gupta, Ramandeep, N. Mani, AK Tripathi, *Some fixed point result involving generalized altering distance function*, Procedia Computer Science, 79 (2016), 112-117.

[12] B. Fisher, P.P. Muthy, *Related fixed point theorems for two pairs of mappings on two metric spaces,* Kyungpook Math. J., 37 (1997), 343-347.

[13] R.K. Jain, H.K. Sahu, B. Fisher, *A related fixed point theorem on three metric spaces,* Novi. Sad J. Math., 26 (1996), 11-17.

[14] R.K. Namdeo, S. Jain, *A related fixed point theorem for two pairs of mappings on two complete metric spaces,* Hacettep J. Math. Stat., 32 (2003), 7-11.

[15] R.K. Namdeo, B. Fisher, *A related fixed point theorem for three pairs of mappings on three metric spaces,* Thai Journal of Mathematics, 7 (2009), 129-135.

[16] N.P. Nung, *A fixed point theorem in three metric spaces,* Math. Seminar Notes, Kobe University, 11 (1985), 77-79.

[17] L. Kikina, K. Kikina, *A related fixed point theorem on four metric spaces,* Int. Journal of Math. Analysis, 3 (2009), 1559-1568.

[18] L. Kikina, K. Kikina, *Generalized fixed point theorem in three metric spaces,* Int. Journal of Math. Analysis, 4 (2010), 1995-2004.

[19] M.K. Mishra, D.B. Ojha, *Fixed point theorems in three metric spaces,* Int. J. of Adv. Engg. Sci. and Tech., 1 (2000), 130-134.

[20] V. Gupta *Related fixed point theorem for four complete metric spaces*, Acta Universitatis Sapientiae, Mathematica, 5 (2013), 147-156.

[21] V. Gupta, A. Kanwar, S.B. Singh, *A fixed point theorem for n-mapping on n-metric spaces,* International journal of Emerging Trends in Engineering and Development, 1 (2012), 257-268.

[22] V. Gupta, A. Kanwar, R. Kaur, *A fixed point theorem on five complete metric spaces,* International journal of Emerging Trends in Engineering and Development, 2 (2012), 202-209.

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Generation of analytic semigroups by a pair of generalized mixed linear regular ordinary differential operators with interface condition

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Abstract. In this paper, we establish with suitable assumptions the analyticity of semigroups generated by a pair of generalized mixed linear regular differential operators

$$\begin{aligned} L_{(n,n)}u(x) &:= \big(L_{1n}u_1(x), L_{2n}u_2(x)\big) \\ &= \Big(\sum_{0\le k\le n} p_k(x)\big(\frac{d}{dx}\big)^k u_1(x), \sum_{0\le k\le n} q_k(x)\big(\frac{d}{dx}\big)^k u_2(x)\Big) \end{aligned}$$

with involving an interface condition in the setting of complex Hilbert space $X = L^2([a,b]) \times L^2([b,c])$. We obtain quite general results that extend previous works by the authors ([3], [10]). The key for showing the generation analytic semigroups will be an inequality of the form

$$Re\langle\big(L_{(n,n)} - \rho I\big)u,\ u\rangle_X + \delta|Im\langle\big(L_{(n,n)} - \rho I\big)u,\ u\rangle_X| \le 0, \forall u \in D(L_{(n,n)})$$

for some constant $\rho > 0$.

Keywords: unbounded operators, Dissipative operators, Adjoint, Interface condition, C_0- semigroups, analytic semigroups.

1. Introduction

Interface problems are a class of problems wherein two different differential equations are defined on two adjacent intervals and the solutions satisfy matching conditions at the point of interface.We encounter these problems in the study of acoustic wave guides [4]. Recently, a new class of problems of the type where different differential operators are defined over two adjacent intervals, involving certain mixed (interface) conditions are studied in [1, 2, 13] and the references

therein. These problems involve a pair of differential operators of the type

$$(1.1)\qquad \begin{cases} L_{1n}u_1 = \displaystyle\sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1 = \lambda u_1, \\ \qquad\qquad \text{defined on the interval } \ I_1=[a,b] \\ L_{2m}u_2 = \displaystyle\sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2 = \lambda u_2, \\ \qquad\qquad \text{defined on the adjacent interval } \ I_2=[b,c], \end{cases}$$

where $-\infty < a < c < +\infty$, λ is an unknown constant (eigenvalue) and the functions $u_1$and u_2 are required to satisfy certain mixed conditions at the interface $x = b$.

In this paper we consider the following differential system

$$(1.2)\qquad \begin{cases} \dfrac{du_1}{dt} - \displaystyle\sum_{1\le k\le n} p_k(x)\frac{d^k u_1}{dx^k} = 0 \text{ in } I_1 \\ \dfrac{du_2}{dt} - \displaystyle\sum_{1\le k\le m} q_k(x)\frac{d^k u_2}{dx^k} = 0 \text{ in } I_2, \end{cases}$$

where $(p_k)_{1\le k\le n}$ (respectively $(q_k)_{1\le k\le m}$) are numerical functions on $[a,b]$ (respectively on $[b,c]$). Our goal is to establish some existence results and regularity for solutions of the initial value problem associated with the system (1.2) i.e.,

$$(1.3)\qquad \begin{cases} \dfrac{du(t)}{dt} - L_{(n,m)}u(t) = 0, \quad t \ge 0 \\ u(0) = u_0 \end{cases}$$

where $L_{(n,m)}$ is the pair of differential operators defined by

(1.4)

$$L_{(n,m)}u = \begin{pmatrix} L_{1n}u_1 \\ L_{2m}u_2 \end{pmatrix} = \begin{cases} L_{1n}u_1 = \displaystyle\sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1 \ \text{ on } \ I_1=[a,b] \\ L_{2m}u_2 = \displaystyle\sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2 \ \text{ on } \ I_2=[b,c]. \end{cases}$$

For $n = m = 2$, a similar works on these problems for regular case have been discussed in [3, 7, 10], and the problem of having singularity at the boundary is discussed in [11].

In [3], T. Bhaskar and R. Kummar have considered the following system on the Hilbert space X:

$$(1.5)\qquad \begin{cases} \dfrac{du_1}{dt} - \tau_1 u_1 = 0 \ \text{ in } I_1, \\ \dfrac{du_2}{dt} - \tau_2 u_2 = 0 \ \text{ in } I_2, \end{cases}$$

where τ_k is the second order differential operator $\tau_k u_k = p_k u_k'' + q_k u_k' + r_k u_k$ on I_k for $k = 1, 2$. They proved that under certain assumptions on the matrices A and B that the operator $((\tau_1, \tau_2) - \omega I)$ generates an analytic semigroup of contractions.

In a similar order of ideas, A.Saddi and the present author in [10], proved that certain differential operators of a single variable on an interval $[a, c]$, with matrix coefficients and interface conditions at a point $b \in]a, c[$, generate analytic semigroups in L^2. More precisely, the following second order operator is considered in the space $X = L^2(a, b)^4 \times L^2(b, c)^4$

$$Lu = L \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \end{pmatrix} = \begin{cases} p_{11}u_{11}'' + q_{11}u_{11}' \\ p_{11}u_{12}'' + q_{11}u_{12}' \\ p_{12}u_{13}'' + q_{12}u_{13}' & \text{in } [a,\ b] \\ p_{12}u_{14}'' + q_{12}u_{14}' \\ \\ p_{21}u_{21}'' + q_{21}u_{21}' \\ p_{21}u_{22}'' + q_{21}u_{22}' \\ p_{22}u_{23}'' + q_{22}u_{23}' & \text{in } [b,\ c] \\ p_{22}u_{24}'' + q_{22}u_{24}' \end{cases}$$

where $p_{ki} \in H^2(a, c)$, $p_{ki}(x) \geq p > 0$ and $q_{ki} \in AC[a, c]$.

It is well know that problem of type (1.3) is well posed in a Banach space X if and only if the operator $(L_{(n,m)}, D(L_{(n,n)}))$ generates a C_0-semigroup $(T_t)_{t\geq 0}$ on X. Here the solution $u(t)$ is given by $u(t) = T_t u_0$ for the initial data $u_0 \in D(L_{(n,n)})$. For operator semigroups we refer to [5, 8, 9].

The paper is organized as follows: In section 2, we start by setting up our different notions and notations which we shall need in the sequel. In section 3, we study the mixed operator $L_{(n,m)}$ and its adjoint $L^*_{(n,m)}$ and investigate some of its properties. In section 4, we study the m-dissipativity of the operator $(L_{(n,n)} - \rho I)$ for some $\rho > 0$. The main results of this section are Lemma 4.1,which is the key to proving our main results, Propositions 4.1 and 4.2 give conditions,when the operator $L_{(n,n)} - \rho I$ and its adjoint are dissipative. Using the results of section 4,we prove in section 5, that under suitable assumptions $L_{(n,n)} - \rho I$ satisfies

$$(1.6)\quad Re\langle \big(L_{(n,n)} - \rho I\big)u,\ ; u\rangle_X + \delta |Im\langle \big(L_{(n,n)} - \rho I\big)u\ u\rangle_X| \leq 0 \quad \forall\, u \in D(L_{(n,n)})$$

with $\rho > 0$, so that L generates an analytic semigroup on X (see [6]).

2. Functional setting of the problem

We shall introduce a few notations and make some assumptions. Let $\mathbb{R}$ and $\mathbb{C}$ represent the real and complex number fields and let $M_n(\mathbb{K})$, $(K = \mathbb{R},\ \text{or}\ \mathbb{C})$ be the space of square $n \times n$ matrix with coefficients in $\mathbb{K}$. For any compact

interval I of $\mathbb{R}$ and for a nonnegative integer k, let $C^k(I,\mathbb{C})$ denote the space of all k-times continuously differentiable complex valued functions defined on I. Let $AC^k(I,\mathbb{C})$ denote the space of all complex valued functions φ which have $(k-1)$ continuous derivatives in I and the $(k-1)^{th}$ derivative of φ absolutely continuous in I. For a function φ, let $\varphi^{(j)}$ denote the j^{th} derivative of φ, if it exists. If φ is any function with $(n-1)$-derivatives, the vector $k[\varphi] = column(\varphi, \varphi', ..., \varphi^{(n-1)})$ is called the Wronskian of φ. For any $n \times m$ matrix A, let A^* denote the adjoint of A. For a square matrix A, A^{-1} denotes the inverse of A, if it exists. For any two nonempty sets(topological spaces) X_1 and X_2, let $X_1 \times X_2$ denote the cartesian product (space equipped with product topology) of X_1 and X_2, taken in that order.

Let $I_1 = [a,\ b]$ and $I_2 = [b,\ c]$, where $-\infty < a < b < c < +\infty$. We introduce the functions spaces $E(I_k,\mathbb{C}) = \{\varphi : I_k \to \mathbb{K}/\ \varphi \ \text{ measurable }\}$ and denote by $L^2(I_k,\ \mathbb{C})$ the Hilbert space defined by

$$L^2(I_k,\mathbb{C}) = \Big\{\varphi \in E(I_k,\ \mathbb{C})/ \int_{I_k} |\varphi(x)|^2 dx < +\infty\Big\},$$

endowed with the inner product $\langle\ .\ \rangle_{L^2(I_k,\ \mathbb{C})}$ and the norm $\|\ .\ \|_{L^2(I_k,\ \mathbb{C})}$ given by

$$\langle \varphi\ , \psi\rangle_{L^2(I_k,\ \mathbb{C})} = \int_{I_k} (\varphi\overline{\psi})(x)dx, \ \text{ for } k = 1,\ 2$$

and

$$\|\varphi\|_{L^2(I_k,\ \mathbb{C})} = \left(\int_{I_k} |\varphi(x)|^2 dx\right)^{\frac{1}{2}} \ \text{ for } k = 1,\ 2.$$

We consider the product Hilbert space $X = L^2(I_1,\ \mathbb{C}) \times L^2(I_2,\ \mathbb{C})$. Notice X is a Hilbert space with the scalar product $\langle\ .\ \rangle_X$ and the norm $\|\ .\ \|_X$ given by

$$\langle u,\ v\ \rangle_X = \langle u_1,\ v_1\rangle_{L^2(I_1,\ \mathbb{C})} + \langle u_2,\ v_2\rangle_{L^2(I_2,\ \mathbb{C})} \tag{2.1}$$

for all $u = (u_1,\ u_2),\ v = (v_1,\ v_2) \in X$ and

$$\|(u_1\ ,\ u_2)\|_X = \left(\ \|u_1\|^2_{L^2(I_1,\mathbb{C})} + \|u_2\|^2_{L^2(I_2,\mathbb{C})}\ \right)^{\frac{1}{2}},\ (u_1,u_2) \in X.$$

We consider the following spaces,

$$\mathcal{H}^n(I_1) = \Big\{\varphi \in AC^m(I_1,\ \mathbb{C}) \ \text{ suth that } \ \varphi \ \text{ and } \ \varphi^{(n)} \ \text{ are both in } \ L^2(I_k,\ \mathbb{C})\Big\}$$

and

$$\mathcal{H}^m(I_2) = \Big\{\varphi \in AC^m(I_2,\ \mathbb{C}) \ \text{ suth that } \ \varphi \ \text{ and } \ \varphi^{(m)} \ \text{ are both in } \ L^2(I_2,\ \mathbb{C})\Big\}.$$

Let $\mathcal{H}(I_1 \times I_2) = \mathcal{H}^n(I_1) \times \mathcal{H}^m(I_2)$ be the cartesian product Hilbert space with the inner product

$$\begin{aligned}\langle u,\ v\rangle_{\mathcal{H}} &= \langle (u_1,u_2)\,,(v_1,v_2)\rangle_{\mathcal{H}} \\ &= \sum_{0\le k\le n}\int_{I_1} u_1^{(k)}\overline{v_1}^{(k)}dx + \sum_{0\le k\le m}\int_{I_2} u_2^{(k)}\overline{v_2}^{(k)}dx\end{aligned}$$

and the associated norm

$$\|u\|_{\mathcal{H}} = \Big(\sum_{0\le k\le n}\int_{I_1}|u_1^{(k)}|^2dx + \sum_{0\le k\le m}\int_{I_2}|u_2^{(k)}|^2dx\Big)^{\frac{1}{2}}.$$

For $u_1 \in \mathcal{H}^n(I_1)$, and $u_2 \in \mathcal{H}^m(I_2)$, we denote

$$k[u_1](b) = Column\Big(u_1(b), u_1'(b), u_1^{(2)}(b)..., u_1^{(n-1)}(b)\Big) \tag{2.2}$$

and

$$k[u_2](b) = Column\Big(u_2(b), u_2'(b), u_2^{(2)}(b)..., u_2^{(m-1)}(b)\Big). \tag{2.3}$$

Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be non singular matrices with complex entries.

Let L_{1n} and L_{2m} be a pair of formal ordinary differential operators of order n and m defined on the intervals I_1 and I_2, respectively, of the form

$$L_{1n}u_1 = \sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1 = \sum_{0\le k\le n} p_k D^k u_1 \ \text{ on } \ I_1$$

and

$$L_{2m}u_2 = \sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2 = \sum_{0\le k\le m} q_k D^k u_2 \ \text{ on } \ I_2,$$

We make the following assumptions

$$(H_1)\begin{cases} p_n \in \mathcal{H}^n(I_1),\ p_n > 0 \ \text{ on } I_1 \\ p_k \in AC^k(I_1), \ \ k = 1,2,....n-1 \\ p_2 > 0 \ \text{ and } \ p_2' > 0 \ \text{ on } I_1 \\ p_0 \ \text{ piecewise continuous on } \ I_1. \end{cases}$$

$$(H_2)\begin{cases} q_m \in \mathcal{H}^m(I_2),\ q_m > 0 \ \text{ on } I_2 \\ q_k \in AC^k(I_2), \ \ k = 1,2,....m-1 \\ q_2 > 0 \ \text{ and } \ q_2' > 0 \ \text{ on } I_2 \\ q_0 \ \text{ piecewise continuous on } \ I_2. \end{cases}$$

and the interface condition at the point x = b, for $(u_1,u_2) \in \mathcal{H}(I_1 \times I_2)$

$$(H_3) \qquad Ak[u_1](b) = Bk[u_2](b).$$

Remark 2.1. If φ and $\psi \in C^k(I_1)$ $\big($or $\varphi,\ \psi \in C^k(I_2)\big)$, then the following formula is easily verified

$$\psi\varphi^{(k)} = (-1)^k\varphi\psi^{(k)} + \frac{d}{dt}\Big(\sum_{0\le j\le k-1}(-1)^j\psi^{(j)}\varphi^{(k-1-j)}\Big) \tag{2.4}$$

3. Mixed operator $(L_{(n,m)}, D(L_{(n,m)}))$ and its adjoint

In order to study the operator $(L_{(n,m)}, D(L_{(n,m)}))$, we introduce its Green formula. We will be able to obtain some characteristic proprieties.

According to [12] the corresponding formal Lagrange adjoint expressions of L_{1n} and L_{2m} are given as

$$L^*_{1n}u_1 = \sum_{0\le k\le n}(-1)^k\Big(\frac{d}{dx}\Big)^k p_k u_1 \ \text{ and } \ L^*_{2m}u_2 = \sum_{0\le k\le m}(-1)^k\Big(\frac{d}{dx}\Big)^k q_k u_2.$$

Consider now the operator $(L_{(n,m)},\ D(L_{(n,m)}))$ given by

$$\begin{cases} D(L_{(n,m)}) = \Big\{(u_1,\ u_2) \in \mathcal{H}(I_1\times I_2)/Ak[u_1](b) = Bk[u_2](b), \\ \qquad\qquad \beta^k_a = 0, k = 1,2,....n-1 \\ \qquad\qquad \beta^j_c = 0, j = 1,2,...,m-1 \Big\} \\ L_{(n,m)}u = \big(L_{1n}u_1, L_{2m}u_2\big) \\ = \Big(\sum\limits_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1, \sum\limits_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2\Big) \end{cases} \tag{3.1}$$

with $\beta^k_a = u_1^{(k)}(a) - \gamma^k_a u_1(a)$ for $k = 1,2,...,n-2$ and $\beta^{n-1}_a = u_1^{(n-1)}(a)$ and $\beta^j_c = u_2^{(j)}(c) - \delta^j_c u_2(c)$ for $j = 1,2,...,m-2$ and $\gamma^{m-1}_c = u_2^{(m-1)}(c)$ γ^k_a and δ^j_c are here fixed real numbers.

Theorem 3.1. *The operator $(L_{(n,m)},\ D(L_{(n,m)}))$ is a densely defined closed unbounded linear operator in X.*

Proof. According to [14], the operators $(L_{1n},\ \mathcal{H}^n(I_1))$ (resp. $(L_{2m}, \mathcal{H}^m(I_2))$) given in (3.1) is densely defined and closed on $L^2(I_1,\ \mathbb{C})$ (resp. $L^2(I_2,\ \mathbb{C})$)). Hence the result (see also [15], Theorem 3.6). □

Green's formula for $L_{(n,m)}$: Let $u = (u_1,\ u_2) \in D(L_{(n,m)})$ and $v = (v_1,\ v_2) \in \mathcal{H}$, the Green's formula takes the form

$$\begin{aligned} \big\langle L_{(n,m)}u, v\big\rangle_X &= \big\langle L_{(n,m)}(u_1,u_2),(v_1,v_2)\big\rangle_X \\ &= \sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}(c) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^n(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}(a)\\
&+\Big\{\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}\\
&- \sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}\Big\}(b)\\
&+\int_a^b\overline{(L_n^*v_1)}u_1(x)dx+\int_b^c\overline{(L_m^*v_2)}u_2(x)dx\\
&=\sum_{1\le k\le m}\Lambda_{ck}^*u_2^{(k-1-j)}(c)-\sum_{1\le k\le n}\Lambda_{ak}^*u_1^{(k-1-r)}(a)\\
&+\Big\{\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}\\
&- \sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}\Big\}(b)\\
&+\int_a^b\overline{(L_n^*v_1)}u_1(x)dx+\int_b^c\overline{(L_m^*v_2)}u_2(x)dx
\end{aligned}
$$

where

$$\Lambda_c^{*k}=\sum_{0\le j\le k-1}(-1)^j\sum_{0\le i\le j}\binom{j}{i}q_k^{(j-i)}(c)\overline{v_2}^{(i)}(c),\quad k=1,2,...,m$$

and

$$\Lambda_a^{*k}=\sum_{0\le r\le k-1}(-1)^r\sum_{0\le i\le r}\binom{r}{i}p_k^{(r-i)}(a)\overline{v_1}^{(i)}(a),\quad k=1,2,...,n.$$

Let us define two matrix $P=(p_{ij})_{1\le i,j\le n}\in M_n(\mathbb{R})$ and $Q=(q_{ij})_{1\le i,j\le n}\in M_m(\mathbb{R})$ as follows

$$P=(p_{ij})_{1\le i,j\le n}=\Big(\sum_{1\le k,r\le n-1}a_{ij}^rp_r^{(k)}(b)\Big)_{1\le i,j\le n},\quad a_{ij}^r\in\mathbb{R}$$

and

$$Q=\big(q_{ij}\big)_{1\le i,\,j\le m}=\Big(\sum_{1\le k,r\le m-1}c_{ij}^rq_r^{(k)}(b)\Big)_{1\le i,j\le m},\quad c_{ij}^r\in\mathbb{R}.$$

Now, by using the matrix P and Q, we verify that

$$
\begin{aligned}
&\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}(b)
\end{aligned}
$$

$$
\begin{aligned}
&= \big(\overline{v_2(b)}, \overline{v_2'(b)}, ..., \overline{v_2^{(m-1)}(b)}\big)\Big(\sum_{1\le k,r\le m-1} c_{ij}^r q_r^{(k)}(b)\Big)_{1\le i,j\le m} \\
&\quad \cdot \big(u_2(b), u_2'(b), ..., u_2^{(m-1)}(b)\big) \\
&- \big(\overline{v_1(b)}, \overline{v_1'(b)}, ..., \overline{v_1^{(n-1)}(b)}\big)\Big(\sum_{1\le k,r\le n-1} a_{ij}^r p_r^{(k)}(b)\Big)_{1\le i,j\le n} \\
&\quad \cdot \big(u_1(b), u_1'(b), ..., u_1^{(n-1)}(b)\big)\big)
\end{aligned}
$$

From (2.2) and (2.3) and the fact that $(u_1, u_2) \in D(L_{(n,m)})$, it follows that,

$$
\begin{aligned}
\langle L_{(n,m)}u,\ v\rangle_X = &\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) \\
&- \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) + \big(\overline{k[v_2]}Qk[u_2] - \overline{k[v_1]}Pk[u_1]\big)(b) \\
&+ \int_a^b \overline{(L_{1n}^* v_1)} u_1(x)dx + \int_b^c \overline{(L_{2m}^* v_2)} u_2(x)dx.
\end{aligned}
$$

Introducing the new matrix

$$M_1 = (A^{-1})^* P^* \text{ and } M_2 = (B^{-1})^* Q^*$$

and using the interface condition (H_3) ,we then transform the Green's formula to

$$
\begin{aligned}
&\langle L_{(n,m)}u,\ v\rangle_{\mathcal{H}} = \sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) \\
&+\big((M_2 k[v_2])^* B k[u_2] - (M_1 k[v_1])^* A k[u_1]\big)(b) + \langle u_1,\ L_{1n}^* v_1\rangle + \langle u_2,\ L_{2m}^* v_2\rangle.
\end{aligned}
$$

The following proposition characterize the adjoint operator of $L_{(n,m)}$.

Proposition 3.1. *Let $(L_{(n,m)},\ D(L_{(n,m)}))$ be the operator given as in* (3.1). *Then its adjoint $(L_{(n,m)}^*,\ D(L_{(n,m)}^*))$ is a densely defined closed unbounded operator defined by*

$$
(3.2)\begin{cases}
D(L_{(n,m)}^*) = \Big\{(v_1,\ v_2) \in \mathcal{H}(I_1 \times I_2)/\ M_1 k[v_1](b) = M_2 k[v_2](b), \\
\qquad\qquad \beta_a^{*k} = 0, k = 1, 2, ...n-1 \\
\qquad\qquad \beta_c^* j = 0, j = 1, 2, ..., m-1 \Big\} \\
L_{(n,m)}^*(v_1,\ v_2) = (L_{1n}^* v_1,\ L_{2m}^* v_2).
\end{cases}
$$

where $\beta_a^{*k} = v_1^{(k)}(a) - \omega_a^k v_1(a),\ k = 1, 2, ..., n-1$ and $\beta_c^{*j} = v_2^{(j)}(c) - \omega_c^j v_2(c),\ j = 1, 2, ..., m-1$ ω_a^k and ω_c^j are here fixed real numbers.

Proof. Let T be the linear operator defined by

$$\begin{cases} D(T) = \Big\{(v_1,\ v_2) \in \mathcal{H}/\ M_1k[v_1](b) = M_2k[v_2](b), \\ \qquad\qquad \beta_a^{*k} = 0, k = 1, 2, ...n-1 \\ \qquad\qquad \beta_c^{*j} = 0, j = 1, 2, ..., m-1 \Big\} \\ T(v_1,\ v_2) = (L_{1n}^* v_1,\ L_{2m}^* v_2) \end{cases}$$

We have to show that $L_{(n,m)}^* = T$. From the Green's formula, it follows that $D(T) \subset D(L_{(n,m)}^*)$. Now we need to prove the following equality

$$\langle L_{(n,m)}(u_1,\ u_2),\ (v_1,\ v_2)\rangle = \langle (u_1,\ u_2),\ L_{(n,m)}^*(v_1,\ v_2)\rangle,$$

for all $(u_1,\ u_2) \in D(L_{(n,m)})$ and $(v_1,\ v_2) \in D(L_{(n,m)}^*)$, which follows if we prove that

$$\begin{aligned} &\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a)) \\ &+ \big((M_2k[v_2])^* BK[u_2] - (M_1k[v_1])^* Ak[u_1]\big)(b)) = 0. \end{aligned}$$

Choose $(u_1,\ u_2) \in D(L_{(n,m)})$ verifying $u_1(a) = u_2(c) = 0$; then we obtain

$$\big((M_2k[v_2])^* BK[u_2] - (M_1k[v_1])^* Ak[u_1]\big)(b)) = 0,$$

Green's formula implies that

$$\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) = 0.$$

An appropriate choice of functions $(u_1,\ u_2) \in D(L_{(n,m)})$, we get $\beta_a^{*k} = \beta_c^{*j} = 0, 1 \le k \le n-1, 1 \le j \le m-1$ and $M_1k[v_1](b) = M_2k[v_2](b)$. This implies that $D(L_{(n,m)}^*) \subset D(T)$ and therefore $L_{(n,m)}^* = T$ as required. □

Lemma 3.1. *For $v = (v_1, v_2) \in D(L_{(n,m)}^*)$ we have*

$$\begin{aligned} \langle L_{(n,m)}^* v,\ v\rangle_X = &\sum_{0\le k\le n} (-1)^k \int_a^b p_k v_1 (\overline{v_1})^{(k)} dx \\ &+ \sum_{1\le k\le n} \sum_{0\le j\le k-1} (-1)^j (\overline{v_1})^{(j)} (p_k v_1)^{(k-1-j)} \Big]_a^b \\ &+ \sum_{0\le k\le m} (-1)^k \int_b^c q_k v_2 (\overline{v_2})^{(k)} dx \\ &+ \sum_{1\le k\le m} \sum_{0\le j\le k-1} (-1)^j (\overline{v_2})^{(j)} (q_k v_2)^{(k-1-j)} \Big]_b^c. \end{aligned}$$

Proof. From (2.1) we have

$$\begin{aligned}\langle L^*_{(n,m)}v,\ v\rangle_X &= \langle L^*_n v_1,\ v_1\rangle + \langle L^*_m v_2,\ v_2\rangle \\ &= \underbrace{\sum_{0\le k\le n}(-1)^k\int_a^b (p_k v_1)^{(k)}\overline{v_1}(x)dx}_{I} \\ &+ \underbrace{\sum_{0\le k\le m}(-1)^k\int_b^c (q_k v_2)^{(k)}\overline{v_2}(x)dx}_{J}\end{aligned}$$

Let us examine the term I,using the identity (2.4) we get for $k\ge 1$

$$\begin{aligned}\overline{v_1}(p_k v_1)^{(k)} &= (-1)^k p_k v_1(\overline{v_1})^{(k)} \\ &+ \frac{d}{dx}\Big(\sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_k v_1)^{(k-1-j)}\Big).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\int_a^b (p_k v_1)^{(k)}\overline{v_1}(x)dx &= (-1)^k\int_a^b p_k v_1(\overline{v_1})^{(k)}(x)dx \\ &+ \sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_k v_1)^{(k-1-j)}\Big]_a^b.\end{aligned}$$

A similar argument applies to the term J gives

$$\begin{aligned}\int_b^c (q_k v_2)^{(k)}\overline{v_2}(x)dx &= (-1)^k\int_b^c q_k v_2(\overline{v_2})^{(k)}(x)dx \\ &+ \sum_{0\le j\le k-1}(-1)^j(\overline{v_2})^{(j)}(q_k v_2)^{(k-1-j)}\Big]_b^c.\end{aligned}$$

□

In the rest of the work, we assume that $n=m\ge 3$.

4. m- Dissipativity of the operator $(L_{(n,n)}, D(L_{(n,n)}))$.

In this section we prove the m-dissipativity of the operator $(L_{(n,n}, D(L_{(n,n)}))$.

Definition 4.1 ([9]). *A linear closed densely defined operator $(T,\ D(T))$ on a complex Hilbert space X is called dissipative if*

$$\forall\ u\in D(T)\subset X,\ \ Re\langle Tu,\ u\rangle\le 0.$$

Definition 4.2 ([9]). *A dissipative operator $(T,\ D(T))$ on a Hilbert space X is called m-dissipative if there exists $\lambda>0$ such that $\mathcal{R}(\lambda I-T)=X$.*

We establish the following technical lemma which is the key to proving our main results.

Lemma 4.1. *If $u \in C^k([a,\ b],\ \mathbb{R})$ and $v \in C^k([a,\ b],\ \mathbb{C})$ then the following properties hold.*

1. *If $k = 1$ then*

$$2Re \int_a^b (u\overline{v}v')(x)dx = \{u(b)|v|^2(b) - u(a)|v|^2(a)\} - \int_a^b u'(x)|v|^2(x)dx$$

2. *If $k \geq 2$, then*

$$\begin{aligned} 2Re \int_a^b (u\overline{v}v^{(k)})(x)dx &= (-1)^k \int_a^b (u^{(k)}|v|^2)(x)dx \\ &+ \sum_{0\leq j\leq k-1} \left[u^{(j)}(|v|^2)^{(k-1-j)}(b) - u^{(j)}(|v|^2)^{(k-1-j)}(a)\right] \\ &- \sum_{1\leq j\leq k-1} \binom{k}{j} \int_a^b uv^{(j)}\overline{v}^{(k-j)}(x)dx. \end{aligned}$$

Proof. 1. Let $Z = \int_a^b (u\overline{v}v')(x)dx$, we have

$$2Re(Z) = \int_a^b \big(u(\overline{v}v' + \overline{v'}v)\big)(x)dx = \int_a^b \big(u(|v|^2)'\big)(x)dx$$

hence 1. follows.

2. It is will know that

$$2Re \int_a^b (u\overline{v}v^{(k)})(x)dx = \int_a^b u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big)(x)dx.$$

On the other hand the classical Leibnitz formula gives

$$\big(v\overline{v}\big)^{(k)} = \sum_{0\leq j\leq k} \binom{k}{j} v^{(j)}(\overline{v})^{(k-j)} = v(\overline{v})^{(k)} + \sum_{1\leq j\leq k-1} \binom{k}{j} v^{(j)}(\overline{v})^{(k-j)} + v^{(k)}\overline{v}.$$

Hence

$$\begin{aligned} u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big) &= u(|v|^2)^{(k)} - \sum_{1\leq j\leq k-1} \binom{k}{j} uv^{(j)}(\overline{v})^{(k-j)} \\ &= (-1)^k u^{(k)}|v|^2 + \frac{d}{dt} \sum_{0\leq j\leq k-1} (-1)^j u^{(j)}(|v|^2)^{(k-1-j)} - \sum_{1\leq j\leq k-1} \binom{k}{j} uv^{(j)}(\overline{v})^{(k-j)}. \end{aligned}$$

It follows that

$$\int_a^b u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big)(x)dx = (-1)^k \int_a^b u^{(k)}|v|^2(x)dx$$
$$+ \sum_{0\le j\le k-1} (-1)^j \big[u^{(j)}(|v|^2)^{(k-1-j)}(x)\big]_a^b - \sum_{1\le j\le k-1} \binom{k}{j} \int_a^b uv^{(j)}(\overline{v})^{(k-j)}(x)dx.$$

□

Proposition 4.1. *Assume that the following conditions are satisfies*

$$(A^{-1})^* P A^{-1} = (B^{-1})^* Q B^{-1}, \tag{4.1}$$
$$\sum_{1\le j\le k-1} \binom{k}{j} \Big(\int_a^b p_k u_1^{(j)} \overline{u_1}^{(k-j)}(x)dx$$
$$+ \int_b^c q_k u_2^{(j)} \overline{u_2}^{(k-j)}(x)dx \Big) = 0,\ k = 3, ..., n, \tag{4.2}$$

for all $(u_1, u_2) \in D(L_{(n,n)})$. *Then there exists a positive constant* $\rho' > 0$ *such that the operators* $(L - \rho' I)$ *is dissipative.*

Proof. We prove that $(L_{n,n} - \rho' I)$ is dissipative for some $\rho' > 0$. The proof requires two steps.

Step 1. From (2.1) we have

$$\begin{aligned} \langle L_{(n,n)}u,\ u\rangle_X &= \langle L_{1n}u_1,\ u_1\rangle_{L^2(I_1)} + \langle L_{2n}u_2,\ u_2\rangle_{L^2(I_2)} \\ &= \sum_{0\le k\le n} \int_a^b p_k \overline{u_1} u_1^{(k)}(x)dx + \sum_{0\le k\le n} \int_b^c q_k \overline{u_2} u_2^{(k)}(x)dx. \end{aligned}$$

Therefore we have

$$Re\langle L_{(n,n)}u,\ u\rangle_X = \int_a^b p_0(x)|u_1|^2(x)dx + Re\Big(\int_a^b p_1 \overline{u_1} u_1'(x)dx \Big)$$
$$+ \sum_{2\le k\le n} Re\Big(\int_a^b p_k \overline{u_1} u_1^{(k)}(x)dx \Big) + \int_b^c q_0(x)|u_2|^2(x)dx$$
$$+ Re\Big(\int_b^c q_1 \overline{u_2} u_2'(x)dx \Big) + \sum_{2\le k\le n} Re\Big(\int_b^c q_k \overline{u_2} u_2^{(k)}(x)dx \Big).$$

Form Lemma 4.1 we deduce that

$$Re\langle L_{(n,n)}u,\ u\rangle_X = \frac{1}{2}\Big\{ \int_a^b 2p_0|u_1|^2(x)dx + \sum_{1\le k\le n}(-1)^k \int_a^b p_k^{(k)}|u_1|^2(x)dx$$
$$+ \sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)}(|u_1|^2)^{(k-1-j)}\big]_a^b - \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k u_1^{(j)}(\overline{u_1})^{(k-j)}dx$$
$$+ \int_b^c 2q_0|u_2|^2(x)dx + \sum_{1\le k\le n}(-1)^k\int_b^c q_k^{(k)}|u_2|^2(x)dx$$
$$+ \sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)}(|u_2|^2)^{(k-1-j)}\big]_b^c - \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k u_2^{(j)}(\overline{u_2})^{(k-j)}dx\Big\}$$
$$= \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,$$

where

$$\mathcal{R}_1 = \frac{1}{2}\Big\{ \sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)}(|u_2|^2)^{(k-1-j)}(c) - \sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)}(|u_1|^2)^{(k-1-j)}(a)\Big\},$$

$$\mathcal{R}_2 = \frac{1}{2}\Big\{ \sum_{1\le k\le n}\sum_{1\le j\le k} p_k^{(j-1)}(|u_1|^2)^{(k-j)}(b) - \sum_{1\le k\le n}\sum_{1\le j\le k} q_k^{(j-1)}(|u_2|^2)^{(k-j)}(b)\Big\}$$

and

$$\mathcal{R}_3 = \frac{1}{2}\Big\{ \int_a^b 2p_0|u_1|^2(x)dx + \sum_{1\le k\le n}(-1)^k\int_a^b p_k^{(k)}|u_1|^2(x)dx$$
$$+ \int_b^c 2q_0|u_2|^2(x)dx + \sum_{1\le k\le m}(-1)^k\int_b^c q_k^{(k)}|u_2|^2(x)dx$$
$$- \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k u_1^{(j)}(\overline{u_1})^{(k-j)}dx$$
$$- \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k u_2^{(j)}(\overline{u_2})^{(k-j)}dx\Big\}.$$

Step 2. Now let us consider each of the above terms separately.
As $(u_1, u_2) \in D(L_{(n,n)})$ we have that

$$\mathcal{R}_1 = \frac{1}{2}\Big\{ \sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)} \sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_c^r\gamma_c^{k-j-r}|u_2(c)|^2$$
$$- \sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)} \sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_a^r\gamma_a^{k-1-j-r}|u_1(a)|^2\Big\}.$$

Let

$$A_k(x) = -\frac{1}{2}\Big(\sum_{0\le j\le k-1}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_a^r\gamma_a^{k-1-j-r}p_k^{(j)}(x)\Big)\frac{b-x}{b-a}$$

for $k = 1, 2...., n$, and $x \in I_1$ and

$$B_k(x) = \frac{1}{2}\Big(\sum_{0\le j\le k-1} \sum_{0\le r\le k-1-j} \binom{k-1-j}{r} \gamma_c^r \gamma_c^{k-1-j-r} q_k^{(j)}(x) \Big) \frac{c-x}{c-b}$$

for $k = 1, 2...., m$, and $x \in I_2$. It follows that

$$\mathcal{R}_1 = \sum_{1\le k\le n} \int_b^c \big(B_k(x)|u_2(x)|^2\big)' dx + \sum_{1\le k\le n} \int_a^b \big(A_k(x)|u_1(x)|^2\big)' dx.$$

Note that for $k = 1, 2, ..., n$

$$\begin{aligned} \big(B_k(x)|u_2(x)|^2\big)' &= B_k'(x)|u_2(x)|^2 + 2B_k(x)Re(\overline{u_2(x)}u_2'(x)) \\ &\le |B_k'(x)||u_2(x)|^2 + 2|B_k(x)||u_2(x)||u_2'(x)|. \end{aligned}$$

Now, not further that for all $\epsilon > 0$, $\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2 \ge 2|u_2(x)||u_2'(x)|$ which implies that $\big(B_k(x)|u_2(x)|^2\big)' \le |B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2(x)|^2 + \frac{1}{\epsilon^2}|u_2'(x)|^2\big)$ Similarly for $k = 1.2, ..., n$ $\big(A_k(x)|u_1(x)|^2\big)' \le |A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\frac{1}{\epsilon^2}|u_1(x)|^2 + \epsilon^2|u_1'(x)|^2.\big)$

$$\begin{aligned} \mathcal{R}_1 \le & \sum_{1\le k\le n} \int_b^c \Big[|B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2\big) \Big] dx \\ & + \sum_{1\le k\le n} \int_a^b \Big[|A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\epsilon^2|u_1'(x)|^2 + \frac{1}{\epsilon^2}|u_1(x)|^2\big) \Big] dx. \end{aligned}$$

We then apply our hypotheses (4.1) and (4.2) to check that

$$\begin{aligned} & Re\langle Lu,\ u\rangle_X \le \\ & \frac{1}{2}\Big\{ \sum_{1\le k\le n} \int_b^c \Big[|B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2\big) \Big] dx \\ & + \sum_{1\le k\le n} \int_a^b \Big[|A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\epsilon^2|u_1'(x)|^2 + \frac{1}{\epsilon^2}|u_1(x)|^2\big) \Big] dx \Big\} \\ & + \frac{1}{2}\Big\{ \overline{k[u_1]}Pk[u_1](b) - \overline{k[u_2]}Qk[u_2](b) \Big\} \\ & + \frac{1}{2}\Big\{ \sum_{1\le k\le n} (-1)^k \int_a^b p_k^{(k)}|u_1|^2(x)dx + \sum_{1\le k\le m} (-1)^k \int_b^c q_k^{(k)}|u_2|^2(x)dx \\ & + 2\int_a^b p_0|u_1|^2(x)dx - 2\int_a^b p_2|u_1'|^2 dx + 2\int_b^c q_0|u_2|^2(x)dx - 2\int_b^c q_2|u_2'|^2 dx \Big\}. \end{aligned}$$

The above inequality implies

$$
\begin{aligned}
&Re\langle L_{(n,n)}u,u\rangle_X\\
&\le \frac{1}{2}\Big\{\int_a^b\Big(\sum_{1\le k\le n}\Big(|A_k'(x)|+\frac{|A_k(x)|}{\epsilon^2}+(-1)^k p_k^{(k)}(x)\Big)+2p_0(x)\Big)|u_1|^2(x)dx\\
&-\int_a^b\Big(2p_2(x)-\epsilon^2\sum_{1\le k\le n}|A_k(x)|^2\Big)|u_1'|^2(x)dx\\
&+\int_b^c\Big(\sum_{1\le k\le n}\Big(|B_k'(x)|+\frac{|B_k(x)|}{\epsilon^2}+(-1)^k q_k^{(k)}(x)\Big)+2q_0(x)\Big)|u_2|^2(x)dx\\
&-\int_b^c\Big(2q_2(x)-\epsilon^2\sum_{1\le k\le m}|B_k(x)|^2\Big)|u_2'|^2(x)dx\Big\}\\
&+\frac{1}{2}\Big\{(Ak[u_1])^*(A^{-1})^*PA^{-1}Ak[u_1](b)-(Bk[u_2])^*(B^{-1})^*QB^{-1}Bk[u_2](b)\Big\}.
\end{aligned}
$$

For sufficiently small $\epsilon>0$ such that

$$\Big(2p_2(x)-\epsilon^2\sum_{1\le k\le n}|A_k(x)|^2\Big)>0$$

and

$$\Big(2q_2(x)-\epsilon^2\sum_{1\le k\le n}|B_k(x)|^2\Big)>0$$

we obtain that $Re\langle L_{(n,n)}u,u\rangle_X\le n\max(\alpha,\beta)\|(u_1,u_2)\|_X$ where

$$\alpha=\sup_{a\le x\le b}\{|A_k'(x)|+\frac{|A_k(x)|}{\epsilon^2}+|p_k^{(k)}(x)|+\frac{2}{n}|p_0(x)|,\ k=1,2,...,n\}$$

and

$$\beta=\sup_{b\le x\le c}\{|B_k'(x)|+\frac{|B_k(x)|}{\epsilon^2}+|q_k^{(k)}(x)|+\frac{2}{n}|q_0(x)|\ \ k=1,2,...,n\}.$$

Hence, $\langle\big(L-n\max(\alpha,\beta)\big)u,u\rangle_X\le 0$. This prove that $(L_{(n,n)}-\rho' I, D(L_{(n,n)}))$ is dissipative. □

Remark 4.1. As above, we can define two matrix $\widetilde{P}$ and $\widetilde{Q}$ in $M_n(\mathbb{R})$ satisfying the following equality

$$
\begin{aligned}
&\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(b)+2(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)\Big)\\
&-\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}2(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)+p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(b)\Big)\\
&=\Big\{\overline{k[v_2](b)}\widetilde{Q}k[v_2](b)-\overline{k[v_1](b)}\widetilde{P}k[v_1](b)\Big\}
\end{aligned}
$$

Proposition 4.2. *Assume that the following conditions are satisfies*

$$(A^{-1})^*\widetilde{P}A^{-1} = (B^{-1})^*\widetilde{Q}B^{-1}, \tag{4.3}$$

$$\sum_{1\le j\le k-1}\binom{k}{j}\left(\int_a^b p_k^{(j)}v_1^{(j)}\overline{v_1}^{(k-j)}(x)dx + \int_b^c q_k^{(j)}v_2^{(j)}\overline{v_2}^{(k-j)}(x)dx\right) = 0, \tag{4.4}$$
$$k = 3, ..., n,$$

for all $(v_1, v_2) \in D(L^*_{(n,n)})$. *Then there exists a positive constant* $\rho'' > 0$ *such that the operators* $(L_{(n,n)} - \rho''I)^*$ *is dissipative.*

Proof. Now to prove that $(L^*_{(n,n)} - \rho''I, D(L^*))$ is dissipative for some constant $\rho'' > 0$,we proceed as in the proof of the previous proposition. From Lemma 3.1 it follows that

$$\begin{aligned}\langle L^*_{(n,n)}v,\ v\rangle_X &= \int_a^b p_0|v_1|^2dx - \int_a^b p_1v_1\overline{v_1}'dx + \sum_{2\le k\le n}(-1)^k\int_a^b p_kv_1(\overline{v_1})^{(k)}dx\\
&+ \sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_kv_1)^{(k-1-j)}\Big]_a^b + \int_b^c q_0|v_2|^2dx - \int_b^c q_1v_2\overline{v_2}'dx\\
&+ \sum_{2\le k\le n}(-1)^k\int_b^c q_kv_2(\overline{v_2})^{(k)}dx + \sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(\overline{v_2})^{(j)}(q_kv_2)^{(k-1-j)}\Big]_b^c\end{aligned}$$

and so that

$$\begin{aligned}&Re\langle L^*_{(n,n)}v,\ v\rangle_X =\\
&\int_a^b p_0|v_1|^2dx - Re\left(\int_a^b p_1v_1\overline{v_1}'dx\right) + \sum_{2\le k\le n}(-1)^kRe\left(\int_a^b p_kv_1(\overline{v_1})^{(k)}dx\right)\\
&+ Re\left(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_kv_1)^{(k-1-j)}\Big]_a^b\right)\\
&+ \int_b^c q_0|v_2|^2dx - Re\left(\int_b^c q_1v_2\overline{v_2}'dx\right)\\
&+ \sum_{2\le k\le n}(-1)^kRe\left(\int_b^c q_kv_2(\overline{v_2})^{(k)}dx\right)\\
&+ Re\left(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j(\overline{v_2})^{(j)}(q_kv_2)^{(k-1-j)}\Big]_b^c\right).\end{aligned}$$

Taking into account Lemma 4.1 we get the following equality

$$\begin{aligned}&2Re\langle L^*_{(n,n)}v,\ v\rangle_X =\\
&\int_a^b 2p_0|v_1|^2dx - \left(p_1(b)|v_1|^2(b) - p_1(a)|v_1|^2(a) - \int_a^b p_1'|v_1|^2(x)dx\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{2\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx\\
&+\sum_{2\le k\le n}(-1)^k\sum_{0\le j\le k-1}\Big(p_k^{(j)}(|v_1|^2)^{(k-1-j)}(b)-p_k^{(j)}(|v_1|^2)^{(k-1-j)}(a)\Big)\\
&-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k^{(j)}v_1^{(j)}\overline{v_1}^{(k-j)}(x)dx\\
&+\int_b^c 2q_0|v_2|^2dx-\Big(q_1(c)|v_2|^2(c)-q_1(b)|v_2|^2(b)-\int_b^c q_1'|v_2|^2(x)dx\Big)\\
&+\sum_{2\le k\le m}\int_b^c q_k^{(k)}|v_2|^2(x)dx\\
&+\sum_{2\le k\le m}(-1)^k\sum_{0\le j\le k-1}\Big(q_k^{(j)}(|v_2|^2)^{(k-1-j)}(c)-q_k^{(j)}(|v_2|^2)^{(k-1-j)}(b)\Big)\\
&-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k^{(j)}v_2^{(j)}\overline{v_2}^{(k-j)}(x)dx\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big)\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big).
\end{aligned}
$$

From (4.3) we deduce that

$$
\begin{aligned}
&2Re\langle L^*_{(n,n)}v,\ v\rangle_X\\
&=\Big\{\int_a^b 2p_0|v_1|^2(x)dx+\sum_{1\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx+\int_b^c 2q_0|v_2|^2(x)dx\\
&+\sum_{1\le k\le n}\int_b^c q_k^{(k)}|v_2|^2(x)dx\Big\}\\
&+\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(c)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(a)\Big)\\
&+\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(b)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(b)\Big)
\end{aligned}
$$

$$+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)$$
$$-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big)$$
$$+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)$$
$$-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big)$$
$$-2\int_a^b p_2'|v_1'|^2(x)dx-2\int_b^c q_2'|v_2'|^2(x)dx=\mathcal{R}_1'+\mathcal{R}_2'+\mathcal{R}_3'$$

where

$$\mathcal{R}_1'=\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(c)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(a)\Big)$$
$$+2Re\Big(\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)$$
$$-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big),$$

$$\mathcal{R}_2'=\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(b)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(b)\Big)$$
$$+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)$$
$$-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big)$$

and

$$\mathcal{R}_3'=\Big\{\int_a^b 2p_0|v_1|^2(x)dx+\sum_{1\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx+\int_b^c 2q_0|v_2|^2(x)dx$$
$$+\sum_{1\le k\le n}\int_b^c q_k^{(k)}|v_2|^2(x)dx\Big\}-2\int_a^b p_2'|v_1'|^2(x)dx-2\int_b^c q_2'|v_2'|^2(x)dx.$$

Now let us consider each of the term in the above equation separately: From easy calculations, it follows that for $(v_1,v_2)\in D(L_{n,n}^*)$

$$\mathcal{R}_1'=\sum_{1\le k\le n}\sum_{0\le j\le k-1}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}$$

$$\cdot \left(\omega_c^r \omega_c^{k-1-j-r} q_k^{(j)}(c) + 2(-1)^j \omega_c^j \omega_c^{k-1-j-r} q_k^{(r)}(c)\right) |v_2(c)|^2$$
$$- \sum_{1\le k\le n} \sum_{0\le j\le k-1} \sum_{0\le r\le k-1-j} \binom{k-1-j}{r}$$
$$\cdot \left(\omega_a^r \omega_a^{k-1-j-r} p_k^{(j)}(a) + 2(-1)^j \omega_a^j \omega_a^{k-1-j-r} p_k^{(r)}(a)\right) |v_1(a)|^2.$$

Put for $k = 1, 2, ..., n$

$$C_k(x) = \sum_{0\le j\le k-1} \sum_{0\le r\le k-1-j} \binom{k-1-j}{r}$$
$$\cdot \left(\omega_a^r \omega_a^{k-1-j-r} p_k^{(j)}(a) + 2(-1)^j \omega_a^j \omega_a^{k-1-j-r} p_k^{(r)}(a)\right) \frac{b-x}{b-a},\ x \in I_1,$$

and

$$D_k(x) = \sum_{0\le j\le k-1} \sum_{0\le r\le k-1-j} \binom{k-1-j}{r}$$
$$\cdot \left(\omega_c^r \omega_c^{k-1-j-r} q_k^{(j)}(c) + 2(-1)^j \omega_c^j \omega_c^{k-1-j-r} q_k^{(r)}(c)\right) \frac{c-x}{c-b},\ x \in I_2.$$

From this we conclude

$$\mathcal{R}_1' = \sum_{1\le k\le n} \left[\int_a^b \left(C_k(x)|v_1(x)|^2\right)' dx + \int_b^c \left(D_k(x)|v^2(x)|^2\right)' dx \right].$$

In same way as in proof of Proposition 4.1,it is possible to prove that

$$\mathcal{R}_1' \le \sum_{1\le k\le n} \left\{ \int_b^c \left[|D_k'(x)||v_2(x)|^2 + |D_k(x)|\left(\epsilon^2 |v_2'(x)|^2 + \frac{1}{\epsilon^2}|v_2(x)|^2\right)\right] dx \right.$$
$$\left. + \int_a^b \left[|C_k'(x)||v_1(x)|^2 + |C_k(x)|\left(\epsilon^2 |v_1'(x)|^2 + \frac{1}{\epsilon^2}|v_1(x)|^2\right)\right] dx \right\}.$$

Concerning $\mathcal{R}_2'$ we have the following identity

$$\mathcal{R}_2' = Re\left\{ \overline{k[v_2](b)}\widetilde{Q}k[v_2](b) - \overline{k[v_1](b)}\widetilde{P}k[v_1](b) \right\}.$$

According to the interface condition and equation (3.4) it is possible to prove that

$$\mathcal{R}_2' = Re\left\{ \overline{k[v_2](b)}\widetilde{Q}k[v_2](b) - \overline{k[v_1](b)}\widetilde{P}k[v_1](b) \right\}$$
$$= Re\left\{ \left(Bk[v_2](b)\right)^* \left(B^{-1}\right)^* \widetilde{Q} B^{-1} Bk[v_2](b) \right.$$
$$\left. - \left(Ak[v_1](b)\right)^* \left(A^{-1}\right)^* \widetilde{P} A^{-1} Ak[v_1](b) \right\} = 0.$$

From the above calculations, we get the following estimation of $Re\,\langle L^*v \mid v\rangle_X$ as

$$\begin{aligned}Re\langle L^*v, v\rangle_X \leq \frac{1}{2}\Big\{ &\int_a^b \Big(\sum_{1\leq k\leq n} \Big(|C_k'(x)| + \frac{|C_k(x)|}{\epsilon^2} + p_k^{(k)}(x)\Big) + 2p_0(x)\Big)|v_1|^2(x)dx \\ &- \int_a^b \Big(2p_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |C_k(x)|^2\Big)|v_1'|^2(x)dx \\ &+ \int_b^c \Big(\sum_{1\leq k\leq n} \Big(|D_k'(x)| + \frac{|D_k(x)|}{\epsilon^2} + q_k^{(k)}(x)\Big) + 2q_0(x)\Big)|v_2|^2(x)dx \\ &- \int_b^c \Big(2q_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |D_k(x)|^2\Big)|v_2'|^2(x)dx\Big\}.\end{aligned}$$

For sufficiently small $\epsilon > 0$ such that

$$\Big(2p_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |C_k(x)|^2\Big) > 0$$

and

$$\Big(2q_2'(x) - \epsilon^2 \sum_{1\leq k\leq m} |D_k(x)|^2\Big) > 0$$

we obtain that $Re\langle L^*v, v\rangle_X \leq \max(\alpha', \beta')\|v\|_X$ where

$$\alpha' = \sup_{a\leq x\leq b} \{|C_k'(x)| + \frac{|C_k(x)|}{\epsilon^2} + |p_k^{(k)}(x)| + \frac{2}{n}|p_0(x)|,\ k = 1, 2, ..., n\}$$

and

$$\beta' = \sup_{b\leq x\leq c} \{|D_k'(x)| + \frac{|D_k(x)|}{\epsilon^2} + |q_k^{(k)}(x)| + \frac{2}{n}|q_0(x)|\ \ k = 1, 2, ..., m\}.$$

Hence,

$$Re\left\langle \big(L^* - \underbrace{n\max(\alpha', \beta')}_{\rho''}\big)v, v\right\rangle_X \leq 0.$$

□

Theorem 4.1. *Assume that the conditions* (4.1) − (4.2) − (4.3) − (4.4) *are satisfies. Then there exists a positive constant* $\gamma > 0$ *such that the operators* $(L - \gamma I)$ *is m-dissipative.*

Proof. From Proposition 4.1 it follows that there exists a constant $\rho' > 0$ such that $Re\langle L_{(n,n)}u,\ u\rangle \leq \rho'\|u\|_X^2$ For all $u \in D(L_{n,n})$. Similarly, by Proposition 4.2 we have for all $v \in D(L_{n,n}^*)$ $Re\langle L_{(n,n)}^*v,\ v\rangle \leq \rho''\|v\|_X^2$ for some $\gamma_0' > 0$. If we put $\gamma = \max(\rho', \rho'')$, we can conclude that $\big(L_{(n,n)} - \gamma I\big)$ and $\big(L_{(n,n)} - \gamma I\big)^*$ are dissipative. □

The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.1. *Under the assumptions of Theorem 4.1, the operator* $(L_{(n,n)}, D(L_{n,n}))$ *generates a contraction semigroup on* X.

5. Analyticity of the semigroup generated by $(L, D(L))$

In this section we shall apply the general results of Section 4. In order to establish the analyticity of the semigroup generated by $(L_{(n,n)}, D(L_{(n,n)})$, we need the following theorem.

Theorem 5.1 ([6])**.** *Let* A *be a densely defined operator in a complex Hilbert space* X *such that; for* $u \in D(A)$, $Re\langle Au\,, u\rangle \pm \delta Im\langle Au\,, u\rangle \leq 0$. *Then* A *generates an analytic semigroup of contractions.*

In analogy with [3, Theorem 2.4] and [10, Theorem 4] we prove the following result.

Theorem 5.2. *Assume that* (4.1), (4.2), (4.3), (4.4) *are hold and suppose in addition that there exist real numbers* $\mu > 0$ *and* $\nu > 0$ *such that*

$$\sum_{2\leq k\leq n}\left(\int_a^b |p_k(x)||u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)||u_2^{(k)}(x)|^2dx\right)$$
$$\leq \mu\int_a^b |u_1'(x)|^2dx + \nu\int_b^c |u_2'(x)|^2dx. \tag{5.1}$$

Then the operator $L_{(n,n)}$ *generates an analytic semigroup.*

Proof.

$$\langle L_{(n,n)}u,\ u\rangle_X = \sum_{0\leq k\leq n}\left(\int_a^b p_k\overline{u_1}u_1^{(k)}(x)dx + \int_b^c q_k\overline{u_2}u_2^{(k)}(x)dx\right)$$
$$= \int_a^b p_0|u_1|^2(x)dx + \int_b^c q_0|u_2|^2(x)dx + \int_a^b p_1\overline{u_1}u_1'(x)dx + \int_b^c q_1\overline{u_2}u_2'(x)dx$$
$$+ \sum_{2\leq k\leq n}\left(\int_a^b p_k\overline{u_1}u_1^{(k)}(x)dx + \int_b^c q_k\overline{u_2}u_2^{(k)}(x)dx\right)$$

We deduce that

$$Im\langle L_{(n,n)}u,\ u\rangle_X = \int_a^b p_1Im\Big(\overline{u_1}u_1'(x)\Big)dx + \int_b^c q_1Im\Big(\overline{u_2}u_2'(x)\Big)dx$$
$$+ \sum_{2\leq k\leq n}\left(\int_a^b p_kIm\Big(\overline{u_1}u_1^{(k)}(x)\Big)dx + \int_b^c q_kIm\Big(\overline{u_2}u_2^{(k)}(x)\Big)dx.\right.$$

Hence we can conclude that

$$
\begin{aligned}
&|Im\left\langle L_{(n,n)}u,\ u\right\rangle_X| \le \int_a^b |p_1||\overline{u_1}||u_1'(x)|dx + \int_b^c |q_1||\overline{u_2}||u_2'(x)|dx\\
&+ \sum_{2\le k\le n}\left(\int_a^b |p_k||\overline{u_1}||u_1^{(k)}|(x)dx + \int_b^c |q_k||\overline{u_2}||u_2^{(k)}|(x)dx\right.\\
&\le \int_a^b |p_1|\big(\epsilon^{-2}|u_1(x)|^2 + \epsilon^2|u_1'(x)|^2\big)dx + \int_b^c |q_1|\big(\epsilon^{-2}|u_2(x)|^2 + \epsilon^2|u_2'(x)|^2\big)dx\\
&+ \sum_{2\le k\le n}\left(\int_a^b |p_k|\epsilon^{-2}|u_1(x)|^2dx + \int_b^c |q_k|\epsilon^{-2}|u_2(x)|^2dx\right)\\
&+ \sum_{2\le k\le n}\left(\int_a^b |p_k(x)|\epsilon^2|u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)|\epsilon^2|u_2^{(k)}(x)|^2dx\right)\\
&= \int_a^b\left(\epsilon^{-2}\sum_{1\le k\le n}|p_k(x)|\right)|u_1(x)|^2dx + \int_a^b \epsilon^2|p_1(x)||u_1'(x)|^2dx\\
&+ \int_b^c\left(\epsilon^{-2}\sum_{1\le k\le n}|q_k(x)|\right)|u_2(x)|^2dx + \int_b^c \epsilon^2|q_1(x)||u_2'(x)|^2dx\\
&+ \sum_{2\le k\le n}\left(\int_a^b |p_k(x)|\epsilon^2|u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)|\epsilon^2|u_2^{(k)}(x)|^2dx\right).
\end{aligned}
$$

From (5.1) it follows that

$$
\begin{aligned}
&|Im\left\langle L_{(n,n)},\ u\right\rangle_X|\\
&\le \int_a^b\left(\epsilon^{-2}\sum_{1\le k\le n}|p_k(x)|\right)|u_1(x)|^2dx + \int_a^b\left(\mu+\epsilon^2|p_1(x)|\right)|u_1'(x)|^2dx\\
&+ \int_b^c\left(\epsilon^{-2}\sum_{1\le k\le n}|q_k(x)|\right)|u_2(x)|^2dx + \int_b^c\left(\nu+\epsilon^2|q_1(x)|\right)|u_2'(x)|^2dx.
\end{aligned}
$$

Then we have for a sufficiently small $\epsilon > 0$

$$
\begin{aligned}
&Re\left\langle L_{n,n}u,\ u\right\rangle_X + \delta|Im\left\langle L_{n,n}u,\ u\right\rangle_X|\\
&\le \frac{1}{2}\Big\{\int_a^b\Big(\sum_{1\le k\le n}\Big(|A_k'(x)|+\frac{|A_k(x)|}{\epsilon^2}\\
&+(-1)^k p_k^{(k)}(x)+\frac{2\delta}{\epsilon^2}|p_k(x)|\Big)+2p_0(x)\Big)|u_1|^2(x)dx\\
&-\int_a^b\left(2p_2(x)-\epsilon^2\sum_{1\le k\le n}|A_k(x)|^2-2\delta\mu-2\delta\epsilon^2|p_1(x)|\right)|u_1'|^2(x)dx
\end{aligned}
$$

$$+\int_b^c\Bigg(\sum_{1\le k\le n}\Bigg(|B_k'(x)|+\frac{|B_k(x)|}{\epsilon^2}$$
$$+(-1)^k q_k^{(k)}(x)+\frac{2\delta}{\epsilon^2}|q_k(x)|\Bigg)+2q_0(x)\Bigg)|u_2|^2(x)dx$$
$$-\int_b^c\left(2q_2(x)-\epsilon^2\sum_{1\le k\le n}|B_k(x)|^2-2\delta\nu-2\epsilon^2\delta|q_1(x)|\right)|u_2'|^2(x)dx\Bigg\}$$
$$+\frac{1}{2}\Big\{(Ak[u_1])^*(A^{-1})^*PA^{-1}Ak[u_1](b)-(Bk[u_2])^*(B^{-1})^*QB^{-1}Bk[u_2](b)\Big\}$$
$$\le\rho\|u\|_X^2$$

where

$$\rho=\max_{1\le k\le n}\left(n\sup_{a\le x\le b}(\alpha_k(x),\ n\sup_{b\le x\le c}(\beta_k(x))\right),$$

where

$$\alpha_k(x)=|A_k'(x)|+\frac{|A_k(x)|}{\epsilon^2}+|p_k^{(k)}(x)|+\frac{2\delta}{\epsilon^2}|p_k(x)|+\frac{2}{n}|p_0(x)|,\ k=1,2,...,n$$

and

$$\beta_k(x)=|B_k'(x)|+\frac{|B_k(x)|}{\epsilon^2}+(-1)^k q_k^{(k)}(x)+\frac{2\delta}{\epsilon^2}|q_k(x)|+\frac{2}{n}|q_0(x)|,\quad k=1,2,...,n.$$

The proof of this theorem is completed. □

References

[1] P.K. Baruah and M. Venkatesulu, *Deficiency indices of a differential operator satisfying certain maching interface conditions*, Electronic Journal of Differential Equations, 38 (2005), 1-9.

[2] T.G. Bhaskar and M. Venkatesulu,*Computation of Green's matrices for boundary value problems associated with a pair of mixed linear regular ordinary differential operators*, Internat. J. Math. Math. Sci., 18 (1995), 789–798

[3] T.G. Bhaskar, R. Kumar,*Analyticity of semigroup generated by a class of differential operators with interface*, Nonlinear Analysis, 39 (2000), 779-791.

[4] C. A. Boyes, *Acoustic waveguides*, Application to Oceanic Sciences, Wiley, New York, 1984.

[5] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, 2000.

[6] H. O. Fattorini, *The Cauchy problem*, Addison Wesley, Reading, MA, 1983.

[7] R. Kumar and T. G. Bhaskar,*m- accretivity of the differential operators with an interface*, Nonlinear Analysis, Theory, Methodes and Applications, 24 (1995), 765–772.

[8] R. Nagel, *One-parameter semigroups of positive operators*, Lecture Notes in Math, Springer-Verlag, 1986.

[9] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. Ser, Vol. 44, Springer, Berlin, 1983.

[10] A. Saddi and O. A. M. Sid Ahmed, *Nalyticity of semigroups generated by a class of differential operators with matrix coefficients and interface*, Semigroup Forum, 71 (2005), 1-17.

[11] A. Saddi, *Analyticity of semigroups generated by degenerate mixed differential operators*, Advances in Pure Mathematics, 1 (2011), 42–48

[12] S. Sun Cheng, *Ordinary differential operators and their convolution adjoints*, Bulletin of the Institute of Mathematics Academia Sinica, 7 (1979), 443–459.

[13] M. Venkatsulu, T. G. Bhaskar, *Selfadjoint boundary value problems associated with a pair of mixed linear ordinary differential equations*, J. Math. Anal. Appl., 144 (1989), 322-341.

[14] U. Tippenhauer, *Generalized boundary value problems for ordinary differential operators and least squares solutions*, J. Math. Anal. App., 110 (1985), 230-246

[15] J. Weidmann, *Spectral theory of ordinary differential operators*, Lecture notes in Mathematics, Springer, Berlin, 1987.

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Geometric results of Han-Banach theorem for functionals on weak hypervector spaces

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Abstract. In this paper we prove some basic results of Han-Banach theorem for functionals on normed normal weak hypervector spaces.
Keywords: normal hypervector space, normed weak hypervector space, weak linear functional.

1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in the last decades and has constructed some other structures such as hypergroups, hypermodules, hyperfields and hypervector spacses. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatories, codes, cryptography, probability and etc. Some of applications of this concepts are given in [1, 3] and [16].

In 1988 , the concept of hypervector space was first introduced by Scafati-Tallini. She later considered more properties of such spaces. Authors, in [6]-[12] considered hypervector spaces in viewpoint of analysis. In mentioned papers, authors introduce concepts such as dimension of hypervectore spaces, normed hypervector spaces, operator on these spaces and another important concepts.

Authors in [7] proved the Han-Banach Theorem for normed normal weak hypervector spaces. In this paper we prove the basic geometric results of Han-Banach theorem for functionals on normed normal weak hypervector spaces.

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2. Preliminaries

Definition 2.1 ([15])**.** A weak or weakly distributive hypervector space over a field F ($\mathbb{C}$ *or* $\mathbb{R}$) is a quadruple $(X, +, o, F)$ such that $(X, +)$ is an abelian group and $o : F \times X \to P*(X)$ is a multivalued product such that

1. $\forall a \in F,\ \forall x, y \in X,\ [ao(x + y)] \cap [aox + aoy] \neq \emptyset$;
2. $\forall a, b \in F,\ \forall x \in X,\ [(a + b)ox] \cap [aox + box] \neq \emptyset$;
3. $\forall a, b \in F,\ \forall x \in X,\ ao(box) = (ab)ox$;
4. $\forall a \in F,\ \forall x \in X,\ ao(-x) = (-a)ox = -(aox)$;
5. $\forall x \in X,\ x \in 1ox$.

The properties 1 and 2 are called weak right and left distributive laws, respectively. Note that the set $ao(box)$ in 3 is of the form $\cup_{y \in box} aoy$.

Definition 2.2. Let X be a weak hypervector space over F, $a \in F$ and $x \in X$. Essential point of $a \circ x$, that we denote it by $e_{a\circ x}$, for $a \neq 0$ is the element of $a \circ x$ such that $x \in a^{-1} \circ e_{a\circ x}$. For $a = 0$, we define $e_{a\circ x} = 0$.

Remark 2.3. Note that $e_{a\circ x}$ is not unique, necessarily. Hence we denote the set of all essential points by $E_{a\circ x}$. When in this note we use $e_{a\circ x}$ in an equation, we mean any element of $E_{a\circ x}$.

Definition 2.4. A weak hypervector space X over the field F is said to be normal if for every $a, b \in F$ and $x, y \in X$ the followings hold:

1. $(E_{a\circ x} + E_{a\circ y}) \cap E_{a\circ(x+y)} \neq \emptyset$
2. $E_{a\circ x} + E_{b\circ x} \subseteq E_{(a+b)\circ x}$

Lemma 2.5 ([6])**.** *Let X be a weak hypervector space over F, $a, b \in F$ and $x \in X$. Then the following properties hold:*

1. $x \in E_{1\circ x}$
2. *If $b \neq 0$, then $a \circ e_{b\circ x} = ab \circ x$*
3. $E_{-a\circ x} = -E_{a\circ x}$
4. *If $a \neq 0$, then there exists an $y \in X$ such that $x \in E_{a\circ y}$.*
5. *If X is normal, then $E_{a\circ x}$ is singleton.*

Definition 2.6. A subset E of a weak hypervector space X is said to be hyperconvex if

$$E_{t\circ x} + E_{(1-t)\circ y} \subseteq E$$

for every $x, y \in E$ and $t \in [0, 1]$.

Definition 2.7 ([6])**.** An additive subgroup M of a weak hypervector space X is called a weak hypervector subspace of X, when $E_{a\circ x} \subseteq M$ for every $a \in F$ and $x \in M$.

Definition 2.8 ([15])**.** A norm on a weak hypervector space X is a mapping $\|\cdot\| : X \to \mathbb{R}$ such that for every $a \in F$ and $x, y \in X$ we have:

1. $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\sup \|a \circ x\| = |a| \cdot \|x\|$

X with a norm is called a normed weak hypervector space.

Definition 2.9 ([7])**.** Let X be a weak hypervector space over F. A map $f : X \to F$ is called weakly linear functional if f is additive and satisfies $f(e_{a\circ x}) = af(x)$, for every $a \in F$ and $x \in X$.
If X is normed weak hypervector space over F, we denote the set of all bounded (see [8]) weakly linear functionals on X by X^*_w.

Lemma 2.10 ([7])**.** *Suppose X is a normal weak hypervector space over $\mathbb{C}$.*
1. *If u is the real part of a complex weakly linear functional f, then u is real weakly linear and*

$$f(x) = u(x) - iu(e_{i\circ x}), \quad x \in X. \tag{2.1}$$

2. *If u is a real weakly linear functional on X and f to be defined by* (2.1)*, then f is a complex weakly linear functional on X.*
3. *If, X in addition, is normed and f to be defined by* (2.1) *, then $\|f\| = \|u\|$.*

3. Main results

Now we prepare to prove some results of Han-Banach Theorem for weakly linear functionals on normed normal hypervector spaces.

Throughout of this section, X is a normed normal weak hypervector space over the field $\mathbb{C}$.

Lemma 3.1. *Let $C \subseteq X$ be an open hyperconvex set with $0 \in C$. For every $x \in C$, set*

$$p(x) = \inf\{\alpha > 0 |\ e_{\alpha^{-1}\circ x} \in C\}$$

the p satisfies

1. $p(x + y) \leq p(x) + p(y),\ \forall x, y \in X$
2. $p(e_{\lambda\circ x}) = \lambda p(x), \qquad x \in X,\ \lambda > 0$
3. *There is a constant M such that $0 \leq p(x) \leq M\|x\|$, for every $x \in M$.*

4. $C = \{x \in X |\ p(x) < 1\}$.

Proof. For 2, Let $\lambda > 0$ and $x \in X$ be given, then

$$\begin{aligned} p(e_{\lambda \circ x}) &= \inf\{\alpha > 0 |\ e_{\alpha^{-1}\cdot(\lambda \circ x)} \in C\} \\ &= \inf\{\alpha > 0 | e_{\alpha^{-1}\lambda \circ x} \in C\} \\ &= \lambda \inf\{\alpha\lambda^{-1} | e_{\alpha^{-1}\lambda \circ x} \in C\} = \lambda p(x). \end{aligned}$$

4. First suppose $x \in C$, since C is open, the continuity of mapping $t \to e_{(1+t)\circ x}$ for each $x \in X$ and $t \in F$, implies that $e_{(1+\epsilon)\circ x} \in C$ for some $\varepsilon > 0$, and therefore $p(x) \leq \frac{1}{1+\varepsilon} < 1$. Conversely, if $p(x) < 1$, there exists $\alpha \in (0, 1)$ such that $e_{\alpha^{-1}\circ x} \in C$ and thus,

$$x = e_{\alpha \circ e_{\alpha^{-1}\circ x}} + e_{(1-\alpha)\circ o} \in C.$$

3. Let $r > 0$ be such that $N_r(0) \subseteq C$, clearly for $x \neq 0$,

$$\|e_{\frac{r}{2\|x\|}\circ x}\| = \frac{r}{2\|x\|}\|x\| = \frac{r}{2} < r.$$

Therefore $e_{\frac{r}{2\|x\|}\circ x} \in C$ and hence $p(x) \leq \frac{2}{r}\|x\|$.
1- Let $x, y \in X$ be given, for $\varepsilon > 0$ we have

$$p(e_{\frac{1}{p(x)+\varepsilon}\circ p(x)}) = \frac{1}{p(x)+\varepsilon}p(x) < 1.$$

Then 4 implies $e_{\frac{1}{p(x)+\varepsilon}\circ p(x)} \in C$, similarly $e_{\frac{1}{p(x)+\varepsilon}\circ p(y)} \in C$ thus hyperconvexity of C with $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon} \in [0, 1]$ implies that

$$e_{\frac{1}{p(x)+p(y)+2\varepsilon}} \circ (x + y) \in C$$

therefore $p\left(e_{\frac{1}{p(x)+p(y)+2\varepsilon}\circ(x+y)}\right) < 1$, and 2 implies
$p(x + y) < p(x) + p(y) + 2\varepsilon$. □

Lemma 3.2. *Let $C \subseteq X$ be a nonempty open hyperconvex set and let $x_0 \in X$ with $x_0 \notin C$, then there exists $f \in X_w^*$ such that*

$$\forall x \in C, \qquad f(x) < f(x_0).$$

Proof. Without loss of generality we may assume that C contains 0.

We may thus introduce the function p of Lemma 3.1 for C. Consider the weak hypervector subspace $M = \{e_{t\circ x} |\ t \in \mathbb{R}\}$, and weakly linear functional

$$\begin{cases} g : M \to \mathbb{R} \\ g(e_{t\circ x}) = t, \quad t \in \mathbb{R}. \end{cases}$$

It is clear that $g \leq p$ on M, for, if $t > 0$, then since $x_0 \notin C$, $p(x_0) \geq 1$, therefore

$$g(e_{t_0 \circ x_0}) = t \geq tp(x_0) = p(e_{t_0 \circ x_0}),$$

and for $t < 0$,

$$g(e_{t_0 \circ x_0}) = t < 0 \leq p(e_{t_0 \circ x_0}).$$

Now it follows from Theorem 3.8 of [7] that there exists a weakly linear functional f on X that extends g and satisfies $f \leq p$ on X.
In particular, we have $f(x_0) = g(x_0) = 1$ and that f is continuous by Lemma 2.3-3, finally we deduce from lemma 2.3-4 that $\forall x \in C, f(x) < 1 = f(x_0)$. □

Theorem 3.3. *Let $A \subseteq X$ and $B \subseteq X$ be two nonempty, hyperconvex subsets such that $A \cap B = \emptyset$. Assume that one of them is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that*

$$Re\ f(x) \leq \alpha \leq Re\ f(y),$$

for every $x \in A$ and $y \in B$.

Proof. Lemma 3.2 implies that it is enough to prove this for real scalars.

Set $C = A - B$, so that C is hyperconvex. C is open and $0 \notin C$ by Lemma 3.2 there exists $f \in X_w^*$ such that $f(z) < 0$ for every $z \in C$, that is $f(x) < f(y)$ for every $x \in A$ and $y \in B$.

Fix a constant α satisfying

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y),$$

this complete the proof. □

Theorem 3.4. *Let $A \subseteq X$ and $B \subseteq X$ be two nonempty, hyperconvex subsets such that $A \cap B = \emptyset$. Assume that A is closed and B is compact, then there exist $f \in X_w^*$ and $B \in \mathbb{R}$ such that*

$$Re\ f(x) < \beta < Re\ f(y)$$

for every $x \in A$ and $y \in B$.

Proof. As in proof of Theorem 3.3 , it is enough to use real scalars.

Set $C = A - B$, so that C is hyperconvex and closed and $0 \notin C$. Hence there is some $r > 0$ such that $N_r(0) \cap C = \emptyset$. By Theorem 3.3 there is some $f \in X_w^*$ such that

$$\forall x \in A, y \in B, z \in N_1(0), \qquad f(x - y) \leq f(e_{r \circ z}).$$

It follows $f(x - y) \leq -r\|f\|$ for every $x \in A$ and $y \in B$, with $\varepsilon = \frac{1}{2} r\|f\| > 0$ we have

$$f(x) + \varepsilon < f(y) - \varepsilon, \qquad x \in A, y \in B.$$

Choosing β s.t

$$\sup_{x \in A} f(x) + \varepsilon \leq \beta \leq \inf_{y \in B} f(y) - \varepsilon,$$

this complete the proof. □

Corollary 3.5. *X_w^* separates points on X.*

Proof. Apply Theorem 3.4 for $A = \{x\}$, $B = \{y\}$ with $x, y \in X$, $x \neq y$. □

Corollary 3.6. *Suppose that E is a weak hypervector subspace of X. If $x_0 \in X$ and $x_0 \notin \bar{E}$, then there exists $f \in X_w^*$ such that $f(x_0) = 1$ and $f = 0$ on E.*

Proof. Using Theorem 3.4 with $A = \bar{E}$ and $B = \{x_0\}$, thus we have $f(x) < \alpha < f(x_0)$ for every $x \in E$ and some $\alpha \in \mathbb{R}$. Now suppose $x \in E$ be given, then

$$\forall \lambda \in \mathbb{R} \qquad \lambda f(x) = f(e_{\lambda \circ x}) < \alpha.$$

Therefore $f(x) < \frac{\alpha}{\lambda}$ for $\lambda > 0$ and $f(x) > \frac{\alpha}{\lambda}$ for $\lambda < 0$, letting $\lambda \to \pm\infty$ we have $f(x) = 0$. This completes the proof. □

Corollary 3.7. *If f is a continuous weakly linear functional on a weak hypervector subspace M of X, then there exists $g \in X_w^*$ such that $g = f$ on M.*

Proof. Assume without loss of generality that $f \neq 0$ on M, put $M_0 = \{x \in M | \, f(x) = 0\}$ and pick $x_0 \in M$ such that $f(x_0) = 1$. Since f is continuous, $x_0 \notin \bar{M_0}^M$ (the closure of M_0 with respect of M), therefore $x_0 \notin \bar{M_0}^X$, Corollary 3.6 therefore assures the existence of a $g \in X_w^*$ such that $g(x_0) = 1$ and $g(M_0) = \{0\}$. If $x \in M$ then $x - e_{f(x) \circ x_0} \in M_0$.

Hence

$$g(x) - f(x) = g(x) - f(x)g(x_0) = g(x - e_{f(x) \circ x_0}) = 0,$$

thus $g = f$ on M. □

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Aviani editore, 1993.

[2] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publishers, Advances in Mathematics (Dordrecht), 2003.

[3] F. Marty, *Sur nue generalizeation de la notion de group*, 8^{th} congress of the Scandinavic Mathematics, Stockholm, 1934, 45–49.

[4] P. Raja and S. M. Vaezpour, *Normed hypervector spaces*, Iranian Journal of Mathematical Sciencesand Informatics (IJMSI), 2 (2007), 35-44.

[5] S. Roy and T. K. Samanta, *Innerproduct hyperspaces*, Accepted in Italian J. of Pure and Appl. Math., 2010.

[6] A. Taghavi and R. Hosseinzadeh, *A note on dimension of weak hypervector spaces*, Italian J. of Pure and Appl. Math., 33 (2014), 7-14.

[7] A. Taghavi and R. Hosseinzadeh, *Hahn-Banach theorem for functionals on hypervector spaces*, The Journal of Mathematics and Computer Science (JMCS), 2 (2011), 682-690.

[8] A. Taghavi and R. Hosseinzadeh, *Operators on normed hypervector spaces*, Southeast Asian Bulletin of Mathematics (SABM), 35 (2011), 367-372.

[9] A. Taghavi and R. Hosseinzadeh, *Operators on weak hypervector spaces*, Ratio Mathematica (RM), 22 (2012), 37-43.

[10] A. Taghavi and R. Hosseinzadeh, *Uniform boundedness principle for operators on hypervector spaces*, Iranian Journal of Mathematical Sciences and Informatics (IJMSI), 7 (2012), 9-16.

[11] A. Taghavi and R. Hosseinzadeh and H. Rohi, *Hyperinner product spacess*, Journal of Algebraic Hyperstructures and it's Applications (JAHA), 1 (2014), 95-100.

[12] A. Taghavi and R. Parvinianzadeh, *Hyperalgebras and quotient hyperalgebras*, Italian J. of Pure and Appl. Math. (JPAM), 26 (2009), 17-24.

[13] A. Taghavi and T. Vougiouklis and R. Hosseinzadeh, *A note on operators on normed finite dimensional weak hypervector spaces*, Scientific bulletin, Series A, 74 (2012), 103-108.

[14] M.Scafati-Tallini, *Characterization of remarkable hypervector space*, Proc. 8^{th} congress on "Algebraic Hyperstructures and Aplications", Samotraki, Greece, (2002), Spanidis Press, Xanthi, (2003), 231-237.

[15] M.Scafati-Tallini, *Weak hypervector space and norms in such spaces*, Algebraic Hyperstructures and Applications Hadronic Press., 1994, 199-206.

[16] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield. Algebraic hyperstructures and applications (Xanthi, 1990)*, World Sci. Publishing, Teaneck, NJ, (1991), 203–211.

[17] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, 1994.

[18] M. M. Zahedi, *A review on hyper k-algebras*, Iranian Journal of Mathematical Sciences and Informatics (IJMSI), 1 (2006), 55-112.

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Projective curvature tensor on generalized (k, μ)-space forms

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Abstract. In this paper, we study the projective curvature tensor on generalized (k, μ)-space forms. Here we study the projectively flat, ξ-projectively flat, pseudoprojectvely flat, h-projectively semisymmetric, ϕ-projectively semisymmetric, and $P \cdot S$ on generalized (k, μ)-space forms.
Keywords: generalized (k, μ)-space form, projective curvature tensor, η-Einstein manifold.

1. Introduction

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exists a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η on M satisfying [4],[5]

$$\phi^2 = -I + \eta \circ \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \quad g(X, \xi) = \eta(X), \tag{1.3}$$

for any vector fields X, Y on M.

The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. It is well known that contact metric manifolds are almost contact metric manifolds such that $\Phi = d\eta$.

The sectional curvature of a manifold determines its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known

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as a real space form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Model for these spaces are Euclidean spaces ($c = 0$), spheres ($c > 0$) and hyperbolic spaces ($c < 0$).

When we come to the case of complex manifolds, a Kaehlerian manifold (M, J, g) with constant holomorphic sectional curvature c is called a complex space form and its curvature tensor satisfies,

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}.$$

Many authors studied complex space forms and their submanifolds in different points of view. On the basis of this, Sasakian space forms were introduced in contact geometry (for more details refer [4]). For such a manifold, the curvature tensor is given by [1, 2];

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.4}$$

where f_1, f_2 and f_3 are constants. If f_1, f_2 and f_3 are differentiable functions then we call such a space as generalized Sasakian space form.

Alegre and Carriazo in [1, 2] studied generalized Sasakian space forms with contact metric and trans-Sasakian structure, its submanifolds and effect of conformal changes of metric on them. Many authors worked on its product submanifolds [3, 20, 31], Legendrian warped product submanifolds [22] and parallel submanifolds [17]. Different curvature tensors and their flatness, symmetries and recurrence properties on this space forms were studied by the geometers in [23, 24, 26].

An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a generalized (k,μ) space form if there exists differential functions $f_1, f_2, f_3, f_4, f_5, f_6$ on $M^{2n+1}(f_1, \ldots, f_6)$, whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.5}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors;

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \\ R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \\ R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \\ R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \end{aligned}$$

where the tensor h is defined by $2h = L_\xi \phi$ is symmetric and satisfies the following conditions

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr(h) = 0, \quad \eta \circ h = 0, \quad h^2 = (k-1)\phi^2, \quad k \leq 1 \tag{1.6}$$

and where L is the usual Lie derivative and X, Y, Z are vector fields on $M^{2n+1}(f_1, \ldots, f_6)$. In particular if $f_4 = f_5 = f_6 = 0$, then generalized (k,μ)-space form $M^{2n+1}(f_1, \ldots, f_6)$ reduces to generalized Sasakian space forms. Moreover, it was proved in [19] that (k,μ)-space forms are natural examples of generalized (k,μ)-space forms for constant functions $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu$.

In [8], after introducing a generalized (k,μ) space forms, some basic identities for generalized (k,μ)-space forms were obtained in an analogous way to those satisfied by Sasakian manifold and they examined in depth the contact metric case with some examples for all possible dimensions. Further they proved that the curvature tensor of a generalized (k,μ)-space form is not unique in 3-dimensional case.

In a generalized (k,μ) space forms, the following relations hold [8];

$$\begin{aligned} QX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ &\quad + ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{1.7}$$

$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \\ &\quad + ((2n-1)f_4 - f_6)g(hX,Y), \end{aligned} \tag{1.8}$$

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\}, \tag{1.9}$$

$$\begin{aligned} R(\xi,X)Y &= (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \\ &\quad + (f_4 - f_6)\{g(hX,Y)\xi - \eta(Y)hX\}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ &\quad + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{1.11}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{1.12}$$

After the conformal curvature tensor, the projective curvature tensor is another important curvature tensor in differential geometry. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n > 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes and it is given by [14]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}. \tag{1.13}$$

Definition 1.1. *An almost contact metric manifold is said to be*

- *Projectively flat if* $P(X,Y)Z = 0$,

- *ξ-Projectively if $P(X,Y)\xi = 0$,*
- *Pseudoprojectively flat if $g(P(\phi X,Y)Z,\phi W) = 0$,*

for all vector fields X, Y, Z.

Definition 1.2. *An almost contact metric manifold is said to be* [12]

- *h-projectively semisymmetric if $P(X,Y)\cdot hZ = 0$,*
- *ϕ-projectively semisymmetric if $P(X,Y)\cdot \phi Z = 0$,*

for all vector fields X, Y, Z.

Definition 1.3. *An almost contact metric manifold M is said to be*

- *Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y)$,*
- *η-Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2\eta(X)\eta(Y)$,*
- *Special type of η-Einstein manifold if $S(X,Y) = \lambda_1\eta(X)\eta(Y)$,*

where S is the Ricci tensor and λ_1 and λ_2 are constants.

2. Projectively flat generalized (k,μ)-space form

Definition 2.1. *A generalized (k,μ)-space forms is said to be projectvely flat if it satisfies*

$$P(X,Y)Z = 0$$

for all vector fields X, Y, Z.

A projectively flat generalized (k,μ)-space forms can be written from (1.13), we have

$$R(X,Y)Z = \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}. \tag{2.1}$$

Inserting $X = \xi$ and using (1.8) in (2.1), we have

$$R(\xi,Y)Z = \frac{1}{2n}\{S(Y,Z)\xi - S(\xi,Z)Y\}. \tag{2.2}$$

Replace Z by ϕZ in (2.2), we get

$$\{(1-2n)f_3 - 3f_2\}g(Y,\phi Z)\xi + \{f_4 - (2n-1)f_6\}g(hY,\phi Z)\xi = 0. \tag{2.3}$$

Since $g(Y,\phi Z) \neq 0$ and $g(hY,\phi Z) \neq 0$, from (2.3) we have

$$f_3 = \frac{3f_2}{(1-2n)} \quad \textit{and} \quad f_6 = \frac{f_4}{(2n-1)}.$$

The generalized Sasakian space form is a projectively flat if and only if $f_3 = \frac{3f_2}{(1-2n)}$ (Theorem 3.1 of [16]). But for the case of generalized (k,μ)-space form $M^{2n+1}(f_1,\ldots,f_6)$, following theorem states;

Theorem 2.1. *If a $(2n+1)$-dimensional generalized (k,μ)-space form is projectively flat if $f_3 = \frac{3f_2}{1-2n}$ and $f_6 = \frac{f_4}{(2n-1)}$.*

3. ξ-projectively flat generalized (k,μ)-space forms

Definition 3.1. *A generalized (k,μ) space forms is said to be ξ-projectvely flat if it satisfies*

$$P(X,Y)\xi = 0$$

for all vector field X, Y

A ξ-projectively flat generalized (k,μ)-space forms can be written from (1.13), we have

$$R(X,Y)\xi = \frac{1}{2n}\{S(Y,\xi)X - S(X,\xi)Y\}. \tag{3.1}$$

Using (1.8) and (1.11) in (3.1), we get

$$(f_4 - f_6)\{\eta(X)hY - \eta(Y)hX\} = 0. \tag{3.2}$$

From (3.2), we can conclude that

$$(f_4 - f_6) = 0 \Rightarrow f_4 = f_6,$$

If $f_4 = f_6$, then it is reduces to generalized Sasakian space forms and observing the above results we can state the following;

Theorem 3.1. *If a ξ-projectively flat generalized (k,μ)-space form is ξ-projectively flat generalized Sasakian space forms if and only if $f_4 = f_6$.*

Remark 3.1. In article [25] author's proved 3-dimensional contact metric generalized (k,μ)-space forms is ξ-projectively flat. The obtained our result of ξ-projectively flat generalized (k,μ)-space forms is stronger compare with the results of paper [25]. Because of generalized (k,μ)-space forms weaker than the contact metric generalized (k,μ)-space forms.

4. Pseudoprojectively flat generalized (k,μ)-space forms

Definition 4.1. *A generalized (k,μ)-space forms is called pseudoprojectively flat if it satisfies the condition*

$$g(P(\phi X,Y)Z,\phi W) = 0$$

for all vector fields X, Y, Z, W.

In view of (1.13), pseudoprojectively flat generalized (k,μ)-space form can be written as

$$g(R(\phi X,Y)Z,\phi W) = \frac{1}{2n}\{S(Y,Z)g(\phi X,\phi W) - S(\phi X,Z)g(Y,\phi W)\}. \tag{4.1}$$

Now we take orthogonal basis $X = W = e_i$ in (4.1), we have

$$g(R(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2n}\{S(Y,Z)g(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i)\}, \tag{4.2}$$

We know that almost contact metric manifold of $(2n+1)$-dimensional, if $\{e_1, e_2, \ldots e_{2n}, \xi\}$ is a local orthonormal basis of a vector fields in M, then $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{4.3}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(e_i, Y)S(e_i, Z) &= \sum_{i=1}^{2n} g(\phi e_i, Y)S(\phi e_i, Z)\\ &= S(Y,Z) - S(\xi, Y)\eta(Z),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(R(e_i, Y)Z, W) &= \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y,Z) - g(R(\xi, Y)Z, \xi)\\ &= S(Y,Z) - 2n(f_1 - f_3)g(\phi Y, \phi Z) + (f_4 - f_6)g(hY, Z).\end{aligned} \tag{4.5}$$

Using (4.3)-(4.5) in (4.2), we have

$$S(Y,Z) = 2n(f_1 - f_3)g(Y,Z) - 2n(f_4 - f_6)g(hY, Z). \tag{4.6}$$

Calculate the value of $g(hY, Z)$ from (1.8) and then inserting value of $g(hY, Z)$ in (4.6), we get

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z).$$

Where

$$A = \left\{\frac{2n(f_1 - f_3) + (2n(f_4 - f_6))((2n-1)f_4 - f_6) + (2nf_1 + 3f_2 - f_3)}{2n(f_4 - f_6) - ((2n-1)f_4 - f_6)}\right\}$$

and

$$B = \left\{\frac{2n(f_4 - f_6)(3f_2 + (2n-1)f_3)}{2n(f_4 - f_6) - ((2n-1)f_4 - f_6)}\right\}.$$

Now we can state the following statement

Theorem 4.1. *If a generalized (k,μ)-space forms is pseudoprojectively flat, then it is an η-Einstein manifold.*

5. h-projectively semisymmetric generalized (k,μ)-space forms

Definition 5.1. *A generalized (k,μ)-space forms is said to be h-projectively semisymmetric if it satisfies*

$$P(X,Y)\cdot hZ = 0,$$

for all vector fields X, Y, Z.

Consider $M^{2n+1}(f_1,\dots,f_6)$ be a h-projectively semisymmetric i.e. $P(X,Y)\cdot hZ=0$, then we have

$$P(X,Y)hZ-h(P(X,Y)Z)=0, \tag{5.1}$$

for all vector fields X,Y,Z.
Now we consider

$$P(X,Y)hZ=R(X,Y)hZ-\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y\}, \tag{5.2}$$

$$h(P(X,Y)Z)=h(R(X,Y)Z)-\frac{1}{2n}\{S(Y,Z)hX-S(X,Z)hY\}. \tag{5.3}$$

Substitute (5.3) and (5.2) in (5.1) and then using (1.5), (5.1) reduces to

$$\begin{aligned}&(f_1-f_3)\{g(Y,hZ)X-g(X,hZ)Y\}+(f_4-f_6)\{g(Y,hZ)hX-g(X,hZ)hY\}\\&\quad-(f_1-f_3)\{g(Y,Z)hX-g(X,Z)hY\}\\&\quad-(f_4-f_6)\{g(Y,Z)h^2X-g(X,Z)h^2Y\}\\&\quad-\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y-S(Y,Z)hX+S(X,Z)hY\}=0.\end{aligned} \tag{5.4}$$

Replace X by hX in (5.4) and then taking inner product with respect to ξ on both side, we have

$$-(f_1-f_3)g(hX,hZ)\eta(Y)=0. \tag{5.5}$$

Again replace $Y=\xi$ in (5.5) and then replace X by hX, Z by hZ, we get

$$(f_1-f_3)(k-1)^2\{g(X,Z)+\eta(X)\eta(Z)\}=0.$$

Now we can state the following theorem;

Theorem 5.1. *If a generalized (k,μ)-space forms is h-projectively semisymmetric if and only if either $f_1=f_2$ or $k=1$.*

6. ϕ-projectively semisymmetric generalized (k,μ)-space forms

Definition 6.1. *A generalized (k,μ)-space forms is said to be ϕ-projectively semisymmetric if it satisfies*

$$P(X,Y)\cdot\phi Z=0,$$

for all vector field X,Y,Z.

Let $M^{2n+1}(f_1,\dots,f_6)$ be a ϕ-projectively semisymmetric, i.e. $P(X,Y)\cdot\phi Z=0$, then we have

$$P(X,Y)\phi Z-\phi(P(X,Y)Z)=0, \tag{6.1}$$

for all vector field X, Y, Z.
Now we consider

$$P(X,Y)\cdot\phi Z = R(X,Y)\phi Z - \frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y\}, \tag{6.2}$$

and

$$\phi(P(X,Y)Z) = \phi(R(X,Y)Z) - \frac{1}{2n}\{S(Y,Z)\phi X - S(X,Z)\phi Y\}. \tag{6.3}$$

In view of (6.2) and (6.3) in (6.1), we get

$$\begin{aligned} P(X,Y)\cdot\phi Z &= R(X,Y)\phi Z - \frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y\} \\ &- \phi(R(X,Y)Z) + \frac{1}{2n}\{S(Y,Z)\phi X - S(X,Z)\phi Y\}. \end{aligned} \tag{6.4}$$

Inserting $Y = \xi$ in (5.4) and with the help of (1.11), one can get

$$\begin{aligned} &- (f_1 - f_3)g(X,\phi Z)\xi - (f_4 - f_6)g(hX,\phi Z)\xi \\ &+ \frac{1}{2n}S(X,\phi Z)\xi + 2(f_1 - f_3)\eta(Z)\phi X = 0. \end{aligned} \tag{6.5}$$

Again inserting $Z = \xi$ in (6.5), we have

$$(f_4 - f_6)\phi X \quad\Rightarrow\quad f_4 - f_6 = 0 \quad or \quad \phi X = 0. \tag{6.6}$$

Theorem 6.1. *If a generalized (k,μ)-space forms is ϕ-projectively semisymmetric if and only if either $f_4 = f_6$ or $\phi X = 0$.*

If a generalized (k,μ)-space forms is ϕ-projectively semisymmetric, then it is reduces to generalized Sasakian space forms. De and et.al proved in [12], a generalized Sasakian space forms is ϕ-projectvely semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.

Corollary 6.1. *A generalized (k,μ)-space forms is ϕ-projectively semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.*

Carriazo et.al shown in [8], If $h = 0$, then it is K-contact manifold and generalized (k,μ)-space forms reduce to generalized Sasakian space forms.

Corollary 6.2. *Let $M^{2n+1}(f_1,...,f_6)$ be a generalized (k,μ)-space form. If M^{2n+1} is a K-contact manifold, then $f_3 = f_1 - 1$. Moreover, M^{2n+1} is Sasakian.*

Corollary 6.3. *Let $M^{2n+1}(f_1,...,f_6)$ be a generalized (k,μ)-space form. If M^{2n+1} is a Sasakian manifold, then $f_2 = f_3 = f_1 - 1$.*

7. Generalized (k,μ)-space forms satisfying $P \cdot S = 0$

Let $M(f_1, \ldots, f_6)$ generalized (k,μ) space form satisfying $P \cdot S = 0$, then

$$P(U,V)S(Y,Z) = S(P(U,V)Y,Z) + S(Y,P(U,V)Z). \tag{7.1}$$

Setting $U = Z = \xi$ in (7.1), we have

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z),$$

where

$$A = -\left\{ \frac{\begin{array}{c}[2n(f_1-f_3) - 2n(f_4-f_6)(k-1)((2n-1)f_4 - f_6) \\ +(-(f_4-f_6)(1+(2nf_1+3f_2-f_3))(2nf_1+3f_2-f_3))]\end{array}}{((2n-1)f_4-f_6)(-1-2n(f_1-f_3)+2n(f_4-f_6)(1+(2nf_1+3f_2-f_3)} \right\}$$

$$B = -\left\{ \frac{\begin{array}{c}[2n(f_4-f_6)((2n-)f_4-f_6)(1+(2nf_1+3f_2-f_3))(3f_2+(2n-1)f_3 \\ +(4n^2(f_1-f_3)(k-1)((2n-1)f_4-f_6)(2n-1)f_4-f_6)]\end{array}}{((2n-1)f_4-f_6)(-1-2n(f_1-f_3)+2n(f_4-f_6)(1+(2nf_1+3f_2-f_3)} \right\}$$

Theorem 7.1. *A generalized (k,μ) space form is satisfying $P \cdot S = 0$, then it is an η-Einstein manifold.*

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References

[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian space forms,* Israel J. Math., 141 (2004), 157-183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian space forms,* Differential Geom. Appl., 26 (2008), 656-666.

[3] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, *Contact CR-warped product submanifolds in generalized Sasakian space forms,* Balkan J. Geom. Appl., 11 (2006), 1-10.

[4] D. E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes in Mathematics, 509 (1976), Springer-Verlag, Berlin.

[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Birkhäuser Boston, 2002.

[6] E. Cartan, *Leons sur la gomtrie des espaces de Riemann,* Gauthier-Villars Paris, (1963).

[7] A. Carriazo and V. M. Molina, *Generalized (k,μ) space forms and D_α homothetic deformations,* Balkan J. Geom. Appl., 16 (2011), 37-47.

[8] A. Carriazo, V. M. Molina and M. M. Tripathi, *Generalized (k,μ) space forms,* Mediterr. J. Math., 10 (2013), 475-496.

[9] M. C. Chaki, *On pseudosymmetric manifolds,* Ann.St.Univ.Al I Cuza Iasi, 33 (1987).

[10] B. Y. Chen, *Geometry of submanifolds,* Marcel Dekker. Ine. New York, 1973.

[11] U. C. De and P. Majhi, *Certain curvature properties of generalized Sasakian space forms,* Facta Universitatis (NIS)Ser. Math. Inform., 27 (2012), 271-282.

[12] U. C. De and P. Majhi, *ϕ-semisymmetric generalized Sasakian space forms,* Arab J. Math Sci., 21 (2015), 170-178.

[13] U. C. De and S. Mallick, *On the existence of generalized quasi Einstein manifolds,* Arch. Math. (Brno), 47 (2011), 279-291.

[14] R. S. Mishra, *Structure on differentiable manifold and their applications,* Chandrama Prakasana, Allahabad, India, 1984.

[15] U. C. De and S. Samui, *The structure of some classes of (k,μ)-contact space forms,* Diffr. Geom. Dyn. Syst., 18 (2016), 1-13.

[16] U.C. De and A. Sarkar, *On the projective curvature tensor of generalized Sasakian-space-forms,* Quaest. Math., 33 (2010), 245-252.

[17] F. Gherib and M. Belkhelfa, *Parallel submanifolds of generalized sasakian space forms,* Bull. Transilv. Univ. Brasov Ser. III, 2 (2009), 185-192.

[18] A. Ghosh, *Killing vector fields and twistor forms on generalized Sasakian space forms,* Mediterr. J. Math., 10 (2013), 1051-1065.

[19] T. Koufogiorgos, *Contact Riemannian manifolds with constant ϕ-sectional curvature,* Tokyo J. Math., 20 (1997), 55-67.

[20] F. Malek and V. Nejadakbary, *Warped product submanifold in generalized sasakian space form,* Acta Math. Acad. Paedagog. Nyhazi., 27 (2011), 325-338.

[21] H. G. Nagaraja and G. Somashekhara, *Curvature tensor in (k,μ)-contact manifolds,* Proc. Est. Acad. Sci., 61 (2012), 20-28.

[22] A. Olteanu, *Legendrian warped product submanifolds in generalized Sasakian space forms,* Acta Math. Acad. Paedagog. Nyhazi., 25 (2009), 137-144.

[23] D. G. Prakasha, *On generalized Sasakian space forms with Weyl-conformal curvature tensor,* Lobachevskii J. Math., 33 (2012), 223-228.

[24] A. Sarkar and U. C. De, *Some curvature properties of generalized Sasakian-space-forms,* Lobachevskii J. Math., 33 (2012), 22-27.

[25] D. G. Prakasha, S. K. Hui and K. Mirji, *On 3-dimensional contact metric generalized (k, μ)-space forms,* Int. J. Math. Math. Sci., 2014, Article ID 797162.

[26] A. Sarkar and M. Sen, *On ϕ-recurrent generalized Sasakian space forms,* Lobachevskii J. Math., 33 (2012), 244-248.

[27] S. Tano, *Ricci curvatures of contact Riemannian manifolds,* Tohoku Math. J., 40 (1988), 441-448.

[28] B. Shanmukha, Venkatesha and S. V. Vishunuvardhana, *Some results on generalized (k, μ)-space forms* NTMSCI, 6 (2018) 48-56.

[29] S. Sular and C. Ozgur, *On some submanifolds of Kenmotsu manifolds,* Chaos Solitons Fractals, 42 (2009), 1990-1995.

[30] Z. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X, Y)\cdot R = 0$. I. the local version,* J. Differ. Geom., 17 (1982), 531-582.

[31] S. Sular and C. Ozgura, *Contact CR-warped product submanifolds in generalized Sasakian space forms,* Int. Electron. J. Geom., 4 (2011), 32-47.

[32] Venkatesha and B. Shanmukha, *W_2-curvature tensor on generalized Sasakian space forms,* Cubo. A Mathematical Journal. Univ. Frontera, Dep. Mat. Estad., 20 2018, 17-29.

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Characterizations of obstinate filters in semihoops

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Abstract. In this paper, we consider fundamental properties of obstinate filters in semihoops and give some characterizations of them. Also, we discuss the relationship between obstinate filters and other types of filters (maximal, implicative, positive implicative, normal and fantastic filters) of semihoops and prove that a filter is an obstinate filter if and only if it is a maximal filter and positive implicative filter. Finally, we give some characterizations of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms.

Keywords: semihoop, obstinate filter, homomorphism, simple semihoop.

1. Introduction

Much of human reasoning and decision making is based on an environment of imprecision, uncertainty, incompleteness of information, partiality of truth and partiality of possibility-in short, on an environment of imperfect information. Hence how to represent and simulate human reasoning become a crucial problem in information science field. For this reason, various logical algebras have been proposed as the semantical systems of non classical logic systems, for example, MV-algebras, BL-algebras, MTL-algebras, residuated latticese, hoops and semihoops. Among these logical algebras, semihoops [1] are very basic algebraic structures and contain all logical algebras based on residuated lattices. Semihoops are generalizations of hoops which were introduced by Bosbach. In the last few years, the theory of hoops has been enriched with deep structure theorems[2, 3, 4, 5, 6, 7]. Many of these results have a strong impact with fuzzy logics. In particular, from the structure theorem of finite basic hoops, one obtains an elegant short proof of the completeness theorem for propositional basic

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logic, which introduced by Häjek [8]. As a more general structure, a semihoop is a hoop without the condition $x \odot (x \to y) = y \odot (y \to x)$. It follows that a semihoop does not satisfy the divisibility condition $x \wedge y = x \odot (x \to y)$. Compared to hoops contains all algebraic structures that induce by continuous t-norms [10], semihoops contains all algebraic structures that induce by left continuous t-norms. Therefore, semihoops play an important role in studying fuzzy logics and the related algebraic structures.

The filter theory of the semihoops plays an important role in studying these algebras and the completeness of the correspondence logic system. From a logic point of view, various filters have natural interpretation as various sets of provable formulas. The filters on semihoops have been widely studied and some important results have been obtained [7, 11, 12]. In particular, Block introduced the idea of filters in semihoops and investigate some important properties of it [7]. After then, the concepts of prime, primary and prefect filters were defined and characterized in semihoops in [11]. In [12], Kondo was the first to systematically study filter theory in hoops, in which the relations between kinds of filters were obtained and and some of their characterizations were presented. In the last few years, in order to study the consequence operators and MP rules in Basic Logic (BL for short), Saeid [14] introduced a new type filter in BL-algebras, called an obstinate filter, and obtained some important result about this filters. Since obstinate filters were successful in several distinct tasks respect to inference rule and provide a sold algebraic foundation for inference rule in fuzzy logic, it has been extended to other logical algebras such as MV-algebras[15], residuated lattices [13] and so on. As we have mentioned in the above, obstinate filters have been widely studied on BL-algebras, residuated lattices and MV-algebras, etc. All the above mentioned algebraic structures are the special case of semihoops. In fact, semihoops are the widest possible residuated structure. Therefore, it is interesting to study the obstinate filters on semihoops for providing a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms. This is the motivation for us to investigate obstinate filters on semihoops.

In this paper, we introduce and study obstinate filters in semihoops and give some characterizations of them. Also, we discuss the relationship between obstinate filters and some types of filters (maximal, implicative, positive implicative, normal and fantastic filters) of semihoops and prove that for any semihoop L and an obstinate filter F, L/F is a Boolean algebra. Besides, we prove that a filter F is an obstinate filter if and only if $x \in F$ or $x^* \in F$, for all $x \in L$. This results are used in the rest of the paper to analyzing various filters in semihoops. Finally, we give a characterization of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms.

2. Preliminaries

In this section, we summarize some definitions and results about semihoops which will be used in the following sections.

Definition 2.1 ([2, 10]). *An algebra $(L, \odot, \to, \wedge, 1)$ of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*

(1) $(L, \wedge, 1)$ *is a $\wedge$-semilattice with upper bounded 1,*

(2) $(L, \odot, 1)$ *is a commutative monoid,*

(3) $(x \odot y) \to z = x \to (y \to z)$, *for all $x, y, z \in L$.*

In what follows, by L we denote the universe of a hoop $(L, \odot, \to, \wedge, 1)$. For any $x \in L$, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any natural number n.

On a semihoop L, we define $x \leq y$ if and only if $x \to y = 1$ for all $x, y \in L$. It is easy to check that $\leq$ is a partial order relation on L and for all $x \in L$, $x \leq 1$. Moreover, an algebra L is a bounded semihoop if L is a semihoop and there exists an element $0 \in L$ such that $0 \leq x$ for all $x \in L$. In a bounded semihoop L, we define the negation $* : x^* = x \to 0$ for all $x \in L$. If $x \odot x = x$, that is, $x^2 = x$ for all $x \in L$, then the semihoop L is said to be idempotent. It is easy to check that an idempotent semihoop is equivalent to a Brouwerian semilattice [11].

Proposition 2.2 ([2, 3, 5, 6, 7, 9]). *In any semihoop L, the following properties hold: for any $x, y, z \in L$,*

(1) $x \leq y \to x$,

(2) $x \to 1 = 1$,

(3) $1 \to x = x$,

(4) $x \leq y \Rightarrow x \to z \geq y \to z$,

(5) $x \leq y \Rightarrow z \to x \leq z \to y$,

(6) $x \to (y \to z) = y \to (x \to z)$,

(7) $x \odot y \leq z$ *iff* $x \leq y \to z$,

(8) $x \odot y \leq x, y$,

(9) $x \odot y \leq x \wedge y$,

(10) $x \to y \leq (z \to x) \to (z \to y)$,

(11) $x \to y \leq (y \to z) \to (x \to z)$,

(12) $(x \to y) \odot (y \to z) \leq x \to z$.

Proposition 2.3 ([2, 3]). *In any bounded semihoop L, the following properties hold: for any $x, y, z \in L$,*

(1) $0^* = 1$, $1^* = 0$,

(2) $x \leq y \Rightarrow x^* \geq y^*$,

(3) $x \odot x^* = 0$.

Definition 2.4 ([2]). *Let L_1 and L_2 be two semihoops. A function $f : L_1 \longrightarrow L_2$ is called a homomorphism of semihoops if and only if:*

(1) $f(1) = 1$,

(2) $f(x \odot y) = f(x) \odot f(y)$,

(3) $f(x \to y) = f(x) \to f(y)$.

Let L be a semihoop. A non-empty set F of L is called a filter of L if it satisfies: (1) $x, y \in F$ implies $x \odot y \in F$; (2) $x \in F$, $y \in H$ and $x \leq y$ implies $y \in F$. A filter of L is called a proper filter if $F \neq L$. A non-empty set F of L is a filter if and only if it satisfies $x, x \to y \in F$ implies $y \in F$, for any $x, y, z \in L$. A proper filter of L is called a prime filter of L, if for any filters F_1,F_2 of L such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$. A proper filter F of L is called a maximal filter of L, if it is not properly contained in any other proper filters of L. A subset $F \subseteq L$ is called an implicative filter if it satisfies: (1) $1 \in F$; (2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a positive implicative filter if it satisfies: (1) $1 \in F$; (2) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a fantastic filter if it satisfies: (1) $1 \in F$; (2) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a normal filter if it satisfies: (1) $1 \in f$; (2) $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply that $(x \to y) \to y \in F$, for all $x, y, z \in L$[11, 12].

Proposition 2.5 ([11, 12]). *Let L be a semihoop and F a filter of L. Then F is a normal filter if and only if $D(F) \subseteq F$, where $D(F) = \{x \in L | x^{**} \in F\}$.*

Proposition 2.6 ([12]). *Let L be a semihoop and F a filter of L. Then F is a positive implicative filter if and only if it is an implicative and fantastic filter.*

Proposition 2.7 ([12]). *Let L be a bounded semihoop and F a filter of L. Then F is a positive implicative filter if and only if L/F is a Boolean algebra.*

3. Obstinate filters in semihoops

Definition 3.1. *Let L be a semihoop and F be a proper filter of L. F is called an obstinate filter of L if it satisfies: $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for any $x, y \in L$.*

The following example shows that any filter may not be an obstinate filter in semihoops.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ with $0 \leq a, b \leq c \leq 1$. Define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. One can easily to check that $F = \{a, c, 1\}$ is an obstinate filter of L. Moreover, the set $G = \{1\}$ is a filter of L while it is not an obstinate filter of L, since $a, b \notin G$ and $a \to b = b \notin G$.

In what follows, some conditions for a proper filter become an obstinate filter of bounded semihoops are given.

Theorem 3.3. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter of L,*

(2) *for any $x \in L$, if $x \notin F$, then there exists $n \geq 1$ such that $(x^*)^n \in F$.*

Proof. (1) $\Rightarrow$ (2) Suppose that F is an obstinate filter and $x \notin F$. Since $0 \notin F$, then $1 = 0 \to x \in F$ and $x^* = x \to 0 \in F$. Therefore $(x^*)^n \in F$ for $n = 1$.

(2) $\Rightarrow$ (1) Suppose that $x, y \notin F$. By hypothesis $(x^*)^n \in F$ and $(y^*)^m \in F$, for some $n, m \geq 1$. We know that $(x^*)^n \leq x^*$ and $(y^*)^m \leq y^*$. By filter property, $x^* \in F$ and $y^* \in F$. By Proposition 2.2, we have $x^* \leq x \to y$ and $y^* \leq y \to x$. Hence, $x \to y \in F$ and $y \to x \in F$. Therefore F is an obstinate filter of L. □

Corollary 3.4. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter of L,*

(2) *$x \in F$ or $x^* \in F$, for any $x \in L$.*

Proof. (1) $\Rightarrow$ (2) Let F be an obstinate filter and $x \notin F$. By Theorem 3.3, we have $(x^*)^n \in F$ for some $n \geq 1$. Further by Proposition 2.2(8), one can obtain that $(x^*)^n \leq x^*$ and hence $x^* \in F$.

(2) $\Rightarrow$ (1) Suppose that $x \notin F$. To prove that F is an obstinate filter of L, we need only to show that $(x^*)^n \in F$ for some $n \geq 1$. By hypothesis, we get that $x^* \in F$. Therefore F is an obstinate filter of L. □

In the following, we discuss the relationship between obstinate filters and other filters in semihoops.

Proposition 3.5. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a maximal filter of L.*

Proof. Suppose that F is an obstinate filter but it is not a maximal filter of L. So there exists a proper filter G strictly greater than F (with respect to set-inclusion). Let $a \in G \backslash F$, then $(a^*)^n \in F$, for some $n \geq 1$. We know that $(a^*)^n \leq a^*$. By the filter property $a^* \in F$ and also $a^* \in G$. So $a \odot a^* = 0 \in G$ and which is a contradiction to G is a proper filter of L. □

The following example shows that the converse of Proposition 3.5 is not true in general.

Example 3.6. Let $L = \{0, a, b, c, 1\}$, where $0 \leq a \leq b \leq c \leq 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	b	b
c	0	0	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop and $F = \{c, 1\}$ is a maximal filter. However, F is not an obstinate filter of L since $a, b \notin F$ and $b \rightarrow a = b \notin F$.

Theorem 3.7. *Let L be a bounded semihoop and $\{1\}$ be an obstinate filter of L. Then L is a simple bounded semihoop.*

Proof. If $\{1\}$ is an obstinate filter of L, by Proposition 3.5, $\{1\}$ is a maximal filter of L and hence $\{1\}$ is the only maximal filter of L. Therefore L is a simple bounded semihoop. □

Theorem 3.8. *The two element chain is the only simple bounded semihoop which has an obstinate filter.*

Proof. It is easy to check that, in the two-element chain, the trivial filter is obstinate filter. Now let L be a simple bounded semihoop which is not the two-element chain. Then L has at least one element x which is different from 0 and 1. Assume by absurdum that L has an obstinate filter, F . Then F is a proper filter of L, hence F is the trivial filter: $F = \{1\}$. So $x \neq 1$, thus $x \notin F$, and F is an obstinate filter, hence $x^* \in F = \{1\}$ by Corollary 3.4. Thus $x \rightarrow 0 = x^* = 1$, that is $x \leq 0$, which means that $x = 0$. This is a contradiction to the choice of x. Thus, L has no obstinate filter. □

Proposition 3.9. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a positive implicative filter of L.*

Proof. Suppose that F is not a positive implicative filter of L. By the definition of positive implicative filter, there exist $x, y \in L$ such that $(x \to y) \to x \in F$ but $x \notin L$. We have $y \in F$ or $y \notin L$. Consider the following cases:

Case 1: let $y \in F$, we have $y \leq x \to y$. Then $x \to y \in F$. Since $(x \to y) \to x \in F$ and F is a filter, then $x \in F$, which is a contradiction to $x \notin L$.

Case 2: let $y \notin F$. Since F is an obstinate filter of H, then $x \to y \in F$. We can obtain that $x \in F$, which is a contradiction to $x \notin L$. Hence F is a positive implicative filter of L. □

The following example shows that the converse of Proposition 3.9 is not true in general.

Example 3.10. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq 1$, $0 \leq a \leq d \leq 1$ and $0 \leq c \leq d \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1\}$ is a positive implicative filter. However, F is not an obstinate filter of L since $b \notin F$ and $c = (b)^* \notin F$.

Corollary 3.11. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is an implicative filter of L.*

Proof. The results follows from Propositions 2.6 and 3.9. □

The following example shows that the converse of Corollary 3.11 is not true in general.

Example 3.12. Let $L = \{0, a, b, c, 1\}$, where $0 \leq c \leq a \leq 1$ and $0 \leq c \leq b \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	c	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop and $F = \{b, 1\}$ is an implicative filter. However, F is not an obstinate filter of L.

Theorem 3.13. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter,*

(2) *F is a maximal and positive implicative filter,*

(3) *F is a maximal and implicative filter.*

Proof. (1) $\Rightarrow$ (2) The results follows from Propositions 3.5 and 3.9.

(2) $\Rightarrow$ (3) The results follows from Proposition 2.6.

(3) $\Rightarrow$ (1) Let $x, y \notin F$, we show that $L_y = \{u \in L \mid y \to u \in F\}$ is a filter containing F. It is easy to check that L_y is a filter. We have $x \leq y \to x$. If $x \in F$, we get $y \to x \in F$, then $x \in L_y$. Hence $F \subseteq L_y \subseteq L$. By hypothesis F is a maximal filter and since $y \notin F$, we get $L_y = L$ and so $y \to x \in F$. The case $x \to y \in F$ is similar. □

Proposition 3.14. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a fantastic filter of L.*

Proof. The results follows from Propositions 2.6 and 3.9. □

The following example shows that the converse of Proposition 3.14 is not true in general.

Example 3.15. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq 1$, $0 \leq a \leq d \leq 1$ and $0 \leq c \leq d \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	1	1	1	a
1	0	a	b	c	d	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1, c\}$ is a fantastic filter. However, F is not an obstinate filter of L.

Proposition 3.16. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a normal filter of L.*

Proof. By Proposition 2.5, to prove that F is a normal filter, we need only to show that $D(F) \subseteq F$. Let $D(F) \nsubseteq F$, then there exists a $x \in D(F)$ F, i.e. $x \in D(F)$ but $x \notin F$. So by Proposition 2.5, we have $x^{**} \in F$ and by Corollary 3.4, we obtain that $x^* \in F$. We know that $x^* \to 0 = (x \to 0) \to 0 = x^{**} \in F$. Therefore by filter property we get that $0 \in F$, which is a contradiction. Hence $D(F) \subseteq F$ and we conclude that F is a normal filter of L. □

Example 3.17. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq b \leq a \leq 1$, $0 \leq d \leq a \leq 1$ and $0 \leq d \leq c \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
1	0	a	b	c	d	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1, c\}$ is a normal filter. However, F is not an obstinate filter of L since $a \notin F$ and $d = (a)^* \notin F$.

Corollary 3.18. *Let F be an obstinate filter of L. Then L/F is a Boolean-algebra.*

Proof. It follows from Propositions 2.7 and 3.9. □

Proposition 3.19. *Let L be a bounded semihoop and F be an obstinate filter of L. Then every filter G containing F is a obstinate filter of L.*

Proof. Let $F \subseteq G$, and $x \notin F$. By Corollary 3.4, we have $x^* \in F$. Since $F \subseteq G$, then $x^* \in G$. Hence G is an obstinate filter. □

Corollary 3.20. *Let L be a bounded semihoop. Then $\{1\}$ is an obstinate filter of L if and only if every filter of L is an obstinate filter.*

Proof. It follows from Proposition 3.19. □

Theorem 3.21. *Let L be a bounded semihoop and F be a filter of L. Then F is an obstinate filter if and only if every filter of the quotient algebra L/F is an obstinate filter.*

Proof. Suppose that F is an obstinate filter of L and $x \in H$ such that $[x] \notin \{[1]\}$, i.e. $[x] \neq [1]$, then $x \notin F$. By Corollary 3.4, we have $x^* \in F$, then $[x^*] = [1]$, i.e. $[x^*] \in \{[1]\}$. Hence $\{[1]\}$ is an obstinate filter of L/F. By Corollary 3.20, we get every filter of the quotient algebra L/F is an obstinate filter.

Conversely, suppose that every filter of the quotient algebra L/F is an obstinate filter and $x \in L$ but $x \notin F$. By hypothesis, we get that $[x] \neq [1]$, i.e. $[x] \notin \{[1]\}$. Since $\{[1]\}$ is an obstinate filter of L/F, then $[x^*] \in \{[1]\}$, i.e. $[x^*] = [1]$. So $x^* \in F$. Hence F is an obstinate filter of L. □

Proposition 3.22. *Let L be a bounded semihoop, F, G and I be filter of L. If I is an obstinate filter and $F \cap G \subseteq I$, then $F \subseteq I$ or $G \subseteq I$.*

Proof. Let $F \cap G \subseteq I$, $F \nsubseteq I$ and $G \nsubseteq I$. We take $a \in F/I$ and $b \in G/I$, then $a \in F$, $a \notin I$ and $b \in G$, $b \notin I$. By Proposition 2.2, we have $a, b \leq a \vee b$. Since F and G are filter. We get $a \vee b \in F$ and $a \vee b \in G$. Therefore $a \vee b \in F \cap G \subseteq I$. Hence $a \vee b \in I$. Since I is an obstinate filter and $a \notin I$, $b \notin I$, we obtain $a^* \in I$ and $b^* \in I$. Since I is a filter, we get $a^* \odot b^* \in I$. By Proposition 2.2, we have $a^* \odot b^* \leq a^* \wedge b^*$. Hence $a^* \wedge b^* \in I$. By Proposition 2.2, we know that $a^* \wedge b^* = (a \vee b)^*$, hence $(a \vee b)^* \in I$. Therefore $(a \vee b) \to 0 = (a \vee b)^* \in I$. Since I is a filter and $a \vee b \in I$, we get that $0 \in I$, which is a contradiction. Therefore $F \subseteq I$ or $G \subseteq I$. □

Corollary 3.23. *Let L be a bounded semihoop, F, G and I be filter of L. If I is an obstinate filter and $I = F \cap G$, then $F = I$ or $G = I$.*

Proof. Suppose that $I = F \cap G$, hence $F \cap G \subseteq I$. By Theorem 3.22, we obtain $F \subseteq I$ or $G \subseteq I$. By hypothesis, we have $I = F \cap G \subseteq F, G$. Hence $F = I$ or $G = I$. □

Proposition 3.24. *Let L_1, L_2 be two bounded semihoops, f be homomorphism from L_1 to L_2 and G be an obstinate filter of L_2. Then the inverse image of G is an obstinate filter of L_1.*

Proof. Let G be an obstinate filter of L_2 and $x \in L_1$ such that $x \notin f^{-1}(G)$. Then $f(x) \notin G$, by hypothesis, $[f(x)]^* \in G$. By Definition 2.3, we have $f(x^*) \in G$. Then we get that $x^* \in f^{-1}(G)$. Hence $f^{-1}(G)$ is an obstinate filter of L_1. □

Theorem 3.25. *Let L be a semihoop and F is an obstinate filter of L. Then there exists a Brouwerian semilattice $G(L) = \{x \in L | x \odot x = x\}$ and a homomorphism from L to $G(L)$ such that $ker(f) = F$.*

Proof. Suppose that F is an obstinate filter of L. We define f as follows

$$f(x) = \begin{cases} 1, & x \in F \\ a, & x \in L \backslash F \end{cases}.$$

Where a is any fixed element of $G(L)$ and $a \neq 1$ and take $f(0) = 0$. In order to verify f is a homomorphism from L to $G(L)$, we will divide our investigation into our four cases.

Case 1: If $x, y \in F$, then by filter property we get that $x \odot y \in F$. Hence $f(x) = 1 = f(y)$ and $f(x \odot y) = 1$. On the other hand, $f(x) \odot f(y) = 1 \odot 1 = 1$. Therefore $f(x \odot y) = f(x) \odot f(y)$. Moreover, since $x, y \in F$, by Proposition 2.2, we have $y \leq x \to y$. We get that $x \to y \in F$. Hence $f(x \to y) = 1$. On the other hand, $f(x) \to f(y) = 1 \to 1 = 1$. Therefore $f(x \to y) = f(x) \to f(y)$.

Case 2: If $x, y \notin F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x) \odot f(y) = a \odot a = a$, since $a \in G(L)$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if $x, y \notin F$, then $x \to y \in F$ because F is an obstinate filter, and so $f(x) \to f(y) = a \to a = 1$. It follows that $f(x \to y) = f(x) \to f(y)$.

Case 3: If $x \notin F$ and $y \in F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x \odot y) = a \odot 1 = 1$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if $x \notin F$ and $y \in F$, then we have $y \leq x \to y$, hence $x \to y \in F$. Then $f(x \to y) = 1$. On the other hand $f(x) \to f(y) = a \to 1 = 1$. It follows that $f(x \to y) = f(x) \to f(y)$.

Case 4: If $x \in F$ and $y \notin F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x) \odot f(y) = 1 \odot a = a$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if we have $x \in F$ and $y \notin F$, then $x \to y \notin F$. So $f(x \to y) = a$. On the other hand $f(x) \to f(y) = 1 \to a = a$. It follows that $f(x \to y) = f(x) \to f(y)$.

Summarizing all the above we know f is a homomorphism from L to $G(L)$. It is clear that $\mathrm{Ker}(f) = f^{-1}(1) = F$. □

4. Conclusion

In this paper, motivated by the previous research of obstinate filters in BL-algebras, we extended the concept of obstinate filters to a more generally algebraic structure semihoops. We have also presented several different characterizations and many important properties of obstinate filters in semihoops. We also proved that if F is an obstinate filter, then L/F is a Boolean algebra. Finally, we give a characterization of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms. In our future work, we will consider the notion of rough obstinate filters in semihoops.

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References

[1] B. Bosbach, K. Halbgruppen, *Axiomatic und aritmetik*, Fundamenta Mathematicae, 64 (1969), 257-287.

[2] B. Bosbach, K. Halbgruppen, *Axiomatic und quotienten*, Fundamenta Mathematicae, 69 (1970), 1-14.

[3] J.T. Wang, X.L. Xin, P.F. He, *Monadic bounded hoops*, Soft Computing, 22 (2018), 1749-1762.

[4] M. Wang, X.L. Xin, J.T. Wang, *Implicative pseudo valuations on Hoops*, Chinese Quarterly Journal of Mathematics, 33 (2018), 51-60.

[5] I.M.A. Ferreirim, *On varieties and quasivarieties of hoops and their reducts*, Ph. D. thesis, Vanderbilt University, Nashvile, Tennessee, 1992.

[6] W.J. Blok, I.M.A. Ferreirim, *Hoops and their implicational reducts*, Algebraic Methods in Logic and Computer Science, Banach Center Publications, 28 (1993), 219-230.

[7] W.J. Blok, I.M.A. Ferreirim, *On the structure of hoops*, Algebr Universalis, 43 (2000), 233-257.

[8] P. Häjek, *Metamathematics of fuzzy logic*, Trends in Logic-Studia Logica Library, 4 (1998), 155-174.

[9] P.F. He, B. Zhao, X.L. Xin, *States and internal states on semihoops*, Soft Computing, 21 (2017), 2941-2957.

[10] F. Esteva, L. Godo, P. Hüjek, F. Montagna, *Hoops and fuzzy logic*, Journal of Logic and Computation, 13 (2003), 532-555.

[11] R.A. Borzooei, M.A. Kologani, *Local and perfect semihoops*, Journal of Intelligent & Fuzzy Systems, 29 (2015), 223-234.

[12] M. Kondo, *Some types of filters in hoops*, Proceedings of the International Symposium on Multiple-Valued Logic, 47 (2011), 50-53.

[13] A.B. Saeid, M. Pourkhatun, *Obstinate filters in residuated lattices*, Bulletin Mathematique De La Societe Des Science Mathematiques De Roumanie, 55 (2012), 413-422.

[14] A.B. Saeid, S. Motamed, *A new filter in BL-algebras*, Journal of Intelligent & Fuzzy Systems, 27 (2014),2949-2957.

[15] F. Foruzesh, E. Eslami, A.B. Saeid, *On obstinate ideals in MV-algebras*, University Politehn Bucharest Science Bulletin Ser. A Appl. Math. Phys., 76 (2014),53-63.

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Some subclasses of meromorphic multivalent functions involving a family of multiplier transforms

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Abstract. Making use of the principle of subordination between analytic functions and a family of multiplier transforms defined on the space of meromorphic functions, we introduce and investigate some new subclasses of meromorphic multivalent functions. Such results as inclusion relationships and integral-preserving properties associated with these subclasses are proved. Several subordination and superordination results involving this family of multiplier transforms are also investigated.

Keywords: analytic functions, meromorphic multivalent functions, subordination and superordination between analytic functions, Hadamard product (or convolution), multiplier transforms.

1. Introduction and preliminaries

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U}\backslash\{0\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \{f \in \mathcal{H}(\mathbb{U}) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

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Let $f,\ g \in \Sigma_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition $\Re(p(z)) > 0 \quad (z \in \mathbb{U})$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write $f(z) \prec g(z), (z \in \mathbb{U})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(\omega(z)), (z \in \mathbb{U})$. Indeed, it is known that $f(z) \prec g(z), (z \in \mathbb{U}) \Longrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In a recent paper, El-Ashwah [6] defined the multiplier transform $\mathcal{D}_{\lambda,p}^{n,l}$ of functions $f \in \Sigma_p$ by

$$\mathcal{D}_{\lambda,p}^{n,l} f(z) := z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\lambda + l(k+p)}{\lambda} \right)^n a_k z^k \tag{1.2}$$
$$(z \in \mathbb{U}^*;\ \lambda > 0;\ l \geqq 0;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ p \in \mathbb{N}).$$

It should be remarked that the operators $\mathcal{D}_{\lambda,1}^{n,1}$ and $\mathcal{D}_{1,1}^{n,1}$ are the multiplier transforms introduced and investigated, respectively, by Sarangi and Uralegaddi [14], and Uralegaddi and Somanatha [18, 19]. Analogous to $\mathcal{D}_{\lambda,p}^{n,l}$, we here define a new multiplier transform $\mathcal{I}_{\lambda,p,\mu}^{n,l}$ as follows.

By setting

$$f_{\lambda,p}^{n,l}(z) := z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\lambda + l(k+p)}{\lambda} \right)^n z^k,$$
$$(z \in \mathbb{U}^*;\ n,\ l \geqq 0;\ \lambda > 0;\ p \in \mathbb{N}), \tag{1.3}$$

we define a new function $f_{\lambda,p,\mu}^{n,l}(z)$ in terms of the Hadamard product (or convolution):

$$f_{\lambda,p}^{n,l}(z) * f_{\lambda,p,\mu}^{n,l}(z) = \frac{1}{z(1-z)^{\mu}} \quad (z \in \mathbb{U}^*;\ \lambda,\ \mu > 0;\ n,\ l \geqq 0;\ p \in \mathbb{N}). \tag{1.4}$$

Then, analogous to $\mathcal{D}_{\lambda,p}^{n,l}$, we have

$$\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) := f_{\lambda,p,\mu}^{n,l}(z) * f(z), \quad (z \in \mathbb{U}^*;\ f \in \Sigma_p), \tag{1.5}$$

where (and throughout this paper unless otherwise mentioned) the parameters $n,\ l,\ p,\ \lambda$ and μ are constrained as follows:

$$n \geqq 0;\ l \geqq 0;\ p \in \mathbb{N},\ \lambda > 0 \text{ and } \mu > 0.$$

We can easily find from (1.3), (1.4) and (1.5) that

$$\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) = z^{-p} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(k+1)!} \left(\frac{\lambda}{\lambda + l(k+p)}\right)^n a_k z^k, \quad (z \in \mathbb{U}^*), \tag{1.6}$$

where $(\mu)_k$ is the Pochhammer symbol defined by

$$(\mu)_k := \begin{cases} 1, & (k = 0), \\ \mu(\mu+1)\cdots(\mu+k-1), & (k \in \mathbb{N}). \end{cases}$$

Clearly, the operator $\mathcal{I}_{\lambda,1,\mu}^{n,1}$ $(n \in \mathbb{N}_0)$ is the well-known Cho-Kwon-Srivastava operator (see, for more details, [2, 3, 8, 13, 15]).

It is readily verified from (1.6) that

$$lz\left(\mathcal{I}_{\lambda,p,\mu}^{n+1,l} f\right)'(z) = \lambda \mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) - (\lambda + p\, l)\mathcal{I}_{\lambda,p,\mu}^{n+1,l} f(z), \tag{1.7}$$

and

$$z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l} f\right)'(z) = \mu \mathcal{I}_{\lambda,p,\mu+1}^{n,l} f(z) - (\mu+1)\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z). \tag{1.8}$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}_p^*(\eta;\phi)$, $\mathcal{MK}_p(\eta;\phi)$, $\mathcal{MC}_p(\eta,\delta;\phi,\psi)$ and $\mathcal{MQC}_p(\eta,\delta;\phi,\psi)$ of the class Σ_p which are defined by

$$\begin{aligned}
\mathcal{MS}_p^*(\eta;\phi) := \Bigg\{ & f \in \Sigma_p :\ \frac{1}{p-\eta}\left(-\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z), \\
& (\phi \in \mathcal{P};\ 0 \leqq \eta < p\,;\ z \in \mathbb{U})\Bigg\}, \\
\mathcal{MK}_p(\eta;\phi) := \Bigg\{ & f \in \Sigma_p :\ \frac{1}{p-\eta}\left(-1 - \frac{zf''(z)}{f'(z)} - \eta\right) \prec \phi(z), \\
& (\phi \in \mathcal{P};\ 0 \leqq \eta < p\,;\ z \in \mathbb{U})\Bigg\}, \\
\mathcal{MC}_p(\eta,\delta;\phi,\psi) := \Bigg\{ & f \in \Sigma_p :\ \exists\ g \in \mathcal{MS}_p^*(\eta;\phi) \\
& \text{such that } \frac{1}{p-\delta}\left(-\frac{zf'(z)}{g(z)} - \delta\right) \prec \psi(z) \\
& (\phi,\ \psi \in \mathcal{P};\ 0 \leqq \eta,\ \delta < p\,;\ z \in \mathbb{U})\Bigg\},
\end{aligned}$$

and

$$\mathcal{MQC}_p(\eta,\delta;\phi,\psi) := \Big\{ f \in \Sigma_p :\ \exists\ g \in \mathcal{MK}_p(\eta;\phi)$$
$$\text{such that}\ \ \frac{1}{p-\delta}\left(-\frac{(zf'(z))'}{g'(z)} - \delta\right) \prec \psi(z), (\phi,\ \psi \in \mathcal{P};\ 0 \leqq \eta,\ \delta < p\,;\ z \in \mathbb{U}) \Big\}.$$

Indeed, the above mentioned function classes are generalizations of the general meromorphic starlike, meromorphic convex, meromorphic close-to-convex and meromorphic quasi-convex functions in analytic function theory (see, for details, [1, 7, 11, 12, 16, 17, 20, 21, 22]).

Next, by using the operator defined by (1.6), we define the following subclasses $\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$, $\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$, $\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ and $\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ of the class Σ_p:

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MS}^*_p(\eta;\phi) \right\},$$
$$\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MK}_p(\eta;\phi) \right\},$$
$$\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MC}_p(\eta,\delta;\phi,\psi) \right\},$$

and

$$\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MQC}_p(\eta,\delta;\phi,\psi) \right\}.$$

Clearly, we know that

$$f \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \Longleftrightarrow -zf' \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi), \tag{1.9}$$

and

$$f \in \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) \Longleftrightarrow -zf' \in \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi). \tag{1.10}$$

In order to prove our main results, we need the following definition and lemmas.

Definition 1. (See [10]) Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 1 ([5])**.** *Let $\kappa, \vartheta \in \mathbb{C}$. Suppose also that $\mathfrak{m}$ is convex and univalent in $\mathbb{U}$ with*

$$\mathfrak{m}(0) = 1, \quad \textit{and} \quad \Re(\kappa\mathfrak{m}(z) + \vartheta) > 0, \quad (z \in \mathbb{U}).$$

If $\mathfrak{u}$ *is analytic in* $\mathbb{U}$ *with* $\mathfrak{u}(0) = 1$*, then the subordination*

$$\mathfrak{u}(z) + \frac{z\mathfrak{u}'(z)}{\kappa\mathfrak{u}(z) + \vartheta} \prec \mathfrak{m}(z), \quad (z \in \mathbb{U})$$

implies that

$$\mathfrak{u}(z) \prec \mathfrak{m}(z), \quad (z \in \mathbb{U}).$$

Lemma 2 ([9])**.** *Let* h *be convex univalent in* $\mathbb{U}$ *and* ζ *be analytic in* $\mathbb{U}$ *with*

$$\Re(\zeta(z)) \geqq 0, \quad (z \in \mathbb{U}).$$

If q *is analytic in* $\mathbb{U}$ *and* $q(0) = h(0)$*, then the subordination*

$$q(z) + \zeta(z)zq'(z) \prec h(z), \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z), \quad (z \in \mathbb{U}).$$

The main purpose of the present paper is to investigate some inclusion relationships and integral-preserving properties of the subclasses

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi),\ \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi),\ \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) \text{ and } \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$$

of meromorphic functions involving the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$. Several subordination and superordination results involving this operator are also investigated.

2. The main inclusion relationships

We begin by presenting our first inclusion relationship given by Theorem 1 below.

Theorem 1. *Let* $0 \leqq \eta < p$ *and* $\phi \in \mathcal{P}$ *with*

$$\max_{z\in\mathbb{U}} \{\Re(\phi(z))\} < \min\left\{\frac{\mu+1-\eta}{p-\eta},\ \frac{\lambda+p\,l-\eta\,l}{(p-\eta)l}\right\}, \quad (z \in \mathbb{U}). \tag{2.1}$$

Then

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \subset \mathcal{MS}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi).$$

Proof. We first prove that

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi). \tag{2.2}$$

Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)$ and suppose that

$$\mathfrak{h}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)} - \eta\right), \tag{2.3}$$

where $\mathfrak{h}$ is analytic in $\mathbb{U}$ with $\mathfrak{h}(0)=1$. Combining (1.8) and (2.3), we find that

$$\mu\frac{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)}=-(p-\eta)\mathfrak{h}(z)-\eta+\mu+1. \tag{2.4}$$

Taking the logarithmical differentiation on both sides of (2.4) and multiplying the resulting equation by z, we get

$$\begin{aligned}&\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z)}-\eta\right)\\ &=\mathfrak{h}(z)+\frac{z\mathfrak{h}'(z)}{-(p-\eta)\mathfrak{h}(z)-\eta+\mu+1}\prec\phi(z).\end{aligned} \tag{2.5}$$

By virtue of (2.1), an application of Lemma 1 to (2.5) yields $\mathfrak{h}\prec\phi$, that is $f\in\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$. Thus, the assertion (2.2) of Theorem 1 holds.

To prove the second part of Theorem 1, we assume that $f\in\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$ and set

$$\mathfrak{g}(z):=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z)}-\eta\right), \tag{2.6}$$

where $\mathfrak{g}$ is analytic in $\mathbb{U}$ with $\mathfrak{g}(0)=1$. Combining (1.7), (2.1) and (2.6) and applying the similar method of proof of the first part, we get $\mathfrak{g}\prec\phi$, that is $f\in\mathcal{MS}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi)$. Therefore, the second part of Theorem 1 also holds. The proof of Theorem 1 is evidently completed. □

Theorem 2. *Let $0\leqq\eta<p$ and $\phi\in\mathcal{P}$ with (2.1) holds. Then*

$$\mathcal{MK}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)\subset\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)\subset\mathcal{MK}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi).$$

Proof. In view of (1.9) and Theorem 1, we find that

$$\begin{aligned}f\in\mathcal{MK}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)&\Longleftrightarrow\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\in\mathcal{MK}_p(\eta;\phi)\\ &\Longleftrightarrow-z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'\in\mathcal{MS}^*_p(\eta;\phi)\\ &\Longleftrightarrow\mathcal{I}^{n,l}_{\lambda,p,\mu+1}\left(-zf'\right)\in\mathcal{MS}^*_p(\eta;\phi)\\ &\Longleftrightarrow-zf'\in\mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)\\ &\Longrightarrow-zf'\in\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)\\ &\Longleftrightarrow\mathcal{I}^{n,l}_{\lambda,p,\mu}\left(-zf'\right)\in\mathcal{MS}^*_p(\eta;\phi)\\ &\Longleftrightarrow-z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'\in\mathcal{MS}^*_p(\eta;\phi)\\ &\Longleftrightarrow\mathcal{I}^{n,l}_{\lambda,p,\mu}f\in\mathcal{MK}_p(\eta;\phi)\\ &\Longleftrightarrow f\in\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi),\end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
f \in \mathcal{MK}_{\lambda,p,\mu}^{n,l}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi) \\
&\Longrightarrow -zf' \in \mathcal{MS}_{\lambda,p,\mu}^{n+1,l}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{I}_{\lambda,p,\mu}^{n+1,l}\left(-zf'\right) \in \mathcal{MS}_{p}^{*}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{I}_{\lambda,p,\mu}^{n+1,l} f \in \mathcal{MK}_{p}(\eta;\phi) \\
&\Longleftrightarrow f \in \mathcal{MK}_{\lambda,p,\mu}^{n+1,l}(\eta;\phi).
\end{aligned} \tag{2.8}
$$

Combining (2.7) and (2.8), we deduce that the assertion of Theorem 2 holds. □

Theorem 3. *Let $0 \leqq \eta < p$, $0 \leqq \delta < p$ and $\phi,\ \psi \in \mathcal{P}$ with (2.1) holds. Then*

$$\mathcal{MC}_{\lambda,p,\mu+1}^{n,l}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}_{\lambda,p,\mu}^{n+1,l}(\eta,\delta;\phi,\psi).$$

Proof. We begin by proving that

$$\mathcal{MC}_{\lambda,p,\mu+1}^{n,l}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi). \tag{2.9}$$

Let $f \in \mathcal{MC}_{\lambda,p,\mu+1}^{n,l}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that

$$\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu+1}^{n,l} f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu+1}^{n,l} g(z)} - \delta\right) \prec \psi(z), \quad (z \in \mathbb{U}) \tag{2.10}$$

with $g \in \mathcal{MS}_{\lambda,p,\mu+1}^{n,l}(\eta;\phi)$.

Moreover, by Theorem 1, we know that $g \in \mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$, which implies that

$$\mathfrak{q}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l} g\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l} g(z)} - \eta\right) \prec \phi(z), \quad (z \in \mathbb{U}). \tag{2.11}$$

We now suppose that

$$\mathfrak{p}(z) := \frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l} f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l} g(z)} - \delta\right), \quad (z \in \mathbb{U}), \tag{2.12}$$

where $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$. Combining (1.8) and (2.12), we find that

$$-[(p-\delta)\mathfrak{p}(z) + \delta]\mathcal{I}_{\lambda,p,\mu}^{n,l} g(z) = \mu \mathcal{I}_{\lambda,p,\mu+1}^{n,l} f(z) - (\mu+1)\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z). \tag{2.13}$$

Differentiating both sides of (2.13) with respect to z and multiplying the resulting equation by z, we get

$$\begin{aligned} &-(p-\delta)z\mathbb{p}'(z)-[(p-\delta)\mathbb{p}(z)+\delta][-(p-\eta)\mathbb{q}(z)-\eta+\mu+1] \\ &=\mu\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}. \end{aligned} \tag{2.14}$$

In view of (1.8), (2.11) and (2.14), we conclude that

$$\begin{aligned} &\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)}-\delta\right) \\ &=\mathbb{p}(z)+\frac{z\mathbb{p}'(z)}{-(p-\eta)\mathbb{q}(z)-\eta+\mu+1}\prec\psi(z),\quad(z\in\mathbb{U}). \end{aligned} \tag{2.15}$$

By noting that (2.1) holds and

$$\mathbb{q}(z)\prec\phi(z),\quad(z\in\mathbb{U}),$$

we know that

$$\Re(-(p-\eta)\mathbb{q}(z)-\eta+\mu+1)>0,\quad(z\in\mathbb{U}).$$

Thus, an application of Lemma 2 to (2.15) yields

$$\mathbb{p}(z)\prec\psi(z),\quad(z\in\mathbb{U}),$$

that is, that $f\in\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$, which implies that the assertion (2.9) of Theorem 3 holds.

By virtue of (1.7) and (2.1), and making use of the similar arguments of the details above, we deduce that

$$\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)\subset\mathcal{MC}^{n+1,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi).$$

The proof of Theorem 3 is thus completed. □

Theorem 4. *Let $0\leqq\eta<p$, $0\leqq\delta<p$ and $\phi,\ \psi\in\mathcal{P}$ with (2.1) holds. Then*

$$\mathcal{MQC}^{n,l}_{\lambda,p,\mu+1}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{n+1,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi).$$

Proof. In view of (1.10) and Theorem 3, and by similarly applying the method of proof of Theorem 2, we conclude that the assertion of Theorem 4 holds. □

3. A set of integral-preserving properties

In this section, we derive some integral-preserving properties involving two families of integral operators.

Theorem 5. *Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and*

$$\Re(\phi(z)) < \frac{\Re(\nu)-\eta}{p-\eta}, \quad (z \in \mathbb{U};\ \Re(\nu) > p). \tag{3.1}$$

Then the integral operator $F_\nu(f)$ defined by

$$F_\nu(f) := F_\nu(f)(z) = \frac{\nu-p}{z^\nu}\int_0^z t^{\nu-1}f(t)dt, \quad (z \in \mathbb{U};\ \Re(\nu) > p) \tag{3.2}$$

belongs to the class $\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$.

Proof. Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$. Then, from (3.2), we find that

$$z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z) + \nu\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z) = (\nu-p)\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z). \tag{3.3}$$

By setting

$$\mathbb{P}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)} - \eta\right), \tag{3.4}$$

we observe that $\mathbb{P}$ is analytic in $\mathbb{U}$ with $\mathbb{P}(0) = 1$. It follows from (3.3) and (3.4) that

$$-(p-\eta)\mathbb{P}(z) - \eta + \nu = (\nu-p)\frac{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)}. \tag{3.5}$$

Differentiating both sides of (3.5) with respect to z logarithmically and multiplying the resulting equation by z, we get

$$\begin{aligned}&\mathbb{P}(z) + \frac{z\mathbb{P}'(z)}{-(p-\eta)\mathbb{P}(z) - \eta + \nu}\\ &= \frac{1}{p-\eta}\left(-\frac{z(\mathcal{I}^{n,l}_{\lambda,p,\mu}f)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)} - \eta\right) \prec \phi(z), \quad (z \in \mathbb{U}).\end{aligned} \tag{3.6}$$

Since (3.1) holds, an application of Lemma 1 to (3.6) yields

$$\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)} - \eta\right) \prec \phi(z),$$

which implies that the assertion of Theorem 5 holds. □

Theorem 6. *Let $f \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$.*

Proof. By virtue of (1.9) and Theorem 5, we easily find that

$$\begin{aligned} f \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \\ &\Longrightarrow F_\nu\left(-zf'\right) \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \\ &\Longleftrightarrow -z\left(F_\nu(f)\right)' \in \mathcal{MS}^{*}_{p}(\eta;\phi) \\ &\Longleftrightarrow F_\nu(f) \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi). \end{aligned}$$

The proof of Theorem 6 is evidently completed. □

Theorem 7. *Let $f \in \mathcal{MC}^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{MC}^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$.*

Proof. Let $f \in \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that there exists a function $g \in \mathcal{MS}^{*}_{p}(\eta;\phi)$ such that

$$\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}-\eta\right) \prec \psi(z), \quad (z\in\mathbb{U}). \tag{3.7}$$

Since $g \in \mathcal{MS}^{*}_{p}(\eta;\phi)$, by Theorem 5, we easily find that $F_\nu(g) \in \mathcal{MS}^{*}_{p}(\eta;\phi)$, which implies that

$$\mathbb{H}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(g)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(g)(z)}-\eta\right) \prec \phi(z). \tag{3.8}$$

We now set

$$\mathbb{Q}(z) := \frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(g)(z)}-\delta\right), \tag{3.9}$$

where $\mathbb{Q}$ is analytic in $\mathbb{U}$ with $\mathbb{Q}(0)=1$. From (3.3) and (3.9), we get

$$-[(p-\delta)\mathbb{Q}(z)+\delta]\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(g)(z)+\nu\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)=(\nu-p)\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we find that

$$\begin{aligned} &-(p-\delta)z\mathbb{Q}'(z)-[(p-\delta)\mathbb{Q}(z)+\delta][-(p-\eta)\mathbb{H}(z)-\eta+\nu] \\ &=(\nu-p)\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(g)(z)}. \end{aligned} \tag{3.11}$$

By virtue of (1.8), (3.8) and (3.11), we deduce that

$$\begin{aligned}&\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}g(z)}-\delta\right)\\ &=\mathbb{Q}(z)+\frac{z\mathbb{Q}'(z)}{-(p-\eta)\mathbb{H}(z)-\eta+\nu}\prec\psi(z),\quad(z\in\mathbb{U}).\end{aligned}\tag{3.12}$$

The remainder of the proof of Theorem 7 is much akin to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{Q}(z)\prec\psi(z),\quad(z\in\mathbb{U}),$$

which implies that $F_\nu(f)\in\mathcal{MC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$. The proof of Theorem 7 is thus completed. □

Theorem 8. *Let $f\in\mathcal{MQC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$ with $\phi\in\mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{MQC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$.*

Proof. In view of (1.10) and Theorem 7, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 8 holds. □

Theorem 9. *Let $f\in\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$ with $\phi\in\mathcal{P}$ and*

$$\Re\left(\sigma-\eta\,\xi-(p-\eta)\xi\,\phi(z)\right)>0,\quad(z\in\mathbb{U};\ \xi\neq0).\tag{3.13}$$

Then the function $\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)\in\Sigma_p$ defined by

$$\begin{aligned}\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f):&=\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)\\ &=\left(\frac{\sigma-p\,\xi}{z^\sigma}\int_0^z t^{\sigma-1}\left(\mathcal{I}_{\lambda,\mu}^{n}f(t)\right)^\xi dt\right)^{1/\xi},\quad(z\in\mathbb{U}^*;\ \xi\neq0)\end{aligned}\tag{3.14}$$

belongs to the class $\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$.

Proof. Let $f\in\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$ and suppose that

$$\mathbb{M}(z):=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)}-\eta\right),\quad(z\in\mathbb{U}).\tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\sigma-\eta\,\xi-(p-\eta)\xi\,\mathbb{M}(z)=(\sigma-p\,\xi)\left(\frac{\mathcal{I}_{\lambda,p,\mu}^{n,l}f(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)}\right)^\xi.\tag{3.16}$$

Making use of (3.14), (3.15) and (3.16), we get

$$\mathbb{M}(z)+\frac{z\mathbb{M}'(z)}{\sigma-\eta\,\xi-(p-\eta)\xi\,\mathbb{M}(z)} = \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)}-\eta\right)\prec\phi(z),\quad(z\in\mathbb{U}). \tag{3.17}$$

Since (3.13) holds, an application of Lemma 1 to (3.17) yields

$$\mathbb{M}(z)\prec\phi(z),\quad(z\in\mathbb{U}),$$

that is, that $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\in\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$. We thus complete the proof of Theorem 9. □

Theorem 10. *Let $f\in\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$ with $\phi\in\mathcal{P}$ and (3.13) holds. Then the function $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\in\Sigma_p$ defined by (3.14) belongs to the class $\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$.*

Proof. By virtue of (1.9) and Theorem 9, and by similarly applying the method of proof of Theorem 6, we conclude that the assertion of Theorem 10 holds. □

Theorem 11. *Let $f\in\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ with $\phi\in\mathcal{P}$ and (3.13) holds.*

Then the function $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\in\Sigma_p$ defined by (3.14) belongs to the class $\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$.

Proof. Let $f\in\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that there exists a function $g\in\mathcal{MS}^{*}_{p}(\eta;\phi)$ such that (3.7) holds. Since $g\in\mathcal{MS}^{*}_{p}(\eta;\phi)$, by Theorem 9, we easily find that $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)\in\mathcal{MS}^{*}_{p}(\eta;\phi)$, which implies that

$$\mathbb{R}(z):=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)(z)}-\eta\right)\prec\phi(z). \tag{3.18}$$

We now set

$$\mathbb{L}(z):=\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)(z)}-\delta\right), \tag{3.19}$$

where $\mathbb{L}$ is analytic in $\mathbb{U}$ with $\mathbb{L}(0)=1$. From (3.14) and (3.19), we get

$$-\xi[(p-\delta)\mathbb{L}(z)+\delta]\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)(z)+\delta\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)(z)=(\delta-p\,\xi)\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z). \tag{3.20}$$

Combining (3.18), (3.19) and (3.20), we find that

$$\begin{aligned} &-\xi(p-\delta)z\mathbb{L}'(z)-[(p-\delta)\mathbb{L}(z)+\delta][-(p-\eta)\xi\mathbb{R}(z)-\eta\,\xi+\delta] \\ &=(\delta-p\,\xi)\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)(z)}. \end{aligned} \tag{3.21}$$

By virtue of (1.8), (3.18) and (3.21), we deduce that

$$\begin{aligned} &\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}-\delta\right) \\ &=\mathbb{L}(z)+\frac{z\mathbb{L}'(z)}{-(p-\eta)\xi\mathbb{R}(z)-\eta\,\xi+\delta}\prec\psi(z),\quad(z\in\mathbb{U}). \end{aligned} \tag{3.22}$$

The remainder of the proof of Theorem 11 is similar to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{L}(z)\prec\psi(z),\quad(z\in\mathbb{U}),$$

which implies that $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\in\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$. The proof of Theorem 11 is thus completed. □

Theorem 12. *Let $f\in\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ with $\phi\in\mathcal{P}$ and* (3.13) *holds. Then the function $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f)\in\Sigma_p$ defined by* (3.14) *belongs to the class $\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$.*

Proof. By virtue of (1.10) and Theorem 11, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 12 holds. □

4. Subordination and superordination results

Finally, we derive some subordination and superordination results associated with the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$. The proofs are much akin to that of the results obtained by Cho *et al.* [4], we here choose to omit the details involved.

Corollary 1. *Let $f,\ g\in\Sigma_p$ and $l>0$. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)>-\varrho,\quad\left(z\in\mathbb{U};\ \varphi(z):=z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)\right), \tag{4.1}$$

where

$$\varrho:=\frac{l^2+\lambda^2-|l^2-\lambda^2|}{4l\lambda}, \tag{4.2}$$

then the following subordination relationship

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g$ *is the best dominant.*

Corollary 2. *Let* $f,\ g \in \Sigma_p$ *. If*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi, \quad \left(z \in \mathbb{U};\ \chi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)\right),$$

where

$$\varpi := \frac{1+\mu^2-|1-\mu^2|}{4\mu}, \tag{4.3}$$

then the following subordination relationship

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g$ *is the best dominant.*

If f is subordinate to $\mathscr{F}$, then $\mathscr{F}$ is superordinate to f. We now derive the following superordination results.

Corollary 3. *Let* $f,\ g \in \Sigma_p$ *and* $l > 0$*. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho, \quad \left(z \in \mathbb{U};\ \varphi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)\right),$$

where ϱ *is given by* (4.2)*, also let the function* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f$ *is univalent in* $\mathbb{U}$ *and* $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f \in Q$*, then the following subordination relationship*

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g$ *is the best subordinant.*

Corollary 4. *Let* $f,\ g \in \Sigma_p$ *. If*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi, \quad \left(z \in \mathbb{U};\ \chi(z) := z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} g(z)\right),$$

where ϖ *is given by* (4.3)*, also let the function* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f$ *is univalent in* $\mathbb{U}$ *and* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f \in Q$*, then the following subordination relationship*

$$z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} g(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g$ *is the best subordinant.*

Combining the above mentioned subordination and superordination results involving the operator $\mathcal{I}^{n,\,l}_{\lambda,p,\mu}$, we get the following "sandwich-type results".

Corollary 5. *Let* $f,\ g_k \in \Sigma_p\ (k = 1,\, 2)$ *and* $l > 0$*. If*

$$\Re\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\varrho, \quad \left(z \in \mathbb{U};\ \varphi_k(z) := z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_k(z)\ \ (k = 1,\, 2)\right),$$

where ϱ *is given by* (4.2)*, also let the function* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f$ *is univalent in* $\mathbb{U}$ *and* $z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} f \in Q$*, then the subordination relationship*

$$z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_1(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_2(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} g_1(z) \prec z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} f(z) \prec z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} g_2(z), \quad (z \in \mathbb{U}).$$

Furthermore, the functions $z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} g_1$ *and* $z^p \mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} g_2$ *are, respectively, the best subordinant and the best dominant.*

Corollary 6. *Let* $f,\ g_k \in \Sigma_p\ (k = 1,\, 2)$*. If*

$$\Re\left(1+\frac{z\chi_k''(z)}{\chi_k'(z)}\right) > -\varpi, \quad \left(z \in \mathbb{U};\ \chi_k(z) := z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} g_k(z)\ \ (k = 1,\, 2)\right),$$

where ϖ *is given by* (4.3)*, also let the function* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f$ *is univalent in* $\mathbb{U}$ *and* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f \in Q$*, then the subordination relationship*

$$z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} g_1(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} g_2(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_1(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f(z) \prec z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_2(z), \quad (z \in \mathbb{U}).$$

Furthermore, the functions $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_1$ *and* $z^p \mathcal{I}^{n,\,l}_{\lambda,p,\mu} g_2$ *are, respectively, the best subordinant and the best dominant.*

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References

[1] R. M. Ali and V. Ravichandran, *Classes of meromorphic α-convex functions,* Taiwanese J. Math., 14 (2010), 1479–1490.

[2] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations,* J. Math. Anal. Appl., 300 (2004), 505–520.

[3] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations,* Integral Transforms Spec. Funct., 16 (2005), 647–659.

[4] N. E. Cho, O. S. Kwon and H. M. Srivastava, *A class of integral operators preserving subordination and superordination for meromorphic functions*, Appl. Math. Comput., 193 (2007), 463–474.

[5] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination,* in General Mathematics 3, International Series of Numerical Mathematics, Vol. 64, Birkhäuser Verlag, Basel, 1983, 339–348; see also Rev. Roumaine Math. Pures Appl., 29 (1984), 567–573.

[6] R. M. El-Ashwah, A note on certain meromorphic p-valent functions, *Appl. Math. Lett.*, 22 (2009), 1756–1759.

[7] R. M. El-Ashwah and M. K. Aouf, *Hadamard product of certain meromorphic starlike and convex functions,* Comput. Math. Appl. , 57 (2009), 1102–1106.

[8] S. Hussain, Z. Shareef and M. Darus, *On a class of functions associated with Sălăgean operator,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 111 (2017), 213–222.

[9] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions,* Michigan Math. J., 28 (1981), 157–171.

[10] S. S. Miller and P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations,* J. Differential Equations, 56 (1985), 297–309.

[11] M. H. Mohd, R. M. Ali, L. S. Keong and V. Ravichandran, *Subclasses of meromorphic functions associated with convolution*, J. Inequal. Appl., 2009 (2009), Article ID 190291, 1–10.

[12] M. Nunokawa and O. P. Ahuja, *On meromorphic starlike and convex functions*, Indian J. Pure Appl. Math., 32 (2001), 1027–1032.

[13] K. Piejko and J. Sokół, *Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator*, J. Math. Anal. Appl., 337 (2008), 1261–1266.

[14] S. M. Sarangi and S. B. Uralegaddi, *Certain differential operators for meromorphic functions,* Bull. Calcutta Math. Soc., 88 (1996), 333–336.

[15] S. Supramaniam, R. Chandrashekar, S. K. Lee and K. G. Subramanian, *Convexity of functions defined by differential inequalities and integral operators,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 111 (2017), 147–157.

[16] H. M. Srivastava, D.-G. Yang and N. Xu, *Some subclasses of meromorphically multivalent functions associated with a linear operator,* Appl. Math. Comput., 195 (2008), 11–23.

[17] T. Thulasiram, K. Suchithra, T. V. Sudharsan and G. Murugusundaramoorthy, *Some inclusion results associated with certain subclass of analytic functions inolving Hohlov operator,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 108 (2014), 711–720.

[18] B. A. Uralegaddi and C. Somanatha, *Certain differential operators for meromorphic functions,* Houston J. Math., 17 (1991), 279–284.

[19] B. A. Uralegaddi and C. Somanatha, *New criteria for meromorphic starlike univalent functions,* Bull. Austral. Math. Soc., 43 (1991), 137–140.

[20] Z.-G. Wang, Y.-P. Jiang and H. M. Srivastava, *Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function,* Comput. Math. Appl., 57 (2009), 571–586.

[21] Z.-G. Wang, Y. Sun and Z.-H. Zhang, *Certain classes of meromorphic multivalent functions,* Comput. Math. Appl., 58 (2009), 1408–1417.

[22] S.-M. Yuan, Z.-M. Liu and H. M. Srivastava, *Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators*, J. Math. Anal. Appl., 337 (2008), 505–515.

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Some types of filters and states on hyper NM-algebras

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Abstract. In this paper, we apply the hyper structures theory to NM-algebras and define the notion of hyper NM-algebras, which is a generalization of NM-algebras, and obtain some related results. We also state that any hyper NM-algebra is a hyper residuated lattice and the converse may not hold. Furthermore, we put forth some types of filters and deductive systems of hyper NM-algebras, such as (weak)h-filters, weak h-deductive systems and (positive) implicative weak h-deductive systems. Especially, we focus on discussing relationships of them. Finally, we give the definitions of sup-state, inf-state and hyper state on hyper NM-algebras and obtain there exists a hyper NM-algebra with sup-state and inf-state.

Keywords: hyper NM-algebra, weak h-filter, weak h-deductive system, sup-state, inf-state.

1. Introduction

In past several years, the study of fuzzy logic and fuzzy reasoning has greatly increased. Some scholars obtained many interesting results on t-norm based fuzzy logics. Hájek's basic logic system BL is an ideal formalization or logical framework of continuous t-norms and their residua [13]. Based on Hájek's work, Esteva and Godo proposed a new formal deductive system MTL, called monoidal t-norm based logic, intended to cope with left-continuous t-norms and

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their residua [9]. As important schematic extensions of MTL, they gave weak nilpotent minimum logic WNM, the involutive monoidal t-norm based logic IMTL and nilpotent minimum logic NM. The system NM is a common schematic extension of WNM and IMTL which is an ideal formalization of Fodor's nilpotent minimums. We know that nilpotent minimums are important t-norms and possess many excellent properties which are proposed by Fodor in 1995 [10].

In order to provide a logic foundation for fuzzy reasoning and to reduce the gap between fuzzy reasoning and artificial intelligence, a new formal deductive system $\mathcal{L}^*$ for fuzzy propositional calculus is proposed by G. J. Wang [22]. D. W. Pei introduced R_0-algebras as the algebraic completeness of $\mathcal{L}^*$. He also proved the standard completeness of $\mathcal{L}^*$ with respect to R_0-algebras [21], and nilpotent minimum algebras are equivalent to R_0-algebras [20]. Consequently, NM-algebras play an important role in studying fuzzy logics and the corresponding algebraic structures.

The hyper structure theory (called also multialgebra) was introduced by F. Marty in 1934 at 8th congress of Scandinavian Mathematicians [18]. In his definition, a function $f : A \times A \to P^*(A)$, of the set $A \times A$ into the set of all nonempty subsets of A, is called a binary hyperoperation, and the pair (A, f) is called a hypergroupoid. If f is associative, A is called a semihypergroup, and it is said to be commutative if f is commutative. Also, an element $1 \in A$ is called the unit or the neutral element if $a \in f(1, a)$, for all $a \in A$. Since then many researchers have worked on this area [1, 6, 7]. Zahiri et al. applied the hyperstructures to lattices and residuated lattice [2, 27]. Borzooei et al. introduced and studied hyper K-algebras [5]. Ghorbani et al. applied the hyperstructures to MV-algebras, named hyper MV-algebras which are a generalization of MV-algebras, and defined some new types of deductive systems on hyper MV-algebras, such as (weak)hyper MV-deductive systems and (weak) implicative hyper MV-deductive systems, and studied their relations [11, 12, 15, 16]. Xin et al. considered the concepts of states, state operators and state-morphism operators on hyper BCI/BCK-algebras [3, 4, 14, 23, 24, 25].

Dvurečenskij et al.[8] introduced hyper effect algebras as a generalization of effect algebras and presented basic notions like states on hyper effect algebras. They also provided a representation of any finite linearly ordered hyper effect algebra. Xin et al.[26] defined the concept of hyper BL-algebras which is a generalization of BL-algebras. In particular, they defined the concept of regular compatible congruence on hyper BL-algebras and construct the quotient structure in hyper BL-algebras. Finally, they discussed the conditions in which a quotient hyper BL-algebra is an MV-algebra.

At present, the research papers about hyper NM-algebras have not been proposed. Hence, in order to enrich the theories of hyper structures, it is meaningful to construct hyper NM-algebras as a generalization of the concept of NM-algebras. We show that hyper NM-algebras are more complicated structures than NM-algebras, because in general, they are not necessarily transitive.

In NM-algebras, the operation of two elements is an element, while in hyper NM-algebras the operation of two elements is a subset.

This paper is structured in five sections. In order to make the paper as self-contained as possible, in Section 2, we recapitulate some basic notions and some results of some hyperstructures. In Section 3, we exhibit an axiom system of hyper NM-algebras and study some properties of them. Also, we state that any hyper NM-algebra is hyper residuate lattice. In Section 4, we investigate and prove some propositions about (weak) h-filters and weak h-deductive systems on hyper NM-algebras. In Section 5, we give three types of states on hyper NM-algebras and study some basic properties of them.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the following sections.

Definition 2.1 ([9]). *An algebra* $(L, \wedge, \vee, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *is called an NM-algebra if it satisfies the following conditions: for all* $x, y, z \in L$,

(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(2) $(L, \odot, 1)$ *is a commutative monoid,*
(3) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,
(4) $(x \to y) \vee (y \to x) = 1$,
(5) $((x \odot y) \to 0) \vee ((x \wedge y) \to (x \odot y)) = 1$,
(6) $(x \to 0) \to 0 = x$.

R_0-algebras were introduced by Prof.Wang as an algebraic structure counterpart of the formal logic $\mathcal{L}^*$ in 1996.

Definition 2.2 ([22]). *Let* M *be an algebra of type* $(\neg, \vee, \to)$. *If* $(M, \leq, 0, 1)$ *is a bounded distributive lattice with a partial order* $\leq$ *(0 and 1 are the least element and the greatest element of* M *with respect to* $\leq$, *respectively),* $\vee$ *is the supermum operator, and* $\neg$ *is an order-reserving involution, then* M *is called an* R_0*-algebra, if the following conditions are satisfied: for all* $x, y, z \in M$,

$(M1)$ $\neg x \to \neg y = y \to x$,
$(M2)$ $1 \to x = x, x \to x = 1$,
$(M3)$ $y \to z \leq (x \to y) \to (x \to z)$,
$(M4)$ $x \to (y \to z) = y \to (x \to z)$,
$(M5)$ $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
$(M6)$ $(x \to y) \wedge ((x \to y) \to (\neg x \vee y)) = 1$.

Remark 2.3. Let L be an NM-algebras. For all $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$. Then R_0-algebras are equivalent to NM-algebras in [20]. Therefore, they have similar properties.

Proposition 2.4 ([22]). *Let* L *be an* R_0*-algebra. The following properties hold: for all* $x, y, z \in L$,

(1) $x \leq y$ *if and only if* $x \to y = 1$,
(2) $x \leq y \to x$,
(3) $\neg x = x \to 0$,
(4) $x \leq y$ *implies* $y \to z \leq x \to z$,
(5) $x \leq y$ *implies* $z \to x \leq z \to y$,
(6) $((x \to y) \to y) \to y = x \to y$,
(7) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(8) $x \odot \neg x = 0$,
(9) $x \odot y \leq x \wedge y$ *and* $x \odot (x \to y) \leq x \wedge y$,
(10) $(x \odot y) \to z = x \to (y \to z)$,
(11) $x \leq y \to (x \odot y)$,
(12) $x \leq y$ *implies* $x \odot z \leq y \odot z$,
(13) $x \to y \leq (y \to z) \to (x \to z)$,
(14) $(x \to y) \odot (y \to z) \leq x \to z$,
(15) $x \to y = x \to (x \wedge y)$,
(16) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Definition 2.5 ([18]). *Let H be a nonempty set and "$\circ$" be a function from H^2 to $P(H) \setminus \{\emptyset\}$. Then "$\circ$" is called a hyper operation on H.*

Note that for any two subsets A and B of H, denote the set $\bigcup_{a \in A, b \in B} \{a \circ b\}$ by $A \circ B$. We use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

In a classical algebraic structure, the operation of two elements is an element, while in an algebraic hyper structure the operation of two elements is a subset.

Definition 2.6 ([19]). *A super lattice is a partially ordered set $(S, \leq)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $x, y, z \in S$,*
$(SL1)$ $x \in (x \vee x) \cap (x \wedge x)$,
$(SL2)$ $x \vee y = y \vee x$, $x \wedge y = y \wedge x$,
$(SL3)$ $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
$(SL4)$ $x \in ((x \vee y) \wedge x) \cap ((x \wedge y) \vee x)$,
$(SL5)$ $x \leq y$ *implies* $y \in x \vee y$ *and* $x \in x \wedge y$,
$(SL6)$ *if* $x \in x \wedge y$ *or* $y \in x \vee y$, *then* $x \leq y$.

Proposition 2.7 ([19]). *Let $(L, \leq)$ be a partially ordered set with the least element 0 and the largest element 1. Define two binary hyperoperations $\vee$ and $\wedge$ on L as follows: $a \vee b = \{c | a \leq c, b \leq c\}$ and $a \wedge b = \{c | c \leq a, c \leq b\}$, for all $a, b \in L$. Then $(L, \vee, \wedge)$ is a bounded super lattice.*

Definition 2.8 ([1]). *Let A be a set, $\odot$ be a binary hyperoperation on A and $1 \in A$. $(A, \odot, 1)$ is called a commutative semihypergroup with 1 as an identity if it satisfies the following properties: for all $x, y, z \in A$,*
$(CSHG1)$ $x \odot (y \odot z) = (x \odot y) \odot z$,
$(CSHG2)$ $x \odot y = y \odot x$,

$(CSHG3)$ $x \in 1 \odot x$.

Moreover if $a \in A$ such that $|a \odot x| = 1$ for all $x \in A$, a is called a scalar element of A, simply a scalar of A.

Definition 2.9 ([27]). *By a hyper residuated lattice, we mean a nonempty set L endowed with four binary hyperoperations $\vee, \wedge, \odot, \to$ and two constants 0 and 1 satisfying the following conditions: for all $x, y, z \in L$,*

$(HRL1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded super lattice,*

$(HRL2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with 1 as an identity,*

$(HRL3)$ $x \odot z \ll y$ *if and only if* $z \ll x \to y$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all nonempty subsets A and B of L.

Definition 2.10 ([17]). *Let L be an R_0-algebra. A state on L is a function $s : L \to [0, 1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s(x \to y) = s(y) + s(y \to x)$.

Proposition 2.11 ([17]). *Let s be a state on R_0-algebra L. Then the following properties hold:*

(1) *if $x \leq y$, then $s(x) \leq s(y)$,*

(2) $s(\neg x) = 1 - s(x)$,

(3) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$,

(4) *if $x \leq \neg y$, then $s(x \oplus y) = s(x) + s(y)$,*

(5) *if $x = \neg x$, then $s(x) = \frac{1}{2}$,*

(6) *$ker(s) = \{x : s(x) = 1\}$ is a proper filter of L.*

Proposition 2.12 ([17]). *Let L be an R_0-algebra and $s : L \to [0, 1]$ be a function. Then s is a state if and only if it satisfies Definition 2.10(1) and Proposition 2.11(3).*

Thanks to R_0-algebras that have been shown to be equivalent to NM-algebras, we can get that the properties of state on NM-algebras are similar to R_0-algebras. In section 5, we will not repeat to introduce the properties of state on NM-algebras.

3. Hyper NM-algebras

In this section, we will introduce the notion and some properties of hyper NM-algebras.

Definition 3.1. *By a hyper NM-algebra we mean a nonempty set L endowed with four binary hyperoperations $\vee, \wedge, \odot, \to$ and two constants 0 and 1 satisfying the following conditions: for all $x, y, z \in L$,*

$(HNM1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded super lattice,*

$(HNM2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with 1 as an identity,*

$(HNM3)$ $x \odot y \ll z$ *if and only if* $x \ll y \to z$,
$(HNM4)$ $1 \in (x \to y) \vee (y \to x)$,
$(HNM5)$ $1 \in ((x \odot y) \to 0) \vee ((x \wedge y) \to (x \odot y))$,
$(HNM6)$ $x \in (x \to 0) \to 0$,
where $A \ll B$ *means that there exist* $a \in A$ *and* $b \in B$ *such that* $a \leq b$, *for all nonempty subsets* A *and* B *of* L.

In the following, we give some examples of hyper NM-algebras.

Example 3.2. Let $L = \{0, a, b, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < b < 1$. Consider the following tables:

$\vee$	0	a	b	1
0	{0, a, b, 1}	{a, b, 1}	{b, 1}	{1}
a	{a, b, 1}	{a, b, 1}	{b, 1}	{1}
b	{b, 1}	{b, 1}	{b, 1}	{1}
1	{1}	{1}	{1}	{1}

$\wedge$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0, a}	{0, a}	{0, a}
b	{0}	{0, a}	{0, a, b}	{0, a, b}
1	{0}	{0, a}	{0, a, b}	{0, a, b, 1}

$\to$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{a, b, 1}	{a, 1}	{a, b, 1}	{1}
b	{a, b, 1}	{a, b, 1}	{b, 1}	{1}
1	{0, a, b, 1}	{a, b, 1}	{a, b, 1}	{0, 1}

Let $\odot = \wedge$. We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Example 3.3. Let $L = [0, 1]$. Define a unary operation "$\neg$" on L by $\neg x = 1 - x$, and four binary hyperoperations $\vee, \wedge, \odot$ and $\to$ as follows: $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x \to y = \begin{cases} 1, & \text{x} \leq \text{y} \\ \neg x \vee y, & \text{x} > \text{y} \end{cases}$, and $x \odot y = [\neg(x \to \neg y), 1]$, for all $x, y \in L$. Then we can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Example 3.4. Let $L = [0, 1]$. Define a unary operation "$\neg$" by $\neg x = 1 - x$, and four binary hyperoperations $\vee, \wedge, \odot$ and $\to$ as follows: $x \vee y = \{z \mid x \leq z, y \leq z\}$, $x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}$, and $x \to y = \neg x \vee y$, for all $x, y \in L$. We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Proposition 3.5. (1) *Let* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *be an NM-algebra. We define* $x \bar{\circ} y = \{x \circ y\}$, *for any* $\circ \in \{\vee, \wedge, \odot, \to\}$. *Then* $(L, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\to}, 0, 1)$ *is a hyper NM-algebra.*

(2) *Let* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *be a hyper NM-algebra satisfying that four hyperoperations* $\vee, \wedge, \odot, \to$ *are all binary operations. Then* L *is an NM-algebra.*

(3) *Any hyper NM-algebra is a hyper residuated lattice.*

Proof. (1) Straightforward.

(2) Let L be a hyper NM-algebra satisfying that all binary hyperoperations are binary operations. Then by (HNM1)-(HNM3), we have L is a residuated lattice. By (HNM4)-(HNM6), we can get $1 = (x \to y) \vee (y \to x)$, $1 = ((x \odot y) \to 0) \vee ((x \wedge y) \to (x \odot y))$, $x = (x \to 0) \to 0$, respectively. Therefore, L is an NM-algebra.

(3) Straightforward. □

Remark 3.6. (1) From Proposition 3.5(1), we have that any NM-algebra can be seen as a hyper NM-algebra. And from Proposition 3.5(2), we know that the notion of hyper NM-algebras is a generalization of the notion of NM-algebras.

(2) From Proposition 3.5(3), we know that any hyper NM-algebra is a hyper residuated lattice. L is a residuated lattice while all hyperoperations of a hyper NM-algebra L are binary operations.

In the following example we show that any hyper NM-algebra is hyper residuated lattice, but the converse may be not.

Example 3.7. Let $L = \{0, a, b, c, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < c < a < b < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ on L as follows: $x \vee y = \{z \mid x \leq z, y \leq z\}$ and $x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}$, for all $x, y \in L$. Now, let $\to$ be a hyperoperation on L defined by the following table.

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0, 1}	{1}	{1}	{c, 1}	{1}
b	{0, 1}	{a, b, 1}	{1}	{c, 1}	{1}
c	{0, 1}	{1}	{1}	{1}	{1}
1	{0, 1}	{a, b, 1}	{b, 1}	{c, 1 }	{1}

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice ([27]). But it isn't a hyper NM-algebra, since $a \notin (a \to 0) \to 0$.

Now, we characterize some basic properties of hyper NM-algebras.

Proposition 3.8. *Let* L *be a hyper NM-algebra. Then the following properties hold: for any* $x, y, z \in L$ *and for any nonempty subsets* A, B, C *of* L*,*

(1) $1 \ll A$ *implies* $1 \in A$*,* $A \ll 0$ *implies* $0 \in A$*,*

(2) $A \odot B \ll C$ *if and only if* $A \ll B \to C$*,*

(3) $A \ll x \ll B$ *implies* $A \ll B$*,*
if $A \cap B \neq \emptyset$*, then* $A \ll B$ *and* $B \ll A$*,*

(4) $x \odot (x \to y) \ll y$*,* $x \ll y \to (x \odot y)$*,*

(5) $x \ll (x \to y) \to y$, $A \ll (A \to B) \to B$,

(6) $x \to y \ll ((x \to y) \to y) \to y$,

(7) $x \leq y$ *implies* $1 \in x \to y$, *and if* 1 *is a scalar element of* L, *the converse holds,*

$A \ll B$ *implies* $1 \in A \to B$,

(8) $1 \in x \to x$, $1 \in x \to 1$, *and* $1 \in 0 \to x$,

(9) *if* 1 *is a scalar element, then* $x \in 1 \to x$ *and* x *is the greatest element of* $1 \to x$,

(10) $(x \odot y) \to z \ll x \to (y \to z)$,

$(x \odot y) \to z \ll y \to (x \to z)$,

(11) $x \to (y \to z) \ll (x \odot y) \to z$,

$y \to (x \to z) \ll (x \odot y) \to z$,

(12) $x \to (y \to z) \ll y \to (x \to z)$,

(13) $x \leq y$ *implies* $x \odot z \ll y \odot z$, $z \to x \ll z \to y$, *and* $y \to z \ll x \to z$,

(14) $x \leq y$ *implies* $y^- \ll x^-$, *where* $x^- = x \to 0$,

(15) $x \odot y \ll x$, $x \odot y \ll y$. *Particularly,* $0 \in x \odot 0$,

(16) $x \ll y \to x$, $1 \in x \to (y \to x)$,

(17) $y \ll (x \to y) \to y$, $B \ll (A \to B) \to B$,

(18) $0 \in x \odot x^-$,

(19) $x \leq y$ *and* $x \leq z$ *imply* $x \ll y \wedge z$,

$y \leq x$ *and* $z \leq x$ *imply* $y \vee z \ll x$,

(20) $x \wedge y \ll x$, $x \ll x \vee y$, $A \wedge B \ll A$, $A \ll A \vee B$,

(21) $x \to (x \wedge y) \ll x \to y$,

(22) $x \ll y^- \Leftrightarrow 0 \in x \odot y$.

Proof. (1) If $1 \ll A$, then $1 \leq a$, for some $a \in A$. Hence $1 = a \in A$. If $A \ll 0$, then $a \leq 0$, for some $a \in A$. Hence $0 = a \in A$.

(2) Let $A \ll B \to C$. Then there exist $a \in A$, $b \in B$, and $c \in C$ such that $a \ll b \to c$. Therefore $a \odot b \ll c$. This means $A \odot B \ll C$. Conversely, let $A \odot B \ll C$. Then $a \odot b \ll c$, for some $a \in A$, $b \in B$, and $c \in C$. Therefore $a \ll b \to c$. We get that $A \ll B \to C$.

(3) Straightforward.

(4) It follows from $x \to y \ll x \to y$ and $x \odot y \ll x \odot y$ that $x \odot (x \to y) \ll y$ and $x \ll y \to (x \odot y)$.

(5) It is clear by (2) and (4).

(6) It is clear by (5).

(7) Let $x \leq y$. Note that $x \in 1 \odot x$. Hence $1 \odot x \ll x$. Thus $1 \odot x \ll y$, $1 \ll x \to y$. Therefore $1 \in x \to y$. Let $1 \in x \to y$. Then $1 \ll x \to y$ and $1 \odot x \ll y$ by (HNM3). By (CSHG3) $x \in 1 \odot x$ and 1 is a scalar, we have $x = 1 \odot x$. Therefore $x \leq y$. Moreover, let $A \ll B$. Then $a \leq b$ for some $a \in A, b \in B$. Hence $1 \in a \to b \subseteq A \to B$.

(8) It is clear by (7).

(9) For any $u \in 1 \to x$, we have $u \ll 1 \to x$. So $\{u\} = u \odot 1 \ll x$. Thus $u \leq x$, and x is the greatest element of $1 \to x$. Since $x \in x \odot 1$, we get $x \odot 1 \ll x$.

Hence $x \ll 1 \to x$. There exists $u \in 1 \to x$ such that $x \leq u$. So $x \leq u \leq x$. Therefore, $x \in 1 \to x$.

(10) Let $u \in (x \odot y) \to z$. Then $u \ll (x \odot y) \to z \Leftrightarrow u \odot (x \odot y) \ll z \Leftrightarrow (u \odot x) \odot y \ll z \Leftrightarrow u \odot x \ll y \to z \Leftrightarrow u \ll x \to (y \to z)$. So $(x \odot y) \to z \ll x \to (y \to z)$. Moreover, we have $(x \odot y) \to z = (y \odot x) \to z \ll y \to (x \to z)$.

(11) Let $t \in x \to (y \to z)$. Then there exixts $u \in y \to z$ such that $t \in x \to u$. Hence we have $t \ll x \to u$, $x \odot t \ll u$. Thus $x \odot t \ll y \to z$. Hence $s \ll y \to z$ for some $s \in x \odot t$. By (HNM3), we have $s \odot y \ll z$. It is clear that $(x \odot t) \odot y \ll z$ or $(x \odot y) \odot t \ll z$. Then $v \odot t \ll z$ for some $v \in x \odot y$. By (HNM3) again we get $t \ll v \to z$. Thus $t \ll (x \odot y) \to z$. This shows that $x \to (y \to z) \ll (x \odot y) \to z$.

Similarly, we have $y \to (x \to z) \ll (x \odot y) \to z$.

(12) Let $u \in x \to (y \to z)$. Then $u \ll x \to (y \to z)$. We get $u \ll x \to a$, for some $a \in y \to z$. Hence $u \odot x \ll a$. So $b \leq a$, for some $b \in u \odot x$. Since $a \in y \to z$, we get $b \ll y \to z$ and $b \odot y \ll z$. Hence $(u \odot x) \odot y = (u \odot y) \odot x \ll z$. Hence $u \odot y \ll x \to z$. Therefore, $u \ll y \to (x \to z)$. Then $x \to (y \to z) \ll y \to (x \to z)$.

(13) Since $y \ll z \to (y \odot z)$ and $x \leq y$, we have $x \ll z \to (y \odot z)$. Hence we get $x \odot z \ll y \odot z$.

Let $u \in z \to x$. Then $u \ll z \to x$ and $u \odot z \ll x$. Since $x \leq y$, we get $u \odot z \ll y$ and $u \ll z \to y$. Therefore, $z \to x \ll z \to y$.

Let $u \in y \to z$. Then $u \ll y \to z$ and $y \ll u \to z$. Since $x \leq y$, we get that $x \ll u \to z$. Hence we get $u \ll x \to z$. Therefore $y \to z \ll x \to z$.

(14) It is clear by (13).

(15) Since $y \leq 1 \in x \to x$, we get $y \ll x \to x$. So $x \odot y \ll x$. Similarly, we have $x \odot y \ll y$. Particularly, $x \odot 0 \ll 0$. Therefore $0 \in x \odot 0$.

(16) Since $x \odot y \ll x$, we get $x \ll y \to x$. Then $x \leq u$ for some $u \in y \to x$. It follows from (7) that $1 \in x \to u$. This shows that $1 \in x \to (y \to x)$.

(17) Since $y \odot (x \to y) \ll y$, we get $y \ll (x \to y) \to y$. Therefore $B \ll (A \to B) \to B$.

(18) Since $x^- \ll x^-$, we have $x \odot x^- \ll 0$. Then $0 \in x \odot x^-$.

(19) Let $x \leq y$ and $x \leq z$. Then $x \in x \wedge y$ and $x \in x \wedge z$ by (SL5). Thus $x \in (x \wedge y) \wedge z = x \wedge (y \wedge z)$. This shows that there exists $a \in y \wedge z$ such that $x \in x \wedge a$. It follows from (SL6) that $x \leq a$. Hence $x \ll y \wedge z$. Similarly we can prove that $y \leq x$ and $z \leq x$ imply $y \vee z \ll x$.

(20) From the properties of super lattices, it is known that $x \wedge y \ll x$ and $x \ll x \vee y$. Since $a \wedge b \ll a$ for any $a \in A, b \in B$, we have $A \wedge B \ll A$. Similarly, $a \ll a \vee b$, for any $a \in A, b \in B$, implies $A \ll A \vee B$.

(21) It is clear by (13) and (20).

(22) Note that $x \ll y^- = y \to 0 \Leftrightarrow x \odot y \ll 0 \Leftrightarrow 0 \in x \odot y$. □

4. Hyper filters on hyper NM-algebras

In this section we set up the theory of hyper filters on hyper NM-algebras.

Definition 4.1. *Let L be a hyper NM-algebra. A nonempty subset F of L satisfying condition*

$$(F) \quad x \leqslant y \text{ and } x \in F \text{ imply } y \in F, \text{for all } x, y \in L,$$

is called

(1) *an h-filter of L if it satisfies: (HF) $x \odot y \subseteq F$, for all $x, y \in F$,*

(2) *a weak h-filter of L if it satisfies: (WHF) $F \ll x \odot y$, for all $x, y \in F$.*

Definition 4.2. *Let D be a nonempty subset of a hyper NM-algebra L satisfying conditions:*

$$\begin{cases} (DS) & 1 \in D, \\ (WDS) & x \in D \text{ and } D \ll x \to y \text{ imply } y \in D. \end{cases}$$

Then D is called a weak h-deductive system.

Remark 4.3. It is easy to see that any h-filter of hyper NM-algebra L is a weak h-filter. Moreover, for any (weak) h-filter F of L, we know that $1 \in F$.

Proposition 4.4. *Let L be a hyper NM-algebra. Then the following properties hold:*

(1) *every weak h-deductive system satisfies (F),*

(2) *if D is a non-empty subset of L containing 1, then D is a weak h-deductive system of L if and only if D satisfies the following condition,*

$$(D) \quad (x \to y) \cap D \neq \emptyset \text{ and } x \in D \text{ imply } y \in D, \text{ for all } x, y \in L,$$

(3) *every weak h-deductive system is a weak h-filter,*

(4) *$\{1\}$ is a weak h-filter of L,*

(5) *if 1 is scalar, then $\{1\}$ is a weak h-deductive system of L.*

Proof. (1) Let F be a weak h-deductive system of L, $x \leq y$ and $x \in F$, for all $x, y \in L$. Then by Proposition 3.8(7), we get $1 \in x \to y$. So $F \ll x \to y$. Now, it follows from (WDS) that $y \in F$. Therefore, F satisfies condition (F).

(2) ($\Rightarrow$) Let D be a weak h-deductive system of L. Then $1 \in D$ by (DS). Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then there exists $a \in (x \to y) \cap D$. So $D \ll x \to y$. By (WDS) we have $y \in D$.

($\Leftarrow$) Let D be a non-empty subset of L containing 1 and satisfying (D). First we prove that D satisfies (F). Let $x \leq y$ and $x \in D$. Then by Proposition 3.8(7), we get $1 \in x \to y$. So $(x \to y) \cap D \neq \emptyset$. Therefore $y \in D$ by (D). This means that (F) holds. Now, let $x \in D$ and $D \ll x \to y$. Then there exist $d \in D$ and $u \in x \to y$ such that $d \leq u$. So by (F), we get $u \in D$. Hence $(x \to y) \cap D \neq \emptyset$ and $y \in D$. Therefore, D is a weak h-deductive system of L.

(3) Let F be a weak h-deductive system of L. Then (F) holds by (1). Now, let $x, y \in F$. By Proposition 3.8(4) $y \ll x \to (x \odot y)$, we get $y \leq u$ for some

$u \in x \to (x \odot y)$. Hence $u \in F$. So $F \ll x \to v$ for some $v \in x \odot y$. Since $x \in F$, we have $v \in F$. Therefore $F \ll x \odot y$.

(4) Straightforward.

(5) Let 1 be scalar and $D = \{1\}$. Clearly (DS) is true. Assume $x \in D$ and $D \ll x \to y$. Then $x = 1$ and $1 \ll x \to y$. Therefore $1 \in x \to y$. By Proposition 3.8(7), we have $x \leq y$. Hence $y = 1$. This shows that $y \in D$, that is, (WDS) holds. □

The following example shows that the converse of Proposition 4.4 (3) may not hold.

Example 4.5. Consider the hyper NM-algebra L given in Example 3.2. Let $F = \{b, 1\}$. It is easy to see that F is a weak h-filter of L but F is not a weak h-deductive system since $b \in F, F \ll b \to a$ but $a \notin F$.

In general, if 1 is not a scalar element of a hyper NM-algebra L, then $\{1\}$ may be not a weak h-deductive system of L. We give the following counter example.

Example 4.6. Consider the hyper NM-algebra L given in Example 3.2, in which 1 is not a scalar element of L. Then $D = \{1\}$ is not a weak h-deductive system of L since $1 \in D$ and $D \ll 1 \to b$ but $b \notin D$.

Proposition 4.7. *Let $\{F_i \mid i \in I\}$ be a family of nonempty subsets of a hyper NM-algebra L. Then the following properties hold,*

(1) *if F_i is a weak h-deductive system, for all $i \in I$, then $\bigcap_{i\in I} F_i$ is a weak h-deductive system of L,*

(2) *let $\{F_i \mid i \in I\}$ be a chain. if F_i is a weak h-filter (weak h-deductive system), for all $i \in I$, then $\bigcup_{i\in I} F_i$ is a weak h-filter (weak h-deductive system) of L.*

Proof. Straightforward.

□

Recall that a weak h-filter F (weak h-deductive system D) is called proper if $F \neq L$ $(D \neq L)$.

Definition 4.8. *Let F be a proper weak h-filter (weak h-deductive system) of a hyper NM-algebra L. Then F is said to be maximal, if $F \subseteq J \subseteq L$ implies $F = J$ or $J = L$, for all weak h-filters (weak h-deductive systems) J of L.*

Proposition 4.9. *Let L be a hyper NM-algebra. Then every proper weak h-filter (weak h-deductive system) of L is contained in a maximal weak h-filter (weak h-deductive system) of L.*

Proof. Let F be a proper weak h-filter of L and S be the collection of all proper weak h-filter of L containing F. Then $F \in S$ and $(S, \subseteq)$ is a poset. Let $\{F_i \mid i \in I\}$ be a chain in S. Then $\bigcup_{i\in I} F_i$ is a weak h-filter of L containing F. If

$0 \in \bigcup_{i\in I} F_i$, then there exists $i \in I$ such that $0 \in F_i$, which is impossible. Hence, $\bigcup_{i\in I} F_i$ is a proper weak h-filter of L containing F and $\bigcup_{i\in I} F_i \in S$. Hence, every chain of elements of S has an upper bounded in S. By Zorn's lemma, S has a maximal element such as M. We shall show that M is a maximal weak h-filter of L. Let $M \subseteq J \subseteq L$, for some weak h-filter J of L. If $J \neq L$, then $J \in S$. Since M is a maximal element of S we get $M = J$. Therefore, M is a maximal weak h-filter of L.

It is similar that the proof of every proper weak h-deductive system of L is contained in a maximal weak h-deductive system of L. □

Recall that a hyper NM-algebra L is called nontrivial, if $L \neq \{1\}$. Using Propositions 4.4 and 4.9, we have the following corollaries.

Corollary 4.10. *Every nontrivial hyper NM-algebra has a maximal weak h-filter.*

Corollary 4.11. *If 1 is a scalar element, every nontrivial hyper NM-algebra has a maximal weak h-deductive system.*

Definition 4.12. *Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a hyper NM-algebra and D be a nonempty subset of L containing 1. Then D is called*

(1) *an implicative weak h-deductive system (IWDS for short) if $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$ imply $(x \to z) \cap D \neq \emptyset$, for all $x, y, z \in L$,*

(2) *a positive implicative weak h-deductive system (PIWDS for short) if $(x \to ((y \to z) \to y)) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$ for all $x, y, z \in L$.*

Example 4.13. Consider the hyper NM-algebra L given in Example 3.2. Let $D = \{a, b, 1\}$. Then D is an IWDS but not a PIWDS. Since $a \in D$ and $(a \to ((0 \to b) \to 0)) \cap D = \{a, b, 1\} \cap D \neq \emptyset$ but $0 \notin D$.

Open Problem: Is there a PIWDS which is not IWDS?

Proposition 4.14. *Let D be a nonempty subset of a hyper NM-algebra L. Then*

(1) if D is an $IWDS$ of L and an upset, and 1 is scalar, then D is a weak h-deductive system,

(2) if D is a $PIWDS$ of L and 1 is scalar, then D is a weak h-deductive system.

Proof. (1) It is clear that $1 \in D$. Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then $(1 \to (x \to y)) \cap D \neq \emptyset$ and $(1 \to x) \cap D \neq \emptyset$ by Proposition 3.8(9). Since D is a $IWDS$ of L, we get $(1 \to y) \cap D \neq \emptyset$. Hence there exists $a \in D$ and $a \in 1 \to y$ such that $a \ll 1 \to y$. So $1 \odot a \ll y$. Since 1 is scalar, $a = 1 \odot a \ll y$. Note that D is an upset, then we get $y \in D$. Therefore, D is a weak h-deductive system.

(2) It is clear that $1 \in D$. Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then by Proposition 3.8(8) and Proposition 3.8(9), $x \to y \subseteq x \to (1 \to y) \subseteq x \to ((y \to 1) \to y)$. So $(x \to ((y \to 1) \to y)) \cap D \neq \emptyset$. Since $x \in D$ and D is a $PIWDS$ of L, we conclude that $y \in D$. Therefore D is a weak h-deductive system by Proposition 4.4(2). □

Proposition 4.15. *Let D be a non-empty subset of a hyper NM-algebra L. Then the following properties hold,*

(1) *if 1 is a scalar, then D is a PIWDS of L if and only if D is a weak h-deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$,*

(2) *D is an IWDS of L if and only if $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ is a weak h-deductive system of L, for all $x \in L$.*

Proof. (1) Let D be a PIWDS. Then by Proposition 4.14(2), D is a weak h-deductive system. Now, let $((x \to y) \to x) \cap D \neq \emptyset$. Then there exists $u \in ((x \to y) \to x) \cap D$. It follows from 1 is scalar and Proposition 3.8(9) that $u \in 1 \to u \subseteq (1 \to ((x \to y) \to x))$. Hence $(1 \to ((x \to y) \to x)) \cap D \neq \emptyset$. Since $1 \in D$ and D is a $PIWDS$, we get $x \in D$.

Conversely, let $(x \to ((y \to z) \to y)) \cap D \neq \emptyset$ and $x \in D$. Since D is a weak h-deductive system and $x \in D$, we get $(y \to z) \to y)) \cap D \neq \emptyset$. So $y \in D$. Therefore D is a PIWDS.

(2) Let D be an IWDS of L and $x \in L$. By Proposition 3.8(8), $1 \in D_x$. Now, let $(a \to b) \cap D_x \neq \emptyset$ and $a \in D_x$, for some $a, b \in L$. Then $(x \to a) \cap D \neq \emptyset$ and $(x \to (a \to b)) \cap D \neq \emptyset$. Since D is a $IWDS$, we get $(x \to b) \cap D \neq \emptyset$. So $b \in D_x$. Hence D_x satisfies (D) and D_x is a weak h-deductive system.

Conversely, let $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ be a weak h-deductive system of L, for all $x \in L$. If $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$, for $x, y, z \in L$, then $y \in D_x$ and $(y \to z) \cap D_x \neq \emptyset$. Since D_x is a weak h-deductive system of L, we get that $z \in D_x$. So $(x \to z) \cap D \neq \emptyset$. Therefore, D is an IWDS of L. □

Theorem 4.16. *Let D be a non-empty subset of a hyper NM-algebra L and 1 is a scalar. Then the following statements are equivalent:*

(1) *D is an IWDS and a maximal weak h-deductive system of L, for all $x, y \in L \setminus D$,*

(2) *D is a weak h-deductive system and $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$, for all $x, y \in L \setminus D$.*

Proof. (1)⇒(2) Let D be an IWDS and a maximal weak h-deductive system of L. By Proposition 4.15(2), we have $1 \in D$, and $(x \to x) \cap D \neq \emptyset$, $(y \to y) \cap D \neq \emptyset$. Thus $x \in D_x$, $y \in D_y$. Since $x, y \in L \setminus D$, we get $D \subset D_x \subseteq L$ and $D \subset D_y \subseteq L$. Moreover, by Proposition 4.15(2) we get D_x and D_y are weak h-deductive systems of L. Hence by assumption $D_x = L = D_y$ and so $y \in D_x, x \in D_y$. Therefore, $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$. Clearly D is a weak h-deductive system.

(2)⇒(1) Let $x, y \in L \setminus D$. Suppose D_a is not a weak h-deductive system of L, for some $a \in D$. Then there exist $x, y \in L$ and $(x \to y) \cap D_a \neq \emptyset$, $x \in D_a$ such that $y \notin D_a$. Hence for some $u \in x \to y$, $(a \to u) \cap D \neq \emptyset$ and $(a \to x) \cap D \neq \emptyset$, but $(a \to y) \cap D = \varnothing$. Since D is a weak h-deductive system, we get $x \in D$ and $u \in D$, for some $u \in x \to y$. Hence $(x \to y) \cap D \neq \emptyset$. It is clear that $y \in D$,

which is a contradiction. Therefore, D_a is a weak h-deductive system of L, for any $a \in L$. By Proposition 4.15(2), we have D is an IWDS.

Now, we show that D_a is the least weak h-deductive system of L containing $D \cup \{a\}$, for all $a \in L \setminus D$. Let D' be a weak h-deductive system of L containing $D \cup \{a\}$, for all $a \in L \setminus D$. Then $(a \to u) \cap D \neq \emptyset$, for all $u \in D_a$. Since $D \subseteq D'$, we get $(a \to u) \cap D' \neq \emptyset$. Hence $u \in D'$. Therefore $D_a \subseteq D'$. That is D_a is the least weak h-deductive system of L containing $D \cup \{a\}$ and $D \subset D_a \subseteq L$. We show that D is a maximal weak h-deductive system of L. Since $a \in L \setminus D$, by assumption we get $D_a = L$. Therefore, D is a maximal weak h-deductive system of L. □

5. Some types of states on hyper NM-algebras

Inspired by the notion of states on hyper effect algebras [8], we will introduce three types of states on hyper NM-algebras such as sup-states, inf-states, and hyper states, furthermore, study the relationships of them in the following.

Definition 5.1. *Let L be a hyper NM-algebra. A sup-state on L is a mapping $s : L \to [0,1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$.

where $s^*(A)$ is defined by $s^*(A) = \sup\{s(t) \mid t \in A\}$, for any $A \subseteq L$.

Example 5.2. Let $L = \{0, a, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$

$\odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{a\}$	$\{a, 1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = \frac{1}{2}$, $s(1) = 1$. It is easy to verify that s is a sup-state on the hyper NM-algebra L.

Proposition 5.3. *Let s be a sup-state on hyper NM-algebras L. Then the following properties hold: for any $x, y \in L$,*

(1) *if $x \leq y$ implies $s(x) = s(y) + s^*(y \to x) - 1$,*

(2) *if $x \leq y$ implies $s(x) \leq s(y)$,*

(3) $s(x) = 1 - s^*(x \to 0)$,

(4) $s(x) = s^*(1 \to x)$,

(5) $s^*(x \to 0) + s^*(1 \to x) = 1$.

Proof. (1) If $x \leq y$, $1 \in x \to y$ by Proposition 3.8(7). We get $s(x) + s(1) = s(y) + s^*(y \to x)$. Therefore $s(x) = s(y) + s^*(y \to x) - 1$.

(2) It follows from (1) and $x \leq y$ that $s^*(y \to x) = s(x) + s(1) - s(y) \leq 1$. Therefore $s(x) \leq s(y)$.

(3) It follows from Proposition 3.8(8) $1 \in 0 \to x$ that $s(x) + s^*(x \to 0) = s(0) + s(1)$. Therefore, $s(x) = 1 - s^*(x \to 0)$.

(4) It follows from Proposition 3.8(8) $1 \in x \to 1$ that $s(x) + s(1) = s(1) + s^*(1 \to x)$. Therefore, $s(x) = s^*(1 \to x)$.

(5) It is clear by (3) and (4). □

Proposition 5.4. *Let s be a sup-state on hyper NM-algebras L. Define $K = Ker(s) = \{a \in L \mid s(a) = 1\}$ which is called the kernel of the sup-state s. Then K is a weak h-deductive system of L.*

Proof. Clearly, $1 \in K$. Let $x \in K$ and $K \ll x \to y$. Then $s(x) = 1$. There exist $v \in K$ and $u \in x \to y$ such that $v \leq u$. Since s is order-preserving, we have $1 = s(v) \leq s(u)$. Hence $s(u) = 1$, i.e. $s^*(x \to y) = 1$. Also note that $x \ll y \to x$, so $1 = s(x) \leq s^*(y \to x)$. This shows that $s^*(y \to x) = 1$. Then we obtain that $s(y) = 1$. Therefore, $y \in K$. □

Now, we define another type of states on hyper NM-algebras.

Definition 5.5. *Let L be a hyper NM-algebra. An inf-state on L is a mapping $s : L \to [0, 1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$.

where $s_*(A)$ is defined by $s_*(A) = \inf\{s(t) \mid t \in A\}$ for any $A \subseteq L$.

Example 5.6. Let $L = \{0, a, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge = \odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{0, a, 1\}$	$\{a, 1\}$	$\{0, 1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0, 1]$ be a mapping and $s(0) = 0$, $s(a) = s(1) = 1$. It is easy to verify that s is an inf-state on the hyper NM-algebra L. But it isn't sup-state, indeed, $s(0) = 0$, $s^*(0 \to 1) = 1$, $s(1) = 1$, $s^*(1 \to 0) = 1$. Therefore $s(0) + s^*(0 \to 1) \neq s(1) + s^*(1 \to 0)$.

Example 5.7. Consider the hyper NM-algebra L given in Example 5.2. We can check that s is both a sup-state and an inf-state on the hyper NM-algebra L.

Proposition 5.8. *Let s be a sup-state and an inf-state on hyper NM-algebra L. For some $x, y \in L$, if $0, 1 \in x \to y$, then $s^*(x \to y) = s^*(y \to x) = 1, s_*(x \to y) = s_*(y \to x) = 0$.*

Proof. Let s be a sup-state and inf-state. Then for some $x, y \in L$, we have $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$, $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$. Since $0, 1 \in x \to y$, it is obvious that $s^*(x \to y) = 1$, $s_*(x \to y) = 0$. We also get that $s(x) + 1 = s(y) + s^*(y \to x)$, $s(x) + 0 = s(y) + s_*(y \to x)$. Hence, $s^*(y \to x) - s_*(y \to x) = 1$. This shows that $s^*(x \to y) = s^*(y \to x) = 1, s_*(x \to y) = s_*(y \to x) = 0$. □

Proposition 5.9. *Let L be a hyper NM-algebra. If s is a sup-state and an inf-state, then $s^*(x \to y) - s^*(y \to x) = s_*(x \to y) - s_*(y \to x)$, for any $x, y \in L$.*

Proof. If s is a sup-state and inf-state, then for any $x, y \in L$, we have $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$, $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$. Hence, $s^*(x \to y) - s^*(y \to x) = s_*(x \to y) - s_*(y \to x)$.

Open problem: How can S be a sup-state and inf-state at the same time? □

We introduce another definition of states on hyper NM-algebras.

Definition 5.10. *Let L be a hyper NM-algebra. A hyper state on L is a mapping $s : L \to [0, 1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) *there exist $a \in x \to y$, $b \in y \to x$, for which $s(x) + s(a) = s(y) + s(b)$.*

Example 5.11. Let $L = \{0, a, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$

$\odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = \frac{1}{2}$, $s(1) = 1$. It is easy to verify that s is a sup-state and a hyper state on the hyper NM-algebra L. But it isn't inf-state, indeed, $s(0) = 0$, $s_*(0 \to a) = 1$, $s(a) = \frac{1}{2}$, $s_*(a \to 0) = 0$. Therefore $s(0) + s_*(0 \to a) \neq s(a) + s_*(a \to 0)$.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = s(1) = 1$. It is easy to verify that s is an inf-state and a hyper state on the hyper NM-algebra L. But it isn't sup-state, indeed, $s(0) = 0$, $s^*(0 \to a) = 1$, $s(a) = 1$, $s^*(a \to 0) = s(a) = 1$, therefore $s(0) + s^*(0 \to a) \neq s(a) + s^*(a \to 0)$.

Proposition 5.12. *Let s be a state on NM-algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ and let $(L_h, \vee_h, \wedge_h, \odot_h, \to_h, 0, 1)$ be a hyper NM-algebra, where $L_h = L$, $x \vee_h y := \{x \vee y\}$, $x \wedge_h y := \{x \wedge y\}$, $x \odot_h y := \{x \odot y\}$, $x \to_h y := \{x \to y\}$. Define $s_h(x) := s(x), x \in L$ on hyper NM-algebra L_h, then s_h is a hyper state on the hyper NM-algebra L_h.*

Proof. Straightforward. □

Let X be a nonempty set and $s : L \to [0,1]$ be a mapping. We say that the mapping s has supremum property, if for all subset A of X, $\sup\{s(a) | a \in A\} = s(t)$, for some $t \in A$. Similarly, we say that the mapping s has infimum property, if for all subset A of X, $\inf\{s(a) | a \in A\} = s(t)$, for some $t \in A$.

Proposition 5.13. *Let s be a hyper state on hyper NM-algebra L. Then the following properties hold,*

(1) *if s with supremum property, then every sup-state is a hyper state,*

(2) *if s with infimum property, then every inf-state is a hyper state.*

Proof. Straightforward. □

The following example shows that a hyper state may be neither sup-state nor inf-state on hyper NM-algebras.

Example 5.14. Consider the hyper NM-algebra L given in Example 3.2. Let $s : L \to [0,1]$ be a mapping and $s(0) = s(a) = 0$, $s(b) = s(1) = 1$. We can check that s is a hyper state rather than sup-state and inf-state on the hyper NM-algebra L, since $s(a) + s^*(a \to b) \neq s(b) + s^*(b \to a)$ and $s(a) + s_*(a \to b) \neq s(b) + s_*(b \to a)$.

6. Conclusions

In this paper, we introduce the concept of hyper NM-algebras, which is a generalization of the concept of NM-algebras. Also, we state that hyper NM-algebras are special hyper residuated lattices. Furthermore, we give some properties, related results and relations between (weak) h-filters and weak h-deductive systems on hyper NM-algebras. Finally, we preliminarily define the notions of sup-states, inf-states and hyper states on hyper NM-algebras. In the next task,

we will focus on some related properties of those states for further studying the algebraic structure of hyper NM-algebras.

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References

[1] P. Bonansinga, P. Corsini, *On semihypergroup and hypergroup homomorphisms,* Unione Matematica Italiana Bollettino B. Serie VI, 1 (1982), 717-727.

[2] R. A. Borzooei, M. Bakhshi, O. Zahiri, *Filter theory on hyper residuated lattice*, Quasigroups and Related Systems, 22 (2014), 33-50.

[3] R. A. Borzooei, M. Bakhshi, *Some results on hyper BCK-algebras*, Quasigroups and Related Systems, 11 (2004), 9-24.

[4] R. A. Borzooei, H. Harizavi, *Regular congruence relations on hyper BCK-algebras*, Scientiae Mathematicae Japonicae, 61(1)(2005): 83-97.

[5] R. A. Borzooei, A. Hasankhani, M. M. Zahedi, Y. B. Jun, *On hyper K-algebras*, Mathematica Japonicae, 52 (2000), 113-121.

[6] B. Davvaz, *Some results on congruences on semihypergroups*, Bulletin of the Malaysian Mathematical Society, 23 (2000), 53-58.

[7] B. Davvaz, N. S. Poursalavati, *Semihypergroups and S-hypersystems*, Pure Mathematics and Applications, 11 (2000), 43-49.

[8] A. Dvurečenskij, M. Hyčko, *Hyper effect algebras*, Fuzzy Sets and Systems, 326 (2017), 34-51.

[9] F. Esteva, L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, 124 (2001), 271-288.

[10] J. C. Fodor, *Contrapositive symmetry of fuzzy implications*, Fuzzy Sets and Systems, 69 (1995), 141-156.

[11] S. Ghorbani, A. Hasankhani, E. Eslami, *Hyper MV-algebras*, Set-Valued Matematics and Applications, 2 (2008), 205-222.

[12] S. Ghorbani, E. E. Eslami, A. Hasankhani, *Quotient hyper MV-algebras*, Scientiae Mathematicae Japonicae, 66 (2007), 371-386.

[13] P. Hájek, *Metamathematics of fuzzy Logic*, Trends in Logic-Studia Logica Library 4. Kluwer Academic Publishers, Dordrecht, 1998.

[14] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematic, 8 (2000), 127-136.

[15] Y. B. Jun, M. S. Kang, H. S. Kim, *New type of hyper MV-deductive system in hyper MV-algebras*, Mathematical Logic Quarterly, 56 (2010), 400-405.

[16] Y. B. Jun, M. S. Kang, H. S. Kim, *Hyper MV-deductive systems of hyper MV-algebras*, Communications of the Korean Mathematical Society, 25 (2010), 537-545.

[17] L. Z. Liu, X. Y. Zhang, *States on R_0 algebras*, Soft Computing, 12 (2008), 1099-1104.

[18] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45-49.

[19] J. Mittas, M. Konstantinidou, *Sur une nouvelle génération de la notion detreillis, Les supertreillis et certaines de certaines de leurs propriťés générales*, Ann. Sci. Univ. Blaise Pascal (Clermont II), Sér. Math. Fasc., 25 (1989), 61-83.

[20] D. W. Pei, *On equivalent forms of fuzzy logic systems NM and IMTL*, Fuzzy Sets and Systems, 138 (2003), 187-195.

[21] D. W. Pei, G. J. Wang, *The completeness and applications of the formal system $\mathcal{L}^*$*, Science in China Series F, 45 (2002), 40-50.

[22] G. J. Wang, H. J. Zhou, *Introduction to mathematical logic and resoultion pinciple,* Alpha Science International Ltd, Oxford, 2009.

[23] X. L. Xin, P. Wang, *States and measures on hyper BCK-algebras*, Journal of Applied Mathematics, (2014), 1-7.

[24] X. L. Xin, *Hyper BCI-algebras*, Discussiones Mathematicae General Algebra and Applications, 26 (2006), 5-19.

[25] X. L. Xin, B. Davvaz, *State operators and state-morphism operators on hyper BCK-algebras*, Journal of Intelligent and Fuzzy Systems, 29 (2015), 1869-1880.

[26] X. L. Xin, Y. X. Zou, J. M. Zhan, *Hyper BL-algebras*, Filomat, 32 (2018), 6675-16689.

[27] O. Zahiri, R. A. Borzooei, M. Bakhshi, *Quotient hyper residuated lattices*, Quasigroups Related Systems, 20 (2012), 125-138.

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Improved reverses Young type inequalities

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Abstract. The main purpose of this paper is to present new reverses Young type inequalities for numbers. Then, we use these inequalities to establish corresponding reverses Young type inequalities for operators.
Keywords: reverse Young type inequality, Kantorovich constant, positive operator.

1. Introduction

Throughout this paper, $\mathcal{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, < \cdot, \cdot >)$ and m, m', M, M' are scalars. An operator A is said to be positive (denoted $A \geq 0$) if $< Ax, x > \geq 0$ for all $x \in H$, and A is said to be strictly positive (denoted $(A > 0)$) if A is positive and invertible.

For $A, B > 0$ and $0 \leq v \leq 1$, the v-weighted arithmetic mean $A\nabla_v B$ and the v-weighted geometric mean $A\sharp_v B$ are defined by

$$A\nabla_v B = (1-v)\,A + vB, A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}},$$

when $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}} B$ and $A\sharp_{\frac{1}{2}} B$, respectively.

It is well-known that

$$A\sharp B \leq H_v(A, B) \leq A\nabla B,$$

where $H_v(A, B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2}$ is called Heinz mean.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)b \tag{1.1}$$

with equality if and only if $a = b$. When $v = \frac{1}{2}$, we have $\sqrt{ab} \leq \frac{a+b}{2}$.

In 2011, Zou et al. [1] proved the refined version of (1.1)

$$va + (1-v)b \geq K(h, 2)^r a^v b^{1-v}, \tag{1.2}$$

where $K(h, 2) = \frac{(h+1)^2}{4h}$ is called Kantorovich constant, $h = \frac{a}{b}$ and $r = \min\{v, 1-v\}$.

Hu [2] obtained the refinement of the inequality (1.1) as follows:

$$v^2a^2+(1-v)^2b^2\geq v^2(a-b)^2+\left[(va)^vb^{1-v}\right]^2,\quad 0\leq v\leq\frac{1}{2} \tag{1.3}$$

and

$$v^2a^2+(1-v)^2b^2\geq(1-v)^2(a-b)^2+\left[a^v\left((1-v)b\right)^{1-v}\right]^2,\quad \frac{1}{2}\leq v\leq 1. \tag{1.4}$$

Moreover, Nasiri and Liao [3] derived the following reverse Young inequalities:

$$(1-v)^{2v}\left[(1-2v)a+2vb\right]+(1-v)^{2-2v}a^{2v}b^{1-2v}K(h,2)^{-r}\geq 2(1-v)\sqrt{ab}, \tag{1.5}$$

$$(1-v)^{2-2v}\left[2va+(1-2v)b\right]+(1-v)^{2v}a^{1-2v}b^{2v}K(h,2)^{-r}\geq 2(1-v)\sqrt{ab}, \tag{1.6}$$

where $0\leq v\leq\frac{1}{2}$ and $r=\min\{2v,1-2v\}$.

By (1.5) and (1.6), Nasiri and Liao [3] gave the following inequalities:

$$\begin{aligned}&(1-v)^{2v}\left(\frac{a+b}{2}\right)+(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\&\geq 2(1-v)\sqrt{ab},\end{aligned} \tag{1.7}$$

$$\begin{aligned}&(1-v)^{2-2v}\left(\frac{a+b}{2}\right)+(1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\&\geq 2(1-v)\sqrt{ab},\end{aligned} \tag{1.8}$$

where $0\leq v\leq\frac{1}{2}$ and $r=\min\{2v,1-2v\}$.

For some related of Young inequalities, the reader is also referred to recent papers [4-8], and references therein.

Motivated by the aforementioned discussion, we will obtain sharper results than (1.5)-(1.8) in this paper .

2. Main results

In this section, we will give the reverses Young type inequalities for numbers and operators.

Theorem 1. *Let $a,b\geq 0$ and $0\leq v\leq\frac{1}{2}$, then*

$$(1-v)^{2v}\left[(1-2v)a+2vb\right]+(1-v)^{2-2v}a^{2v}b^{1-2v}\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab} \tag{2.1}$$

and

$$(1-v)^{2-2v}\left[2va+(1-2v)b\right]+(1-v)^{2v}a^{1-2v}b^{2v}\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}, \tag{2.2}$$

where $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{a}{b}$ *and* $r=\min\{2v,1-2v\}$.

Proof. For $0\leq v\leq\frac{1}{2}$, by inequality (1.2), we have

$$\begin{aligned}
&(1-v)^{2v}\left[(1-2v)a+2vb\right]K(h,2)^{-\frac{r}{2}}\\
&+(1-v)^{2-2v}a^{2v}b^{1-2v}K(h,2)^{-\frac{r}{2}}-2(1-v)\sqrt{ab}\\
&\geq(1-v)^{2v}a^{1-2v}b^{2v}K(h,2)^{\frac{r}{2}}+(1-v)^{2-2v}a^{2v}b^{1-2v}K(h,2)^{-\frac{r}{2}}-2(1-v)\sqrt{ab}\\
&=\left[(1-v)^{v}a^{\frac{1-2v}{2}}b^{v}K(h,2)^{\frac{r}{4}}-(1-v)^{1-v}a^{v}b^{\frac{1-2v}{2}}K(h,2)^{-\frac{r}{4}}\right]^2\\
&\geq 0.
\end{aligned}$$

Thus, (2.1) holds.

Similarly, we have

$$\begin{aligned}
&(1-v)^{2-2v}\left[2va+(1-2v)b\right]K(h,2)^{-\frac{r}{2}}\\
&+(1-v)^{2v}a^{1-2v}b^{2v}K(h,2)^{-\frac{r}{2}}-2(1-v)\sqrt{ab}\\
&\geq(1-v)^{2-2v}a^{2v}b^{1-2v}K(h,2)^{\frac{r}{2}}+(1-v)^{2v}a^{1-2v}b^{2v}K(h,2)^{-\frac{r}{2}}-2(1-v)\sqrt{ab}\\
&=\left[(1-v)^{1-v}a^{v}b^{\frac{1-2v}{2}}K(h,2)^{\frac{r}{4}}-(1-v)^{v}a^{\frac{1-2v}{2}}b^{v}K(h,2)^{-\frac{r}{4}}\right]^2\\
&\geq 0.
\end{aligned}$$

Thus, (2.2) holds.

This completes the proof. □

Corollary 1. *Let* $a,b\geq 0$ *and* $0\leq v\leq\frac{1}{2}$, *then*

$$(1-v)^{2v}\left[(1-v)a+vb\right]+(1-v)^{2-2v}a^{v}b^{1-v}\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}$$

and

$$(1-v)^{2-2v}\left[va+(1-v)b\right]+(1-v)^{2v}a^{1-v}b^{v}\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab},$$

where $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{a}{b}$ *and* $r=\min\{v,1-v\}$.

Corollary 2. *Let* $a,b\geq 0$ *and* $0\leq v\leq\frac{1}{2}$, *then*

$$\begin{aligned}
&(1-v)^{2v}\left(\frac{a+b}{2}\right)+(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)\\
&\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab} \qquad (2.3)
\end{aligned}$$

and

$$\begin{aligned}
&(1-v)^{2-2v}\left(\frac{a+b}{2}\right)+(1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)\\
&\geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}, \qquad (2.4)
\end{aligned}$$

where $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{a}{b}$ *and* $r=\min\{2v,1-2v\}$.

Remark 1. Our results are better than (1.5)-(1.8). In fact, we have

$$\begin{aligned}
&(1-v)^{2v}\left(\frac{a+b}{2}\right)+(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\
&-(1-v)^{2v}\left(\frac{a+b}{2}\right)K(h,2)^{-\frac{r}{2}}-(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}\\
&=(1-v)^{2v}\left(\frac{a+b}{2}\right)\left[1-K(h,2)^{-\frac{r}{2}}\right]\\
&-(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}\left[1-K(h,2)^{-\frac{r}{2}}\right]\\
&=\left[1-K(h,2)^{-\frac{r}{2}}\right]\left[(1-v)^{2v}\left(\frac{a+b}{2}\right)\right.\\
&\left.-(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}\right]\\
&\geq\left[1-K(h,2)^{-\frac{r}{2}}\right]\left[(1-v)^{2v}\left(\frac{a+b}{2}\right)-(1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)\right]\\
&\geq 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&(1-v)^{2-2v}\left(\frac{a+b}{2}\right)+(1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\
&\geq(1-v)^{2-2v}\left(\frac{a+b}{2}\right)K(h,2)^{-\frac{r}{2}}+(1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}.
\end{aligned}$$

Theorem 2. *Let* A,B *be two operators satisfy* $0<mI\leq A\leq m'I<M'I\leq B\leq MI$ *where* m,m',M',M *are positive real numbers such that* $m<m'<M'<M$. *Then for* $0\leq v\leq\frac{1}{2}$,

$$(1-v)^{2v}(A\nabla B)+(1-v)^{2-2v}H_{2v}(A,B)\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B \tag{2.5}$$

and

$$(1-v)^{2-2v}(A\nabla B)+(1-v)^{2v}H_{2v}(A,B)\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B, \tag{2.6}$$

where $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{M}{m}$, $h'=\frac{M'}{m'}$ *and* $r=\min\{2v,1-2v\}$.

Proof. If $0\leq v\leq\frac{1}{2}$ and $x\geq 0$, by inequality (2.1), we have

$$(1-v)^{2v}\left[(1-2v)+2vx\right]+(1-v)^{2-2v}x^{1-2v}\geq 2(1-v)K(x,2)^{\frac{r}{2}}\sqrt{x}.$$

Let $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. It is easy to know that

$$0<h'I\leq X\leq hI.$$

By the monotonicity of operator functions and (2.1), we have

$$(1-v)^{2v}\left[(1-2v)+2vA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right]+(1-v)^{2-2v}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-2v}$$
$$\geq 2(1-v)K(h,2)^{\frac{r}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}$$

and so

$$(2.7)\quad (1-v)^{2v}\left[(1-2v)A+2vB\right]+(1-v)^{2-2v}A\sharp_{1-2v}B\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B.$$

Replacing A and B by B and A respectively in (2.7), we have

$$(2.8)\quad (1-v)^{2v}\left[(1-2v)B+2vA\right]+(1-v)^{2-2v}A\sharp_{2v}B\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B.$$

Summing (2.7) and (2.8), we have

$$(1-v)^{2v}(A\nabla B)+(1-v)^{2-2v}H_{2v}(A,B)\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B.$$

Similarly, (2.6) holds.

This completes the proof. □

References

[1] H. Zuo, G. Shi, M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., 5 (2011), 551-556

[2] X. Hu, *Young type inequalities for matrices*, J. East China Norm. Univ. Natur. Sci. Ed., 4 (2012), 12-17

[3] L. Nasiri, W. Liao, *The new reverses of Young type inequalities for numbers, operators and matrices*, Oper. Matrices, 12 (2018), 1063-1071

[4] X. Hu, J. Xue, *A note on reverses of Young type inequalities*, J. Inequal. Appl., 98 (2015).

[5] X. Hu, F. Yang, J. Xue, *On improved Young type inequalities for matrices*, Ital. J. Pure Appl. Math., 34 (2015), 413-420.

[6] Y. Peng, *Young type inequalities for matrices,* Ital. J. Pure Appl. Math., 32 (2014), 515-518.

[7] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., 361 (2010), 262-269.

[8] Y. Al-Manasrah, F. Kittaneh, *A generalization of two refined Young inequalities,* Positivity, 19 (2015), 757-768.

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Structure of finite groups with two real conjugacy class sizes

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Abstract. In this short note, we deteremine the structure of a finite group G satisfying $cs_r(G) = \{1, 2\}$, where $cs_r(G)$ denotes the set of real conjugacy class sizes of G.
Keywords: finite groups, real elements, real conjugacy class sizes.

1. Introduction

All groups considered in this paper are finite. We called $x \in G$ a real element, if $x^g = x^{-1}$ for some $g \in G$, moreover, x^G is called a real conjugacy class size of G. We denote by $Re(G)$ the set of all real elements of G, and by $cs_r(G)$ the set of the real conjugacy class sizes of G. All unexplained notation and terminology are standard(see [4]).

There are many results illustrating the relationship between the structure of a group and the arithmetic property of real conjugacy class sizes. In [6], S. Dolfi, E. Pacifici and L. Sanus gave the structure of group G when $cs_r(G) = \{1, 2\}$:

Theorem A. *Let G be a group. Then $cs_r(G) = \{1, 2\}$ if and only if $G = A \times O$, where $cs_r(O) = \{1\}$, and either*

(a) A is a 2-group with $cs_r(A) = \{1, 2\}$; or

(b) $A = MP$, where M is a normal abelian 2-complement of A and P is a Sylow 2-subgroup of A, $C = C_P(M)$ has index 2 in P, and $Re(P) \subseteq Z(C)$.

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In this note, we point out that in case (b) of Theorem A above, M can be replaced by a subgroup generating by all real elements of odd order of G. Also, we will give a new proof of Theorem A.

2. Preliminaries

In this section we list some lemmas which will be used in the sequel.

Lemma 2.1 ([5, Theorem C]). *Let G be a group. All real classes of G have 2-power size if and only if G has a normal 2-complement K and $Re(G) \subseteq C_G(K)$.*

Lemma 2.2 ([1, Proposition 6.4]). *Let G be a group. Then every nontrivial real element in G has even order if and only if G has a normal Sylow 2-subgroup.*

Lemma 2.3 ([3, Theorem B]). *Let K be a group of odd order that acts on a 2-group P, and assume that K fixes all elements of order 2 in P and all real elements of order 4. Then K acts trivially on P.*

Lemma 2.4 ([2, Lemma 2.2]). *Let $N \trianglelefteq G$ and suppose that Nx is a real element in G/N. Assume that $|N|$ or the order of Nx in G/N is odd. Then $Nx = Ny$ for some real element y of G (of odd order if the order of Nx is odd).*

3. Proof of the Theorem A

Proof. Since $cs_r(G) = \{1, 2\}$, Lemma 2.1 implies that G has a normal 2-complement, say H. Hence H is solvable, so is G.

Let P be a Sylow 2-subgroup of G. Assume first P is normal in G. Now we consider the action of H on P. Clearly, $|G : C_G(x)| = 1$ or 2 for every real element x of P, forcing $H \leq C_G(x)$. By Lemma 2.3, it follows that H acts on P trivially. Hence $G = P \times H$. Further, $Re(H) = \{1\}$ by Lemma 2.2, Statement (a) of Theorem A holds.

Now suppose that P is not normal in G. By Lemma 2.2, there exists at least one non-trivial real element of odd order. Let Ω be the set of all non-trivial odd order real elements of G. Then $|\Omega| \geq 1$. Let $M := \langle \Omega \rangle$. Obviously, $M \trianglelefteq G$. Further, for every $w_1, w_2 \in \Omega$, $|G : C_G(w_i)| = 1$ or 2 for $i = 1, 2$. Hence $w_i \in C_G(w_j)$ for $i, j = 1, 2$, showing $M \leq O_{2'}(G)$ and $M \leq C_G(w)$. Consequently, M is abelian.

Let $\widetilde{G} := G/M$. By Lemma 2.4, we have that $\widetilde{G}$ has no non-trivial odd order real element. Therefore, $\widetilde{P} \trianglelefteq \widetilde{G}$ by Lemma 2.2. As the same in the first paragraph of our proof, we see that $\widetilde{G} = \widetilde{P} \times \widetilde{H}$, where $\widetilde{H}$ is the Hall $2'$-subgroup of $\widetilde{G}$ and $MP \trianglelefteq G$. Then $H \trianglelefteq G$ and $G = H \rtimes P$.

Now consider the action of P on H. By [4, 8.2.7], we have $H = [H, P]C_H(P)$. Note that $[H, P] \leq M$ and $M \leq Z(H)$, leading to $[H, P, H] \leq [M, H] = 1$. Thus $[P, H, H] = 1$. By the Three Subgroups Lemma, we ontain that $[H, H, P] = 1$, leading to $H' \leq C_H(P)$ and thus $C_H(P) \trianglelefteq H$. Note that $\widetilde{H}$ has only one real element $\widetilde{1}$, we have that $M \leq [H, P]$. Therefore, $M = [H, P]$. Since $M \leq Z(H)$,

we have that $C_M(P) \leq Z(G)$. By [4, 8.4.2], we have $M = [M, P] \times C_M(P)$. Since $C_M(P) \leq Z(G)$, we may assume that $C_M(P) = 1$. Hence $M = [M, P]$. Now we have $M \cap C_H(P) = 1$ and $H = M \times C_H(P)$. Then $G = MP \times C_H(P)$.

Let $C := C_P(M)$. We claim that $C_G(a) = C_G(b)$ for every $a, b \in \Omega \setminus Z(G)$. Assume there exist $a, b \in \Omega \setminus Z(G)$ such that $C_G(a) \neq C_G(b)$. Then $G = C_G(a)C_G(b)$. Hence $(ab)^G = a^G b^G$ and $ab \in \Omega$ by [6, Lemma 2.5(i)]. It is easy to see that $ab \notin Z(G)$. Since $|(ab)^G| = |a^G| = |b^G|$, we have that $(ab)^G = \{ab, (ab)^{-1}\}$, $a^G = \{a, a^{-1}\}$ and $b^G = \{b, b^{-1}\}$. So we have that $a^G b^G = \{ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}\}$. It follows that $ab = ab^{-1}$ or $a^{-1}b$, that is, $b^2 = 1$ or $a^2 = 1$, a contradiction. This proves that $C_G(a) = C_G(b)$ for every $a, b \in \Omega \setminus Z(G)$, as required.

Furthermore, $C_P(a) = C_P(b) = C$, which implies that $|P : C| = 2$. It is easy to see that $Re(P) \subseteq C$. We assert that $C_G(z) = C_G(e)$ for every $z \in Re(P) \setminus Z(G), e \in \Omega \setminus Z(G)$. Otherwise, by the same reason as above, we $o(z) = 2$. Since $|G : C_G(z)| = 2$, we get that $\langle z \rangle \trianglelefteq G$. In particular, $z \in Z(G)$, a contradiction. Hence $C_G(z) = C_G(e)$. This shows that $C_P(z) = C$ and thus $z \in Z(C)$. So we get that $Re(P) \subseteq Z(C)$. □

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References

[1] S. Dolfi, G. Navarro, P. H. Tiep, *Primes dividing the degrees of the real characters*, Math. Z., 259 (2008), 755774.

[2] R. Guralnick, G. Navarro, P. H. Tiep, *Real class sizes and real character degrees*, Math. Proc. Cambridge Philos. Soc., 150 (2011), 4771.

[3] I. M. Isaacs, G. Navarro, *Normal p-complements and fixed elements*, Arch. Math., 95 (2010), 207211.

[4] H. Kurzweil, B. Stellmacher, *The theory of finite groups. An introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 2004.

[5] G. Navarro, L. Sanus, P. H. Tiep, *Real characters and degrees*, Israel J. Math., 171 (2009), 157173.

[6] S. Dolfi, E. Pacifici, L. Sanus, *Finite groups with real conjugacy classes of prime size*, Israel J. Math., 175 (2010), 179189.

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On normalizers of Sylow subgroups and p-nilpotency of finite groups

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Abstract. In this paper, we prove the p-nilpotency of a finite group under the assumption that some subgroups of Sylow subgroups are partial CAP-or c-supplemented in their normalizers. Our results unify and generalize some earlier results.

Keywords: partial CAP-subgroup, c-supplemented subgroup, maximal subgroup, p-nilpotency, formation.

1. Introduction

In this paper, all groups considered are finite and G stands for a finite group. Let $\pi(G)$ stand for the set of all prime divisors of the order of G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ denote the class of supersolvable groups. The other notation and terminology are standard(see [9]).

Let H be a subgroup of G and A/B a G-chief factor. We say, following Gaschütz in [5], that H covers A/B if $HA = HB$; H avoids A/B if $H \cap A = H \cap B$. H is said to be cover-avoiding in G, in brevity, H is a CAP-subgroup of G, if H either covers or avoids any G-chief factor. In [2], H is a partial CAP-subgroup of G if there exists a chief series Γ_A of G such that H either covers or avoids each factor of Γ_A. Partial CAP-subgroups have also been called semi CAP-subgroups in [4]. Many authors presented some conditions for a group to be p-nilpotent or supersolvable under the condition that some subgroups of Sylow subgroups are partial CAP-subgroups, for details, the readers are referred to [7], [11] and [13]. Recall that a subgroup H is said to be c-supplemented (in G) if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. Many authors presented some conditions for a group to be solvable, p-nilpotent

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and supersolvable under the condition that some subgroups of Sylow subgroups are c-supplemented subgroups (see[3], [6] and [14]).

On the other hand, the normalizers of Sylow subgroups of a group play an important role in the structure of groups. A question of particular interest in studying finite groups is the influence of the structure and embedding of the normalizers of the Sylow subgroups on the structure of the group. A nice example is the celebrated Burnside′s Theorem.

Theorem 1.1 (Burnside)**.** *Let P be a Sylow p-subgroup of G. If $N_G(P) = C_G(P)$, then G is p-nilpotent.*

The following extension of Burnside′s Theorem due to Hall is also interesting.

Theorem 1.2 ([8])**.** *Let P be a Sylow p-subgroup of G. If p'-elements of $N_G(P)$ commute with the elements of P and the nilpotency class of P is less than p, then G is p-nilpotent.*

Wielandt, Ballester-Bolinches and Esteban-Romero proved the following results respectively.

Theorem 1.3 ([15])**.** *A group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Theorem 1.4 ([1])**.** *A group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

The main purpose of the present paper is to analyze the structural consequences of the c-supplementation or the partial cover and avoidance property of the maximal subgroups of the Sylow subgroups in their normalizers.

2. Preliminary results

A subgroup H is said to be π-quasinormal (s-quasinormal or s-permutable) in G if $PH = HP$ for all Sylow subgroups P of G (see [10]).

Lemma 2.1 ([[10] and [12, Lemma 2.2]])**.** *Suppose that U is a π-quasinormal subgroup of G, $H \leq G$ and $K \trianglelefteq G$. Then*

(a) *If $U \leq H$, then U is π-quasinormal in H;*

(b) *UK is π-quasinormal in G and UK/K is π-quasinormal in G/K;*

(c) *Let $K \leq H$. If H/K is π-quasinormal in G/K, then H is π-quasinormal in G;*

(d) *If P is a π-quasinormal subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

The lemma presented below is crucial in the sequel. The proof is a routine check, and we omit its details.

Lemma 2.2. *Let H be a subgroup of G. Then*

(1) *Let $N \trianglelefteq G$ and $N \leq H$. If H is a partial CAP-or c-supplemented subgroup of G, then H/N is a partial CAP-or c-supplemented subgroup of G/N;*

(2) *Let π be a set of primes, H a π-subgroup and N a normal $\pi^{'}$-subgroup of G. If H is a partial CAP-or c-supplemented subgroup of G, then HN/N is a partial CAP-or c-supplemented subgroup of G/N;*

(3) *Let $H \leq K$. If H is a partial CAP-or c-supplemented subgroup of G, then H is a partial CAP-or c-supplemented subgroup of K;*

(4) *Let $N \trianglelefteq G$. If H is a CAP-subgroup of G, then HN/N is a CAP-subgroup of G/N.*

Lemma 2.3 ([18, Theorem 3.1])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a solvable normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every Sylow subgroup of $F(N)$ is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.4 ([17, Theorem 3.1])**.** *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of G, then G is p-nilpotent.*

3. Main results

Theorem 3.1. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is partial CAP-or c-supplemented subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is p-nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

(1) Let L be a normal subgroup of G contained in P. Then G/L satisfies the hypothesis.

It is clear that $(|G/L|, p-1) = 1$. For any maximal subgroup P_1/L of P/L, $p = |P/L : P_1/L| = |P : P_1|$, so P_1 is a maximal subgroup of P. By the hypothesis, P_1 is a partial CAP-or c-supplemented subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G. By Lemma 2.2 (1), P_1/L is a partial CAP-or c-supplemented subgroup of $N_G(P)/L = N_{G/L}(P/L)$, and $(P/L)^{'} = P^{'}L/L$ is π-quasinormal in G/L by Lemma 2.1 (b), thus we have (1).

(2) $1 \neq P^{'} \leq O_p(G)$ and G is solvable.

For any $Q \in Syl_q(N_G(P))$, where $q \neq p$. It is easy to see that all maximal subgroups of P are partial CAP-or c-supplemented subgroups of PQ by Lemma

2.2 (3). Thus PQ satisfies the hypothesis of Lemma 2.4, so PQ is p-nilpotent, hence $Q \leq C_G(P)$. Assume that P is abelian, then $N_G(P) = C_G(P)$, hence G is p-nilpotent by Burnside$'$s Theorem, a contradiction. So $P' \neq 1$. By the hypothesis, P' is π-quasinormal in G, thus $P' \lhd \lhd G$, hence $O_p(G) \neq 1$. By (1), we have $G/O_p(G)$ is p-nilpotent. Since $(|G|, p-1) = 1$, we conclude that $G/O_p(G)$ is solvable, thus G is solvable.

(3) End of the proof.

Let $\{G_r \mid r \in \pi(G)\}$ be a Sylow system of G and $H = G_pG_r$ for any $r \in \pi(G)$ with $r \neq p$. By Lemma 2.1(a) and Lemma 2.2 (3), the hypothesis is still true for H. If $|\pi(G)| > 2$, then $G_r \trianglelefteq H$, which implies that G_p normalizes G_r for any $r \in \pi(G)$, hence G is p-nilpotent, a contradiction. Thus we may assume that $|G| = p^aq^b$.

Let L be a minimal normal subgroup of G. Since P' is π-qusinormal in G, by Lemma 2.1(b), we have $P'L$ is π-qusinormal in G and $P'L/L$ is π-qusinormal in G/L. If L is a q-group, then we consider the quotient group G/L. Evidently, $PL/L \in Syl_p(G/L)$. For any maximal subgroup T/L of PL/L, we have $p = |(PL/L) : (T/L)|$, and $T = PL \cap T = (P \cap T)L$. Let $P_1 = P \cap T$. Then $P_1 \cap L = P \cap T \cap L = P \cap L$, so

$$p = |PL : T| = |PL : (P \cap T)L| = |P : P \cap T| = |P : P_1|.$$

Thus P_1 is a maximal subgroup of P. By the hypothesis, P_1 is a partial CAP- or c-supplemented subgroup of $N_G(P)$. If P_1 is a partial CAP-subgroup of $N_G(P)$, by Lemma 2.2, it is easy to check that P_1L/L is also a partial CAP-subgroup of $N_{G/L}(PL/L) = N_G(P)L/L$. If P_1 is a c-supplemented subgroup of $N_G(P)$, then there exists a subgroup K of $N_G(P)$ such that $N_G(P) = P_1K$ and $P_1 \cap K \leq (P_1)_{N_G(P)}$. Obviously, we have $N_G(P)L/L = (P_1L/L)(KL/L)$. Since $(|P_1|, |L|) = 1$, we get

$$|P_1 \cap KL| = \frac{|P_1| \cdot |KL|_p}{|P_1KL|_p} = \frac{|P_1| \cdot |K|_p}{|N_G(P)L|_p} = \frac{|P_1| \cdot |K|_p}{|N_G(P)|_p} = |P_1 \cap K|.$$

This implies that $P_1 \cap KL = P_1 \cap K$, thus

$$\begin{aligned}(P_1L/L) \cap (KL/L) &= (P_1L \cap KL)/L = (P_1 \cap KL)L/L \\ &= (P_1 \cap K)L/L \leq (P_1)_{N_G(P)}L/L \leq (P_1L/L)_{N_{G/L}(PL/L)}.\end{aligned}$$

So P_1L/L is a partial CAP-or c-supplemented subgroup of $N_{G/L}(PL/L)$. By the minimality of G, G/L is p-nilpotent and so is G, a contradiction. Hence L is a p-group and $L \leq P$. By (1), G/L is p-nilpotent. Similarly, if N is another minimal normal subgroup of G, then $N \leq P$ and so G/N is also p-nilpotent. Now it follows that $G \cong G/N \cap L$ is p-nilpotent, a contradiction. Thus L must be the unique minimal normal subgroup of G. Since the class of p-nilpotent groups is a saturated formation, $L \nleq \Phi(G)$, hence $\Phi(G) = 1$. By [16, Thorem 5.3], we get $O_p(G) = F(G) = L$.

Since $P^{'}$ is π-quasinormal in G, we have $N_G(P^{'}) \geq O^p(G)$ by Lemma 2.1 (d), and P normalizes $P^{'}$, we get $P^{'} \trianglelefteq G$, so $P^{'} = O_p(G) = L$ by the unique minimal normality of L. By (1), we have $G/O_p(G)$ is p-nilpotent, hence $O_p(G)Q \trianglelefteq G$, where $Q \in Syl_q(G)$. Since $O_p(G)Q \cap P = O_p(G) = P^{'} \leq \Phi(P)$, $O_p(G)Q$ is p-nilpotent by J. Tate Theorem([9, Theorem 4.4.7]). Thus $Q \trianglelefteq O_p(G)Q \trianglelefteq G$, which implies that $Q \trianglelefteq G$, so G is p-nilpotent, a contradiction. This final contradiction completes our proof. □

Theorem 3.2. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

By Lemma 2.1 (a) and Lemma 2.2 (3), it is easy to see that every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of $N_H(P)$ and $P^{'}$ is π-quasinormal in H, then H is p-nilpotent by Theorem 3.1. Let M be a normal p-complement of H. Then $M \trianglelefteq G$. Assume that $M \neq 1$. We consider the quotient group G/M. Similar to the proof of (3) in Theorem 3.1, it is easy to see that the hypothesis is still true for $(G/M, H/M)$, hence G/M is p-nilpotent and so is G, a contradiction. Thus we conclude that $M = 1$. Now $H = P$ is a p-subgroup. Let T/P be a normal p-complement of G/P. It is clear that every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of $N_T(P)$ and $P^{'}$ is π-quasinormal in T, then T is p-nilpotent by Theorem 3.1, so $T_{p^{'}} \trianglelefteq T \trianglelefteq G$ and $T_{p^{'}}$ is also a Hall $p^{'}$-subgroup of G, thus $T_{p^{'}} \trianglelefteq G$, hence G is p-nilpotent, a contradiction. This final contradiction completes our proof. □

Corollary 3.1. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is a partial CAP-subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is p-nilpotent.*

Corollary 3.2. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is a c-supplemented subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is p-nilpotent.*

Theorem 3.3. *Let G be a group. For any prime factor p of $|G|$, there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

By Theorem 3.1, we know that G is a Sylow tower group of supersolvable type, so G is solvable. Let L be a minimal normal subgroup of G. Then L is an elementary r-subgroup, where $r \in \pi(G)$. By Lemma 2.2 (1) and (2), we have G/L satisfies the hypothesis, thus G/L is supersolvable by the minimal choice of G. Since the class of supersolvable subgroups is a saturated formation, we may assume that L is the unique minimal normal subgroup of G and $L \nleq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$, thus $L \leq Q$ by the unique minimal normality of L. Since $Q = O_q(G) \leq F(G) \leq C_G(L)$, L and M normalize $Q \cap M$, thus $Q \cap M \lhd G$. So $Q \cap M = 1$ or $L \leq Q \cap M$. If the later happens, then $L \leq M$, that is, $G = LM = M$, a contradiction. So $Q \cap M = 1$, and $L \cap M = 1$. This implies that $|Q| = |G : M| = |L|$, hence $L = Q$ and $N_G(Q) = G$. Let Q_1 be a maximal subgroup of Q. By the hypothesis, Q_1 is a partial CAP-or c-supplemented subgroup of $N_G(Q)$. Assume that Q_1 is a partial CAP-subgroup of $N_G(Q)$. By the unique minimal normality of L, $L/1$ is a chief factor of every chief series. Since $L = Q$, clearly, Q_1 can not covers $Q/1$, so Q_1 avoids $Q/1$, that is, $Q_1 \cap Q = 1$, thus $Q_1 = 1$, hence $|L| = |Q| = q$. By Lemma 2.3, we have G is supersolvable, a contradiction. So Q_1 is c-supplemented in $N_G(Q)$. By $N_G(Q) = G$, there exists a subgroup K of G such that $G = Q_1K$ and $Q_1 \cap K \leq (Q_1)_G$. By the unique minimal normality of L, we have $(Q_1)_G = 1$, so $Q_1 \cap K = 1$. On the other hand, since $Q \cap K \trianglelefteq \langle Q, K\rangle = G$, we have $Q \cap K = L = Q$ by the unique minimal normality of L, so $Q \leq K$. Thus $K = G$. So $Q_1 = Q_1 \cap K \leq (Q_1)_G = 1$, thus $|L| = |Q| = q$. By Lemma 2.3, we have G is supersolvable, a contradiction. This final contradiction completes our proof. □

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. For any prime factor p of $|H|$, there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is a CAP-or c-supplemented subgroup of $N_G(P)$ and P' is π-quasinormal in G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

Since a CAP-subgroup is also a partial CAP-subgroup, H satisfies the hypothesis of Theorem 3.3, hence H is supersolvable. Let $p = max\pi(H)$ and $P \in Syl_p(H)$. Then $P \trianglelefteq G$. We consider the quotient group G/P, then $G/H \cong (G/P)/(H/P) \in \mathcal{F}$. By Lemma 2.1 (a) and Lemma 2.2 (4), we have $(G/P, H/P)$ satisfies the hypothesis, thus $G/P \in \mathcal{F}$. Hence we may assume that $H = P$.

Let N be a minimal normal subgroup of G contained in P. By Lemma 2.1 (a) and Lemma 2.2 (4), the hypothesis is still true for $(G/N, P/N)$, so $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, $N \nleq \Phi(G)$. So there exists a

maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. On the other hand, we can conclude that $\Phi(P) = 1$. Otherwise, we have $G/\Phi(P) \in \mathcal{F}$, then $G/\Phi(G) \cong (G/\Phi(P))/(\Phi(G)/\Phi(P)) \in \mathcal{F}$, so $G \in \mathcal{F}$, a contradiction. Therefore, P is an elementary abelian subgroup.

By $N \leq P$, we have $G = NM = PM$ and $P \cap M \lhd G$. If $P \cap M \neq 1$, then $N \leq P \cap M$, $N \leq M$, we get $G = NM = M$, a contradiction. Thus $P \cap M = 1$, so $P = N$ is a minimal normal subgroup of G. Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 is a CAP-or c-supplemented subgroup of $N_G(P)$. Assume that P_1 is c-supplemented in $N_G(P)$. By $G = N_G(P)$, there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_G$. By the minimal normality of N and $N = P$, we have $(P_1)_G = 1$, $P_1 \cap K = 1$. On the other hand, we have $P = P \cap P_1K = P_1(P \cap K)$. Since $P \cap K$ is normal in K, P is abelian and $G = PK$, we have $P \cap K$ is a normal subgroup of G. The minimality of N implies that $P \cap K = P$, thus $K = G$. So $P_1 = 1$. This shows that P is a cyclic subgroup of order p, thus $G \in \mathcal{F}$ by Lemma 2.3, a contradiction. Hence P_1 is a CAP-subgroup of $N_G(P)$, then P_1 either covers or avoids the chief factor $N/1 = P/1$ of G. Clearly, P_1 can not cover $P/1$, so P_1 avoids $P/1$, that is, $P_1 \cap P = 1$, thus $P_1 = 1$, hence $|P| = p$. By Lemma 2.3, we have G is supersolvable, a contradiction. This final contradiction completes our proof. □

Recall that a group G was called an A-group if all of its Sylow subgroups are abelian. Let G be an A-group. Then for any $P \in Syl_p(G)$, we have $P' = 1$, of course, it is π-quasinormal in G, so we have the following corollaries.

Corollary 3.3 ([13, Theorem 4.3])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal A-subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup P of H is cover-avoiding in $N_G(P)$, then $G \in \mathcal{F}$.*

Corollary 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal A-subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup P of H is a c-supplemented subgroup of $N_G(P)$, then $G \in \mathcal{F}$.*

4. Remarks

Remark 4.1. The following example illustrates that the hypothesis in Theorem 3.1 that "P' is π-quasinormal in G" can not be removed.

Example Let $G = PSL_2(q)$, where $q \equiv \pm 1 \pmod 8$. Let P be a Sylow 2-subgroup of G. By [9, II, Theorem 8.27], we have the Sylow 2-subgroup of $PSL_2(q)$ is selfnormalizing in $PSL_2(q)$. Evidently, every maximal subgroup of P is normal in $N_G(P) = P$, so every maximal subgroup of P is a partial CAP-or c-supplemented subgroup of $N_G(P)$. However, G is not 2-nilpotent.

Remark 4.2. Even if G is a solvable group and p is an odd prime, the hypothesis in Theorem 3.1 that "P' is π-quasinormal in G" could not be omitted, either.

Example Let $H = Z_3 \times Z_3 \times Z_3$ be an elementary abelian 3-group. It is clear that $Aut(H)$ has a subgroup $Z_{13} \rtimes Z_3$. Now suppose that

$$G = (Z_3 \times Z_3 \times Z_3) \rtimes (Z_{13} \rtimes Z_3).$$

Let P_3 be a Sylow 3-subgroup of G. It is clear that $N_G(P_3) = P_3$, so every maximal subgroup of P_3 is a partial CAP-or c-supplemented subgroup of $N_G(P_3) = P_3$. However, G is not 3-nilpotent.

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References

[1] A. Ballester-Bolinches, R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups*, J. Algebra, 251 (2002), 727-738.

[2] A. Ballester-Bolinches, L.M. Ezquerro, A.N. Skiba, *Local embeddings of some families of subgroups of finite groups*, Acta Math. Sin. (Engl. Ser.), 25 (2009), 869-882.

[3] A. Ballester-Bolinches, Y.M. Wang, X.Y. Guo, *c-supplemented subgroups of finite groups*, Glasg. Math. J., 42 (2000), 383-389.

[4] Y. Fan, X.Y. Guo, K.P. Shum, *Remarks on two generalizations of normality of subgroups*, Chin. Ann. Math. (Chin. Ser.), 27 (2006), 169-176.

[5] W. Gaschütz, *Praefrattinigruppen*, Arch. Math., 13 (1962), 418-426.

[6] X.Y. Guo, K.P. Shum, *On p-nilpotency of finite groups with some subgroups c-supplemented*, Algebra Colloq., 10 (2003), 259-266.

[7] X.Y. Guo, P.F. Guo, K.P. Shum, *On semi cover-avoiding subgroups of finite groups,* J. Pure Appl. Algebra, 209 (2007), 151-158.

[8] P. Hall, *On a theorem of Frobenius*, Proc. London Math. Soc., 40 (1936), 468-501.

[9] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-New York, 1967.

[10] O. Kegel, *Sylow-gruppen and subnormalteiler endlicher gruppen*, Math. Z., 78 (1962), 205-211.

[11] X.H. Li, Y.W. Yang, *Semi CAP-subgroups and the structure of finite groups*, Acta Math. Sin. (Chin. Ser.), 51 (2008), 1181-1187.

[12] Y.M. Li, Y.M. Wang, H.Q. Wei, *Influence of π-quasinormality of some subgroups of a finite group,* Arch. Math., 81 (2003), 245-252.

[13] L.L. Wang, G.Y. Chen, *Some properties of finite groups with some (semi-p-)cover-avoiding subgroups*, J. Pure Appl. Algebra, 213 (2009), 686-689.

[14] Y.M. Wang, *Finite groups with some subgroups of Sylow subgroups c-supplemented,* J. Algebra, 224 (2000), 467-478.

[15] H. Wielandt, *p-sylowgruppen und p-faktorgruppen,* J. Math., 182 (1940), 180-193.

[16] M.Y. Xu, *A introduction to finite group,* Science Publisher, Beijing, 1999 (in Chinese).

[17] Y. Xu, X.H. Li, *Maximal subgroups of a Sylow p-subgroup and p-nilpotency of finite groups*, J. Algebra Appl., 9 (2010), 383-391.

[18] Y. Xu, T. Zhao, X. Li, *On $CISE$-normal subgroups of finite groups,* Turish J. Math., 36 (2012), 231-243.

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Derivations and differential filters on semihoops

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Abstract. In this paper, we introduce the notions of derivations and differential filters on semihoops and investigate some related properties of them. Also we discuss related properties of some particular derivations and give some characterizations of ideal derivations. Then we obtain that the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop, the set of all fixed points of a $\vee$-hoop for an ideal derivation is an ideal. And for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. Finally we find some relations between differential filters and other filters. It is proved that every maximal differential filter is a prime differential filter and the set of all differential filters of an ideal differential $\vee$-semihoop is a Heyting algebra.
Keywords: semihoop, derivation, differential filter, fixed point.

1. Introduction

As we all known, various fuzzy logic algebras have been widely introduced and studied as non-classical logic semantic systems. These are logical algebras based on continuous t-norm, such as: MV-algebras, BL-algebras, Hoops. From the point of view of algebras, hoops are naturally ordered commutative residuated integral monoids, introduced by B.Bosbach in [5, 6]. In 2003, semihoop was introduced by Francesc Esteva and Lluis Godo in [9], it was a generalization of a hoop. In recent years, the theory of hoops and semihoops had a lot of achievements. These conclusions were helpful to further study fuzzy logic. As a more general structure, semihoops have no the separability, that is, $x \wedge y = x \odot (x \to y)$ does not hold. Semihoops have attracted the attention and research of many scholars at home and abroad since it was put forward.

The notion of derivations introduced from the analytic theory, which is helpful for the research of structures and properties in algebraic systems. Several authors [3, 1, 2, 15] have studied derivations in rings and near rings. Jun and Xin applied the notion of derivation in rings and near rings theory to BCI-algebras in [12], and as a result they introduced a new concept, called a regular derivation, in BCI-algebras. Xin introduced the concept of derivations in lattices and

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characterized modular lattices and distributive lattices by isotone derivations in [16]. Later, He, Xin and Zhan studied the relations between derivations and their fixed point sets in residuated lattice in [11]. Recently Liang and Xin discussed the derivations on EQ-algebra in [14]. Although a number of researchers have been studied derivations in algebraic systems. However, none of them apply derivations to characterize algebraic structure of semihoops. Inspired by this, we want to introduce and study derivations in semihoops. Therefore it is meaningful to study this.

The filter theory of the logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas. Hence, we define the differential filter by derivation and study the relationships among different differential filters.

This paper is organized as follows: in Section 2, we review some necessary notions and properties, which will be used in the next section. In Section 3, we define derivation, ideal derivation and regular derivation on semihoops, investigate some properties and relations of them. Moreover, we discuss the structures and properties of the fixed point sets of ideal derivations. For example, the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop; the set of all fixed points of a $\vee$-hoop for an ideal derivation is an ideal; and for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. In Section 4, we define the differential filter on semihoops and discuss some relations of different differential filters, we prove that every maximal differential filter is a prime differential filter and the set of all differential filters of an ideal differential $\vee$-semihoop forms a Heyting algebra.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1 ([9]). *An algebra $(A, \odot, \to, \wedge, 1)$ of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*

$(SH1)$ $(A, \wedge, 1)$ *is a $\wedge$-semilattice with upper bound 1;*

$(SH2)$ $(A, \odot, 1)$ *is a commutative monoid;*

$(SH3)$ $(x \odot y) \to z = x \to (y \to z)$, *for all $x, y, z \in A$.*

On a semihoop A we define a binary relation $'' \leq''$, where $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $'' \leq''$ is a partial order relation on A and 1 is the upper bound, that is, for any $x \in A, x \leq 1$.

A semihoop A is bounded if there exists an element $0 \in A$, such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$ for all $n \in N$. In a bounded semihoop A, we define the negation $''*''$ on A by $x^* = x \to 0$ for all $x \in A$. A semihoop is called a hoop if it satisfies $x \odot (x \to y) = y \odot (y \to x)$, for all $x, y \in A$.(see [9])

Proposition 2.2 ([9]). *Let A be a semihoop. Then the following properties hold for all $x, y, z \in A$:*

(1) $x \odot y \leq z$ *if and only if* $x \leq y \to z$;
(2) $x \leq y$ *implies* $x \odot z \leq y \odot z$;
(3) $x \odot (x \to y) \leq y$;
(4) $x \to 1 = 1$, $1 \to x = x$, $x \to x = 1$;
(5) $x \odot y \leq x, y$;
(6) $x \leq (x \to y) \to y$;
(7) $x \leq y$ *implies* $z \to x \leq z \to y$, $y \to z \leq x \to z$.

Proposition 2.3 ([7]). *Let A be a bounded semihoop. Then the following properties hold for all $x \in A$:*

(1) $1^* = 0$, $0^* = 1$;
(2) $x \leq x^{**}$;
(3) $x \odot x^* = 0$;
(4) $x^{***} = x^*$.

Proposition 2.4 ([7]). *Let A be a semihoop. We define for any $x, y \in A$:*

$$x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Then the following conditions are equivalent:

(1) $\sqcup$ *is associative;*
(2) $x \leq y$ *implies* $x \sqcup z \leq y \sqcup z$ *for all* $x, y, z \in A$;
(3) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup z)$ *for all* $x, y, z \in A$;
(4) $\sqcup$ *is the join operation on* A.

Definition 2.5 ([7]). *A $\vee$-semihoop is a semihoop if it satisfies one of the equivalent conditions of Proposition 2.4.*

Proposition 2.6 ([10]). *Let A be a $\vee$-semihoop. For any $x, y, z \in A$, suppose that $x \vee y$ exists. Then:*

(1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(2) *For any* $n \in N_+$, $(x \vee y)^n \to z = \bigwedge\{(x_1 \odot \cdots \odot x_n) \to z | x_i \in \{x, y\}, i \in N_+\}$

Definition 2.7 ([7]). *Let A be a semihoop and F be a non-empty subset of A. F is called a filter of A if it satisfies the following conditions for every $x, y \in A$:*

($F1$) $x, y \in F$ *implies* $x \odot y \in F$;
($F2$) $x \in F$ *and* $x \leq y$ *imply* $y \in F$.

We denote the set of all filters of A by $\mathcal{F}(A)$.

Proposition 2.8 ([7]). *Let A be a semihoop and F be a nonempty subset of A. Then F is a filter if and only if the following conditions hold:*

(1) $1 \in F$;
(2) *For all* $x, y \in A$, *if* $x, x \to y \in F$ *then* $y \in F$.

Definition 2.9 ([8]). *An ideal is a non-void subset I of a lattice L with the properties for every $x, y \in L$:*

(1) *$x \in I$ and $y \leq x$ imply $y \in I$;*

(2) *$x, y \in I$ implies $x \vee y \in I$.*

Moreover, an ideal I is called a prime ideal if it satisfies the following condition:

(3) *$x \wedge y \in I$ implies $x \in I$ or $y \in I$.*

Proposition 2.10 ([10]). *Let A be a $\vee$-hoop. Then $(A, \wedge, \vee)$ is a distributive lattice.*

Definition 2.11 ([7]). *Let A be a semihoop and $X \subseteq A$. The filter of A generated by X will be denoted by $\langle X \rangle$, where $\langle X \rangle = \{a \in A | a \geq x_1 \odot x_2 \odot \cdots \odot x_n$, for some $x_i \in X$ and some $n \in N_+\}$.*

Proposition 2.12 ([7]). *Let A be a semihoop, $F \in \mathcal{F}(A)$ and $x \in A$. Then $\langle F \cup \{x\} \rangle = \{a \in A | y \odot x^n \leq a$, for some $n \in N, y \in F\}$.*

Definition 2.13 ([4]). *A lattice $(L, \vee, \wedge)$ is called a Heyting algebra if for any $x, y \in L$, there exists an element $x \to y \in L$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$ for all $z \in L$.*

From now on, we denote the semihoop $(A, \odot, \to, \wedge, 1)$ by A, unless otherwise stated.

3. Derivations on semihoop

In this section, we introduce the notion of derivations in semihoops and investigate some related properties of regular, isotone, contractive and ideal derivations, respectively. Also, we give some equivalent conditions under which a derivation is an ideal derivation. Moreover, we discuss the structures and properties of the fixed point sets of ideal derivations.

In order to study derivations, we first discuss some properties of the $''\sqcup''$ operation.

Proposition 3.1. *Let A be a bounded semihoop. Then the following properties hold for any $x, y \in A$:*

(1) *$x \sqcup y \geq x, y$;*

(2) *$x \leq y$ if and only if $x \sqcup y = y$;*

(3) *$x \sqcup 1 = 1$, $x \sqcup x = x$;*

(4) *$x \sqcup y = 0$ if and only if $x = 0$ and $y = 0$.*

Proof. (1) Since $y \to x \leq 1 = x \to x$ for all $x, y \in A$, by Proposition 2.2(1) we have $x \odot (y \to x) \leq x$, that means $x \leq (y \to x) \to x$. Again, by Proposition 2.2(6) $x \leq (x \to y) \to y$. Hence we obtain $x \leq ((x \to y) \to y) \wedge ((y \to x) \to x) = x \sqcup y$. Similarly $y \leq x \sqcup y$.

(2) Suppose that $x \leq y$ for some $x, y \in A$. Then by Proposition 2.2(4), (6) we have $x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x) = (1 \to y) \wedge ((y \to x) \to x) = y \wedge ((y \to x) \to x) = y$. Conversely, if $x \sqcup y = y$, by (1) $x \sqcup y \geq x$ we have $y \geq x$.

(3) By (2) we can get $x \sqcup 1 = 1$, $x \sqcup x = x$.

(4) Let $x \sqcup y = 0$. By (1) $x \sqcup y \geq x, y$, we have $x \leq 0$, $y \leq 0$, that is, $x = 0$, $y = 0$. Conversely, if $x = 0$ and $y = 0$, we can obtain that $x \sqcup y = 0 \sqcup 0 = 0$. □

The following example shows that $x \sqcup y$ is just an upper bound of $\{x, y\}$, not the supremum.

Example 3.2 ([13])**.** Suppose $A = \{0, a, b, c, 1\}$, with $0 < a$, $b < c < 1$. We define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(A, \odot, \to, \wedge, 1)$ is a bounded semihoop, where $x \wedge y = x \odot y$ for all $x, y \in A$. By calculation we have $a \sqcup b = 1$, while $a \vee b = c$.

Now we begin to introduce the definition of derivations and study some related properties.

Definition 3.3. *Let A be a semihoop and $d : A \to A$ be a function. d is called a derivation on A, if it satisfies the condition: for any $x, y \in A$,*

$$d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y)).$$

If we define a function $d : A \to A$ by $d(x) = 0$, for all $x \in A$, then d is a derivation on A, which is called a zero derivation. If we define $d(x) = x$, for all $x \in A$, then d is a derivation on A, which is called an identity derivation.

Here we give some examples of derivations on a semihoop A.

Example 3.4. (1) In Example 3.2, for all $x \in A$, we define:

$$d_1(x) = \begin{cases} 0, & \text{if } x = 0, 1 \\ a, & \text{if } x = a \\ b, & \text{if } x = b \\ c, & \text{if } x = c \end{cases}$$

$$d_2(x) = \begin{cases} 0, & \text{if } x = 0, a \\ b, & \text{if } x = b, c, 1. \end{cases}$$

It is easy to verify that d_1, d_2 are all derivations on A.

(2) Given that $I = [0, 1]$ is the unit interval. For all $x, y \in I$, we define $\odot$ and $\to$ as follows:

$$x \odot y = x \wedge y = \min\{x, y\}$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

Then $(I, \odot, \to, \wedge, 1)$ is a bounded semihoop. We define $d(x) = a \odot x$ for all $a \in [0, 1]$, one can easily check that d is a derivation on I.

Proposition 3.5. *Let A be a bounded semihoop and d be a derivation on A. Then the following properties hold for all $x, y \in A$:*

(1) $d(0) = 0$;
(2) $d(x) \odot x^* = x \odot (d(x))^* = 0$;
(3) $x \odot d(1) \leq d(x) \leq x^{**}$;
(4) $d(x^n) = x^{n-1} \odot d(x)$, *for any* $n \in N_+$;
(5) $x \leq y^*$ *implies* $d(y) \leq x^*$ *and* $d(x) \leq y^*$;
(6) $d(x^*) \leq (d(x))^*$.

Proof. (1) Since d is a derivation on A, we conclude that $d(0) = d(0 \odot 0) = (d(0) \odot 0) \sqcup (0 \odot d(0)) = 0 \sqcup 0 = 0$.

(2) By Proposition 2.3(3) $x \odot x^* = 0$ we have $0 = d(0) = d(x \odot x^*) = (d(x) \odot x^*) \sqcup (x \odot (d(x))^*)$. So we get $d(x) \odot x^* = 0$ and $x \odot (d(x))^* = 0$.

(3) For all $x \in A$, we have $d(x) = d(x \odot 1) = (d(x) \odot 1) \sqcup (x \odot d(1)) = d(x) \sqcup (x \odot d(1))$. Thus by Proposition 3.1(2) we can get $x \odot d(1) \leq d(x)$. By (2) $d(x) \odot x^* = 0$ we obtain $d(x) \leq x^{**}$.

(4) Let $n = 1$ we have $d(x) = x^0 \odot d(x) = d(x)$. Suppose the equation is established when $n = k$. That is, $d(x^k) = x^{k-1} \odot d(x)$. Now we will prove that the equation is established when $n = k+1$. By Definition 3.3 we have $d(x^{k+1}) = d(x^k \odot x) = (d(x^k) \odot x) \sqcup (x^k \odot d(x)) = ((x^{k-1} \odot d(x)) \odot x) \sqcup (x^k \odot d(x)) = x^k \odot d(x)$. Therefore we can obtain $d(x^n) = x^{n-1} \odot d(x)$.

(5) For all $x, y \in A$, assume that $x \leq y^*$. By Proposition 2.2(1), we have $x \odot y = 0$. Then $0 = d(0) = d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$. It follows from Proposition 3.1(4) that $d(x) \odot y = 0$ and $x \odot d(y) = 0$. Thus we have $d(x) \leq y^*$ and $d(y) \leq x^*$.

(6) By Proposition 2.3(2) $x \leq x^{**}$, we infer $d(x) \leq x^{**}$ by the statement (5), which implies $d(x^*) \leq x^{***}$, by Proposition 2.2(7) we can obtain $x^{***} \leq (d(x))^*$. Therefore, we conclude that $d(x^*) \leq (d(x))^*$. □

In what follows, we introduce the ideal derivation in semihoops and investigate some related properties of them.

Definition 3.6. *Let d be a derivation on a semihoop A.*

(1) *d is called an isotone derivation provided that $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in A$;*

(2) *d is called a contractive derivation provided that $d(x) \leq x$ for all $x \in A$;*

(3) *d is called an ideal derivation provided that d is both an isotone and a contractive derivation;*

(4) *d is called an idempotent derivation provided that $d^2 = d$, where $d^2(x) = d(d(x))$ for all $x \in A$.*

Example 3.7. (1) In Example 3.4(1), d_1 is a contractive derivation, however, d_1 is not an isotone derivation, since $b \leq 1$, $b = d_1(b) \not\leq d_1(1) = 0$. One can easily check that d_2 is an ideal derivation.

(2) Let $A = \{0, a, b, 1\}$ and $0 < a < b < 1$. For all $x, y \in A$, we define $x \wedge y = min\{x, y\}$, $\odot$ and $\rightarrow$ as follows,

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	b	b	1	1
1	0	a	b	1

Then $(A, \odot, \rightarrow, \wedge, 1)$ is a bounded semihoop. We define:

$$d_1(x) = \begin{cases} 0, & \text{if } x = 0 \\ b, & \text{if } x = a, b, 1 \end{cases}$$

$$d_2(x) = \begin{cases} 0, & \text{if } x = 0, 1 \\ a, & \text{if } x = b \\ b, & \text{if } x = a. \end{cases}$$

It is easy to verify that d_1 is an isotone derivation but it is not a contractive derivation, since $b = d_1(a) \not\leq a$. One can easily check that d_2 is a derivation on A, but it is neither an isotone derivation nor a contractive derivation.

Theorem 3.8. *Let A be a semihoop. For $a \in A$, $d_a : A \rightarrow A$ is a map defined by $d_a(x) = a \odot x$, for all $x \in A$. Then d_a is an ideal derivation on A, which is called a principal ideal derivation.*

Proof. For all $x, y \in A$, $d_a(x \odot y) = a \odot x \odot y = (a \odot x \odot y) \sqcup (x \odot a \odot y) = (d_a(x) \odot y) \sqcup (x \odot d_a(y))$, so we have d_a is a derivation. Moreover, if $x \leq y$, by Proposition 2.2(2) we have $d_a(x) = a \odot x \leq a \odot y = d_a(y)$, thus d_a is an isotone derivation. Since $d_a(x) = a \odot x \leq x$, we get d_a is a contractive derivation. Hence d_a is an ideal derivation on A. □

We denote the set of all idempotent elements of the monoid $(A, \odot, 1)$ by $G(A)$, that is, $G(A) = \{x \in A | x \odot x = x\}$.

Definition 3.9. *Let d be a derivation on a semihoop A. Then d is called a regular derivation if $d(1) \in G(A)$.*

Example 3.10. All derivations in Example 3.4 are regular derivations. But not every derivation is a regular derivation. In Example 3.7(2), d_1 is not a regular derivation, since $0 = d(1) \odot d(1) \neq d(1) = b$.

Theorem 3.11. *Let A be a semihoop. Then $a \in G(A)$ if and only if d_a is a regular derivation on A.*

Proof. For all $a \in G(A)$, we have $d_a(1) = a \odot 1 = a \odot a \odot 1 \odot 1 = (a \odot 1) \odot (a \odot 1) = d_a(1) \odot d_a(1)$, hence $d_a(1) \in G(A)$, therefore d_a is a regular derivation. Conversely, if d_a is a regular derivation, then $d_a(1) \odot d_a(1) = d_a(1)$, that is, $(a \odot 1) \odot (a \odot 1) = a \odot 1$, which implies $a \odot a = a$, thus we can obtain $a \in G(A)$. □

Example 3.4(1) shows that the derivation on a semihoop A is not unique, thus it is necessary to discuss the existence of derivations on semihoops.

Corollary 3.12. *Let A be a semihoop with finite elements. Then:*

(1) *A has at least $|A|$ (ideal) derivations ($|A|$ represents the number of elements contained in A);*

(2) *A has at least $|G(A)|$ regular (ideal) derivations.*

Proof. (1) Define a map $d_a : A \to A$ by $d_a(x) = a \odot x$, for all $a \in A$. By Theorem 3.8 we have d_a is a derivation on A. Let $a \neq b$, then we show that $d_a \neq d_b$. Suppose that $d_a = d_b$, then $d_a(x) = d_b(x)$, for all $x \in A$. Thus $a \odot x = b \odot x$. Now if $x = 1$, then $a \odot 1 = b \odot 1$. We get $a = b$, which is contradictory. Therefore if $a \neq b$, then $d_a \neq d_b$. So A has at least $|A|$ (ideal) derivations.

(2) It follows from Theorem 3.11. □

Corollary 3.12 shows that a semihoop must have principal ideal derivations. The following example shows that there exists other (ideal) derivations besides the principal (ideal) derivations.

Example 3.13. In Example 3.4(1), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a \\ b, & \text{if } x = b, c, 1. \end{cases}$$

It is easy to verify that d is an (ideal) derivation rather than a principal (ideal) derivation on a bounded semihoop A.

Proposition 3.14. *Let d be an isotone derivation on a semihoop A. Then for all $x, y, z \in A$ we have:*

(1) *$z \leq x \to y$ implies $z \leq d(x) \to d(y)$ and $x \leq d(z) \to d(y)$,*

(2) *$x \to y \leq d(x) \to d(y)$, $d(x \to y) \leq x \to d(y)$.*

Proof. (1) By Proposition 2.2(1) we have $z \leq x \to y$ implies $x \odot z \leq y$ for all $x, y, z \in A$. Since d is an isotone derivation, we conclude that $d(x \odot z) \leq d(y)$. It follows from Definition 3.3 that $d(x \odot z) = (d(x) \odot z) \sqcup (x \odot d(z))$. Then we have $(d(x) \odot z) \sqcup (x \odot d(z)) \leq d(y)$, which implies $d(x) \odot z \leq d(y)$ and $x \odot d(z) \leq d(y)$. Thus we get $z \leq d(x) \to d(y)$ and $x \leq d(z) \to d(y)$.

(2) By Proposition 2.2(3) $x \odot (x \to y) \leq y$ for all $x, y \in A$ and d is an isotone derivation. We can obtain $d(x \odot (x \to y)) \leq d(y)$. By Definition 3.3 $d(x \odot (x \to y)) = (d(x) \odot (x \to y)) \sqcup (x \odot d(x \to y)) \leq d(y)$, which implies $d(x) \odot (x \to y) \leq d(y)$ and $x \odot d(x \to y) \leq d(y)$. Therefore, we have $x \to y \leq d(x) \to d(y)$ and $d(x \to y) \leq x \to d(y)$. □

Proposition 3.15. *Let d be a contractive derivation on a semihoop A. Then for all $x, y \in A$ we have:*

(1) $d(x) \odot d(y) \leq d(x \odot y)$,

(2) *d is an isotone derivation implies* $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to y$,

(3) $d(1) = 1$ *implies d is an identity derivation.*

Proof. (1) For all $x, y \in A$, since d is a contractive derivation, we have $d(x) \odot d(y) \leq x \odot d(y)$ and $d(x) \odot d(y) \leq d(x) \odot y$. It follows that $d(x) \odot d(y) \leq (d(x) \odot y) \sqcup (x \odot d(y)) = d(x \odot y)$.

(2) By Proposition 2.2(3) $x \odot (x \to y) \leq y$ for all $x, y \in A$ and d is an isotone derivation. We deduce $d(x \odot (x \to y)) \leq d(y)$. From the statement (1), we conclude that $d(x \to y) \odot d(x) \leq d(x \odot (x \to y))$. It follows that $d(x \to y) \odot d(x) \leq d(y)$, which implies $d(x \to y) \leq d(x) \to d(y)$. On the other hand, from $d(y) \leq y$, we infer $d(x) \to d(y) \leq d(x) \to y$ by Proposition 2.2(7). Therefore, we obtain $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to y$.

(3) By Proposition 3.5(3), we have $x \odot d(1) \leq d(x)$ for all $x \in A$. Assume that $d(1) = 1$, we get $x = x \odot d(1) \leq d(x) \leq x$ showing that $d(x) = x$. Therefore, d is an identity derivation. □

Proposition 3.16. *Let d be a derivation on a semihoop A. Then the following conditions are equivalent:*

(1) *d is an ideal idempotent derivation;*

(2) *d satisfies $d(x) \to d(y) = d(x) \to y$ for all $x, y \in A$.*

Proof. (1) $\Rightarrow$ (2) Suppose that d is an ideal idempotent derivation. Then we have that $d(y) \leq y$ for all $y \in A$, it follows that $d(x) \to d(y) \leq d(x) \to y$ by Proposition 2.2(7). On the other hand, let $t \leq d(x) \to y$ for all $t \in A$. We can obtain $d(x) \odot t \leq y$. Since d is an isotone derivation, we have $d(d(x) \odot t) \leq d(y)$ for all $x, y, t \in A$. By Definition 3.3 $d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$, we get $d(x) \odot y \leq d(x \odot y)$. It follows that $d(d(x)) \odot t \leq d(d(x) \odot t)$. Combining $d^2 = d$, we have $d(x) \odot t \leq d(d(x) \odot t) \leq d(y)$. Hence $t \leq d(x) \to d(y)$, which implies $d(x) \to y \leq d(x) \to d(y)$. Therefore we obtain $d(x) \to d(y) = d(x) \to y$.

(2) $\Rightarrow$ (1) Assume that $d(x) \to d(y) = d(x) \to y$ for all $x, y \in A$. By Proposition 2.2(4) we have $1 = d(x) \to d(x) = d(x) \to x$. That means $d(x) \leq x$. Thus, d is a contractive derivation. Moreover, suppose that $x \leq y$ for some $x, y \in A$, we have $d(x) \odot 1 = d(x) \leq x \leq y$. Thus by Proposition 2.2(1) we have $1 \leq d(x) \to y = d(x) \to d(y)$, which implies $d(x) \odot 1 \leq d(y)$, that is, $d(x) \leq d(y)$. Therefore, d is an isotone derivation. Hence d is an ideal derivation. Finally, for all $x \in A$, since $d(x) \odot 1 \leq d(x)$, we conclude that $1 \leq d(x) \to d(x) = d(x) \to$

$d(d(x))$. Then $d(x)\odot 1 \leq d(d(x))$, we get $d(x) \leq d(d(x))$. Since d is a contractive derivation we have $d(d(x)) \leq d(x)$. Thus, we obtain $d(d(x)) = d(x)$ for all $x \in A$, that means, $d^2 = d$. Therefore, d is an ideal idempotent derivation. □

Next, we will discuss the structures and properties of the fixed points set of ideal derivations.

Let d be a derivation on a semihoop A. We denote the set of all fixed points of A for d by $Fix_d(A)$, that is, $Fix_d(A) = \{x \in A | d(x) = x\}$.

Proposition 3.17. *Let d be a contractive derivation on a semihoop A. Then for all $x, y \in Fix_d(A)$, $x \odot y \in Fix_d(A)$, that is, $Fix_d(A)$ is closed under $\odot$.*

Proof. For all $x, y \in Fix_d(A)$, we have $d(x) = x$ and $d(y) = y$. It follows from Proposition 3.15 (1) that $x \odot y = d(x) \odot d(y) \leq d(x \odot y) \leq x \odot y$, which implies that $d(x \odot y) = x \odot y$. Thus, $x \odot y \in Fix_d(A)$. □

Theorem 3.18. *Let d be an ideal idempotent derivation on a semihoop A. Then $(Fix_d(A), \odot, \mapsto, \sqcap, \overline{1})$ is a semihoop, where $x \sqcap y = d(x \wedge y)$, $x \mapsto y = d(x \to y)$, $\overline{1} = d(1)$ for all $x, y \in Fix_d(A)$.*

Proof. By Proposition 3.17 $d(x \odot y) = x \odot y$ we have $d(1) = d(1 \odot 1) = 1 \odot 1 = 1$.
(1) First, we show that $(Fix_d(A), \sqcap, \overline{1})$ is a $\wedge$-semilattice with upper bound $\overline{1}$. For all $x, y \in Fix_d(A)$, $d(x \wedge y) \leq d(x) = x$, $d(x \wedge y) \leq d(y) = y$, so $d(x \wedge y)$ is a lower bound of x and y in $Fix_d(A)$. Now let $z \in Fix_d(A)$ and $z \leq x, y$. Then $z \leq x \wedge y$, so we have $z = d(z) \leq d(x \wedge y)$, thus the infimum of x and y exists in $Fix_d(A)$ and is $d(x \wedge y)$. By $d(1) = 1$ we get $d(d(1)) = d(1)$, that is, $d(1) \in Fix_d(A)$. For all $x \in Fix_d(A)$, $x = d(x) \leq d(1)$. Therefore $(Fix_d(A), \sqcap, \overline{1})$ is a $\wedge$-semilattice with upper bound $\overline{1}$.

(2) Next, we prove that $(Fix_d(A), \odot, \overline{1})$ is a commutative monoid with $\overline{1}$ as unit element. By Proposition 3.17 we get $x \odot y \in Fix_d(A)$. For all $x \in Fix_d(A)$, $x \odot d(1) = x$. so $(Fix_d(A), \odot, \overline{1})$ is a commutative monoid.

(3) Finally, we prove that $(x \odot y) \mapsto z = x \mapsto (y \mapsto z)$ for all $x, y, z \in Fix_d(A)$. Since $d(y \to z) \leq y \to z$, we have $x \to d(y \to z) \leq x \to (y \to z)$, then $(x \odot y) \mapsto z = d((x \odot y) \to z) = d(x \to (y \to z)) \geq d(x \to d(y \to z)) = x \mapsto (y \mapsto z)$. That is $(x \odot y) \mapsto z \geq x \mapsto (y \mapsto z)$. On the other hand, by Proposition 3.14 we have $d((x \odot y) \to z) = d(x \to (y \to z)) \leq x \to d(y \to z)$, since d is an ideal idempotent derivation we have $d((x \odot y) \to z) = d(d((x \odot y) \to z)) \leq d(x \to d(y \to z)) = x \mapsto (y \mapsto z)$, that is $(x \odot y) \mapsto z \leq x \mapsto (y \mapsto z)$. Hence $(x \odot y) \mapsto z = x \mapsto (y \mapsto z)$. Thus $(Fix_d(A), \odot, \mapsto, \sqcap, \overline{1})$ is a semihoop. □

The following example shows that $Fix_d(A)$ is not an ideal of a semihoop A.

Example 3.19. In Example 3.7(2), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a, b \\ c, & \text{if } x = c, 1. \end{cases}$$

It is easy to verify that d is an ideal derivation and $Fix_d(A) = \{0, a, c\}$. One can check that $Fix_d(A)$ is not an ideal because $c \in Fix_d(A)$ and $b \leq c$, but $b \notin Fix_d(A)$.

In general semihoops, $Fix_d(A)$ is not an ideal. However, if the conditions are strengthened, we can obtain the following result.

Theorem 3.20. *Let d be an ideal derivation on a $\vee$-hoop A. Then $Fix_d(A)$ is a lattice ideal on A.*

Proof. From Proposition 2.10 we have $\vee$-hoop is a distributive lattice. Next we will prove that if $x \in Fix_d(A)$ and $y \leq x$ then $y \in Fix_d(A)$. Let $x \in Fix_d(A)$ and $y \leq x$. It follows from $y \leq x$ that $x \wedge y = y$. So $d(y) = d(x \wedge y) = d(x \odot (x \to y)) = (d(x) \odot (x \to y)) \vee (x \odot d(x \to y)) = (x \odot (x \to y)) \vee (x \odot d(x \to y)) = (x \wedge y) \vee (x \odot d(x \to y)) = y \vee (x \odot d(x \to y)$, which implies $d(y) \geq y$. Combining $d(y) \leq y$ we obtain $d(y) = y$. Therefore $y \in Fix_d(A)$. Finally we will prove that if $x, y \in Fix_d(A)$ then $x \vee y \in Fix_d(A)$. Since d is an ideal derivation on A, we have $x \vee y = d(x) \vee d(y) \leq d(x \vee y) \leq x \vee y$. Hence we can obtain $d(x \vee y) = x \vee y$, that is, $x \vee y \in Fix_d(A)$. Thus $Fix_d(A)$ is an ideal on A. □

The following is a counter example showing that the converse of Theorem 3.20 may not hold.

Example 3.21. In Example 3.7 (2), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a, b, 1. \end{cases}$$

One can easily check that d is an ideal derivation on a bounded semihoop A and $Fix_d(A) = \{0, a\}$ is an ideal of A. However, A does not a $\vee$-hoop. Since $a = a \odot (a \to b) \neq b \odot (b \to a) = 0$.

It is natural to ask for every non-trivial ideal I of a general semihoop A whether there exists an ideal derivation d such that $Fix_d(A) = I$. The following theorem proves that the conclusion for prime ideals is established.

Theorem 3.22. *Let I be a non-trivial prime ideal of a semihoop A. Then there exists a derivation d such that $Fix_d(A) = I$.*

Proof. Define a function d as follows:

$$d(x) = \begin{cases} x, & \text{if } x \in I \\ a \odot x, & \text{if } x \in A \backslash I, \end{cases}$$

where $a \in I$.

We claim that d is a derivation. In fact, if $x, y \in I$, then by Proposition 2.2(5) $x \odot y \leq x$ and I is an ideal we have $x \odot y \in I$. Hence we can see that

$d(x\odot y)=x\odot y=(x\odot y)\sqcup(x\odot y)=(d(x)\odot y)\sqcup(x\odot d(y))$. If $x\in I$, $y\in A\backslash I$, then $x\odot y\leq x$ and so $x\odot y\in I$. Thus $d(x\odot y)=x\odot y$, since $x\odot a\odot y\leq x\odot y$, by Proposition 3.1(2) we can get $(d(x)\odot y)\sqcup(x\odot d(y))=(x\odot y)\sqcup(x\odot a\odot y)=x\odot y$. This shows that $d(x\odot y)=(d(x)\odot y)\sqcup(x\odot d(y))$. If $x,y\in A\backslash I$, then $x\odot y\in A\backslash I$ since A is a prime ideal. Hence, $d(x\odot y)=a\odot x\odot y$, $(d(x)\odot y)\sqcup(x\odot d(y))=(a\odot x\odot y)\sqcup(x\odot a\odot y)=a\odot x\odot y$, thus $d(x\odot y)=(d(x)\odot y)\sqcup(x\odot d(y))$. By the above argument we can get that d is a derivation. Clearly, $Fix_d(A)=I$. □

4. Differential filters in differential semihoop

In this section, we introduce differential filters of differential semihoops. We focus on algebraic structures of the set $\mathcal{DF}(A)$ of all differential filters in the differential semihoop (A,d). Moreover, we study the relationships between different differential filters.

Definition 4.1. *A pair (A,d) is called a differential semihoop, if A is a semihoop and d is a derivation on A.*

If d is an ideal derivation on A, then (A,d) is called an ideal differential semihoop.

Definition 4.2. *Let (A,d) be a differential semihoop. A filter F of A is called a differential filter of (A,d) if $x\in F$ implies $d(x)\in F$ for all $x\in A$.*

We will denote the set of all differential filters of (A,d) by $\mathcal{DF}(A)$.
A differential filter F of (A,d) is called a proper differential filter if $F\neq A$.

Example 4.3. In Example 3.4(1), one can easily check that the differential filters of (A,d_2) are $\{b,c,1\}$ and A. Proper differential filter is $\{b,c,1\}$.

Theorem 4.4. *Let (A,d) be an ideal differential semihoop. Then $Ker(d)=\{x\in A|d(x)=1\}$ is a differential filter of (A,d).*

Proof. Let $x,y\in Ker(d)$. Then by Proposition 3.15 we have $d(x\odot y)\geq d(x)\odot d(y)=1\odot 1=1$, so $d(x\odot y)=1$, we get $x\odot y\in Ker(d)$. Let $x\in Ker(d)$, $x\leq y$. Then $1=d(x)\leq d(y)$, so $d(y)=1$, $y\in Ker(d)$. Thus $Ker(d)$ is a filter. We can obtain $1\in Ker(d)$ which implies $d(1)=1$. If $x\in Ker(d)$, then $d(x)=1$, $d(d(x))=d(1)=1$, we have $d(x)\in Ker(d)$. Therefore $Ker(d)$ is a differential filter. □

Let (A,d) be a differential semihoop. For any nonempty subset X of A, we denote by $\langle X\rangle_d$ the differential filter of (A,d) generated by X, that is, $\langle X\rangle_d$ is the smallest differential filter of (A,d) containing X. If F is a differential filter of (A,d) and $x\notin F$, put $\langle F,x\rangle_d=\langle F\cup\{x\}\rangle_d$.

Theorem 4.5. *Let (A,d) be an ideal differential semihoop and X be a nonempty subset of A. Then $\langle X\rangle_d=\{x\in A|x\geq d^{l_1}(y_1)\odot\cdots\odot d^{l_n}(y_n),\ y_i\in X,\ \text{for some } l_i,n\in N_+\}$.*

Proof. (1) For all $x \in X$, d is an ideal derivation, we have $x \geq d(x)$, that means $x \in \langle X \rangle_d$, thus we get $X \subseteq \langle X \rangle_d$.

(2) First $1 \in \langle X \rangle_d$. Then, let $x, y \in \langle X \rangle_d$. We have $x \geq d(x)$, $y \geq d(y)$, we can conclude that $x \odot y \geq d(x) \odot d(y)$ which implies $x \odot y \in \langle X \rangle_d$; if $x \in \langle X \rangle_d$ and $x \leq y$, then $d(x) \leq x \leq y$, so $y \in \langle X \rangle_d$. Therefore $\langle X \rangle_d$ is a filter.

(3) Let $x \in \langle X \rangle_d$. We have $x \geq d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)$, for some $l_i \in N_+, y_i \in X$, then by Proposition 3.15 we obtain $d(x) \geq d(d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)) \geq d^{l_1+1}(y_1) \odot \cdots \odot d^{l_n+1}(y_n)$, so we get $d(x) \in \langle X \rangle_d$. Thus $\langle X \rangle_d$ is a differential filter.

(4) If there exists a differential filter T, $X \subseteq T$, then we will show that $\langle X \rangle_d \subseteq T$. Let $t \in \langle X \rangle_d$. We have $t \geq d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)$, for some $l_i \in N_+, y_i \in X$, by $X \subseteq T$ we have $y_i \in T$, then $d(y_i) \in T$, we get $d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n) \in T$, so we can obtain $t \in T$. Hence $\langle X \rangle_d \subseteq T$.

Therefore we show that $\langle X \rangle_d$ is the smallest differential filter of (A, d) containing X. □

If $X = \{x\}$, then $\langle x \rangle_d = \{a \in A | a \geq d^n(x), \text{ for some } n \in N\}$.

Proposition 4.6. *Let (A, d) be an ideal differential semihoop, $F \in \mathcal{DF}(A)$ and $x \in A$. Then $\langle F, x \rangle_d = \{a \in A | a \geq d^m(y) \odot d^n(x)$, for some $m, n \in N, y \in F\}$.*

Proof. Let $B = \{a \in A | a \geq d^m(y) \odot d^n(x)$, for some $m, n \in N, y \in F\}$. Similar to the proof of Theorem 4.5, $B \in \mathcal{DF}(A)$. Now let $y \in F$, and $m, n \in N$. By $d^m(y) \odot d^n(x) \leq d(y) \leq y$ we have $y \in B$. Also $d^m(y) \odot d^n(x) \leq d(x) \leq x$ we have $x \in B$, so $F \cup \{x\} \subseteq B$. Next to prove that B is the smallest differential filter that include $F \cup \{x\}$. Let $C \in \mathcal{DF}(A)$ such $F \cup \{x\} \subseteq C$. Then, for any $z \in B$, there exist $m, n \in N$ and $y \in F$, such $d^m(y) \odot d^n(x) \leq z$. Thus $d^m(y) \leq d^n(x) \to z$. Since $C \in \mathcal{DF}(A)$ and $F \subseteq C$, we have $y \in C$, then $d^m(y) \in C$, so $d^n(x) \to z \in C$. Moreover, since $C \in \mathcal{DF}(A)$ and $x \in C$, we have $d^n(x) \in C$, thus $z \in C$, hence $B \subseteq C$. Therefore $\langle F, x \rangle_d = B$. □

Proposition 4.7. *Let (A, d) be an ideal differential semihoop and $F_1, F_2 \in \mathcal{DF}(A)$. Then $\langle F_1 \cup F_2 \rangle_d = \{a \in A | a \geq d^m(y_1) \odot d^n(y_2)$, for some $m, n \in N, y_1 \in F_1, y_2 \in F_2\}$.*

Proof. It follows from by Proposition 4.6. □

Proposition 4.8. *Let (A, d) be an ideal differential semihoop, $F \in \mathcal{DF}(A)$ and $x \in Fix_d(A)$. Then $\langle F \cup \{x\} \rangle$ is also a differential filter of (A, d).*

Proof. Assume that $y \in \langle F \cup \{x\} \rangle$. By Proposition 2.12 there exists $n \in N, c \in F$ such that $c \odot x^n \leq y$. Thus $d(y) \geq d(c \odot x^n) \geq d(c) \odot d(x^n) = d(c) \odot x^n$ by Proposition 3.15(1). Thus $d(y) \in \langle F \cup \{x\} \rangle$, hence $\langle F \cup \{x\} \rangle$ is a differential filter of (A, d). □

Proposition 4.9. *Let (A, d) be an ideal differential $\vee$-semihoop. Then $\langle x \vee y \rangle_d = \langle x \rangle_d \cap \langle y \rangle_d$ for all $x, y \in A$.*

Proof. It is obvious that $x \in \langle x\rangle_d$, $y \in \langle y\rangle_d$. Since $x, y \leq x \vee y$, it follows that $x\vee y \in \langle x\rangle_d$ and $x\vee y \in \langle y\rangle_d$. So $x\vee y \in \langle x\rangle_d \cap \langle y\rangle_d$. Hence $\langle x\vee y\rangle_d \subseteq \langle x\rangle_d \cap \langle y\rangle_d$.

Conversely, let $z \in \langle x\rangle_d \cap \langle y\rangle_d$. Then $d^n(x) \leq z$, $d^m(y) \leq z$, for some $n, m \in N$. Then we will show that $z \in \langle x \vee y\rangle_d$, that is, $z \geq d^{n+m}(x \vee y) \geq d((x \odot y))^{n+m} \geq (d(x) \odot d(y))^{n+m}$. By Proposition 2.6 we have $(d(x) \vee d(y))^{n+m} \to z = \bigwedge\{(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z | d(x_i) \in \{d(x), d(y)\}\}$. Consider $d(x_1), ..., d(x_{n+m}) \in \{d(x), d(y)\}$. Denote by r the number of occurrences of $d(x)$ in the sequence $d(x_1), ..., d(x_{n+m})$, and by s the number of occurrences of $d(y)$ in the sequence $d(x_1), ..., d(x_{n+m})$. Of course, $r+s = n+m$. By Proposition 2.2(5) we have that $d(x_1)\odot\cdots\odot d(x_{n+m}) \leq (d(x))^r$, and $d(x_1)\odot\cdots\odot d(x_{n+m}) \leq (d(y))^s$. Then by Proposition 2.2(7) we get $(d(x))^r \to z \leq (d(x_1) \odot \cdots \odot d(x_{n+m})) \to z$ and $(d(y))^s \to z \leq (d(x_1) \odot \cdots \odot d(x_{n+m})) \to z$. If $r \leq n$, then $s \geq m$, so $(d(y))^s \leq (d(y))^m$. Thus $(d(y))^s \to z \geq (d(y))^m \to z = 1$, that is $(d(y))^s \to z = 1$, we have $(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z = 1$. Similarly if $r > n$, we can obtain $(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z = 1$. So $(d(x) \vee d(y))^{n+m} \to z = 1$. Thus $(d(x) \odot d(y))^{n+m} \leq (d(x) \vee d(y))^{n+m} \leq z$. Therefore $z \in \langle x \vee y\rangle_d$. Hence $\langle x \vee y\rangle_d = \langle x\rangle_d \cap \langle y\rangle_d$. □

Let (A, d) be a differential semihoop. Setting $D((A, d)) = \{x \in A | (d(x))^n > 0$, for any $n \in N_+\}$.

Proposition 4.10. *Let (A, d) be a differential semihoop and F be a proper differential filter of (A, d). Then $F \subseteq D((A, d))$.*

Proof. Assume that F be a proper differential filter of (A, d) and $x \in F$. Then we have $d(x) \in F$, so $(d(x))^n \in F$, for any $n \in N_+$. Since $0 \notin F$ we have $(d(x))^n > 0$ for any $n \in N_+$. Thus $x \in D((A, d))$. Therefore $F \subseteq D((A, d))$. □

Proposition 4.11. *Let (A, d) be an ideal differential semihoop. If $(d(x))^n > 0$ and $(d(y))^n > 0$ imply $(d(x))^n \odot (d(y))^n > 0$, then $D((A, d))$ is a differential filter of (A, d).*

Proof. By $(d(1))^n > (d(0))^n = 0$ for any $n \in N_+$, we have $1 \in D((A, d))$. Let $x, x \to y \in D((A, d))$. Then $(d(x))^n > 0$, $(d(x \to y))^n > 0$ for any $n \in N_+$, by Proposition 3.15(1) we have $(d(y))^n \geq (d(x \odot (x \to y)))^n \geq (d(x))^n \odot (d(x \to y))^n > 0$. So $y \in D((A, d))$. Finally if $x \in D((A, d))$, then $(d(x))^n > 0$, $d(d(x))^n > d(d(0))^n = 0$, so $d(x) \in D((A, d))$. Therefore $D((A, d))$ is a differential filter of (A, d). □

Definition 4.12. *A proper differential filter F of a differential semihoop (A, d) is called a prime differential filter, if for any $H, G \in \mathcal{DF}(A)$ such that $H \cap G \subseteq F$, then $H \subseteq F$ or $G \subseteq F$.*

Proposition 4.13. *Let (A, d) be an ideal differential $\vee$-semihoop and F be a proper differential filter of (A, d). Then the following conditions are equivalent:*

(1) *F is a prime differential filter of (A, d);*

(2) *If $x \vee y \in F$ for some $x \in F$ or $y \in F$.*

Proof. (1) $\Rightarrow$ (2) Let $x \vee y \in F$ for some $x, y \in A$. By Proposition 4.9 $\langle x\rangle_d \cap \langle y\rangle_d = \langle x \vee y\rangle_d \subseteq F$. Since F is a prime differential filter, we have $\langle x\rangle_d \subseteq F$ or $\langle y\rangle_d \subseteq F$, thus we get $x \in F$ or $y \in F$.

(2) $\Rightarrow$ (1) Suppose that $H, G \in \mathcal{DF}(A)$ such that $H \cap G \subseteq F$, $H \nsubseteq F$ and $G \nsubseteq F$. Then there are $x \in H$ and $y \in G$ such that $x, y \notin F$. Since $x, y \leq x \vee y$ and $H, G \in \mathcal{DF}(A)$, we have $x \vee y \in H \cap G \subseteq F$. By (2) we have $x \in F$ or $y \in F$, which is contradiction. Hence, F is a prime differential filter of (A, d). □

Definition 4.14. *A proper differential filter F of a differential semihoop (A, d) is called a maximal differential filter, if it is not properly contained in the any other proper differential filter.*

We denote the set of all maximal differential filters of (A, d) by $\mathcal{DM}(A)$.

Example 4.15. In Example 3.4 (1), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a \\ b, & \text{if } x = b \\ c, & \text{if } x = c, 1. \end{cases}$$

Then d is an ideal derivation on A. The all differential filters are $F_1 = \{c, 1\}$, $F_2 = \{a, c, 1\}$, $F_3 = \{b, c, 1\}$ and A. F_2, F_3 are not only prime differential filters also maximal differential filters.

Proposition 4.16. *Let (A, d) be an ideal differential semihoop and F be a proper differential filter of (A, d). Then the following conditions are equivalent:*

(1) *$F \in \mathcal{DM}(A)$;*

(2) *If $x \notin F$, then $\langle F, x\rangle_d = A$.*

Proof. (1) $\Rightarrow$ (2) If $x \notin F$ then we get $F \subset \langle F, x\rangle_d$. Since $F \in \mathcal{DM}(A)$, we have $\langle F, x\rangle_d = A$.

(2) $\Rightarrow$ (1) Let G be a proper differential filter of (A, d) such $F \subseteq G$ and $F \neq G$. Then there exists $x \in G$ such that $x \notin F$. By (2) we have $\langle F, x\rangle_d = A$. Now since $A = \langle F, x\rangle_d \subseteq G$, we get $G = A$, which is a contradiction. Therefore, $F \in \mathcal{DM}(A)$. □

Proposition 4.17. *Let (A, d) be an ideal differential bounded semihoop, $M \in \mathcal{DM}(A)$ and $x \in A$. Then $x \notin M$ if and only if there exists $n \in N$ such that $(d^n(x))^* \in M$.*

Proof. On the one hand, let $M \in \mathcal{DM}(A)$ and $x \in A$. Suppose that $x \notin M$, by Proposition 4.16 $A = \langle M, x\rangle_d$, and so we have $0 \in \langle M, x\rangle_d$. Then by Proposition 4.6, there exist $n \in N$ and $y \in M$ such that $d^m(y) \odot d^n(x) \leq 0$. Thus $d^m(y) \leq d^n(x) \to 0$. Moreover, since $M \in \mathcal{DF}(A)$ and $y \in M$, we have $d^m(y) \in M$, so $d^n(x) \to 0 \in M$. Hence $(d^n(x))^* \in M$.

On the other hand, let $x \in A$ and there exists $n \in N$ such that $(d^n(x))^* \in M$. If $x \in M$, then $d^n(x) \in M$, for any $n \in N$. Since $M \in \mathcal{DF}(A)$, by Proposition 2.3(3) $0 = d^n(x) \odot (d^n(x))^* \in M$, which is contradictory with M is a proper differential filter. □

Proposition 4.18. *Let (A, d) be an ideal differential $\vee$-semihoop. Then every maximal differential filter of (A, d) is a prime differential filter.*

Proof. Let $M \in \mathcal{DM}(A)$. Suppose that $H, G \in \mathcal{DF}(A)$, such that $H \cap G \subseteq M$. Let $H, G \nsubseteq M$. Then there are $x \in H$ and $y \in G$ such that $x, y \notin M$. Since $M \in \mathcal{DM}(A)$, by Proposition 4.16, $\langle M, x\rangle_d = \langle M, y\rangle_d = A$. Moreover, since $x, y \leq x \vee y$, we have $x \vee y \in H \cap G \subseteq M$. Thus by Proposition 4.9 we have $A = \langle M, x\rangle_d \cap \langle M, y\rangle_d = \langle M, x \vee y\rangle_d = M$, which is a contradiction. Therefore M is a prime differential filter. □

Proposition 4.19. *Let (A, d) be an ideal differential $\vee$-semihoop. Then $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra in which the operations are given by $J \wedge_d K = J \cap K$, $J \vee_d K = \{x \in A | x \geq d^n(j) \odot d^m(k), j \in J, k \in K$, for some $n, m \in N\}$, $J \to_d K = \{x \in A | d(j) \vee x \in K$, for any $j \in J\}$, for all $J, K \in \mathcal{DF}(A)$.*

Proof. First we show that $(\mathcal{DF}(A), \vee_d, \wedge_d)$ is a lattice. It is clear that if J and K are differential filters of (A, d) then so is $J \cap K$, which is the largest differential filters of (A, d) and is contained in both J and K. Hence $J \wedge_d K$ exists in $\mathcal{DF}(A)$ and is $J \cap K$. Then, we observe that $J \vee_d K$ is a differential filters of (A, d). Clearly, $1 \in J \vee_d K$. Let $a \in J \vee_d K$, $a \leq b$. Then there exist $j \in J, k \in K$ such that $a \geq d^n(j) \odot d^m(k)$ for some $n, m \in N$. We have $b \geq a \geq d^n(j) \odot d^m(k)$, hence $b \in J \vee_d K$. Let $a, b \in J \vee_d K$. Then there exist $j_1, j_2 \in J, k_1, k_2 \in K$ such that $a \geq d^n(j_1) \odot d^m(k_1)$, $b \geq d^n(j_2) \odot d^m(k_2)$ for some $n, m \in N$. Since $d^n(j_1 \odot j_2) \leq d^n(j_1)$ and $d^m(k_1 \odot k_2) \leq d^m(k_1)$, we have $d^n(j_1 \odot j_2) \odot d^m(k_1 \odot k_2) \leq d^n(j_1) \odot d^m(k_1) \leq a$. Similarly we get $d^n(j_1 \odot j_2) \odot d^m(k_1 \odot k_2) \leq b$. Thus $a \odot b \geq d^{2n}(j_1 \odot j_2) \odot d^{2m}(k_1 \odot k_2)$. That means $a \odot b \in J \vee_d K$. Therefore $J \vee_d K$ is a filter. If $x \in J \vee_d K$, then there exist $j \in J, k \in K$ such that $d(x) \geq d(d^n(j) \odot d^m(k)) \geq d^{n+1}(j) \odot d^{m+1}(k)$. We have $d(x) \in J \vee_d K$. Thus $J \vee_d K$ is a differential filter. Now, by Proposition 4.7 we can see that $J \vee_d K = \langle J \cup K\rangle_d$, thus $J \vee_d K$ is the supremum of J and K. Therefore $(\mathcal{DF}(A), \vee_d, \wedge_d)$ is a lattice.

In order to prove that $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra, we need to prove that $J \wedge_d K \subseteq L$ if and only if $J \subseteq K \to_d L$. Assume that $J \wedge_d K \subseteq L$. Let $j \in J$ then $d(j) \in J$ and for any $k \in K$, we have $d(k) \vee j \geq j$, $d(k) \vee j \geq d(k)$. Hence $d(k) \vee j \in J \wedge_d K$. So $j \in K \to_d L$. Thus $J \subseteq K \to_d L$. Conversely, assume that $J \subseteq K \to_d L$. Let $x \in J \wedge_d K$. Then $x \in K \to_d L$. For any $k \in K$, we have $d(k) \vee x \in L$. Taking $k = x \in K$, we have $x \vee d(x) = x \in L$. Thus $J \wedge_d K \subseteq L$.

Therefore $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra. □

5. Conclusion

Derivations play an important role for researches in the theory of algebraic structures. In the paper, we extend the concept of derivations on semihoops and get some more general conclusions. It is proved that the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop. And for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. Finally, we focus on algebraic structures of the set $\mathcal{DF}(A)$ of all differential filters in differential semihoops, and prove that $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra.

In our further work, the following topics should be considered:

(1) For every ideal I of a semihoop A, whether there exists a derivation d such that $Fix_d(A) = I$.

(2) We can also introduce differential congruences, and then study the related properties and construct quotient semihoops by differential congruences.

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References

[1] H.E .Bell, G. Mason, *On derivations in near-rings and near-fields*, North-Holland Mathematics Studies, 137 (1987), 31-35.

[2] H.E. Bell, G. Mason, *On derivations in near-rings and rings*, Mathematical Journal of Okayama University, 34 (1992), 135-144.

[3] H.E. Bell, L.C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Mathematica Hungarica, (3-4) 53 (1989), 339-346.

[4] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer London, 2005.

[5] B. Bosbach, *Komplementäre Halbgruppen: Axiomatik und Arithmetik*, Fundamenta Mathematicae, 64 (1969), 257-287.

[6] B. Bosbach, *Komplementäre Halbgruppen. Kongruenzen und Quatienten*, Fundamenta Mathematicae, 69 (1970), 1-14.

[7] R.A. Borzooei, M. Aaly Kologani, *Local and perfect semihoops*, Journal of Intelligent and Fuzzy Systems, 29 (2015), 223-234.

[8] G. Brikhoof, *Lattice Theory*, American Mathematical Society Colloquium, 1940.

[9] F. Esteva, L. Godo, P. Hajek, F. Montagna, *Hoops and fuzzy logic*, Journal of Logic and Computation, 13 (2003), 532-555.

[10] G. Georgescu, L. Leustean, V. Preoteasa, *Pseudo-hoops*, Journal of Multiple-Valued logic and soft Computing, 11 (2005), 153-184.

[11] P.F. He, X.L. Xin, J.M. Zhan, *On derivations and their fixed point sets in residuated lattices*, Fuzzy Sets and Systems, 303 (2016), 97-113.

[12] Y.B. Jun, X.L. Xin, *On derivations of BCI-algebras*, Information Sciences, 159 (2004), 167-176.

[13] R.T. Khorami, A.B. Saeid, *Some Unitary Operators on Hoop-algebras*, Fuzzy information and Engineering, 9 (2017), 208-209.

[14] J. Liang, X.L. Xin, *On derivations of EQ-algebras*, Journal of Intelligent and Fuzzy Systems, (5) 35 (2018), 5573-5583.

[15] E. Posner, *Derivations in prime rings*, Proceeding of The America Mathematical Society, 8 (1957), 1093-1100.

[16] X.L. Xin, T.Y. Li, J.H. Lu, *On derivations of lattices*, Information Sciences, 178 (2008), 307-316.

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Comparing study between simplex method and Lagrange method in a linear programming problem

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Abstract. This study aims to discuss a different way to solve a linear programming problems. Two methods are discussed in this paper to determine a suitable method to solve these problems, and to determine which one is the easiest. We used : Simplex method and Lagrange method. Two methods were applied in general system to evaluate the result and compare between them. After that the researchers applying the numerical example to find the degree of readers satisfaction for these methods. The results of this study indicated that a two methods were a better, but the second method is easier than the first method.

Keywords: operation research, Lagrange multiplier, simplex method.

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1. Introduction

Statistical sciences specially operations research is related with different fields as mathematics, statistics, economics, psychology, engineering such as (Alsaraireh, et al., 2018; AL Wadi, et al., 2018). Also operation research used to make a new decision. Recently operation research became a professional science with respect to other science. The commerce field is developing and growing rapidly, and increasing the aware competency between all organizations in the world, and striving to be more successes, survival and achieving to a competitive advantage. Therefore, all organizations seek to minimize the cost and time, and increase their margin profit. For these reasons, this study aims to link between the solution of a linear programming problem by two methods (Simplex method and Lagrange method). For more details about the lagrange theorem refer to (Mangazarian, 1969), and (Rockefeller, 1970). In (Bussotti, P., 2003), Joseph-Louis Lagrange introduced the multiplier method in classical mechanics in his book. In optimization the multipliers are used in solving constrained optimization problems and study the stationary points of the Lagrange function.

2. Literature review

Operations research as a science has been used to help solve decision problems using mathematical and statistical models for a long time, and it has been developed in many scientific fields such as : Mathematics, Engineering and Management. It is one of the areas that has contributed to solving many of mathematical problems and management problems: simplex method and Lagrange method. The first book of operations researches appeared in 1946 As "Methods of Research Operations" for Morris and Campbell's. Some of the scientists developed a method of problem solving in the Simplex Method and other methods. In (Khachian, 1979) proposed a new method of solving the linear program, but theoretically only. In (Karmarkar, 1984) developed a new polynomial-time algorithm for linear programming and introduced an algebraic method with high results but the rest of the simplex is the easiest. Also (Holman and Robert, 1995) introduced a new application about the linear programming and applied a different branch in operation research. The applications continued to be widely period-intensive until (Hillier, 2001), (Hamdy, 2007), and others presented a new application in operation research. In (Vaidya and Kasturiwale, 2016) discussed a new approach while solving two phase simplex method, and they discussed this subject with respect to a number of iteration. Lagrange multipliers is a strategy for finding the local maxima and minima of a objective function subject to constraints. The method is very important because it allows the optimization to be solved without explicit parametrization in terms of the constraints. The method of Lagrange multipliers is used to solve challenging constrained optimization problems. In (C. Simon and L. Blume, 1994), the function f is a production function, it is related with several constraints and

Lagrange multipliers, and the Lagrange multipliers are interpreted to imputed value or shadow prices for production. Also, for more comparing between Simplex method and other methods see (Ahmed Alsaraireh and et al, 2018), and (Mohammad Almasarweh and et. Al, 2018). There are an excellent studies for discussing the solution as the gradient method, Hessian and other types of analysis, if we found a maximum, minimum or saddle point. For more details, see (Powell, M.J.D., 1978), (Liu, D.C. and Nocedal, J.,1989), (Fletcher, R. 1987) and (Ford, J.A. and Tharmlikit, S., 2003). Since the applications remained separate and all studies discussed the evolution of simplex method, this study attempts to investigate the application of more than one mathematical model in problem solving and work on discussion. In the pricing problem, the multipliers are not unique when the solution involves bunching (where the firm has not enough dimensions to differentiate the product). Through the previous studies were based on separate applications and did not apply some of the methods on the same problem. Therefore, it is necessary to apply mathematical methods in order to compare the results and determine the most appropriate method of the solution.

The paper is organized in two sections, section 3 is devoted to general formulate and discuss some steps of methods. In section 4, numerical example for problems are discussed.

3. Problem formulation

A linear programming problem is very important in a different fields as: Mathematics, Engineering, management, and others. Any linear programming problems consists from an objective function with a single variable or multivariable's, and the constraints with linear equalities or linear inequalities. The computation is a simple of a linear programming problem with respect to a nonlinear programming problem, but we used Lagrange method for a linear programming problem to explain that method gives the result in a short steps. Now , Denote P1 , P2 and P3 are systems of linear programming problems.

$$p_1 : \begin{cases} \text{Max.}: \ f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n \geq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0 \end{cases}$$

$$p_2 : \begin{cases} \text{Max.}: \ f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n \leq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0 \end{cases}$$

and we can discuss a mixture system such that the constraints consists from equalities and inequalities.

$$p_1 : \begin{cases} \text{Max. : } f(x_1, x_2, ..., x_n) \\ \text{Constraints:} \\ g_i(x_1, x_2, ..., x_n \leq b_i, i = 1, 2, ..., m \\ h_i(x_1, x_2, ..., x_n \leq k_l, l = 1, 2, ..., r \\ Q_i(x_1, x_2, ..., x_n = t_t, t = 1, 2, ..., w \\ x_1, x_2, ..., x_n \geq 0 \end{cases} , n, m, r, w \in R^+.$$

To find the solution of these systems, we can use a Simplex method and Lagrange method. So, we will present a general solution for P1 by two methods.

Method 1: Simplex method

In this section we will discuss a general method to solve a linear programming problem by a famous method in operation research that is called Simplex method.

1.1. Minimization problem

We will discuss all steps of this type to find an optimal solution. 1. Prepare a problem in a standard form

$$Z - f(x_1, x_2, ..., x_n) = 0, \quad g_i(x_1, x_2, ..., x_n) + S_i - u_i = b_i, i = 1, 2, ..., m.$$

2. Fill the problem in a Simplex table. 3. Determine the pivot column and row,

Basis	X_1	x_2 .... x_n	s_1	s_2 ..... s_m	u_1 .... u_m	RHS
u_1	a_{11}	a_{12} .... a_{1n}	1	0 ... 0	-1 0	b_1
.						
.						
u_m	a_{m1}	a_{m2} .... a_{mn}	0	0 1	0 0	b_m
z	c_1	c_2 .... c_n	0	0 0	0 0	Z_0

then apply basic steps to solve this table. From previous method we notice that ,we need a long calculations.

1.2. Maximization problem

In this section we will discuss a general method to solve a linear programming problem by a famous method in operation research that is called Simplex method.

1. Prepare a problem in a standard form

$$Z - f(x_1, x_2, ..., x_n) = 0, \quad g_i(x_1, x_2, ..., x_n) + S_i = b_i, i = 1, 2, ..., m$$

Basis	X_1	x_2	....	x_n	s_1	s_2	.....	s_m	RHS
u_1	a_{11}	a_{12}	....	a_{1n}	1	0	...	0	b_1
.									
.									
u_m	a_{m1}	a_{m2}	....	a_{mn}	0	0	....	1	b_m
z	c_1	c_2	....	c_n	0	0	....	0	Z_0

2. Fill the problem in a Simplex table

3. Determine the pivot column and row, then apply basic steps to solve this table. From previous method we notice that ,we need a long calculations.

Method 2: Lagrange methods

2.1. Minimization problem Suppose we are given a general optimization problem,

$$p_1 : \begin{cases} MIN. : f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n) \geq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0, \end{cases}$$

where $x_1, x_2, ..., x_n \in R^n, b_i \in R^m$. The Lagrangian is $L(x_j, \lambda_i) = f(x_j) + \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i], with \lambda_i \in R^m$. Each component of λ_i is called a Lagrange multiplier.

2.2. Maximization problem Suppose we are given a general optimization problem,

$$p_1 : \begin{cases} Max. : f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n) \leq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0, \end{cases}$$

where $x_1, x_2, ..., x_n \in R^n, b_i \in R^m$. The Lagrangian is $L(x_j, \lambda_i) = f(x_j) - \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i], with \lambda_i \in R^m$.

Each component of λ_i is called a Lagrange multiplier.

The following theorem is simple to prove, and extremely useful in practice.

Theorem (Lagrangian sufficiency theorem)**.** *If x^* and λ_i^* exist such that x^* is feasible for p_1 and ?????? , for all ???????, then ??????????? is optimal for ??????.*

Now, we will discuss a general steps to apply Lagrange method. we now verify that the Kuhn Tucker conditions hold.

1. Write the Lagrange form

$$L(x_j, \lambda_i) = f(x_j) + \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i].$$

2. Find

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial}{\partial x_j} \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i] = 0.$$

3. Find

$$\frac{\partial L}{\partial \lambda_i} = \frac{\partial f}{\partial \lambda_i} + \frac{\partial}{\partial \lambda_i} \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i] = 0.$$

4. Find

$$\bar{\lambda_i}[g_i(\bar{x}_j)] = 0.$$

After these steps , discuss all cases of λ_i or we can apply a simple matrix.

4. Numerical example

The following system represents a problem in three variable and three constraints, where the variables represent the different items in a management sector.

$$p : \begin{cases} Z = 30x_1 + 50x_2 \\ \text{Constraints:} \\ 2x_1 + x_2 \leq 60, \\ x_1 + x_2 \leq 40, \\ 2x_1 + 3x_2 \leq 70 \\ x_1, x_2 \geq 0 \end{cases} .$$

the first solution will be by Simplex method.

The solution is: $x_1 = 0, x_2 = \frac{70}{3}$.

Now we use Lagrange multipliers to find the solution in a simple steps. This method is very important, because it is connected between two important courses in a management as: Mathematics for Business and Operation research.

$$p : \begin{cases} Z = 30x_1 + 50x_2 \\ \text{Constraints:} \\ 2x_1 + x_2 \leq 60, \\ x_1 + x_2 \leq 40, \\ 2x_1 + 3x_2 \leq 70 \\ x_1, x_2 \geq 0 \end{cases} .$$

	x_1	x_2	S_1	S_2	S_3	R.H.S
S_1	2	1	1	0	0	60
S_2	1	1	0	1	0	40
S_3	2	3	0	0	1	70
z	-30	-50	0	0	0	0

	x_1	x_2	S_1	S_2	S_3	R.H.S
S_1	$\frac{4}{3}$	1	1	0	$-\frac{1}{3}$	$\frac{110}{3}$
S_2	$\frac{1}{3}$	0	0	1	$-\frac{1}{3}$	$\frac{50}{3}$
S_3	$\frac{2}{3}$	0	-1	0	$\frac{1}{3}$	$\frac{70}{3}$
z	$\frac{10}{3}$	0	0	0	$\frac{50}{3}$	$\frac{3500}{3}$

The previous example consists a linear objective function and three constraints with two variables.

The Lagrangian function of this problem is:

$$L(x_j, \lambda_i) = 30x_1 + 50x_2 - \lambda_1[2x_1 + x_2 - 60] - \lambda_2[x_1 + x_2 - 40] - \lambda_3[2x_1 + 3x_2 - 70].$$

Next, find the gradient ∇L equal to zero. First, we handle the partial derivative with respect to x_1, x_2 and λ_i

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, \\ \dfrac{\partial L}{\partial x_2} = 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0, \end{cases}$$

$$\begin{cases} \dfrac{\partial L}{\partial \lambda_1} = 2x_1 + x_2 - 60 = 0, \\ \dfrac{\partial L}{\partial \lambda_2} = x_1 + x_2 - 40 = 0 \\ \dfrac{\partial L}{\partial \lambda_3} = 2x_1 + 3x_2 - 70 = 0, \end{cases}$$

$$\begin{cases} \lambda_1[2x_1 + x_2 - 60] = 0, \\ \lambda_1[x_1 + x_2 - 40] = 0, \\ \lambda_1[2x_1 + 3x_2 - 70] = 0, \end{cases}$$

complementary slackness conditions.Now we will discuss all cases of λ_i.

Case 1. If $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0,$

$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_2 = 20\lambda_3 = -10$, reject.

Case 2. If $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = -2.5, \lambda_3 = 17.5$.
No feasible points in this case.

Case 3. If $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = -20, \lambda_2 = 70$.
No feasible points in this case.

Case 4. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_3 = 16$, accept.
The feasible point to be a solution in this case is $(0, 23.3)$,because it is satisfied a conditions.$Z(0, 23.3) = 1165$.

Case 5. If $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_2 = 40$, reject.
No feasible points in this case.

Case 6. If $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = 80/3$, reject.
The feasible point to be a solution in this case is $(30, 0)$,because it is satisfied a conditions.$Z(30, 0) = 900$.

Case 7. If $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_i > 0$, accept.
The feasible point to be a solution in this case is $(27.5, 5)$,because it is satisfied a conditions.$Z(27.5, 5) = 1075$.

Case 8. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$.
Here we will use the original constraints.

From previous cases, we notice that the optimal solution of this problem at $x_1 = 0, x_2 = 23.3$ For more application, the reader can apply the following problem.

$$p : \begin{cases} Z = 3x_1 + 4x_2 + x_3 \\ \text{Constraints:} \\ x_1 + 2x_2 + x_3 \leq 6, \\ 2x_1 + 2x_3 \leq 4, \\ 3x_1 + x_2 + x_3 \leq 9 \\ x_1, x_2, x_3 \geq 0 \end{cases} .$$

5. Conclusion

In this study we discussed the application of simplex method and Lagrange method in Linear programming problem. To use simplex method, a given linear programming problem model needs to be in standard form, but in Lagrange

model, we can solve a linear programming problem in any form. We presented the basic techniques of the previous methods. Also, we explained the main steps and the important conditions in Lagrange model. Lagrange model is important to solve linear and non-linear programming problem. Also we discussed an example on a linear programming problem.

References

[1] L.G. Khachian, *Polynomial algorithm for linear programming*, Doklady Akademia Nauk USSR, Mathematica, 244 (1979), 1093-1096.

[2] N. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica, 4 (1984), 373-395.

[3] Kolman and Robert, *Elementary linear programming with application*, 2nd edition, Academic Press Inc, United States, 1995.

[4] Hillier, *Introduction to operations research*, 7th edition, McGraw-Hill Science/ Engineering/Math, 7 edition, 2001.

[5] Wayne Winston, *Operations research: applications and algorithms*, (4th edition), Duxbury, 2004.

[6] A. Hamdy, *Operations research an introduction*, 8th edition, Prentice-Hall, Inc. Upper Saddle River, NJ, USA, 2007.

[7] N.V. Vaidya and N.N. Kasturiwale, British Journal of Mathematics and Computer Science, 16, 1-15, Article no. BJMCS. 24440, (2016).

[8] O.L. Mangsarian, *Nonlinear Programming*, New York: McGraw-Hill, 1969.

[9] R.T. Rockafellar, *Convex analyses* Princeton University press, 1970.

[10] A.M. Rubinov, X. Q. Yang, *Lagrange-type functions in constrained non-convex optimization*, Kluwer Academic Publishers, Dordrecht, The Netherlands, (2003).

[11] D.P. Bertsekas, A. E. Ozdaglar, *Pseudonormality and a Lagrange multiplier theory for constrained optimization*, J. Optim. Theory Appl. 114 (2002), 287343.

[12] C. Simon, L. Blume, *Mathematics for Economists*, W. W. Norton and Company, New York, (1994).

[13] P. Bussotti, *On the genesis of the Lagrange multipliers*, J. Optim. Theory Appl. 117 (2003), 453459.

[14] A.A. Alsaraireh, M. Almasarweh, M.B. Alnawaiseh, S. Al Wadi, V. Bhama, *The Effect of methods of operation research in obtaining the best result in the* , Italian Journal of Pure and Applied Mathematics, 40 (2018), 501-509.

[15] Mohammad Almasarweh, Ahmed Atallah Alsaraireh, Raed Masadeh, *A Statistical study to determine the production capacity of jordanian pharmaceutical companies based on the number of working hours using the assignment problem*, Modern and Applied science, 12 (2018), 301-808.

[16] S. Al Wadi, Mohammad Almasarweh and Ahmed Atallah Alsaraireh, *Predicting closed price time series data using ARIMA Model*, Modern Applied Science, 12 (2018), 181-185.

[17] M.J.D. Powell, *Algorithms for nonlinear constraints that use Lagrange functions*, Math. Programming, 14 (1978), 224248.

[18] D.C. Liu, J. Nocedal, *On the limited memory BFGS method for large scale optimization*, Math. Programming (Series B), 45 (1989), 503528.

[19] R. Fletcher, *Practical methods of optimization*, (2nd edition), Wiley, Chichester, England, Reprinted in 2000, (1987).

[20] J.A. Ford, S. Tharmlikit, *New implicit updates in multi-step quasi-Newton methods for unconstrained optimization*, J. Compt. Appl. Math., 152 (2003), 133146.

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Hybrid-based mathematical method for enhancing the quantitative research

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Abstract. This study aims to discuss a special structure that can be exploited in the construction of efficient solution. The advantage of their methods usually has been the need to solve larger problems than otherwise would be possible to solve with computer technology. Two methods are discussed in this paper to determine a best method to solve these problems, and to determine which one is has a best result. We used : Network work method and Transportation problem method. Two methods were applied in general system to evaluate the result and compare between them. Also, the researcher discussed an example on a transportation problems. The results of this study indicated which one of the method has a simple steps in the solution.

Keywords: network work method, transportation problem, efficient solution.

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1. Introduction

Operations Research is related with different fields as mathematics, statistics, economics, psychology, engineering. Also operation research used to make a new decision. The commerce field is developing and growing rapidly, and increasing the aware competency between all organizations in the world, and striving to be more successes, survival and achieving to a competitive advantage. The rst of these special method to be analyzed was the transportation problem, which is a particular type of network problem. Also, all organizations need to minimize the cost and time, and increase their margin profit. Recently, the development of method to find the solution of the transportation problem in a simple method has become a major idea in applied mathematical programming For these reasons, this study aims to link between the solution of a transportation problem transshipment problem, and simplex method. For more details about these methods refer to (Mangazarian, 1969), and (Rockefeller, 1970).

2. Literature review

Operations research as a science has been used to help solve decision problems using mathematical and statistical models for a long time, and it has been developed in many scientific fields such as : Mathematics, Engineering and Management. It is one of the areas that has contributed to solving many of mathematical problems and management problems: transportation problem and Network method.

Network method is the most important of the special method in linear programming problems. In this paper, we present a general formulation of network and transportation problems, and we formulate some example to the connect between two methods. So, we used to convert the formulation of these methods to a simplex formulation.

In 1736, Euler discussed a new solution of the problem that such a path does not exist. This solution is considered to be the first theorem of graph theory, specifically of planar graph theory, but the first book of operations researches appeared in 1946 As "Methods of Research Operations" for Morris and Campbell's. Some of the scientists developed a method of problem solving in the Simplex Method and other methods. In (Khachian, 1979) proposed a new method of solving the linear program, but theoretically only. In (Karmarkar,1984) developed a new polynomial-time algorithm for linear programming and introduced an algebraic method with high results but the rest of the simplex is the easiest, and in (C. Simon and L. Blume, 1994) you can study about anything in Mathematics for Economists. Also (Holman and Robert, 1995) introduced a new application about the linear programming and applied a different branch in operation research. There many application are related with these method for example in (K. Fagerholt, 1999), he dicuss an optimal design in a ship routing problem.

The applications continued to be widely period-intensive until (Hillier, 2001), (Hamdy, 2007), and other spresented a new application in operation research. In (Reeb, J. and S. Leavengood, 2002), they studied a special for linear programming problems in a transportation problem. (Wayne Winston, 2004), presented an applications and algorithms in some of operation research models. Also, in (Ahuja, R. K., Magnanti, T. L.Orlin, J. B.,1993), (Baillon, J.B., Cominetti, R., 2008), (Fosgerau, M., Frejinger, E., Karlstrom, A., 2013),(Heuberger, C.,2004), and (Airoldi, E. M., Blocker, A. W., 2013) discussed a different ideas in these topics.

In (Vaidya and Kasturiwale, 2016) discussed a new approach while solving two phase simplex method, and they discussed this subject with respect to a number of iteration. Also, for more studies between Simplex method and other methods see (Ahmed Alsaraireh and et al, 2018), and (Mohammad Almasarweh and et. Al, 2018).

All applications remained separate and all studies discussed the development of simplex method, network and transportation problem, this study discussed the link between all mathematical models in problem solving Through the previous studies were based on separate applications and did not apply some of the methods on the same problem. Therefore, it is necessary to apply mathematical methods in order to compare the results and determine the most appropriate method of the solution. The paper is organized in two sections, section 3 is devoted to general formulate and discuss some steps of methods. In section 4, numerical example for problems are discussed.

3. Problem formulation

A linear programming problem is very important in different fields as Mathematics, Engineering, management, and others. Any linear programming problems consist from an objective function with a single variable or multi-variables,and the constraints with linear qualities or linear inequalities. The computation is a simple of a linear programming problem with respect to a nonlinear programming problem.

Here we will present a general formulation of the transportation problem, formulation of network models to connect the previous models with a simplex model. This formulation is very important for all researcher in an application science, especially in management and business.

Now, Denote the following figure presents a general model a transportation problem. Where $b_1, b_2, ..., b_n \equiv$ Demand quantities, $a_1, a_2, ..., a_m \equiv$ Supply quantities, $c_{ij} \equiv$ unit cost from i to j unit and $a_1, a_2, ..., a_m \equiv$ Number of units transferred.

We must check if the model is balanced or not. If the model is not balanced, add a dummy row or column.

The cost of each cell in a dummy row or column will be zero.

Sources	Destinations				Supply
	1	2	⋯	n	
1	C_{11}	C_{12}		C_{1n}	a_1
2	C_{21}	C_{22}		C_{2n}	a_2
3	C_{31}	C_{32}		C_{3n}	a_3
⋮	⋮	⋮	⋮	⋮	⋮
m	C_{m1}	C_{m2}		C_{mn}	a_m
Demand	b_1	b_2		b_n	**Sum**

Also, the following model presents an equivalent model for transportation problems by simplex model. Also, the following model presents an equivalent model for transportation problems by simplex model.

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \text{ Subject to: } \begin{cases} \sum_{j=1}^{n} x_{ij} \leqslant a_i \\ \sum_{i=1}^{m} x_{ij} \geq b_j \\ x_{ij} \geq 0 \end{cases}$$

If $\sum a_i = \sum b_j$, then model will be

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \text{ Subject to: } \begin{cases} \sum_{j=1}^{n} x_{ij} = a_i \\ \sum_{i=1}^{m} x_{ij} = b_j \\ x_{ij} \geq 0 \end{cases}$$

If $\sum a_i \neq \sum b_j$, then model will be

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \text{ Subject to: } \begin{cases} \sum_{j=1}^{n+1} x_{ij} = a_i \\ \sum_{i=1}^{m} x_{ij} = b_j \\ x_{ij} \geq 0 \end{cases}$$

or

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \text{ Subject to: } \begin{cases} \sum_{j=1}^{n} x_{ij} = a_i \\ \sum_{i=1}^{m+1} x_{ij} = b_j \\ x_{ij} \geq 0 \end{cases}$$

From previous models, we used all cases to convert from a transportation model to simplex model.

The transport problem studies the method of transport from the source site to the requested site directly, but in some cases, the transport goes through stages. Therefore, we connect this problem the mathematical programming. so, we can present the transport and shipping data in terms of costs, demand quantities, and supply quantities.

The following figure presents the general model of the transshipment model and transportation network in two ways (Direct path and un-direct path), but we will discuss in this paper un-directed path.

Figure (1): Directed Network

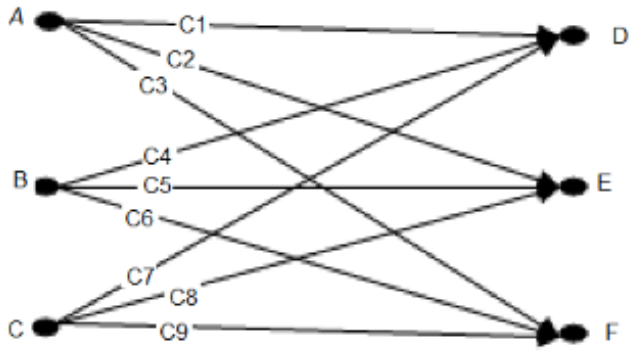
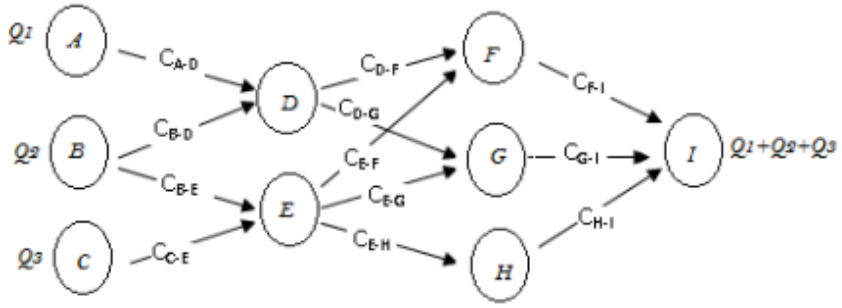


Figure (2): Undirected Network



In this paper, we will discuss undirected path and will try to connect it with a transportation table and simplex table. The first step, we will present the previous network by a linear-programming formulation through the simplex table. From the table we can write an equivalent a transportation problems to

Nodes	x_{AD}	x_{BD}	x_{BE}	x_{CE}	x_{DF}	x_{DG}	x_{EF}	x_{EG}	x_{EH}	x_{FI}	x_{GI}	x_{HI}	*RHS*
A	1												Q_1
B	1		1										Q_2
C				1									Q_3
D	−1	−1			1	1							0
E		−1		−1			1	1	1				0
F					−1		−1			1			0
G						−1		−1			1		0
H									−1			1	0
I										−1	−1	−1	$-\sum_{i=1}^{3} Q_i$
Obj. fun.	c_{AD}	c_{BD}	c_{BE}	c_{CE}	c_{DF}	c_{DG}	c_{EF}	c_{EG}	c_{EH}	c_{FI}	c_{GI}	c_{HI}	

the first model by the following equations. At each node we will depend on the following formula:

$$Flow\ out\ of\ a\ node - Flow\ into\ of\ a\ node = 0$$

The following equation presents the initial points A,B,C:

$$x_{AD} = Q_1, \tag{1}$$

$$x_{BD} + x_{BE} = Q_2, \tag{2}$$

$$x_{CE} = Q_3, \tag{3}$$

But the following equation presents the other nodes D,E,..,H:

$$x_{AD} + x_{BD} = x_{DF} + x_{DG}, \tag{4}$$

$$\vdots$$

$$x_{EH} = x_{HI}. \tag{5}$$

The equation of the last node I:

$$x_{FI} + x_{GI} + x_{HI} = Q_1 + Q_2 + Q_3. \tag{6}$$

Entering a dummy variable with a nonnegative variable and suppose a positive quantity(w) for the internal nodes to make all the equations are balanced, as:

$$x_{AD} + x_{BD} = x_{DF} + x_{DG} = W, \tag{7}$$

$$\vdots$$

$$x_{EH} = x_{HI} = W. \tag{8}$$

Now, we will display the equations in a transportation table to find an optimal solution of the system, and we notice that the table represents the general model to display any network in a transportation table. From the previous table, there

	D	E	F	G	H	I	**Supply**
A	C_{AD}	k	k	k	k	k	Q_1
B	C_{BD}	C_{BE}	k	k	k	k	Q_2
C	k	C_{CE}	k	k	k	k	Q_3
D	0	k	C_{DF}	C_{DG}	k	k	W
E	K	0	C_{EF}	C_{EG}	C_{EH}	k	W
F	k	k	0	k	k	C_{FI}	W
G	k	k	k	0	k	C_{GI}	W
H	k	k	k	k	0	C_{HI}	W
Demand	W	W	W	W	W	$\sum_1^3 Q_i$	$\sum_1^3 Q_i + 5W$

exist some variable such that,

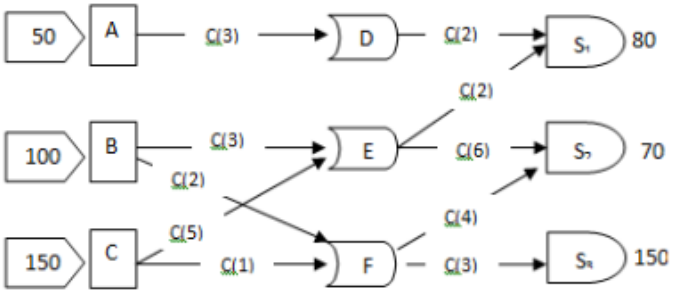
W ≡ The large positive number to make a problem is balanced,

K ≡ The large cost to fill all missing cost.

To find the solution to this problem, we can write this problem in a simplex model, or we can solve it directly by transportation methods.

4. Numerical example

The following table presents the distribution plan from three companies A(50 units), B(100 units), and C(150 units) to four customers S1(80 units), S2(70 units), and S3(150 units). The distribution through four intermediary companies. Find the optimal plan for distribution from companies to customers such that we get the lowest transfer cost. Now, we will display the previous problem



in a simplex model

$$x_{AD} = 50, \tag{9}$$

$$x_{BE} + x_{BF} = 100, \tag{10}$$

$$x_{CE} + x_{CF} = 150 \tag{11}$$

and,

$$x_{DS_1} = x_{AD}, \tag{12}$$

$$x_{ES_1} + x_{ES_1} = x_{BE} + x_{CE}, \tag{13}$$

$$x_{FS_2} + x_{FS_3} = x_{BF} + x_{CF}, \tag{14}$$

$$x_{DS_1} + x_{ES_1} = 80, \tag{15}$$

$$x_{ES_2} + x_{FS_2} = 80, \tag{16}$$

$$x_{FS_3} = 150. \tag{17}$$

	x_{AD}	x_{BE}	x_{BF}	x_{CE}	x_{CF}	x_{DS_1}	x_{ES_1}	x_{ES_2}	x_{FS_2}	x_{FS_3}	R.H.S
A	1										50
B		1	1								100
C				1	1						150
D	-1					1					0
E		-1		-1			1	1			0
F			-1		-1				1	1	0
S_1						-1	-1				-80
S_2								-1	-1		-70
S_3										-1	-150
Obj. Fun.	3	3	2	5	1	2	2	6	4	3	

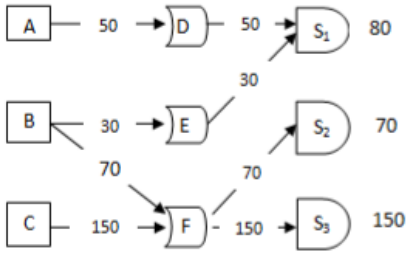
The following table represents the previous problem in a transportation table such that, k means missing link. We have a different methods to solve this

	D	E	F	S_1	S_2	S_3	Supply
A	3	k	k	k	k	k	50
B	k	3	2	k	k	k	100
C	k	5	1	k	k	k	150
D	0	k	k	2	k	k	300
E	K	0	k	2	6	k	300
F	k	k	0	k	4	3	300
Demand	300	300	300	80	70	150	

problem and to compare it with a solution of network.So, we used a maximum flow to solve this problem.

	From	To	Shipment
1	A	D	50
2	B	E	30
3	B	F	70
4	C	F	150
5	D	D	250
6	D	S1	50
7	E	E	270
8	E	S1	30
9	F	F	80
10	F	S2	70
11	F	S3	150

Total Obj. Fun. = 1420



5. Conclusion

In this study we discussed the link between network method, simplex method and transportation problem. Also, we presented the main idea to write any transportation problem in a network and simplex model. The general of formulation for three models is related to gather in a simple procedure. Network method and simplex method are suitable methods to solve any transportation problem. Also, we explained the main steps and the important conditions in simplex model and network model. Simplex model is important to write a transportation problem and network in a simplex model. Also we discussed an example on a linear programming problem. This study attempts to investigate the application of more than one mathematical model in problem solving and work on discussion. From the result of this paper we notice that the three methods are related to gather in a strong procedure, and every one can used these steps to compare it with another methods.

References

[1] A. Hamdy, *Operations research an introduction*, 8th edition. Prentice-Hall, Inc. Upper Saddle River, NJ, USA, 2007.

[2] Ahmed Alsaraireh and et. al, *The Effect of methods of operation research in obtaining the best result in the best result in the trade,* Italian journal of Pure and Applied Mathematics, 40 (2018), 501-509.

[3] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, *Network flows: theory, algorithms and applications,* Pearson, 1993.

[4] E.M. Airoldi, A.W. Blocker, *Estimating latent processes on a network from indirect measurements*, Journal of the American Statistical Association, 108 (2013), 149-164.

[5] J.B. Baillon, R. Cominetti, *Markovian traffic equilibrium*, Mathematical Programming, 111 (2008), 3356.

[6] C. Simon and L. Blume, *Mathematics for economists*, W. W. Norton & Company, New York, 1994.

[7] M. Fosgerau, E. Frejinger, A. Karlstrom, *A link based network route choice model with unrestricted choice set*, Transportation Research Part B, 56 (2013), 70-80.

[8] C. Heuberger, *Inverse combinatorial optimization: a survey on problems, methods and results*, Journal of Combinatorial Optimization, 8 (2004), 329-361.

[9] Frederick S. Hillier, *Introduction to operations research*, 7th edition, McGraw-Hill Science/ Engineering/Math;7 edition, 2001.

[10] K. Fagerholt, *Optimal fleet design in a ship routing problem,* International Transactions in Operational Research, 6 (1999), 453464.

[11] Kolman and Robert, *Elementary linear programming with application*, 2nd edition, Academic Press Inc, United States, 1995.

[12] L.G. Khachian, *Polynomial algorithm for linear programming,* Doklady Akademia Nauk USSR, Mathematica, 244 (1979), 1093-1096.

[13] O.L. Mangsarian, *Nonlinear programming*, New York: McGraw-Hill, 1969.

[14] Mohammad Almasarweh and et. al, *A statistical study to determine the production capacity of jordanian pharmaceutical companies based on the number of working hours using the assignment problem*, Modern and Applied science, 12 (2018),301-808).

[15] N. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica, 4 (1984), 373-395.

[16] N.V. Vaidya, N.N. Kasturiwale, British Journal of Mathematics and Computer Science, 16, 1-15, Article no.BJMCS.24440, 2016.

[17] J. Reeb, S. Leavengood, *Transportation problem: a special case for linear programming problems*, Operations Research, Oregon State University, `http://extension.oregonstate.edu/catalog/pdf/em/em8779e.pdf`.

[18] R.T. Rockafellar, *Convex analysls*, Princeton University Press, 1970.

[19] Wayne Winston, *Operations research: applications and algorithms*, (4th edition), Duxbury, 2004.

[20] Mohammad Almasarweh, Ahmed Atallah Alsaraireh & Raed Masadeh, *A statistical study to determine the production capacity of jordanian pharmaceutical companies based on the number of working hours using the assignment problem*, Modern and Applied science, 12 (2018), 301-808.

[21] S. AL Wadi, Mohammad Almasarweh, Ahmed Atallah Alsaraireh, *Predicting closed price time series data using ARIMA Model*, Modern Applied Science, 12 (2018), 181-185.

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Intellectual capital and effect on marketing performance an empirical study in Jordanian pharmaceutical industrial firms

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Abstract. This study focuses on intellectual capital and its effect on marketing Performance through an economy based on knowledge. This study aims to determine the role of intellectual capital and its different dimensions (human capital ; structural capital and technological capital) in the company of our study. A questionnaire was developed to ensure the relationship between the independent and dependent variables (marketing Performance).

The hypotheses were tested using SPSS statistical software, the analyzes results to several outcomes including the existence of a statistically significant relationship between the intellectual capital and marketing Performance. One of the most important recommendations of the study is to increase attention to and management of intellectual capital, and it must be managed as an important source To achieve excellence, and the need to deal with intellectual capital as the most important strategic resource possessed

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by the company, and maintain the continuity of the active component in light of the tremendous technological development that the business environment.

Keywords: intellectual capital, human capital, structural capital, technological capital, marketing performance.

1. Introduction

Intellectual Capital is an important and modern concept, which scientists and researchers in the field of administration have started discussing widely. This underscore the meaning of human capital in building organizations and bringing operational success. there are many definitions of human capital.

Today, business organizations belong to a highly dynamic and changing environment, making it difficult for organizations to control them. Even the current age of organizations is the age of knowledge economy, globalization, contemporary production, Competition and the revolution in information and communication systems. As a result of the pressures imposed by these events was Organizations need to shift towards new organizational patterns that are flexible, focused on knowledge and on assets Intangible or so-called intellectual capital in generating value.

Intellectual capital represents the outstanding capabilities of human resources and associated structural assets Organizational structures and procedures, and the client capital of relations between the Organization and its clients.

The focus of knowledge-based and technology-based organizations has been to manage their intellectual assets as wealth Its true, because the physical assets erode over time and their market value falls, reversing assets Intellectual, human, organizational or interrelated, are the only basis for building and developing competitive capacities Adding value to the organization and achieving its competitive budget, which is the basis on which performance is based Organizations, because they are allowed to add value to the customer and achieve excellence and maintain market share and increase, And achieve effectiveness, and this will be achieved only in light of its ability to satisfy the needs of its customers, and this will be achieved only by Effective way to capitalize on its intellectual capital.

As a result, organizations today are looking for the meaning and content of intellectual capital, and ways The best to invest, especially for its intangible moral character, and uncertainty of future returns due to difficulty Prediction of its useful life, difficulty in measuring and evaluating it.

This research comes to address Intellectual Capital and effect on marketing Performance For the Jordanian pharmaceutical companies, and how to build an intellectual capital capable of achieving marketing performance, and the extent of availability Its requirements in the Jordanian pharmaceutical companies, and what methods can be used for valuing the head Intellectual money by activating its role in order to achieve marketing performance.

The problem of the study

The problem of the modern study stems from the following questions: 1. Is there a relationship between Intellectual capital (human capital ; structural capital and technological capital) and marketing Performance in Jordanian pharmaceutical companies?

2. Is there a trace of Intellectual capital (human capital ; structural capital and technological capital) in marketing Performance in Jordanian pharmaceutical companies?

The importance of the study

This study stem from the following methodical and practical considerations:

1. Highlight the importance of Intellectual capital in Jordanian pharmaceutical industry and their importance in achieving high levels of performance, raising the level of these companies, and contributing to the achievement of the objectives in the long run.

2. add to the developmental work of Jordanian pharmaceutical companies mechanism as well as maintain the continuity for public good.

3. This study contributes to opening up other research areas on the subject of Intellectual capital and spurring interest in this kind of capital, especially in the Jordanian pharmaceutical companies in order to accomplish the strategic objectives of these companies.

Objectives of the study

The aim of this study is to learn of the impact of Intellectual capital on marketing Performance of the Jordanian pharmaceutical industry. The independent variable is Intellectual capital, whereas the dependent variable is marketing Performance in Jordanian pharmaceutical companies.

The study also aims to identify the impact of Intellectual capital (human capital ; structural capital and technological capital) on marketing Performance in Jordanian pharmaceutical companies.

Hypotheses of the study

Hypothesis HO 1

There is no statistically significant effect of Intellectual capital (human capital ; structural capital and technological capital) on marketing Performance in Jordanian pharmaceutical companies at a level of significance of 0.05. This give rise to the following sub-hypotheses:

Hypothesis HO1-1

There is no statistically significant effect of human capital on marketing Performance in Jordanian pharmaceutical companies at a level of significance of 0.05.

Hypothesis HO1-2

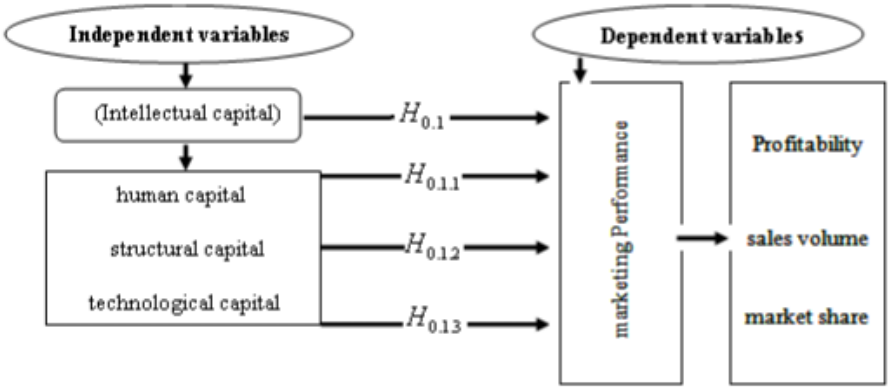
There is no statistically significant effect of structural capital on marketing Performance in Jordanian pharmaceutical companies at a level of significance of 0.05.

Hypothesis HO1-3

There is no statistically significant effect of technological capital on marketing Performance in Jordanian pharmaceutical companies at a level of significance of 0.05.

Model study

The dimensions of Intellectual capital (human capital ; structural capital and technological capital) were determined based on the study by Ukenna et al. (2010). However, the marketing performance has been relying on (Eusebio etal 2006)



The study procedural definition of terms

Intellectual Capital: The strategic capabilities of the organization through the skills, talents and practical knowledge necessary to accomplish the activities required by the organization's strategy (Kaplan and Norton, 2004).

Marketing performance

Is the level of achievement of the company's goals set in the marketing plan, and will be based on the measurement of marketing performance on the (profitability, sales volume, market share) (Eusebio et al,2006).

Literature review

Intellectual Capital (IC) is known as the new invisible asset and the most widely used definition of intellectual capital is knowledge that is of value to an organization. From this definition it is suggested that the management of knowledge (what is known) generate IC , Bayburina and Golovko (2009) explain that IC includes human capital, process capital, client capital, innovation capital and network capital. The complexity of these character represents the competitive

advantage originated within and within the company. Therefore, survival and competitive success of a firm depends highly on the strategic management of its IC as compare to financial resources. Unfortunately, one of the challenge in managing IC is when the managers are not clear that their firms possess valuable resources, which might lead to success of new strategies. The idea of IC is broad and is in general split into a few characteristics, which are human capital, customer capital, structural capital, social capital, technological capital and spiritual capital.

Human capital (HC)

Human can be consider valuable asset or liability to an organization (Khan, Farooq and Hussain, 2010). Human capital refers to expert or workforce skills, knowledge and experience shared with their organization in order to add value (Baron,2011).

Structural capital means the capability of organization to accommodate their customers demand. Recent evidence suggests that a good organization structure, together with skilled employees provided that efficient and quality service will cause greater performance of an institution (Amrizah and Nawal, 2013).

Technological capital

Information technology (IT), research and development (R&D) and improvement can be consider as technological capital. In order for knowledge to be shared swiftly and available to others, the existence technological capital is important. Collection, storing and distribution of information will be easier with the help of technological capital (Ngah, 2011).

Marketing performance refer the level of achievement of the company for its objectives in the marketing plan"(Ambler, etal,2004).

Market share according to (Farris, et. al, 2010) the percentage of the total market or piece of the market being serviced or the total sales of the company divided by total sales.

Profitability refers to the level of profits achieved by the organization compared to competitors(Chandra, 1997,p: 538) ,Profitability is a measurement of return in the form of profits received (shareholders for their investments in the organization) ,(Armstrong, 2003, p101).

Previous studies

Karahan and Tetik,(2012) The Determination of the Effect Level on Employee Performance of TQM Practices with Artificial Neural Networks: A Case Study on Manufacturing Industry Enterprises in Turkey. The aim was to test the impact of the use of the total quality tools on the performance of workers in three industries that are ISO 9000 certified in Turkey. The study sample consisted of (352) industrial establishments operating in Turkey. The study found

many results, most notably that the applications of total quality affect the performance of employees, and that the performance of workers affects the nature of the work of artificial neural networks.

Dizgah, et. al, (2011) human capital character and Organizational performance . The mean was to examine the relationship between human capital characteristics and organizational performance in banks operating in Guilin Province. The study sample include (22) government banks. The study found a statistically significant relationship at (0.05) level between human capital and organizational performance.

(Chin and sofian, 2011) The Impact of human capital and total quality management on corporate performance: a reviw. The objective of the study was to confirm the impact of human capital and TQM on corporate performance. A practical framework for integration between human capital and TQM has been developed which improves the performance of organizations.

Almasarweh, M. (2016) The Impact of Human Capital on Competitive Performance: An Empirical Study on Jordanian Pharmaceutical Companies, The study sample include of Jordanian human medicines manufacturing companies listed in the Jordanian union of Pharmaceutical Manufacturers records of manufacturing companies for 2015. This amounted to 15 companies. The study recommended that thought be given to human capital in the companies, which will lead to an enhancement in the level of the company, as well as an growth of its market contribute to and maximizing of its strengths.

Hashim, and et al (2015) Effect of Intellectual Capital on Organizational Performance, The purpose of this empirical study is to investigate the relationship between six elements of intellectual capital such as human capital, structural capital, customer capital, social capital, technological capital and spiritual capital with organizational performance in Malaysia.

Study approach

The current study can be regarded as an exploratory study. It is based on the descriptive analytical technique where the questionnaire will be used to gather data and information about the supported variables of the study sample. It is the descriptive analytical way that is most suitable to achieve the objectives of the present study.

Study population and sample

The sample of the research is obtain from the study population, which include of Jordanian Association of Pharmaceutical Manufacturers records of pharmaceutical manufacturing companies for 2015. 15 companies have been intentionally selected.

Stability study tool

The stability or the internal consistency of the instrument use in this research to evaluate responses to paragraphs was assured using Cronbach equation alpha

(Cronbach Alpha). The end result is acceptable statistically if the value is bigger than 0.70 (Sekaran, 2006). In addition, the results show that the coefficient values stability is acceptable.

N	Dimension	Questions	(α)
1	Intellectual capital	13	0.864
1-1	human capital	5	0.808
2 - 1	structural capital	5	0.811
3 - 1	technological capital	5	0.836
2	marketing Performance	5	0.811
	Questionnaire	33	0.811

As the above table shows, The Alpha Cronbach indicators on the suitability of the above study tool is generally high by a factor of stability and thus achieves the purposes of the study.

Results and test hypotheses

Hypothesis HO 1

There is no statistically significant effect of Intellectual capital (human capital ; structural capital and technological capital) on marketing Performance in Jordanian pharmaceutical companies at a level of significance of 0.05.

Table 2. The results of multiple regression contrasts the impact of (human capital ; structural capital and technological capital) analysis on marketing performance (The dependent variable is the marketing performance).

Table 2

Dependent Variable														
marketing performance														
R	R^2	df			Mean square		F	B			T			Sig
0.811	0.658	Regression	Residual	Total	Regression	Residual	Regression	human capital	structural capital	technological capital	human capital	structural capital	technological capital	0.000
		4	56	60	2.45	0.024	165.23	0.321	0.245	0.11	3.12	3.27	0.64	

The results of Table 2 shows the value of the coefficient of determination to be $R^2 = 0.658$. This means that the independent variable (Intellectual capital) explains what percentage of discrepancy (66%) is in the dependent variable (the marketing performance). Because the value of F is 165.23 at the significance level of < 0.05, we accept the hypothesis that there is a statistically significant effect ($0.05 \leq \alpha$) of Intellectual capital on marketing performance.

Hypothesis HO 1-1

There is no statistically significant effect of human capital on marketing performance in Jordanian pharmaceutical companies at a level of significance of 0.05.

Table 3 The results of simple regression contrasts human capital performance impact analysis (dependent variable is the marketing performance).

Table 3

Dependent Variable										
marketing performance										
R	R^2	df			Mean square		F	B	T	Sig
0.789	0.623	Regression	Residual	Total	Regression	Residual	Regression	0.789	38.13	0.000
		3	57	60	4.23	0.321	321.61			

The results of Table 3 shows that the value of the coefficient of determination R^2= 0.623. This explains the rate (62%) of the discrepancy in the dependent variable (the marketing performance). And because the value of F is 321.61 at a significance level of ¡0.05, we accept the hypothesis that there is a statistically significant effect ($0.05 \leq \alpha$) to learn the marketing performance.

Sub-third hypothesis HO 1-2

Table 4,There is no statistically significant effect of structural capital on marketing performance in Jordanian pharmaceutical companies at a level of significance of 0.05. The results of simple regression contrasts the impact of structural capital on marketing performance analysis (dependent variable is the marketing performance) Results of table 4 showed the value of the coefficient

Table 4

Dependent Variable										
marketing performance										
R	R^2	df			Mean square		F	B	T	Sig
0.722	0.521	Regression	Residual	Total	Regression	Residual	Regression	0.722	29.37	0.000
		3	57	60	5.73	1.23	397.3			

of determination to be $R^2 = 0.521$. This means that the independent variable structural capital explains what percentage of (52%) discrepancy is existent in the dependent variable (the marketing performance). And because the value of F = 397.3 at a level significance¡0.05, we accept the hypothesis that there is a statistically significant effect ($0.05 \leq \alpha$) of structural capital on the marketing performance.

Sub-third hypothesis HO 1-3

There is no statistical effect of technological capital on marketing performance indication in Jordanian pharmaceutical companies at a level of signifi-

cance of 0.05. The results of simple regression contrasts the influence of technological capital on the marketing performance analysis (dependent variable is the marketing performance). Results showed, through the table 5, the value of the

Table 5

Dependent Variable										
marketing performance										
R	R^2	df			Mean square		F	B	T	Sig
0.567	0.321	Regression	Residual	Total	Regression	Residual	Regression	0.432	21.34	0.000
		3	57	60	3.77	0.76	79.35			

coefficient of determination to be $R^2 = 0.321$. This means that the independent variable technological capital explains what percentage of (32%) discrepancy is present in the dependent variable (the marketing performance). And because the value of F = 79.35 at a significance level of ¡ 0.05, we accept the hypothesis that there is a statistically significant effect ($0.05 \leq \alpha$) of skill on competitive performance.

2. Conclusion

The study aims to investigate the influence of intellectual capital on marketing performance in Pharmaceutical firms. The significance of intellectual capital for organizational sustainable expansion cannot be over emphasized in today's dynamic world. This specially becomes more important in intense market competition where corporations strives to present maximum value added good and services to its customers. Therefore, firms are paying attention to intellectual capital to enhance marketing performance. This study investigated how intellectual capital affects marketing performance. The study found overall significant influence of intellectual capital on marketing performance.The study also found significant association between marketing performance and all three dimensions of intellectual capital (human capital; structural capital and technological capital). The study propose that organizational should ensure systematic use of its intellectual capital including (human capital; structural capital and technological capital). In this method organizations can make best use of their human resources, better satisfy their customers and register more patents as compare to their competitors.The study proposes that efficient intellectual capital management can result in high level of motivation and dedication in employees, larger market share, leading place in the market place.The study provide important information to corporate decision makers and future researchers on this topic.

References

[1] R. Eusebio, J. Andreu, P. Belbeze, *Measures of marketing performance: a comparative study from Spain*, International Journal of Contemporary Hospitality Management, (on-line), 18 (2006), file://A:emerald insight.com.

[2] M. Hashima, I. Osmanb, S. Alhabshic, *Effect of intellectual capital on organizational performance*, 2nd Global Conference on Business and Social Science-2015, GCBSS-2015, 17-18 September 2015, Bali, Indonesia.

[3] Robert S. Kaplan, David P. Norton, *Strategy maps: converting intangible assets into tangible outcomes*, Harvard Business School Press, Boston, Massachusetts, 2004.

[4] Mehmet Karahan, Nevzat Tetik, *The determination of the effect level on employee performance of TQM practices with artificial neural networks: a case study on manufacturing industry enterprises in Turkey*, International Journal of Business and Social Science, 3 (2012), 133-142.

[5] Morad Rezaee Dizgah, Hamid Reza Alipour, Mehrdad Goudarzv Chegini, Amene Falahati, *Human capital characteristics and organizational performance*, Australian Journal of Basic and Applied Sciences, 5 (2001), 803-809.

[6] Khor Saw Chin, Saudah Sofian, *The impact of human capital and total quality management on corporate performance: a review*, Interdisciplinary Journal of Contemporary Research in Business, 3 (2011), 10911100.

[7] Uma Sekran, *Research methods for business: a skill-building approach*, 4th ed; New York, John Wiley and Sons Inc., 2006.

[8] E. Bayburina, T. Golovko, *Design of sustainable development: intellectual value of large BRIC companies and factors of electronic journal of knowledge management*, 7 (2009), 535558.

[9] B. Khan, A. Farooq, Z. Hussain, *Human resource management: an Islamic perspective*, Asia-Pacific Journal of Business Administration, 2 (2010), 1734.

[10] A. Baron, *Measuring human capital*, Strategic HR Review, 10 (2011), 3035.

[11] K. Amrizah, K. Nawal, *The Relationship between human resource management and islamic microfinance providers performance: the medi ating role of human capital*, International Journal of Business and Social Science, 4 (2003), 5257.

[12] R. Ngah, *The relationship of intellectual capital*, Knowledge Sharing, Innovation and Organizational Performance of Malaysian SME's, 2011.

[13] Mohammad Almasarweh, *The impact of human capital on competitive performance: an empirical study on Jordanian pharmaceutical companies*, European Scientific Journal, 12 (2016), 1857-7881.

[14] T. Ambler, F. Kokkinaki, S. Puntoni, *Assessing marketing performance: reasons for metrics selection*, Journal of Marketing Management, (on-line), 20, 2004, file://A:search.epnet.com.

[15] Paul W. Farris, T. Bendle Neil,; Phillip E. Pfeifer; David J. Reibstein, *Marketing metrics: the definitive guide to measuring marketing performance*, Upper Saddle River, New Jersey, Pearson Education, Inc., 2010.

[16] Prasama Chandra, *Financial management: theory and practice*, 4th ed., Tata McGrawHill Delhi. Armstrong, M., "A Handbook of Management Techniques", British Library Cataloguing-in-Publication Data, 2003, 101, 1997.

[17] A.A. Alsaraireh, M. Almasarweh, M.B. Alnawaiseh, S. Al Wadi, V. Bhama, *The effect of methods of operation research in obtaining the best result in the trade*, Italian Journal of Pure and Applied Mathematics, 40 (2018), 501-509.

[18] Mohammad Almasarweh, Ahmed Atallah Alsaraireh, Ra'ed Masa'deh, *A statistical study to determine the production capacity of jordanian pharmaceutical companies based on the number of working hours using the assignment problem*, Modern and Applied Science, 12 (2018),301-808).

[19] S. AL Wadi, Mohammad Almasarweh and Ahmed Atallah Alsaraireh, *Predicting closed price time series data using ARIMA model*, Modern Applied Science, 12 (2018), 181-185.

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Modeling the volatility insurance time series data using Wavelet transform

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Abstract. Nowadays, the volatility of stock market data have contributed an essential section in risk study. Volatility is measured by standard deviation of the return. This study explores volatility event for the insurance time series data from Amman Stock Exchange (ASE). Wavelet models (WT) have used in order to study the events in the volatility data that collected from 2010 to 2018. Therefore, the researchers found that WT is a suitable model in studying the volatility data.
Keywords: volatility, insurance time series data, wavelet transform.

1. Introduction

Statistical science has widely applications in various application such as forecasting and operation research (Alsaraireh, et al., 2018; Mohammad Almasarweh

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et al., 2018; Al Wadi, 2018) volatility is one of the concerning points for many researchers in financial institutions. Therefore, they have focus in the volatility field such as forecasting, detecting and modeling volatility since the volatility plays an significant part in risk management. For example, the financial markets use the forecasting to predict the future risks which will effect in the decisions regarding the financial institutions (Bollersley, et al. 2016). Nowadays, many models of volatility models have been proposed in the literature reviews in order to make testing the fundamental trade-off between return and risk of financial assets.

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is the first and famous model in volatility field. This model is usually implemented for approximating and predicting volatility for the financial time series data for more details refer to (Babu and Reddy, 2015).As a new contribution in this article the volatility data will be modeled using WT in order to detect the events (such as outlier values) in the insurance section in content of ASE from the period between 2010-2018.

The organization of this article as the following. In section 2 the mathematical review will be presented. Section 3 the statistical data description will be presented also. While section 4 signs the results.Finally, section 5 is the conclusion.

2. Literature review

2.1 Volatility model

As we mentioned previously, many researchers have focused on the volatility data since all the Financial time series data are highly volatile. Therefore, the data trend and accuracy necessities can be met simultaneously (Babu and Reddy, 2015) this reason force the researchers to improve many methods for modeling for forecasting volatility data such as (Bentes, 2015) discuss the GARCH model in describing the gold volatility behavior. (Joukar and Nahmens, 2015) implemented ARCH and GARCH models to determine the persistency volatilities, for more details about the volatility data in content of modeling and predicting refer to (Kambouroudis and McMillan, 2015; Kristjanpoller and Minutolo, 2015; Abounoori, et al.,2016; Byun , 2016; Bollerslev et al., 2016). The definition of volatility WT methods will be storied in this section. In literature , volatility is the variance of the continuously compounded return per day (Baillie and DeGennaro, 1990). A variable's volatility, σ_t,is defined as the standard deviation of $R_i's$ at time t.

$$\sigma_t = \sqrt{\frac{\sum_1^n (R_i - \overline{R})^2}{n-1}}.$$

$\overline{R}$: arithmetic mean of $R_i's$. for more details refer to (Hull, 2012; Jaber, 2017).

$$\psi^H(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2} \\ -1, & \frac{1}{2} \le t \le 1 \\ 0, & Otherwise \end{cases}$$

Where:

$$l_k = \sqrt{2} \int_{-\infty}^{\infty} \phi(t)\phi(2t-k)dt = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases},$$

$$\int_{-\infty}^{\infty} \psi(t)dt = 0, \int_{-\infty}^{\infty} |\psi(t)| < \infty, \int_{-\infty}^{\infty} \frac{|\psi_1(\omega)|^2}{|\omega|} d\omega < \infty.$$

2.2 Wavelet transform

Wavelet transform was defined as:

$$\psi_{i,j}(t) = 2^{j/2}\psi(2^j t - k), i, k \in Z, Z = 0, 1, 2, ...$$

ψ:is the real-valued function having compactly supported (Chiann and Moretin, 1998), (Gencay et al, 2002). Generally, WT were calculated by using dilation equations and defined as:

$$\phi(t) = \sqrt{2}\sum l_k \phi(2t-k), \psi(t) = \sqrt{2}\sum h_k \phi(2t-k).$$

$\phi(2t-k), \psi(t)$:are the father and mother wavelets respectively, these two functions are producing the approximation and details coefficients respectively (Gencay et al, 2002). In this research the simplest and oldest WT will be used which is Haar WT (HWT) this model is defined as: In HWT the mother WT should stratifies: Where $\phi_1(\omega)$ presents WT (Gencay et al, 2002). For more details about WT and its application refer to (Al Wadi and Tahir Ismail, 2011; Al Wadi, 2017; Al Wadi et al., 2011; Al Wadi, 2013).

3. Methodology and results

In order to modeling the volatility data for the insurance time sereis data the daily time series data from the insurance sector is used from 2010-2018. HWT

transforms the time series data into two sets; approximation and details coefficients. The details coefficients will be adopted in this research since this series has the fluctuations and the behavior of the data used.

3.1 Volatility data decomposition

The volatility data is decomposed as presented using HWT in Fig.1 for Jordanian insurance sector. It is noticeable that, S= a5+d1+d2+d3+d4+d5 where a5 represents the approximation coefficients while from d1 to d5 represent the details coefficients which will be used in this article. Therefore, we can adopt d1 since the fluctuations are very clear so stating from the first observations (2010) then the data is very fluctuated, nonlinear, non-stationary. Therefore, the percentage of volatility is high in this year because the crisis 2009. The high fluctuations are continued until the observation number 700 which means the year number 2013 since the political situation in the neighbor's countries. Then the sector become more stable and the volatility is less than before. In general, Insurance sector is a main mechanisms of the financial system. It protects properties and individuals from risks, also plays a main record in supporting economic development. This sector contributes the GDP to reach about 2.14% in 2016.Since this sector is important , therefore many regulations in year 2016 have approved as follows: Apprising the supervisory frameworks of the insurance business, developing the financial solvency of the insurance companies, establishing regulatory requirements, applying the prudential regulatory requirements for the investment policies of the insurance companies and improving corporate governance requirements for the insurance companies. Therefore, the mentioned regulations help the sector to be more stable with less fluctuations.

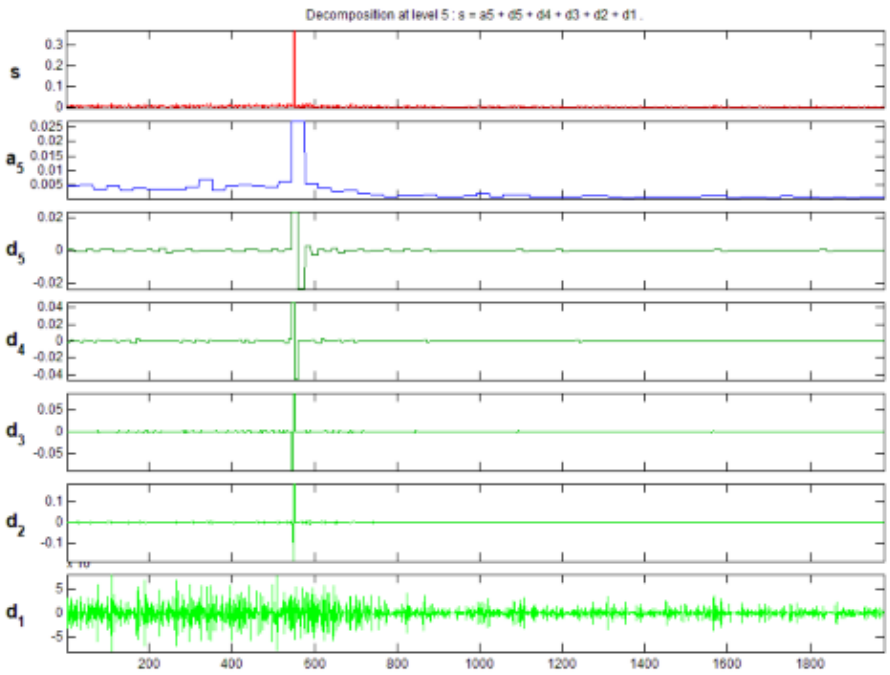
4. Conclusion

In this paper, the volatility of the insurance time series data is modeled using Haar wavelet transform (HWT). Therefore, the especial event, outlier values abrupt change and regime shits were detected and discussed based on the HWT's transforms decompositions. As a result the authors have found that HWT is suitable method for decomposition and studying the volatility behavior.

References

[1] T. Bollerslev, A. Patton, R. Quaedvlieg, *Exploiting the errors: a simple approach for improved volatility forecasting*, Journal of Econometrics, 192 (2016), 1-18.

[2] A.A. Alsaraireh, M. Almasarweh, M.B. Alnawaiseh, S. Al Wadi, V. Bhama, *The effect of methods of operation research in obtaining the best result in the*, Italian Journal of Pure and Applied Mathematics, 40 (2018), 501-509.



[3] Mohammad Almasarweh, Ahmed Atallah Alsaraireh, Ra'ed Masa'deh, *A statistical study to determine the production capacity of Jordanian pharmaceutical companies based on the number of working hours using the assignment problem,* Modern and Applied Science, 12 (2018), 301-808.

[4] S. Al Wadi, Mohammad Almasarweh and Ahmed Atallah Alsaraireh *Predicting closed price time series data using ARIMA model*, Modern Applied Science, 12 (2018), 181-185.

[5] C.N. Babu, B.E. Reddy, *Prediction of selected Indian stock using a partitioninginterpolation based ARIMAGARCH model*, Applied Computing and Informatics, 11 (2015), 130-143.

[6] E. Abounoori, Z. Elmi, Y. Nademic, *Forecasting tehran stock exchange volatility; Markov switching GARCH approach,* Physica A: Statistical Mechanics and its Applications, 445 (2016), 264-282.

[7] S. Bentes, *Modeling and forecasting volatility in gold returns under the GARCH, IGARCH and FIGARCH frameworks: New evidence*, Physica A: Statistical Mechanics and its Applications, 48 (2015), 355-364.

[8] S. Byun, *The usefulness of cross-sectional dispersion for forecasting aggregate stock price volatility*, Journal of Empirical Finance, 36 (2016), 162-180.

[9] R. Gencay, F. Seluk, B. Whitcher, *An introduction to wavelets and other filtering methods in finance and economics*, Academic Press, New York, 2002.

[10] A. Joukar, I. Nahmens, *Volatility forecast of construction cost index using general autoregressive conditional heteroskedastic method*, Journal of Construction Engineering and Management, 142 (2015), 1943-7862.

[11] D. Kambouroudis, D. McMillan, *Does VIX or volume improve GARCH volatility forecasts?,* Journal of Applied Economics, 48 (2015), 1210-1228.

[12] W. Kristjanpoller, M. Minutolo, *Gold price volatility: a forecasting approach using the artificial neural networkgarch model*, International Journal: Expert Systems with Applications, 42 (2015), 7245-7251.

[13] J. Hull, *Risk management and financial institutions*, John Wiley & Sons, 2012.

[14] A. Al-Khazaleh, S. Alwadi, F. Ababneh, *Wavelet transform asymmetric winsorized mean in detecting outlier value*, Far East Journal of Mathematical Sciences, 2015.

[15] M.T.I.S. Al Wadia, M. Tahir Ismail, *Selecting wavelet transforms model in forecasting financial time series data based on ARIMA model*, Applied Mathematical Sciences, 5 (2011), 315-326.

[16] S. Al Wadi, A. Hamarsheh, H. Alwadi, *Maximum overlapping discrete wavelet transform in forecasting banking sector*, Applied Mathematical Sciences, 7 (2013), 3995-4002.

[17] R.M. Alrumaih, M.A. Al-Fawzan, *Time series forecasting using wavelet denoising: an application to Saudi stock index*, Journal of King Saud University, Engineering Sciences, 2 (2002), 221-234.

[18] S. Al Wadi, *Improving Volatility Risk Forecasting Accuracy in Industry Sector*, International Journal of Mathematics and Mathematical Sciences, 2017.

[19] S. Al Wadi, M.T. Ismail, S.A. Abdul Kari, *Discovering Structure breaks in Amman stocks market*, Journal of Applied Sciences, 11 (2011), 1273-1278.

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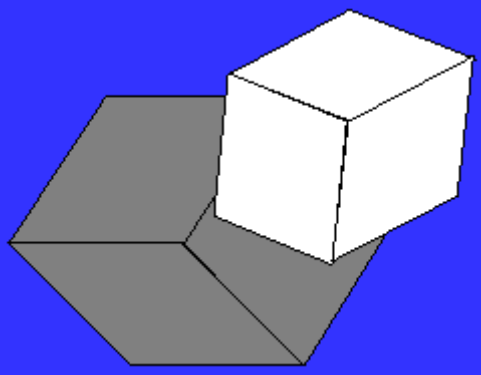
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