N° 40 – July 2018

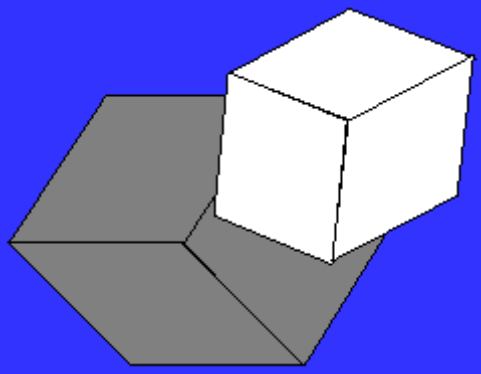
Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

EDITOR-IN-CHIEF
Piergiulio Corsini

Editorial Board

Saeid Abbasbandy
Praveen Agarwal
Bayram Ali Ersoy
Reza Ameri
Luisa Arlotti
Alireza Seyed Ashrafi
Krassimir Atanassov
Vadim Azhmyakov
Malvina Baica
Federico Bartolozzi
Rajabali Borzooei
Carlo Cecchini
Gui-Yun Chen
Domenico Nico Chillemi
Stephen Comer
Irina Cristea
Mohammad Reza Darafsheh
Bal Kishan Dass
Bijan Davvaz
Mario De Salvo
Alberto Felice De Toni
Mostafa Eslami
Franco Eugeni
Giovanni Falcone
Yuming Feng
Antonino Giambruno
Furio Honsell
Luca Iseppi
James Jantosciak
Tomas Kepka
David Kinderlehrer
Sunil Kumar
Andrzej Lasota
Violeta Leoreanu-Fotea
Maria Antonietta Lepellere
Mario Marchi
Donatella Marini
Angelo Marzollo
Antonio Maturo
Fabrizio Maturo
Sarka Hozkova-Mayerova
Vishnu Narayan Mishra
M. Reza Moghadam
Syed Tauseef Mohyud-Din
Petr Nemec
Vasile Oproiu
Livio C. Piccinini
Goffredo Pieroni
Flavio Pressacco
Vito Roberto
Ivo Rosenberg
Gaetano Russo
Paolo Salmon
Maria Scafati Tallini
Kar Ping Shum
Alessandro Silva
Florentin Smarandache
Sergio Spagnolo
Stefanos Spartalis
Hari M. Srivastava
Yves Sureau
Carlo Tasso
Ioan Tofan
Aldo Ventre
Thomas Vougiouklis
Hans Weber
Shanhe Wu
Xiao-Jun Yang
Yunqiang Yin
Mohammad Mehdi Zahedi
Fabio Zanolin
Paolo Zellini
Jianming Zhan



F O R U M

EDITOR-IN-CHIEF

Piergiulio Corsini
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
piergiuliocorsini@gmail.com

VICE-CHIEFS

Violeta Leoreanu
Maria Antonietta Lepellere

MANAGING BOARD

Domenico Chillemi, CHIEF
Piergiulio Corsini
Irina Cristea
Alberto Felice De Toni
Furio Honsell
Violeta Leoreanu
Maria Antonietta Lepellere
Elena Mocanu
Livio Piccinini
Flavio Pressacco
Luminita Teodorescu
Norma Zamparo

EDITORIAL BOARD

Saeid Abbasbandy
Dept. of Mathematics, Imam Khomeini International University,
Ghazvin, 34149-16818, Iran
abbasbandy@yahoo.com

Praveen Agarwal
Department of Mathematics, Anand International College of Engineering
Jaipur-303012, India
goyal.praveen2011@gmail.com

Bayram Ali Ersoy
Department of Mathematics, Yildiz Technical University
34349 Beşiktaş, Istanbul, Turkey
ersoya@gmail.com

Reza Ameri
Department of Mathematics
University of Tehran, Tehran, Iran
reza_ameri@yahoo.com

Luisa Arlotti
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
luisa.arlotti@dic.uniud.it

Alireza Seyed Ashrafi
Department of Pure Mathematics
University of Kashan, Kāshān, Isfahan, Iran
ashrafi_1385@yahoo.co.in

Krassimir Atanassov
Centre of Biomedical Engineering, Bulgarian Academy of Science
BL 105 Acad. G. Bontchev Str.
1113 Sofia, Bulgaria
krat@argo.bas.bg

Vadim Azhmyakov
Department of Basic Sciences, Universidad de Medellin,
Medellin, Republic of Colombia
vazhmyakov@udem.edu.co

Malvina Baica
University of Wisconsin-Whitewater
Dept. of Mathematical and Computer Sciences
Whitewater, W.I. 53190, U.S.A.
baicam@uww.edu

Federico Bartolozzi
Dipartimento di Matematica e Applicazioni
via Archirafi 34 - 90123 Palermo, Italy
bartolozzi@math.unipa.it

Rajabali Borzooei
Department of Mathematics
Shahid Beheshti University, Tehran, Iran
borzooei@sbu.ac.ir

Carlo Cecchini
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
cecchini@dimi.uniud.it

Gui-Yun Chen
School of Mathematics and Statistics,
Southwest University, 400715, Chongqing, China
gychen1963@sina.com

Domenico (Nico) Chillemi
Executive IT Specialist, IBM z System Software
IBM Italy SpA
Via Sciangai 53 – 00144 Roma, Italy
nicochillemi@it.ibm.com

Stephen Comer
Department of Mathematics and Computer Science
The Citadel, Charleston S. C. 29409, USA
comers@citadel.edu

Irina Cristea
CSIT, Centre for Systems and Information Technologies
University of Nova Gorica
Vipavska 13, Rožna Dolina, SI-5000 Nova Gorica, Slovenia
irinacri@yahoo.co.uk

Mohammad Reza Darafsheh
School of Mathematics, College of Science
University of Tehran, Tehran, Iran
darafsheh@ut.ac.ir

Bal Kishan Dass
Department of Mathematics
University of Delhi, Delhi - 110007, India
dassbk@rediffmail.com

Bijan Davvaz
Department of Mathematics,
Yazd University, Yazd, Iran
bdavvaz@yahoo.com

Mario De Salvo
Dipartimento di Matematica e Informatica
Viale Ferdinando Stagno d'Alcontres 31, Contrada Papardo
98166 Messina
desalvo@unime.it

Alberto Felice De Toni
Udine University, Rector
Via Palladio 8 - 33100 Udine, Italy
detoni@uniud.it

Franco Eugeni
Dipartimento di Metodi Quantitativi per l'Economia del Territorio
Università di Teramo, Italy
eugenif@tin.it

Mostafa Eslami
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran, Babolsar, Iran
mostafa.eslami@umz.ac.ir

Giovanni Falcone
Dipartimento di Metodi e Modelli Matematici
viale delle Scienze Ed. 8
90128 Palermo, Italy
gfalcone@unipa.it

Yuming Feng
College of Math. and Comp. Science, Chongqing Three-Gorges University,
Wanzhou, Chongqing, 404000, P.R.China
yumingfeng25928@163.com

Antonino Giambruno
Dipartimento di Matematica e Applicazioni
via Archirafi 34 - 90123 Palermo, Italy
giambruno@math.unipa.it

Furio Honsell
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
honsell@dimi.uniud.it

Luca Iseppi
Department of Civil Engineering and Architecture,
section of Economics and Landscape
Via delle Scienze 206 - 33100 Udine, Italy
luca.iseppi@uniud.it

James Jantosciak
Department of Mathematics, Brooklyn College (CUNY)
Brooklyn, New York 11210, USA
jsjbc@cunyvm.cuny.edu

Tomas Kepka
MFF-UK
Sokolovská 83
18600 Praha 8,Czech Republic
kepka@karlin.mff.cuni.cz

David Kinderlehrer
Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA15213-3890, USA
davidk@andrew.cmu.edu

Sunil Kumar
Department of Mathematics, National Institute of Technology
Jamshedpur, 831014, Jharkhand, India
skiitbhu28@gmail.com

Andrzej Lasota
Silesian University, Institute of Mathematics
Bankova 14
40-007 Katowice, Poland
lasota@ux2.math.us.edu.pl

Violeta Leoreanu-Fotea
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
leoreanu2002@yahoo.com

Maria Antonietta Lepellere
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
maria.lepellere@uniud.it

Mario Marchi
Università Cattolica del Sacro Cuore
via Trieste 17, 25121 Brescia, Italy
geomet@bs.unicatt.it

Donatella Marini
Dipartimento di Matematica
Via Ferrata 1- 27100 Pavia, Italy
marini@imati.cnr.it

Angelo Marzollo
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
marzollo@dimi.uniud.it

Antonio Maturo
University of Chieti-Pescara, Department of Social Sciences,
Via dei Vestini, 31
66013 Chieti, Italy
amaturo@unich.it

Fabrizio Maturo
University of Chieti-Pescara,
Department of Management and Business Administration,
Viale Pindaro, 44
65127 Pescara, Italy
f.maturo@unich.it

Sarka Hoskova-Mayerova
Department of Mathematics and Physics
University of Defence
Kounicova 65, 662 10 Brno, Czech Republic
sarka.mayerova@seznam.cz

Vishnu Narayan Mishra
Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
395 007, Surat, Gujarat, India
vishnunarayanmishra@gmail.com

M. Reza Moghadam
Faculty of Mathematical Science, Ferdowsi University of Mashhadh
P.O.Box 1159 - 91775 Mashhad, Iran
moghadam@math.um.ac.ir

Syed Tauseef Mohyud-Din
Faculty of Sciences, HITEC University Taxila
Cantt Pakistan
syedtauseefs@hotmail.com

Petr Nemec
Czech University of Life Sciences, Kamycka' 129
16521 Praha 6, Czech Republic
nemec@tf.czu.cz

Vasile Oproiu
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
voproiu@uaic.ro

Livio C. Piccinini
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
piccinini@uniud.it

Goffredo Pieroni
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
pieroni@dimi.uniud.it

Flavio Pressacco
Dept. of Economy and Statistics
Via Tomadini 30
33100, Udine, Italy
flavio.pressacco@uniud.it

Vito Roberto
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
roberto@dimi.uniud.it

Ivo Rosenberg
Departement de Mathematique et de Statistique
Université de Montreal
C.P. 6128 Succursale Centre-Ville
Montreal, Quebec H3C 3J7 - Canada
rosenb@DMS.UMontreal.CA

Gaetano Russo
Department of Civil Engineering and Architecture
Via delle Scienze 206
33100 Udine, Italy
gaetano.russo@uniud.it

Paolo Salmon
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
salmon@dm.unibo.it

Maria Scafati Tallini
Dipartimento di Matematica "Guido Castelnuovo"
Università La Sapienza
Piazzale Aldo Moro 2 - 00185 Roma, Italy
maria.scafati@gmail.com

Kar Ping Shum
Faculty of Science
The Chinese University of Hong Kong
Hong Kong, China (SAR)
kpshum@ynu.edu.cn

Alessandro Silva
Dipartimento di Matematica "Guido Castelnuovo"
Università La Sapienza
Piazzale Aldo Moro 2 - 00185 Roma, Italy
silva@mat.unirom1.it

Florentin Smarandache
Department of Mathematics, University of New Mexico
Gallup, NM 87301, USA
smarand@unm.edu

Sergio Spagnolo
Scuola Normale Superiore
Piazza dei Cavalieri 7 - 56100 Pisa, Italy
spagnolo@dm.unipi.it

Stefanos Spartalis
Department of Production Engineering and Management,
School of Engineering, Democritus University of Thrace
V.Sofias 12, Prokat, Bdg A1, Office 308
67100 Xanthi, Greece
sspart@pme.duth.gr

Hari M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia
V8W3P4, Canada
hmsri@uwm.univic.ca

Yves Sureau
27, rue d'Aubiere
63170 Perignat, Les Sarlieve - France
hysuroq@orange.fr

Carlo Tasso
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
tasso@dimi.uniud.it

Ioan Tofan
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
tofan@uaic.ro

Aldo Ventre
Seconda Università di Napoli, Fac. Architettura, Dip. Cultura del Progetto
Via San Lorenzo s/n
81031 Aversa (NA), Italy
aldoventre@yahoo.it

Thomas Vougiouklis
Democritus University of Thrace,
School of Education,
681 00 Alexandroupolis. Greece
tvougiou@eled.duth.gr

Hans Weber
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
weber@dimi.uniud.it

Shanhe Wu
Department of Mathematics, Longyan University,
Longyan, Fujian, 364012, China
shanhewu@gmail.com

Xiao-Jun Yang
Department of Mathematics and Mechanics,
China University of Mining and Technology,
Xuzhou, Jiangsu, 221008, China
dyangxiaojun@163.com

Yunqiang Yin
School of Mathematics and Information Sciences,
East China Institute of Technology, Fuzhou, Jiangxi
344000, P.R. China
yunqiangyin@gmail.com

Mohammad Mehdi Zahedi
Department of Mathematics, Faculty of Science
Shahid Bahonar, University of Kerman
Kerman, Iran
zahedi_mm@mail.uk.ac

Fabio Zanolin
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
zanolin@dimi.uniud.it

Paolo Zellini
Dipartimento di Matematica, Università degli Studi Tor Vergata
via Orazio Raimondo (loc. La Romanina)
00173 Roma, Italy
zellini@axp.mat.uniroma2.it

Jianming Zhan
Department of Mathematics, Hubei Institute for Nationalities
Enshi, Hubei Province,445000, China
zhanjianming@hotmail.com

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

Web Site

http://ijpam.uniud.it/journal/home.html

Twitter

@ijpamitaly

https://twitter.com/ijpamitaly

EDITOR-IN-CHIEF

Piergiulio Corsini
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
piergiuliocorsini@gmail.com

Vice-CHIEFS

Violeta Leoreanu-Fotea
Maria Antonietta Lepellere

Managing Board

Domenico Chillemi, CHIEF
Piergiulio Corsini
Irina Cristea
Alberto Felice De Toni
Furio Honsell
Violeta Leoreanu-Fotea
Maria Antonietta Lepellere
Elena Mocanu
Livio Piccinini
Flavio Pressacco
Luminita Teodorescu
Norma Zamparo

Editorial Board

Saeid Abbasbandy
Praveen Agarwal
Bayram Ali Ersoy
Reza Ameri
Luisa Arlotti
Alireza Seyed Ashrafi
Krassimir Atanassov
Vadim Azhmyakov
Malvina Baica
Federico Bartolozzi
Rajabali Borzooei
Carlo Cecchini
Gui-Yun Chen
Domenico Nico Chillemi
Stephen Comer
Irina Cristea
Mohammad Reza Darafsheh
Bal Kishan Dass
Bijan Davvaz
Mario De Salvo
Alberto Felice De Toni
Franco Eugeni
Mostafa Eslami
Giovanni Falcone
Yuming Feng
Antonino Giambruno
Furio Honsell
Luca Iseppi
James Jantosciak
Tomas Kepka
David Kinderlehrer
Sunil Kumar
Andrzej Lasota
Violeta Leoreanu-Fotea
Maria Antonietta Lepellere
Mario Marchi
Donatella Marini
Angelo Marzollo
Antonio Maturo
Fabrizio Maturo
Sarka Hozkova-Mayerova
Vishnu Narayan Mishra
M. Reza Moghadam
Syed Tauseef Mohyud-Din
Petr Nemec
Vasile Oproiu
Livio C. Piccinini
Goffredo Pieroni
Flavio Pressacco
Vito Roberto
Ivo Rosenberg
Gaetano Russo
Paolo Salmon
Maria Scafati Tallini
Kar Ping Shum
Alessandro Silva
Florentin Smarandache
Sergio Spagnolo
Stefanos Spartalis
Hari M. Srivastava
Yves Sureau
Carlo Tasso
Ioan Tofan
Aldo Ventre
Thomas Vougiouklis
Hans Weber
Shanhe Wu
Xiao-Jun Yang
Yunqiang Yin
Mohammad Mehdi Zahedi
Fabio Zanolin
Paolo Zellini
Jianming Zhan

Forum Editrice Universitaria Udinese Srl
Via Larga 38 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

Table of contents

Sanja Jancic Rasovic, Vucic Dasic
On generalization of division near-rings 1-8

H. Mirabdollahi, S.M. Anvariyeh, S. Mirvakili
Basic notions of partially ordered hypermodules 9-27

Xuesha Wu
Inequalities of unitarily invariant norms for matrices 28-33

Kuldip Raj, Charu Sharma
Ideal convergent generalized difference sequence spaces of infinite matrix and Orlicz function 34-46

R. Maritz, J.M.W. Munganga
On the role of the Stokes problem in second grade fluid flow in regions with permeable interfaces 47-60

Ibraheem Abu-Falahah
Soliton solutions for non-linear dispersive wave equations with variable-coefficients 61-67

Junling Sun, Jie Yang, Lei Sun
A dissipative hyperbolic systems approach to image restoration 68-81

Aynur Keskin Kaymakci, Wan Aunin Mior Othman, Cenap Ozel
On partially topological groups: extension closed properties 82-89

Ahmad Yousefian Darani, Masoomeh Shabani
On weak McCoy modules over commutative rings 90-97

H. Faramarzi, F. Rahbarnia, M. Tavakoli
Some results on distanced-balanced and strongly distance-balanced graphs 98-107

Yu-Hsien Liao, Tsu-Yin Chen, Ling-Yun Chung
A power index and its normalization under fuzzy multicriteria situation 108-121

Q.J. Kong, S. Wang
Some sufficient conditions implying nilpotency of finite groups 122-125

Xiaohui Wang, Xumeng Li, Xingjie Wu
Global exponential stability of Cohen-Grossberg neural networks with time-varying delays 126-140

Hamed M. Obiedat, Ameer A. Jaber
F-contractive mappings of Hardy-Rogers-type in G-metric spaces 141-148

S.A. Khafagy
On positive weak solutions for a class of nonlinear systems 149-156

Z. Fattahi, A. Erfanian, A. Azimi
A bipartite graph associated to a BI-module of a ring 157-163

Zhang Qiu-Ju
An improved clustering method based on density and division method 164-171

Zhenluo Lou
Existence of many non-radial solutions of an elliptic system 172-179

K. Rahman, F. Hussain, M.S. Ali Khan
Pythagorean fuzzy hybrid averaging aggregation operator and its application to multiple attribute decision making 180-187

Renario G. Hinampas Jr., Sergio R. Canoy Jr.
1-movable doubly connected domination in graphs 188-199

A.F. Sayed, Jamshaid Ahmad
Fixed point theorems for fuzzy soft contractive mappings in fuzzy soft metric spaces 200-214

Moa'ath N. Oqielat
Comparison of surface fitting methods for modelling leaf surface 215-226

Zhi-Jie Jiang
Product-type operators from area Nevanlinna spaces to Bloch-Orlicz spaces 227-243

Jiayin Feng, Dongyan Jia, Li Cui, Jing Cao, Zhuo Lin, Min Zhang
Comparison of SVM algorithm and BP algorithm: study on the evaluation index system of scientific research performance of vocational colleges 244-255

Morteza Jafari, Akbar Golchin, Hossein Mohammadzadeh Saany
On characterization of monoids by properties of generators II 256-276

P.L. Rama Kameswari, V.S. Bhagavan
Certain generating functions of generalized hypergeometric 2D polynomials from Truesdell's method 277-285

A. Tajmouati, M. El Berrag
d-mixing and d-universal J-class operators 286-293

Ruhul Amin, Sahadat Hossain
Pairwise connectedness in fuzzy bitopological spaces in quasi-coincidence sense 294-300

Ping Cai
Hopf bifurcation analysis and amplitude control of a new 4D hyper-chaotic system 301-310

Javid Iqbal, Rustam Abass, Puneet Kumar
Solution of linear and nonlinear singular boundary value problems using Legendre wavelet method 311-328

A.R. Hassan, R. Maritz, M. Mbehou
On the application of the adomian decomposition method to solve non-linear boundary value problems of a steady state flow of a liquid film 329-338

Sumera Naz, Samina Ashraf, Faruk Karaaslan
Energy of a bipolar fuzzy graph and its application in decision making 339-352

X. Zhang, F. Smarandache, M. Ali, X. Liang
Commutative neutrosophic triplet group and neutro-homomorphism basic theorem 353-375

Gurninder S. Sandhu, Deepak Kumar
Derivable mappings and commutativity of associative rings 376-393

Z. Mustafa, S.U. Khan, M.M.M. Jaradat, M. Arshad, H.M. Jaradat
Fixed point results of F-rational cyclic contractive mappings on 0-complete partial metric spaces 394-409

J.B. Bacani, J.F.T. Rabago
Class of admissible perturbations of special expressions involving completely monotonic functions 410-423

S. Shokrolahi Yancheshmeh
The topological indices of the Cayley graphs of dihedral group D_{2n} and the generalized quaternion group Q_{2^n} 424-433

B.G. Sidharth
Going beyond the standard model 434-437

Ould Ahmed Mahmoud Sid Ahmed
On the joint (m,q)-partial isometries and the joint m-invertible tuples of commuting operators on a Hilbert space 438-463

S. Shanthi, N. Rajesh
Separation axioms in topological ordered spaces 464-473

Sushil Kumar, Rajendra Prasad
Conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds 474-500

A.A. Alsaraireh, M. Almasarweh, M. B. Alnawaiseh, S. Al Wadi, V. Bhama
The effect of methods of operation research in obtaining the best results in the trade 501-509

Yao Zhang, Tingsong Du, Hao Wang
Some new k-fractional integral inequalities containing multiple parameters via generalized (s,m)-preinvexity 510-527

Jianming Xue
Some operator α-geometric mean inequalities 528-534

M. Zuriqat
The homo separation analysis method for solving the partial differential equation 535-543

Mahmood Parsamanesh
Global dynamics of an SIVS epidemic model with bilinear incidence rate 544-557

Fengwen Zhai, Jianwu Dang, Yangping Wang, Jing Jin
Using multi-scale auto convolution moments to get image affine invariant features 558-571

J. Moori, P. Perumal
On the double Frobenius group of the form $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ 572-599

Pengfei Guo, Yue Yang
Finite groups whose all proper subgroups are GPST-groups 600-606

S. Al Wadi, Ahmed Atallah Alsaraireh
Industrial data forecasting using discrete wavelet transform 607-614

Kewalee Suebyat, Nopparat Pochai
Three-dimensional air quality assessment simulations inside sky train platform with airflow obstacles on heavy traffic road 615-632

Barbora Batíková, Tomáš Kepka and Petr Němec
A construction of congruence-simple semirings 633-655

Abdul Haseeb, Mobin Ahmad, Sheeba Rizvi
On the conformal curvature tensor of ϵ-Kenmotsu manifolds 656-670

Xingkai Hu, Linru Nie
Exponential stability of nonlinear systems via alternate control 671-678

Moin Akhtar Ansari, Ali N.A. Koam
Rough approximations in KU-algebras 679-691

Ze Gu
On hyperideals of ordered semihypergroups 692-698

Ghassan K. Abufoudeh, Raed R. Abu Awwad
Bayesian estimation and prediction based on exponential residual type II censored life data 699-710

Essam R. El-Zahar, Abdelhalim Ebaid
On computing differential transform of nonlinear non-autonomous functions and its applications 711-723

Liang Liu, Ling Zhang, Xiangguang Dai, Yuming Feng
NAGSC: Nesterov's accelerated gradient methods for sparse coding 724-735

Manisha Shrivastava, Takashi Noiri, Purushottam Jha
Contra weakly-θ_I-precontinuous functions in ideal topological spaces 736-747

Lamairia Abd Elhakim, Haouam Kamel, Rebiai Belgacem
Nonexistence of global solutions to a fractional nonlinear ultra-parabolic system 748-755

Rakesh Kumar, Om Parkash
A new intuitionistic fuzzy divergence measure and its applications to handle fault diagnosis of turbine 756-771

Mohammad Hadi Zahedi, Abbas Ali Rezaee, Zeinab Dehghan
Fuzzy protection method for flood attacks in software defined networking (SDN) 772–789

Exchanges

Up to December 2015 this journal is exchanged with the following periodicals:

1.	Acta Cybernetica - Szeged	H
2.	Acta Mathematica et Informatica Universitatis Ostraviensis	CZ
3.	Acta Mathematica Vietnamica – Hanoi	VN
4.	Acta Mathematica Sinica, New Series – Beijing	RC
5.	Acta Scientiarum Mathematicarum – Szeged	H
6.	Acta Universitatis Lodziensis – Lodz	PL
7.	Acta Universitatis Palackianae Olomucensis, Mathematica – Olomouc	CZ
8.	Actas del tercer Congreso Dr. Antonio A.R. Monteiro - Universidad Nacional del Sur Bahía Blanca	AR
9.	AKCE International Journal of Graphs and Combinatorics - Kalasalingam	IND
10.	Algebra Colloquium - Chinese Academy of Sciences, Beijing	PRC
11.	Alxebra - Santiago de Compostela	E
12.	Analele Ştiinţifice ale Universităţii “Al. I Cuza” - Iaşi	RO
13.	Analele Universităţii din Timişoara - Universitatea din Timişoara	RO
14.	Annales Academiae Scientiarum Fennicae Mathematica - Helsinki	SW
15.	Annales de la Fondation Louis de Broglie - Paris	F
16.	Annales Mathematicae Silesianae – Katowice	PL
17.	Annales Scientif. Université Blaise Pascal - Clermont II	F
18.	Annales sect. A/Mathematica – Lublin	PL
19.	Annali dell’Università di Ferrara, Sez. Matematica	I
20.	Annals of Mathematics - Princeton - New Jersey	USA
21.	Applied Mathematics and Computer Science -Technical University of Zielona Góra	PL
22.	Archivium Mathematicum - Brnö	CZ
23.	Atti del Seminario di Matematica e Fisica dell’Università di Modena	I
24.	Atti dell’Accademia delle Scienze di Ferrara	I
25.	Automatika i Telemekhanika - Moscow	RU
26.	Boletim de la Sociedade Paranaense de Matematica - San Paulo	BR
27.	Bolétin de la Sociedad Matemática Mexicana - Mexico City	MEX
28.	Bollettino di Storia delle Scienze Matematiche - Firenze	I
29.	Buletinul Academiei de Stiinte - Seria Matem. - Kishinev, Moldova	CSI
30.	Buletinul Ştiinţific al Universităţii din Baia Mare - Baia Mare	RO
31.	Buletinul Ştiinţific şi Tecnic-Univ. Math. et Phyis. Series Techn. Univ. - Timişoara	RO
32.	Buletinul Universităţii din Braşov, Seria C - Braşov	RO
33.	Bulletin de la Classe de Sciences - Acad. Royale de Belgique	B
34.	Bulletin de la Societé des Mathematiciens et des Informaticiens de Macedoine	MK
35.	Bulletin de la Société des Sciences et des Lettres de Lodz - Lodz	PL
36.	Bulletin de la Societé Royale des Sciences - Liege	B
37.	Bulletin for Applied Mathematics - Technical University Budapest	H
38.	Bulletin Mathematics and Physics - Assiut	ET
39.	Bulletin Mathématique - Skopje Macedonia	MK
40.	Bulletin Mathématique de la S.S.M.R. - Bucharest	RO
41.	Bulletin of the Australian Mathematical Society - St. Lucia - Queensland	AUS
42.	Bulletin of the Faculty of Science - Assiut University	ET
43.	Bulletin of the Faculty of Science - Mito, Ibaraki	J
44.	Bulletin of the Greek Mathematical Society - Athens	GR
45.	Bulletin of the Iranian Mathematical Society - Tehran	IR
46.	Bulletin of the Korean Mathematical Society - Seoul	ROK
47.	Bulletin of the Malaysian Mathematical Sciences Society - Pulau Pinang	MAL
48.	Bulletin of Society of Mathematicians Banja Luka - Banja Luka	BiH
49.	Bulletin of the Transilvania University of Braşov - Braşov	RO
50.	Bulletin of the USSR Academy of Sciences - San Pietroburgo	RU
51.	Busefal - Université P. Sabatier - Toulouse	F
52.	Calculus CNR - Pisa	I
53.	Chinese Annals of Mathematics - Fudan University – Shanghai	PRC

54.	Chinese Quarterly Journal of Mathematics - Henan University	PRC
55.	Classification of Commutative FPF Ring - Universidad de Murcia	E
56.	Collectanea Mathematica - Barcelona	E
57.	Collegium Logicum - Institut für Computersprachen Technische Universität Wien	A
58.	Colloquium - Cape Town	SA
59.	Colloquium Mathematicum - Instytut Matematyczny - Warszawa	PL
60.	Commentationes Mathematicae Universitatis Carolinae - Praha	CZ
61.	Computer Science Journal of Moldova	CSI
62.	Contributi - Università di Pescara	I
63.	Cuadernos - Universidad Nacional de Rosario	AR
64.	Czechoslovak Mathematical Journal - Praha	CZ
65.	Demonstratio Mathematica - Warsawa	PL
66.	Discussiones Mathematicae - Zielona Gora	PL
67.	Divulgaciones Matemáticas - Universidad del Zulia	YV
68.	Doctoral Thesis - Department of Mathematics Umea University	SW
69.	Extracta Mathematicae - Badajoz	E
70.	Fasciculi Mathematici - Poznan	PL
71.	Filomat - University of Nis	SRB
72.	Forum Mathematicum - Mathematisches Institut der Universität Erlangen	D
73.	Functiones et Approximatio *Commentarii Mathematici* - Adam Mickiewicz University	L
74.	Funkcialaj Ekvaciaj - Kobe University	J
75.	Fuzzy Systems & A.I. Reports and Letters - Iaşi University	RO
76.	General Mathematics - Sibiu	RO
77.	Geometria - Fasciculi Mathematici - Poznan	PL
78.	Glasnik Matematicki - Zagreb	CRO
79.	Grazer Mathematische Berichte – Graz	A
80.	Hiroshima Mathematical Journal - Hiroshima	J
81.	Hokkaido Mathematical Journal - Sapporo	J
82.	Houston Journal of Mathematics - Houston - Texas	USA
83.	IJMSI - Iranian Journal of Mathematical Sciences & Informatics, Tarbiat Modares University, Tehran	IR
84.	Illinois Journal of Mathematics - University of Illinois Library - Urbana	USA
85.	Informatica - The Slovene Society Informatika - Ljubljana	SLO
86.	Internal Reports - University of Natal - Durban	SA
87.	International Journal of Computational and Applied Mathematics – University of Qiongzhou, Hainan	PRC
88.	International Journal of Science of Kashan University - University of Kashan	IR
89.	Iranian Journal of Science and Technology - Shiraz University	IR
90.	Irish Mathematical Society Bulletin - Department of Mathematics - Dublin	IRL
91.	IRMAR - Inst. of Math. de Rennes - Rennes	F
92.	Israel Mathematical Conference Proceedings - Bar-Ilan University - Ramat -Gan	IL
93.	Izvestiya: Mathematics - Russian Academy of Sciences and London Mathematical Society	RU
94.	Journal of Applied Mathematics and Computing – Dankook University, Cheonan – Chungnam	ROK
95.	Journal of Basic Science - University of Mazandaran – Babolsar	IR
96.	Journal of Beijing Normal University (Natural Science) - Beijing	PRC
97.	Journal of Dynamical Systems and Geometric Theory - New Delhi	IND
98.	Journal Egyptian Mathematical Society – Cairo	ET
99.	Journal of Mathematical Analysis and Applications - San Diego California	USA
100.	Journal of Mathematics of Kyoto University - Kyoto	J
101.	Journal of Science - Ferdowsi University of Mashhad	IR
102.	Journal of the Bihar Mathematical Society - Bhangalpur	IND
103.	Journal of the Faculty of Science – Tokyo	J
104.	Journal of the Korean Mathematical Society - Seoul	ROK
105.	Journal of the Ramanujan Mathematical Society - Mysore University	IND
106.	Journal of the RMS - Madras	IND
107.	Kumamoto Journal of Mathematics - Kumamoto	J
108.	Kyungpook Mathematical Journal - Taegu	ROK
109.	L'Enseignement Mathématique - Genève	CH
110.	La Gazette des Sciences Mathématiques du Québec - Université de Montréal	CAN
111.	Le Matematiche - Università di Catania	I
112.	Lecturas Matematicas, Soc. Colombiana de Matematica - Bogotà	C
113.	Lectures and Proceedings International Centre for Theorical Phisics - Trieste	I

114.	Lucrările Seminarului Matematic – Iaşi	RO
115.	m-M Calculus - Matematicki Institut Beograd	SRB
116.	Matematicna Knjiznica - Ljubljana	SLO
117.	Mathematica Balcanica – Sofia	BG
118.	Mathematica Bohemica - Academy of Sciences of the Czech Republic Praha	CZ
119.	Mathematica Macedonica, St. Cyril and Methodius University, Faculty of Natural Sciences and Mathematics - Skopje	MK
120.	Mathematica Montisnigri - University of Montenegro - Podgorica	MNE
121.	Mathematica Moravica - Cacak	SRB
122.	Mathematica Pannonica - Miskolc - Egyetemvaros	H
123.	Mathematica Scandinavica - Aarhus - Copenhagen	DK
124.	Mathematica Slovaca - Bratislava	CS
125.	Mathematicae Notae - Universidad Nacional de Rosario	AR
126.	Mathematical Chronicle - Auckland	NZ
127.	Mathematical Journal - Academy of Sciences - Uzbekistan	CSI
128.	Mathematical Journal of Okayama University - Okayama	J
129.	Mathematical Preprint - Dep. of Math., Computer Science, Physics – University of Amsterdam	NL
130.	Mathematical Reports - Kyushu University - Fukuoka	J
131.	Mathematics Applied in Science and Technology – Sangyo University, Kyoto	J
132.	Mathematics Reports Toyama University - Gofuku	J
133.	Mathematics for Applications - Institute of Mathematics of Brnö University of Technology, Brnö	CZ
134.	MAT - Prepublicacions - Universidad Austral	AR
135.	Mediterranean Journal of Mathematics – Università di Bari	I
136.	Memoirs of the Faculty of Science - Kochi University - Kochi	J
137.	Memorias de Mathematica da UFRJ - Istituto de Matematica - Rio de Janeiro	BR
138.	Memorie linceee - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
139.	Mitteilungen der Naturforschenden Gesellschaften beider Basel	CH
140.	Monografii Matematice - Universitatea din Timişoara	RO
141.	Monthly Bulletin of the Mathematical Sciences Library – Abuja	WAN
142.	Nagoya Mathematical Journal - Nagoya University,Tokyo	J
143.	Neujahrsblatt der Naturforschenden Gesellschaft - Zürich	CH
144.	New Zealand Journal of Mathematics - University of Auckland	NZ
145.	Niew Archief voor Wiskunde - Stichting Mathematicae Centrum – Amsterdam	NL
146.	Nihonkai Mathematical Journal - Niigata	J
147.	Notas de Algebra y Analisis - Bahia Blanca	AR
148.	Notas de Logica Matematica - Bahia Blanca	AR
149.	Notas de Matematica Discreta - Bahia Blanca	AR
150.	Notas de Matematica - Universidad de los Andes, Merida	YV
151.	Notas de Matematicas - Murcia	E
152.	Note di Matematica - Lecce	I
153.	Novi Sad Journal of Mathematics - University of Novi Sad	SRB
154.	Obzonik za Matematiko in Fiziko - Ljubljana	SLO
155.	Octogon Mathematical Magazine - Braşov	RO
156.	Osaka Journal of Mathematics - Osaka	J
157.	Periodica Matematica Hungarica - Budapest	H
158.	Periodico di Matematiche - Roma	I
159.	Pliska - Sofia	BG
160.	Portugaliae Mathematica - Lisboa	P
161.	Posebna Izdanja Matematickog Instituta Beograd	SRB
162.	Pre-Publicaçoes de Matematica - Univ. de Lisboa	P
163.	Preprint - Department of Mathematics - University of Auckland	NZ
164.	Preprint - Institute of Mathematics, University of Lodz	PL
165.	Proceeding of the Indian Academy of Sciences - Bangalore	IND
166.	Proceeding of the School of Science of Tokai University - Tokai University	J
167.	Proceedings - Institut Teknology Bandung - Bandung	RI
168.	Proceedings of the Academy of Sciences Tasked – Uzbekistan	CSI
169.	Proceedings of the Mathematical and Physical Society of Egypt – University of Cairo	ET
170.	Publicaciones del Seminario Matematico Garcia de Galdeano - Zaragoza	E
171.	Publicaciones - Departamento de Matemática Universidad de Los Andes Merida	YV
172.	Publicaciones Matematicas del Uruguay - Montevideo	U
173.	Publicaciones Mathematicae - Debrecen	H

174.	Publicacions mathematiques - Universitat Autonoma, Barcelona	E
175.	Publications de l'Institut Mathematique - Beograd	SRB
176.	Publications des Séminaires de Mathématiques et Informatiques de Rennes	F
177.	Publications du Departmenet de Mathematiques, Université Claude Bernard - Lyon	F
178.	Publications Mathematiques - Besançon	F
179.	Publications of Serbian Scientific Society - Beograd	SRB
180.	Publikacije Elektrotehnickog Fakulteta - Beograd	SRB
181.	Pure Mathematics and Applications - Budapest	H
182.	Quaderni di matematica - Dip. to di Matematica – Caserta	I
183.	Qualitative Theory of Dynamical Systems - Universitat de Lleida	E
184.	Quasigroups and Related Systems - Academy of Science - Kishinev Moldova	CSI
185.	Ratio Mathematica - Università di Pescara	I
186.	Recherche de Mathematique - Institut de Mathématique Pure et Appliquée Louvain-la-Neuve	B
187.	Rendiconti del Seminario Matematico dell'Università e del Politecnico – Torino	I
188.	Rendiconti del Seminario Matematico - Università di Padova	I
189.	Rendiconti dell'Istituto Matematico - Università di Trieste	I
190.	Rendiconti di Matematica e delle sue Applicazioni - Roma	I
191.	Rendiconti lincei - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
192.	Rendiconti Sem. - Università di Cagliari	I
193.	Report series - Auckland	NZ
194.	Reports Math. University of Stockholm - Stockholm	SW
195.	Reports - University Amsterdam	NL
196.	Reports of Science Academy of Tajikistan – Dushanbe	TAJ
197.	Research Reports - Cape Town	SA
198.	Research Reports - University of Umea - Umea	SW
199.	Research Report Collection (RGMIA) Melbourne	AUS
200.	Resenhas do Instituto de Matemática e Estatística da universidadae de São Paulo	BR
201.	Review of Research, Faculty of Science, Mathematics Series - Institute of Mathematics University of Novi Sad	SRB
202.	Review of Research Math. Series - Novi Sad	YN
203.	Revista Ciencias Matem. - Universidad de la Habana	C
204.	Revista Colombiana de Matematicas - Bogotà	C
205.	Revista de Matematicas Aplicadas - Santiago	CH
206.	Revue Roumaine de Mathematiques Pures et Appliquées - Bucureşti	RO
207.	Ricerca Operativa AIRO - Genova	I
208.	Ricerche di Matematica - Napoli	I
209.	Rivista di Matematica - Università di Parma	I
210.	Sains Malaysiana - Selangor	MAL
211.	Saitama Mathematical Journal - Saitama University	J
212.	Sankhya - Calcutta	IND
213.	Sarajevo Journal of Mathematics	BIH
214.	Sciences Bulletin, DPRK, Pyongyang	KR
215.	Scientific Rewiev - Beograd	SRB
216.	Scientific Studies and Research, Vasile Alecsandri University Bacau	RO
217.	Semesterbericht Funktionalanalysis - Tübingen	D
218.	Séminaire de Mathematique - Université Catholique, Louvain la Neuve	B
219.	Seminario di Analisi Matematica - Università di Bologna	I
220.	Serdica Bulgaricae Publicaciones Mathematicae - Sofia	BG
221.	Serdica Mathematical Journal - Bulgarian Academy of Sciences, University of Sofia	BG
222.	Set-Valued Mathematics and Applications – New Delhi	IND
223.	Sitzungsberichte der Mathematisch Naturwissenschaflichen Klasse Abteilung II – Wien	A
224.	Southeast Asian Bulletin of Mathematics - Southeast Asian Mathematical Society	PRC
225.	Studia Scientiarum Mathematica Hungarica – Budapest	H
226.	Studia Universitatis Babes Bolyai - Cluj Napoca	RO
227.	Studii şi Cercetări Matematice - Bucureşti	RO
228.	Studii şi Cercetări Ştiinţifice, ser. Matematică - Universitatea din Bacău	RO
229.	Sui Hak - Pyongyang DPR of Korea	KR
230.	Tamkang Journal of Mathematics - Tamsui - Taipei	TW
231.	Thai Journal of Mathematics – Chiang Mai	TH
232.	Task Quarterly	PL
233.	The Journal of the Academy of Mathematics Indore	IND

234.	The Journal of the Indian Academy of Mathematics - Indore	IND
235.	The Journal of the Nigerian Mathematical Society (JNMS) - Abuja	WAN
236.	Theoretical and Applied Mathematics – Kongju National University	ROK
237.	Thesis Reprints - Cape Town	SA
238.	Tohoku Mathematical Journal – Sendai	J
239.	Trabalhos do Departamento de Matematica Univ. - San Paulo	BR
240.	Travaux de Mathematiques – Bruxelles	B
241.	Tsukuba Journal of Mathematics - University of Tsukuba	J
242.	UCNW Math. Preprints Prifysgol Cymru - University of Wales – Bangor	GB
243.	Ukranii Matematiskii Journal – Kiev	RU
244.	Uniwersitatis Iagiellonicae Acta Mathematica – Krakow	PL
245.	Verhandlungen der Naturforschenden Gesellschaft – Basel	CH
246.	Vierteljahrsschrift der Naturforschenden Gesellschaft – Zürich	CH
247.	Volumenes de Homenaje - Universidad Nacional del Sur Bahía Blanca	AR
248.	Yokohama Mathematical Journal – Yokohama	J
249.	Yugoslav Journal of Operations Research – Beograd	SRB
250.	Zbornik Radova Filozofskog – Nis	SRB
251.	Zbornik Radova – Kragujevac	SRB
252.	Zeitschrift für Mathematick Logic und Grundlagen der Math. – Berlin	D

ON GENERALIZATION OF DIVISION NEAR-RINGS

Sanja Jancic Rasovic*
Department of Mathematics
Faculty of Natural Science and Mathematics
University of Montenegro
Podgorica, Montenegro
sabu@t-com.me

Vucic Dasic
Department of Mathematics
Faculty of Natural Science and Mathematics
University of Montenegro
Podgorica, Montenegro
mihajlo0107@gmail.com

Abstract. In this paper we introduce the class of D-division near-rings as a subclass of near-rings with a defect D and that one of division near-rings. We introduce the notion of D-division near-ring and we state necessary and sufficient condition under which a near-ring with defect of distributivity D is a D-division near-ring.
Keywords: near-ring, division near-ring.

1. Introduction and preliminaries

The interest in near-rings and near-fields started at the beginning of the 20th century when Dickson wanted to know if the list of axioms for skew fields id redundant. He found in [3] that there do exist "near-fields" which fulfill all axioms for skew fields except one distributive law. Since 1950, the theory of near-rings had applications to several domains, for instance in area of dynamical systems, graphs, homological algebra, universal algebra, category theory, geometry and so on.

A comprehensive review of the theory of near-rings and its applications appears in Pilz [10], Meldrun [8], Clay [1], Wahling [14], Scot [12], Ferrero [4], Vukovic [13], and Satyanarayana and Prasad [11].

Let $(R, +, \cdot)$ be a left near-ring, i.e. $(R, +)$ is a group (not necessarily commutative) with the unit element 0, $(R, \cdot)$ is a semigroup and the left distributivity holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for any $x, y, z \in R$. It is clear that $x \cdot 0 = 0$, for any $x \in R$, while it might exists $y \in R$ such that $0 \cdot y \neq 0$. If 0 is a bilaterally absorbing element, that is $0 \cdot x = x \cdot 0 = 0$, for any $x \in R$, then R is called a zero-symmetric near-ring. Obviously, if $(R, +, \cdot)$ is a left near-ring then $x \cdot (-y) = -(xy)$ for any $x, y \in R$.

∗. Corresponding author

A normal subgroup I of $(R,+)$ is called an ideal of a near-ring $(R,+,\cdot)$ if:
1) $RI=\{r\cdot i|r\in R,\quad i\in I\}\subseteq I$.
2) $(r+i)r'-r\cdot r'\in I$, for all $r,r'\in R$ and $i\in I$.

Obviously, if I is an ideal of zero-symmetric near-ring R, then $IR\subseteq I$ and $RI\subseteq I$. In particular, if $(R,+,\cdot)$ is a left near-ring that contains a multiplicative semigroup S, whose elements generate $(R,+)$ and satisfy $(x+y)\cdot s=x\cdot s+y\cdot s$, for all $x,y\in R$ and $s\in S$, then we say that R is a distributively generated near-ring (d.g. near-ring). Regarding the classical example of a near-ring, that one represented by the set of the functions from an additive group G into itself with the pointwise addition and the natural composition of functions, if S is the multiplicative semigroup of the endomorphisms of G and R' is the subnear-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [5]. A near-ring containing more than one element is called a division near-ring, if the set $R\backslash\{0\}$ is a multiplicative group [7]. Several examples of division near-rings are given in [5]. It is well known that every division ring is a division near-ring, while there are division nearrings which are not division rings.

Ligh [7] give necessary and sufficient condition for a d.g. near-ring to be a division ring.

Lemma 1.1 ([7]). *If R is a d.g. near-ring, then $0\cdot x=0$, for all $x\in R$.*

Theorem 1.1 ([7]). *A necessary and sufficient condition for a d.g. near-ring with more than one element to be division ring is that for all non-zero elements $a\in R$, it holds $a\cdot R=R$.*

Lemma 1.2 ([7]). *The additive group $(R,+)$ of a division near-ring R is abelian.*

Another example of division ring is given by the following result.

Lemma 1.3. *Every d.g. division near-ring R is a division ring.*

Proof. By Lemma 1.2, the additive group $(R,+)$ of a division near-ring is abelian. It follows ([5], p.93) that every element of R is right distributive, i.e. $(x+y)\cdot z=x\cdot z+y\cdot z$, for all $x,y,z\in R$. Thereby, if R is d.g. near-ring, then R is a division near-ring if and only if R is a division ring. □

In [2] Dasic introduced the notion of a near-ring with defect of distributivity as a generalization of d.g. near-ring.

Definition 1.1 ([2]). *Let R be a zero-symmetric (left) near-ring. A set S of generators of R is a multiplicative subsemigroup $(S,\cdot)$ of the semigroup $(R,\cdot)$, whose elements generate $(R,+)$. The normal subgroup D of the group $(R,+)$ which is generated by the set $D_S=\{d\in R|d=-(x\cdot s+y\cdot s)+(x+y)\cdot s, x,y\in R, s\in S\}$ is called the defect of distributivity of the near-ring R.*

In other words, if $s\in S$, then for all $x,y\in R$, there exists $d\in D$ such that $(x+y)\cdot s=x\cdot s+y\cdot s+d$. This expresses the fact that the elements of S are

distributive with the defect D. When we want to stress the set S of generators, we will denote the near-ring by the couple (R, S). In particular, if $D = 0$, then R is a distributively generated near-ring. The following lemma is easy to verify.

Lemma 1.4. *Let (R, S) be a near-ring with the defect D.*

i) If $s \in S$ and $x \in R$, then there exists $d \in D$ such that $(-x)s = -(xs) + d$.

ii) If $s \in S$, and $x, y \in R$,then there exists $d \in D$ such that that $(x - y) \cdot s = x \cdot s - y \cdot s + d$.

The main properties of this kind of near-rings are summarized in the following results [2].

Theorem 1.2. *i) Every homomorphic image of a near-ring with the defect D is a near-ring with the defect $f(D)$, when f is a homomorphism of near-rings.*

ii) Every direct sum of a family of near-rings R_i with the defects D_i, respectively, is a near-ring whose defect is a direct sum of the defects D_i.

iii) The defect D of the near-ring R is an ideal of R.

iv) Let R be a near-ring with the defect D and A be an ideal of R. The quotient near-ring R/A has the defect $D = \{d + A | d \in D\}$. Moreover, R/A is distributively generated if and only if $D \subseteq A$.

Following this idea, Jancic Rasovic and Cristea [6], introduce the concept of hypernear-ring with a defect of distributivity, and present several properties of this class of hypernear-rings, in connection with their direct product, hyperhomomorphisms, or factor hypernear-rings.

In this paper we introduce the class of $D-$division near-rings as a subclass of near-rings with a defect D and that one of division near-rings. Then we state necessary and sufficient condition under which a near-ring with defect of distributivity D is a $D-$ division near-ring. On the end, we show that Ligh's theorem proved for distributively generated near-rings is a corollary of our result.

2. D-division near-rings

Definition 2.1. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. The structure $(R\backslash D, \cdot)$ is a $D-$ multiplicative group of the near -ring R if:*

i) The set $R\backslash D$ is closed under the multiplication.

ii) There exists $e \in R\backslash D$ such that, for each $x \in R$ it holds $x \cdot e = x + d_1$ and $e \cdot x = x + d_2$, for some $d_1, d_2 \in D$. A such element e is called the identity element.

iii) For each $x \in R\backslash D$ there exists $x' \in R\backslash D$ and $d_1, d_2 \in D$, such that: $x \cdot x' = e + d_1$ and $x' \cdot x = e + d_2$.

Definition 2.2. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. We say that R is a $D-$division near-ring (a near-ring of $D-$ fractions) if $(R\backslash D, \cdot)$ is a $D-$multiplicative group.*

Obviously, if (R, S) is a near-ring with defect of distributivity $D \neq R$, such that $(R\backslash D, \cdot)$ is a multiplicative group, then $(R\backslash D, \cdot)$ is a $D-$multiplicative group. Also, if R is a distributively generated such that R is a division near-ring, then R is an example of $D-$division near ring with defect of distributivity $D = \{0\}$.

Now we present another examples of $D-$division near-rings.

Example 2.1. Let $(R, +) = (Z_6, +)$, be the additive group of integers modulo 6, and define on R the multiplication as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	4	3	2	1
2	0	1	2	3	4	5
3	0	0	0	0	0	0
4	0	5	4	3	2	1
5	0	1	2	3	4	5

It is simple to check that the multiplication is associative, so $(R, \cdot)$ is a semigroup, having 0 as two-sided absorbing element. Moreover, the multiplication distributes over addition, so for any $x, y, z \in R$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$ (we let these part to the reader as a simple exercice). For example, $1 \cdot (4 + 2) = 1 \cdot 0 = 0 (0 = 2 + 4 = 1 \cdot 4 + 1 \cdot 2.)$ Take $S = \{0, 2, 3\}$ a system of generators of the hypergroup $(R, +)$. We also notice that $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$. Now we determine the set $D_S : D_S = \{d \in R | d = -(x \cdot s + y \cdot s) + (x + y) \cdot s, x, y \in R, s \in S\} = \{-(x \cdot 0 + y \cdot 0) + (x + y) \cdot 0 | x, y \in R\} \cup \{-(x \cdot 2 + y \cdot 2) + (x + y) \cdot 2 | x, y \in R\} \cup \{-(x \cdot 3 + y \cdot 3) + (x + y) \cdot 3 | x, y \in R\} = \{0\} \cup \{0\} \cup \{0, 3\} = \{0, 3\}$. The table of the hypercomposition $x \cdot 3 + y \cdot 3$ is the following one:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	0	0	3	0	0
2	3	0	0	3	0	0
3	0	3	3	0	3	3
4	3	0	0	3	0	0
5	3	0	0	3	0	0

It follows that the table of $-(x \cdot 3 + y \cdot 3)$ is:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	0	0	3	0	0
2	3	0	0	3	0	0
3	0	3	3	0	3	3
4	3	0	0	3	0	0
5	3	0	0	3	0	0

Similarly, the table of the hypercomposition $(x + y) \cdot 3$ is:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	3	0	3	3	0
2	3	0	3	3	0	3
3	0	3	3	0	3	3
4	3	3	0	3	3	0
5	3	0	3	3	0	3

We obtain that $A = \{-(x \cdot 3 + y \cdot 3) + (x + y) \cdot 3 | x, y \in R\} = \{0, 3\}$.

It follows that the defect of distributivity of the near-ring R is $D = \{0, 3\}$.

It can be easily verified that $(R\backslash D, \cdot)$ is a $D-$multiplicative group. Indeed, $R\backslash D = \{1, 2, 4, 5\}$ is closed under the multiplication. Moreover, $e = 2$ is the identity element. Finally, for any $a \in R\backslash D$, there exists $d \in D$ such that $a \cdot a = 2 + d$, meaning that the inverse of each element $a \in R\backslash D$ is a itself. So (R, S) is a $D-$division near-ring.

Example 2.2. Let $(R, +) = (Z_4, +)$ be the additive group of the integers modulo 4 and define on R the multiplication as follows:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	3	2	1

Then, $(R, \cdot)$ is a semigroup, having 0 as a bilaterally absorbing element. It can be veried that, for any $x, y, z \in R$, it holds $x \cdot (y + z) = x \cdot y + x \cdot z$, meaning that $(R, +, \cdot)$ is a near-ring. Take $S = \{1\}$. Obviously, S is a subsemigroup of $(R, \cdot)$ and it generates $(R, +)$. Since the set $D_S = \{-(x \cdot 1 + y \cdot 1) + (x + y) \cdot 1 | x, y \in R\} = \{0, 2\}$, we conclude that the the defect of distributivity of the near-ring R is $D = \{0, 2\}$.

We can see that the multiplicative structure$(R\backslash D, \cdot)$ is a group, so $R\backslash D$ is a $D-$multiplicative group, i.e. (R, S) is a $D-$division near-ring.

Definition 2.3. *Let (R, S) be a near-ring with the defect of distributivity D. We say that (R, S) is a near-ring without $D-$ divisors if, for all $x, y \in R, x \cdot y \in D$ implies that $x \in D$ or $y \in D$. Otherwise, we say that R has $D-$divisors if there exist $x, y \in R\backslash D$ such that $x \cdot y \in D$.*

Proposition 2.1. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. If $a \cdot (R\backslash D) + D = R\backslash D + D$, for all $a \in R\backslash D$, then R is a near-ring without $D-$divisors.*

Proof. Suppose there exist $x, y \in R\backslash D$ such that $x \cdot y \in D$. Since $x \in R\backslash D \subseteq R\backslash D + D = x \cdot (R\backslash D) + D$, it follows that there exists $x' \in R\backslash D$ and $d_1 \in D$ such that $x = x \cdot x' + d_1$. Moreover, from $x' \in R\backslash D \subseteq R\backslash D + D = y \cdot (R\backslash D) + D$, it follows that there exist $y' \in R\backslash D$ and $d_2 \in D$ such that $x' = y \cdot y' + d_2$. Therefore, $x = x \cdot (y \cdot y' + d_2) + d_1 = x \cdot y \cdot y' + x \cdot d_2 + d_1$. Since D is an ideal of R, and R

is a zero-symmetric near-ring, then $(x \cdot y) \cdot y' \in D$ as $x \cdot y \in D$, and $x \cdot d_2 \in D$, as $d_2 \in D$. It follows that $(x \cdot y) \cdot y' + x \cdot d_2 + d_1 \in D$, i.e. $x \in D$. It contradicts the initial assumption. Therefore, R is a near-ring without $D-$divisors. □

Corollary 2.1. *If (R, S) is a near-ring with the defect of distributivity $D \neq R$, such that $a \cdot (R\backslash D) + D = R\backslash D + D$, for all $a \in R\backslash D$, then the set $R\backslash D$ is closed under the multiplication.*

Proof. It follows immediately from the previous proposition. □

Theorem 2.1. *Let (R, S) be a near-ring with the defect $D \neq R$. A necessary and sufficient condition for the near-ring R to be a $D-$division near-ring is that $a \cdot (R\backslash D) + D = R\backslash D + D$, for all $a \in R\backslash D$.*

Proof. Sufficiency. Let $a \cdot (R\backslash D) + D = R\backslash D + D$, for all $a \in R\backslash D$. By Corollary 2.1, it follows that the set $R\backslash D$ is closed under the multiplication. Note that there exists $s \in R\backslash D$ such that $s \in S$. To the contrary, if $S \subseteq D$, then $R = \langle S \rangle \subseteq D$, meaning that $R = D$, which contradicts our assumption. Thus, let $s \in R\backslash D$ such that $s \in S$. Since $s \in (R\backslash D) + D = s \cdot (R\backslash D) + D$, it follows that there exists $e \in R\backslash D$ and $d_1 \in D$ such that $s = s \cdot e + d_1$. Hence, $s \cdot (e \cdot s - s) = (s \cdot e) \cdot s - s \cdot s = (s - d_1) \cdot s - s \cdot s \in D$, since D is an ideal in R. By Proposition 2.1, R is a near-ring without $D-$divisors and since $s \in R\backslash D$, we get $e \cdot s - s \in D$, i.e. $e \cdot s \in D + s = s + D$, and so there exists $d_2 \in D$ such that $es = s + d_2$. If $x \in R\backslash D$, then for some $d_3 \in D$ it holds: $(x \cdot e - x) \cdot s = x \cdot (e \cdot s) - x \cdot s + d_3 = x \cdot (s + d_2) - x \cdot s + d_3 = x \cdot s + x \cdot d_2 - x \cdot s + d_3 \in x \cdot s + D - x \cdot s + D \subseteq D + D = D$. Since $s \notin D$, we have $x \cdot e - x \in D$, meaning that $x \cdot e \in D + x = x + D$. So, there exists $d_4 \in D$ such that $xe = x + d_4$. Besides, $s \cdot (e \cdot x - x) = (s \cdot e) \cdot x - s \cdot x = (s - d_1) \cdot x - s \cdot x \in D$, since D is an ideal. Again, since $s \notin D$, we obtain $e \cdot x - x \in D$, implying that $e \cdot x \in D + x = x + D$. Thus, there exists $d_5 \in D$ such that $ex = x + d_5$ .Thereby e is the identity element.

Suppose now that $a \in R\backslash D$. Since $e \in R\backslash D \subseteq R\backslash D + D = a \cdot (R\backslash D) + D$, then there exist $a' \in R\backslash D$ and $d \in D$ such that $e = a \cdot a' + d$. Besides, $a \cdot (a' \cdot a - e) = (a \cdot a') \cdot a - a \cdot e = (e - d) \cdot a - (a + d_1)$, for some $d_1 \in D$. Since D is an ideal of R, we have $(e - d) \cdot a - e \cdot a \in D$, i.e. $(e - d) \cdot a \in D + e \cdot a = e \cdot a + D$. Therefore, $a \cdot (a' \cdot a - e) \in e \cdot a + D - (a + d_1) = e \cdot a + D - d_1 - a$. Besides, $e \cdot a = a + d_2$, for some $d_2 \in D$ and thus $a \cdot (a' \cdot a - e) \in a + d_2 + D - d_1 - a \subseteq a + D - a \subseteq D$. Since $a \notin D$, it follows that $a' \cdot a - e \in D$, meaning that $a' \cdot a \in D + e = e + D$ i.e $a' \cdot a = e + d_4$ for some $d_4 \in D$. Hence, we have shown that $R\backslash D$ is a $D-$multiplicative group, implying that (R, S) is a $D-$division near-ring.

Necessity. Let $R\backslash D$ be a $D-$multiplicative group with the identity element e. Let $a \in R\backslash D$. Obviously, $a \cdot (R\backslash D) + D \subseteq R\backslash D + D$. We prove now the other inclusion $R\backslash D + D \subseteq a \cdot (R\backslash D) + D$. Suppose $x \in R\backslash D$. Since $R\backslash D$ is a $D-$multiplicative group, it follows that there exist $a' \in R\backslash D$ and $d_1 \in D$ such that $a \cdot a' = e + d_1$. Besides there exists $d_2 \in D$ such that $x = e \cdot x + d_2 = (a \cdot a' - d_1) \cdot x + d_2$. Since D is an ideal of R, we have $(aa' - d_1) \cdot x - (a \cdot a') \cdot x \in D$,

and therefore $(a\cdot a'-d_1)\cdot x=(aa'-d_1)\cdot x-(a\cdot a')\cdot x+(a\cdot a')\cdot x\in D+(a\cdot a')\cdot x$. It follows that $x\in D+a\cdot(a'\cdot x)+d_2=a\cdot(a'\cdot x)+D\subseteq a\cdot(R\backslash D)+D$. Therefore, $R\backslash D\subseteq a\cdot(R\backslash D)+D$, i.e. $R\backslash D+D\subseteq a\cdot(R\backslash D)+D$. □

Now we will show that Theorem 1.1 [7] follows from the previous theorem.

Corollary 2.2. *A necessary and sufficient condition for a d.g. near-ring with more than one element to be division ring is that for all non-zero elements $a\in R$, it holds $a\cdot R=R$.*

Proof. If R is a d.g. near-ring, then by Lemma 1.1, R is a zero symmetric near-ring, with the defect of disrtributivity $D=\{0\}$. From the previous theorem, it follows that a necessary and sucient condition for a d.g. near-ring R with more than one element to be a division near-ring is that $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Now we prove that if R is a d.g. near-ring with more than one element, then $a\cdot R=R$, for all $a\in R\backslash\{0\}$, if and only if $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Obviously, $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$ implies that $a\cdot R=R$, for all $a\in R\backslash\{0\}$.

Suppose now that we have $a\cdot R=R$, for all $a\in R\backslash\{0\}$. First we prove that $a\cdot R\backslash\{0\}\subseteq R\backslash\{0\}$, for $a\neq 0$. If there exist $a\neq 0, b\neq 0$, such that $a\cdot b=0$, then since $a\cdot R=R$ and $b\cdot R=R$ it follows that there exist $x,y\in R$ such that $a=a\cdot x$ and $x=b\cdot y$. Therefore, by Lemma 1.1, we have $0=0\cdot y=a\cdot b\cdot y=a\cdot x=a$, which is a contradiction. Thus $a\cdot R\backslash\{0\}\subseteq R\backslash\{0\}$. On the other side, for all $a\in R\backslash\{0\}$, it holds $R\backslash\{0\}\subseteq a\cdot R=R$ and since $a\cdot 0=0$ it follows that $R\backslash\{0\}\subseteq a\cdot(R\backslash\{0\})$. Therefore, $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Thus, from Lemma 1.3, we obtain Corollary 2.2. □

3. Conclusion and future work

In our future research we intend to extend to the case of hypernear-rings the notions that were studied in this paper. Jancic- Rasovic and Cristea have recently started [6] the study of hypernear-rings with a defect of distributivity D. Our aim is to continue in the same direction, introducing the class of $D-$division hypernear-rings as a subclass of hypernear-rings with a defect D, and that one of division hypernear-rings. Another aim is to state a necessary and sufficient condition under which a hypernear-ring with a defect of distributivity D is a $D-$division hypernear-ring.

References

[1] J. Clay, *Nearrings: Geneses and Application*, Oxford Univ. Press, Oxford, 1992.

[2] V. Dasic, *A defect of distributivity of the near-rings* , Math. Balkanica, 8 (1978), 63-75.

[3] L. Dickson, *Definitions of a group and a field by independent postulates*, Trans. Amer. Math. Soc., 6 (1905), 198-204.

[4] G. Ferrero, C. Ferrero-Cotti, *Nearrings. Some Developments Linked to Semigroups and Groups*, Kluwer, Dordrecht, 2002.

[5] A. Fronlich, *Distributively generated near-rings (I. Ideal Theory)*, Proc. London Math. Soc., (3) 8 (1958), 76-94.

[6] S. Jancic Rasovic, I. Cristea, *Hypernear-rings with a defect of distributivity*, submitted.

[7] S. Ligh, *On distributively generated near-rings*, Proc. Edinburg Math. Soc., 16, Issue 3 (1969), 239-242.

[8] J. Meldrum, *Near-Rings and their Links with Groups*, Pitman, London, 1985.

[9] B.H. Neumann, *On the commutativity of addition*, London Math. Soc. 15 (1940), 203-208.

[10] G. Pilz, *Near-rings*, North-Holland Publ.Co., rev.ed. 1983.

[11] B. Satyanarayana, K.S Prasad, *Near-Rings, Fuzzy Ideals, and Graph Theory*, CRC Press, New York, 2013.

[12] S. Scott, *Tame Theory*, Amo Publishing, Auckland, 1983.

[13] V. Vukovic, *Nonassociative Near-Rings*, Univ. of Kragujevac-Studio Plus, Belgrade, 1996.

[14] H. Wahling, *Theorie der Fastkörper*, Thales-Verlag, Essen, 1987.

Accepted: 24.01.2018

BASIC NOTIONS OF PARTIALLY ORDERED HYPERMODULES

H. Mirabdollahi
Department of Mathematics
Yazd University
Yazd
Iran
h.mirabdollahi@stu.yazd.ac.ir

S.M. Anvariyeh*
Department of Mathematics
Yazd University
Yazd
Iran
anvariyeh@yazd.ac.ir

S. Mirvakili
Department of Mathematics
Payame Noor University (PNU)
Yazd
Iran
saeed_mirvakili@pnu.ac.ir

Abstract. In this paper, we construct the ring-like hyperstructures derived from a (partially) quasi ordered ring R, and we study some basic properties to this class. Then, we introduce the new class of (partially) ordered hypermodules by using of the (partially) ordered modules. Moreover, we study some basic properties of this new class and the essential differences between this class and the earlier one (i.e. ordered modules) are also investigated.

Keywords: (partially) ordered hypermodule, (partially) ordered ring, (partially) ordered module, (good) hyperring, (good) hypermodule

1. Introduction and preliminaries

Hyperstructures theory was born in 1934, when Marty at the 8th congress of Scandinavian mathematicians, gave the definition of a hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and rational functions [16]. He defined the concept of hypergroups, as a natural generalization of groups, based on the notion of hyperoperation. Since then, a number of different hyperstructures have been widely studied by many mathematicians. One of the first books about hypergroups was written

∗. Corresponding author

by P. Corsini in 1993 [6]. Also the book "Hyperstructures and Their Representations" was published in 1994 by T. Vougiouklis [26]. The other book on these topics is "Applications of Hyperstructure Theory", by P. Corsini and V. Leoreanu, published in 2003 [5]. Another book, devoted especially to the study of hyperring theory, is "Hyperring Theory and Applications", written by B. Davvaz and V. Leoreanu-Fotea [9]. A recent book on hyperstructures [8] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. In this book basic definitions and notions concerning hyperstructure theory can be found.

First of all, we present some basic definitions and ideas from the hyperstructures theory. The hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a *hyperoperation*. A nonempty set H, endowed with a hyperoperation, $+ : H \times H \longrightarrow \wp^*(H)$ is called a *hypergroupoid*. $\wp^*(H)$ denotes the set of all nonempty subsets of H. A hypergroupoid which verifies the condition $(x + y) + z = x + (y + z)$, for all $x, y, z \in H$, is called a *semihypergroup*. A semihypergroup H which verifies reproduction axioms, $x + H = H = H + x$, for all $x \in H$, is called a *hypergroup* [8].

Since we deal with the theory of ordered structures, we recall that a *quasi ordered (semi)group* is a triple $(G, +, \leq)$, where $(G, +)$ is a (semi)group and $\leq$ is a reflexive and transitive binary relation on G such that for any triple $a, b, c \in G$ with the property $a \leq b$ also $a + c \leq b + c$ and $c + a \leq c + b$ hold. We call the (semi)group *partially ordered* if the relation $\leq$ is moreover antisymmetric [13]. In a partially ordered group, an element x of G is called *positive element* if $0 \leq x$. The set of elements $0 \leq x$ is often denoted with G_+, and it is called the positive cone of G. So, we have $a \leq b$ if and only if $-a + b \in G_+$.

For a general group G, the existence of a positive cone specifies an order on G [1]. A group G is a *partially ordered group* if and only if there exists a subset H which is G_+ of G such that:

i) $0 \in G_+$;

ii) if $a, b \in G_+$ then $a + b \in G_+$;

iii) if $a \in G_+$ then $-x + a + x \in G_+$ for any x of G;

iv) if $a \in G_+$ and $-a \in G_+$ then $a = 0$;

v) G is totally ordered when it satisfies also $G_+ \cup G_- = G$, in which $G_- = \{-x | x \in G_+\}$.

The (partially) ordered group $(G, \leq)$ is denoted by (G, G_+). In addition $[a)_\leq := \{x \in H | a \leq x\}$ is a principal end generated by $a \in G$. The definition of partially ordered ring and modules will be presented later in sections 2 and 3, respectively.

Recall that a *ring* consists of a set R equipped with two binary operations $+$ and . such that (i) $(R, +)$ is an abelian group and (ii) $(R, .)$ is a semigroup,

and (iii) $a.(b+c) = a.b + a.c$ and $(a+b).c = a.c + b.c$ for all $a, b, c \in R$. A ring R is called *unitary* whenever, there exists $1 \in R$ such that $1.a = a.1 = a$ for all $a \in R$. Let R is a ring and 1 is its multiplicative identity. A *left R-module M* consists of an abelian group $(M, +)$ and an operation $. : R \times M \longrightarrow M$ such that for all $a, b \in R$ and $x, y \in M$, we have (i') $a.(x + y) = a.x + a.y$, (ii') $(a + b).x = a.x + b.x$, (iii') $(ab).x = a.(b.x)$, and (iv') $1.x = x$. The operation of the ring R on M is called *scalar multiplication.*

The relation of ordered sets and algebraic hyperstructures was first studied by Vougiouklis in 1987 [27]. Then the connection between hyperstructures and ordered sets have been analyzed by many researchers, such as Vougiouklis [29, 28], Corsini [7], Hoskova [15], Ghazavi and et al [11, 12, 13], Heidari and Davvaz [14] and Novak [18, 19]. One special aspect of this issue, known as EL-hyperstructures, was touched upon by Chvalina [3]. He investigated quasi ordered sets and hypergroups. Also, Rosenberg in [25], Hoskova in [15], Rackova in [24] and Novak in [17, 20, 21, 22] extended some results on the ordered semigroups and ordered groups connected with EL-hyperstructures. EL-hyperstructures, mainly studied by M. Novak, are hyperstructures constructed from a (partially) quasi-ordered (semi)groups. More exactly, Novak in [21] considered subhyperstructures of EL-hyperstructures and in [18], he discussed some interesting results of important elements in this family of hyperstructures. Then, in [19] Novak studied some basic properties of EL-hyperstructures like invertibility, normality, being closed (ultra closed) and etc.

A number of articles and contributions in the hyperstructures theory discussed about the creation of hyperstructures from a (partially) quasi-ordered (semi)groups. This results known as the "Ends lemma"and first used in [4] as theorems 1.3 and 1.4 (chapter 4; and the remark mentioned in the following), would be presented in the following:

Theorem 1.1 ([4], Theorem 1.3, p. 146)**.** *Let $(G, +, \leq)$ be a partially ordered semigroup. Binary hyperoperation $\oplus : G \times G \longrightarrow \wp^*(G)$ defined by $a \oplus b = [a+b)_{\leq}$ is associative. The semihypergroup $(G, \oplus)$ is commutative if and only if the semigroup $(G, +)$ is commutative.*

Theorem 1.2 ([4], Theorem 1.4, p. 147)**.** *Let $(G, +, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

i) For any pair $a, b \in G$, there exists a pair $c, c' \in G$ such that $b + c \leq a$ and $c' + b \leq a$.

ii) The associated semihypergroup $(G, \oplus)$ is a hypergroup.

Remark 1 ([4])**.** If $(G, +, \leq)$ is a partially ordered group, then if we take $c = -b + a$ and $c' = a - b$, then condition (i) is valid. Therefore, if $(G, +, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.

If the condition of quasi ordered is replaced by the condition of partially ordered, then the proofs are valid.

In this paper, we intend to build the ring-like hyperstructures using of a ring H. Also, the ring H with order relation, on which two operations $+$ and $.$ and the quasi ordered semigroups (groups) $(H, +, \leq)$ and $(H, ., \leq)$ has studied. In addition, we introduce a new class of hypermodules from a given (partially) quasi-ordered modules as a generalization of Ends lemma based on hyperstructures. Then, the essential differences between this class and the ordered modules are also investigated.

2. (Partially) ordered (semi)hyperrings

Since a ring R is a algebraic structure that endow two operations $+$ and $.$, such that $(R, +)$ is an abelian group and $(R, .)$ is a semigroup, therefore an ordered ring is defined as:

Definition 2.1 ([10])**.** Let R be a ring with unit element $1 \neq 0$. We say that R is *partially ordered* when there exists a partial order $\leq$ on the underlying set R such that for any a, b, c in R:

i) $a \leq b$ implies $a + c \leq b + c$

ii) $0 \leq a$ and $0 \leq b$ imply that $0 \leq a.b$.

If any two elements $a, b \in R$ are comparable, then R is *ordered*.

The additive group of a partially ordered ring is a partially ordered group. Furthermore, The set of elements x for which $0 \leq x$ (the set of non-negative elements) of a partially ordered ring, is closed under addition and multiplication, i.e. if P is the set of non-negative elements of a partially ordered ring, then $P + P \subseteq P$, and $P.P \subseteq P$. Furthermore, $P \cap (-P) = \{0\}$.

If there exists a subset R_+ of R such that:

i) $0 \in R_+$;

ii) $R_+ \cap (R_-) = \{0\}$, in which $R_- = \{-x | x \in R_+\}$;

iii) $R_+ + R_+ \subseteq R_+$;

iv) $R_+.R_+ \subseteq R_+$;

v) R is ordered when it satisfies also $R_+ \cup R_- = R$,

then the relation $\leq$ where $a \leq b$ if and only if $b - a \in R_+$ defines a compatible partial order on R (i.e. $(R, \leq)$ is a partially ordered ring). Also, R is ordered when it satisfies also $R_+ \cup R_- = R$.

Example 1.
i) The ring $(\mathbb{Z}, \leq)$ with the usually order relation is an ordered ring.

ii) The ring $\mathbb{Z}^I$ of integral-valued functions on a set, with pointwise order, is partially ordered (when I has at least two elements).

In any ring R, the *absolute value* $\mid x \mid$ of an element x can be defined as following:

$$\mid x \mid = \begin{cases} x, & 0_R \leq x \\ -x, & x \leq 0_R \end{cases}.$$

The order relation of partially ordered ring R is compatible with the addition of the abelian group $(R, +)$. Thus for the construction of hyperstructures based Ends Lemma, the hyperoperations are defined:

Definition 2.2. Let R be an ordered ring. For all $a, b \in R$, we define:

$$a \oplus b = [\mid a \mid + \mid b \mid)_{\leq} \tag{1}$$

$$a \odot b = [\mid a \mid\mid b \mid)_{\leq}. \tag{2}$$

Lemma 2.3. *Let R be an ordered ring. By definitions are presented in* (1) *and* (2)*, $(R, \oplus)$ is an commutative semihypergroup, and $(R, \odot)$ is a semihypergroup.*

Proof. Let $a_1, a_2, a_3 \in R$ such that $a_1, a_2 \in R_-$ and $a_3 \in R_+$. Now, we show that $a_1 \oplus (a_2 \oplus a_3) = (a_1 \oplus a_2) \oplus a_3$. Therefore, by Definition 2.2. we have:

$$\begin{aligned} a_1 \oplus (a_2 \oplus a_3) &= \cup\{a_1 \oplus a' | a' \geq \mid a_2 \mid + \mid a_3 \mid\} \\ &= \cup\{a_1 \oplus a' | a' \geq -a_2 + a_3\} \\ &= \{a | a \geq \mid a_1 \mid + \mid a' \mid, a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1 + a', a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1 + a' \geq -a_1 + (-a_2 + a_3)\} \\ &= \{a | a \geq -a_1 - a_2 + a_3\}. \end{aligned}$$

On the other hand

$$\begin{aligned} (a_1 \oplus a_2) \oplus a_3 &= \cup\{a' \oplus a_3 | a' \geq \mid a_1 \mid + \mid a_2 \mid\} \\ &= \cup\{a' \oplus a_3 | a' \geq -a_1 - a_2\} \\ &= \{a | a \geq \mid a' \mid + \mid a_3 \mid, a' \geq -a_1 - a_2\} \\ &= \{a | a \geq a' + a_3, a' \geq -a_1 - a_2\} \\ &= \{a | a \geq a' + a_3 \geq (-a_1 - a_2) + a_3\} \\ &= \{a | a \geq -a_1 - a_2 + a_3\}. \end{aligned}$$

For $(R, \odot)$, we have:

$$\begin{aligned} a_1 \odot (a_2 \odot a_3) &= \cup\{a_1 \odot a' | a' \geq \mid a_2 \mid\mid a_3 \mid\} \\ &= \cup\{a_1 \odot a' | a' \geq (-a_2)a_3\} \\ &= \{a | a \geq \mid a_1 \mid\mid a' \mid, a' \geq -a_2 a_3\} \\ &= \{a | a \geq (-a_1)a', a' \geq -a_2 a_3\} \\ &= \{a | a \geq -a_1 a' \geq -a_1(-a_2 a_3)\} \\ &= \{a | a \geq a_1 a_2 a_3\}. \end{aligned}$$

On the other hand

$$\begin{aligned}(a_1 \odot a_2) \odot a_3 &= \cup\{a' \odot a_3 | a' \geq\mid a_1 \mid\mid a_2 \mid\} \\ &= \cup\{a' \odot a_3 | a' \geq (-a_1)(-a_2)\} \\ &= \{a | a \geq\mid a' \mid\mid a_3 \mid, a' \geq a_1 a_2\} \\ &= \{a | a \geq a' a_3, a' \geq a_1 a_2\} \\ &= \{a | a \geq a' a_3 \geq (a_1 a_2) a_3\} \\ &= \{a | a \geq a_1 a_2 a_3\}.\end{aligned}$$

Therefore, in this case $(R, \oplus)$ and $(R, \odot)$ are semihypergroup. The other cases are proved, similarly. The commutativity of hyperoperation $\oplus$ is the direct consequence of the commutativity of operation $+$, according to Theorem 1.1. □

The hyperstructures endowed with two internal hyperoperations, is called a *hyperringoid*. Recall that $(R, +, .)$ is a *hyperring* (*semihyperring*) in the general sense if; (1) $(R, +)$ is a commutative hypergroup (semihypergroup); (2) . is associative hyperoperation and; (3) the distributive law $a.(b + c) \subseteq (a.b) + (a.c)$ and $(a + b).c \subseteq (a.c) + (b.c)$ is satisfied for every a, b, c of R. If the equality in the distributive law is valid, then the hyperring (semihyperring) is called *good*.

Theorem 2.4. *Let R be an ordered ring. Then $(R, \oplus, \odot)$ is a semihyperring.*

Proof. It is sufficient the condition (3) of definition of semihyperring is checked. So, let $a, b \in R_-$ and $c \in R_+$. Then

$$\begin{aligned}a_1 \odot (a_2 \oplus a_3) &= \cup\{a_1 \odot a' | a' \geq |a_2| + |a_3|\} \\ &= \cup\{a_1 \odot a' | a' \geq -a_2 + a_3\} \\ &= \{a | a \geq |a_1||a'|, a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1 a', a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1(-a_2 + a_3)\} \\ &= \{a | a \geq a_1 a_2 - a_1 a_3)\}.\end{aligned}$$

On the other hand

$$\begin{aligned}(a_1 \odot a_2) \oplus (a_1 \odot a_3) &= \cup\{a \oplus b | a \geq |a_1||a_2|, b \geq |a_1||a_3|\} \\ &= \cup\{a \oplus b | a \geq a_1 a_2, b \geq -a_1 a_3\} \\ &= \{a' | a' \geq |a| + |b|, a \geq a_1 a_2, b \geq -a_1 a_3\} \\ &= \{a' | a' \geq a + b, a \geq a_1 a_2, b \geq -a_1 a_3\} \\ &= \{a' | a' \geq a_1 a_2 - a_1 a_3\}.\end{aligned}$$

The proof of distributive property on the right is done, similarly. The proof of the other cases is done easily, too. □

Notice that if R is not ordered, then there exists an element $a_0 \in R$ such that $a_0 \notin R_+ \cup R_-$, so $|a_0|$ is meaningless. Therefore, the definitions of hyperoperations $\oplus$ and $\odot$ in (1) and (2) respectively, is not efficient for $a \oplus b$ or $a \odot b$ when, at least one of a or b is not belonging to $R_+ \cup R_-$. So, we modify the definitions of $\oplus$ and $\odot$ in (1) and (2) respectively, in the following way:

Definition 2.5. Let $(R, +, ., \leq)$ be a partially ordered ring. For $a, b \in R$, we define:

$$a \oplus_1 b = [\mid a \mid + \mid b \mid)_{\leq} \cup \{a, b\}, \tag{3}$$

$$a \odot_1 b = [\mid a \mid\mid b \mid)_{\leq} \cup \{a, b\}. \tag{4}$$

With the hyperoperations $\oplus_1$ and $\odot_1$ presented in (3) and (4), for any a, b of the partially ordered ring R, we have $a \oplus_1 b \subseteq \wp^*(R)$ and $a \odot_1 b \subseteq \wp^*(R)$.

Recall that $(R, +, .)$ is a hyperring in the general sense if $(R, +)$ is a commutative hypergroup, . is associative hyperoperation and the distributive law $a.(b + c) \subseteq (a.b) + (a.c)$, $(a + b).c \subseteq (a.c) + (b.c)$ is satisfied for any a, b, c of R. If the equality in the distributive law is valid, then the hyperring is called *good*.

Example 2. The hyperstructure $R = (\{a, b\}, \oplus, \odot)$ defined as follows:

$\oplus$	a	b
a	a	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$

$\odot$	a	b
a	a	$\{a, b\}$
b	a	$\{a, b\}$

is a good hyperring.

Theorem 2.6. *Let $(R, +, ., \leq)$ be an ordered ring. Then $(R, \oplus_1, \odot)$ is a good hyperring.*

Proof. Let $a, b, c \in R$. First of all, we show that $(a \oplus_1 b) \oplus_1 c = a \oplus_1 (b \oplus_1 c)$. To prove the associative property of $\oplus$, there are eight different cases. All cases, particularly the cases in which $a, b, c \in R_+$ or $a, b, c \in R_-$, can easily be proved. Here we prove the case in which $a \in R_+$ and $b, c \in R_-$.

$$\begin{aligned}
(a \oplus_1 b) \oplus_1 c &= ([\mid a \mid + \mid b \mid)_{\leq} \cup \{a, b\}) \oplus_1 c \\
&= (\{r' | r' \geq \mid a \mid + \mid b \mid\} \cup \{a, b\}) \oplus_1 c \\
&= (\{r' | r' \geq a - b\} \cup \{a, b\}) \oplus_1 c \\
&= \cup\{r' \oplus_1 c | r' \geq a - b\} \cup a \oplus_1 c \cup b \oplus_1 c \\
&= \{r | r \geq |r'| + |c|, r' \geq a - b\} \cup \{r' | r' \geq a - b\} \cup \{c\} \\
&\quad \cup\{a, c\} \cup \{r | r \geq |a| + |c|\} \cup \{b, c\} \cup \{r | r \geq |b| + |c|\} \\
&= \{r | r \geq r' - c, r' \geq a - b\} \cup \{r' | r' \geq a - b\} \\
&\quad \cup\{r | r \geq a - c\} \cup \{r | r \geq -b - c\} \cup \{a, b, c\} \\
&= \{r | r \geq (a - b) - c\} \cup \{r' | r' \geq a - b\} \cup \{r | r \geq a - c\} \\
&\quad \cup\{r | r \geq -b - c\} \cup \{a, b, c\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \oplus_1 (b \oplus_1 c) &= a \oplus_1 ([|\, b \mid + \mid c \,|)_{\leq} \cup \{b, c\}) \\
&= a \oplus_1 (\{r' | r' \geq |\, b \mid + \mid c \,|\} \cup \{b, c\}) \\
&= a \oplus_1 (\{r' | r' \geq -b - c\} \cup \{b, c\}) \\
&= \cup\{a \oplus_1 r' | r' \geq -b - c\} \cup a \oplus_1 b \cup a \oplus_1 c \\
&= \{r | r \geq |a| + |r'|, r' \geq -b - c\} \cup \{r' | r' \geq -b - c\} \\
&\quad \cup\{a\} \cup \{a, b\} \cup \{r | r \geq |a| + |b|\} \cup \{a, c\} \cup \{r | r \geq |a| + |c|\} \\
&= \{r | r \geq a + r', r' \geq -b - c\} \cup \{r' | r' \geq -b - c\} \\
&\quad \cup\{r | r \geq a - b\} \cup \{r | r \geq a - c\} \cup \{a, b, c\} \\
&= \{r | r \geq a + (-b - c)\} \cup \{r' | r' \geq -b - c\} \cup \{r | r \geq a - b\} \\
&\quad \cup\{r | r \geq a - c\} \cup \{a, b, c\}.
\end{aligned}$$

In the following, we show that the reproduction principles. Let $a \in R$, then we have:

$$R \supseteq a \oplus_1 R = \cup\{a \oplus_1 b | b \in R\} \supseteq \cup\{a, b | b \in R\} = R,$$

as well as

$$R \supseteq R \oplus_1 a = \cup\{b \oplus_1 a | b \in R\} \supseteq \cup\{b, a | b \in R\} = R.$$

So $R \oplus_1 a = a \oplus_1 R = R$, for all $a \in R$. Therefore, $(R, \oplus_1)$ is a hypergroup. Now, we show that $(R, \odot)$ is a semihypergroup. Again, consider the case in which $a \in R_+$ and $b, c \in R_-$, then

$$\begin{aligned}
(a \odot b) \odot c &= ([|\, a \parallel b \,|)_{\leq}) \odot c \\
&= (\{r' | r' \geq -ab\}) \odot c \\
&= \cup\{r' \odot c | r' \geq -ab\} \\
&= \{r | r \geq |\, r' \parallel c \,|, a' \geq -ab\} \\
&= \{r | r \geq -r'c, r' \geq -ab\} \\
&= \{r | r \geq r'(-c) \geq (-ab)(-c)\} \\
&= \{r | r \geq (ab)c\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \odot (b \odot c) &= a \odot ([|\, b \parallel c \,|)_{\leq}) \\
&= a \odot \{r' | r' \geq (-b)(-c)\} \\
&= \cup\{a \odot r' | r' \geq bc\} \\
&= \{r | r \geq |\, a \parallel r' \,|, r' \geq bc\} \\
&= \{r | r \geq ar', r' \geq bc\} \\
&= \{r | r \geq ar' \geq a(bc)\} \\
&= \{r | r \geq a(bc)\}.
\end{aligned}$$

To investigate the distribution $\odot$ by the ratio of $\oplus_1$ of the left, the case $a \in R_+$ and $b, c \in R_-$ are considered. So we have:

$$\begin{aligned} a \odot (b \oplus_1 c) &= a \odot (\{r' | r' \geq \mid b \mid + \mid c \mid\} \cup \{b, c\}) \\ &= \cup\{a \odot r' | r' \geq -b - c\} \cup a \odot b \cup a \odot c \\ &= \{r | r \geq \mid a \mid\mid r' \mid, r' \geq (-b - c)\} \\ &\quad \cup\{r | r \geq \mid a \mid\mid b \mid\} \cup \{r | r \geq \mid a \mid\mid c \mid\} \\ &= \{r | r \geq ar', r' \geq -(b + c)\} \cup \{r | r \geq -ab\} \cup \{r | r \geq -ac\} \\ &= \{r | r \geq -ab - ac\} \cup \{r | r \geq -ab\} \cup \{r | r \geq -ac\}. \end{aligned}$$

On the other hand

$$\begin{aligned} (a \odot b) \oplus_1 (a \odot c) &= \cup\{r \oplus_1 r' | r \geq \mid a \mid\mid b \mid, r' \geq \mid a \mid\mid c \mid\} \\ &= \{r | r \geq -ab\} \cup \{r | r' \geq -ac\} \\ &\quad \cup\{r'' | r'' \geq |r| + |r'|, r \geq -ab, r' \geq -ac\} \\ &= \{r | r \geq -ab\} \cup \{r | r' \geq -ac\} \\ &\quad \cup\{r'' | r'' \geq r + r', r \geq -ab, r' \geq -ac\} \\ &= \{r | r \geq -ab\} \cup \{r | r' \geq -ac\} \cup \{r'' | r'' \geq -ab - ac\}. \end{aligned}$$

The distribution $\odot$ by the ratio of $\oplus_1$ of the right is proved similarly. Therefore, $(R, \oplus_1, \odot)$ is a good hyperring. □

Theorem 2.7. *Let $(R, +, \leq)$ be a partially ordered group. Then $(R, \oplus_1)$ is a hypergroup.*

Proof. According to the previous theorems, it is sufficient to show the associative property of hyperoperation $\oplus_1$ and the reproduction principles for the case in which at least one element is not belonging to $R_+ \cup R_-$. Let $a \notin R_+ \cup R_-$, and $b, c \in R_+$. Then we have:

$$\begin{aligned} (a \oplus_1 b) \oplus_1 c &= ([\mid a \mid + \mid b \mid)_{\leq} \cup \{a, b\}) \oplus_1 c \\ &= (\emptyset \cup \{a, b\}) \oplus_1 c \\ &= a \oplus_1 c \cup b \oplus_1 c \\ &= \{a, c\} \cup \{r | r \geq |a| + |c|\} \cup \{b, c\} \cup \{r | r \geq |b| + |c|\} \\ &= \emptyset \cup \{r | r \geq b + c\} \cup \{a, b, c\} \\ &= \{r | r \geq b + c\} \cup \{a, b, c\}. \end{aligned}$$

On the other hand

$$\begin{aligned} a \oplus_1 (b \oplus_1 c) &= a \oplus_1 ([\mid b \mid + \mid c \mid)_{\leq} \cup \{b, c\}) \\ &= a \oplus_1 (\{r' | r' \geq \mid b \mid + \mid c \mid\} \cup \{b, c\}) \\ &= a \oplus_1 (\{r' | r' \geq b + c\} \cup \{b, c\}) \\ &= \cup\{a \oplus_1 r' | r' \geq b + c\} \cup a \oplus_1 b \cup a \oplus_1 c \end{aligned}$$

$$\begin{aligned} &= \{r|r \geq |a| + |r'|, r' \geq b + c\} \cup \{r'|r' \geq b + c\} \\ &\quad \cup\{a\} \cup \{a, b\} \cup \{r|r \geq |a| + |b|\} \cup \{a, c\} \cup \{r|r \geq |a| + |c|\} \\ &= \emptyset \cup \{r'|r' \geq b + c\} \cup \emptyset \cup \{a, b\} \cup \emptyset \cup \{a, c\} \\ &= \{r|r \geq b + c\} \cup \{a, b, c\}. \end{aligned}$$

But, for $a \in R$ such that $a \notin R_+ \cup R_-$, we have:

$$a \oplus_1 R = \cup\{a \oplus_1 b|b \in R\} = \cup\{a, b|b \in R\} = R,$$

as well as

$$R \oplus_1 a = \cup\{b \oplus_1 a|b \in R\} = \cup\{b, a|b \in R\} = R.$$

So $R \oplus_1 a = a \oplus_1 R = R$, for all $a \in R$. Therefore, $(R, \oplus_1)$ is a hypergroup. □

Example 3. Let $G = (\mathbb{Z}[i] = \{a + bi|a, b \in \mathbb{Z}\}, +)$ with ordinary addition of complex numbers. Put $\mathbb{Z}[i]_+ = \{a|a \in \mathbb{Z}, a \geq 0\} \cup \{bi|b \in \mathbb{Z}, b \leq 0\}$. Then $(\mathbb{Z}[i], \mathbb{Z}[i]_+)$ is a partially ordered group.

Also, we can see that:

Theorem 2.8. *Let $(R, ., \leq)$ be a partially ordered group. Then $(R, \odot_1)$ is a hypergroup.*

According to the previous lemmas, The following theorems satisfying:

Theorem 2.9. *Let $(R, +, ., \leq)$ be a partially ordered ring. Then $(R, \oplus_1, \odot_1)$ is a hyperring.*

Proof. First, we show the distribution $\odot_1$ by the ratio of $\oplus_1$ of the left, for the case in which $a \in R_+$ and $b, c \in R_-$.

$$\begin{aligned} a \odot_1 (b \oplus_1 c) &= a \odot_1 (\{r'|r' \geq\mid b \mid + \mid c \mid\} \cup \{b, c\}) \\ &= \cup\{a \odot_1 r'|r' \geq -b - c\} \cup a \odot_1 b \cup a \odot_1 c \\ &= \{a\} \cup \{r'|r' \geq -b - c\} \cup \{r|r \geq\mid a \mid\mid r' \mid, r' \geq (-b - c)\} \\ &\quad \cup\{a, b\} \cup \{r|r \geq\mid a \mid\mid b \mid\} \cup \{a, c\} \cup \{r|r \geq\mid a \mid\mid c \mid\} \\ &= \{r|r \geq ar', r' \geq -(b + c)\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\} \\ &\quad \cup\{a, b, c\} \cup \{r'|r' \geq -b - c\} \\ &= \{r|r \geq -ab - ac\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\} \\ &\quad \cup\{a, b, c\} \cup \{r'|r' \geq -b - c\}. \end{aligned}$$

On the other hand

$$\begin{aligned} (a \odot_1 b) \oplus_1 (a \odot_1 c) &= \cup\{r \oplus_1 r'|r \in a \odot_1 b, r' \in a \odot_1 c\} \\ &= \cup\{r \oplus_1 r'|r \geq |a||b|, r = a, b, r' \geq |a||c|, r' = a, c\} \\ &\supseteq \cup\{r \oplus_1 r'|r \geq -ab, r' \geq -ac\} \cup b \oplus_1 c \cup \{a\} \\ &= \{r|r \geq -ab\} \cup \{r'|r' \geq -ac\} \\ &\quad \cup\{r''|r'' \geq |r| + |r'|, r \geq -ab, r' \geq -ac\} \end{aligned}$$

$$\begin{aligned} & \cup\{r|r \geq |b| + |c|\} \cup \{b, c\} \cup \{a\} \\ = \quad & \{r|r \geq -ab\} \cup \{r'|r' \geq -ac\} \cup \{r''|r'' \geq -ab - ac\} \\ & \cup\{r|r \geq -b - c\} \cup \{a, b, c\}. \end{aligned}$$

Then, consider the case in which $a, c \in R_+$ and $b \notin R_+ \cup R_-$.

$$\begin{aligned} a \odot_1 (b \oplus_1 c) &= a \odot_1 (\{r'|r' \geq | \, b \, | + | \, c \, |\} \cup \{b, c\}) \\ &= a \odot_1 (\emptyset \cup \{b, c\}) \\ &= a \odot_1 b \cup a \odot_1 c \\ &= \{a, b\} \cup \{r|r \geq |a||c|\} \cup \{a, c\} \\ &= \{a, b, c\} \cup \{r|r \geq ac\}. \end{aligned}$$

On the other hand

$$\begin{aligned} (a \odot_1 b) \oplus_1 (a \odot_1 c) &= \cup\{r \oplus_1 r'|r \in a \odot_1 b, r' \in a \odot_1 c\} \\ &= \cup\{r \oplus_1 r'|r \geq |a||b|, r = a, b, r' \geq |a||c|, r' = a, c\} \\ &= \cup\{r \oplus_1 r'|r = a, b, r' \geq ac, r' = a, c\} \\ &\supseteq \{a, b, c\} \cup \{r'|r' \geq ac\}. \end{aligned}$$

The other cases are proved, similarly. therefore $(R, \oplus_1, \odot_1)$ is a hyperring. □

Example 4. Consider the partially ordered ring $\mathbb{Z}^I$ of integral-valued functions on a set $I = \{a, b\}$, with pointwise order. Then $(\mathbb{Z}^I, \oplus_1, \odot)$ is a good hyperring.

3. (Partially) ordered hypermodules

In this section, we intend to build module-like hyperstructures using the definitions presented in previous section and the definitions would be presented in following.

Definition 3.1 ([23])**.** Let R be a partially ordered ring. A *partially ordered* (left) R-module is a (left) R-module $(M, .)$ (by the function $. : R \times M \longrightarrow M$), together with a compatible partial order, i.e. a partial order $\leq_M$ on the underlying set M that is compatible with the operation of the abelian group M and the operation ., in the sense that it satisfies:

i) $x \leq y$ implies $x + z \leq y + z$

ii) $0_R \leq a$ and $0_M \leq x$ imply that $0_M \leq a.x$,

for any $x, y, z \in M$ and $a \in R$. If any two elements $a, b \in R$ are comparable and any two elements $x, y \in M$ too, then M is *ordered*.

If there exists a subset N (which is M_+) of M such that:

i) $0_M \in M_+$;

ii) $M_+ \cap M_- = \{0\}$ in which $M_- = \{-m | m \in m \in M_+\}$;

iii) $M_+ + M_+ \subseteq M_+$;

iv) $R_+.M_+ \subseteq M_+$;

v) M is ordered when it satisfies also $M_+ \cup M_- = M$.

Then the relation $\leq_M$ where $x \leq y$ if and only if $y - x \in N$ defines a compatible partial order on M (i.e. $(M, \leq_M)$ is a partially ordered R-module).

Example 5. Consider $\mathbb{Z}$-module $\mathbb{Z}$ with the usually order relation. Then M_+ equals to all non-negative integer numbers, and M_- equals to all non-positive integer numbers. Therefore $M = M_+ \cup M_-$.

In the following, we present an example of an ordered module in which the order relation is partially only.

Example 6. Let I be a non-empty set and $M := \mathbb{Z}^I$ be the $\mathbb{Z}$- module of all functions from I to $\mathbb{Z}$, with pointwise order, where the order relation is the usually order relation on integer numbers. Then M_+ consists of all functions from I to $\mathbb{Z}^+ \cup \{0\}$. Now, Let me put $I := \{a, b\}$. Then, assuming $f(a) = 1$ and $f(b) = -1$, we have $f \in M$, while $f \notin M_+ \cup M_-$.

In any module, the *absolute value* $\mid x \mid$ of an element x can be defined as following:

$$\mid x \mid = \begin{cases} x, & 0_M \leq x \\ -x, & 0_M \geq x \end{cases}.$$

Now we are trying to present an appropriate hyperoperations associated to the operations and order relation of the R-module M. The order relation of partially ordered R-module M is compatible with the addition of the abelian group M and abelian group R. Thus for the construction of hyperstructures based Ends Lemma, the hyperoperations are defined:

Definition 3.2. Let R be a partially ordered ring and M be a partially ordered (left) R-module. For all $a \in R$ and $x, y \in M$, we define the hyperoperations as follows:

$$x \star y = [\mid x \mid + \mid y \mid)_{\leq}; \tag{5}$$

$$x \star_1 y = [\mid x \mid + \mid y \mid)_{\leq} \cup \{x, y\}; \tag{6}$$

$$a \circ x = [\mid a \mid . \mid x \mid)_{\leq}; \tag{7}$$

$$a \circ_1 x = [|a|.|x|)_{\leq} \cup \{x\}. \tag{8}$$

Let $(R, +, .)$ be a hyperring and $\circ : R \times M \longrightarrow \wp^*(M)$ be the scalar hyperoperation. Then M is a *left R- hypermodule* whenever, $(M, +)$ is a commutative hypergroup and for all $a, b \in R$ and $x, y \in M$, i) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$; ii) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$, and iii) $(a.b) \circ x = a \circ (b \circ x)$. If R is a good hyperring and the equalities in the (i) and (ii) are valid, then the hypermodule is called *good* [2].

Example 7 ([2])**.** Let A be a ring and M be an $R-$module. If M' is a submodule of M and one defines the following scalar hyperoperation

$$\forall (a,x) \in A \times M,\ a \circ x = ax + M'.$$

Then M is an $R-$hypermodule.

Remark 2. With an argument similar to that presented in the previous section, we can see that $(M, \star)$ is a commutative semihypergroup and $(M, \star_1)$ is a commutative hypergroup.

Theorem 3.3. *Let R be an ordered ring, and M be an ordered module over R. We apply hyperoperations $\oplus_1$, $\odot$, $\star_1$ and $\circ$ on the ordered R-module M. Then M is an good R-hypermodule.*

Proof. It is sufficient that we show the validity of equality in conditions (i), (ii) and (iii) of definition good R-hypermodule. To show equality in conditions (i), (ii) and (iii); we consider cases in which $a \in R_-, x \in M_+, y \in M_-$, $a, b \in R_-, x \in M_-$ and $a \in R_+, b \in R_-, x \in M_-$, respectively. To prove (i), we have:

$$\begin{aligned}
a \circ (x \oplus_1 y) &= a \circ (\{z | z \geq |x| + |y|\} \cup \{x, y\}) \\
&= a \circ (\{z | z \geq x - y\} \cup \{x, y\}) \\
&= \cup \{a \circ z | z \geq x - y\} \cup a \circ x \cup a \circ y \\
&= \{z' | z' \geq |a|.|z|, z \geq x - y\} \cup \{z' | z' \geq |a|.|x|\} \cup \{z' | z' \geq |a|.|y|\} \\
&= \{z' | z' \geq -a.z, z \geq x - y\} \cup \{z' | z' \geq -a.x\} \cup \{z' | z' \geq a.y\} \\
&= \{z' | z' \geq a.(y - x)\} \cup \{z' | z' \geq -a.x\} \cup \{z' | z' \geq a.y\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \circ x \star_1 a \circ y &= \cup \{z_1 \star_1 z_2 | z_1 \geq |a|.|x|, z_2 \geq |a|.|y|\} \\
&= \{z' | z' \geq |z_1| + |z_2|, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
&\quad \cup \{z_2 | z_2 \geq a.y\} \\
&= \{z' | z' \geq z_1 + z_2, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
&\quad \cup \{z_2 | z_2 \geq a.y\} \\
&= \{z' | z' \geq z_1 + z_2 \geq -a.x + a.y\} \cup \{z_1 | z_1 \geq -a.x\} \cup \{z_2 | z_2 \geq a.y\} \\
&= \{z' | z' \geq a.(y - x)\} \cup \{z_1 | z_1 \geq -a.x\} \cup \{z_2 | z_2 \geq a.y\}.
\end{aligned}$$

To prove (ii),

$$\begin{aligned}
(a \oplus_1 b) \circ x &= (\{c | c \geq |a| + |b|\} \cup \{a, b\}) \circ x \\
&= \cup \{c \circ x | c \geq -a - b\} \cup a \circ x \cup b \circ x \\
&= \{y | y \geq |c|.|x|, c \geq -a - b\} \cup \{y | y \geq a.x\} \cup \{y | y \geq b.x\} \\
&= \{y | y \geq -c.x, c \geq -a - b\} \cup \{y | y \geq a.x\} \cup \{y | y \geq b.x\} \\
&= \{y | y \geq (a + b).x\} \cup \{y | y \geq a.x\} \cup \{y | y \geq b.x\},
\end{aligned}$$

and

$$\begin{aligned} a\circ x\star_1 b\circ x &= \cup\{y_1\star_1 y_2|y_1\geq |a|.|x|, y_2\geq |b|.|x|\}\\ &= \{y'|y'\geq |y_1|+|y_2|, y_1\geq a.x, y_2\geq b.x\}\cup\{y_1|y_1\geq a.x\}\\ &\quad \cup\{y_2|y_2\geq b.x\}\\ &= \{y'|y'\geq y_1+y_2, y_1\geq a.x, y_2\geq b.x\}\cup\{y_1|y_1\geq a.x\}\\ &\quad \cup\{y_2|y_2\geq b.x\}\\ &= \{y'|y'\geq y_1+y_2\geq a.x+b.x\}\cup\{y_1|y_1\geq a.x\}\cup\{y_2|y_2\geq b.x\}\\ &= \{y'|y'\geq a.x+b.x\}\cup\{y_1|y_1\geq a.x\}\cup\{y_2|y_2\geq b.x\}. \end{aligned}$$

In the following for proving (iii), we have

$$\begin{aligned} (a\odot b)\circ x &= \{c|c\geq |a||b|\}\circ x\\ &= \cup\{c\circ x|c\geq -ab\}\\ &= \{y|y\geq |c|.|x|, c\geq -ab\}\\ &= \{y|y\geq c.(-x), c\geq -ab\}\\ &= \{y|y\geq c.(-x)\geq (-ab).(-x)\}\\ &= \{y|y\geq (ab).x\}, \end{aligned}$$

and

$$\begin{aligned} a\circ (b\circ x) &= a\circ\{y|y\geq |b|.|x|\}\\ &= \cup\{a\circ y|y\geq b.x\}\\ &= \{y'|y'\geq |a|.|y|, y\geq b.x\}\\ &= \{y'|y'\geq a.y, y\geq b.x\}\\ &= \{y'|y'\geq a.(b.x)\}. \end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. □

Theorem 3.4. *Let R be an ordered ring, and M be an ordered module over R. We apply hyperoperations $\oplus_1$, $\odot_1$, $\star_1$ and $\circ_1$ on the ordered R-module M. Then M is an R-hypermodule.*

Proof. We showed that $(R,\oplus_1,\odot_1)$ is a hyperring, and we know that $(M,\star_1)$ is a commutative hypergroup. Now, we prove that for all $a,b\in R$ and $x,y\in M$, (i) $a\circ_1(x\star_1 y)\subseteq (a\circ_1 x)+(a\circ_1 y)$; (ii) $(a\oplus_1 b)\circ_1 x\subseteq (a\circ_1 x)+(b\circ_1 x)$, and (iii) $(a\odot_1 b)\circ_1 x = a\circ_1(b\circ_1 x)$. For the proving of the case (i), suppose that $x\in M_+$, $y\in M_-$ and $a\in R_-$. So, we have:

$$\begin{aligned} a\circ_1(x\star_1 y) &= a\circ_1(\{z: z\geq |x|+|y|\}\cup\{x,y\})\\ &= \cup\{a\circ_1 z: z\geq x-y\}\cup a\circ_1 x\cup a\circ_1 y\\ &= \{z': z'\geq |a|.|z|, z: z\geq x-y\}\cup\{z: z\geq x-y\} \end{aligned}$$

$$
\begin{aligned}
& \cup\{z' : z' \geq |a|.|x|\} \cup \{z' : z' \geq |a|.|y|\} \cup \{x, y\} \\
= \; & \{z' : z' \geq (-a).z, (-a).z \geq (-a).(x-y)\} \cup \{z : z \geq x - y\} \\
& \cup\{z' : z' \geq (-a).x\} \cup \{z' : z' \geq (-a).(-y)\} \cup \{x, y\} \\
= \; & \{z' : z' \geq (-a).z \geq (-a).(x-y)\} \cup \{z : z \geq x - y\} \\
& \cup\{z' : z' \geq (-a).x\} \cup \{z' : z' \geq (-a).(-y)\} \cup \{x, y\} \\
= \; & \{z' : z' \geq a.(y-x)\} \cup \{z : z \geq x - y\} \cup \{z' : z' \geq -a.x\} \\
& \cup\{z' : z' \geq a.y\} \cup \{x, y\}, \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
a \circ_1 x \star_1 a \circ_1 y \; = \; & \cup\{z_1 \star_1 z_2 | z_1 \geq |a|.|x|, z_1 = x, z_2 \geq |a|.|y|, z_2 = y\} \\
\supseteq \; & \{z' | z' \geq |z_1| + |z_2|, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
& \cup\{z_2 | z_2 \geq a.y\} \cup x \star_1 y \\
= \; & \{z' | z' \geq z_1 + z_2, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
& \cup\{z_2 | z_2 \geq a.y\} \cup \{z | z \geq |x| + |y|\} \cup \{x, y\} \\
= \; & \{z' | z' \geq z_1 + z_2, z_1 + z_2 \geq (-a.x + a.y)\} \cup \{z_1 | z_1 \geq -a.x\} \\
& \cup\{z_2 | z_2 \geq a.y\} \cup \{z | z \geq x - y\} \cup \{x, y\} \\
= \; & \{z' | z' \geq a.(y-x)\} \cup \{z_1 | z_1 \geq -a.x\} \cup \{z_2 | z_2 \geq a.y\} \\
& \cup\{z | z \geq x - y\} \cup \{x, y\}. \qquad (10)
\end{aligned}
$$

The proof of (ii), for $a \in R_+$, $b \in R_-$ and $x \in M_-$,

$$
\begin{aligned}
(a \oplus_1 b) \circ_1 x \; = \; & (\{c | c \geq |a| + |b|\} \cup \{a, b\}) \circ_1 x \\
= \; & \cup\{c \circ_1 x | c \geq a - b\} \cup a \circ_1 x \cup b \circ_1 x \\
= \; & \{y | y \geq |c|.|x|, c \geq a - b\} \cup \{y | y \geq -a.x\} \cup \{y | y \geq b.x\} \cup \{x\} \\
= \; & \{y | y \geq c.(-x), c.(-x) \geq (a-b).(-x)\} \cup \{y | y \geq -a.x\} \\
& \cup\{y | y \geq b.x\} \cup \{x\} \\
= \; & \{y | y \geq (b-a).x\} \cup \{y | y \geq -a.x\} \cup \{y | y \geq b.x\} \cup \{x\},
\end{aligned}
$$

and

$$
\begin{aligned}
a \circ_1 x \star_1 b \circ_1 x \; = \; & \cup\{y_1 \star_1 y_2 | y_1 \geq |a|.|x|, y_1 = x, y_2 \geq |b|.|x|, y_2 = x\} \\
\supseteq \; & \{y' | y' \geq |y_1| + |y_2|, y_1 \geq a.(-x), y_2 \geq (-b).(-x)\} \\
& \cup x \star_1 x \cup \{y_1 | y_1 \geq a.(-x)\} \cup \{y_2 | y_2 \geq (-b).(-x)\} \\
\supseteq \; & \{y' | y' \geq y_1 + y_2, y_1 + y_2 \geq -a.x + b.x\} \cup \{x\} \\
& \cup\{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \\
= \; & \{y' | y' \geq (-a+b).x\} \cup \{x\} \cup \{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \\
= \; & \{y' | y' \geq (b-a).x\} \cup \{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \cup \{x\}.
\end{aligned}
$$

Finally, for $a, b \in R_+$ and $x \in M_-$, we have

$$\begin{aligned}(a \odot_1 b) \circ_1 x &= (\{c|c \geq |a||b|\} \cup \{a, b\}) \circ_1 x \\ &= \cup\{c \circ_1 x|c \geq ab\} \cup a \circ_1 x \cup b \circ_1 x \\ &= \{y|y \geq |c|.|x|, c \geq ab\} \cup \{y|y \geq |a|.|x|\} \cup \{y|y \geq |b|.|x|\} \cup \{x\} \\ &= \{y|y \geq c.(-x), c.(-x) \geq (ab).(-x)\} \cup \{y|y \geq a.(-x)\} \\ &\quad \cup\{y|y \geq b.(-x)\} \cup \{x\} \\ &= \{y|y \geq (ab).(-x)\} \cup \{y|y \geq -a.x\} \cup \{y|y \geq -b.x\} \cup \{x\},\end{aligned}$$

and

$$\begin{aligned}a \circ_1 (b \circ_1 x) &= a \circ_1 (\{y|y \geq |b|.|x|\} \cup \{x\}) \\ &= \cup\{a \circ_1 y|y \geq b.(-x)\} \cup a \circ_1 x \\ &= \{y'|y' \geq |a|.|y|, y \geq b.(-x)\} \cup \{y|y \geq b.(-x)\} \\ &\quad \cup\{y|y \geq a.(-x)\} \cup \{x\} \\ &= \{y'|y' \geq a.y, a.y \geq a.(-b.x)\} \cup \{y|y \geq -b.x\} \\ &\quad \cup\{y|y \geq -a.x\} \cup \{x\} \\ &= \{y'|y' \geq -a.(b.x)\} \cup \{y|y \geq -b.x\} \cup \{y|y \geq -a.x\} \cup \{x\}.\end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. □

Example 8. Consider the abelian group $(\mathbb{Z}[x], +)$ with the ordinary addition of polynomials. Let $p(x) = a_m x^m + a_{m+1}x^{m+1} + \cdots + a_n x^n$ be a polynomial with $a_m, a_n \neq 0$ and $m \leq n$. Define $p(x) \in \mathbb{Z}[x]_+$ if and only if $a_m \in \mathbb{Z}_+$. Then $(\mathbb{Z}[x], \mathbb{Z}[x]_+)$ is an ordered abelian group, and $\mathbb{Z}$- module $\mathbb{Z}[x]$ is an ordered module. Applying Theorem 3.3, the resulting hyperstructure $(\mathbb{Z}[x], \oplus, \star_1)$ will be a good hypermodule.

Theorem 3.5. *Let R be a partially ordered ring, and M be a partially ordered module over R. We apply hyperoperations $\oplus_1$, $\odot_1$, $\star_1$ and $\circ_1$ on the partially ordered R-module M. Then M is an R-hypermodule.*

Proof. It is sufficient that we show inclusion in scalar conditions for the case in which there is at least an element $a \in R$ or $x \in M$ such that $a \notin R_+ \cup R_-$ or $x \notin M_+ \cup M_-$, respectively. So for condition (i), in the case $a \in R_+$, $x \in M_-$ and $y \notin M_+ \cup M_-$, we have:

$$\begin{aligned}a \circ_1 (x \star_1 y) &= a \circ_1 (\{z|z \geq |x| + |y|\} \cup \{x, y\}) \\ &= a \circ_1 (\emptyset \cup \{x, y\}) \\ &= a \circ_1 x \cup a \circ_1 y \\ &= \{z|z \geq |a|.|x|\} \cup \{z|z \geq |a|.|y|\} \cup \{x, y\} \\ &= \{z|z \geq a.(-x)\} \cup \emptyset \cup \{x, y\} \\ &= \{z|z \geq -a.x\} \cup \{x, y\}.\end{aligned}$$

On the other hand

$$\begin{aligned} a \circ_1 x \star_1 a \circ_1 y &= (\{z|z \geq |a|.|x|\} \cup \{x\}) \star_1 (\{z|z \geq |a|.|y|\} \cup \{y\}) \\ &= (\{z|z \geq a.(-x)\} \cup \{x\}) \star_1 (\emptyset \cup \{y\}) \\ &= \cup\{z \star_1 y|z \geq a.(-x)\} \cup x \star_1 y \\ &= \{z'|z' \geq |z| + |y|, z \geq -a.x\} \cup \{z|z \geq -a.x\} \\ &\quad \cup\{z'|z' \geq |x| + |y|\} \cup \{x, y\} \\ &= \emptyset \cup \{z|z \geq -a.x\} \cup \emptyset \cup \{x, y\} \\ &= \{z|z \geq -a.x\} \cup \{x, y\}. \end{aligned}$$

In condition (ii), for the case in which $a \notin R_+ \cup R_-$, $b \in R_-$ and $x \in M_+$, we have

$$\begin{aligned} (a \oplus_1 b) \circ_1 x &= (\{c|c \geq |a| + |b|\} \cup \{a, b\}) \circ_1 x \\ &= (\emptyset \cup \{a, b\}) \circ_1 x \\ &= a \circ_1 x \cup b \circ_1 x \\ &= \{y|y \geq |a|.|x|\} \cup \{y|y \geq |b|.|x|\} \cup \{x\} \\ &= \emptyset \cup \{y|y \geq -b.x\} \cup \{x\} \\ &= \{y|y \geq -b.x\} \cup \{x\}. \end{aligned}$$

On the other hand

$$\begin{aligned} a \circ_1 x \star_1 b \circ_1 x &= (\{y_1|y_1 \geq |a|.|x|\} \cup \{x\}) \star_1 (\{y_2|y_2 \geq |b|.|x|\} \cup \{x\}) \\ &= (\emptyset \cup \{x\}) \star_1 (\{y_2|y_2 \geq |b|.|x|\} \cup \{x\}) \\ &= \cup\{x \star_1 y_2|y_2 \geq (-b).x\} \cup x \star_1 x \\ &\supseteq \{y|y \geq |x| + |y_2|, y_2 \geq -b.x\} \cup \{y_2|y_2 \geq -b.x\} \cup \{x\} \\ &\supseteq \{y_2|y_2 \geq -b.x\} \cup \{x\}. \end{aligned}$$

Finally, for $a \in R_+$, $b \notin R_- \cup R_+$ and $x \in M_-$, we have

$$\begin{aligned} (a \odot_1 b) \circ_1 x &= (\{c|c \geq |a||b|\} \cup \{a, b\}) \circ_1 x \\ &= (\emptyset \cup \{a, b\}) \circ_1 x \\ &= a \circ_1 x \cup b \circ_1 x \\ &= \{y|y \geq |a|.|x|\} \cup \{y|y \geq |b|.|x|\} \cup \{x\} \\ &= \{y|y \geq a.(-x)\} \cup \emptyset \cup \{x\} \\ &= \{y|y \geq -a.x\} \cup \{x\}, \end{aligned}$$

and

$$\begin{aligned} a \circ_1 (b \circ_1 x) &= a \circ_1 (\{y|y \geq |b|.|x|\} \cup \{x\}) \\ &= a \circ_1 (\emptyset \cup \{x\}) \\ &= a \circ_1 x \end{aligned}$$

$$\begin{aligned} &= \{y|y \geq |a|.|x|\} \cup \{x\} \\ &= \{y|y \geq a.(-x)\} \cup \{x\} \\ &= \{y|y \geq -a.x\} \cup \{x\}. \end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. □

References

[1] N.G. Alimov, *On ordered semigroups*, Izvestiya Akad. Nauk SSSR., 14 (1950), 569-576.

[2] S. M. Anvariyeh and B. Davvaz, *Strongly transitive geometric spaces associated to hypermodules*, J. Algebra, 322 (2009), 1340-1359.

[3] J. Chvalina, S. Hoskova-Mayerova, *On certain proximities and preorderings on the transposition hypergroups of linear first-order partial differential operators*, An. St. Univ. Ovidius Constanta, 22(1) (2014), 85-103.

[4] J. Chvalina, *Functional graphs, quasi-ordered sets and commutative hypergroups*, Masaryk University, Brno (in Czech), 1995.

[5] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer, Dordrecht, 2003.

[6] P. Corsini, *Prolegomena of hypergroup theory*, 2nd edition, Aviani editor, 1993.

[7] P. Corsini, *Hyperstrutures associated with ordered sets*, Bull Greek Math. Soc. 48, (2003) 7-18.

[8] B. Davvaz, *Polygroup theory and related system*, World Scientific Publishing, Singapore CrossRef. Math., 2012.

[9] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, 115, Palm Harber, USA, 2007.

[10] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Company, Inc., Princeton, 1960.

[11] S. H. Ghazavi, S. M. Anvariyeh, *EL–hyperstructures associated to n-ary relations*, Soft Comput. DOI 10.1007/s00500-016-2165-3, (2016), 1-10.

[12] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, *Ideals in EL-semihypergroups associated to ordered semigroups*, J. Algebraic Syst., 3(2), (2016), 109-125.

[13] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, *EL-hyperstructures derived from (partially) quasi ordered hyperstructures*, Iran J. Math. Sci. Inf., 10(2), (2015), 99-114.

[14] D. Heidari, B. Davvaz, *On ordered hyperstructures*, U. P. B. Sci. Bull. Series A, Vli. 73, Iss.2, (2011), 85-96.

[15] S. Hoskova, *Order Hypergroups-The unique square root condition for quasi-order hypergroups*, Set-valued Mathematics and Applications, 1, (2008), 1-7.

[16] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, (1934), 45-49.

[17] M. Novak, *EL-hyperstructures: an overview*, Ratio Math., 23 (2012), 85-96.

[18] M. Novak, *Important elements of EL-hyperstructures*, APLIMAT: 10th International Conference, STU in Bratislava, Bratislava, (2011), 151-158.

[19] M. Novak, *Some basic properties of EL-hyperstructure*, Eur. J. Combin., 34 (2013), 446-459.

[20] M. Novak, *The notion of subhyperstructure of Ends Lemma-based hyperstructures*, Aplimat. J. Appl. Math., 3(II), (2010), 237-247.

[21] M. Novak, *On EL-semihypergroups*, Eur. J. Combin. Part B, 44 (2015), 274-286.

[22] M. Novak, *Potential of the Ends Lemma to create ring-like hyperstructures from quasi-ordered (semi)groups*, South Bohemia Mathematical Letters, Volume 17, 1 (2009), 39-50.

[23] P. Ribenboim, *On ordered modules*, Crelles J., 225 (2009), 120-146.

[24] P. Rackova, *Hypergroups of symmetric matrices*, 10th International Congress of Algebraic Hyperstructures and Applications, Proceeding of AHA, 2008.

[25] I. G. Rosenberg, *Hypergroups and join spaces determined by relations*, Ital. J. Pure. Appl. Math., 4 (1998), 93-101.

[26] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Palm Harbour, 1994.

[27] T. Vougiouklis, *Generalization of P-hypergroups*, Rend Circ Mat. Palermo, 36(II) (1987), 114-121.

[28] T. Vougiouklis, *Representation of hypergroups by generalized permutations*, Math Sci. Net. CrossRef. Math., 1992, 172-183.

[29] T. Vougiouklis, *On some representations of hypergroups*, Ann. Sci. Univ. Clermont-Ferrand II Math., 26 (1990), 21-29.

Accepted: 21.01.2016

INEQUALITIES OF UNITARILY INVARIANT NORMS FOR MATRICES

Xuesha Wu
School of Marxism and General Education
Chongqing College of Electronic Engineering
Chongqing, 401331
P.R. China
xuesha_wu@163.com

Abstract. We present some inequalities of unitarily invariant norms for matrices by using majorization, Fan dominance principle and some existing inequalities of singular values and unitarily invariant norms for matrices. Our results are refinements or generalizations of ones shown by Audenaert, Al-khlyleh, and Kittaneh.
Keywords: singular values, unitarily invariant norms, fan dominance principle.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and suppose that $s_1(A) \geq \cdots \geq s_n(A) \geq 0$ are the singular values of A, which is the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. For $A \in M_n$, by singular value decomposition of A, we know that the trace norm $\|A\|_1 = \sum_{j=1}^n s_j(A) = tr|A|$ and the Frobenius norm $\|A\|_F = (\sum_{j=1}^n s_j^2(A))^{1/2} = (tr|A|^2)^{1/2}$ are both unitarily invariant.

Let $A, B \in M_n$. Recently, Audenaert proved in [1] that if $v \in [0,1]$, then

$$\|AB^*\|^2 \leq \|vA^*A + (1-v)\,B^*B\|\,\|(1-v)\,A^*A + vB^*B\|\,, \tag{1.1}$$

which is unity of the arithmetic-geometric mean and Cauchy-Schwarz inequalities for unitarily invariant norms.

Let $A, X, B \in M_n$. Zou proved in [2] that if $v \in [0,1]$, then

$$\|AXB^*\|^2 \leq \|vA^*AX + (1-v)\,XB^*B\|\,\|(1-v)\,A^*AX + vXB^*B\|\,, \tag{1.2}$$

which is a generalization of inequality (1.1).

Let $A, X, B \in M_n$. Very recently, Al-Manasrah and Kittaneh proved in [3] that if $v \in [0,1]$, then

$$\begin{aligned}\|AXB^*\|_F^2 \leq &\left(\|vA^*AX + (1-v)\,XB^*B\|_F^2 - v_0^2\,\|A^*AX - XB^*B\|_F^2\right)^{1/2} \\ &\times \left(\|(1-v)\,A^*AX + vXB^*B\|_F^2 - v_0^2\,\|A^*AX - XB^*B\|_F^2\right)^{1/2},\end{aligned} \tag{1.3}$$

where $v_0 = \min\{v,\ 1-v\}$. Inequality (1.3) is a refinement of inequality (1.2) for the Frobenius norm.

Lin [4] gave a new proof of inequality (1.1). The authors of [5, 6] showed some generalizations of inequality (1.2).

In this short note, following the idea of Lin [4], Al-Manasrah and Kittaneh [3], we first present an improvement of inequality (1.1) for the trace norm. Meanwhile, we also give a generalization of inequality (1.3).

2. Main results

In this section, we will show the main results of this paper. To do this, we need the following lemmas.

Lemma 2.1 ([7]). *Let $A \in M_n$. Then for any $k = 1, \cdots, n$, we have*

$$\prod_{j=1}^{k} |\lambda_j(A)| \leq \prod_{j=1}^{k} s_j(A).$$

Lemma 2.2 ([7]). *Let $A, B \in M_n$. Then for any $k = 1, \cdots, n$, we have*

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A)\, s_j(B).$$

Lemma 2.3 ([8]). *Let $A, B \in M_n$ be positive semidefinite. If $v \in [0,1]$, then*

$$\left\|A^v B^{1-v}\right\|_1 \leq \|vA + (1-v)B\|_1 - v_0\left(\sqrt{\|A\|_1} - \sqrt{\|B\|_1}\right)^2,$$

where $v_0 = \min\{v,\ 1-v\}$.

Lemma 2.4 ([9]). *Let $A, X, B \in M_n$ such that A, B are positive semidefinite. If $v \in [0,1]$, then*

$$\begin{aligned}\left\|A^v X B^{1-v}\right\|_F \leq \big(&\|vAX + (1-v)XB\|_F^p \\ &-v_0^p\left(\|AX + XB\|_F^p - 2^p\left\|A^{1/2}XB^{1/2}\right\|_F^p\right)\big)^{1/p},\end{aligned}$$

where $v_0 = \min\{v,\ 1-v\}$.

Theorem 2.1. *Let $A, B \in M_n$. If $v \in [0,1]$, then*

$$\begin{aligned}\|AB^*\|_1^2 \leq\ & \left(\|vA^*A + (1-v)B^*B\| - v_0\left(\sqrt{\|A^*A\|_1} - \sqrt{\|B^*B\|_1}\right)^2\right) \\ \times\ & \left(\|(1-v)A^*A + vB^*B\| - v_0\left(\sqrt{\|A^*A\|_1} - \sqrt{\|B^*B\|_1}\right)^2\right),\end{aligned} \tag{2.1}$$

where $v_0 = \min\{v,\ 1-v\}$.

Proof. By Lemmas 2.1 and 2.2, we know that for any $k = 1, \cdots, n$, we have

$$\begin{aligned}
\prod_{j=1}^{k} s_j^2 (AXB^*) &= \prod_{j=1}^{k} \lambda_j (BX^*A^*AXB^*) \\
&= \prod_{j=1}^{k} \lambda_j (A^*AXB^*BX^*) \\
&= \prod_{j=1}^{k} \lambda_j \left((A^*A)^v X (B^*B)^{1-v} (B^*B)^v X^* (A^*A)^{1-v} \right) \\
&\leq \prod_{j=1}^{k} s_j \left((A^*A)^v X (B^*B)^{1-v} (B^*B)^v X^* (A^*A)^{1-v} \right) \\
&\leq \prod_{j=1}^{k} s_j \left((A^*A)^v X (B^*B)^{1-v} \right) s_j \left((B^*B)^v X^* (A^*A)^{1-v} \right).
\end{aligned}$$

That is

$$\prod_{j=1}^{k} s_j(AXB^*) \leq \prod_{j=1}^{k} s_j^{1/2}((A^*A)^v X(B^*B)^{1-v}) s_j^{1/2}((B^*B)^v X^*(A^*A)^{1-v}). \tag{2.2}$$

Let

$$Y_1 = diag \left(s_1^{1/2} \left((A^*A)^v X (B^*B)^{1-v} \right), \cdots, s_n^{1/2} \left((A^*A)^v X (B^*B)^{1-v} \right) \right),$$

$$Y_2 = diag \left(s_1^{1/2} \left((B^*B)^v X^* (A^*A)^{1-v} \right), \cdots, s_n^{1/2} \left((B^*B)^v X^* (A^*A)^{1-v} \right) \right).$$

Then, it follows from (2.2) that

$$\prod_{j=1}^{k} s_j (AXB^*) \leq \prod_{j=1}^{k} s_j (Y_1) s_j (Y_2) = \prod_{j=1}^{k} s_j (Y_1 Y_2).$$

Since weak log-majorization implies weak majorization, we obtain

$$\sum_{j=1}^{k} s_j (AXB^*) \leq \sum_{j=1}^{k} s_j (Y_1 Y_2). \tag{2.3}$$

By Fan's dominance principle [7], we know that inequality (2.3) is equivalent to

$$\|AXB^*\| \leq \|Y_1 Y_2\|. \tag{2.4}$$

Putting $X = I$ and $v = 0$ or $v = 1$ in inequality (1.1), we get

$$\|Y_1 Y_2^*\|^2 \leq \|Y_1^* Y_1\| \, \|Y_2^* Y_2\|.$$

It follows from (2.4) and this last inequality that

$$\|AXB^*\|^2 \leq \left\|(A^*A)^v X (B^*B)^{1-v}\right\| \left\|(B^*B)^v X^* (A^*A)^{1-v}\right\|. \tag{2.5}$$

Since trace is unitarily invariant, we have

$$\|AB^*\|_1^2 \leq \left\|(A^*A)^v (B^*B)^{1-v}\right\|_1 \left\|(B^*B)^v (A^*A)^{1-v}\right\|_1.$$

Lemma 2.3 and the above inequality complete the proof. □

Remark 2.1. Obviously, inequality (2.1) is a refinement of inequality (1.1).

Remark 2.2. Putting $X = I$ in (2.2), we have

$$\prod_{j=1}^{k} s_j^2 (AB^*) \leq \prod_{j=1}^{k} s_j \left((A^*A)^v (B^*B)^{1-v}\right) s_j \left((B^*B)^v (A^*A)^{1-v}\right). \tag{2.6}$$

Ando proved in [10] that if $v \in [0,1]$, then

$$s_j \left(A^v B^{1-v}\right) \leq s_j (vA + (1-v) B), j = 1, \cdots, n.$$

Combining inequality (2.6) with Ando's result, we get

$$\prod_{j=1}^{k} s_j^2 (AB^*) \leq \prod_{j=1}^{k} s_j (vA^*A + (1-v) B^*B) s_j ((1-v) A^*A + vB^*B),$$

which implies inequality (1.1).

Next, we shall give a generalization of inequality (1.3).

Theorem 2.2. *Let $A, X, B \in M_n$. If $v \in [0,1]$, then*

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left(\|vA^*AX + (1-v) XB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p} \\ &\times \left(\|(1-v) A^*AX + vXB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p}, \end{aligned} \tag{2.7}$$

where

$$v_0 = \min\{v,\ 1-v\},$$

$$f(A,X,B,p) = \|A^*AX + XB^*B\|_F^p - 2^p \left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^p.$$

Proof. Note that for any $Y \in M_n$, we have $\|Y\| = \|Y^*\|$. Since Frobenius norm is unitarily invariant, it follows from inequality (2.5) and Lemma 2.5, we obtain

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left\|(A^*A)^v X (B^*B)^{1-v}\right\|_F \left\|(B^*B)^v X^* (A^*A)^{1-v}\right\|_F \\ &= \left\|(A^*A)^v X (B^*B)^{1-v}\right\|_F \left\|(A^*A)^{1-v} X (B^*B)^v\right\|_F \\ &\leq \left(\|vA^*AX + (1-v) XB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p} \\ &\times \left(\|(1-v) A^*AX + vXB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p}, \end{aligned}$$

where

$$v_0 = \min\{v,\ 1-v\},$$

$$f(A,X,B,p) = \|A^*AX + XB^*B\|_F^p - 2^p \left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^p.$$

This completes the proof. □

Remark 2.3. Putting $p = 2$ in inequality (2.7), we obtain

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left(\|vA^*AX + (1-v)XB^*B\|_F^2 - v_0^2 f(A,X,B,2)\right)^{1/2} \\ &\times \left(\|(1-v)A^*AX + vXB^*B\|_F^2 - v_0^2 f(A,X,B,2)\right)^{1/2}. \end{aligned} \tag{2.8}$$

Note that

$$\|A^*AX + XB^*B\|_F^2 = \|A^*AX - XB^*B\|_F^2 + 4\left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^2,$$

then, we can rewrite inequality (2.8) as follows

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left(\|vA^*AX + (1-v)XB^*B\|_F^2 - v_0^2 \|A^*AX - XB^*B\|_F^2\right)^{1/2} \\ &\times \left(\|(1-v)A^*AX + vXB^*B\|_F^2 - v_0^2 \|A^*AX - XB^*B\|_F^2\right)^{1/2}. \end{aligned}$$

This is inequality (1.3) and so we know that inequality (2.7) is a generalization of inequality (1.3).

Acknowledgements

The author wishes to express her heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] K.M.R. Audenaert, *Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Oper. Matrices, 9 (2015), 475-479.

[2] L. Zou, Y. Jiang, *A note on interpolation between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, J. Math. Inequal., 10 (2016), 1119-1122.

[3] M. Al-khlyleh, F. Kittaneh, *Interpolating inequalities related to a recent result of Audenaert*, Linear Multilinear Algebra, 65 (2017), 922-929.

[4] M. Lin, *Remarks on two recent results of Audenaert,* Linear Algebra Appl., 489 (2016), 24-29.

[5] M. Bakherad, R. Lashkaripour, M. Hajmohamadi, *Extensions of interpolation between the arithmetic-geometric mean inequality for matrices*, J. Inequal. Appl, 2017 (2017), 209.

[6] M. Alakhrass, *A note on Audenaert interpolation inequality,* Linear Multilinear Algebra. In Press, doi: 10.1080/03081087.2017.1376614.

[7] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[8] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., 361 (2010), 262-269.

[9] Y. Al-Manasrah, F. Kittaneh, *Further Generalizations, Refinements, and Reverses of the Young and Heinz Inequalities*, Results in Mathematics, 71 (2017), 1063-1072.

[10] T. Ando, *Matrix Young inequality*, Oper. Theory Adv. Appl., 75 (1995), 33-38.

Accepted: 21.01.2016

IDEAL CONVERGENT GENERALIZED DIFFERENCE SEQUENCE SPACES OF INFINITE MATRIX AND ORLICZ FUNCTION

Kuldip Raj*

Charu Sharma
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K
India
kuldipraj68@gmail.com
charu145.cs@gmail.com

Abstract. In this paper we introduce some generalized difference sequence spaces by using Musielak-Orlicz function, ideal convergence and an infinite matrix defined on n-normed spaces. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.
Keywords: Musielak-Orlicz function, ideal convergence, solid, infinite matrix, n-normed space.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan ([11], [12]) and Gunawan and Mashadi [13] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the real field $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;
2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \ \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$, and
4. $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called an n-normed space over the field $\mathbb{R}$.

*. Corresponding author

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$, where script E denotes Euclidean space. Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} \|x_k - x_i, z_1, \cdots, z_{n-1}\| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notions of statistical convergence and convergence in density for sequences has been in the literature, under different guises, since the early part of the last century. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Statistical convergence was recently investigated by Fast [9] and Schoenberg [29] independently.

The concept of ideal convergence was first introduced by P. Kostyrko et al. [16] as a generalization of statistical convergence which was further studied in topological spaces by Das et al. [1]. More applications of ideals can be seen in ([1], [2]). We continue in this direction and introduce I-convergence of generalized sequences in more general setting.

A family $I \subset 2^X$ of subsets of a non empty set X is said to be an ideal in Xif

1. $\phi \in I$;
2. $A, B \in I$ imply $A \cup B \in I$;
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in I$, while an admissible ideal I of X further satisfies $\{x\} \in I$ for each $x \in X$ (see [14]).

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to I (see [13]). A non empty family of sets $F \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin F$, for $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and $A \subseteq B$ implies $B \in F$. An ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of 2^X can be found in [16].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|(x_k)\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [18] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([19], [24]).

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \leq a$.

The notion of difference sequence spaces was introduced by Kızmaz [17], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [8] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [8]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [17].For more details about sequence spaces (see [3], [4], [5], [6], [7], [21], [22], [23], [25], [26], [27], [28], [30], [31]) and reference therein.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}).$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of above equention converges for each $n \in \mathbb{N}$ and every $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}.$$

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
Let I be an admissible ideal of $\mathbb{N}$, let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (a_{nk})$ be an infinite matrix. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $(X, \|., ..., .\|)$ be an n-normed space. Suppose $\Lambda = (\lambda_n)$ is a non-decreasing sequence of positive real numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$. Further $w(n-x)$ denotes the space of all X-valued sequences. For every $z_1, z_2, ..., z_{n-1} \in X$, for each $\epsilon > 0$ and for some $\rho > 0$ we define the following sequence spaces:

$$W^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$$

$$= \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I,$$

$$\text{for } L \in X \text{ and } s \geq 0 \Bigg\},$$

$$W_0^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I, \text{ for } s \geq 0 \Bigg\}$$

and

$$W_\infty^I\big[\Lambda,\Delta_n^m,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq K\Big\}\in I,\ \text{for } s\geq 0\Bigg\},$$

where $I_n=[n-\lambda_n+1,n]$. Some special cases of the above defined sequence spaces are arises: If $m=0$, then we obtain the spaces as follows:

$$W^I\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_kx_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,$$

$$\text{for } L\in X \text{ and } s\geq 0\Bigg\},$$

$$W_0^I\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_kx_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{for } s\geq 0\Bigg\}$$

and

$$W_\infty^I\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_kx_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq K\Big\}\in I,\ \text{for } s\geq 0\Bigg\}.$$

If $m=n=1$, then the above spaces are as follows:

$$W^I\big[\Lambda,\Delta,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta x_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,$$

$$\text{for } L\in X \text{ and } s\geq 0\Bigg\},$$

$$W_0^I\big[\Lambda,\Delta,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given } \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{for } s\geq 0\Bigg\}$$

and

$$W_\infty^I\big[\Lambda,\Delta,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq K\Big\}\in I,\ \text{for}\ s\geq 0\Bigg\}.$$

If $s=0$ and $\mathcal{M}(x)=x$ for all $x\in[0,\infty)$, then we have

$$W^I\big[\Lambda,\Delta_n^m,A,u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given}\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big(\|\frac{u_k\Delta_n^m x_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)^{p_k}\geq\epsilon\Big\}\in I,$$

$$\text{for}\ \ L\in X\ \ \text{and}\ s\geq 0\Bigg\},$$

$$W_0^I\big[\Lambda,\Delta_n^m,A,u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given}\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big(\|\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)^{p_k}\geq\epsilon\Big\}\in I,\ \text{for}\ \ s\geq 0\Bigg\}$$

and

$$W_\infty^I\big[\Lambda,\Delta_n^m,A,u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big(\|\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)^{p_k}\geq K\Big\}\in I,\ \text{for}\ s\geq 0\Bigg\}.$$

If $p=(p_k)=1$ for all k, then the above spaces are as follows

$$W^I\big[\Lambda,A,\Delta_n^m,\mathcal{M},u,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given}\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m x_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]$$

$$\geq\epsilon\Big\}\in I,\ \ \text{for}\ \ L\in X\ \ \text{and}\ s\geq 0\Bigg\},$$

$$W_0^I\big[\Lambda,\Delta_n^m,A,\mathcal{M},u,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{for given}\ \epsilon>0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]$$

$$\geq \epsilon \Big\} \in I, \text{ for } s \geq 0 \Bigg\}$$

and

$$W_\infty^I \big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \exists\, K > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big] \geq K \Big\} \in I, \text{ for } s \geq 0 \Bigg\}.$$

If $A = (C, 1)$, the Cesàro matrix, then the above spaces are as follows:

$$W^I \big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I,$$

$$\text{for } L \in X \text{ and } s \geq 0 \Bigg\},$$

$$W_0^I \big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I, \text{ for } s \geq 0 \Bigg\}$$

and

$$W_\infty^I \big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \exists\, K > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq K \Big\} \in I, \text{ for } s \geq 0 \Bigg\}.$$

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. We finally arrived, let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k_{r-1} < k < k_r \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $W^I\big[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\big]$, $W_0^I\big[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\big]$ and $W_\infty^I\big[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\big]$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{1}$$

for all k and $a_k, b_k \in \mathbb{R}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{R}$.

The main aim of this paper is to introduce some generalized difference sequence spaces defined by ideal convergence, Musielak-Orlicz function and an infinite matrix. We have also make an effort to study some inclusion relations and their topological properties.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $W^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$, $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ and $W_\infty^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ are linear spaces over the real field $\mathbb{R}$.*

Proof. We shall prove the result for the space $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$. Let $x = (x_k)$ and $y = (y_k)$ be two elements of $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ and for $z_1, z_2, ..., z_{n-1} \in X$ such that

$$A_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I.$$

Let $\alpha, \beta \in \mathbb{R}$. Since $\|., ..., .\|$ is a n-norm, Δ_n^m is linear and the contributing of $\mathcal{M} = (M_k)$, the following inequality holds:

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k_{-s} M_k\Big(\|\frac{u_k \Delta_n^m(\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq DK \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ DK \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k},
\end{aligned}$$

where $K = \max\{1, \frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\}$.

From the above relation, we get

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m(\alpha x_k+\beta y_k)}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}$$
$$\subseteq \Big\{n \in \mathbb{N} : DK\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : DK\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. Similarly we can prove other cases. This completes the proof of the theorem.

Theorem 2.2. *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ be two Musielak-orlicz functions. Then we have $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|., ..., .\|\big] \cap W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|., ..., .\|\big] \subseteq W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}'+\mathcal{M}'', u, p, \|., ..., .\|\big]$.*

Proof. Let $x = (x_k) \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|., ..., .\|\big] \cap W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|., ..., .\|\big]$. Then we get the result by the following inequality:

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}(M_k'+M_k'')\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}$$
$$\leq D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k'\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}$$
$$+ D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k''\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}.$$

Hence,

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}(M_k'+M_k'')\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}$$
$$\subseteq \Big\{n \in \mathbb{N} : D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k'\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k''\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. This completes the proof of the theorem.

Theorem 2.3. *The inclusions $Z\big[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|., ..., .\|\big] \subseteq Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ are strict for $m \geq 1$. In general $Z\big[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|., ..., .\|\big] \subseteq Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$, for $m = 0, 1, 2, ...$ where $Z = W^I, W_0^I, W_\infty^I$.*

Proof. We give the proof for $W_0^I\left[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|.,...,.\|\right]$ only. The others can be proved by similar argument. Let $x = (x_k)$ be any element in the space $W_0^I\left[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|.,...,.\|\right]$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the set

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex for every k, it follows that

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}\\
&= \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1} - u_k\Delta_n^{m-1}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}\\
&\leq D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[\frac{1}{2}k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}\\
&+ D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[\frac{1}{2}k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}\\
&\leq DH\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}\\
&+ DH\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k},
\end{aligned}$$

where $H = \max\Big\{1, (\frac{1}{2})^G\Big\}$. Thus we have

$$\begin{aligned}
&\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}\\
&\subseteq \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}\\
&\cup \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Since both the sets in right hand side of the above relation belongs to I, therefore we get the set

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

This inclusion is strict follows from the following example.

Example. Let $M_k(x) = x$, for all $k \in \mathbb{N}$, $u_k = p_k = 1$ for all $k \in \mathbb{N}$, $s = 0$, $\lambda_n = 1$ and $A = (C, 1)$, the Cesaro matrix. Now consider a sequence $x =$

$(x_k) = (k^t)$. Then for $n = 1, x = (x_k)$ belongs to $W_0^I\big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big]$ but does not belongs to $W_0^I\big[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|., ..., .\|\big]$, because $\Delta_n^m x_k = 0$ and $\Delta_n^{m-1} x_k = (-1)^{m-1}(m-1)!$.

Theorem 2.4. *For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n-norms $\|., ..., .\|_1$ and $\|., ..., .\|_2$ on X, we have the following $Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|_1\big] \cap Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, q, \|., ..., .\|_2\big] \neq \phi$ where $Z = W^I, W_0^I$ and W_∞^I.*

Proof. Since the zero element belongs to both the classes of sequences, so the intersection is non-empty.

Theorem 2.5. *The sequence spaces $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ and $W_\infty^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ are normal as well as monotone.*

Proof. We shall prove the theorem for $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$. Let $x = (x_k) \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ and $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then for given $\epsilon > 0$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m (\alpha_k x_k)}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\}$$

$$\subseteq \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m (x_k)}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I.$$

Hence, $\alpha_k x_k \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u,\ p, \|., ..., .\|\big]$. Thus, the space $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u,\ p, \|., ..., .\|\big]$ is normal. Therefore, $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ is monotone also (see [15]). Similarly, we can prove the theorem for other case. This completes the proof of the theorem.

References

[1] P. Das, P. Kostyrko, W. Wilczynski and P. Malik, *I and I^* convergence of double sequences*, Math. Slovaca, 58 (2008), 605-620.

[2] P. Das and P. Malik, *On the statistical and I-variation of double sequences*, Real Anal. Exchange, 33 (2007-2008), 351-364.

[3] A. Esi and B. C. Tripathy, *On some new difference sequence spaces*, Commun. Fac. Sci. Univ. Ank. Series A1, 53 (2004), 57-66.

[4] A. Esi and M. Işık, *Some Generalized Difference Sequence Spaces*, Thai J. Math., 3 (2005), 241-247.

[5] A. Esi, *Some Classes of Generalized difference paranormed sequence spaces associated with multiplier sequences*, J. Comput. Anal. Appl., 11 (2009), 536-545.

[6] A. Esi, *On some generalized difference sequence spaces of invariant means defined by a sequence of Orlicz functions*, J. Comput. Anal. Appl., 11 (2009), 524-535.

[7] A. Esi, *Generalized difference sequence spaces defined by Orlicz functions*, Gen. Math., 17 (2009), 53-66.

[8] M. Et and R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math., 21 (1995), 377-386.

[9] H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), 241-244.

[10] S. Gähler, *Linear 2-normietre Rume*, Math. Nachr., 28 (1965), 1-43.

[11] H. Gunawan, *On n-inner product, n-norms, and the Cauchy-Schwartz inequality*, Sci. Math. Jpn., 5 (2001), 47-54.

[12] H. Gunawan, *The space of p-summable sequence and its natural n-norm*, Bull. Aust. Math. Soc., 64 (2001), 137-147.

[13] H. Gunawan and M. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci., 27 (2001), 631-639.

[14] M. Gurdal, and S. Pehlivan, *Statistical convergence in 2-normed spaces*, Southeast Asian Bull. Math., 33 (2009), 257-264.

[15] P. K. Kamthan and M. Gupta, *Sequence spaces and series*, Marcel Dekkar, New York, 1981.

[16] P. Kostyrko, T. Salat and W. Wilczynski, *I-Convergence*, Real Anal. Exchange, 26 (2000), 669-686.

[17] H. Kızmaz, *On certain sequence spaces*, Canad. Math-Bull., 24 (1981), 169-176.

[18] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., 10 (1971), 345-355.

[19] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics, 5, Polish Academy of Science, 1989.

[20] A. Misiak, *n-inner product spaces*, Math. Nachr., 140 (1989), 299-319.

[21] M. Mursaleen, *On statistical convergence in random 2-normed spaces*, Acta sci. Math., 76 (2010), 101-109.

[22] M. Mursaleen and A. Alotaibi, *On I-convergence in random 2-normed spaces*, Math. Slovaca, 61 (2011), 933-940.

[23] M. Mursaleen and S. A. Mohiuddine, *Statistical convergence of double sequences in intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals, 41 (2009), 2414-2421.

[24] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034 (1983).

[25] K. Raj and S. Jamwal, *On some generalized statistical convergent sequence spaces*, Kuwait J. Sci., 42(3) (2015), 86-104.

[26] K. Raj and A. Kilicman, *On certain generalized paranormed spaces*, J. Inequal. Appl., 37 (2015).

[27] K. Raj and S. K. Sharma, *Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function*, Acta Univ. Sapientiae Math., 3 (2011), 97-109.

[28] K. Raj, S. Jamwal and S. K. Sharma, *Some double lacurnary sequence spaces*, Italian J. Pure and Appl. Math., 32 (2014), 347-358.

[29] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 (1959), 361-375.

[30] B. C. Tripathy, A. Esi and B. Tripathy, *On a New Type Of Generalized Difference Cesaro Sequence Spaces*, Soochow J. Math., 31 (2005), 333-340.

[31] B. C. Tripathy and A. Esi, *A new type difference sequence spaces*, Int. J. Sci. Technol., 1 (2006), 11-14.

Accepted: 20.04.2016

ON THE ROLE OF THE STOKES PROBLEM IN SECOND GRADE FLUID FLOW IN REGIONS WITH PERMEABLE INTERFACES

R. Maritz
Department of Mathematical Sciences
University of South Africa
Science Campus, Florida
Unisa 0003
South Africa

J.M.W. Munganga*
Department of Mathematical Sciences
University of South Africa
Science Campus, Florida
Unisa 0003
South Africa
mungajmw@unisa.ac.za

Abstract. The role of the Stokes problem in conducting a theory of existence of solutions, weak as well as classical, is under investigation in this research paper. We model the motion of incompressible flows of non–Newtonian fluids through permeable boundaries in terms of the Stokes equation. The equation of motion in the region $\Omega \in \mathbb{R}^3$ is coupled with the dynamic boundary condition through the permeable boundary Γ, in order to design a transport equation in a product space $\mathbf{Y}_m$. The existence of a unique solution for the Stokes problem is proved by using a special Helmholtz projection and employing the results from a paper of Agmon et al [1, Theorem 10.5 p.78]. The modelling is done for a special case where a "shear flow" is assumed.

Keywords: Stokes problem, Helmholtz projection, weak solution, non-Newtonian fluids, permeable boundary, dynamic boundary conditions, kinematic boundary condition.

1. Introduction

Existence of solutions to the general initial-boundary-value problem for an incompressible second grade fluid in a bounded domain, with no slip, has first been addressed by the pioneer A.P. Oskolkov [13], who proved the global (in time) existence and uniqueness of a *general solution* to a simplified version of the problem by formulating it in terms of

$$\mathbf{u} \equiv \mathbf{v} - \alpha \Delta \mathbf{v},$$

and applying the Faedo-Galerkin method. Cioranescu and Girault [5] followed a similar approach, but used the quantity $\mathbf{curl}(\mathbf{v} - \alpha_1 \Delta \mathbf{v})$ and applied the

*. Corresponding author

Faedo-Galerkin method to the full problem in order to obtain a unique solution, global in time if Ω is in $\mathbb{R}^2$ and local in time if Ω is in $\mathbb{R}^3$, for flows with $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$.

Certain one-dimensional flows of a so-called power-law fluid of grade two, with shear-dependent viscosity, were studied by Man [9], where similar existence results were established.

In the proof of existence the Galdi-Grobbelaar-Sauer method [6] of reducing the equation of conservation of linear momentum to a transport equation and a Stokes-type equation is modified so that the dynamical boundary condition is combined with the transport equation. In [6], Galdi et.al. formulated the problem of existence of classical solutions as a problem of the existence of a fixed point of a certain mapping by considering the change of variable $\mathbf{v} \rightarrow \mathbf{u} = \mathbf{v} - \alpha_1 \Delta \mathbf{v}$. This was the first proof of existence and uniqueness of classical solutions. The restriction that $\alpha_1 > 0$ as well as α_1 sufficiently large was imposed for the global existence result to hold. No restriction on α_2 was imposed. Here, orthogonal projections on solenoidal fields were used to annihilate the pressure term in the equation of motion in the region Ω. They considered a fixed point problem in $\mathbf{w}$, based on the Helmholtz decomposition $\Delta \mathbf{v} = \mathbf{w} + \nabla \pi$.

Stokes flow in region with stretching boundaries was investigated by numerous researchers like Wang [17], Andersson [2] and Ariel et Al [3]. Wang [17] proved an exact solution for a flow due to a stretching boundary, where as Andersson and Ariel et Al[2, 3] gave an analysis of slip flow past stretching surfaces. Makinde et Al [8] investigated MHD steady flow in a channel with slip at permeable boundaries, and Sekhar et Al [4] discussed Stokes flow inside a porous spherical shell stress jump boundary condition.

For traditional boundary conditions, the classical Helmholtz decomposition is often useful. The Helmholtz decomposition results from orthogonal projections from the space of square integrable vector fields onto the subspaces consisting of gradients and solenoidal fields [16, Thm. 1.5, p. 16].

To answer the question: "What will cause the fluid to move through the permeable boundary?", we consider the boundary as a continuum with its own physical properties. The motion of the boundary material is determined by normal forces exerted on it by the fluid in the container as well as forces due to the contact between the fluid in the container and the boundary surfaces. In section 2 we present the problem where the Frenet-Serret formulae, [11, 19, 18] is used to formulate the permeability through the boundary. In this section two additional assumptions were made concerning the shape of Ω. Section 3 is devoted to definitions of spaces, operators, norms and projections. In section 4 we formulate the auxiliary problems in terms of a Stokes-like expression for the fluid in the region Ω as well as the fluid flow through the boundary. The existence for the Stokes Problem is presented in section 5. The transport problem in terms of the Stokes expression is given in section 6 to be followed by the Conclusion in section 7.

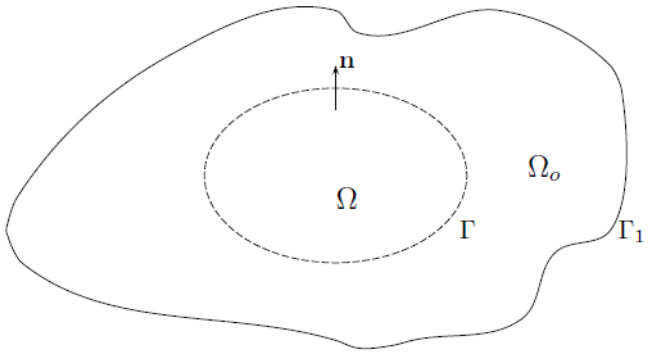


Figure 1: Profile for normal flow through the permeable wall Γ.

2. Presentation

In our case, we consider a canister (Ω), filled with incompressible non-Newtonian fluid, immersed in fluid of the same kind. It is assumed that the wall of the canister admits normal flow through it.

Figure 1 illustrates the situation where the curvature of the boundary Γ of Ω is non negative.

Permeability of the walls of the container is described by assuming that at the boundary Γ, the flow $\mathbf{v}$ has the direction of the normal (see Figure 1):

$$\gamma_o \mathbf{v}(x,t) = -\eta(x,t)\mathbf{n}(x). \tag{1}$$

The velocity component η is treated as an unknown and an evolution equation has to be found for it.

The Frenet-Serret [11, 19, 18], formulae are used to model the dynamic boundary condition, and we ask the reader to familiarize himself with this modelling from [10] or [11].

Modelling of the situation, has led to the equations which are written as follows [10]:

$$\left.\begin{array}{ll} \rho^{1/2}\mathbf{v}_t + \rho^{1/2}(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{-1/2}\nabla\cdot\mathbf{T}^* & \text{in} \quad \Omega \\ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega & \\ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \quad \text{on} \quad \Gamma; \quad \mathbf{v} = 0 \text{ on } \Gamma_1 & \\ \gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta_v\mathbf{M} \quad \text{on} \quad \Gamma & \\ \rho^{1/2}\partial_t(\rho\eta_v - \alpha\Delta_s\eta_v) + \rho^{-1/2}\frac{\gamma_o p}{\delta} = \rho^{-1/2}\mu\Delta_s\eta_v \quad \text{on} \quad \Gamma. & \end{array}\right\} \tag{2}$$

where ρ denotes the density of the fluid and δ the "thickness of the boundary" and has a physical dimension of length. $\mathbf{T}^*$ is the part of the stress tensor which depends only on velocity, that is

$$\mathbf{T}^* = \mu\mathbf{A} + \alpha D_t\mathbf{A} + \frac{\alpha}{2}(\mathbf{AW} - \mathbf{WA}).$$

In what follows, we assume that the constants μ and α are strictly positive. Please see [11] equation (2.16) for an explanation of the tensor $\mathbf{M}$ which only depends on the geometry of the surface Γ. $\mathbf{M}$ is defined by

$$\mathbf{M} := [K(\mathbf{n} \otimes \mathbf{n}) - \kappa_1(\tau_1 \otimes \tau_1) - \kappa_2(\tau_2 \otimes \tau_2)] \tag{3}$$

We have chosen $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$ and $\mathbf{n}$ so that $\boldsymbol{\tau_1} \wedge \boldsymbol{\tau_2} = \mathbf{n}$. Apart from the smoothness of Γ we make two additional assumptions regarding the shape of Ω, namely that the curvatures κ_1, κ_2 and K are constrained in the following way:

1. There exist constants g and G such that

$$0 < g \leq K(x) \leq G \text{ for all } x \in \Gamma. \tag{4}$$

2. There exists a constant H such that

$$0 \leq \kappa_1^2 + \kappa_2^2 \leq H^2 \text{ on } \Gamma. \tag{5}$$

When considering dynamic boundary conditions where interaction between the boundary and the fluid is taken into account, we need a modified projection theorem for the same results. The projection we construct is designed to keep the pair $\langle \rho^{\frac{1}{2}}\mathbf{v}, \rho^{1/2}\eta\rangle$ intact and at the same time eliminate the pressure term $\langle \rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p\rangle$ [14]. The following: $\langle \mathbf{v}, \eta\rangle \in \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ has the orthogonal (therefore, unique) decomposition

$$\begin{aligned} \langle \mathbf{v}, \eta\rangle &= \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\mathbf{n} \cdot \gamma_o\mathbf{w}\rangle + \langle \rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o q\rangle \\ \nabla \cdot \mathbf{w} &= 0 \end{aligned}$$

with $\mathbf{w} \in \mathbf{H}^k(\Omega)$, $q \in H^{k+1}(\Omega)$, provided that $\mathbf{v} \in \mathbf{H}^k(\Omega)$ and $\eta \in H^{k-1/2}(\Gamma)$.

The orthogonal projection associated with the term $\langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\gamma_o\mathbf{w} \cdot \mathbf{n}\rangle$ will be denoted by P.

3. Some spaces, operators and definitions

All spaces of vector fields are denoted by boldface letters. We define the space $\mathbf{V}(\Omega)$ as

$$\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{C}^\infty(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \text{ on } \Gamma,$$

$$\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta_v\mathbf{M} \text{ on } \Gamma, \ \gamma_o\mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Note that $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}^2(\Omega)$.

From now on, the notation $\gamma_o\mathbf{v}$ will be used to denote the restriction of $\mathbf{v}$ to Γ and Γ_1.

We define the following Hilbert spaces:

$$\mathbf{X}_m = Cl_{\mathbf{H}^m(\Omega)}(\mathbf{V}(\Omega)); \quad \mathbf{X}_m^o = \{\mathbf{v} \in \mathbf{X}_m : \gamma_o\mathbf{v} = \mathbf{0}\}, \quad m = 1, 2, 3 \ldots.$$

The inner products of the above spaces are the usual inner products defined for the Sobolev space $\mathbf{H}^m(\Omega)$.

The metric of a product space is defined in the usual way: For $\mathbf{X}$ and $\mathbf{Y}$ Hilbert spaces, the scalar product in $\mathbf{X} \times \mathbf{Y}$ is defined as

$$(\langle \mathbf{p}, \mathbf{q} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle)_{X \times Y} = (\mathbf{p}, \mathbf{r})_X + (\mathbf{q}, \mathbf{s})_Y.$$

The corresponding norm is then defined by

$$\|\langle \mathbf{p}, \mathbf{q} \rangle\|^2_{X \times Y} = \|\mathbf{p}\|^2_X + \|\mathbf{q}\|^2_Y.$$

We define the canonical operators $\mathbf{C}_o$ and $\mathbf{C}_m$ in the following way:

Let

$$\mathbf{Y}_o = \mathbf{L}^2(\Omega) \times L^2(\Gamma), \quad \text{and} \quad \mathbf{Y}_m = \mathbf{X}_m \times H^{m-\frac{1}{2}}(\Gamma).$$

The canonical operator $\mathbf{C}_o : \mathbf{X}_1 \to \mathbf{Y}_o := \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ is defined by

$$\mathbf{C}_o \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle. \tag{6}$$

According to the definition (6) above, we have

$$\begin{aligned} (\mathbf{C}_o \mathbf{v}, \mathbf{C}_o \mathbf{w})_{Y_o} &:= \rho(\mathbf{v}, \mathbf{w})_{L^2(\Omega)} + (\rho^{1/2} \eta_v, \rho^{1/2} \eta_w)_{L^2(\Gamma)} \\ \|\mathbf{C}_o \mathbf{v}\|^2_{Y_o} &:= \rho \|\mathbf{v}\|^2_{L^2(\Omega)} + \rho \|\eta_v\|^2_{L^2(\Gamma)}. \end{aligned} \tag{7}$$

The canonical operator $\mathbf{C}_m : \mathbf{X}_m \to \mathbf{Y}_m \quad (m \geq 1)$ is defined by

$$\mathbf{C}_m \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle. \tag{8}$$

With (7) in mind, the norm of $\mathbf{C}_m \mathbf{v}$ in $\mathbf{Y}_m$ is given by

$$\|\mathbf{C}_m \mathbf{v}\|^2_{Y_m} = \rho \|\mathbf{v}\|^2_{X_m} + \rho \|\eta_v\|^2_{H^{m-1/2}(\Gamma)}.$$

In terms of the Trace Theorem, $\mathbf{C}_o$ and $\mathbf{C}_m$ are continuous linear mappings.

Let $I = [0, T]$. For Y, a Banach space with norm $\| \cdot \|_Y$, and for $1 \leq p < \infty$ let

$$L^p(I; Y) = \{v : t \to v(t) \in Y; t \in I, \ v \ \text{ measurable, and } \int_o^T \|v(t)\|^p_Y \, dt < \infty\}.$$

and denote by $W^{m,p}(I; Y)$ the space of functions such that the distributional time derivatives of order up to and including m are in $L^p(I; Y)$. For $p = \infty$, we denote by $L^\infty(I; Y)$ the Banach space of measurable and essentially bounded functions defined on I with values in Y. The norms in $W^{k,\infty}(I; \mathbf{H}^m(\Omega))$ and in $W^{k,\infty}(I; H^{m-1/2}(\Gamma))$, $k \geq 0$, are denoted by $\| \cdot \|_{k,m,T}$ and $\| \cdot \|_{k,m-1/2,T,\Gamma}$, respectively. For $k = 0$, we write $\| \cdot \|_{m,T}$ and $\| \cdot \|_{m-1/2,T,\Gamma}$.

The linear operator $\mathbf{B}_m : \mathbf{X}_{m+2} \to \mathbf{Y}_m$ is defined by

$$\mathbf{B}_m \mathbf{v} := \langle \rho^{-1/2}(\rho \mathbf{v} - \alpha \Delta \mathbf{v}), \delta \rho^{-1/2}(\rho \eta_v - \alpha \Delta_s \eta_v) \rangle.$$

Now for $\mathbf{v} \in \mathbf{X}_{m+2}$, we define

$$\begin{aligned} \mathbf{u}_1 &= \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}) \\ u_2 &= \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) \\ \mathbf{u} &= \langle \mathbf{u}_1, u_2 \rangle \in \mathbf{Y}_m, \end{aligned}$$

so that

$$\mathbf{B}_m\mathbf{v} = \mathbf{u}.$$

We note that since $\mathbf{X}_m$ is embedded in $\mathbf{X}_1$ and $\mathbf{Y}_m$ is embedded in $\mathbf{Y}_o$, $\mathbf{C}_m\mathbf{v} = \mathbf{C}_o\mathbf{v}$, $\mathbf{v} \in \mathbf{X}_m$ for $m \geq 1$.

In motivation of what is to follow, we obtain a formal energy identity by taking the scalar product in $\mathbf{Y}_o$ of $\mathbf{u} = \mathbf{B}_m\mathbf{v}$ with $\mathbf{C}_m\phi$. This identity holds for ϕ, $\mathbf{v} \in \mathbf{X}_m$, $m \geq 1$:

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_m\phi)_{Y_o} &= (\mathbf{B}_m\mathbf{v}, \mathbf{C}_o\phi)_{Y_o} \\ &= (\langle \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}), \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v)\rangle, \langle \rho^{1/2}\phi, -\rho^{1/2}\gamma_o\phi \cdot \mathbf{n}\rangle)_{Y_o} \\ &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} - \alpha(\Delta\mathbf{v}, \phi)_{L^2(\Omega)} + \delta\rho(\eta_v, \eta_\phi)_{L^2(\Gamma)} - \delta\alpha(\Delta_s\eta_v, \eta_\phi)_{L^2(\Gamma)}.. \end{aligned}$$

But with $\gamma_o\phi = -\eta_\phi\mathbf{n}$ and $\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} = -2K\eta_v\mathbf{n}$, we have

$$\begin{aligned} -(\Delta\mathbf{v}, \phi)_{L^2(\Omega)} &= -(\nabla \cdot \mathbf{A}(\mathbf{v}), \phi)_{L^2(\Omega)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - (\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n}, \gamma_o\phi)_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - (-2K\eta_v\mathbf{n}, -\eta_\phi\mathbf{n})_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - 2(K\eta_v, \eta_\phi)_{L^2(\Gamma)}. \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_o\phi)_{Y_o} &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} + \delta\rho(\eta_v, \eta_\phi)_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} \\ &\quad -2\alpha(K\eta_v, \eta_\phi)_{L^2(\Gamma)} + \delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \end{aligned}$$

for any ϕ, $\mathbf{v} \in \mathbf{X}_m$; $m \geq 1$.

For $\phi = \mathbf{v}$,

$$(\mathbf{u}, \mathbf{C}_o\mathbf{v})_{Y_o} = \rho\|\mathbf{v}\|^2_{L^2(\Omega)} + \|(\delta\rho - 2\alpha K)\eta_v\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|^2_{L^2(\Omega)} + \alpha\|\nabla_s\eta_v\|^2_{L^2(\Gamma)}.$$

We define, accordingly, the bilinear form b_1 by

$$\begin{aligned} b_1(\mathbf{v}, \phi) &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} + ((\delta\rho - 2\alpha K)\gamma_o\mathbf{v}, \gamma_o\phi)_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} \\ &\quad + \delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \quad \text{for} \quad \mathbf{v}, \phi \in \mathbf{X}_1. \end{aligned} \tag{9}$$

We assume that the parameter $p_2 \in (0, 1/2)$, as in [10], and that $p_2 = \frac{\alpha K}{\delta\rho}$.

We define the following two operators: the operator $\mathbf{N}_v$ by

$$\mathbf{N}_v\mathbf{u} = \langle(\mathbf{v}\cdot\nabla)\mathbf{u}_1, 0\rangle \tag{10}$$

with $\mathbf{u}_1 = \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v})$,
and the pressure operator Π is defined by

$$\Pi p = \langle\rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p\rangle. \tag{11}$$

According to [15] on Helmholtz projections, there exists a unique $q \in H^{k+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^k(\Omega)$ such that

$$\langle\mathbf{v}, \eta_v\rangle = \mathbf{C}_o\mathbf{w} + \Pi q$$

if $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $q \in H^{1/2}(\Gamma)$, i.e. $P\mathbf{C}_o\mathbf{w} = \mathbf{C}_o\mathbf{w}$, $P\Pi q = 0$.

4. The auxiliary problems

Let us write the equations of (2) in the form

$$\left.\begin{aligned} \partial_t\left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] + (\mathbf{v}\cdot\nabla)\left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] \\ +\rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \ \text{ in } \Omega\times(0,T) \\ \mathbf{v} = 0 \ \text{ on } \ \Gamma_1\times(0,T) \\ \gamma_0\mathbf{v} = -\eta_v\mathbf{n} = -\eta\mathbf{n} \ \text{ on } \Gamma\times(0,T) \\ \gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta\mathbf{M} \quad \text{on} \quad \Gamma\times(0,T) \\ \delta\partial_t(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_0 p = s^*(\eta), \ \text{ on } \Gamma \end{aligned}\right\} \tag{12}$$

with

$$\left.\begin{aligned} \mathbf{S}(\mathbf{v}) = \rho^{-1/2}[\tfrac{\alpha}{2}\nabla\cdot[\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})]+ \\ \alpha\nabla\cdot(\nabla\mathbf{v}\mathbf{A}(\mathbf{v})) + \mu\Delta\mathbf{v}] \\ s^*(\eta) = \rho^{-1/2}\delta\mu\Delta_s\eta_v \end{aligned}\right\} \tag{13}$$

Note that we refer to $\gamma_o[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2K\eta_v\mathbf{n}$ as the kinematic boundary condition.

Under the substitution

$$\left.\begin{aligned} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} = \mathbf{u}_1 \ \text{ in } \ \Omega \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) = u_2 \ \text{ on } \ \Gamma \\ \mathbf{v} \in \mathbf{X}_{m+2}, \quad m \geq 1 \end{aligned}\right\} \tag{14}$$

the first and last equations of (12) become

$$\left.\begin{aligned} \partial_t\mathbf{u}_1 + (\mathbf{v}\cdot\nabla)\mathbf{u}_1 + \rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \ \text{ in } \ \Omega \\ \partial_t u_2 + \rho^{-1/2}\gamma_o p = s^*(\eta) \ \text{ on } \ \Gamma. \end{aligned}\right\} \tag{15}$$

5. The stokes problem ($\mathcal{SP}$)

The problem (14) with $\mathbf{u}_1$ and u_2 given, leads to the Stokes Problem. Since $\mathbf{v}$ has to be divergence-free, we shall write the system (14) as a Stokes-like system. To this end, we use the Helmholtz-decomposition [15] in the following form:

Theorem 1. *Let $\mathbf{u}_1 \in \mathbf{H}^k(\Omega)$ and $u_2 \in H^{k-1/2}(\Gamma)$; $k \geq 1$. Then there exists $q \in H^{k+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^k(\Omega)$, such that*

$$\left.\begin{array}{l} \mathbf{u}_1 = \rho^{1/2}\mathbf{w} + \rho^{-1/2}\nabla q \ \text{ in } \ \Omega \\ u_2 = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} + \rho^{-1/2}\gamma_o q \ \text{ on } \ \Gamma \\ \nabla\cdot\mathbf{w} = 0 \ \text{ in } \ \Omega. \end{array}\right\}$$

See [15] for proof of Theorem 1.

The operator $P : \langle \mathbf{u}_1, u_2\rangle \in \mathbf{L}^2(\Omega)\times L^2(\Gamma) \to \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w}\rangle \in \mathbf{L}^2(\Omega)\times L^2(\Gamma)$ is an orthogonal projection.

Remark 2. *Note that for $\mathbf{v} \in \mathbf{X}_m$; $m \geq 1$, $P\mathbf{C}_m\mathbf{v} = \mathbf{C}_m\mathbf{v}$, (see (8)).*

Indeed, we have the orthogonal decomposition

$$\begin{aligned} \mathbf{u} = \langle \mathbf{u}_1, u_2\rangle &= \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\gamma_o\mathbf{n}\cdot\mathbf{w}\rangle + \langle \rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o q\rangle \\ &= \mathbf{C}_o\mathbf{w} + \Pi q. \end{aligned}$$

From this result, we can rewrite (14) in the form

$$\left.\begin{array}{r} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{1/2}\mathbf{w} \ \text{ in } \Omega\times(0,T) \\ \nabla\cdot\mathbf{v} = 0 \ \text{ in } \Omega\times(0,T) \\ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \ \text{ on } \Gamma\times(0,T); \ \ \mathbf{v} = 0 \text{ on } \Gamma_1\times(0,T) \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_o p = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} \ \text{ on } \Gamma\times(0,T) \\ \gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} = -2\eta_v K\mathbf{n} \ \text{ on } \ \Gamma\times(0,T) \end{array}\right\} \tag{16}$$

(with $p = -q$), which is Stokes-like. We shall refer to (16) as the Stokes Problem ($\mathcal{SP}$).

Note that the kinematical boundary condition is incorporated in the $\mathcal{SP}$.

According to the definition of the operators $\mathbf{C}_m$, $\mathbf{B}_m$ and Π, (16) may also be written in the form

$$\mathbf{B}_m\mathbf{v} + \Pi p = \mathbf{C}_m\mathbf{w}, \quad \mathbf{v} \in \mathbf{X}_{m+2}, \ \ m \geq 1.$$

Led by the discussion above, we show the existence and uniqueness for the $\mathcal{SP}$:

Theorem 3. *Let Γ be of class C^∞, and suppose that $p_2 < 1/2$. Let $m \geq 0$ and let there be given $\mathbf{w} \in \mathbf{W}^{k,\infty}(I;\mathbf{H}^m(\Omega))$ and $\gamma_o\mathbf{w}\cdot\mathbf{n} \in W^{k,\infty}(I;H^{m+1/2}(\Gamma))$. Then the problem (16) has a solution for which $\mathbf{v}$ is unique and ∇p is unique. $\mathbf{v} \in \mathbf{W}^{k,\infty}(I;\mathbf{X}_{m+2})$, $\eta_v \in W^{k,\infty}(I;H^{m+3/2}(\Gamma))$ and $p \in H^{m+1}(\Omega)$.*

Proof. Consider the system (16) of equations

$$\left.\begin{aligned} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{1/2}\mathbf{w} \ \text{ in } \ \Omega\times(0,T)\\ \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_o p = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} \ \text{ on } \ \Gamma\times(0,T)\\ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \ \text{ on } \ \Gamma\times(0,T)\\ \gamma_o[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2\eta_v K\mathbf{n} \ \text{ on } \ \Gamma\times(0,T). \end{aligned}\right\} \tag{17}$$

We prove first that (17) has a unique weak solution: We use the projection P in [15] to eliminate p. If we formally take the $\mathbf{L}^2(\Omega)$-inner product of the first two equations of (17) with $\mathbf{C}_o\phi$, we obtain the weak formulation of the problem

$$b_1(\mathbf{v},\phi) = \rho(\mathbf{w},\phi) + \rho(\mathbf{n}\cdot\gamma_o\mathbf{w},\eta_\phi)_{L^2(\Gamma)}, \tag{18}$$

with b_1 defined in (9).

Since all the spaces $\mathbf{X}_m$ have a common dense subspace $\mathbf{V}(\Omega)$, we define a weak solution of (16) if $\mathbf{v}\in\mathbf{X}_1$ such that (18) holds for all $\phi\in\mathbf{X}_1$. The bilinear form b_1, being equivalent to the metric in $\mathbf{H}^1(\Omega)$, is clearly positive definite on $\mathbf{X}_1$, and therefore, a unique weak solution exists. □

Regularity results for the $\mathcal{SP}$ need to be investigated. A classical result is that the solution $\mathbf{u}\in\mathbf{H}^1_o(\Omega)$ for the Dirichlet problem $-\Delta\mathbf{u}+\mathbf{u}=\mathbf{f}$ belongs to $\mathbf{H}^{m+2}(\Omega)$ whenever $\mathbf{f}\in\mathbf{H}^m(\Omega)$(with Ω sufficiently smooth). The question is whether similar results exist for the $\mathcal{SP}$.

A priori estimates for the solution $\mathbf{v}$ of (17) is obtained in a proposition with the use of Theorem 10.5 of [1] in the following way:

Proposition 4. *Let Ω be an open bounded set in $\mathbf{R}^3$, with boundary $\Gamma\cup\Gamma_1$ of class C^{m+3}, m a non-negative integer. Suppose that $\mathbf{v}\in\mathbf{X}_2$ is a weak solution of the Stokes-like problem* (16)*. If $\mathbf{w}\in\mathbf{H}^m(\Omega)$ and $\gamma_o\mathbf{w}\cdot\mathbf{n}\in H^{m-1/2}(\Gamma)$, then $\mathbf{v}\in\mathbf{H}^{m+2}(\Omega)$, and there exists a $p\in H^{m+1}(\Omega)$ and a constant $C_o(\alpha,\rho,m,\Omega)$ such that*

$$\|\mathbf{v}\|_{m+2} + \|p\|_{m+1} \le C_o\{\|\mathbf{w}\|_m + \|\gamma_o\mathbf{w}\cdot\mathbf{n}\|_{H^{m-1/2}(\Gamma)}\}. \tag{19}$$

Proof. We write equation (16), after multiplying both sides with $\rho^{1/2}$, in the form

$$\begin{array}{cccc} (\Delta-\frac{\rho}{\alpha})v_1 & \cdot & \cdot & -\frac{1}{\alpha}p_{,1} = -\frac{\rho^{1/2}}{\alpha}(y_1)_1\\ \cdot & (\Delta-\frac{\rho}{\alpha})v_2 & \cdot & -\frac{1}{\alpha}p_{,2} = -\frac{\rho^{1/2}}{\alpha}(y_1)_2\\ \cdot & \cdot & (\Delta-\frac{\rho}{\alpha})v_3 & -\frac{1}{\alpha}p_{,3} = -\frac{\rho^{1/2}}{\alpha}(y_1)_3\\ v_{1,1} & +v_{2,2} & +v_{3,3} & = 0 \end{array}$$

Let $v_4 = -\frac{\rho}{\alpha}$ and $\mathbf{f} = (-\frac{\rho^{1/2}}{\alpha}(y_1)_1, -\frac{\rho^{1/2}}{\alpha}(y_1)_2, -\frac{\rho^{1/2}}{\alpha}(y_1)_3, 0)$, to become

$$\sum_{j=1}^{4}\ell_{ij}(\partial)v_j(\mathbf{x}) = f_i(\mathbf{x}) \ \text{ in } \ \Omega, \ \ i=1,2,3,4$$

where $\partial = (\partial_1, \partial_2, \partial_3)$ and the matrix $[\ell_{ij}(\xi)], \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, is given by

$$\begin{aligned} \ell_{ij}(\xi) &= |\xi|^2\delta_{ij} - \frac{\rho}{\alpha}, \quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad i,j = 1,2,3, \\ \ell_{4j}(\xi) &= -\ell_{j4}(\xi) = \xi_j, \quad j = 1,2,3, \\ \ell_{4,4}(\xi) &= 0. \end{aligned}$$

In accordance with the proof of Proposition 2.2 [16, p.34], we define two systems of weights by $s_1 = s_2 = s_3 = 0$, $s_4 = -1$, and $t_1 = t_2 = t_3 = 2$, $t_4 = 1$. Then $s_i \leq 0$ and degree$(\ell_{ij}(\xi)) \leq s_i + t_j$, as required by [1, p.38]. The matrix $[\ell'_{ij}(\xi)]$, where $\ell'_{ij}(\xi)$ consists of the terms in $\ell_{ij}(\xi)$ that are of order $s_i + t_j$ in ξ, is identical to the corresponding matrix in [16]:

$$[\ell'_{ij}(\xi)] = \begin{bmatrix} |\xi|^2 & 0 & 0 & -\xi_1 \\ 0 & |\xi|^2 & 0 & -\xi_2 \\ 0 & 0 & |\xi|^2 & -\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{bmatrix}$$

It is easily shown that $\mathcal{L}(\xi) \equiv \det[\ell'_{ij}(\xi)] = |\xi|^6$, so that $\mathcal{L}(\xi) \neq 0$ for non-zero real ξ, that is (16) is *elliptic*. Moreover, the *supplementary condition on* $\mathcal{L}$ is satisfied: $\mathcal{L}(\xi)$ is of even degree 6, and for every pair of linearly independent real vectors ξ, ξ', in particular for each point $\mathbf{x}$ on Γ, ξ is a tangent and ξ' is a normal at $\mathbf{x}$; the polynomial $\mathcal{L}(\xi + \tau\xi')$ in τ has exactly 3 roots with positive imaginary part, namely $\tau^+(\xi, \xi') = i|\xi|/|\xi'|$:

$$\begin{aligned} \mathcal{L}(\xi + \tau\xi') &= [(\xi + \tau\xi') \cdot (\xi + \tau\xi'^3 \\ &= (|(\xi|^2 + |\xi'^2\tau^2))^3 \\ &= |\xi|^6(\tau - i|\xi|/|\xi'^6(\tau + i|\xi|/|(\xi'^6. \end{aligned}$$

Now, concerning the boundary conditions, we set

$$\begin{aligned} \gamma_o v_{1,3} + \gamma_o v_{3,1} &= 0 \\ \gamma_o v_{2,3} + \gamma_o v_{3,2} &= 0 \\ (\frac{\rho}{\alpha} - \Delta_s)v_3 + \gamma_o p &= y_2. \end{aligned}$$

These boundary conditions can be expressed as

$$[\mathcal{B}_{hj}(\mathbf{x}, \xi)] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 \\ 0 & 0 & \frac{\rho}{\alpha} - \xi_3^2 & 1 \end{bmatrix}$$

in other words

$$\begin{aligned}
\mathcal{B}_{hj} &= \xi_3\delta_{hj} \quad \text{for} \quad h=1,2,3 \quad \text{and} \quad j=1,2\\
\mathcal{B}_{h3} &= \xi_h, \quad \text{for} \quad h=1,2,\\
\mathcal{B}_{h4} &= 0, \quad \text{for} \quad h=1,2\\
\mathcal{B}_{34} &= 1\\
\mathcal{B}_{33} &= \frac{\rho}{\alpha}-\xi_3^2.
\end{aligned}$$

Take $r_1=r_2=-1, r_3=0$ and $t_1=t_2=t_3=2$, then $\text{degree}(\mathcal{B}_{hj}) \le r_h+t_j$ and $[\mathcal{B}'_{hj}]=[\mathcal{B}_{hj}]$, where $\mathcal{B}'_{hj}(\mathbf{x},\xi)$ consists of the terms in $\mathcal{B}_{hj}(\mathbf{x},\xi)$ that are of order r_h+t_j in ξ.

$$[\mathcal{B}'_{hj}(\mathbf{x},\xi)] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0\\ 0 & \xi_3 & \xi_2 & 0\\ 0 & 0 & \xi_3^2 & 0 \end{bmatrix}$$

Now it remains to check the *complementing boundary condition*: For an arbitrary $\mathbf{x}\in\Gamma$, let $\mathbf{n}$ denote the outward unit normal vector at $\mathbf{x}$, let ξ be any non-zero real tangent vector to Γ at $\mathbf{x}$ and define $\mathcal{L}^{jk}(\cdot)\equiv \ell'_{jk}(\cdot)$, $j,k=1,2,3,4$. Then$[\mathcal{B}'_{hj}(\mathbf{x},\xi+\tau\mathbf{n})\mathcal{L}^{jk}(\xi+\tau\mathbf{n})] =$

$$\begin{bmatrix} \tau[|\xi|^2+\tau^2] & 0 & \xi_1[|\xi|^2+\tau^2] & -2\xi_1\tau\\ 0 & \tau[|\xi|^2+\tau^2] & \xi_2[|\xi|^2+\tau^2] & -2\xi_2\tau\\ 0 & 0 & \tau^2[|\xi|^2+\tau^2] & -\tau^3 \end{bmatrix}.$$

The rows of the latter matrix are required to be linearly independent modulo $\mathcal{M}^+$. Let $\tau^+=\tau^+(\xi,\mathbf{n})=i|\xi|$ and set $\mathcal{M}^+=(\tau-\tau^+)^3$, and suppose that $\mathcal{C}=(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ is a constant vector with the property that, as polynomials in τ,

$$\sum_{h=1}^{3}\mathcal{C}_h\Big(\sum_{j=1}^{4}\mathcal{B}'_{hj}\mathcal{L}^{jk}\Big)\equiv 0 \pmod{\mathcal{M}^+}, \quad k=1,2,3,$$

that is

$$\begin{aligned}
\mathcal{C}_1\tau(|\xi|^2+\tau^2) &= 0\\
\mathcal{C}_2\tau(|\xi|^2+\tau^2) &= 0\\
\mathcal{C}_1\xi_1(|\xi|^2+\tau^2)+\mathcal{C}_2\xi_2(|\xi|^2+\tau^2)+\mathcal{C}_3\tau^2(|\xi|^2+\tau^2) &= 0.
\end{aligned}$$

Now it is easy to verify that $\mathcal{C}=\mathbf{0}$ and that the *complementing condition* holds.

We then apply Theorem 10.5, on [1, p.78] in order to get the final result. (For a similar application, see [7].) □

Proposition 5. *Let $\Gamma\cup\Gamma_1$ be of class C^∞. Let $k\ge 0$ and let $\mathbf{w}\in\mathbf{W}^{k,\infty}(I;\mathbf{H}^m(\Omega))$ and $\gamma_o\mathbf{w}\cdot\mathbf{n}\in W^{k,\infty}(I;H^{m-1/2}(\Gamma))$ be given. Then there exists a unique vector field $\mathbf{v}\in\mathbf{W}^{k,\infty}(I;\mathbf{X}_{m+2})$ satisfying (17). Moreover, there is a constant $C_1=C_{1(\Omega,m,\rho,\alpha)}$ such that*

$$\|\mathbf{v}\|_{k,m+2,T}\le C_1\{\|\mathbf{w}\|_{k,m,T}+\|\gamma_o\mathbf{w}\cdot\mathbf{n}\|_{W^{k,\infty}(I;H^{m-1/2}(\Gamma))}\}.$$

Proof. By Proposition 4, it is only necessary to prove that $\mathbf{v} \in \mathbf{X}_2$. This was proven for the general $\mathbf{L}^p$ case by Miyakawa in [12].

This concludes the proof of Theorem 3. □

Let the mapping $f : \mathbf{Y}_m \to \mathbf{X}_{m+2}$ be defined as

$$f : \mathbf{u} \to \mathbf{w} \to \mathbf{v}^* = f(\mathbf{u}),$$

where $\mathbf{v}^* \in \mathbf{X}_{m+2}$ is the unique solution of (16) for a given $\mathbf{u} \in \mathbf{Y}_m$.

Theorem 6. *f is a bounded linear operator.*

Proof. The result follows from (19) and the fact that $\mathbf{w}$ is obtained from $\mathbf{u}$ by the orthogonal projection P. □

6. Conclusion

We modelled the flow of a second grade fluid in the region Ω which is emerged in fluid of the same kind. The interface of Ω is permeable and fluid flow is modelled through the boundary in the direction of the outer normal. The ultimate goal would be to prove the existence of a weak and classical solution. In this article, the $\mathcal{TP}$ was modelled in terms of the $\mathcal{SP}$, for which the existence of a unique solution was proved. This article lays the foundation for a existence proof.

The existence of a weak and/or classical solution of the $\mathcal{TP}$ can be proven and should roughly be based on the following algorithm:

1. With $\mathbf{v}$ given, at least one solution $\mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle$ of (15) under the initial condition $\mathbf{u}(x, 0) := \mathbf{u}_0(x)$ must be found.

2. With each $\mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle$ in hand, a unique solution $\mathbf{v}^*$ of (14) must be found.

3. It must be proved that the composite mapping

$$\mathbf{\Phi} : \mathbf{v} \xrightarrow{g} \mathbf{u} \xrightarrow{f} \mathbf{v}^*$$

has a fixed point, which is a solution of (12).

The rough algorithm described above, should be developed in detail, after some refinement of the equations (14) and (15).

References

[1] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Commun. Pure Appl. Math., 17 (1964), 3592.

[2] Helge I. Andersson, *Slip flow past a stretching surface*, Acta Mech., 158 (2002), 121125.

[3] P. D. Ariel, T. Hayat, S. Asghar, *The flow of an elastico-viscous fluid past a stretching sheet with partial slip*, Acta Mech., 187 (2006), 2935.

[4] Anindita Bhattacharyya and G. P. Raja Sekhar, *Stokes flow inside a porous spherical shell: Stress jump boundary condition*, Zeitschrift fur Angew. Math. und Phys., 56 (2005), 475496.

[5] D. Cioranescu and V. Girault, *Solutions variationnelles et classiques d'une famille de fluides de grade deux*, C.R. Acad. Sci. Paris, 322 (1996), 11631168.

[6] G.P. Galdi, M. Grobbelaar-Van Dalsen, N. Sauer, *Existence and Uniqueness of Classical Solutions of the Equations of a Second-Grade Fluid*, Arch. Rat. Mech. Anal., 124 (1992), 221235.

[7] Christian Le Roux, *Second Grade Fluids with Slip Boundary Conditions*, PhD. thesis, University of Pretoria, 1997.

[8] O.D. Makinde and E. Osalusi, *MHD steady flow in a channel with slip at the permeable boundaries*, Rom. Journ. Phys., 51 (2006), 319328.

[9] Chi Sing Man, *Nonsteady channel flow of ice as a modified second-order fluid with power-law viscosity*, Arch. Ration. Mech. Anal., 119 (1992), 3557.

[10] Riette Maritz and Emile Franc Doungmo Goufo, *Newtonian and Non-Newtonian Fluids through Permeable Boundaries*, Math. Probl. Eng., 2014, 114.

[11] Riette Maritz and Niko Sauer, *On Boundary Permeation in Navier Stokes and Second Grade Incompressible Fluids*, Math. Model. Methods Appl. Sci., 16 (2006), 5975.

[12] Tetsuro Miyakawa, *The L^p approach to the Navier-Stokes equations with the Neumann boundary condition*, 10 (1980), 517537.

[13] A.P. Oskolkov, *Solvability in the large of the first boundary-value problem for a quasilinear third-order system pertaining to the motion of a viscous fluid*, V.A. Steklova Akad. Nauk SSSR, 27 (1972), 145160.

[14] Niko Sauer, *On Helmholtz Projections*, In R. Salvi, editor, Navier-Stokes equations Theory Numer. methods, 257-263, Pitman-Longman, London, 1998. Research Notes in Maths Series 338.

[15] Niko Sauer, Alna Van, and Der Merwe, *Eigenvalue problems with the spectral parameter also in the boundary condition*, Quaest. Math., 5 (1982), 127.

[16] Roger Temam, *Navier-Stokes Equations*, North-Holland Publishing Company, Amsterdam, 2nd edition, 1977.

[17] C.Y. Wang, *Flow due to a stretching boundary with partial slip an exact solution of the Navier Stokes equations,* 2002.

[18] C. E. Weatherburn, *Differential Geometry of three dimensions*, Vol II. Cambridge University Press, Cambridge, 1930.

[19] C.E. Weatherburn, *Differential Geometry of three dimensions*, Vol I. Cambridge University Press, Cambridge, 1931.

Accepted: 15.11.2016

SOLITON SOLUTIONS FOR NON-LINEAR DISPERSIVE WAVE EQUATIONS WITH VARIABLE-COEFFICIENTS

Ibraheem Abu-Falahah
Department of Mathematics
Hashemite University
Zarqa 13115
Jordan
iabufalahah@hu.edu.jo

Abstract. In this paper, we study the solitary wave solution for the variable-coefficient non-linear dispersive wave equation. We develop a simplified bilinear method to construct the multi-soliton solutions for such an equation. We prove that the proposed scheme is effective and easy to implement. Finally, effects of the inhomogeneities of media on the soliton behavior are discussed with the aid of the characteristic curve method and graphical analysis.
Keywords: dispersive wave equations, multiple soliton solution, hirota bilinear method.

Introduction

Nowadays, attention has been focused on soliton equations due to its scientific applications. There has been growing interest in studying the variable-coefficient nonlinear evolution equations (NLEEs), which are often considered to be more realistic than their constant-coefficient counterparts in modeling a variety of complex nonlinear phenomena under different physical backgrounds. Since those variable-coefficient NLEEs are of practical importance, it is meaningful to construct various exact analytic solutions, including the soliton solutions and study the integrable properties. However, the inverse scattering transform [1], the Hirota direct method [6, 7], Bäcklund transformation method [12], and the Darboux transformations method [10] are applied for the soliton solutions to NLEEs.

For describing the propagation of solitonic waves in inhomogeneous media, the variable-coefficient non-linear dispersive wave equation

$$u_t + \alpha(t)u_x + \beta(t)uu_x + \gamma(t)u_{xxx} = 0 \tag{0.1}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are time-dependent functions, has been derived from many physical applications in plasma physics, fluid dynamics and other fields [9, 11, 13].

Obviously, equation (0.1) contains quite a number of variable-coefficient models arising from various branches of physics. A currently important example of which can be seen in [3]. The forced variable-coefficient KdV equation,

which has the relevant physical applications not only in the internal solitary waves dynamics [3], but also in the arterial mechanics [11, 13]. For instance, in [11], the forced variable-coefficient KdV equation has been derived by means of the reductive perturbation as studying the propagation of the weakly nonlinear waves in a composite medium, in which the arteries and blood are treated as the prestressed thin walled elastic tubes with a stenosis and the Newtonian fluid, respectively.

In this paper, by virtue of a newly developed simplified bilinear method, we would like to study equation (0.1) and devote ourselves to its soliton solutions and some important properties under nonlinearly dependent constraints among the variable coefficients.

The structure of the present paper is as follows. In section 1, with certain coefficient conditions, we will construct the analytic one- and two-soliton solutions for equation (0.1) by employing the variable-coefficient simplified bilinear method [2, 8] and Hirota method [6, 7]. Moreover, new singular soliton solutions will be constructed. Section 2 will analyze the effects of the coefficient functions on the stabilities and propagation characteristics of the solitonic waves through figure drawing. Section 3 will offer the conclusion.

1. Solitary wave solutions for equation (0.1)

It is well known that the Hirota method is an important analytic tool for dealing with NLEEs and relevant soliton problems [7]. Through bilinearization for a given NLEE, one can not only construct its multi-soliton solutions, but also derive the bilinear auto-Bäcklund transformation, nonlinear superposition formula, Lax pair and some other remarkable properties.

In this method, a given nonlinear equation is rewritten into the bilinear equation by the dependent variable transformation to enable us deriving the auxiliary function. But it is not easy for us to find such a transformation for many equations. However, Hereman et al. [4, 5], introduced a simplified algorithm to derive the auxiliary functions without using the bilinear forms; instead it assumes that the N-soliton solutions can be expressed as polynomials of exponential functions. The authors in [2] modified Hereman's simplified method to be able to solve variable-coefficient NLEEs. In the rest of this section, we will employ the method in [2, 8] to obtain multiple soliton solutions for equation (0.1).

1.1 Analytic one-soliton solution

Taking as our ansatz the plane wave solution

$$u(x,t) = e^{k_i x - \omega_i(t)}. \tag{1.1}$$

Substituting (1.1) into the linear terms of (0.1), we get the following non-linear dispersion relation

$$\omega_i(t) = \phi_i(t) = k_i^2 \int \gamma(t)dt. \tag{1.2}$$

and as a result we obtain the following phase variables

$$\theta_i = k_i x - \int \left[k_i^3\gamma(t) + k_i\alpha(t)\right] dt, \qquad i = 1, 2, \ldots. \tag{1.3}$$

Our practice with the perturbation method suggests that making the substitution

$$u(x,t) = R\,(\ln f)_{xx} \tag{1.4}$$

in (0.1) to obtain a bilinear form may be a fruitful undertaking.

To find the one-soliton solution, take

$$f(x,t) = 1 + \varepsilon e^{k_1 x - \omega_1(t)}, \tag{1.5}$$

where $\varepsilon = \pm 1$. Substituting (1.4) and (**??**) into equation (0.1) and solving for R we find

$$R = 12\frac{\gamma(t)}{\beta(t)}$$

with the constraint on coefficients

$$\gamma(t) = \lambda\beta(t).$$

Here $\lambda \neq 0$ is an arbitrary constant. That is to say, the one-soliton solution of equation (0.1) under condition (1.5) is derived as

$$u(x,t) = 12\lambda\varepsilon k_1^2 \frac{e^{\theta_1}}{(1+\varepsilon e^{\theta_1})^2}. \tag{1.6}$$

Upon using $\varepsilon = 1$ we obtain the one-soliton solution

$$u(x,t) = 3\lambda k_1^2 \sec h^2\left(\frac{\theta_1}{2}\right),$$

which can be seen as the bell-shaped profile, while $\varepsilon = -1$ leads to the singular one-soliton solution

$$u(x,t) = 3\lambda k_1^2 \csc h^2\left(\frac{\theta_1}{2}\right),$$

which gives a profile of an explosive pulse.

1.2 Multiple soliton solutions

To generate the two-soliton solution, we expand the function f as

$$f(x,t) = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1+\theta_2}, \tag{1.7}$$

where $\varepsilon = \pm 1$ and θ_i are given by (1.4).

Upon substituting (1.7) and (1.4), into equation (0.1) and solving for the phase shift we obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

and hence we can generalize for other phase shifts by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \qquad 1 \le i < j \le 3. \tag{1.8}$$

In this case, the two-soliton solution of equation (0.1) subject to condition (1.5) and using $\varepsilon_1 = \epsilon_2 = 1$ is found to be

$$u(x,t) = 12\gamma \frac{F(\theta_1, \theta_2)}{G(\theta_1, \theta_2)} \tag{1.9}$$

where

$$\begin{aligned} F(\theta_1, \theta_2) &= k_1^2 e^{\theta_1} \left(1 + a_{12} e^{2\theta_2}\right) + k_2^2 e^{\theta_2} \left(1 + a_{12} e^{2\theta_1}\right) \\ &\quad + \left[(k_1 - k_2)^2 + a_{12} (k_1 + k_2)^2\right] e^{\theta_1+\theta_2}, \\ G(\theta_1, \theta_2) &= \left(1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}\right)^2, \end{aligned}$$

where θ_1 and θ_2 are defined in (1.3). It is noted that solution (1.9) represents the interaction of two solitonic waves. On the other hand, using $\varepsilon_1 = \epsilon_2 = -1$ in (1.7), gives the singular two-soliton solution

$$u(x,t) = -12\gamma \frac{H(\theta_1, \theta_2)}{I(\theta_1, \theta_2)},$$

where

$$\begin{aligned} H(\theta_1, \theta_2) &= k_1^2 e^{\theta_1} \left(1 + a_{12} e^{2\theta_2}\right) + k_2^2 e^{\theta_2} \left(1 + a_{12} e^{2\theta_1}\right) \\ &\quad - \left[(k_1 - k_2)^2 + a_{12} (k_1 + k_2)^2\right] e^{\theta_1+\theta_2} \end{aligned}$$

and

$$I(\theta_1, \theta_2) = \left(1 - e^{\theta_1} - e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}\right)^2.$$

For the three kink solutions, we set the auxiliary function

$$f(x,t) = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1\varepsilon_2 a_{12} e^{\theta_1+\theta_2} + \varepsilon_1\varepsilon_3 a_{13} e^{\theta_1+\theta_3} + \varepsilon_2\varepsilon_3 a_{23} e^{\theta_2+\theta_3} + \varepsilon_1\varepsilon_2\varepsilon_3 a_{123} e^{\theta_1+\theta_2+\theta_3}, \tag{1.10}$$

where $\varepsilon_i = \pm 1$, $i = 1, 2, 3$, and θ_i are given by (1.3). Substituting (1.10) and (1.4) into equation (0.1), we find that

$$a_{123} = a_{12}a_{13}a_{23}.$$

This shows that equation (0.1) has N-soliton solution which can be obtained for finite N, where $N > 1$. Proceeding as before, the three-soliton solutions and the singular three-soliton solutions can be obtained using calculations with Mathematica.

1.3 Propagation characteristics of the soliton solutions

From the one-soliton solution (1.6), it is obvious that the amplitude of the front is $3\lambda k_i^2$ which keeps invariant during the propagation. Following the characteristic-line method [14], the characteristic curve for each solitary solution can be defined by

$$x - \int \left[k_i^2\gamma(t) + \alpha(t)\right] dt = 0, \qquad i = 1, 2, \ldots,$$

which can be derived from relations (1.2) and (1.3). Correspondingly, the velocity of the front at time t can be expressed as

$$\nu_x = k_i^2\gamma(t) + \alpha(t). \tag{1.11}$$

From the explicit expression (1.11), we can see that the functions $\alpha(t)$ and $\gamma(t)$ affect the propagation velocity of the solitonic fronts significantly. For example, taking $\gamma(t) = \sin t$ and $\alpha(t) = 0$, solution (1.6) becomes

$$u(x,t) = 6\sec h^2\left(\frac{x + \cos t}{2}\right), \tag{1.12}$$

which has the non-zero boundary value 0 as $t,\ x \to \infty$. Its velocity along the x direction is given by

$$\nu_x = \sin t$$

and during one periodic time from 0 to 2π, the solitonic wave moves along the positive x direction when $t \in (0, \pi)$ and along the negative x direction when $t \in (\pi, 2\pi)$. Moreover, the wave accelerates in the intervals $(0, \pi/2) \cup (3\pi/2, 2\pi)$ while decelerates in the interval $(\pi/2, 3\pi/2)$.

The characteristic curve in Fig. 1 can be expressed by

$$x + \cos t = 0.$$

The profile figure for solution (1.6) in an inhomogeneous medium in the $x-t$ plane with: $\alpha(t) = 0$, $\gamma(t) = sint$, $k_1 = 1$. As a result, the front reveals the snake-like type propagation trajectory with the unalterable amplitude but continuously changeable velocity.

In Fig 2, the characteristic curve is given in the form

$$x + 2\cos t - \frac{t^3}{3} = 0.$$

When t approaches zero, i.e. $\cos t >> t^3$, the trajectory is snake-like type with periodic oscillation. Otherwise, when t is far from the origin, the trajectory is t^3-like type one. From which we conclude that, besides the periodic oscillation of the solitons in the local region, the large scale propagation trajectories for such a structure are the t^3-typed curves. Likewise, if the variable coefficients are taken as the other forms, the corresponding characteristic curves will present different characters.

The profile figure for solution (1.6) in an inhomogeneous medium in the $x-t$ plane with: $\alpha(t) = t^2$, $\gamma(t) = sint$, $\lambda = 2$, $k_1 = 1$.

2. Conclusions

In the present work, we have studied the variable-coefficient non-linear dispersive wave equation (0.1) from the soliton analysis point of view. Of physical interest, we have specially presented the multiple soliton solutions and multiple singular soliton solutions for equation (0.1). Through the figures for some sample solutions, we have discussed the structures of solitonic waves with inhomogeneities of media and nonuniformities of boundaries effects.

Further work for this class of equations is possible to be performed, for example, one can extend the present results to the corresponding three-dimensional variable-coefficient evolution equations.

References

[1] M.J. Ablowitz and H. Segur, Soliton and the inverse scattering transform, SIAM, Philadelphia, 1981.

[2] F. Awawdeh, H.M. Jaradat and S. Al-Sharáh , Applications of a simplified bilinear method to ion-acoustic solitary waves in plasma, Eur. Phys. J. D., vol. 66, (2012.

[3] R.H. Grimshaw, E. Pelinovsky and T. Talipova, Modelling internal solitary waves in the coastal ocean. Surv. Geophys., vol. 28, 2007.

[4] W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, Math. Comput. Simul., vol. 43, pp. 13-27, 1997.

[5] W. Hereman and W. Zhuang, Symbolic computation of solitons with Macsyma, Comput. Appl. Math. II: Differen. Equat., pp. 287-296, 1992.

[6] R. Hirota, Direct methods in soliton theory, in: R.K. Bullough, P.J. Caudrey (Eds.), Solitons, Springer, Berlin, 1980.

[7] R. Hirota, The direct method in soliton theory, Cambridge University Press, Cambridge, 2004.

[8] H.M. Jaradat, S. Al-Shara, F. Awawdeh and M. Alquran, Variable coefficient equations of the Kadomtsev-Petviashvili hierarchy: multiple soliton solutions and singular multiple soliton solutions, Phys. Scr., vol. 85, 2012.

[9] Y. Liu, Y.T. Gao, Z.Y. Sun, and X. Yu, Multi-soliton solutions of the forced variable-coefficient extended Korteweg-de Vries equation arisen in fluid dynamics of internal solitary waves, Nonlinear Dyn., vol. 66, pp. 575-587, 2011.

[10] V.B. Matveev and M.A. Salle, Darboux transformations and solitons. Berlin, Heidelberg: Springer-Verlag; 1991.

[11] J.W. Miles, On internal solitary waves II. Tellus, vol. 33, pp. 397, 1981.

[12] C. Rogers and W.K. Schief, Bäcklund and Darboux transformations, Cambridge University Press, 2002.

[13] B. Tian, G.M. Wei, C.Y. Zhang, W.R. Shan and Y.T.Gao, Transformations for a generalized variable-coefficient Korteweg-de Vries model from blood vessels, Bose-Einstein condensates, rods and positons with symbolic computation, Phys. Lett. A., vol. 356, pp. 8, 2006.

[14] A. Veksler and Y. Zarmi, Wave interactions and the analysis of the perturbed Burgers equation, Physica D, vol. 211, pp. 57-73, 2005.

Accepted: 22.11.2016

A DISSIPATIVE HYPERBOLIC SYSTEMS APPROACH TO IMAGE RESTORATION

Junling Sun
School of Information Engineering
WuHan University of Technology
Wuhan, 430070
People's Republic of China

Jie Yang*
School of Information Engineering
WuHan University of Technology
Wuhan, 430070
People's Republic of China
yangjie196002@163.com

Lei Sun
School of Mathematics and Information Science
Henan University of Polytechnic
Jiaozuo, Henan 454003
People's Republic of China

Abstract. We present here a new dissipative hyperbolic systems to image restoration. The existence of global dissipative solutions of this system under the Dirichlet boundary conditions and initial condition is shown. To this end, an experimental results are provided to show the efficiency of this kind of model.
Keywords: image restoration, wave equation, dissipative solution.

1. Introduction

In this paper, we consider the following dissipative hyperbolic systems

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \operatorname{div}(g(|\nabla G_\rho * u|)\nabla u) = 0, \tag{1.1}$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) - (1-\lambda)(|\nabla u| - v) = 0, \tag{1.2}$$

subject to the initial condition and Dirichlet boundary conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \frac{\partial v}{\partial t}(x,0) = 0 \quad x \in \Omega \tag{1.3}$$

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial v}{\partial n}|_{\partial\Omega} = 0 \quad 0 < t < T, \tag{1.4}$$

∗. Corresponding author

where Ω is a bounded domain of $\mathbf{R}^n$ with appropriately smooth boundary, n is the unit outer normal to Ω, $T > 0$ and $\lambda > 0$. The nonlinear term $g(s)$,

$$g(s) = \frac{1}{1 + (\frac{s}{K})^2}, \quad or \quad g(s) = |s|^{-1} \tag{1.5}$$

with $K > 0$.

Here $u = u(t, x)$ denotes the original image describing a real scene, $u_0(x)$ denotes the observed image, $G_\rho(x)$ is the Gaussian kernel

$$G_\rho(x) = \frac{1}{(4\pi\rho)^{\frac{M}{2}}} e^{\frac{-|x|^2}{4\rho}},$$

and

$$|\nabla G_\rho * u| = [\sum_{i=1}^{M} (\frac{\partial G_\rho}{\partial x_i} * u)^2]^{\frac{1}{2}}.$$

Partial differential equations based image restoration is a powful method to deal with the trade-off between noise removal and edge preservation. This method is now a well researched area within the image processing community. One of powerful models is the following nonlinear parabolic model with variable coefficient

$$\frac{\partial u}{\partial t} - \mathrm{div}(c(x, y)\nabla u) = 0,$$

where the degree of denoising and preservation of singularities can be determined by changing $c(x, y)$. There exist other type parabolic equation, such as anisotropic diffusion models [1, 18], complex diffusion models [13], fourth order equation models [7, 20], total variation models [9, 14, 24]. In Perona-Malik [18] the denoising capabilities of the linear diffusion can be better, let $c(x, y) = g(|\nabla u|)$ and initial data $u(0) = u_0$. Here the diffusion smooth function $g : [0, \infty) \longrightarrow [0, \infty)$ is important in controlling the smoothing and even enhancement of edges. They mainly considered the following two diffusion functions

$$g(s) = \frac{1}{1 + (\frac{s}{K})^2}, \quad \text{or} \quad g(s) = e^{-(\frac{s}{K})^2} \text{ with } K > 0.$$

Catte et al. [3] first introduced a new modification and proved its wellposedness to make the gradient computation robust outliers and provide a smooth edge map for the diffusion operator. This makes the Perona-Malik type PDE better.

$$\frac{\partial u}{\partial t} - \mathrm{div}(g(|G_\sigma \star \nabla u|))\nabla u) = 0,$$

where

$$G_\sigma(x) = (2\pi\sigma)^{-1} e^{-\frac{|x|^2}{2\sigma}}$$

is the Gaussian kernel and the operation means convolution.

A coupled parabolic equations was introduced to create better edge maps (see [5, 6]), which has the following form:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) = 0,\\ &\frac{\partial v}{\partial t} - \triangle v = 0.\end{aligned}$$

In order to localizing denoising effects in the diffusion process based scheme, Nitzberg and Shiota [17] introduced the following relaxation model:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) = 0,\\ &\frac{\partial v}{\partial t} - \lambda G_\sigma \star |\nabla u|^2 - \lambda v = 0,\end{aligned}$$

where $\lambda > 0$ is the relaxation parameter.

Surya Prasath and Vorotnikov [19] improved the above model and provided some new modifications. One of them has caught our attention, as follows:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) = 0,\\ &\frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) - (1-\lambda)(|\nabla u| - v) = 0,\end{aligned}$$

where $g(s) = \frac{1}{1+(\frac{s}{K})^2}$ (Perona-Malik type diffusion function) or $g(s) = |s|^{-1}$ (total variation diffusion function). $0 \leq \lambda \leq 1$ is a balancing parameter. The first equation is usually used in the Perona-Malik type PDEs. In their discussion, above model is in favor of preservation of edges. However, when the noise is very large, the preservation of edges will be unstable, which is similar to that of the Perona-Malik model.

Another of powerful models is the following nonlinear hyperbolic model with variable coefficient

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(c(x,y)\nabla u) = 0,$$

where the degree of denoising and preservation of singularities can be determined by changing $c(x, y)$. Ratner and Zeevi [21] introduced a new telegraph-diffusion model

$$\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} - \operatorname{div}(c(x,y)\nabla u) = 0$$

to discribe the contraction and fluctuation of the image create denoising and edge preserving effect. This model is based on viewing the image as an elastic sheet.

Inspired by the work of [19, 21], we consider a coupled hyperbolic systems (1.1)-(1.2) as a method based on viewing the image as an elastic sheet to improve the quality of the detected edges. As we known, most of these schemes use the absolute value of the gradient image as a guiding road map in the diffusion process to restore noisy images. One can see [2, 3, 4, 8, 12, 16, 23] for more details.

This paper is organized as follows. In section 2 we study the existence and uniqueness of solutions of the problem (1.1)-(1.5). In section 3 we give some numerical experiments.

2. Existence of dissipative solutions and weak solutions

In this section, we establish the existence, uniqueness, regularity of dissipative solutions to the problem (1.1)-(1.5) following the arguments in [19]. Recall that the standard notations used throughout the paper. Let $L^p(\Omega)$, $W_p^m(\Omega)$ and $H^m(\Omega)$ for the Lebesgue and Sobolev spaces. We will keep the function space symbol and omit Ω. The Euclidean norm in finite-dimensional spaces is denoted by $|\cdot|$. The symbol $\|\cdot\|$ will stand for the Euclidean norm in $L^2(\Omega)$. The corresponding scalar products is denoted by a $\cdot$ and parentheses $(\cdot,\cdot)$. Let $H_0^1(\Omega)$ be the closure of the smooth set, and it is compactly supported in Ω . By virtue of Friedrichs inequality, the Euclidean norm $\|\cdot\|_1$ corresponding to the scalar product $(u,v)_1 = (\nabla u, \nabla v)$ is a norm in H_0^1. Then there are standard Sobolev inequality

The usual Sobolev inequality

$$\|u\|_{L_\infty} \leq C(\Omega)\|u\|_2, \quad \forall u \in V_2,$$

and the Ladyzhenskaya inequality

$$\|u^2\| \leq \sqrt{2}\|u\|\|\nabla u\|, \quad \forall u \in H_0^1.$$

Let V_r be the closure of V_2 in W_r^1 with $1 < r < 2$, where $V_2 = H_0^1(\Omega) \cap H^2(\Omega)$ is a Hilbert space with the scalar product

$$(u,v)_2 = (u,v)_1 + \sum_{|\alpha|=2} (D^\alpha u, D^\alpha v).$$

We will consider our problem in the following space

$$W_1 = W_1(\Omega, T) = \{u \in L_2(0,T;V_2),\ u' \in L_2(0,T;V_2^*)\}$$

with the norm

$$\|u\|_{W_1} = \|u\|_{L_2(0,T;V_2)} + \|u'\|_{L_2(0,T;V_2^*)},$$

and

$$W_2 = W_2(\Omega, T) = \{u \in L_2(0,T;H_0^1),\ u' \in L_2(0,T;H^{-1})\}$$

with the norm

$$\|u\|_{W_2} = \|u\|_{L_2(0,T;H_0^1)} + \|u'\|_{L_2(0,T;H^{-1})}.$$

We also need the following class of pairs of functions

$$\begin{aligned}\mathbb{R} \;=\; & L_{4,loc}(0,\infty;V_2)\cap L_\infty(0,\infty;W_\infty^1)\cap W_{4,loc}^1(0,\infty;L_2)\times L_{2,loc}(0,\infty;V_2)\\ & \cap L_\infty(0,\infty;L_\infty)\cap W_{2,loc}^1(0,\infty;L_2).\end{aligned}$$

Define

$$\begin{aligned}
&E_1(u,v,\nu) = -\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} - \nu\,\mathrm{div}(g(|\nabla G_\rho * u|)\nabla u),\\
&E_2(u,v,\nu) = -\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t} + \lambda\,\mathrm{div}(\nabla v) + \nu(1-\lambda)(|\nabla u| - v) + (1-\delta)(\nabla v, \nabla\lambda),\\
&E_1(u,v) = E_1(u,v,1),\\
&E_2(u,v) = E_2(u,v,1),
\end{aligned}$$

where ν is a positive constant and $\forall (u,v)\in\mathbb{R}$.

Definition 2.1. *A pair of functions $(u,v)\in C_w([0,\infty);L_2)$ is called a dissipative solutions of the problem (1.1)-(1.2), if $\forall$ test functions $(\psi,\phi)\in\mathbb{R}$, one has*

$$\begin{aligned}
&\gamma^{\|u(t)\|^2}[\|u(t)-\psi(t)\|^2 + \|u'(t)-\psi'(t)\|^2 + \|v(t)-\phi(t)\|^2 + \|v'(t)-\phi'(t)\|^2]\\
&\le \gamma^{2t+\|u_0\|^2}\{\|u(0)-\psi(0)\|^2 + \|u'(0)-\psi'(0)\|^2 + \|v(0)-\phi(0)\|^2\\
&+ \|v'(0)-\phi'(0)\|^2 + \int_0^t 2\gamma^{-s}|(E_1(\psi,\phi)(s), u(s)-\psi(s))\\
&+ (E_2(\psi,\phi)(s), v(s)-\phi(s))|\},
\end{aligned}$$

where $u'(t)=\frac{du}{dt}$, $v'(t)=\frac{dv}{dt}$, $u_0, v_0, v_0'\in L_2(\Omega)$ and $\gamma>1$ is a certain function of Ω, g, λ, ψ and ϕ.

Definition 2.2. *A pair of functions $(u,v)\in W_1\times W_2$ is called a weak solutions of the problem (1.1)-(1.2), if $\forall$ test functions $(\psi,\phi)\in V_2\times H_0^1$,*

$$\begin{aligned}
&\frac{d}{dt}(u',\psi') + \frac{d}{dt}(u,\psi) + \epsilon(u,\psi)_2 + \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla\psi) = 0, \qquad (2.1)\\
&\frac{d}{dt}(v',\phi') + \frac{d}{dt}(v,\phi) + \lambda(\nabla v,\nabla\phi) + \nu(\nabla v, \phi\nabla\nu)\\
&-\nu(1-\lambda)(|\nabla u| - v, \phi) = 0 \qquad (2.2)
\end{aligned}$$

holds almot everywhere in $(0,T)$.

Now we state our main result.

Theorem 2.3. *The problem* (1.1)-(1.2) *with conditions* (1.3)-(1.4) *admits a dissipative solution* $(u, v) \in H^1(0, \infty; H_0^1) \times H^1(0, \infty; H_0^1)$ *with the initial data* $u_0, u_0', v_0, v_0' \in L_2$ *and* $0 < \epsilon < \frac{1}{3}$. *Moreover, if there exists a strong solution* (u_T, v_T) *of problem* (1.1)-(1.2), *then the restriction of any dissipative solution to* $(0, T)$ *coincides with* (u_T, v_T) $(T > 0)$. *Every strong solution* $(u, v) \in \mathbb{R}$ *is a unique dissipative solution.*

To prove our main result, we introduce the following auxiliary problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \nu \operatorname{div}(g(|\nabla G_\rho * u|)\nabla u), \tag{2.3}$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) = \nu(1-\lambda)(|\nabla u| - v) + (1-\delta)(\nabla v, \nabla \lambda), \tag{2.4}$$

with the condition

$$u(x,0) = \nu u_0(x), \quad u'(x,0) = \nu u_0'(x) \quad v(x,0) = \nu v_0(x),$$
$$v'(x,0) = \nu v_0'(x), x \in \Omega, \tag{2.5}$$

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial v}{\partial n}|_{\partial\Omega} = 0, \quad 0 < t < T. \tag{2.6}$$

Lemma 2.4. *Let* $(u_0, v_0, u_o', v_0') \in L_2 \times L_2$ *and* T *be positive constant. Then the problem* (2.3)-(2.4) *admits a weak solution with* $\nu = 1$.

Proof. We define the operators $\mathcal{A}$ and $\mathcal{B}$ from $W_1 \times W_2$ to $L_2(0, T; V_2^*) \times H^1(0, T; H^{-1}) \times L_2 \times L_2 \times L_2$ by

$$\begin{aligned}(\mathcal{A}(u,v),(\psi,\phi)) &= (\frac{d}{dt}(u',\psi') + \frac{d}{dt}(u,\psi) + \epsilon(u,\psi)_2, \frac{d}{dt}(v',\phi') + \frac{d}{dt}(v,\phi)\\ &\quad + (\lambda\nabla v, \nabla\phi), u|_{t=0}, v|_{t=0}, v'|_{t=0}),\\ (\mathcal{B}(u,v),(\psi,\phi)) &= (-(g(|\nabla G_\rho * u|)\nabla u, \nabla\psi), -(c(x)\nabla v, \phi\nabla\lambda)\\ &\quad + (1-\lambda)(|\nabla u| - v, \phi), u_0, v_0),\end{aligned}$$

where $(\psi, \phi) \in H_0^1 \times H_0^1$ is a pair of test function.

Then we can rewrite the problem (2.3)-(2.4) as weak statement

$$\mathcal{A}(u, v) = \nu \mathcal{B}(u, v). \tag{2.7}$$

Note that $\mathcal{B}$ is continuous and compact, $W_1 \subset L_p(0, T; W_p^1)$ is compact for some $p > 2$ and $W_2 \subset L_p(0, T; L_2^1)$ (see [22]). If

$$(u_n, v_n) \longrightarrow (u_0, v_0), \quad \text{weakly in } W_1 \times W_2,$$

then we have

$$(u_n, v_n) \longrightarrow (u_0, v_0), \quad \text{strongly in } H^1(0, T; L_2) \times H^1(0, T; L_2).$$

By Krasnoselskii's theorem ([15]), we have

$$g(v_n) \longrightarrow g(v_0), \quad \text{in } L_p(0,T;L_q)\ \forall q < +\infty.$$

So there holds

$$g(|\nabla G_\rho * u_n|)\nabla u_n \longrightarrow g(|\nabla G_\rho * u_0|)\nabla u_0, \quad in\ L_p(0,T;L_2).$$

Since the linear operator $\mathcal{A}$ is continuous and invertible (see [19]), we can rewrite (2.7) as

$$(u,v) = \nu\mathcal{A}^{-1}\mathcal{B}(u,v) \ \text{ in } \ W_1 \times W_2. \tag{2.8}$$

Now we derive the following estimate.

$$\begin{aligned}
&\gamma^{\|u(t)\|^2}[\|u(t)-\psi(t)\|^2 + \|u'(t)-\psi'(t)\|^2 \\
&+ \|v(t)-\phi(t)\|^2 + \|v'(t)-\phi'(t)\|^2 \\
&+ 2\epsilon\int_0^t \|u(s)-\psi(s)\|_2^2 ds + \lambda_0\int_0^t \|v(s)-\phi(s)\|_1^2 ds] \\
&\le \gamma^{2t+\nu\|u_0\|^2}\{\|\nu u(0)-\psi(0)\|^2 + \|\nu u'(0)-\psi'(0)\|^2 \\
&+ \|\nu v(0)-\phi(0)\|^2 + \|\nu v'(0)-\phi'(0)\|^2 \\
&+ \int_0^t 2\gamma^{-s}|(E_1(\psi,\phi,\nu)(s), u(s)-\psi(s)) \\
&+ (E_2(\psi,\phi,\nu)(s), v(s)-\phi(s)) - \epsilon(\psi(s), u(s)-\psi(s))_2|ds\},
\end{aligned} \tag{2.9}$$

where $\gamma > 1$ is a certain function of Ω, g, λ, μ, ψ and ϕ.

To prove above estimate, we need to carry out energy estimate. Let $\psi(t) = u(t)$ and $\phi(t) = v(t)$ in (2.1)-(2.2), respectively, and we have

$$\frac{d}{dt}(u',u') + \frac{1}{2}\frac{d}{dt}(u,u) + \epsilon(u,u)_2 + \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla u) = 0, \tag{2.10}$$

$$\begin{aligned}
&\frac{d}{dt}(v',v') + \frac{1}{2}\frac{d}{dt}(v,v) + \lambda(\nabla v, \nabla v) + \nu(\nabla v, v\nabla\nu) \\
&- \nu(1-\lambda)(|\nabla u| - v, v) = 0.
\end{aligned} \tag{2.11}$$

Summing up (2.12)-(2.13) and integrating over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}(\|u\|^2 + \|u'\|^2 + \|v\|^2 + \|v'\|^2) + \int_0^t \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla u)ds \\
&+ \int_0^t \lambda(\nabla v, \nabla v) + \nu(\nabla v, v\nabla\nu) - \nu(1-\lambda)(|\nabla u| - v, v)ds \\
&\le \frac{\nu}{2}(\|u_0\|^2 + \|u_0'\|^2 + \|v_0\|^2 + \|v_0'\|^2).
\end{aligned} \tag{2.12}$$

On the other hand, $\forall(\eta,\theta)$ test function in $H_0^1 \times H_0^1$, we have

$$\begin{aligned} &\frac{d}{dt}(\psi',\eta') + \frac{d}{dt}(\psi,\eta) + \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla\eta) \\ &+ (E_1(\psi,\phi,\nu),\eta) + \epsilon(\psi,\eta)_2 = \epsilon(\psi,\eta), \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\frac{d}{dt}(\phi',\theta') + \frac{dt}{dt}(\phi,\theta) + (\lambda\nabla\phi,\theta) + \nu(\nabla\phi,\theta\nabla\lambda) \\ &- \nu(1-\lambda)(|\nabla\psi| - \phi,\theta) + (E_2(\psi,\phi,\lambda),\theta) = 0. \end{aligned} \tag{2.14}$$

Let $\eta = u - \psi$ and $\theta = v - \phi$. Summing up (2.13)-(2.14) and noticing (2.12)-(2.13), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&((\eta,\eta) + (\theta,\theta) + (\eta',\eta') + (\theta',\theta')) + \nu(g(|\nabla G_\rho * v|)\nabla\eta,\nabla\eta) \\ &+ \epsilon(\eta,\eta)_2 + (\lambda\nabla\theta,\nabla\theta) + \nu((1-\lambda)\theta,\theta) \\ &= -\nu([g(|\nabla G_\rho * v|) - g(|\nabla G_\rho * \phi|)]\nabla\theta,\nabla\eta) \\ &+ \nu(1-\lambda)(|\nabla u| - |\nabla\theta|,\theta) - \nu(\nabla\theta,\theta\nabla\lambda) \\ &+ (E_1(\psi,\phi,\nu),\eta) + (E_2(\psi,\phi,\lambda),\theta) - \epsilon(\eta,\theta)_2. \end{aligned} \tag{2.15}$$

It is easy to derive that

$$\begin{aligned} -\nu([g(|\nabla G_\rho * v|) - g(|\nabla G_\rho * \phi|)]\nabla\theta,\nabla\eta) &+ \nu(1-\lambda)(|\nabla u| - |\nabla\theta|,\theta) \\ &\le C(\psi,g)\nu(|v-\phi|,|\nabla\eta|) \\ &\le \|\sqrt{\nu g(|\nabla G_\rho * v|)}\nabla\eta\|^2 + C(\psi,\phi,g)\|\theta\|^2 \\ &+ C(\psi,g)(\theta^2, \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|), \end{aligned} \tag{2.16}$$

and

$$-\nu(\nabla\theta,\theta\nabla\lambda) \le \frac{\lambda_0}{4}\|\theta\|_1^2 + C(\lambda)\|\theta\|^2. \tag{2.17}$$

Thus, applying (2.16)-(2.17) to (2.15), we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}((\eta,\eta) + (\eta',\eta') + (\theta,\theta) + (\theta',\theta')) + \epsilon(\eta,\eta)_2 + \frac{3\lambda_0}{4}\mu^{-1}\|\theta\|_1^2 \\ &\le C(\theta,\phi,\lambda,g)(\theta^2, 1 + \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|) \\ &+ (E_1(\psi,\phi,\nu),\eta) + (E_2(\psi,\phi,\lambda),\theta) - \epsilon(\eta,\theta)_2. \end{aligned} \tag{2.18}$$

Denote $\Phi(t) = \|1 + \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|\|$. Then it follows from (2.18) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\eta\|^2 + \|\theta\|^2 &+ \|\theta'\|^2) + 2\epsilon\|\eta\|_2^2 + \lambda_0\mu^{-1}\|\nabla\theta\|^2 \\ &\le C(\theta,\phi,\lambda,g)\Phi^2\|\theta\|^2 + 2(E_1(\psi,\phi,\nu),\eta) + 2(E_2(\psi,\phi,\lambda),\theta) \\ &\quad -2\epsilon(\eta,\theta)_2. \end{aligned} \tag{2.19}$$

By (2.13), we have

$$t|\Omega| \le \int_0^t \Phi^2(s)ds \le 2t|\Omega| + \nu\|u_0\|^2 - \|u(t)\|^2. \tag{2.20}$$

Hence, using (2.19)-(2.20) and a Growall-type inequality, we obtain (2.9).

Let $\eta = \theta = 0$. It is easy to see that

$$\|u\|_{L_\infty(0,T;L_2)} + \|u'\|_{L_2(0,T;H_0^1)} + \|v\|_{L_\infty(0,T;L_2)} + \|v'\|_{L_2(0,T;H_0^1)} \le C, \tag{2.21}$$

$$\|u\|_{L_\infty(0,T;V_2)} + \|u'\|_{L_2(0,T;H_0^1)} \le C\epsilon^{-\frac{1}{2}}, \tag{2.22}$$

where C is a constant independent of ϵ and ν.

It follows from (1.5) that

$$\frac{1}{\sqrt{g(s)}} \le |\frac{1}{\sqrt{g(s)}} - \frac{1}{\sqrt{g(0)}}| + \frac{1}{\sqrt{g(0)}} \le C(g)(1+|s|). \tag{2.23}$$

So by (2.23) and (2.15), we have

$$\|\nabla u\|_{L_2(0,T;L_1)} \le \|\sqrt{\nu g(|\nabla G_\rho * v|)}\nabla u\|_{L_2(0,T;L_2)} \|\sqrt{g(|\nabla G_\rho * v|)}^{-1}\|_{L_\infty(0,T;L_2)} \le C.$$

By Sobolev embedding $H_0^1 \subset L_p$ for any $p < \infty$ and Holder inequality, we derive

$$\|\nabla u\|_{L_2(0,T;L_1)} + \|\nabla u\|_{L_1(0,T;L_r)} + \|\nabla u\|_{L_{\frac{4}{3}}(0,T;L_{-\epsilon+\frac{4}{3}})} \le C.$$

Furthermore, by (2.13)-(2.14) and (2.15), for $1 < r < 2$ and $0 < \epsilon < \frac{1}{3}$, we have the following estimates

$$\begin{aligned}
&\|\nabla u\|_{L_2(0,T;V_2^*)} + \|v\|_{L_2(0,T;H^{-2})} \le C(1+\sqrt{\epsilon}),\\
&\|u\|_{H^1(0,T;H^{-1})} \le C(1+\frac{1}{\sqrt{\epsilon}}),\\
&\|v\|_{H^1(0,T;H^{-1})} \le C(1+\frac{1}{\sqrt{\epsilon}}),
\end{aligned}$$

where C is a constant independent of ϵ and ν.

Hence the above estimates imply that

$$\|u\|_{W_1} + \|u'\|_{W_1} + \|v\|_{W_2} + \|v'\|_{W_2} \le C,$$

where C depends on ϵ but not ν.

Applying Schaeffer's theorem (see [11], p. 539), we know that there exists a fixed point of (2.8),which is a solution of (2.7). This completes the proof. □

The following convergence Proposition is tanken from [19].

Proposition 2.5. *Let G be a measurable set in a finite dimensional space, $y_n : G \longrightarrow R$ be a sequence of functions and $\mathcal{X} : R \longrightarrow R$ be a continuous function. Assume that $\{y_n\}$ is uniformly bounded in $L_\infty(G)$ and $y_m \longrightarrow y_0$ in $L_q(G)$ with $q \geq 1$. Then*

$$\mathcal{X}(y_n) \longrightarrow \mathcal{X}(y_0)$$

in $L_p(G)$ for any $p < \infty$.

Now we are ready to prove Theorem 2.1. The proof is similar to that of theorem 1 in [19]. For completeness of our paper, we sketch the proof. Based on Lemma 2.1, we can proceed with the sketch of the proof of Theorem 2.1. We refer to [19] for the details of the technique, and mainly focus on the new issues. The existence of dissipative solutions, one passes the limit in (2.9) with $\nu = 1$ as $\epsilon = \epsilon_m \longrightarrow 0$ on every interval $(0, T)$ with $T > 0$. Let (u_n, v_n) be the weak solution to problem (2.5)-(2.6) with ϵ_n in Lemma 2.1. Using Sobolev embedding $W_1 \subset L_2$, we derive

$$\begin{aligned} u_m &\longrightarrow u \ \ in \ \ H^1(0,T;L_2), \\ v_m &\longrightarrow v \ \ in \ \ H^1(0,T;L_2). \end{aligned}$$

Then (2.21) and Proposition 2.1,

$$\begin{aligned} &\gamma^{\|u_n(t)\|^2} \longrightarrow \gamma^{\|u(t)\|^2} \ \ in \ \ L_2(0,T), \\ &\|u_n(t) - \psi(t)\|^2 \longrightarrow \|u(t) - \psi(t)\|^2 \ \ in \ \ L_2(0,T), \\ &\|u_n'(t) - \psi'(t)\|^2 \longrightarrow \|u'(t) - \psi'(t)\|^2 \ \ in \ \ L_2(0,T), \\ &\|v_n(t) - \phi(t)\|^2 \longrightarrow \|v(t) - \phi(t)\|^2 \ \ in \ \ L_2(0,T), \\ &\|v_n'(t) - \phi'(t)\|^2 \longrightarrow \|v'(t) - \phi'(t)\|^2 \ \ in \ \ L_2(0,T). \end{aligned}$$

So we have

$$\begin{aligned} &\gamma^{\|u_n(t)\|^2}(\|u_n(t) - \psi(t)\|^2 + \|u_n'(t) - \psi'(t)\|^2 + \|v_n(t) - \phi(t)\|^2 + \|v_n'(t) - \phi'(t)\|^2) \\ &\to \gamma^{\|u(t)\|^2}(\|u(t) - \psi(t)\|^2 + \|u'(t) - \psi'(t)\|^2 + \|v(t) - \phi(t)\|^2 + \|v'(t) - \phi'(t)\|^2) \end{aligned}$$

in $L_1(0, T)$. Thus, we can pass to the limit in the right-hand side of (2.9) as well and the last summand (the one with ϵ) goes to zero. Therefore, we conclude that Theorem 2.1 holds.

3. Numerical experiments

In this section we present some experimental results on pictures in the two dimensions case by using an explicit numerical scheme based on Rothe's method in time discretization and finite difference method in spatial discretization. To proceed with the discrete numerical algorithm, we subdivide the time interval $(0, T)$ by points $t_n = n\tau$ with $\tau = \frac{T}{L}$, $n = 0, 1, 2, \ldots, N$, where N is

a positive integer. Assume that the image $(u(t), v(t))$ is defined in the lattice $\{1h, 2h, \ldots, Mh\} \times \{1h, 2h, \ldots, Lh\}$, where h is the space stepsize. Denote $(u^n_{i,j}, v^n_{i,j})$ an approximation of $(u(n\tau, ih, jh), v(n\tau, ih, jh))$. We define the discrete approximation

$$\begin{aligned}
&\nabla^+_x u^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i,j}}{h}, \quad \nabla^+_x v^n_{i,j} = \frac{v^n_{i+1,j} - v^n_{i,j}}{h},\\
&\nabla^-_x u^n_{i,j} = \frac{u^n_{i-1,j} - u^n_{i,j}}{h}, \quad \nabla^-_x v^n_{i,j} = \frac{v^n_{i-1,j} - v^n_{i,j}}{h},\\
&\nabla^+_y u^n_{i,j} = \frac{u^n_{i,j+1} - u^n_{i,j}}{h}, \quad \nabla^+_y v^n_{i,j} = \frac{v^n_{i,j+1} - v^n_{i,j}}{h},\\
&\nabla^-_y u^n_{i,j} = \frac{u^n_{i,j-1} - u^n_{i,j}}{h}, \quad \nabla^-_y v^n_{i,j} = \frac{v^n_{i,j-1} - v^n_{i,j}}{h},\\
&\delta u^n_{i,j} = \frac{u^n_{i,j} - u^{n-1}_{i,j}}{\tau}, \quad \delta v^n_{i,j} = \frac{v^n_{i,j} - v^{n-1}_{i,j}}{\tau},\\
&\delta^2 u^n_{i,j} = \frac{\delta u^n_{i,j} - \delta u^{n-1}_{i,j}}{\tau}, \quad \delta^2 v^n_{i,j} = \frac{\delta v^n_{i,j} - \delta v^{n-1}_{i,j}}{\tau},\\
&g^n_{i,j} = \frac{1}{1 + (\frac{v^n_{i,j}}{K})^2}, \quad |\nabla G_\rho * u^n_{i,j}| = [\sum_{i=1}^{M} (\delta G_\rho * u^n_{i,j})^2]^{\frac{1}{2}}.
\end{aligned}$$

Then the discrete explicit scheme of the problem (1.1)-(1.2) can be obtained

$$\begin{aligned}
\delta^2 u^{n+1}_{i,j} + \delta u^{n+1}_{i,j} \quad - \quad & \frac{1}{2}[(g^n_{i+1,j} + g^n_{i,j})\nabla^+_x u^n_{i,j} + (g^n_{i-1,j} + g^n_{i,j})\nabla^-_x u^n_{i,j}\\
& +(g^n_{i,j+1} + g^n_{i,j})\nabla^+_y u^n_{i,j} + (g^n_{i,j-1} + g^n_{i,j})\nabla^-_y u^n_{i,j}] = 0,
\end{aligned}$$

$$\begin{aligned}
\delta^2 v^{n+1}_{i,j} + \delta v^{n+1}_{i,j} \quad - \quad & \frac{1}{2}[(c^n_{i+1,j} + c^n_{i,j})\nabla^+_x v^n_{i,j} + (c^n_{i-1,j} + c^n_{i,j})\nabla^-_x v^n_{i,j}\\
& +(c^n_{i,j+1} + c^n_{i,j})\nabla^+_y v^n_{i,j} + (c^n_{i,j-1} + c^n_{i,j})\nabla^-_y v^n_{i,j}]\\
& -(1-\lambda)(|\nabla^+_x u^n_{i,j}| + |\nabla^-_x u^n_{i,j}| - v^n_{i,j}) = 0,
\end{aligned}$$

with the condition

$$\begin{aligned}
&u^0_{i,j} = u_0(i,j), \quad \delta u^0_{i,j} = \delta u_0(i,j) \quad v^0_{i,j} = v_0(i,j),\\
&\delta v^0_{i,j} = \delta v_0(i,j) \quad 1 \le i \le M, \quad 1 \le j \le L,\\
&u^n_{0,j} = u^n_{1,j}, \quad u^n_{M,j} = u^n_{M+1,j}, \quad u^n_{i,0} = u^n_{i,1}, \quad u^n_{i,L} = u^n_{i,L+1},\\
&v^n_{0,j} = v^n_{1,j}, \quad v^n_{M,j} = v^n_{M+1,j}, \quad v^n_{i,0} = v^n_{i,1}, \quad v^n_{i,L} = v^n_{i,L+1}.
\end{aligned}$$

We show numerical results which are obtained by applying the above scheme to two artificial heavily noised images. In order to stabilize the numerical scheme, we rescale the images so that the theoretical results obtained in the previous section can be applied. Our experiments depend on three parameters: the "scale" of diffusion λ, the diffusive coefficient $c(x)$ and the threshold K. In

general, we measure the amount of noise by its standard deviation. The better quality image will have a higher SNR. Define the signal noise ratio (SNR) as

$$SNR = 10\log_{10}(\frac{\sum_{\Omega}(u_{i,j} - \bar{u})^2}{\sum_{\Omega}(n_{i,j})^2}),$$

where $\bar{u}$ denotes the mean of the signal $u_{i,j}$ and $n_{i,j}$ is the noise.

References

[1] L. Alvarez, L. Mazorra, *Signal and image restoration using shock filters and anisotropic diffusion*, SIAM J. Numer. Anal., 31 (2) (1994) 590-605.

[2] L. Ambrosio, V.M. ortorelli, *Approximation of functionals depending on jumps by elliptic functionals via-convergence*, Commun. Pure Appl. Math., 43(8) (1990), 999-1036.

[3] V. Catte, P.L. Lions, J.M. Morel, T. Coll, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal., 29(1) (1992), 182-193.

[4] M. Ceccarelli, *A finite Markov random field approach to fast edge-preserving image recovery*, Image Vis. Comput., 25(6) (2007), 792- 804.

[5] Y. Chen, C.A.Z. Barcelos, B.A. Mair, *Smoothing and edge detection by time-varying coupled nonlinear diffusion equations*, Comput. Vis. Image Underst., 82(2) (2001), 85-100.

[6] Y. Chen, P. Bose, *On the incorporation of time-delay regularization into curvature-based diffusion*, J. Math. Imaging Vis., 14(2) (2001) 149-164.

[7] T.F. Chan, A. Marquina, P. Mulet, *High-order total variation based image restoration*, SIAM J. Sci. Comput., 22 (2) (2000), 503-516.

[8] P. Charbonnier, L. Blanc-Feraud, G. Aubert, M. Barlaud, *Deterministic edge-preserving regularization in computed imaging*, IEEE Trans. Image Process, 6(2) (1997), 298-311.

[9] T.F. Chan, G.H. Golub, P. Mulet, *A nonlinear primal-dual method for total variation-based image restoration*, SIAM J. Sci. Comput., 20 (6) (1999), 1964-1977.

[10] E. Erdem, S. Tari, *Mumford-Shah regularizer with contextual feedback*, J. Math. Imaging Vis., 33(1) (2009), 67-84 .

[11] L.C. Evans, *Partial Differential Equations*, 2nd edn. Graduate Studies in Mathematics, 19. Am. Math. Soc., Providence, 2010.

[12] S. Geman, D. Geman, *Stochastic relaxation, Gibbs distribution and the Bayesian restoration of images*, IEEE Trans. Pattern Anal. Mach. Intell., 6(6) (1984), 721-741.

[13] G. Gilboa, N. Sochen, Y.Y. Zeevi, *Image enhancement and denoising by complex diffusion process*, IEEE Trans. Pattern Anal. Machine Intell., 26 (8) (2004), 1020-1036.

[14] L.I. Rudin, S. Osher, E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D., 60 (1992), 259-268.

[15] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, Elmsford, 1964, Translated by A. H. Armstrong; translation edited by J. Burlak.

[16] D. Mumford, J. Shah, *Optimal approximations by piecewise smooth functions and associated variational problems*, Commun. Pure Appl. Math., 42(5) (1989), 577-685.

[17] M. Nitzberg, T. Shiota, *Nonlinear image filtering with edge and corner enhancement*, IEEE Trans. Pattern Anal. Mach. Intell., 14(8) (1992), 826-833.

[18] P. Perona, J. Malik, *Scale-space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Machine Intell., 12 (7) (1990), 629-639.

[19] V.B. Surya Prasath, D. Vorotnikov, *On a system of adaptive coupled PDEs for image restoration*, J. Math. Imaging Vis, 48 (2014), 35-52.

[20] L. Qiang, Z.A. Yao, Y.Y. Ke, *Entropy solutions for a fourth-order nonlinear degenerate problem for noise removal*, Nonlinear Anal. TMA, 2007 (67) (2007), 1908-1918.

[21] V. Ratner, Y.Y. Zeevi, *Image enhancement using elastic manifolds*, ICIAP 2007, 14th International Conference on Image Analysis and Processing, 2007, 769-774.

[22] J. Simon, *Compact sets in the space* $Lp(0,T;B)$, Ann Mat. Pura Appl., 146 (1987), 65-96.

[23] R. Szeliski, R. Zabih, D. Scharstein, O. Veksler, V. Kolmogorov, A. Agarwala, M. Tapen, C. Rother, *A comparative study of energy minimization methods for Markov random fields with smoothness based priors*, IEEE Trans. Pattern Anal. Mach. Intell., 30(6) (2008), 1068-1080.

[24] C.R. Vogel, M.E. Oman, *Iterative methods for total variation denoising*, SIAM J. Sci. Comput., 17 (1) (1996), 227-238.

[25] V.G. Zvyagin, D.A. Vorotnikov, *Topological Approximation Methods for Evolutionary Problems of Nonlinear Hydrodynamics*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 12, de Gruyter, Berlin, 2008.

Accepted: 29.11.2016

ON PARTIALLY TOPOLOGICAL GROUPS: EXTENSION CLOSED PROPERTIES

Aynur Keskin Kaymakci
Department of Mathematics
Selcuk University
Konya
Turkey
akeskin@selcuk.edu.tr

Wan Aunin Mior Othman
Institute of Mathematical Science
Faculty of Sciences
University of Malaya
50603, Kuala Lumpur
Malaysia
wanainun@um.edu.my

Cenap Ozel*
Department of Mathematics
King Abdulaziz University
P.O.Box 80203 Jeddah 21589
Saudi Arabia
and
Department of Mathematics
Dokuz Eylul University
Tinaztepe Campus Buca, Izmir, 35160
Turkey
cenap.ozel@gmail.com

Abstract. The partially (para)topological groups were defined in [9]. In this paper, we give more results in partially topological groups in the sense of H. Delfs and M. Knebusch and we prove extension closed property for connectedness, compactness, and separability of partially topological groups.

Keywords: extension closed property, Delfs-Knebusch generalized topology, partially topological groups.

1. Introduction and preliminaries

In 1985, H. Delfs and M. Knebusch [3] introduced a notion of generalized topological space. Morphisms between such spaces were named strictly continuous. In 2013, a more convenient definition of a generalized topological space (gts) was

*. Corresponding author

introduced and basic theory of gtses was developed by A. Piękosz [10, 11]. Here we will work in the category $\mathbf{GTS}_{pt}$ of partially topological gtses and strictly continuous mappings (introduced in [10]). From now on, we will use notations from [10].

Generalized topology in the sense od H. Delfs and M. Knebusch is an unknown chapter of general topology. In fact, it is a generalization of the classical concept of topology. The aim of this paper is to continue the systematic study of generalized topology in this sense. Do not mix with other meanings of generalized topology appearing in the literature.

Now we need some assumptions in this paper. We assume the existence of a universe $\mathbb{U}$ (cf. pages 22 and 23 of [8]). Sets in the sense of [8] can be called *totalities* or *collections*. A *class* is a collection $u \subseteq \mathbb{U}$. A *proper class* is a class u such that $u \notin \mathbb{U}$. Elements of the universe $\mathbb{U}$ are called $\mathbb{U}$-small sets in [8]. We denote by *ZF* the system of axioms which consists of the existence of a universe and axioms 0–8 from pages 9–10 of [7] for sets in the sense of [8]. From now on, a totality u will be called a *set* if and only if u is a $\mathbb{U}$-small set. All other set-theoretic axioms applied here, defined in [6] and independent of *ZF*, concern only $\mathbb{U}$-small sets. Axioms independent of *ZF* that are used in this work will be denoted as in [6]. When we do not involve proper classes, we can use directly *ZF* instead of **ZF** and still use the notation **ZF** of [6] for *ZF*. That is why we write $\mathbf{ZF} + \mathbf{AC} = \mathbf{ZFC}$. We recall that, in view of Exercise E2 in Section 1.1 of [6], given a finite collection of sets $X_1, \ldots, X_n$, where $n \in \omega \setminus \{0\}$, the statement that every X_i with $i \in \{1, ..., n\}$ is non-void is equivalent in **ZF** to the statement that the collection $\{X_1, \ldots X_n\}$ has a choice function.

The basic set-theoretic assumption of this work is **ZF**. If a theorem is unprovable in **ZF** or if we give its proof not in **ZF**, we shall clearly denote the system of axioms we use in the proof. In all other cases, theorems and their proofs are in **ZF**.

Definition 1.1. Let X be any set, τ_X be a topology on X. A family of open families $\mathrm{Cov}_X \subseteq \mathcal{P}(\tau_X)$ will be called a *partial topology* if the following conditions are satisfied:

(i) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{U}$ is finite, then $\mathcal{U} \in \mathrm{Cov}_X$;

(ii) if $\mathcal{U} \in \mathrm{Cov}_X$ and $V \in \tau_X$, then $\{U \cap V : U \in \mathcal{U}\} \in \mathrm{Cov}_X$;

(iii) if $\mathcal{U} \in \mathrm{Cov}_X$ and, for each $U \in \mathcal{U}$, we have $\mathcal{V}(U) \in \mathrm{Cov}_X$ such that $\bigcup \mathcal{V}(U) = U$, then $\bigcup_{U \in \mathcal{U}} \mathcal{V}(U) \in \mathrm{Cov}_X$;

(iv) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{V} \in \mathrm{Cov}_X$ are such that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ and, for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$, then $\mathcal{U} \in \mathrm{Cov}_X$.

Elements of τ_X are called *open sets*, and elements of Cov_X are called *admissible families*. We say that (X, Cov_X) is a *partially topological generalized*

topological space or simply *partially topological space*. We shall denote a partially topological space (X, Cov_X) by X when no confusion will arise. Since $\tau_X = \bigcup \mathrm{Cov}_X$, we can omit τ_X in notation.

Let X and Y be partially topological spaces and let $f : X \to Y$ be a function. Then f is called *strictly continuous* if $f^{-1}(\mathcal{U}) \in \mathrm{Cov}_X$ for any $\mathcal{U} \in \mathrm{Cov}_Y$. A bijection $f : X \to Y$ is called a *strict homeomorphism* if both f and f^{-1} are strictly continuous functions. If we have a strict homeomorphism between X and Y we say that they are *strictly homeomorphic* and we denote that by $X \cong Y$.

Remark 1.2. The above notion of partial topology is a special case of the notion of generalized topology in the sense of H. Delfs and M. Knebusch considered in [3, 10, 11, 12, 13], when the family Op_X of open sets of the generalized topology forms a topology. The category $\mathbf{GTS}_{pt}$ of partially topological spaces and their strictly continuous mappings is a topological construct (Theorem 4.4 of [13]).

Definition 1.3. Let (X, Cov_X) be a partially topological space and let Y be a subset of X. Then the partial topology

$$\mathrm{Cov}_Y = (\langle \mathrm{Cov}_X \cap_2 Y \rangle_Y)_{pt},$$

that is: the smallest partial topology containing $\mathrm{Cov}_X \cap_2 Y$, is called a *subspace partial topology* on Y, and (Y, Cov_Y) is a *subspace* of (X, Cov_X). (It is also the smallest generalized topology containing $\mathrm{Cov}_X \cap_2 Y$.)

Fact 1.4. *Let $\varphi : X \to X'$ be a mapping between partially topological spaces and let Y be a subspace of X. Then the following are equivalent:*

a) *φ is strictly continuous,*

b) *the restriction map $\varphi|_Y : Y \to X'$ is strictly continuous.*

Definition 1.5. Let (X, Cov_X) and (Y, Cov_Y) be two partially topological spaces. The *product partial topology* on $X \times Y$ is the partial topology $\mathrm{Cov}_{X\times Y} = (\langle \mathrm{Cov}_X \times_2 \mathrm{Cov}_Y \rangle_{X\times Y})_{pt}$ in the notation of Definition 4.6 of [13]; in other words: the smallest partial topology in $X \times Y$ that contains $\mathrm{Cov}_X \times_2 \mathrm{Cov}_Y$.

2. Partially topological groups

The fundamental reference for topological groups and their properties is [1]; see also [15]. In [9], the partially (para)topological groups in the sense of H. Delfs and M. Knebusch were defined. Now we discuss partially topological groups with more details and recall their basic properties.

Definition 2.1. A *partially paratopological group* G is an ordered pair $((G, *), \mathrm{Cov}_G)$ such that $(G, *)$ is a group, while Cov_G is a generalized topology on G such that $\bigcup \mathrm{Cov}_G$ is a T_1 topology on G and the multiplication map of $(G \times G, \mathrm{Cov}_{G\times G})$ into (G, Cov_G), which sends ordered pair $(x, y) \in G \times G$ to $x * y$, is strictly

continuous. If also the inverse map from (G, Cov_G) into (G, Cov_G), which sends each $x \in G$ to x^{-1}, is strictly continuous, then $((G,*), \mathrm{Cov}_G)$ is called a *partially topological group*. For simplicity, when this does not lead to misunderstanding, we shall denote a partially (para)topological group $((G,*), \mathrm{Cov}_G)$ by G or by (G, Cov_G), or by $(G,*)$.

Definition 2.2. (i) A *small partially (para)topological group* is a partially (para)-topological group $((G,*), \mathrm{Cov}_G)$ such that
$\mathrm{Cov}_G = EssFin(\bigcup \mathrm{Cov}_G)$.

(ii) A *smallification* of a partially (para)topological group $G = ((G,*), \mathrm{Cov}_G)$ is the partially (para)topological group

$$G_{sm} = ((G,*), EssFin(\bigcup \mathrm{Cov}_G)).$$

(Cf. Definintion 2.3.16 of [10]).

Fact 2.3. *Let $(G,*)$ be a group and let τ be a topology on G. Let us apply the functor of smallification to (G,τ) by putting $Cov_{G_{sm}} = EssFin(\tau)$. Then $G = ((G,*),\tau)$ is a (para)topological group if and only if $G_{sm} = ((G,*), Cov_{G_{sm}})$ is a partially (para)topological group.*

The group G_{sm} can also be denoted by G_{st}, applying the conventions from Definition 1.2 of [13]. Similarly, we have

Example 2.4. Since $(\mathbb{R}^n, +)$ is a group and the localized smallified euclidean space $\mathbb{R}^n_{lst}$ (the localization of $\mathbb{R}^n_{st}$; see Definition 2.1.15 in [11]) is still partially topological, we get partially topological groups $(\mathbb{R}^n_{lst}, +)$ for $n \in \mathbb{N}$.

Example 2.5. The usual topological as well as smallified Sorgenfrey lines (see Definition 4.3 of [14]) with addition as group action form partially paratopological groups $(\mathbb{R}^S_{ut}, +)$, $(\mathbb{R}^S_{st}, +)$.

Definition 2.6. Any subgroup H of a partially (para)topological group G is a partially (para)topological group again, and is called a *partially (para)topological subgroup of G*.

Definition 2.7. Let $\varphi : G \to G'$ be a function. Then φ is called a *morphism of partially (para)topological groups* if φ is both strictly continuous and a group homomorphism. Moreover, φ is an *isomorphism* if it is a strict homeomorphism and group isomorphism.

If we have an isomorphism between two partially (para)topological groups G and G' then we say that they are isomorphic and we denote that by $G \cong G'$.

Remark 2.8. It is obvious that composition of two morphisms of partially (para)topological groups is again a morphism. Also, the identity map is an isomorphism. So partially paratopological groups and their morphisms form a category **PPTGr**, while partially topological groups and their morphisms form a category **PTGr**.

3. Extension closed properties of partially topological groups

In this section, we prove extension closed property for connectedness, compactness, and separability of partially topological groups.

From [9] we know that Cartesian product of compact(small) sets is compact.

Definition 3.1. A partially topological space (X, Cov_X) is *topologically compact (admissibly compact)* if each open (admissible) cover of X admits a finite subcover, and is *small* if it is hereditary admissibly compact.

Fact 3.2. *Let (X, Cov_X) and (Y, Cov_Y) be two partially topological spaces and $f : X \to Y$ a surjective strictly continuous function. If (X, Cov_X) is*

i) topologically compact,

ii) admissibly compact,

iii) small

then so is (Y, Cov_Y).

The famous Tychonoff's theorem may be stated in the following way

Theorem 3.3. *Assume* **ZF**. *Let X be the product of partially topological spaces $X_k, k \in K$, where K is a finite set. If every X_k is topologically compact (small, respectively), then so is X.*

Remark 3.4. It is still unknown whether admissible compactness is finitely productive even in **ZFC**.

Corollary 3.5. *Assume* **ZF**. *Let G be a partially topological group and A and B subsets of G. If A and B are topologically compact (small, respectively), then so is AB.*

Theorem 3.6. *For any two compact(small) subsets E and F of a partial topological group G, their product EF in G is a compact(small) subspace of G.*

Proof. Since multiplication is strictly continuous, the subspace EF of G is a strictly continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is compact(small), the space EF is compact. □

Theorem 3.7. *Assume* **ZF**. *Let G be a normal compact partial topological group, F a compact subset of G, and P a closed subset of G. Then the sets FP and PF are closed.*

Proof. Since G is compact then $P \times F$ is compact and by previous result PF and FP are $\mathcal{G}$-compact. Since G is normal, FP and PF are closed. □

From[9] we know that Cartesian product of connected sets is connected.

Definition 3.8. Let X be a partially topological space and let $U, V \subseteq X$. Then we say that the pair U, V is *separated* if $Cl(U) \cap V = Cl(V) \cap U = \emptyset$. We say that X is *connected* if it can not be written as a union of two separated sets.

Definition 3.9. A function $f : X \to Y$ between partially topological spaces is called *open*(*closed*) if for every open (closed) set $U \subseteq X$, $f(U)$ is open (closed) in Y.

Fact 3.10. *Let $f : X \to Y$ be an open and injective function between partially topological spaces and let $A \subseteq X$. If $f(A)$ is connected, so is A.*

Theorem 3.11. *For any two connected subsets E and F of a partial topological group G, their product EF in G is a connected subspace of G.*

Proof. Since multiplication is strictly continuous, the subspace EF of G is a strictly continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is connected from [9], the space EF is connected. □

Definition 3.12. A closed strictly continuous mapping with compact preimages of points is called perfect.

Theorem 3.13. *The quotient mapping π of G onto the quotient space G/H is perfect where H is a compact subgroup of a normal partial topological group G.*

Proof. Take any closed subset P of G. Then, by Theorem 3.7, PH is closed in G. However, PH is the union of cosets, that is $PH = \pi^{-1}\pi(P)$. It follows by definition of a quotient mapping, that the set $\pi(P)$ is closed in the quotient space G/H. Thus π is a closed mapping. In addition, if $y \in G/H$ and $\pi(x) = y$ for some $x \in G$, then $\pi^{-1}(y) = xH$ is a compact subset of G. Hence the fibers of π are compact and π is perfect. □

Corollary 3.14. *Let H be a compact subgroup of a normal partial topological group G such that the quotient space G/H is compact. Then G is also compact.*

Theorem 3.15. *Let $f : G \to H$ be a strictly continuous mapping of partial topological spaces. If G is compact and H is normal, then f is closed.*

Proof. Let K be a closed set in G. Since G is compact then K is compact. So by strictly continuity of f, $f(K)$ is compact in H. By assumption, H is normal, so $f(K)$ is closed. □

Definition 3.16. Let $X$and Y be partially topological spaces. A strictly continuous onto mapping $f : X \to Y$ is called *identification map* if f is open or closed.

Theorem 3.17. *Let $f : G \to H$ be a strictly continuous onto homomorphism of partial topological groups. If G is compact and H is normal, then f is open.*

Proof. By Theorem 3.15, the mapping f is closed, and hence it is quotient. Let K be the kernel of f. If U is open in G, then $f^{-1}(f(U)) = KU$ is open in G. Since f is quotient, it follows that the image $f(U)$ is open in H. Therefore, f is an open mapping. □

Lemma 3.18. *Suppose that $f : X \to Y$ is an open strictly continuous mapping of a space X onto a space Y, $x \in X$, $B \subseteq Y$, and $f(x) \in Cl(B)$ where $Cl(B)$ is closure of B. Then $x \in f^{-1}(Cl(B))$.*

Proof. Take $y = f(x)$, and let O be an open neighborhood of x. Then $f(O)$ is an open neighborhood of y. Therefore, $f(O) \cap B \neq \emptyset$ and, hence $O \cap f^{-1}(B) \neq \emptyset$. It follows that $x \in Cl(f^{-1}(B))$. Equality is evident. □

Theorem 3.19. *Let H be a closed subgroup of a partial topological group G. If the spaces H and G/H are separable, then the space G is also separable.*

Proof. Let π be the natural homomorphism of G onto the quotient space G/H. Since G/H is separable, we can fix a dense countable subset B of G/H. Since H is separable and every coset xH is strictly homeomorphic to H, we can fix a dense countable subset M_y of $\pi^{-1}(y)$, for each $y \in B$. Put $M = \bigcup\{M_y : y \in B\}$. Then M is a countable subset of G and M is dense in $\pi^{-1}(B)$. Since π is an open mapping of G onto G/H, it follows from Lemma 3.18 that $Cl(\pi^{-1}(B)) = G$. Hence, M is dense in G and G is separable. □

Theorem 3.20. *Let H be a closed invariant subgroup of a partial topological group G. If H and G/H are connected, then so is G.*

Proof. Suppose that H and G/H are connected and $f : G \to \{0, 1\}$ be an arbitrary strictly continuous map. We have to show that f is constant. The restriction of f to H must be constant and since each coset gH is connected, f must be constant on gH as well taking value $f(g)$. Thus we have a well-defined map $\tilde{f} : G/H \to \{0, 1\}$ such that $\tilde{f} \circ \pi = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is strictly continuous and so must be constant since G/H is connected. Hence f is also constant and we conclude that G is connected. □

Acknowledgements

Authors would like to express their appreciation to the referees for their comments and valuable suggestions.

References

[1] A. Arhangel'skii and M. Tkachenko, *Topological groups and related structures*, World Scientific, 2008.

[2] T. Banakh and A. Ravsky, *Each regular paratopological group is completely regular*, to appear in Proc. Amer. Math. Soc.

[3] H. Delfs and M. Knebusch, *Locally Semialgebraic Spaces*, Lecture Notes in Math., 1173, Springer, Berlin-Heidelberg, 1985.

[4] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[5] C. Good and I. J. Tree, *Continuing horrors of topology without choice*, Topology Appl., 63 (1995), 79-90.

[6] H. Herrlich, *Axiom of Choice*, Lecture Notes in Mathematics 1876, Springer-Verlag, Berlin-Heidelberg, 2006.

[7] K. Kunen, *The Foundations of Mathematics*, Studies in Logic 19, College Publications, London, 2009.

[8] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin-Heidelberg, 1971.

[9] C. Ozel, A. Piękosz, M. A. Al Shumrani and E. Wajch, *Partial paratopological groups*, Submitted to Topology Appl., 2016.

[10] A. Piękosz, *On generalized topological spaces I*, Ann. Polon. Math., 107 (2013), 217-241.

[11] Piękosz, A. *On generalized topological spaces II*, Ann. Polon. Math., 108 (2013), 185-214.

[12] A. Piękosz, *O-minimal homotopy and generalized (co)homology*, Rocky Mountain J. Math., 43 (2013), 573-617.

[13] A. Piękosz and E. Wajch, *Compactness and compactifications in generalized topology*, Topology Appl., 194 (2015), 241-268.

[14] A. Piękosz and E. Wajch, *Quasi-metrizability in generalized topology in the sense of Delfs and Knebusch*, arxiv:1505.04442v2, 2015.

[15] L. Pontryagin, *Topological groups*, CRC Press, 3 edition, 1987.

[16] A. Ravsky, *Paratopological groups I*, Mat. Stud., 16 (2001), 37-48.

[17] A. Ravsky, *Paratopological groups II*, Mat. Stud., 17 (2002) 93-101.

[18] M. Sanchis and M. Tkachenko, *Recent progress in paratopological groups*, a preliminary version of the article published in Quaderni Math. in: Assymetric Topology and its Applications, 26 (2012), 247-300, 2012

Accepted: 7.12.2016

ON WEAK MCCOY MODULES OVER COMMUTATIVE RINGS

Ahmad Yousefian Darani
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil
Iran
yousefian@uma.ac.ir

Masoumeh Shabani
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil
Iran
m.shabani@uma.ac.ir

Abstract. Let M be a module over a commutative ring R. In this paper we generalize some annihilator conditions from rings to modules. Denote by $Nil(M)$ the set of all nilpotent elements of M. M is said to be weak McCoy if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^{k} m_i x^i \in R[x]\backslash\{0\}$ and $m(x) = \sum_{j=0}^{n} a_j x^j \in M[x]\backslash\{0\}$, then $sm_i \in Nil(M)$, for every $1 \leq i \leq k$ and for some $s \in R\backslash\{0\}$. We prove that the class of weak McCoy modules is closed under direct limit, finite direct product and localization. We show that if R is a uniform R-module, then the direct sum of weak McCoy R-modules is again weak McCoy. We prove that over a domain D, the D-module M is weak McCoy provided that if $T(M)$ is weak McCoy, where $T(M)$ is the torsion submodule of M.

Keywords: McCoy ring, McCoy module, weak McCoy ring, weak McCoy module.

1. Introduction

Throughout this paper all rings are considered to be commutative with a nonzero identity and all modules are unitary unless otherwise stated. Let R be a ring, M an R-module and N a submodule of M. We denote by $(N :_R M)$ the set of all r in R such that $rM \subseteq N$. The *annihilator* of M, denoted by $ann_R(M)$, is then $(0 :_R M)$. We denote by $R[x]$ the polynomial ring over the ring R and $M[x]$ the polynomial module over the module M.

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R (Barnard, 1981). Note that since $I \subseteq (N :_R M)$, then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. A submodule N of M is called *pure* if $IN = N \cap IM$ for every ideal I of R (Ribenboim, 1972). It is proved in [2, Theorem 1.1] that every pure submodule of a multiplication module is again a multiplication module. A submodule N of

M is *prime* whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$.

We recall that an ideal I of R is *nilpotent* if $I^k = 0$ for some positive integer k and an element r of R is *nilpotent* if $r^k = 0$ for some $k \in \mathbb{N}$. We denote by $Nil(R)$ the set of all nilpotent elements of R. According to [1], a submodule N of M is called *nilpotent* if $(N :_R M)^k N = 0$ for some positive integer k. An element $m \in M$ is said to be *nilpotent* if Rm is a nilpotent submodule of M. Clearly, the zero submodule of M is nilpotent and hence the zero element of M is nilpotent. Denote by $Nil(M)$ the set of all nilpotent elements of M. $Nil(M)$ is not necessarily a submodule of M, but if M is faithful, then $Nil(M)$ is a submodule of M [1, Theorem 6]. Moreover, if M is a faithful multiplication R-module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M.

By [1, Proposition 4(2)], if I is a nilpotent ideal of R, then IM is nilpotent in M and the converse is true if M is faithful. Also if $K \subseteq N$ and N is nilpotent in M, then $\frac{N}{K}$ is nilpotent in $\frac{M}{K}$ as an R-module and the converse is true if K is nilpotent in M and M is faithful. Also, a submodule N of M is *idempotent* if $N = (N :_R M)N$ (Ali and Smith, 2004). By [1, Theorem 3], if N is a direct summand of a multiplication module M, then N is multiplication and idempotent. An R module M is called *uniform* if any two nonzero submodules of M have a nonzero intersection (Faith, 1981).

Let R be an associative ring with identity. R is said to be *right McCoy* (respectively *left McCoy*) if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (respectively $rg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy. Also, we say that a ring R is *right weak McCoy* whenever $f(x) = a_0 + a_1x + \ldots + a_mx^m, g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then $a_is \in Nil(R)$ for each $i = 0, 1, \ldots, m$ and for some $s \in R\setminus\{0\}$. *Left weak McCoy* rings are defined similarly. If a ring R is both left and right weak McCoy, then we say that R is a *weak McCoy ring*.

By those definitions, clearly, McCoy rings are weak McCoy. Also, the name "McCoy" was chosen because McCoy in [7] had noted that every commutative ring satisfies this condition.

An R-module M is said to be *McCoy* if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^{n} a_ix^i \in R[x]\setminus\{0\}$ and $m(x) = \sum_{j=0}^{k} m_jx^j \in M[x]$, implies that there exists a nonzero element $r \in R$ such that $rm(x) = 0$ (Cui and Chen, 2011).

In this paper, we generalize the concept of weak McCoy rings defined on associative rings to modules over commutative rings. Let R be a commutative ring. An R-module M is called a *weak McCoy module* if $f(x)m(x) = 0$, where $f(x) \in R[x]\setminus\{0\}$ and $m(x) = \sum_{i=0}^{n} m_ix^i \in M[x]\setminus\{0\}$, then $sm_i \in Nil(M)$, for every $0 \leq i \leq n$ and for some $s \in R\setminus\{0\}$.

2. Weak McCoy modules

In this section we study some basic properties of weak McCoy modules and investigate the relations between weak McCoy rings and weak McCoy modules.

Definition 2.1. *Let R be a commutative ring. An R-module M is called weak McCoy whenever $f(x) \in R[x]\backslash\{0\}$ and $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]\backslash\{0\}$ satisfy $f(x)m(x) = 0$, then $sm_i \in Nil(M)$, for every $0 \leq i \leq n$ and for some $s \in R\backslash\{0\}$.*

Proposition 2.2. *Let R be a ring. Then every McCoy R-module M is weak McCoy and the converse is true if M has no nilpotent elements.*

Proof. It is trivial. □

Corollary 2.3. *Let M be a finitely generated faithful multiplication R-module. Then, the $\frac{R}{Nil(R)}$-module $\frac{M}{Nil(M)}$ is McCoy if and only if it is weak McCoy.*

Proof. First note that $Nil(M)$ is a submodule of M and $Nil(M) = Nil(R)M$ by [1, Theorem 6]. So we may form the $\frac{R}{Nil(R)}$-module $\frac{M}{Nil(M)}$. By [1, Corollary 7], $\frac{M}{Nil(M)}$ has no nilpotent elements. Now the result follows from Proposition 2.2. □

Proposition 2.4. *Let M be an R-module. Then the direct limit of every direct system of weak McCoy submodules of M is weak McCoy.*

Proof. Let I be a directed set and $\{M_i; i \in I\}$ a direct system of weak McCoy submodules of M. Set $N = \varinjlim_{i \in I} M_i$. We know that $N = \sum_{i \in I} M_i = \bigcup_{i \in I} M_i$, by [8, Example 5.32]. Let $m(x) = \sum_{j=0}^{n} m_j x^j \in N[x]\backslash\{0\}$ and $f(x) = \sum_{k=0}^{m} a_k x^k \in R[x]\backslash\{0\}$ with $f(x)m(x) = 0$. There exists $t \in I$ such that $m_j \in M_t$ for each $0 \leq j \leq n$. So $m(x) \in M_t[x]$. Since M_t is weak McCoy, there exists $s \in R\backslash\{0\}$ such that $sm_j \in Nil(M_t)$. Thus, there exists an $l_j \in \mathbb{N}$ such that $(R(sm_j) :_R M_t)^{l_j} R(sm_j) = 0$, for each $j = 0, 1, \ldots, n$. But $(R(sm_j) :_R N) \subseteq (R(sm_j) :_R M_t)$, hence $(R(sm_i) :_R N)^{l_j} R(sm_j) = 0$ and $sm_j \in Nil(N)$, for each $j = 0, 1, \ldots, n$. □

Corollary 2.5. *If every finitely generated submodule of an R-module M is weak McCoy, then M is weak McCoy.*

Proof. By , by [8, Example 5.32], every module is the direct limit of it's finitely generated submodules. Now the result is trivial by Proposition 2.4. □

Definition 2.6. *A ring R is said to be a strongly weak McCoy ring if every R-module is weak McCoy.*

It is clear that if an R-module M is not faithful, then it is McCoy, and so is weak McCoy. Hence to study the weak McCoy condition on a module, it is sufficient to consider the faithful case.

Proposition 2.7. *A ring R is strongly weak McCoy if and only if every finitely generated R-module is weak McCoy.*

Proof. $\Rightarrow$) It is trivial.

$\Leftarrow$) Let M be a faithful R-module. We know that M is a direct limit of its finitely generated submodules. Hence by assumption and Proposition 2.4, M is weak McCoy. □

Let M be an R-module and S a multiplicatively closed subset of R. Then $S^{-1}M$ has an $S^{-1}R$-module structure. M is called *S-torsion free if $sm \neq 0$, for each $0 \neq m \in M$ and $s \in S$.*

Lemma 2.8. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated R-module. Then $m \in Nil_R(M)$ if and only if $\frac{m}{s} \in Nil_{S^{-1}R}(S^{-1}M)$, for every $s \in S$.*

Proof. Let $m \in M$ and $k \in \mathbb{N}$. Since M is finitely generated, we have

$$\begin{aligned} S^{-1}((Rm :_R M)^k Rm) &= S^{-1}((Rm :_R M)^k)S^{-1}(Rm) \\ &= (S^{-1}(Rm :_R M))^k S^{-1}(Rm) \\ &= (S^{-1}(Rm) :_{S^{-1}R} S^{-1}M)^k S^{-1}(Rm). \end{aligned}$$

On the other hand, $S^{-1}(Rm) = S^{-1}R(\frac{m}{s})$, for each $s \in S$. So the proof is complete. □

Theorem 2.9. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated S-torsion free R-module. then M is weak McCoy if and only if $S^{-1}M$ is a weak McCoy $S^{-1}R$-module.*

Proof. First suppose that M is a weak McCoy R-module.
Let $f(x) = \sum_{i=0}^{m} \alpha_i x^i \in S^{-1}R[x]\backslash\{0\}$ and $m(x) = \sum_{j=0}^{n} \beta_j x^j \in S^{-1}M[x]\backslash\{0\}$ with $f(x)m(x) = 0$.

We can assume that $\alpha_i = \frac{r_i}{u}$ and $\beta_j = \frac{m_j}{v}$ for some $r_i \in R$, $m_j \in M$ and $u, v \in S$. Note that if $\alpha_i = \frac{r_i}{u_i}$ and $\beta_j = \frac{m_j}{v_j}$, then we can assume $u = u_1 u_2 \dots u_m$ and $v = v_1 v_2 \dots v_n$.

Therefore $f(x)m(x) = 0$ implies that $0_{S^{-1}R} = \frac{r_0 m_0}{uv} = \frac{r_0 m_1 + r_1 m_0}{uv} = \frac{r_0 m_2 + r_1 m_1 + r_2 m_0}{uv} = \cdots$. But M is S-torsion free, so $0 = r_0 m_0 = r_0 m_1 + r_1 m_0 = r_0 m_2 + r_1 m_1 + r_2 m_0 = \cdots$.

Now, we set $f_1(x) = \sum_{i=0}^{m} r_i x^i$ and $m_1(x) = \sum_{j=0}^{n} m_j x^j$. Clearly $f_1(x) \in R[x]\backslash\{0\}$, $m_1(x) \in M[x]\backslash\{0\}$ and $f_1(x)m_1(x) = 0$. Since M is weak McCoy, there exists $c \in R\backslash\{0\}$ such that $cm_j \in Nil_R(M)$, for each $0 \leq j \leq n$. It is clear that $c \in S^{-1}R\backslash\{0\}$ and $c(\frac{m_j}{v}) \in Nil(S^{-1}M)$, for each $0 \leq j \leq n$, by Lemma 2.8. Hence, $S^{-1}M$ is a weak McCoy $S^{-1}R$-module.

Now assume that $S^{-1}M$ is a weak McCoy $S^{-1}R$-module. Let $g(x) = \sum_{i=0}^{q} a_i x^i \in R[x]\backslash\{0\}$ and $n(x) = \sum_{j=0}^{p} n_j x^j \in M[x]\backslash\{0\}$ with $g(x)n(x) = 0$. But $n(x) \in S^{-1}M[x]\backslash\{0\}$ and $g(x) \in S^{-1}R[x]\backslash\{0\}$. By assumption, there exists $\frac{r}{s} \in S^{-1}R[x]\backslash\{0\})$ such that $\frac{r}{s}(m_j) \in Nil_{S^{-1}R}(S^{-1}M)$, for each $0 \leq j \leq p$.

Therefore, $rm_j \in Nil_R(M)$, for each $0 \leq j \leq p$ and hence M is a weak McCoy R-module. □

We recall that $R[x, x^{-1}]$ denotes the Laurent polynomial ring over R. For an R-module M, let $M[x, x^{-1}] = \{\sum_{i=k}^{n} m_i x^i; n, k \in \mathbb{Z}, m_i \in M\}$. $M[x, x^{-1}]$ is an $R[x, x^{-1}]$-module under the addition operation and the following scaler product operation. For $m(x) = \sum_i m_i x^i \in M[x, x^{-1}]$ and $f(x) = \sum_j a_j x^j \in R[x, x^{-1}]$, then $f(x)m(x) = \sum_k (\sum_{i+j=k} a_j m_i) x^k$ (see [6]).

Corollary 2.10. *Let R be a ring and M a finitely generated R-module. Then $M[x]$ is a weak McCoy $R[x]$-module if and only if $M[x, x^{-1}]$ is a weak McCoy $R[x, x^{-1}]$-module.*

Proof. Let $S = \{1, x, x^2, ...\}$. It is clear that S is a multiplicatively closed subset of $R[x]$ and $R[x]$-module $M[x]$ is S-torsion free. By Theorem 2.9, $M[x]$ is a weak McCoy $R[x]$-module if and only if $S^{-1}M[x]$ is a weak McCoy $S^{-1}R[x]$-module. But $S^{-1}R[x] = R[x, x^{-1}]$ and $S^{-1}M[x] = M[x, x^{-1}]$. Now the result follows from Theorem 2.9. □

Corollary 2.11. *Let R be a ring, S a multiplicatively closed subset of R and every R-module is finitely generated and S-torsion free. Then the localization $S^{-1}R$ is strongly weak McCoy if and only if R is strongly weak McCoy.*

Proposition 2.12. *Every pure submodule of a weak McCoy multiplication module is weak McCoy.*

Proof. Let N be a faithful pure submodule of the weak McCoy multiplication R-module M. It is clear that N is multiplication. Let $f(x)m(x) = 0$, where $m(x) = \sum_{i=1}^{n} m_i x^i \in N[x] \setminus \{0\}$ and $f(x) \in R[x] \setminus \{0\}$. Since $m(x) \in M[x]$ and M is weak McCoy, there exists $s \in R\setminus\{0\}$ such that $sm_i \in Nil(M)$. By [1, Proposition 4], $(R(sm_i) :_R M)$ is a nilpotent ideal of R and hence $(R(sm_i) :_R M) \subseteq Nil(R)$, for each $i = 0, 1, \ldots, n$. By [1, Theorem 6], $Nil(R)M = M$ and $Nil(R)N = N$, since N and M are multiplication. Also, N is a pure submodule of M and $R(sm_i) = (R(sm_i) :_R M)M$. So $sm_i \in N \cap (R(sm_i) :_R M)M = (R(sm_i) :_R M)N \subseteq Nil(R)N = Nil(N)$, for each $i = 0, 1, \ldots, n$. Therefore, these elements are nilpotent in N and hence N is weak McCoy. □

Proposition 2.13. *Let R be a ring, I a directed set and $\{M_i\}_{i\in I}$ a family of faithful weak McCoy R-modules. If R is uniform as an R-module, then the direct sum $M = \coprod_{i\in I} M_i$ is a weak McCoy R-module.*

Proof. Let $f(x) \in R[x]\setminus\{0\}$ and $m(x) = \sum_{k=0}^{n} m_k x^k \in M[x]\setminus\{0\}$ satisfy $f(x)m(x) = 0$, where $m_k = (m_{ik})_{i\in I}$, for $k = 0, 1, \ldots, n$.

Set $m_i(x) = \sum_{k=0}^{n} m_{ik} x^k \in M_i[x]$. Clearly, $f(x)m_i(x) = 0$. Since M_i's are weak McCoy, there exists $s_i \in R\setminus\{0\}$ such that $s_i m_{ik} \in Nil(M_i)$ for each $i \in I$ and $k = 0, 1, \ldots, n$. We know that the set $I' = \{i \in I; m_i(x) \neq 0\}$ is a finite set. Set $U = \bigcap_{i\in I'} Rs_i$. Since R is a uniform R-module, $U \neq 0$.

Let $s \in U\backslash\{0\}$. Then $s = r_i s_i$, for each $i \in I'$ and for some $r_i \in R$. Also $R(sm_{ik}) = R(r_i s_i m_{ik}) = (Rr_i)(Rs_i m_{ik})$. But $R(s_i m_{ik})$ is a nilpotent submodule of M_i. Hence $R(sm_{ik})$ is nilpotent in M_i, for each $i \in I'$ and $k = 0, 1, \ldots, n$, by [1, Proposition]. Therefore, $(R(sm_{ik}) :_R M_i)$ is a nilpotent ideal of R, for each $i \in I'$, since M_i's are faithful. But $R(sm_k) \subseteq \coprod_{i\in I} R(sm_{ik})$ and also we have

$$\begin{array}{rcl}(R(sm_k) : \coprod_{i\in I} M_i) & \subseteq & (\coprod_{i\in I} R(sm_{ik}) :_R \coprod_{i\in I} M_i) \\ & \subseteq & (\coprod_{i\in I'} R(sm_{ik}) :_R \coprod_{i\in I'} M_i) \\ & = & \bigcap_{i\in I}(R(sm_{ik}) :_R M_i).\end{array}$$

Thus, $sm_k \in Nil(M)$, for each $k = 0, 1, \ldots, n$, since finite intersection of nilpotent ideals is nilpotent. Consequently, $M = \coprod_{i\in I} M_i$ is weak McCoy. □

Proposition 2.14. *Every finite direct product of faithful weak McCoy modules is weak McCoy.*

Proof. Let $M = \prod_{k=1}^{n} M_k$, where M_k's are faithful weak McCoy R-modules. So M is a faithful R-module. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]\backslash\{0\}$ and $f(x) \in R[x]\backslash\{0\}$ with $f(x)m(x) = 0$. Since $m_i \in M$, for each i, $m_i = (m_{i1}, \ldots, m_{in})$, for some $m_{ik} \in M_k$.

Set $m_k(x) = \sum_{i=0}^{p} m_{ik} x^i \in M_k[x]$, for each $1 \leq k \leq n$. Hence $f(x)m_k(x) = 0$, for each $1 \leq k \leq n$. By assumption, there exists $c_k \in R\backslash\{0\}$ such that $c_k m_{ik} \in Nil(M_k)$. Therefore, $R(c_k m_{ik})$ is nilpotent in M_k. So the ideal $(R(c_k m_{ik}) :_R M_k)$ is nilpotent, for each $1 \leq k \leq n$ and $0 \leq i \leq p$, by [1, Proposition 4]. Consider $c = c_1 c_2 \ldots c_n$. Clearly, $(R(cm_{ik}) :_R M_k) = (R(c_k m_{ik}) :_R M_k)$. Thus $(R(m_{ik}) :_R M_k)$ is a nilpotent ideal, for each $i = 0, 1, \ldots, p$ and $k = 0, 1, \ldots, n$.

On the other hand,

$$(R(cm_i) :_R M) = (\prod_{k=1}^{n} R(cm_{ik}) :_R \prod_{k=1}^{n} M_k) = \bigcap_{k=1}^{n}(R(cm_{ik}) :_R M_k).$$

Intersection of finite number of nilpotent ideals is nilpotent. This implies that $(R(cm_i) :_R M)$ is a nilpotent ideal and therefore $cm_i \in Nil(M)$, for each $i = 0, 1, \ldots, p$. □

Proposition 2.15. *A ring R is weak McCoy if and only if it is weak McCoy as an R-module.*

Proof. We know R is a multiplication R-module. So by [1, Theorem 6], $Nil(R) = Nil(R)R$. Therefore the proof is obvious. □

Proposition 2.16. *Let R be a ring and N a submodule of an R-module M such that $\frac{M}{N}$ is a faithful weak McCoy R-module. Then M is weak McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]\backslash\{0\}$ and $f(x) = \sum_{j=0}^{k} a_j x^j \in R[x]\backslash\{0\}$ with $f(x)m(x) = 0$. Set $\overline{m}(x) = \sum_{i=0}^{n} \overline{m_i} x^i = \sum_{i=0}^{n} (m_i + N)x^i \in (\frac{M}{N})[x]\backslash\{0\}$. Hence $f(x)\overline{m}(x) = \bar{0} = N$. By hypothesis, there exists $c \in R\backslash\{0\}$ such that

$c\overline{m_i} \in Nil(\frac{M}{N})$, for every $0 \leq i \leq n$. Since R-module $\frac{M}{N}$ is faithful, by [1, Proposition 4], $(R(c\overline{m_i}) :_R \frac{M}{N})$ is a nilpotent ideal of R, for each $0 \leq i \leq n$. But $(R(cm_i) :_R M) \subseteq (R(c\overline{m_i}) :_R \frac{M}{N})$. Thus $(R(cm_i) :_R M)$ is a nilpotent ideal and hence $cm_i \in Nil(M)$ for each i. Therefore, M is weak McCoy. □

For a commutative domain R and an R-module M, the torsion submodule of M is defined by $T(M) = \{x \in M | ann_R(x) \neq 0\}$. In this case, $T(M)$ is a submodule of M, called the torsion part of M. An R-module M is called a *torsion* module if $T(M) = M$.

Proposition 2.17. *Let D be a domain and M a D-module. If D-module $T(M)$ is weak McCoy, then M is also weak McCoy.*

Proof. Suppose that $T(M)$ is weak McCoy. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in D[x]\backslash\{0\}$ and $m(x) = \sum_{j=0}^{n} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. We have $a_0 m_0 = a_0 m_1 + a_1 m_0 = a_0 m_2 + a_1 m_1 + a_2 m_0 = \ldots = a_k m_n = 0$.

Since $f(x) \neq 0$, we may assume that $a_0 \neq 0$. Multiplying $a_0 m_1 + a_1 m_0 = 0$ by a_0 implies that $a_0^2 m_1 = 0$. Similarly, multiplying $a_0 m_2 + a_1 m_1 + a_2 m_0 = 0$ by a_0^2 implies that $a_0^3 m_2 = 0$. Continuing this process, we have $a_0^{j+1} m_j = 0$ and so $m_j \in T(M)$, for $j = 0, 1, \ldots, n$. Hence $m(x) \in T(M)[x]$. Since $T(M)$ is a weak McCoy D-module, there exists $d \in D\backslash\{0\}$ such that $dm_j \in Nil(T(M))$. Therefore, for each $0 \leq j \leq n$, there exists positive integer k_j such that $(D(dm_j) :_D T(M))^{k_j} D(dm_j) = 0$.

But $(D(dm_j) :_D M) \subseteq (D(dm_j) :_D T(M))$. Thus, $(D(dm_j) :_D M)^{k_j} D(dm_j) = 0$, for $j = 0, 1, \ldots, n$. Therefore, $dm_j \in Nil(M)$, for $j = 0, 1, \ldots, n$ and hence M is a weak McCoy D-module. □

Theorem 2.18. *Let R be an integral domain and Q the quotient field of R. If M is a weak McCoy Q-module, then M is weak McCoy as an R-module and the converse is true if M is finitely generated.*

Proof. Suppose that M is a weak McCoy Q-module. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{n} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. First note that every element $q \in Q$ has the form rs^{-1} for some regular element $s \in R$. Since M is weak McCoy as an Q-module, there exists a nonzero element $q \in Q$ such that $qm_j \in Nil_Q(M)$, for every $j = 0, 1, \ldots, k$. We have $q = rs^{-1}$. So $rm_j = sqm_j \in Nil_Q(M)$, for every $j = 0, 1, \ldots, k$. Moreover, M is a faithful Q-module, since $M \neq 0$. Hence the ideal $(Q(rm_j) :_Q M)$ is a nilpotent ideal of Q, for each $0 \leq j \leq k$, by [1, Theorem 4(1)]. Hence M is a weak McCoy R-module.

For the converse, suppose that M is a finitely generated weak McCoy R-module. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in Q[x]\backslash\{0\}$ and $m(x) = \sum_{j=0}^{n} m_j x^j$ belongs to the $Q[x]$-module $M[x]$. We may assume that $a_i = a_i' s_i^{-1}$, for every $0 \leq i \leq n$ and for some $a_i' \in R$ and regular element $s_i \in R$. Setting $s = s_0 s_1 \ldots s_n$, we have $a_i = a_i' s^{-1}$, for every $0 \leq i \leq n$. Then $0 = f(x)m(x) = \sum_{i=0}^{n} \sum_{j=0}^{k} s^{-1} a_i m_j x^{i+j} = s^{-1} f(x)m(x)$. So $f'(x)m(x) = 0$.

Since M is weak McCoy as an R-module, there exists $r \in R\backslash\{0\}(\subseteq Q\backslash\{0\})$ such that $rm_j \in Nil(M)$. By assumption, M is a faithful R-module. Thus $(R(rm_j) :_R M)$ is a nilpotent ideal of R, for each $j = 0, 1, \ldots, k$. But $Q(R(rm_j) :_R M) = (Q(rm_j) :_Q M)$, since M is finitely generated. Hence $(Q(rm_j) :_Q M)$ is a nilpotent ideal of Q and so $rm_j \in Nil_Q(M)$, for every $0 \leq j \leq k$. Therefore, M is a weak McCoy Q-module. □

References

[1] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra, 36 (2008), 4620–4642.

[2] M. M. Ali and D. J. Smith, *Pure submodules of multiplication modules*, Contributions to Algebra and Geometry, 45(1) (2004), 61–74.

[3] J. Cui and J. Chen, *On McCoy modules*, Bull. Korean Math. Soc., 48 (2011), 23–33.

[4] C. Faith, *Algebra I, Rings, Modules and Categories*, Springer-Verlag, New York, (1981), 205-207.

[5] E. Hashemi, *Annihilator conditions on polynomials over modules*, Mediterr. J. Math., 8 (2011), 207-214.

[6] T. K. Lee and Y. Zhou, *Reduced Modules, Rings, modules, algebras and abelian groups*, Lect. Notes Pure Appl. Math., 236, Dekker, New york, (2004) 365-377.

[7] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly, 49 (1942), 286-295.

[8] J. J. Rotman, *An Introduction to Homological Algebra*, Springer, New York, second edition, 2009.

Accepted: 9.12.2016

SOME RESULTS ON DISTANCED-BALANCED AND STRONGLY DISTANCE-BALANCED GRAPHS

H. Faramarzi
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R.
Iran
hamed.faramarzi@gmail.com

F. Rahbarnia
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R.
Iran
rahbarnia@um.ac.ir

M. Tavakoli
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R.
Iran
m_tavakoli@um.ac.ir

Abstract. Distance-balanced graphs are graphs in which for every edge $e = uv$ the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. The graph is strongly distance-balanced if the relation holds for every distance of i which $0 \leq i \leq diam(G)$. In this paper, we study some local properties of them and also under some graph operations.

Keywords: distance-balanced graph, strongly distance-balanced graph, graph invariant, graph operation.

1. Introduction and preliminaries

Let G be a simple undirected graph. The distance $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. For a pair of adjacent vertices $a, b \in V(G)$ let W_{ab} denote the set of all vertices of G closer to a than to b and let ${}_aW_b$ denote the set of all vertices of G that are at the same distance to a and b. For each $i \geq 0$ let W_{ab}^i and ${}_aW_b{}^i$ be the subsets of W_{ab} and ${}_aW_b$ respectively, of all the vertices at distance i to a. Therefore

$$W_{ab} = \{x \in V(G) | d(a, x) < d(b, x)\}, {}_aW_b = \{x \in V(G) | d(a, x) = d(b, x)\},$$

$$W_{ab}^i = \{x \in W_{ab} | d(a, x) = i\}, {}_aW_b{}^i = \{x \in {}_aW_b | d(a, x) = i\}.$$

Distance-balanced graphs were introduced as graphs that $|W_{ab}| = |W_{ba}|$ for every pair of adjacent vertices $a, b \in V(G)$. A graph G is strongly distance-balanced if $W^i_{ab} = W^i_{ba}$ holds for every pair of adjacent vertices $a, b \in V(G)$ and every $i \geq 0$. For more information, we recommend the readers to look at [4, 1, 2, 5]. Note that every strongly distance-balanced graph is distance-balanced. Throughout this article, graphs are supposed to be simple and connected. Moreover, we denote the neighbours of v by $N(v)$.

2. Local properties

Let G be a regular graph with v vertices and degree k. G is said to be strongly regular with (v, k, μ, λ) if there are also integers λ and μ such that:

1. Every two adjacent vertices have μ common neighbours.
2. Every two non-adjacent vertices have λ common neighbours.

Theorem 2.1. *Let G is a strongly regular graph with (v, k, μ, λ), then G is strongly distance-balanced and hence distance-balanced.*

Proof. Let $\lambda = 0$, then every $v \in V(G)$ is connected to the neighbours of $N(v)$, therefore G is a complete graph and strongly distance-balanced. On the other hand, if $\lambda \neq 0$, then for every $uv \in E(G)$, $|W^1_{uv}| = |W^1_{vu}| = k - \mu$ and since $dim(G) \leq 2$ then $|W^2_{uv}| = |W^2_{vu}| = |V(G)| - (k - \mu) - |_uW^2_v|$. Therefore G is strongly distance-balanced and hence distance-balanced.

Let $u \in V(G)$. We define $d(u, G)$ and $d_i(u, G)$ by

$$d(u, G) = \sum_{v \in V(G)} d(u, v)$$

and

$$d_i(u, G) = \sum_{v \in V(G), d(u,v)=i} d(u, v) \ ,$$

respectively, which $1 \leq i \leq diam(G)$. Moreover, $diam(G)$ is the length of the longest shortest path between any two graph vertices of G.

Theorem 2.2. *G is strongly distance-balanced if and only if $d_i(u, G) = d_i(v, G)$ for each $1 \leq i \leq diam(G)$ and for each $u, v \in V(G)$.*

Proof. Suppose $d_i(u, G) = d_i(v, G)$ for each $1 \leq i \leq diam(G)$ and for each $u, v \in V(G)$. It is clear that if $d_1(u, G) = d_1(v, G)$ then $|W^1_{uv}| = |W^1_{vu}|$. Now By induction, suppose the statement holds for $i \leq k$:

$$d_i(u, G) = d_i(v, G) \rightarrow |W^i_{uv}| = |W^i_{vu}|, 1 \leq i \leq k.$$

Let $d_{k+1}(u, G) = d_{k+1}(v, G)$, then

$$d_{k+1}(u, G) = d_{k+1}(u, W^{k+1}_{uv}) + (d_k(v, W^k_{vu}) + |W^k_{vu}|) + d_{k+1}(u, {}_uW_v{}^{k+1}) =$$

$$d_{k+1}(v,G) = d_{k+1}(v, W_{vu}^{k+1}) + (d_k(u, W_{uv}^k) + |W_{uv}^k|) + d_{k+1}(v,_v W_u{}^{k+1}) .$$

Therefore

$$d_{k+1}(u, W_{uv}^{k+1}) = d_{k+1}(v, W_{vu}^{k+1}),$$

which implies that

$$|W_{uv}^{k+1}| = |W_{vu}^{k+1}|.$$

For sufficiency, let G be a strongly distance-balanced graph, which means that for each $1 \leq i \leq diam(G)$ and $uv \in E(G)$, $|W_{uv}^i| = |W_{vu}^i|$. Therefore

$$|W_{uv}^i| \times i + |W_{vu}^{i-1}| \times (i-1) + |_u W_v{}^i| \times i =$$

$$|W_{vu}^i| \times i + |W_{uv}^{i-1}| \times (i-1) + |_v W_u{}^i| \times i,$$

which shows that

$$d_i(u,G) = d_i(v,G).$$

Let G be a graph and $e \in E(G)$, then by $G.e$ we mean contraction of e that is the operation which removes the edge from the graph while simultaneously merging the two vertices that it previously joined. By $G.[e_1, \cdots, e_n]$, we mean $(((G.e_1).e_2).\cdots.e_n)$.

Theorem 2.3. *If G is regular, then G is distance-balanced if and only if*

$$W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) = W_{vu}(G.[vq_j]_{vu \neq vq_j \in E(G)})$$

for each $u, v \in V(G)$, which $[up_i]_{uv \neq up_i \in E(G)}$ and $[vq_j]_{vu \neq vq_j \in E(G)}$ are all edges connected to u and v respectively, except for uv and vu.

Proof. Since

$$|W_{uv}| = |W_{uv}^1| + |W_{p_1 u}| + \cdots + |W_{p_n u}| - C_{uv},$$

where $p_i (1 \leq i \leq n)$ are vertices connected to u that $p_i \neq v$ and C_{uv} consists of members x such that

1. if $x \in W_{p_i u}$ but $x \in W_{vu}$ or $x \in_u W_v$.
2. $x \in W_{p_i u}$ for more than one i.

Since members of C_{uv} and $W_{p_i u} (p_i \in N(u) \backslash v)$ are not related to neighbours of u and v, therefore

$$W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) = (|W_{uv}| - |W_{uv}^1|) + |_u W_v|$$

Hence

$$\begin{aligned}
&|W_{uv}| = |W_{vu}|,\\
&|W_{uv}^1| + |W_{up_1}| + \cdots + |W_{up_i}| - |C_{uv}| + |_u W_v|\\
&= |W_{vu}^1| + |W_{vq_1}| + \cdots + |W_{vq_i}| - |C_{vu}| + |_v W_u|\\
&W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) = W_{vu}(G.[vq_j]_{vu \neq vq_j \in E(G)}).
\end{aligned}$$

Graph G is a chordal graph if all its cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Theorem 2.4. *Let G be a chordal graph. G is complete if G is distance-balanced.*

Proof. If G be a chordal graph on 3 vertices and distance-balanced, then G is complete. By induction, suppose G is a chordal graph on $n > 3$ vertices and distance-balanced, then G is complete. Now let G be a graph on $n+1$ vertices, T be the left corner triangle in G, $u \in V(T)$ and x, y be other vertices of T. If there is no other vertex connected to u, therefore G is not distance-balanced ($|W_{ux}| \neq |W_{xu}|$). Since the graph is chordal, suppose p_1 be a vertex such that $up_1, xp_1 \in E(G)$. If there is no vertex connected to p_1, then G is not distance-balanced ($|W_{p_1u}| \neq |W_{up_1}|$). By chordal property, let p_2 be a vertex on such that $p_1p_2, up_2 \in E(G)$. By continuing this method, u is connected to all vertices of G. By removing u from G, $G - \{u\}$ is again chordal and distance-balanced. Therefore, $G - \{u\}$ is complete which implies that the statement holds for each graph with $n > 3$ vertices.

Let G be a graph, $u \in V(G)$, $N(u) = \{p_1, p_2, \cdots, p_n\}$ and $1 \leq i < j \leq n$. We denote the set $\{W_{up_1} \cap \cdots \cap W_{up_{i-1}} \cap W_{up_{i+1}} \cap \cdots \cap W_{up_{j-1}} \cap W_{up_{j+1}} \cap \cdots \cap W_{up_n}\}$ by $S_u(p_i, p_j)$.

Lemma 2.5. *Let G be a distance-balanced graph which $diam(G) > 1$. Then G is triangle-free if and only if for each $uv, uw \in E(G)$, $|S_u(v,w) \cap W_{wu} \cap_u W_v| = |S_u(v,w) \cap W_{vu} \cap_u W_w|$.*

Proof. Let $uv, uw \in E(G)$, then by the property of triangle-free, we have

$$_uW_v \cap W_{wu} \cap S_u(v,w) = \{x \in S_u(v,w) | d(w,x) = d(v,x) + 1\},$$

$$_uW_w \cap W_{vu} \cap S_u(v,w) = \{x \in S_u(v,w) | d(p_k,x) = d(v,x) - 1\}.$$

Now, we define the function

$$f :_u W_v \cap W_{wu} \cap S_u(v,w) \to_u W_w \cap W_{vu} \cap S_u(v,w),$$

$$x \to y \ \ (y \in p(v, \cdots, y, x)),$$

where $p(v, \cdots, y, x)$ is the shortest path between v and x. Since w is not connected to v (triangle-free), the function is well-defined and bijective. Therefore

$$|W_{wu} \cap_u W_v \cap S_u(v,w)| = |W_{vu} \cap_u W_w \cap S_u(v,w)|.$$

For the sufficiency, suppose $N(u) = \{v, w, p_1, p_2, \cdots, p_n\}$ and for each $uv, uw \in E(G)$, $|S_u(v,w) \cap W_{wu} \cap_u W_v| = |S_u(v,w) \cap W_{vu} \cap_u W_w|$. By contrary, if $vw \in E(G)$, then $|S_u(v,p_i) \cap W_{p_iu} \cap_u W_v| = 0$ for each $1 \leq i \leq n$. Therefore $vp_i \in E(G)$ for each $1 \leq i \leq n$, and consequently neighbours of any vertex are connected together and by theorem 2.4, the graph is complete.

Theorem 2.6. *Let G be a distance-balanced graph which $diam(G) > 1$. Then for each $uv, ab \in E(G)$, $|W_{vu}| = |W_{ab}|$ if and only if G is triangle-free and for each $u \in V(G)$ and $uv, uw \in E(G)$, $|S_u(v,w) \cap W_{vu}| = |S_u(v,w) \cap W_{wu}|$.*

Proof. Let G be a distance-balanced graph which $diam(G) > 1$, $uv, uw \in E(G)$ and for each $uv, ab \in E(G)$, $|W_{vu}| = |W_{ab}|$. Then

$$|W_{uv}| = deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}| - |_uW_v|,$$

$$|W_{uw}| = deg(u) + |W_{uw}G.[up_i]_{uw \neq uq_j \in E(G)}| - |_uW_w|.$$

Therefore we have

$$|W_{uv}| + |_uW_v| = deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}|,$$

$$|W_{uw}| + |_uW_w| = deg(u) + |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|$$

and

$$|V(G)| - |W_{uv}| = deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}|,$$

$$|V(G)| - |W_{uw}| = deg(u) + |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|\ ,$$

which implies that

$$|W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}| = |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|.$$

Therefore

$$|(W_{p_1u} \cup \cdots \cup W_{p_nu} \cup W_{wu}) \cap (W_{uv} \cup_u W_v)| =$$
$$|(W_{p_1u} \cup \cdots \cup W_{p_nu} \cup W_{vu}) \cap (W_{uw} \cup_u W_w)|\ ,$$

which shows that

$$|(W_{uv} \cup_u W_v)| - |(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uw})| =$$

$$|(W_{uw} \cup_u W_w)| - |(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uv})|.$$

Therefore we have

$$|(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uw})| =$$
$$|(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uv})|\ ,$$

which implies that

$$|(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uw} \cap W_{vu})| =$$

$$|(W_{up_1} \cap \cdots \cap W_{up_n} \cap W_{uv} \cap W_{wu})|\ ,$$

and finally

$$|(S_u(v,w) \cap W_{uw} \cap W_{vu})| =$$
$$|(S_u(v,w) \cap W_{uv} \cap W_{wu})|.$$

On the other hand,

$$|W_{wu}| = |W_{wu} \cap_u W_v| + |W_{wu} \cap W_{uv}| + |W_{wu} \cap W_{vu}|$$

$$|W_{vu}| = |W_{vu} \cap_u W_w| + |W_{vu} \cap W_{uw}| + |W_{vu} \cap W_{wu}|.$$

Therefore

$$|S_u(v,w) \cap W_{wu}| - |S_u(v,w) \cap W_{wu} \cap_u W_v| =$$
$$|S_u(v,w) \cap W_{vu}| - |S_u(v,w) \cap W_{vu} \cap_u W_w|.$$

Since $(S_u(v,w) \cap W_{wu} \cap_u W_v) \subseteq (S_u(v,w) \cap W_{wu})$ and $(S_u(v,w) \cap W_{vu} \cap_u W_w) \subseteq (S_u(v,w) \cap W_{vu})$, we have

$$|S_u(v,w) \cap W_{wu} \cap_u W_v| = |S_u(v,w) \cap W_{vu} \cap_u W_w|$$

and

$$|S_u(v,w) \cap W_{wu}| = |S_u(v,w) \cap W_{vu}|.$$

Therefore by lemma 2.5, the graph is triangle-free and $|S_u(v,w) \cap W_{wu}| = |S_u(v,w) \cap W_{vu}|$ for each $u \in V(G)$ and $v, w \in N(u)$. The sufficiency is straightforward with regard to the reverse of the proof and lemma 2.5.

Corollary 2.7. *Let G be a k-regular distance-balanced and triangle-free graph with $diam(G) = 2$. Then $|W_{ab}| = k$ for each $ab \in E(G)$.*

Lemma 2.8. *Let G be a connected graph with at least one leaf and $n > 2$ vertices. Then G is not distance-balanced.*

Proof. Let $uv \in E(G)$ which v be a leaf of G. Then $|W_{vu}| = 0$ which implies that $|W_{vu}| < |W_{uv}|$.

Lemma 2.9. *If G is a connected distance-balanced graph, then $\kappa(G) \geq 2$.*

Proof. By contrary, suppose that $\kappa(G) = 1$ and $u \in V(G)$ be the cut-vertex, and $C_1, C_2, \cdots, C_n$ be components of $G - \{u\}$. Let C_p be the component such that

$$|V(C_p)| \leq |V(C_i)| \ \ (1 \leq i \leq n).$$

Then by selecting $e = uv$ such that $v \in V(C_p)$, we have $|W_{uv}| > |W_{vu}|$.

Theorem 2.10. *Let G be a connected distance-balanced graph. Then for each $u, v \in V(G)$, there exists a cycle containing u and v.*

Proof. The proof is clear up to lemma 2.8 and lemma 2.9.

Wiener index W is a graph index defined for a graph on n vertices by

$$W = 1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij},$$

where d_{ij} is the graph distance matrix. In [7], it has been conjectured that C_n has the most wiener value among distance-balanced graphs.

Theorem 2.11. *Let G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$W(G) \leq W(C_n).$$

Proof. Since $W(G+e) \leq W(G)$, then suppose G does not contain C_n as its subgraph. Moreover, since $W(G) = \frac{n \times d(u,G)}{2}$, and the fact that between every $u, v \in V(G)$, there exists a cycle in G containing u and v, thus

$$d(u, G) \leq d(x, C_n),$$

where $x \in V(C_n)$ and therefore

$$W(G) \leq W(C_n).$$

3. Strongly distance-balanced graphs under product graphs

All of the graph products constructed from two graphs G and H have vertex set $V(G) \times V(H)$. Let (a, u) and (b, v) be two vertices in $V(G) \times V(H)$. They are adjacent in the Cartesian product $G \square H$ if they are equal in one coordinate and adjacent in the other and are adjacent in the direct product $G \times H$ if they are adjacent in both coordinates. The edge set of the strong product $G \boxtimes H$ is $E(G \times H) \cup E(G \square H)$. These vertices are adjacent in the lexicographic product $G \circ H$ if $ab \in E(G)$ or if $a = b$ and $uv \in E(H)$. The edge set of disjunction $G \vee H$ of graphs G and H is such that (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$, see [3]. Moreover, the edge corona of two graphs G and H is denoted by $G \diamond H$ is obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertices of i-th edge of G to every vertex in the i-th copy of H. We have denoted $ab \in E(G \diamond H)$ if both a and b belong to G and $(ab, x) \in E(G \diamond H)$ if $x \in V(H)$ is connected to $a, b \in V(G)$ $(ab \in E(G))$ in $G \diamond H$. For more information, refer to [3].

By G_i and H_j, we mean i-th copy and j-th copy of G and H, respectively. Moreover, $|W_{ab}|_G$, $|_aW_b|_G$ and $|_aW_b{}^1|_G$ denotes $|W_{ab}|$ and $|_aW_b|$ in graph G.

Theorem 3.1. *$G \vee H$ is strongly distance-balanced if and only if G and H are strongly distance-balanced.*

Proof. Let G, H be strongly distance-balanced graphs on n and m vertices, respectively. Then

$$\begin{aligned} |W^1_{(a,x)(b,y)}| &= |W^1_{ab}|_{G_1} + \cdots |W^1_{ab}|_{G_m} + |W^1_{xy}|_{H_1} + \cdots |W^1_{xy}|_{H_n} \\ &\quad - deg(a)deg(x) - |_{(a,x)}W^1_{(b,y)}| \\ |W^1_{(b,y)(a,x)}| &= |W^1_{ba}|_{G_1} + \cdots |W^1_{ba}|_{G_m} + |W^1_{yx}|_{H_1} + \cdots |W^1_{yx}|_{H_n} \\ &\quad - deg(b)deg(y) - |_{(a,x)}W^1_{(b,y)}|, \end{aligned}$$

and consequently,

$$|W^2_{(a,x)(b,y)}| = |V(G)| \times |V(H)| - |W^1_{(a,x)(b,y)}| - |_{(a,x)}W^2_{(b,y)}|.$$

Therefore,

$$|W^1_{(a,x)(b,y)}|=m(deg(a)-|_aW^1_b|_G)+n(deg(x)-|_xW^1_y|_H)-deg(a)deg(x)-|_{(a,x)}W^1_{(b,y)}|,$$

$$|W^1_{(b,y)(a,x)}|=m(deg(b)-|_aW^1_b|_G)+n(deg(y)-|_xW^1_y|_H)-deg(b)deg(y)-|_{(a,x)}W^1_{(b,y)}|.$$

Hence $G \vee H$ is strongly distance-balanced. Now, suppose $G \vee H$ be a strongly distance-balanced. Since $|W^1_{(a,x)(b,y)}| = |W^1_{(b,y)(a,x)}|$, then by summing up over all $ab \in E(G)$, we have:

$$\begin{aligned}
&\sum_{ab\in E(G)} m(deg(a) - deg(b)) + n(deg(x) - deg(y))\\
&= \sum_{ab\in E(G)} (deg(b)deg(y) - deg(a)deg(x))\\
&mn\frac{|E(G)|}{2} - mn\frac{|E(G)|}{2} + n^3(deg(x) - deg(y))\\
&= n\frac{|E(G)|}{2}deg(y) - n\frac{|E(G)|}{2}deg(x)\\
&n^2(deg(x) - deg(y)) = \frac{|E(G)|}{2}(deg(y) - deg(x)).
\end{aligned}$$

Since $|E(G)| < 2n^2$, therefore $deg(y) = deg(x)$, $deg(b) = deg(a)$ and G, H are strongly distance-balanced.

Theorem 3.2. *$G \diamond H$ is strongly distance-balanced and hence distance-balanced if and only if $G = P_2$ and H is complete.*

Proof. Let $G = P_2$ and H be a complete graph, $ab \in E(G)$ and $xy \in E(H)$. Then

$$|W^1_{(ab,x)(ab,y)}| = |W^1_{xy\in H}|,$$

$$|W^2_{(ab,x)(ab,y)}| = |W^2_{(ab,y)(ab,x)}| = 0.$$

Therefore $|W_{(ab,x)(ab,y)}| = |W^1_{(ab,x)(ab,y)}| = deg_H(x) - |_xW^1_y|_H = |W_{(ab,y)(ab,x)}| = |W^1_{(ab,y)(ab,x)}| = deg_H(y) - |_xW^1_y|_H$. Moreover,

$$|W^1_{a(ab,x)}| = (deg_G(a) - 1) + (deg_G(a) - 1)|V(H)|,$$

$$|W_{a(ab,x)}| = (|V(G)|-2)+(|E(G)|-1)|V(H)|+(|V(H)|-deg_H(x)-1)-|_aW_{(ab,x)}|,$$

$$|W_{(ab,x)a}| = 0.$$

Then

$$|W_{a(ab,x)}| = |W_{(ab,x)a}| = 0$$

and finally

$$|W_{ab}|_{G\diamond H} = |W_{ab}|_G + \sum_{c,d\in W_{ab} \text{ in } G, cd\in E(G)} |V(H)|,$$

which implies that

$$|W_{ab}|_{G\diamond H} = |W_{ba}|_{G\diamond H} = 0.$$

On the other hand, suppose that $G \diamond H$ is (strongly) distance-balanced graph. Then

$$|W^1_{(ab,x)(ab,y)}| = deg_H(x) - |_xW^1_y|_H = |W^1_{(ab,y)(ab,x)}| = deg_H(y) - |_xW^1_y|_H,$$

which implies that H is regular and $(deg_G(a) - 1) + (deg_G(a) - 1)|V(H)| = 0$ shows that $deg_G(a) = 1$ for each $a \in V(G)$. Moreover, $|W_{a(ab,x)}| = 0$ implies that $deg_H(x) = |V(H)| - 1$ for each $x \in V(H)$.

4. Minimum edges for distance-balanced

Let G be a graph and let $b(G)$ be the smallest number of edges that can be added to G such that the obtained graph is distance-balanced.

Lemma 4.1. *Let G, H be graphs. Then*

1. $b(G\square H) = b(G) \times |V(H)| + b(H) \times |V(G)|$.
2. $b(G \circ H) = b(G) \times |V(H)| + r(H) \times |V(G)|$.

where $r(H) = \frac{deg_{max}(H)+1)\times(V(H)-1)}{2} + 1$ if $deg_{max}(H)$ and $V(H) - 1$ are odd and $r(H) = \frac{deg_{max}(H)\times(V(H)-1)}{2}$, otherwise.

Proof.

1. Since $G\square H$ is distance-balanced if and only if G and H are distance-balanced, the proof is clear.
2. Since $G \circ H$ is distance-balanced if and only if G is distance-balanced and H is regular, the proof is clear.

Theorem 4.2. *Let G is a graph on n vertices with $diam(G) = 2$. then*

$$1/2 \sum_{uv\in E(G)} |deg(u) - deg(v)| \leq b(G) \leq |E(K_n)| - E(G).$$

Proof. Let $uv \in E(G)$, then for G to be distanced-balanced, we should have $d(u, G) = d(v, G)$. Therefore,

$$d(u, G) = deg(u) + 2 \times (|V(G)| - deg(u) - 1),$$
$$d(v, G) = deg(v) + 2 \times (|V(G)| - deg(v) - 1),$$

$$\begin{aligned} b(G) \geq & \sum_{uv\in E(G)} 1/2|(deg(u) + 2 \times (|V(G) - deg(u) - 1)) \\ & - (deg(v) + 2 \times (|V(G)| - deg(v) - 1))| \\ & = 1/2 \sum_{uv\in E(G)} |deg(u) - deg(v)|. \end{aligned}$$

On the other hand, it is clear that $b(G) \leq |E(K_n)| - |E(G)|$.

References

[1] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl and A. R. Subhamathi, *Strongly distance-balanced graphs and graph products,* European J. Combin., 30 (2009), 1048-1053.

[2] A. Ilić, S. Klavžar and M. Milanović, *On distance-balanced graphs*, European J. Combin., 31 (2010), 733-737.

[3] S. Klavžar and W. Imrich, *Product graphs: structure and recognition*, Wiley, New York, 2000.

[4] S. Klavžar, J. Jerebic and D.F. Rall, *Distance-balanced graphs*, Anna. Comb., 12 (2008), 71-79.

[5] M.H. Khalifeh, H. Yousefi-Azari, A.R. Ashrafi, and S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin., 30 (2009), 1149-1163.

[6] H. Narumi and M. Katayama, *Simple Topological Index: A Newly Devised Index Characterizing The Topological Nature of Structural Isomers of Saturated Hydrocarbons*, Memoirs of the Faculty of Engineering, Hokkaido University, 16 (1984) 209-214.

[7] M. Tavakoli, F. Rahbarnia and A.R. Ashrafi, *Further results on Distance-Balanced graphs*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 75 (2013), 77-84.

Accepted: 11.12.2016

A POWER INDEX AND ITS NORMALIZATION UNDER FUZZY MULTICRITERIA SITUATION

Yu-Hsien Liao*
Department of Applied Mathematics
National Pingtung University
Pingtung 900
Taiwan
twincos@ms25.hinet.net

Tsu-Yin Chen
Department of Cosmetic Science
Chang Gung University of Science and Technology
Taiwan
tychen@gw.cgust.edu.tw

Ling-Yun Chung
Graduate School of Technological and Vocational Education
National Yunlin University of Science and Technology
Taiwan
darlingyun@gmail.com

Abstract. By considering the supreme-utilities among fuzzy level vectors, we propose an index and its normalization in the framework of multicriteria fuzzy transferable-utility (TU) games. We show that there exists a reduced game that could be adopted to analyze these two indexes. Further, alternative formulation for the normalized index is also proposed by applying excess function. Based on reduced game and excess function, we introduce different dynamic processes for the normalized index.

Keywords: multicriteria fuzzy TU games, supreme-utilities, reduced game, excess function, dynamic process.

1. Introduction

In the framework of transferable-utility (TU) games, the power indexes may be defined to measure the political power of each member of a voting system. A member in a voting system is, e.g., a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Banzhaf [3], Dubey and Shapley [8], Haller [9], Lehrer [11], van den Brink and van der Laan [5] and so on. The theory of *fuzzy TU games* commenced with the investigation of Aubin [1, 2] where the opinions of a fuzzy TU game and the fuzzy core are introduced. Many fuzzy solutions have been applied wildly, e.g.,

∗. Corresponding author

Borkotokey and Mesiar [4], Butnariu and Kroupa [6], Hwang [10], Li and Zhang [12], Meng and Zhang [14], Tijs et al. [19], and so on.

In the axiomatic formulation of cooperative games, *consistency* is a crucial characteristic of viable solutions. Consistency states the independence of a value with respect to fixing some agents with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some agents are settled on. It has been introduced in different ways depending upon how the payoffs of the agents that "leave the bargaining" are defined. This property has been investigated in various classes of problems by applying *reduced games* always. In addition to axiomatizations for solutions, *dynamic processes* also can be described that lead the agents to that solution, starting from an arbitrary efficient payoff vector. The foundation of a dynamic theory was laid by Stearns [18].

The above pre-existing results raise one motivation in the framework of fuzzy TU games:

- whether the power indexes could be extended under fuzzy behavior and multicriteria situation simultaneously.

The paper is devoted to answer the question. Different from the framework of fuzzy TU games, we consider the framework of *multicriteria fuzzy TU games* in Section 2. A power index and its normalization on multicriteria fuzzy TU games, the *fuzzy marginal index* and the *fuzzy normalized index*, are further introduced by considering the supreme-utilities among fuzzy level vectors. In order to present the rationality for these two indexes, we propose an extended reduction to analyze these two indexes in Section 3. In order to establish the dynamic processes of the fuzzy normalized index, we present alternative formulation for the fuzzy normalized index in terms of *excess functions*. In Section 4, we adopt reduced game and excess function to show that the fuzzy normalized index can be reached by agents who start from an arbitrary efficient payoff vector.

2. The fuzzy marginal index and its normalization

Let U be the universe of agents. For $i \in U$ and $b_i \in (0,1]$, $B_i = [0, b_i]$ could be treated as the level (decision) space of agent i and $B_i^+ = (0, b_i]$, where 0 denotes no participation. Let $B^N = \prod_{i\in N} B_i$ be the product set of the level (decision) spaces of all agents of N. For all $T \subseteq N$, we define $\theta^T \in B^N$ is the vector with $\theta_i^T = 1$ if $i \in T$, and $\theta_i^T = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy TU game**[1] is a triple (N, b, v), where N is a non-empty and finite set of agents, $b = (b_i)_{i\in N} \in \prod_{i\in N} B_i^+$ is the vector that presents the highest

1. A fuzzy TU game, which is defined by Aubin [1, 2], is a pair (N, v^a), where N is a coalition and v^a is a mapping such that $v^a : [0,1]^N \longrightarrow \mathbb{R}$ and $v^a(0_N) = 0$. In fact, $(N, v^a) = (N, \theta^N, v)$.

levels for each agent, and $v : B^N \to \mathbb{R}$ is a characteristic mapping with $v(0_N) = 0$ which assigns to each $\alpha = (\alpha_i)_{i\in N} \in B^N$ the worth that the agents can gain when each agent i participates at level α_i. Given a fuzzy TU game (N, b, v) and $\alpha \in B^N$, we write $A(\alpha) = \{i \in N |\ \alpha_i \neq 0\}$ and α_T to be the restriction of α at T for each $T \subseteq N$. Further, we define $v_*(T) = \sup_{\alpha\in B^N}\{v(\alpha)|A(\alpha) = T\}$ is the **supreme-utility**[2] among all action vector α with $A(\alpha) = T$. A **multicriteria fuzzy TU game** is a triple (N, b, V^m), where $m \in \mathbb{N}$, $V^m = (v^t)_{t\in\mathbb{N}_m}$ and (N, b, v^t) is a fuzzy TU game for all $t \in \mathbb{N}_m$.

Denote the collection of all multicriteria fuzzy TU games by Γ. Let $(N, b, V^m) \in \Gamma$. A **payoff vector** of (N, b, V^m) is a vector $x = (x^t)_{t\in\mathbb{N}_m}$ and $x^t = (x_i^t)_{i\in N} \in \mathbb{R}^N$, where x_i^t denotes the payoff to agent i in (N, b, v^t) for all $t \in \mathbb{N}_m$ and for all $i \in N$. A payoff vector x of (N, b, V^m) is **multicriteria efficient** if $\sum_{i\in N} x_i^t = v_*^t(N)$ for all $t \in \mathbb{N}_m$. The collection of all multicriteria efficient vector of (N, b, V^m) is denoted by $E(N, b, V^m)$. A **solution** is a map σ assigning to each $(N, b, V^m) \in \Gamma$ an element

$$\sigma\big(N, b, V^m\big) = \big(\sigma^t\big(N, b, V^m\big)\big)_{t\in\mathbb{N}_m},$$

where $\sigma^t\big(N, b, V^m\big) = \big(\sigma_i^t\big(N, b, V^m\big)\big)_{i\in N} \in \mathbb{R}^N$ and $\sigma_i^t\big(N, b, V^m\big)$ is the payoff of the agent i assigned by σ in $\big(N, b, v^t\big)$.

Next, we provide the fuzzy marginal index and the fuzzy normalized index under multicriteria situation.

Definition 1. *The* **fuzzy marginal index (FMI)**, *β, is defined by*

$$\beta_i^t(N, b, V^m) = v_*^t(N) - v_*^t(N \setminus \{i\})$$

for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. Under the solution β, all agents receive their **marginal contributions** *related to supreme-utilities in N respectively.*

A solution σ satisfies **multicriteria efficiency (MEFF)** if for all $(N, b, V^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i\in N} \sigma_i(N, b, V^m) = v_*^t(N)$. Property MEFF asserts that all agents allocate all the utility completely. It is easy to check that the FMI violates EFF. Therefore, we consider an efficient normalization as follows.

Definition 2. *The* **fuzzy normalized index (FNI)**, *$\overline{\beta}$, is defined by*

$$\overline{\beta_i^t}(N, b, V^m) = \frac{v_*^t(N)}{\sum\limits_{k\in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m)$$

for all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, where $\Gamma^* = \{(N, b, V^m) \in \Gamma \mid \sum\limits_{i\in N} \beta_i^t(N, b, V^m) \neq 0$ for all $t \in \mathbb{N}_m\}$.

2. From now on we consider bounded fuzzy TU games, defined as those games (N, b, v) such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \leq K_v$ for all $\alpha \in B^N$. We adopt it to ensure that $v_*(T)$ is well-defined.

Lemma 1. *The FNI satisfies MEFF on* Γ^*.

Proof. For all $(N, b, V^m) \in \Gamma^*$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}\sum_{i\in N} \overline{\beta_i^t}(N, b, V^m) &= \sum_{i\in N} \frac{v_*^t(N)}{\sum\limits_{k\in N} \beta_k^t(N,b,V^m)} \cdot \beta_i^t(N, b, V^m) \\ &= \frac{v_*^t(N)}{\sum\limits_{k\in N} \beta_k^t(N,b,V^m)} \sum_{i\in N} \beta_i^t(N, b, V^m) \\ &= v_*^t(N).\end{aligned}$$

Thus, the FNI satisfies MEFF on Γ^*. □

Here we provide a brief application of multicriteria fuzzy TU games in the setting of "management". This kind of problem can be formulated as follows. Let $N = \{1, 2, \cdots, n\}$ be a set of all agents of a grand management system (N, b, V^m). The function v^t could be treated as an utility function which assigns to each level vector $\alpha = (\alpha_i)_{i\in N} \in B^N$ the worth that the agents can obtain when each agent i participates at operation strategy $\alpha_i \in B_i$ in the sub-management system (N, b, v^t). Modeled in this way, the grand management system (N, b, V^m) could be considered as a multicriteria fuzzy TU game, with v^t being each characteristic function and B_i being the set of all operation strategies of the agent i. In the following sections, we would like to show that the FMI and the FNI could provide "*optimal allocation mechanisms*" among all agents, in the sense that this organization can get payoff from each combination of operation strategies of all agents under fuzzy behavior and multicriteria situation.

3. Axiomatizations

In this section, we show that there exists a corresponding reduced game that could be adopted to analyze the FMI and the FNI.

First, we present alternative formulation for the FNI in terms of *excess*. Let $(N, b, V^m) \in \Gamma^*$, $S \subseteq N$ and x be a payoff vector in (N, b, V^m). Define that $x^t(S) = \sum_{i\in S} x_i^t$ for all $t \in \mathbb{N}_m$. The **excess** of a coalition $S \subseteq N$ at x is the real number

$$e(S, V^m, x) = (e(S, v^t, x^t))_{t\in\mathbb{N}_m} \text{ and } e(S, v^t, x^t) = v_*^t(S) - x^t(S). \tag{1}$$

The value $e(S, v^t, x^t)$ can be treated as the **complaint** of coalition S when all agents receive their payoffs from x^t in (N, b, v^t).

Lemma 2. *Let* $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$, $x \in E(N, b, V^m)$ *and* $a^t = \frac{v_*^t(N)}{\sum\limits_{k\in N} \beta_k^t(N,b,V^m)}$. *Then*

$$e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) = e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) \ \ \forall\, i, j \in N \Longleftrightarrow x = \overline{\beta}(N, b, V^m).$$

Proof. Let $(N, b, V^m) \in \Gamma^*$ and $x \in E(N, b, V^m)$. For all $t \in \mathbb{N}_m$ and for all $i, j \in N$,

$$
\begin{aligned}
& e(N \setminus \{j\}, v^t, \tfrac{x^t}{a^t}) = e(N \setminus \{i\}, v^t, \tfrac{x^t}{a^t}) \\
\iff & v^t_*(N \setminus \{j\}) - \tfrac{x^t(N\setminus\{j\})}{a^t} = v^t_*(N \setminus \{i\}) - \tfrac{x^t(N\setminus\{i\})}{a^t} \\
\iff & v^t_*(N \setminus \{j\}) - \tfrac{x^t_i}{a^t} = v^t_*(N \setminus \{i\}) - \tfrac{x^t_j}{a^t} \\
\iff & x^t_i - x^t_j = a^t \cdot [v^t_*(N \setminus \{j\}) - v^t_*(N \setminus \{i\})].
\end{aligned} \tag{2}
$$

By definition of $\overline{\beta}$,

$$
\overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m) \quad = a^t \cdot [v^t_*(N \setminus \{j\}) - v^t_*(N \setminus \{i\})]. \tag{3}
$$

By equations (2) and (3), for all $i, j \in N$,

$$
x^t_i - x^t_j = \overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m).
$$

Hence,

$$
\sum_{j \neq i} [x^t_i - x^t_j] = \sum_{j \neq i} [\overline{\beta^t_i}(N, b, V^m) - \overline{\beta^t_j}(N, b, V^m)].
$$

That is, $(|N| - 1) \cdot x^t_i - \sum_{j \neq i} x^t_j = (|N| - 1) \cdot \overline{\beta^t_i}(N, b, V^m) - \sum_{j \neq i} \overline{\beta^t_j}(N, b, V^m)$. Since $x \in E(N, b, V^m)$ and $\overline{\beta}$ satisfies MEFF, $|N| \cdot x^t_i - v^t_*(N) = |N| \cdot \overline{\beta^t_i}(N, b, V^m) - v^t_*(N)$. Therefore, $x^t_i = \overline{\beta^t_i}(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in N$, i.e., $x = \overline{\beta}(N, b, V^m)$. □

Remark 1. It is easy to check that $e(N \setminus \{i\}, V^m, \beta(N, b, V^m)) = e(N \setminus \{j\}, V^m, \beta(N, b, V^m))$ for all $(N, b, V^m) \in \Gamma$ and for all $i, j \in N$.

Inspired by the complement-reduced game due to Moulin [15], we introduced a fuzzy extension and related consistency as follows. Let ψ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **reduced game** $(S, b_S, V^m_{S,\psi})$ is defined by $V^m_{S,\psi} = (v^t_{S,\psi})_{t \in \mathbb{N}_m}$ and

$$
v^t_{S,\psi}(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0_S, \\ v_*\big(A(\alpha) \cup (N \setminus S)\big) - \sum\limits_{i \in N \setminus S} \sigma_i(N, b, v), & \text{otherwise.} \end{cases}
$$

ψ satisfies **consistency (CON)** if $\psi^t_i(S, b_S, V^m_{S,\psi}) = \psi^t_i(N, b_S, V^m)$ for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$ with $|S| = 2$, for all $t \in \mathbb{N}_m$ and for all $i \in S$. Unfortunately, it is easy to check that $\sum_{k \in S} \beta^t_k(N, b, v) = 0$ for some $(N, b, V^m) \in G$, for some $t \in \mathbb{N}_m$ and for some $S \subseteq N$, i.e., $\overline{\beta}(S, b_S, V^m_{S,\psi})$ doesn't exist for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$. Therefore, we consider the *resilient consistency* as follows. A solution ψ satisfies **resilient consistency (RCON)** if $(S, b_S, V^m_{S,\psi})$ and $\psi(S, b_S, V^m_{S,\psi})$ exist for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$ with $|S| = 2$, it holds that $\psi^t_i(S, b_S, V^m_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in S$.

Lemma 3.

1. *The FMI satisfies CON on* Γ.
2. *The FNI satisfies RCON on* Γ^*.

Proof. To verify result 1, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$ and $S = \{i, j\}$ for some $i, j \in N$. For all $t \in \mathbb{N}_m$ and for all $i \in S$,

$$
\begin{aligned}
& \beta_i^t(S, b_S, V^m_{S,\beta}) \\
= \; & (v^t_{S,\beta})_*(S) - (v^t_{S,\beta})_*(S \setminus \{i\}) \\
= \; & \sup_{\alpha \in B^S} \{v^t_{S,\beta}(\alpha) | A(\alpha) = S\} - \sup_{\alpha \in B^S} \{v^t_{S,\beta}(\alpha) | A(\alpha) = S \setminus \{i\}\} \\
= \; & v^t_*(N) - v^t_*(N \setminus \{i\}) \\
= \; & \beta_i^t(N, b, V^m).
\end{aligned} \tag{4}
$$

Thus, the FMI satisfies CON.

To prove result 2, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$. If $S = \{i, j\}$ for some $i, j \in N$ and $(S, b_S, v^m_{S,\overline{\beta}}) \in \Gamma^*$. Similar to the proof of equation (4), for all $t \in \mathbb{N}_m$ and for all $i \in S$,

$$
\beta_i^t(S, b_S, V^m_{S,\overline{\beta}}) = \beta_i^t(N, b, V^m). \tag{5}
$$

By definition of $\overline{\beta}$ and equation (5),

$$
\begin{aligned}
& \overline{\beta_i^t}(S, b_S, V^m_{S,\overline{\beta}}) \\
= \; & \frac{(v^t_{S,\overline{\beta}})_*(S)}{\sum\limits_{k \in S} \beta_k^t(S, b_S, V^m_{S,\overline{\beta}})} \cdot \beta_i^t(S, b_S, V^m_{S,\overline{\beta}}) \\
= \; & \frac{v^t_*(N) - \sum\limits_{k \in N \setminus S} \overline{\beta_k^t}(N, b, V^m)}{\sum\limits_{k \in S} \beta_k^t(S, b_S, V^m_{S,\overline{\beta}})} \cdot \beta_i^t(S, b_S, V^m_{S,\overline{\beta}}) && \textbf{(by definition of } V^m_{S,\overline{\beta}}\textbf{)} \\
= \; & \frac{v^t_*(N) - \sum\limits_{k \in N \setminus S} \overline{\beta_k^t}(N, b, V^m)}{\sum\limits_{k \in S} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) && \textbf{(by equation (5))} \\
= \; & \frac{\sum\limits_{k \in S} \overline{\beta_k^t}(N, b, V^m)}{\sum\limits_{k \in S} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) && \textbf{(by MEFF of } \overline{\beta}\textbf{)} \\
= \; & a^t \cdot \beta_i^t(N, b, V^m), \text{ where } a^t = \frac{v^t_*(N)}{\sum\limits_{k \in N} \beta_k^t(N, b, V^m)} \\
= \; & \overline{\beta_i^t}(N, b, V^m).
\end{aligned}
$$

Thus, the FNI satisfies RCON on Γ^*. □

Next, we characterize the FMI and the FNI by applying the properties of CON and RCON.

- A solution ψ satisfies **marginal-standard for games (MSG)** if $\psi(N,b,v) = \beta(N,b,v)$ for all $(N,b,v) \in \Gamma$ with $|N| \leq 2$.
- A solution ψ satisfies **normalized-standard for games (NSG)** if $\psi(N,b,v) = \overline{\beta}(N,b,v)$ for all $(N,b,v) \in \Gamma^*$ with $|N| \leq 2$.

Lemma 4. *If a solution ψ satisfies NSG and RCON on Γ^*, then it also satisfies MEFF on Γ^*.*

Proof. Let $(N,b,V^m) \in \Gamma^*$. If $|N| \leq 2$, then ψ satisfies MEFF on Γ^* by NSG. Suppose that $|N| > 2$. Assume, on the contrary, that there exists $(N,b_S,V^m) \in \Gamma^*$ such that $\sum_{i\in N}\psi_i^t(N,b,V^m) \neq v_*^t(N)$ for some $t \in \mathbb{N}_m$. This means that there exist $i \in N$ and $j \in N$ such that $[v_*^t(N) - \sum_{k\in N\setminus\{i,j\}}\psi_k^t(N,b,V^m)] \neq [\psi_i^t(N,b,V^m) + \psi_j^t(N,b,V^m)]$. By RCON and ψ satisfies MEFF for two-person games, this contradicts with

$$\begin{aligned}\psi_i^t(N,b,V^m) + \psi_j^t(N,b,V^m) &= \psi_i^t(\{i,j\},V^m_{\{i,j\},\psi}) + \psi_j^t(\{i,j\},V^m_{\{i,j\},\psi}) \\ &= v_*^t(N) - \sum_{k\in N\setminus\{i,j\}}\psi_k^t(N,b,V^m).\end{aligned}$$

Hence ψ satisfies MEFF. □

Theorem 1.

1. *On Γ, the FMI is the only solution satisfying MSG and CON.*
2. *On Γ^*, the FNI is the only solution satisfying NSG and RCON.*

Proof. By Lemma 3, β and $\overline{\beta}$ satisfy CON and RCON on Γ and Γ^* respectively. Clearly, β and $\overline{\beta}$ satisfy MSG and NSG on Γ and Γ^* respectively.

To prove uniqueness of result 1, suppose ψ satisfies CON and MSG on Γ. Let $(N,b,V^m) \in \Gamma$. If $|N| \leq 2$, then $\psi(N,b,V^m) = \beta(N,b,V^m)$ by MSG. Suppose that $|N| > 2$. Let $t \in \mathbb{N}_m$ and $i \in N$. Assume that $S \subseteq N$ with $|S| = 2$ and $i \in S$. Then,

$$\begin{aligned}\psi_i^t(N,b,V^m) &= \psi_i^t(S,b_S,V^m_{S,\psi}) \ \textbf{(by CON of } \psi\textbf{)} \\ &= \beta_i^t(S,b_S,V^m_{S,\psi}) \ \textbf{(by MSG of } \psi\textbf{)} \\ &= (v^t_{S,\psi})_*(S) - (v^t_{S,\psi})_*(S\setminus\{i\}) \\ &= v_*^t(N) - v_*^t(N\setminus\{i\}) \\ &= \beta_i^t(N,b,V^m).\end{aligned}$$

Hence, $\psi(N,b,V^m) = \beta(N,b,V^m)$ for all $(N,b,V^m)\Gamma$.

To prove uniqueness of result 2, suppose ψ satisfies RCON and NSG on Γ^*. By Lemma 4, ψ satisfies MEFF on Γ^*. Let $(N,b,V^m) \in \Gamma^*$. We will complete the proof by induction on $|N|$. If $|N| \leq 2$, it is trivial that $\psi(N,b,V^m) = \overline{\beta}(N,b,V^m)$ by NSG. Assume that it holds if $|N| \leq r-1$, $r \geq 3$. The case $|N| = r$: Let $t \in \mathbb{N}_m$ and $i,j \in N$ with $i \neq j$. By Definition 2, $\overline{\beta_k^t}(N,b,V^m) =$

$\frac{v_*^t(N)}{\sum\limits_{h\in N}\beta_h^t(N,b,V^m)}\cdot\beta_k^t(N,b,V^m)$ for all $k\in N$. Assume that $\alpha_k^t=\frac{\beta_k^t(N,b,v)}{\sum\limits_{h\in N}\beta_h^t(N,b,v)}$ for all $k\in N$. Therefore,

$$
\begin{aligned}
\psi_i^t(N,b,V^m) &= \psi_i^t\big(N\setminus\{j\},V^m_{N\setminus\{j\},\psi}\big) && \textbf{(by RCON of } \psi\textbf{)}\\
&= \overline{\beta_i^t}\big(N\setminus\{j\},V^m_{N\setminus\{j\},\psi}\big) && \textbf{(by NSG of } \psi\textbf{)}\\
&= \frac{(v^t_{N\setminus\{j\},\psi})_*\big(N\setminus\{j\}\big)}{\sum\limits_{k\in N\setminus\{j\}}\beta_k^t\big(N\setminus\{j\},V^m_{N\setminus\{j\},\psi}\big)}\cdot\beta_i^t\big(N\setminus\{j\},V^m_{N\setminus\{j\},\psi}\big) && (6)\\
&= \frac{v_*^t(N)-\psi_i^t(N,b,V^m)}{\sum\limits_{k\in N\setminus\{j\}}\beta_k^t(N,b,V^m)}\cdot\beta_i^t(N,b,V^m) && \textbf{(by equation (5))}\\
&= \frac{v_*^t(N)-\psi_i^t(N,b,V^m)}{-\beta_j^t(N,b,V^m)+\sum\limits_{k\in N}\beta_k^t(N,b,V^m)}\cdot\beta_i^t(N,b,V^m).
\end{aligned}
$$

By equation (6),

$$
\begin{aligned}
& \psi_i^t(N,b,V^m)\cdot[1-\alpha_j^t]=[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot\alpha_j^t\\
\Longrightarrow\ & \sum_{i\in N}\psi_i^t(N,b,V^m)\cdot[1-\alpha_j^t]=[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot\sum_{i\in N}\alpha_j^t\\
\Longrightarrow\ & v_*^t(N)\cdot[1-\alpha_j^t]=[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot 1 \quad \textbf{(by MEFF of } \psi\textbf{)}\\
\Longrightarrow\ & v_*^t(N)-v_*^t(N)\cdot\alpha_j^t=v_*^t(N)-\psi_j^t(N,b,V^m)\\
\Longrightarrow\ & \overline{\beta_j^t}(N,b,V^m)=\psi_j^t(N,b,V^m).
\end{aligned}
$$

The proof is completed. □

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

Example 1. Define a solution ψ by for all $(N,b,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$, $\psi_i^t(N,b,V^m)=0$. Clearly, ψ satisfies CON and RCON on Γ and Γ^*, but it violates MSG and NSG on Γ and Γ^*.

Example 2. Define a solution ψ by for all $(N,b,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,b,V^m)=\begin{cases}\beta_i^t(N,b,V^m), & \text{if } |N|\le 2,\\ 0, & \text{otherwise.}\end{cases}$$

On Γ, ψ satisfies MSG, but it violates CON.

Example 3. Define a solution ψ by for all $(N,b,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,b,V^m)=\begin{cases}\overline{\beta}_i(N,b,V^m), & \text{if } |N|\le 2,\\ 0, & \text{otherwise.}\end{cases}$$

On Γ^*, ψ satisfies NSG, but it violates RCON.

4. Dynamic processes

In this section, we adopt excess function and reduction to propose dynamic processes for the FNI.

In order to establish the dynamic processes of the FNI, we firstly define correction function by means of excess functions. The correction function is based on the idea that, each agent shortens the complaint relating to his own and others' nonparticipation, and adopts these regulations to correct the original payoff.

Definition 3. *Let $(N,b,V^m)\in\Gamma^*$ and $i\in N$. The* **correction function** *is defined to be $f=(f^t)_{t\in\mathbb{N}_m}$, where $f^t=(f_i^t)_{i\in N}$ and $f_i^t: E(N,b,V^m)\to\mathbb{R}$ is define by*

$$f_i^t(x)=x_i^t+w\sum_{j\in N\setminus\{i\}}a^t\cdot\Big(e(N\setminus\{j\},v^t,\frac{x^t}{a^t})-e(N\setminus\{i\},v^t,\frac{x^t}{a^t})\Big),$$

where $a^t=\frac{v_*^t(N)}{\sum\limits_{k\in N}\beta_k^t(N,b,V^m)}$ and $w\in\mathbb{R}$ is a fixed postive number, which reflects the assumption that agent i does not ask for full correction (when $w=1$) but only (usually) a fraction of it. Define $[x]^0=x,\ [x]^1=f([x]^0),\cdots,\ [x]^q=f([x]^{q-1})$ for all $q\in\mathbb{N}$.

Lemma 5. *$f(x)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^*$ and for all $x\in E(N,b,V^m)$.*

Proof. Let $(N,b,V^m)\in\Gamma^*$, $t\in\mathbb{N}_m$, $i,j\in N$ and $x\in E(N,b,V^m)$.

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big)\\
=&\sum_{j\in N\setminus\{i\}}a^t\cdot\Big(v^t(N\setminus\{j\})-\tfrac{x^t(N\setminus\{j\})}{a^t}-v^t(N\setminus\{i\})+\tfrac{x^t(N\setminus\{i\})}{a^t}\Big)\qquad(7)\\
=&\sum_{j\in N\setminus\{i\}}a^t\cdot\Big(v^t(N\setminus\{j\})-v^t(N\setminus\{i\})-\tfrac{x_i^t}{a^t}+\tfrac{x_j^t}{a^t}\Big).
\end{aligned}$$

By definition of $\overline{\beta}$,

$$\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_j^t}(N,b,V^m)=a^t\cdot\Big(v^t(N\setminus\{j\})-v^t(N\setminus\{i\})\Big).\qquad(8)$$

By equations (7) and (8),

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v,\tfrac{x^t}{a^t})\Big)\\
=&\sum_{j\in N\setminus\{i\}}\Big(\overline{\beta_i^t}(N,b,V^m)-\overline{\beta_j^t}(N,b,V^m)-x_i^t+x_j^t\Big)\\
=&\Big((|N|-1)\big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\big)-\sum_{j\in N\setminus\{i\}}\overline{\beta_j^t}(N,b,V^m)+\sum_{j\in N\setminus\{i\}}x_j^t\Big)\qquad(9)\\
=&\Big(|N|\big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\big)-v_*^t(N)+v_*^t(N)\Big)\\
&\qquad\qquad\textbf{(by MEFF of }\overline{\beta}\textbf{, }x\in E(N,b,V^m))\\
=&\ |N|\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
& \sum_{i\in N}\sum_{j\in N\setminus\{i\}} a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big)\\
= & \sum_{i\in N}|N|\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big)\\
= & |N|\cdot\Big(\sum_{i\in N}\overline{\beta_i^t}(N,b,V^m)-\sum_{i\in N}x_i^t\Big)\\
= & |N|\cdot\big(v_*^t(N)-v_*^t(N)\big) \qquad \textbf{(by MEFF of } \overline{\beta}\textbf{, } x\in E(N,b,V^m)\textbf{)}\\
= & \ 0.
\end{aligned}
\tag{10}
$$

So we have that

$$
\begin{aligned}
\sum_{i\in N} f_i^t(x) &= \sum_{i\in N}\Big[x_i^t+w\sum_{j\in N\setminus\{i\}} a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big)\Big]\\
&= \sum_{i\in N}x_i^t+w\sum_{i\in N}\sum_{j\in N\setminus\{i\}} a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big)\\
&= v_*^t(N)+0 \qquad \big(\textbf{by equation (10) and } x\in E(N,b,V^m)\big)\\
&= v_*^t(N).
\end{aligned}
$$

Hence, $f(x)\in E(N,b,V^m)$ if $x\in E(N,b,V^m)$. □

Theorem 2. *Let $(N,b,V^m)\in\Gamma^*$. If $0<t<\frac{2}{|N|}$, then $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*

Proof. Let $(N,b,V^m)\in\Gamma^*$, $t\in\mathbb{N}_m$, $i\in N$ and $x\in E(N,b,V^m)$. By equation (9) and definition of f,

$$
\begin{aligned}
f_i^t(x)-x_i^t &= w\sum_{j\in N\setminus\{i\}} a^t\cdot\Big(e(N\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(N\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big)\\
&= w\cdot|N|\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\overline{\beta_i^t}(N,b,V^m)-f_i^t(x) &= \overline{\beta_i^t}(N,b,V^m)-x_i^t+x_i^t-f_i^t(x)\\
&= \overline{\beta_i^t}(N,b,V^m)-x_i^t-w\cdot|N|\cdot(\overline{\beta_i^t}(N,b,V^m)-x_i^t)\\
&= \Big(1-w\cdot|N|\Big)\Big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big].
\end{aligned}
$$

So, for all $q\in\mathbb{N}$,

$$
\overline{\beta}(N,b,V^m)-[x]^q=\Big(1-w\cdot|N|\Big)^q\Big[\overline{\beta}(N,b,V^m)-x\Big].
$$

If $0<w<\frac{2}{|N|}$, then $-1<\Big(1-w\cdot|N|\Big)<1$ and $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta}(N,b,V^m)$. □

Inspired by Maschler and Owen [13], we will find a dynamic process under reductions.

Definition 4. *Let ψ be a solution, $(N,b,V^m)\in\Gamma^*$, $S\subseteq N$ and $x\in E(N,b,V^m)$. The (x,ψ)-**reduced game** $(S,b_S,V^m_{\psi,S,x})$ is given by $V^m_{\psi,S,x}=(v^t_{\psi,S,x})_{t\in\mathbb{N}_m}$ and for all $T\subseteq S$,*

$$v^t_{\psi,S,x}(\alpha)=\begin{cases} v^t_*(N)-\sum\limits_{i\in N\setminus S}x^t_i, & A(\alpha)=S,\\ v^t_{S,\psi}(\alpha), & \text{otherwise.}\end{cases}$$

Inspired by Maschler and Owen [13], we also define different correction function as follow. The **R-correction function** to be $g=(g^t)_{t\in\mathbb{N}_m}$, where $g^t=(g^t_i)_{i\in N}$ and $g^t_i:E(N,b,V^m)\to\mathbb{R}$ is define by

$$g^t_i(x)=x^t_i+w\sum_{k\in N\setminus\{i\}}\Big(\overline{\beta^t_i}\big(\{i,k\},v^t_{\overline{\beta},\{i,k\},x}\big)-x^t_i\Big).$$

Define $[\eta]^0=x,\ [\eta]^1=g([\eta]^0),\cdots,\ [\eta]^q=g([\eta]^{q-1})$ for all $q\in\mathbb{N}$.

Lemma 6. *$g\big(x\big)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^*$ and for all $x\in E(N,b,V^m)$.*

Proof. Let $(N,b,V^m)\in\Gamma^*$, $t\in\mathbb{N}_m$, $i,k\in N$ and $x\in X(N,b,v)$. Let $S=\{i,k\}$, by MEFF of $\overline{\beta}$ and Definition 4,

$$\overline{\beta^t_i}(S,b_S,V^m_{\overline{\beta},S,x})+\overline{\beta^t_k}(S,b_S,V^m_{\overline{\beta},S,x})=x^t_i+x^t_k.$$

By RCON and NSG of $\overline{\beta}$,

$$\begin{aligned}\overline{\beta^t_i}(S,b_S,v^{com}_{\overline{\beta},S,x})-\overline{\beta^t_k}(S,b_S,V^m_{\overline{\beta},S,x}) &=(v^t_{\overline{\beta},S,x})_*(\{i\})-(v^t_{\overline{\beta},S,x})_*(\{k\})\\ &=(v^t_{S,\overline{\beta}})_*(\{i\})-(v^t_{S,\overline{\beta}})_*(\{k\})\\ &=\overline{\beta^t_i}(S,b_S,V^m_{S,\overline{\beta}})-\overline{\beta^t_k}(S,b_S,V^m_{S,\overline{\beta}})\\ &=\overline{\beta^t_i}(N,b,V^m)-\overline{\beta^t_k}(N,b,V^m).\end{aligned}$$

Therefore,

$$2\cdot\Big[\overline{\beta^t_i}(S,b_S,V^m_{\overline{\beta},S,x})-x^t_i\Big]=\overline{\beta^t_i}(N,b,V^m)-\overline{\beta^t_k}(N,b,V^m)-x^t_i+x^t_k. \tag{11}$$

By definition of g and equation (11),

$$\begin{aligned} g^t_i(x) &= x^t_i+\tfrac{w}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\beta^t_i}(N,b,V^m)-\sum_{k\in N\setminus\{i\}}x^t_i\\ &\qquad -\sum_{k\in N\setminus\{i\}}\overline{\beta^t_k}(N,b,V^m)+\sum_{k\in N\setminus\{i\}}x^t_k\Big]\\ &= x^t_i+\tfrac{w}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\beta^t_i}(N,b,V^m)-\big(|N|-1\big)x^t_i\\ &\qquad -\sum_{k\in N\setminus\{i\}}\overline{\beta^t_k}(N,b,v)+\big(v^t_*(N)-x^t_i\big)\Big] \qquad (12)\\ &= x^t_i+\tfrac{w}{2}\cdot\Big[\big(|N|-1\big)\overline{\beta^t_i}(N,b,V^m)-\big(|N|-1\big)x^t_i\\ &\qquad -\big(v^t_*(N)-\overline{\beta^t_i}(N,b,V^m)\big)+\big(v^t_*(N)-x^t_i\big)\Big]\\ &= x^t_i+\tfrac{|N|\cdot w}{2}\cdot\Big[\overline{\beta^t_i}(N,b,V^m)-x^t_i\Big].\end{aligned}$$

So we have that

$$\begin{aligned}\sum_{i\in N} g_i^t(x) &= \sum_{i\in N}\Big[x_i^t + \tfrac{|N|\cdot w}{2}\cdot\big[\overline{\beta_i^t}(N,b,V^m) - x_i^t\big]\Big]\\ &= \sum_{i\in N} x_i^t + \tfrac{|N|\cdot w}{2}\cdot\big[\sum_{i\in N}\overline{\beta_i^t}(N,b,V^m) - \sum_{i\in N} x_i^t\big]\\ &= v_*^t(N) + \tfrac{|N|\cdot w}{2}\cdot\big[v_*^t(N) - v_*^t(N)\big]\\ &= v_*^t(N).\end{aligned}$$

Thus, $g\big(x\big) \in E(N,b,V^m)$ for all $x \in E(N,b,V^m)$. □

Theorem 3. *Let $(N,b,V^m) \in \Gamma^*$. If $0 < \alpha < \frac{4}{|N|}$, then $\{[\eta]^q\}_{q=1}^{\infty}$ converges to $\overline{\beta}(N,b,V^m)$ for each $x \in E(N,b,V^m)$.*

Proof. Let $(N,b,V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$ and $x \in E(N,b,V^m)$. By equation (12), $g_i^t(x) = x_i^t + \frac{|N|\cdot w}{2}\cdot\Big[\overline{\beta_i^t}(N,b,V^m) - x_i^t\Big]$ for all $i \in N$. Therefore,

$$\big(1 - \frac{|N|\cdot w}{2}\big)\cdot\big[\overline{\beta_i^t}(N,b,V^m) - x_i^t\big] = \big[\overline{\beta_i^t}(N,b,V^m) - g_i^t(x)\big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\beta}(N,b,V^m) - [\eta]^q = \Big(1 - \frac{|N|\cdot w}{2}\Big)^q\Big[\overline{\beta}(N,b,V^m) - x\Big].$$

If $0 < w < \frac{4}{|N|}$, then $-1 < \Big(1 - \frac{|N|\cdot w}{2}\Big) < 1$ and $\{[\eta]^q\}_{q=1}^{\infty}$ converges to $\overline{\beta}(N,b,v)$ for all $(N,b,V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. □

5. Conclusions

In this paper, we investigate the normalizations for the fuzzy marginal index and the fuzzy normalized index. Based on reduced game, two axiomatizations for these two indexes are proposed. By applying reduction and excess function, we also introduce alternative formulations and related dynamic processes for the normalized index. One should compare our results with related pre-existing results:

- The fuzzy marginal index and the fuzzy normalized index are introduced initially in the framework of multicriteria fuzzy TU games.
- The idea of our correction functions in Definitions 3, 4 and related dynamic processes are based on that of Maschler and Owen's [13] dynamic results for the Shapley value [17]. The major difference is that our correction functions in Definition 4 are based on "excess function", and Maschler and Owen's [13] correction function is based on "reduced games".

These mentioned above raise one question:

- Whether there exist other normalizations and related results for some more solutions in the framework of multicriteria fuzzy TU games.

To our knowledge, these issues are still open questions.

Acknowledgements

The authors are very grateful to the Editor and the anonymous referees for valuable comments which much improved the paper.

References

[1] J.P. Aubin, *Coeur et valeur des jeux flous á paiements latéraux*, Comptes Rendus de l'Académie des Sciences, 279 (1974), 891-894

[2] J.P. Aubin, *Cooperative fuzzy games*, Mathematics of Operations Research, 6 (1981), 1-13.

[3] J.F. Banzhaf, JF (1965) *Weighted voting doesn't work: A mathematical analysis*, Rutgers Law Review, 19 (1965), 317-343, Problemy Kybernetiki 10, 119-139 (in Russian).

[4] S. Borkotokey, R. Mesiar, *The Shapley value of cooperative games under fuzzy settings: a survey*, International Journal of General Systems, 43 (2014), 75-95.

[5] R. van den Brink, G. van der Lann, *Axiomatizations of the normalized banzhaf value and the Shapley value*, Social Choice and Welfare, Kluwer Academic Publishers, Dordrecht, 15 (1998), 567-582.

[6] D. Butnariu, T. Kroupa, *Shapley mappings and the cumulative value for n-person games with fuzzy coalitions*, Eur. J. Oper. Res., 186 (2008), 288-299, Mathematical Social Sciences 34, 175-190

[7] M. Davis, M. Maschler, *The kernel of a cooperative game*, Naval Research Logistics Quarterly, 12 (1965), 223-259.

[8] P. Dubey, L.S. Shapley, *Mathematical properties of the Banzhaf power index*, Mathematics of Operations Research, 4 (1979), 99-131 Econometrica, 57, 589-614.

[9] H. Haller, *Collusion properties of values*, International Journal of Game Theory, 23 (1994), 261-281.

[10] Y.A. Hwang, *Fuzzy games: a characterization of the core*, Fuzzy Sets and Systems, 158 (2007), 2480-2493, International Journal of Game Theory, 29, 597-623.

[11] E. Lehrer, *An axiomatization of the Banzhaf value*, International Journal of Game Theory, 17 (1988), 89-99.

[12] S. Li, Q. Zhang, *A simplified expression of the Shapley function for fuzzy game*, Eur. J. Oper. Res., 196 (2009), 234-245.

[13] M. Maschler, G. Owen, *The consistent Shapley value for hyperplane games*, International Journal of Game Theory, 18 (1989), 389-407.

[14] F. Meng, Q. Zhang, *The Shapley value on a kind of cooperative fuzzy games*, Journal of Computational Information Systems, 7 (2011), 1846-1854.

[15] H. Moulin, *On additive methods to share joint costs*, The Japanese Economic Review, 46 (1995), 303-332.

[16] S. Muto, S. Ishihara, S. Fukuda, S. Tijs and R. Branzei, *Generalized cores and stable sets for fuzzy games*, International Game Theory Review, 8 (2006), 95-109.

[17] L.S. Shapley, *A value for n-person game*, In: Kuhn, H.W., Tucker, A.W. (eds.), Contributions to the Theory of Games II, Princeton, 307-317, 1953.

[18] R.E. Stearns, *Convergent transfer schemes for n-person games*, Trans. Amer. Math. Soc., 134 (1968), 449-459.

[19] S. Tijs, R. Branzei, S. Ishihara, S. Muto, *On cores and stable sets for fuzzy games*, Fuzzy Sets and Systems, 146 (2004), 285-296.

Accepted: 22.01.2017

SOME SUFFICIENT CONDITIONS IMPLYING NILPOTENCY OF FINITE GROUPS

Qingjun Kong*
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300387
People's Republic of China
kqj2929@163.com

Shuai Wang
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300387
People's Republic of China

Abstract. Let G be a finite group. For a weak n-Engel condition, we mean that $[x,_n y] \in Z(G)$ for two elements x and y of G, where n is a positive integer. In this paper, we mainly study the influence of the weak n-Engel condition on the nilpotence and p-nilpotence of finite groups. Our results generalize some well-known results.
Keywords: weak n-Engel condition, nilpotent groups, p-nilpotent group.

1. Introduction

As we know, the n-Engel condition is that $[x,_n y] = 1$ for two elements x and y of G, where n is a positive integer. Obviously, a finite group which satisfies the n-Engel condition must satisfy the weak n-Engel condition, but the inverse is not true. Many scholars investigate the influence of the n-Engel condition on the nilpotence of finite groups. In [1], Huppert. B proved that let x and y be arbitrary element of G, if there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent. In [2], It is proved that let $G = AB$, where A and B are nilpotent subgroups of G. If $[A, B]$=1, then G is nilpotent. In this paper, we mainly discuss the influence of the weak n-Engel condition on the nilpotence and p-nilpotence of finite groups and generalize some well-known results.

We shall use the following notation for commutators:
$[x, y] = x^{-1}y^{-1}xy$,
$[x_1, x_2, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n](n \geq 3)$,
$[x,_0 y] = x$,
$[x,_n y] = [[x,_{n-1} y], y](n \geq 1)$

The rest of the notation is standard (see [2]), all groups mentioned are assumed to be finite groups.

∗. Corresponding author

2. Basic definitions and preliminary results

In this section, we give one definition and some results that are needed in this paper.

Definition 2.1. *Let G be a finite group. For a weak n-Engel condition, we mean that $[x,_n y] \in Z(G)$ for two elements x and y of G, where n is a positive integer.*

Lemma 2.2 ([2])**.** *Let $G = AB$, where A and B are nilpotent subgroups of G. If $[A, B]$=1, then G is nilpotent.*

Lemma 2.3 ([2])**.** *If $G/\Phi(G)$ is nilpotent, then G is nilpotent.*

3. Main results

Theorem 3.1. *Let $G = AB$, where A is a normal nilpotent subgroup of G and B is a nilpotent subgroup of G with$(|A|, |B|) = 1$. If for each element x in A and each element y in B there is a positive integer n such that $[x,_n y] \in Z(G)$, then G is nilpotent.*

Proof. Since $[x, y] \in Z(G)$ means $[x,_2 y] \in Z(G)$, we may assume $n \geq 2$. Let $a = [x,_{n-2} y]$, by the hypothesis we have $[(y^{-1})^a y, y] = [a, y, y] \in Z(G)$. It follows that $(y^{-1})^a yy = y(y^{-1})^a yz$ for some $z \in Z(G)$, that is, $(y^{-1})^a y = y(y^{-1})^a z$, i,e., $[(y^{-1})^a, y] \in Z(G)$. We consider the following cases:

(i) If $|B|_2 = 1$ or $|B|_2 > 1$ and q is an odd prime, then $o(y)|C^2_{o(y)}$ and $((y^{-1})^a)^{o(y)} = ((y^{-1})^a)^{o(y)} y^{o(y)} ([y, (y^{-1})^a]^{o(y)})^{C^2_{o(y)}/o(y)} = 1$. Hence $(y^{-1})^a y$ is a q-element of G.

(ii) If $|B|_2 > 1$ and q is 2, then $((y^{-1})^a y)^{2o(y)} = 1$ and $(y^{-1})^a y$ is a 2-element of G.

On the other hand, since A is a normal subgroup of G, $(y^{-1})^a y = [a, y] \in A$. And $(y^{-1})^a y$ is a p-element. Therefore, by the above discussion, $[a, y] = (y^{-1})^a y = 1$. That is, $[x,_{n-1} y] = 1$. Now, a simple induction on n, we have $[x, y]$=1. Since $(|A|, |B|) = 1$, it follows that $[A, B]$=1. By Lemma 2.2 we have G is nilpotent.

If we drop the restriction that A is a normal subgroup of G, we can prove the following result.

Theorem 3.2. *Let $G = AB$, where A and B are nilpotent subgroups of G with $(|A|, |B|) = 1$. If for each element x in A and each element y in B there is a positive integer n such that $[x,_n y] \in Z(G)$, then G is nilpotent.*

Proof. Let M be an arbitrary maximal subgroup of G. Since G is solvable, it follows that $|G : M| = p^n$ for some prime p. Since $(|A|, |B|) = 1$, we can assume that p does not divide, say, $|B|$. Let M_1 be a p'-Hall subgroup of M. By

π-Sylow Theorem [1, Theorem 4.1, p.231], we have $B \leq M_1^x$ for some x in G. Since M^x has the same properties as M, we can replace M by M^x and so we can assume without loss of generality that $B \leq M$. Since $G = AB$, it follows that $M = B(A \cap M)$. Clearly, B and $A \cap M$ are nilpotent subgroups of M, $(|B|, |A \cap M|) = 1$. By the hypothesis, for each element x in $A \cap M$ and each element y in B there is a positive integer n such that $[x,_n y] \in Z(G) \cap M \leq Z(M)$. By induction on $|G|$, M is nilpotent. If G is not nilpotent, then G is a minimal non-nilpotent group. By Itô′s Theorem [1, Theorem 5.2, p.281], $G = PQ$, where P is normal in G, and P is a Sylow p-subgroup of G, Q is non-normal cyclic Sylow q-subgroup of G, $p \neq q$. We can assume that $P = A$ and $Q = B$, so that $A \lhd G$. Due to each element x in A and each element y in B there is a positive integer n such that $[x,_n y] \in Z(G)$, Theorem 3.1 implies that G is nilpotent, a contradiction. This is impossible as G is minimal non-nilpotent group. This completes the proof of the Theorem.

Theorem 3.3. *Let $N \trianglelefteq G$, N and G/N are nilpotent groups. If for each element x in N and each element y in G, there is a positive integer n such that $[x,_n y] \in Z(G)$, then G is nilpotent.*

Proof. We first prove that there is a nilpotent subgroup A of G such that $G = NA$. Let $\mathcal{F} = \{A | A \subseteq G, G = NA\}$. Clearly, $G \in \mathcal{F}$, then $\mathcal{F}$ is a non-empty set. Now let A be a minimal element in the set $\mathcal{F}$, we show that A is nilpotent. Assume that $N \cap A \nsubseteq \Phi(A)$, there exists a maximal subgroup B of A such that $N \cap A \nsubseteq B$. Since $N \cap A \lhd A$, we have $A = (N \cap A)B$. Since $G = NA = N(N \cap A)B = NB$, then $B \in \mathcal{F}$, a contradiction. Hence $N \cap A \subseteq \Phi(A)$. Since $G = NA$, so that $G/N \simeq A/N \cap A$. But $A/N \cap A \sim (A/N \cap A)/(\Phi(A)/N \cap A)$, then $G/N \sim (A/N \cap A)/(\Phi(A)/N \cap A) \simeq A/\Phi(A)$. This implies that $G/N \sim A/\Phi(A)$. G/N is nilpotent implies that $A/\Phi(A)$ is nilpotent. By Lemma 2.3 we have A is nilpotent. By the hypothesis, $G = NA$, N and A are nilpotent subgroups of G, so for each element x in N and each element y in A there is a positive integer n such that $[x,_n y] \in Z(G)$. By Theorem 3.1 we know that G is nilpotent.

Theorem 3.4. *Let G be a finite group, x is an arbitrary element of order p or order 4(p=2) and y is an arbitrary p'-element of G. If there is positive integer n such that $[x,_n y] \in Z(G)$, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counter-example of minimal order. Then G is not p-nilpotent. But each of whose proper subgroup of G is p-nilpotent, by Itô′s Theorem [1, Theorem 5.2, p.281], $G = PQ$, $P \in \mathrm{Sly}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, $P \trianglelefteq G$, $\exp(P) = p$ or 4. Let $x \in P$, $y \in Q$. By the hypothesis, there is a positive integer n such that $[x,_n y] \in Z(G)$. Theorem 3.1 implies that G is nilpotent.

This Theorem may be considered as a generalization of the following well-known result of Itô: Let G be a finite group. If each element of G of order p or order $4(p{=}2)$ lies in $Z(G)$, then G is p-nilpotent[1, Theorem 5.5, p.435].

Theorem 3.5. *Let G be a finite group and P be an arbitrary p-subgroup of G. Let x be an arbitrary element of P of order p or order $4(p{=}2)$ and y be an arbitrary p'-element of $N_G(P)$. If there is a positive integer n such that $[x,_n y] \in Z(G)$, then G is p-nilpotent.*

Proof. Let y be an arbitrary p'-element of $N_G(P)$. We can consider the group $P\langle y\rangle$. Let x be an arbitrary element of $P\langle y\rangle$ of order p order 4. Clearly, $x \in P$, since $P\langle y\rangle \leq N_G(P)$. Thus an arbitrary p'-element in $P\langle y\rangle$ that it is an arbitrary p'-element in $N_G(P)$. We can assume that y' is an arbitrary p'-element in $N_G(P)$. By the hypothesis, there is a positive integer n such that $[x,_n y'] \in Z(G)$. Theorem 3.4 implies that $P\langle y'\rangle$ is p-nilpotent. It follows that $\langle y'\rangle \lhd P\langle y'\rangle$. Hence $P\langle y'\rangle = P \times \langle y'\rangle$, $N_G(P)/C_G(P)$ is p-group. By Frobenius' Theorem, G is p-nilpotent. This completes the proof of the Theorem.

This Theorem may be considered as a generalization of Theorem 10.24 [1, Theorem 10.24, p.124].

4. Acknowledgements

The paper is dedicated to Professor Xiuyun Guo for his 60th birthday.

The research of the author is supported by the National Natural Science Foundation of China(11301378).

References

[1] B. Huppert, *Endliche Gruppen I*, Berlin-Heidelberg-New York, 1967.

[2] D.J.S Robinson, *A course in the theory of groups*, Springer-Verlag, New york, 1993.

Accepted: 19.02.2017

GLOBAL EXPONENTIAL STABILITY OF COHEN-GROSSBERG NEURAL NETWORKS WITH TIME-VARYING DELAYS

Xiaohui Wang
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China

Xumeng Li*
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China
xm.li@hunau.edu.cn

Wenxin Zhou
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China

Abstract. In this paper, without the assumptions for boundedness, monotonicity, and differentiability on activation functions and symmetry of interconnections, a class of Cohen-Grossberg neural networks with time-varying delays is studied. A new useful criteria on the uniqueness of equilibrium is obtained by utilizing the nonlinear measure. Combining with Dini derivatives and Young inequality, new sufficient condition for the global exponential stability is established by directly estimating the upper bound of solutions of the system. All results are presented in M-matrix form, which extended and generalized the corresponding results in previous literature.

Keywords: Cohen-Grossberg neural networks, time-varying delays, unique equilibrium, global exponential stability, nonlinear measure, M-matrix.

1. Introduction

The classic Cohen-Grossberg neural network was initially proposed and studied by Cohen and Grossberg in 1983([1]), which can be described by the following ordinary differential equations:

$$\dot{x}_i(t) = a_i(x_i(t))\{-b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + I_i\}, i = 1, 2, \ldots, n. \tag{1}$$

*. Corresponding author

Here $n \geq 2$ is the number of neurons in the network, $x_i(t)$ is the state of neuron i at time t, $a_i(\cdot)$ represents an amplification function, $b_i(\cdot)$ is an appropriately behaved function to keep the solutions of system (1) bounded, the activation function $f_j(\cdot)$ shows how neuron i reacts to the input, $W = (w_{ij})_{n\times n}$ is a real constant matrix and denotes the normal weights of the neuron interconnections. System(1) is a very general neural network model. Models such as the Hopfield neural networks, cellular neural networks, and bidirectional associative memory neural networks are its special cases(see for instance [4],[7],[8]). All these neural neural networks have attracted much attention for they successful or promising potential applications in the pattern recognition, associative memory,signal processing,and optimization([5],[9],[10]).

In reality, however, time delays universally exist in biological and artificial neural networks due to the finite switching speed of neurons and amplifiers. It is well known that, with symmetric connection matrix and the so called sigmoid activation functions, system (1) has the property of absolute stability([1]), i.e. given any initial conditions, the solution of system (1) converges to some equilibrium of the system. On the other hand, the existence of time delays is frequently a source of oscillation and instability([6]). Marcus and Westervelt ([2]) first introduced a single delay into the model and observed sustained oscillations even with symmetric connections. Moreover, the delays in artificial neural networks are usually time-varying([3]). Therefore, it is natural and important to incorporate time-varying delays into the model. A general Cohen-Grossberg neural network with time-varying delays can be described by the following retarded differential difference equations:

$$\begin{aligned}\dot{x}_i(t) = a_i(x_i(t))\{ &- b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} w_{ij}^d f_j(x_j(t - d_{ij}(t))) + I_i\}, i = 1, 2, \ldots, n\end{aligned} \tag{2}$$

where n, $x_i(t)$, $a_i(\cdot)$, $b_i(\cdot)$ and $W = (w_{ij})_{n\times n}$ are the same as these in system (1), $W^d = (w_{ij}^d)_{n\times n}$ is a real constant matrix and denotes the delayed weights of the neuron interconnections, $d_{ij}(t)$ is the time delay required in processing and transmitting signals from neuron j to neuron i at time t.

The stability of neural networks is a prerequisite for almost all applications. In applications of neural networks to parallel computation, signal processing and other problems involving the solutions of optimization problems, it is frequently required that the network have a unique global attractive equilibrium. Meanwhile, in designing and implementing a network, it is preferable and desirable that the neural network not only converges to an equilibrium, but also converges as fast as possible. It is well known that the exponential stability gives a fast convergence rate to the equilibrium. Thus, the global exponential stability of system(1), system (2) and their special cases are of great importance,

and has been widely investigated. Many useful results have been obtained by some authors in the previous literature([14][16], [17],[18],[19],[20],[21],[22], [23],[24],[25],[26]).

In applications, the activation functions maybe just continuous. On the other hand, the assumption of symmetry connections also lays a restriction on the connection topology of the networks. In this paper, without any assumption on the boundedness, monotonicity, and differentiability of activation functions and symmetry of interconnections, we will study the existence and global exponential stability of an equilibrium for System (2). A new useful criteria on the uniqueness of equilibrium is obtained by utilizing the nonlinear measure, which was initially introduced in [11]. Combining with Dini derivatives and Young inequality, new sufficient conditions for the global exponential stability are established by inroducing many real paremeters and directly estimating the upper bound of solutions of the system. This mathod has been utilized by some authors (see for instance,[14],[22],[17]), but we do introduce a different type parameters. Our results extend and generalize the corresponding results in previous literature. By the way, for the common use of M-matrix in the study of qualitative properties of various neural networks, we represent our results in M-matrix form.

The remainder of this paper is organized as follows. In section 2, the basic notations, definitions, and some useful lemmas are introduced. Some assumptions used in the main results are listed there, too. In section 3, the main results of this paper are proposed and proved. Some remarks and corollaries are given. In section 4, some examples are given to demonstrate the main results.

2. Preliminaries

In this section, we state some notations, definitions and lemmas.

Let $N = \{1, 2, \ldots, n\}$, R denote the set of real numbers, R^n denote the n-dimensional real vector space, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_r$ $(r \geq 1)$ denote the usual inner product and l^r norm of vectors in R^n respectively. For $x \in R^n$, x_i denotes the ith coordinate of x, x^T denotes the transpose of x, $sign(x) = (sign(x_1), \ldots, sign(x_n))^T$ denotes the sign vector of x, where $sign(\cdot)$ is the sign function of real numbers. $R^n_+ = \{x | x \in R^n, x_i > 0, for\ i \in N\}$. $R^{n\times n}$ denotes the set of all $n \times n$ real matrices, $diag\{a_1, \ldots, a_n\}$ denotes the usual diagonal matrix. For $A = (a_{ij}) \in R^{n\times n}$, $|A| = (|a_{ij}|)$, A^T denotes the transpose of A, A^{-1} denotes the inverse of A (if it has). For two matrices $A = (a_{ij}), B = (b_{ij}) \in R^{n\times n}$, we say $A \leq B$ if $a_{ij} \leq b_{ij},\ for\ i, j \in N$.

Throughout this paper, we always assume that for $i, j \in N, a_i, b_i, f_i, d_{ij} : R \to R$ are continuous functions, and there exist a real number d such that $0 \leq d_{ij}(t) \leq d, t \in R$. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ denote the solution of system (2). System (2) is supplemented with initial values of the type

$$x(t) = \varphi(t), \varphi(t) \in C([-d, 0], R^n),\ t \in [-d, 0],$$

where $C([-d,0],R^n)$ denotes the space of continuous functions $\varphi:[-d,0]\to R^n$.

Definition 1. *Suppose that $x^*\in R^n$ is an equilibrium of system (2), x^* is said to be globally exponentially stable, if there exist $\lambda>0$ and $C>0$ such that for any solution $x(t)$ of system (2), we have*

$$|x_i(t)-x_i^*|\le C\sum_{i=1}^{n}\sup_{t\in[-d,0]}|\varphi_i(t)-x_i^*|e^{-\lambda t},\ for\ t\ge 0,\ i\in N.$$

In order to obtain the uniqueness of equilibrium of system (2), we introduce some results about nonlinear measure from J.Peng,H.Qiao and Z.Xu([11]).

Definition 2 ([11])**.** *Suppose that Ω is an open subset of R^n, F is an operator from Ω to R^n, the constant*

$$m_\Omega(F)=\sup_{x,y\in\Omega,x\ne y}\frac{\langle F(x)-F(y),sign(x-y)\rangle}{\|x-y\|_1}$$

is called the nonlinear measure of F on Ω.

Lemma 2.1 ([11])**.** *If $m_\Omega(F)<0$, then F is injective on Ω. If in addition $\Omega=R^n$, then F is a homeomorphism of R^n.*

For the study of the global exponential stability of an equilibrium by our methods, we introduce the Dini derivatives and the Young inequality now.

Definition 3. *Suppose that $V(t)$ is a real function, the left upper Dini derivative and the left lower Dini derivative of $V(t)$, denoted by $D^-V(t)$ and $D_-V(t)$ respectively , are defined by*

$$D^-V(t)=\limsup_{h\to 0^-}\frac{V(t+h)-V(t)}{h},\ D_-V(t)=\liminf_{h\to 0^-}\frac{V(t+h)-V(t)}{h}.$$

Lemma 2.2 ([12],Young inequality)**.** *Assume that $a\ge 0,b\ge 0,p>1,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then we have inequality*

$$ab\le\frac{1}{p}a^p+\frac{1}{q}b^q.$$

For the common use of M-matrix in the study of qualitative properties of various neural networks, we represent all of our results in the M-matrix form. The definition of M-matrix and some useful results about it are given as follows.

Definition 4 ([14])**.** *Let $A=(a_{ij})\in R^{n\times n}$, and $a_{ij}\le 0$ for $i\ne j$, $i,j\in N$. A is called M-matrix if there exist $P\in R_+^n$, such that $AP>0$ or $P^TA>0$.*

Remark 2.3. From the definition, it is obvious that A is a M-matrix if and only if for every diagonal matrix D with positive diagonal elements, DA and AD are M-matrices.

Lemma 2.4 ([13]). *Let $A = (a_{ij}) \in R^{n\times n}$, and $a_{ij} \leq 0$ for $i \neq j$, $i, j \in N$. Then the following conditions are equivalent:*

(1) *A is a M-matrix;*

(2) *All the leading principal minors of A are positive;*

(3) $A^{-1} \geq 0$.

Remark 2.5. From lemma 2.4 (2), we can obtain that A is a M-matrix if and only if A^T is. Thus in the definition, The statement " there exist $P \in R^n_+$, such that $AP > 0$ or $P^TA > 0$ " is equivalent to the statement " there exist P_1, $P_2 \in R^n_+$, such that $AP_1 > 0$ and $P_2^TA > 0$ ".

Lemma 2.6 ([25]). *Let $B = (b_{ij}) \in R^{n\times n}$, and $b_{ij} \leq 0$ for $i \neq j$, $i, j \in N$. If $B \geq A$ and A is a M-matrix, then B is a M-matrix.*

At the end of this section, we list some assumptions which will be used in the main results of system (2).

(H_1) For each $i \in N$, $a_i(s) > 0, for\ s \in R$.

(H_1') For each $i \in N$, there exist positive real numbers $\underline{\alpha}_i$ and $\overline{\alpha}_i$ such that

$$\underline{\alpha}_i \leq a_i(s) \leq \overline{\alpha}_i, for\ s \in R.$$

(H_2) For each $i \in N$, b_i is global left Lipschitz continuous, i.e. there exists a positive constant $\beta_i > 0$ such that

$$\frac{b_i(s) - b_i(t)}{s - t} \geq \beta_i, for\ s, t \in R\ and\ s \neq t.$$

(H_3) For each $i \in N$, f_i is global Lipschitz continuous, i.e. there exists a positive constant L_i such that

$$|f_i(s) - f_i(t)| \leq L_i|s - t|, for\ s, t \in R.$$

3. Main results

In this section, a new criteria on uniqueness of equilibrium of system (2) is firstly proposed and proved. The remainder is mainly concerned with the global exponential stability.

On the existence of equilibrium of system (2), we refer to the powerful result initially proposed recently in K. Lu, D. Xu and Z. Yang ([14]).

Theorem 3.1 ([14]). *Assume that (H_1) holds, and there are nonnegative constants such that*

$$b_i(s) \geq \beta_i^0|s| - \xi_i, |f_i(s)| \leq L_i^0|s| + \mu_i, for\ s \in R,\ i \in N.$$

If

$$M_0 \triangleq B_0 - |W + W^d|L_0 \tag{3}$$

is a M-matrix, where $B_0 = diag\{\beta_1^0, \ldots, \beta_n^0\}$ and $L_0 = diag\{L_1^0, \ldots, L_n^0\}$, then System (2) has at least a equilibrium x^*. Furthermore, $|x^*| \leq M_0^{-1}(|W+W^d|\xi + \mu + |I|)$, where $\xi = (\xi_1, \ldots, \xi_n)^T$, and $\mu = (\mu_1, \ldots, \mu_n)^T$.

Now, we propose and prove our result on uniqueness of equilibrium of system (2).

Theorem 3.2. *Assume that* (H_1), (H_2), (H_3) *hold. If*

$$M \triangleq B - |W + W^d|L \tag{4}$$

is a M-matrix, where $B = diag\{\beta_1, \ldots, \beta_n\}$ *and* $L = diag\{L_1, \ldots, L_n\}$, *then System* (2) *has a unique equilibrium* x^*. *Furthermore, for any* $x \in R^n$, *let* $y = x - x^*$, *we have* $|y| \leq M^{-1}J(x)$, *where* $J : R^n \to R^n$ *is defined by*

$$J_i(x) = \sum_{j=1}^{n} |w_{ij} + w_{ij}^d||f_j(x_j)| + |I_i - b_i(x_i)|, \; for \; i \in N.$$

In particular, $|x^*| \leq M^{-1}J(0)$.

Proof. By (H_1), x^* is an equilibrium of System (2) if and only if $x = x^*$ is a solution of equations

$$F_i(x) \triangleq -b_i(x_i) + \sum_{j=1}^{n}(w_{ij} + w_{ij}^d)f_j(x_j) + I_i = 0, \; for \; i \in N. \tag{5}$$

Using (4), there exists $P \in R_+^n$ such that $P^T M > 0$, i.e.

$$P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i > 0, \; for \; i \in N. \tag{6}$$

Define $F : R^n \to R^n$ by $F(x) = (P_1F_1(x), \ldots, P_nF_n(x))^T, \forall x \in R^n$. Then x^* is an equilibrium of System (2) if and only if $F(x^*) = 0$.

We now prove that $m_{R^n}(F) < 0$. For all $x, y \in R^n$, Using (H_2) and (H_3),

$$\begin{aligned}
&\langle F(x) - F(y), sign(x - y)\rangle \\
&= \sum_{i=1}^{n} P_i\{-(b_i(x_i) - b_i(y_i)) + \sum_{j=1}^{n}|w_{ij} + w_{ij}^d|(f_j(x_j) - f_j(y_j))\}sign(x_i - y_i) \\
&\leq \sum_{i=1}^{n} P_i(-\beta_i|x_i - y_i| + \sum_{j=1}^{n}|w_{ij} + w_{ij}^d|L_j|x_j - y_j|) \\
&= -\sum_{i=1}^{n}(P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i)|x_i - y_i| \\
&\leq -\min_{i\in N}\{P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i\}\sum_{i=1}^{n}|x_i - y_i|.
\end{aligned}$$

Therefore, by means of (6),

$$m_{R^n}(F) \leq -\min_{i\in N}\{P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i\} < 0.$$

According to lemma 2.1, F is a homeomorphism of R^n, which indicates that $F(x) = 0$ has a unique solution x^*, and thus system (2) has a unique equilibrium x^*. To estimate the existence region of x^*, for $\forall x \in R^n$ and $i \in N$, using (H_2) , (H_3) and (5), we can obtain

$$\begin{aligned}
&\beta_i|x_i^* - x_i| \leq (b_i(x_i^*) - b_i(x_i))sign(x_i^* - x_i)\\
&= (\sum_{j=1}^{n}(w_{ij} + w_{ij}^d)f_j(x_j^*) + I_i - b_i(x_i))sign(x_i^* - x_i)\\
&\leq |\sum_{j=1}^{n}(w_{ij} + w_{ij}^d)f_j(x_j^*) + I_i - b_i(x_i)|\\
&\leq |\sum_{j=1}^{n}|w_{ij} + w_{ij}^d|L_j|x_j^* - x_i| + \sum_{j=1}^{n}|w_{ij} + w_{ij}^d||f_j(x_j)| + |I_i - b_i(x_i)|,
\end{aligned}$$

namely, $M|y| \leq J(x)$. Utilizing lemma 2.4, $M^{-1} \geq 0$, then we have $|y| \leq M^{-1}J(x)$. In particular, Let $x = 0$, $|x^*| \leq M^{-1}J(0)$. The proof is completed.□

Remark 3.3. As far as existence of equlibrium concerned, theorem 3.2 is just a direct corollary of theorem 3.1. But theorem 3.1 can not ensure the uniqueness of equilibrium. To the best of our knowledge, there are few results about the uniqueness of equilibrium ([21]), most of the existing results in the previous literature just deal with the existence ([18], [16]). Moreover, our estimate of the existence region of equilibrium can be used many times by setting different x to obtain more exact estimation.

For the global exponential stability of equilibrium of system (2), we have the following main result.

Theorem 3.4. *Assume that all the conditions in theorem 3.1 hold. Moreover, (H_1'),(H_2) and (H_3) hold. If there exist $r \geq 1$, $P = (P_1, \ldots, P_n)^T \in R_+^n$, and real numbers h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, for $i, j \in N$, such that*

$$M_{r,P,h,l,h^*,l^*} \triangleq (m_{ij}) \tag{7}$$

is a M-matrix, where for $i, j \in N$,

$$\begin{aligned}
m_{ii} &= \{r\frac{\alpha_i}{\overline{\alpha}_i}\beta_i P_i - (r-1)\sum_{k=1}^{n}(|w_{ik}|^{\frac{r-h_{ik}}{r-1}}L_k^{\frac{r-l_{ik}}{r-1}} + |w_{ik}^d|^{\frac{r-h_{ik}^*}{r-1}}L_k^{\frac{r-l_{ik}^*}{r-1}})P_k\\
&\quad - (|w_{ii}|^{h_{ii}}L_i^{l_{ii}} + |w_{ii}^d|^{h_{ii}^*}L_i^{l_{ii}^*})P_i\},\\
m_{ij} &= -(|w_{ij}|^{h_{ij}}L_j^{l_{ij}} + |w_{ij}^d|^{h_{ij}^*}L_j^{l_{ij}^*})P_j, i \neq j,
\end{aligned}$$

then System (2) has a unique equilibrium, which is globally exponentially stable.

Proof. From theorem 3.1, system (2) has at least one equilibrium, say x^*. Uniqueness of equilibrium can be induced directly from the global exponential stability of x^*. So we just need to prove the global exponential stability of x^*. Let $y(t) = x(t) - x^*$, substitute $x(t) = y(t) + x^*$ into system (2), for each $i \in N$,

$$\dot{y}_i(t) = A_i(y_i(t))\{-B_i(y_i(t)) + \sum_{j=1}^{n} w_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} w_{ij}^d g_j(y_j(t - d_{ij}(t)))\}. \tag{8}$$

Here, for each i, $j \in N$, $A_i(y_i(t)) = a_i(y_i(t) + x_i^*)$, $B_i(y_i(t)) = b_i(y_i(t)) - b_i(x_i^*)$, $g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$.

We will show that $y(t) = 0$ is globally exponentially stable. Firstly, using (H_1'), (H_2) and (H_3),

$$\underline{\alpha}_i \le A_i(s) \le \overline{\alpha}_i,\ \frac{B_i(s)}{s} \ge \beta_i,\ |g_i(s)| \le L_i|s|,\ for\, s \in R,\ i \in N. \tag{9}$$

Using (7), let $\overline{A}$=$diag\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$, there exists $Q \in R_+^n$ such that $M_{r,P,h,l,h^*,l^*}\overline{A}Q > 0$, i.e.

$$\begin{aligned} &r\underline{\alpha}_i\beta_i P_i Q_i - (r-1)\overline{\alpha}_i \sum_{j=1}^{n} (|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}}) P_j Q_i \\ &- \sum_{j=1}^{n} |w_{ij}|^{h_{ij}} L_j^{l_{ij}} \overline{\alpha}_j P_j Q_j - \sum_{j=1}^{n} |w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*} \overline{\alpha}_j P_j Q_j > 0,\ for\ i \in N. \end{aligned} \tag{10}$$

So, we can choose a real number $0 < \lambda \ll 1$ such that

$$\begin{aligned} &r\underline{\alpha}_i\beta_i P_{i_0} Q_i - (r-1)\overline{\alpha}_i \sum_{j=1}^{n} (|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}}) P_j Q_i \\ &- \sum_{j=1}^{n} |w_{ij}|^{h_{ij}} L_j^{l_{ij}} \overline{\alpha}_j P_j Q_j - \sum_{j=1}^{n} |w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*} \overline{\alpha}_j P_j Q_j e^{\lambda d} - P_i Q_i \lambda > 0, \end{aligned} \tag{11}$$

for $i \in N$.

Let $y_i(t) = P_i z_i(t)$, $U_i(t) = \left| \int_0^{z_i(t)} \frac{|s|^{r-1}}{A_i(P_i s)} ds \right|$. It follows From (9) that

$$\frac{|z_i(t)|^r}{r\overline{\alpha}_i} \le U_i(t) \le \frac{|z_i(t)|^r}{r\underline{\alpha}_i} \tag{12}$$

For any $\varepsilon > 0$, let $V(t) = m\sum_{j=1}^{n} (\sup_{s\in[-d,0]} |z_j(s)|^r + \varepsilon) e^{-\lambda t}$, $V_i(t) = Q_i V(t)$, where $m \gg 1$ is a constant such that $mQ_i e^{-\lambda t} > \frac{1}{r\underline{\alpha}_i}$, $for\ t \in [-d, 0]\, i \in N$. Then $U_i(t) < V_i(t)$, $for\ t \in [-d, 0], i \in N$. We claim that

$$U_i(t) < V_i(t), for\ i \in N, t \in [-d, \infty). \tag{13}$$

Contrarily, there must exist $i_0 \in N$ and $t_0 > 0$ such that

$$U_{i_0}(t_0) = V_{i_0}(t_0),\ and\ U_j(t) < V_j(t),\ for j \in N,\ t \in [-d, t_0), \tag{14}$$

and we have

$$D_- U_{i_0}(t_0) \geq \dot{V}_{i_0}(t_0) = -\lambda V_{i_0}(t_0). \tag{15}$$

Now, By (8), (9), and (12), for each $i \in N$, we estimate $D^- U_i(t)$. Noting that $\frac{|s|^{r-1}}{A_i(P_i s)} > 0$, for each $i \in N$, we have

$$\begin{aligned} D^- U_i(t) &= D^- \{sign(z_i(t)) \int_0^{z_i(t)} \frac{|s|^{r-1}}{A_i(P_i s)} dx\} \\ &\leq \frac{1}{P_i} |z_i(t)|^{r-1} \{-B_i(P_i z_i(t)) + \sum_{j=1}^{n} w_{ij} g_j(P_j z_j(t)) \\ &+ \sum_{j=1}^{n} w_{ij}^d g_j(P_j z_j(t - d_{ij}(t)))\} sign(z_j(t)) \\ &\leq \frac{1}{P_i} \{-\beta_i P_i |z_i(t)|^r + \sum_{j=1}^{n} |w_{ij}| P_j L_j |z_j(t)| |z_i(t)|^{r-1} \\ &+ \sum_{j=1}^{n} |w_{ij}^d| P_j L_j |z_j(t - d_{ij}(t))| |z_i(t)|^{r-1}\} \end{aligned}$$

According to the Young inequality (lemma 2.2), for $i, j \in N$ and $r > 1$,

$$\begin{aligned} &|w_{ij}| L_j |z_j(t)| |z_i(t)|^{r-1} \\ &= \{|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} |z_i(t)|^r\}^{\frac{r-1}{r}} \{|w_{ij}|^{h_{ij}} L_j^{l_{ij}} |z_j(t)|^r\}^{\frac{1}{r}} \\ &\leq \frac{r-1}{r} |w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} |z_i(t)|^r + \frac{1}{r} |w_{ij}|^{h_{ij}} L_j^{l_{ij}} |z_j(t)|^r, \end{aligned}$$

similarly,

$$\begin{aligned} &|w_{ij}^d| L_j |z_j(t - d_{ij}(t))| |z_i(t)|^{r-1} \\ &\leq \frac{r-1}{r} |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}} |z_i(t)|^r + \frac{1}{r} |w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*} |z_j(t - d_{ij}(t))|^r. \end{aligned}$$

We regulate it that $0 \cdot \infty = 0$,then the preceding two inequality hold trivially for $r = 1$. Thus, for each $i \in N$,

$$\begin{aligned} D^- U_i(t) \leq &\frac{1}{P_i} \{-\beta_i P_i |z_i(t)|^r \\ &+ \frac{r-1}{r} \sum_{j=1}^{n} (|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}}) P_j |z_i(t)|^r \\ &+ \frac{1}{r} \sum_{j=1}^{n} |w_{ij}|^{h_{ij}} L_j^{l_{ij}} P_j |z_j(t)|^r + \frac{1}{r} \sum_{j=1}^{n} |w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*} P_j |z_j(t - d_{ij}(t))|^r\}. \end{aligned} \tag{16}$$

In light of (14) and (16), we estimate $D_-U_{i_0}(t_0) - \dot{V}_i(t_0)$ as follows.

$$
\begin{aligned}
&D_-U_{i_0}(t_0) - \dot{V}_i(t_0) \le D^-U_{i_0}(t_0) - \dot{V}_i(t_0)\\
&\le \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}|z_{i_0}(t)|^r + \frac{r-1}{r}\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}}\\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j|z_{i_0}(t)|^r + \frac{1}{r}\sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j|z_j(t)|^r\\
&+ \frac{1}{r}\sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j|z_j(t-d_{i_0j}(t))|^r\} + \lambda V_{i_0}(t_0)\\
&\le \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}U_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}}\\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}U_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jU_j(t_0)\\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jU_j(t_0-d_{i_0j}(t_0)) + P_{i_0}\lambda V_{i_0}(t_0)\}\\
&\le \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}V_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}}\\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}V_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jV_j(t_0)\\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jV_j(t_0-d_{i_0j}(t_0)) + P_{i_0}\lambda V_{i_0}(t_0)\}\\
&\le \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}V_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}}\\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}V_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jV_j(t_0)\\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jV_j(t_0-d) + P_{i_0}\lambda V_{i_0}(t_0)\}\\
&= -\frac{1}{P_{i_0}}\{r\underline{\alpha}_{i_0}\beta_{i_0}P_{i_0}Q_{i_0} - (r-1)\overline{\alpha}_{i_0}\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}}\\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_jQ_{i_0} - \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}\overline{\alpha}_jP_jQ_j\\
&- \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}\overline{\alpha}_jP_jQ_je^{\lambda d} - P_{i_0}Q_{i_0}\lambda\}V(t_0).
\end{aligned}
$$

Using (11), we obtain that $D_-U_{i_0}(t_0) < \dot{V}_i(t_0)$, which contradicts (15). Hence, (13) holds. With the help of (12),

$$\begin{aligned}|y_i(t)| &= P_i|z_i(t)| < (r\overline{\alpha}_i V_i(t))^{\frac{1}{r}} \\ &= (mr\overline{\alpha}_i Q_i)^{\frac{1}{r}}(\sum_{j=1}^{n}(\sup_{s\in[-d,0]}|y_j(s)|^r + \varepsilon))^{\frac{1}{r}}e^{-\frac{\lambda}{r}t}, \text{ for } i \in N.\end{aligned}$$

For the arbitrariness of ε, it follows that

$$|y_i(t)| \le P_i(mr\overline{\alpha}_i Q_i)^{\frac{1}{r}}(\sum_{j=1}^{n}\sup_{s\in[-d,0]}|y_j(s)|^r)^{\frac{1}{r}}e^{-\frac{\lambda}{r}t}. \tag{17}$$

Noting that all norms in R^n are equivalent, then there exist $C_r > 0$ such that,

$$\|x\|_r = (\sum_{j=1}^{n}|x_j|^r)^{\frac{1}{r}} \le C_r\|x\|_1 = C_r\sum_{j=1}^{n}|x_j|, \text{ for } x \in R^n \tag{18}$$

By virtue of (17) and (18), we can obtain that

$$|y_i(t)| \le C\sum_{j=1}^{n}\sup_{s\in[-d,0]}|y_j(s))|e^{-\frac{\lambda}{r}t}, \text{ for } i \in N,$$

where $C = \max_{i\in N}\{P_i(m\overline{\alpha}_i Q_i)^{\frac{1}{r}}C_r\}$. Namely, system(8) and thus system(2) is globally exponentially stable.

Remark 3.5. From the proof of theorem 3.4, (H_2) and (H_3) are only use to ensure that "$\frac{B_i(s)}{s} \ge \beta_i,\ |g_i(s)| \le L_i|s|,\ for\, s \in R,\ i \in N$" in (9). If we have known the equlibrium x^* of system (2), the global left Lipschitz continuity of $b_i(t)$ and the global Lipschitz continuity of $f_i(t)$ can be relaxed to the so called global left quasi-Lipschtiz continuity and global quasi-Lipschtiz continuity at x_i^*, respectively. Namely,

$$\frac{b_i(s)-b_i(x_i^*)}{s-x_i^*} \ge \beta_i,\ |f_i(s) - f_i(x_i^*)| \le L_i|s - x_i^*| for\ s \ne x_i^* \in R,\ i \in N.$$

Remark 3.6. When theorem 3.4 ensures global exponential stability of system (2), it really ensures the robust global exponential stability in the following meaning: changing the involved system parameters (namely, all the parameters in theorem 3.4 except r, P, and h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, $for\ i, j \in N$) small enough has no harm on the global exponential stability. This can be infered by lemma 2.4 (2) and the well-known fact that the determinant of every matrix in $R^{n\times n}$ is a continuous function of its elements.

By letting $P = (1, \ldots, 1)^T$ in theorem 3.4, we can obtain the following result.

Corollary 3.7. *Assume that all the conditions in theorem 3.1 hold. Moreover, $(H_1^{'})$,(H_2) and (H_3) hold. If there exist $r \geq 1$, real numbers h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, for $i, j \in N$, such that*

$$M_{r,h,l} \triangleq (m_{ij}) \tag{19}$$

is a M-matrix, where for $i, j \in N$,

$$\begin{aligned} m_{ii} &= \{r\frac{\alpha_i}{\overline{\alpha}_i}\beta_i - (r-1)\sum_{k=1}^{n}(|w_{ik}|^{\frac{r-h_{ik}}{r-1}}L_k^{\frac{r-l_{ik}}{r-1}} + |w_{ik}^d|^{\frac{r-h_{ik}^*}{r-1}}L_k^{\frac{r-l_{ik}^*}{r-1}}) \\ &\quad - (|w_{ii}|^{h_{ii}}L_i^{l_{ii}} + |w_{ii}^d|^{h_{ii}^*}L_i^{l_{ii}^*})\ \}, \\ m_{ij} &= -(|w_{ij}|^{h_{ij}}L_j^{l_{ij}} + |w_{ij}^d|^{h_{ij}^*}L_j^{l_{ij}^*}), i \neq j, \end{aligned}$$

then System (2) has a unique equilibrium, which is globally exponentially stable.

Remark 3.8. Because coditions in theorem 3.1 are always satisfied by letting $L_0 = diag\{0, \ldots, 0\}$ and $\mu = (\sup_{s\in R} f_1(s) \ldots, \sup_{s\in R} f_n(s))^T$ for bounded activation functions. Corollary 3.7 contains the corresponding results in [22]. In [22] the activation functions are assumed to be bounded, and (H_2) is replaced by the assumption $(H_2^{'})$: "For each $i \in N$, $b_i \in C^1(R, R)$, $\dot{b}_i(\cdot) > 0$, $b_i(\cdot)$ and $b_i^{-1}(\cdot)$ is global Lipschtiz continuous." Obviously, we have relaxed this condition. Moreover, Corollary 3.7 can be used when the activation functions are unbounded.

Corollary 3.9. *Assume that all the conditions in theorem 3.1 hold. Moreover, $(H_1^{'})$,(H_2) and (H_3) hold. If there exist $r \geq 1$ and $P = (P_1, \ldots, P_n)^T \in R_+^n$ such that*

$$M_{r,P} \triangleq (m_{ij}) \tag{20}$$

is a M-matrix, where for $i, j \in N$,

$$\begin{aligned} m_{ii} &= r\{\frac{\alpha_i\beta_i}{\overline{\alpha}_i L_i} - (|w_{ii}| + |w_{ii}^d|)\}P_i - (r-1)\sum_{k=1,k\neq i}^{n}(|w_{ik}| + |w_{ik}^d|)P_k, \\ m_{ij} &= -(|w_{ij}| + |w_{ij}^d|)P_j, i \neq j, \end{aligned}$$

then System (2) has a unique equilibrium, which is globally exponentially stable.

Proof. Let $P = LP^{'}$ with $L = diag\{L_1, \ldots, L_n\}$, then we can see that $M_{r,LP^{'}}$ is the same as $M_{r,P^{'},h,l,h^*,l^*}$ in theorem 3.4 with $h_{ij} = l_{ij} = h_{ij}^* = l_{ij}^* = 1$, $for\ i, j \in N$. Using (20), $M_{r,P^{'},h,l,h^*,l^*}$ is also a M-matrix. Thus, according to theorem 3.4, corollary 3.9 is proved. □

Corollary 3.10. *Assume that $(H_1^{'})$, (H_2) and (H_3) hold. If*

$$M_1 \triangleq A - (|W| + |W^d|) \tag{21}$$

is a M-matrix, where $A = diag\{\frac{\alpha_1\beta_1}{\overline{\alpha}_1 L_1}, \ldots, \frac{\alpha_n\beta_n}{\overline{\alpha}_n L_n}\}$, then System (2) has a unique equilibrium, which is globally exponentially stable.

Proof. From (21), M_1L is a M-matrix. Noting that $M \geq M_1L$ with $L = diag\{L_1, \ldots, L_n\}$, by lemma 2.6, M is a M-matrix, too. Then theorem 3.2 ensures a unique equilibrium. In the corollary 3.9, let $r = 1$ and $P = (1, \ldots, 1)^T$, we immediately get the result. □

Remark 3.11. Corollary 3.10 contains the corresponding results in [17]. But in [17], (H_2) is replaced by (H_2').

4. Acknowledgements

This work was supported by the State's Key R & D Program of China (Grant No.2017YFD0301507), the Natural Science Foundation of Hunan Province, China (Grant No.2018JJ3227) and Key R & D Program in Hunan Province, China (No.2017NK2382).

References

[1] M. Cohen, S. Grossberg, *Absolute stability and global pattern formation and parallel memory storage by competitive neural networks*, IEEE Transactions on Systems, Man and Cybernetics, SMC, 13 (1983), 815-826.

[2] C. Marcus, R. Westervelt, *Stability of analog neural networks with delay*, Physical Review A, 39 (1989), 347-359.

[3] Z. Liu, L. Liao, *Existence and global exponential stability of periodic solution of cellular neural networks with time-varying delays*, J. Math. Anal. Appl., 290 (2004), 247-262.

[4] J. Hopfield, *Neural networks and physical systems with emergent collective computational ablities*, Proceedings of the National Academy of Sciences, 71 (1982), 2252-2258.

[5] J. Hopfield, D. Tank, *Computing with neural circuits: a model, Science*, 233 (1986), 625-633.

[6] P. Baldi, A. Atiya, *How delays affect neural dynamics and leaning*, IEEE Transactions on Neural Networks, 5 (1994), 612-621.

[7] K. Gopalsamy, X. He, *Delay-independent stability in bidirectional associative memory networks*, IEEE Transactions on Neural Networks, 5 (1994), 998-1002.

[8] L. Chua, L. Yang, *Cellular neural networks: theory*, IEEE Transactions on Circuits and Systems, 35 (1988), 1257-1272.

[9] L. Chua, L. Yang, *Cellular neural networks: applications*, IEEE Transactions on Circuits and Systems, 35 (1988), 1273-1290.

[10] M. Kennedy, L. Chua, *Neural networks for nonlinear programming*, IEEE Transactions on Circuits and Systems, 35 (1988), 554-562.

[11] J. Peng, H. Qiao, Z. Xu, *A new approach to stability of neural networks with time-varying delays*, Neural Networks, 15 (2002), 95-103.

[12] G. Hardy, H. Littlewood, G. Polya, *Inequality*, Cambridge University Press, Cambridge, 1934.

[13] A. Berman, R. Plemmons, *Nonnegative matrices in the mathematical sciences*, New York: Academic, 1979.

[14] K. Lu, D. Xu, Z. Yang, *Global attraction and stability for Cohen-Grossberg neural networks with delays*, Neural Networks, 19 (2006), 1538-1549.

[15] D. Xu, Z. Xu, *Stability analysis of linear delay-differential systems*, Control Theory and Advanced Technology, 7 (1991), 629-642.

[16] X. Liao, C. Li, K. Wong, *Criteria for exponential stability of Cohen-Grossberg neural networks*, Neural Networks, 17 (2004), 1401-1404.

[17] C. Huang and L. Huang, *Dynamics of a class of Cohen-Grossberg neural networks with time-varying delays*, Nonlinear Analysis: Real World Applications, 8 (2007), 40-52.

[18] L. Wang, X. Zou, *Exponential stability of Cohen-Grossberg neural networks,* Neural Networks, 15 (2002), 415-422.

[19] L. Wang, X. Zou, *Harmless delays in Cohen-Grossberg neural networks*, Physica D, 170 (2002), 162-173.

[20] T. Chen, L. Rong, *Robust global exponential stability of Cohen-Grossberg neural networks with time delays*, IEEE Transactions on Neural Networks, 15 (2004), 203-206.

[21] A. Wan, M. Wang, J. Peng, H. Qiao, *Exponential stability of Cohen-Grossberg neural networks with a general class of activation functions*, Physics Letters A, 350 (2006), 96-102.

[22] H. Jiang, J. Cao, Z. Teng, *Dynamics of Cohen-Grossberg neural networks with time-varying delays*, Physics Letters A, 354 (2006), 414-422.

[23] S. Mandal, N. C. Majee, A. B. Roy, *Global exponential stability of periodic solution to Cohen-Grossberg type neural network with both ordinary and neutral type discrete time varying delays: an LMI approach,* Applied Mathematical Sciences, 7 (2013), 2979 - 2991.

[24] Zhengqiu Zhang, Jinde Cao, Dongming Zhou, *Novel LMI-based condition on global asymptotic stability for a class of Cohen-Grossberg BAM networks with extended activation functions*, IEEE Trans on Neural Networks and Learning System, 256 (2014), 1161-1172.

[25] C. Xu, Q. Zhang, *Existence and global exponential stability of anti-periodic solutions for BAM neural networks with inertial term and delay*, Neurocomputing, 153 (2015), 207-215.

[26] Z. Zhang, S. Yu, *Global asymptotic stability for a class of complex-valued Cohen-Grossberg neural network with time delays*, Neurocomputing, 171 (2016), 1158-1166.

Accepted: 25.02.2017

F-CONTRACTIVE MAPPINGS OF HARDY-ROGERS-TYPE IN G-METRIC SPACES

Hamed M. Obiedat*
Department of Mathematics
Hashemite University
P.O.Box150459
Zarqa13115-Jordan
hobiedat@hu.edu.jo

Ameer A. Jaber
Department of Mathematics
Hashemite University
P.O.Box150459
Zarqa13115-Jordan
ameerj@hu.edu.jo

Abstract. In this paper, we prove some fixed point results for F-contractive mappings of Hardy-Rogers-Type in the setting of $G-$metric spaces.
Keywords: fixed point theorems, G-metric spaces, F-contractive mappings of Hardy-Rogers-type.

1. Introduction

The class of $G-$metric spaces introduced by Z. Mustafa and B. Sims (See [5]) was to provide a new class of generalized metric spaces and to extend the fixed point theory for a variety of mappings. Moreover, many theorems were proved in this new setting with most of them recognizable as counterparts of a well-known metric space theorems (See [1], [2], [3]).

In [9], a new type of contractive mappings was introduced, called $F-$contractive mappings, to provide a generalization of Banach's contraction mapping principle. Recently, in ([4]), the authors proved some fixed point results for $F-$contractive mappings of Hardy-Rogers-Type. Due to the importance of the fixed point results of Hardy-Rogers-Type, it has been improved, generalized, extended and used in many applications such as logic programming semantics, ordinary differential equations of first order, and dislocated topology (see[8], [7], [6]).

In this paper, we prove some fixed point results for $F-$contractive mappings of Hardy-Rogers-Type in the setting of $G-$metric spaces. Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

∗. Corresponding author

Definition 1 ([5]). *G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$, we have:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$;

(G2) $0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$;

(G4) $G(x, y, z) = G(p\{x, z, y\})$ *(symmetry in all three variables)*;

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *(rectangle inequality).*

The function G is called a $G-$metric on X .

Definition 2 ([5]). *A sequence (x_n) in a $G-$metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is $G-$convergent to x.*

Proposition 3. *Let X be $G-$metric space. Then the following statements are equivalent.*

1. *(x_n) is $G-$convergent to x.*
2. *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
3. *$G(x_n, x, x) \to 0$, as $n \to \infty$.*
4. *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

In a $G-$metric space X, a sequence (x_n) is said to be $G-$Cauchy if given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N_\varepsilon$.

Proposition 4 ([5]). *In a $G-$metric space X, the following statements are equivalent.*

1. *The sequence (x_n) is $G-$Cauchy.*
2. *For every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 5 ([5]). *Let X be a $G-$metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables. A $G-$metric space X is said to be complete if every $G-$Cauchy sequence in X is $G-$convergent in X.*

2. Preliminaries

In this section, we give definitions and results that we will use later. We start with $F-$contractive mappings as stated in [9].

Definition 6. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

(F1) F is strictly increasing;

(F2) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers $\lim \alpha_n = 0$ if and only if $\lim F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim \alpha_n^k F(\alpha_n) = 0$.

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F1) - (F3)$.

Definition 7. *Let $(X; G)$ be a $G-$metric space. A self-mapping T on X is called an $F-$contraction if there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that*

$$\rho + F(G(Tx; Ty, Tz)) \leq F(G(x; y, z));$$

for all $x, y \in X$ with $G(Tx, Ty, Tz) > 0$.

Definition 8. *Let $(X; G)$ be a $G-$metric space. A self-mapping T on X is called an F-contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that*

$$\begin{aligned} \rho + F(G(Tx, Ty, Tz)) \quad \leq \quad & F(\alpha_1 G(x, y, z) + \alpha_2 G(x, Tx, Tx) \\ & + \alpha_3 G(y, Ty, Ty) + \alpha_4 G(z, Tz, Tz) + \alpha_5 G(x, Ty, Tz) \\ & + \alpha_6 G(y, Tx, Tz) + \alpha_7 G(y, Ty, Tx)), \end{aligned}$$

for all $x, y \in X$ with $G(Tx, Ty, Tz) > 0$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(a_5 + \alpha_6 + \alpha_7) = 1$ and $\alpha_3 + \alpha_4 \neq 1$.

3. Fixed points for F-contractions of Hardy-Rogers-type

In this section, we give some fixed point results for F-contractions of Hardy-Rogers-type in complete $G-$metric spaces.

Theorem 9. *Let $(X; G)$ be a complete metric space and let T be a self-mapping on X. Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that*

$$\begin{aligned} (1) \qquad \rho + F(G(Tx, Ty, Tz)) \quad \leq \quad & F(\alpha_1 G(x, y, z) + \alpha_2 G(x, Tx, Tx) \\ & + \alpha_3 G(y, Ty, Ty) + \alpha_4 G(z, Tz, Tz)) \\ & + \alpha_5 G(x, Ty, Tz) + \alpha_6 G(y, Tx, Tz) \\ & + \alpha_7 G(y, Ty, Tx)), \end{aligned}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) \neq 0$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(a_5 + \alpha_6 + \alpha_7) = 1$ and $\alpha_3 + \alpha_4 \neq 1$. Then T has a fixed point. Moreover, if $\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7 \leq 1$, then the fixed point of T is unique.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0 = T(T^{n-1}x_0) = T\ x_{n-1}$ for $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then it is clear that x_n is a fixed point. Suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ and set $x = x_{n-1}$ and $y = z = x_n$. Then by applying the inequality (1) with $x = x_{n-1}$ and $y = z = x_n$. We obtain

$$\begin{aligned}
&\rho + F(G(Tx_{n-1}, Tx_n, Tx_n)) = \rho + F(G(x_n, x_{n+1}, x_{n+1}))\\
&\leq F(\alpha_1 G(x_{n-1}, x_n, x_n) + \alpha_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \alpha_3 G(x_n, Tx_n, Tx_n)\\
&+ \alpha_4 G(x_n, Tx_n, Tx_n) + \alpha_5 G(x_{n-1}, Tx_n, Tx_n) + \alpha_6 G(x_n, Tx_{n-1}, Tx_n)\\
&+ \alpha_7 G(x_n, Tx_{n-1}, Tx_n))\\
&= F(\alpha_1 G(x_{n-1}, x_n, x_n) + \alpha_2 G(x_{n-1}, x_n, x_n) + \alpha_3 G(x_n, x_{n+1}, x_{n+1})\\
&+ \alpha_4 G(x_n, x_{n+1}, x_{n+1}) + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + \alpha_6 G(x_n, x_n, x_{n+1})\\
&+ \alpha_7 G(x_n, x_n, x_{n+1}))\\
&= F((\alpha_1 + \alpha_2) G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4) G(x_n, x_{n+1}, x_{n+1})\\
&+ \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1})).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 < \rho \leq\ & F((\alpha_1 + \alpha_2) G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4) G(x_n, x_{n+1}, x_{n+1})\\
&+ \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1})) - F(G(x_n, x_{n+1}, x_{n+1})).
\end{aligned}$$

Since F is strictly increasing, which produce

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) \leq\ & (\alpha_1 + \alpha_2) G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4) G(x_n, x_{n+1}, x_{n+1})\\
&+ \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1}).
\end{aligned}$$

This leads to

$$\begin{aligned}
&(1 - (\alpha_3 + \alpha_4)) G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha_1 + \alpha_2) G(x_{n-1}, x_n, x_n)\\
&+ \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1})\\
&\leq (\alpha_1 + \alpha_2) G(x_{n-1}, x_n, x_n) + \alpha_5 G(x_{n-1}, x_n, x_n)\\
&+ \alpha_5 G(x_n, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1}),
\end{aligned}$$

which implies

$$\begin{aligned}
&(1 - (\alpha_3 + \alpha_4 + a_5)) G(x_n, x_{n+1}, x_{n+1})\\
&\leq (\alpha_1 + \alpha_2 + a_5) G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7) G(x_n, x_n, x_{n+1})\\
&\leq (\alpha_1 + \alpha_2 + a_5) G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7) G(x_n, x_{n+1}, x_{n+1})\\
&+ (\alpha_6 + \alpha_7) G(x_{n+1}, x_n, x_{n+1})\\
&= (\alpha_1 + \alpha_2 + a_5) G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7) G(x_n, x_{n+1}, x_{n+1})\\
&+ (\alpha_6 + \alpha_7) G(x_n, x_{n+1}, x_{n+1})\\
&= (\alpha_1 + \alpha_2 + a_5) G(x_{n-1}, x_n, x_n) + 2(\alpha_6 + \alpha_7) G(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Therefore,

$$(1-(\alpha_3+\alpha_4+a_5+2a_6+2a_7))G(x_n,x_{n+1},x_{n+1}) \leq (\alpha_1+\alpha_2+a_5)G(x_{n-1},x_n,x_n).$$

Hence,

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{(\alpha_1+\alpha_2+a_5)}{(1-(\alpha_3+\alpha_4+a_5+2a_6+2a_7))}G(x_{n-1},x_n,x_n).$$

Since $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2(a_5+\alpha_6+\alpha_7)=1$ with $\alpha_3+\alpha_4 \neq 1$, we obtain $1-(\alpha_3+\alpha_4+a_5+2a_6+2a_7)>0$. We have

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) \\ &\leq \frac{(\alpha_1+\alpha_2+a_5)}{(1-(\alpha_3+\alpha_4+a_5+2a_6+2a_7))}G(x_{n-1},x_n,x_n) = G(x_{n-1},x_n,x_n). \end{aligned}$$

Therefore,

$$\begin{aligned} F(G(x_n,x_{n+1},x_{n+1})) &\leq F(G(x_{n-1},x_n,x_n))-\rho \\ &\leq F(G(x_{n-2},x_{n-1},x_{n-1}))-2\rho \\ &\leq F(G(x_{n-3},x_{n-2},x_{n-2}))-3\rho. \end{aligned}$$

Continuing in the same argument, we deduce

$$F(G(x_n,x_{n+1},x_{n+1})) \leq F(G(x_0,x_1,x_1))-n\rho. \tag{2}$$

This implies that $\lim F(G(x_n,x_{n+1},x_{n+1}))=-\infty$. By Definition 6 property (F2), we get $\lim G(x_n,x_{n+1},x_{n+1})=0$. Also, if we choose $k\in(0,1)$ and multiply both sides of (2) by $G^k(x_n,x_{n+1},x_{n+1})$, we obtain

$$\begin{aligned} &G^k(x_n,x_{n+1},x_{n+1})F(G(x_n,x_{n+1},x_{n+1}))-G^k(x_n,x_{n+1},x_{n+1})F(G(x_0,x_1,x_1)) \\ &\leq -n\rho G^k(x_n,x_{n+1},x_{n+1}) \leq 0. \end{aligned}$$

Using Definition 6 property (F2), we get

$$\lim G^k(x_n,x_{n+1},x_{n+1})F(G(x_n,x_{n+1},x_{n+1}))=0,$$

which implies that

$$\lim(n\rho-F(G(x_0,x_1,x_1)G^k(x_n,x_{n+1},x_{n+1})) \leq 0.$$

But $\lim G(x_n,x_{n+1},x_{n+1})=0$. Hence, $\lim(nG^k(x_n,x_{n+1},x_{n+1}))=0$, and then we deduce that $\lim(n^{\frac{1}{k}}G(x_n,x_{n+1},x_{n+1}))=0$. Now we claim that the sequence $\{x_n\}$ is $G-$Cauchy. To show this given $\varepsilon>0$, then there exists a natural number K such that $\left|n^{\frac{1}{k}}G(x_n,x_{n+1},x_{n+1})\right|<\varepsilon$ for all $n\geq K$. That is, $G(x_n,x_{n+1},x_{n+1})<\varepsilon n^{-\frac{1}{k}}$ for all $n\geq K$ which implies that the series

$\sum G(x_n, x_{n+1}, x_{n+1})$ is convergent by comparison test. Hence, the sequence $\{x_n\}$ is G-Cauchy in a complete $G-$metric space. Then there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. This implies that $u = Tu$ and hence u is a fixed point of T. To show this, assume on the contrary that $u \neq Tu$. Then by inequality(1), we have

$$\begin{aligned}\rho + F(G(x_{n+1}, Tu, Tu)) &\leq F(\alpha_1 G(x_n, u, u)\\ &+ \alpha_2 G(x_n, x_{n+1}, x_{n+1}) + \alpha_3 G(u, Tu, Tu)\\ &+ \alpha_4 G(u, Tu, Tu)) + \alpha_5 G(x_n, x_{n+1}, Tu)\\ &+ \alpha_6 G(u, x_{n+1}, Tu) + \alpha_7 G(u, x_{n+1}, Tu)).\end{aligned}$$

By Definition 6 property (F1), we have

$$\begin{aligned}G(x_{n+1}, Tu, Tu) &\leq \alpha_1 G(x_n, u, u)\\ &+ \alpha_2 G(x_n, x_{n+1}, x_{n+1}) + \alpha_3 G(u, Tu, Tu) + \alpha_4 G(u, Tu, Tu))\\ &+ \alpha_5 G(x_n, x_{n+1}, Tu) + \alpha_6 G(u, x_{n+1}, Tu) + \alpha_7 G(u, x_{n+1}, Tu).\end{aligned}$$

By letting $n \to \infty$, we have

$$\begin{aligned}G(u, Tu, Tu) &\leq \alpha_1 G(u, u, u) + \alpha_2 G(u, u, u) + \alpha_3 G(u, Tu, Tu)\\ &+ \alpha_4 G(u, Tu, Tu)) + \alpha_5 G(u, u, Tu) + \alpha_6 G(u, u, Tu) + \alpha_7 G(u, u, Tu)\\ &\leq \alpha_3 G(u, Tu, Tu) + \alpha_4 G(u, Tu, Tu)) + \alpha_5 G(u, u, Tu)\\ &+ \alpha_6 G(u, u, Tu) + \alpha_7 G(u, u, Tu).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(1 - (\alpha_3 + \alpha_4) G(u, Tu, Tu) &\leq (\alpha_5 + \alpha_6 + \alpha_7) G(u, u, Tu)\\ &= (\alpha_5 + \alpha_6 + \alpha_7) G(Tu, u, u)\\ &\leq (\alpha_5 + \alpha_6 + \alpha_7) G(u, Tu, Tu) + (\alpha_5 + \alpha_6 + \alpha_7) G(Tu, u, Tu)\\ &= 2(\alpha_5 + \alpha_6 + \alpha_7) G(u, Tu, Tu).\end{aligned}$$

Thus

$$(1 - (\alpha_3 + \alpha_4 - 2(\alpha_5 + \alpha_6 + \alpha_7)) G(u, Tu, Tu) \leq 0.$$

But $(1-(\alpha_3+\alpha_4-2(\alpha_5+\alpha_6+\alpha_7)) \geq 0$ which implies that $G(u, Tu, Tu) = 0$, and therefore, $Tu = u$. Now, we prove the uniqueness of the fixed point. Assume that $v \in X$, is another fixed point of T such that $u \neq v$. This means that $G(u, v, v) > 0$. Taking $x = u$ and $y = z = v$ in (1), we have

$$\begin{aligned}\rho + F(G(u, v, v)) &\leq F(\alpha_1 G(u, v, v) + \alpha_2 G(u, u, u) + \alpha_3 G(v, v, v)\\ &+ \alpha_4 G(v, v, v)) + \alpha_5 G(u, v, v) + \alpha_6 G(v, u, v) + \alpha_7 G(v, u, v))\\ &\leq F(\alpha_1 G(u, v, v) + \alpha_5 G(u, v, v) + \alpha_6 G(v, u, v) + \alpha_7 G(v, u, v))\\ &= F(\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7) G(u, v, v)).\end{aligned}$$

By Definition 6 property (F1), we have

$$G(u,v,v) \leq (\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7)G(u,v,v).$$

The condition $\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7 \leq 1$ implies that $G(u,v,v) = 0$, and hence $u = v$. This completes the proof of Theorem 9. □

If we put $\alpha_5 = \alpha_6 = \alpha_7 = 0$ and $\sum_{i=1}^{4} a_i < 1$, we obtain a version of Reich ([2], Theorem 2.1) in $G-$metric spaces setting.

Corollary 10. *Let $(X;G)$ be a complete metric space and let T be a self-mapping on X. Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that*

$$\rho + F(G(Tx,Ty,Tz)) \leq F(\alpha_1 G(x,y,z) + \alpha_2 G(x,Tx,Tx) + \alpha_3 G(y,Ty,Ty) + \alpha_4 G(z,Tz,Tz)),$$

for all $x,y,z \in X$ where $\sum_{i=1}^{4} a_i < 1$. Then T has a unique fixed point in X.

Corollary 11. *Let $(X;G)$ be a complete metric space and let T be a self-mapping on X. Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that the mapping T satisfy the following condition for some $m \in \mathbb{N}$*

$$\begin{aligned} \rho + F(G(T^m x, T^m y, T^m z)) \quad \leq \quad & F(\alpha_1 G(x,y,z) + \alpha_2 G(x,T^m x,T^m x) \\ & + \alpha_3 G(y,T^m y,T^m y) + \alpha_4 G(z,T^m z,T^m z)), \end{aligned}$$

for all $x,y,z \in X$ where $\sum_{i=1}^{4} a_i < 1$. Then T has a unique fixed point in X.

Proof. From Theorem 9 we deduce that the mapping T^m has a unique fixed point (say u). Then $T^m u = u$ but $Tu = T^{m+1}u = T^m(Tu)$. This implies that Tu is another fixed point of T^m. But T^m has a unique fixed point. Hence, $Tu = u$. □

References

[1] Mustafa, Z, *A new structure for generalized metric spaces-with applications to fixed point theory*, PhD thesis, the University of Newcastle, Australia (2005).

[2] Z. Mustafa, H. Obiedat, *A fixed point theorem of Reich in G-metric spaces*, CUBO, 12 (2010), 83-93.

[3] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory Appl., 2008, Article ID 189870 (2008).

[4] M. Cosentino, P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat, 28 (2014), 715-722.

[5] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., 7 (2006), 289-297.

[6] J.J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Mathematica Sinica, English Series, 23 (2007), 2205-2212.

[7] A.C.M. Ran, M.C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proceedings.

[8] I. R. Sarma and P. S. Kumari, *On dislocated metric spaces,* International Journal of Mathematical Archive, 3 (2012), 72-77.

[9] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, 94 (2012).

Accepted 1.03.2017

ON POSITIVE WEAK SOLUTIONS FOR A CLASS OF NONLINEAR SYSTEMS

S. A. Khafagy
Mathematics Department
Faculty of Science in Zulfi
Majmaah University
Zulfi 11932, P.O. Box 1712
Saudi Arabia
and
Mathematics Department
Faculty of Science
Al-Azhar University
Nasr City (11884), Cairo
Egypt
el_gharieb@hotmail.com

Abstract. We study the positive weak solutions for the system

$$\left.\begin{array}{rl} -\Delta_{P,p}u = \lambda a(x)f(v) & \text{in } \Omega, \\ -\Delta_{P,p}v = \lambda b(x)g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\}$$

where $\lambda > 0$ is a parameter, $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, $a(x)$, $b(x)$ are weight functions and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. We discuss the existence of positive weak solutions for large λ when

$$\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0, \qquad \text{for every } \ M > 0.$$

In particular, we do not assume any sign-changing conditions on $a(x)$ or $b(x)$. Our approach depends on the method of sub–supersolutions.

Keywords: weak solution, p-Laplacian, nonlinear system, sub-supersolutions.

1. Introduction

In this paper, we study the existence of positive weak solutions for λ large for the following nonlinear system

$$\left.\begin{array}{rl} -\Delta_{P,p}u = \lambda a(x)f(v) & \text{in } \Omega, \\ -\Delta_{P,p}v = \lambda b(x)g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.1}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $a(x)$ and $b(x)$ are weight functions and that there exist positive constants a_0, b_0 such that $a(x) \geq a_0$, $b(x) \geq b_0$, f and g are given functions and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. Our approach is based on the method of sub-supersolutions (see e.g. [2]).

Recently many results concerning the existence of positive weak solutions for the nonlinear systems involving Laplacian, p-Laplacian or weighted p-Laplacian operators were obtained by various authors with the help of the sub-supersolutions method (see [1, 3, 6],[8]-[16]).

Dalmasso [4] have been studied system (1.1) when $p = 2$, $P(x) = a(x) = b(x) = 1$, f and g are increasing functions and $f, g \geq 0$. Results of [4] extended in [7] to the case when no sigh conditions on $f(0)$ or $g(0)$ were required and without assuming monotonicity conditions on f or g.

This paper is organized as follows:

In section 2, we introduce some technical results and notations, which are established in [5]. In section 3, we give some assumptions on the functions f, g to insure the validity of the existence of the positive weak solutions for system (1.1) in a suitable weighted Sobolev space. Also, we prove the existence of positive weak solutions for system (1.1) by using the method of sub–supersolutions. In section 4, we give some related results and examples.

2. Technical results

Now, we introduce some technical results to the weighted homogeneous eigenvalue problem (see [5])

$$\left.\begin{array}{cc} -\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega. \end{array}\right\} \tag{2.1}$$

The function $P(x)$ is a weight function (measurable and positive a.e. in Ω), satisfying the conditions

$$\begin{aligned} &P(x),\ (P(x))^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ \text{with}\ p > 1, (P(x))^{-s} \in L^1(\Omega), \\ &\text{with}\ s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \end{aligned} \tag{2.2}$$

and $a(x)$ is a measurable function satisfies

$$a(x) \in L^{\frac{k}{k-r}}(\Omega), \tag{2.3}$$

with some k satisfies $p < k < p^*_s$ where $p^*_s = \frac{Np_s}{N-p_s}$ with $p_s = \frac{ps}{s+1} < p < p^*_s$ and $meas\ \{x \in \Omega : a(x) > 0\} > 0$. Examples of functions satisfying (2.2) are mentioned in [5].

Lemma 1 ([5]). *There exists the first eigenvalue $\lambda_{1p} > 0$ and at least one corresponding eigenfunction $\phi_{1p} \geq 0$ a.e. in Ω of the eigenvalue problem* (2.1).

Theorem 2 ([5]). *Let $P(x)$ satisfies* (2.2) *and $a(x)$ satisfies* (2.3), *then* (2.1) *admits a positive eigenvalue λ_{1p}. Moreover, it is characterized by*

$$\lambda_{1p}\int_\Omega a(x)|\phi_{1p}|^p \le \int_\Omega P(x)|\nabla\phi_{1p}|^p. \tag{2.4}$$

Moreover, let us consider the weighted Sobolev space $W^{1,p}(P,\Omega)$ which is the set of all real valued functions u defined in Ω with the norm

$$\|u\|_{W^{1,p}(P,\Omega)} = \left(\int_\Omega |u|^p + \int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.5}$$

and the space $W_0^{1,p}(P,\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(P,\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(P,\Omega)} = \left(\int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.6}$$

which is equivalent to the norm given by (2.5). The two spaces $W^{1,p}(P,\Omega)$ and $W_0^{1,p}(P,\Omega)$ are well defined reflexive Banach Spaces.

3. Existence results

In this section, we prove the existence of positive weak solutions (u,v) for system (1.1) via the method of sub-supersolutions. We shall establish our results by constructing a subsolution $(\psi_1,\psi_2) \in (W_0^{1,p}(P,\Omega))^2$ and a supersolution $(z_1,z_2) \in (W_0^{1,p}(P,\Omega))^2$ of (1.1) such that $\psi_i \le z_i$ for $i = 1,2$. That is, ψ_i, $i = 1,2$, satisfies

$$\begin{aligned}\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx &\le \lambda\int_\Omega a(x)f(\psi_2)\zeta dx\\ \int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\nabla\zeta dx &\le \lambda\int_\Omega b(x)g(\psi_1)\zeta dx\end{aligned}$$

and z_i, $i = 1,2$, satisfies

$$\begin{aligned}\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\nabla\zeta dx &\ge \lambda\int_\Omega a(x)f(z_2)\zeta dx\\ \int_\Omega P(x)|\nabla z_2|^{p-2}\nabla z_2\nabla\zeta dx &\ge \lambda\int_\Omega b(x)g(z_1)\zeta dx,\end{aligned}$$

for all test functions $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \ge 0$. Then the following result holds:

Lemma 3 ([2]). *Suppose there exist sub and supersolutions (ψ_1,ψ_2) and (z_1,z_2) respectively of system (1.1) such that $(\psi_1,\psi_2) \le (z_1,z_2)$. Then system* (1.1) *has a solution (u,v) such that $(u,v) \in [(\psi_1,\psi_2),(z_1,z_2)]$.*

We give the following hypotheses:

$(\mathbf{H}_1)$ $f, g : [0,\infty) \longrightarrow [0,\infty)$ are C^1 nondecreasing functions such that $f(s), g(s) > 0$ for $s > 0$.

$(\mathbf{H}_2)$ For all $M > 0$,

$$\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0.$$

Theorem 4. *Let* $(H.1)$ *and* $(H.2)$ *hold. Then system* (1.1) *has a positive weak solution* $(u, v) \in (W_0^{1,p}(P,\Omega))^2$ *for* λ *large.*

Proof. Let λ_{1p} be the first eigenvalue of the eigenvalue problem (2.1) and ϕ_{1p} the corresponding positive eigenfunction with $\|\phi_{1p}\|_\infty = 1$. Let k_0, m, $\delta > 0$ be such that $f(x), g(x) \geq -k_0$ for all $x \geq 0$, $P(x)|\nabla\phi_{1p}|^p - \lambda_{1p}a(x)\phi_{1p}^p \geq m$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x,\partial\Omega) \leq \delta\}$. We shall verify that $(\psi_1,\psi_2) = (\frac{p-1}{p}(\frac{\lambda a_0 k_0}{m})^{\frac{1}{p-1}}\phi_{1p}^{\frac{p}{p-1}}, \frac{p-1}{p}(\frac{\lambda b_0 k_0}{m})^{\frac{1}{p-1}}\phi_{1p}^{\frac{p}{p-1}})$ is a subsolution of (1.1) for λ large. Let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$.

A calculation shows that

$$\begin{aligned}
&\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta dx\\
&= \frac{\lambda a_0 k_0}{m}\int_\Omega P(x)\phi_{1p}|\nabla\phi_{1p}|^{p-2}\nabla\phi_{1p}\cdot\nabla\zeta dx\\
&= \frac{\lambda a_0 k_0}{m}\left\{\int_\Omega P(x)|\nabla\phi_{1p}|^{p-2}\nabla\phi_{1p}\nabla(\phi_{1p}\zeta)dx - \int_\Omega P(x)|\nabla\phi_{1p}|^p\zeta dx\right\}\\
&= \frac{\lambda a_0 k_0}{m}\int_\Omega(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p)\zeta dx.
\end{aligned}$$

Similarly, we have

$$\int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\cdot\nabla\zeta dx = \frac{\lambda b_0 k_0}{m}\int_\Omega(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p)\zeta dx.$$

Now, on $\overline{\Omega}_\delta$, we have $P(x)|\nabla\phi_{1p}|^p - \lambda_{1p}a(x)\phi_{1p}^p \geq m$. Hence,

$$\frac{\lambda a_0 k_0}{m}(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p) \leq -\lambda a_0 k_0 \leq \lambda a(x) f(\psi_2).$$

A similar argument shows that

$$\frac{\lambda b_0 k_0}{m}(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p) \leq -\lambda b_0 k_0 \leq \lambda b(x) g(\psi_1).$$

Next, on $\Omega - \overline{\Omega}_\delta$, we have $\phi_{1p} \geq \mu$ for some $\mu > 0$. Also $f(\psi_2)$ and $g(\psi_1)$ are depending on λ and nondecreasing functions and therefore for λ large we have, using (2.4),

$$\begin{aligned}
f(\psi_2) &\geq \frac{k_0}{m}\lambda_{1p} \geq \frac{k_0}{m}(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p),\\
g(\psi_1) &\geq \frac{k_0}{m}\lambda_{1p} \geq \frac{k_0}{m}(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p).
\end{aligned}$$

Hence

$$\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta dx \le \lambda\int_\Omega a(x)f(\psi_2)\zeta dx.$$

Similarly, we have

$$\int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\cdot\nabla\zeta dx \le \lambda\int_\Omega b(x)g(\psi_1)\zeta dx,$$

i.e. (ψ_1,ψ_2) is a subsolution of (1.1) for λ large.

Next, let e_p be the solution of (see [17])

$$-\Delta_{P,p}e_p = 1 \quad \text{in } \Omega, \qquad e_p = 0 \quad \text{on } \partial\Omega.$$

Let

$$(z_1,z_2) = \left(\frac{C}{\mu_p}\lambda^{\frac{1}{p-1}}e_p, (l_b\lambda)^{\frac{1}{p-1}}\ [g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}}e_p\right)$$

where $\mu_p = \|e_p\|_\infty$, $l_b = \|b(x)\|_\infty$and $C>0$ is a large number to be chosen later, We shall verify that (z_1,z_2) is a supersolution of (1.1) for λ large. To this end, let $\zeta\in W_0^{1,p}(P,\Omega)$ with $\zeta\ge 0$. Then we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx &= \lambda\left(\frac{C}{\mu_p}\right)^{p-1}\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta dx\\ &= \frac{1}{\mu_p^{p-1}}\left(C\lambda^{\frac{1}{p-1}}\right)^{p-1}\int_\Omega\zeta dx.\end{aligned}$$

By $(\mathbf{H}_2)$, we can choose C large enough so that

$$(C\lambda^{\frac{1}{p-1}})^{p-1} \ge (\mu_p^{p-1}l_a\lambda)f([(l_b\lambda)^{\frac{1}{p-1}}\mu_p][g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}}),$$

where $l_a = \|a(x)\|_\infty$, and therefore,

$$\begin{aligned}\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx &\ge \lambda l_a\int_\Omega f([(l_b\lambda)^{\frac{1}{p-1}}\mu_p][g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}})\zeta dx\\ &\ge \lambda\int_\Omega a(x)f(z_2)\zeta dx.\end{aligned}$$

Next, we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z_2|^{p-2}\nabla z_2\cdot\nabla\zeta dx &= \lambda l_b g(C\lambda^{\frac{1}{p-1}})\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta dx\\ &= \lambda l_b g(C\lambda^{\frac{1}{p-1}})\int_\Omega\zeta dx\\ &\ge \lambda l_b\int_\Omega g(C\mu_p^{-1}\lambda^{\frac{1}{p-1}}e_p)\zeta dx\\ &\ge \lambda\int_\Omega b(x)g(z_1)\zeta dx\end{aligned}$$

i.e. (z_1,z_2) is a supersolution of (1.1) with $z_i\ge\psi_i$ for C large, $i=1,2$. Thus, there exists a positive weak solution (u,v) of (1.1) with $\psi_1\le u\le z_1,\ \psi_2\le v\le z_2$. This completes the proof.

4. Example and related result

4.1 Example

Many illustrative examples for the results obtained in this paper can be easily constructed. We just give the following one below.

Let

$$f(x) = ax^r,\ g(x) = bx^s,$$

where, $a, b, r, s > 0$ and $rs < (p-1)^2$. Then it is easy to see that f and g satisfy $(\mathbf{H}_1)$, $(\mathbf{H}_2)$.

4.2 Related result

Existence results obtained in this article can be established in a similar way for the following nonlinear system

$$\left.\begin{array}{rl} -\Delta_{P,p}u = \lambda a(x)v^{\beta} & \text{in } \Omega, \\ -\Delta_{P,p}v = \lambda b(x)u^{\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\}$$

under the assumptions that

$(\boldsymbol{A}_1)$ $a(x)$ and $b(x)$ are weight functions such that $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$.

$(\boldsymbol{A}_2)$ $0 < \alpha < p-1$ and $0 < \beta < p-1$.

Remark 5. Existence results of positive weak solutions for system (1.1) still hold if we replace the condition $\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0$, for every $M > 0$, given in (H_2), by the condition $\lim_{x\to+\infty} \frac{f[M(g(x))^{\frac{1}{p-1}}]}{x^{p-1}} = 0$, for every $M > 0$.

Acknowledgements

The author would like to express his gratitude to Professor H. M. Serag (Mathematics Department, Faculty of Science, AL- Azhar University) for continuous encouragement during the development of this work.

References

[1] G. Afrouzi, S. Ala, *An existence result of positive solutions for a class of Laplacian system,* Int. Journal of Math. Analysis, 4 (42) (2010), 2075-2078.

[2] A. Canada, P. Drabek, J. Games, *Existence of Positive solutions for some problems with nonlinear diffusion*, Trans. Amer. Math. Soc., 349 (1997), 4231-4249.

[3] M. Chhetri, D. Hai, R. Shivaji, *On positive solutions for classes of p-Laplacian semipositone system*, Discrete and Dynamical Systems, 9(4) (2003), 1063-1071.

[4] R. Dalmasso, *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal., 39 (2000), 559-568.

[5] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear elliptic equation with degenerations and singularities*, Walter de Gruyter, Bertin, New York, 1997.

[6] D. Hai, R. Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems*, Nonlinear Anal., 56 (2004), 1007-1010.

[7] D. Hai, R. Shivaji, *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), 137-141.

[8] S. Khafagy, *Existence and nonexistence of positive weak solutions for a class of (p, q)-Laplacian with different weights*, Int. J. Contemp. Math. Sciences, 6 (48) (2011), 2391-2400.

[9] S. Khafagy, *Existence and nonexistence of positive weak solutions for a class of weighted (p, q)-Laplacian nonlinear system*, Global Journal of Pure and Applied Mathematics, 9(4) (2013), 379-387.

[10] S. Khafagy, *Maximum principle and existence of weak solutions for nonlinear system involving weighted (p, q)-Laplacian,* Southeast Asian Bulletin of Mathematics, 40 (2016), 353-364.

[11] S. Khafagy, *Maximum principle and existence of weak solutions for nonlinear system involving singular p-Laplacian operators,* Journal of Partial differential equations, 29 (2) (2016), 89-101.

[12] S. Khafagy, *Non-existence of positive weak solutions for some weighted p-Laplacian system*, Journal of Advanced Research in Dynamical and Control Systems, 7 (2015), 71-77.

[13] S. Khafagy, *On positive weak solutions for nonlinear elliptic system involving singular p-Laplacian operator,* Journal of Mathematical Analysis, 7 (5) (2016), 10-17.

[14] S. Khafagy, *On the stability of positive weak solution for weighted p-Laplacian nonlinear system*, New Zealand Journal of Mathematics, 45 (2015), 39-43.

[15] E. Lee, R. Shivaji, J. Ye, *Positive solutions for elliptic equations involving nonlinearities with falling zeroes*, Applied Mathematics Letters, 22 (2009), 846-851.

[16] S. Rasouli, Z. Halimi, Z. Mashhadban, *A note on the existence of positive solution for a class of Laplacian nonlinear system with sign-changing weight*, The Journal of Mathematics and Computer Science, 3 (3) (2011), 339-354.

[17] H. Serag, S. Khafagy, *On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear systems involving degenerated p-Laplacian operator*, Turk. J. M., 34 (2010), 59-71.

Accepted: 9.03.2017

A BIPARTITE GRAPH ASSOCIATED TO A BI-MODULE OF A RING

Z. Fattahi
Ferdowsi University of Mashhad International Campus
Mashhad
Iran
z_fattahi82@yahoo.com

A. Erfanian*
Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
Mashhad
Iran
erfanian@math.um.ac.ir

A. Azimi
Department of Mathematics
University of Neyshabur
P. O. Box 91136-899, Neyshabur
Iran
ali.azimi61@gmail.com

Abstract. Let R be a ring, M be a left and right R-module. We associate a bipartite graph to R-module M of ring R, denoted by $\Gamma_{R,M}$ as undirected simple graph whose two parts of vertices are $R \setminus C_R(M)$ and $M \setminus C_M(R)$ and two distinct vertices x and y are adjacent if $xy \neq yx$, where $C_R(M)$ is the set of elements of R that commute with all elements in M. Some graph theoretical properties of this graph stated in this paper.
Keywords: Bi-module, diameter, girth, bipartite, planar, vertex and edge connectivity.

1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last ten years, leading to many interesting results. There are many papers on assigning a graph to a group of ring, see [1-6]. Now we are going to define a graph which is associated to a ring R and R-module M. Assume that $C_R(M)$ is the set of elements of R which commutes with all elements of M and similarly $C_M(R)$ is the set of elements of M that commutes with all elements of R. Then we define a bipartite undirected simple graph with vertex sets $V_R \cup V_M$, where $V_R = R \setminus C_R(M)$ and $V_M = M \setminus C_M(R)$ and

*. Corresponding author

two vertices $x \in V_R$ and $y \in V_M$ are adjacent if $xy \neq yx$. We denoted this graph by $\Gamma_{R,M}$. In the rest of this section, we remind some basic definitions in graph theory which are necessary in the paper. Section 2 deals with connectivity, diameter, girth and planarity of this graph. In section 3, vertex and edge connectivity of $\Gamma_{R,M}$ will be considered.

Now we remind some basic definitions and concepts in graph theory as following. Let X be a graph. X is a simple graph if it does not have loop and multiple edges. $V(X)$ is the set of vertices of X and the degree of a vertex v of X, denoted by $deg(v)$, is the number of edges incident to v. A path P is an alternating sequence vertices and edges, $v_1e_1v_2e_2v_3 \dots v_ke_kv_{k+1}$, such that for each i, $1 \leq i \leq k$, e_i is edge between v_i and v_{i+1}. If edges between vertices in path is not important we show them by $v_1 \sim v_2 \sim v_3 \sim ... \sim v_k \sim v_{k+1}$. We denote the minimum and maximum degrees of the vertices of X ,respectively, by δ and Δ and we use the notaion $N(v)$ for the set of neighbours of vertex v. For two vertices x and y, $\mathrm{d}(x, y)$ denotes the length of the shortest path between x and y and if there is no such path $d(x, y) = \infty$. The diameter of X is defined by diam(X)=max$\{d(x, y) : x$ and y are distinct vertices of $X\}$. X is a connected graph if there is a path between every two distinct vertices of X.

The girth of X is the length of the shortest cycle in X and it is denoted by $\mathrm{gr}(X)$ and if X does not have cycles $gr(X) = \infty$. A bipartite graph is a graph whose vertices can be divided into two disjoint parts V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2 .A complete bipartite graph is a special kind of bipartite graph where every vertex of the first part is connected to every vertex of the second part. A complete bipartite graph with parts of size $\mid V_1 \mid = m$ and $\mid V_2 \mid = n$ is denoted by $K_{m,n}$.

A dominating set in a graph X is a subset S of $V(X)$ such that every vertex not in S is joined to at least one member of S. The domination number $\gamma(X)$ of X is the number of vertices in the smallest dominating set for X. An independent set of X is a subset of vertices that no each pair of distinct vertices are adjacent. A maximal independent set is an independent set such that adding any other vertex to the set forces the set to contain some edges.The notation $\alpha(X)$ is the order of maximal independent set of X . A planar graph is a graph that can be embedded in the plane so that no two edges intersect expect at the end vertices. uv-paths P and Q in a graph are internally disjoint if they have no internal vertices in common, that is, if $V(P_1) \cap V(P_2) = \{u, v\}$. The local connectivity between distinct vertices u and v is the maximum number of pairwise internally disjoint uv-paths, denoted $p(u, v)$; the local connectivity is undefined when $u = v$. A non trivial graph X is k-connected if $p(u, v) \geq k$ for any distinct vertices u and v . The connectivity $\kappa(X)$ of X is the maximum value of k for which X is k-connected. Two uv-paths P and Q in a graph are internally edge-disjoint if they have no internal edges in common, that is, if $E(P_1) \cap E(P_2) = \emptyset$. All notations and terminology are standard here and we refer to [7] and [8]. Moreover, we always assume that R is finite ring, M is finite R-module.

2. Basic properties of $\Gamma_{R,M}$

In this section, we start with the definition of graph $\Gamma_{R,M}$

Definition. Let R be a ring and M be a left and right R-module such that M and R are distinct and have different object. The non-commutative graph of R with respect to R-module M, denoted by $\Gamma_{R,M}$, is defined as a bipartite graph with vertex sets $R \setminus C_R(M)$, $M \setminus C_M(R)$ as its parts such that $C_R(M) = \{x \in R \mid xy = yx \text{ for all } y \in M\}$ and $C_M(R) = \{x \in M | xy = yx \text{ for all } y \in R\}$ in such a way that two vertices $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$ are adjacent if $xy \neq yx$.

It is clear that if we have one of the cases $R = \{0\}$, $M = \{0\}$, $R = C_R(M)$ or $M = C_M(R)$ then $V_R = \emptyset$ or $V_M = \emptyset$ and so the graph is null. Moreover, if R is a commutative ring, then graph is null. Thus we always assume that $R \neq \{0\}$ and $M \neq \{0\}$. We may also assume that R and M are finite and we do not mention it in the lemmas and theorems. If R is infinite, then we will state specifically. Let us state the following simple lemmas.

Lemma 2.1. *Graph $\Gamma_{R,M}$ does not have an isolated vertex.*

Proof. Assume that $V(\Gamma_{R,M}) = V_R \cup V_M$ where $V_R = R \setminus C_R(M)$ and $V_M = M \setminus C_M(R)$. If $x \in V_R$ is an arbitrary vertex in the first part, then we have $x \in R$ and $x \notin C_R(M)$ thus there exists an element $y \in M$ such that $xy \neq yx$. Hence $y \in M \setminus C_M(R)$ which it implies that y is vertex in V_M. Therefore x and y are adjacent. Similary, for every vertex of V_M there exists a vertex in V_R. □

Lemma 2.2. *Let x be a vertex in graph $\Gamma_{R,M}$. Then $deg(x) = |M| - |C_M(x)|$ if $x \in R \setminus C_R(M)$ and $deg(x) = |R| - |C_R(x)|$ if $x \in M \setminus C_M(R)$.*

Proof. Let $x \in R \setminus C_R(M)$ then $\deg(x) = |\{y \in M \setminus C_M(R)$ such that $xy \neq yx\}| = |M| - |\{y \in M$ such that $xy = yx\}| = |M| - |C_M(x)|$. If $x \in M \setminus C_M(R)$ thus it is similar to above. □

Lemma 2.3. *The graph $\Gamma_{R,M}$ is connected.*

Proof. Let x and y be two arbitrary vertices. We have two cases. The first case is that x and y are in the same part, for instance assume that $x, y \in V_R$. Then by Lemma 2.2, there are vertices $m, n \in V_M$ such that $x \sim m$, $y \sim n$. If $y \sim m$ or $x \sim n$ then we have a path between x and y and the proof is done. Suppose that $y \nsim m$ and $x \nsim n$ then put $z = x + y$ and we have $x \sim m \sim z \sim n \sim y$. The second case is that $x, \in V_R$ and $y \in V_M$. If x is adjacent to y then the proof follows. Otherwise, there are vertices $r \in V_R$ such that $r \sim y$, by Lemma 2.2. So we have path between x and r by the previous case and $r \sim y$. Therefore we have a path between x and y and the proof is completed. □

Lemma 2.4. *Let $a \in R$, $b \in M$ and $a + C_R(M)$, $b + C_M(R)$ are two cosets of R and R-module M,respectively, and a is adjacent to b if and only if every $x \in a + C_R(M)$ and every $y \in b + C_M(R)$ are adjacent.*

Proof. Assume that $x \in a+C_R(M)$ and $y \in b+C_M(R)$ then there are elements $z \in C_R(M)$ and $m \in C_M(R)$ such that $x = a + z$ and $y = b + m$. If x is not adjacent to y, then we have $xy = yx$ or $(a + z)(b + m) = (b + m)(a + z)$ or $ab = ba$ which is a contradiction. Hence x is adjacent to y as required. □

By the above lemma, one can see that induce subgraph of $\Gamma_{R,M}$ to the cosets $a + C_R(M)$ and $b + C_M(R)$ is a complete bipartite graph, where $a \in V_R$ and $b \in V_M$.

Lemma 2.5. *diam* $(\Gamma_{R,M}) \leq 3$.

Proof. Suppose that x and y are two arbitrary vertices. Then we have the following cases:

Case1: $x, y \in R\backslash C_R(M)$. Then there are vertices n and m in part $M\backslash C_M(R)$ such that $x \sim n$ and $y \sim m$. If $x \sim m$ or $y \sim n$., then $x \sim m \sim y$ or $x \sim n \sim y$ and so $d(x, y) = 2$ otherwise, we will have $x \sim n + m \sim y$ which implies that $d(x, y) = 2$ thus in case 1, $d(x, y) = 2$.

Case 2: $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$. If $x \sim y$ then $d(x, y) = 1$. So assume that $x \nsim y$. By Lemma 2.2, there are vertices $n \in R \setminus C_R(M)$ such that $n \sim y$. Hence $d(x, y) \leq d(x, n) + d(n, y) \leq 3$. Thus $d(x, y) \leq 3$.

The case that $x, y \in M \setminus C_M(R)$, then the proof is very similar to case 1. □

Lemma 2.6. *If diam*$(\Gamma_{R,M}) = 2$ *then* $\Gamma_{R,M}$ *is complete bipartite graph.*

Proof. If there are two vertices that not adjacent so $\mathrm{diam}(\Gamma_{R,M}) > 2$ which is a contradiction. □

In the following lemma, we compute the girth of $\Gamma_{R,M}$.

Lemma 2.7. *gr*$(\Gamma_{R,M}) = 4$ *or* 6.

Proof. We have three cases to follow:

Case 1: If $\mid V_R \mid = 1$ or $\mid V_M \mid = 1$ then $\mathrm{gr}(\Gamma_{R,M}) = \infty$. So $\mid V_R \mid \geq 2$ and $\mid V_M \mid \geq 2$.

Case 2: If r_1, r_2 are in one part. Assume that $r_1, r_2 \in V_R$. By lemma 2.2 there exist $m_1, m_2 \in V_M$ such that if $r_1 \sim m_1, r_2 \sim m_2$ then $\mathrm{gr}(\Gamma_{R,M}) = 4$.

Case 3: If $r_1 \in V_R$ and $m_1 \in V_M$. By lemma 2.2 there exist $r_2 \in V_R$ and $m_2 \in V_M$ such that $r_1 \sim m_1, r_2 \sim m_2$ so we will have $r_1 \sim m_1 + m_2 \sim r_2 \sim m_2 \sim r_1 + r_2 \sim m_1 \sim r_1$. The proof is completed. □

Lemma 2.8. $R = \cup_{a\in R} a + C_R(M)$.

Proof. The proof follows the fact that $C_R(M)$ is an additive subgroup of R. □

Lemma 2.9. *If* $x \in V(\Gamma_{R,M})$, *if* $x \in V_R$ *then* $d(x) \geq \lceil \frac{|M\backslash C_M(R)|}{2} \rceil$. *or if* $x \in V_M$ *then* $d(x) \geq \lceil \frac{|R\backslash C_R(M)|}{2} \rceil$.

Proof. Case 1: Assume that $x \in V_R$. Let x be adjacent to y in $M \setminus C_M(R)$ and the vertex m_i not be adjacent to x in $M \setminus C_M(R)$, for each $1 \leq i \leq k$. Obviously, the vertices y, $y + m_1$, $y + m_2$, $\cdots$, $y + m_k$ are in $M \setminus C_M(R)$ and are adjacent to x. Since they are distinguished, then x is adjacent to $k + 1$ vertices.

Case 2: If $x \in V_M$ then the proof is similar above. □

Lemma 2.10. $\alpha(\Gamma_{R,M}) = max\{\mid R \setminus C_R(M) \mid, \mid M \setminus C_M(R) \mid\}$.

Proof. Without loss of generality, suppose that $\mid M \setminus C_M(R) \mid \leq \mid R \setminus C_R(M) \mid = k$. Let H be the largest independent set and $V(H) = V_1 \cup V_2$, where $V_1 \subseteq V(R \setminus C_R(M))$ and $V_2 \subseteq V(M \setminus C_M(R))$. Clearly, $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$. By Lemma 2.9, each vertex of V_1 or V_2 is adjacent to at least $\lceil \frac{|M \setminus C_M(R)|}{2} \rceil$ or $\lceil \frac{|R \setminus C_R(M)|}{2} \rceil$ vertices, respectively. Therefore,

$$\mid V_1 \mid + \mid V_2 \mid \leq \lceil \frac{\mid M \setminus C_M(R) \mid}{2} \rceil + \lceil \frac{\mid R \setminus C_R(M) \mid}{2} \rceil.$$

Thus, $V(H)$ is lower than k. □

Lemma 2.11. *If graph $\Gamma_{R,M}$ has a pendent then $\Gamma_{R,M}$ is star graph.*

Proof. Suppose that $x \in R \setminus C_R(M)$ and $\deg(x) = 1$ then by Lemma 2.3, $|M| - |C_M(x)| = 1$ or $|C_M(x)| = |M| - 1$. Since $C_M(x) \subset M$ so $|C_M(x)||M|$ or $|M| - 1||M|$ then there exists a positive integer k such that $|M| = k(|M| - 1)$. Thus $k = \frac{|M|}{|M|-1} \in Z$ and we should have $|M| - 1 = 1$ consequently $|M| = 2$ or $|M| - 1 = |M|$ that is a contradicthion. We know that $C_M(R) \subseteq M$ then we have two cases: $C_M(R) = M$ that it is a contradiction and $C_M(R) = \{0\}$ then $M \setminus C_M(R) = \{a\}$ so $|M \setminus C_M(R)| = 1$ therefore $\Gamma_{R,M}$ is star graph. □

Theorem 2.12. *If $|C_R(M)| \geq 3$ and $|C_M(R)| \geq 3$ then $\Gamma_{R,M}$ is not planar graph.*

Proof. Since $|C_R(M)| \geq 3$ and $|C_M(R)| \geq 3$ then the order of cosets of $C_R(M)$ and $C_M(R)$ are at least 3. Let $x \in C_R(M)$ so by Lemma 2.2, there exists $a \in M \setminus C_M(R)$ such that $x \sim a$ and by Lemma 2.5, x is adjacent to all members of coset $a + C_M(R)$ and because the order of cosets of R is at least 3 so we have a subgraph $K_{3,3}$.Therefore $\Gamma_{R,M}$ is not planar. □

Corollary 2.13. *If $|C_R(M)| \geq 2$ and $|C_M(R)| \geq 3$ then $\Gamma_{R,M}$ is not 1-plannar graph.*

Proof. It is similar to proof of Theorem 2.12. □

3. Vertex and edge connectivity

In this section, we discuss about vertex and edge k-connectivity of $\Gamma_{R,M}$. Let us state the following lemma which plays an important role in the proof of next theorem.

Lemma 3.1. *Every connected graph G satisfies the inequalities $\kappa(G) \leq \kappa'(G) \leq \delta$.*

Definition. For two vertices x and y which are in the same part, put $Z = N(x) \cap N(y)$, $X = N(x) \setminus Z$ and $Y = N(y) \setminus Z$.

Lemma 3.2. *Let x and y be the vertices in the same part. Then $|Z| > |X|$ and $|Z| > |Y|$.*

Proof. If $X = \emptyset$ or $Y = \emptyset$, then the assertion holds by Lemma 2.9. So, suppose that $1 \leq |X| \leq |Y|$ and $x_1 \in X$. If $Y = \{y_1, y_2, \cdots, y_k\}$, then $\{y_1 + x_1, y_2 + x_1, \cdots, y_k + x_1\} \subseteq Z$. □

Theorem 3.3. *Graph $\Gamma_{R,M}$ is δ-edge-connected.*

Proof. Let x, y be two vertices of $\Gamma_{R,M}$. We show that there exists $\min\{d(x), d(y)\}$ edge disjoint paths between x and y. Consider the two following cases:

Case 1: x and y belong to the same part. Assume that $x, y \in R \setminus C_R(M)$ and $|X| \leq |Y|$. Put $X = \{x_1, x_2, \cdots, x_k\}$, $Y = \{y_1, y_2, \cdots, y_l\}$ and $Z = \{z_1, z_2, \cdots, z_t\}$. For vertices x_i and y_i, there is vertex u_i such that $x_i \sim u_i \sim y_i$, $1 \leq i \leq k$. Now, consider the paths $x \sim y_i \sim y$ $(1 \leq i \leq l)$ and $x \sim x_i \sim u_i \sim y_i \sim y$ $(1 \leq i \leq k)$. Therefore, there exists $d(x)$ edge disjoint paths between x and y.

Case 2: x and y are in the different parts. Let $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$ and consider the following cases:

Case 2.1: $x \nsim y$. Put H the induced bipartite subgraph on $A \cup B$, where $A = N(x) \cup \{y\}$ and $B = N(y) \cup \{x\}$. For each set $S \subseteq A$, we claim that $N(S) \geq S$. If $|S| < |A|/2$, then we have $N(S) \geq S$, by Lemma 2.9. If $|S| \geq |A|/2$ and $a \in B \setminus N(S)$, then we have $N(a) \subset A \setminus S$, that it is a contradiction by Lemma 2.9. From Hall's Theorem we can conclude that graph H has a perfect matching. If vertex x is not in perfect matching, then assume that $x_i \sim y_i$, for every integer i, $1 \leq i \leq k$. In this case, there are k disjoint paths $x \sim x_i \sim y_i \sim y$. If vertex x is in perfect matching, then consider $x \sim x_1$ and $x_i \sim y_i$ and $y \sim y_{k+1}$, for each $2 \leq i \leq k$. Consider the following cases:

Case 2.1.1: y_{k+1} is adjacent to x_1. Then consider the paths $x \sim x_i \sim y_i \sim y$, for $2 \leq i \leq k$ and $x \sim x_1 \sim y_{k+1} \sim y$.

Case 2.1.2: y_{k+1} is not adjacent to x_1. By Lemma 2.9, there is integer t such that $2 \leq t \leq k$, $x_1 \sim y_t$ and $y_{k+1} \sim x_t$. Now, there is k disjoint paths between x to y as $x \sim x_1 \sim y_t \sim y$, $x \sim x_t \sim y_{k+1} \sim y$, and $x \sim x_i \sim y_i \sim y$, for integers $2 \leq i \leq k$ except $i = t$.

Case 3: $x \sim y$. Each vertex of the set A or B is adjacent to half of vertices of other part, by Lemma 2.9. It is not difficult to see that similar to Case

2.1, graph H has a perfect matching by Hall's Theorem. Let the edge xy be in the matching. Without of lose of generality, assume that $x_i \sim y_i$, for every $1 \leq i \leq k$.. Thus, $x \sim y$ and $x \sim x_i \sim y_i \sim y$ are the disjoint xy-paths. If the edge xy is not in the matching, then similar to Case 1, we have k edge disjoint paths. □

The following theorem gives the maximum value of k for which $\Gamma_{R,M}$ is k-connected.

Theorem 3.4. $\kappa(\Gamma_{R,M}) = \delta$.

Proof. Let x, y be two vertices of $\Gamma_{R,M}$. If x and y are in different parts, then there are δ disjoint paths between x and y, by Theorem 3.3. Now, suppose that x and y are in the same part. Without lose of generality, assume that $x, y \in R \setminus C_R(M)$ and $d(x) = \delta$. Put $X = \{x_1, x_2, \cdots, x_k\}$, $Y = \{y_1, y_2, \cdots, y_l\}$ and $Z = \{z_1, z_2, \cdots, z_t\}$. By Lemma 3.2, $t > l$ and $t > k$. Put $A_i = N(x_i) \cap N(y_i)$, for $1 \leq i \leq k$. Since $d(x) = \delta$, we have $|A_i| > k$. Thus, there exist k distinguished vertices $h_1, h_2, \cdots, h_k$ such that $x_i \sim h_i \sim y_i$, $1 \leq i \leq k$. Now, consider xy-paths $x \sim z_i \sim y$, $1 \leq i \leq t$ and $x \sim x_i \sim h_i \sim y_i \sim y$, $1 \leq i \leq k$. Thus, there is δ disjoint xy-paths. If $d(x) \neq \delta$, then it is enough to put δ neighbours of x and do like above. □

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298 (2006), 468-492.

[2] S. Akbari and A. Mohammadian, *Zero-divisor graphs of non-commutative rings*, J. Algebra, 296 (2006), 462-479.

[3] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (2) (1999) 434-447.

[4] P. J. Cameron, *The power graph of a finite group*, J. Group Theory, 13 (2010), 779-783.

[5] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen,*Intersection graphs of ideals of rings*, Discrete Math., 309 (2009), 5381-5392.

[6] H. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, 319 (2008), 1801-1808.

[7] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge University Press, Cambridge, 1990.

[8] West D.B., *Introduction to Graph Theory*, 2000.

Accepted: 11.03.2017

AN IMPROVED CLUSTERING METHOD BASED ON DENSITY AND DIVISION METHOD

Zhang Qiu-Ju
School of Management and Economics
Beijing Institute of Technology, Beijing
100081, China
zhangqj1983@hotmail.com

Abstract. Combining partitioning and density - based clustering method, an improved clustering method is proposed in this paper on the basis of objective clustering algorithm. Firstly, the points with greater density which are distant from each other were selected as the initial centers of K-means clustering. Then, Use K-means was used to roughly determine the elements contained in each class. Afterwards, the points with the largest density in each class were searched and taken as the centers to re-conduct K-means clustering. If a class has more than one maximum density points, then the class will have multiple clustering centers, which make the shape of the class not round any longer and facilitate the classification of irregular shapes. Finally, by the dipole idea of the objective clustering algorithm, the optimal number of clusters was determined. The improved algorithm proposed in this paper achieved very good results in the clustering tests on random data set and UCI data set

Keywords: clustering analysis, K-means, density-based clustering method, objective clustering algorithm.

1. Introduction

In the analysis and description of the world, class represents a group of objects with public characteristics. People divide objects into different classes to explore the commonality between the objects of the same class and the gap between the objects of different classes, which are the information that can not be obtained by other methods. In the case of data analysis, clusters are potential classes and clustering analysis is a technique for discovering these classes automatically.

Clustering analysis refers to the analysis process which divides the sets of physical or abstract objects into classes made up of similar objects, with a goal to collect data on a similar basis to classify. Clustering has been applied and developed in many fields, including mathematics, computer science, statistics, biology and economics, which is mainly used for describing data, measuring the similarity between different data sources and classifying data sources into different clusters.

Traditional clustering analysis methods are mainly partitioning method, hierarchical method, density-based method, grid-based method and model-based method, with algorithms that are widely used in each method, such as K-means

clustering algorithm in partitioning method, cohesive hierarchical clustering algorithm in hierarchical method, neural network clustering algorithm in model-based method, etc.

In general, a partitioning-based clustering method firstly requires a given number of clusters to create an initial division and then applies an iterative relocation technique to move the objects between partitions to achieve the final division. Each class contains at least one sample point, each of which belongs to and can only belong to one class. The typical division methods are K-means algorithm, CLARANS algorithm and FREM algorithm. Partitioning-based clustering requires the size of the number of clusters to be specified in advance. If the values are not appropriate, the clustering rationality will be affected. Besides, partitioning-based clustering basically belongs to spherical clustering and its division effect on classes with irregular shapes is not obvious.

Hierarchical clustering methods combine data objects into a clustered tree, which can be further divided into agglomerative type (with bottom-up hierarchical decomposition) and divisive type (with top-down hierarchical decomposition) hierarchical clustering. Aggregated hierarchical clustering initially treats each sample point as a class and then merges the different classes in each iteration process until the preset clustering effect is reached. Split hierarchical clustering initially treats all sample points as a class and then splits them in each iteration process until the termination condition is met. The representative hierarchical clustering algorithms are BIRCH algorithm and CURE algorithm. However, the two methods are time-consuming and their clustering complexity is too high.

The density - based clustering methods consider a cluster as a high-density region in the data space that is separated by a low-density region. Its main idea is to continue clustering as long as the density of the neighboring area (the number of objects or data points) exceeds a certain threshold. That is, for each data point in a given class, there must be at least a certain number of points in a region with a given range. The methods can be used to filter "noise", isolate point data and find clusters of arbitrary shapes. Typical density-based clustering methods are DBSCAN and OPTICS. However, density-based clustering requires two parameters to be preset: the radius of adjacent threshold and the number of data points. As the clustering results are highly sensitive to the two parameters, improper parameter setting can lead to poor classification results.

The grid-based clustering methods quantify the object space into a finite number of units to form a grid structure, where all the clustering operations are performed. Represented by STING algorithm and CLIQUE algorithm, the grid-based clustering methods have fast processing speed and their processing time is independent of the number of data objects and is only related to the number of units in each dimension in the quantization space. However, the clustering effect of the methods is sensitive to the division of the grid.

The model - based clustering methods assume a model for each cluster, looking for data to best fit the given models. Model-based clustering algorithms may locate clustering by constructing a spatial density function that reflects

the spatial distribution of data points. It also automatically determines the number of clusters based on standard statistics, taking into account "noise" data or isolated points, resulting in robust clustering methods. Neural networks and decision trees are typical model-based clustering methods, which often have strict assumptions on data distribution. Therefore, model - based clustering methods have their own limitations.

In this paper, an improved clustering algorithm is proposed based on an analysis of the advantages and disadvantages of various clustering methods. On the one hand, the objective clustering analysis algorithm is applied to determine the number of clusters. On the other hand, the idea of density - based clustering methods is introduced to realize the clustering of classes of various shapes, breaking the limitation that division-based clustering is only suitable for spherical shape.

2. The basic principle of objective clustering algorithm (OCA)

One of the major problems to be solved in clustering is the number of classes that should be divided. One of the basic principles of clustering is to ensure that the similarity of the samples within the class is high and the gap between classes is obvious. At present, the algorithms that can automatically determine the number of clusters are grid-based clustering algorithm, density-based clustering algorithm and model-based clustering algorithm. However, as mentioned above, the clustering results of grid-based clustering methods and density-based clustering methods rely heavily on input parameters while the model-based clustering method is relatively strict on the assumption of data distribution. Partitioning-based clustering and hierarchical clustering often determine the number of clusters by establishing an objective function, for example, taking the average distance of the elements in each class to the various centroids as the objective function, and the number of clusters which makes the objective function reaches the minimum value is determined as the optimal number of clusters. However, in the case that the clustering point is not clear, the situation that the objective function gets smaller and smaller with the increase in the number of clusters tends to appear, resulting in a best clustering result that each sample becomes a class.

The objective clustering algorithm is a nonparametric clustering method proposed by Academician A. G. Ivakhnenko [11, 12] of the Ukrainian Academy of Sciences that can automatically and objectively determine the number of classification categories. The method uses two criteria for clustering: inner criteria are used to generate classes, and outer criteria (consistency criteria) are used to determine the optimal number of classes.

The basic principle firstly uses the "dipole" idea to classify the sample data into two corresponding subsets A and B, where clustering is carried out based on the distance between the sample points, respectively, so that each class has similarity. Then, by applying the idea that the nearest two points should belong

to the same category, whether the corresponding dipole samples in the two sets are assigned to the same class is taken as a criterion for testing the rationality of clustering in order to determine the optimal number of clusters.

The specific method is as follows:

Step 1: Calculation of the distance between all the data samples in the training set. The distance between the samples is sorted from small to large, with each distance corresponding to a dipole.

Step 2: Assuming that the sample size is n, take a number of dipoles that do not have a common sample with each other and put them into set A and B. Similarly, sets C and D are obtained from the remaining dipoles and taken as detection sets.

Step 3: The samples in sets A and B are numbered and the two samples from the same dipole use the same number in sets A and B to correspond to each other.

Step 4: for $i = 1 : r - 2$.

Step 5: Find the classification scheme with the highest value. If there are multiple schemes with the same vale, then Step 4 is repeated on detection sets C and D. Afterwards, which scheme has the highest value on C and D among the schemes that have the highest value on A and B is considered and the clustering of this scheme is the optimal clustering and the corresponding number of clusters is the optimal number of clusters.

Obviously, objective clustering analysis is essentially a hierarchical clustering algorithm, so there exists the ubiquitous defects that hierarchical clustering algorithm have. When the sample size is very large, the number of times needed for clustering is large, resulting in very low calculation efficiency. Moreover, the algorithm is only suitable for the clustering of spherical clusters and can not find clusters of arbitrary shapes. Therefore, this paper combines OCA with K-means and density-based clustering method to propose a new clustering algorithm.

3. Improved objective clustering algorithm

The improved objective clustering algorithm takes into account the fact that dividing the data into too many classes has actually lost the meaning of clustering, so it sets one parameter to be the maximum number of classes. The data is clustered using the K-means method. The criteria for determining the best number of classes refer to the OCA algorithm. The specific algorithm is as follows:

Step 1: The corresponding two sets A and B, as well as C and D, are generated using the methods mentioned in the objective clustering algorithm, and each set contains one sample.

Step 2: Let the number of clusters be an integer which is large enough.

Step 3: Select the initial clustering center.

The Euclidean distance between the samples in set A is calculated to represent the Euclidean distance between points and points. The average distance

between all points is calculated and taken as the neighborhood threshold. Assuming that the two most distant sample points in the set are E and F, put them into the initial cluster center set; the distance between the remaining points in the set and point E and F is tested and corresponding points are selected and denoted by G and put into the set; the distance between the remaining points in the set and point E, F and G is tested and corresponding points are selected and denoted by H and put into the set. This process goes on until the point is finally found out.

Step 4: Select the previous point in the set as the initial cluster center. The data in set A is clustered as a class using the K-means algorithm. The average distance of the samples in this class is calculated as the neighborhood threshold, and the points with the highest density in each class are found as the new clustering centers. If there is more than one point with the maximum density, it suggests that the class is of irregular shape and the points should be taken as the new clustering centers of the class. K-means clustering is re-conducted with the new clustering centers to re-determine which class the sample points belong to, and to gradually obtain the final clustering center through iteration. The final cluster centers of each class are used as the initial centers of set B to conduct K-means clustering and set B is also clustered as a class.

Step 5: command.

Step 6: command, if, execute Step 4.

Step 7: The corresponding classification scheme is the optimal clustering scheme. If there are multiple schemes with the same value, clustering is carried out on the sets C and D based on these schemes. The clustering of the scheme with the highest value on C and D is the optimal clustering and the corresponding number of clusters is the optimal number of clusters.

Step 8: The clustering center point of the optimal clustering scheme in Step7 is used as the initial clustering center, and the whole data set is clustered to obtain the final clustering result.

4. Numerical experiment

The improved algorithm proposed in this paper is applied to the clustering of two random data sets, which is compared to the OCA algorithm and K-mean algorithm. The number of clusters of the K-mean algorithm is set to three. The clustering results are shown in Fig. 1 and Fig. 2.

It can be seen from Figure 1 and Figure 2 that the improved-OCA algorithm inherits the advantages of density-based clustering algorithm for class contour recognition, and has obvious advantages on non-spherical data clustering compared with other partition-based algorithms. Meanwhile, it is not as much sensitive as the density-based algorithm to input parameters and its clustering accuracy and number of clusters are independent from parameter setting. For example, the clustering process of the above random data sets verifies that the obtained clustering results are exactly the same, whether the K value is set to be

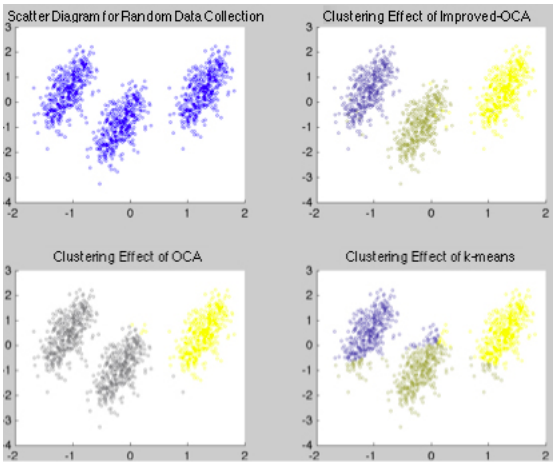


Figure 1. Comparison of clustering algorithms on random data set 1

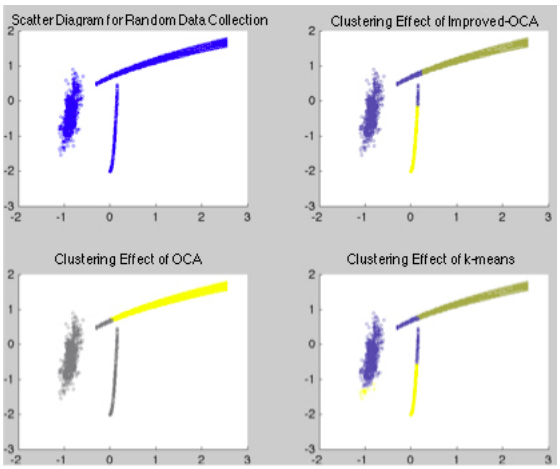


Figure 2. Comparison of clustering algorithms on random data set 2

20, 15 or 10. The improved-OCA algorithm automatically determines the number of categories and identifies the category outline. In addition, when the K value is set to be small, the number of operations is small, and the classification efficiency can be improved.

Among the judgments on the classes of the 1200 samples on data set 1 by the improved-OCA algorithm, there are only 16 wrong judgments, with an accuracy rate of 98.67%. While the accuracy rate of the judgment on the 1510 samples on data set 2 is 90.73%.

Then, the Iris dataset and the Breastcancerw dataset provided on the UCI dataset are clustered. The Iris dataset contains 150 sample points and is divided into three classes, each containing 50 sample points. Class 1 is completely separated from class 2 and class 3 while class 2 and class 3 are crossed. There-

fore, it is a reasonable division to divide the dataset into 2 or 3 classes. The improved-OCA algorithm classifies the first 50 sample points into one class and the remaining 100 sample points into another class, with a correct rate of 100diagnosis, contains 444 class 1 data and 239 class 2 data, which are correctly classified into 435 class 1 data and 168 class 2 data by the improved-OCA algorithm, with an accuracy rate of 88.29

The above random data set and UCI dataset clustering test proved the advantage of the improved-OCA algorithm in discovering clusters of arbitrary shapes relative to partitioning and hierarchical methods. Besides, the improved-OCA algorithm is not as sensitive as the density-based clustering method to input parameters. The only input parameter of the improved algorithm is the maximum number of classes permitted and the final clustering result does not depend on this parameter.

5. Conclusion

This study improved the OCA algorithm based on hierarchical clustering based on the analysis of the advantages and disadvantages of various algorithms by preserving its consistency criteria for determining the number of clustered categories, introduced the idea of density - based clustering algorithm, found out the point with the largest density in each class and took them as the centers of the classes to reconduct clustering so as to ensure that the classes are of arbitrary shapes, with no need of inputting parameters which may influence the clustering results. Finally, a test was carried out on the random data set and UCI data set, which proves the effectiveness of the improved-OCA algorithm proposed in this paper.

However, there are some shortcomings in the improved-OCA algorithm. It is essentially exhaustive K-means clustering in a small range to determine the value of the number of clusters, which is bound to affect the efficiency of clustering when the sample size is very large. How to judge the optimal number of clusters more quickly and efficiently is the problem which remains to be further studied.

References

[1] I.S. Dhillon, D.S. Modha, *Concept decompositions for large sparse text data using clustering*, Machine learning, 2001, 42(1-2), 143-175.

[2] A.K. Jain, M.N. Murty, P.J. Flynn, *Data clustering: a review*, ACM computing surveys (CSUR), 1999, 31(3), 264-323.

[3] C. Elkan, *Using the triangle inequality to accelerate k-means*, Fawcett T., Mishra N. Proceedings of the Twentieth International Conference on Machine Learning (ICML-2003). Washington DC,USA: The AAAI Press, 2003, 3, 147-153.

[4] D. Arthur, S. Vassilvitskii, *k-means++: The advantages of careful seeding*, Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms. Philadelphia, PA, USA: Association for Computing Machinery, 2007, 1027-1035.

[5] P.S. Bradley, U.M. Fayyad, *Refining Initial Points for K-Means Clustering*, Shavlik J. Proceedings of the 15th International Conference on Machine Learning (ICML98), San Francisco, USA: Morgan Kaufmann, 1998,98, 91-99.

[6] Zhao Yanchang, Song Mei, Xie Fan, et al., *Clustering Datasets Containing Clusters of Various Densities*, Journal of Beijing University of Posts and Telecommunications, 2003, 26(2), 42-47.

[7] L. Ertoz, M. Steinbach, V. Kumar, *Finding Clusters of Different Sizes, Sharps, Densities in Noisy, High Dimensional Data*, Proc of International Conference on Data Mining San Francisco, USA SIAM Press, 2003, 1-12.

[8] E. Martin, H.P. Kriegel, *A Density-based Algorithm for Discovering Clusters in Large Spatial Databases with Noise*, Proc of the 2nd International Conference on Knowledge Discovery and Data Mining, Portland, OR, USA, 1996, 226-231.

[9] M. Patwary, D. Palsetia, A. Agrawal et al., *A new scalable parallel DBSCAN algorithm using the disjointset data structure*, High Performance Computing, Networking, Storage and Analysis Salt Lake City, Utah, USA, IEEE, 2012, 10-16

[10] T.N. Thanh, K. Drab, M. Daszykowski, *Revised DBSCAN algorithm to cluster data with dense adjacent clusters*, Chemo metrics and Intelligent Laboratory Systems, 2013, 120, 92-96.

[11] A.G. Ivakhnenko, *Heuristic self-organizing in problems of engineering cybernetics*, Automatic, 1967, 6, 207-219.

[12] A.G. Ivakhnenko, *The Review of Problems Solvable by Algorithms of the Group Method of Data Handling (GMDH)*, Pattern Recognition and Image Analysis, 1995, 5 (4), 527 - 535.

Accepted: 31.03.2017

EXISTENCE OF MANY NON-RADIAL SOLUTIONS OF AN ELLIPTIC SYSTEM

Zhenluo Lou
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang, Henan, 471023
P.R. China
louzhenluo@amss.ac.cn

Abstract. In this paper, we consider the following elliptic system

$$\begin{cases} -\Delta u = \mu_1 u^{p-1} + \beta u^{\frac{p}{2}-1} v^{\frac{p}{2}}, & x \in \Omega \\ -\Delta v = \mu_2 v^{p-1} + \beta u^{\frac{p}{2}} v^{\frac{p}{2}-1}, & x \in \Omega \\ u, v > 0, \ x \in \Omega, \quad u = v = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 4)$ is an annulus, $\mu_1, \mu_2 > 0$, $\beta > 0$ and $2 < p < \frac{2N-2}{N-3}$. By variational and rescaling method, we prove that the system has many non-radial solutions.
Keywords: elliptic system, variational method, non-radial solutions.

1. Introduction

In this paper, we study the following elliptic system

$$\begin{cases} -\Delta u = \mu_1 u^{p-1} + \beta u^{\frac{p}{2}-1} v^{\frac{p}{2}}, & x \in \Omega \\ -\Delta v = \mu_2 v^{p-1} + \beta u^{\frac{p}{2}} v^{\frac{p}{2}-1}, & x \in \Omega \\ u, v > 0, \ x \in \Omega, \quad u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in \mathbb{R}^N : \ r^2 < |x|^2 < (r+d)^2\}$ is an annulus, $N > 4$, $\mu_1, \ \mu_2 > 0$, $\beta > 0$, $2 < p < \frac{2N-2}{N-3}$, and d is a fixed positive number.

In [6, 7], Gidas-Ni-Nirenberg studied the scalar equation

$$-\Delta u + u = u^p, \quad x \in \Omega, \quad u > 0, \ x \in \Omega, \quad u = 0, \ x \in \partial\Omega, \tag{1.2}$$

where $1 < p < 2^* = \frac{2N}{N-2}$. By moving plane method, they proved that the positive solutions of equation (1.2) are radially symmetric. After that there are many results about the symmetric properties of solutions. In fact, if moving plane method can not be used, then the solutions can be symmetry-breaking. We point out that if the domain Ω is an annulus, then it is not convex, thus the moving plane method can not be used. In [9], Y. Li considered the non-radial solutions of equation (1.2), the author also studied the supercritical case, see also [11, 15] and the reference therein.

In recently decades, several researchers have studied the symmetric results of elliptic systems. In [3], by maximal principle and Morse index, the authors studied the symmetric and symmetry-breaking results of elliptic system, see [14, 17, 18] for symmetry-breaking results of system and the references therein. In particular, for $p = 3$ of system, the author [12] studied the non-radial solution of system (1.1), where they proved that the system has at least a non-radial solution.

Noticing that $\frac{2N-2}{N-3} > 2^*$, the exponent p may be super critical exponent. In this paper, by a compact embedding in [9], we study the multiple solutions of system (1.1), we have that

Theorem 1.1. *Let d be a fixed positive number, $2 < p < \frac{2N-2}{N-3}(N > 4)$. If $r > d$ is large enough, then system* (1.1) *has at least $[\frac{N}{2}] - 1$ non-radially symmetric solutions.*

This paper is organized as follows. In section 2, we give several preliminaries and lemmas; in section 3, by variational and rescaling method, we give the proof of Theorem 1.1.

2. Some preliminaries and lemmas

Firstly, we define the working space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm as

$$\|(u, v)\| = \left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx \right)^{1/2},$$

where $H_0^1(\Omega)$ is the standard Sobolev space.

We define the functional of system (1.1) as

$$I(u, v) = \frac{1}{2} \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \frac{1}{p} \int_\Omega \mu_1 u^p + \mu_2 v^p dx - \frac{2}{p} \beta \int_\Omega u^{\frac{p}{2}} v^{\frac{p}{2}} dx,$$

and the following quotation

$$J(u, v) = \frac{\left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx \right)^{p/2}}{\int_\Omega \mu_1 u^p + \mu_2 v^p dx + \frac{p}{2} \int_\Omega u^{\frac{p}{2}} v^{\frac{p}{2}} dx}.$$

Now we define

$$\Lambda_0 = \{(u, v) \in H \backslash \{(0, 0)\} : u,\ v \text{ are radial functions}\},$$

and

$$\Lambda_k = \{(u, v) \in H \backslash \{(0, 0)\} : u(x) = u(|y_1|, |y_2|), v(x) = v(|y_1|, |y_2|)\},$$

where $y_1 = (x_1, \cdots, x_k)$, $y_2 = (x_{k+1}, \cdots, x_N)$, $k = 0, 1, \cdots, [\frac{N}{2}]$.

We define the minimum value as

$$\lambda_k = \inf_{(u,v) \in \Lambda_k} J(u, v),\ k = 0, 1, \cdots, [\frac{N}{2}].$$

Now we list a inequality, see [9].

Lemma 2.1. *Let $\Omega = \{x \in \mathbb{R}^N : r^2 < |x|^2 < (r+d)^2\}$ be an annulus, $N > 4$, $2 < p < \frac{2N-2}{N-3}$ and d is a fixed positive number, let*

$$\bar{\Lambda}_k = \{u \in H_0^1(\Omega) : u(x) = u(|y_1|, |y_2|)\},$$

where $y_1 = (x_1, \cdots, x_k)$, $y_2 = (x_{k+1}, \cdots, x_N)$. Then for $r > 10d$ large enough, we have

$$\frac{\left(\int_\Omega |\nabla u|^2 dx\right)^{p/2}}{\int_\Omega u^p dx} \geq C r^{\frac{p-2}{2}},$$

where C is a positive constant independent of r.

Remark 2.2. *If $1 < p < 2^*$, by Sobolev embedding theorem, we have that $\bar{\Lambda}_k \hookrightarrow L^p$ is compact, see [8, 20, 22], by previous lemma, we have that $\bar{\Lambda}_k \hookrightarrow L^{\frac{2N-2}{N-3}}$ is continuous, thus for $2 < p < \frac{2N-2}{N-3}$, we have that $\bar{\Lambda}_k \hookrightarrow L^p$ is compact.*

Lemma 2.3. *Suppose $2 \leq k < \bar{k} \leq [\frac{N}{2}]$, then*

$$\bar{\Lambda}_k \cap \bar{\Lambda}_{\bar{k}} \subset \bar{\Lambda}_0.$$

Next section we give the proof of the main result.

3. The proof of Theorem 1.1

We divide the proof into two parts. Firstly we have that

Lemma 3.1. *There exists a positive constant C depending on μ_1, μ_2, β, d, such that*

$$\lambda_0 \geq C r^{(N-1)(\frac{p}{2}-1)}. \tag{3.1}$$

Proof. Let $(u, v) \in \Lambda_0$, i.e. $u(x) = u(|x|), v(x) = v(|x|)$, then

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &= |\partial\Omega| \int_r^{r+d} |u'_\rho|^2 \rho^{N-1} d\rho \\ &\geq C(\Omega) r^{N-1} \int_r^{r+d} |u'_\rho|^2 d\rho. \end{aligned} \tag{3.2}$$

Similarly we have that

$$\int_\Omega |\nabla v|^2 dx \geq C(\Omega) r^{N-1} \int_r^{r+d} |v'_\rho|^2 d\rho.$$

By Sobolev embedding results, we have that

$$\begin{aligned} \int_\Omega u^p dx &= |\partial\Omega| \int_r^{r+d} u^p(\rho) \rho^{N-1} d\rho \\ &\leq C(\Omega, d) r^{N-1} \int_r^{r+d} u^p d\rho \\ &\leq C(\Omega, d) r^{N-1} \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{p/2}, \end{aligned} \tag{3.3}$$

and

$$\int_\Omega v^p dx \le C(\Omega,d) r^{N-1} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{p/2}.$$

By Hölder and Sobolev inequalities, we have

$$\begin{aligned} \int_\Omega u^{\frac{p}{2}} v^{\frac{p}{2}} dx &\le \left(\int_\Omega u^p dx \right)^{1/2} \left(\int_\Omega v^p dx \right)^{1/2} \\ &\le C(\Omega,d) r^{N-1} \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}}. \end{aligned} \tag{3.4}$$

Then we obtain that

(3.5)

$$\begin{aligned} \int_\Omega \mu_1 u^p + \mu_2 v^p + \frac{p\beta}{2} u^{\frac{p}{2}} v^{\frac{p}{2}} dx \le C r^{N-1} \Bigg\{ & \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}} \\ & + 2 \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}} \Bigg\} \\ \le & C r^{N-1} \left\{ \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}} \right\}^2 \\ \le & 4C r^{N-1} \left\{ \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}} \right\}, \end{aligned}$$

where C is a positive constant depending on μ_1, μ_2, β, d.

Noticing that $2 < p < \frac{2N-2}{N-3}$, then we obtain that

$$\left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} \le \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}.$$

Then we have that

$$\begin{aligned} \lambda_0 &\ge \frac{C r^{\frac{p}{2}(N-1)} \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}}{4C r^{N-1} \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}} \\ &\sim C(\mu_1, \mu_2, \beta, d) r^{(N-1)(\frac{p}{2}-1)}. \end{aligned} \tag{3.6}$$

In conclusion, we complete the proof. □

Following the idea of Li [9]. we have that

Lemma 3.2. *There exists a constant $C > 0$ depending on μ_1, μ_2, β, d, such that*

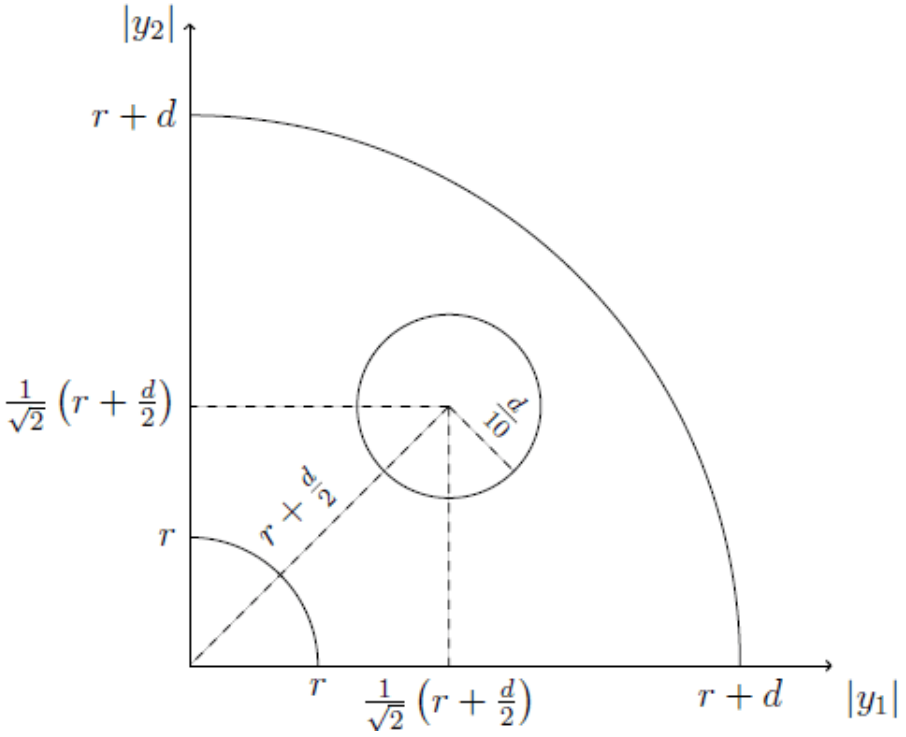
$$\lambda_k \le C r^{(N-2)(\frac{p}{2}-1)}, \text{ for } r > 10d \text{ large enough,} \tag{3.7}$$

where $k = 2, \cdots, [\frac{N}{2}]$.

Proof. Choose a nonnegative function $\omega(x) \in C_0^\infty(B_{d/10})\backslash\{0\}$, where $B_{d/10} \subset \mathbb{R}^2$, and let

$$u_k(x) = v_k(x) = \omega(|y_1| - \frac{1}{\sqrt{2}}(r + \frac{d}{2}), |y_2| - \frac{1}{\sqrt{2}}(r + \frac{d}{2})),$$

where $y_1 = (x_1, \cdots, x_k), y_2 = (x_{k+1}, \cdots, x_N)$. One can prove that $(u_k(x), v_k(x)) \in H$, see the following figure



Then we have that

$$\begin{aligned} \int_\Omega |\nabla u_k|^2 dx &\le C \int_{r^2 \le t^2+s^2 \le (r+d)^2} |\nabla\omega|^2 t^{k-1} s^{N-k-1} dtds \\ &\le C(r+d)^{N-2} \int_{r^2 \le t^2+s^2 \le (r+d)^2} |\nabla\omega|^2 dtds \\ &\le C(d,\Omega) r^{N-2} \int_{r^2 \le t^2+s^2 \le (r+d)^2} |\nabla\omega|^2 dtds, \end{aligned} \tag{3.8}$$

and

$$\int_\Omega |\nabla v_k|^2 dx \le C(d,\Omega) r^{N-2} \int_{r^2 \le t^2+s^2 \le (r+d)^2} |\nabla\omega|^2 dtds.$$

Next we have that

$$\begin{aligned}\int_\Omega u_k^p dx &= |\partial\Omega| \int_{r^2\le t^2+s^2\le (r+d)^2} u_k^p t^{k-1} s^{N-k-1} dtds \\ &\ge Cr^{N-2} \int_{r^2\le t^2+s^2\le (r+d)^2} \omega^p dtds,\end{aligned} \tag{3.9}$$

and

$$\int_\Omega v_k^p dx \ge Cr^{N-2} \int_{r^2\le t^2+s^2\le (r+d)^2} \omega^p dtds. \tag{3.10}$$

Then

$$\int_\Omega u_k^{\frac{p}{2}} v_k^{\frac{p}{2}} dx = \int_\Omega \omega^p dx \ge Cr^{N-2} \int_{r^2\le t^2+s^2\le (r+d)^2} \omega^p dtds. \tag{3.11}$$

By the previous inequalities and noticing $supp\ \omega(t,s) \subset \mathbb{R}^2$, we have that

$$\begin{aligned}\lambda_k &\le \frac{Cr^{\frac{p}{2}(N-2)} \left(\int_{r^2\le t^2+s^2\le (r+d)^2} |\nabla\omega|^2 dtds\right)^{\frac{p}{2}}}{C(\beta,\mu_1,\mu_2) \int_{r^2\le t^2+s^2\le (r+d)^2} \omega^p dtds} \\ &\le Cr^{(\frac{p}{2}-1)(N-2)},\end{aligned}$$

where $C>0$ is constant depending on β,μ_1,μ_2. In conclusion, we complete the proof. □

Next we give the proof of main result.

The Proof of Theorem 1.1. By Lemma 2.1, we have that $\lambda_k, k = 0,1,\cdots,[\frac{N}{2}]-1$ is achieved, then by standard rescaling method and critical theory, we can find $(u_0,v_0)\in\Lambda_0$ and $(u_k,v_k)\in\Lambda_k$ which are solutions of system (1.1). If $r>0$ is large enough, by (3.1) and (3.7), it is a contraction, thus (u_k,v_k) is non-radial, $k=2,\cdots,[\frac{N}{2}]$. Notice that Lemma 2.3, we prove that system (1.1) has at least $[\frac{N}{2}]-1$ non-radial solutions. In conclusion, we complete the proof. □

Remark 3.3. If $y_1=(x_1,\cdots,x_k)$ $y_2=(x_{k+1},\cdots,x_N)$ and $\bar{y}_1=(x_{i_1},\cdots,x_{i_k})$ $\bar{y}_2=(x_{i_{k+1}},\cdots,x_{i_N})$, $i_1,\cdots,i_N$ is a permutation of $1,\cdots,N$, we call that $u(|y_1|,|y_2|)\sim u(|\bar{y}_1|,|\bar{y}_2|)$.

Acknowledgements

This work is supported by NSFC 11571339, HAUST 13480051.

References

[1] C.O. Alves, D.C. de Morais Filho, M.A.S. Souto, *On system of elliptic equations involving subcritical or critical Sobolev exponents,* Nonl. Anal., 42 (2000), 771-787.

[2] C. V. Coffman, *A nonlinear boundary value problem with many positive solutions*, J. Diff. Equa., 54 (1984), 429-437.

[3] L. Damascelli, F. Pacella, *Symmetry results for coorperative elliptic systems via linearization*, SIAM J. Math. Anal., 45 (2013), 1003-1026.

[4] E.N. Dancer, K. Wang, Z. Zhang, *The limit equation for the Gross-Pitaevskii equations and S. Terracini's conjecture,* J. Func. Anal., 262 (2012), 1087-1131.

[5] E.N. Dancer, J. Wei, T. Weth, *A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system*, Ann. I.H. Poincaré-AN, 27 (2010), 953-969.

[6] B. Gidas, W. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68 (1979), 209-243.

[7] B. Gidas, W. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Mathematical Analysis and Applications, Part A, Adv. in Math. Suppl. Stud., 7a, 369-402, Academic Press, 1981.

[8] D. Gilbarg, N. Trdinger, *Elliptic partial differential equations of second order*, Springer Verlag, 1998.

[9] Y. Li, *Existence of many positive solutions of semilinear elliptic equations on annulus*, J. Diff. Equa., 83 (1990), 348-367.

[10] K. Li, Z. Zhang, *A perturbation result for system of Schrödinger equations of Bose-Einstein condensates in* $\mathbb{R}^3$*,* Disc. Cont. Dyna. Syst., 36 (2016), 851-860.

[11] S. S. Lin, *Existence of many positive nonradial solutions for nonlinear elliptic equations on an annulus*, J. Diff. Equa., 103 (1993), 338-349.

[12] Z. Lou, *Existence of non-radial solutions of an elliptic system*, Appl. Math. Lett., 68 (2017), 157-162.

[13] L.A. Maia, E. Montefuso, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, J. Diff. Equa., 229 (2006), 743-767.

[14] S. Peng, Z.-Q. Wang, *Segregated and synchronized vector solutions for nonlinear Schrödinger systems,* Arch. Rational Mech. Anal., 208 (2013), 305-339.

[15] D. Smets, J. Su, M. Willem, *Non-radial ground states for the Hénon equation*, Comm. in Cont. Math., 4 (2002), 467-480.

[16] S. Terracini, G. Verzini, *Multipulse phases in K-Mixtures of Bose-Einstein condensates*, Arch. Rational Mech. Anal., 194 (2009), 717-741.

[17] H. Tavares, T. Weth, *Existence and symmetry results for competing variational systems*, NoDEA Nonl. Diff. Equa. Appl., 20 (2013), 715-740.

[18] Z.-Q. Wang, M. Willem, *Partial symmetry of vector solutions for elliptic systems*, J. Anal. Math., 122 (2014), 69-85.

[19] K. Wang, Z. Zhang, *Some new results in competing systems with many species*, Ann. I. H. Poincaré-AN, 27 (2010), 739-761.

[20] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and Their Applications, 24, Birkhäuser, Boston, 1996.

[21] J. Wei, T. Weth, *Radial solutions and phase separation in a system of two coupled Schrödinger equations*, Arch. Rational Mech. Anal., 190 (2008), 83-106.

[22] Z. Zhang, *Variational, topological, and partial order methods with their applications*, Springer, 2013.

Accepted: 4.04.2017

PYTHAGOREAN FUZZY HYBRID AVERAGING AGGREGATION OPERATOR AND ITS APPLICATION TO MULTIPLE ATTRIBUTE DECISION MAKING

K. Rahman*
Department of Mathematics
Hazara University Mansehra
KPK, Pakistan
khaista355@yahoo.com

F. Hussain
Department of Mathematics
Abbottabad University of Science and Technology Abbottabad
KPK, Pakistan
fawadhussacin998@hotmail.com

M. S. Ali Khan
Department of Mathematics
Hazara University Mansehra
KPK, Pakistan
sajjadalimath@yahoo.com

Abstract. In this paper, we introduce the notion of Pythagorean fuzzy hybrid averaging operator, which is the generalization of Pythagorean fuzzy weighted averaging operator and Pythagorean fuzzy ordered weighted averaging operator. We also study several properties of the propose operator. At the last we apply the the proposed operator to deal with MAGDM under Pythagorean fuzzy information.
Keywords: pythagorean fuzzy set, PFHA operator, some properties of PFHA operator, decision making.

1. Introducation

In [1] Zadeh introduced the notion of fuzzy set characterized by a membership function. In [2] Atanassov generalized the notion of fuzzy set and introduced the idea of intuitionistic fuzzy set characterized by a membership function and a non-membership function. In [3] Yager generalized the concept of intuitionistic fuzzy set and introduced the concept of Pythagorean fuzzy set. Pythagorean fuzzy set is more powerful tool to solve uncertain problems. Like intuitionistic fuzzy aggregation operators, Pythagorean fuzzy aggregation operators are also become an interesting and important area for research, after the advent of Pythagorean fuzzy set theory. In [4] Yager and Abbasov introduced the notion of PFWA operator, PFOWA operator. In [5] X. Zeng and Z. S. Xu introduced the notion

∗. Corresponding author

of Topsis method using Pythagorean fuzzy numbers. In [6, 7] H. Garg used the Einstein sum and Einstein product and introduced the notion of several arithmetic and geometric aggregation operators and also applied them to group decision making. In [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] K. Rahman et al. introduced the notion of many aggregation operators based on PFNs and applied them to group decision making. Thus, in this paper, our aim to introduce the notion of Pythagorean fuzzy hybrid averaging operator.

The remainder of this paper is structured as follows. In section 2, we give some basic definitions and results. In section 3, we develop Pythagorean fuzzy hybrid averaging operator. In section 4, we apply the proposed operator to deal with MAGD based on PFNs. In section 5, we construct a numerical example. In section 6, we have conclusion.

2. Preliminaries

Definition 1 ([3]). *Let Z be a fixed set, then PFS can be defined as:*

$$P = \{(z, \mu_P(z), \nu_P(z)) \,|z \in Z\}, \tag{1}$$

where $\mu_p(z)$ and $\nu_p(z)$ are mappings from z to $[0,1]$, with the condition $0 \leq \mu_p^2(z) + \nu_p^2(z) \leq 1$, for all $z \in Z$. Let $\pi_p(z) = \sqrt{1 - \mu_p^2(z) - \nu_p^2(z)}$, then it is called the Pythagorean fuzzy index of element $z \in Z$ to set P, which shows the degree of hesitation or indeterminacy of z to p. It is clear that $0 \leq \pi_p(z) \leq 1$, for every $z \in Z$.

Definition 2 ([5]). *Let $\beta = (\mu_\beta, \nu_\beta), \beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ be the three PFVs, then*

$$\begin{aligned}
(1)\ \beta_1 \oplus \beta_2 &= \left(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \mu_{\beta_2}^2}, \nu_{\beta_1}\nu_{\beta_2}\right), \\
(2)\ \beta_1 \otimes \beta_2 &= \left(\mu_{\beta_1}\mu_{\beta_2}, \sqrt{\nu_{\beta_1}^2 + \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \nu_{\beta_2}^2}\right), \\
(3)\ \lambda\beta &= \left(\sqrt{1 - \left(1 - \mu_\beta^2\right)^\lambda}, \nu_\beta^\lambda\right), \lambda > 0, \\
(4)\ \beta^\lambda &= \left(\mu_{\bar{a}}^\lambda, \sqrt{1 - \left(1 - \nu_\beta^2\right)^\lambda}\right), \lambda > 0.
\end{aligned}$$

Definition 3 ([5]). *Let $\beta = (\mu_\beta, \nu_\beta)$ be a PFN, then the score function and accuracy degree of β can be denoted by: $s(\beta) = \mu_\beta^2 - \nu_\beta^2$, $s(\beta) \in [-1, 1]$ and $h(\beta) = \mu_\beta^2 + \nu_\beta^2$ where $h(\beta) \in [0, 1]$.*

Definition 4 ([4]). *Let $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})\,(i = 1, 2, ..., n)$ be a collection of PFVs, then a $PFWA$ operator of dimension n is a mapping $PFWA : \Omega^n \to \Omega$, that has an associated vector $w = (w_1, w_2, w_3, ..., w_n)^T$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Furthermore,*

$$PFWA_w(\beta_1, \beta_2, \beta_3, ..., \beta_n) = w_1\beta_1 \oplus w_2\beta_2 \oplus w_3\beta_3 \oplus ... \oplus w_n\beta_n. \tag{2}$$

Definition 5 ([4]). *Let $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})\,(i = 1, 2, ..., n)$ be a collection of PFVs, then a PFOWA operator of dimension n is a mapping $PFOWA : \Omega^n \to \Omega$, that has an associated vector $w = (w_1, w_2, w_3, ..., w_n)^T$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore*

(3) $PFOWA_w\,(\beta_1, \beta_2, \beta_3, ..., \beta_n) = w_1\beta_{\sigma(1)} \oplus w_2\beta_{\sigma(2)} \oplus w_3\beta_{\sigma(3)} \oplus ... \oplus w_n\beta_{\sigma(n)}$,

where $(\sigma\,(1), \sigma\,(2), \sigma\,(3), ..., \sigma\,(n))$ is any permutation of $(1, 2, 3, ..., n)$, such that $\beta_{\sigma(i-1)} \geq \beta_{\sigma(i)}$ for all i.

3. Pythagorean fuzzy hybrid averaging aggregation operator

Definition 6. *A Pythagorean fuzzy hybrid averaging operator of dimension n is a mapping $PFHA : \Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, ..., w_n)^T$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore*

$$PFHA_{\omega,w}\,(\beta_1, \beta_2, ..., \beta_n) = w_1\dot{\beta}_{\sigma(1)} \oplus w_2\dot{\beta}_{\sigma(2)} \oplus ... \oplus w_n\dot{\beta}_{\sigma(n)}, \tag{4}$$

where $\dot{\beta}_{\sigma(i)}$ is the ith largest of the weighted PFVs $\dot{\beta}_{\sigma(i)}$ $\left(\dot{\beta}_{\sigma(i)} = n\omega_i\beta_i\right)$. $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weighted vector of $\beta_i\,(i = 1, 2, 3, ..., n)$ such that $\omega_i \in [0, 1]$, $\sum_{i=1\omega i}^{n} = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $w = (w_1, w_2, ..., w_n)^T$ approaches $\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the vector $(n\omega_1\beta_1, n\omega_2\beta_2, ..., n\omega_n\beta_n)^T$ approaches $(\beta_1, \beta_2, ..., \beta_n)^T$.

Theorem 1. *Let $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})\,(i = 1, 2, 3, ..., n)$ be a collection of PFVs, then their aggregated value by using the PFHA operator is also a PFV, and*

$$PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) = \left(\sqrt{1 - \prod_{i=1}^{n}(1 - \mu^2_{\dot{\beta}_{\sigma(i)}})^{w_i}}, \prod_{i=1}^{n}(\nu^2_{\dot{\beta}_{\sigma(i)}})^{w_i}\right). \tag{5}$$

Proof. By mathematical induction we can prove that equation (5) holds for all n. First we show that equation (5) holds for $n = 2$. Since

$$w_1\dot{\beta}_{\sigma(1)} = \left(\sqrt{1 - (1 - \mu^2_{\dot{\beta}_{\sigma(1)}})^{w_1}}, \nu^{w_1}_{\dot{\beta}_{\sigma(1)}}\right), w_2\dot{\beta}_{\sigma(2)} = \left(\sqrt{1 - (1 - \mu^2_{\dot{\beta}_{\sigma(2)}})^{w_2}}, \nu^{w_2}_{\dot{\beta}_{\sigma(2)}}\right).$$

Thus

$$\begin{aligned} PFHA_{\omega,w}\,(\beta_1, \beta_2) &= w_1\dot{\beta}_{\sigma(1)} \oplus w_2\dot{\beta}_{\sigma(2)} \\ &= \left(\sqrt{1 - \prod_{i=1}^{2}\left(1 - \mu^2_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}}, \prod_{i=1}^{2}\nu^{w_i}_{\dot{\beta}_{\sigma(i)}}\right). \end{aligned}$$

Thus equation (5) holds for $n = 2$. Now we show that equation (5) holds for $n = k$. Thus

$$PFHA_{\omega,w}\,(\beta_1, \beta_2, \beta_3, ..., \beta_k) = \left(\sqrt{1 - \prod_{i=1}^{k}\left(1 - \mu^2_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}}, \prod_{i=1}^{k}\left(\nu_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}\right).$$

If equation (5) true for $n = k$, then we show that equation (5) holds for $n = k+1$. Thus

$$PFHA_{\omega,w}(\beta_1, \beta_2, ..., \beta_{k+1}) = \left(\sqrt{1 - \prod_{i=1}^{k} \left(1 - \mu^2_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}}, \prod_{i=1}^{k} \left(\nu_{\dot{\beta}_{\sigma(i)}}\right)^{w_i} \right)$$
$$+ \left(\sqrt{1 - \left(1 - \mu^2_{\dot{\beta}_{\sigma(k+1)}}\right)^{w_{k+1}}}, \left(\nu_{\dot{\beta}_{\sigma(k+1)}}\right)^{w_{k+1}} \right)$$
$$= \left(\sqrt{1 - \prod_{i=1}^{k+1} \left(1 - \mu^2_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}}, \prod_{i=1}^{k+1} \left(\nu_{\dot{\beta}_{\sigma(i)}}\right)^{w_i} \right).$$

Thus equation (5) holds for $n = k + 1$. Thus equation (5) holds for n. □

Theorem 2. *Let $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})\,(i = 1, 2, 3, ..., n)$ be a collection of PFVs, then the following conditions always true.*

(1) (Idempotency): If $\dot{\beta}_{\sigma(i)} = \dot{\beta}$, for all i, then

$$PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) = \dot{\beta}. \tag{6}$$

(2) (Boundary):

$$\dot{\beta}^-_{\sigma(i)} \preceq PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) \preceq \dot{\beta}^+_{\sigma(i)}, \tag{7}$$

where $\dot{\beta}^+_{\sigma(i)} = (\max_i(\mu_{\dot{\beta}_{\sigma(i)}}), \min_i(\nu_{\dot{\beta}_{\sigma(i)}}))$, $\dot{\beta}^-_{\sigma(i)} = (\min_i(\mu_{\dot{\beta}_{\sigma(i)}}), \max_i(\nu_{\dot{\beta}_{\sigma(i)}}))$.

(3) (Monotonicity): Let $\dot{\beta}^*_{\sigma(i)} = (\mu_{\dot{\beta}^*_{\sigma(i)}}, \nu_{\dot{\beta}^*_{\sigma(i)}})(i = 1, 2, 3, ..., n)$ a collection of PFVs, if $\mu_{\dot{\beta}_{\sigma(i)}} \preceq \mu_{\dot{\beta}^*_{\sigma(i)}}$ and $\nu_{\dot{\beta}_{\sigma(i)}} \succeq \nu_{\dot{\beta}^*_{\sigma(i)}}$ for all i, then

$$PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) \preceq PFHA_{\omega,w}(\beta^*_1, \beta^*_2, \beta^*_3, ..., \beta^*_n). \tag{8}$$

Proof. Idempotency: Since $PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) = w_1\dot{\beta}_{\sigma(1)} \oplus w_2\dot{\beta}_{\sigma(2)} \oplus ... \oplus w_n\dot{\beta}_{\sigma(n)} = (w_1 \oplus w_2 \oplus ... \oplus w_n)\dot{\beta} = \dot{\beta}$.

Boundedness: Since $\dot{\beta}^-_{\sigma(j)} \preceq PFHA_{\omega,w}(\beta_1, \beta_2, \beta_3, ..., \beta_n) \preceq \dot{\beta}^+_{\sigma(j)}$. Since

$$\Leftrightarrow \min_i \left(\mu_{\dot{\beta}_{\sigma(i)}}\right) \preceq \mu_{\dot{\beta}_{\sigma(i)}} \preceq \max_i \left(\mu_{\dot{\beta}_{\sigma(i)}}\right)$$
$$\Leftrightarrow \min_i \left(\mu_{\dot{\beta}_{\sigma(i)}}\right) \preceq \sqrt{1 - \prod_{i=1}^{n} \left(1 - \mu^2_{\dot{\beta}_{\sigma(i)}}\right)^{w_i}} \preceq \max_i \left(\mu_{\dot{\beta}_{\sigma(i)}}\right). \tag{9}$$

Again

$$\Leftrightarrow \min_i \left(\nu_{\dot{\beta}_{\sigma(i)}}\right) \preceq \nu_{\dot{\beta}_{\sigma(i)}} \preceq \max_i \left(\nu_{\dot{\beta}_{\sigma(i)}}\right)$$
$$\Leftrightarrow \min_i \left(\nu_{\dot{\beta}_{\sigma(i)}}\right) \preceq \prod_{i=1}^{n} \nu^{w_i}_{\dot{\beta}_{\sigma(i)}} \preceq \max_i \left(\nu_{\dot{\beta}_{\sigma(i)}}\right). \tag{10}$$

Let $PFHA_{\omega,w}(\beta_1,\beta_2,...,\beta_n)=\dot{\beta}_{\sigma(i)}=(\mu_{\dot{\beta}_{\sigma(i)}},\nu_{\dot{\beta}_{\sigma(i)}})$. Then $s(\dot{\beta}_{\sigma(i)})\preceq s(\dot{\beta}^{+}_{\sigma(i)})$. Again $s(\dot{\beta}_{\sigma(i)})\succeq s(\dot{\beta}^{-}_{\sigma(i)})$. Thus

$$\dot{\beta}^{-}_{\sigma(i)}\prec PFWA_{\omega,w}\left(\beta_1,\beta_2,\beta_3,...,\beta_n\right)\prec\dot{\beta}^{+}_{\sigma(i)}. \tag{11}$$

If $s(\dot{\beta}_{\sigma(i)})=s(\dot{\beta}^{+}_{\sigma(i)})$. Then $h(\dot{\beta}_{\sigma(i)})=\mu^2_{\dot{\beta}_{\sigma(i)}}+\nu^2_{\dot{\beta}_{\sigma(i)}}=\max_i(\mu_{\dot{\beta}_{\sigma(i)}})^2+\min_i(\nu_{\dot{\beta}_{\sigma(i)}})^2=h(\dot{\beta}^{+}_{\sigma(i)})$. Thus

$$PFHA_{\omega,w}\left(\beta_1,\beta_2,\beta_3,...,\beta_n\right)=\dot{\beta}^{+}_{\sigma(i)}. \tag{12}$$

If $s(\dot{\beta}_{\sigma(i)})=s(\dot{\beta}^{-}_{\sigma(i)})$. Then $h(\dot{\beta}_{\sigma(i)})=\mu^2_{\dot{\beta}_{\sigma(i)}}+\nu^2_{\dot{\beta}_{\sigma(i)}}=\min_i(\nu_{\dot{\beta}_{\sigma(i)}})^2+\max_i(\mu_{\dot{\beta}_{\sigma(i)}})^2=h(\dot{\beta}^{-}_{\sigma(i)})$. Thus

$$PFHA_{\omega,w}\left(\beta_1,\beta_2,\beta_3,...,\beta_n\right)=\dot{\beta}^{-}_{\sigma(i)}. \tag{13}$$

Thus from equation (11) to (13), we have $\dot{\beta}^{-}_{\sigma(i)}\preceq PFHA_{\omega,w}(\beta_1,\beta_2,\beta_3,...,\beta_n)\preceq\dot{\beta}^{+}_{\sigma(i)}$.

(3) (Monotonicity): Follows the proof of above. □

Theorem 3. *The PFWA operator is a special case of the PFHA operator.*

Proof. Let $w=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^T$, then

$\text{PFHA}_{\omega,w}(\beta_1,\beta_2,\beta_3,...,\beta_n)=w_1\dot{\beta}_{\sigma(1)}\oplus w_2\dot{\beta}_{\sigma(2)}\oplus...\oplus w_n\dot{\beta}_{\sigma(n)}=\frac{1}{n}(\dot{\beta}_{\sigma(1)}\oplus\dot{\beta}_{\sigma(2)}\oplus...\oplus\dot{\beta}_{\sigma(n)})=\omega_1\beta_1\oplus\omega_2\beta_2\oplus...\oplus\omega_n\beta_n=\text{PFWA}_{\omega}(\beta_1,\beta_2,\beta_3,...,\beta_n)$.

Theorem 4. *The PFOWA operator is a special case of the PFHA operator.*

Proof. $\omega=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^T$, then $\dot{\beta}_{\sigma(i)}=\beta_{\sigma(i)}\,(i=1,2,3,...,n)$. Thus $\text{PFHA}_{\omega,w}(\beta_1,\beta_2,\beta_3,...,\beta_n)=w_1\dot{\beta}_{\sigma(1)}\oplus w_2\dot{\beta}_{\sigma(2)}\oplus...\oplus w_n\dot{\beta}_{\sigma(n)}=w_1\beta_{\sigma(1)}\oplus w_2\beta_{\sigma(2)}\oplus...\oplus w_n\beta_{\sigma(n)}=\text{PFOWA}_{w}(\beta_1,\beta_2,\beta_3,...,\beta_n)$.

4. Approaches to multiple attribute decision making with pythagorean fuzzy information

Algorithm 1. *Let $A=(A_1,A_2,...,A_m)$ be the set of alternatives and $C=(C_1,C_2,...,C_n)$ be the set of attributes. Let $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ be the weighted vector of attributes, $C_j\,(j=1,2,...,n)$, such that $\omega_j\in[0,1]$ and $\sum_{j=1\omega i}^{n}=1$. Let us suppose that, $D=(d_{ij})_{m\times n}=(\mu_{ij},\nu_{ij})_{m\times n}$ be the Pythagorean fuzzy decision matrix.*

Step 1: Construct decision matrix.
Step 2: Calculating $\dot{A}_{ij}=nw_jA_{_{ij}}$.
Step 3: Apply PFHA operator to derive the overall preference values.
Step 4: Calculate the score function.
Step 5: Ranking to the given alternatives accoring to their scores function.

5. Numerical example

Suppose a man wants to invest money, for the investment the man has four possible options (1) A_1 : Mobile Company (2) A_2 : Car Company (3) A_3 : Fan Company (4) A_4 : Laptop Company. There are many factors that must be considered while selecting a suitable company for investment, but here, we consider the following four criteria. whose weighted vector is $w = (0.10, 0.20, 0.30, 0.40)^T$.

(1) C_1 : is the risk analysis,
(2) C_2 : is the growth analysis,
(3) C_3 : is the social political impact analysis,
(4) C_4 : is the environmental impact analysis.

Step 1. The decision makers give his decision in the following table.

Table 1 : Pythagorean Fuzzy Decision Making D

	C_1	C_2	C_3	C_4
A_1	$(0.60, 0.40)$	$(0.70, 0.50)$	$(0.80, 0.30)$	$(0.70, 0.40)$
A_2	$(0.70, 0.60)$	$(0.60, 0.40)$	$(0.70, 0.40)$	$(0.60, 0.50)$
A_3	$(0.60, 0.60)$	$(0.70, 0.40)$	$(0.80, 0.40)$	$(0.70, 0.50)$
A_4	$(0.70, 0.40)$	$(0.60, 0.50)$	$(0.70, 0.30)$	$(0.60, 0.40)$

Using $\dot{\beta}_{ij} = n\omega_j\beta_{ij}$, where $\omega = (0.10, 0.20, 0.30, 0.40)^T$ we have

$$\begin{aligned}
\dot{\beta}_{11} &= (0.40, 0.69), \dot{\beta}_{12} = (0.64, 0.57), \dot{\beta}_{13} = (0.84, 0.23), \dot{\beta}_{14} = (0.81, 0.23) \\
\dot{\beta}_{21} &= (0.48, 0.81), \dot{\beta}_{22} = (0.54, 0.48), \dot{\beta}_{23} = (0.74, 0.33), \dot{\beta}_{24} = (0.71, 0.32) \\
\dot{\beta}_{31} &= (0.40, 0.81), \dot{\beta}_{32} = (0.64, 0.48), \dot{\beta}_{33} = (0.84, 0.33), \dot{\beta}_{34} = (0.81, 0.32) \\
\dot{\beta}_{41} &= (0.48, 0.69), \dot{\beta}_{42} = (0.54, 0.57), \dot{\beta}_{43} = (0.74, 0.23), \dot{\beta}_{44} = (0.71, 0.23).
\end{aligned}$$

By the score function we have

$$\begin{aligned}
&\dot{\beta}_{\sigma(11)} = (0.84, 0.23), \dot{\beta}_{\sigma(12)} = (0.81, 0.23), \dot{\beta}_{\sigma(13)} = (0.64, 0.57), \dot{\beta}_{\sigma(14)} = (0.40, 0.69) \\
&\dot{\beta}_{\sigma(21)} = (0.74, 0.33), \dot{\beta}_{\sigma(22)} = (0.71, 0.32), \dot{\beta}_{\sigma(23)} = (0.54, 0.48), \dot{\beta}_{\sigma(24)} = (0.48, 0.81) \\
&\dot{\beta}_{\sigma(31)} = (0.84, 0.33), \dot{\beta}_{\sigma(32)} = (0.81, 0.32), \dot{\beta}_{\sigma(33)} = (0.64, 0.48), \dot{\beta}_{\sigma(34)} = (0.40, 0.81) \\
&\dot{\beta}_{\sigma(41)} = (0.74, 0.23), \dot{\beta}_{\sigma(42)} = (0.71, 0.23), \dot{\beta}_{\sigma(43)} = (0.54, 0.57), \dot{\beta}_{\sigma(44)} = (0.48, 0.69).
\end{aligned}$$

Step 2. Using Pythagorean fuzzy hybrid averaging operator, where $\omega = (0.10, 0.20, 0.30, 0.40)^T$ we have $d_1 = (0.65, 0.47), d_2 = (0.60, 0.53), d_3 = (0.60, 0.47), d_4 = (0.45, 0.41)$.

Step 3. Calculating scores function $s(d_i)(i = 1, 2, 3, 4)$, $s(d_1) = 0.14, s(d_2) = 0.08, s(d_3) = 0.15, s(d_4) = -0.01$.

Step 4. Thus A_3 : fan company is the best option for a man to invest his money.

6. Conclusion

In this paper, we have defined Pythagorean fuzzy hybrid averaging operator, which is the generalization of PFWA operator and PFOWA operator. At the

last we applied the proposed operator to MCDM problem, based on Pythagorean fuzzy information.

References

[1] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., 20 (1986), 87-96.

[3] R. R. Yager, *Pythagorean membership grades in multi-criteria decision making*, IEEE Trans Fuzzy Syst., 22 (2014), 958-965.

[4] R. R. Yager, A. M. Abbasov, *Pythagorean membeship grades, complex numbers and decision making*, Int. J. Intell Syst., 28 (2013), 436–452.

[5] X. Zang, Z. S. Xu, *Extension of TOPSIS to multiple criteria decision making with pythagorean fuzzy sets*, international journal of intelligent systems, 29 (2014), 1061–1078.

[6] H. Garg, *Generalized pythagorean fuzzy geometric aggregation operators using einstein t-norm and t-conorm for multicriteria decision-making process*, International Journal of Intelligent Systems, 2016, 1–34.

[7] H. Garg, *A new generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making*, International Journal of Intelligent Systems, (2011), 1–35.

[8] K. Rahman, S. Abdullah, M. S. Ali Khan, M. Shakeel, *Pythagorean fuzzy hybrid geometric operator and their application to multiple attribute decision making*, International Journal of Computer Science and Information Security, 14 (2016), 837-854.

[9] K. Rahman, S. Abdullah, F. Hussain, M. S. Ali Khan, *Approaches to pythagorean fuzzy geometric aggregation operators*, International Journal of Computer Science and information security, 14 (2016), 174-200.

[10] K. Rahman, M.S.A. Khan, M. Ullah, A. Fahmi, *Multiple attribute group decision making for plant location selection with pythagorean fuzzy weighted geometric aggregation operator*, The Nucleus, 54 (2017), 66-74.

[11] K. Rahman, S. Abdullah, F. Hussain, M. S. Ali Khan, *Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making*, J. Appl. Environ. Biol. Sci., 7 (2017), 67-83.

[12] K. Rahman, A. Ali, M. Shakeel, M.S. A. Khan, Murad Ullah, *Pythagorean fuzzy weighted averaging aggregation operator and its application to decision making theory*, The Nucleus, 54 (2017), 190-196.

[13] K. Rahman, M. S. A. Khan, M. Ullah, *New Approaches to pythagorean fuzzy averaging aggregation operators*, Mathematics Letters, 3 (2017), 29-36.

[14] K. Rahman, M. S. Ali Khan, S. Abdullah, F. Husain, M. Ibrar, *Some properties of pythagorean fuzzy hybrid averaging aggregation operator*, J. Appl. Environ. Biol. Sci., 7 (2017), 122-133.

[15] K. Rahman, S. Abdullah, A. Ali, F. Amin, *Some induced averaging aggregation operators based on pythagorean fuzzy numbers*, Mathematics Letters, 3 (2017), 40-45.

[16] K. Rahman, S. Abdullah, R. Ahmed, M. Ullah, *Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making*, Journal of Intelligent & Fuzzy Systems, 33 (2017), 635–647.

[17] K. Rahman, S. Abdullah, M. Shakeel, M. S. Ali Khan, M. Ullah, *Interval-valued pythagorean fuzzy geometric aggregation operators and their application to group decision making problem*, Cogent Mathematics, 4 (2017), 1338638.

[18] K. Rahman, S. Abdullah, M. S. Ali Khan M. Ibrar, F. Husain, *Some basic operations on pythagorean fuzzy sets*, J. Appl. Environ. Biol. Sci., 7 (2017), 111-119.

[19] K. Rahman, A. Ali, M. S. Ali Khan, *Some interval-valued pythagorean fuzzy weighted averaging aggregation operators and their application to multiple attribute decision making*, Punjab University Journal of Mathematics, 50 (2018), 113-129.

[20] K. Rahman, S. Abdullah, M. S. Ali Khan, *Some interval-valued pythagorean fuzzy einstein weighted averaging aggregation operators and their application to group decision making*, J. Intell. Syst., (2018), 1-16.

[21] K. Rahman, S. Abdullah, *Generalized interval-valued Pythagorean fuzzy aggregation operators and their application to group decision-making*, Granular Computing, https://doi.org/10.1007/s41066-018-0082-9

[22] K. Rahman, S. Abdullah, A. Ali, *Some induced aggregation operators based on interval-valued pythagorean fuzzy numbers*, Granular Computing, https://doi.org/10.1007/s41066-018-0091-8.

Accepted: 6.04.2017

1-MOVABLE DOUBLY CONNECTED DOMINATION IN GRAPHS

Renario G. Hinampas, Jr.*
College of Teacher Education
Bohol Island State University-Main Campus
CPG North Avenue, 6300 Tagbilaran City, Bohol
Philippines
renariojr.hinampas@yahoo.com
and
renariojr.hinampas@bisu.edu.ph

Sergio R. Canoy, Jr.
Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga Highway, 9200 Iligan City
Philippines
serge_canoy@yahoo.com

Abstract. This paper presents some characterizations involving the concept of 1-movable doubly connected domination and investigates the 1-movable doubly connected dominating sets in the join of graphs. Moreover, the 1-movable doubly connected domination number of the join of graphs is determined.
Keywords: Doubly connected domination, 1-movable doubly connected domination, internal and external private neighbors, join.

1. Introduction

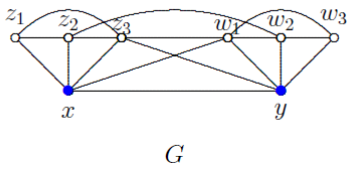
Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ-set of G. If S is a dominating set of G, then a vertex w is a *private neighbor* of $v \in S$ with respect to S if $N(w) \cap S = \{v\}$. If $w \in S$, then w is an *internal private neighbor* of $v \in S$, otherwise, w is an *external private neighbor* of $v \in S$. The set of private neighbors of $v \in S$ is denoted by $pn(v; S)$, the set of internal private neighbors of $v \in S$ is denoted by $ipn(v; S)$

∗. Coressponding author

and the set of external private neighbors of $v \in S$ is denoted by $epn(v; S)$. A dominating set S of $V(G)$ is a *connected dominating set* of G if the subgraph $\langle S \rangle$ induced by S is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the smallest cardinality of a connected dominating set of G. A connected dominating set of G with cardinality equal to $\gamma_c(G)$ is called a γ_c-*set* of G. A non-empty subset S of $V(G)$ is a *doubly connected dominating set* of G if S is a connected dominating set and the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The *doubly connected domination number* of G, denoted by $\gamma_{cc}(G)$, is the smallest cardinality of a doubly connected dominating set of G. A doubly connected dominating set of G with cardinality equal to $\gamma_{cc}(G)$ is called a γ_{cc}-*set* of G. Doubly connected domination in graphs was studied in [1], [2], [5] and [8].

A non-empty set $S \subseteq V(G)$ is a *1-movable doubly connected dominating set* of G if (i) $S = V(G)$ and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of G or (ii) S is a doubly connected dominating set of G and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of G or $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of G for some $u \in (V(G) \setminus S) \cap N_G(v)$. The *1-movable doubly connected domination number* of a graph G, denoted by $\gamma^1_{mcc}(G)$, is the smallest cardinality of a 1-movable doubly connected dominating set of G. A 1-movable doubly connected dominating set of G with cardinality equal to $\gamma^1_{mcc}(G)$ is called γ^1_{mcc}-*set* of G.

Consider for example the graph as shown in Figure 1. The sets $S = \{x, y\}$, $(S \setminus \{x\}) \cup \{z_3\} = \{y, z_3\}$ and $(S \setminus \{y\}) \cup \{w_1\} = \{x, w_1\}$ are doubly connected dominating sets of G. Moreover, S is a γ^1_{mcc}-set of G. Hence, $\gamma^1_{mcc}(G) = |S| = 2$.



G

Figure 1: A graph G with a 1-movable doubly connected dominating set

The movability of sets in some variants of domination were introduced and studied in [3], [6], [7], [9] and [10].

2. Results

Remark 2.1 ([4])**.** Every connected dominating set in a connected graph contains every cut-vertex of G.

The next result characterizes those graphs with 1-movable doubly connected dominating sets.

Theorem 2.2. *A connected nontrivial graph G has a 1-movable doubly connected dominating set if and only if G has no cut-vertices.*

Proof. Suppose that G has a 1-movable doubly connected dominating set, say S. Then clearly, S is a connected dominating set of G. Suppose further that G has a cut-vertex, say v. Then by Remark 2.1, $v \in S$. Thus, $\langle S \setminus \{v\}\rangle$ and $\langle (S \setminus \{v\}) \cup \{u\}\rangle$ are not connected subgraphs of G for every $u \in (V(G) \setminus S) \cap N_G(v)$. Hence, S is not a 1-movable doubly connected dominating set of G. This is a contradiction to the assumption. Therefore, G has no cut-vertex.

For the converse, suppose that G has no cut-vertex. Let $S = V(G)$ and let $v \in S$. From the assumption, v is not a cut-vertex. Hence, $S \setminus \{v\}$ is a connected dominating set of G and $\langle V(G) \setminus (S \setminus \{v\})\rangle \cong K_1$ is a connected subgraph of G. Hence, $S \setminus \{v\}$ is a doubly connected dominating set of G. Thus, $S = V(G)$ is a 1-movable doubly connected dominating set of G. □

The next result follows from the proof of Theorem 2.2.

Corollary 2.3. *If G is a connected nontrivial graph without cut-vertices, then $V(G)$ is a 1-movable doubly connected dominating set of G.*

Clearly, every 1-movable doubly connected dominating set $S \neq V(G)$ of a graph G is a doubly connected dominating set of G. Hence, $\gamma_{cc}(G) \leq \gamma^1_{mcc}(G)$ for any graph G without cut-vertices. Moreover, for any graph G without cut-vertices, $1 \leq \gamma^1_{mcc}(G) \leq n$ and these bounds are sharp. Consider the complete graph K_{10} and the cycle C_{10} as shown in Figure 2. In these graphs, $\gamma^1_{mcc}(K_{10}) = 1$ and $\gamma^1_{mcc}(C_{10}) = 10$.



Figure 2: The complete graph K_{10} and the cycle C_{10}.

The next result characterizes all graphs without cut-vertices which attained the lower bound.

Theorem 2.4. *Let G be a connected nontrivial graph without cut-vertices. Then $\gamma^1_{mcc}(G) = 1$ if and only if $G = K_2$ or $G \cong K_2 + H$ for some graph H.*

Proof. Suppose that $\gamma^1_{mcc}(G) = 1$. Let $S = \{x\}$ be a γ^1_{mcc}-set of G for some $x \in V(G)$. If $|V(G)| = 2$, then $G = K_2$. Suppose that $|V(G)| \geq 3$. Since S is

a dominating set of G, $xv \in E(G)$ for all $v \in V(G) \setminus \{x\}$. Also, since S is a 1-movable doubly connected dominating set of G, there exists $u \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{u\} = \{u\}$ is a doubly connected dominating set of G. This implies that $uw \in E(G)$ for all $w \in V(G) \setminus \{u\}$. This means that $xu \in E(G)$. Let $H = \langle V(G) \setminus \{x, u\} \rangle$. Then $G = \langle \{x, u\} \rangle + H \cong K_2 + H$. For the converse, suppose first that $G = K_2$. Then, clearly, $\gamma^1_{mcc}(G) = 1$. Suppose that $G = K_2 + H$. Let $V(K_2) = \{x, y\}$ and let $S = \{x\}$. Then S is a connected dominating set of G and $\langle V(G) \setminus S \rangle = \langle V(G) \setminus \{x\} \rangle = \langle \{y\} \rangle + H$ is connected. Hence, S is a doubly connected dominating set of G. Furthermore, since $y \in (V(G) \setminus S) \cap N_G(x)$ and $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a connected dominating set of G and $\langle V(G) \setminus \{y\} \rangle = \langle \{x\} \rangle + H$ is connected, it follows that $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a doubly connected dominating set of G. Consequently, S is a 1-movable doubly connected dominating set of G and hence, a γ^1_{mcc}-set of G. Therefore $\gamma^1_{mcc}(G) = |S| = 1$. □

Corollary 2.5. $\gamma^1_{mcc}(K_n) = 1$ *for all* $n \geq 2$.

The next result characterizes the concept of 1-movable doubly connected domination in graphs in terms of the concept of private neighbors.

Theorem 2.6. *Let G be a connected graph without cut-vertices. A proper subset S of $V(G)$ is a 1-movable doubly connected dominating set of G if and only if S is a doubly connected dominating set of G and for each $v \in S$, either*

(i) $epn(v; S) = ipn(v; S) = \emptyset$ *and* $\langle (V(G) \setminus S) \cup \{v\} \rangle$ *is connected or*

(ii) *there exists* $u \in (V(G) \setminus S) \cap N_G(v)$ *such that* $\langle [V(G) \setminus (S \cup \{u\})] \cup \{v\} \rangle$ *is connected,* $epn(v; S) \cup ipn(v; S) \subseteq N(u)$ *and* $u \in N_G[(S \setminus \{v\}) \setminus ipn(v; S)]$ *whenever* $S \setminus \{v\} \neq ipn(v; S)$.

Proof. Suppose S is a 1-movable doubly connected dominating set of G, where $S \neq V(G)$. Then S is a doubly connected dominating set of G. Let $v \in S$. Since S is a 1-movable doubly connected dominating set of G, $S \setminus \{v\}$ or $S_v = (S \setminus \{v\}) \cup \{u\}$, for some $u \in (V(G) \setminus S) \cap N_G(v)$, is a doubly connected dominating set of G. Suppose that $S \setminus \{v\}$ is a doubly connected dominating set of G. Then $\langle V(G) \setminus (S \setminus \{v\}) \rangle = \langle (V(G) \setminus S) \cup \{v\}) \rangle$ is connected and every vertex w in $(V(G) \setminus S) \cap N_G(v)$ is adjacent to some vertex in $S \setminus \{v\}$. This implies that $epn(v; S) = \emptyset$. Moreover, since $\langle S \setminus \{v\} \rangle$ is connected, $ipn(v; S) = \emptyset$. Suppose $S \setminus \{v\}$ is not a doubly connected dominating set of G. Then there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that S_v is a doubly connected dominating set of G. Hence, $\langle V(G) \setminus S_v \rangle = \langle [V(G) \setminus (S \cup \{u\})] \cup \{v\} \rangle$ is connected. Let $z \in epn(v; S)$. Then $z \in N(u)$ since S_v is a dominating set of G. Thus, $epn(v; S) \subseteq N(u)$. Also, if $y \in ipn(v; S)$, then $y \in N(u)$ since $\langle S_v \rangle$ is connected. Thus, $epn(v; S) \cup ipn(v; S) \subseteq N(u)$. Now, suppose that $S \setminus \{v\} \neq ipn(v; S)$. Suppose further that $u \notin N_G[(S \setminus \{v\}) \setminus ipn(v; S)]$. Then $\langle (S \setminus \{v\}) \setminus ipn(v; S) \rangle$

and $\langle\{u\} \cup ipn(v;S)\rangle$ are components of $\langle S_v\rangle$, contrary to the fact that $\langle S_v\rangle$ is connected. Therefore, $u \in N_G[(S\setminus\{v\})\setminus ipn(v;S)$ whenever $S\setminus\{v\} \neq ipn(v;S)$.

For the converse, suppose that S is a doubly connected dominating set of G satisfying one of the given conditions. Let $v \in S$. If (i) holds, then $epn(v;S) = ipn(v;S) = \emptyset$. Hence, $S\setminus\{v\}$ is a connected dominating set of G. Furthermore, since $\langle(V(G)\setminus S)\cup\{v\}\rangle = \langle V(G)\setminus(S\setminus\{v\})\rangle$ is connected, $S\setminus\{v\}$ is a doubly connected dominating set of G. Suppose that (ii) holds. Then there exists $u \in (V(G)\setminus S)\cap N_G(v)$ such that $epn(v;S)\cup ipn(v;S) \subseteq N(u)$. Set $S_v = (S\setminus\{v\})\cup\{u\}$ and let $x \in V(G)\setminus S_v$. If $x = v$ or $x \in epn(v;S)$, then $xu \in E(G)$. If $x \notin \{v\}\cup epn(v;S)$, then $xy \in E(G)$ for some $y \in S\setminus\{v\}$ since S is a dominating set of G. Thus, S_v is a dominating set of G. If $S\setminus\{v\} = ipn(v;S)$, then $S\setminus\{v\} \subseteq N_(u)$. Thus, $\langle S_v\rangle$ is connected. If $S\setminus\{v\} \neq ipn(v;S)$, then by assumption, $u \in N_G[(S\setminus\{v\})\setminus ipn(v;S)\rangle$. Hence, since $ipn(v;S) \subseteq N_G(u)$ also, $\langle S_v\rangle$ is connected. Thus, S_v is a connected dominating set of G. Furthermore, by assumption, $\langle[V(G)\setminus(S\cup\{u\})]\cup\{v\}\rangle = \langle V(G)\setminus[(S\setminus\{v\})\cup\{u\}]\rangle = \langle V(G)\setminus S_v\rangle$ is connected. Hence, S_v is a doubly connected dominating set of G. Accordingly, S is a 1-movable doubly connected dominating set of G. □

Theorem 2.7. *Let G be a connected graph without cut-vertices and let its order be $n \geq 3$. Then $\gamma^1_{mcc}(G) = 2$ if and only if the following conditions hold:*

- (i) *G is not isomorphic to $K_2 + H$ for any graph H and*
- (ii) *there exist two adjacent vertices x and y that dominate G such that*
  - (a) *$\langle V(G)\setminus\{x,y\}\rangle$ is connected*
  - (b) *$epn(x;\{x,y\}) = \emptyset$ or $\emptyset \neq epn(x;\{x,y\}) \subseteq N_G(z)$ for some $z \in N_G(x)\cap N_G(y)$ and $\langle V(G)\setminus\{z,y\}\rangle$ is connected and*
  - (c) *$epn(y;\{x,y\}) = \emptyset$ or $\emptyset \neq epn(y;\{x,y\}) \subseteq N_G(w)$ for some $w \in N_G(x)\cap N_G(y)$ and $\langle V(G)\setminus\{x,w\}\rangle$ is connected.*

Proof. Suppose that $\gamma^1_{mcc}(G) = 2$. Then by Theorem 2.4, G is not isomorphic to $K_2 + H$ for any graph H. Thus, (i) holds. Let $S = \{x,y\}$ be a γ^1_{mcc}-set of G. Since S is a doubly connected dominating set of G, x and y are two adjacent vertices that dominate G and $\langle V(G)\setminus\{x,y\}\rangle$ is connected. Suppose first that $S\setminus\{x\}$ is a doubly connected dominating set of G. Then $epn(x;\{x,y\}) = \emptyset$. Suppose that $S\setminus\{x\}$ is not a doubly connected dominating set of G. Then $(S\setminus\{x\})\cup\{z\} = \{y,z\}$ is a doubly connected dominating set of G for some $z \in (V(G)\setminus S)\cap N_G(x)$. This implies that $z \in N_G(y)$ and $\langle V(G)\setminus\{z,y\}\rangle$ is a connected subgraph of G. Let $v \in epn(x;\{x,y\})$. Since $vy \notin E(G)$ and since $\{z,y\}$ is a dominating set of G, it follows that $v \in N_G(z)$. Since v was arbitrarily chosen, it follows that $epn(x;\{x,y\}) \subseteq N_G(z)$. Similarly, (c) holds. Hence, (ii) holds.

For the converse, suppose that (i) and (ii) hold. Then by Theorem 2.4, $\gamma^1_{mcc}(G) \geq 2$. Let $S = \{x,y\}$ such that x and y are vertices in G satisfying (ii). Then S is a doubly connected dominating set of G. Now by (b), suppose that

$epn(x;\{x,y\}) = \emptyset$. Then $S \setminus \{x\} = \{y\}$ is a connected dominating set of G. Since y is not a cut-vertex, $\langle V(G) \setminus \{y\}\rangle$ is connected. Thus, $S \setminus \{x\}$ is a doubly connected dominating set of G. Suppose that $\emptyset \neq epn(x;\{x,y\}) \subseteq N_G(z)$ for some $z \in (V(G) \setminus S) \cap [N_G(x) \cap N_G(y)]$ and $\langle V(G) \setminus \{z,y\}\rangle$ is connected. Then $(S \setminus \{x\}) \cup \{z\} = \{y,z\}$ is a doubly connected dominating set of G. Similarly, $S \setminus \{y\} = \{x\}$ or $(S \setminus \{y\}) \cup \{w\} = \{x,w\}$ is a doubly connected of G for some $w \in N_G(x) \cap N_G(y)$. Hence, S is a 1-movable doubly connected dominating set of G and thus a γ^1_{mcc}-set of G. Therefore, $\gamma^1_{mcc}(G) = |S| = 2$. □

We now characterize the 1-movable doubly connected dominating sets in the join of two connected nontrivial graphs.

Theorem 2.8 ([2])**.** *Let G and H be any graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G+H)$ is a doubly connected dominating set of $G+H$ if and only if at least one of the following holds:*

(*i*) *$S \subseteq V(G)$ is a connected dominating set of G, where H is connected if $S = V(G)$;*

(*ii*) *$S \subseteq V(H)$ is a connected dominating set of H, where G is connected if $S = V(H)$;*

(*iii*) *$V(G) \subseteq S$, $V(H) \cap S \neq \emptyset$, and $\langle V(H) \setminus (S \cap V(H))\rangle$ is a connected subgraph of H;*

(*iv*) *$V(H) \subseteq S$, $V(G) \cap S \neq \emptyset$, and $\langle V(G) \setminus (S \cap V(G))\rangle$ is a connected subgraph of G;*

(*v*) *$1 \leq |S \cap V(G)| < m$ and $1 \leq |S \cap V(H)| < n$.*

Theorem 2.9. *Let G and H be connected graphs of order $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G+H)$ is a 1-movable doubly connected dominating set of $G+H$ if and only if one of the following statements is true:*

(*i*) *$S = V(G+H)$.*

(*ii*) *S is a connected dominating set of G such that if $|S| = 1$, then either S is a 1-movable connected dominating set of G or there exists $u \in V(H)$ such that $\{u\}$ is a connected dominating set of H.*

(*iii*) *S is a connected dominating set of H such that if $|S| = 1$, then either S is a 1-movable connected dominating set of H or there exists $w \in V(G)$ such that $\{w\}$ is a connected dominating set of G.*

(*iv*) *$S = S_1 \cup S_2 (\neq V(G+H)$ where $\emptyset \neq S_1 \subseteq V(G)$ and $\emptyset \neq S_2 \subseteq V(H)$ such that*

(1) *if $S_1 = V(G)$, then $\langle V(H) \setminus S_2\rangle$ is connected and for every $v \in S_2$, either $\langle V(H) \setminus (S_2 \setminus \{v\})\rangle$ or $\langle V(H) \setminus [(S_2 \setminus \{v\}) \cup \{u\}]\rangle$ is connected for some $u \in (V(H) \setminus S_2) \cap N_H(v)$ and*

(2) *if* $S_2 = V(H)$, *then* $\langle V(G) \setminus S_1\rangle$ *is connected and for every* $v \in S_1$, *either* $\langle V(G) \setminus (S_1 \setminus \{v\})\rangle$ *or* $\langle V(G) \setminus [(S_1 \setminus \{v\}) \cup \{u\}]\rangle$ *is connected for some* $u \in (V(G) \setminus S_1) \cap N_G(v)$.

Proof. Suppose that S is a 1-movable doubly connected dominating set of $G + H$. If $S = V(G + H)$, then (i) holds. Suppose that $S \neq V(G + H)$. Then S is a doubly connected dominating set of $G + H$. Suppose that $S \subseteq V(G)$. Then by Theorem 2.8(i), S is a connected dominating set of G. Suppose that $|S| = 1$, say $S = \{x\}$ for some $x \in V(G)$. If $\gamma(H) = 1$, then there exists $u \in V(H)$ such that $\{u\}$ is a dominating set of H. Suppose that $\gamma(H) \neq 1$. Since S is a 1-movable doubly connected dominating set of $G + H$, there exists $w \in (V(G + H) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a doubly connected dominating set of $G + H$. Since $\gamma(H) \neq 1$, it follows that $w \in V(G) \setminus S$. Hence, $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a connected dominating set of G. Thus, S is a 1-movable connected dominating set of G. This proves (ii). Similarly, (iii) holds if $S \subseteq V(H)$. Next, suppose that $S_1 = S \cap V(G) \neq \emptyset$ and $S_2 = S \cap V(H) \neq \emptyset$. Then $S = S_1 \cup S_2$. Suppose that $S_1 = V(G)$. Since $S \neq V(G + H)$, $S_2 \neq V(H)$. Since S is a doubly connected dominating set of $G + H$, $\langle V(G + H) \setminus S\rangle = \langle V(H) \setminus S_2\rangle$ is connected. Let $v \in S_2$. With the assumption, suppose first that $S \setminus \{v\}$ is a doubly connected dominating set of $G+H$. Then $\langle V(G+H) \setminus (S \setminus \{v\})\rangle = \langle V(H) \setminus (S_2 \setminus \{v\})\rangle$ is connected. Suppose that $S \setminus \{v\}$ is not a doubly connected dominating set of $G + H$. Then there exists $u \in (V(G + H) \setminus S) \cap N_{G+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G+H$. Since $S_1 = V(G)$, $u \in (V(H) \setminus S_2) \cap N_H(v)$. Thus, $\langle V(G + H) \setminus [(S \setminus \{v\}) \cup \{u\}]\rangle = \langle V(H) \setminus [(S_2 \setminus \{v\}) \cup \{u\}]\rangle$ is connected. Showing that (1) holds. Similarly, (2) holds if $S_2 = V(H)$ Hence, (iv) holds.

For the converse, suppose first that (i) holds i.e., $S = V(G + H)$. Since $G + H$ has no cut-vertices, S is 1-movable doubly connected dominating set of $G + H$ by Corollary 2.3. Suppose that (ii) holds. Then by Theorem 2.8(i), S is a doubly connected dominating set of $G + H$. Let $v \in S$. If $|S| \geq 2$, then $S \setminus \{v\} \neq \emptyset$ and there exists $u \in V(H) \cap N_{G+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Suppose that $|S| = 1$. Suppose further that S is a 1-movable connected dominating set of G. Then there exists $w \in V(G) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{w\} = \{w\}$ is a connected dominating set of G. By Theorem 2.8(i), $(S \setminus \{v\}) \cup \{w\}$ is a doubly connected dominating set of $G + H$. Suppose that S is not a 1-movable connected dominating set of G. Then by assumption, there exists $u \in V(H)$ such that $\{u\}$ is a connected dominating set of H. Hence, $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(ii). Therefore, S is a 1-movable doubly connected dominating set of $G + H$. Similarly, if (iii) holds, then S is a 1-movable doubly connected dominating set of $G+H$. Suppose that (iv) holds. Then $S = S_1 \cup S_2 (\neq V(G + H)$, where $\emptyset \neq S_1 \subseteq V(G)$ and $\emptyset \neq S_2 \subseteq V(H)$. Consider the following cases:

Case 1: $S_1 \neq V(G)$ and $S_2 = V(H)$ or $S_1 = V(G)$ and $S_2 \neq V(H)$.

Suppose that $S_1 \neq V(G)$ and $S_2 = V(H)$. Then by assumption, $\langle V(G) \setminus S_1 \rangle$ is connected. Hence, by Theorem 2.8(iv), S is a doubly connected dominating set of $G + H$. Let $v \in S$. Suppose that $v \in S_1$. If $|S_1| \geq 2$, then $S_1 \setminus \{v\} \neq \emptyset$. If $\langle (V(G) \setminus (S_1 \setminus \{v\})) \rangle$ is connected, then $S \setminus \{v\} = (S_1 \setminus \{v\}) \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(iv). Suppose that $\langle (V(G) \setminus (S_1 \setminus \{v\})) \rangle$ is not connected. Then by assumption, there exists $u \in (V(G) \setminus S_1) \cup N_G(v)$ such that $\langle V(G) \setminus [(S_1 \setminus \{v\}) \cup \{u\}] \rangle$ is connected. Hence, $(S \setminus \{v\}) \cup \{u\} = [(S_1 \setminus \{v\}) \cup \{u\}] \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(iv). Suppose that $|S_1| = 1$. Then $S \setminus \{v\} = S_2 = V(H)$ and $\langle V(G + H) \setminus (S \setminus \{v\}) \rangle = \langle V(G) \rangle$ is connected. Hence, $S \setminus \{v\}$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(ii). Next, suppose that $v \in S_2$. Since $S_2 = V(H)$, $\emptyset \neq S_2 \setminus \{v\} \neq V(H)$. Since $S_1 \neq V(H)$, $S \setminus \{v\} = S_1 \cup (S_2 \setminus \{v\})$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Therefore, S is a 1-movable doubly connected dominating set of $G + H$. Similarly, if $S_1 = V(G)$ and $S_2 \neq V(H)$, then S is a 1-movable doubly connected dominating set of $G + H$.

Case 2: $S_1 \neq V(G)$ and $S_2 \neq V(H)$.

Then by Theorem 2.8(v), $S = S_1 \cup S_2$ is a doubly connected dominating set of $G + H$. Let $v \in S$. Suppose that $v \in S_1$. If $|S_1| \geq 2$, then $S_1 \setminus \{v\} \neq \emptyset$. Hence, $S \setminus \{v\} = (S_1 \setminus \{v\}) \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Suppose that $|S_1| = 1$. Since G is a connected nontrivial graph, there exists $u \in V(G)$ such that $uv \in E(G)$. Hence, $|(S_1 \setminus \{v\}) \cup \{u\}| = 1$. Thus, $(S \setminus \{v\}) \cup \{u\} = [(S_1 \setminus \{v\}) \cup \{u\}] \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Similarly, if $v \in S_2$, then either $S \setminus \{v\}$ or $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G + H$ for some $u \in (V(G+H) \setminus S) \cap N_{G+H}(v)$. Consequently, S is a 1-movable doubly connected dominating set of $G + H$. □

Corollary 2.10. *Let G and H be connected nontrivial graphs. Then*

$$\gamma^1_{mcc}(G + H) = \begin{cases} 1, & \text{if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma^1_{mc}(G) = 1 \text{ or } \gamma^1_{mc}(H) = 1 \\ 2, & \text{otherwise} \end{cases}$$

Theorem 2.11. *Let H be a connected graph of order $n \geq 2$ and $K_1 = \langle \{x\} \rangle$. Then $S \subseteq V(K_1 + H)$ is a 1-movable doubly connected dominating set of $K_1 + H$ if and only if one of the following statements holds:*

(i) *$S = V(K_1)$ and there exists $w \in V(H)$ such that $\{w\}$ is a connected dominating set of H.*

(ii) *$S = V(K_1) \cup S_1$ where $\emptyset \neq S_1 \subseteq V(H)$ such that if $S_1 \neq V(H)$, then*

(1) *$\langle V(H) \setminus S_1 \rangle$ is connected;*

(2) *either S_1 is a connected dominating set of H or $S_1 \cup \{a\}$ is a connected dominating set of H for some $a \in V(H) \setminus S_1$; and*

(3) *for every* $v \in S_1$ *, either* $\langle V(H)\setminus(S_1\setminus\{v\})\rangle$ *or* $\langle V(H)\setminus[(S_1\setminus\{v\})\cup\{u\}]\rangle$ *is connected for some* $u \in (V(H) \setminus S_1) \cap N_H(v)$.

(*iii*) *S is a connected dominating set of H such that for every $v \in S$, $S\setminus\{v\}$ or $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of H for some $u \in (V(H) \setminus S) \cap N_H(v)$ or $\langle V(H) \setminus (S \setminus \{v\})\rangle$ is connected.*

Proof. Suppose that $S \subseteq V(K_1 + H)$ is a 1-movable doubly connected dominating set of $K_1 + H$. Consider the following cases:

Case 1: $x \in S$.

Suppose that $S = \{x\}$. Since S is a 1-movable doubly connected dominating set of $K_1 + H$, there exists $w \in V(H)$ such that $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a doubly connected dominating set of $K_1 + H$. It follows that $\{w\}$ is a connected dominating set of $K_1 + H$ (and hence of H). This proves (i). Suppose that $S = V(K_1) \cup S_1$, where $\emptyset \neq S_1 \subseteq V(H)$. Suppose that $S_1 \neq V(H)$. Since S is a doubly connected dominating set of $K_1 + H$, it follows that $\langle V(K_1 + H) \setminus S\rangle = \langle V(H) \setminus S_1\rangle$ is connected. Suppose that $S\setminus\{x\}$ is a doubly connected dominating set in $K_1 + H$. Then $S \setminus \{x\} = S_1$ is a connected dominating set in H. Next, suppose that $S \setminus \{x\}$ is not a doubly connected dominating set of $K_1 + H$. Since S is a 1-movable doubly connected dominating set of $K_1 + H$, there exists $a \in V(H) \setminus S_1$ such that $(S \setminus \{x\}) \cup \{a\} = S_1 \cup \{a\}$ is a doubly connected dominating set of $K_1 + H$. This implies that $S_1 \cup \{a\}$ is a connected dominating set of H. Let $v \in S_1$. Suppose that $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Then $\langle V(K_1 + H) \setminus (S \setminus \{v\})\rangle = \langle V(H) \setminus (S_1 \setminus \{v\})\rangle$ is connected. Suppose that $S \setminus \{v\}$ is not a doubly connected dominating set of $K_1 + H$. Then by assumption, $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$ for some $u \in V(H) \setminus S_1) \cap N_H(v)$. Hence, $\langle V(K_1 + H) \setminus [(S \setminus \{v\}) \cup \{u\}]\rangle = \langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{v\}]\rangle$ is connected. This proves (ii).

Case 2: $x \notin S$.

Then $S \subseteq V(H)$. By assumption, S is a connected dominating set of H. Let $v \in S$. If $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$, then $S \setminus \{v\}$ is a connected dominating set of H. Suppose that $S \setminus \{v\}$ is not a doubly connected dominating set of $K_1 + H$. Then by assumption, there exists $u \in (V(K_1 + H) \setminus S) \cap N_{K_1+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$. If $u \in V(H) \setminus S$, then $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of H. If $u = x$, then $\langle V(H) \setminus (S \setminus \{v\})\rangle$ is connected. Thus, (iii) holds.

For the converse, suppose first that (i) holds. Then $S = \{x\} = V(K_1)$ is a connected dominating set of $K_1 + H$ and $\langle V(K_1 + H) \setminus S\rangle = H$ is connected. Thus, S is a doubly connected dominating set of $K_1 + H$. By assumption, there exists $w \in V(H)$ such that $\{w\}$ is a connected dominating set of H. This implies that $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a connected dominating set of $K_1 + H$ and $\langle V(K_1 + H) \setminus \{w\}\rangle = \langle (V(H) \setminus \{w\}) \cup \{x\}\rangle$ is connected. Hence, $(S\setminus\{x\})\cup\{w\} = \{w\}$ is a doubly connected dominating set of K_1+H. Therefore, S is a 1-movable doubly connected dominating set of $K_1 + H$. Suppose that (ii)

holds. Suppose first that $S = V(K_1 + H)$. Since $K_1 + H$ has no cut-vertices, by Corollary 2.3, S is a 1-movable doubly connected dominating set of $K_1 + H$. Suppose that $S_1 \neq V(H)$. Then by (1), S is a doubly connected dominating set of $K_1 + H$. Let $v \in S$. Consider the following cases:

Case 1: $v = x$.

Suppose that S_1 is a connected dominating set of H. Then $S \setminus \{v\} = S \setminus \{x\} = S_1$ is a connected dominating set of $K_1 + H$. Since $\langle V(K_1 + H) \setminus S_1 \rangle = \langle \{x\} \rangle + \langle V(H) \setminus S_1 \rangle$ is connected, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that S_1 is not a connected dominating set of H. Then by assumption, $S_1 \cup \{a\}$ is a connected dominating set of H for some $a \in V(H) \setminus S_1$. It follows that $(S \setminus \{x\}) \cup \{a\} = S_1 \cup \{a\}$ is a connected dominating set of $K_1 + H$. Moreover, since $\langle V(K_1 + H) \setminus [(S_1 \setminus \{v\}) \cup \{a\}] \rangle = \langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{a\}] \cup \{x\} \rangle$ is connected, $(S \setminus \{x\}) \cup \{a\}$ is a doubly connected dominating set of $K_1 + H$.

Case 2: $v \neq x$.

Then $v \in S_1$. Clearly, $S \setminus \{v\} = \{x\} \cup (S_1 \setminus \{v\})$ is a connected dominating set of $K_1 + H$. If $\langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is connected, then $\langle V(K_1 + H) \setminus (S \setminus \{v\}) \rangle = \langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is connected. Thus, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $\langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is not connected. Then by assumption, $\langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{u\} \rangle$ is connected for some $u \in (V(H) \setminus S_1) \cap N_H(v)$. Hence, $\langle V(K_1 + H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle = \langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{u\} \rangle$ is connected. Moreover, $(S \setminus \{v\}) \cup \{u\} = \{x\} \cup [(S_1 \setminus \{v\}) \cup \{u\}]$ is a connected dominating set of $K_1 + H$. Thus, $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$. Therefore, S is a 1-movable doubly connected dominating set of $K_1 + H$.

Finally, suppose that (iii) holds. Since S is a connected dominating set of H, it is a connected domonating set of $K_1 + H$. Moreover, since $\langle V(K_1 + H) \setminus S \rangle = \langle \{x\} \rangle + \langle V(H) \setminus S \rangle$ is connected, S is a doubly connected dominating set of $K_1 + H$. Now, let $v \in S$. Suppose that $S \setminus \{v\}$ is a connected dominating set of H. Since $\langle V(K_1 + H) \setminus (S \setminus \{v\}) \rangle = K_1 + \langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of H for some $u \in (V(H) \setminus S) \cap N_H(v)$. Then it is a connected dominating set of $K_1 + H$. Since $\langle V(K_1 + H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle = K_1 + \langle V(H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle$ is connected, $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $\langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected in H. Let $S^* = (S \setminus \{v\}) \cup \{x\}$. Then S^* is a connected dominating set of $K_1 + H$. Since $\langle V(K_1 + H) \setminus S^* \rangle = \langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected in $K_1 + H$, it follows that S^* is a doubly connected dominating set of $K_1 + H$. Thus, S is a 1-movable doubly connected dominating set of $K_1 + H$. □

Corollary 2.12. *Let H be a connected nontrivial graph. Then the following holds:*

(i) $\gamma^1_{mcc}(K_1 + H) = 1$ *if and only if* $\gamma(H) = 1$.

(ii) If H has no cut-vertices, then $\gamma^1_{mcc}(K_1 + H) \leq \gamma^1_{mc}(H)$.

Corollary 2.13. *Let n be a positive integer. Then*

$$\gamma^1_{mcc}(F_n) = \begin{cases} 1, & if\, 1 \leq n \leq 3 \\ n, & if\, n \geq 4 \end{cases}$$

and

$$\gamma^1_{mcc}(W_n) = \begin{cases} 1, & if\, n = 3 \\ 2, & if\, n = 4 \\ 3, & if\, n = 5 \\ n, & if\, n \geq 6 \end{cases}$$

Theorem 2.14. *Let $m \geq 2$ and $n \geq 2$ be integers. Then $S \subseteq V(K_{m,n})$ is a 1-movable doubly connected dominating set of $K_{m,n} = \bar{K}_m + \bar{K}_n$ if and only if $S = S_1 \cup S_2$, where $\emptyset \neq S_1 \subseteq V(\bar{K}_m)$ with $|S_1| \geq 2$ and $\emptyset \neq S_2 \subseteq V(\bar{K}_n)$ with $|S_2| \geq 2$ such that $S_1 = V(\bar{K}_m)$ if and only if $S_2 = V(\bar{K}_n)$.*

Proof. Suppose that $S \subseteq V(K_{m,n})$ is a 1-movable doubly connected dominating set of $K_{m,n}$. Let $S_1 = S \cap V(\bar{K}_m)$ and $S_2 = S \cap V(\bar{K}_n)$. Then $S = S_1 \cup S_2$. Since S is a (doubly) connected dominating set of $K_{m,n}$, $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Suppose that $|S_1| = 1$, say $S_1 = \{x\}$ for some $x \in V(\bar{K}_m)$. Then $\langle S \setminus \{x\}\rangle = \langle S_2\rangle$ is an empty subgraph of $\bar{K}_n$. Hence, $S \setminus \{x\}$ is not a (doubly)connected dominating set of $K_{m,n}$. Also, $\langle (S \setminus \{x\}) \cup \{u\}\rangle = \langle S_2 \cup \{u\}\rangle$ is an empty subgraph of $\bar{K}_n$ for all $u \in V(\bar{K}_n) \setminus S_2$. Hence, $(S \setminus \{x\}) \cup \{u\}$ is not a (doubly)connected dominating set of $K_{m,n}$ for all $u \in V(\bar{K}_n) \setminus S_2$. This implies that S is not a 1-movable doubly connected dominating set of $K_{m,n}$, contrary to our assumption. Hence, $|S_1| \geq 2$. Similarly, $|S_2| \geq 2$. Suppose that $S_1 = V(\bar{K}_m)$. Since S is a doubly connected dominating set of $K_{m,n}$, $\langle V(K_{m,n}) \setminus S\rangle = \langle V(\bar{K}_n) \setminus S_2\rangle$ is connected if $S_2 \neq V(\bar{K}_n)$. Since this is not possible, $S_2 = V(\bar{K}_n)$. Similarly, $S_1 = V(\bar{K}_m)$ if $S_2 = V(\bar{K}_n)$.

For the converse, suppose that $S = S_1 \cup S_2$, where S_1 and S_2 are the sets satisfying the given conditions. If $S_1 = V(\bar{K}_m)$ and $S_2 = V(\bar{K}_n)$, then $S = V(K_{m,n})$. Since $m \geq 2$ and $n \geq 2$, $K_{m,n}$ has no cut-vertices. Hence by Corollary 2.3, S is a 1-movable doubly connected dominating set of $K_{m,n}$. Suppose that $S_1 \neq V(\bar{K}_m)$ or $S_2 \neq V(\bar{K}_n)$. Then by assumption, $2 \leq |S_1| < m$ and $2 \leq |S_2| < n$. Hence, by Theorem 2.8(v), $S = S_1 \cup S_2$ is a doubly connected dominating set of $K_{m,n}$. Let $v \in S$. Suppose that $v \in S_1$. Since $2 \leq |S_1| < m$, $S_1 \setminus \{v\} \neq \emptyset$. Since $1 \leq |S_1 \setminus \{v\}| < m$, it follows from Theorem 2.8(v) that $S \setminus \{v\}$ is a doubly connected dominating set of $K_{m,n}$. Similarly, $S \setminus \{v\}$ is a doubly connected dominating set of $K_{m,n}$ if $v \in S_2$. Consequently, S is a 1-movable doubly connected dominating set of $K_{m,n}$. □

Corollary 2.15. *Let $m \geq 2$ and $n \geq 2$ be integers. Then*

$$\gamma^1_{mcc}(K_{m,n}) = \begin{cases} n+2, & if \quad m = 2 \,and\, n \geq 2 \\ m+2, & if \quad n = 2 \,and\, m \geq 2 \\ 4, & otherwise \end{cases}$$

References

[1] M.H. Akhbari, R. Hasni, O. Favaron, H. Karami and S.M. Sheikholeslami, *Inequalities of nordhaus-gaddum type for doubly connected domination number*, Discrete Applied Mathematics, 158 (2010), 1465-1470.

[2] B. Arriola and S.R. Canoy Jr., *Doubly connected domination in the join and Cartesian product of some graphs*, Asian-Europian J. Math., 7 (2014).

[3] J. Blair, R. Gera and S. Horton, *Movable dominating sensor sets in networks*, Journal of Combinatorial Mathematics and Combinatorial Computing, 77 (2011), 103-123.

[4] M. Chellali, F. Maffray and K. Tablennehas, *Connected domination DOT-critical graphs*, Contributions to Discrete Mathematics, 5 (2010), 11-25.

[5] J. Cyman, M. Lemanska and J. Raczek, *On the doubly connected domination number of a graph*, Central European Journal of Mathematics, 4 (2006), 34-45.

[6] R.G. Hinampas, Jr. and S.R. Canoy, Jr., *1-movable domination in graphs*, Applied Mathematical Sciences, 8 (2014), 8565-8571.

[7] R.G. Hinampas, Jr. and S.R. Canoy, Jr., *1-movable independent domination in graphs*, International Journal of Mathematical Analysis, 9 (2015), 73-80.

[8] H. Karami, R. Khoeilar, S.M. Sheikholeslami, *Doubly connected domination subdivision numbers of graphs*, Matematicki Vesnik, 64 3(2012), 232-239.

[9] J. Lomarda and S.R. Canoy, Jr., *1-movable total dominating sets in graphs*, International journal of Mathematical Analysis, 8 (2014), 2703-2709.

[10] J. Lomarda and S.R. Canoy, Jr., *1-movable connected dominating sets in graphs*, Applied Mathematical Sciences, 9 (2015), 507-514.

Accepted: 19.05.2017

FIXED POINT THEOREMS FOR FUZZY SOFT CONTRACTIVE MAPPINGS IN FUZZY SOFT METRIC SPACES

A.F. Sayed*
Mathematics Department
Al-Lith University College
Umm Al-Qura University
P.O. Box 112, Al-Lith 21961
Makkah Al Mukarramah
Kingdom of Saudi Arabia
dr.afsayed@hotmail.com

Jamshaid Ahmad
Department of Mathematics
University of Jeddah
P.O.Box 80327, Jeddah 21589
Kingdom of Saudi Arabia
jamshaid_jasim@yahoo.com

Abstract. The aim of this paper is to examine some important properties of fuzzy soft metric spaces. Fuzzy soft continuous mappings are introduced and some their properties are investigated. Finally, we prove some fixed point theorems of fuzzy soft contractive mappings in fuzzy soft metric spaces.

Keywords: soft set, fuzzy set, fuzzy soft set, fuzzy soft metric space, fuzzy soft contractive mapping.

1. Introduction

In daily life, the problems in many fields deal with uncertain data and are not successfully modelled in classical mathematics. There are two types of mathematical tools to deal with uncertainties namely fuzzy set theory introduced by Zadeh [15] and the theory of soft sets initiated by Molodstov [9], which helps to solve problems in all areas. Maji et al. [8], introduced several operations in soft sets and has also coined fuzzy soft sets. In [12], Beaula and Gunaseelib were introduced a definition of the fuzzy soft metric space.

The concept of fuzzy soft topology firstly introduced by Tanay and Kandemir [11], they defined the concept of fuzzy soft topology as a topology over the given fuzzy soft set f_A. So a fuzzy soft topology in the sense of Tanay and Kandemir [11], is the collection τ of fuzzy soft subsets of f_A closed under arbitrary supremum and finite infimum. It also contains $\tilde{\phi}$ and $\tilde{E}$. But Roy and

∗. Corresponding Author

Samanta [10], redefined the concept of fuzzy soft topology. The most significant reason for such a change is to make sure that the DeMorgan Laws hold in the new definition of fuzzy soft topology. Here we recall the definition of fuzzy soft topology as introduced in [10].

The aim of this paper is to examine some important properties of fuzzy soft metric spaces. Fuzzy soft continuous mappings are introduced and some their properties are investigated. Finally, we prove some fixed point theorems of fuzzy soft contractive mappings in fuzzy soft metric spaces.

2. Preliminaries

In this section we present some basic definitions of fuzzy soft set and fuzzy soft metric space.

Throughout our discussion, X refers to an initial universe, E the set of all parameters for X, $P(X)$ denotes the power set of X and $[0,1] = I$.

Definition 2.1 ([15]). *A fuzzy set A in a non-empty set X is characterized by a membership function $\mu_A : X \to I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for every $x \in X$. Let I^X denotes the family of all fuzzy sets on X.*

A member A in I^X is contained in a member B of I^X, denoted $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$ (see [15]).

Let $A, B \in I^X$, we have the following properties on fuzzy sets (see [15]).

(1) $A = B$ if and only if $\mu_A(x) = \mu_B(x) \forall x \in X$. (Equality),

(2) $C = A \wedge B \in I^X$ by $\mu_C(x) = min\{\mu_A(x), \mu_B(x)\} \forall x \in X$. (Intersction),

(3) $D = A \vee B \in I^X$ by $\mu_C(x) = max\{\mu_A(x), \mu_B(x)\} \forall x \in X$.(Union),

(4) $E = A^c \in I^X$ by $\mu_E(x) = 1 - \mu_A(x) \forall x \in X$. (Complement).

Definition 2.2 ([15]). *An empty fuzzy set denoted by $\tilde{0}$ is a function which maps each $x \in X$ to 0. That is, $\tilde{0}(x) = 0$ for all $x \in X$. A universal fuzzy set denoted by $\tilde{1}$ is a function which maps each $x \in X$ to 1. That is, $\tilde{1}(x) = 1$ for all $x \in X$.*

Definition 2.3 ([9]). *Let $A \subseteq E$. A pair (F, A) is called a soft set over X if F is a mapping $F : A \to P(X)$.*

In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A), or as the set of ϵ-approximate elements of the soft set.

Definition 2.4 ([7]). *A pair (f, A), denoted by f_A, is called a fuzzy soft set over X, where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ where*

$$\mu^e_{f_A} = \begin{cases} \tilde{0}, & \text{if } e \notin A; \\ otherwise, & \text{if } e \in A. \end{cases}$$

$\widetilde{(X, E)}$ denotes the class of all fuzzy soft sets over (X, E) and is called a fuzzy soft universe ([8]).

Definition 2.5 ([2]). *A fuzzy soft set F_A over X is said to be:*

(a) *NULL fuzzy soft set, denoted by $\tilde{\phi}$, if for all $e \in A, f_A(e) = \tilde{0}$.*

(b) *absolute fuzzy soft set, denoted by $\tilde{E}$, if for all $e \in A, f_A(e) = \tilde{1}$.*

Definition 2.6 ([10]). *The complement of a fuzzy soft set f_A, denoted by f^c_A, where $f^c_A : E \to I^X$ is a mapping given by $\mu^e_{f^c_A} = \tilde{1} - \mu^e_{f_A}$, for all $e \in E$ and where $\tilde{1}(x) = 1$, for all $x \in X$. Clearly $(f^c_A)^c = f_A$.*

Definition 2.7 ([10]). *Let $f_A, g_B \in \widetilde{(X, E)}$. f_A is fuzzy soft subset of g_B, denoted by $f_A \tilde{\subseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \leq \mu^e_{g_B}$ for all $e \in A$, i.e. $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)$ for all $x \in X$ and for all $e \in A$.*

Definition 2.8 ([10]). *Let $f_A, g_B \in \widetilde{(X, E)}$. The union of f_A and g_B is the fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C, h_C(e) = \mu^e_{h_c} = \mu^e_{f_A} \vee \mu^e_{g_B}$. Here we write $h_C = f_A \tilde{\cup} g_B$.*

Definition 2.9 ([10]). *Let $f_A, g_B \in \widetilde{(X, E)}$. The intersection of f_A and g_B is the fuzzy soft set d_C, where $C = A \cap B$ and for all $e \in C, d_C(e) = \mu^e_{d_c} = \mu^e_{f_A} \wedge \mu^e_{g_B}$. Here we write $d_C = f_A \tilde{\cap} g_B$.*

Definition 2.10 ([6]). *The fuzzy soft set $f_A \in \widetilde{(X, E)}$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha (0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.*

Definition 2.11 ([6]). *The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set g_A, denoted by $x^e_\alpha \tilde{\in} g_A$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x)$.*

Definition 2.12 ([12]). *Let f_A be fuzzy soft set over X. The two fuzzy soft points $f_{e_1}, f_{e_2} \in f_A$ are said to be equal if $\mu_{f_{e_1}}(x) = \mu_{f_{e_2}}(x)$ for all $x \in X$. Thus $f_{e_1} \neq f_{e_2}$ if and only $\mu_{f_{e_1}}(x) \neq \mu_{f_{e_2}}(x)$ for all $x \in X$.*

Proposition 2.13 ([12]). *The union of any collection of fuzzy soft points can be considered as a fuzzy soft set and every fuzzy soft set can be expressed as the union of all fuzzy soft points.*

$$f_A = \{\tilde{\cup}_{f_e \tilde{\in} f_A} f_e : e \in E\}$$

Proposition 2.14 ([12]). *Let f_A, f_B be two fuzzy soft sets then $f_A\tilde{\subseteq}f_B$ if and only if $f_e\tilde{\in}f_A$ implies $f_e\tilde{\in}f_B$ and hence $f_A = f_B$ if and only if $f_e\tilde{\in}f_A$ if and only if $f_e\tilde{\in}f_B$.*

Definition 2.15 ([4]). *Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters, $A \subseteq E$. Then a mapping $f : A \to B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0, \tilde{1}(e) = 1$ for all $e \in E$ respectively.*

The set of all soft real numbers is denoted by $\mathcal{R}(A)$ and the set of all non-negative soft real numbers by $\mathcal{R}(A)^*$

Definition 2.16 ([3]). *A (non negative) fuzzy soft real number is a fuzzy set on the set of all (non negative) soft real numbers $\mathcal{R}(A)$, that is, a mapping $\tilde{\tilde{\lambda}} : \mathcal{R}(A) \to [0, 1]$, associating with each (non negative) soft real number $\tilde{t}$, its grade of membership $\tilde{\tilde{\lambda}}(\tilde{t})$ satisfying the following conditions:*

(*i*) $\tilde{\tilde{\lambda}}$ *is convex that is,* $\tilde{\tilde{\lambda}}(\tilde{t}) \geq min\ (\tilde{\tilde{\lambda}}(\tilde{s}), \tilde{\tilde{\lambda}}(\tilde{r}))$ *for* $\tilde{s} \subseteq \tilde{t} \subseteq \tilde{r}$,

(*ii*) $\tilde{\tilde{\lambda}}$ *is normal that is, there exists* $\tilde{t_0} \in \mathcal{R}(A)^*$ *such that* $\tilde{\tilde{\lambda}}(\tilde{t_0}) = 1$,

(*iii*) $\widetilde{\widetilde{\lambda}}$ *is upper semi continuous provided for all* $\widetilde{t} \in \mathcal{R}(A)$ *and* $\alpha \in [0, 1]$, $\widetilde{\widetilde{\lambda}}(\widetilde{t}) < \alpha$, *there is a* $\delta > 0$ *such that* $\|\widetilde{s} - \widetilde{t}\| \leq \delta$ *implies that* $\widetilde{\widetilde{\lambda}}(\widetilde{s}) < \alpha$.

Definition 2.17 ([10]). *A fuzzy soft topology τ on X is a family of fuzzy soft sets over X satisfying the following properties:*

(*i*) $\tilde{\phi}, \tilde{E} \in \tau$,

(*ii*) *if* $f_A, g_B \in \tau$, *then* $f_A\tilde{\cap}g_B \in \tau$.

(*iii*) *if* $f_{A_\alpha} \in \tau$ *for all* $\alpha \in \Delta$ *an index set, then* $\tilde{\bigcup}_{\alpha\in\Delta} f_{A_\alpha} \in \tau$.

Definition 2.18 ([10]). *If τ is a fuzzy soft topology on X, then the triple (X, E, τ) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (X, E, τ).*

The complement of a fuzzy soft open set is a fuzzy soft closed set.

Definition 2.19 ([11]). *Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft closure of f_A is defined as the intersection of all fuzzy soft closed sets which contained f_A and is denoted by $\bar{f_A}$ or $cl(f_A)$ we write*

$$cl(f_A) = \tilde{\bigcap}\{g_B : g_B \text{ is fuzzy soft closed and } f_A\tilde{\subseteq}f_B\}.$$

Definition 2.20 ([11]). *Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft interior of f_A denoted by f_A^o is the union of all fuzzy soft open subsets of f_A. Clearly, f_A^o is the largest fuzzy soft open set over X which contained in f_A.*

Definition 2.21 ([5]). *Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft boundary of f_A denoted by ∂f_A is defined as $\partial f_A = \bar{f}_A \tilde{\cap} \bar{f}'_A$.*

Definition 2.22 ([6]). *A fuzzy soft topological space $(X, E, \tau))$ is said to be a fuzzy soft normal space if for every pair of disjoint fuzzy soft closed sets h_A and k_A, $\exists$ disjoint fuzzy soft open sets g_{1_A}, g_{2_A} such that $h_A \tilde{\subseteq} g_{1_A}$ and $k_A \tilde{\subseteq} g_{2_A}$.*

Definition 2.23 ([1]). *Let $\widetilde{(X, E)}$ and $\widetilde{(Y, E')}$ be classes of fuzzy soft sets over X and Y with attributes (the set of all parameters) from E and E' respectively. Let $\varphi : X \longrightarrow Y$ and $\psi : E \longrightarrow E'$ be two mappings. Then $\varphi_\psi = (\varphi, \psi) : \widetilde{(X, E)} \longrightarrow \widetilde{(Y, E')}$ is called a fuzzy soft mapping from $\widetilde{(X, E)}$ to $\widetilde{(Y, E')}$.*

If φ and ψ is injective then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be injective.

If φ and ψ is surjective then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be surjective.

If φ and ψ is constant then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be constant.

Definition 2.24 ([14]). *Let (X, E, τ_1) and (X, E, τ_2) be two fuzzy soft topological spaces.*

(i) *A fuzzy soft mapping $\varphi_\psi = (\varphi, \psi) : \widetilde{(X, E, \tau_1)} \longrightarrow \widetilde{(X, E, \tau_2)}$ is called fuzzy soft continuous if $\varphi_{\psi^{-1}}(g_B) \in \tau_1$, for all $g_B \in \tau_2$.*

(ii) *A fuzzy soft mapping $\varphi_\psi = (\varphi, \psi) : \widetilde{(X, E, \tau_1)} \longrightarrow \widetilde{(X, E, \tau_2)}$ is called fuzzy soft open if $\varphi_\psi \tilde{\in} \tau_2$, for all $f_A \in \tau_1$.*

Let $A \subseteq E$. The collection of all fuzzy soft points of a fuzzy soft set f_A over X be denoted by $FSC(f_A)$.

Let $\mathcal{R}(A)^*$ be the set of all non negative fuzzy soft real numbers. The fuzzy soft metric using fuzzy soft points is defined as follows:

Definition 2.25 ([12]). *Let $A \subseteq E$ and $\tilde{E}$ be the absolute fuzzy soft set. A mapping $\tilde{d} : FSC(\tilde{E}) \times FSC(\tilde{E}) \to \mathcal{R}(A)^*$ is said to be a fuzzy soft metric on $\tilde{E}$ if $\tilde{d}$ satisfies the following conditions:*

$(FSM_1) : \tilde{d}(f_{e_1}, f_{e_2}) \tilde{\geq} \tilde{0}$ for all $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,

$(FSM_2) : \tilde{d}(f_{e_1}, f_{e_2}) = \tilde{0}$ if and only if $f_{e_1} = f_{e_2}$ for al $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,

$(FSM_3) : \tilde{d}(f_{e_1}, f_{e_2}) = \tilde{d}(f_{e_2}, f_{e_1})$ for all $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,

$(FSM_4) : \tilde{d}(f_{e_1}, f_{e_3}) \tilde{\leq} \tilde{d}(f_{e_1}, f_{e_2}) + \tilde{d}(f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3} \tilde{\in} \tilde{E}$).

The fuzzy soft set $\tilde{E}$ with the fuzzy soft metric $\tilde{d}$ is called the fuzzy soft metric space and is denoted by $(\tilde{E}, \tilde{d})$.

Definition 2.26 ([13]). *Let $(\tilde{E},\tilde{d})$ be a fuzzy soft metric space and $\tilde{\tilde{t}}$ be a fuzzy soft real number and $\tilde{\epsilon}\in(0,1)$. A fuzzy soft open ball centered at the fuzzy point $f_e\tilde{\in}\tilde{E}$ and radius $\tilde{\tilde{t}}$ is a collection of all fuzzy soft points g_e of $\tilde{E}$ such that $\tilde{d}(g_e,f_e)\tilde{<}\tilde{\tilde{t}}$. It is denoted by $\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon})$ where $\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon})=\{g_e\tilde{\in}\tilde{E}|\tilde{d}(g_e,f_e)\tilde{<}\tilde{\tilde{t}}\}$ with $\mid\mu^a_{g_e}(x)-\mu^a_{f_e}(x)\mid<\tilde{\epsilon}$ for all $a\in E,x\in X$.*

The fuzzy soft closed ball is denoted by $\tilde{\tilde{B}}[f_e,\tilde{\tilde{t}},\tilde{\epsilon}]=\{g_e\tilde{\in}\tilde{E}|\tilde{d}(g_e,f_e)\tilde{\leq}\tilde{\tilde{t}}\}$ with $\mid\mu^a_{g_e}(x)-\mu^a_{f_e}(x)\mid\leq\tilde{\epsilon}$ for all $a\in E,x\in X$.

Definition 2.27 ([13]). *A sequence $\{f_{e_n}\}$ in a fuzzy soft metric space $(\tilde{E},\tilde{d})$ is said to converge to $f_{e'}$ if $\tilde{d}(f_{e_n},f_{e'})\to\tilde{0}$ as $n\to\infty$ for every $\tilde{\epsilon}\tilde{>}\tilde{0}$ there exists $\tilde{\delta}\tilde{>}\tilde{0}$ and a positive integer $N=N(\tilde{\epsilon})$ such that $\tilde{d}(f_{e_n},f_{e'})\tilde{<}\tilde{\delta}$ implies $\mid\mu^a_{f_{e_n}}(x)-\mu^a_{f_{e'}}(x)\mid<\tilde{\epsilon}$ whenever $n\geq N,a\in E$ and $x\in X$. It is usually denoted as $lim_{\,n\to\infty}f_{e_n}=f_{e'}$.*

Definition 2.28 ([13]). *A sequence $\{f_{e_n}\}$ in a fuzzy soft metric space $(\tilde{E},\tilde{d})$ is said to be a Cauchy sequence if to every $\tilde{\epsilon}\tilde{>}\tilde{0}$ there exists $\tilde{\delta}\tilde{>}\tilde{0}$ and a positive integer $N=N(\tilde{\epsilon})$ such that $\tilde{d}(f_{e_n},f_{e_m})\tilde{<}\tilde{\delta}$ implies $\mid\mu^a_{f_{e_n}}(x)-\mu^a_{f_{e_m}}(x)\mid<\tilde{\epsilon}$ for all $n,m\geq N,a\in E$ and $x\in X$ that is $\tilde{d}(f_{e_n},f_{e_m})\to\tilde{0}$ as $n,m\to\infty$.*

Definition 2.29 ([13]). *A fuzzy soft metric space $(\tilde{E},\tilde{d})$ is said to be complete if every cauchy sequence in $\tilde{E}$ converges to some fuzzy soft point of $\tilde{E}$.*

Definition 2.30 ([13]). *A fuzzy soft set f_A in a fuzzy soft metric space $(\tilde{E},\tilde{d})$ is said to be fuzzy soft open if for each fuzzy soft point f_e of f_A there exist a fuzzy soft open ball $\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon})\tilde{\subseteq}f_A$.*

Lemma 2.31 ([13]). *Let $(\tilde{E},\tilde{d})$ be a fuzzy soft metric space then the fuzzy soft open ball $\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon})$ is a fuzzy soft open set.*

Theorem 2.32 ([13]). *Given a fuzzy soft metric space $(\tilde{E},\tilde{d})$. Let $\Im$ denote the set of all fuzzy soft open sets in $\tilde{E}$. Then $\Im$ has the following properties:*

(i) $\tilde{\phi},\tilde{E}\in\Im$,

(ii) *if $f_A,g_B\in\Im$ then $f_A\tilde{\cap}g_B\in\Im$,*

(iii) *if $f^\alpha_{A_\alpha}\in\Im$ for all $\alpha\in\Delta$ an index set, then $\tilde{\cup}_{\alpha\in\Delta}f^\alpha_{A_\alpha}\in\Im$.*

$\Im$ is called the topology determined by the fuzzy soft metric $\tilde{d}$.

3. Fuzzy soft topology generated by fuzzy soft metric

In this section, we study some important results of fuzzy soft metric spaces.

Let $\tilde{E}$ be the absolute fuzzy soft set over X and E be a parameter set and $\tilde{E}_e$ be a family of fuzzy soft points i.e. $\tilde{E}_e=\{f_e=(e,\tilde{1}):\tilde{1}(x)=1,\forall x\in X,e\in E\}$. Then there exists a bijective mapping between the fuzzy soft set $\tilde{E}$ and the set X. If $e\neq\acute{e}\in E$, then $\tilde{E}_e\tilde{\cap}\tilde{E}_{\acute{e}}=\tilde{\phi}$, and $FSC(\tilde{E})=\tilde{\cup}_{e\in E}\tilde{E}_e$.

Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space. It is clear that $(\tilde{E}_e, \tilde{d}_e)$ is a fuzzy soft metric space for $e \in E$. Then by using the fuzzy soft metric $\tilde{d}_e$, we define a metric on X as $d_e(x, y) = \tilde{d}_e(f_e, g_e)$, for all $x, y \in X$ and $f_e, g_e \tilde{\in} \tilde{E}_e$.

Note that $e \neq \acute{e} \in E$, then d_e and $d_{\acute{e}}$ on X are generally different metrics.

Proposition 3.1. *Every fuzzy soft metric space is a family of parameterized metric spaces.*

Proof. It is obvious from above. □

The converse of Proposition 3.1 may not be true in general. This is shown by the following example.

Example 3.2. Let $E = R$ be a parameter set and (X, d) be a metric space.

We define the function $\tilde{d} : FSC(\tilde{E}) \times FSC(\tilde{E}) \to R$ by

$$\tilde{d}(f_e, g_{\acute{e}}) = d(x, y)^{\tilde{1}+|\mu^a_{f_e}(x)-\mu^a_{g_{\acute{e}}}(y)|},$$

for all $f_e, g_{\acute{e}} \in FSC(\tilde{E})$. Then for all $e \in E, d_e$ is a metric on X.

If $\tilde{d}(f_e, g_{\acute{e}}) = \tilde{0}$, then this does not always mean that $f_e = g_{\acute{e}}$, so $\tilde{d}$ is not a fuzzy soft metric on $\tilde{E}$.

Proposition 3.3. *Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space and $\tau_{\tilde{d}_e}$ be a fuzzy soft topology generated by the fuzzy soft metric $\tilde{d}$. Then for every $e \in E$, the topology $(\tau_{\tilde{d}})_e$ on X is the topology τ_{d_e} generated by the metric d_e on X.*

Proof. It is obvious. □

Lemma 3.4. *Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space. Then the following expressions are true:*

(*i*) $f_e \tilde{\in} \overline{f_A} \Leftrightarrow \tilde{d}(f_e, f_A) = \tilde{0}$;

(*ii*) $f_e \tilde{\in} f^o_A \Leftrightarrow \tilde{d}(f_e, f^c_A) \tilde{>} \tilde{0}$;

(*iii*) $f_e \tilde{\in} \partial f_A \Leftrightarrow \tilde{d}(f_e, f_A) = \tilde{d}(f_e, f^c_A) = \tilde{0}$.

Proof. It is clear. □

Note that if f_A is a fuzzy soft closed set in the fuzzy soft metric space $(\tilde{E}, \tilde{d})$ and $f_e \tilde{\notin} (f_A)$, then there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ such that $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}) \tilde{\cap} f_A = \tilde{\phi}$.

Theorem 3.5. *Every fuzzy soft metric space is a fuzzy soft normal space.*

Proof. Let h_A and k_B be two disjoint fuzzy soft closed sets in the fuzzy soft metric space $(\tilde{E}, \tilde{d})$. For every fuzzy soft points $f_e \tilde{\in} h_A$ and $g_{\acute{e}} \tilde{\in} k_B$, we choose fuzzy soft open balls $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ and $\tilde{\tilde{B}}(f_{\grave{e}}, \tilde{\tilde{r}}, \tilde{\grave{\epsilon}})$ such that $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}) \tilde{\cap} k_B = \tilde{\phi}$ and $\tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}) \tilde{\cap} h_A = \tilde{\phi}$.

Thus, we have $h_A \tilde{\subseteq} \widetilde{\bigcup} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3) = u_A$ and $k_B \tilde{\subseteq} \widetilde{\bigcup} \tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3) = v_B$. We want to show that $u_A \tilde{\cap} v_B = \tilde{\phi}$.

Assume that $u_A \tilde{\cap} v_B \neq \tilde{\phi}$. Then there exists a fuzzy soft point $w_{\acute{e}}$ such that $w_{\acute{e}} \tilde{\in} u_A \tilde{\cap} v_B$. Therefore, there exist fuzzy soft open balls $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3)$ and $\tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3)$ such that $w_{\acute{e}} \tilde{\in} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3)$ and $w_{\acute{e}} \tilde{\in} \tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3)$.

Here, we have $\tilde{d}(f_e, w_{\acute{e}}) \tilde{<} \epsilon/3$ and $\tilde{d}(g_{\acute{e}}, w_{\acute{e}}) \tilde{<} \acute{\epsilon}/3$. If we get $\max\{\tilde{\epsilon}/3,\ \tilde{\acute{\epsilon}}/3\} = \tilde{\epsilon}/3$, then we have

$$\tilde{d}(f_e, g_{\acute{e}}) \tilde{\leq} \tilde{d}(f_e, w_{\acute{e}}) + \tilde{d}(w_{\acute{e}}, g_{\acute{e}}) \tilde{\leq} \epsilon/3 + \acute{\epsilon}/3 \tilde{\leq} \epsilon$$

and so $g_{\acute{e}} \tilde{\in} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ and which contradicts with our assumption. Therefore, $u_A \tilde{\cap} v_B = \tilde{\phi}$. □

4. Fuzzy soft contractive mappings

In this section we shall prove some fixed point theorems of fuzzy contractive mappings.

Definition 4.1. *Let $(\tilde{E}, \tilde{d})$ and $(\tilde{E}', \tilde{\rho})$ be two fuzzy soft metric spaces. The mapping $\varphi_\psi = (\varphi, \psi) : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is a fuzzy soft mapping, if $\varphi : \tilde{E} \to \tilde{E}'$ and $\psi : E \to E'$ are two mappings.*

Proposition 4.2. *For each fuzzy soft point $f_e \tilde{\in} FSC(\tilde{E}), \varphi_\psi(f_e)$ is a fuzzy soft point in $\tilde{E}'$.*

Proof. Let $f_e \tilde{\in} FSC(\tilde{E})$ be a fuzzy soft point.

Then $\varphi_\psi(f_e)(\acute{e}) = \widetilde{\bigcup}_{e \in \psi^{-1}(\acute{e})} \varphi(f_e(e)) = (\varphi(f_e))_{\psi(e)}$. □

Definition 4.3. *Let $(\tilde{E}, \tilde{d})$ and $(\tilde{E}', \tilde{\rho})$ be two fuzzy soft metric spaces and $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is a fuzzy soft continuous mapping at the fuzzy soft point $f_e \tilde{\in} FSC(\tilde{E})$ if for every fuzzy soft open ball $\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\acute{\epsilon}})$ of $(\tilde{E}', \tilde{\rho})$, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ of $(\tilde{d}, \tilde{E})$ such that $\varphi(\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})) \tilde{\subseteq} \tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{r}}, \tilde{\acute{\epsilon}})$.*

If $\varphi_\psi(f_e)$ is a fuzzy soft continuous mapping at every fuzzy soft point f_e of $(\tilde{E}, \tilde{d})$, then it is said to be fuzzy soft continuous mapping on $(\tilde{E}, \tilde{d})$.

Now, this definition can be expressed using $\tilde{\varepsilon} - \tilde{\delta}$ as follows:

Definition 4.4. *The mapping $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is said to be a fuzzy soft continuous mapping at the fuzzy soft point $f_e \tilde{\in} FSC(\tilde{E})$ if for every $\tilde{\varepsilon} \tilde{>} \tilde{0}$ there exists a $\tilde{\delta} \tilde{>} \tilde{0}$ such that $\tilde{d}(f_e, g_{\acute{e}}) \tilde{<} \tilde{\delta}$ implies that $\tilde{\rho}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}})) \tilde{<} \tilde{\varepsilon}$.*

Theorem 4.5. *Let $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ be a fuzzy soft mapping. Then the following conditions are equivalent:*

(1) *$\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is a fuzzy soft continuous mapping,*

(2) *For each fuzzy soft open set g_B in $(\tilde{E}', \tilde{\rho})$, $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set in $(\tilde{E}, \tilde{d})$,*

(3) *For each fuzzy soft closed set h_C in $(\tilde{E}', \tilde{\rho})$, $(\varphi_\psi)^{-1}(h_C)$ is a fuzzy soft closed set in $(\tilde{E}, \tilde{d})$,*

(4) *For each fuzzy soft set f_A in $(\tilde{E}, \tilde{d})$, $\varphi_\psi(\overline{f_A})\tilde{\subset}\overline{(\varphi_\psi(f_A))}$ is a fuzzy soft closed set in $(\tilde{E}', \tilde{\rho})$,*

(5) *For each fuzzy soft set g_B in $(\tilde{E}', \tilde{\rho})$, $\overline{((\varphi_\psi)^{-1}(g_B))}\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B})$,*

(6) *For each fuzzy soft set g_B in $(\tilde{E}', \tilde{\rho})$, $(\varphi_\psi)^{-1}(g_B^o)\tilde{\subset}((\varphi_\psi)^{-1}(g_B))^o$.*

Proof. (1) $\Rightarrow$ (2) Let φ_ψ be a fuzzy soft continuous mapping and g_B be a fuzzy soft open set in $(\tilde{E}', \tilde{\rho})$. Consider the fuzzy soft set $(\varphi_\psi)^{-1}(g_B)$. If $(\varphi_\psi)^{-1}(g_B) = \tilde{\phi}$, then the proof is completed. Let $(\varphi_\psi)^{-1}(g_B) \neq \tilde{\phi}$. In this case there exists at least one fuzzy soft point $f_e\tilde{\in}(\varphi_\psi)^{-1}(g_B)$. Then we have $\varphi_\psi(f_e)\tilde{\in}g_B$. Since g_B is a fuzzy soft open set, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})$ such that $\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})\tilde{\subset}\tilde{\subset}g_B$ holds. Also since φ_ψ is a fuzzy soft continuous mapping, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\epsilon})$ such that $\varphi_\psi(\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\epsilon}))\tilde{\subset}\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})$. Thus,

$$\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\epsilon})\tilde{\subset}(\varphi_\psi)^{-1}(\varphi_\psi)(\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\epsilon}))\tilde{\subset}(\varphi_\psi)^{-1}(\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})\tilde{\subset}(\varphi_\psi)^{-1}(g_B).$$

Consequently, $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set.

(2) $\Rightarrow$ (3) Let h_C be any fuzzy soft set in $(\tilde{E}', \tilde{\rho})$ Then h_C^c is a fuzzy soft open set. From (2), we have $(\varphi_\psi)^{-1}(h_C)^c$ is a fuzzy soft open set in $(\tilde{E}, \tilde{d})$. Thus $(\varphi_\psi)^{-1}(h_C)$ is a fuzzy soft closed set.

(3) $\Rightarrow$ (4) Let f_A be a fuzzy soft set in $(\tilde{E}, \tilde{d})$. Since $f_A\tilde{\subset}(\varphi_\psi)^{-1}\varphi_\psi(f_A)$ and $\varphi_\psi(f_A)\tilde{\subset}\overline{(\varphi_\psi(f_A))}$,we have $f_A\tilde{\subset}(\varphi_\psi)^{-1}\varphi_\psi(f_A)\tilde{\subset}(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$. By part (3), since $(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$ is a fuzzy soft closed set in $(\tilde{E}, \tilde{d})$, $\bar{f_A}\tilde{\subset}(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$.

Thus $\varphi_\psi(\bar{f_A})\tilde{\subset}\varphi_\psi((\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))}))$ is obtained.

(4) $\Rightarrow$ (5) Let g_B be a fuzzy soft set in $(\tilde{E}', \tilde{\rho})$ and $(\varphi_\psi)^{-1}(g_B) = f_A$. By part (4), we have $\varphi_\psi(\bar{f_A}) = \varphi_\psi(\overline{((\varphi_\psi)^{-1}(g_B))})\tilde{\subset}\overline{(\varphi_\psi(\varphi_\psi)^{-1}(g_B))}\tilde{\subset}\bar{g_B}$. Then

$\overline{((\varphi_\psi)^{-1}(g_B))} = \bar{f_A}\tilde{\subset}(\varphi_\psi)^{-1}(\varphi_\psi(\bar{f_A}))\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B})$.

(5) ⇒ (6) Let g_B be a fuzzy soft set in $(\tilde{E}', \tilde{\rho})$. Substituting g_B^c for condition in (5). Then $\overline{((\varphi_\psi)^{-1}(g_B^c))}\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B^c})$. Since $g_B^o = (\bar{g_B^c})^c$, then we have

$$\begin{aligned}(\varphi_\psi)^{-1}(g_B^o) &= (\varphi_\psi)^{-1}((\bar{g_B^c})^c)\\ &= ((\varphi_\psi)^{-1}(\bar{g_B^c}))^c\tilde{\subset}(\overline{((\varphi_\psi)^{-1}(g_B^c))})^c\\ &= (\overline{((\varphi_\psi)^{-1}(g_B))^c})^c\\ &= ((\varphi_\psi)^{-1}(g_B))^o.\end{aligned}$$

(6) ⇒ (1) Let g_B be a fuzzy soft open set in $(\tilde{E}', \tilde{\rho})$.

Then since $((\varphi_\psi)^{-1}(g_B))^o\tilde{\subset}(\varphi_\psi)^{-1}g_B = (\varphi_\psi)^{-1}(g_B^o)\tilde{\subset}((\varphi_\psi)^{-1}(g_B))^o$, $((\varphi_\psi)^{-1}(g_B))^o = (\varphi_\psi)^{-1}(g_B)$ is obtained. This implies that $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set. □

Definition 4.6. *The fuzzy soft mapping $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is said to be fuzzy soft sequentially continuous at the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ iff for every sequence of fuzzy soft points $\{f_{e_n}\}$ converging to the fuzzy soft point f_e in the fuzzy soft metric space $(\tilde{E}, \tilde{d})$, the sequence $\varphi_\psi(\{f_{e_n}\})$ in $(\tilde{E}', \tilde{\rho})$ converges to a fuzzy soft point $\varphi_\psi(f_e)\tilde{\in}FSC(\tilde{E}')$.*

Theorem 4.7. *Fuzzy soft continuity is equivalent to fuzzy soft sequential continuity in fuzzy soft metric spaces.*

Proof. Let $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ be a fuzzy soft continuous mapping and $\{f_{e_n}\}$ be any sequence of fuzzy soft points converging to the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$. Let $\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})$ be a fuzzy soft open ball in $(\tilde{E}', \tilde{\rho})$. By fuzzy soft continuity of φ_ψ choose a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}})$ containing f_e such that $\varphi_\psi(\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}))\tilde{\subseteq}\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})$. Since $\{f_{e_n}\}$ converges to f_e there exists $n_0 \in \mathbb{N}$ such that $\{f_{e_n}\}\tilde{\in}\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}})$ for all $n \geq n_0$. Therefore for all $n \geq n_0$ we have $\varphi_\psi(\{f_{e_n}\})\tilde{\in}\varphi_\psi(\tilde{\tilde{B}}(f_e, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}))\tilde{\subseteq}\tilde{\tilde{B}}(\varphi_\psi(f_e), \tilde{\tilde{t}}, \tilde{\epsilon})$, as required.

Conversely, assume for contradiction that $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is fuzzy soft sequential continuous but not fuzzy soft continuous mapping. Since φ_ψ is not fuzzy soft continuous at the fuzzy soft point f_e, there exists such that $\tilde{\varepsilon}\tilde{>}\tilde{0}$ for all $\tilde{\delta}\tilde{>}\tilde{0}$ there exists $g_{\acute{e}}\tilde{\in}FSC(\tilde{E})$ such that $\tilde{d}(f_e, g_{\acute{e}})\tilde{<}\tilde{\delta}$ and $\tilde{\rho}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}}))\tilde{>}\tilde{\epsilon}_0$. For $n \geq 1(n \in \mathbb{N})$, define $\tilde{\delta_n} = 1/n$. For $n \geq 1$ we may choose $\{g_{e'_n}\}$ in $(\tilde{E}, \tilde{d})$ such that $\tilde{d}(f_{e_n}, g_{\acute{e}n})\tilde{<}\tilde{\delta_n}$ and $\tilde{\rho}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}}))\tilde{>}\tilde{\epsilon}_0$.

Therefore, by definition the sequence $\{g_{e'_n}\}(n \geq 1)$ converges to f_e. However, by definition the sequence $\{\varphi_\psi(g_{e'_n})\}(n \geq 1)$ does not converge to $\varphi_\psi(f_e)$. That is, φ_ψ is not fuzzy soft sequentially continuous at f_e. □

Definition 4.8. *Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space. A function $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}, \tilde{d})$ is called a fuzzy soft contraction mapping if there exists a soft real number $\tilde{\alpha}$ with $\tilde{0}\tilde{\leq}\tilde{\alpha}\tilde{<}\tilde{1}$ such that for every fuzzy soft points $f_e, g_{\acute{e}}\tilde{\in}FSC(\tilde{E})$ we have $\tilde{d}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}}))\tilde{\leq}\tilde{\alpha} \cdot \tilde{d}(f_e, g_{\acute{e}})$.*

Proposition 4.9. *Every fuzzy soft contraction mapping is a fuzzy soft continuous mapping.*

Proof. Let $f_e\tilde{\in}FSC(\tilde{E})$ be any fuzzy soft point and $\tilde{\epsilon}\tilde{>}\tilde{0}$ be arbitrary.

If we choose $\tilde{d}(f_e, g_{\acute{e}})\tilde{<}\tilde{\delta}\tilde{<}\tilde{\epsilon}$, then since φ_ψ is a fuzzy soft contraction mapping, we have
$\tilde{d}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}}))\tilde{\leq}\tilde{d}(f_e, g_{\acute{e}})\tilde{<}\tilde{\alpha}\cdot\tilde{\delta}\tilde{<}\tilde{\epsilon}$ and so φ_ψ is a soft continuous mapping. □

Theorem 4.10. *Let $(\tilde{E}, \tilde{d})$ be a complete fuzzy soft metric space. If the mapping $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}, \tilde{d})$ is a fuzzy soft contraction mapping on a complete fuzzy soft metric space, then there exists a unique fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ such that $\varphi_\psi(f_e) = f_e$.*

Proof. Let f_e^0 be any fuzzy soft point in $FSC(\tilde{E})$.

Set $f_{e_1}^1 = \varphi_\psi(f_e^0) = (\varphi(f_e^0))_{\psi(e)}, f_{e_2}^2 = \varphi_\psi(f_{e_1}^1) = (\varphi^2(f_e^0))_{\psi^2(e)}, ..., f_{e_{n+1}}^{n+1} = \varphi_\psi(f_{e_n}^n) = (\varphi^{n+1}(f_e^0))_{\psi^{n+1}(e)}, ...$

We have

$$\begin{aligned}\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) &= \tilde{d}(\varphi_\psi(f_{e_n}^n), \varphi_\psi(f_{e_{n-1}}^{n-1}))\tilde{\leq}\tilde{\alpha}\cdot\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1})\\ &\tilde{\leq}\tilde{\alpha}^2\cdot\tilde{d}(f_{e_{n-1}}^{n-1}, f_{e_{n-2}}^{n-2})\tilde{\leq}...\tilde{\leq}\tilde{\alpha}^n\cdot\tilde{d}(f_{e_1}^1, f_{e_0}^0).\end{aligned}$$

So for $n > m$

$$\begin{aligned}\tilde{d}(f_{e_n}^n, f_{e_m}^m)&\tilde{\leq}\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}) + \tilde{d}(f_{e_{n-1}}^{n-1}, f_{e_{n-2}}^{n-2}) + ... + \tilde{d}(f_{e_{m+1}}^{m+1}, f_{e_m}^m)\\ &\tilde{\leq}(\tilde{\alpha}^{n-1} + \tilde{\alpha}^{n-2} + ... + \tilde{\alpha}^m)\cdot\tilde{d}(f_{e_1}^1, f_{e_0}^0)\\ &\tilde{\leq}\frac{\tilde{\alpha}^m}{\tilde{1}-\tilde{\alpha}}\cdot\tilde{d}(f_{e_1}^1, f_{e_0}^0).\end{aligned}$$

We get $\tilde{d}(f_{e_n}^n, f_{e_m}^m)\tilde{\leq}\frac{\tilde{\alpha}^m}{\tilde{1}-\tilde{\alpha}}\cdot\tilde{d}(f_{e_1}^1, f_e^0)$. This implies $\tilde{d}(f_{e_n}^n, f_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$.

Hence $\{f_{e_n}^n\}$ is a fuzzy soft Cauchy sequence, by the completeness of $(\tilde{E}, \tilde{d})$, there is a fuzzy soft point $f_e^0\tilde{\in}FSC(\tilde{E})$ such that $f_{e_n}^n \to f_e^0$ as $(n \to \infty)$.

Since

$$\begin{aligned}\tilde{\tilde{d}}(\varphi_\psi(f_e^0), f_e^0)&\tilde{\leq}\tilde{\tilde{d}}(\varphi_\psi(f_{e_n}^n), \varphi_\psi(f_e^0)) + \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^0)\\ &\tilde{\leq}\tilde{\alpha}\cdot(\tilde{d}(f_{e_n}^n, f_e^0) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^0)),\\ \tilde{d}(\varphi_\psi(f_e^0), f_e^0)&\tilde{\leq}\tilde{\alpha}\cdot(\tilde{\alpha}\cdot\tilde{d}(f_{e_n}^n, f_e^0) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^0)) \to \tilde{0}.\end{aligned}$$

Hence, $\tilde{d}(\varphi_\psi(f_e^0), f_e^0) \to \tilde{0}$. This implies $\varphi_\psi(f_e^0) = f_e^0$. So the fuzzy soft point f_e^0 is a fixed fuzzy soft point of the mapping φ_ψ. Now, if $g_{\acute{e}}^0$ is another fixed fuzzy soft point of φ_ψ, then $\tilde{d}(f_e^0, g_{\acute{e}}^0) = \tilde{d}(\varphi_\psi(f_e^0), \varphi_\psi(g_{\acute{e}}^0))\tilde{\leq}\tilde{\alpha}\cdot\tilde{d}(f_e^0, g_{\acute{e}}^0)$.

Hence, for $\tilde{\alpha}\tilde{\leq}\tilde{1}, \tilde{d}(f_e^0, g_{\acute{e}}^0) = \tilde{0} \Rightarrow f_e^0 = g_{\acute{e}}^0$.

Therefore, the fixed fuzzy soft point of φ_ψ is unique. □

Theorem 4.11. *Let $(\tilde{E},\tilde{d})$ be a complete fuzzy soft metric space. Suppose the fuzzy soft mapping $\varphi_\psi : (\tilde{E},\tilde{d}) \to (\tilde{E},\tilde{d})$satisfies the fuzzy soft contractive condition:*

$$\tilde{d}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{\leq}\tilde{\tilde{\alpha}}\cdot[\tilde{d}(\varphi_\psi(f_e),f_e)+\tilde{d}(\varphi_\psi(g_{\acute{e}}),g_{\acute{e}})],$$

for all $f_e, g_{\acute{e}}\tilde{\in}FSC(\acute{E}$, where $\tilde{\tilde{\alpha}}\tilde{\in}[\tilde{0},\frac{\tilde{1}}{2})$ is a fuzzy soft constant. Then φ_ψ has a unique fixed fuzzy soft point in $FSC(\acute{E})$

Proof. Choose f_e^0 be any fuzzy soft point in $FSC(\tilde{E})$. Set

$$f_{e_1}^1=\varphi_\psi(f_e^0)=(\varphi(f_e^0))_{\psi(e)}, f_{e_2}^2=\varphi_\psi(f_{e_1}^1)=(\varphi^2(f_e^0))_{\psi^2(e)},...,$$
$$f_{e_{n+1}}^{n+1}=\varphi_\psi(f_{e_n}^n)=(\varphi^{n+1}(f_e^0))_{\psi^{n+1}(e)},....$$

We have

$$\begin{aligned}\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n)&=\tilde{d}(\varphi_\psi(f_{e_n}^n),\varphi_\psi(f_{e_{n-1}}^{n-1}))\\&\tilde{\leq}\tilde{\tilde{\alpha}}\cdot[\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_n}^n)+\tilde{d}(\varphi_\psi(f_{e_{n-1}}^{n-1}),f_{e_{n-1}}^{n-1})]\\&=\tilde{\tilde{\alpha}}\cdot[\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n)+\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1})].\end{aligned}$$

$$\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n)\tilde{\leq}\frac{\tilde{\tilde{\alpha}}}{\tilde{1}-\tilde{\tilde{\alpha}}}\cdot\tilde{d}(f_{e_{n+1}}^n,f_{e_{n-1}}^{n-1})=\tilde{\tilde{h}}\cdot\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n),$$

where $\tilde{\tilde{h}}=\frac{\tilde{\tilde{\alpha}}}{\tilde{1}-\tilde{\tilde{\alpha}}}$. For $n>m$

$$\begin{aligned}\tilde{d}(f_{e_n}^n,f_{e_m}^m)&\tilde{\leq}\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1})+\tilde{d}(f_{e_{n-1}}^{n-1},f_{e_{n-2}}^{n-2})+...+\tilde{d}(f_{e_{m+1}}^{m+1},f_{e_m}^m)\\&\tilde{\leq}(\tilde{\tilde{h}}^{n-1}+\tilde{\tilde{h}}^{n-2}+...+\tilde{\tilde{h}}^m)\cdot\tilde{d}(f_{e_1}^1,f_e^0)\tilde{\leq}\frac{\tilde{\tilde{h}}^m}{\tilde{1}-\tilde{\tilde{h}}}\cdot\tilde{d}(f_{e_1}^1,f_e^0).\end{aligned}$$

We get $\tilde{d}(f_{e_n}^n,f_{e_m}^m)\tilde{\leq}\frac{\tilde{\tilde{h}}^m}{\tilde{1}-\tilde{\tilde{h}}}\cdot\tilde{d}(f_{e_1}^1,f_e^0)$. This implies $\tilde{d}(f_{e_n}^n,f_{e_m}^m)\to\tilde{0}$ as $(n,m\to\infty)$.

Hence $\{f_{e_n}^n\}$ is a fuzzy soft Cauchy sequence, by the completeness of $(\tilde{E},\tilde{d})$, there is a fuzzy soft point $f_e^*\tilde{\in}FSC(\tilde{E})$ such that $f_{e_n}^n\to f_e^*$ as $(n\to\infty)$.

Since

$$\begin{aligned}\tilde{d}(\varphi_\psi(f_e^*),f_e^*)&\tilde{\leq}\tilde{\tilde{d}}(\varphi_\psi(f_{e_n}^n),\varphi_\psi(f_e^*))+\tilde{d}(\varphi_\psi(f_{e_n}^n),f_e^*)\\&\tilde{\leq}\tilde{\tilde{\alpha}}\cdot[\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_n}^n)+\tilde{d}(\varphi_\psi(f_{e_n}^n),f_e^*)]+\tilde{d}(f_{e_{n+1}}^{n+1},f_e^*),\\&\tilde{\leq}\frac{\tilde{1}}{\tilde{1}-\tilde{\alpha}}[\tilde{\tilde{\alpha}}\cdot\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_n}^n)+\tilde{d}(f_{e_{n+1}}^{n+1},f_e^*)],\end{aligned}$$
$$\tilde{d}(\varphi_\psi(f_e^*),f_e^*)\tilde{\leq}\tilde{\tilde{\alpha}}\cdot\frac{\tilde{1}}{\tilde{1}-\tilde{\tilde{\alpha}}}[\tilde{\tilde{\alpha}}\cdot\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n)+\tilde{d}(f_{e_{n+1}}^{n+1},f_e^*)]\to\tilde{0}.$$

Hence, $\tilde{d}(\varphi_\psi(f_e^*),f_e^*)\to\tilde{0}$.

This implies $\varphi_\psi(f_e^*)=f_e^*$. So the fuzzy soft point f_e^* is a fixed fuzzy soft point of the mapping φ_ψ. Now, if $g^*\acute{e}$ is another fixed fuzzy soft point of φ_ψ, then $\tilde{d}(f_e^*,g_{\acute{e}}^*)=\tilde{d}(\varphi_\psi(f_e^*),\varphi_\psi(g_{\acute{e}}^*))\tilde{\leq}\tilde{\tilde{\alpha}}\cdot[\tilde{d}(\varphi_\psi(f_e^*),f_e^*)+\tilde{d}(\varphi_\psi(g_{\acute{e}}^*),g_{\acute{e}}^*)]=\tilde{0}$.

Hence, for $\tilde{\tilde{\alpha}}\tilde{\leq}\tilde{1}, \tilde{d}(f_e^*, g_{\acute{e}}^*) = \tilde{0} \Rightarrow f_e^* = g_{\acute{e}}^*$.

Therefore, the fixed fuzzy soft point of φ_ψ is unique. □

Theorem 4.12. *Let $(\tilde{E}, \tilde{d})$ be a complete fuzzy soft metric space. Suppose the fuzzy soft mapping $\varphi_\psi : (\tilde{E}, \tilde{d}) \to (\tilde{E}, \tilde{d})$satisfies the fuzzy soft contractive condition:*

$$\tilde{d}(\varphi_\psi(f_e), \varphi_\psi(g_{\acute{e}}))\tilde{\leq}\tilde{\tilde{\alpha}} \cdot [\tilde{d}(\varphi_\psi(f_e), g_{\acute{e}}) + \tilde{d}(\varphi_\psi(g_{\acute{e}}), f_e)],$$

for all $f_e, g_{\acute{e}}\tilde{\in}FSC(\acute{E}$, where $\tilde{\tilde{\alpha}}\tilde{\in}[\tilde{0}, \frac{\tilde{1}}{2})$ is a fuzzy soft constant. Then φ_ψ has a unique fixed fuzzy soft point in $FSC(\acute{E})$

Proof. Choose f_e^0 be any fuzzy soft point in $FSC(\tilde{E})$. Set

$$f_{e_1}^1 = \varphi_\psi(f_e^0) = (\varphi(f_e^0))_{\psi(e)}, f_{e_2}^2 = \varphi_\psi(f_{e_1}^1) = (\varphi^2(f_e^0))_{\psi^2(e)}, ...,$$
$$f_{e_{n+1}}^{n+1} = \varphi_\psi(f_{e_n}^n) = (\varphi^{n+1}(f_e^0))_{\psi^{n+1}(e)}, ...$$

We have

$$\begin{aligned}\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) &= \tilde{d}(\varphi_\psi(f_{e_n}^n), \varphi_\psi(f_{e_{n-1}}^{n-1}))\tilde{\leq}\tilde{\tilde{\alpha}} \cdot [\tilde{d}(\varphi_\psi(f_{e_n}^n), f_{e_{n-1}}^{n-1}) + \tilde{d}(\varphi_\psi(f_{e_{n-1}}^{n-1}), f_{e_n}^n)] \\ &= \tilde{\tilde{\alpha}} \cdot [\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) + \tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1})].\end{aligned}$$

So $\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n)\tilde{\leq}\frac{\tilde{\tilde{\alpha}}}{\tilde{1}-\tilde{\tilde{\alpha}}} \cdot \tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}) = \tilde{\tilde{h}} \cdot \tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1})$, where $\tilde{\tilde{h}} = \frac{\tilde{\tilde{\alpha}}}{\tilde{1}-\tilde{\tilde{\alpha}}}$. For $n > m$

$$\begin{aligned}\tilde{d}(f_{e_n}^n, f_{e_m}^m)&\tilde{\leq}\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}) + \tilde{d}(f_{e_{n-1}}^{n-1}, f_{e_{n-2}}^{n-2}) + ... + \tilde{d}(f_{e_{m+1}}^{m+1}, f_{e_m}^m) \\ &\tilde{\leq}(\tilde{\tilde{h}}^{n-1} + \tilde{\tilde{h}}^{n-2} + ... + \tilde{\tilde{h}}^m) \cdot \tilde{d}(f_{e_1}^1, f_e^0)\tilde{\leq}\frac{\tilde{\tilde{h}}^m}{\tilde{1} - \tilde{\tilde{h}}} \cdot \tilde{d}(f_{e_1}^1, f_e^0).\end{aligned}$$

We get $\tilde{d}(f_{e_n}^n, f_{e_m}^m)\tilde{\leq}\frac{\tilde{\tilde{h}}^m}{\tilde{1}-\tilde{\tilde{h}}} \cdot \tilde{d}(f_{e_1}^1, f_e^0)$. This implies $\tilde{d}(f_{e_n}^n, f_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$.

Hence $\{f_{e_n}^n\}$ is a fuzzy soft Cauchy sequence, by the completeness of $(\tilde{E}, \tilde{d})$, there is a fuzzy soft point $f_e^*\tilde{\in}FSC(\tilde{E})$ such that $f_{e_n}^n \to f_e^*$ as $(n \to \infty)$.

Since

$$\begin{aligned}\tilde{d}(\varphi_\psi(f_e^*), f_e^*)&\tilde{\leq}\tilde{d}(\varphi_\psi(f_{e_n}^n), \varphi_\psi(f_e^*)) + \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^*) \\ &\tilde{\leq}\tilde{\tilde{\alpha}} \cdot [\tilde{d}(\varphi_\psi(f_e^*), f_{e_n}^n) + \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^*)] + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*) \\ &\tilde{\leq}\tilde{\tilde{\alpha}} \cdot [\tilde{d}(\varphi_\psi(f_e^*), f_e^*) + \tilde{d}(f_{e_n}^n, f_e^*) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*)] + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*) \\ &\tilde{\leq}\frac{\tilde{1}}{\tilde{1} - \tilde{\tilde{\alpha}}}[\tilde{\tilde{\alpha}} \cdot (\tilde{d}(f_{e_n}^n, f_e^*) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*)],\end{aligned}$$

$$\tilde{d}(\varphi_\psi(f_e^*), f_e^*)\tilde{\leq}\tilde{\tilde{\alpha}} \cdot \frac{\tilde{1}}{\tilde{1} - \tilde{\tilde{\alpha}}}[\tilde{\tilde{\alpha}} \cdot (\tilde{d}(f_{e_n}^n, f_e^*)) + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^*)].$$

Hence, $\tilde{d}(\varphi_\psi(f_e^*), f_e^*) \to \tilde{0}$.

This implies $\varphi_\psi(f_e^*) = f_e^*$. So the fuzzy soft point f_e^* is a fixed fuzzy soft point of the mapping φ_ψ.

Now, if $g^*\acute{e}$ is another fixed fuzzy soft point of φ_ψ, then $\tilde{d}(f_e^*, g_{\acute{e}}^*) = \tilde{d}(\varphi_\psi(f_e^*), \varphi_\psi(g_{\acute{e}}^*)) \tilde{\leq} \tilde{\alpha} \cdot [\tilde{d}(\varphi_\psi(f_e^*), g_{\acute{e}}^*) + \tilde{d}(\varphi_\psi(g_{\acute{e}}^*), f_e^*)] = \tilde{0}$. Hence, for $\tilde{\alpha} \tilde{\leq} \tilde{1}, \tilde{d}(f_e^*, g_{\acute{e}}^*) = \tilde{0} \Rightarrow f_e^* = g_{\acute{e}}^*$.

Therefore, the fixed fuzzy soft point of φ_ψ is unique. □

Acknowledgements

The authors are very grateful to the editor and the reviewers for their valuable suggestions.

References

[1] A. Aygunoglu and H. Aygun, *Introduction to fuzzy soft groups,* Comput. Math. Appl., 58(2009), 1279-1286.

[2] P.K. Maji, R. Biswas and A.R. Roy, *Soft set theory, Computers and Mathematics with Applications*, 45 (2003), 555-562.

[3] S. Christinal Gunaseeli, *Some Contributions to Special Fuzzy Topological Spaces*, Thesis submitted to the Bharathidasan University, Tiruchirappalli in partial fulfillment of the requirements for the Degree of Doctor OF Philosophy in Mathematics, Ref.No.16690/Ph.D./Mathematics/Full-Time/July 2012

[4] S. Das and S.K. Samanta, *Soft real sets, soft real numbers and their properties*, J. Fuzzy Math., 20 (2012), 551-576.

[5] C. Gunduz and S. Bayramov, *Some results on fuzzy soft topological spaces,* Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2013, Article ID 835308, 10 pages.

[6] J. Mahanta and P.K. Das, *Results on fuzzy soft topological spaces*, arXiv:1203.0634v1,2012.

[7] P.K. Maji, R. Biswas and A.R. Roy, *Fuzzy soft sets, Journal of Fuzzy Mathematics*, 9(3) (2001), 589-602.

[8] P.K. Maji, A.R. Roy and R. Biswas, *An application of soft sets in a decision making problem,* Comput. Math. Appl., 44(8-9) (2002), 1077–1083.

[9] D. Molodtsov, *Soft set theory First results,* Comput. Math. Appl., 8(4/5) (1999), 19–31.

[10] S. Roy and T.K. Samanta, *A note on fuzzy soft topological spaces,* Annals of Fuzzy Mathematics and informatics, 3 (2) (2012), 305–311.

[11] B. Tanay and M.B. Kandemir, *Topological structures of fuzzy soft sets,* Comput. Math. Appl., 61 (2011), 412-418.

[12] T. Beaulaa and C. Gunaseelib, *On fuzzy soft metric spaces, Malaya J. Mat.*, 44(8-9) (2014), 197–202.

[13] T. Beaulaa and R. Raja, *Completeness in Fuzzy Soft Metric Space,* Malaya J. Mat., S(2,(2015), 438-442.

[14] B.P. Varol and H. Aygun, *Fuzzy soft topology,* Hacet. J. Math. Stat., 41(3)(2012), 407-419.

[15] L.A. Zadeh, *Fuzzy sets, Information and Control*, 8 (1965), 338–353.

Accepted: 20.05.2017

COMPARISON OF SURFACE FITTING METHODS FOR MODELLING LEAF SURFACE

Moa'ath N. Oqielat
Department of Mathematics
Faculty of Science
Al-balqa' Applied University
Al-Salt 19117
Jordan
moaathoqily@bau.edu.jo

Abstract. A novel hybrid method for modelling leaf surface that combines Gaussian radial basis function (RBF) and Clough-Tocher (CT) methods to achieve a continuous surface is proposed by the author [20]. In this paper, we demonstrate the accuracy of the hybrid Gaussion RBF-CT approach by applying it to model the surface of frangipani and anthurium leaves. Furthermore, a comparison between the hybrid Gaussion RBF-CT method and the hybrid multiquadric RBF-CT method introduced by the author Oqielat et al. [17] is presented.

The development of the algorithm has been made to assist the understanding of leaf surface properties. It is found that the hybrid multiquadric RBF-CT surface fitting methodology produces more accurate and realistic leaf surface representation than using the hybrid Gaussion RBF-CT method.

Keywords: finite elements methods, interpolation, radial basis functions, Clough-Tocher method, virtual plants.

1. Introduction

The modelling of virtual plant has been researched extensively over the last decades [24] and models of leaf surfaces have generally been generated recently with accurate accuracy and level of detail by [5,9,10,11,19]. Leaves play an important role in the development of a plant, and therefore leaf model is required. Loch [13] proposed two methods to accurately model leaf surfaces.

Surface data can often be collected at a discrete set of points and a key problem is to reconstruct the surface, or perhaps capture important features of the surface from a discrete set of measurements. This representation may be used for visualization purposes only [13] or may be used to study biological processes such as photosynthesis and canopy light environments.

The minority of the leaf models that presented in the past were based on accurate measurements until 3D digitizers and faster computers with improved graphic capabilities became available. Virtual leaf models may be presented in an abstract way, where the leaf is represented by a disk [25] or more realistically, by a surface model that captures the surface shape and boundary [22]. Boundary

algorithms were applied by [16] for modelling lobed leaves. Maddonni [15] used piecewise linear triangles to represent the leaf surface, where vertices along the boundary are estimated by allometric relationships.

Loch [13,14] presented two methods to model accurate leaf surfaces in three dimensions based on finite elements methods (piecewise linear triangular and piecewise cubic Clough-Tocher triangular). Loch modelled the leaf surfaces for Flame,Frangipani, Elephant's ear and Anthurium leaves.

Our research in this paper introduces a new surface fitting method that combine Clough-Tocher with Gaussian radial basis techniques for modelling the leaf surface. Then, a comparison of our method with a hybrid method introduced by the author [17] is presented. Finally, the proposed hybrid technique is applied to a large number of three-dimensional data points captured from an Frangipani and Anthurium leaf surface and we found that the hybrid method gives accurate representation of the leaf surface, see figure 1. This work form the bases for future research , for example, accurate leaf surface representation can be used in the context of modelling surface droplet movement.

The research is presented over four sections. In section 1, we overview of interpolation surface fitting methods based on the Clough-Tocher and the radial basis function method which are already outlined by the auother in [19,20]. In section 2 a new surface fitting method is proposed that combines the CT and Gaussian RBF methods for modelling leaf surfaces. In section 3, the application of the new method to a Frangipani leaf and Anthurium leaf is presented. Future work and further applications of the model are discussed in section 4.

2. Clough-Tocher finite element method

The Clough-Tocher method (CTM) is an interpolating finite element method that was introduced originally by Clough and Tocher [4]. This method is used to minimize the degree of the polynomial interpolant without losing the continuity of the gradient over the whole domain.

The CTM is a seamed element approach, whereby each triangle is treated as a macro-element that is split into subtriangles, which are called micro-elements. The CTM, has the advantage that it results in a smooth surface over the whole domain. It approximates the surface as an interpolating cubic polynomial constructed on each subtriangle which enables a bivariate piecewise cubic interpolant to be devised over the entire triangle that is continuously differentiable. The key result is that only twelve degrees of freedom are required for the CTM, namely the function values and the gradient at each vertex, as well as the normal derivative along the edges. The CT interpolant has the form:

$$\varphi(x,y) = \sum_{i=1}^{3} \left(f_i b_i + (c_i, d_i)^T \cdot \nabla f_i \right) + \sum_{j=1}^{3} \frac{\partial f}{\partial n_j} e_j. \tag{1}$$

In this representation the twelve functions $b_i(x,y), c_i(x,y), d_i(x,y)$ and $e_j(x,y)$, $i = 1,2,3$ are cardinal basis functions (see Lancaster [12]), having the property that just one of them is unity and the reminder zero at each of the node points.

In the modelling of leaf surface, the function value is assigned at the triangle vertices. However, the derivative information at the vertices and at the midpoints of each side is unavailable and needs to be estimated. The gradient at the triangle vertices are estimated from neighbouring data information and thereafter the edge normal derivatives are determined as the mean of the normal estimated at the two vertices associated with the edge. This approximation is based on the assumption that the normal slope along the sides of the triangle changes linearly. The author [1,26] presented analysis for the least square gradient estimation method as well as an error bounds for the method. A more detailed description of CTM including the list of cardinal basis functions for the standard triangular element can be found in (Lancaster [12]).

1.2 Radial basis functions

A Radial Basis Function (RBF) approximation to f is a function S of the form:

$$S(x) = \sum_{i=1}^{n} a_i \Phi_i(x), \quad x \in \mathrm{R}^2, \tag{2}$$

where $\Phi_i(x) = R\left(\|x - x_i\|\right)$, $R(r)$ is a non-negative real-valued function with non-negative argument r and $\|.\|$ denotes the Euclidean norm. The points $\{x_i\}$ belonging to R^2 are called the centres of the RBF approximation. The expansion coefficients $\{a_i\}$ are determined by satisfying some approximation criterion; in this application by interpolation (see equation 4).

Radial basis function method has found applications in areas such as geodesy and medical imaging, the theory of the RBF approximation is given by Powell [21]. A main problem of the radial basis function method concerns large sets of data points where the computational costs involved in fitting and evaluating the RBF can become time-consuming.

Well known examples of radial basis function methods include Gaussian RBF which is adopted in this paper:

$$R\left(\|x - x_i\|\right) = \exp -c^2 \|x - x_i\|^2 . \tag{3}$$

The parameter c is specified by the user, however, it is well known that the accuracy for interpolating scattered data with radial basis functions depends strongly on this parameter, see for example [3,6,23]. For some values of c the problem may become ill-conditioned. Franke [6] used $c = 1.25\frac{D}{\sqrt{n}}$ where D is the diameter of the minimal circle enclosing all data points.

Rippa [23] proposed an algorithm for selecting a good value for the parameter c based on minimizing a cost function that represents the error between the interpolating radial basis function and the unknown function (RMS), see

equation 6. Rippa considered two sets of data points and nine different test functions defined on the unit square. A data vector $f = (f_1, f_2, ..., f_n)^T$ was constructed by evaluating each test function over the set of data points so that

$$S(x_j) = f_j, j = 1, 2, ..., n. \tag{4}$$

3. Hybrid Gaussian radial basis function Clough-Tocher method

The CT method requires derivative estimates at the vertices and midpoints of the elements for its evaluation. We propose a new hybrid approach [20] for surface fitting based on using Gaussian RBF (either local or global) to estimate the gradient at the vertices and mid-points of the Clough-Tocher triangle. The Gaussian RBF interpolant $Q(x)$ is given by equation 2. The gradient of Q is then given by

$$\nabla Q(x) = \sum_{i=1}^{n} a_i \nabla \Psi_i(x). \tag{5}$$

Where

$$\nabla \Psi_i(x) = \nabla R\left(\|x - x_i\|\right) = (dR/dx_k, dR/dy_k) = \\ (-2(x_k - x_i)c^2 \exp -r^2c^2, -2(y_k - y_i)c^2 \exp -r^2c^2).$$

The procedure that uses this hybrid approach for the purpose of surface fitting is summarised in the following algorithm:

Algorithm 1: Construction Surfaces using the Hybrid RBF-CT Method

INPUT:N data points $\{(x_i, f_i), i = 1, \ldots, N\}$

Step 1: *Choose a subset of n data points from the given N points to triangulate the surface.*

Step 2: *Using either a global multiquadric RBF interpolant constructed from the ntriangulation points OR, a local multiquadric RBF interpolant constructed on each triangle using a local subset of m points, generate the RBF linear system.*

Step 3: *Approximately solve this linear system using the TSVD method.*

Step 4: *Use the RBF coefficients to construct either the global or local gradient.*

Step 5: *Apply the hybrid CT-RBF method to construct the surface using either*$\nabla S_n(x)$ *(global) or* $\nabla S_m(x)$ *(local) to provide the necessary derivative information for the construction of the CT interpolant.*

The **local hybrid approach** applied here is based on choosing the set of 5 nearest neighbours to each vertex and to the center of the triangle. Next, a local radial basis function is built from the 20 points for each triangle, which is then used to estimate the directional derivative at the triangle vertices and midpoints. A **global hybrid approach** is also applied, which is based on

building one single global RBF from the triangulation points and then using it to evaluate the gradients at the vertices and midpoints of all triangles. The parameter c in both cases was estimated globally using the triangulation points following the Rippa framework [23].

The selection of the local set of points to use for the construction of the RBF is important. For Frangipani leaf, we used the 5 points closest points to each vertex and to the centroid of the triangle for the construction of the local RBF. Using this point set produced the best results and we found that using more points, for example 10 or 20, did not improve the fit. For Anthurium leaf using closest 5 points struck problems because we did not produce enough points of the triangular element to enable a sufficient sampling of function values to produce reasonable gradient estimates. To avoid this problem we construct the local RBF by select the closest 30 points to each vertex and to the centroid of the triangle. This selection process ensured that the local RBF contains a sufficient set of data that enables a more accurate gradient to be produced.

4. Application of hybrid method for the frangipani and anthurium leaves

A set of representative data points sampled from the surface are required to reconstruct the shape of a leaf using surface fitting techniques. The process of sampling data points from the leaf surface using a measuring device is called digitizing such that the visible exterior data points of the leaf are enough to capture the surface of the leaf. Loch [14] collected data points for different types of leaves (such as, Frangipani, Anthurium, Flame and Elephant's Ear) using a laser scanner. The boundary points were selected by hand from the complete set of points using the PointPicker, software written by Hanan [7].

To assess the accuracy of the hybrid Clough-Tocher Radial basis function interpolation method, we applied the method to the laser scanned Frangipani and Anthurium leaf data taken from (Loch [14]) to construct the surface of those two leaves. The Anthurium leaf data set consists of a set containing 4,688 points, which represent the entire leaf surface points and a second set containing 79 points representing the boundary points of the Anthurium leaf surface. The Frangipani leaf data set contains two subsets of data. The first set consists of 3,388 points, which represents the entire leaf surface scanned points; while the second set consists of 17 points representing the boundary points of the Frangipani leaf surface. These point sets are displayed in Figures 1 (a) and (b). Now, to apply the hybrid method to the leaf data, preprocessingsteps are essential which includes determination a new plane for the leaf data and then tringulation of the leaf surface. These two steps are discussed previously by the author [17,19,20] and will explain briefly in the following two subsections.

4.1 Leaf reference plane

The laser scanner returns the coordinate system of points on the leaf, These coordinates may not be suitable for interpolation due to the possibility of multivalued and vertical surfaces. To solve this problem we can use use a reference plane that is a least squares fit to these data points. We construct a reference plane by making a linear least squares fit to the data and rotating the coordinate system so that the reference plane becomes the $xy-$plane. This rotation can be achieved by rotating the normal vector of the reference plane about the $y-$axis into the $yz-$ plane and then about the $x-$axis into the $xz-$ plane (Oqielat [18]). If the vertical height of the data points is single valued in the transformed coordinate system, then the procedure is successful.

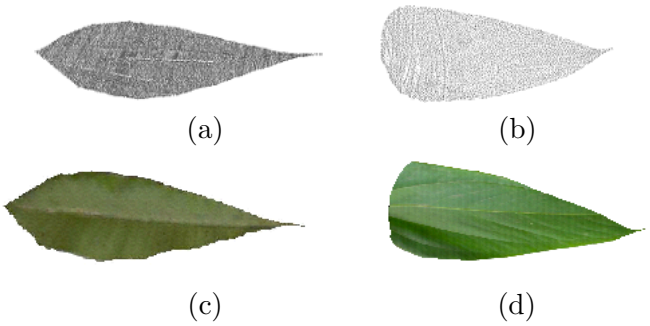


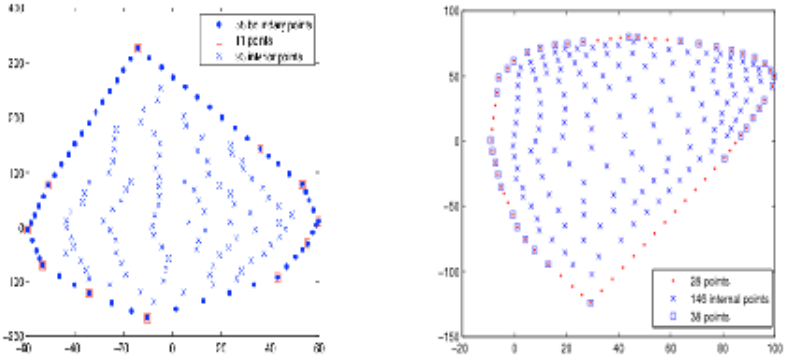
Figure 1: Photos of the scanned (a) Frangipani and (b) Anthurium leaves and corresponding (c) Frangipani and (d) Anthurium leaf surface models for these point sets.

5. Triangulation of the leaf surface

To apply our method to the leaf data points, we need to construct a triangulation to the leaf surface. The data points that represent the leaf surface is large, so the computational expense for surface fitting can be reduced by selecting only a subset of these data to generate a triangulation of the leaf surface. In this research we constructed the triangulation of the leaf using the *EasyMesh* generator, which is software written in the C language by Bojan Niceno [2]. *EasyMesh* generates two-dimensional *Delaunay* and constrained *Delaunay* triangulations in general domains. This software return a good quality triangulation if the domain is convex. However, because the piecewise linear boundary defined by the boundary points do not enclose a convex set, *EasyMesh* was unable to produce the required triangulation. To solve this problem, an algorithm was used to generate a convex hull from the entire set of leaf data points.

Moreover, *EasyMesh* produce better shaped triangles if we define either a horizontal, or vertical, line Inside the convex hull (leaf surface). For the Frangi-

pani and Anthurium leaves (Oqielat [17,18]) it appears that the vertical line produces a more suitable triangulation than the horizontal line. The final triangulation is given in figures 2 and 3. The hybrid Clough-Tocher- radial basis function method is then applied to construct the leaf surface after the triangulation of the leaf surface is constructed.

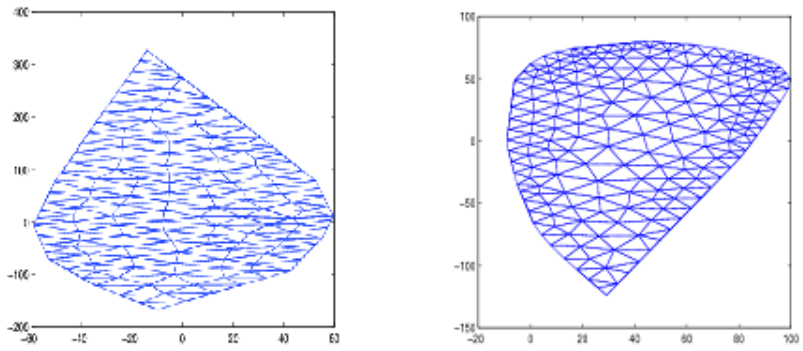


(a) (b)

Figure 2: (a) The vertices of the mesh structure generated using Easymesh. The square points represent the 11 boundary points that are given to Easymesh; the dot points represent the 58 extra points added by Easymesh, while the x points represent the 93 internal points. (b)The vertices of the mesh structure generated using Easymesh. The square points represent the 38 boundary points that are given to Easymesh; the dot points represent the 28 extra points added by Easymesh, while the x points represent the 146 internal points.

6. Numerical experiments

The results of comparisons and applying the hybrid methods to the Frangipani and Anthurium leaf data is present in this section. First, the triangulation points were selected, then the rest of the m data points (denoted by $z_k = z(x_k), k = 1, ..., m$) from the leaf data set were used to measure the quality of the approximation of the hybrid methods. Some of the m data points occurred outside of the virtual leaf mesh so these points were ignored in the quality analysis. The hybrid method then applied to estimate the surface values for the data points occurring inside the triangulation to construct the leaf surface, see Figure 1 (c) and (d).



(a) (b)

Figure 3: (a) Triangulation of 151 points of Frangipani leaf surface generated using EasyMesh. (b) Triangulation of 212 points of Anthurium leaf surface generated using EasyMesh.

The root mean square error RMS, is used in this paper as an error metric to assess the accuracy of our methods which is given by:

$$RMS = \sqrt{\frac{\sum_{k=1}^{k=m}[Q(x_k) - z_k]^2}{m}}. \tag{6}$$

$Q(x_k)$ represents the CT estimated value at the m data points and z_k represents the given function values at the same data points. The second error metric measured the quality in terms of the maximum error associated with the surface fit in relation to the maximum variation in z.

$$\text{Scaled Max. Error} = \frac{\max\left(|Q(x_k) - z_k|\right)}{\max(z_k) - \min(z_k)}, k = 1, 2, ..., m.$$

Tables 1 and 2 show a comparison between the scaled maximum errors and the scaled RMS $= \frac{RMS}{\max(z_k) - \min(z_k)}$ for the Frangipani and the Anthurium leaf data sets respectively using the local and global hybrid multiquadric method and hybrid Gaussian method. For the Anthurium leaf there were a total of 4,460 data points used to assess the accuracy of the surface. Note the *EasyMesh* triangulation comprised 212 vertices. There were about 59 points ignored in the analysis because these points were deemed to lie outside the leaf mesh structure.

One observes for the Frangipani leaf that using the local hybrid multiquadric RBF method produced slightly more accurate RMS value than using the global hybrid multiquadric RBF method while it is the converse for the maximum error. The trends depicted in Table 1 for the Frangipani leaf appear consistent with observations from Table 1 for the Athurium leaf. Moreover, the hybrid (local and global) multiquadric RBF method produced more accurate RMS and maximum error than using the hybrid (local and global) Gaussian RBF method.

Table 1: A comparison of RMS computed using hybrid local and global RBF for the Frangipani leaf data points as well as the maximum error associated with the surface fit.

	Hybrid multiquadric local	Hybrid Gaussian local	Hybrid multiquadratic global	Hybrid Gaussian global
	RBF	RBF	RBF	RBF
Scaled RMS	0.0086	0.0218	0.0139	0.1163
Scaled maximum error	0.0700	0.3603	0.0655	0.4749
Boundary points	48	48	48	48
Points tested	3155	3155	3155	3155
Triangulation points	141	141	141	141
Outside points	104	104	104	104
No. of triangles	257	257	257	257

Table 2: A comparison of RMS computed using hybrid local and global Gaussian RBF for the Anthurium leaf data points as well as the maximum error associated with the surface fit.

	Hybrid multiquadric local	Hybrid Gaussian local	Hybrid multiquadratic global	Hybrid Gaussian global
	RBF	RBF	RBF	RBF
Scaled RMS	0.0043	0.02313	0.0068	0.0100
Scaled maximum error	0.0537	0.8209	0.0435	0.0561
Boundary points	66	66	66	66
Points tested	4460	4460	4460	4460
Triangulation points	212	212	212	212
Outside points	59	59	59	59
No. of triangles	387	387	387	387

6.1 Conclusion and future research

The research described here proposed a new mathematical surface fitting technique for modelling the leaf surface based on a hybrid CT-Gaussion-RBF methodology. The hybrid method has been successfully applied and compared with another interpolation method introduced by the author (Oqielat [17]) and shown to produce a good accuracy for the leaf surface representation compared with the other method.

Our method allows the user to construct an accurate leaf surface based on three-dimensional data points. Moreover, the research provides a basis on which future research can be built. Surface representations can be extended to models determining a water droplet, or pesticide paths along a leaf surface before it falls from or comes to a standstill on the surface; for example, the simulation of a pesticide application to plant surfaces presented by Hanan [5,8,11]. An advantage of the leaf models described in this paper is that they may be used in different plant modelling environments such as AMAP, xfrog or L-Studio.

References

[1] J. Belward, I. Turner I. and M. Oqielat, *Numerical Investigation of Linear Least Square Methods for Derivatives Estimation*, CTAC 08 Computational Techniques and applications conference, Australia, 2008.

[2] Bojan Niceno, *www.dinma.univ.trieste.it/nirftc/research/easymesh*, 2002.

[3] R.E. Carlson, and T.A. Foley, *The parameter R2 in multiquadric interpolation*, Mathematics with Applications, 21 (1991), 29-42.

[4] R.W. Clough and J.L. Tocher, *Finite element stiffness matrices for analysis of plate bending*, In Proceedings of the Conference on Matrix Methods in Structural Mechanics. Wright-Patterson A.F.B., Ohio, 515-545, 1965.

[5] G.J. Dorr, D. Kempthorne, L.C. Mayo, W.A. Forster, J.A. Zabkiewicz, S.W. McCue, J.A. Belward, I.W. Turner, J. Hanan, *Towards a model of spray–canopy interactions: interception, shatter, bounce and retention of droplets on horizontal leaves*, Ecol.Model., 290 (2014), 94-101.

[6] R. Franke, R. (1982) Scattered data interpolation: tests of some methods, *Mathematics of Computation*, 38 (157), 1982.

[7] J. Hanan, B. Loch, T. McAleer, *Processing laser scanner plant data to extract structural information*, Proceedings of the 4th International Workshop on Functional-Structural Plant Models, 7-11, June 2004, Montpellier, France, pages 9-12, 2004.

[8] J. Hanan, M. Renton, E. Yorston, *Simulating and visualising spray deposition in plant canopies,* ACM GRAPHITE 2003, Melbourne, Australia, 259-260, 11-14 February 2003.

[9] D.M. Kempthorne, I.W. Turner, J.A. Belward, *A comparison of techniques for the reconstruction of leaf surfaces from scanned data*, SIAM J. Sci., 2015.

[10] D.M. Kempthorne, I.W. Turner, J.A. Belward, S.W. McCue, M. Barry, J. Young, G.J. Dorr, J. Hanan, J.A. Zabkiewicz, *Surface reconstruction of wheat leaf morphology from three-dimensionalscanned data*, Funct. Plant Biol., 42 (2015), 444-451.

[11] C.M.Lisa, S. McCue, T. Moroney, W. Alison, D. Kempthorne1, J. Belward and I. Turner, *Simulating droplet motion on virtual leaf surfaces*, Royal society open science, 10, 2016.

[12] P. Lancaster and K. Salkauskas, *Curve and Surface Fitting, an introduction*, Academic Press, London, Orlando, 1986.

[13] B. Loch, *Surface fitting for the modelling of plant leaves*, PhD Thesis, University of Queensland, http://www.sci.usq.edu.au/staff/lochb, 2004.

[14] B. Loch, J. Belward and J. Hanan, *Application of surface fitting techniques for the representation of leaf surfaces*, MODSIM 2005 International Congress on Modelling and SimulatMion, 12-15 Dec 2005, Melbourne, Australia, 2005.

[15] G. Maddonni, M. Chelle, J.-L. Drouet and B. Andrieu, *Light interception of contrasting azimuth canopies under square and rectangular plant spatial distributions: simulations and crop measurements*, Field Crops Research, 13 (2001), 1-13.

[16] L. Mundermann, P. MacMurchy, J. Pivovarov and P. Prusinkiewicz, *Modeling lobed leaves*, In Computer Graphics International, Proceedings, Tokyo, 13 (2003), 60-67, 9-11.

[17] M.N. Oqielat, J.A. Belward and I.W. Turner, *A Hybrid Clough-Tocher method for surface fitting with application to leaf data*, Applied Mathematical Modelling, 33 (2009), 2582-2595.

[18] M.N. Oqielat, J.A. Belward, I.W. Turner, B.I. Loch, *A hybrid Clough-Tocher radial basis function method for modelling leaf surfaces*, MODSIM 2007 International Congress on Modelling and Simulation. Modelling and Simulation Society of Australia and New Zealand, 400-406, 2007.

[19] M.N. Oqielat, I.W. Turner, J.A. Belward, S.W. McCue, *Modelling water droplet movements on a leaf surface*, Math. Comput. Simulat., 81 (2011), 1553-1571.

[20] M.N. Oqielat, *Surface Fitting Methods for Modelling Leaf surface from scanned data*, Journal of King Saud University-Science, 2017.

[21] M.N. Oqielat, *Scattered data approximation using radial basis function with a cubic polynomial reproduction for modelling leaf surface*, Journal of Taibah University for Science, 2018.

[22] M.N. Oqielat, *Application of interpolation finite element methods to a real 3D leaf data*, Journal of King Saud University-Science, 2018.

[23] M.J.D. Powell, *Advances in numerical analysis, wavelets, subdivision algorithms and radial functions*, W. Light, Oxford University Press, Oxford, UK, 105-210, 1991.

[24] P. Prusinkiewicz, A. Lindenmayer, J.S. Hanan, F.D. Fracchia, D.R. Fowler, M.J.M. de Boer and L. Mercer, *The algorithmic beauty of plants*, Springer Verlag, New York, Berlin, 1990.

[25] S. Rippa, *An algorithm for selecting a good value for the parameter c in radial basis function interpolation*, Advances in Computational Mathematics, 11 (1999), 193-210.

[26] P. Room, J. Hanan and P. Prunsinkiewicz, *Virtual plants: new perspectives for ecologists, pathologists and agricultural scientist*, Trends in Plant Science, 1 (1996), 33-38.

[27] A.R. Smith, *Plants, fractals and formal languages,* Computer Graphics, 18 (1984), 1-10.

[28] I. Turner, J. Belward and M. Oqielat, *Error Bounds for Least square Gradients Estimates*, SIAM Journal on Scientic Computing, 2008.

Accepted: 23.05.2017

PRODUCT-TYPE OPERATORS FROM AREA NEVANLINNA SPACES TO BLOCH-ORLICZ SPACES

Zhi-Jie Jiang
School of Science
Sichuan University of Science and Engineering
Zigong, Sichuan
643000
P. R. China
matjzj@126.com

Abstract. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. By constructing some more effective test functions in area Nevanlinna space, in this paper we characterize the boundedness and compactness of product-type operators $D^n M_u C_\varphi$, $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from area Nevanlinna spaces to Bloch-Orlicz spaces.

Keywords: Area Nevanlinna space, product-type operator, boundednedss, compactness.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u} f(z) = u(z) f(\varphi(z)), \ z \in \mathbb{D}.$$

If $u \equiv 1$, it is reduced to the composition operator, usually denoted by C_φ. While if $\varphi(z) = z$, it is reduced to the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_u C_\varphi$, it can be regarded as a product-type operator. It is a standard problem how to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator (see, e.g., [2, 4, 5, 18, 21, 23] and the references therein).

Let D be the differentiation operator on $H(\mathbb{D})$, that is

$$Df(z) = f'(z), \ z \in \mathbb{D}.$$

A systematic study of other product-type operators started with Stević et al. in [14, 16]. Before that there were a few papers in the topic, e.g., [6]. The next two product-type operators DC_φ and $C_\varphi D$, attracted some attention first (see, e.g., [17, 19, 27, 29] and the references therein). The publication of [16] attracted some attention in product-type operators involving integral-type ones (see, e.g.,

[10, 28, 30, 31, 34] and the references therein). Since that time there has been a great interest in various product-type operators. For example, the following six operators

$$M_uC_\varphi D,\ C_\varphi M_u D,\ M_u DC_\varphi,\ C_\varphi DM_u,\ DC_\varphi M_u,\ DM_uC_\varphi \tag{1}$$

were studied by Sharma in [25]. The product-type operators $W_{\varphi,u}D$ and $DW_{\varphi,u}$, which were considered by Jiang in [7, 8], are included in (1) as the first and sixth operators, respectively. For some other product-type operators, we refer the reader to [12, 13, 15, 20, 22, 24, 39] and the references therein.

The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^n f(z) = f^{(n)}(z),\ z \in \mathbb{D}.$$

Zhu in [38] introduced the so-called generalized weighted composition operator:

$$D^n_{\varphi,u} f(z) = u(z) f^{(n)}(\varphi(z)),\ z \in \mathbb{D}.$$

Since $D^n_{\varphi,u} = M_uC_\varphi D^n$, it is also a product-type operator.

The product-type operator $M_uC_\varphi D^n$ from area Nevanlinna space to Bloch-type and Zygmund spaces was studied by Yang et al. in [35, 36]. The weighted composition operator from weighted Bergman-Nevanlinna space to Zygmund and Bloch-type spaces was also studied in [11, 26]. It must be mentioned that in these studies, there is no need to construct more complex test functions in area Nevanlinna space or weighted Bergman-Nevanlinna space. But we find that the used test functions become invalid in the study of the following product-type operators

$$D^nM_uC_\varphi,\ D^nC_\varphi M_u,\ C_\varphi D^nM_u,\ M_uD^nC_\varphi,\ M_uC_\varphi D^n,\ C_\varphi M_uD^n. \tag{2}$$

By constructing some more suitable test functions in area Nevanlinna space, in this paper we characterize the boundedness and compactness of the operators in (2) from area Nevanlinna space to Bloch-Orlicz space. This paper can be regarded as a continuation of the investigation of concrete operators between these spaces.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$ and $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ the weighted Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$ and $p \geq 1$, the area Nevanlinna space $\mathcal{N}^p_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{N}^p_\alpha} = \int_{\mathbb{D}} \big[\log(1+|f(z)|)\big]^p dA_\alpha(z) < \infty.$$

From [3], we see that the area Nevanlinna space $\mathcal{N}^p_\alpha$ is a Fréchet space with the translation invariant metric given by $d(f,g) = \|f-g\|_{\mathcal{N}^p_\alpha}$. If $p=1$, it becomes the weighted Bergman-Nevanlinna space, usually denoted by $\mathcal{A}^\alpha_{\log}$ (see, [11, 26]).

Let Ψ be a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ was introduced in [21] by Ramos Fernández, is the class of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Ramos Fernández in [21] proved that $\mathcal{B}^\Psi$ is isometrically equal to μ_Ψ-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})}, \quad z \in \mathbb{D}.$$

Hence, $\mathcal{B}^\Psi$ is a Banach space with the norm given by $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + b_{\mu_\Psi}(f)$, where

$$b_{\mu_\Psi}(f) = \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

This space generalizes some other spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then the space $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}_\alpha$, where $\alpha = 1/p$. Also, if $\Psi(t) = t\log(1+t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [1]).

Let X be a topological vector space whose topology is given by the translation invariant metric d_X. A linear operator $L : X \to \mathcal{B}^\Psi$ is metrically bounded if there exists a positive constant K such that

$$\|Lf\|_{\mathcal{B}^\Psi} \le K d_X(f, 0)$$

for all $f \in X$. The operator $L : X \to \mathcal{B}^\Psi$ is metrically compact if it maps bounded sets into relatively compact sets.

In this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). Constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \le Cb$.

2. Auxiliary results

In [33], Stević used the Faà di Bruno's formula of the following version

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z))B_{n,k}(\varphi'(z), \dots, \varphi^{(n-k+1)}(z)), \tag{3}$$

where $B_{n,k}(x_1, \dots, x_{n-k+1})$ is the Bell polynomial. For $n \in \mathbb{N}$, the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), \dots, \varphi^{(n+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0} = 1$.

Now we present some useful information of the Bell polynomials from [9]. The Bell polynomials are associated with set partitions. To the partition $\{1\}$ we associate the monomial x_1; this is the only partition of the set $\{1\}$, and we define

$B_{1,1}(x_1) = x_1$. The set $\{1,2\}$ has the two partitions $\{1,2\}$ and $\{1\}$, $\{2\}$, the former with one block and the latter with two, and we associate to them the monomials x_2 and x_1^2, respectively. Then $B_{2,1}(x_1,x_2) = x_2$ and $B_{2,2}(x_1,x_2) = x_1^2$. There are five partitions of the set $\{1,2,3\}$. Three of these have two blocks, namely $\{1,2\}$, $\{3\}$ and $\{1,3\}$, $\{2\}$ and $\{1\}$, $\{2,3\}$; we associate the monomial x_1x_2 to each of these, and so $B_{3,2}(x_1,x_2) = 3x_1x_2$. The other Bell polynomials of order three are $B_{3,3}(x_1) = x_1^3$, corresponding to $\{1\}$, $\{2\}$, $\{3\}$; and $B_{3,1}(x_1,x_2,x_3) = x_3$, corresponding to $\{1,2,3\}$. In general,

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \frac{1}{k!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_i\geq 1}} \binom{n}{j_1,j_2,\dots,j_k} x_{j_1}\cdots x_{j_k}.$$

The sum is effectively over set partitions of $\{1,2,\dots,n\}$ with block sizes $j_1,\dots,j_k$, with the factor $1/k!$ correcting for the multiple counting inside the sum.

From (3) and the Leibnitz formula the next result follows.

Lemma 2.1. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\begin{aligned}&\big(u(z)f(\varphi(z))\big)^{(m)}\\ &= \sum_{k=0}^{m} f^{(k)}(\varphi(z)) \sum_{j=k}^{m} C_m^j u^{(m-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big).\end{aligned} \tag{4}$$

To find some useful test functions, we first introduce the following functions. For a fixed $w \in \mathbb{D}$ and $i \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we define the function

$$k_{w,i}(z) = \frac{(1-|w|^2)^{\frac{\alpha+2}{p}+i}}{(1-\overline{w}z)^{\frac{2(\alpha+2)}{p}+i}}, \quad z \in \mathbb{D}.$$

Then from Lemma 4.2.2 in [37], it follows that

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |k_{w,i}(z)|^p dA_\alpha(z) \lesssim 1. \tag{5}$$

By using the functions $k_{w,i}$, the following result provides some useful test functions.

Lemma 2.2. *Let $w \in \mathbb{D}$ and $m \in \mathbb{N}$. Then for each fixed $k \in \{0,1,\dots,m\}$, there exist two groups of constants $a_{0,k}$, $a_{1,k}$,..., $a_{m,k}$ and $b_{0,k}$, $b_{1,k}$,..., $b_{m,k}$, such that the function*

$$h_{w,k}(z) = \sum_{i=0}^{m} a_{i,k} k_{w,i}(z) \exp \sum_{i=0}^{m} b_{i,k} k_{w,i}(z) \tag{6}$$

satisfies

$$h_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\frac{\alpha+2}{p}+k}} \exp \frac{1}{(1-|w|^2)^{\frac{\alpha+2}{p}}} \quad \text{and} \quad h_{w,k}^{(j)}(w) = 0 \tag{7}$$

for each $j \in \{0, 1, \ldots, m\} \setminus \{k\}$. *Moreover,*

$$\sup_{w\in\mathbb{D}} \|h_{w,k}\|_{\mathcal{N}^p_\alpha} \lesssim 1.$$

Proof. First we prove that if the function $h_{w,k}$ has the expression (6), then $h_{w,k} \in \mathcal{N}^p_\alpha$ and $\sup_{w\in\mathbb{D}} \|h_{w,k}\|_{\mathcal{N}^p_\alpha} \lesssim 1$. Indeed, from the facts that

$$\log(1+xy) \le \log(1+x) + \log(1+y) \ \text{ and } \ \log(1+x) \le x,\ x,\ y > 0,$$

we have

$$\begin{aligned}
\log(1+|h_{w,k}(z)|) &= \log\Big(1 + \big|\sum_{i=0}^m a_{i,k}k_{w,i}(z)\big|\big|\exp\sum_{i=0}^m b_{i,k}k_{w,i}(z)\big|\Big)\\
&\le \big|\sum_{i=0}^m a_{i,k}k_{w,i}(z)\big| + \log\Big(1+\exp\big|\sum_{i=0}^m b_{i,k}k_{w,i}(z)\big|\Big)\\
&\le 1 + \sum_{i=0}^m (|a_{i,k}| + |b_{i,k}|)|k_{w,i}(z)|.
\end{aligned}$$

Then from this, (5) and a elementary inequality, we get

$$\begin{aligned}
\|h_{w,k}\|_{\mathcal{N}^p_\alpha} &\le \int_{\mathbb{D}} \Big(1 + \sum_{i=0}^m (|a_{i,k}| + |b_{i,k}|)|k_{w,i}(z)|\Big)^p dA_\alpha(z)\\
&\le (m+2)^p \Big(1 + \sum_{i=0}^m (|a_{i,k}| + |b_{i,k}|)^p \int_{\mathbb{D}} |k_{w,i}(z)|^p dA_\alpha(z)\Big)\\
&\le C.
\end{aligned}$$

From this, the desired result follows.

Write

$$f_{w,k}(z) := \sum_{i=0}^m a_{i,k}k_{w,i}(z) \ \text{ and } \ g_{w,k}(z) := \sum_{i=0}^m b_{i,k}k_{w,i}(z).$$

We first consider the case of $k = 0$. By some calculation, we obtain that the function $h_{w,0}$ satisfies (7) if and only if the functions $f_{w,0}$ and $g_{w,0}$ satisfy the following systems

$$\text{(8)} \qquad \begin{cases} f_{w,0}(w) = \frac{1}{(1-|w|^2)^a} \\ f'_{w,0}(w) = 0 \\ f''_{w,0}(w) = 0 \\ \quad \cdots\cdots \\ f^{(m)}_{w,0}(w) = 0 \end{cases}$$

and

$$\begin{cases} g_{w,0}(w) = \frac{1}{(1-|w|^2)^a} \\ g'_{w,0}(w) = 0 \\ g''_{w,0}(w) = 0 \\ \quad \cdots\cdots \\ g^{(m)}_{w,0}(w) = 0, \end{cases} \tag{9}$$

respectively, where $a := (\alpha + 2)/p$. According to the expressions of $f_{w,0}$ and $g_{w,0}$, by calculating $f^{(j)}_{w,0}(w)$ and $g^{(j)}_{w,0}(w)$ we see that $f_{w,0}$ and $g_{w,0}$ satisfy the systems (8) and (9), respectively, if and only if unknowns $a_{0,0}$, $a_{1,0}$, $\ldots$, $a_{m,0}$, $b_{0,0}$, $b_{1,0}$, $\ldots$, $b_{m,0}$, satisfy the following linear system of equations

$$\begin{cases} \sum\limits_{i=0}^{m} a_{i,0} = 1 \\ \sum\limits_{i=0}^{m} (2a+i) a_{i,0} = 0 \\ \sum\limits_{i=0}^{m} (2a+i)(2a+i+1) a_{i,0} = 0 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m} (2a+i+j-1) a_{i,0} = 0 \end{cases} \tag{10}$$

and

$$\begin{cases} \sum\limits_{i=0}^{m} b_{i,0} = 1 \\ \sum\limits_{i=0}^{m} (2a+i) b_{i,0} = 0 \\ \sum\limits_{i=0}^{m} (2a+i)(2a+i+1) b_{i,0} = 0 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m} (2a+i+j-1) b_{i,0} = 0, \end{cases} \tag{11}$$

respectively. Hence we only need to prove that (10) and (11) have nonzero solutions, respectively. Indeed, if we can show that the determinants of the systems (10) and (11) are nonzero, then (10) and (11) have nonzero solutions, respectively. By Lemma 3 in [33], the determinants of the systems (10) and (11) equal $\prod_{j=1}^{m} j!$, respectively, which is different from zero. This finishes the proof of the lemma for the case $k = 0$.

We next consider the case of $k \neq 0$. From (4) we have

$$\begin{aligned} h_{w,k}^{(k)}(z) &= \sum_{j=0}^{k}\sum_{i=j}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\ &= f_{w,k}^{(k)}(z) \exp g_{w,k}(z) \\ &\quad + \sum_{j=1}^{k}\sum_{i=j}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\ &\quad + \sum_{i=1}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z). \end{aligned} \tag{12}$$

From (12) we see that if $f_{w,k}^{(l)}(w) = 0$ for all $l < k$, then $h_{w,k}^{(l)}(w) = 0$. On the other hand, for all $s > k$, from (4) we have

$$\begin{aligned} h_{w,k}^{(s)}(z) &= \sum_{j=0}^{s}\sum_{i=j}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\ &= C_s^k f_{w,k}^{(k)}(z) \exp g_{w,k}(z) \sum_{j=0}^{s-k} B_{s-k,j}(g'_{w,k}(z), \dots, g_{w,k}^{(s-k-j+1)}(z)) \\ &\quad + \sum_{j=0}^{s-k}\sum_{i=j, i\neq s-k}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\ &\quad + \sum_{j=s-k+1}^{s}\sum_{i=j}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z), \dots, g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z). \end{aligned} \tag{13}$$

From (13), for each $s > k$ we see that if $g'_{w,k}(w) = 0$, $\dots$, $g_{w,k}^{(s-k+1)}(w) = 0$, $f_{w,k}(w) = 0$, $f'_{w,k}(w) = 0$, $\dots$, $f_{w,k}^{(k-1)}(w) = 0$, $f_{w,k}^{(k+1)} = 0$, $\dots$, $f_{w,k}^{(s)}(w) = 0$, then $h_{w,k}^{(s)}(w) = 0$.

Now letting $s = k+1, \dots, m$, and noticing condition (7), we see that if we can prove that there exist two groups of constants $a_{0,k}$, $a_{1,k}$, $\dots$, $a_{m,k}$ and $b_{0,k}$, $b_{1,k}$, $\dots$, $b_{m,k}$, such that the following systems hold

$$\begin{cases} \sum\limits_{i=0}^{m} a_{i,k} = 0, \quad \sum\limits_{i=0}^{m} (2a+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{m} (2a+i)(2a+i+1) a_{i,k} = 0 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{k} (2a+i+j-1) a_{i,k} = 1 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m} (2a+i+j-1) a_{i,k} = 0 \end{cases} \tag{14}$$

and

$$
\begin{cases}
\sum\limits_{i=0}^{m} b_{i,k} = 1 \\
\sum\limits_{i=0}^{m} (2a+i) b_{i,k} = 0 \\
\sum\limits_{i=0}^{m} (2a+i)(2a+i+1) b_{i,k} = 0 \\
\qquad\qquad \cdots\cdots \\
\sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m-k+1} (2a+i+j-1) b_{i,k} = 0,
\end{cases} \tag{15}
$$

then this finishes the proof of the lemma for the case of $k \neq 0$. From Lemma 3 in [33] and a calculation, we get that the determinant of the system (14) equals $(-1)^{k-1} \prod_{j=1}^{m} j!$, which is different from zero. By this, there must exist $a_{0,k}$, $a_{1,k}$, …, $a_{m,k}$ such that the system (14) holds. Since the number of equations in the system (15) is less than the number of variables, there must exist $b_{0,k}$, $b_{1,k}$, …, $b_{m,k}$, such that the system (15) holds. □

To characterize the compactness, we need the following result, which is proved in a standard way (see [23]). So the proof is omitted.

Lemma 2.3. *Let $T \in \{D^n M_u C_\varphi, D^n C_\varphi M_u, C_\varphi D^n M_u, M_u D^n C_\varphi, M_u C_\varphi D^n, C_\varphi M_u D^n\}$. Then the bounded operator $T : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}$ in $\mathcal{N}_\alpha^p$ such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that*

$$\lim_{j\to\infty} \|T f_j\|_{\mathcal{B}^\Psi} = 0.$$

We need also the following estimate for derivative of functions in area Nevanlinna spaces. We refer the reader to [36] for a complete proof.

Lemma 2.4. *For a fixed $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists a positive constant $C_k = C(\alpha, p, k)$ independent of $f \in \mathcal{N}_\alpha^p$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{1}{(1-|z|^2)^k} \exp \frac{C_k \|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}.$$

3. Boundedness and compactness

We first characterize the boundedness and compactness of $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent*:

(*i*) *The operator $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is bounded.*

(*ii*) *The operator* $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \dots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$I_k = \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_k(z) < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) I_k(z)}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$I_k(z) = \Big| \sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|.$$

Proof. (*i*) $\Rightarrow$ (*iii*). Let $h_k(z) = z^k \in \mathcal{N}^p_\alpha$, $k = 0, 1, \dots, n+1$. Applying the operator $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ to the function h_0, we have

$$\begin{aligned} b_{\mu_\Psi}(D^n M_u C_\varphi h_0) &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| (D^n M_u C_\varphi h_0)'(z) \Big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=0}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \dots, \varphi^{(j+1)}(z)) \Big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_0(z) = I_0. \end{aligned} \tag{16}$$

Since the operator $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ is bounded, we have

$$b_{\mu_\Psi}(D^n M_u C_\varphi h_0) \le \|D^n M_u C_\varphi h_0\| \le C \|D^n M_u C_\varphi\|. \tag{17}$$

From (16) and (17), we obtain that $I_0 < \infty$.

Assume now that we have proved the following inequalities

$$I_l = \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_l(z) \le C \|D^n M_u C_\varphi\| \tag{18}$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \le n+1$. Applying Lemma 2.1 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned} \big(D^n M_u C_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C^i_{n+1} u^{(n+1-i)}(z) \cdot \\ &\quad \cdot B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)) \\ &= \sum_{j=0}^{k} k \cdots (k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C^i_{n+1} u^{(n+1-i)}(z) \cdot \\ &\quad \cdot B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)). \end{aligned} \tag{19}$$

From (19), using the boundedness of $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$, the boundedness of $\varphi(z)$ and the triangle inequality, noticing that the coefficient at

$$\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))$$

is independent of z, finally using hypothesis (18) we easily obtain

$$I_k = \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_k(z) \le C\|D^n M_u C_\varphi\|. \tag{20}$$

By induction we get that (20) holds for each $k \in \{0, 1, \dots, n+1\}$.

Let $w \in \mathbb{D}$ and $c > 0$. Then for a fixed $k \in \{0, 1, \dots, n+1\}$, by Lemma 2.2 there exists a function $h_{\varphi(w),k} \in \mathcal{N}^p_\alpha$ such that

$$h^{(k)}_{\varphi(w),k}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \quad \text{and} \tag{21}$$
$$h^{(j)}_{\varphi(w),k}(\varphi(w)) = 0$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. Hence, from the boundedness of $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} \mu_\Psi(w)\big|(D^n M_u C_\varphi h_{\varphi(w)})'(w)\big| &= \frac{\mu_\Psi(w)|\varphi(w)|^k I_k(w)}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \\ &\le \|D^n M_u C_\varphi h_{\varphi(w)}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{22}$$

Then from (22) we get

$$\frac{\mu_\Psi(w)|\varphi(w)|^k I_k(w)}{(1-|\varphi(w)|^2)^k} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \le C(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}. \tag{23}$$

Taking the limit in (23) as $|\varphi(w)| \to 1$ gives the following

$$\lim_{|\varphi(w)|\to 1} \frac{\mu_\Psi(w) I_k(w)}{(1-|\varphi(w)|^2)^k} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

This shows that the statement (i) implies (iii).

$(iii) \Rightarrow (ii)$. In order to prove that the operator $D^n M_u C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 2.3 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{N}^p_\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{N}^p_\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

Notice that the second condition in (iii) holds for all $c > 0$. Hence, for arbitrary $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$\frac{\mu_\Psi(z) I_k(z)}{(1-|\varphi(z)|^2)^k} \exp\frac{C_k M}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} < \varepsilon, \tag{24}$$

where C_k is the constant in Lemma 2.4. For such chosen ε and η, by using (24) and Lemma 2.4, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)\cdot \\
&\cdot B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \qquad (25)\\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| I_k(z) \\
&\le \Big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\Big)\mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| I_k(z) \\
&\le (n+2)\varepsilon + \sum_{k=0}^{n+1} I_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big|.
\end{aligned}$$

Since $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$ implies that for each $k\in\mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$, from (25) we get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0.$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (26)$$

Consequently, from (25) and (26) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0. \qquad (27)$$

Hence, from Lemma 2.3 we see that the operator $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact.

$(ii) \Rightarrow (i)$. This implication is obvious. This finishes the proof of the theorem. □

Since $D^n C_\varphi M_u = D^n M_{u\circ\varphi} C_\varphi$, by Faà di Bruno's formula and Theorem 3.1 we obtain the characterizations of the boundedness and compactness for the operator $D^n C_\varphi M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ in the following result.

Corollary 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent*:

(*i*) *The operator $D^n C_\varphi M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is bounded.*

(*ii*) *The operator $D^n C_\varphi M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \ldots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z) J_k(z) < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) J_k(z)}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$J_k(z) = \Big| \sum_{j=k}^{n+1} \sum_{i=0}^{n+1-j} C_{n+1}^j u^{(i)}(\varphi(z)) B_{n+1-j,i}\big(\varphi'(z), \ldots, \varphi^{(n+2-j-i)}(z)\big) \cdot B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big|.$$

Since

$$(C_\varphi D^n M_u f)'(z) = \sum_{k=0}^{n+1} C_{n+1}^k u^{(n+1-k)}(\varphi(z)) \varphi'(z) f^{(k)}(\varphi(z)),$$

we have the following result, whose proof is similar to that of Theorem 3.1. So it is omitted.

Theorem 3.2. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $C_\varphi D^n M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $C_\varphi D^n M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \ldots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z) |u^{(n+1-k)}(\varphi(z))| |\varphi'(z)| < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) |u^{(n+1-k)}(\varphi(z))| |\varphi'(z)|}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Next we characterize the boundedness and compactness of $M_u D^n C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$.

Theorem 3.3. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $M_u D^n C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $M_u D^n C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \dots, n\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1} < \infty,$$
$$\sup_{z\in\mathbb{D}} \mu_\Psi(z) L_k(z) < \infty,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) L_k(z)}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$L_k(z) = \big|u'(z) B_{n,k}(\varphi'(z), \dots, \varphi^{(n-k+1)}(z)) + u(z) B_{n+1,k}(\varphi'(z), \dots, \varphi^{(n-k+2)}(z))\big|,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Proof. By some calculation, we have

$$\begin{aligned}(M_u D^n C_\varphi f)'(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) \Big(& u'(z) B_{n,k}\big(\varphi'(z), \dots, \varphi^{(n-k+1)}(z)\big) \\ & + u(z) B_{n+1,k}\big(\varphi'(z), \dots, \varphi^{(n-k+2)}(z)\big)\Big) \\ & + u(z)(\varphi'(z))^{n+1} f^{(n+1)}(\varphi(z)).\end{aligned}$$

By this formula, the proof can be given similar to that of Theorem 3.1. So we omit it. □

Since $(M_u C_\varphi D^n f)'(z) = u'(z) f^{(n)}(\varphi(z)) + u(z)\varphi'(z) f^{(n+1)}(\varphi(z))$, we have

Theorem 3.4. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $M_u C_\varphi D^n : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $M_u C_\varphi D^n : \mathcal{N}^p_\alpha \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$, *the functions* u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u'(z)| < \infty,$$
$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)| < \infty,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Noticing that $C_\varphi M_u D^n = M_{u\circ\varphi} C_\varphi D^n$, by Theorem 3.4 we have

Corollary 3.2. *Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent*:

(*i*) *The operator $C_\varphi M_u D^n : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is bounded.*

(*ii*) *The operator $C_\varphi M_u D^n : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact.*

(*iii*) *For all $c > 0$, the functions u and φ satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u'(\varphi(z))||\varphi'(z)| < \infty,$$
$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(\varphi(z))||\varphi'(z)| < \infty,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u'(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Acknowledgments. The author would like to thank the anonymous referee very much for providing valuable suggestions for the improvement of this paper. This work was supported by the Open Program of Key Laboratory of Mathematics and Interdiscriplinary Science of Guangdong Higher Education Institutes, Guangzhou University, the Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2016QZJ01) and the Cultivation Project of Sichuan University of Science and Engineering (No.2015PY04).

References

[1] K. Attele, *Toeplitz and Hankel operators on Bergman spaces*, Hokkaido Math. J., 21 (1992), 279-293.

[2] F. Colonna, S. Li, *Weighted composition operators from the minimal Möbius invariant space into the Bloch space*, Mediter. J. Math., 10 (1) (2013), 395-409.

[3] B. Choe, H. Koo, W. Smith, *Carleson measure for the area Nevalinna spaces and applications*, J. Anal. Math., 104 (2008), 207-233.

[4] C. C. Cowen, B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Roton, 1995.

[5] K. Esmaeili, M. Lindström, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integral Equ. Oper. Theory., 75 (2013), 473-490.

[6] R. A. Hibschweiler, N. Portnoy, *Composition followed by differentiation between Bergman and Hardy spaces*, Rocky Mountain J. Math., 35 (3) (2005), 843-855.

[7] Z. J. Jiang, *On a class of operators from weighted Bergman spaces to some spaces of analytic functions*, Taiwan. J. Math., 15 (5) (2011), 2095-2121.

[8] Z. J. Jiang, *On a product-type operator from weighted Bergman-Orlicz space to some weighted type spaces*, Appl. Math. Comput., 256 (2015), 37-51.

[9] W. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly., 109 (3) (2002), 217-234.

[10] S. Krantz, S. Stević, *On the iterated logarithmic Bloch space on the unit ball*, *Nonlinear Anal. TMA* **71** (2009), 1772-1795.

[11] P. Kumar, S. D. Sharma, Weighted composition operators from weighted Bergman Nevanlinna spaces to Zygmund spaces, Int. J. Mod. Math. Sci., 3 (1) (2012), 31-54.

[12] Y. Liu, Y. Yu, *Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball*, J. Math. Anal. Appl., 423(1) (2015), 76-93.

[13] H. Li, Z. Guo, *On a product-type operator from Zygmund-type spaces to Bloch-Orlicz spaces,* J. Inequal. Appl., Vol. 2015, Article no. 132, (2015), 18 pages.

[14] S. Li, S. Stević, *Composition followed by differentiation between Bloch type spaces*, J. Comput. Anal. Appl., 9 (2) (2007), 195-205.

[15] S. Li, S. Stević, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl., 338 (2008), 1282-1295.

[16] S. Li, S. Stević, *Products of composition and integral type operators from H^∞ to the Bloch space*, Complex Var. Elliptic Equ., 53 (5) (2008), 463-474.

[17] S. Li, S. Stević, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput., 217 (2010), 3144-3154.

[18] K. Madigan, A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc., 347 (1995), 2679-2687.

[19] S. Ohno, *Products of composition and differentiation on Bloch spaces*, Bull. Korean Math. Soc., 46 (6) (2009), 1135-1140.

[20] C. Pan, *Generalized composition operators from μ-Bloch spaces into mixed norm spaces*, Ars Combin., 102 (2011), 263-268

[21] J. C. Ramos Fernández, *Composition operators on Bloch-Orlicz type spaces*, Appl. Math. Comput., 217 (2010), 3392-3402.

[22] Y. Ren, *On an integral-type operator from mixed norm spaces to Zygmund-type spaces*, Bulletin of Mathematical Analysis and Applications, 4 (3) (2012), 71-77.

[23] H. J. Schwartz, *Composition operators on H^p*, Thesis, University of Toledo, 1969.

[24] B. Sehba, S. Stević, *On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces*, Appl. Math. Comput., 233 (2014), 565-581.

[25] A. K. Sharma, *Products of composition multiplication and differentiation between Bergman and Bloch type spaces*, Turkish. J. Math., 35 (2011), 275-291.

[26] A. K. Sharma, Z. Abbas, *Weighted composition operators between weighted Bergman-Nevanlinna and Bloch-type spaces*, Appl. Math. Sci., 41 (4) (2010), 2039-2048.

[27] S. Stević, *Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞_μ*, Appl. Math. Comput., 207 (2009), 225-229.

[28] S. Stević, *On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces*, Nonlinear Anal. TMA, 71 (2009), 6323-6342.

[29] S. Stević, *Products of composition and differentiation operators on the weighted Bergman space*, Bull. Belg. Math. Soc., 16 (2009), 623-635.

[30] S. Stević, *Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces*, Siberian Math. J., 50 (4) (2009), 726-736.

[31] S. Stević, *On an integral operator from the Zygmund space to the Bloch-type space on the unit ball*, Glasg. J. Math., 51 (2009), 275-287.

[32] S. Stević, *On an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball*, Appl. Math. Comput., 215 (2010), 3817-3823.

[33] S. Stević, *Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk*, Appl. Math. Comput., 216 (2010), 3634-3641.

[34] S. Stević, S. Ueki, *Integral-type operators acting between weighted-type spaces on the unit ball*, Appl. Math. Comput., 215 (2009), 2464-2471.

[35] W. Yang, W. Yan, *Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces*, Bull. Korean Math. Soc., 48 (6) (2011), 1195-1205.

[36] W. Yang, X. Zhu, *Generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces*, Taiwan. J. Math., 16 (3) (2012), 869-883.

[37] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2005.

[38] X. Zhu, *Products of differentiation, composition and multiplication operator from Bergman type spaces to Bers spaces*, Integral Transforms Spec. Funct., 18 (2007), 223-231.

[39] X. Zhu, *Generalized weighted composition operators from Bloch spaces into Bers-type spaces*, Filomat., 26 (2012), 1163-1169.

Accepted: 1.06.2017

COMPARISON OF SVM ALGORITHM AND BP ALGORITHM: STUDY ON THE EVALUATION INDEX SYSTEM OF SCIENTIFIC RESEARCH PERFORMANCE OF VOCATIONAL COLLEGES

Jiayin Feng*
Dongyan Jia
Li Cui
Jing Cao
Zhuo Lin
Min Zhang
Hebei Normal University of Science & Technology
Qinhuangdao Haigang District
Heibei, 066000
China
feng_ada2001@163.com

Abstract. As the education in China develops rapidly, the scientific research of higher vocational education has gradually drawn extensive attentions. In order to construct a reasonable evaluation model of scientific research performance to further enhance the research enthusiasm of teachers, this study constructed a model based on relevant theories of the support vector machine (SVM) algorithm and the back propagation (BP) algorithm. In addition, the simulation of the model was performed and the accuracy rate and errors of these two algorithms were compared and analyzed. Then the most appropriate algorithm was applied to the evaluation index system. The simulation results showed that, simplified data of scientific research evaluation could be applied as the input data of the SVM algorithm to accurately and effectively construct an evaluation index system of scientific research performances of vocational colleges. Thus a more reasonable and accurate evaluation system was constructed.

Keywords: SVM algorithm, BP algorithm, vocational colleges, scientific research, performance evaluation.

1. Introduction

The classification of industries refines day by day. The performance evaluation is not only the recognition to the achievements of bottom workers, but also a reflection of top managers' ability. Therefore, patents for invention of the comprehensive performance evaluation system of enterprises occur [1]. However, the corporate image can also be affected by safety problems, thus patents of the safety performance evaluation system of manufacturing enterprises come

∗. Corresponding author

into being [2]. Medical institutions also have the same service performance evaluations [3].

Under the impetus of China's education, the higher vocational education also develops rapidly. Therefore, the evaluation of the performance of teachers in vocational colleges becomes very important [4]. As the reflection of the teaching level and educational quality of vocational colleges, the performance of teachers has become the competitive force of each college [5]. Current vocational colleges are transformed from secondary technical schools, thus their management systems and teaching models are still in old modes. Teachers are the main force of scientific researches, thus the correct evaluation of teachers' capacity for scientific research becomes the important issue of each vocational college [6]. Teachers with strong capacity for scientific researches are more likely to participate in scientific and research activities, thus the scientific research ability of colleges can be improved [7]. As the discipline classification increases, it has been unreasonable to evaluate teachers of different disciplines using the same evaluation standard [8]. Thus evaluation systems that are more accurate and fair are needed. Current evaluation methods include Delphi method, analytic hierarchy process (AHP), fuzzy comprehensive evaluation, etc. [9]. Heping Y et al. [10] formulated a new evaluation index system for scientific research achievements in universities by introducing value engineering and then analyzed the system using AHP method and multi-level fuzzy comprehensive evaluation method to obtain comprehensive and objective evaluation conclusions. Xiaoping W et al. [11] designed the framework of university teacher performance assessment indicators, established a hierarchical model of comprehensive quality evaluation using AHP to figure out the weight of each indicator, set up a fuzzy comprehensive evaluation model and provides the emamples to suggest the feasibility and effectiveness of the method. Combining scientificity with systematicness, hierarchy with independence and maneuverability with easy quantization, this study established evaluation models of scientific research performance based on theories of the support vector machine (SVM) algorithm and the back propagation (BP) algorithm respectively. The simulation was performed after obtaining model results. The comparison showed that the SVM algorithm could establish the evaluation index system of scientific research performance more accurately and reasonbly in vocational colleges.

2. Relevant theories

2.1 SVM and particle swarm optimization algorithms

The SVM keeps the balance of a small range of data samples in model training complexity and learning ability based on Vapnik-Chervonenkis (VC) dimension and theories of structual risk minimization [12]. The SVM can be linearly separable, linearly inseparable and multi-classification [13]. Its fitting function is

$$(g)_n = f_n(m) = \{\omega_n, \varphi_n(m)\} + \alpha_n. \tag{1}$$

In equation (1), ω_n shows the complexity of $f_n(m)$ and α_n is a constant. If

$$(f)_k - \{\omega_n, \varphi_n(n)\} - \alpha_n = (\xi_n)_k, (k = 1, 2, 3, \ldots, d), \tag{2}$$

the minimum cost functional is

$$P_n = 0.5\{\omega_n, \omega_n\} + 0.5W \sum_{k=1}^{d} [(\xi_n)_k]^2 \tag{3}$$

where W is the penalty factor and ξ refers to the slack variable. The constructed optimizing function is:

$$\begin{aligned} K_n(\omega_n, \alpha_n, \xi_n, \mu_n) = {} & 0.5\{\omega_n, \omega_n\} + 0.5W \sum_{k=1}^{d} [(\xi_n)_k]^2 \\ & + \sum_{k=1}^{d} (\mu_n)_k [\{\omega_n \varphi_n(t_k)\} + \alpha_n + (\xi_n) - (IMFn)_k]. \end{aligned} \tag{4}$$

In equation (4), $(\mu_n)_k$ refers to the multiplier and final equation (1) is

$$f_n(m) = \sum_{k=1}^{d} (\mu_n)_k K_n(m, m_k) + \alpha_n. \tag{5}$$

If the $K_n(m, m_k)$ in equation (5) is the SVM function, then

$$K_n(m, m_k) = \exp[\frac{\|m - m_k\|^2}{2\vartheta^2 n}], \tag{6}$$

which is the most common kernel function. In equation (6), ϑ_n is the standard deviation of f_n in the nth group.

In particle swarm optimization (PSO), each individual is a particle and each particle is a potential solution [14]. The initial particle is expressed by position (Q), speed (V) and adaptive value (F). The adaptive value determines the performance of the particle. There is an optimal particle position in all particles, which is called the population extremum F_{best}. Meanwhile, there is an individual extremum G_{best} at the optimal position of individual fitness value. Suppose the X-dimensional solution space is composed of h populations composed of particles, $Y = (Y1, Y2, \ldots, Yh)$. The nth particle represents a vector quantity as well as the search location in the X-dimensional solution space. After obtaining the fitness value, the speed of the nth particle is solved: $V_n = (V_{n1}, V_{n2}, \ldots, V_{nX})^T$. The individual extremum is $G_n = (G_{n1}, G_{n2}, \ldots, G_{nX})^T$; the population extremum is $F = (G_{\sigma 1}, G_{\sigma 2}, \ldots, G_{\sigma X})^T$. Through the iteration, particles can adjust and update by themselves according to the speed and the position of individual extremum and population extremum. The result is $V_{nx}^{i+1} = \tau V_{nx}^i + l_1 r_1 (D_{nx}^i - X_{nd}^i) + l_2 r_2 (D_{\alpha x}^i - X_{\sigma x}^i)$, $V_{nd}^{i+1} = Y_{nx}^i + V_{nd}^{i+1}$, in which i refers to the times of iteration and τ is the inertia weight; $x = 1, 2, \ldots, X$; $n = 1, 2, \ldots, h$; V_{nx} is the particle speed; l_1 and l_2 are accelerated factors; r_1 and r_2 are random values in $[0, 1]$.

2.2 BP algorithm

BP algorithm is used to estimate errors of the leading layer based on errors of the output layer; then based on this, errors of the upper layer were estimated, until errors of all layers were estimated [15]. The specific algorithm is as follows. Random values are given to the weight matrix and the threshold vector and the error is designed as 0; the accuracy value is E and the maximum times of learning is U. Then sample data are input for calculation of the output results of each layer. The actual network output and neuron partial derivative are calculated. Then based on this, the neuron partial derivative of the hidden layer is calculated. The overall errors of network and error signals of each layer are calculated, and the weight of each layer is adjusted. Finally, it is checked whether all samples are tested and whether the overall errors of network reach the requirements.

3. Simulation experiment

3.1 Data collection

3.1.1 Index selection for the evaluation system of scientific research

As shown in table 1, the system is composed of 5 level-I indexes and 28 level-II indexes. Due to the lack of unified assessment criteria, this study took the engineering higher vocational schools as examples to establish a system for evaluation of scientific research performance of teachers.

3.1.2 Data acquirement

Data were collected from representative scientific research data of teachers. Taking the performance scoring of scientific researches formulated by vocational colleges as the standard, we calculated the scientific research activities of teachers. Teachers' capacity for scientific research was expressed as Z; one to four points was used to represent the capability, 1 point for weak capacity, 2 for moderate capacity, 3 for good capacity and 4 for strong capacity. As shown in table 2, $P1 \sim P11$, $P12 \sim P18$, $P19 \sim P23$ and $P24 \sim P28$ were data of teachers with strong capacity, good capacity, moderate capacity and weak capacity respectively.

The missing data were recorded as 0 to reduce experimental errors.

3.1.3 Pretreatment of experimental data

The data in table 2 were pretreated and the equation was:

$$a_{n\gamma} = \frac{a_{n\gamma} - \operatorname{Min} a_{\kappa\gamma}}{\operatorname{Max} a_{\kappa\gamma} - \operatorname{Min} a_{\kappa\gamma}}, \quad n = 1, 2, 3, \ldots, t, \quad \gamma = 1, 2, 3, \ldots, \rho, 1 \leq \kappa \leq t.$$

In the equation, $\operatorname{Min} a_{\kappa\gamma}$ refers to the minimum value of the column where the attribute of variate a_γ locates and $\operatorname{Max} a_{\kappa\gamma} - \operatorname{Min} a_{\kappa\gamma}$ refers to the range

Target layer	Grade-I index	Grade-II index
Indexes of teachers' capacity for scientific research	Scientific research project	State-level major project study L1
		State-level key project study L2
		State-level general project study L3
		Provincial major project study L4
		Provincial key project study L5
		Provincial general project study L6
		Bureau-level project study L7
		Cross-sectional project study L8
		School-level project study L9
	Research paper	Retrieval paper L10
		A-level journal article L11
		B1-level journal article L12
		B2-level journal article L13
		C2-level Chinese core journal L14
		C3-level foreign language periodical and international conference L15
		C4-level published papers and national conference L16
	Work	Monograph national press L17
		Monograph other levels of press L18
		Non-monograph national press L19
		Non-monograph other levels of press L20
	Award grade	National award L21
		Provincial award L22
		Bureau-level award L23
		Social force establishment L24
	Patent grade	International patent for invention L25
		National patent for invention L26
		Patent for utility models L27
		Design patent L28

Table 1. Index system of teachers' capacity for scientific research

value of variate a_γ. However, these two values might be 0, thus the index was eliminated to guarantee the accuracy of the experiment. L1∼L11 were selected as the experimental sample data and the obtained sample data are as shown in table 3.

3.2 Construction of the evaluation model based on SVM algorithm

In order to achieve the balance between the tolerance degree and the complexity of sample classification errors, this study adopted penalty factor W to define the degree of misclassification of penalty samples. The model included the training function and the forecasting function. The scientific research evaluation model after training was SVMtrain (labels, research, $-w8g0.02t2$), in which the labels represented teachers' capacity for scientific research (1∼4 points); research was the sample data; $-w, g$ and t were the penalty factor, kernel parameter and

	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11
P1	569.7	64	312.4	0	422	13.9	0	1114.2	0	0	0
P2	260.5	31	124.6	0	95	95.2	0	1240.3	0	0	0
P3	0	1130.5	144.7	0	991.3	1.4	0	27	0	135.7	28
P4	0	597	117.1	0	1022.9	10.8	0	4822.2	0	0	55.4
P5	0	239	8	22	285	65	0	4822.2	0	0	0
P6	0	421.8	165.3	0	10	10.8	0	12.4	0	9.8	8.2
P7	0	197.3	264.7	0	99	4	0	1.3	0	97	50.5
P8	0	237	51.9	0	252.1	165	5	287	0	0	0
P9	0	138.1	474	0	187	117.5	3	52.4	0	0	0
P10	211	23.8	124	0	50	56.8	0	966.3	0	0	0
P11	0	0	142.6	23.9	157.3	96.2	0	335.5	0	1042.1	886
P12	0	0	37.2	0	0	44.7	0	224.8	0	0	0
P13	0	0	0	0	0	0	34	346.7	0	0	0
P14	0	0	33	0	9.4	30.3	0	502.6	0	0	0
P15	0	0	82.8	0	0	0	0	448.3	3.7	0	0
P16	0	0	0	0	0	200	0	148	0	0	0
P17	0	0	63.9	0	0	51.7	0	173	0	0	0
P18	0	0	63.9	0	0	52.9	0	173	0	0	0
P19	0	0	0	0	0	1.4	0	86.1	8.7	0	0
P20	0	0	0	0	0	9.3	0	56.7	3.3	0	0
P21	0	0	0	0	0	0	5	76	0	0	0

Table 2. Partial original data of the evaluation index system of teachers scientific research

kernel function respectively. The range of the penalty factor and the kernel parameter was $[2^{-10}, 2^{10}]$.

The classification model of the predicted function was (New label, accuracy)=SVMpredict(labels, research, model), in which "New label" refers to the new capacity score, "accuracy" was the accurate rate of classification; "labels" represented the forecasting of the capacity for scientific research, "research" refers to the predicted sample data and "model" refers to the scientific research evaluation model after training.

	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11
P1	1	0.054	0.651	0	0.411	0.051	0	0.187	0	0	0
P2	0.468	0.021	0.259	0	0.089	0.476	0	0.255	0	0	0
P3	0	1	0.308	0	0.943	0.008	0	0.006	0	0.122	0.031
P4	0	0.512	0.241	0	1	0.044	0	1	0	0	0.057
P5	0	0.216	0.018	0.847	0.268	0.328	0	1	0	0	0
P6	0	0.364	0.374	0	0.033	0.059	0	0.007	0	0.012	0.009
P7	0	0.221	0.571	0	0.092	0.018	0	0.011	0	0.108	0.048
P8	0	0.221	0.108	0	0.231	0.843	0.122	0.621	0	0	0
P9	0	0.124	1	0	0.179	0.641	0.009	0.017	0	0	0
P10	0.372	0.031	0.207	0	0.037	0.281	0	0.202	0	0	0
P11	0	0	0.347	1	0.148	0.493	0	0.071	0	1	1
P12	0	0	0.084	0	0	0.238	0	0.046	0	0	0
P13	0	0	0	0	0	0	1	0.071	0	0	0
P14	0	0	0.074	0	0.079	0.151	0	0.112	0	0	0
P15	0	0	0.168	0	0	0	0	0.120	0.060	0	0
P16	0	0	0	0	0	1	0	0.027	0	0	0
P17	0	0	0.134	0	0	0.235	0	0.029	0	0	0
P18	0	0	0.134	0	0	0.235	0	0.029	0	0	0
P	0	0	0	0	0	0.00	0	0.0	1	0	0

Table 3. Experimental sample data after pretreatment

3.3 Construction of the evaluation model based on BP algorithm

During the training process, the corresponding weight of the neuron was constantly adjusted to narrow the gap between the forecasting output results and the actual results. The number of samples, input nodes and output layers was 11, 11 and 4 respectively. The learning rate was set as 0.01, the maximum training times was as 200, and the training accuracy was $E-3$. The model was net=new ff(research, labels, 2); net=train(net, research, labels); sim=sim(net, labels), in which research represented sample data and labels represented the network predicting capacity for scientific research.

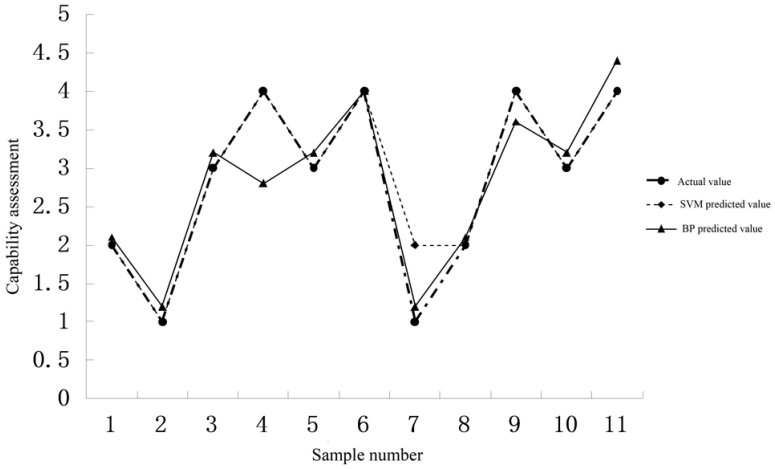


Figure 1. Comparison of the predicted results and the actual results.

4. Simulation results

Figure 1 shows the comparison between predicted values obtained from the above two models and the actual values. It can be noted from the figure that most of BP predicted values were basically similar to the actual values, but some of them had large error, and the error value of the maximum capability assessment reached 1.5 approximately; SVM predicted values had higher accuracy and smaller error compared to the predicted values; except sample 7, the errors of other samples could be ignored. Therefore, the SVM algorithm could evaluate the scientific research performance of teachers more accurately.

5. Improvement of the evaluation system based on SVM algorithm

The SVM model was optimized and improved by combining the SVM algorithm with PSO. It was obtained by the proof method that w was 8 and g was 0.25. In iterative solution, w was 11.5 and g was 1.5 under the optimal accuracy. The obtained predicted values are shown in figure 2.

Compared to the SVM algorithm, the accuracy of the PSO optimized SVM algorithm reached 100%, indicating the penalty factor and kernel parameter of the PSO optimized SVP algorithm ensured the accuracy of prediction, avoid errors caused by individual samples, and effectively improved the accuracy of predicted results.

The optimized SVM algorithm was applied to the computer system, and the detection results were the same as the actual results. Its application in the evaluation indicator system for the research performance in higher vocational colleges had a 100% accuracy.

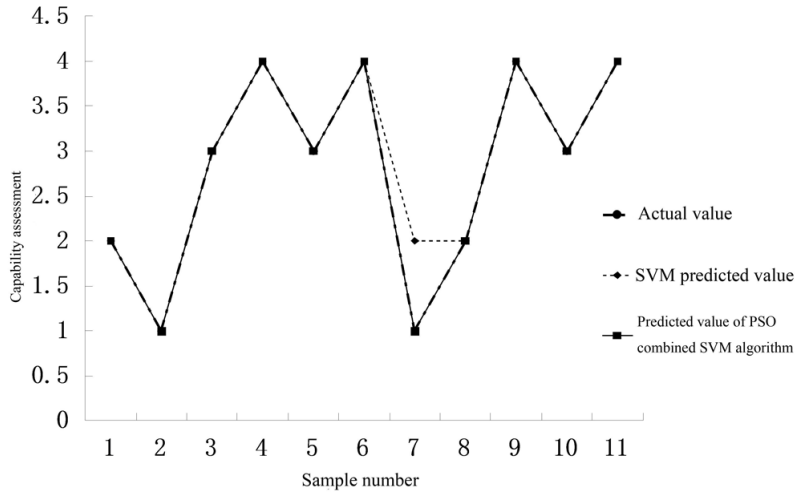


Figure 2. Comparison between predicted values of the PSO optimized SVM algorithm, predicted values of the SVM algorithm and actual values.

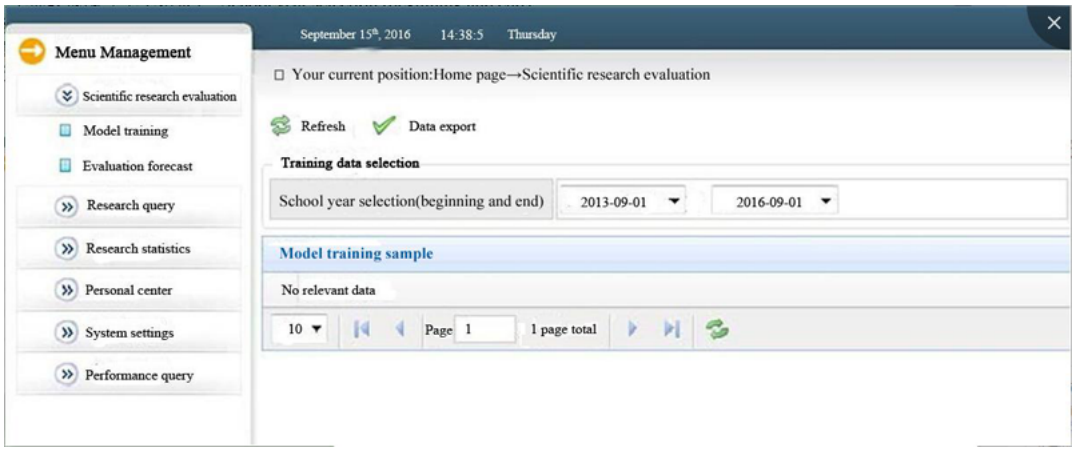
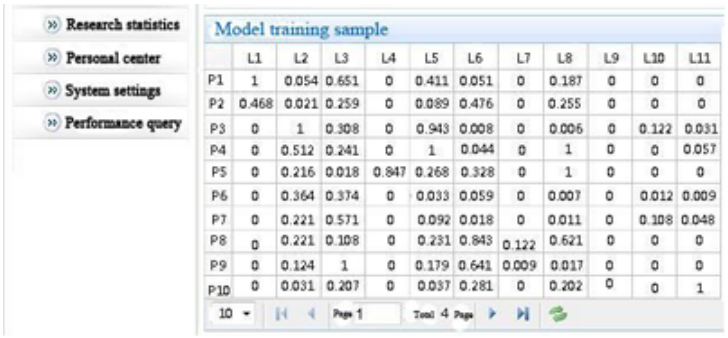


Figure 3. Computer interface.

6. Discussion and conclusion

Currently, the evaluation of the scientific research performance of vocational colleges is still at the starting stage. There are plenty of theoretical researches concerning core factors, evaluation indexes and methods that can affect the evaluation of the scientific research performance; however, there are still some deficiencies. Compared with universities, the vocational colleges still need to further improve their evaluation on the scientific research performance. Current evaluation methods in vocational colleges focus more on quantized evaluations instead of qualitative evaluations. This study compared the BP algorithm and



	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11
P1	1	0.054	0.651	0	0.411	0.051	0	0.187	0	0	0
P2	0.468	0.021	0.259	0	0.089	0.476	0	0.255	0	0	0
P3	0	1	0.308	0	0.943	0.008	0	0.006	0	0.122	0.031
P4	0	0.512	0.241	0	1	0.044	0	1	0	0	0.057
P5	0	0.216	0.018	0.847	0.268	0.328	0	1	0	0	0
P6	0	0.364	0.374	0	0.033	0.059	0	0.007	0	0.012	0.009
P7	0	0.221	0.571	0	0.092	0.018	0	0.011	0	0.108	0.048
P8	0	0.221	0.108	0	0.231	0.843	0.122	0.621	0	0	0
P9	0	0.124	1	0	0.179	0.641	0.009	0.017	0	0	0
P10	0	0.031	0.207	0	0.037	0.281	0	0.202	0	0	1

Figure 4. Pretreatment data.

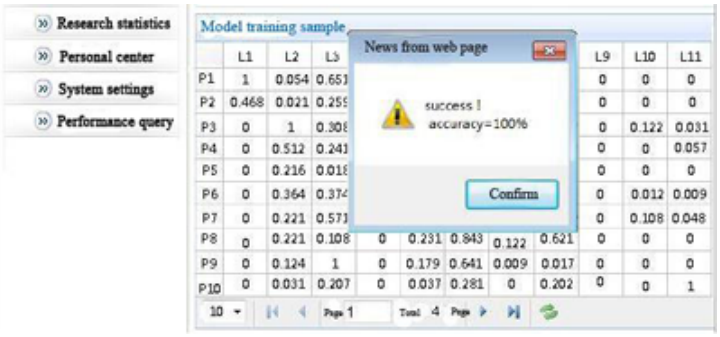


Figure 5. Assessment after the improvement of SVM algorithm.

the SVM algorithm, established evaluation models, and made simulation. The results demonstrated that the SVM algorithm was more accurately in processing small samples and multiple indexes than the BP algorihtm. Currently, the BP algorihtm develops well in related fields, but there were some deficiencies. The SVM algorithm tactfully avoids those deficiencies and moreover achieves global optimization. Therefore, the evaluation of SVM algorithm is more scientific and accurate in this study. Based on it, the SVM algorithm was further optimized and improved and combined with PSO. Then the accuracy of the experimental results reached 100%. However, only one vocational college was selected for study; the research performance was evaluated using the optimized SVM algorithm to further prove the accuracy of the experimental results. Due to the diversity of different vocational colleges, further studies are required in the future. In conclusion, the application of SVM algorithm to the evaluation of scientific research performance in vocational colleges should be promoted.

This study is supported by Study on the new method of practical teaching in SSH course,(JYZD201606).

Acknowledgements

Research project on the development of social science in Hebei province 2018 The people's livelihood project No.201803040106.

References

[1] *Enterprise integrated performance evaluation system*, CN Patent 2,00,810,080,220.4, December 25, 2008.

[2] *State grid corporation of its electric power company in zhejiang province taizhou power supply company*, Production enterprise security performance appraisal system, CN Patent 2,01,410,008,561.6, January 08, 2014.

[3] Wave software Co, LTD, *A method of medical institutions in hospital service performance evaluation*, CN Patent 2,01,410,426,131.6, August 27, 2014.

[4] Hunan institute of finance and economy, *A device used for evaluation of university teachers' performance*, CN Patent2,01,520,902,556.X, November 13, 2015.

[5] L. Qian, *vAnalysis of incentive theory of research management in Higher Vocational Colleges and universities research management enlightenment*, J. Education and Teaching Forum, 47 (2015), 19-20.

[6] Z. Hui, L. Caigao, H. Chaocai, *Innovation of teaching supervision system in Higher Vocational Colleges- the construction of all-around supervision mode*, J. Hunan Social Sciences, 6 (2015), 194-198.

[7] S. Jing, P. Huaiqing, X. Jing, *The strategy of promoting the scientific research quality of Higher Vocational College Teachers*, J. Journal of Hebei Normal University, Education Science Edition, 110-113, 2015.

[8] S. Yuelan, *Research on the structure and evaluation system of teachers' vocational ability in Higher Vocational Colleges*, J. Journal of Qingyuan Polytechnic, 4 (2014), 26-29.

[9] L. Ziyu, *Research on Evaluation of scientific research performance management in Higher Vocational Colleges*, J. Journal of Nanyang Institute of Technology, 4 (2012), 62-65.

[10] Y. Heping, J. Hongxia, D. Lianjie, *Evaluation on the Performance of Scientific Research in Universities Based on VE*, 5 (2013), 843-849.

[11] W. Xiaoping, J. Li, J.M. Zhong, *Study on Fuzzy Comprehensive Evaluation Model of Teacher's Performance*, Applied Mechanics & Materials, 701-702, 1352-1358, 2015.

[12] C. Chingshan, C. Minyuan, W. Yuwei, *Seismic assessment of school buildings in Taiwan using the evolutionary support vector machine inference system*, J. Expert Systems with Applications, 39 (2012), 4102-4110.

[13] L. State, C. Cocianu, C. Uscatu et al., *Extensions of the SVM Method to the Non-Linearly Separable Data*, J. Informatica Economica, 17(2/2013), 173-182.

[14] Q. Quande, L. Li, S. Cheng, L. Rongjun, *Particle swarm optimization algorithm for interactive learning*, J. Journal of Intelligent Systems, 7 (2012), 547-553.

[15] Z. Lina, D. Qian, *Performance prediction model based on BP algorithm*, J. Journal of Shenyang Normal University (Natural Science Edition), 29 (2011), 226-229.

Accepted: 29.06.2017

ON CHARACTERIZATION OF MONOIDS BY PROPERTIES OF GENERATORS II

Morteza Jafari
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
mjdm@pgs.usb.ac.ir

Akbar Golchin*
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
agdm@math.usb.ac.ir

Hossein Mohammadzadeh Saany
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
hmsdm@math.usb.ac.ir

Abstract. Kilp and Knauer in (Communications In Algebra, 20(7), 1841-1856, 1992) gave a characterizations of monoids when all generators in the category of right S-acts(S is a monoid) satisfy properties such as freeness, projectivity, strong flatness, Condition (P), principal weak flatness, principal weak injectivity, weak injectivity, injectivity, divisibility, strong faithfulness and torsion freeness. Sedaghatjoo in (Semigroup Forum, 87: 653-662, 2013) gave a characterizations of monoids when all generators in the category of right S-acts satisfy properties such as weak flatness, Condition (E) and regularity. Continuing this study the authors (On characterization of monoids by properties of generators, submitted) investigated the corresponding problem for (finitely generated, cyclic, monocyclic) right acts. To our knowledge the problem has not been yet studied for properties such as GP-flatness, strongly (P)-cyclic, (P)-regularity and Conditions $(EP), (E'P), (E'), (P_E), (PWP), (PWP_E), WPF, WKF, PWKF, TKF, (WP)$ etc. In this article we answer the question corresponding to these properties.
Keywords: generator, GP-flat, strongly (P)-cyclic, (P)-regular, condition (PWP).

1. Introduction

For a monoid S, a non-empty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \longrightarrow A$,

∗. Corresponding author

$(a, s) \longmapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A_S$ and all $s, t \in S$. Throughout this article, S will always stand for a monoid and A_S is a right S-act. For basic definitions and terminology relating semigroups and acts over monoids, we refer the reader to [11] and [13].

Let $\mathbf{C}$ be a category. An object $G \in \mathbf{C}$ is called a *generator* in $\mathbf{C}$ if the functor $Mor_{\mathbf{C}}(G, -)$ is faithful, i.e, for any $X, Y \in \mathbf{C}$ and any $f, g \in Mor_{\mathbf{C}}(X, Y)$ with $f \neq g$ there exists $\alpha \in Mor_{\mathbf{C}}(G, X)$ such that $f\alpha \neq g\alpha$.

We recall from [13, II, 3.16] that G_S is a generator if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$. Hence S_S is a generator in $\mathbf{Act}$-S. Recall from [13], [8] and [7] that:

An S-act A_S satisfies *Condition* (P), if for all $a, a' \in A_S, s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A_S)(\exists u, u' \in S)(a = a''u,\ a' = a''u'$ and $us = u's')$. A_S satisfies *Condition* (E), if for all $a \in A_S, s, s' \in S$, $as = as' \Rightarrow (\exists a' \in A_S)(\exists u \in S)(a = a'u$ and $us = us')$.

A_S satisfies *Condition* (EP), if for all $a \in A_S, s, s' \in S$, $as = as' \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u'$ and $us = u's')$.

A_S satisfies *Condition* $(E'P)$, if for all $a \in A_S, s, s', z \in S$, $as = as', sz = s'z \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u'$ and $us = u's')$.

Also we recall that a monoid S is called *left(right) collapsible* if for every $s, t \in S$ there exists $u \in S$ such that $us = ut(su = tu)$. Let S be a monoid and $x, y \in S$ then $l(x, y) = \{z \in S \mid zx = zy\}$. Evidently, $l(x, y) = \emptyset$ or $l(x, y)$ is a left ideal. If S is a left collapsible monoid, then for every $x, y \in S$, $l(x, y) \neq \emptyset$, and so $l(x, y)$ is a left ideal. Similarly, for every $x, y \in S$, we define $r(x, y) = \{t \in S \mid xt = yt\}$. Clearly $r(x, y) = \emptyset$ or $r(x, y)$ is a right ideal of S. If S is a right collapsible monoid, then for every $x, y \in S$, $r(x, y) \neq \emptyset$ and so $r(x, y)$ is a right ideal.

We use the following abbreviations

weak pullback flatness = WPF

weak kernel flatness = WKF

principal weak kernel flatness = $PWKF$

translation kernel flatness = TKF

principal weak homoflatness = (PWP)

torsion freeness = TF

Theorem 1.1 (12, Theorem 1.3)**.** *Let S be a monoid and α be an act property which is preserved under retraction. Then the following statements are equivalent:*

1. *all generators satisfy property α;*
2. *$S_S \times A_S$ satisfies property α for right S-act A_S;*

3. *$S_S \times A_S$ satisfies property α for every generator A_S;*

4. *a right S-act A_S satisfies property α if $Hom(A_S, S_S) \neq \emptyset$.*

It is obvious that all properties under discussion here are preserved under retraction.

2. Monoids over which all generators are GP-flat, strongly (P)-cyclic, (P)-regular

A monoid S is called *regular*, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$. We recall from 18 that a monoid S is called a *generally regular*, if for every $s \in S$, there exist a natural number n and $x \in S$ such that $s^n = sxs^n$. It is clear that the class of generally regular monoids contains all regular monoids. Also we recall from [18] that an act A_S is called GP-flat, if for every $s \in S$ and $a, a' \in A_S, a \otimes s = a' \otimes s$ in $A_S \otimes {}_SS$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SSs^n$. It is obvious that every principally weakly flat act is GP-flat, but not the converse, also every GP-flat act is torsion free.

First see the following result.

Lemma 2.1 (18, Lemma 2.2)**.** *Let S be a monoid and A_S be a right $S-$act. Then The following statements are equivalent:*

1. *A_S is GP-flat;*

2. *for every $s \in S$, and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes S$ implies that there exist $m, n \in \mathbb{N}$ and elements $a_1, \cdots a_m \in A_S, s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 s^n = t_1 s^n \\ a_2 t_2 = a_3 s_3 & s_2 s^n = t_2 s^n \\ \vdots & \vdots \\ a_m t_m = a' & s_m s^n = t_m s^n. \end{array}$$

Now we begin our investigations for monoids over which all (finitely generated) generators are GP-flat.

Theorem 2.2. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are GP-flat;*

2. *all finitely generated generators right S-acts are GP-flat;*

3. *all generators right S-acts generated by at most three elements are GP-flat;*

4. *$S \times A_S$ is GP-flat for every generator right S-act A_S;*

5. *$S \times A_S$ is GP-flat for every finitely generated generator right S-act A_S;*

6. *$S \times A_S$ is GP-flat for every generator right S-act A_S generated by at most three elements;*

7. *$S \times A_S$ is GP-flat for every right S-act A_S;*

8. *$S \times A_S$ is GP-flat for every finitely generated right S-act A_S;*

9. *$S \times A_S$ is GP-flat for every right S-act A_S generated by at most two elements;*

10. *a right S-act A_S is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S-act A_S is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a right S-act A_S generated by at most two elements is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

13. *all right S-acts are GP-flat;*

14. *all finitely generated right S-acts are GP-flat;*

15. *S is generally regular.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) and (13) $\Leftrightarrow$ (14) $\Leftrightarrow$ (15) are clear from Theorem 1.1 and [18, Theorem 3.4].

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12) and (13) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and suppose $\pi : A_S \longrightarrow S_S$ is an epimorphism. Let $a \otimes s = a' \otimes s$ in $A \otimes S$, for $a, a' \in A_S, s \in S$, then $as = a's$ in A_S. Since π is an epimorphism, there exists $a'' \in A_S$ such that $\pi(a'') = 1$. If $A' = aS \cup a'S \cup a''S$, Then $as = a's$ in A_S implies $as = a's$ in A', and so $a \otimes s = a' \otimes s$ in $A' \otimes S$. But A' is a generator and so A' is GP-flat by assumption. Thus there exists a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A'_S \otimes {}_SSs^n \subseteq A_S \otimes {}_SSs^n$. Hence A_S is GP-flat.

(6) $\Rightarrow$ (4). Let A_S be a generator and suppose $(l_1, a) \otimes s = (l_2, a') \otimes s$ in $(S \times A)_S \otimes {}_SS$, for $l_1, l_2, s \in S$ and $a, a' \in A_S$. Let $A' = aS \cup a'S \cup a''S$ is as in the proof of (3) $\Rightarrow$ (1), then $S \times A'$ is GP-flat by assumption. Thus there exists a natural number n such that $(l_1, a) \otimes s^n = (l_2, a') \otimes s^n$ in $(S \times A')_S \otimes {}_SSs^n \subseteq (S \times A)_S \otimes {}_SSs^n$, and so the result follows.

(7) $\Rightarrow$ (13) Let A_S be a right S-act and suppose $a \otimes s = a' \otimes s$ in $A \otimes S$, for $a, a' \in A_S$ and $s \in S$. Then $as = a's$ in A_S and so $(1, a)s = (1, a')s$ in $S \times A_S$. Thus $(1, a) \otimes s = (1, a') \otimes s$ in $(S \times A) \otimes S$. Since $S \times A_S$ is GP-flat, by lemma 2.1 there exist $m, n \in \mathbb{N}, s_1, t_1, s_2, t_2, \cdots, s_m, t_m \in S$ and

$(l_1, a_1), (l_2, a_2), \cdots, (l_m, t_m) \in S \times A_S$ such that:

$$\begin{array}{rl} (1, a) = (l_1, a_1)s_1 & \\ (l_1, a_1)t_1 = (l_2, a_2)s_2 & s_1 s^n = t_1 s^n \\ (l_2, a_2)t_2 = (l_3, a_3)s_3 & s_2 s^n = t_2 s^n \\ \vdots & \vdots \\ (l_m, a_m)t_m = (1, a') & s_m s^n = t_m s^n. \end{array}$$

The above tossing implies that

$$\begin{array}{rl} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 s^n = t_1 s^n \\ a_2 t_2 = a_3 s_3 & s_2 s^n = t_2 s^n \\ \vdots & \vdots \\ a_m t_m = a' & s_m s^n = t_m s^n \ , \end{array}$$

and so A_S is GP-flat by Lemma 2.1.

(9) $\Rightarrow$ (7). Let A_S be a right S-act and suppose $(l_1, a) \otimes s = (l_2, a') \otimes s$, for $a, a' \in A_S$ and $l_1, l_2, s \in S$. Let $A' = aS \cup a'S$. Then $S \times A'_S$ is GP-flat by assumption and so there exists a natural number n such that $(l_1, a) \otimes s^n = (l_2, a') \otimes s^n$ in $(S \times A')_S \otimes {}_S Ss^n \subseteq (S \times A)_S \otimes {}_S Ss^n$ and so the result follows.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and suppose $a \otimes s = a' \otimes s$, for $a, a' \in A_S$ and $s \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists a homomorphism $f : A_S \longrightarrow S_S$. If $A' = aS \cup a'S$ and $f' = f \mid_{A'}$, then A' is GP-flat by assumption and so $a \otimes s = a' \otimes s$ in $A' \otimes S$ implies that there exists a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A' \otimes Ss^n \subseteq A \otimes Ss^n$, and so the result follows. □

We recall from [19] that a right congruence ρ on S_S is called *right subannihilator congruence* if $\rho \leq ker\lambda_s$ for some $s \in S$. Also we recall from [10] that S is a *right PCP monoid*, if all principal right ideals of S satisfy Condition (P) and a right S-act A_S is called *strongly (P)-cyclic*, if for every $a \in A_S$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P).

Now similar to Theorem 2.2, we give equivalents of when all (finitely generated) generators right S-acts are strongly (P)-cyclic.

Theorem 2.3. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are strongly (P)-cyclic;*
2. *all finitely generated generators right S-acts are strongly (P)-cyclic;*
3. *all generators right S-acts generated by at most two elements are strongly (P)-cyclic;*
4. *$S \times A_S$ is strongly (P)-cyclic for every generator right S-act A_S;*

5. *$S \times A_S$ is strongly (P)-cyclic for every finitely generated generator right S-act A_S;*

6. *$S \times A_S$ is strongly (P)-cyclic for every generator right S-acts generated by at most two elements;*

7. *$S \times A_S$ is strongly (P)-cyclic for every right S-act A_S;*

8. *$S \times A_S$ is strongly (P)-cyclic for every finitely generated right S-act A_S;*

9. *$S \times A_S$ is strongly (P)-cyclic for every cyclic right S-act A_S;*

10. *a right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a cyclic right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

13. *for every right subannihilator congruence ρ, S/ρ is strongly (P)-cyclic.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$ then A' is a generator and so it is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a) \in S \times A$, for $a \in A_S$ and $l \in S$. Let $A' = aS \cup a'S$ be as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_{(l,a)} = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows.

(9) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$. Let $a \in A_S$ and let $f : A_S \longrightarrow S_S$ be an S-homomorphism. Consider $(f(a), a) \in S \times aS$. Since $S \times aS$ is strongly (P)-cyclic by assumption, there exists $z \in S$ such that zS satisfy Condition (P) and $ker\lambda_{(f(a),a)} = ker\lambda_z$. So we have:

$$ker\lambda_z = ker\lambda_{(f(a),a)} = ker\lambda_{f(a)} \cap ker\lambda_a \subseteq ker\lambda_a.$$

Hence $ker\lambda_z \subseteq ker\lambda_a$. Now we show that $ker\lambda_a \subseteq ker\lambda_z$. Let $(l_1, l_2) \in ker\lambda_a$ for $l_1, l_2 \in S$, then $(l_1, l_2) \in ker\lambda_{f(a)}$. Thus $(l_1, l_2) \in ker\lambda_a \cap ker\lambda_{f(a)} = ker\lambda_z$ and so $ker\lambda_a \subseteq ker\lambda_z$. Therefore, $ker\lambda_a = ker\lambda_z$ and zS satisfy Condition (P), as required.

(12) $\Rightarrow$ (13). Let ρ be a right subannihilator congruence. Thus there exists $s \in S$ such that $\rho \leq ker\lambda_s$. Define $f : S/\rho \longrightarrow S$ by $f([t]_\rho) = st$. Clearly f is

an S-homomorphism. So $Hom(S/\rho, S) \neq \emptyset$. Thus S/ρ is strongly (P)-cyclic by assumption.

(13) $\Rightarrow$ (1). Let A_S be a generator and $a \in A_S$. Thus, there exists an epimorphism $\pi : A_S \longrightarrow S$. Let $\pi(a) = t$ and let $(l_1, l_2) \in ker\lambda_a$, for $l_1, l_2 \in S$. Then $(l_1, l_2) \in ker\lambda_t$. So $ker\lambda_a \leq ker\lambda_t$. Thus $ker\lambda_a$ is a right subannihilator congruence and so $aS \cong S/ker\lambda_a$ is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows. □

Corollary 2.4. *Let S be a monoid over which all generators right S-acts are strongly (P)-cyclic. Then for any nonempty family $\{A_i \mid i \in I\}$ of strongly (P)-cyclic right S-acts, $\prod_I A_i$ is strongly (P)-cyclic.*

Proof. Since S_S is a generator, it is strongly (P)-cyclic by assumption and so S is right PCP by [10, Theorem 2.2]. Thus by Theorem 2.3, $(S_S)^I$ is strongly (P)-cyclic for any nonempty set I and so the result follows by [16, Corollary 5.5]. □

Definition 2.5 ([3])**.** *Let S be a monoid. A right S-act A_S is called (P)-regular if all cyclic subacts of A_S satisfy Condition (P).*

It is obvious that every regular right act is (P)-regular, but not the converse.

Theorem 2.6 ([3], Theorem 2.2)**.** *Let S be a monoid and A_S be a right S-act. Then A_S is (P)-regular if and only if for every $a \in A$ and $x, y \in S, ax = ay$ implies that there exist $u, v \in S$ such that $a = au = av$ and $ux = vy$.*

Similar to Theorem 2.3 we have the following result for (P)-regularity.

Theorem 2.7. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are (P)-regular;*
2. *all finitely generated generators right S-acts are (P)-regular;*
3. *all generators right S-acts generated by at most two elements are (P)-regular;*
4. *$S \times A_S$ is (P)-regular for every generator right S-act A_S;*
5. *$S \times A_S$ is (P)-regular for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ is (P)-regular for every generator right S-act A_S generated by at most two elements;*
7. *$S \times A_S$ is (P)-regular for every right S-act A_S;*
8. *$S \times A_S$ is (P)-regular for every finitely generated right S-act A_S;*
9. *$S \times A_S$, is (P)-regular for every cyclic right S-act A_S;*

10. a right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;

11. a finitely generated right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;

12. a cyclic right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;

13. for every right subannihilator congruence ρ, S/ρ is (P)-regular.

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$ then A' is a generator and so it is (P)-regular by assumption. Thus aS satisfies Condition (P) and the result follows.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a) \in S \times A$, for $a \in A_S$ and $l \in S$. Let $A' = aS \cup a'S$ be as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ is (P)-regular by assumption. Thus $(l, a)S$ satisfies Condition (P) and the result follows.

(9) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$. Let $a \in A_S$ and let $f : A_S \longrightarrow S_S$ be an S-homomorphism such that $ax = ay$, for $x, y \in S$. Thus $(f(a), a)x = (f(a), a)y$. Since $S \times aS$ is (P)-regular, the last equality implies that there exist $u, v \in S$ such that $(f(a), a) = (f(a), a)u = (f(a), a)v$ and $ux = vy$. Thus $a = au = av$ and $ux = vy$, and so A_S is (P)-regular by Theorem 2.6.

(12) $\Rightarrow$ (13). Let ρ be a right subannihilator congruence. Thus $Hom(S/\rho, S) \neq \emptyset$ and so S/ρ is (P)-regular by assumption.

(13) $\Rightarrow$ (1). Let A_S be a generator and let $a \in A_S$. Similar to (13) $\Rightarrow$ (1) in Theorem 2.3, $aS \cong S/ker\lambda_a$ is (P)-regular and so aS satisfy Condition (P), thus the result follows. □

3. Monoids over which all generators satisfy Condition (EP), $(E'P)$, (E'), (P_E), (PWP_E), (PWP), (P), (WP), WPF, WKF, $PWKF$, TKF or (P')

In this section by using Theorem 1.1, we give a characterization of monoids over which all generators satisfy each of the above conditions.

Lemma 3.1 ([12], Lemma 3.3)**.** *Suppose for every $x, y \in S$,($l(x, y) = \emptyset \vee l(x, y) = S \vee xS \cup yS = S$). If $x, x' \in S$ such that $xx' = 1$, then $xx' = 1 = x'x$.*

Now we investigate monoids over which all generators satisfy Condition (EP).

Theorem 3.2. *For any monoid S the following statements are equivalent:*

1. all generators right S-acts satisfy Condition (EP);

2. *all finitely generated generators right S-acts satisfy Condition (EP);*
3. *all generators right S-acts generated by at most two elements satisfy Condition (EP);*
4. *$S \times A_S$ satisfies Condition (EP) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (EP) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (EP) for every generator right S-act A_S generated by at most two elements*
7. *$S \times A_S$ satisfies Condition (EP) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (EP) for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (EP) for every cyclic right S-act A_S;*
10. *$S \times A_S$ satisfies Condition (EP) for every monocyclic right S-act A_S;*
11. *a right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a finitely generated right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *a cyclic right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
14. *a monocyclic right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
15. *$(\forall x, y \in S)\ (l(x,y) = \emptyset \ \vee \ l(x,y) = S \ \vee \ xS \cup yS = S)$;*
16. *for every right subannihilator congruence ρ, S/ρ satisfies Condition (EP);*
17. *$(\forall x, y \in S)(\ l(x,y) = \emptyset \vee S/\rho(x,y)$ satisfies Condition $(EP))$;*
18. *$(\forall x, y \in S)(\ l(x,y) = \emptyset \vee (\exists u, v \in S,\ ux = vy \ \wedge \ 1\ \rho(x,y)\ u\ \rho(x,y)\ v))$.*
19. *$(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ satisfies Condition $(EP))$;*
20. *$(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee (\exists u, v \in S,\ t\ \rho(tx,ty)\ u\ \rho(tx,ty)\ v \ \wedge \ ux = vy))$.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (11) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) and (17) $\Rightarrow$ (19) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $as = at$, for $a \in A_S$ and $s, t \in S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$, then A' is a generator and

so it satisfies Condition (EP) by assumption. Thus $as = at$ in A' implies the existence of $a'' \in A' \subseteq A$ and $u, v \in S$ such that $a = a''u = a''v$ and $us = vt$. That is, A_S satisfies Condition (EP), as required.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a)s = (l, a)t$, for $a \in A_S$ and $l, s, t \in S$. If $A' = aS \cup a'S$, then A' is a generator and so $S \times A'$ satisfies Condition (EP) by assumption. Thus there exist $(l', a'') \in S \times A' \subseteq S \times A$ and $u, v \in S$ such that $(l, a) = (l', a'')u = (l', a'')v$ and $us = vt$. Thus $S \times A$ satisfies Condition (EP).

(11) $\Rightarrow$ (16). Let ρ be a right subannihilator congruence. Thus $Hom(S/\rho, S) \neq \emptyset$, and so S/ρ satisfies Condition (EP) by assumption.

(16) $\Rightarrow$ (17). Let $x, y \in S$ and suppose $l(x, y) \neq \emptyset$. Then there exists $s \in S$ such that $sx = sy$, and so $(x, y) \in ker\lambda_s$, that is, $\rho(x, y) \leq ker\lambda_s$. Thus $\rho(x, y)$ is a right subannihilator congruence and so $S/\rho(x, y)$ satisfies Condition (EP) as required.

(17) $\Rightarrow$ (18). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then $S/\rho(x, y)$ satisfies Condition (EP) and so $[1]_\rho x = [1]_\rho y$ implies that there exist $\alpha, u_1, u_2 \in S$, such that $[1]_\rho = [\alpha]_\rho u_1 = [\alpha]_\rho u_2$ and $u_1 x = u_2 y$. If $\alpha u_1 = u$ and $\alpha u_2 = v$, then $[1]_\rho = [u]_\rho = [v]_\rho$ and $ux = vy$. Hence $1\ \rho(x, y)\ u\ \rho(x, y)\ v$ and $ux = vy$.

(18) $\Rightarrow$ (20). Let $x, y \in S$ such that $l(tx, ty) \neq \emptyset$. By assumption there exist $u_1, u_2 \in S$ such that $u_1 tx = u_2 ty, 1\ \rho(tx, ty)\ u_1\ \rho(tx, ty)\ u_2$. Therefore, $t\ \rho(tx, ty)\ u_1 t\ \rho(tx, ty)\ u_2 t$ and $u_1 tx = u_2 ty$. If $u_1 t = u$ and $u_2 t = v$, then $t\ \rho(tx, ty)\ u\ \rho(tx, ty)\ v$ and $ux = vy$ as required.

(19) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$, and let $\rho(tx, ty) = \rho$. Since $[t]_\rho x = [t]_\rho y$ and S/ρ satisfy Condition (EP), there exist $\alpha, u_1, u_2 \in S$, such that $[t]_\rho = [\alpha]_\rho u_1 = [\alpha]_\rho u_2$ and $u_1 x = u_2 y$. If $\alpha u_1 = u$ and $\alpha u_2 = v$, then $t\ \rho\ u\ \rho\ v$ and $ux = vy$.

(20) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$ and suppose $l(x, y) \neq S$. Since $l(x, y) \neq S$, thus $x \neq y$. Let $K_S = xS \cup yS$. Clearly, K_S is a right ideal and $|K_S| \geq 2$. Since $\rho_{K_S} = (K_S \times K_S) \cup 1_S$, we have $(x, y) \in \rho_{K_S}$ and so $\rho(x, y) \leq \rho_{K_S}$. By assumption there exists $u, v \in S$ such that $1\ \rho(x, y)\ u\ \rho(x, y)\ v$ and $ux = vy$. Now $\rho(x, y) \leq \rho_{K_S}$ implies that, $1\ \rho_{K_S}\ u\ \rho_{K_S} v$ and $ux = vy$. If $K_S \neq S$ then $1\ \rho_{K_S}\ u\ \rho_{K_S}\ v$ implies that $u = v = 1$ and so $x = y$, a contradiction. Hence $K_S = xS \cup yS = S$.

(14) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Define the mapping $f : S/\rho(x, y) \longrightarrow S_S$ by $f([t]_{\rho(x,y)}) = zt$, for $t \in S$. Clearly f is a well defined S-homomorphism. Therefore, $Hom(S/\rho(x, y), S_S) \neq \emptyset$ and so by assumption $S/\rho(x, y)$ satisfies Condition (EP), as required.

(14) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$ and $l(x, y) \neq S$. Thus $x \neq y$ and there exists $z \in S$ such that $zx = zy$, which implies $(x, y) \in ker\lambda_z$ and so $\rho(x, y) \leq ker\lambda_z$. Let $\rho = \rho(x, y)$. Thus similarly as (14) $\Rightarrow$ (17), $Hom(S/\rho, S_S) \neq \emptyset$ and so by assumption S/ρ satisfies Condition (EP). Since $x\ \rho\ y$ and S/ρ satisfies Condition (EP), $[1]_\rho x = [1]_\rho y$ implies that there exist $w, u, v \in S$ such that $[1]_\rho = [w]_\rho u = [w]_\rho v$ and $ux = vy$. If $1 = wu = wv$ then

$x = y$, which is a contradiction. Thus without loss of generality we suppose $1 \neq wu$. Using [13, Lemma I, 4.37], we get the following sequence of equalities:

$$\begin{array}{ccccc} 1 = p_1w_1 & & q_2w_2 = p_3w_3 & \cdots & q_nw_n = wu \\ & q_1w_1 = p_2w_2 & & \cdots & \end{array}$$

where $p_i, q_i \in \{x, y\}, w_i \in S$ for each $1 \leq i \leq n$. Thus $1 = p_1w_1$ implies that $xS \cup yS = S$.

(15) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Suppose $as = as'$, for $a \in A_S$ and $s, s' \in S$. Thus $\pi(a)s = \pi(a)s'$ implies that $l(s, s') \neq \emptyset$ and so $l(s, s') = S$ or $sS \cup s'S = S$ by assumption. If $l(s, s') = S$ then $s = s'$ and so the result follows. Otherwise, $sS \cup s'S = S$. Without loss of generality we suppose $1 \in sS$, thus there exists $x \in S$ such that $sx = 1$. Then $xs = 1$ by Lemma 3.1 and so $xss' = s'1 = s'xs$. Let $xs = u$ and $s'x = v$. Thus $a = au = av$ and $us' = vs$, as required.

(10) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Suppose $\rho(x, y) = \rho$ and let $l_1 \ \rho \ l_2$ for $l_1, l_2 \in S$. Then $zl_1 = zl_2$. Thus $(z, [1]_\rho)l_1 = (z, [1]_\rho)l_2$ in $S \times S/\rho$. The last equality implies that there exist $(w, [a]_\rho) \in S \times S/\rho$ and $u, v \in S$ such that $(z, [1]_\rho) = (w, [a]_\rho)u = (w, [a]_\rho)v, ul_1 = vl_2$. If $au = u'$ and $av = v'$. Thus $S/\rho(x, y)$ satisfies Condition (EP) by [7, Theorem 3.2]. □

Since $(E) \Longrightarrow (EP)$, the following can be concluded immediately.

Corollary 3.3 ([19], Corollary 2.6)**.** *Let S be a monoid over which all generators satisfy Condition (E). Then for each pair $(x, y) \in S \times S$, $l(x, y) = \emptyset$ or $l(x, y) = S$ or $xS \cup yS = S$.*

Now using an argument similar to that of the proof of Theorem 3.2, we have the following.

Theorem 3.4. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition $(E'P)$;*
2. *all finitely generated generators right S-acts satisfy Condition $(E'P)$;*
3. *all generators right S-acts generated by at most two elements satisfy Condition $(E'P)$;*
4. *$S \times A_S$ satisfies Condition $(E'P)$ for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition $(E'P)$ for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition $(E'P)$ for every generator right S-act A_S generated by at most two elements*
7. *$S \times A_S$ satisfies Condition $(E'P)$ for every right S-act A_S;*

8. *$S \times A_S$ satisfies Condition $(E'P)$ for every finitely generated right S-act A_S;*

9. *$S \times A_S$ satisfies Condition $(E'P)$ for every cyclic right S-act A_S;*

10. *$S \times A_S$ satisfies Condition $(E'P)$ for every monocyclic right S-act A_S;*

11. *a right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a finitely generated right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

13. *a cyclic right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

14. *a monocyclic right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

15. *$(\forall x, y \in S)$ $(l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee l(x,y) = r(x,y) = S \vee xS \cup yS = S)$;*

16. *for every right subannihilator congruence ρ, S/ρ satisfies Condition $(E'P)$;*

17. *$(\forall x, y \in S)(\ l(x,y) = \emptyset \vee S/\rho(x,y)$ satisfies Condition $(E'P))$;*

18. *$(\forall x, y \in S)(\ l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee (\exists u, v \in S,\ ux = vy\ \wedge\ 1\ \rho(x,y)\ u\ \rho(x,y)\ v))$;*

19. *$(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ satisfies Condition $(E'P))$;*

20. *$(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee\ \ r(tx,ty) = \emptyset \vee\ \ (\exists u, v \in S,\ t\ \rho(tx,ty)\ u\ \rho(tx,ty)\ v \wedge\ ux = vy))$.*

Corollary 3.5 ([8], Corollary 2.12)**.** *If S is a right collapsible monoid, then for cyclic acts Conditions (P) and $(E'P)$ coincide.*

Theorem 3.6. *If S is a collapsible monoid, then the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition $(E'P)$;*

2. *all right S-acts satisfy Condition $(E'P)$;*

3. *all finitely generated right S-acts satisfy Condition $(E'P)$;*

4. *all cyclic right S-acts satisfy Condition $(E'P)$;*

5. *$S = \{1\}$ or $S = \{1, 0\}$.*

Proof. Implications (2) ⇒ (3) ⇒ (4), (2) ⇒ (1) and (5) ⇒ (2) are obvious.

(1) ⇒ (2). Let A_S be a right S-act and $as = at, sz = tz$, for $a \in A_S, s, t, z \in S$. By Theorem 3.4, $S \times A_S$ satisfies Condition $(E'P)$. Since S is left collapsible, there exists $u \in S$ such that $us = ut$. Thus $(u, a)s = (u, a)t, sz = tz$ implies that there exist $(w, a') \in S \times A_S, u, v \in S$ such that $(u, a) = (w, a')u = (w, a')v, us = vt$. Hence $a = a'u = a'v, us = vt$ and so A_S satisfies Condition $(E'P)$.

(4) ⇒ (5). By Corollary 3.5, all cyclic right S-acts satisfy Condition (P) and so by [13, IV, 9.9] S is a group or a group with a zero adjoined. Since S is right collapsible, $S = \{1\}$ or $S = \{1, 0\}$. □

Now for a monoid S we answer the question of when all generators right S-acts satisfy Condition (E'). Recall from [15] that A_S satisfies Condition (E'), if for all $a \in A_S, s, s', z \in S$

$as = as', sz = s'z \Rightarrow (\exists a' \in A_S)(\exists u \in S)(a = a'u \text{ and } us = us')$.

Theorem 3.7. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (E');*
2. *all finitely generated generators right S-acts satisfy Condition (E');*
3. *all generators right S-acts generated by at most two elements satisfy Condition (E');*
4. *$S \times A_S$ satisfies Condition (E') for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (E') for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (E') for every generator right S-act A_S generated by at most two elements;*
7. *$S \times A_S$ satisfies Condition (E') for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (E') for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (E') for every cyclic right S-act A_S;*
10. *$S \times A_S$ satisfies Condition (E') for every monocyclic right S-act A_S;*
11. *a right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a finitely generated right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
13. *a cyclic right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
14. *a monocyclic right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
15. *$(\forall x, y \in S)(\ l(x, y) = \emptyset \vee\ r(x, y) = \emptyset \vee (\exists e \in E(S),\ \rho(x, y) = ker\lambda_e))$;*

16. *for every right subannihilator congruence ρ, S/ρ satisfies Condition (E');*

17. $(\forall x, y \in S)(\ l(x,y) = \emptyset \vee S/\rho(x,y)$ *satisfies Condition* $(E'))$;

18. $(\forall x, y \in S)(\ l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee (\exists u \in S, ux = uy\ \wedge\ 1\ \rho(x,y)\ u))$;

19. $(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ *satisfies Condition* $(E'))$;

20. $(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee\ r(tx,ty) = \emptyset \vee (\exists u \in S,\ t\ \rho(tx,ty)\ u\ \wedge\ ux = uy))$;

21. $(\forall x, y \in S)(\ l(x,y) = \emptyset \vee\ r(x,y) = \emptyset \vee (\exists e \in E(S),\ ex = ey \wedge 1\ \rho(x,y)\ e))$;

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (11) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) and (17) $\Rightarrow$ (19) are obvious.

(20) $\Rightarrow$ (21). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. If $t = 1$, then there exists $u \in S$ such that $ux = uy$ and $1\ \rho(x,y)\ u$. If $\rho = \rho(x,y)$, then $(x,y) \in ker\lambda_u$ implies that $\rho \subseteq ker\lambda_u$. Since $1\ \rho\ u$ we have $(1,u) \in ker\lambda_u$, that is, $u = u^2$ and so u is an idempotent. If $u = e$, then we are done.

(21) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. If $\rho = \rho(x,y)$, then by assumption there exists $e \in E(S)$ such that $ex = ey$ and $1\ \rho\ e$. Thus $ex = ey$ implies $\rho \subseteq ker\lambda_e$. Let $l_1, l_2 \in S$, such that $(l_1, l_2) \in ker\lambda_e$. Then $el_1 = el_2$, and that $1\ \rho\ e$ we have $l_1\ \rho\ el_1$, $l_2\ \rho\ el_2$ and so $l_1\ \rho\ l_2$. Thus, $ker\lambda_e \subseteq \rho$, and so $ker\lambda_e = \rho$ as required.

(15) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S, x, y, z \in S$ and suppose $ax = ay$ and $xz = yz$. Then $\pi(a)x = \pi(a)y$ and so $l(x,y) \neq \emptyset$. Also $xz = yz$ implies that $r(x,y) \neq \emptyset$. Thus by assumption there exists $e \in E(S)$ such that $\rho(x,y) = ker\lambda_e$. On the other hand $ax = ay$ implies that $(x,y) \in ker\lambda_a$ and so $ker\lambda_e = \rho(x,y) \subseteq ker\lambda_a$. Since $(1,e) \in ker\lambda_e \subseteq ker\lambda_a$ so $a = ae$. Also $(x,y) \in \rho(x,y) = ker\lambda_e$ implies that $ex = ey$, and so A_S satisfies Condition (E') as required.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Suppose $as = at, sz = tz$, for $a \in A_S$ and $s, t, z \in S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$, then A' is a generator and so by assumption satisfies Condition (E'). Thus $as = at, sz = tz$ implies the existence of $a'' \in A' \subseteq A$ and $u \in S$ such that $a = a''u$ and $us = ut$. That is A_S satisfies Condition (E'), as required.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l,a)s = (l,a)t, sz = tz$, for $a \in A_S$ and $l, s, t, z \in S$. If $A' = aS \cup a'S$ is as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ satisfies Condition (E') by assumption. Thus there exist $(l', a'') \in S \times A' \subseteq S \times A$ and $u \in S$ such that $(l,a) = (l', a'')u$ and $us = ut$. Therefore $S \times A$ satisfies Condition (E').

(17) $\Rightarrow$ (18). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. Since $r(x,y) \neq \emptyset$, there exists $z \in S$ such that $xz = yz$. Then $S/\rho(x,y)$ satisfies Condition (E') and so $[1]_\rho x = [1]_\rho y,\ xz = yz$ implies that there exist $\alpha, u_1 \in S$

such that $[1]_\rho = [\alpha]_\rho u_1$ and $u_1 x = u_1 y$. If $\alpha u_1 = u$, then $[1]_\rho = [u]_\rho$ and $ux = uy$. Hence $1\ \rho(x,y)\ u$ and $ux = uy$.

(18) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$ and $r(tx, ty) \neq \emptyset$. Then by assumption there exists $u_1 \in S$ such that $u_1 tx = u_1 ty$ and $1\ \rho(tx, ty)\ u_1$. Thus $t\ \rho(tx, ty)\ u_1 t$ and $u_1 tx = u_1 ty$. If $u_1 t = u$, then $t\ \rho(tx, ty)\ u$ and $ux = uy$ as required.

(19) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$ and $r(tx, ty) \neq \emptyset$. Since $r(tx, ty) \neq \emptyset$, there exists $z \in S$ such that $txz = tyz$. Suppose $\rho(tx, ty) = \rho$. Since $[1]_\rho tx = [1]_\rho ty, txz = tyz$, there exist $\alpha, u \in S$, such that $[1]_\rho = [\alpha]_\rho u_1$ and $u_1 tx = u_1 ty$. If $\alpha u_1 t = u$ then $t\ \rho(tx, ty)\ u$ and $ux = uy$ and so we are done.

(14) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Therefore, $Hom(S/\rho(x, y), S_S) \neq \emptyset$ and so by assumption $S/\rho(x, y)$ satisfies Condition (E'), as required.

(10) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $l \in S$ such that $lx = ly$ and so $\rho(x, y) \leq ker\lambda_l$. Suppose $\rho(x, y) = \rho$ and let $l_1, l_2, z \in S$ such that $l_1 \rho\ l_2$ and $l_1 z = l_2 z$, then $ll_1 = ll_2$. Thus $(l, [1]_\rho)l_1 = (l, [1]_\rho)l_2$ in $S \times S/\rho$ and $l_1 z = l_2 z$. By assumption there exist $(w, [a]_\rho) \in S \times S/\rho$ and $v \in S$ such that $(l, [1]_\rho) = (w, [a]_\rho)v$ and $vl_1 = vl_2$. If $av = u$, then $1\rho(x, y)u$ and $ul_1 = ul_2$. Thus $S/\rho(x, y)$ satisfies Condition (E') by [17, Lemma 2.2]. □

Recall from [1] and [4] that an act A_S satisfies *Condition* (PWP), if for all $a, a' \in A_S, s \in S$, $as = a's \Rightarrow (\exists a'' \in A_S)(\exists u, v \in S)(a = a''u,\ a' = a''v$ and $us = vs)$. Also we say that A_S satisfies *Condition* (PWP_E), if for all $a, a' \in A_S, s \in S$, $as = a's \Rightarrow (\exists a'' \in A_S)(\exists u, v \in S)(\exists e, f \in E(S))(ae = a''ue,\ a'f = a''vf, es = s = fs$ and $us = vs)$.

Theorem 3.8. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (PWP);*
2. *all finitely generated generators right S-acts satisfy Condition (PWP);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (PWP);*
4. *$S \times A_S$ satisfies Condition (PWP) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (PWP) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (PWP) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (PWP) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (PWP) for every finitely generated right S-act A_S;*

9. *$S \times A_S$ satisfies Condition (PWP) for every right S-act A_S generated by at most two elements;*

10. *a right S-act A_S satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S act A_S satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a right S-act A_S generated by at most two elements satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

13. *all right S-acts satisfy Condition (PWP).*

14. *S is group.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

Implications (13) $\Rightarrow$ (14) and (14) $\Rightarrow$ (1) follows from [1, Proposition 9].

(9) $\Rightarrow$ (7). Let A_S be a right S-act and $(s, x)c = (t, y)c$, for $s, t, c \in S$ and $x, y \in A_S$. If $A' = xS \cup yS$, then $S \times A'$ satisfies Condition (PWP) by assumption. Thus there exist, $(w, z) \in S \times A' \subseteq S \times A$ for $w \in S, z \in A'$ and $u, v \in S$ such that $(s, x) = (w, z)u, (t, y) = (w, z)v$ and $us = vs$. That is, $S \times A_S$ satisfies Condition (PWP) as required.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and suppose $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists homomorphism $f : A_S \longrightarrow S_S$. If $A' = aS \cup a'S$ and $f' = f \mid_{A'}$, the result follows by assumption.

(3) $\Rightarrow$ (1). Let A_S be a generator. Then there exists an epimorphism $\pi : A_S \longrightarrow S_S$. Suppose now that $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since π is an epimorphism, there exists $c \in A_S$ such that $\pi(c) = 1$. Let $A' = aS \cup a'S \cup cS$ and so $\pi \mid_{A'}: A' \longrightarrow S$ is an epimorphism. Thus A' is a generator and so A' satisfies Condition (PWP) by assumption. Hence $as = a's$ implies that there exist $a'' \in A' \subseteq A, u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs$, thus the result follows.

(6) $\Rightarrow$ (1). Let A_S be a generator and let $as = a's$, for $a, a' \in A_S$ and $s \in S$. If $A' = aS \cup a'S \cup cS$ is as in the proof of (3) $\Rightarrow$ (1), then $(1, a)s = (1, a')s$. Clearly, A' is a generator and so $S \times A'$ satisfies Condition (PWP) by assumption. Thus $(1, a)s = (1, a')s$ implies that there exist $(w, z) \in S \times A' \subseteq S \times A_S$, $w \in S, a \in A' \subseteq A$ and $u, v \in S$ such that $(1, a) = (w, z)u, (1, a') = (w, z)v$ and $us = vs$. Thus $a = zu, a' = zv$ and $us = vs$ and so the result follows.

(7) $\Rightarrow$ (13). Let A_S be a right S-act and $as = a's$, for $a, a' \in A_S, s \in S$. Thus $(1, a)s = (1, a')s$. Since $S \times A$ satisfies Condition (PWP), there exist $(w, a'') \in S \times A_S, u, v \in S$ such that $(1, a) = (w, a'')u, (1, a') = (w, a'')v$ and $us = vs$. Thus $a = a''u, a' = a''v$ and $us = vs$, as required. $\square$

From Theorems 3.2, 3.4, 3.7, 3.8 and [16, Theorems 3.21, 3.22, 3.23], and [16, Proposition 3.16], we have the following corollary.

Corollary 3.9. *Let S be a monoid over which all generators satisfy Condition (EP)(Condition $(E'P)$, Condition (E'), Condition (PWP)). Then for any nonempty family $\{A_i \mid i \in I\}$ of right S-acts satisfying Condition (EP)(Condition $(E'P)$, Condition (E'), Condition (PWP)), $\prod_I A_i$ satisfies Condition (EP) (Condition $(E'P)$, Condition (E'), Condition (PWP)).*

By a similar argument as in the proof of Theorem 3.8 and using [4, Theorem 3.1] we can show the following theorem.

Theorem 3.10. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (PWP_E);*
2. *all finitely generated generators right S-acts satisfy Condition (PWP_E);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (PWP_E);*
4. *$S \times A_S$ satisfies Condition (PWP_E) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (PWP_E) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (PWP_E) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (PWP_E) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (PWP_E) for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (PWP_E) for every right S-act A_S generated by at most two elements;*
10. *a right S-act A_S satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
11. *a finitely generated right S act A_S satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a right S act A_S generated by at most two elements satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *all right S-acts satisfy Condition (PWP_E);*
14. *S is regular.*

We recall from [5] and [13] that a right $S-$act A_S satisfies *Condition* (P_E), if for all $a, a' \in A, s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A)(\exists u, u' \in S)(\exists e, f \in E(S))(ae = a''ue,\ a'f = a''u'f, es = s, fs' = s'$ and $us = u's')$ also S is called *right reversible*, if for any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$.

We recall from [19], a right S-act A_S is called *almost weakly flat* if A_S is principally weakly flat and satisfies Condition

(W') If $as = a't$, and $Ss \cap St \neq \emptyset$, for $a, a' \in A_S, s, t \in S$, then there exists $a'' \in A_S$, $u \in Ss \cap St$ such that $as = a't = a''u$. It is proved in [19, Theorem 3.4] that all generators are weakly flat if and only if all right S-acts are almost weakly flat.

Theorem 3.11. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (P_E);*
2. *all finitely generated generators right S-acts satisfy Condition (P_E);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (P_E);*
4. *$S \times A_S$ satisfies Condition (P_E) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (P_E) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (P_E) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (P_E) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (P_E) for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (P_E) for every right S-act A_S generated by at most two elements;*
10. *a right S-act A_S satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
11. *a finitely generated right S act A_S satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a right S act A_S generated by at most two elements satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *all right S-acts are almost weakly flat.*
14. *S is regular and for every $s, t \in S$ with $Ss \cap St \neq \emptyset$, there exists $w \in Ss \cap St$ such that $1\ (ker\lambda_s \vee ker\lambda_t)\ w$.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(1) $\Rightarrow$ (13). Since Condition (P_E) implies weak flatness by [9, Theorem 2.3], the result follows by [19, Theorem 3.4].

(13) $\Rightarrow$ (14). It follows from [19, Theorems 3.4, 3.8].

(3) $\Rightarrow$ (1). It is similar to (3) $\Rightarrow$ (1) of Theorem 3.8.

(14) $\Rightarrow$ (1). Since S is regular, it is left PP and so by [9, Theorem 2.5] Condition (P_E) and weak flatness are the same and so the result follows by [19, Theorem 3.8] .

(9) $\Rightarrow$ (7). Let A_S be a right S-act and $(w_1, a)s = (w_2, a')t$, for $w_1, w_2 \in S, a, a' \in A$. If $A' = aS \cup a'S$, then $S \times A'$ satisfies Condition (P_E) by assumption. Hence there exists $(w, a'') \in S \times A' \subseteq S \times A_S$ and $u, v \in S, e, f \in E(S)$, such that $(w_1, a)e = (w, a'')ue, (w_2, a')f = (w, a'')vf, es = s, ft = t$ and $us = vt$, as required.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and $as = a't$, for $a, a' \in A_S, s, t \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists an S-homomorphism $f : A_S \longrightarrow S_S$. Let $A' = aS \cup a'S$ and $f' = f \mid_{A'}$. Thus $as = a't$ in A' implies that there exists $a'' \in A' \subseteq A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s, ft = t, us = vt$, and so A_S satisfies Condition (P_E) as required.

(6) $\Rightarrow$ (1). Let A_S be a generator and let $as = a's'$ for $a, a' \in A_S, s, s' \in S$. If $A' = aS \cup a'S \cup cS$ is as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.8, then $(\pi(a), a)s = (\pi(a'), a')s'$. Clearly, A' is a generator and so $S \times A'$ satisfies Condition (P_E) by assumption. Thus the last equality implies that there exist $(w, a'') \in S \times A' \subseteq S \times A, u, v \in S$ and $e, f \in E(S)$ such that $(\pi(a), a)e = (w, a'')ue, (\pi(a'), a')f = (w, a'')vf, es = s, fs' = s'$ and $us = u's'$ and so the result follows. □

Corollary 3.12. *Let S be a right reversible monoid. Then the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (P_E);*
2. *all right S-acts satisfy Condition (P_E);*
3. *S is regular and satisfies Condition (R).*

$$(R) : (\forall s, t \in S)(\exists w \in Ss \cap St),\ w\ \rho(s, t)\ s$$

Proof. (2) $\Leftrightarrow$ (3) By [5,Theorem 2.1].

(2) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (3). Since S is right reversible, weak flatness and almost weak flatness are equivalent. By Theorem 3.11, all S-acts are weakly flat, and so the result follows from [13, IV,7.5]. □

From Theorems 3.10, 3.11 and [16, Theorems 3.18, 3.15] we have the following corollary.

Corollary 3.13. *Let S be a commutative monoid over which all generators satisfy Condition $(PWP_E)((P_E))$. If $A_1, \cdots A_n, n \in \mathbb{N}$ satisfy Condition (PWP_E) $((P_E))$, then $\prod_{i=1}^{n} A_i$ satisfies Condition $(PWP_E)((P_E))$.*

Definition 3.14 ([20])**.** *An act A_S is called strongly torsion free (STF) if for any $a, b \in A$ and any $s \in S$ the equality $as = bs$ implies $a = b$.*

It is obvious that $STF \Rightarrow Condition(PWP) \Rightarrow PWF \Rightarrow TF$.

Corollary 3.15. *Let S be a monoid and (U) be any of the conditions or properties $WPF, WKF, PWKF, TKF, (P), (WP), (P'), STF$ of right S-acts, then the following statements are equivalent:*

1. *all generators right S-acts satisfy (U);*
2. *$S \times A_S$ satisfies (U) for every generator right S-act A_S;*
3. *$S \times A_S$ satisfies (U) for every right S-act A_S;*
4. *a right S-act A_S satisfies (U) if $Hom(A_S, S_S) \neq \emptyset$;*
5. *all right S-acts satisfy (U).*
6. *S is group.*

Proof. Implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are clear from Theorem 1.1 and (5) $\Leftrightarrow$ (6) follows from [1, Proposition 9], [2, Theorem 2.5] and [20, Theorem 3.2].

(6) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (6). Since $(U) \Longrightarrow (PWP)$ the result follows from Theorem 3.8. □

References

[1] S. Bulman-Fleming, M. Kilp, V. Laan, *Pullbacks and flatness properties of acts II*, Comm. Algebra, 29 (2001), 851-878.

[2] A. Golchin, H. Mohammadzadeh, *On Condition (P')*, Semigroup Forum., 86 (2012), 413-430.

[3] A. Golchin, H. Mohammadzadeh, P. Rezaei, *On (P)-regularity of acts*, Advances in Pure Mathematics, 2 (2012), 104-108.

[4] A. Golchin, H. Mohammadzadeh, *On Condition (PWP_E)*, Southeast Asian Bulletin of Mathematics, 33 (2009), 245-256.

[5] A. Golchin, H. Mohammadzadeh, *On homological classification of monoids by Condition (P_E) of right S-acts*, Italian Journal of Pure and Applied Mathematics, 25 (2009), 175-186.

[6] A. Golchin, H. Mohammadzadeh, *On homological classification of monoids by Condition* (E') *of right* S*-acts*, Yokohama Mathematical, 54 (2007), 79-88.

[7] A. Golchin, H. Mohammadzadeh, *On Condition* (EP), International Mathematical Forum., 19 (2007), 911-918.

[8] A. Golchin, H. Mohammadzadeh, *On Condition* $(E'P)$, Journal of Sciences, Islamic Republic of Iran, 17 (2006), 343-349.

[9] A. Golchin, J. Renshaw, *A flatness property of acts over monoids*, Conference on Semigroup, University of St. Andrews, July 1997, 1998, 72-77.

[10] A. Golchin, P. Rezaei, H. Mohammadzadeh, *On strongly* (P)*-cyclic acts*, Czechoslovak Mathematical Journal, 59 (2009), 595-611.

[11] J.M. Howie, *Fundamentals of Semigroup Theory*, London Mathematical Society Monographs, London, Oxford University Press.

[12] M. Jafari, A. Golchin, H. Mohammadzadeh Saany, *On charaterization of monoids by properties of generators*, (submitted).

[13] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[14] M. Kilp, U. Knauer, *Charaterization of monoids by properties of generators*, Communication in Algebra, 20 (1992), 1841-1856.

[15] V. Laan, *Pullbacks and flatness properties of acts*, Ph.D Thesis, University of Tartu, Tartu, Estonia, 1999.

[16] L. Nouri, A. Golchin, H. Mohammadzadeh, *On properties of product acts over monoids*,Comm. Algebra., 43 (2015), 1854-1876.

[17] H. Qiao, Z. Liu, *Monoids characterized by Condition* (E'), Pure Mathematics and Applications, 19(2005), 165-171.

[18] H. Qiao, C. Wei, *On a generalization of principal weak flatness property*, Semigroup Forum., 85 (2012), 147-159.

[19] H. Sedaghatjoo, *On monoids over which all generators satisfy a flatness property*, Semigroup Forum, 87 (2013), 653-662.

[20] A. Zare, A. Golchin, H. Mohammadzadeh, *Strongly torsion free acts over monoids*, Asian-European Journal of Mathematics, 6 (2013), 1350049 (22 pages).

Accepted: 23.07.2017

CERTAIN GENERATING FUNCTIONS OF GENERALIZED HYPERGEOMETRIC 2D POLYNOMIALS FROM TRUESDELL'S METHOD

P.L. Rama Kameswari*
Department of Mathematics
K. L. University
Guntur Dt., A.P.
India
and
Department of Mathematics
Swarnandhra College of Engineering and Technology
Seetharampuram, Narsapuram-534 280
West Godawari Dt., A.P.
India
ramaravikumar.i@gmail.com

V.S. Bhagavan
Department of Mathematics
K. L. University
Guntur Dt., A.P.
India
drvsb002@kluniversity.in

Abstract. In this paper, the generating functions for generalized Hypergeometric 2D polynomials $U_n(\beta, \gamma, x, y)$ are obtained by using the Truesdell's method giving a suitable interpretation to the index n. Further, a pair of linearly independent differential recurrence relations are used in order to derive generating functions for $U_n(\beta, \gamma, x, y)$. The principal interest in our results lies in the fact that, how the Truesdell's method is utilized in an effective and suitable way to generalized Hypergeometric 2D polynomials in order to derive two generating functions independently from ascending and descending recurrence relations respectively.

Keywords: special functions, generalized hypergeometric 2D polynomials $U_n(\beta, \gamma, x, y)$ generating functions.

1. Introduction

Generating functions play a very important role in the investigation of various properties of the sequences, which they generate. They are used with good effect for the determination of the asymptotic behaviour of the generated sequence $\{f_n\}$ as $n \longrightarrow \infty$[2].In recent years, the development of advanced computers has made it necessary to study the hypergeometric polynomials with series represen-

∗. Corresponding author

tations from the numerical point of view. Because of the important role which hypergeometric polynomials play important role in problems of applied mathematics, the theory of generating functions has been developed various directions and found wide applications in different branches of science and technology.

The aim of present paper is to derive the generating functions for the generalized hypergeometric 2D polynomials by using the Truesdell's method , giving suitable interpretation to the index n. It is worth recalling that this method yields two generating functions for the generalized hypergeometric 2D polynomials, independently from ascending and descending recurrence relations, where as the simultaneous use of these recurrence relations in other group theoretic methods. The results obtained for generalized hypergeometric 2D polynomials are new in the theory of special functions.

The generalized hypergeometric polynomials $U_n(\beta, \gamma, x, y)$ satisfy the following descending and ascending recurrence relations, respectively:

$$DU_n(\beta, \gamma, x, y) = nU_{n-1}(\beta, \gamma, x, y). \tag{1.1}$$

$$\begin{aligned} DU_n(\beta, \gamma, x, y) = \frac{1}{y(x-y)}\{&(\gamma+n)U_{n+1}(\beta, \gamma, x, y) \\ &+ [(n+\beta)x - (\gamma+2n)y]U_n(\beta, \gamma, x, y)\}. \end{aligned} \tag{1.2}$$

These two independent differential recurrence relations determine the second order linear ordinary differential equation

$$\begin{aligned} &y(x-y)D^2U_n(x, y) - [(n+\beta-1)x - (\gamma+2n-2)y]DU_n(x, y) \\ &- n(\gamma+n-1)U_n(x, y) = 0, \end{aligned} \tag{1.3}$$

where $D = \frac{d}{dy}$. The proof of these results are obvious.

2. Generating funcion derived from the ascending recurrence relation

We shell use the Truesdell's F-equation

$$\frac{\partial}{\partial t}F(z, \alpha) = F(t, \alpha+1). \tag{2.1}$$

To find the generating function for the ser of polynomials $U_n(\beta, \gamma, x, y)$ as follows:

The polynomials $U_n(\beta, \gamma, x, y)$ satisfies the asending recurrence relation

$$\begin{aligned} \frac{d}{dy}U_n(\beta, \gamma, x, y) = \frac{1}{y(x-y)}\{&(\gamma+n)U_{n+1}(\beta, \gamma, x, y) \\ &+ [(n+\beta)x - (\gamma+2n)y]U_n(\beta, \gamma, x, y)\}. \end{aligned} \tag{2.2}$$

Let $f(z,\alpha) = U_n(\beta,\gamma,x,y)$, so that we have

$$\frac{\partial}{\partial z}f(z,\alpha) = \frac{[(\alpha+\beta)-(\gamma+2\alpha)z]}{z(x-z)}U_\alpha(\beta,\gamma,x,y)\}$$
$$+\frac{(\gamma+\alpha)}{z(x-z)}U_{\alpha+1)}(\beta;\gamma;x,y). \tag{2.3}$$

This equation is called the f-type equation and can be written as

$$\frac{\partial}{\partial z}f(z,\alpha) = A(z,\alpha)f(z,\alpha) + B(z,\alpha)f(z,\alpha+1),$$

with $A(z,\alpha) = \frac{[(\alpha+\beta)-(\gamma+2\alpha)z]}{z(x-z)}$ and $B(z,\alpha) = \frac{(\gamma+\alpha)}{z(x-z)}$.

$$\frac{\partial}{\partial z}g(z,\alpha) = C(z,\alpha)g(z,\alpha+1)$$

by supposing

$$g(z,\alpha) = f(z,\alpha)\exp\left\{-\int_{z_0}^{z}A(\nu,\alpha)d\nu\right\}$$
$$= f(z,\alpha)z_0^{\alpha+\beta}z^{-\alpha-\beta}(x-z_0)^{\alpha-\beta+\gamma}(x-z)^{-\alpha+\beta-\gamma}$$

and by choosing $z_0 = 1$, then we get

$$g(z,\alpha) = (x-1)^{\alpha-\beta+\gamma}z^{-\alpha-\beta}(x-z)^{\beta-\gamma-\alpha}f(z,\alpha). \tag{2.4}$$

Now, it can easily verified that this satisfies g-type equation

$$\frac{\partial}{\partial z}g(z,\alpha) = (\alpha+\gamma)(x-1)^{-1}g(z,\alpha+1). \tag{2.5}$$

Let $C(z,\alpha)$ denote the factorable coefficient of $g(z,\alpha+1)$ in (2.5), then

$$C(z,\alpha) = (\alpha+\gamma)(x-1)^{-1}$$

with $C(z,\alpha) = A(\alpha)Z(z)$. Where $A(\alpha) = (\alpha+\gamma)(x-1)^{-1}$ and $Z(z) = 1$. We effect the transformation of $f(z,\alpha)$ into $F(t,\alpha)$ by letting

$$t = -\int_{z_1}^{z}Z(\nu)d\nu = z - z_1 \tag{2.6}$$

and $F_0F(t,\alpha) = g(z,\alpha)\exp\left\{\int_{\alpha_0}^{\alpha}\log A(\nu)\Delta\nu\right\}$ on choosing $\alpha_0 = -\gamma$, we get $\int_{-\gamma}^{\alpha}\left\{(V+\gamma)(x-1)^{-1}\right\}\Delta V = \int_{-\gamma}^{\alpha}\log\left((x-1)^{-1}\right) + \log(V+\gamma)\Delta V$ Since

$$\int_0^x \log z\Delta z = \log\Gamma(x) - \log\sqrt{2\pi} \tag{2.7}$$

we have $\int_{-\gamma}^{\alpha}\left\{(V+\gamma)(x-1)^{-1}\right\}\Delta V = \log\left[\frac{\Gamma(\alpha+\gamma)\{(x-1)\}^{-(\alpha+\gamma)}}{\sqrt{2\pi}}\right]$. This implies

$$F_0F\,(t,\alpha) = g\,(z,\alpha)\exp\left\{\log\left[\frac{\Gamma(\alpha+\gamma)\,\{(x-1)\}^{-(\alpha+\gamma)}}{\sqrt{2\pi}}\right]\right\}.$$

Now by choosing $\alpha_0 = -\gamma, z_1 = v$ and $F_0 = \frac{1}{\sqrt{2\pi}}$, we get

$$F(t,\alpha) = \Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}g(t+v,\alpha). \tag{2.8}$$

To show that $F(t,\alpha)$ does indeed satisfy the F-equation we determine $\frac{\partial}{\partial t}F(t,\alpha)$ as follows:

$$\begin{aligned} F(t,\alpha) &= \Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}\frac{\partial g(t+v,\alpha)}{\partial(t+v)}\frac{\partial g(t+v)}{\partial t} \\ &= \Gamma(\alpha+\gamma+1)(x-1)^{-(\alpha+\gamma+1)}g(v+t,\alpha+1). \end{aligned} \tag{2.9}$$

Therefore,

$$\frac{\partial}{\partial t}F(t,\alpha) = F(t,\alpha+1).$$

For later use express $F(t,\alpha)$ in the following form:

$$\begin{aligned} F(t,\alpha) &= \Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}g(t+v,\alpha) \\ &= \Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}(x-1)^{\alpha+\beta-\gamma}(v+t)^{-\alpha-\beta} \\ &\quad (x-v-t)^{-(\alpha-\beta+\gamma)}f(v+t,\alpha) \\ &= \Gamma(\alpha+\gamma)(x-1)^{-\beta}(v+t)^{-(\alpha+\beta)} \\ &\quad (x-v-t)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t). \end{aligned} \tag{2.10}$$

Thus

$$\begin{aligned} F(t+z,\alpha) &= \Gamma(\alpha+\gamma)(x-1)^{-\beta}(v+t+z)^{-(\alpha+\beta)} \\ &\quad (x-v-t-z)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t+z). \end{aligned} \tag{2.11}$$

$$\begin{aligned} F(t,\alpha+n) &= \Gamma(\alpha+n+\gamma)(x-1)^{-\beta}(v+t)^{-(\alpha+n+\beta)} \\ &\quad (x-v-t)^{-(\alpha-\beta+\gamma-n)}U_{\alpha+n}(\beta;\gamma;x,v+t). \end{aligned} \tag{2.12}$$

Now let us apply Truesdell's generating function theorem, if the function $F(t,\alpha)$. Satisfies the F-equation and $f(t+z,\alpha)$ possesses a Taylor's series in power of z, then this series may be expressed as

$$F(t+z,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{n!}F(t,\alpha+n). \tag{2.13}$$

It follows that

$$(v+t+z)^{-(\alpha+\beta)}(x-v-t-z)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t+z)$$
$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n+\gamma)}{\Gamma(\alpha+\gamma)}(v+t)^{-(\alpha+n+\beta)}(x-v-t)^{-(\alpha-\beta+\gamma-n)}U_{\alpha+n}(\beta;\gamma;x,v+t).$$

Now replacing $v+t$ by y and z by $yt(x-y)$, we get the generating relation

$$(1-yt)^{-(\alpha-\beta+\gamma)}\{1+t(x-y)\}^{-(\alpha+\beta)}U_\alpha[\beta;\gamma;x,y+yt(x-y)]$$
$$\sum_{n=0}^{\infty}\frac{(\gamma+\alpha)_n}{n!}U_{\alpha+n}(\beta;\gamma;x,y)t^n. \tag{2.14}$$

Which is a generating relation for $U_n(\beta;\gamma;x,y)$.

3. Generating function derived from the descending recurrence relation

Similarly, by using the Truesdell's G-equation

$$\frac{\partial}{\partial t}G(t,\alpha)=G(t,\alpha-1). \tag{3.1}$$

We have derived a generating relation for the set of polynomials $U_{\alpha-n}(\beta;\gamma;x,y)$ as follows:

The decsending recurrence relation for $U_n(\beta;\gamma;x,y)$ is

$$\frac{d}{dy}U_n(\beta;\gamma;x,y)=nU_{n-1}(\beta;\gamma;x,y). \tag{3.2}$$

Let $f(z,\alpha)=U_n(\beta;\gamma;x,y)$, so that we have

$$\frac{\partial}{\partial z}f(z,\alpha)=\alpha f(z,\alpha-1). \tag{3.3}$$

This equation is called the F-type equation and can be written as

$$\frac{\partial}{\partial z}f(z,\alpha)=A(z,\alpha)f(z,\alpha)+B(z,\alpha)f(z,\alpha-1)$$

with $A(z,\alpha)=0$ and $B(z,\alpha)=\alpha$.

Now let us transform $f(z,\alpha)$ into $g(z,\alpha)$ so that

$$\frac{\partial}{\partial z}g(z,\alpha)=C(z,\alpha)g(z,\alpha-1).$$

Let us suppose that

$$g(z,\alpha)=f(z,\alpha)\exp\left\{-\int_{z_0}^{z}A(\nu,\alpha)d\nu\right\}=\exp(c)f(z,\alpha),$$

where c being an integration constant.

Now in particular, if we write $z_0 = 0$, then we get

$$g(z, \alpha) = \exp(c) f(z, \alpha).$$

Now

$$\begin{aligned} g(z, \alpha) &= \exp(c) f(z, \alpha) \\ &= \alpha \exp(c) f(z, \alpha - 1) \\ &= \alpha g(z, \alpha - 1). \end{aligned} \tag{3.4}$$

Let $C(z, \alpha)$ denote the factorable coefficient of $g(z, \alpha - 1)$,then $C(z, \alpha) = \alpha$ with $Z(z) = 1$ and $A(\alpha) = \alpha$. Further let us effect the transformation of $g(z, \alpha)$ into $g(t, \alpha)$ by letting

$$t = -\int_{z_1}^{z} Z(\nu) d\nu = z - z_1$$

and

$$\begin{aligned} G_0 G(t, \alpha) &= g(z, \alpha) \exp\left\{ -\int_{\alpha_0}^{\alpha+1} \log A(\nu) \Delta\nu \right\} \\ &= g(z, \alpha) \exp\left\{ -\int_{\alpha_0}^{\alpha+1} \log \nu \Delta\nu \right\}. \end{aligned}$$

In particular if we choose $\alpha_0 = 0$ then we have

$$\begin{aligned} G_0 G(t, \alpha) &= g(z, \alpha) \exp\left\{ -\int_{0}^{\alpha+1} \log A(\nu) \Delta\nu \right\} \\ &= g(z, \alpha) \exp\left\{ -\log \frac{\Gamma(\alpha + 1)}{\sqrt{2\pi}} \right\} \\ &= g(z, \alpha) \frac{\sqrt{2\pi}}{\Gamma(\alpha + 1)}. \end{aligned}$$

Suppose $z_1 = \nu, \alpha_0 = 0$ and $G_0 = \sqrt{2\pi}$ then

$$G(t, \alpha) = \frac{\exp c}{\Gamma(\alpha + 1)} g(\nu + t, \alpha).$$

To show that $G(t, \alpha)$ does indeed satisfy the G-equation we determine $\frac{\partial}{\partial t} G(t, \alpha)$ as follows:

$$\begin{aligned} \frac{\partial}{\partial t} G(t, \alpha) &= \frac{\exp c}{\Gamma(\alpha + 1)} \frac{\partial g(\nu + t, \alpha)}{\partial(\nu + t)} \frac{\partial(\nu + t)}{\partial t} \\ &= \frac{\exp c}{\Gamma(\alpha + 1)} g(\nu + t, \alpha - 1). \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} G(t - \alpha).$$

For later use let us express $G(t,\alpha)$ in the following form:

$$\begin{aligned} G(t,\alpha) &= \frac{\exp c}{\Gamma(\alpha+1)} g(\nu+t,\alpha) \\ &= \frac{\exp(2c)}{\Gamma(\alpha+1)} f(\nu+t,\alpha) \\ &= \frac{\exp c}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t). \end{aligned}$$

Also

$$G(t+z,\alpha) = \frac{\exp(2c)}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t+z)$$

and

$$G(t,\alpha-n) = \frac{\exp(2c)}{\Gamma(\alpha-n+1)} U_{\alpha-n}(\beta;\gamma;x,\nu+t).$$

By Tailor's series in powers of z

$$G(t+z,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{n!} G(t,\alpha-n). \tag{3.5}$$

Which implies that

$$\begin{aligned} &\frac{\exp(2c)}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t+z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\exp(2c)}{\Gamma(\alpha-n+1)} U_{\alpha-n}(\beta;\gamma;x,\nu+t) \end{aligned}$$

or

$$U_\alpha(\beta;\gamma;x,\nu+t+z) = \sum_{n=0}^{\infty} \frac{(-1)^n(-\alpha)_n}{n!} U_{\alpha-n}(\beta;\gamma;x,\nu+t).$$

Now replacing $\nu+t$ by y and z by $-t$, we get

$$U_\alpha(\beta;\gamma;x,y-t) = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} U_{\alpha-n}(\beta;\gamma;x,y) t^n.$$

which is the another generating relation for

$$U_n(\beta;\gamma;x,y). \tag{3.6}$$

4. Applications

From the relations (2.14) and (3.6),we can derive the following generating functions for Laguerre polynomials of two variables:

$$\begin{aligned} &(1-yt)^{-n-\alpha-1} \exp\left(\frac{-tx}{(1-ty)}\right) L_n^{(\alpha)}\left(x, \frac{x}{1-ty}\right) \\ &= \sum_{l=0}^{n} \frac{(1+n)_l}{l!} L_{n+l}^{(\alpha)}(x) t^l. \end{aligned} \tag{1}$$

$$(1-t)^n L_n^{(\alpha)}\left(x, \frac{1-t}{y}\right) = \sum_{l=0}^{n} \frac{(\alpha-n)_l}{l!} L_{n-1}^{(\alpha)}(x,y)t^l. \tag{2}$$

5. Conclusion

Generating functions involving generalized Hypergeometric 2D polynomials are derived by Truesdell's Method. Certain known generating relations to two variable Laguerre polynomials are discussed as applications. The applications of generalized Hypergeometric 2D polynomials in communications including wireless, mobile, and satellite communications with new ideas and approaches to design communications system with high performance in comparison with employed communication systems is the further scope of this research.

References

[1] I.K. Khanna, V.S. Bhagavan, *Lie group theoretic origins of certain generating functions of the generalised hypergeometric polynomials*, Integral Transforms Spec. Functions, 11(2) (2001), 177-188.

[2] I.K. Khanna, V.S. Bhagavan, *Weisner's method to obtain generating relations for the generalised hypergeometric polynomial set*, J. Phys. A: Gen and Math, 32 (1999), 989-998.

[3] S. Khan, M.A. Pathan, G. Yasmin, *Representation of lie algebra $G(0,1)$ and three variable generalized Hermite polynomials $H_{n(x,y,z)}$*, Integral Transforms Special Functions, 13 (2002), 59-64.

[4] S. Khan, G. Yasmin, *Lie-theoretic generating relations of two variable laguerre polynomials*, Reports on Mathematical Physics, 51 (2013), 1-7.

[5] Manic Chandra Mukherjee, *Generating functions on extended Jacobi polynomials from Lie group view point*, Publications Matematiques, 40 (1996), 3-13.

[6] E.B. Mc Bride, *Obtaining Generating Functions*, Springer Verlag, New York, 1971.

[7] W. Miller Jr., *Lie theory and special functions*, Academic Press, New York, 1968.

[8] E.D. Rainville, *Special functions*, Macmillan Co., New York, 1960.

[9] K.P. Samanta, B.C. Chandra, C.S. Bera, *Some generating functions of modified Gegenbauer polynomials by Lie algebraic method*, Mathematics and Statistics, 2(4) (2014), 172-178.

[10] A.K. Shukla, S.K. Meher, *Group-theoretic origin of some generating functions for lagurre polynomials of two variables*, Appl. Math. Sci., 5(61) (2011), 775-784.

[11] H.M. Srivastava, H.L. Manocha, *A treatise on generating functions*, Halsted/Wiley, New York, 1984.

[12] T. Srinivasulu, V.S. Bhagavan, *Irreducible representation of $SL(2, C)$ and generating relations for the generalized hypergeometric functions*, Far East J. Math. Sci., 83(2) (2013), 127-144.

[13] Subuhi Khan, Ghazale Yasmin, *Lie theoretic generating relation of two variable Laguerre Polynomials*, Rep. Math. Phys., 51 (2003), 1-7.

Accepted: 25.07.2017

d-MIXING AND d-UNIVERSAL J-CLASS OPERATORS

A. Tajmouati*

M. El Berrag
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Fez
Morocco
abdelaziz.tajmouati@usmba.ac.ma
mohammed.elberrag@usmba.ac.ma

Abstract. In this paper, we characterize the notion of d-universal extended (respectively d-universal extended mixing) limit set and we give a relation between d-universal (or d-hypercyclic sequences) and d-universal extended limit set.
Keywords: tuple of sequence, hypercyclic sequences,d-topologically transitive, d-J^{univer}-class operators, d-J^{u-mix}- class operators.

1. Introduction

For an infinite-dimensional separable complex Banach space X, $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X. For $x \in X$, the orbit of x under $(T_n)_n$ is the set $Orb(T_n, x) = \{T_n x : n \in \mathbb{Z}_+\}$. A sequence $(T_n)_n$ of operators is called hypercyclic or universal if there is some x whose orbit under $(T_n)_n$ is dense in X. In such a case, x is called a hypercyclic or universal vector for $(T_n)_n$. A sequence $(T_n)_n$ of operators is called topologically transitive if for every nonempty open subsets U and V of X there is some $n \geq 0$ such that $T_n(U) \cap V \neq \emptyset$. For some sources on these topics; see [1, 6, 7, 8, 10].

The notion of disjoint hypercyclicity, a strengthening of hypercyclicity, concerning a tuple of linear operators, was introduced independently by Bernal [2] and by Bès and Peris [4] in 2007.

For any integer $N \geq 2$, the tuple $(T_1, T_2, ..., T_N)$ of operators, acting on the same topological vector space X, is said disjoint hypercyclic, or d-hypercyclic for short, provided there exists $(z, \ldots, z)$ in X^N, such that $\{(T_1^n z, T_2^n z, ..., T_N^n z) : n \in \mathbb{Z}_+\}$ is dense in X^N. Such a vector z is called a $d-$hypercyclic vector for the tuple $(T_1, T_2, \ldots, T_N)$.

We say that the operators $T_1, T_2, \ldots, T_N$ in $\mathcal{B}(X)$ with $N \geq 2$ are d-topologically transitive if for any non-empty open subsets $V_0, V_1, \ldots, V_N$ in X, there exists a positive integer n so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \ldots \cap T_N^{-n}(V_N)$.

∗. Corresponding author

Definition 1.1 ([4], Definition 2.1)**.** *We say that $N \geq 2$ sequences of operators $(T_{1,n})_{n=1}^{\infty}$, $(T_{2,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ in $\mathcal{B}(X)$ are d-topologically transitive (respectively d-mixing) provided for every non-empty open subsets $V_0, V_1, ..., V_N$ of X there exists a positive integer m so that $\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap T_{2,m}^{-1}(V_2) \cap \ldots \cap T_{N,m}^{-1}(V_N)$ (respectively so that $\emptyset \neq V_0 \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \ldots \cap T_{N,n}^{-1}(V_N)\}$ for every $n \geq m$).*

Definition 1.2 ([4], Definition 2.2)**.** *We say that $N \geq 2$ sequences $(T_{1,n})_{n=1}^{\infty}$, $(T_{2,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ in $\mathcal{B}(X)$ are d-universal if $\{(T_{1,n}z, T_{2,n}z, ..., T_{N,n}z) : n \in \mathbb{Z}_+\}$ is dense in X^N for some vector $z \in X$. We call vector z a d-universal for $(T_{1,n})_{n=1}^{\infty}$, $(T_{2,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$.*

Recall that S. Shkarin in [9] gave a short proof of existence of disjoint hypercyclic tuples of operators of any given length on any separable infinite dimensional Fréchet space. Similar argument provides disjoint dual hypercyclic tuples of operators of any length on any infinite dimensional Banach space with separable dual. For recent results on disjoint hypercyclicity (see [3]).

Definition 1.3. *Let $T : X \to X$ be a bounded linear operator on a Banach space X. For every $x \in X$ the sets $J(x) = \{y \in X :$ there exist a strictly increasing sequence of positive integers k_n and a sequence $x_n \subset X$ such that $x_n \to x$ and $T^{k_n}x_n \to y\}$ $J^{mix}(x) = \{y \in X :$ there exist a sequence $x_n \subset X$ such that $x_n \to x$ and $T^n x_n \to y\}$ will be called the extended limit set of x under T and the extended mixing limit set of x under T respectively.*

The notions of the limit and extended limit sets are well known in the theory of topological dynamics. In [5] G. Costakis and A. Manoussos, defined the J-sets, examined some basic properties of these sets and investigated the relation between hypercyclicity and J-sets. In particular they showed that T is hypercyclic if and only if there exists a cyclic vector $x \in X$ such that $J(x) = X$. Recall that a vector x is cyclic for T if the linear span of the orbit $Orb(T, x)$ is dense in X. In [12] we localized the notion of M-extended semigroup(resp. M-extended semigroup mixing) limit set of x under $\mathcal{T} = (T_t)_{t\geq 0}$ and we gave sufficient conditions of being M-hypercyclic for this semigroup. Then by this result, we proved that $(T_t^{-1})_{t\geq 0}$ is a M-hypercyclic.

On other hand, let $T_1, T_2, \ldots, T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets d-$J_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: for every neighborhood $V_0, V_1, \ldots, V_N$ of $x_0, x_1, \ldots, x_N$ respectively, there exists a positive integer n so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \ldots \cap T_N^{-n}(V_N)\}$ and d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: for every neighborhood $V_0, V_1, \ldots, V_N$ of $x_0, x_1, \ldots, x_N$ respectively, there exists a positive integer m so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \ldots \cap T_N^{-n}(V_N)$ for every $n \geq m\}$, will be called the extended limit set of x_0 under $T_1, T_2, \ldots, T_N$ and the extended mixing limit set of x_0 under $T_1, T_2, \ldots, T_N$ respectively. We have introduced this notion in [11], and we extended some results known for a single operator to a tuple of sequences of operators.

In this paper, we characterize the notion of d-universal extended (respectively d-universal extended mixing) limit set and we give a relation between d-universal (or d-hypercyclic sequences) and d-universal extended limit set.

2. d-universal J-class

Definition 2.1. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets*

d-$L^{univer}(x_0) := \{(x_1, \ldots, x_N) \in X^N$: there exists a strictly increasing sequence of positive integers k_n such that $T_{i,k_n} x_0 \to x_i$ for all $1 \leq i \leq N\}$

d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: for every neighborhood $V_0, V_1, \ldots, V_N$ of $x_0, x_1, \ldots, x_N$ respectively, there exists a positive integer n so that $\emptyset \neq V_0 \cap T^{-1}_{1,n}(V_1) \cap T^{-1}_{2,n}(V_2) \cap \ldots \cap T^{-1}_{N,n}(V_N)\}$ will be called the d-universal limit set and d-universal extended limit set of x_0 under $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ respectively.

Remark 2.2. An equivalent definition for the set d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$ is the following:

d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_{i,k_n} x_n \to x_i$ for all $1 \leq i \leq N\}$.

Theorem 2.3. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. *For every $x_0 \in X$, d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$.*
2. *$T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ is d-transitive.*
3. *The set of d-universal vectors for $(T_{1,n})^\infty_1, (T_{2,n})^\infty_1, \ldots, (T_{N,n})^\infty_1$ is a dense G_δ.*

Proof. We first prove: (1) implies (2). Let $V_0, V_1, \ldots, V_N$ the nonempty open. Consider $x_i \in V_i$ $(0 \leq i \leq N)$. Since d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$, so there exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^N T^{-1}_{i,n}(V_i)$. By definition $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ is d-transitive.

We will show that (2) implies (1). Let $x_i \in V_i$ $(0 \leq i \leq N)$ and $V_0, V_1, \ldots, V_N$ be relatively open subsets of X. So there exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^N T^{-1}_{i,n}(V_i)$. So $(x_1, \ldots, x_N) \in$ d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$, and consequently d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$.

For (2) $\Leftrightarrow$ (3) see [4, Proposition 2.3]. □

Proposition 2.4. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then*

$$d - J^{univer}_{(\lambda_1 T_{1,n},\lambda_2 T_{2,n},\ldots,\lambda_N T_{N,n})}(0) = d - J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(0),$$

for every $|\lambda_i| = 1 \quad (1 \leq i \leq N)$.

Proof. Let $(x_1, \ldots, x_N) \in \text{d-}J^{univer}_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \ldots, \lambda_N T_{N,n})}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $\lambda_i T_{i,k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1$ then $\lambda_i x_n \to 0$ and since $T_{i,k_n}(\lambda_i x_n) \to x_i$ it follows that $(x_1, \ldots, x_N) \in \text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(0)$.

Let $(x_1, \ldots, x_N) \in \text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $T_{i,k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1 \quad (1 \leq i \leq N)$, without loss of generality we can assume that $\lambda_i \to \mu_i$ for some $|\mu_i| = 1$. Hence $\lambda_i T_{i,k_n}(\frac{x_n}{\mu_i}) \to x_i$ for all $1 \leq i \leq N$ and since $\frac{x_n}{\mu_i} \to 0$ then $(x_1, \ldots, x_N) \in \text{d-}J^{univer}_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \ldots, \lambda_N T_{N,n})}(0)$. □

Lemma 2.5. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$ and $x_{0,n}$, $x_{1,n}$,. . . , $x_{N,n}$ be the sequences in X such that $x_{0,n} \to x_0$ and $x_{i,n} \to x_i$ for some $x_0, x_i \in X$ $(1 \leq i \leq N)$. If $(x_{1,n}, ..., x_{N,n}) \in d\text{-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_{0,n})$ for every $n = 1, 2, ...,$ then $(x_1, ..., x_N) \in d\text{-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0)$.*

Proof. For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{0,k_1} - x_0\| < \tfrac{1}{2} \quad \text{and} \quad \|x_{i,k_1} - x_i\| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Since $(x_{1,k_1}, \ldots, x_{N,k_1}) \in \text{d-}J_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_{0,k_1})$ we may find a positive integer l_1 and $z_1 \in X$ such that

$$\|z_1 - x_{0,k_1}\| < \tfrac{1}{2} \quad \text{and} \quad \|T_{i,l_1} z_1 - x_{i,k_1}\| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Therefore,

$$\|z_1 - x_0\| < 1 \quad \text{and} \quad \|T_{i,l_1} z_1 - x_i\| < 1 \ (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers l_n and a sequence z_n in X, such that

$$\|z_n - x_0\| < \tfrac{1}{n} \quad \text{and} \quad \|T_{i,l_n} z_n - x_i\| < \tfrac{1}{n} \ (1 \leq i \leq N).$$

This complete the proof. □

Proposition 2.6. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. If $\sup_n \|T_{i,n}\| < \infty$ for all $1 \leq i \leq N$, then $d\text{-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0) = d\text{-}L^{univer}(x_0)$ For every $x_0 \in X$.*

Proof. Since $\sup_n \|T_{i,n}\| < \infty$ for all $1 \leq i \leq N$ there exists a positive number M such that $\|T_{i,n}\| \leq M$ for every positive integer n and $1 \leq i \leq N$. Let $x_0 \in X$. If $\text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0) = \emptyset$ there is nothing to prove. Therefore assume that $\text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0) \neq \emptyset$.

Since the inclusion $\text{d-}L^{univer}(x_0) \subset \text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0)$ is always true, it suffices to show that $\text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0) \subset \text{d-}L^{univer}(x_0)$. Take $(x_1, ..., x_N) \in \text{d-}J^{univer}_{(T_{1,n}, T_{2,n}, \ldots, T_{N,n})}(x_0)$. There exist a strictly increasing sequence of positive

integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_{i,k_n}x_n \to x_i$ for all $1 \leq i \leq N$. Then we have

$$\begin{aligned} \|T_{i,k_n}x_0 - x_i\| &\leq \|T_{i,k_n}x_0 - T_{i,k_n}x_n\| + \|T_{i,k_n}x_n - x_i\| \\ &\leq M\|x_0 - x_n\| + \|T_{i,k_n}x_n - x_i\| \end{aligned}$$

and letting n goes to infinity to the above inequality, we get that $(x_1, \ldots, x_N) \in$ d-$L^{univer}(x_0)$. □

Proposition 2.7. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the set $A = \{x_0 \in X : d\text{-}J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N\}$ is closed.*

Proof. By Lemma 2.5 A is closed. □

Theorem 2.8. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. *$(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \ldots, (T_{N,n})_1^\infty$ is d-universal or d-hypercyclic sequences.*
2. *For every $x_0 \in X$, $d\text{-}J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$.*
3. *The set $A = \{x_0 \in X : d\text{-}J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N\}$ is dense in X^N.*

Proof. We first prove that (1) implies (2). Let $x_i \in V_i$ $(0 \leq i \leq N)$. Since the set of d-universal vectors for $(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \ldots, (T_{N,n})_1^\infty$ is a dense G_δ in X^N. Hence there exist a d-universal (or sequences of hypercyclic) vectors $(x_n) \subset X$ and a strictly increasing sequence of positive integers k_n such that $x_n \to x_0$ and $T_{i,k_n}x_n \to x_i$ for all $1 \leq i \leq N$. Hence $(x_1, \ldots, x_N) \in$ d-$J^{univer}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$.

The implication (2) $\Rightarrow$ (3) is trivial. Conversely (3) $\Rightarrow$ (2) By Lemma 2.5. □

3. d-mixing J-class

Definition 3.1. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets $d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: for every neighborhood $V_0, V_1, \ldots, V_N$ of $x_0, x_1, \ldots, x_N$ respectively, there exists a positive integer m so that $\emptyset \neq V_0 \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \ldots \cap T_{N,n}^{-1}(V_N)\}$ for every $n \geq m\}$ will be called the d-universal extended mixing limit set of x_0 under $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$.*

Proposition 3.2. *An equivalent definition for the set $d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$ is the following:*

$d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = \{(x_1, \ldots, x_N) \in X^N$: there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_{i,n}x_n \to x_i$ for all $1 \leq i \leq N\}$.

Proof. Let us prove that d-$J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) \subset \{(x_1, \ldots, x_N) \in X^N$: there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_{i,n}x_n \to x_i$ for all $1 \leq i \leq N\}$.

Let $(x_1, \ldots, x_N) \in \text{d-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$ and consider the open balls $V_{0,n} = B(x_0, \frac{1}{n}), V_{i,n} = B(x_i, \frac{1}{n})$ centered at $x_0, x_i \in X$ and with radius $1/n$ for $n = 1, 2, \ldots.$ and $1 \leq i \leq N$. Then there exists a positive integer m so that $\emptyset \neq V_{0,n} \cap T^{-1}_{1,n}(V_{1,n}) \cap T^{-1}_{2,n}(V_{2,n}) \cap \ldots \cap T^{-1}_{N,n}(V_{N,n})$ for every $n \geq m$. Hence there exists $x_n \in V_{0,n} = B(x_0, \frac{1}{n})$ such that $x_n \in \bigcap_{i=1}^{N} T^{-1}_{i,n}(V_{i,n})$, this implies that $T_{i,n}(x_n) \in V_{i,n}$ for all $i = 1, \ldots, N$. Therefore there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_{i,n}x_n \to x_i$ for all $1 \leq i \leq N$. The converse is obvious. □

Theorem 3.3. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. *For every $x_0 \in X$, $d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$.*
2. *$T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ is d-mixing.*

Proof. The proof goes similarly as in Theorem 2.3. □

Proposition 3.4. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then*

$$d - J^{u-mix}_{(\lambda_1 T_{1,n},\lambda_2 T_{2,n},\ldots,\lambda_N T_{N,n})}(0) = d - J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(0),$$

for every $|\lambda_i| = 1$, $(1 \leq i \leq N)$.

Proof. The proof goes similarly as in Proposition 2.4. □

Lemma 3.5. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$ and $x_{0,n}, x_{1,n}, \ldots, x_{N,n}$ be the sequences in X such that $x_{0,n} \to x_0$ and $x_{i,n} \to x_i$ for some $x_0, x_i \in X$ $(1 \leq i \leq N)$. If $(x_{1,n}, \ldots, x_{N,n}) \in d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_{0,n})$ for every $n = 1, 2, \ldots,$ then $(x_1, \ldots, x_N) \in d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0)$.*

Proof. For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{0,k_1} - x_0\| < \tfrac{1}{2} \quad \text{and} \quad \|x_{i,k_1} - x_i\| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Since $(x_{1,k_1}, \ldots, x_{N,k_1}) \in \text{d-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_{0,k_1})$ we may find a positive integer l_1 and $z_n \in X$ such that

$$\|z_n - x_{0,k_1}\| < \tfrac{1}{2} \quad \text{and} \quad \|T_{i,n}z_n - x_{i,k_1}\| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

For every $n \geq l_1$. Therefore,

$$\|z_n - x_0\| < 1 \quad \text{and} \quad \|T_{i,n}z_n - x_i\| < 1 \ (1 \leq i \leq N).$$

For every $n \geq l_1$. Proceeding inductively we find a strictly increasing sequence of positive integers $l_2 > l_1$ and a sequence $w_n \subset X$ such that

$$\|w_n - x_0\| < \tfrac{1}{2} \quad \text{and} \quad \|T_{i,n}w_n - x_i\| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

For every $n \geq l_2$. Set $v_n = z_n$ for every $l_1 \leq n \leq l_2$, hence

$$\|v_n - x_0\| < 1 \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < 1 \ (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers n_k and a sequence $v_n \subset X$ such that if $n \geq k$ then

$$\|v_n - x_0\| < \tfrac{1}{k} \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < \tfrac{1}{k} \ (1 \leq i \leq N).$$

Take any $\varepsilon > 0$. There exists a positive integer k_0 such that $\frac{1}{k_0} < \varepsilon$. Hence for every $n \geq n_{k_0}$ we get

$$\|v_n - x_0\| < \tfrac{1}{k_0} < \varepsilon \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < \tfrac{1}{k_0} < \varepsilon \ (1 \leq i \leq N).$$

This completes the proof . □

Proposition 3.6. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the set $B = \{x_0 \in X : d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N\}$ is closed.*

Proof. By Lemma 3.5 B is closed. □

Theorem 3.7. *Let $T_{1,n}, T_{2,n}, \ldots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. *$(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \ldots, (T_{N,n})_1^\infty$ is d-mixing.*
2. *For every $x_0 \in X$, $d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N$.*
3. *The set $B = \{x_0 \in X : d\text{-}J^{u-mix}_{(T_{1,n},T_{2,n},\ldots,T_{N,n})}(x_0) = X^N\}$ is dense in X^N*

Proof. For (1) implies (2) we use the same argument as in the proof of Theorem 3.3.

The implication (2) ⇒ (3) is trivial. Conversely (3) ⇒ (2) by Lemma 3.5. □

Acknowledgement

The authors thank the referees for his suggestions and comments thorough reading of the manuscript.

References

[1] S.I. Ansari, *Existance of hypercyclic operatos on topological vector space,* J. F. Anal., 148 (1997), 384-390.

[2] L. Bernal-González, *Disjoint hypercyclic operators,* Studia Math, 182 (2) (2007), 113-130.

[3] J. Bès, Ö. Martin, A. Peris and S. Shkarin, *Disjoint mixing operators,* J. Funct. Anal., 263 (2012), 1283-1322.

[4] J. Bès and A. Peris, *Disjointness in hypercyclicity,* J. Math. Anal. Appl., 336 (2007), 297-315.

[5] G. Costakis, A. Manoussos, *J-class operators and hypercyclicity,* J. Operator Theory, 67 (1) (2012), 101-119.

[6] K. Goswin and G. Erdmann, *Universal families and hypercyclic operators.* Bulletin of the American Mathematical Society, 36 (1999), 345-381.

[7] Karl-G. Grosse-Erdmann, Alfred Peris Manguillot, *Linear chaos*, Springer-Verlag London Limited, 2011.

[8] F. Leon-Saavedra, V. Muller, *Hypercyclic sequences of operators*, Studia Math., 175 (1) (2006), 1-18.

[9] S. Shkarin, *A short proof of existence of disjoint hypercyclic operators,* J. Math. Anal. Appl., 367 (2) (2010), 713-715.

[10] A.Tajmouati, M. El Berrag, *Some results on hypercyclicity of tuple of operators,* Italian Journal of Pure and Applied Mathematics, 35-2015, 487-492.

[11] A. Tajmouati, M. El Berrag, *Disjoint J-class operators,* Italian Journal of Pure and Applied Mathematics, 37 (2017), 19-28.

[12] A. Tajmouati, M. El Berrag, *J-class semigroup operators,* International Journal of Pure and Applied Mathematics, 109, 4 (2016), 861-868.

Accepted: 28.07.2017

PAIRWISE CONNECTEDNESS IN FUZZY BITOPOLOGICAL SPACES IN QUASI-COINCIDENCE SENSE

Ruhul Amin*
Department of Mathematics
Faculty of Science
Begum Rokeya University
Rangpur, Rangpur-5404
Bangladesh
ruhulbru1611@gmail.com

Sahadat Hossain
Department of Mathematics
Faculty of Science
University of Rajshahi
Rajshahi-6205
Bangladesh

Abstract. In this paper, we have defined a new notion of fuzzy connectedness in fuzzy bitopological spaces in sense of quasi-coincidence sense. We have found the relations among our and other such notions. We have observed that our notion is stronger than some other such notions. We have shown that the pairwise fuzzy connectedness is preserved under the FP-continuous mapping. Moreover, we have obtained productivity and some other properties of this new concept.

Keywords: quasi-coincidence, fuzzy bitopological spaces, fuzzy pairwise connectedness.

1. Introduction

Zadeh [17] introduced the concept of a fuzzy set in his classical paper. Using this notion Chang [3] introduced the concept of fuzzy topological space. Since then, many authors have studied successfully to generalize several concepts of general topology to the fuzzy setting. The notion of bitopological spaces was initially introduced by Kelly [8] in 1963. Kandil and El-Shafee [7] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. The concepts of fuzzy connectedness have been introduced earlier by Lowen and Srivastava [9], Wuyts [16], Dewan Muslim Ali [1], Tapi and Deole [13], Fatteh and Bassan [4] and G. Jager [6]. In [11], Park and Lee generalized the concept of fuzzy extremally disconnected spaces due to Ghosh [5] into a fuzzy bitopoogical setting and discuss some of its properties. In 2002,

∗. Corresponding author

Chandrasekar and Balasubramanian [2] defined weaker forms of connectedness and stronger forms of disconnectedness in fuzzy bitopological spaces.

The purpose of this paper is to introduce a new notion of fuzzy connectedness in fuzzy bitopological space in the light of quasi-coincidence sense and compare with [2]. We investigate that pairwise fuzzy connectedness is preserved under FP-continuous mapping. Moreover, we obtain some other properties of this new concept.

2. Priliminaries

For the purpose of the main results, we need to introduce some definitions and notations. Through this paper, X will be a nonempty set, $I = [0, 1]$ and FP stands for fuzzy pairwise. The notations (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space respectively and μ_A will be denoted the characteristics function on the subset A of X.

Definition 2.1 ([17]). *A fuzzy set μ in a set X is a function from X into the closed unit interval $I = [0, 1]$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of x. A member of I^X may also be called fuzzy subset of X.*

Definition 2.2 ([17]). *Let f be a mapping from a set X into a set Y and u a fuzzy set in X. Then the image of u, written as $f(u)$, is a fuzzy set in Y whose membership function is given by*

$$f(u)(y) = \begin{cases} \sup\{u(x)\}, & \text{if } f^{-1}[\{y\}] \neq \Phi,\ x \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 ([17]). *Let f be a mapping from a set X into a set Y and v be a fuzzy set in Y. Then the inverse of v, denoted by $f^{-1}(v)$ a fuzzy set in X, is defined by $f^{-1}(v)(x) = v(f(x))$, for all $x \in X$.*

Definition 2.4 ([15]). *A fuzzy set μ in X is called a fuzzy singleton if and only if $\mu(x) = r$, $(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. We call x_r is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in X is denoted by $S(X)$.*

Definition 2.5 ([3]). *A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called t-closed (or simply closed) fuzzy set if $1 - \mu \in t$.*

Definition 2.6 ([3]). *A function f from a fuzzy topological space (X, t) into a fuzzy topological space (Y, s) is called fuzzy continuous if and only if for every $u \in s$, $f^{-1}(u) \in t$.*

Definition 2.7 ([7]). *A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q \mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q} \mu$.*

Definition 2.8 ([8]). *Let X be any non empty set and S and T be any two general topologies on X then the triple (X, S, T) is called bitopological space.*

Definition 2.9 ([7]). *A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X.*

Definition 2.10 ([10]). *A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called FP-continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.*

Definition 2.11 ([14]). *Let $\{(X_i, s_i, t_i), i \in \bigwedge\}$ be a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called product fuzzy bitopological space of the family $\{(X_i, s_i, t_i), i \in \bigwedge\}$, where $\prod s_i$, $\prod t_i$ respectively denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \bigwedge\}$ and $\{\prod t_i : i \in \bigwedge\}$ of the fuzzy topologies on X.*

A fuzzy topological property P is called productive if the product of a family of fbts, each having property P, has property P.

3. The main results

The aim of this section is to introduce a new notion of fuzzy pairwise connectedness in fuzzy bitopological spaces by using quasi-coincidence sense.

Definition 3.1 ([2]). *An fbts (X, s, t) is called pairwise fuzzy connected if X has no proper fuzzy sets $u \in s$, $v \in t$ such that $u + v = 1$.*

Definition 3.2 ([2]). *An fbts (X, s, t) is called pairwise fuzzy strongly connected if it has no proper fuzzy sets u, $v \in s^c \cup t^c$ such that $u + v \leq 1$.*

Definition 3.3. *An fbts (X, s, t) is called pairwise fuzzy q-connected if it has no proper fuzzy sets $u \in s^c$, $v \in t^c$ such that $u \bar{q} v$.*

Theorem 3.1. *In a fuzzy bitopological space, the following implications hold:*

Pairwise fuzzy strongly connectedness

$\Downarrow$

Pairwise fuzzy q-connectedness

$\Downarrow$

Pairwise fuzzy connectedness.

Proof. The proof is obvious.

However, the converses are not true in general as shown in the following examples.

Example. Let $X = \{a, b\}$ and let s be indiscrete fuzzy topology on X and t be discrete fuzzy topology on X. Then (X, s, t) is pairwise fuzzy q-connected but not pairwise strongly fuzzy connected.

Example. Let $X = [0, 1]$. Let s be a fuzzy topology on X generated by $\{u\}$ where $u(x) = 0.7$ for all $x \in X$. Again, let t be a fuzzy topology on X generated by $\{v\}$ where $v(x) = 0.6$ for all $x \in X$. Then (X, s, t) is pairwise fuzzy connected but not pairwise fuzzy q-connected.

Theorem 3.2. *An fbts (X, s, t) is pairwise fuzzy q-connected iff it has no proper fuzzy sets $u \in s$, $v \in t$ such that $u + v \geq 1$.*

Proof. Suppose (X, s, t) is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that $u\bar{q}v$. That is, $u(x) + v(x) \leq 1$ for all $x \in X$. Put $\alpha = 1 - u$, $\beta = 1 - v$. Then $\alpha \in s$, $\beta \in t$ such that

$$\alpha(x) + \beta(x) \geq 1 \text{ for all } x \in X.$$

Hence $\alpha + \beta \geq 1$ which is a contradiction.

Conversely, suppose that (X, s, t) contains fuzzy sets $u \in s$, $v \in t$ such that $u + v \geq 1$. Put $\lambda = 1 - u$, $\mu = 1 - v$. Then $\lambda \in s^c$, $\mu \in t^c$. Now

$$\lambda(x) + \mu(x) = 1 - u(x) + 1 - v(x) = 2 - (u(x) + v(x)) \leq 1,$$

since $u + v \geq 1$. Thus (X, s, t) is not pairwise fuzzy q-connected which is a contradiction and hence the proof is complete.

Theorem 3.3. *An fbts (X, s, t) is pairwise fuzzy q-connected if X contains no proper fuzzy set u such that u is both s-open and t-closed or both t-open and s-closed.*

Proof. Suppose that X contains no proper fuzzy set u such that u is both s-open and t-closed. Then by definition, $1 - u$ is s-closed fuzzy set. Now,

$$(1 - u)(x) + u(x) = 1 \leq 1 \text{ for all } x \in X.$$

That is, $(1 - u)\bar{q}u$. Hence (X, s, t) is not pairwise fuzzy q-connected

Theorem 3.4. *Let (X, s, t) be a fuzzy topological space, $A \subset X$. If A is a pairwise fuzzy q-connected subset of X, then for any fuzzy sets $u \in s$, $v \in t$, $\mu_A \leq u + v$ implies either $\mu_A \leq u$ or $\mu_A \leq v$.*

Proof. Suppose that A is not pairwise fuzzy q-connected subset of X. Then there exist fuzzy sets $\lambda \in s^c, \delta \in t^c$ such that (i) $\lambda/A \neq 0$, (ii) $\delta/A \neq 0$ and $(iii)\,(\lambda/A)\bar{q}(\delta/A)$. Now, (iii) implies that

$$(\lambda/A)(x) + (\delta/A)(x) \leq 1 \quad \ldots\ldots\ldots \qquad (iv)$$

Now, if we put $u = 1 - \lambda$, $v = 1 - \delta$ then $u/A = 1 - \lambda/A$ and $v/A = 1 - \delta/A$. Hence (i), (ii) and (iv) imply that $\mu_A \leq u + v$ but $\mu_A \not\leq u$ and $\mu_A \not\leq v$.

Theorem 3.5. *If F is a subset of an fbts (X, s, t) such that μ_F is both s-open and t-closed in X, then X is pairwise fuzzy q-connected implies that F is a pairwise q-connected subset of X.*

Proof. Suppose F is not pairwise q-connected subset of X. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that

$$(i)\, u/F \neq 0,\ (ii) v/F \neq 0 \text{ and } (iii)\,(u/F)\bar{q}(v/F).$$

Now (iii) implies that

$$(u/F)(x) + (v/F)(x) \leq 1 \text{ for all } x \in F.$$

Since μ_F is s-open, then $1 - \mu_F$ is s-closed. Now we have

$$(u \cap (1 - \mu_F))(x) + (v \cap \mu_F)(x) \leq 1, \text{ for all } x \in X.$$

Also by (i) and (ii), we get

$$u \cap (1 - \mu_F) \neq 0 \text{ and } v \cap \mu_F \neq 0.$$

So, X is not pairwise fuzzy q-connected which is a contradiction.

Theorem 3.6. *If A and B are two subsets of a fuzzy bitopological space (X, s, t) and $\mu_A \leq \mu_B \leq \bar{\mu}_A$ and A is a pairwise fuzzy q-connected subset of X, then B is a pairwise fuzzy q-connected subset of X.*

Proof. Suppose B is not pairwise fuzzy q-connected subset of X. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that

$$(i)\, u/B \neq 0,\ (ii) v/B \neq 0 \text{ and } (iii)\,(u/B)\bar{q}(v/B).$$

We first show that $u/A \neq 0$. Suppose $u/A = 0$. Then it is clear that

$$u(x) + \mu_A(x) \leq 1 \text{ for all } x \in A.$$

This implies that

$$u(x) + \bar{\mu}_A(x) \leq 1 \text{ for all } x \in A.$$

So,

$$u(x) + \mu_B(x) \leq 1 \text{ for all } x \in A$$

since $\mu_B \leq \bar{\mu}_A$. Hence $u/B = 0$ which is a contradiction as $u/B \neq 0$. Therefore, $u/A \neq 0$. Similarly, we can show that $v/A \neq 0$.

Now from (iii) we get

$$u/B(x) + v/B(x) \leq 1 \text{ for all } x \in B.$$

So, we have $u/A(x)+v/A(x) \leq 1$ for all $x \in A$. as $\mu_A \leq \mu_B$. Hence $(u/A)\bar{q}(v/A)$. Therefore A is not pairwise fuzzy q-connected subset of X which is a contradiction.

In following theorem, we observe here that our concept is preserved under fuzzy continuous mapping.

Theorem 3.7. *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f : X \to Y$ be FP-continuous. Then X is pairwise fuzzy q-connected implies Y is pairwise fuzzy q-connected.*

Proof. Suppose Y is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \in s_1^c$, $v \in t_1^c$ such that $u\bar{q}v$. That is, $u(y) + v(y) \leq 1$ for all $y \in Y$. Since f is continuous, then $f^{-1}(u)$ and $f^{-1}(v)$ are non-zero s-open and t-closed respectively. Now

$$f^{-1}(u)(x) + f^{-1}(v)(x) = u(f(x)) + v(f(x)) \leq 1,$$

for all $x \in X$ since $u\bar{q}v$. Hence $f^{-1}(u)\bar{q}f^{-1}(v)$. Therefore X is not pairwise fuzzy q-connected which is a contradiction.

In the following theorem, we show that the productivity property holds in pairwise fuzzy q-connected spaces.

Theorem 3.8. *Product space is pairwise fuzzy q-connected if coordinate spaces are pairwise fuzzy q-connected.*

Proof. Let (X, s_1, t_1) and (Y, s_2, t_2) be two fuzzy bitopological spaces. Suppose the product space $(X \times Y, s_1 \times s_2, t_1 \times t_2)$ is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \times v \in t_1 \times t_2$, $\alpha \times \beta \in t_1 \times t_2$ such that $u \times v \neq 1$, $\alpha \times \beta \neq 1$ and for every $x \in X$, $y \in Y$

$$(u \times v)(x) + (\alpha \times \beta)(x, y) \geq 1.$$

That is, $\min\{u(x), v(y)\} + \min\{\alpha(x), \beta(y)\} \geq 1$. Hence $u(x) + \alpha(x) \geq 1$ and $v(y) + \beta(y) \geq 1$. Therefore, the coordinate spaces are not pairwise fuzzy q-connected which is a contradiction.

References

[1] D. M. Ali, *Some other types of Fuzzy connectedness*, Fuzzy Sets and Systems, 46 (1992), 55-61.

[2] V. Chandrasekar and G. Balasubramanian, *Weaker forms of connectedness and stronger forms of disconnectedness in fuzzy bitopological spaces,* Indian J. pure appl. Math., 33 (2002), 955-965.

[3] C. L. Chang, *Fuzzy topological spaces,* Journal of Mathematical Analysis and Applications, 24 (1968), 190-201.

[4] U. V. Fatteh and D. S. Bassan, *Fuzzy connectedness and Its stronger forms,* Journal of Mathematical Analysis and Applications, 111 (1985), 449-464.

[5] B. Ghosh, *Fuzzy extremally disconnected space*, Fuzzy Sets and Systems, 46 (1992), 245-250.

[6] G. Jager, *Compactness and connectedness as absolute properties in fuzzy topological spaces*, Fuzzy Sets and Systems, 94 (1998), 405-410.

[7] A. Kandil and M. E. El-Shafee, *Separation axioms for fuzzy bitopological space,* Journal of Institute of Mathematics and Computer Sciences, 4 (3) (1991), 373-383.

[8] J. C. Kelly, *Bitopological space,* Proc. London Math. Soc., 13 (1963), 71-89.

[9] R. Lowen and A. K. Srivastava, *On Preuss' connectedness concept in FTS*, Fuzzy Sets and Systems, 47 (1992), 99-104.

[10] A. Mukherjee, *Completely induced bifuzzy topological spaces,* Indian J. pure appl Math., 33 (6) (2002), 911-916.

[11] J. H. Park and B. Y. Lee, *Fuzzy pairwise extremally disconnected spaces,* Fuzzy sets and systems, 98 (1998), 201-206.

[12] A. K. Srivastava and D. M. Ali, *A note on K. K. Azads fuzzy Hausdorffness concepts*, Fuzzy sets and Systems, 42 (1991), 363-367.

[13] U. D. Tapi and B. A. Deole, *Strongly connectedness in fuzzy closure spaces*, Annals of Pure and Applied Mathematics 8 (1) (2014) 77-82.

[14] C. K. Wong, *Fuzzy topology: product and quotient theorems,* Journal of Mathematical Analysis and Applications, 45 (1974), 512-521.

[15] C. K. Wong, *Fuzzy points and local properties of fuzzy topology,* Journal of Mathematical Analysis and Applications, 46 (1974), 316-328.

[16] P. Wuyts, *Fuzzy path and fuzzy connectedness*, Fuzzy Sets and Systems, 24 (1987), 127-128.

[17] L. A. Zadeh, *Fuzzy sets,* Information and Control, 8 (1965), 338-353.

Accepted: 13.08.2017

HOPF BIFURCATION ANALYSIS AND AMPLITUDE CONTROL OF A NEW 4D HYPER-CHAOTIC SYSTEM

Ping Cai
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou 363000
China
caiping0596@163.com

Abstract. Hopf bifurcation and amplitude control in a new 4D hyper-chaotic system are investigated in this paper. Theoretical analysis shows that the system will exhibit Hopf bifurcation at equilibrium when the Hopf bifurcation conditions are satisfied. Relationship between the amplitude and control gains is given. Hence the amplitude of the limit cycle can be controlled by choosing suitable control gains, ensuring the stability of the bifurcating period solution. Finally, some applications of the amplitude control are carried out to illustrate the effectiveness of the main theoretical results. The accuracy of different kinds of control function are also compared.

Keywords: 4D hyper-chaotic system, Hopf bifurcation, limit cycle, amplitude control.

1. Introduction

Dynamics of nonlinear system is very rich in terms of bifurcation and chaos, and they have great potential applications in many areas of science, biology and engineering. Bifurcation analysis and control have been studied as early as in 60s of the last century, which play an important role in modern nonlinear dynamics [1, 2, 3, 4, 5]. In general, bifurcation control deals with designing a control law to modify the bifurcation characteristics. More specifically, in dynamic bifurcation control, Hopf bifurcation control has an essential role. During the last few years, great efforts have been devoted to investigating chaotic systems, such as Lorenz system [6], Chua's system [7], Chen system [8], Lü system [9], Liu system [10], T system [11] and other new chaotic system [12]. The problem of amplitude control of the bifurcated solution is becoming more and more widely concerned by researchers [13, 14, 15]. On the one hand, decreasing the amplitude can inhibit the harmful vibration behavior of the system. On the other hand, increasing the amplitude can make the vibration used by people. 4D hyper-chaotic system has more complicated dynamical behavior, which has recently become a hot topic [16, 17, 18, 19]. Base on Lorenz system, a new four-dimensional quadratic

autonomous hyper-chaotic attractors is present in Ref.[20],

$$\begin{cases} \dot{x_1} = a(x_2 - x_1), \\ \dot{x_2} = bx_1 - x_2 + ex_4 - x_1x_3, \\ \dot{x_3} = -cx_3 + x_1x_2 + x_1^2, \\ \dot{x_4} = -dx_2, \end{cases} \tag{1}$$

where x_1, x_2, x_3, x_4 are the state variables, a, b, c, d, e are positive real parameters. Several properties of system (1) were investigated, including analysis of Hopf bifurcation and estimation of ultimate bound. In this paper, we would like to investigate Hopf bifurcation and amplitude control of the system.

The rest of this paper is organized as follows. In Section 2, the local stability and Hopf bifurcation at the equilibrium are analyzed, then the bifurcation behavior in the model is given. In Section 3, a control strategy based on the introduction control parameters with quadratic nonlinearities is applied to the model. The relationship between the amplitude of the limit cycle and the control gains is given by using the normal form theory and the center manifold theorem. In Section 4, some applications of the amplitude control are given, and the effectiveness of the control strategy is verified through numerical simulation. The accuracy for different cases of control functions are compared. Conclusions are given in Section 5 finally.

2. Local stability and Hopf bifurcation analysis

Obviously, system (1) has only one equilibrium at $S_0(0, 0, 0, 0)$. The Jacobian matrix of system (1) at the equilibrium S_0 is given by

$$J = \begin{pmatrix} -a & a & 0 & 0 \\ b & -1 & 0 & e \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}. \tag{2}$$

And the characteristic equation is

$$\lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4 = 0, \tag{3}$$

where

$$\begin{aligned} k_1 &= 1 + a + c, \\ k_2 &= a - ab + c + ac + de, \\ k_3 &= ac - abc + ade + cde, \\ k_4 &= acde. \end{aligned}$$

Computing the following determinants:

$$\Delta_1 = k_1,$$
$$\Delta_2 = \begin{vmatrix} k_1 & 1 \\ k_3 & k_2 \end{vmatrix} = k_1k_2 - k_3,$$
$$\Delta_3 = \begin{vmatrix} k_1 & 1 & 0 \\ k_3 & k_2 & k_1 \\ 0 & k_4 & k_3 \end{vmatrix} = k_3(k_1k_2 - k_3) - k_1^2k_4,$$
$$\Delta_4 = \begin{vmatrix} k_1 & 1 & 0 & 0 \\ k_3 & k_2 & k_1 & 1 \\ 0 & k_4 & k_3 & k_2 \\ 0 & 0 & 0 & k_4 \end{vmatrix} = k_4\Delta_3.$$

If $k_1 > 0, k_3 > 0, k_4 > 0$ and $k_3(k_1k_2 - k_3) - k_1^2k_4 > 0$, then $\Delta_i > 0 (i = 1, 2, 3, 4)$. Based on Routh-Hurwitz criteria, all roots of the characteristic equation have negative real parts. Thus, S_0 is locally asymptotically stable. If $k_3(k_1k_2 - k_3) - k_1^2k_4 \leq 0$, and $k_i > 0 (i = 1, 2, 3, 4)$, S_0 is unstable and non-hyperbolic. Taking b as the Hopf bifurcation parameter, by the equation

$$k_3(k_1k_2 - k_3) - k_1^2k_4 = 0, \tag{4}$$

we get the critical value

$$b_0 = \frac{a + a^2 + de}{a + a^2}. \tag{5}$$

When $b = b_0$, the Jacobian matrix J has a pair of imaginary eigenvalues as follows:

$$\lambda_{1,2} = \pm i\omega_0 = \pm i\sqrt{\frac{ade}{1 + a}}. \tag{6}$$

The other two eigenvalues are

$$\lambda_3 = -1 - a < 0 \tag{7}$$

and

$$\lambda_4 = -c < 0. \tag{8}$$

Under these conditions, the following transversality condition is also satisfied:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda=i\omega_0}) = \frac{(1 + a)(3adek_1k_5 + 2c\omega_0k_6k_7)}{ade(9k_8 + 4k_9)} \neq 0, \tag{9}$$

where

$$
\begin{aligned}
k_5 &= cde + a(c + de + cde) + a^2(c + 2de),\\
k_6 &= c + 2ac + a^2c - ade,\\
k_7 &= de + (1 + \omega_0)(a^2 + a),\\
k_8 &= ade(1 + a)(1 + a + c)^2,\\
k_9 &= ((1 + a)^2c - ade)^2.
\end{aligned}
$$

Therefore, system (1) undergoes Hopf bifurcation at $S_0(0, 0, 0, 0)$ based on Hopf bifurcation theory [21]. The bifurcation results a family of limit cycles emerging from the equilibrium S_0 at the sufficiently small neighborhood of b_0. Next, a control strategy is applied to the model to control the amplitude of the limit cycle.

3. Relationship between the amplitude of limit cycle and control gains

In this section, a control strategy is applied to the model. The control functions are introduced for the quadratic nonlinearities of system (1), as shown below:

$$
\begin{cases}
\dot{x_1} = a(x_2 - x_1),\\
\dot{x_2} = bx_1 - x_2 + ex_4 - f(m) * x_1x_3,\\
\dot{x_3} = -cx_3 + g(n) * (x_1x_2 + x_1^2),\\
\dot{x_4} = -dx_2,
\end{cases}
\tag{10}
$$

where $f(m)$ and $g(n)$ are control functions. In general, the threshold of bifurcation is determined by the linear parts of the system, and the stability of bifurcating solution is determined by the non-linear parts of the system. So, the control approach do not shift the bifurcation critical value. And the original equilibrium $S_0(0, 0, 0, 0)$ is also preserved. By the linear transform $(x_1, x_2, x_3, x_4)^T = P(X_1, X_2, X_3, X_4)^T$, where

$$
P = \begin{pmatrix}
\frac{(a^2+a)\omega_0}{d(a^2+a+de)} & \frac{-ae}{a^2+a+de} & \frac{-a^2-a}{d} & 0\\
\frac{\omega_0}{d} & 0 & \frac{1+a}{d} & 0\\
0 & 0 & 0 & 1\\
0 & 1 & 1 & 0
\end{pmatrix},
\tag{11}
$$

then system (10) has the following normal form:

$$
\begin{cases}
\dot{X}_1 = -\omega_0 X_2 + F_1(X_1, X_2, X_3, X_4),\\
\dot{X}_2 = \omega_0 X_1 + F_2(X_1, X_2, X_3, X_4),\\
\dot{X}_3 = \lambda_3 X_3 + F_3(X_1, X_2, X_3, X_4),\\
\dot{X}_4 = \lambda_4 X_4 + F_4(X_1, X_2, X_3, X_4),
\end{cases}
\tag{12}
$$

where F_1, F_2, F_3, F_4 are high order nonlinear functions about X_1, X_2, X_3, X_4 , which are shown as follows:

$$
\begin{aligned}
&F_1(X_1, X_2, X_3, X_4)\\
&= \frac{(1+a)(a^2+2a+de+1)f(m)(-a^2\omega_0 X_1X_4 + a\omega_0^2 X_2X_4 + (a^4+a^3+a^2de)X_3X_4)}{\omega_0(a^2+de+a)(a^3+3a^2+3a+ade+1)},\\
&F_2(X_1, X_2, X_3, X_4)\\
&= \frac{af(m)((1+a)^2\omega_0 X_1X_4 - (1+a)deX_2X_4 - (a^4+3a^3+3a^2+a+(1+a)^2de)X_3X_4)}{a^5+4a^4+2a^3(3+de)+4a^2(1+de)+a(1+3de+d^2e^2)+de},\\
&F_3(X_1, X_2, X_3, X_4)\\
&= \frac{(a^2+a)f(m)(-(1+a)\omega_0 X_1X_4 + deX_2X_4 + (1+a)(a+a^2de)X_3X_4)}{a^5+4a^4+2a^3(3+de)+4a^2(1+de)+a(1+3de+d^2e^2)+de},\\
&F_4(X_1, X_2, X_3, X_4)\\
&= \frac{ag(n)((1+a)(2a+2a^2+de)\omega_0^2X_1^2 - (3a^2de+3ade+d^2e^2)\omega_0X_1X_2 + ad^2e^2X_2^2)}{d(a+a^2+de)}\\
&+ a(a+1)g(n)\left(\frac{(1-de-2a-3a^2)\omega_0}{d}X_1X_3 + ae(2a-1)X_2X_3\right.\\
&\left.+\frac{(a^2-1)(a^2+a+de)}{d}X_3^2\right).
\end{aligned}
$$

Based on the center manifold theory and normal form reduction, the curvature coefficient is expressed by the following according to Ref. [21]:

$$\sigma_1 = Re\left\{\frac{g_{20}g_{11}}{2\omega_0}i + G^1_{110}w^1_{11} + G^2_{110}w^2_{11} + \frac{G_{21}+G^1_{101}w^1_{20}+G^2_{101}w^2_{20}}{2}\right\}. \tag{13}$$

The characteristic quantities can be calculated from system (12) as follows:

$$g_{11}=0, g_{20}=0, G^1_{110}=0, G^1_{101}=0, G_{21}=0, w^1_{11}=0, w^1_{20}=0,$$

$$
\begin{aligned}
G^2_{110} &= \frac{-(a(a+1)^2+i(a^2+a)\omega_0)f(m)}{2(a^3+3a^2+3a+ade+1)},\\
G^2_{101} &= \frac{-a^5-3a^3-3a^2-a^2de+ade-a+i\omega_0(a^3+4a^2+5a+ade+de+2)}{2(a^2+a+de)(a^3+3a^2+3a+ade+1)},\\
w^2_{11} &= \frac{a^2eg(n)}{cd(a^2+a+de)},\\
w^2_{20} &= \frac{((-2a^3-2a^2)i+(3a^2+3a+de)\omega_0)aeg(n)}{2d(a^2+a+de)^2(2\omega_0-ic)}.
\end{aligned}
$$

So an explicit expression of the curvature coefficient is written as:

$$\sigma_1 = -\frac{a^2e\sqrt{1+a}f(m)g(n)(k_{10}+k_{11})}{8cdk_{12}k_{13}}, \tag{14}$$

where

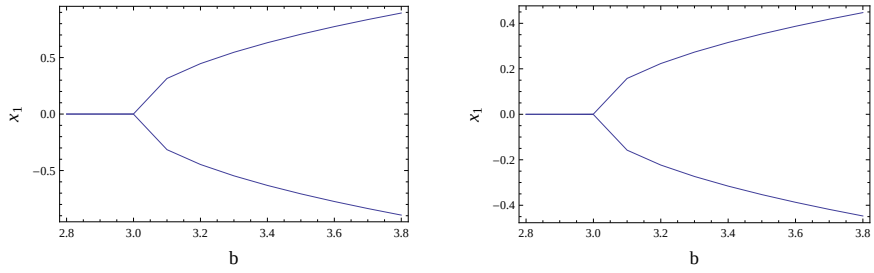
$$
\begin{aligned}
k_{10} &= \sqrt{1+a}(-2c\omega_0^2 + ac^2(6+de) + 2a^3(3c^2+8de)),\\
k_{11} &= 2a^2\sqrt{1+a}(6c^2+8de+c\omega_0^2),\\
k_{12} &= (1+a)c^2 + 4ade,\\
k_{13} &= a^5 + 4a^4 + 2a^3(3+de) + 4a^2(1+de) + a(1+3de+d^2e^2) + de.
\end{aligned}
$$

Therefore, the approximate amplitude in close vicinity to the Hopf bifurcation point is

$$
\begin{aligned}
r &= \sqrt{-\frac{\alpha'(0)}{\sigma_1}(b-b_0)}\\
&= \sqrt{\frac{8\sqrt{1+a}ack_{12}k_{13}(3adek_1k_5+2cw_0k_6k_7)}{a^3e^2f(m)g(n)(9k_8+4k_9)(k_{10}+k_{11})}(b-b_0)}, \quad |b-b_0| \ll 1.
\end{aligned} \tag{15}
$$

4. Application of amplitude control

It should be pointed out that Eq.(15) describes the relationship between the amplitude of the state variable x_1 and the control functions. Other linear transformation can be chosen to describe the amplitude of the other state variables. We choose $a = 2, c = 1, d = 3, e = 4$, then $b_0 = 3, \alpha'(0) = 0.484654, \sigma_1 = -0.0546643f(m)g(n)$. Obviously, σ_1 degrade into the curvature coefficient of the original system(1) when $f(m)g(n) = 1$. If $f(m)g(n) > 0$, the bifurcated limit cycle is stable, and then the parameter $\mu_2 = -\frac{\sigma_1}{\alpha'(0)} > 0$, where the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $b > b_0$. Since this control strategy does not change the bifurcation critical value of the system, it means that both the original system and the controlled system bifurcated at $b_0 = 3$. The bifurcation figures are shown in Fig.1.



(a) the original system (b) the controlled system

Fig. 1. Bifurcation diagrams of the original system and the controlled system at equilibrium S_0.

Setting $b = 3.10$, time displacement curves of the period solutions under different value of control gains are shown in Fig.2 and Fig.3, respectively. The solid lines denote the period solutions of the original system while the dashed lines represent the period solutions of the controlled system. It can be seen that under different values of the control functions, the amplitude can be large or small. Other values can be similar.

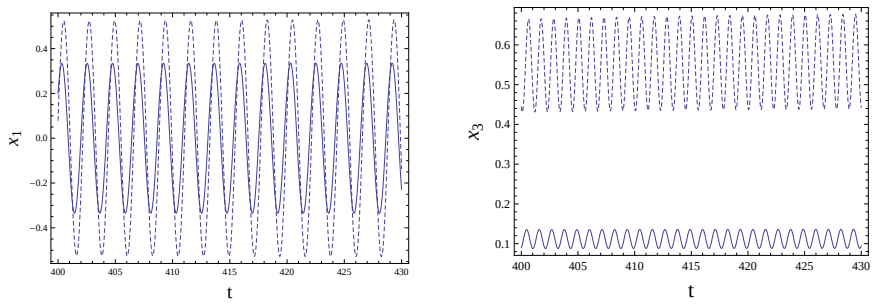


Fig. 2. Time displacement curves of period solutions ($0 < f(m)g(n) < 1, 0 < f(m) < 1$).

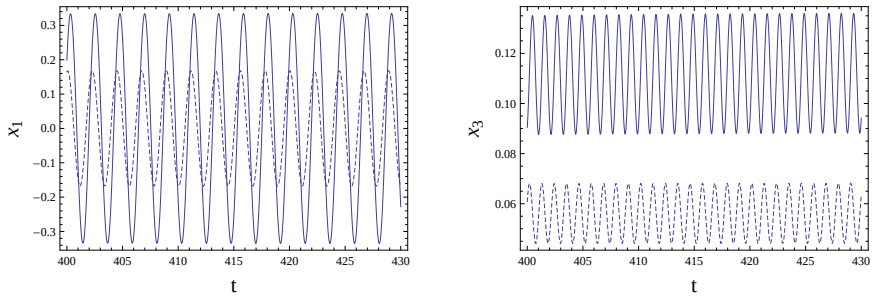
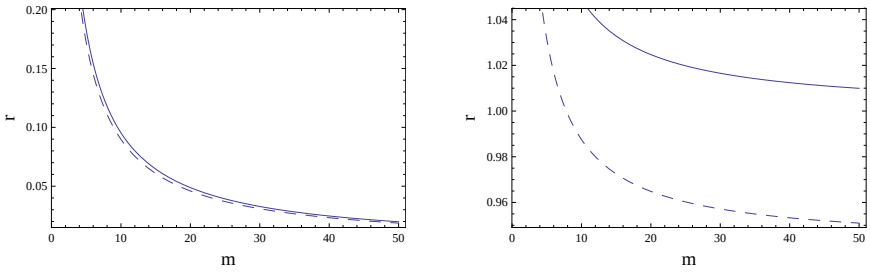


Fig. 3. Time displacement curves of period solutions ($f(m)g(n) > 1, f(m) > 1$).

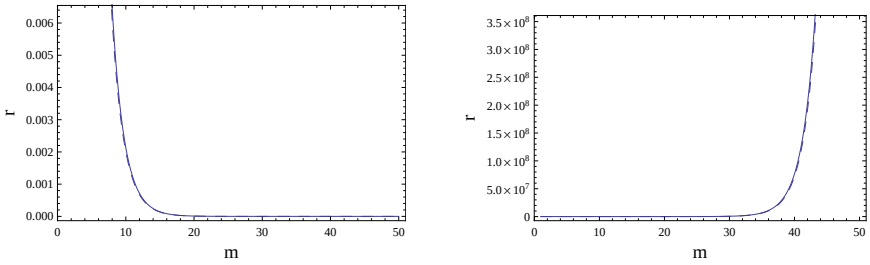
For simplicity, suppose $m = n$. Let $f(m) = m$, while $g(m) = m + 1, g(m) = \frac{1}{m+1}, g(m) = e^m$ and $g(m) = \frac{1}{e^m}$, the accuracy of approximate solution of amplitude(15) and numerical solution is compared in Fig.4. In this case, the dotted lines represent the approximate solution, and the solid lines represent the numerical solution.

5. Conclusion

This paper is concerned about Hopf bifurcation and amplitude control of a hyper-chaotic system. Applying Routh-Hurwitz criterion, the stability of the equilibrium is investigated. Then the existence of Hopf bifurcation is given based on Hopf bifurcation theory. A control approach is applied to the system, which not only keeps the equilibrium structure of the original system, but also not change the Hopf bifurcation critical value. By the normal form theory and the center manifold theorem, the relationship between the amplitude of the limit



(a) $f(m) = m, g(m) = m + 1$ (b) $f(m) = m, g(m) = \frac{1}{m+1}$



(c) $f(m) = m, g(m) = e^m$ (d) $f(m) = m, g(m) = e^{-m}$

Fig. 4. The accuracy of approximated solution of amplitude and numerical solution.

cycle and the control gains is presented. Numerical simulations show, in case of the stability of the bifurcating periodic solutions, the control law can increase or decrease the amplitude of the periodic solution effectively under different control gains. The accuracy of different kinds of control functions is also compared.

Acknowledgment

The author wishes to thank the anonymous reviewers for their valuable comments and suggestions.

Project supported by Natural Science Foundation of Fujian Province of China (No. 2017J05012); and National Natural Science Foundation of China for Youth (Nos. 61403181, 61603174); and the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Date Science and Statistics.

References

[1] C.X. Liang, J.S. Tang, *Equilibrium points and bifurcation control of a chaotic system*, Chin. Phys. B, 17 (2008), 0135-0139.

[2] S.H. Liu, J.S. Tang, J.Q. Qin and X.B. Yin, *Bifurcation analysis and control of periodic solutions changing into invariant tori in Langford system*, Chin. Phys. B, 17 (2008), 1691-1697.

[3] G.R. Jiang, B.G. Xu and Q.G. Yang, *Bifurcation control and chaos in a linear impulsive system,* Chin. Phys. B, 18 (2009), 5235-5241.

[4] J. Zhou, *Bifurcation analysis of the Oregonator model*, Appl. Math. Lett., 52 (2016), 192-198.

[5] D.W. Ding, X.Y. Zhang, J.D. Cao, N. Wang and D. Liang, *Bifurcation control of complex networks model via pd controller*, Neurocomputing, 175 (2016), 1-9.

[6] H.W. Li and M. Wang, *Hopf bifurcation analysi in a Lorenz-type system*, Nonlinear Dyn., 71 (2013), 235-240.

[7] J.H. Yang and L.Q. Zhao, *Bifurcation analysis and chaos control of the modified Chua's circuit system*, Chaos Soliton Fract., 77 (2015), 332-339.

[8] Z.S. Cheng, *Anti-control of Hopf bifurcation for Chen's system through washout filters*, Neurocomputing, 73 (2010), 3139-3146.

[9] Z.S. Lü and L.X. Duan, *Control of codimension-2 bautin bifurcation in chaotic lu system,* Commun. Theor. Phys., 52 (2009), 631-636.

[10] Z.S. Lü and L.X. Duan, *Anti-control of Hopf bifurcation in the chaotic Liu system with symbolic computation*, Chin. Phys. Lett., 26 (2009), 050504.

[11] R.Y. Zhang, *Bifurcation analysis for T system with delayed feedback and its application to control of chaos,* Nonlinear Dyn., 72 (2013), 629-641.

[12] P. Cai and Z.Z. Yuan, *Hopf bifurcation and chaos control in a new chaotic system via hybrid control strategy*, Chinese J. Phys., 55 (2017), 64-70.

[13] J.S. Tang, F. Han, H. Xiao and X. Wu, *Amplitude control of a limit cycle in a coupled van der Pol system*, Nonlinear Anal.-Theor., 71 (2009), 2491-2496.

[14] C. Yan, S.H. Liu, J.S. Tang and Y.M. Meng, *Amplitude control of limit cycles in Langford system,* Chaos Soliton Fract., 42 (2009), 335-340.

[15] P. Cai, J. S. Tang, *Control of amplitude of limit cycles in a class of strongly nonlinear oscilation systems*, J. Vib. Shock (in Chinese), 32 (2013), 110-112.

[16] C.L. Li, K.L. Su, J. Zhang and D.Q. Wei, *Robust control for fractional-order four-wing hyperchaotic system using LMI*, Optik, 12 (2013), 5807-5810.

[17] W. Xue, G.Y. Qi, J.J. Mu, H.Y. Jia and Y.L. Guo, *Hopf bifurcation analysis and circuit implementation for a novel four-wing hyper-chaotic system,* Chin. Phys. B, 22 (2013), 080504.

[18] P. Cai, J.S. Tang and Z.B. Li, *Controlling Hopf bifurcation of a new modified hyperchaotic system,* Math. Prob. Eng., 2015, 614135.

[19] L.L. Zhou, Z.Q. Chen, Z.L. Wang and J.Z. Wang, *On the analysis of local bifurcation and topological horseshoe of a new 4D hyper-chaotic system,* Chaos Soliton Fract., 91 (2016), 148-156.

[20] A. Zarei and S. Tavakoli, *Hopf bifurcation analysis and ultimate bound estimation of a new 4-D quadratic autonomous hyper-chaotic system,* Appl. Math. Comput., 291 (2016), 323-339.

[21] B.D. Hassard, N.D. Kazarinoff and Y. Wan, *Theory and Applications of Hopf Bifurcation,* Theory and Applications of Hopf Bifurcation, London Cambridge Univ., 1981.

Accepted: 1.09.2017

SOLUTION OF LINEAR AND NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING LEGENDRE WAVELET METHOD

Javid Iqbal*
Department of Mathematical Sciences
BGSB University
Rajouri-185234, J & K
India
javid2iqbal@gmail.com

Rustam Abass
Department of Mathematical Sciences
BGSB University
Rajouri-185234, J & K
India
rustamabass13@gmail.com

Puneet Kumar
Department of Applied Sciences
Dronacharya College
Greater Noida-201308, U.P.
India
puneetkmath@gmail.com

Abstract. In this paper, we utilize a robust and precise method for solving both linear and nonlinear singular initial or boundary value problems. We use Legendre wavelets to construct operational matrix of integration and product operational matrix to solve the problems. This method reduces the problems into algebraic equations and gives a fast convergent series of easily computable components. Illustrative examples are incorporated to show the productivity and exactness of the technique. The outcomes obtained by the utilized method demonstrate that the proposed way is entirely sensible when compared with exact solution.

Keywords: Legendre wavelets, operational matrix of integration, product operational matrix, singular value problems, MATLAB.

1. Introduction

The investigation of singular initial as well as boundary value problems of the second order ordinary differential equations (ODEs) have attracted a special attention of many mathematicians and physicists. One of the equations describing

*. Corresponding author

this type is formulated as

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad a < t < b, \tag{1.1}$$

with the initial conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2$$

or boundary conditions

$$y(a) = \alpha_1, \quad y(b) = \alpha_2,$$

where a, b, α_1 and α_2 are finite constants, whereas the coefficients $p(t), q(t)$ and $f(t)$ are given continuous functions. Eq.(1.1) with singularities appears to many phenomena in mathematical physics, astrophysics, chemistry and mechanics such as in the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents. While making the model in these fields, singularity typically occurs at the end of the interval of integration which causes the computed solution in losing its accuracy in the vicinity of the singular points. A systematic study of the formulation of these models and the physical structure of the solutions can be found in [5, 7, 10, 12]. Due to the presence of singularity, it is not easy to derive the analytical or exact solutions of most of the singular boundary value problems, so that it requires an efficient and accurate numerical method for solving these kind of problems. A number of mathematicians and physicists have attempted to solve these sort of problems which are available in the literature. Biazar et al. [3] used He's variational iteration method for solving linear cum nonlinear ordinary differential equations. Lu [18] employed the variational iteration method for solving a nonlinear system of second-order boundary value problems. Rashidinia et al. [22] suggested a convergence of cubic-spline technique for solving boundary-value problems, while B-spline approach has been used by Caglar et al. [4]. Later on, Tatari et al. [26] generalized He's variational iteration method for solving the same systems of singular initial value problems. Homotopy perturbation with reproducing kernel has been applied by Geng and Cui [11] whereas Akinfenwa et al. [1] have employed the continuous block backward differentiation formula for solving the same behaviour Stiff differential equations. Again Dehghan and Nikpour [9] showed their interest to solve singular initial value problem by differential quadrature collocation method based on local basis functions. Moreover, Kadalbajoo and Aggarwal [14] have also employed a method based on Chebyshev polynomials coupled with B-spline for solving singular boundary value problems in which they have suggested the economized expansion procedure to avoid the singularity from the singular BVPs. Although all these methods has successfully solved these type of problems but still small convergence rate and high computational efforts compel us to use a high accuracy and low computational method. For more details, refer [16] and the references therein.

In recent years, wavelet theory [8] have received a special attention by researchers from both theoretical and practical point of view in different fields of science and engineering. It possesses several beautiful properties like orthogonality, compact support, accurate representation of concern polynomial at a certain degree and ability to represent different type of functions at different levels of resolution. In addition, wavelet method has made a connection with fast numerical algorithms [2]. From the last decades many wavelet methods such as Haar wavelet method [6, 7, 17], Daubechies wavelet method [8, 20], Haar wavelet collocation method [25], Chebyshev wavelet method [13, 15], Legendre wavelet method [23, 24, 28] and Laguerre wavelet method [29] were used to solve differential equations but among them Legendre wavelet method attracted much attention. The key strength of Legendre wavelet method (LWM) is to convert the differential equations into a system of algebraic equations by the operational matrices of integral.

In this paper, we use Legendre wavelet method to solve the singular initial or boundary value problems by using operational matrix of integration along with product operational matrix. This paper is organized as follows. In Section 2, we present the Legendre wavelets, their approximation and the operational matrix of integration. The derivation of product operational matrix and the convergence analysis of Legendre wavelets is discussed in Section 3. Several numerical examples are included in Section 4 to show the efficiency and accuracy of our proposed method. In the end, conclusions are appended.

2. Definition and properties of Legendre wavelets

In this section, we state some necessary definitions and preliminaries of Legendre wavelet theory which are required for establishing our results.

2.1 Wavelets and Legendre wavelets

For last two decades, wavelets have found their ways towards recent fields of science and technology. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet $\psi(t)$ as:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right) \quad a,\, b \in \mathbb{R},\ a \neq 0.$$

If we choose $a = a^{-k}$ and $b = nba^{-k}$ where $a > 1$, $b > 0$ and $n, k \in \mathbb{Z}^+$, then we get the following family of discrete wavelets as

$$\psi_{k,n}(t) = |a|^{\frac{k}{2}}\, \psi(a^k t - nb),$$

where $\psi_{k,n}(t)$ forms a basis for $L^2(\mathbb{R})$. In particular, when $a = 2$ and $b = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{n,m} = \psi(k, \acute{n}, m, t)$ have four arguments; $k = 2, 3, \ldots,$ $\acute{n} = 2n - 1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, m is the order of Legendre polynomials and t de-

notes the time. Legendre wavelets defined in [24] are as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)} 2^{\frac{k}{2}} T_m(2^k t - 2n + 1), & t \in [\frac{2n-2}{2^k}, \frac{2n}{2^k}], \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $m = 0, 1, 2, \ldots, M-1$. The coefficient $\sqrt{\left(m+\frac{1}{2}\right)}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \acute{n}2^{-k}$. Here, $T_m(x)$ are well-known Legendre polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ on the interval [-1,1] and can be determined with the aid of the following recurrence formulae:

$$T_0(t) = 1, T_1(t) = t,$$

$$T_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t T_m(t) - \left(\frac{m}{m+1}\right) T_{m-1}(t), \quad m = 1, 2, 3, ...$$

The set of Legendre wavelets $\psi_{n,m}(t)$ also forms an orthonormal set with respect to the weight function $w(t) = 1$.

2.2 Function approximation

Any function f defined over $[0, 1)$ can be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \tag{2.2}$$

where

$$c_{n,m} = \langle f(t), \psi_{n,m}(t) \rangle = \int_0^t f(t) \psi_{n,m}(t) dt. \tag{2.3}$$

The series in Eq.(2.2) is truncated, which can then be written as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t), \tag{2.4}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices are given by

$$\begin{aligned} C = [&c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, \\ &c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}M-1}]^T, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Psi(t) = [&\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \psi_{21}(t), \ldots, \\ &\psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \psi_{2^{k-1}1}(t), \ldots, \psi_{2^{k-1}M-1}(t)\Big]^T. \end{aligned} \tag{2.6}$$

The suitable collocation points depends on resolution are as follows:

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M. \tag{2.7}$$

2.3 Operational matrix of integration

The integration of $\Psi(t)$ defined in Eq.(2.6) can be approximated by Legendre wavelet series with Legendre wavelet coefficient matrix P as

$$\int_0^t \Psi(t)dt = P\Psi(t). \tag{2.8}$$

Now we construct the structure of operational matrix of integration for $\psi(t)$ which is defined in Eq.(2.1). To illustrate the calculation procedure, we choose $M = 3$, $k = 2$. On the basis of chosen values we get the six basis function which are given by

$$\Psi_1 = \psi_{1,0} = \begin{cases} 2^{1/2}, & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}, \quad \Psi_2 = \psi_{1,1} = \begin{cases} 6^{1/2}(4t-1), & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

$$\Psi_3 = \psi_{1,2} = \begin{cases} (10)^{1/2}\left[\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right], & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases},$$

$$\Psi_4 = \psi_{2,0} = \begin{cases} 2^{1/2}, & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases}$$

$$\Psi_5 = \psi_{2,1} = \begin{cases} 6^{1/2}(4t-3), & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases},$$

$$\Psi_6 = \psi_{2,2} = \begin{cases} (10)^{1/2}\left[\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right], & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\Psi_{6\times 6} = \begin{bmatrix} \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \frac{\sqrt{10}}{2} & 0 & 0 & 0 \\ \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & \frac{\sqrt{10}}{2} \\ 0 & 0 & 0 & \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \end{bmatrix}. \tag{2.9}$$

So, integrating the above defined basis $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6$ from 0 to t and using wavelet coefficients, we obtain

$$\int_0^t \Psi_1(t)dt = \begin{cases} 2^{1/2}t, & t \in [0, \frac{1}{2}) \\ \dfrac{2^{1/2}}{2}, & t \in [\frac{1}{2}, 1) \end{cases} = \left[\frac{1}{4}, \frac{2^{1/2}}{4 \times 6^{1/2}}, 0, \frac{1}{2}, 0, 0\right]^T \Psi_6(t),$$

$$\int_0^t \Psi_2(t)dt = \begin{cases} 2 \times 6^{1/2}t^2 - 6^{1/2}t, & t \in [0, \frac{1}{2}) \\ 0, & t \in [\frac{1}{2}, 1) \end{cases}$$
$$= \left[-\frac{3^{1/2}}{12}, 0, \frac{3^{1/2}}{12 \times 5^{1/2}}, 0, 0, 0\right]^T \Psi_6(t),$$

Similarly, we have

$$\int_0^t \Psi_3(t)dt = \left[0, -\frac{5^{1/2}}{20 \times 3^{1/2}}, 0, 0, 0, 0\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_4(t)dt = \left[0, 0, 0, \frac{1}{4}, \frac{2^{1/2}}{4 \times 5^{1/2}}, 0\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_5(t)dt = \left[0, 0, 0, \frac{-3^{1/2}}{12}, 0, \frac{3^{1/2}}{12 \times 5^{1/2}}\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_6(t)dt = \left[0, 0, 0, 0, \frac{-5^{1/2}}{20 \times 3^{1/2}}, 0, \right]^T \Psi_6(t).$$

Due to the support of $\Psi_i,\ i = 1, 2, \ldots 6$, it is obvious that we have matrix P in [24] as

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} 1 & \frac{2^{1/2}}{6^{1/2}} & 0 & 2 & 0 & 0 \\ \frac{-3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3 \times 5^{1/2}} & 0 & 0 & 0 \\ 0 & \frac{-5^{1/2}}{5 \times 3^{1/2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2^{1/2}}{6^{1/2}} & 0 \\ 0 & 0 & 0 & \frac{-3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3 \times 5^{1/2}} \\ 0 & 0 & 0 & 0 & \frac{-5^{1/2}}{5 \times 3^{1/2}} & 0 \end{bmatrix}. \tag{2.10}$$

For general case, we have

$$P_{2^{k-1}M\times 2^{k-1}M} = \frac{1}{2^k}\begin{bmatrix} L & F & F & \ldots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & F \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \ldots & O & O & L \end{bmatrix}, \tag{2.11}$$

where L, F and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & \dots & \dots & 0 & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

3. The product operational matrix

The following properties of product of two Legendre wavelet functions are also used for solving differential as well as integral equations:

$$C^T \Psi(t) \Psi^T(t) \approx \tilde{C} \Psi^T(t), \tag{3.1}$$

where C and $\Psi(t)$ are given in Eq.(2.5) and Eq.(2.6), respectively, and $\tilde{C}$ is $(2^{k-1}M) \times (2^{k-1}M)$ product operational matrix. To illustrate the calculation procedure, we choose $M = 3$, $k = 2$ and using $\Psi(t)$ as defined in Eq.(2.6), we have

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21} & \psi_{10}\psi_{22} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21} & \psi_{11}\psi_{22} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} & \psi_{12}\psi_{20} & \psi_{12}\psi_{21} & \psi_{12}\psi_{22} \\ \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{12} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21} & \psi_{20}\psi_{22} \\ \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{12} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21} & \psi_{21}\psi_{22} \\ \psi_{22}\psi_{10} & \psi_{22}\psi_{11} & \psi_{22}\psi_{12} & \psi_{22}\psi_{20} & \psi_{22}\psi_{21} & \psi_{22}\psi_{22} \end{bmatrix}. \tag{3.2}$$

As we know, the support of $\psi_{m,n}$, the entries of vector $\Psi(t)$, are the intervals $\left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right]$, therefore $\psi_{ij}\psi_{kl} = 0$ if $i \neq k$. We also have $\psi_{i0}\psi_{ij} = 2^{1/2}\psi_{ij}$, $\psi_{i1}\psi_{i1} = \frac{4}{10^{1/2}}\psi_{i2} + 2^{1/2}\psi_{i0}$.

If we retain only the elements of $\Psi(t)$, then we have

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} 2^{1/2}\psi_{10} & 2^{1/2}\psi_{11} & 2^{1/2}\psi_{12} & 0 & 0 & 0 \\ 2^{1/2}\psi_{11} & 2^{1/2}\psi_{10} + \frac{4}{10^{1/2}}\psi_{12} & \frac{4}{10^{1/2}}\psi_{11} & 0 & 0 & 0 \\ 2^{1/2}\psi_{12} & \frac{4}{10^{1/2}}\psi_{11} & 2^{1/2}\psi_{10} + \frac{20}{7\times 10^{1/2}}\psi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{1/2}\psi_{20} & 2^{1/2}\psi_{21} & 2^{1/2}\psi_{22} \\ 0 & 0 & 0 & 2^{1/2}\psi_{21} & 2^{1/2}\psi_{20} + \frac{4}{10^{1/2}}\psi_{22} & \frac{4}{10^{1/2}}\psi_{21} \\ 0 & 0 & 0 & 2^{1/2}\psi_{22} & \frac{4}{10^{1/2}}\psi_{21} & 2^{1/2}\psi_{20} + \frac{20}{7\times 10^{1/2}}\psi_{22} \end{bmatrix}.$$

Therefore the 6×6 matrix $\tilde{C}$ in Eq.(3.1) can be written as

$$\tilde{C} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{3.3}$$

where $B_i,\ i = 1, 2$, are 3×3 matrices given by

$$B_i = \begin{bmatrix} 2^{1/2}c_{i0} & 2^{1/2}c_{i1} & 2^{1/2}c_{i2} \\ 2^{1/2}c_{i1} & 2^{1/2}c_{i0} + \frac{4}{10^{1/2}}c_{i2} & \frac{4}{10^{1/2}}c_{i1} \\ 2^{1/2}c_{i2} & \frac{4}{10^{1/2}}c_{i1} & 2^{1/2}c_{i0} + \frac{20}{7\times 10^{1/2}}c_{i2} \end{bmatrix}$$

where $c_{i,d}$, $d = 0, 1, 2$ are taken from Eq.(2.5), for more information, one can refer to [24].

Now, we state the following result regarding the convergence of Legendre wavelet method for expansion of any continuous function $f(t) \in L^2(\mathbb{R})$.

Theorem 3.1 ([27]). *The series solution defined in Eq.* (2.4) *of Eq.* (1.1) *using Legendre wavelet method converges to* $y(t)$.

4. Numerical experiment and discussion

In order to show the effectiveness of Legendre wavelet method (LWM), we implement LWM to many linear and nonlinear singular ordinary differential equations. All the numerical experiments were carried out with MATLAB R2010b and MAPLE 14 codes.

Example 4.1 ([19]). Consider the linear singular initial value problem

$$y''(t) + \frac{2}{t}y'(t) - 10y(t) = 12 - 10t^4, \quad 0 < t < 1, \tag{4.1}$$

with initial conditions as

$$y(0) = 0,\ \ y'(0) = 0.$$

First we assume that the unknown function $y''(t)$ is given by

$$y''(t) = C^T\Psi(t). \tag{4.2}$$

Integrating Eq.(4.2) from 0 to t and using boundary conditions, we have

$$y'(t) = C^T P\Psi(t) + y'(0). \tag{4.3}$$

Again integrating Eq.(4.3) from 0 to t, we obtain

$$y(t) = C^T P^2\Psi(t) + ty'(0) + y(0). \tag{4.4}$$

We can express $\frac{2}{t}$ and $12 - 10t^4$ as

$$\begin{aligned}\frac{2}{t} &= \left[\frac{32}{5}\sqrt{2},\ \frac{-12}{5}\sqrt{6},\ \frac{24}{25}\sqrt{10},\ \frac{320}{231}\sqrt{2},\ \frac{-12}{77}\sqrt{6},\ \frac{8}{382}\sqrt{10}\right]^T \Psi(t)\\ &= H_1^T\Psi(t),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}12 - 10t^4 &= \left[\frac{41047}{6912}\sqrt{2}, \frac{-65}{1728}\sqrt{6}, \frac{-29}{1728}\sqrt{10}, \frac{28087}{6912}\sqrt{2}, \frac{-425}{576}\sqrt{6}, \frac{-245}{1728}\sqrt{10}\right]^T \Psi(t)\\ &= H_2^T\Psi(t).\end{aligned} \tag{4.6}$$

Now substituting Eq.(4.2) to Eq.(4.6) in Eq.(4.1), we get

$$C^T\Psi(t) + H_1^T\Psi(t)C^T P\Psi(t) - 10C^T P^2\Psi(t) = H_2^T\Psi(t).$$

Thus with the orthonormality of Legendre wavelet and Eq.(3.1), we have

$$CI_{6\times 6} + \tilde{H}_1 PC - 10P^{2T}C = H_2^T. \tag{4.7}$$

where $\tilde{H}_1$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.7) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.1). The obtained numerical solution of Eq.(4.1) is presented in comparison with the ADM [19] and exact solution $y(t) = 2t^2 + t^4$ in Table 4.1 and graphically shown in Figure 4.1.

t	LWM Result	Ref.[19]	Exact
0.1	0.02020000	0.02009999	0.02010000
0.2	0.08162990	0.08159992	0.08160000
0.3	0.18811922	0.18809781	0.18810000
0.4	0.34557010	0.34557801	0.34560000
0.5	0.56260300	0.56236789	0.56250000
0.6	0.84980100	0.84902639	0.84960000
0.7	1.22051000	1.21810873	1.22010000
0.8	1.68980100	1.68372918	1.68960000
0.9	2.27620000	2.26081676	2.27610000

Table 4.1 Comparison of the approximate solution of Example 4.1 against the exact and ADM [19] solutions for $M = 3, k = 2$.

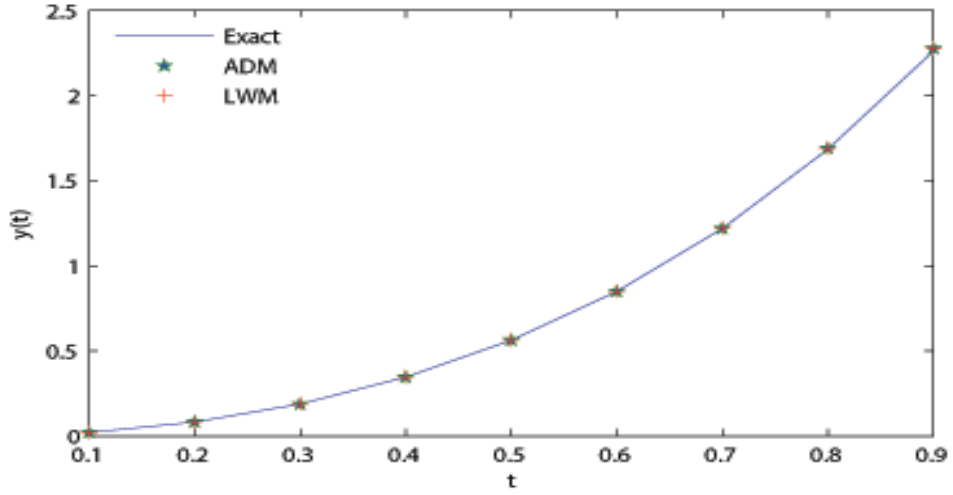


Figure 4.1 Comparison of numerical and exact solution.

Example 4.2 ([19])**.** Consider the linear singular initial value problem

$$y''(t) + \frac{4}{t}y'(t) + \frac{2}{t^2}y(t) = 12, \quad 0 < t < 1, \tag{4.8}$$

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{4}{t}$ and $\frac{2}{t^2}$ as

$$\begin{aligned}\frac{4}{t} &= \left[\frac{64}{5}\sqrt{2}, \frac{-24}{5}\sqrt{6}, \frac{48}{25}\sqrt{10}, \frac{640}{231}\sqrt{2}, \frac{-24}{77}\sqrt{6}, \frac{16}{385}\sqrt{10}\right]^T \Psi(t) \\ &= H_3^T \Psi(t). \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\frac{2}{t^2} &= \left[\frac{1504}{25}\sqrt{2}, \frac{-864}{25}\sqrt{6}, \frac{2208}{125}\sqrt{10}, \frac{106336}{53361}\sqrt{2}, \frac{-2592}{5929}\sqrt{6}, \frac{7648}{88935}\sqrt{10}\right]^T \Psi(t) \\ &= H_4^T \Psi(t). \end{aligned} \tag{4.10}$$

We follow the procedure outlined in Example 4.1 and obtain the algebraic equation as

$$C^T\Psi(t) + H_3^T\Psi(t)C^T P\Psi(t) + H_4^T C^T P^2 \Psi(t) = 12.$$

Thus using the orthonormality of Legendre wavelet and Eq.(3.1),we have

$$CI_{6\times 6} + \tilde{H}_3 PC + \tilde{H}_4 P^{2T} C = 12. \tag{4.11}$$

Where $\tilde{H}_3$ and $\tilde{H}_4$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.11) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.8). The obtained numerical solution of Eq.(4.8) is compared with the ADM [19] and exact solution $y(t) = t^2$ in Table 4.2 and graphically shown in Figure 4.2.

t	LWM Result	Ref.[19]	Exact
0.1	0.010000	0.010000	0.010000
0.2	0.040000	0.040000	0.040000
0.3	0.090000	0.090000	0.090000
0.4	0.160000	0.160000	0.160000
0.5	0.250000	0.250000	0.250000
0.6	0.360000	0.360000	0.360000
0.7	0.490000	0.490000	0.490000
0.8	0.490000	0.490000	0.490000
0.9	0.810000	0.810000	0.810000

Table 4.2 Comparison of the approximate solution of Example 4.2 against the exact and ADM [19] solutions for $M = 3, k = 2$.

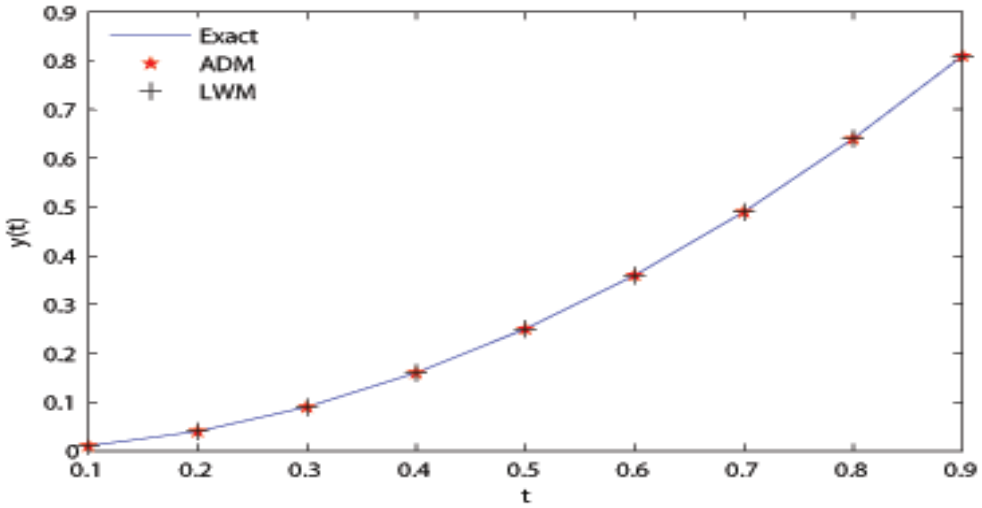


Figure 4.2 Comparison of numerical and exact solution.

Example 4.3 ([19])**.** Consider the linear singular initial value problem

$$y''(t) + \frac{4}{t}y'(t) + \left(\frac{2}{t^2} + t\right) y(t) = 20t + t^4, \quad 0 < t < 1, \tag{4.12}$$

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{4}{t}$ and $\left(\frac{2}{t^2}+t\right)$ as

$$\frac{2}{t^2+t} = \Big[60.2850\sqrt{2}, 34.5183\sqrt{6}, 17.6640\sqrt{10}, 2.3678\sqrt{2}, 0.3955\sqrt{6}, \\ 0.0860\sqrt{10}\Big]\,\Psi(t) = H_5^T\Psi(t). \tag{4.13}$$

$$\left(20t+t^4\right) = \Big[2.5061\sqrt{2}, 0.8371\sqrt{6}, 0.0017\sqrt{10}, 7.6936\sqrt{2}, \\ 0.9071\sqrt{6}, 0.0142\sqrt{10}\Big]\,\Psi(t) = H_6^T\Psi(t). \tag{4.14}$$

The given algebraic equation follows from the procedure configured in Example 4.1 as

$$C^T\Psi(t) + H_3^T\Psi(t)C^TP\Psi(t) + H_5^TC^TP^2\Psi(t) = H_6^T\Psi(t).$$

Applying the orthonormality condition of Legendre wavelet and Eq.(3.1),we have

$$CI_{6\times 6} + \tilde{H}_3PC + \tilde{H}_5P^{2T}C = H_6^T. \tag{4.15}$$

Where $\tilde{H}_3$ and $\tilde{H}_5$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.15) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.12). The obtained numerical solution of Eq.(4.12) is presented in comparison with the ADM [19] and exact solution $y(t) = t^3$ in Table 4.3 and graphically in Figure 4.3.

t	LWM Results	Ref.[19]	Exact
0.1	0.001000	0.001000	0.001000
0.2	0.008003	0.008001	0.008000
0.3	0.027008	0.027013	0.027000
0.4	0.064045	0.064073	0.064000
0.5	0.125131	0.125279	0.125000
0.6	0.216534	0.216834	0.216000
0.7	0.345017	0.345107	0.343000
0.8	0.513002	0.516702	0.512000
0.9	0.730000	0.738553	0.729000

Table 4.3 Comparison of the approximate solution of Example 4.3 against the exact and ADM [19] solutions for $M = 3, k = 2$.

Example 4.4 ([21])**.** Consider the linear singular initial value problem

$$y''(t) + \frac{1}{t}y'(t) = \left(\frac{8}{8-t^2}\right)^2, \quad 0 < t < 1, \tag{4.16}$$

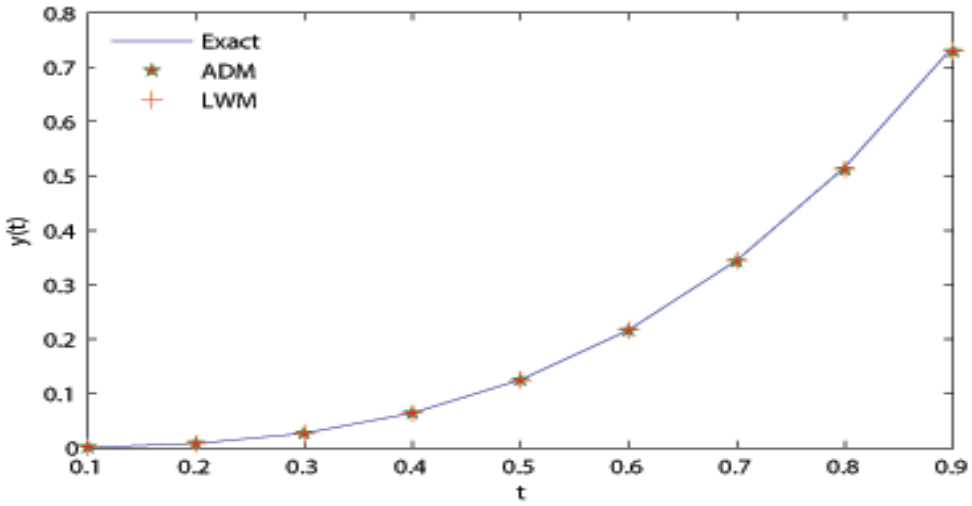


Figure 4.3 Comparison of numerical and exact solution.

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{1}{t}$ and $(\frac{8}{8-t^2})^2$ as

$$\begin{aligned} \frac{1}{t} &= \left[\frac{16}{5}\sqrt{2}, -\frac{6}{5}\sqrt{6}, \frac{12}{25}\sqrt{10}, \frac{160}{231}\sqrt{2}, -\frac{6}{77}\sqrt{6}, \frac{4}{385}\sqrt{10}\right]\Psi(t) \\ &= H_7^T\Psi(t). \end{aligned} \tag{4.17}$$

$$\begin{aligned} \left(\frac{8}{8-t^2}\right)^2 &= \left[0.5107\sqrt{2},\, 0.0054\sqrt{6},\, 0.0011\sqrt{10},\, 0.5832\sqrt{2},\right. \\ &\left. 0.0197\sqrt{6},\, 0.0019\sqrt{10}\right]\Psi(t) = H_8^T\Psi(t). \end{aligned} \tag{4.18}$$

We follow the strategy defined in Example 4.1 and obtain the algebraic equation as

$$C^T\Psi(t) + H_7^T\Psi(t)C^TP\Psi(t) = H_8^T\Psi(t).$$

Eq.(3.1) and orthonormality of Legendre wavelet together give putting, we have

$$CI_{6\times 6} + \tilde{H}_7PC = H_8^T. \tag{4.19}$$

Where $\tilde{H}_7$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.19) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.16). The obtained numerical solution of Eq.(4.16) is presented in comparison with the exact solution $y(t) = 2\log(\frac{7}{8-t^2})$ in Table 4.4 and graphically shown in Figure 4.4.

t	LWM Results	Ref.[21]	Exact
0.1	-0.26456123	-0.26456122	-0.26456122
0.2	-0.25703772	-0.25703770	-0.25703770
0.3	-0.24443526	-0.24443526	-0.24443526
0.4	-0.22665738	-0.22665736	-0.22665737
0.5	-0.20356540	-0.20356538	-0.20356538
0.6	-0.17497491	-0.17497491	-0.17497490
0.7	-0.14065064	-0.14065063	-0.14065063
0.8	-0.10029958	-0.10029956	-0.10029956
0.9	-0.53562045	-0.53562045	-0.53562045

Table 4.4 Comparison of the approximate solution of Example 4.4 against the exact and Chebyshev wavelet [21] solutions for $M = 3, k = 2$.

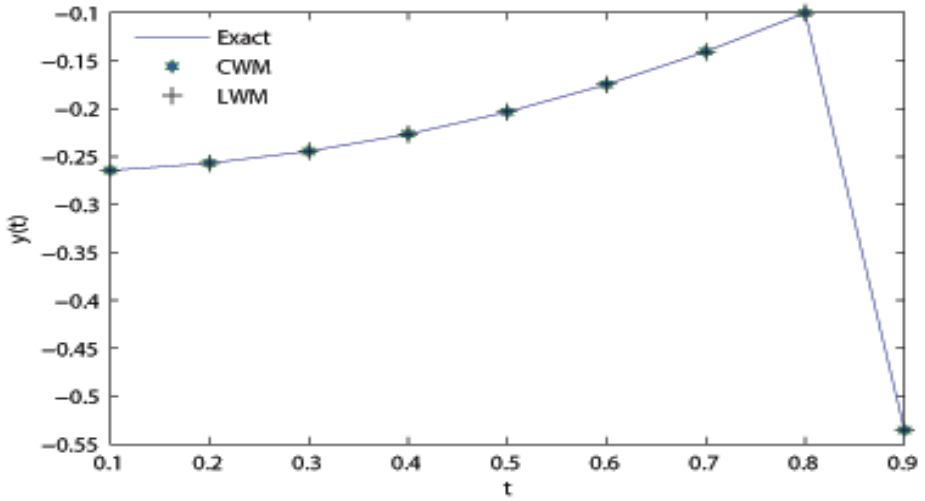


Figure 4.4 Comparison of numerical and exact solution.

Example 4.5 ([21])**.** Consider the nonlinear singular boundary value problem

$$y''(t) + \frac{2}{t}y'(t) + y^5(t) = f(t), \quad 0 < t \leq 1, \tag{4.20}$$

with boundary conditions as $y'(0) = 0$, $y(1) = \frac{\sqrt{3}}{2}$ where $f(t) = 0$ and the exact solution is $y(t) = \frac{1}{\sqrt{1+\frac{t^2}{2}}}$ which explains the equilibrium of isothermal gas sphere.

For this case, we also follow the same procedure as we did in previous examples to approximate $y''(t)$, $y'(t)$ and $y(t)$ with the given boundary conditions using basis functions $\Psi(t)$. We collocate the obtained algebraic equation at suitable collocation points in Eq.(2.7). So, we have a nonlinear system of algebraic

equations with the same number of unknowns which can be easily solved by classical Newton method to obtain the vector C. Now we substitute the wavelet coefficient C into $y(t)$ to get the approximation solution for the Eq.(4.20) which is shown numerically in Table 4.5 and graphically in Figure 4.5. We observe from Table 4.5 and Figure 4.5 that the proposed technique is in full agreement with the exact and Chebyshev wavelet method [21] solutions.

t	LWM Results	Ref.[21]	Exact
0.1	0.99503719	0.99503720	0.99503719
0.2	0.98058069	0.98058067	0.98058067
0.3	0.95782631	0.95782628	0.95782628
0.4	0.92847672	0.92847670	0.92847669
0.5	0.89442718	0.89442718	0.89442719
0.6	0.85749294	0.85749292	0.85749292
0.7	0.81923195	0.81923192	0.81923192
0.8	0.78086881	0.78086880	0.78086880
0.9	0.74329414	0.74329414	0.74329414

Table 4.5 Comparison of the approximate solution of Example 4.5 against the exact and CWM [21] solutions for $M = 3, k = 2$.

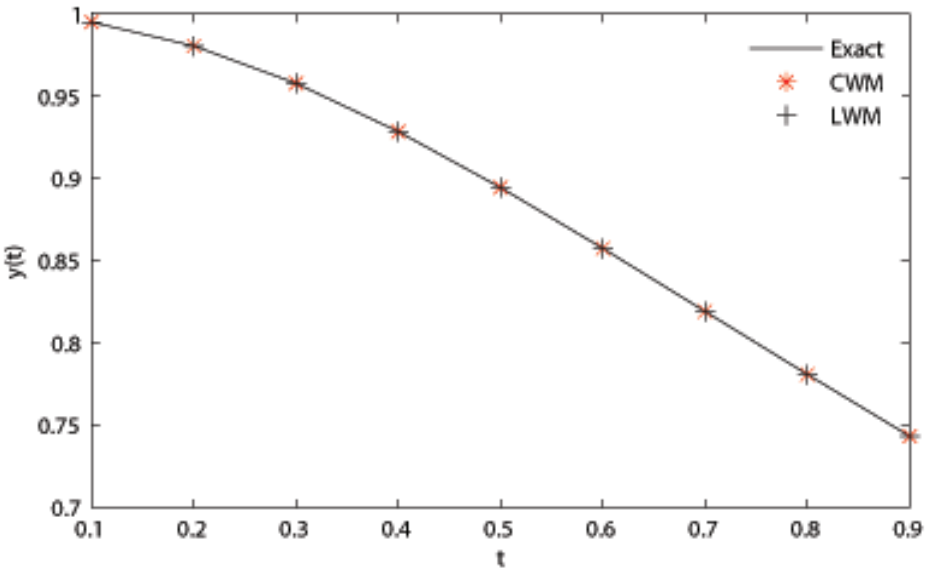


Figure 4.5 Comparison of numerical and exact solution.

5. Conclusion

In this study, we have utilized Legendre wavelet method and its properties for solving linear and nonlinear singular ordinary differential equations. This method also reduces the problem to the solution of algebraic equations which can be solved easily by classical numerical methods. The advantage of this method is that the values of k and M are adjustable as well to yield more accurate numerical solutions. The given numerical examples support the claim that only a small number of Legendre wavelets are needed to accomplish a satisfactory result. A symbolic and numerical calculation software package like MATLAB R2010a and MAPLE 14 are used. The results corroborate our belief that the proposed method is a reliable technique to handle these types of problems. It can therefore be concluded that this method is quite suitable, accurate, and efficient in comparison to other classical methods.

References

[1] O.A. Akinfenwa, S.N. Jator and N.M. Yao, *Continuous block backward differentiation formula for solving stiff ordinary differential equations*, Comput. Math. Appl., 65 (2013), 996-1005.

[2] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms*, Commun. Pure Appl. Math., 4 (1991), 141-183.

[3] J. Biazar and H. Ghazvini, *He's variational iteration method for solving linear and non-linear systems of ordinary differential equations*, Appl. Math. Comput., 191 (2007), 287-297.

[4] N. Caglar and H. Caglar, *B-spline method for solving linear system of second-order boundary value problems*, Comput. Math. Appl., 57 (2009), 757-762.

[5] C. Canuto et al., *Spectral methods in fluid dynamics*, Springer-Verlag, Berlin, 1988.

[6] F.C. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proc. Control Theory Appl., 144 (1997), 87-94.

[7] C.F. Chen and C.H. Hsiao, *A Walsh series direct method for solving variational problems*, J. Franklin Inst., 30 (1997), 265-280.

[8] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[9] M. Dehghan and A. Nikpour, *Numerical solution of the system of second-order boundary value problems using the local radial basis functions based*

differential quadrature collocation method, Appl. Math. Model., 37 (2013), 8578-8599.

[10] L. Fox and I.B. Parker, *Chebyshev polynomials in numerical analysis*, Clarendon Press, Oxford, 1968.

[11] F. Geng and M. Cui, *Homotopy perturbation-reproducing kernel method for nonlinear systems of second order boundary value problems*, J. Comput. Appl. Math., 235 (2011), 2405-2411.

[12] D. Gottelieb and S. Orszag, *Numerical analysis of spectral methods, Theory and Applications*, SIAM, Philadelphia, PA, 1977.

[13] J. Iqbal and R. Abass, *Numerical Solution of Klein/Sine-Gordon Equations by Spectral Method Coupled with Chebyshev Wavelets*, Applied Mathematics, 7 (2016), 2097-2109.

[14] M.K. Kadalbajoo and V.K. Aggarwal, *Numerical solution of singular boundary value problems via Chebyshev polynomial and B-spline*, Appl. Math. Comput., 160 (2005), 851-863.

[15] M. T. Kajania, A. H. Vencheha and M. Ghasemi, *The Chebyshev wavelets operational matrix of integration and product operation matrix*, Int. J. Comp. Math., 86 (2009), 1118-1125.

[16] M. Kumar and N. Singh, *A collection of computational techniques for solving singular boundary-value problems*, Adv. Eng. Softw., 40 (2009), 288-297.

[17] U. Lepik, *Haar wavelet method for solving stiff differential equations*, Math. Modell. Anal., 14 (2009), 467-481.

[18] J. Lu, *Variational iteration method for solving a nonlinear system of second-order boundary value problems*, Comput. Math. Appl., 54 (2007), 1133-1138.

[19] M. Mahmoudi and H. Jafari, *Modified Laplace decomposition method for singular IVPs in the second-order ordinary differential equations*, Casp. J. Math. Sci., 3 (2014), 105-113.

[20] K. Maleknejad and H. Derili, *The collocation method for Hammerstein equations by Daubechies wavelets*, Appl. Math. Comput., 172 (2006), 846-864.

[21] A. K. Nasab et al., *Wavelet analysis method for solving linear and nonlinear singular boundary value problems*, Appl. Math. Model., 37 (2013), 5876-5886.

[22] J. Rashidinia et al., *Convergence of cubic-spline approach to the solution of a system of boundary-value problems*, Appl. Math. Comput., 192 (2007), 319-331.

[23] M. Razzaghi and S. Yousefi, *Legendre wavelets method for the solution of nonlinear problems in the calculus of variations*, Math. Comput. Model., 34 (2001), 45-54.

[24] M. Razzaghi and S. Yousefi, *The Legendre wavelets operational matrix of integration*, Int. J. Syst. Sci., 32 (2001), 495-502.

[25] F. A. Shah, R. Abass and J. Iqbal, *Numerical Solution of Singularly Perturbed Problems using Haar Wavelet Collocation Method*, Cogent Mathematics, 3 (2016), 1-13.

[26] M. Tatari and M. Dehghan, *Improvement of He's variational iteration method for solving systems of differential equations*, Comput. Math. Appl., 58 (2009), 2160-2166.

[27] S.G. Venkatesh, S.K. Ayyaswamy and S. R. Balachandar, *The Legendre wavelet method for solving initial value problems of Bratu-type*, Comput. Math. Appl., 63 (2012), 1287-1295.

[28] S. Yousefi and M. Razzaghi, *Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations*, Math. Comput. Simul., 70 (2005), 1-8.

[29] F. Zhou and X. Xu, *Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets*, Adv. Diff. Eqs., 17 (2016), 1-17.

Accepted: 6.09.2017

ON THE APPLICATION OF THE ADOMIAN DECOMPOSITION METHOD TO SOLVE NON-LINEAR BOUNDARY VALUE PROBLEMS OF A STEADY STATE FLOW OF A LIQUID FILM

A.R. Hassan*
Department of Mathematical Sciences
University of South Africa
Pretoria 0003
South Africa
hassaar@unisa.ac.za
anthonyahssan72@yahoo.co.uk
and
Department of Mathematics
Tai Solarin University of Education Ijagun
Ogun State Nigeria

R. Maritz
Department of Mathematical Sciences
University of South Africa Pretoria 0003
South Africa

M. Mbehou
Department of Mathematical Sciences
University of South Africa Pretoria 0003
South Africa
and
Department of Mathematics
University of Yaounde 1
Cameroon

Abstract. This paper shows the reliability of the Adomian Decomposition Method (ADM) for solving a non-linear boundary value problem in a steady state flow of a liquid film. The solutions of the momentum and energy equations are solved through ADM and the results were compared with previously obtained results by the Homotopy Perturbation Method(HPM) and Hermite - Pade Approximation method (HPA). It is observed that solutions obtained by the ADM takes the form of a convergent series that is capable of greatly reducing the size of computation and solve a large class of non-linear equations effectively and accurately. The results of the boundary value problem are presented in tables and graphs.

Keywords: Adomian Decomposition Method (ADM), Variable Viscosity, Homotopy Perturbation Method (HPM) and Hermite - Pade Approximation (HPA).

∗. Corresponding author

1. Introduction

The use of the Adomian Decomposition Method (ADM) has been applied to a wide class of problems in the sciences. The method is useful for obtaining the closed form and numerical approximations of linear or non linear differential equations as in [1-3]. Also, the method has been established to obtain formal solutions of stochastic and deterministic problems in sciences and engineering involving algebraic, differential, integro - differential, differential delay, integral and partial differential equations as in [4]. This method was first introduced by the American Mathematician, George Adomian (1923 - 1996), in search of a solution in the form of a series and on decomposing the non linear operator into a series in which the terms are calculated recursively using Adomian polynomials [5-7]. In addition to that, some modification were introduced recently to extend the reliability of the ADM in solving problems involving differential equations as described in [8-10] where the proposed modification accelerate the rapid convergence of the series solution if compared with the standard Adomian method. Hence, the various applications of the ADM schemes have been established in [11-17].

In industries, issues like productivity and competitiveness require engineering solutions which heavily rely on Mathematical models. These models are used to determine and investigate fluid flow properties like viscous heating, internal heat generation, entropy generation rates, thermal stability and so on as mentioned in literature [10-18]. It is mentioned in [19] that, one of the the major goals in industries is to understand the fluid behaviour and heat transfer accurately in order to predict the flow regime.

In order to have clear understanding of the fluid dynamics in a channel flow between walls, it is extremely important to critically find and choose an approximate as well as accurate method to solutions of different mathematical models under different conditions. However, critical examination into comparative studies relating to the use of the ADM with other methods has been extensively established, for example, [2] compared ADM and Tau Methods, [3] modified ADM to obtain Taylor series, [4] compared ADM with Picard iteration method and [20] made use of the ADM to solve generalized Riccati differential equations.

Meanwhile, there are other methods that have not been compared with the ADM, like the differential transform method [21], the traditional perturbation method [18] and the Homotopy perturbation method [22]. Hence, the purpose of this paper is to compare and show the reliability of ADM in solving a nonlinear boundary value problem in a steady state flow of a liquid film. The present result from ADM shall be compared with the results of [22] where the Homotopy Perturbation Method (HPM) and the Hermite - Pade Approximation (HPA) were previously used.

2. Review of the Adomian Decomposition Method (ADM)

The ADM provides a closed form of the solution where a general functional equation is given as described in [2,17,23]:

$$y - N(y) = f \tag{1}$$

where N is a nonlinear operator, f is a function that is known with the solution y satisfying (1) which is assumed to have a unique solution. We introduce an approximate solution of (1) in the form of

$$y = \sum_{n=0}^{\infty} y_n \tag{2}$$

and decomposing the nonlinear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n are the Adomian Polynomials of (3) as $y_0, y_1, ..., y_n$ that is determined formally from the relation as:

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d^n}}{\mathrm{dz^n}}\left(N\sum_{i=0}^{\infty} z^i y_i\right)\right]_{z=0}. \tag{4}$$

Therefore, we can identify

$$y_0 = f \tag{5}$$

$$y_{n+1} = A_n\left(y_0, y_1, ..., y_n\right), \qquad n = 0, 1, 2, ... \tag{6}$$

3. Application of the Adomian Decomposition Method (ADM)

The dimensionless governing equations of the non-linear boundary value problem in a steady flow of a liquid film together with the boundary conditions in [22] are given as:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda\left(1-y\right)^2 \mathrm{e}^{\beta\theta} = 0 \tag{7}$$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \left(1-y\right)\mathrm{e}^{\beta\theta} \tag{8}$$

$$\text{with} \quad u = \theta = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}y} = 0 \quad \text{on} \quad y = 1, \tag{9}$$

where θ is the absolute temperature, u is the velocity, λ is the variable viscosity parameter and β is the Brinkmann number.

For simplicity, we expanded the exponential function in (7) and (8) and take the approximation to be:

$$e^{\beta\theta} \simeq 1 + \beta\theta + \frac{(\beta\theta)^2}{2}. \tag{10}$$

Inserting (10) into (7) and (8) together with the boundary conditions in (9), we obtained the following solution for the energy equation:

$$\theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta\int_0^y \left(\int_0^y (1-y)^2\theta(Y)\mathrm{d}Y\right)\mathrm{d}Y$$
$$- \frac{\lambda\beta^2}{2}\int_0^y \left(\int_0^y (1-y)^2[\theta(Y)]^2\mathrm{d}Y\right)\mathrm{d}Y \tag{11}$$

$$u(y) = y - \frac{y^2}{2} + \beta\int_0^y (1-y)\theta(y)\mathrm{d}Y + \frac{\beta^2}{2}\int_0^y (1-y)(\theta(Y))^2\mathrm{d}Y, \tag{12}$$

where $a_0 = \frac{\mathrm{d}\theta}{\mathrm{d}y}(0)$ is to be determined by using the boundary condition (9).

We now introduce a series solution of the form

$$\theta(y) = \sum_{n=0}^{\infty}\theta_n(Y) \qquad \text{and} \qquad u(y) = \sum_{n=0}^{\infty} u_n(Y). \tag{13}$$

Substituting (13) into (11) and (12) respectively, then we have:

$$\theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta\int_0^y \left(\int_0^y (1-y)^2\sum_{n=0}^{\infty}\theta_n(Y)\mathrm{d}Y\right)\mathrm{d}Y$$
$$- \frac{\lambda\beta^2}{2}\int_0^y \left(\int_0^y (1-y)^2\left[\sum_{n=0}^{\infty}\theta_n(Y)\right]^2\mathrm{d}Y\right)\mathrm{d}Y \tag{14}$$

$$u(y) = y - \frac{y^2}{2} + \beta\int_0^y (1-y)[\sum_{n=0}^{\infty}\theta_n(Y)]\mathrm{d}Y$$
$$+ \frac{\beta^2}{2}\int_0^y (1-y)\left[\sum_{n=0}^{\infty}\theta_n(Y)\right]^2\mathrm{d}Y. \tag{15}$$

We let

$$\left[\sum_{n=0}^{\infty}\theta_n(Y)\right]^2 = \sum_{n=0}^{\infty} A_n, \tag{16}$$

where $A_0, A_1, A_2, ...$ are called Adomian polynomials.

Substituting (16) into (14) and (15), we obtain the following:

$$\theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta\int_0^y\left(\int_0^y (1-y)^2\sum_{n=0}^{\infty}\theta_n(Y)\mathrm{d}Y\right)\mathrm{d}Y$$

$$-\frac{\lambda\beta^2}{2}\int_0^y\left(\int_0^y (1-y)^2\sum_{n=0}^{\infty}A_n\mathrm{d}Y\right)\mathrm{d}Y \tag{17}$$

$$u(y)=y-\frac{y^2}{2}+\beta\int_0^y(1-y)[\sum_{n=0}^{\infty}\theta_n(Y)]\mathrm{d}Y+\frac{\beta^2}{2}\int_0^y\left((1-y)\sum_{n=0}^{\infty}A_n\right)\mathrm{d}Y. \tag{18}$$

The few Adomian polynomials of (16) are given as follows:

$$\begin{aligned} A_0 &= \theta_0(y)^2, \\ A_1 &= 2\theta_0(y)\theta_1(y), \\ A_2 &= 2\theta_0(y)\theta_2(y) + \theta_1(y)^2, \qquad \text{and so on.} \end{aligned} \tag{19}$$

Then, the zeroth component of (17) and (18) following the modification of ADM in [8-10, 15, 16, 23] are given as follows:

$$\theta_0(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) \qquad \text{and} \qquad u_0(y) = y - \frac{y^2}{2} \tag{20}$$

$$\theta_{n+1}(y) = -\lambda\beta\int_0^y\left(\int_0^y (1-y)^2\theta_n(Y)\mathrm{d}Y\right)\mathrm{d}Y$$

$$-\frac{\lambda\beta^2}{2}\int_0^y\left(\int_0^y (1-y)^2 A_n\mathrm{d}Y\right)\mathrm{d}Y \tag{21}$$

$$u_{n+1}(y) = \beta\int_0^y(1-y)\theta_n(Y)\mathrm{d}Y + \frac{\beta^2}{2}\int_0^y\left((1-y)A_n\right)\mathrm{d}Y, \text{ for } n \geq 0. \tag{22}$$

Hence, the approximate solutions for temperature and velocity profiles of the boundary value problem using ADM are obtained as follows:

$$\begin{aligned} \theta(y) &= a_0(y) + \lambda\left(\frac{1}{6}a_0\beta y^4 - \frac{1}{6}a_0\beta y^3 - \frac{y^4}{12} + \frac{y^3}{3} - \frac{y^2}{2}\right) \\ &\quad + \lambda\left(-\frac{1}{20}a_0\beta y^5 - \frac{1}{120}a_0^2\beta^2(2(y-3)y+5)y^4\right) \\ &\quad + \lambda^2\left(\frac{\beta y^8}{672} - \frac{\beta y^7}{84} + \frac{\beta y^6}{24} - \frac{\beta y^5}{15} + \frac{\beta y^4}{24}\right) \\ &\quad \frac{+\lambda^2\left(-\left(a_0\beta^2 y^5(y(1344 - 5y(y(7y-54)+180)) - 756)\right)\right)}{30240} + O(\lambda^3) \end{aligned} \tag{23}$$

$$\begin{aligned} u(y) &= y - \frac{y^2}{2} + \frac{1}{504}a_0^2\beta^2(84-63y)y^3 - \frac{1}{3}a_0\beta y^3 + \frac{1}{2}a_0\beta y^2 \\ &\quad + \lambda\left(\frac{1}{504}a_0\beta^2(y(y(6y-35)+84)-63)y^4 + \frac{\beta y^6}{72} - \frac{\beta y^5}{12} + \frac{5\beta y^4}{24} - \frac{\beta y^3}{6}\right) \\ &\quad - \frac{\beta^2\lambda^2 y^5(y(y(7y((y-10)y+45)-760)+980)-504)}{20160} + O(\lambda^3). \end{aligned} \tag{24}$$

4. Previous works

4.1 The Homotopy Perturbation Method (HPM)

The approximate analytical solutions for temperature and velocity fields from HPM in [22] are given as:

$$\theta(y) = \frac{\lambda}{12}\left[1-(1-y)^4\right] + \left(\frac{11\lambda^2\beta}{2016} + \frac{101\lambda^3\beta^2}{1330560}\right) - \frac{\lambda\beta^2}{72}\left(\frac{(1-y)^4}{12} - \frac{(1-y)^8}{56}\right)$$

(25)
$$- \frac{\lambda^3\beta^2}{288}\left(\frac{(1-y)^5}{20} + \frac{(1-y)^12}{132} - \frac{(1-y)^8}{28}\right) \quad \text{and}$$

$$u(y) = \left(y - \frac{y^2}{2}\right) + \frac{\lambda\beta}{72}\left[\left(6y - 3y^2 + (1-y)^6\right) - \left(1 + \frac{7\beta}{120}\right)\right]$$

(26)
$$+ \frac{\lambda^2\beta^2}{8640}\left[30y - 15y^2 - 3(1-y)^{10} + 10(1-y)^6\right].$$

4.2 The Hermite - Pade Approximation (HPA)

The solution for the temperature and velocity fields using HPA in [22] are also given thus:

$$\theta(y) = -\frac{\lambda}{12}y(y-2)(y^2-2y+2)$$

(27)
$$+ \frac{\lambda^2\beta}{2016}y(y-2)(y^2-2y+2)(3y^4-12y^3+18y^2-12y-8) + O(\lambda^3)$$

$$u(y) = -\frac{1}{2}y(y-2) + \frac{\lambda\beta}{72}y^2(y-2)^2(y^2-2y+3)$$

(28)
$$- \frac{\lambda^2\beta^2}{6048}y^2(y-2)^2(3y^6-18y^5+51y^4+7y^2-32y-12) + O(\lambda^3).$$

5. Results and discussion

The main interest of this study is to establish the reliability and accuracy of the ADM in the solution of a non linear boundary value problem in a steady state and to compare the present result of the ADM with the previously obtained results from the HPM and the HPA in [22].

In tables 1 and 2, the numerical values of solutions from the temperature and velocity profiles obtained from the HPM, HPA and ADM are respectively given with the absolute errors in comparison with the present result from ADM. The numerical solutions showed that the ADM is also another convenient method to get an approximate solution of non linear boundary value problems. The present results from ADM showed the reliability and validity of the method with sizeable number of iterations compared with previously obtained results from HPM and HPA. The absolute errors obtained in each comparison evidently showed

that the present ADM results are almost identical with an average difference of order 10^{-4}. This is evident that, results from ADM can conveniently be used as an alternative method to solve non linear problems where exact solutions are not known. Moreso, the assurance of the existence and uniqueness of results from ADM is analysed in [17,24]

Table 1: Comparison of temperature profile between ADM and previous results(HPM and HPA)

$\beta = 1, \lambda = 1$					
y	$\theta_{HPM}(y)$	$\theta_{HPA}(y)$	$\theta_{ADM}(y)$	$\lvert\theta_{HPM}(y) - \theta_{ADM}(y)\rvert$	$\lvert\theta_{HPA}(y) - \theta_{ADM}(y)\rvert$
0.0	0.00454696	0.00000000	0.00000000	4.54696×10^{-3}	0
0.1	0.03330329	0.03019901	0.03040397	2.89932×10^{-3}	2.04959×10^{-4}
0.2	0.05399523	0.05206157	0.05245361	1.54162×10^{-3}	3.92044×10^{-4}
0.3	0.06828202	0.06719977	0.06774765	5.34382×10^{-4}	5.47873×10^{-4}
0.4	0.07764161	0.07711468	0.07778099	1.39372×10^{-4}	6.66309×10^{-4}
0.5	0.08336393	0.08315313	0.08390098	5.37054×10^{-4}	7.47845×10^{-4}
0.6	0.08654554	0.08647955	0.08727721	7.31679×10^{-4}	7.97668×10^{-4}
0.7	0.08808602	0.08805853	0.08888207	7.96050×10^{-4}	8.23544×10^{-4}
0.8	0.08868590	0.08864524	0.08947920	7.93293×10^{-4}	8.33956×10^{-4}
0.9	0.08884568	0.08878065	0.08961719	7.71510×10^{-4}	8.36535×10^{-4}
1.0	0.08886559	0.08878968	0.08962640	7.60811×10^{-4}	8.36717×10^{-4}

Table 2: Comparison of the velocity profile between ADM and previous results (HPM and HPA)

$\beta = 1, \lambda = 1$					
y	$u_{HPM}(y)$	$u_{HPA}(y)$	$u_{ADM}(y)$	$\lvert u_{HPM}(y) - u_{ADM}(y)\rvert$	$\lvert u_{HPA}(y) - u_{ADM}(y)\rvert$
0.0	0.00000000	0.00000000	0.00000000	0	0
0.1	0.09642260	0.09649557	0.09650809	8.54821×10^{-5}	1.25197×10^{-5}
0.2	0.18483294	0.18509391	0.18515536	3.22413×10^{-4}	6.14393×10^{-5}
0.3	0.26419672	0.26471145	0.26487414	6.77422×10^{-4}	1.62687×10^{-4}
0.4	0.33377860	0.33456914	0.33489208	1.11348×10^{-3}	3.22944×10^{-4}
0.5	0.39308777	0.39414324	0.39467937	1.59160×10^{-3}	5.36129×10^{-4}
0.6	0.44182085	0.44310865	0.44389284	2.07199×10^{-3}	7.84194×10^{-4}
0.7	0.47980842	0.48128330	0.48232310	2.51468×10^{-3}	1.03980×10^{-3}
0.8	0.50696856	0.50857908	0.50984842	2.87987×10^{-3}	1.26934×10^{-3}
0.9	0.52326969	0.52496199	0.52639763	3.12794×10^{-3}	1.43564×10^{-3}
1.0	0.52870370	0.53042328	0.53192311	3.21941×10^{-3}	1.49983×10^{-3}

The effects of the Brinkmann number (λ) and the viscosity parameter (β) on the temperature profiles are displayed in figures 1 and 2. The result is approximately the same obtained in [22]. The results showed that the fluid temperature increases as both Brinkmann number (λ) and the viscosity parameter (β) increase. Similarly, figures 3 and 4 displayed the velocity profile. The results also showed that the maximum velocity are obtained as both Brinkmann number (λ) and the viscosity parameter (β) increase in values. Both temperature and

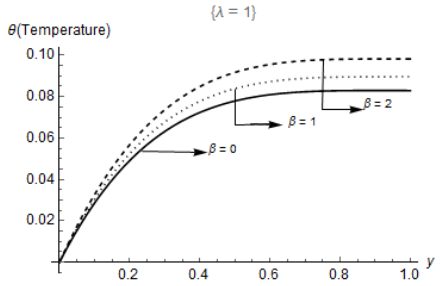


Figure 1: Effect of β on temperature profile

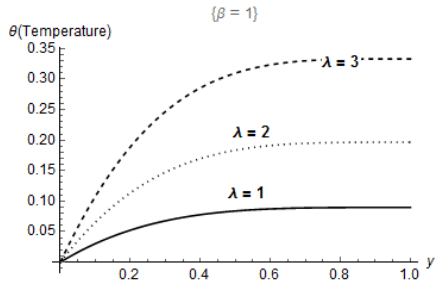


Figure 2: Effect of λ on temperature profile

velocity profiles agreed with the previously obtained results where HPM and HAM were used.

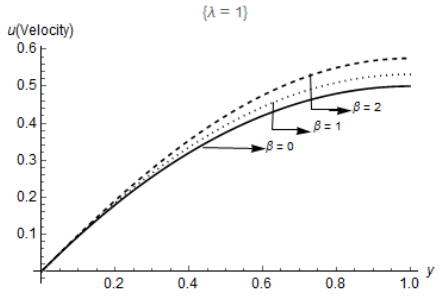


Figure 3: Effect of β on velocity profile

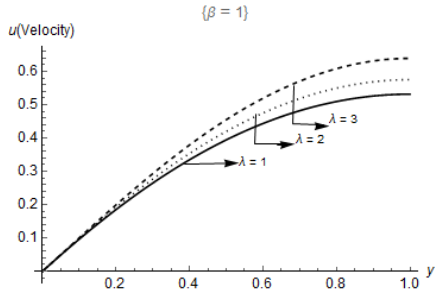


Figure 4: Effect of λ on velocity profile

6. Conclusion

In this study, the ADM has been applied to obtain the solution of a non linear boundary value problem in a steady state flow of a liquid film. The solutions of both temperature and velocity from the ADM were compared with the previously obtained results from [22] where the HPM and the HPA were formerly used. The results showed that the ADM can also be used as an alternative method of getting an approximate and reliable solutions to linear and nonlinear differential equations, and therefore can be applied to wide range of problems in the fields of science and engineering.

References

[1] N. Bildik and M. A. Inc, *A comparison between adomian decomposition and tau methods*, Abstract and Applied Analysis, 2013:15, 2013.

[2] S. Somali and G. Gokmen, *Adomian decomposition method for nonlinear sturm-liouville problems*, Surveys in Mathematics and its Applications, 2:1120, 2007.

[3] E. Kutafina, *Taylor series for the adomian decomposition method*, International Journal of Computer Mathematics, 88(17):36773684, 2011.

[4] E. A. Ibijola and B. J. Adegboyegun, *A comparison of adomians decomposition method and picard iterations method in solving nonlinear differential equations*, Global Journal of Science Frontier Research Mathematics and Decision Sciences, 12(7):3642, 2012.

[5] G. Adomian, *A comparison review of the decomposition method in applied mathematics*, Journal of Mathematical Analysis and Application, 135:501544, 1988.

[6] G. Adomian, *Solving frontier problems of Physics: The decomposition method*, Kluwer academic Publishers, Boston, 1994.

[7] G. Adomian and R. Rach, *Solution of nonlinear ordinary and partial differential equations of physics*, Journal of Mathematics and Physical Sciences, 25:703718, 1991.

[8] M. Almazmumy, F. A. Hendi, H. O. Bakodah, and H. Alzumi, *Recent modifications of adomian decomposition method for initial value problem in ordinary differential equations*, American Journal of Computational Mathematics, 2:228234, 2012.

[9] Y. Q. Hasan and L. M. Zhu, *Modified adomian decomposition method for singular initial value problems in the second-order ordinary differential equations*, Surveys in Mathematics and its Applications, 3:183193, 2008.

[10] A. M. Wazwaz and S. M. El-Sayed, *A new modification of the adomian decomposition method for linear and nonlinear operators*, Applied Maths Computation, 122:393405, 2001.

[11] A. R. Hassan and O. J. Fenuga, *Flow of a maxwell fluid through a porous medium induced by a constantly accelerating plate*, Journal of the Nigerian Association of Mathematical Physics, 19:249254, 2011.

[12] J. A. Gbadeyan and A. R. Hassan, *Multiplicity of solutions for a reactive variable viscous couette flow under arrhenius kinetics*, Mathematical Theory and Modelling, 2(9):3949, 2012.

[13] A. R. Hassan and J. A. Gbadeyan, *The effect of heat absorption on a variable viscosity reactive couette flow under arrhenius kinetics*, Theoretical Mathematics and Applications, 3(1):145159, 2013.

[14] A. R. Hassan and J. A. Gbadeyan, *On multiplicity of solutions for a reactive variable viscous couette flow under arrhenius kinetics in the presence of activation energy*, International Journal of Energy and Technology, 5(26):16, 2013.

[15] A. R. Hassan and J. A. Gbadeyan, *Thermal stability analysis of a reactive hydromagnetic fluid flow through a channel*, American Journal of Applied Mathematics, 2(1):1420, 2014.

[16] A. R. Hassan and J. A. Gbadeyan, *Entropy generation analysis of a reactive hydromagnetic fluid flow through a channel*, U. P. B. Sci. Bull. Series A, 77(2):285296, 2015.

[17] S. O. Adesanya and J. A. Gbadeyan, *Adomian decomposition approach to steady visco elastic fluid flow with slip through a planer channel*, International Journal of Nonlinear Science, 11(1):86 94, 2011.

[18] A. R. Hassan and J. A. Gbadeyan, *A reactive hydromagnetic internal heat generating fluid flow through a channel*, International Journal of Heat and Technology, 33(3):4350, 2015.

[19] D. A. Kamenettski Frank, *Diffusion and heat transfer in chemical kinetics*, Plenum Press, New York, 1969.

[20] T. R. Ramesh Rao, *The use of adomian decomposition method for solving generalised riccati differential equations*, Proceedings of 6th IMT-GT Conference on Mathematics, Statistics and its Applications,(ICMSA2010), pages 935941, 2010.

[21] J. Ali, *One dimensional differential transform method for some higher order boundary value problems in finite domain*, International Journal of Comptemporary Mathememetical Sciences, 7(6):263272, 2012.

[22] V. Ananthaswamy, S. P. Ganesan, and L. Rajendran, *Approximate analytical solution of nonlinear boundary value problem in steady state flow of a liquid films: Homotopy perturbation method*, International Journal of Applied Sciences and Engineering Research, 2(5):569578, 2013.

[23] A. M. Wazwaz, *A new algorithm for calculating adomian polynomials for nonlinear operators*, Abstract and Applied Analysis, 111(1):5369, 2000.

[24] S. S. Ray, *New approach for general convergence of the adomian decomposition method*, World Applied Sciences Journal, 32(11):22642268, 2014.

Accepted: 13.09.2017

ENERGY OF A BIPOLAR FUZZY GRAPH AND ITS APPLICATION IN DECISION MAKING

Sumera Naz*
Department of Mathematics
University of the Punjab
Quaid-e-Azam Campus
Lahore-54590
Pakistan
nsumeranaz@gmail.com

Samina Ashraf
Quality Enhancement Cell
Lahore College For Women University
Lahore
Pakistan
saminaa561@gmail.com

Faruk Karaaslan*Department of Mathematics*
Faculty of Sciences
Çankiri Karatekin University
18100 Çankiri
Turkey
fkaraaslan@karatekin.edu.tr

Abstract. In many domains of information processing, bipolarity is a core feature to be considered: positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. If the information is moreover endowed with vagueness and imprecision, then bipolar fuzzy sets (BFSs) constitute an appropriate knowledge representation framework. In this paper, we introduce the novel concepts of energy of a graph in the context of a bipolar fuzzy environment and investigate some of their properties. We show that if $\mathcal{G}$ is a bipolar fuzzy graph (BFG) on n vertices, then $E(\mathcal{G}) \leq \frac{n}{2}(1+\sqrt{n})$ must hold. Moreover, we introduce the concept of energy of bipolar fuzzy digraphs (BFDGs) along with its application in decision making problem.

Keywords: bipolar fuzzy graph, spectrum, energy, decision making.

1. Introduction

Zhang [21] introduced the concept of BFS characterized by a positive membership function and a negative membership function as an extension of traditional fuzzy set [20] whose basic component is only a membership function. This domain has recently motivated work in several directions, for instance for

∗. Corresponsing author

applications in preference modeling, knowledge representation, argumentation, cooperative games and multi-criteria decision analysis. The range of membership degree of BFSs is $[-1, 1]$. In a BFS, the positive membership degree (0, 1] of an element indicates that the element somewhat satisfies the corresponding property, the negative membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property and the membership degree 0 of an element means that the element is irrelevant to the property [11].

In real life, many situations can be simply abstracted as the graphics problems containing points and connection. For instance, in the Internet, a router can be represented as a vertex and an edge connects two routers with optical fiber. The theory of graphs was first introduced in 1736, when Euler published his paper on graph theory, and solved the problem of the Konigsberg's bridges, which gave birth to a new branch of mathematics. The energy of a graph was originally investigated by Gutman in 1978 [8] and has a wide rang of applications in different fields, including, computer science, physics, chemistry and other branches of mathematics. Fuzzy graphs are designed to represent structures of relationships between objects such that the existence of a concrete object (vertex) and relationship between two objects (edge) are matters of degree. The concept of fuzzy graphs was initiated by Kaufmann [10], based on Zadeh's fuzzy relations. Later, another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges was introduced by Rosenfeld [17] and obtaining analogs of several graph theoretical concepts such as paths, cycles and connectedness etc, he developed the structure of fuzzy graphs. Energy of a fuzzy graph was investigated in [5] by Anjali and Mathew. Akram et al. originally proposed the concept of BFGs, and made a lot of studies on this extension of fuzzy graphs [1, 2, 3, 4, 18]. Naz et al. put forward some new concepts concerning the extended structures of fuzzy graphs and provided their applications in decision-making [6, 13, 14, 15]. Borzooei and Rashmanlou [7] defined the energy of a vague graph. However, to the best of our knowledge, no work addressing the energy in bipolar fuzzy setting is in literature. So, the main purpose of this paper is to introduce the concept of energy of a BFG and BFDG.

The paper is structured as follows: Section 2 contains a brief background about BFSs and BFGs. Section 3 mainly proposes the concept of the energy of a BFG, and investigates its properties. Section 4 introduces the concept of energy of BFDGs along with its application in decision making problem, and finally conclusions are given in Section 5.

Throughout this paper, $\mathcal{V}$ represents a crisp universe of generic elements, G stands for the crisp (undirected, simple) graph and $\mathcal{G}$ is the BFG.

2. Preliminaries

In the following, some basic concepts on BFSs and BFGs are reviewed to facilitate next sections.

A graph $G = (V,\ E)$ is a mathematical structure consisting of a set of vertices V and a set of edges E, where each edge is an unordered pair of distinct vertices. If G is a graph with n vertices and m edges, its adjacency matrix $A(G)$ is the $n \times n$ matrix whose ij-th entry is the number of edges joining vertices i and j. The eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, of the adjacency matrix of G are the eigenvalues of G. The spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the adjacency matrix of G is the spectrum, $\mathrm{Spec}(G)$, of G. The eigenvalues of a graph satisfy the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

Definition 2.1 ([8, 9])**.** The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of G, i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. A graph with all isolated vertices K_n^c has zero energy while the complete graph K_n with n vertices has energy $2(n-1)$.

Definition 2.2 ([16])**.** The energy of a digraph D, denoted by $E(D)$, is defined as the sum of the absolute values of the real part of eigenvalues of D, i.e., $E(D) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)|$.

In 1965, Zadeh [20] originally introduced the fuzzy set, characterized by a membership function in $[0, 1]$, which is very useful in dealing with uncertainty, imprecision and vagueness.

Definition 2.3 ([20])**.** A fuzzy set υ on a set $\mathcal{V}$ is defined through its membership function $\upsilon : \mathcal{V} \to [0, 1]$, where $\upsilon(x)$ represents the degree to which point $x \in \mathcal{V}$ belongs to the fuzzy set. The smallest element and the largest element are the function constantly equal to 0 and 1, respectively.

Definition 2.4 ([19])**.** A fuzzy preference relation R on a set of alternatives $\mathcal{V} = \{x_1, x_2, \ldots, x_n\}$ is characterized by a membership function $\eta_R : \mathcal{V} \times \mathcal{V} \to [0, 1]$. A fuzzy preference relation can be conveniently represented by the $n \times n$ matrix $R = (r_{ij})_{n\times n}$, where r_{ij} indicates the degree of preference of alternative x_i over x_j with $r_{ij} \in [0, 1],\ r_{ij} + r_{ji} = 1,\ r_{ij} = 0.5$ for all $i, j = 1, 2, \ldots, n$.

Definition 2.5 ([21])**.** A BFS X in a non-empty set $\mathcal{V}$ is an object having the following form $X = \{(x, \eta_X^P(x), \eta_X^N(x)) \mid x \in \mathcal{V}\}$ which is characterized by a positive membership function η_X^P and a negative membership function η_X^N, where $\eta_X^P : \mathcal{V} \to [0, 1],\ x \in \mathcal{V} \to \eta_X^P(x) \in [0, 1],\ \eta_X^N : \mathcal{V} \to [-1, 0],\ x \in \mathcal{V} \to \eta_X^N(x) \in [-1, 0]$. If $\eta_X^P(x) \neq 0$ and $\eta_X^N(x) = 0$, then x is regarded as having only positive satisfaction for X. If $\eta_X^P(x) = 0$ and $\eta_X^N(x) \neq 0$, then x does not satisfy the property of X but somewhat satisfies the counter property of X. Finally, if $\eta_X^P(x) \neq 0$ and $\eta_X^N(x) \neq 0$, then the membership function of the property overlaps that of its counter property over some portion of $\mathcal{V}$.

By introducing the concept of BFSs into the theory of graphs, Akram [1] put forward the notion of the BFGs as follows.

Definition 2.6 ([1])**.** A BFG with a finite set $\mathcal{V}$ as the underlying set is a pair $\mathcal{G} = (X, Y)$, where $X = (\eta_X^P, \eta_X^N)$ is a BFS on $\mathcal{V}$ and $Y = (\eta_Y^P, \eta_Y^N)$ is a bipolar fuzzy relation on $\mathcal{V}$ such that $\eta_Y^P(xy) \leq \min\{\eta_X^P(x), \eta_X^P(y)\}$ and $\eta_Y^N(xy) \geq \max\{\eta_X^N(x), \eta_X^N(y)\}$ for all $x, y \in \mathcal{V}$, We call X the bipolar fuzzy vertex set of $\mathcal{G}$ and Y the bipolar fuzzy edge set of $\mathcal{G}$.

3. Energy of a bipolar fuzzy graph

In this section, based on the extension of the energy of a fuzzy graph [5], we define the concept of energy of a BFG, which can be used in real scientific and engineering applications.

Definition 3.1. The adjacency matrix $A(\mathcal{G})$ of a BFG $\mathcal{G} = (X, Y)$ is defined as a square matrix $A(\mathcal{G}) = [a_{ij}]$, $a_{ij} = (\eta_Y^P(u_iu_j), \eta_Y^N(u_iu_j))$, where $\eta_Y^P(u_iu_j)$ and $\eta_Y^N(u_iu_j)$ represent the strength of positive relationship and strength of negative relationship between u_i and u_j, respectively.

Example 3.1. Consider a graph $G = (V, E)$, where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_4u_5\}$. Let $\mathcal{G} = (X, Y)$ be a BFG of a graph G, as shown in Fig. 1. Tabular representation of a BFG is given in Table 1. The adjacency matrix of a BFG given in Fig. 1, is

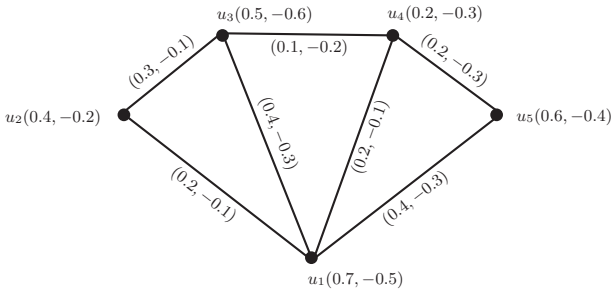


Figure 1: Bipolar fuzzy graph.

Table 1: Tabular representation of a BFG.

X	u_1	u_2	u_3	u_4	u_5
η_X^P	0.7	0.4	0.5	0.2	0.6
η_X^N	-0.5	-0.2	-0.6	-0.3	-0.4

Y	u_1u_2	u_1u_3	u_1u_4	u_1u_5	u_2u_3	u_3u_4	u_4u_5
η_Y^P	0.2	0.4	0.2	0.4	0.3	0.1	0.2
η_Y^N	-0.1	-0.3	-0.1	-0.3	-0.1	-0.2	-0.3

$$A(\mathcal{G}) = \begin{bmatrix} (0,0) & (0.2,-0.1) & (0.4,-0.3) & (0.2,-0.1) & (0.4,-0.3) \\ (0.2,-0.1) & (0,0) & (0.3,-0.1) & (0,0) & (0,0) \\ (0.4,-0.3) & (0.3,-0.1) & (0,0) & (0.1,-0.2) & (0,0) \\ (0.2,-0.1) & (0,0) & (0.1,-0.2) & (0,0) & (0.2,-0.3) \\ (0.4,-0.3) & (0,0) & (0,0) & (0.2,-0.3) & (0,0) \end{bmatrix}.$$

Definition 3.2. The spectrum of adjacency matrix of a BFG $A(\mathcal{G})$ is defined as (S,T), where S and T are the sets of eigenvalues of $A(\eta_Y^P(u_iu_j))$ and $A(\eta_Y^N(u_iu_j))$, respectively.

Definition 3.3. The energy of a BFG $\mathcal{G} = (X,Y)$ is defined as

$$E(\mathcal{G}) = \big(E(\eta_Y^P(u_iu_j)), E(\eta_Y^N(u_iu_j))\big) = \left(\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} |\lambda_i|, \sum_{\substack{i=1\\ \delta_i\in T}}^{n} |\delta_i|\right).$$

Theorem 3.1. *Let $\mathcal{G} = (X,Y)$ be a BFG and $A(\mathcal{G})$ be its adjacency matrix. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$ are the eigenvalues of $A(\eta_Y^P(u_iu_j))$ and $A(\eta_Y^N(u_iu_j))$, respectively. Then*

1. $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i = 0$ and $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i = 0$.
2. $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$ *and* $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$.

Proof. 1. Since $A(\mathcal{G})$ is a symmetric matrix with zero trace, so its eigenvalues are real with sum equal to zero.

2. By trace properties of a matrix, we have $tr((A(\eta_Y^P(u_iu_j)))^2) = \sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2$, where

$$\begin{aligned} tr((A(\eta_Y^P(u_iu_j)))^2) &= \big(0 + (\eta_Y^P(u_1u_2))^2 + \ldots + (\eta_Y^P(u_1u_n))^2\big) \\ &\quad + \big((\eta_Y^P(u_2u_1))^2 + 0 + \ldots + (\eta_Y^P(u_2u_n))^2\big) \\ &\quad \vdots \\ &\quad + \big((\eta_Y^P(u_nu_1))^2 + (\eta_Y^P(u_nu_2))^2 + \ldots + 0\big) \\ &= 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2. \end{aligned}$$

Hence $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$.

Similarly, we can show that $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$. □

Example 3.2. The spectrum and the energy of a BFG $\mathcal{G}$, given in Fig. 1 are as follows: $Spec(\mathcal{G}) = \{(-0.5661,-0.6219),(-0.2767,-0.1029),(-0.1504,0.0814),(0.2075,0.1361),(0.7857,0.5074)\}$, $E(\mathcal{G}) = (1.9864, 1.4498)$.

Further, $\sum_{\substack{i=1\\ \lambda_i\in S}}^{5}\lambda_i = -0.5661-0.2767-0.1504+0.2075+0.7857 = 0, \sum_{\substack{i=1\\ \delta_i\in T}}^{5}\delta_i = -0.6219-0.1029+0.0814+0.1361+0.5074 = 0. \sum_{\substack{i=1\\ \lambda_i\in S}}^{5}\lambda_i^2 = 1.0800 = 2(0.54) = 2\sum_{1\le i<j\le 5}(\eta_Y^P(u_iu_j))^2, \sum_{\substack{i=1\\ \delta_i\in T}}^{5}\delta_i^2 = 0.6800 = 2(0.34) = 2\sum_{1\le i<j\le 5}(\eta_Y^N(u_iu_j))^2.$

We now find upper and lower bounds of the energy of a BFG $\mathcal{G}$, in terms of the number of vertices and the sum of squares of positive membership and negative membership values of edges.

Theorem 3.2. *Let $\mathcal{G} = (X, Y)$ be a BFG on n vertices and $A(\mathcal{G}) = (A(\eta_Y^P(u_iu_j)), A(\eta_Y^N(u_iu_j)))$ be the adjacency matrix of $\mathcal{G}$. Then*

(i) $\sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}$
$\le E(\eta_Y^P(u_iu_j)) \le \sqrt{2n\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}$;

(ii) $\sqrt{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^N(u_iu_j)))|^{\frac{2}{n}}}$
$\le E(\eta_Y^N(u_iu_j)) \le \sqrt{2n\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}$.

Proof. (i) Upper bound: Applying Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ with n entries, we get

$$\sum_{i=1}^{n}|\lambda_i| \le \sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{3.1}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\le i<j\le n}\lambda_i\lambda_j \tag{3.2}$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^{n}(\lambda-\lambda_i) = |A(\mathcal{G}) - \lambda I|,$$

we have

$$\sum_{1\le i<j\le n}\lambda_i\lambda_j = -\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2. \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\sum_{i=1}^{n}|\lambda_i|^2 = 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2. \tag{3.4}$$

Substituting (3.4) in (3.1), we obtain

$$\sum_{i=1}^{n}|\lambda_i| \leq \sqrt{n}\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2} = \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}.$$

Therefore,

$$E(\eta_Y^P(u_iu_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}$$

Lower bound:

$$\begin{aligned}(E(\eta_Y^P(u_iu_j)))^2 &= \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leq i<j\leq n}|\lambda_i\lambda_j| \\ &= 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + \frac{2n(n-1)}{2}AM\{|\lambda_i\lambda_j|\}\end{aligned}$$

Since $AM\{|\lambda_i\lambda_j|\} \geq GM\{|\lambda_i\lambda_j|\}$, $1 \leq i < j \leq n$, so,

$$E(\eta_Y^P(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)GM\{|\lambda_i\lambda_j|\}}$$

also since

$$\begin{aligned}GM\{|\lambda_i\lambda_j|\} = \left(\prod_{1\leq i<j\leq n}|\lambda_i\lambda_j|\right)^{\frac{2}{n(n-1)}} &= \left(\prod_{i=1}^{n}|\lambda_i|^{n-1}\right)^{\frac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\frac{2}{n}} = |\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}\end{aligned}$$

so,

$$E(\eta_Y^P(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}.$$

Thus, $\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}$
$\leq E(\eta_Y^P(u_iu_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}$.

Similarly, we can show that
$\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^N(u_iu_j)))|^{\frac{2}{n}}} \leq E(\eta_Y^N(u_iu_j))$
$\leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}$. □

The following result gives us upper bound of the energy of a BFG, with the conditions $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$ and $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$.

Theorem 3.3. *Let $\mathcal{G} = (X, Y)$ be a BFG on n vertices and $A(\mathcal{G}) = (A(\eta_Y^P(u_iu_j)), A(\eta_Y^N(u_iu_j)))$ be the adjacency matrix of $\mathcal{G}$. If $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$ and $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$. Then*

(i) $E(\eta_Y^P(u_iu_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}{n}$
$$+\sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}{n}\right)^2\right\}};$$

(ii) $E(\eta_Y^N(u_iu_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}{n}$
$$+\sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}{n}\right)^2\right\}}.$$

Proof. If $A = [a_{ij}]_{n\times n}$ is a symmetric matrix with zero trace. Then $\lambda_{\max} \geq \frac{2\sum_{1\leq i<j\leq n} a_{ij}}{n}$, where, $\lambda_{\max}$ is the maximum eigenvalue of A. If $A(\mathcal{G})$ is the adjacency matrix of a BFG $\mathcal{G}$, then $\lambda_1 \geq \frac{2\sum_{1\leq i<j\leq n}\eta_Y^P(u_iu_j)}{n}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Moreover, since

$$\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$$

$$\sum_{i=2}^{n}\lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2 \tag{3.5}$$

Applying Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ with $n-1$ entries, we get

$$E(\eta_Y^P(u_iu_j)) - \lambda_1 = \sum_{i=2}^{n}|\lambda_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}|\lambda_i|^2}. \tag{3.6}$$

Substituting (3.5) in (3.6), we must have

$$E(\eta_Y^P(u_iu_j)) - \lambda_1 \leq \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2\right)}$$

$$E(\eta_Y^P(u_iu_j)) \leq \lambda_1 + \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2\right)}. \tag{3.7}$$

Now, since the function $F(x) = x + \sqrt{(n-1)(2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 - x^2)}$ decreases on the interval $(\sqrt{\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}}, \sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2})$. Also $n \le 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$, $1 \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}$. So,
$\sqrt{\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}} \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n} \le \frac{2\sum_{1\le i<j\le n}\eta_Y^P(u_iu_j)}{n} \le \lambda_1$
$\le \sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}$. Therefore, (3.7) implies $E(\eta_Y^P(u_iu_j))$
$\le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}$
$+ \sqrt{(n-1)\{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 - (\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n})^2\}}$.
Similarly, $E(\eta_Y^N(u_iu_j)) \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}{n}$
$+ \sqrt{(n-1)\{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2 - (\frac{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}{n})^2\}}$. □

Theorem 3.4. *Let $\mathcal{G} = (X, Y)$ be a BFG on n vertices. Then $E(\mathcal{G}) \le \frac{n}{2}(1 + \sqrt{n})$.*

Proof. Suppose that $\mathcal{G} = (X, Y)$ is a BFG with n vertices.

If $n \le 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 = 2y$, then by routine calculus, it is easy to show that $f(y) = \frac{2y}{n} + \sqrt{(n-1)(2y - (\frac{2y}{n})^2)}$ is maximized when $y = \frac{n^2+n\sqrt{n}}{4}$. Substituting this value of y in place of $y = \sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$ in Theorem 3.3, we must have $E(\eta_Y^P(u_iu_j)) \le \frac{n}{2}(1 + \sqrt{n})$. Similarly, it is easy to show that $E(\eta_Y^N(u_iu_j)) \le \frac{n}{2}(1 + \sqrt{n})$. Hence $E(\mathcal{G}) \le \frac{n}{2}(1 + \sqrt{n})$. □

4. Energy of a bipolar fuzzy digraph

In this section, we generalize the concept of energy to BFDGs.

Definition 4.1. Let $\mathcal{D} = (X, \overrightarrow{Y})$ be a BFDG on n vertices. The energy of $\mathcal{D}$ is defined as

$$E(\mathcal{D}) = \left(E(\eta_{\overrightarrow{Y}}^P(u_iu_j)), E(\eta_{\overrightarrow{Y}}^N(u_iu_j))\right) = \left(\sum_{\substack{i=1\\ z_i\in S}}^{n} |\mathrm{Re}(z_i)|, \sum_{\substack{i=1\\ w_i\in T}}^{n} |\mathrm{Re}(w_i)|\right),$$

where $\mathrm{Re}(z_i)$ and $\mathrm{Re}(w_i)$ represent the real part of eigenvalues z_i and w_i, respectively.

Example 4.1. Consider a digraph $D = (V, \overrightarrow{E})$, where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $\overrightarrow{E} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_3u_1, u_5u_4\}$. Let $\mathcal{D} = (X, \overrightarrow{Y})$ be a BFDG of (crisp) digraph D, as given in Fig. 2.
The corresponding adjacent matrix R is as follows:

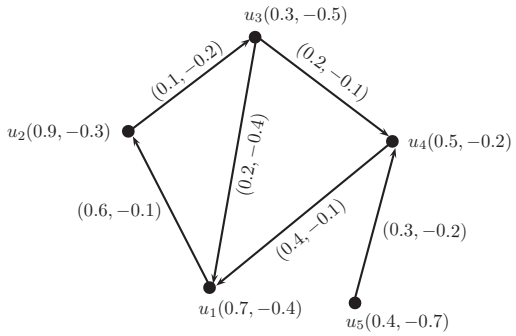


Figure 2: Bipolar fuzzy digraph.

$$R = \begin{bmatrix} (0,0) & (0.6,-0.1) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.1,-0.2) & (0,0) & (0,0) \\ (0.2,-0.4) & (0,0) & (0,0) & (0.2,-0.1) & (0,0) \\ (0.4,-0.1) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.3,-0.2) & (0,0) \end{bmatrix}$$

The spectrum and the energy of a BFDG $\mathcal{D}$, given in Fig. 2 are $Spec(\mathcal{G}) = \{(0,0), (0.3031,-0.2077), (-0.0432+0.2669i, 0.0914+0.1739i), (-0.0432-0.2669i, 0.0914-0.1739i), (-0.2166, 0.0250)\}$ and $E(\mathcal{G}) = (0.6061, 0.4154)$, respectively.

Definition 4.2. A bipolar fuzzy preference relation R on a set of alternatives $\mathcal{V} = \{x_1, x_2, \ldots, x_n\}$ is defined as a matrix $R = (b_{ij})_{n\times n} \subset \mathcal{V} \times \mathcal{V}$ where $b_{ij} = (\eta^P(x_i x_j), \eta^N(x_i x_j))$ for all $i, j = 1, 2, ..., n$. Let $b_{ij} = (\eta^P_{ij}, \eta^N_{ij})$ is a bipolar fuzzy value, composed by the certainty degree η^P_{ij} to which x_i is positively preferred to x_j and the certainty degree η^N_{ij} to which x_i is negatively preferred to x_j with $0 \leq \eta^P_{ij} \leq 1, -1 \leq \eta^N_{ij} \leq 0, \eta^P_{ij} + \eta^P_{ji} = 1, \eta^N_{ij} + \eta^N_{ji} = -1$ and $b_{ii} = (0.5, -0.5)$ for all $i, j = 1, 2, \ldots, n$.

4.1 Application of energy of a BFDG in decision making problem

In modern warfare, it is very important to maintain the communication smoothly. Thus, the performance of the communication equipment plays a key role in campaign victory and defeat. It is necessary for communication units to keep regular communication drills. Suppose that the headquarters are drawing up a plan of communication drill next round. According to the consultations with different simulation environments, there are four possible training venues (alternatives) $x_i (i = 1, 2, 3, 4)$ to choose from. The leaders of the communication unit invite a decision group which contains six experts $e_k (k = 1, 2, \ldots, 6)$ to evaluate all venues so as to make the most reasonable choice. Based on their experiences, the experts compare each pair of alternatives and give individual judgments using the following bipolar fuzzy preference relations $R_k = (r^{P(k)}_{ij}, r^{N(k)}_{ij})_{4\times 4}$ $(k = 1, 2, \ldots, 6)$:

$$R_1 = \begin{bmatrix} (0.50,-0.50) & (0.23,-0.17) & (0.57,-0.67) & (0.63,-0.5) \\ (0.77,-0.83) & (0.50,-0.50) & (0.17,-1) & (0.84,-0.67) \\ (0.43,-0.33) & (0.83,0) & (0.50,-0.50) & (0.42,-0.17) \\ (0.37,-0.50) & (0.16,-0.33) & (0.58,-0.83) & (0.50,-0.50) \end{bmatrix}$$

$$R_2 = \begin{bmatrix} (0.50,-0.50) & (0.38,-0.38) & (0.51,-0.58) & (0.27,-0.84) \\ (0.62,-0.62) & (0.50,-0.50) & (0.60,-0.69) & (0.75,-0.90) \\ (0.49,-0.42) & (0.40,-0.31) & (0.50,-0.50) & (0.14,-0.80) \\ (0.73,-0.16) & (0.25,-0.10) & (0.86,-0.20) & (0.50,-0.50) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} (0.50,-0.50) & (0.57,-0.10) & (0.40,-0.60) & (0.46,-0.70) \\ (0.43,-0.90) & (0.50,-0.50) & (0.61,-0.80) & (0.19,-0.40) \\ (0.60,-0.40) & (0.39,-0.20) & (0.50,-0.50) & (0.80,-0.90) \\ (0.54,-0.30) & (0.81,-0.60) & (0.20,-0.10) & (0.50,-0.50) \end{bmatrix}$$

$$R_4 = \begin{bmatrix} (0.50,-0.50) & (0.30,-0.33) & (0.42,-0.17) & (0.26,-0.67) \\ (0.70,-0.67) & (0.50,-0.50) & (0.90,-0.33) & (0.72,-0.17) \\ (0.58,-0.83) & (0.10,-0.67) & (0.50,-0.50) & (0.81,-1) \\ (0.74,-0.33) & (0.28,-0.83) & (0.19,0) & (0.50,-0.50) \end{bmatrix}$$

$$R_5 = \begin{bmatrix} (0.50,-0.50) & (0.70,-0.34) & (0.16,-0.20) & (0.41,-0.96) \\ (0.30,-0.66) & (0.50,-0.50) & (0.80,-0.33) & (0.29,-0.98) \\ (0.84,-0.80) & (0.20,-0.67) & (0.50,-0.50) & (0.63,-0.99) \\ (0.59,-0.04) & (0.71,-0.02) & (0.37,-0.01) & (0.50,-0.50) \end{bmatrix}$$

$$R_6 = \begin{bmatrix} (0.50,-0.50) & (0.23,-0.50) & (1.0,-0.70) & (0.30,-1.0) \\ (0.77,-0.50) & (0.50,-0.50) & (0.60,-0.80) & (0.36,-0.60) \\ (0,-0.30) & (0.40,-0.20) & (0.50,-0.50) & (0.72,-0.80) \\ (0.70,0) & (0.64,-0.40) & (0.28,-0.20) & (0.50,-0.50) \end{bmatrix}$$

The BFDGs $\mathcal{D}_i$ corresponding to bipolar fuzzy preference relations given in matrices R_i are shown in Fig. 3. The energy of each BFDG is $E(R_1) = (2.9357, 2.3961), E(R_2) = (2.8289, 2.5249), E(R_3) = (2.9602, 2.8495), E(R_4) = (2.7304, 2.6413), E(R_5) = (2.9699, 1.9252), E(R_6) = (2.9510, 2.5897)$. Then the weights can be calculated as:

$$w_k = (w_k^P, w_k^N) = \left(\frac{E(R_k^P)}{\sum_{l=1}^m E(R_l^P)}, \frac{E(R_k^N)}{\sum_{l=1}^m E(R_l^N)} \right), \quad k = 1, 2, \ldots, m,$$

Here $w_1 = (0.1690, 0.1605), w_2 = (0.1628, 0.1692), w_3 = (0.1704, 0.1909), w_4 = (0.1571, 0.177), w_5 = (0.1709, 0.1290), w_6 = (0.1698, 0.1735)$.

The collective bipolar fuzzy preference relation aggregated from the six bipolar fuzzy preference relations is determined as:

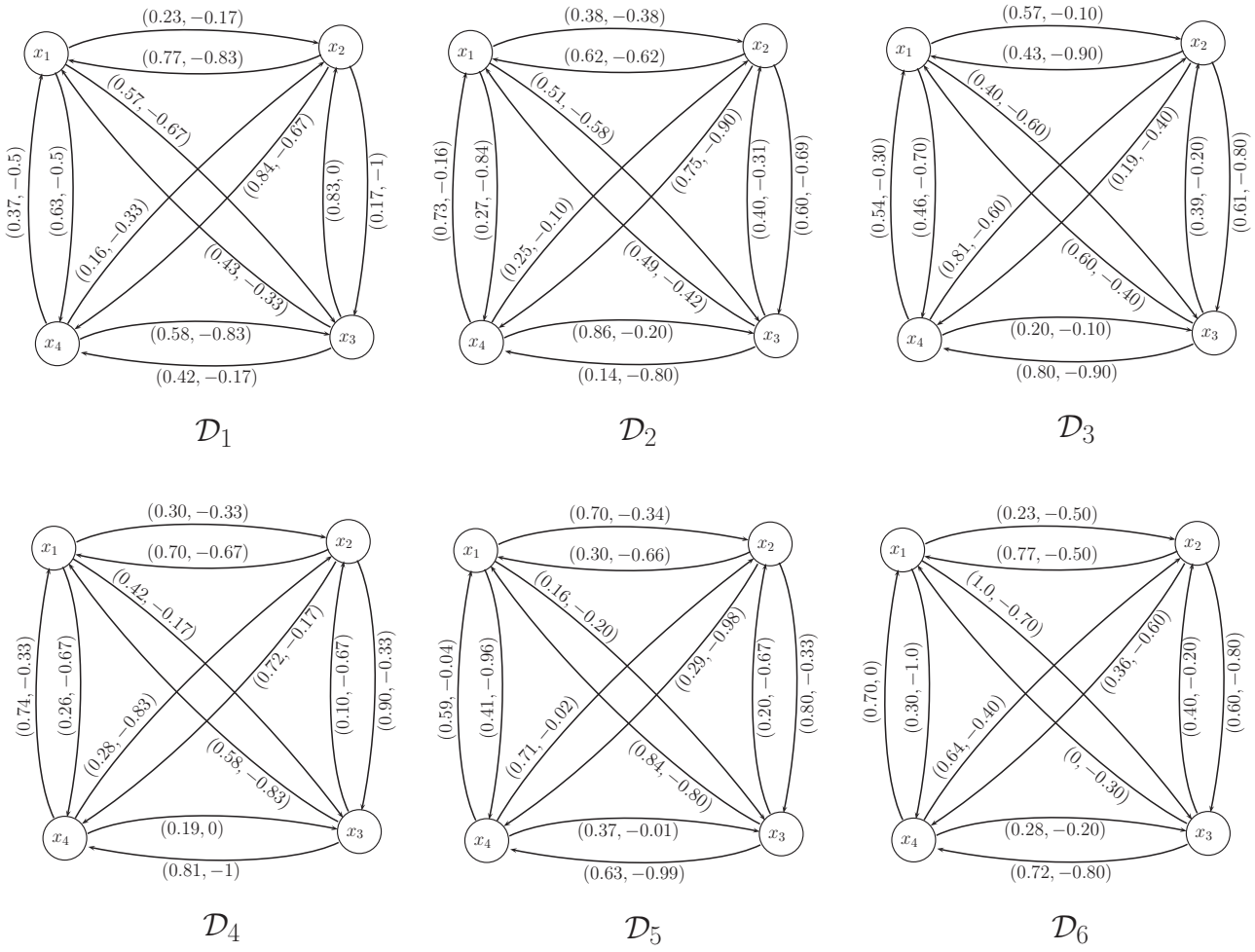


Figure 3: Bipolar fuzzy digraphs

$$R = \sum_{k=1}^{6} w_k R_k =$$
$$\begin{bmatrix} (0.5,-0.5) & (0.4037,-0.2997) & (0.5106,-0.4976) & (0.3907,-0.7719) \\ (0.5963,-0.7004) & (0.5,-0.5) & (0.6103,-0.6697) & (0.5202,-0.5968) \\ (0.4894,-0.5025) & (0.3897,-0.3304) & (0.5,-0.5) & (0.5873,-0.7780) \\ (0.6093,-0.2282) & (0.4798,-0.4033) & (0.4127,-0.2221) & (0.5,-0.5) \end{bmatrix}.$$

Calculate their scores using the score function $s_{ij} = \eta_{ij}^{-} + \eta_{ij}^{+}$ [12]:

$$R = \begin{bmatrix} 0 & 0.1040 & 0.0130 & -0.3812 \\ -0.1041 & 0 & -0.0594 & -0.0766 \\ -0.0131 & 0.0593 & 0 & -0.1907 \\ 0.3811 & 0.0765 & 0.1906 & 0 \end{bmatrix}.$$

The net flow of x_i, i.e., the net degree of preference of x_i over the other alternatives is

$$\phi(x_i) = \sum_{k=1}^{m} w_k \left(\sum_{j=1 j \neq i}^{n} (r_{ij}^{(k)} - r_{ji}^{(k)}) \right), \ i = 1, 2, \ldots, n.$$

So, the net flows of the four alternatives are $\phi(x_1) = -0.5281, \phi(x_2) = -0.4799$, $\phi(x_3) = -0.2887, \phi(x_4) = 1.2967$, which give the ranking of $x_4 \succ x_3 \succ x_2 \succ x_1$. Thus, the best choice is x_4.

5. Conclusions

A bipolar fuzzy model provides more precision, flexibility, and compatibility to the system as compared to the classical and fuzzy model. In this paper, we have introduced the concept of energy of a graph in bipolar fuzzy setting and investigated its properties. We have derived the maximal energy of BFGs. We have also introduced the concept of energy of a BFDG along with its application in decision making problem. In further work, it is necessary and meaningful to extend the energy of BFGs to (1) Pythagorean fuzzy graphs, (2) Interval-valued Pythagorean fuzzy graphs, (3) Hesitant fuzzy graphs, and (4) Hesitant Pythagorean fuzzy graphs.

References

[1] M. Akram, *Bipolar fuzzy graphs*, Information Sciences, 181 (2011), 5548-5564.

[2] M. Akram, *Bipolar fuzzy graphs with applications,* Knowledge-Based Systems, 39 (2013), 1-8.

[3] M. Akram, N. Waseem, *Novel applications of bipolar fuzzy graphs to decision making problems*, Journal of Applied Mathematics and Computing, 56 (2018), 73-91 .

[4] M. Akram, N. Alshehri, B. Davvaz, A. Ashraf, *Bipolar fuzzy digraphs in decision support systems*, Journal of Multiple-Valued Logic and Soft Computing, 27(5-6) (2016), 531-551.

[5] N. Anjali, S. Mathew, *Energy of a fuzzy graph*, Annals Fuzzy Maths and Informatics, 6 (2013), 455–65.

[6] S. Ashraf, S. Naz, H. Rashmanlou, M.A. Malik, *Regularity of graphs in single valued neutrosophic environment*, Journal of Intelligent and Fuzzy Systems, 33(1)(2017), 529–542.

[7] R.A. Borzooei, H. Rashmanlou, *New concepts of vague graphs,* International Journal of Machine Learning and Cybernetics, 8(4) (2017), 1081-1092.

[8] I. Gutman, *The energy of a graph,* Ber Math Stat Sekt Forsch Graz, 103 (1978), 1–22.

[9] I. Gutman, *The energy of a graph: old and new results*, Algebraic Combinatorics and Applications, Springer Berlin Heidelberg, (2001), 196–211.

[10] A. Kaufmann, *Introduction a la Theorie des Sour-ensembles Flous*, Masson et Cie, 1 (1973).

[11] K.M. Lee, *Bipolar-valued fuzzy sets and their basic operations*, in: Proceedings of the International Conference, Bangkok 2000, Thailand, 307–317.

[12] T. Mahmood, et al., *Multiple criteria decision making based on bipolar valued fuzzy sets*, Annals of Fuzzy Mathematics and Informatics, 11 (6) (2016), 1003–1009.

[13] S. Naz, H. Rashmanlou, M.A. Malik, *Operations on single valued neutrosophic graphs with application*, Journal of Intelligent and Fuzzy Systems, 32(3) (2017), 2137–2151.

[14] S. Naz, M.A. Malik, H. Rashmanlou, *Hypergraphs and transversals of hypergraphs in interval-valued intuitionistic fuzzy setting,* The Journal of Multiple-Valued Logic and Soft Computing, 30 (4-6) (2018), 399-417.

[15] S. Naz, S. Ashraf, M. Akram, *A novel approach to decision-making with Pythagorean fuzzy information,* 6 (2018), 1-28.

[16] I. Pea, J. Rada, *Energy of digraphs*, Linear and Multilinear Algebra, 56(5) (2008), 565–579.

[17] A. Rosenfeld, *Fuzzy graphs,* Fuzzy Sets and their Applications (L. A. Zadeh, K. S. Fu, M. Shimura, Eds.) Academic Press, New York, (1975), 77–95.

[18] M. Sarwar, M. Akram, *Certain algorithms for computing strength of competition in bipolar fuzzy graphs*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 25(6) (2017), 877-896.

[19] Y.M. Wang, Z. P. Fan, *Fuzzy preference relations: Aggregation and weight determination*, Computers & Industrial Engineering, 53(1) (2007), 163–172.

[20] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8(3) (1965), 338–353.

[21] W-R. Zhang, *Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis*, Proceedings of IEEE conference, (1994), 305–309.

Accepted: 27.10.2017

COMMUTATIVE NEUTROSOPHIC TRIPLET GROUP AND NEUTRO-HOMOMORPHISM BASIC THEOREM

Xiaohong Zhang*
Department of Mathematics
Shaanxi University of Science and Technology
Xi'an 710021
P. R. China
and
Department of Mathematics
Shanghai Maritime University
Shanghai, 201306
P. R. China
zhangxiaohong@sust.edu.cn
zhangxh@shmtu.edu.cn

Florentin Smarandache
Department of Mathematics
University of New Mexico
Gallup, NM 87301
USA
smarand@unm.edu

Mumtaz Ali
Department of Mathematics
Quaid-i-Azam University
Islamabad, 45320
Pakistan
mumtazali7288@gmail.com

Xingliang Liang
Department of Mathematics
Shaanxi University of Science and Technology
Xi'an, 710021
P. R. China
liangxingl@sust.edu.cn

Abstract. Recently, the notions of neutrosophic triplet and neutrosophic triplet group are introduced by Florentin Smarandache and Mumtaz Ali. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operations. The neutrosophic triplet group is completely different from the classical group in the structural properties. In this paper, we further study neutrosophic triplet group. First, to avoid confusion, some new symbols are introduced, and several basic properties of neutrosophic triplet group are rigorously proved (because the original proof is flawed),

*. Corresponding author

and a result about neutrosophic triplet subgroup is revised. Second, some new properties of commutative neutrosophic triplet group are funded, and a new equivalent relation is established. Third, based on the previous results, the following important propositions are proved: from any commutative neutrosophic triplet group, an Abel group can be constructed; from any commutative neutrosophic triplet group, a BCI-algebra can be constructed. Moreover, some important examples are given. Finally, by using any neutrosophic triplet subgroup of a commutative neutrosophic triplet group, a new congruence relation is established, and then the quotient structure induced by neutrosophic triplet subgroup is constructed and the neutro-homomorphism basic theorem is proved.

Keywords: neutrosophic triplet, neutrosophic triplet group, Abel group, BCI-algebra, neutro-homomorphism basic theorem.

1. Introduction

From a philosophical point of view, Florentin Smarandache introduced the concept of a neutrosophic set (see [12, 13, 14]). The neutrosophic set theory is applied to many scientific fields and also applied to algebraic structures (see [1, 3, 7, 10, 11, 15, 17, 19]). Recently, Florentin Smarandache and Mumtaz Ali in [16], for the first time, introduced the notions of neutrosophic triplet and neutrosophic triplet group. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operation. The neutrosophic triplet group is completely different from the classical group in the structural properties. In 2017, Florentin Smarandache has written the monograph [15] which is present the last developments in neutrodophic theories (including neutrosophic triplet and neutrosophic triplet group).

In this paper, we further study neutrosophic triplet group. We discuss some new properties of commutative neutrosophic triplet group, and investigate the relationships among commutative neutrosophic triplet group, Abel group (that is, commutative group) and BCI-algebra. Moreover, we establish the quotient structure and neutro-homomorphism basic theorem.

As a guide, it is necessary to give a brief overview of the basic aspects of BCI-algebra and related algebraic systems. In 1966, K. Iseki introduced the concept of BCI-algebra as an algebraic counterpart of the BCI-logic (see [5, 24]). The algebraic structures closely related to BCI algebra are BCK-algebra, BCC-algebra, BZ-algebra, BE-algebra, and so on (see [2, 8, 20, 21, 22, 25]). As a generalization of BCI-algebra, W. A. Dudek and Y. B. Jun [4] introduced the notion of pseudo-BCI algebras. Moreover, pseudo-BCI algebra is also as a generalization of pseudo-BCK algebra (which is close connection with various non-commutative fuzzy logic formal systems, see [18, 22, 23, 24]). Recently, some articles related filter theory of pseudo-BCI algebras are published (see [26, 27, 28, 29]). As non-classical logic algebras, BCI-algebras are closely related to Abel groups (see [9]); similarly, BZ-algebras (pseudo-BCI algebras) are closely related general groups (see [20, 26]), and some results in [9, 20] will be applied in this paper.

2. Some basic concepts

2.1 On neutrosophic triplet group

Definition 2.1 ([16]). Let N be a set together with a binary operation $*$. Then, N is called a *neutrosophic triplet set* if for any $a \in N$, there exist a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

The elements a, $neut(a)$ and $anti(a)$ are collectively called as neutrosophic triplet, and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean neutral of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet. For the same element "a" in N, there may be more neutrals to it $neut(a)$ and more opposites of it $anti(a)$.

Definition 2.2 ([16]). The element b in $(N, *)$ is the second component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.3 ([16]). The element c in $(N, *)$ is the third component, denoted as $anti(\cdot)$, of a neutrosophic triplet, if there exist other elements a and b in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.4 ([16]). Let $(N, *)$ be a neutrosophic triplet set. Then, N is called a *neutrosophic triplet group*, if the following conditions are satisfied:

(1) If $(N, *)$ is well-defined, i.e. for any $a, b \in N$, one has $a * b \in N$.

(2) If $(N, *)$ is associative, i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

Definition 2.5 ([16]). Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a *commutative neutrosophic triplet group* if for all $a, b \in N$, we have $a * b = b * a$.

Definition 2.6 ([16]). Let $(N, *)$ be a neutrosophic triplet group under $*$, and let H be a subset of N. Then, H is called a neutrosophic triplet subgroup of N if H itself is a *neutrosophic triplet group* with respect to $*$.

Remark 2.7. In order to include richer structure, the original concept of neutrosophic triplet is generalized to neutrosophic extended triplet by Florentin Smarandache. A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the neutral of x (called "extended neutral") is allowed

to also be equal to the classical algebraic unitary element (if any). Therefore, the restriction "different from the classical algebraic unitary element if any" is released. As a consequence, the "extended opposite" of x, is also allowed to be equal to the classical inverse element from a classical group. Thus, a neutrosophic extended triplet is an object of the form $(x, neut(x), anti(x))$, for $x \in N$, where $neut(x) \in N$ is the extended neutral of x, which can be equal or different from the classical algebraic unitary element if any, such that: $x * neut(x) = neut(x) * x = x$, and $anti(x) \in N$ is the extended opposite of x such that: $x * anti(x) = anti(x) * x = neut(x)$. In this paper, "neutrosophic triplet" means that "neutrosophic extended triplet".

2.2 On BCI-algebras

Definition 2.8 ([5, 23])**.** A BCI-algebra is an algebra $(X; \to, 1)$ of type $(2, 0)$ in which the following axioms are satisfied:

(i) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(ii) $x \to x = 1$,
(iii) $1 \to x = x$,
(iv) if $x \to y = y \to x = 1$, then $x = y$.

In any BCI-algebra $(X; \to, 1)$ one can define a relation $\leq$ by putting $x \leq y$ if and only if $x \to y = 1$, then $\leq$ is a partial order on X.

Definition 2.9 ([9, 26])**.** Let $(X; \to, 1)$ be a BCI-algebra. The set $\{x | x \leq 1\}$ is called the *p-radical* (or BCK-part) of X. A BCI-algebra X is called *p-semisimple* if its p-radical is equal to $\{1\}$.

Proposition 2.10 ([9])**.** *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*

(i) X is p-semisimple,
(ii) $x \to 1 = 1 \Rightarrow x = 1$,
(iii) $(x \to 1) \to 1 = x$, $\forall x \in X$,
(iv) $(x \to 1) \to y = (y \to 1) \to x$ for all $x, y \in X$.

Proposition 2.11 ([26])**.** *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*

$(S1)$ X is p-semisimple,
$(S2)$ $x \to y = 1 \Rightarrow x = y$ for all $x, y \in X$,
$(S3)$ $(x \to y) \to (z \to y) = z \to x$ for all $x, y, z \in X$,
$(S4)$ $(x \to y) \to 1 = y \to x$ for all $x, y \in X$,
$(S5)$ $(x \to y) \to (a \to b) = (x \to a) \to (y \to b)$ for all $x, y, a, b \in X$.

Proposition 2.12 ([9, 26])**.** *Let $(X; \to, 1)$ be p-semisimple BCI-algebra; define $+$ and $-$ as follows: for all $x, y \in X$,*

$$x + y \overset{def}{=} (x \to 1) \to y, \quad -x \overset{def}{=} x \to 1.$$

Then $(X; +, -, 1)$ is an Abel group.

Proposition 2.13 ([9, 26]). *Let* $(X; +, -, 1)$ *be an Abel group. Define* $(X; \leq, \to, 1)$*, where*

$$x \to y = -x + y,\ x \leq y \text{ if and only if } -x + y = 1,\ \forall x, y \in X.$$

Then, $(X; \leq, \to, 1)$ *is a BCI-algebra.*

3. Some properties of neutrosophic triplet group

As mentioned earlier, for a neutrosophic triplet group $(N, *)$, if $a \in N$, then $neut(a)$ may not be unique, and $anti(a)$ may not be unique. Thus, the symbolic $neut(a)$ sometimes means one and sometimes more than one, which is ambiguous. To this end, this paper introduces the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;
$\{neut(a)\}$: denote the set of all neutral of a.
Similarly,
$anti(a)$: denote any certain one of opposite of a;
$\{anti(a)\}$: denote the set of all opposite of a.

Remark 3.1. In order not to cause confusion, we always assume that: (1) for the same a, when multiple $neut(a)$ (or $anti(a)$) are present in the same expression, they are always are consistent. Of course, if they are neutral (or opposite) of different elements, they refer to different objects (for example, in general, $neut(a)$ is different from $neut(b)$). (2) if $neut(a)$ and $anti(a)$ are present in the same expression, then they are match each other.

Proposition 3.2. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$*. Then*

$$neut(a) * neut(a) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$a * neut(a) = a,\ neut(a) * a = a.$$

From this, using associative law, we can get

$$a * (neut(a) * neut(a)) = (neut(a) * neut(a)) * a = a.$$

By Definition 2.1, it follows that $(neut(a) * neut(a))$ is a neutral of a. That is, $neut(a) * neut(a) \in \{neut(a)\}$. □

Remark 3.3. This proposition is a revised version of Theorem 3.21(1) in [16]. If $neut(a)$ is unique, then they are same. But, if $neut(a)$ is not unique, they are different. For example, assume $\{neut(a)\} = \{s, t\}$, then $neut(a)$ denote any one of s, t. Thus $neut(a) * neut(a)$ represents one of $s * s$, and $t * t$. Moreover, Proposition 3.2 means that $s * s,\ t * t \in \{neut(a)\} = \{s, t\}$, that is,

$$s * s = s, \text{ or } s * s = t;\ \ t * t = s, \text{ or } t * t = t.$$

And, in this case, the equation $neut(a) * neut(a) = neut(a)$ means that $s * s = s$, $t * t = t$. So, they are different.

Proposition 3.4. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. If*

$$neut(a) * neut(a) = neut(a).$$

Then

$$neut(a) * anti(a) \in \{anti(a)\};$$
$$anti(a) * neut(a) \in \{anti(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

From this, using associative law, we can get

$$a * (neut(a) * anti(a)) = (a * neut(a)) * anti(a) = a * anti(a) = neut(a).$$

And,

$$(neut(a) * anti(a)) * a = neut(a) * (anti(a) * a) = neut(a) * neut(a) = neut(a).$$

By Definition 2.1, it follows that $(neut(a) * anti(a))$ is a opposite of a. That is, $neut(a) * anti(a) \in \{anti(a)\}$. In the same way, we can get $anti(a) * neut(a) \in \{anti(a)\}$. □

Proposition 3.5. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and let $a, b, c \in N$. Then*

(1) *$a * b = a * c$ if and only if $neut(a) * b = neut(a) * c$.*

(2) *$b * a = c * a$ if and only if $b * neut(a) = c * neut(a)$.*

Proof. Assume $a * b = a * c$. Then $anti(a) * (a * b) = anti(a) * (a * c)$. By associative law, we have

$$(anti(a) * a) * b = (anti(a) * a) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$.

Conversely, assume $neut(a) * b = neut(a) * c$. Then $a * (neut(a) * b) = a * (neut(a) * c)$. By associative law, we have

$$(a * neut(a)) * b = (a * neut(a)) * c.$$

Using Definition 2.1 we get $a * b = a * c$. That is, (1) holds.

Similarly, we can prove that (2) holds. □

Proposition 3.6. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and let $a, b, c \in N$.*

(1) *If $anti(a) * b = anti(a) * c$, then $neut(a) * b = neut(a) * c$.*

(2) *If* $b * anti(a) = c * anti(a)$, *then* $b * neut(a) = c * neut(a)$.

Proof. Assume $anti(a) * b = anti(a) * c$. Then $a * (anti(a) * b) = a * (anti(a) * c)$. By associative law, we have

$$(a * anti(a)) * b = (a * anti(a)) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$. It follows that (1) holds.

Similarly, we can prove that $b * anti(a) = c * anti(a) \Rightarrow b * neut(a) = c * neut(a)$. □

Theorem 3.7. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$neut(a) * neut(neut(a)) = neut(a);$$
$$neut(neut(a)) * neut(a) = neut(a).$$

Then

$$a * (neut(a) * neut(neut(a))) = a * neut(a);$$
$$(neut(neut(a)) * neut(a)) * a = neut(a) * a.$$

By associative law and Definition 2.1, we have

$$a * neut(neut(a)) = a;$$
$$neut(neut(a)) * a = a.$$

From this, by Definition 2.1, $neut(neut(a)) \in \{neut(a)\}$. □

Theorem 3.8. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(anti(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * neut(anti(a)) = anti(a);$$
$$neut(anti(a)) * anti(a) = anti(a).$$

Then

$$a * (anti(a) * neut(anti(a))) = a * anti(a);$$
$$(neut(anti(a)) * anti(a)) * a = anti(a) * a.$$

Using associative law and Definition 2.1,

$$neut(a) * neut(anti(a)) = neut(a);$$
$$neut(anti(a)) * neut(a) = neut(a).$$

It follows that $a * neut(anti(a)) = a$, $neut(anti(a)) * a = a$. That is, $neut(anti(a)) \in \{neut(a)\}$. □

Theorem 3.9. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. Then*

$$neut(a) * anti(anti(a)) = a.$$

where, $neut(a) \in \{neut(a)\}$, $anti(a) \in \{anti(a)\}$, and $neut(a)$ matches $anti(a)$, that is, $a * anti(a) = anti(a) * a = neut(a)$.

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * anti(anti(a)) = neut(anti(a)).$$

Then

$$\begin{gathered} a * (anti(a) * anti(anti(a))) = a * neut(anti(a)). \\ (a * anti(a)) * anti(anti(a)) = a * neut(anti(a)). \\ neut(a) * anti(anti(a)) = a * neut(anti(a)). \end{gathered}$$

On the other hand, by Theorem 3.8, $neut(anti(a)) \in \{neut(a)\}$. By Definition 2.1, it follows that $a*neut(anti(a))=a$. Therefore, $neut(a)*anti(anti(a))=a$. □

Theorem 3.10. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. Then*

$$anti(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$\begin{gathered} neut(a) * anti(neut(a)) = neut(neut(a)); \\ anti(neut(a)) * neut(a) = neut(neut(a)). \end{gathered}$$

Thus

$$\begin{gathered} a * (neut(a) * anti(neut(a))) = a * neut(neut(a)); \\ (anti(neut(a)) * neut(a)) * a = neut(neut(a)) * a. \end{gathered}$$

Applying associative law and Definition 2.1,

$$\begin{gathered} a * anti(neut(a)) = a * neut(neut(a)); \\ anti(neut(a)) * a = neut(neut(a)) * a. \end{gathered}$$

On the other hand, by Theorem 3.7, $neut(neut(a)) \in \{neut(a)\}$. It follows that

$$a * neut(neut(a)) = neut(neut(a)) * a = a.$$

Therefore,

$$a * anti(neut(a))) = anti(neut(a)) * a = a.$$

This means that $anti(neut(a)) \in \{neut(a)\}$. □

Theorem 3.11. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and $a, b \in N$. Then*

$$neut(a * a) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$(a * a) * neut(a * a) = a * a.$$

From this and applying the associativity of operation $*$ and Definition 2.1 we get

$$\begin{aligned}(anti(a) * a) * a * neut(a * a) &= (anti(a) * a) * a.\\ neut(a) * a * neut(a * a) &= neut(a) * a.\\ a * neut(a * a) &= a.\end{aligned}$$

Similarly, we can prove $neut(a * a) * a = a$. This means that $neut(a * a) \in \{neut(a)\}$. □

Now, we note that Proposition 3.18 in [16] is not true.

Example 3.12. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group under the binary operation $\sharp$ with Table 1.

Table 1 Cayley table of neutrosophic triplet group $(Z_{10}, \sharp)$

$\sharp$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

For each $a \in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$\begin{aligned}&neut(0) = 0, neut(1) = 7, neut(2) = 2, neut(3) = 7, neut(4) = 2,\\ &neut(5) = 5, neut(6) = 2, neut(7) = 7, neut(8) = 2, neut(9) = 7.\end{aligned}$$

Let $H = \{0, 2, 5, 7\}$, then $(H, \sharp)$ is a neutrosophic triplet subgroup of $(Z_{10}, \sharp)$, but

$$\begin{aligned}anti(5) \in \{1, 3, 5, 7, 9\} \not\subset H,\\ anti(0) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \not\subset H.\end{aligned}$$

Therefore, Proposition 3.18 in [16] should be revised to the following form.

Proposition 3.13. *Let $(N, *)$ be a neutrosophic triplet group and H be a subset of N. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold:*

(1) $a * b \in H$ *for all* $a, b \in H$.

(2) *there exists* $neut(a) \in H$ *for all* $a \in H$.

(3) *there exists* $anti(a) \in H$ *for all* $a \in H$.

4. New properties of commutative neutrosophic triplet group

Theorem 4.1. *Let* $(N,)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $a, b \in N$. *Then*

$$\{neut(a)\} * \{neut(b)\} \subseteq \{neut(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * neut(a) * neut(b) * b = (a * neut(a)) * (neut(b) * b) = a * b.$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(neut(a) * neut(b)) * (a * b) = (a * b) * (neut(a) * neut(b)) = a * b.$$

This means that $neut(a)*neut(b) \in \{neut(a*b)\}$, that is, $\{neut(a)\}*\{neut(b)\} \subseteq \{neut(a * b)\}$. □

Proposition 4.2. *Let* $(N, *)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $H = \{neut(a) \mid a \in N\}$. *Then* H *is a neutrosophic triplet subgroup of* N *such that* $(\forall a \in N)$ $neut(a) \in H$ *and* $unit(h) \in H$ *for any* $h \in N$.

Proof. For any $h_1, h_2 \in N$, by the definition of H, there exists $a, b \in N$ such that $h_1 = neut(a)$, $h_2 = neut(b)$. Then, by Theorem 4.1 we have

$$h_1 * h_2 = neut(a) * neut(b) \in \{neut(a * b)\} \subseteq H.$$

Moreover, applying Theorem 3.7 and 3.10,

$$neut(h_1) = neut(neut(a)) \in \{neut(a)\} \subseteq H.$$
$$anti(h_1) = anti(neut(a)) \in \{neut(a)\} \subseteq H.$$

Using Proposition 3.13 we know that H is a neutrosophic triplet subgroup of N, and it satisfies

$$(\forall a \in N)\ neut(a) \in H, \text{and } unit(h) \in H \text{ for any } h \in N.$$

□

Theorem 4.3. *Let* $(N, *)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $a, b \in N$. *Then*

$$\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * anti(a) * anti(b) * b = (a * anti(a)) * (anti(b) * b) = neut(a) * neut(b).$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(anti(a) * anti(b))(a * b) = (a * b) * (anti(a) * anti(b)) = neut(a) * neut(b).$$

Applying Theorem 4.1, $neut(a) * neut(b) \in \{neut(a * b)\}$. Hence, by Definition 2.1, $anti(a) * anti(b) \in \{anti(a * b)\}$, that is, $\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}$. □

Theorem 4.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$. Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, and $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2, $neut(a) * neut(a) \in \{neut(a)\}$. Using Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in \{neut(a)\}.$$

Then, $a \approx_{neut} a$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a * anti(b) * neut(p) = neut(q). \tag{C1}$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in neut(p)$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$$\begin{aligned}
&b * anti(a) * x = b * anti(a) * anti(neut(p)) \\
&= anti(a) * b * anti(neut(p)) && (by\ Definition\ 2.5)\\
&= anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p)) && (by\ Theorem\ 3.9)\\
&= (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b) && (by\ Definition\ 2.4 and\ 2.5)\\
&\in \{anti(a * anti(b) * neut(p))\} * neut(b) && (by\ Theorem\ 4.3)\\
&\subseteq \{anti(neut(q))\} * neut(b) && (by\ the\ above\ result\ (C1))\\
&\subseteq \{neut(q)\} * neut(b) && (by\ Theorem\ 3.10)\\
&\subseteq \{neut(q * b)\} && (by\ Theorem\ 4.1)
\end{aligned}$$

This means that $b \approx_{neut} a$. □

Definition 4.5. Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a neutrosophic triplet group with condition (AN) if for all $a, b \in N$, we have

(AN) $\{anti(a * b)\} \subseteq \{anti(a)\} * \{anti(b)\}$.

Proposition 4.6. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. Then*

$$neut(a * b) \in \{neut(a)\} * \{neut(b)\}.$$

Proof. For any $a, b \in N$, by Definition 4.5, there exists $anti(a) \in \{anti(a)\}$, $anti(b) \in \{anti(b)\}$ such that

$$anti(a * b) = anti(a) * anti(b).$$

Then

$$\begin{aligned} neut(a * b) &= (a * b) * anti(a * b) = (a * b) * (anti(a) * anti(b)) \\ &= (a * anti(a)) * (b * anti(b)) = neut(a) * neut(b). \end{aligned}$$

This means that $neut(a * b) \in \{neut(a)\} * \{neut(b)\}$. □

Lemma 4.7. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. If there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, $neut(p) \in \{neut(p)\}$ and $neut(q) \in \{neut(q)\}$ such that*

$$a * anti(b) * neut(p) = neut(q).$$

Then for any $x \in \{anti(b)\}$, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that

$$a * x * neut(p_1) = neut(q_1).$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b * x = x * b = y$. Thus, from $a * anti(b) * neut(p) = neut(q)$ we get

$$\begin{aligned} &a * x * (neut(b) * neut(p)) \\ &= a * x * (anti(b) * b) * neut(p) \\ &= (a * anti(b) * neut(p)) * (x * b) \\ &= neut(q) * y \\ &\in neut(q) * \{neut(b)\} \\ &\subseteq \{neut(q * b)\} \qquad (by\ Theorem\ 4.1) \end{aligned}$$

Therefore, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that $a * x * neut(p_1) = neut(q_1)$. □

Theorem 4.8. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is an equivalent relation on N.

Proof. By Theorem 4.4, we only prove that $\approx_{neut}$ is transitive. Assume that $a \approx_{neut} b$ and $b \approx_{neut} c$, then there exists $p, q, r, s \in N$ such that

(C1) $$a * anti(b) * neut(p) = neut(q).$$

(C2) $$b * anti(c) * neut(r) = neut(s).$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$, $neut(r) \in \{neut(r)\}$, $neut(s) \in \{neut(s)\}$. Using Theorem 3.10 and Theorem 4.1, we have

$$neut(p)*neut(c)*anti(neut(s)) \in \{neut(p)\}*\{neut(c)\}*\{neut(s)\} \subseteq \{neut(p*s*c)\}.$$

Denote $y = neut(p) * neut(c) * anti(neut(s)) \in \{neut(p * s * c)\}$, then

$a * anti(c) * y = a * anti(c) * neut(p) * neut(c) * anti(neut(s))$
$= a * anti(c) * neut(p) * anti(neut(s)) * neut(c)$ (by Definition 2.5)
$= a * anti(c) * neut(p) * anti(b * anti(c) * neut(r)) * neut(c)$ (by the above result (C2))
$\in a * anti(c) * neut(p) * \{anti(b) * anti(anti(c)) * anti(neut(r))\} * neut(c)$ (by Definition 4.5)
$\subseteq a * anti(c) * neut(p) * \{anti(b) * c * anti(neut(r))\}$ (by Definition 2.4, 2.5 and Theorem 3.9)
$\subseteq a * neut(p) * \{anti(b) * neut(r) * (anti(c) * c)\}$ (by Theorem 3.10, Definition 2.4 and 2.5)
$= a * neut(p) * \{anti(b) * neut(r) * neut(c)\}$ (by Definition 2.1)
$\subseteq \{(a * anti(b) * neut(p)) * neut(r) * neut(c)\}$ (by Definition 2.1)
$\subseteq \{neut(q_1)*neut(r)*neut(c)\}$ (by the above result (C1) and Lemma 4.7)
$\subseteq \{neut(q_1 * r * c)\}$ (by Theorem 4.1)

This means that $a \approx_{neut} c$. □

5. Commutative neutrosophic triplet group and Abel group with BCI-algebra

Theorem 5.1. *Let $(N, *)$ be a commutative neutrosophic triplet group condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the following statements are hold:*

(1) $a, b, c \in N$, $a \approx_{neut} b \Rightarrow a * c \approx_{neut} b * c$.

(2) $a \approx_{neut} b \Rightarrow neut(a) \approx_{neut} neut(b)$.

(3) $a \approx_{neut} b \Rightarrow anti(a) \approx_{neut} anti(b)$.

(4) $a, b \in N$, $neut(a) \approx_{neut} neut(b)$.

Proof. (1) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$(C1)$ $$a * anti(b) * neut(p) = neut(q),$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Thus,

$$
\begin{aligned}
&(a*c)*anti(b*c)*neut(p)\\
&\in (a*c)*\{anti(b)\}*\{anti(c)\}*neut(p) && (by\ Definition\ 4.5)\\
&\subseteq \{a*anti(b)*neut(p)\}*\{c*anti(c)\} && (by\ Definition\ 2.4\ and\ 2.5)\\
&= \{a*anti(b)*neut(p)\}*\{neut(c)\} && (by\ Definition\ 2.1)\\
&\subseteq \{neut(q_1)\}*\{neut(c)\} && (by\ the\ above\ result\ (C1)\ and\ Lemma\ 4.7)\\
&\subseteq \{neut(q_1*c)\} && (by\ Theorem\ 4.1)
\end{aligned}
$$

It follows that $a*c \approx_{neut} b*c$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Then, applying Theorem 3.8 and Theorem 4.1 we have

$$neut(a)*anti(neut(b))*neut(p) \in \{neut(a)\}*\{neut(b)\}*\{neut(p)\} \subseteq \{neut(a*b*p)\}.$$

This means that $neut(a) \approx_{neut} neut(b)$.

(3) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10,

$$anti(neut(p)) \in \{neut(p)\},\ anti(neut(q)) \in \{neut(q)\}.$$

Applying Theorem 4.3 we have

$$
\begin{aligned}
anti(a)*anti(anti(b))*anti(neut(p)) &\in \{anti(a*anti(b)*neut(p))\}\\
&\subseteq \{anti(neut(q))\} \subseteq \{neut(q)\}.
\end{aligned}
$$

It follows that $anti(a) \approx_{neut} anti(b)$.

(4) $\forall a, b \in N$, since

$$
\begin{aligned}
&neut(a)*anti(neut(b))*neut(a)\\
&\in neut(a)*\{neut(b)\}*neut(a) && (by\ Theorem\ 3.10)\\
&\subseteq \{neut(a*b*a)\} && (by\ Theorem\ 4.1)
\end{aligned}
$$

This means that $neut(a) \approx_{neut} neut(b)$. □

Theorem 5.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the quotient $N/\approx_{neut}$ is an Abel group with respect to the following operation:*

$$\forall\ a, b \in N,\ [a]_{neut} \bullet [b]_{neut} = [a*b]_{neut}.$$

where $[a]_{neut}$ is the equivalent class of a, the unit elment of $(N/\approx_{neut}, \bullet)$ is $1_{neut} = [neut(a)]_{neut}$, $\forall a \in N$, $neut(a) \in \{neut(a)\}$.

Proof. By Theorem 5.1 (1) $\sim$ (3) we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_{neut}, \bullet)$ is a commutative neutrosophic triplet group.

Moreover, by Theorem 5.1 (4) we get

$$\forall a, b \in N,\ [neut(a)]_{neut} = [neut(b)]_{neut}.$$
$$\forall a, b \in N,\ neut([a]_{neut}) = neut([b]_{neut}).$$

This means that $neut(\cdot)$ is unique. Denote

$$1_{neut} = [neut(a)]_{neut},\ \forall\ a \in N,\ neut(a) \in \{neut(a)\}.$$

Then 1_{neut} is the unit element of $(N/\approx_{neut}, \bullet)$. Moreover, by Theorem 5.1 (3) we get that $anti([a]_{neut})$ is unique, $\forall a \in N$. Therefore, $(N/\approx_{neut}, \bullet)$ is an Abel group. □

Theorem 5.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. If define a new operation "$\to$" on the quotient $N/\approx_{neut}$ as following:*

$$\forall a, b \in N,\ [a]_{neut} \to [b]_{neut} = [a]_{neut} \bullet anti([b]_{neut}).$$

Then $(N/\approx_{neut}, \to, 1_{neut})$ is a BCI-algebra, where 1_{neut}=$[neut(a)]_{neut}$, $\forall a \in N$.

Proof. By Theorem 5.2 and Proposition 2.13 we can get the result. □

Example 5.4. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). For each $a \in N$, we have $neut(a)$ in N. That is,

$$neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in N$, $anti(a)$ in N. That is,

$$anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut}$= $\{[2]_{neut}, [1]_{neut}, [3]_{neut}, [4]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $(Z_4, +)$, where

$$[2]_{neut} = \{2, 7\},\ [1]_{neut} = \{1, 6\},\ [3]_{neut} = \{3, 8\},\ [4]_{neut} = \{4, 9\}.$$

Table 2 Cayley table of neutrosophic triplet group $(N, *)$

$*$	1	2	3	4	6	7	8	9
1	3	6	9	2	8	1	4	7
2	6	2	8	4	6	2	8	4
3	9	8	7	6	4	3	2	1
4	2	4	6	8	2	4	6	8
6	8	6	4	2	8	6	4	2
7	1	2	3	4	6	7	8	9
8	4	8	2	6	4	8	2	6
9	7	4	1	8	2	9	6	3

Table 3 Cayley table of Abel group $((N/\approx_{neut}, \bullet)$

$\bullet$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[2]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[1]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$
$[3]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$
$[4]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$

Table 4 Cayley table of Abel group $(Z_4, +)$

$+$	0	1	3	4
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Example 5.5. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group with condition (AN), the binary operation $\sharp$ is defined in Table 1. For each $\in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$neut(0) = 0,\ neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(5) = 5,\ neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in Z_{10}$, $anti(a)$ in Z_{10}. That is,

$$anti(0) \in \{0,1,2,3,4,5,6,7,8,9\},\ anti(1) = 9,\ anti(2) \in \{2,7\},$$
$$anti(3) = 3,\ anti(4) \in \{1,6\},\ anti(5) \in \{1,3,5,7,9\},$$
$$anti(6) \in \{4,9\},\ anti(7) = 7,\ anti(8) \in \{3,8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut} = \{1_{neut} = [0]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $\{1\}$,where

$$[0]_{neut} = 1_{neut} = \{0,1,2,3,4,5,6,7,8,9\}.$$

6. Quotient structure and neutro-homomorphism basic theorem

Definition 6.1 ([16])**.** Let $(N_1, *_1)$ and $(N_2, *_2)$ be two neutrosophic triplet groups. Let $f : N_1 \to N_2$ be a mapping. Then, f is called *neutro-homomorphism* if for all $a, b \in N_1$, we have:

(1) $f(a *_1 b) = f(a) *_2 f(b)$;
(2) $f(neut(a)) = neut(f(a))$;
(3) $f(anti(a)) = anti(f(a))$.

Theorem 6.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$, H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(a) * neut(a) \in \{neut(a)\} \subseteq H.$$

By Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in H.$$

Then, $a \approx_H a$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$a * anti(b) * neut(p) \in H. \tag{C2}$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Moreover, by the hypothesis ($anti(a) \in H$ for any $a \in H$), we have

$$anti(a * anti(b) * neut(p)) \in H. \tag{C3}$$

Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$b * anti(a) * x$
$= b * anti(a) * anti(neut(p))$
$= anti(a) * b * anti(neut(p))$ (by Definition 2.5)
$= anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p))$ (by Theorem 3.9)
$= (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b)$ (by Definition 2.4 and 2.5)
$\in \{anti(a * anti(b) * neut(p))\} * neut(b)$ (by Theorem 4.3)
$\subseteq H$ (by (C3), the hypothesis and Proposition 3.13 (1))

This means that $b \approx_H a$. □

Lemma 6.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), $a, b \in N$, and H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. If there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that*

$$a * anti(b) * neut(p) \in H.$$

Then for any $x \in \{anti(b)\}$, there exists $p_1 \in N$, and $neut(p_1) \in \{neut(p_1)\}$ such that

$$a * x * neut(p_1) \in H.$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b*x = x*b = y$. Since $(\forall a \in N)\ neut(a) \in H$, then $y \in H$. Thus, from $a*anti(b)*neut(p) \in H$ we get

$$\begin{aligned}
&a * x * (neut(b) * neut(p)) \\
&= a * x * (anti(b) * b) * neut(p) \\
&= (a * anti(b) * neut(p)) * (x * b) \\
&= (a * anti(b * neut(p)) * y \\
&\in H \qquad\qquad (by\ Proposition\ 3.13)
\end{aligned}$$

□

Theorem 6.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is an equivalent relation on N.

Proof. By Theorem 6.2, we only prove that $\approx_H$ is transitive. Assume that $a \approx_H b$ and $b \approx_H c$, then there exists $p, r \in N$ and $q, s \in N$ such that

$$\text{(C3)} \qquad a * anti(b) * neut(p) = q \in H.$$

$$\text{(C4)} \qquad b * anti(c) * neut(r) = s \in H.$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(r) \in \{neut(r)\}$. Using Theorem 4.1 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(p) * neut(s) * neut(c) \in neut(p * s * c) \subseteq H.$$

Denote $y = neut(p) * neut(s) * neut(c) \in neut(p * s * c)$, then

$a * anti(c) * y$

$= a * anti(c) * neut(p) * neut(s) * neut(c)$

$= a * anti(c) * neut(p) * (s * anti(s)) * neut(c)$ (by Definition 2.1)

$= a * anti(c) * neut(p) * s * anti(b * anti(c) * neut(r)) * neut(c)$ (by the above result (C4))

$\in a*anti(c)*neut(p)*s*\{anti(b)\}*\{anti(anti(c))\}*\{anti(neut(r))\}neut(c)$ (by Definition 4.5)

$= a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{anti(neut(r))\}$ (by Theorem 3.9)

$\subseteq a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{neut(r)\}$ (by Theorem 3.10)

$\subseteq \{a * anti(b) * neut(p)\} * s * (anti(c) * c) * \{neut(r)\}$ (by Definition 2.4 and 2.5)

$\subseteq H * s * neut(c) * \{neut(r)\}$

(by Definition 2.1, the above result (C3) and Lemma 6.3)
$\subseteq H$ (by (C4), the hypothesis and Proposition 3.13 (1))
It follows that $a \approx_H c$. □

Theorem 6.5. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then the following statements are hold:
(1) $a, b, c \in N$, $a \approx_H b \Rightarrow a * c \approx_H b * c$.
(2) $a \approx_H b \Rightarrow neut(a) \approx_H neut(b)$.
(3) $a \approx_H b \Rightarrow anti(a) \approx_H anti(b)$.

Proof. (1) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$(C2) \qquad a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. We have
$(a * c) * anti(b * c) * neut(p)$
$\in (a * c) * \{anti(b)\} * \{anti(c)\} * neut(p)$ (by Definition 4.5)
$\subseteq \{a * anti(b) * neut(p)\} * \{c * anti(c)\}$ (by Definition 2.4 and 2.5)
$= \{a * anti(b) * neut(p)\} * neut(c)$ (by Definition 2.1)
$\in H$. (by (C2), the hypothesis, Lemma 6.3 and Proposition 3.13 (1))
It follows that $a * c \approx_H b * c$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that $a * anti(b) * neut(p) \in H$, where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying Theorem 3.8 and Theorem 4.1 we have
$neut(a) * anti(neut(b)) * neut(p) \in neut(a) * \{neut(b)\} * neut(p)$
$\subseteq \{neut(a*b*p)\} \subseteq H$. (by the hypothesis, $neut(a) \in H$ for any $a \in N$)
It follows that $neut(a) \approx_H neut(b)$.

Assume $a \approx_H b$, then there exists $p \in N$ such that

$$a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying the hypothesis ($(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$) and Theorem 3.10,

$$anti(a * anti(b) * neut(p)) \in H.$$
$$anti(neut(p)) \in \{neut(p)\} \subseteq H.$$

Moreover, by Theorem 4.3 we have

$$anti(a) * anti(anti(b)) * anti(neut(p)) \in \{anti(a * anti(b) * neut(p))\} \subseteq H.$$

Hence, $anti(a) \approx_H anti(b)$. □

Theorem 6.6. *Let $(N,*)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. Then the quotient $N/\approx_H$ is a commutative neutrosophic triplet group with respect to the following operation:*

$$\forall a, b \in N,\ [a]_H \bullet [b]_H = [a * b]_H.$$

where $[a]_H$ is the equivalent class of a with respect to $\approx_H$. Moreover, $(N,*)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the following mapping:

$$f:\ N \to N/\approx_H;\ \text{and}\ \forall a \in N,\ f(a) = [a]_H.$$

Proof. By Theorem 6.5 we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_H, \bullet)$ is a commutative neutrosophic triplet group.

By the definitions of operation "$\bullet$" and mapping f we have

$$\forall a, b \in N,\ f(a * b) = [a * b]_H = [a]_H \bullet [b]_H = f(a) \bullet f(b).$$

Moreover, by Theorem 6.5 (2) and (3) we get

$$\forall a \in N, f(neut(a)) = [neut(a)]_H = neut([a]_H) = neut(f(a)).$$
$$\forall a \in N, f(anti(a)) = [anti(a)]_H = anti([a]_H) = anti(f(a)).$$

Therefore, $(N,*)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the mapping f. □

Theorem 6.7. *Let $(N,*)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. If define a new operation "$\to$" on the quotient $N/\approx_H$ as following: $\forall a, b \in N$, $[a]_H \to [b]_H = [a]_H \bullet anti([b]_H)$. Then $(N/\approx_H, \to, 1_H)$ is a BCI-algebra, where $1_H = [neut(a)]_H$, $\forall a \in N$.*

Proof. By Theorem 6.7 and Proposition 2.13 we can get the result. □

Example 6.8. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N,*)$ is a neutrosophic triplet group with condition (AN). We can get the following equation

$$\begin{gathered} neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2, \\ neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7; \\ anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\}, \\ anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1. \end{gathered}$$

Denote $H = \{2, 3, 7, 8\}$, it is easy to verify that H is a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Moreover, $N/\approx_H = \{H = [2]_H, [1]_H\}$ and $(N/\approx_H, \bullet)$ is isomorphism to $(Z_2, +)$, where

$$[2]_H = \{2, 3, 7, 8\},\ [1]_H = \{1, 4, 6, 9\}.$$

Table 5 Cayley table of Abel group $(N/\approx_H, \bullet)$

$\bullet$	$[2]_H$	$[1]_H$
$[2]_H$	$[2]_H$	$[1]_H$
$[1]_H$	$[1]_H$	$[2]_H$

Table 6 Cayley table of Abel group $(Z_2, +)$

$+$	0	1
0	0	1
1	1	0

The following example shows that the basic theorem of neutro-homomorphism (Theorem 6.7) is a natural and substantial generalization of the basic theorem of group-homomorphism.

Example 6.9. Let $(N, *)$ be a commutative group. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). Obviously, if H is a subgroup of N, then binary relation $\approx_H$ on N is the relation induced by subgroup H, that is,

$$\forall a, b \in N,\ a \approx_H b \text{ if and only if } a * b^{-1} \in H.$$

Thus, $(N, *)$ is group-homomorphism to $(N/\approx_H, \bullet) = (N/H, \bullet)$.

7. Conclusion

This paper is focus on neutrosophic triplet group. We proved some new properties of (commutative) neutrosophic triplet group, and constructed a new equivalent relation on any commutative neutrosophic triplet group with condition (AN). Based on these results, for the first time, we have described the inner link between commutative neutrosophic triplet group with condition (AN) and Abel group with BCI-algebra. Furthermore, we establish the quotient structure by neutrosophic triplet subgroup, and prove the basic theorem of neutro-homomorphism, which is a natural and substantial generalization of the basic theorem of group-homomorphism. Obviously, these results will play an important role in the further study of neutrosophic triplet group.

Acknowledgment

This work was supported by National Natural Science Foundation of China (Grant No. 61573240).

References

[1] A. A. A. Agboola, B. Davvaz, F. Smarandache, *Neutrosophic quadruple algebraic hyperstructures*, Annals of Fuzzy Mathematics and Informatics, 14(2017), 29-42.

[2] S.S. Ahn, J.M. Ko, *Rough fuzzy ideals in BCK/BCI-algebras*, Journal of Computational Analysis and Applications, 25(2018), 75-84.

[3] R.A. Borzooei, H. Farahani, M. Moniri, *Neutrosophic deductive filters on BL-algebras*, Journal of Intelligent and Fuzzy Systems, 26(2014), 2993-3004.

[4] W.A. Dudek, Y.B. Jun, *Pseudo-BCI algebras*, East Asian Mathematical Journal, 24(2008), 187-190.

[5] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad., 42(1966), 26-29.

[6] Y. B. Jun, H.S. Kim, J. Neggers, *On pseudo-BCI ideals of pseudo-BCI algebras*, Matematicki Vesnik, 58(2006), 39-46.

[7] Y. B. Jun, *Neutrosophic subalgebras of several types in BCK/BCI-algebras*, Annals of Fuzzy Mathematics and Informatics, 14(2017), 75-86.

[8] H. S. Kim, Y. H. Kim, *On BE-algebras*, Sci. Math. Japon., 66(2007), 113-116.

[9] T. D. Lei, C. C. Xi, *p-radical in BCI-algebras*, Mathematica Japanica, 30(1985), 511-517.

[10] A. Rezaei, A.B. Saeid, F. Smarandache, *Neutrosophic filters in BE-algebras*, Ratio Mathematica, 29(2015), 65-79.

[11] A. B. Saeid, Y. B. Jun, *Neutrosophic subalgebras of BCK/ BCI-algebras based on neutrosophic points*, Annals of Fuzzy Mathematics and Informatics, 14(2017), 87-97.

[12] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic*, Amer. Res. Press, Rehoboth, USA, 1998.

[13] F. Smarandache, *Neutrosophy and Neutrosophic Logic, Information Sciences First International Conference on Neutrosophy, Neutrosophic Logic, Set*, Probability and Statistics University of New Mexico, Gallup, USA, 2002.

[14] F. Smarandache, *Neutrosophic setCa generialization of the intuituionistics fuzzy sets*, International Journal of Pure and Applied Mathematics, 24(2005), 287-297.

[15] F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*, Pons Publishing House, Brussels, 2017.

[16] F. Smarandache, M. Ali, *Neutrosophic triplet group*, Neural Computing and Applications, 2017, DOI 10.1007/ s00521-016-2535-x

[17] H. Wang, F. Smarandache, Y.Q. Zhang et al, *Single valued neutrosophic sets, Multispace and Multistructure*. Neutrosophic Transdisciplinarity, 4(2010), 410-413.

[18] X. L. Xin, Y. J. Li, Y. L. Fu, *States on pseudo-BCI algebras*, European Journal of Pure And Applied Mathematics, 10(2017), 455-472.

[19] J. Ye, *Single valued neutrosophic cross-entropy for multicriteria decision making problems*, Applied Mathematical Modelling, 38(2014), 1170-1175.

[20] X. H. Zhang, R. F. Ye, *BZ-algebra and group*, J. Math. Phys. Sci., 29(1995), 223-233.

[21] X. H. Zhang, Y. Q. Wang, W. A. Dudek, *T-ideals in BZ-algebras and T-type BZ-algebras*, Indian Journal Pure and Applied Mathematics, 34(2003), 1559-1570.

[22] X. H. Zhang, W. H. Li, *On pseudo-BL algebras and BCC-algebra*, Soft Computing, 10(2006), 941-952.

[23] X. H. Zhang, *Fuzzy Logics and Algebraic Analysis*, Science Press, Beijing, 2008.

[24] X. H. Zhang, W. A. Dudek, *BIK+-logic and non-commutative fuzzy logics*, Fuzzy Systems and Mathematics, 23(2009), 8-20.

[25] X. H. Zhang, *BCC-algebras and residuated partially-ordered groupoid*, Mathematica Slovaca, 63(2013), 397-410.

[26] X. H. Zhang, Y. B. Jun, *Anti-grouped pseudo-BCI algebras and anti-grouped pseudo-BCI filters*, Fuzzy Systems and Mathematics, 28(2014), 21-33.

[27] X. H. Zhang, *Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras*, Journal of Intelligent and Fuzzy Systems, 33(2017), 1767-1774.

[28] X. H. Zhang, Y. T. Wu, X. H. Zhai, *Neutrosophic filters in pseudo-BCI algebras*, submitted, 2017.

[29] X. H. Zhang, Y. C. Ma, F. Smarandache, *Neutrosophic regular filters and fuzzy regular filters in pseudo-BCI algebras*, Neutrosophic Sets and Systems, 17(2017), 10-15.

Accepted: 27.10.2017

DERIVABLE MAPPINGS AND COMMUTATIVITY OF ASSOCIATIVE RINGS

Gurninder S. Sandhu*
Department of Mathematics
Punjabi University
Patiala
gurninder_rs@pbi.ac.in

Deepak Kumar
Department of Mathematics
Punjabi University
Patiala
deep_math1@yahoo.com

Abstract. Let R be a ring with center $Z(R)$. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative (generalized)-derivation of R if it is uniquely determined by a mapping $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for each $x, y \in R$. In the present paper, we investigate the commutativity of a semiprime (prime) ring via studying a number polynomial constraints involving multiplicative (generalized)-derivations. Moreover, some annihilator conditions are also examined.

Keywords: Prime ring, Semiprime ring, Derivation, Generalized derivation, Multiplicative (generalized)-derivation.

1. Introduction

All through this paper R be an associative ring with center $Z(R)$. A ring R is said to be a prime ring if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and semiprime if $aRa = (0)$ implies that $a = 0$. Obviously, every prime ring is semiprime. For any nonempty subset S of R the right annihilator $r_R(S)$ of S in R is the set of all $r \in R$ such that $Sr = (0)$. Accordingly, the left annihilator $l_R(S)$ is the set of all $r \in R$ such that $rS = (0)$. The intersection of right and left annihilators of S in R *i.e.*

$$Ann_R(S) = \{r \in R : sr = 0 \text{ and } rs = 0 \text{ for all } s \in S\}$$

is called an annihilator of S in R . Recall that, for any $x, y \in R$ the commutator and anti-commutator are denoted by the symbols $[x, y] = xy - yx$ and $x \circ y = xy + yx$ respectively. We shall frequently use the basic commutator identities:

$$[xy, z] = x[y, z] + [x, z]y, [x, yz] = y[x, z] + [x, y]z,$$

∗. Corresponding author

for all $x, y, z \in R$. For any nonempty subset Q of R, a mapping $f : R \to R$ is said to be centralizing on Q if $[f(x), x] \in Z(R)$ and commuting if $[f(x), x] = 0$ for all $x \in Q$. A derivation (or left multiplier) of R is a map such that $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ (or $d(xy) = d(x)y$) for all $x, y \in R$. The notion of derivation was extended to generalized derivation by Brešar [10]. A generalized derivation of R is an additive map uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Inspired by Martindale's [22] remarkable paper on the additivity of multiplicative bijective mappings, Daif [12] introduced multiplicative derivation, which is a map $d : R \to R$ satisfying Leibnitz rule and not necessarily additive on R. The complete description of such mappings was explained by Goldmann and Šemrl [18]. Daif and Tammam-El-Sayiad [14] extended this notion to multiplicative generalized derivation by dropping the additivity assumption of generalized derivation F. Recently, Dhara and Ali [16] made a slight generalization in this definition of multiplicative generalized derivation by relaxing the conditions on d and call it multiplicative (generalized)-derivation, which is a map $F : R \to R$ (not necessarily additive) along with a map $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ where $x, y \in R$. Observe that every multiplicative derivation is a multiplicative (generalized)-derivation, so multiplicative (generalized)-derivation covers both the concepts of multiplicative derivation (if $F = d$) and multiplicative left multiplier (if $d = 0$). In this way, multiplicative (generalized)-derivation is a more satisfactory generalization of multiplicative derivation.

2. Some preliminary results

Throughout this paper, we shall use the following well known lemmas to prove our results:

Lemma 1 (Lemma 2, [13]). *If R is a prime ring containing nonzero central ideal, then R is commutative.*

Lemma 2 (Corollary 2, [20]). *If R is a semiprime ring and I is an ideal of R, then $I \cap Ann_R(I) = (0)$.*

Lemma 3 (Lemma 2.3, [17]). *If R is a prime ring, I a nonzero ideal and d is derivation of R. If for some $0 \neq a \in R$, $[ad(x), x] = 0$ for all $x \in I$, then $d = 0$ or R is commutative.*

Lemma 4 (Corollary, [20]). *Let R be a semiprime ring and let I be a nonzero right ideal of R. If I is commutative as a ring, then $I \subseteq Z(R)$.*

Lemma 5 (Theorem, [21]). *Let g be a polynomial in n noncommuting variables $u_1, u_2..., u_n$ with relatively prime integer coefficients. Then the following are equivalent:*

(*i*) *Every ring satisfying the polynomial identity* $g = 0$ *has nil commutator ideal.*

(*ii*) *Every semiprime ring satisfying* $g = 0$ *is commutative.*

(*iii*) *For every prime* p *the ring of* 2×2 *matrices over* Z_p *fails to satisfy* $g = 0$.

Throughout this paper, R will denote a semiprime ring with nonzero ideal I, unless otherwise stated.

3. Main results

3.1 On central value conditions

During the last seven decades, there has been a large amount of results concerning the conditions that force a ring to be commutative. In this direction, Posner [23] proved a classical result: *Every prime ring admitting a nonzero centralizing derivation is commutative.* This theorem has been generalized in many ways. Towards the commutativity of prime rings with derivations Ashraf et al. [5] proved: *Let* R *be a prime ring and* I *be a nonzero ideal of* R. *Suppose that* d *is a nonzero derivation of* R *such that* $d(xy) \pm xy \in Z(R)$ *where* $x, y \in I$, *then* R *is commutative.* In [3], Ashraf et al. extend these results for generalized derivations and obtained the following theorem: *Let* R *be a prime ring and* I *a nonzero ideal of* R. *Suppose* F *is a generalized derivation associated with a derivation* d *on* R. *If one of the following:*

(*i*) $F(xy) \pm xy \in Z(R)$,

(*ii*) $F(xy) \pm yx \in Z(R)$,

(*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds on* I, *then* R *is commutative.*

After that, Atteya [6] studied these situations on semiprime rings and obtained the following results: *Let* R *be a semiprime ring and* I *be a nonzero ideal of* R. *If* R *admits a generalized derivation* F *associated with a derivation* d *such that any one of the following:*

(*i*) $F(xy) \pm xy \in Z(R)$,

(*ii*) $F(xy) \pm yx \in Z(R)$,

(*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds on* I, *then* R *contains a nonzero central ideal.*

It is a fact of interest to generalize these results to multiplicative (generalized)-derivations. In this line of investigation Dhara and Ali [16] studied the following identities: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(xy) \pm xy \in Z(R)$, (iv) $F(x)F(y) \pm yx \in Z(R)$, where x, y varies over some suitable subset of semiprime ring R. In this section, we study central valued conditions involving multiplicative (generalized)-derivations and consequently give a generalized version of some known results.

Theorem 1. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(xy) + xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. For each $x, y \in I$, we consider

$$F(xy) + xy + [\phi(x), y] \in Z(R). \tag{3.1}$$

Replace y by yz in (3.1), where $z \in I$ and we get $(F(xy) + xy + [\phi(x), y])z + xyd(z) + y[\phi(x), z] \in Z(R)$. On commuting with z and using our hypothesis, we find

$$[xyd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.2}$$

Again replace y by zy in (3.2), we have

$$[xzyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.3}$$

Left multiply (3.2) by z and subtract from (3.3) in order to obtain

$$[[x, z]yd(z), z] = 0. \tag{3.4}$$

Since I is an ideal of R so we substitute $xd(z)$ in place of x in (3.4) and get

$$[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0. \tag{3.5}$$

Now, substitute $d(z)y$ instead of y in (3.4) and subtract from (3.5) to obtain

$$[x[d(z), z]yd(z), z] = 0. \tag{3.6}$$

Putting $x = d(z)x$ in (3.6) and we obtain $d(z)[x[d(z), z]yd(z), z]+[d(z), z]x[d(z), z]yd(z) = 0$. Relation (3.6) reduces it to $[d(z), z]x[d(z), z]yd(z) = 0$. That is, $(I[d(z), z])^3 = (0)$. But R has no nonzero nilpotent ideal, hence $I[d(z), z] = (0)$. Clearly, $[d(z), z] \in I$ as well as $[d(z), z] \in Ann_R(I)$. That means $[d(z), z] \in I \cap Ann_R(I)$. Therefore, Lemma 2 implies that $[d(z), z] = 0$ for each $z \in I$. This process also shows that every nonzero ideal of a semiprime ring is a semiprime ring itself.

Next, we assume that ϕ is an automorphism of R. Replacing y by yz in (3.2), we get

$$[xyzd(z), z] + [yz[\phi(x), z], z] = 0. \tag{3.7}$$

Multiplying (3.2) from right by z, we get

$$[xyd(z)z, z] + [y[\phi(x), z]z, z] = 0. \tag{3.8}$$

Subtracting (3.7) from (3.8) and we find $[xy[d(z),z],z]+[y[[\phi(x),z],z],z]=0$. Since $[d(z),z]=0$, we left with the expression

$$[y[[\phi(x),z],z],z]=0. \tag{3.9}$$

Putting $y=ty$ in (3.9), where $t\in I$, we have $t[y[[\phi(x),z],z],z]+[t,z]y[[\phi(x),z],z]=0$ for each $x,y,z,t\in I$. Use of Eq. (3.9) gives

$$[t,z]y[[\phi(x),z],z]=0. \tag{3.10}$$

Replace t by $t[\phi(x),z]$ in (3.10) and we obtain

$$t[[\phi(x),z],z]y[[\phi(x),z],z]+[t,z][\phi(x),z]y[[\phi(x),z],z]=0. \tag{3.11}$$

Replace y by $[\phi(x),z]y$ in (3.10) and combine with (3.11) in order to find $t[[\phi(x),z],z]y[[\phi(x),z],z]=0$. In particular, we have $y[[\phi(x),z],z]Ry[[\phi(x),z],z]=(0)$. Hence, we obtain $y[[\phi(x),z],z]=0$. That is, $I[[\phi(x),z],z]=(0)$. Thus, semiprimeness of I assures that, for each $x,z\in I$

$$[[\phi(x),z],z]=0. \tag{3.12}$$

Linearizing (12) w.r.t.z, we get

$$[[\phi(x),t],z]+[[\phi(x),z],t]=0. \tag{3.13}$$

Substituting zt in place of z in (3.13), where $t\in I$. We obtain

$$[[\phi(x),t],z]t+z[[\phi(x),t],t]+[[\phi(x),z],t]t+[z[\phi(x),t],t]=0. \tag{3.14}$$

Using (3.12) and (3.13) in (3.14), it follows that

$$[z,t][\phi(x),t]=0. \tag{3.15}$$

Replace x by $x\phi^{-1}(z)$ in (3.15), we obtain $[z,t]\phi(x)[z,t]=0$ for any $x,z,t\in I$. Since ϕ is an automorphism of R so $\phi(I)$ is an ideal of R. Thus, we may infer that I is commutative as a ring. Hence, by Lemma 4 we infer that $I\subseteq Z(R)$.

On substituting $-\phi$ in place of ϕ in (3.1) and following the same argument with necessary variations, we get the same conclusions for the situation $F(xy)+xy-[\phi(x),y]\in Z(R)$. □

Theorem 2. *Let $F:R\to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d:R\to R$. If ϕ is a mapping of R such that $F(xy)-xy\pm[\phi(x),y]\in Z(R)$ for all $x,y\in I$, then $[d(x),x]=0$ for all $x\in I$.*

Furthermore, if ϕ is an automorphism of R, then $I\subseteq Z(R)$.

Proof. On replacing F by $-F$ and d with $-d$ in Theorem 1, we can get the desired results. □

Theorem 3. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(x)F(y) + xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. For any $x, y \in I$, we consider

$$F(x)F(y) + xy + [\phi(x), y] \in Z(R). \tag{3.16}$$

On replacing y by yz in (3.16), where $z \in I$, we find $(F(x)F(y)+xy+[\phi(x), y])z+ F(x)yd(z)+y[\phi(x), z] \in Z(R)$. On commuting with z, our hypothesis forces that

$$[F(x)yd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.17}$$

Put $y = zy$ in (3.17) and we get

$$[F(x)zyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.18}$$

Left multiply (3.17) by z and subtract from (3.18), we have

$$[[F(x), z]yd(z), z] = 0. \tag{3.19}$$

Replace x by xz in (3.19) and we obtain

$$[[F(x), z]zyd(z), z] + [[xd(z), z]yd(z), z] = 0. \tag{3.20}$$

Replace y by zy in (3.19) and subtract from (3.20) to obtain $[[xd(z), z]yd(z), z] = 0$. That is, $[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0$. This expression is same as (3.5), so the similar arguments imply that $[d(z), z] = 0$ for each z in I. Now, we replace y by yz in (3.17) and get

$$[F(x)yzd(z), z] + [yz[\phi(x), z], z] = 0. \tag{3.21}$$

Right multiply (3.17) by z, we get

$$[F(x)yd(z)z, z] + [y[\phi(x), z]z, z] = 0. \tag{3.22}$$

Combining relations (3.21) and (3.22), we have $[F(x)y[d(z), z], z]+[y[[\phi(x), z], z], z] = 0$. Utilizing the fact $[d(z), z] = 0$, for all $z \in I$, we get $[y[[\phi(x), z], z], z] = 0$. This expression is same as equation (3.9), again the proof follows from Theorem 1.

On substituting $-\phi$ in place of ϕ in (3.16) and following the same technique with necessary variations, we get the same conclusions for the situation $F(x)F(y) + xy - [\phi(x), y] \in Z(R)$. □

Theorem 4. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(x)F(y) - xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. On replacing F by $-F$ and d by $-d$ in Theorem 3, we can get the desired results. □

Now, we extend some theorems of Tiwari et al. [25].

Theorem 5. *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If ϕ is a mapping of R such that $G(xy) + F(x)F(y) \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ and $[g(x), x] = 0$ for all $x \in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Proof. For each $x, y \in I$, we consider

$$G(xy) + F(x)F(y) + [\phi(x), y] \in Z(R). \tag{3.23}$$

Putting $y = yz$ in (3.23), where $z \in I$, we get $(G(xy) + F(x)F(y) + [\phi(x), y])z + xyg(z) + F(x)yd(z) + y[\phi(x), z] \in Z(R)$. On commuting with z, our hypothesis yields

$$[xyg(z), z] + [F(x)yd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.24}$$

Replace y by zy in (3.24) and we get

$$[xzyg(z), z] + [F(x)zyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.25}$$

Left multiply (3.24) by z and subtract from (3.25), we have

$$[[x, z]yg(z), z] + [[F(x), z]yd(z), z] = 0. \tag{3.26}$$

On replacing x by xz in (3.26), we get

$$[[x, z]zyg(z), z] + [[F(x), z]zyd(z), z] + [[xd(z), z]yd(z), z] = 0. \tag{3.27}$$

Replace y by zy in (3.26) and subtract from (3.27) to find

$$[[xd(z), z]yd(z), z] = 0. \tag{3.28}$$

That is, $[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0$. This equation is same as (3.5), so similar arguments imply that $[d(z), z] = 0$ for each $z \in I$. Now, we substitute yz instead of y in (3.26) in order to obtain

$$[[x, z]yzg(z), z] + [[F(x), z]yzd(z), z] = 0. \tag{3.29}$$

Right multiply (3.26) by z, we get

$$[[x, z]yg(z)z, z] + [[F(x), z]yd(z)z, z] = 0. \tag{3.30}$$

Subtract (3.29) from (3.30), we obtain $[[x,z]y[g(z),z],z]+[[F(x),z]y[d(z),z],z] = 0$. Utilizing the fact $[d(z),z]=0$, for all $z\in I$, we find

$$[[x,z]y[g(z),z],z]=0. \tag{3.31}$$

Put $x=xg(z)$ in (3.31), we get

$$[x[g(z),z]y[g(z),z],z]+[[x,z]g(z)y[g(z),z],z]=0. \tag{3.32}$$

Put $y=g(z)y$ in (3.31) and subtract from (3.32) in order to get

$$[x[g(z),z]y[g(z),z],z]=0. \tag{3.33}$$

Substituting $g(z)x$ for x in (3.33) and we get $g(z)[x[g(z),z]y[g(z),z],z]+[g(z),z]x[g(z),z]y[g(z),z],z]=0$. Eq. (3.33) reduces it to $[g(z),z]x[g(z),z]y[g(z),z]=0$. It implies that $(I[g(z),z])^3=(0)$. Since R has no nonzero ideal, we have $I[g(z),z]=(0)$. Semiprimeness of I yields that for each $z\in I$, $[g(z),z]=0$.

Next, let us assume that R is a prime ring and ϕ is an automorphism of R. Replace y by yz in (3.24) and we find

$$[xyzg(z),z]+[F(x)yzd(z),z]+[yz[\phi(x),z],z]=0. \tag{3.34}$$

Right multiply (3.24) by z in order to get

$$[xyg(z)z,z]+[F(x)yd(z)z,z]+[y[\phi(x),z]z,z]=0. \tag{3.35}$$

Subtracting (3.34) from (3.35) and we find

$$[xy[g(z),z],z]+[F(x)y[d(z),z],z]+[y[[\phi(x),z],z],z]=0. \tag{3.36}$$

Since d and g are commuting on I, Eq. (3.36) reduces to $[y[[\phi(x),z],z],z]=0$. This is same as equation (3.9), again from Theorem 1, we get $I\subseteq Z(R)$. By Lemma 1, R is commutative.

On substituting $-\phi$ in place of ϕ in (3.23) and following the same argument with necessary variations, we get the same conclusions for the identity $G(xy)+F(x)F(y)-[\phi(x),y]\in Z(R)$. □

Corollary 1 (Theorem 1, [25])**.** *Let $F,G:R\to R$ be multiplicative (generalized)-derivations of R together with mappings d,g respectively. If ϕ is a mapping of R such that $G(xy)+F(x)F(y)\pm[\phi(x),y]=0$ for all $x,y\in I$, then $[d(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Theorem 6. *Let $F,G:R\to R$ be multiplicative (generalized)-derivations of R together with mappings d,g respectively. If ϕ is a mapping of R such that $G(xy)-F(x)F(y)\pm[\phi(x),y]\in Z(R)$ for all $x,y\in I$, then $[d(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Proof. On replacing G by $-G$ and g by $-g$ in Theorem 5, we can get the desired results. □

Corollary 2 (Theorem 2, [25])**.** *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If ϕ is a mapping of R such that $G(xy) - F(x)F(y) \pm [\phi(x), y] = 0$ for all $x, y \in I$, then $[d(x), x] = 0$ and $[g(x), x] = 0$ for all $x \in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Corollary 3. *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If any of the following condition*

(*i*) $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$

(*ii*) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$

(*iii*) $G(xy) \pm F(x)F(y) \in Z(R)$

(*iv*) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$

holds on R. Then R is commutative.

Proof. (i) Firstly, we consider $G(xy) + F(x)F(y) \pm [x, y] \in Z(R)$ for each $x, y \in R$. In particular, for $\phi = i_d$ (identity map), Theorem 5 gives us that $[y[[x, z], z], z] = 0$ where $x, y, z \in R$. From Theorem 5 commutativity of R easily follows. We also can prove the same conclusion with an alternative way. Since for each $x, y, z \in I$, we have $[y[[z, x], z], z] = 0$, which is a polynomial identity in noncommuting three variables on R. If possible assume that, for some prime integer p the ring $M_2(GF(p))$ satisfies the polynomial identity $[y[[z, x], z], z] = 0$. But, if we choose $x = e_{11}$, $y = e_{12}$, and $z = e_{12} + e_{21}$, where e_{ij} denotes the 2×2 matrix with 1 in $(ij)^{th}$-entry and 0 elsewhere. With these choices we see that $[y[[z, x], z], z] = 2(e_{11} - e_{22})$, which is a contradiction. Hence by Lemma 5, R must be commutative.

Similarly, we can prove the commutativity of R for the constraint $G(xy) - F(x)F(y) \pm [x, y] \in Z(R)$.

The proof of (ii), (iii) and (iv) is straight forward from the fact that if G is a multiplicative (generalized)-derivation of R associated with a mapping g, then so is $G \pm i_d$, where i_d is the identity map of R. □

Immediately after Theorem 5 and Theorem 6 with Corollary 4.2 of [9], we give the following result:

Corollary 4. *Let $F, G : R \to R$ be multiplicative generalized derivations of R together with derivations d, g respectively. If for any map ϕ on R, $G(xy) \pm F(x)F(y) \pm [\phi(x), y] \in Z(R)$ where $x, y \in R$, then there exist $\lambda_1, \lambda_2 \in C$ and additive mappings $\zeta_1, \zeta_2 : R \to C$ respectively such that $d(x) = \lambda_1 x + \zeta_1(x)$ and $g(x) = \lambda_2 x + \zeta_2(x)$ for all $x \in R$.*

Next, we give a generalization of Theorem 2.7 of [3] as a consequence of above results in the setting of generalized derivations:

Remark 1. Let I be a nonzero ideal of a prime ring R. If F and G are generalized derivations of R together with derivations d and g, then the following conditions are equivalent:

(i) $G(xy) + F(x)F(y) \pm [x, y] \in Z(R)$ or $G(xy) - F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in I$.

(ii) $G(xy) + F(x)F(y) \pm yx \in Z(R)$ or $G(xy) - F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$.

(iii) $G(xy) + F(x)F(y) \in Z(R)$ or $G(xy) - F(x)F(y) \in Z(R)$ for all $x, y \in I$.

(iv) $G(xy) + F(x)F(y) \pm xy \in Z(R)$ or $G(xy) - F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$.

(v) R is commutative.

Proof. Clearly, $(v) \Rightarrow (i), (v) \Rightarrow (ii), (v) \Rightarrow (iii)$ and $(v) \Rightarrow (iv)$.

$(i) \Rightarrow (v)$ Let $x \in I$ be a fixed element. Let $A_x = \{y \in I : G(xy) + F(x)F(y) \pm [x, y] \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \pm [x, y] \in Z(R)\}$. Since F and G are additive mappings so both A_x and B_x are additive subgroups of I such that $I = A_x \cup B_x$. Therefore, Brauer's trick forces that either $I = A_x$ or $I = B_x$. Now, for some fixed $y \in I$, let $A_y = \{x \in I : G(xy) + F(x)F(y) \pm [x, y] \in Z(R)\}$ and $B_y = \{x \in I : G(xy) - F(x)F(y) \pm [x, y] \in Z(R)\}$. By the same arguments as above, we find that either $I = A_y$ or $I = B_y$. Hence, the commutativity of R follows from Theorem 5 and Theorem 6 with $\phi = i_d$ the identity map.

$(ii) \Rightarrow (v)$ By substituting $\phi = i_d$ and $G = G \mp i_d$ together with g in Theorem 5 and Theorem 6, we may infer that R is commutative if any one of

(a) $G(xy) + F(x)F(y) \pm yx \in Z(R)$

(b) $G(xy) - F(x)F(y) \pm yx \in Z(R)$

holds on I. For a fixed element $x \in I$ we set $A_x = \{y \in I : G(xy) + F(x)F(y) \pm yx \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \pm yx \in Z(R)\}$. Further, by repeating the same arguments we can get the required results.

$(iii) \Rightarrow (v)$ By substituting $\phi = 0$ in Theorem 5 and Theorem 6, we infer that R is commutative if any one of

(a) $G(xy) + F(x)F(y) \in Z(R)$

(b) $G(xy) - F(x)F(y) \in Z(R)$

holds on I. For a fixed element $x \in I$ we set $A_x = \{y \in I : G(xy) + F(x)F(y) \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \in Z(R)\}$. Again, by repeating the same arguments we can get the desired results.

$(iv) \Rightarrow (v)$ As we just shown that if either $G(xy) + F(x)F(y) \in Z(R)$ or $G(xy) - F(x)F(y) \in Z(R)$ holds on I, then R is commutative. By replacing G by $G \pm i_d$ in these equations, we can easily get the desired conclusion. □

3.2 On annihilator conditions

Let S be any subset of R. A derivation d is said to be acting as a homomorphism or as an anti-homomorphism on a set S if $d(xy) - d(x)d(y) = 0$ for all $x, y \in S$ or $d(xy) - d(y)d(x) = 0$ for all $x, y \in S$ respectively. Study of the derivations acting as homomorphisms or as anti-homomorphisms on associative rings was initiated by Bell and Kappe in [7]. After that a number of results has been obtained with various types of derivations acting as homomorphisms or as anti-homomorphisms on some appropriate subsets of associative rings (see [1], [2], [15], [17], [19], [24] and references therein). In [19], Gusic proved the following: *Let R be an associative prime ring, let d be any function on R (not necessarily a derivation nor an additive function), let F be any function on R (not necessarily additive) satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and let I be a non-zero ideal in R.*

Assume that $F(xy) - F(x)F(y) = 0$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for any $x \in R$.

Assume that $F(xy) - F(y)F(x) = 0$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for any $x \in R$ (in this case R should be commutative) . Ali et al. [2] studied the same functional identities on square closed Lie ideal of 2-torsion free prime ring. In [27], Dhara et al. extend this notion by studying the algebraic identities $F(x)G(y) \pm H(xy) \in Z(R)$ and $F(x)G(y) \pm H(yx) \in Z(R)$ on square-closed Lie ideals of prime ring of char $\neq 2$, where F, G, H are generalized derivations of R. Further, Rehman and Raza in [26] gave a study of generalized derivations acting as homomorphism or anti-homomorphism on Lie ideals (without the assumption of square-closeness) of 2-torsion free prime ring.

Recently, Dhara et al. [17] obtained the following result: *Let R be a prime ring, I a nonzero ideal of R and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $d : R \to R$. For some $0 \neq a \in R$, suppose that $a(F(xy) \pm F(x)F(y)) = 0$ for each $x, y \in I$. Then one of the following hold:*

1. *$d(R) = 0$ and $aF(R) = 0$.*
2. *$d(R) = 0$ and $F(r) = \mp r$, where $r \in R$.*

Following this line of investigation, in this section we studied the situations $a(F(xy) \pm F(x)F(y)) \in Z(R)$ and $a(F(xy) \pm F(y)F(x)) = 0$.

Theorem 7. *Let $(0, I_d \neq) F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If for some $0 \neq a \in R$, $a(F(xy) \pm F(x)F(y)) \in Z(R)$ for all $x, y \in I$, then $[ad(z), z] = 0$ for all $z \in I$.*

Furthermore, if R is prime and d is a derivation on R, then R is commutative.

Proof. For each $x, y \in I$, we consider

$$a(F(xy) \pm F(x)F(y)) \in Z(R). \tag{3.37}$$

Replace y by yt in (3.37), where $t \in I$, we find $a(F(xy)\pm F(x)F(y))t+a(xyd(t)\pm F(x)yd(t)) \in Z(R)$. On commuting with t and using (3.37), we get

$$[a(xyd(t) \pm F(x)yd(t)), t] = 0. \tag{3.38}$$

Put $y = ty$ in (3.38) and we obtain

$$[a(xtyd(t) \pm F(x)tyd(t)), t] = 0. \tag{3.39}$$

On replacing x by xt in (3.38), we get

$$[a(xtyd(t) \pm F(x)tyd(t)), t] \pm [a(xd(t)yd(t)), t] = 0. \tag{3.40}$$

Subtracting (3.39) from (3.40), we get $[a(xd(t)yd(t)), t] = 0$. Substituting $d(t)x$ in place of x, we obtain $[ad(t)xd(t)yd(t), t] = 0$. That is,

$$ad(t)xd(t)yd(t)t - tad(t)xd(t)yd(t) = 0. \tag{3.41}$$

Replacing x by $xad(t)z$ in (3.41), where $z \in I$, we find

$$ad(t)xad(t)zd(t)yd(t)t - tad(t)xad(t)zd(t)yd(t) = 0.$$

Making use of (3.41), we get

$$ad(t)xtad(t)zd(t)yd(t) - ad(t)xad(t)zd(t)tyd(t) = 0.$$

$$ad(t)x[ad(t)zd(t), t]yd(t) = 0. \tag{3.42}$$

Putting $x = zd(t)x$ in (3.42) in order to get

$$ad(t)zd(t)x[ad(t)zd(t), t]yd(t) = 0. \tag{3.43}$$

Replacing x by tx in (3.43), we get

$$ad(t)zd(t)tx[ad(t)zd(t), t]yd(t) = 0. \tag{3.44}$$

Left multiply (3.43) by t and subtract it from (3.44), we left with

$$[ad(t)zd(t), t]x[ad(t)zd(t), t]yd(t) = 0.$$

In this way, we obtain

$$[ad(t)zd(t), t]x[ad(t)zd(t), t]y[ad(t)zd(t), t] = 0.$$

That is, for each $z, t \in I$ we have$(I[ad(t)zd(t), t])^3 = (0)$. Semiprimeness of R forces that $I[ad(t)zd(t), t] = (0)$. Hence, for each $t, z \in I$, we get $[ad(t)zd(t), t] = 0$. That is,

$$ad(t)zd(t)t - tad(t)zd(t) = 0. \tag{3.45}$$

Substitute $zad(t)w$ for z in (3.45), where $w \in I$, we get

$$ad(t)zad(t)wd(t)t - tad(t)zad(t)wd(t) = 0. \tag{3.46}$$

By using (3.45), equation (3.46) can be written as

$$\begin{aligned} 0 =& ad(t)ztad(t)wd(t) - ad(t)zad(t)twd(t) \\ =& ad(t)z[t, ad(t)]wd(t). \end{aligned} \tag{3.47}$$

Replacing z by tz and w by wt in (3.47) in order to get

$$ad(t)tz[t, ad(t)]wtd(t) = 0. \tag{3.48}$$

Multiply t on both sides of (3.47), we have

$$tad(t)z[t, ad(t)]wd(t)t = 0. \tag{3.49}$$

Subtracting (3.48) and (3.49) to obtain $[ad(t), t]z[ad(t), t]w[ad(t), t] = 0$. That means, $(I[ad(t), t])^3 = (0)$. Hence, by the same reasons we obtain $[ad(t), t] = 0$ for any $t \in I$, as desired.

Further, if R is a prime ring and d is derivation of R, then by Lemma 3, either $d = 0$ or R is commutative. If $d = 0$ then our hypothesis gives,

$$aF(x)(y - F(y)) \in Z(R). \tag{3.50}$$

Replacing y by yk in (3.50), where $k \in I$, we get

$$aF(x)(y - F(y))k \in Z(R). \tag{3.51}$$

On commuting both sides by $j \in I$, we find

$$aF(x)(y - F(y))kj - jaF(x)(y - F(y))k = 0. \tag{3.52}$$

By using (3.51) and (3.52), we get $j(aF(x)(y - F(y)))k \in Z(R)$. Put $r = aF(x)(y-F(y))$ and our assumption implies that $r \neq 0$ so, we have $IrI \subseteq Z(R)$. That means R contains a nonzero central ideal. Hence, by Lemma 1, R is commutative. □

Example 1. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$, be a ring over integers modulo 2 and let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$, be an ideal of R. We

define maps $F, d : R \to R$ by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & nb \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & (n-1)b \\ 0 & 0 \end{pmatrix}$, where n is any positive integer. Clearly,F is a multiplicative (generalized)-derivation associated with the map d and for any $0 \neq a \in R$ it is easy to see that the identities $a(F(xy) + F(x)F(y)) \in Z(R)$ and $a(F(xy) - F(x)F(y)) \in Z(R)$ hold for each $x, y \in I$. Here R is not semiprime ring because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$. But neither $[ad(z), z] \neq (0)$ for all $z \in I$ nor R is commutative. Hence, the condition of semiprimeness and primeness in Theorem 7 is not superfluous.

Recently, in [11] Camci and Aydin proved that: *If F is a multiplicative (generalized)-derivation of a semiprime (prime) ring R together with a map f, then f must be multiplicative derivation of R.* In the following theorem, we are taking f as a left multiplier instead of a multiplicative derivation.

Theorem 8. *Let R be a non-commutative prime ring and I be a nonzero ideal of R. Let $F : R \to R$ be a mapping (not necessarily additive) of R such that $F(xy) = F(x)y + xd(y)$, where d is a left-multiplier of R. If for some $0 \neq a \in R$, $a(F(xy) \pm F(y)F(x)) = 0$ for all $x, y \in I$, then either $aF(R) = (0)$ or $F : R \to Z(R)$.*

Proof. For each $x, y \in I$, we consider

$$a(F(xy) - F(y)F(x)) = 0. \tag{3.53}$$

Replacing x by xy in (3.53), we get $a(F(xy)y + xyd(y) - F(y)F(x)y - F(y)xd(y)) = 0$ for all $x, y \in I$. Our hypothesis forces that

$$axyd(y) = aF(y)xd(y). \tag{3.54}$$

Putting ax in place of x in (3.54), we find

$$a^2xyd(y) = aF(y)axd(y). \tag{3.55}$$

Left multiply (3.54) by a and subtract from (3.55) and we get $a[F(y), a]xd(y) = 0$. Primeness of R implies that either $d(I) = (0)$ or $a[F(I), a] = (0)$.

We assume that

$$a[F(y), a] = 0, \text{ for all } y \in I. \tag{3.56}$$

On substitution of yx for y in (3.56), where $x \in I$, we have $aF(y)[x, a] + a[yd(x), a] = 0$. Replacing x by zx, where $z \in I$, in this expression and using it, we obtain $aF(yz)[x, a] = 0$. For some $r \in R$, substitute xr in place of x to get $aF(yz)x[r, a] = 0$. Replace x by px, where $p \in R$ and we get $aF(yz)Rx[r, a] = (0)$. Therefore, either $aF(I^2) = 0$ or $I[r, a] = (0)$. If $I[r, a] = (0)$, then $a \in Z(R)$.

Since we know that center of a prime ring contains no zero divisor, so Eq. (3.54) gives that $(xy - F(y)x)d(y) = 0$ for each $x, y \in I$. For some $t \in I$, replacing x by tx, we find

$$(txy - F(y)tx)d(y) = 0 \tag{3.57}$$

On pre-multiplying the expression $(xy - F(y)x)d(y) = 0$ by t, we obtain

$$(txy - tF(y)x)d(y) = 0 \tag{3.58}$$

Now we combine (3.57) and (3.58) in order to get $[F(y), t]xd(y) = 0$. It implies that either $d(I) = (0)$ or $[F(I), I] = (0)$. Further, if for any $x, y \in I$, $[F(x), y] = 0$. Putting $x = xy$, we find $x[d(y), y] + [x, y]d(y) = 0$ for any $x, y \in I$. Again we put wx instead of x in the last relation, for all $w \in I$, we obtain $[w, y]xd(y) = 0$. For some $r \in R$, we replace x by rx and obtain $[w, y]Rxd(y) = (0)$ where $x, y, w \in I$. It implies that either $[I, I] = (0)$ or $d(I) = (0)$. Since R is assumed to be non-commutative, by Lemma 1, $[I, I] \neq (0)$, so we have $d(I) = (0)$. On the other side, if $aF(yz) = 0$ for each $y, z \in I$, then for some $t \in I$, substitution of zt for z yields that $ayzd(t) = 0$. Since $a \neq 0$ and I a nonzero ideal of the prime ring R, we have $d(I) = (0)$. Therefore, each of our case gives $d(I) = 0$.

Next, we see effect of this outcome $d(I) = (0)$ on the behavior of the mapping F. We consider, $d(I) = (0)$ our hypothesis implies

$$aF(x)y = aF(y)F(x). \tag{3.59}$$

Replacing y by yt in (3.59) where $t \in I$, we get

$$aF(x)yt = aF(y)tF(x). \tag{3.60}$$

Right multiply (3.59) by t and subtract form (3.60) in order to get $aF(y)[F(x), t] = 0$. Put $y = ry$ in the last expression, where $r \in R$, we find $aF(r)y[F(x), t] = 0$. For some $s \in R$ again we replace y be sy in order to get $aF(R)RI[F(I), I] = (0)$. Primeness of R implies that either $aF(R) = (0)$ or $I[F(I), I] = (0)$. Assume that $I[F(I), I] = (0)$. That means for each $x, t \in I$, we have

$$[F(x), t] = 0. \tag{3.61}$$

Putting $x = rx$ where $r \in R$, in the above relation to obtain $[F(r), t]x + F(r)[x, t] = 0$. In particular, we obtain $[F(r), t]t = 0$. Linearizing the last relation w.r.t.t and we get

$$[F(r), t]y + [F(r), y]t = 0. \tag{3.62}$$

Substitute ys for y in (3.62), where $s \in R$, we obtain

$$[F(r), t]ys + [F(r), y]st + y[F(r), s]t = 0. \tag{3.63}$$

Combining (3.62) and (3.63) and we have

$$[F(r), y][s, t] + y[F(r), s]t = 0. \tag{3.64}$$

Replace t by tz in (3.64), where $z \in I$, we get $[F(r), y][s, t]z + [F(r), y]t[s, z] + y[F(r), s]tz = 0$. Eq. (3.64) reduces it to $[F(r), y]t[s, z] = 0$, for all $y, t, z \in I$ and $r, s \in R$. In particular, putting $s = F(r)$ and $y = z$, we obtain $[F(r), z]I[F(r), z] = (0)$. Thus, by primeness of I for each $r \in R$ and $z \in I$, we have $[F(r), z] = 0$. Evidently, $[F(r), s] = 0$, where $r, s \in R$ i.e. $F(R) \subseteq Z(R)$. Hence, F sends R into $Z(R)$.

On replacing F by $-F$ and d by $-d$ in the proof given above, we can get the same conclusions for the situation $a(F(xy) + F(y)F(x)) = 0$. Hence, it proves the theorem. □

We conclude with the following example, which is showing that the Theorem 8 can't be extended to multiplicative (generalized)-derivations.

Example 2. Consider $R = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} : m, n, p, q \in \mathbb{Z}_2 \right\}$, be a ring over integers modulo 2. Since a matrix ring over an integral domain is a prime ring, so R is a non-commutative prime ring. Let $I = \left\{ \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} : m, n \in \mathbb{Z}_2 \right\}$, be an ideal of R. We define maps $F, d : R \to R$ by $F \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} m & 0 \\ p & 0 \end{pmatrix}$, $d \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} 0 & n \\ p & 0 \end{pmatrix}$. Note that F is a multiplicative (generalized)-derivation associated with the map d. For any $0 \neq a \in R$, it is easy to verify that the identities $a(F(xy) + F(y)F(x)) = 0$ and $a(F(xy) - F(y)F(x)) = 0$ are satisfied on I, but neither $aF(R) = (0)$ nor $F(R) \subseteq Z(R)$. Hence, the restrictions imposed in Theorem 8 are crucial.

References

[1] Ali, A., Rehman, N., Ali, S., *On Lie ideals with derivations as homomorphisms and anti-homomorphisms*, Acta Math. Hung., 101 (2003), 79-82.

[2] Ali, S., Dhara, B., Dar, N. A., Khan, A. N., *On Lie ideals with multiplicative (generalized)-derivations in prime and semiprime rings*, Beitr Algebra Geom., 56(1) (2014), 325-337.

[3] Ashraf, M., Ali, A., Ali, S., *Some commutativity theorems for rings with generalized derivations*, Southeast Asian Bull. Math., 31 (2007), 415-421.

[4] Ashraf, M., Ali, A., Rani, R., *On generalized derivations of prime rings*, Southeast Asian Bull. Math., 29(4) (2005), 669-675.

[5] Ashraf, M., Rehman, N., *On derivations and commutativity in prime rings*, East-West J. Math., 3(1) (2001), 87-91.

[6] Atteya, M. J., *On generalized derivations of semiprime rings*, Int. J. Algebra, 4(12) (2010), 591-598.

[7] Bell, H. E., Kappe, L. C., *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hung., 53 (1989), 339-346.

[8] Bell, H. E., Martindale III, W. S., *Centralizing mappings of semiprime rings*, Canad. Math. Bull., 30(1) (1987), 92-101.

[9] Brešar, M., *On certain pair of functions of semiprime rings*, Proc. Amer. Math. Soc., 120(3) (1994), 709-713.

[10] Brešar, M., *On the distance of composition of two derivations to the generalized derivation*, Glasgow Math. J., 33 (1991), 89-93.

[11] Camci, D. K., Aydin, N., *On multiplicative (generalized)-derivations in semiprime rings*, Commun. Fac. Sci. Univ. Ank. Sér. Al Math. Stat., 66(1) (2017), 153-164.

[12] Daif, M. N., *When is a multiplicative derivation additive?* Internat. J. Math. Math. Sci., 14(3) (1991), 615-618.

[13] Daif, M. N., Bell, H. E., *Remarks on derivations on semiprime rings*, Internat. J. Math. Math. Sci., 15(1) (1992), 205-206.

[14] Daif, M. N., Tammam-El-Sayiad, M. S., *Multiplicative generalized derivations which are additive*, East-West J. Math., 9(1) (1997), 33-37.

[15] Dhara, B., *Generalized derivations acting as a homomorphism or anti-homomorphism in semiprime rings*, Beitr. Algebra Geom., 53 (2012), 203-209.

[16] Dhara, B., Ali, S., *On multiplicative (generalized)-derivations in prime and semiprime rings*, Aequations Math. 86(1-2) (2013), 65-79.

[17] Dhara, B., Pradhan, K. G., *A note on multiplicative (generalized)-derivations with annihilator conditions*, Georgian Math. J., 23(2) (2016), 191-198.

[18] Goldmann, H., Sěmrl, P., *Multiplicative derivation on C(X)*, Monatsh. Math., 121(3) (1996), 189-197.

[19] Gusic, I., *A note on generalized derivations of prime rings*, Glasnik Mate., 40(60) (2005), 47-49.

[20] Herstein, I. N., *Rings with involution*, The University of Chicago Press, Chicago, 1976.

[21] Kezlan, T. P., *A note on commutativity of semiprime PI-rings*, Math. Japonica., 27(2) (1982), 267-268.

[22] Martindale III, W. S., *When are multiplicative maps additive?*, Proc. Amer. Math. Soc., 21 (1969), 695-698.

[23] Posner, E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc., 8(1957), 1093-1100.

[24] Rehman, N., *On generalized derivations as homomorphisms and anti-homomorphisms*, Glasnik Mate., 39(59) (2004), 27-30.

[25] Tiwari, S. K., Sharma, R. K., Dhara, B., *Multiplicative (generalized)-derivation in semiprime rings*, Beitr. Algebra Geom., DOI 10.1007/s13366-015-0279.

[26] Rehman, N., Raza, M. A., *Generalized derivations as homomorphism or anti-homomorphims on Lie ideals*, Arab J. Math. Sci., 22(1) (2016), 22-28.

[27] Dhara, B., Rehman, N., Raza, M. A., *Lie ideals and action of generalized derivations in rings*, Miskolc Math. Notes., 16(2) (2015), 769-779.

Accepted: 2.11.2017

FIXED POINT RESULTS OF F-RATIONAL CYCLIC CONTRACTIVE MAPPINGS ON 0-COMPLETE PARTIAL METRIC SPACES

Zead Mustafa
Department of Mathematics
Statistics and Physics
Qatar University
Doha-Qatar
and
Department of Mathematics
The Hashemite University
P.O. 330127, Zarqa 13115
Jordan
zead@qu.edu.qa
zmagablh@hu.edu.jo

Sami Ullah Khan
Department of Mathematics
International Islamic University
H-10, Islamabad - 44000
Pakistan
gomal85@gmail.com

M.M.M. Jaradat
Department of Mathematics
Statistics and Physics
Qatar University
Doha-Qatar
mmjst4@qu.edu.qa

Muhammad Arshad
Department of Mathematics
International Islamic University
H-10, Islamabad - 44000, Pakistan
marshad_zia@yahoo.com

H.M. Jaradat
Department of Mathematics
Al al-Bayt University
Jordan
husseinjaradat@yahoo.com

Abstract. Wardowski [19] introduced a new concept of contraction which called F-contraction and proved a fixed point theorem on complete metric space. Following this direction of research, in this paper, we introduce an $F-$rational cyclic contraction on

partial metric spaces and we present new fixed point results for such cyclic contraction in 0-complete partial metric spaces. An example is given to illustrate the main result, also an application to integral equation is given to show the usability of our results.
Keywords: fixed point, F-contractions, (0−complete) partial metric space.

1. Introduction

In 1994, S. G. Matthews [14] introduced the notion of partial metric spaces and obtained various fixed point theorems. In fact, he showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

Later on, Romaguera [16] introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

In 2012, Wardowski [19] introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction. Furthermore, Abbas et al.,[1] generalized the concept of $F-$contraction and proved certain fixed and common fixed point results. Afterwards Secelean [17] proved fixed point theorems consisting of $F-$contractions by iterated function systems. Piri et al.,[15] proved a fixed point result for F-Suzuki contractions for some weaker conditions on the self map of a complete metric space which generalizes the result of Wardowski. Lately, Acar et al.,[5] introduced the concept of generalized multivalued $F-$contraction mappings. Further Altun et al.,[4] extended multivalued mappings with $\delta-$Distance and established fixed point results in complete metric space. Sgroi et al.,[18] established fixed point theorems for multivalued $F-$contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler's. Recently Ahmad et al.,[6],[9] recalled the concept of $F-$contraction to obtain some fixed point, and common fixed point results in the context of complete metric spaces.

In this paper, we introduce an $F-$rational cyclic contraction on partial metric spaces and we present new fixed point results for such cyclic contraction in 0-complete partial metric spaces. An example is given to illustrate the main result, also an application to integral equation is given to show the usability of our results.

2. Preliminaries

First we recall some definitions and properties of partial metric spaces.

Definition 2.1. [14] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$;

(P2) $p(x,x) \leq p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric spaces is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x,y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

Example 2.2. [14] If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x,y) = \max\{x,y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

For more examples of partial metric spaces, we refer the reader to [8] and the references therein.

Each partial metric p on X generates a T_0 topology $\tau(p)$ on X which has a base topology of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ and $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon + p(x,x)\}$.

A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to $\tau(p)$ to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to $\tau(p)$ to $fx \in X$.

Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

(ii) A sequence $\{x_n\}$ in partial metric space (X,p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$. The space $(X,\ p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iii) A sequence $\{x_n\}$ in partial metric space (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in $\tau(p)$) to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 2.3. *Let (X,p) be a partial metric space.*

(a) *[2],[12] If $p(x_n,z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n,y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*

(b) *[16] If (X,p) is complete, then it is $0-$complete.*

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete. The following example shows that the converse assertion of (b) need not hold.

Example 2.4 ([16])**.** The space $X = [0,+\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

Definition 2.5 ([11]). Let (X, d) be a metric space and $f : X \to X$ be a mapping. Then it is said that f satisfies the orbital condition if there exists a constant $k \in (0, 1)$ such that

$$d\left(fx, f^2x\right) \leq k\ d\left(x, fx\right), \tag{2.1}$$

for all $x \in X$.

Theorem 2.6 ([3]). *Let (X, p) be a $0-$complete partial metric space and $f : X \to X$ be continuous such that*

$$p\left(fx, f^2x\right) \leq k\ p\left(x, fx\right) \tag{2.2}$$

holds for all $x \in X$, where $k \in (0, 1)$. Then there exists $z \in X$ such that $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$.

Definition 2.7 ([11]). Let (X, p) be a partial metric space and $f : X \to X$ be a mapping with fixed point set $Fix(f) \neq \phi$. Then f has property (P) if $Fix(f^n) = Fix(f)$, for each $n \in N$.

Lemma 2.8 ([11]). *Let (X, p) be a partial metric space, $f : X \to X$ be a self map such that Fix $(f) \neq \phi$. Then f has the property (P) if (2.2) holds for some $k \in (0, 1)$ and either (i) for all $x \in X$, or (ii) for all $x \neq fx$.*

One of the remarkable generalizations of Banach's contraction principle was reported by Kirk et al.,[13] via cyclic contraction.

Theorem 2.9 ([13]). *Let $\{A_i\}_{i=1}^m$ be a nonempty closed subset of a complete metric space (X, d) and suppose $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a mapping satisfying the following conditions:*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \leq i \leq m$, *where* $A_{m+1} = A_1$.

(2) $d\left(fx,\ fy\right) \leq \psi\left(d\left(x, y\right)\right)$, *for all* $x \in A_i,\ y \in A_{i+1},\ i \in \{1, 2, \cdots, m\}$,

where $A_{m+1} = A_1$ and $\psi : [0, 1) \to [0, 1)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then, f has a fixed point $z \in \bigcap_{i=1}^m A_i$.

Wardowski [19] defined the $F-$contraction as follows.

Definition 2.10 ([19]). Let (X, d) be a complete metric space. A self mapping $f : X \to X$ is said to be an $F-$contraction if there exists a constant $\lambda > 0$ such that

$$\forall\ x, y \in X,\ d(fx, fy) > 0 \Rightarrow \lambda + F\left(d(fx, fy)\right) \leq F\left(d(x, y)\right). \tag{1.3}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if, $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F1) - (F3)$.

Example 2.11 ([19])**.** The Family of $\mathcal{F}$ is not empty.

1. $F(x) = \ln(x); x > 0$;
2. $F(x) = x + \ln(x); x > 0$;
3. $F(x) = \ln(x^2 + x); x > 0$;
4. $F(x) = \frac{-1}{\sqrt{x}}; x > 0$.

Theorem 2.12 ([19])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a $F-$contraction. If f or F is continuous, then we have*

(1) f has a unique fixed point $x^* \in X$.
(2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

3. Main results and discussion

Let (X, p) be a partial metric space, through out of this paper we mean by Δ_p be the set of all nonempty closed subsets of X.

Definition 3.1. Let (X, p) be a partial metric space, $V_i \in \Delta_p$ for $i = 1, 2, \cdots, m$, $E = \bigcup_{i=1}^{m} V_i$ where $m \in N$. A mapping $f : E \to E$ is called an F-rational cyclic contraction if there exists $F \in \mathcal{F}$ and $\lambda \in \mathbb{R}_+$ such that

1. $f(V_i) \subseteq V_{i+1}, i = 1, 2, ..., m,$ where $V_{m+1} = V_1$,
2. For $x \in V_i$, $y \in V_{i+1}$, $i = 1, 2, ..., m$, with $p(fx, fy) > 0$, we have

$$\lambda + F(p(fx, fy)) \leq F(\mathcal{H}_f(x, y)), \tag{3.1}$$

where

$$\begin{aligned} \mathcal{H}_f(x, y) &= ap(x, y) + bp(x, fx) + cp(y, fy) + dp(x, fy) + ep(y, fx) \\ &\quad + l\frac{p(x, fx).p(y, fy)}{1 + p(x, y)}, \end{aligned} \tag{3.2}$$

and

$$a, b, c, d, e, l \geq 0 \text{ with } a + b + c + d + e + l < 1. \tag{3.3}$$

The main result of this section is the following.

Theorem 3.2. *Let (X,p) be a 0-complete partial metric space, $V_i \in \Delta_p$; $i = 1,2,\cdots,m$ where $m \in \mathbb{N}$ and $E = \bigcup_{i=1}^{m} V_i$. Suppose that $f : E \to E$ is an F-rational cyclic contraction. Then,*

1. *f has a unique fixed point $z \in E$.*
2. *$p(z,z) = 0$ and $z \in \bigcap_{i=1}^{m} V_i$.*
3. *for any $x_0 \in E$, the sequence $x_n = f^n x_0$, converges to z in topology $\tau(p)$.*

Proof. Let $x_0 \in E$ be an arbitrary point. Then there exists i_0 such that $x_0 \in V_{i_0}$, so there is $x_1 \in V_{i_0+1}$ where $x_1 = fx_0$. Continue in this process we can construct a sequence $x_n = fx_{n-1} = f^n x_0 \in V_{i_0+n}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. From now on assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$ and let $p_n = p(x_n, x_{n+1})$, so $p_n > 0$ for all $n \in \mathbb{N}$. Since $f : E \to E$ is an F-rational cyclic contraction. So, from (3.1) and (3.2) we have that

$$\begin{aligned}
&\lambda + F(p_n) = \lambda + F(p(x_n, x_{n+1})) \\
&= \lambda + F(p(fx_{n-1}, fx_n)) \\
&\leq F\left(\begin{array}{c} ap(x_{n-1},x_n) + bp(x_{n-1},x_n) + cp(x_n,x_{n+1}) + dp(x_{n-1},x_{n+1}) \\ +ep(x_n,x_n) + l\frac{p(x_{n-1},x_n).p(x_n,x_{n+1})}{1+p(x_{n-1},x_n)} \end{array} \right).
\end{aligned}$$

Since $p(x_{n-1},x_{n+1}) \leq p(x_{n-1},x_n) + p(x_n,x_{n+1}) - p(x_n,x_n)$, F is strictly increasing and $\frac{p(x_{n-1},x_n).p(x_n,x_{n+1})}{1+p(x_{n-1},x_n)} < p(x_n,x_{n+1})$, then the above inequality becomes

$$\lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n,x_n)). \tag{3.4}$$

Since $\lambda > 0$, then

$$F(p_n) \leq \lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n,x_n)).$$

But, F is strictly increasing, so we deduce that

$$p_n \leq (a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n,x_n)). \tag{3.5}$$

By symmetry of $p(x_{n+1},x_n) = p(x_n,x_{n+1})$, and using similar argument as above one can deduce that

$$\begin{aligned}
\lambda + F(p(x_{n+1},x_n)) &= \lambda + F(p(fx_n, fx_{n-1})) \\
&\leq F((a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n,x_n)).
\end{aligned}$$

Thus,

$$F(p_n) \leq \lambda + F(p_n) \leq F((a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n,x_n))$$

which implies that

$$p_n \leq (a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n,x_n). \tag{3.6}$$

Adding up, equations (3.5) and (3.6) we get $p_n \leq \beta p_{n-1}$, where $\beta = \frac{2a+b+c+d+e}{2-b-c-d-e-2l} < 1$, which is a consequence of (3.3). Hence,

$$p_n < p_{n-1}, \text{ for all } n \in \mathbb{N} \tag{3.7}$$

Using property(P2) of partial metric, equations (3.4), (3.7) and the property of strictly increasing of F we get

$$\begin{aligned} \lambda + F(p_n) &\leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)) \\ &\leq F((a+b+d)p_{n-1} + (c+d+l)p_{n-1} + (e-d)p_{n-1}) \\ &= F((a+b+c+d+e+l)p_{n-1}) \\ &\leq F(p_{n-1}). \end{aligned}$$

Hence, $\lambda + F(p_n) \leq F(p_{n-1})$ for all $n \in \mathbb{N}$. This implies

$$F(p_n) \leq F(p_{n-1}) - \lambda \leq \cdots \leq F(p_0) - n\lambda, \text{ for all } n \in \mathbb{N} \tag{3.8}$$

and so $\lim_{n\to+\infty} F(p_n) = -\infty$. By the property $(F2)$, we get that $p_n \to 0$ as $n \to +\infty$.

Now, by (F3) there exist $k \in (0,1)$ such that $\lim_{n\to+\infty} p_n^k F(p_n) = 0$. By (3.8), the following holds for all $n \in \mathbb{N}$:

$$p_n^k F(p_n) - p_n^k F(p_0) \leq -n\lambda p_n^k \leq 0. \tag{3.9}$$

Letting $n \to +\infty$ in (3.9) we deduce that

$$\lim_{n\to+\infty} n p_n^k = 0. \tag{3.10}$$

By using the continuous function $g(x) = x^{\frac{1}{k}}$; $x \in (0, \infty)$, we get that

$$\lim_{n\to+\infty} n^{\frac{1}{k}} p_n = \lim_{n\to+\infty} g(n p_n^k) = 0. \tag{3.11}$$

Now, by using the limit comparison test with $a_n = p_n$, $b_n = n^{\frac{-1}{k}}$ and equation (3.10) we ensure that the series $\sum_{n=1}^{+\infty} p_n$ is convergent. This implies that $\{x_n\}$ is a $0-$Cauchy sequence. Since E is closed in a 0-complete partial metric (X, p), then E is also 0-complete and there exist $z \in E = \bigcup_{i=1}^{m} V_i$ such that

$$\lim_{n\to\infty} p(x_n, z) = 0 = p(z, z). \tag{3.12}$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in V_i for each $i = 1, ..., m$. Hence, there is a subsequence of $\{x_n\}$ in each V_i, $i = 1, ..., m$, which converges to z. Using that each V_i, $i = 1, ..., m$, is closed, we conclude that $z \in \bigcap_{i=1}^{m} V_i$.

We shall prove that z is a fixed point of f. Using the triangle inequality (p4) of partial metric space and (3.2) (which is possible since z belongs to each V_i) to obtain

$$\begin{aligned} p(z,fz) &\leq p(z,x_{n+1}) + p(x_{n+1},fz) - p(x_{n+1},x_{n+1}) \\ &\leq p(z,x_{n+1}) + p(fx_n,fz) \\ &\leq p(z,x_{n+1}) + ap(x_n,z) + bp(x_n,x_{n+1}) + cp(z,fz) + dp(x_n,fz) \\ &\quad + ep(x_{n+1},z) + l\ \frac{p(x_n,x_{n+1}).p(z,fz)}{1+p(x_n,z)}. \end{aligned} \tag{3.13}$$

Using Lemma 2.3 part (a) and passing to the limit when $n \to \infty$ in (3.13), we obtain that

$$(1-c-d)\,p(z,fz) \leq 0,$$

and hence

$$p(z,fz) = 0. \tag{3.14}$$

Now by using triangle inequality (P4), (3.14) and (3.12) we deduce that $p(fz,fz) = 0$. Therefore, by (P1) we get $f(z) = z$.

Finally, we will prove the uniqueness, let u be another fixed point of f in E, with $p(u,z) \neq 0$. By the cyclic character of f, we have $u, z \in \bigcap_{i=1}^{m} V_i$. Since f is an F-rational cyclic contraction and using the property (P2) of partial metric, we have

$$\begin{aligned} \lambda + F(p(u,z)) &= \lambda + F(p(fu,fz)) \\ &\leq F\left(\begin{array}{c} ap(u,z) + bp(u,u) + cp(z,z) + dp(u,z) + ep(u,z) \\ +l\frac{p(u,fu).p(z,fz)}{1+p(u,z)} \end{array}\right) \\ &\leq F((a+b+c+d+e)\,p(u,z)), \end{aligned}$$

which is a contradiction deduced from the strictly increasing property of F and being $a+b+c+d+e<1$, hence $z=u$. Thus z is a unique fixed point of f. □

By taking $F(\alpha) = \alpha + \ln(\alpha)$ in Theorem 3.2 we get the following corollary.

Corollary 3.3. *Let (X,p) be a 0-complete partial metric space, $V_i \in \Delta_p$; $i = 1,2,\cdots,m$ where $m \in \mathbb{N}$ and $E = \bigcup_{i=1}^{m} V_i$. Suppose that $f : E \to E$ and the following conditions are hold:*

1. *$f(V_i) \subseteq V_{i+1}, i = 1,2,...,m$, where $V_{m+1} = V_1$,*
2. *There exist $\lambda > 0$ such that for $x \in V_i$, $y \in V_{i+1}$, $i = 1,2,...,m$, with $p(fx,fy) > 0$, we have*

$$\begin{aligned}\lambda + \ln(p(fx,fy)) &\leq \left(\begin{array}{c} ap\,(x,y) + bp\,(x,fx) + cp\,(y,fy) + dp\,(x,fy) \\ +ep\,(y,fx) + l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array} \right) \\ &+ \ln \left(\begin{array}{c} ap\,(x,y) + bp\,(x,fx) + cp\,(y,fy) + dp\,(x,fy) \\ +ep\,(y,fx) + l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array} \right),\end{aligned}$$

where $a,b,c,d,e,l \geq 0$ *and* $a+b+c+d+e+l<1$. *Then,*

1. f *has a unique fixed point* $z \in E$.
2. $p\,(z,z)=0$ *and* $z \in \bigcap_{i=1}^{m} V_i$.
3. *for any* $x_0 \in E$, *the sequence* $x_n = f^n x_0$, *converges to* z *in topology* $\tau(p)$.

By taking $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$ in Theorem 3.2 we get the following corollary.

Corollary 3.4. *Let* (X,p) *be a* 0*-complete partial metric space,* $V_i \in \Delta_p$; $i = 1,2,\cdots,m$ *where* $m \in \mathbb{N}$ *and* $E = \bigcup_{i=1}^{m} V_i$. *Suppose that* $f : E \to E$ *and the following conditions are hold:*

1. $f(V_i) \subseteq V_{i+1}, i = 1,2,...,m,$ *where* $V_{m+1} = V_1$,
2. *There exist* $\lambda > 0$ *such that for* $x \in V_i$, $y \in V_{i+1}$, $i = 1,2,...,m$, *with* $p\,(fx,fy) > 0$, *we have*

$$\lambda + \frac{-1}{\sqrt{p(fx,fy)}} \leq \frac{-1}{\sqrt{\left(\begin{array}{c} ap\,(x,y) + bp\,(x,fx) + cp\,(y,fy) + dp\,(x,fy) + ep\,(y,fx) \\ +l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array} \right)}}$$

where $a,b,c,d,e,l \geq 0$ *and* $a+b+c+d+e+l<1$. *Then,*

1. f *has a unique fixed point* $z \in E$.
2. $p\,(z,z)=0$ *and* $z \in \bigcap_{i=1}^{m} V_i$.
3. *for any* $x_0 \in E$, *the sequence* $x_n = f^n x_0$, *converges to* z *in topology* $\tau(p)$.

Example 3.5. Let $X = \mathbb{R}$ be equipped with the usual partial metric $p\,(x,y) = \max\{|x|,|y|\}$. Then, clearly (X,p) is 0−complete. Suppose $V_1 = \left[0, \frac{1}{2}\right]$, $V_2 = \left[\frac{-1}{6}, 0\right]$, $V_3 = \left[0, \frac{1}{18}\right]$, $V_4 = \left[\frac{-1}{54}, 0\right]$ and $E = \bigcup_{i=1}^{4} V_i$. Define $f : E \to E$ such that $fx = \frac{-x}{8}$ for all $x \in E$. It is clear that $f(V_i) \subseteq V_{i+1}$.

Take $\lambda = ln(4)$, $a = \frac{1}{2}$ and $b = c = d = e = l = \frac{1}{11}$. Let $x \in V_i$ and $y \in V_{i+1}$ such that either $x \neq 0$ or $y \neq 0$, then

$$\begin{aligned} p(fx, fy) &= \max\{|\frac{-x}{8}|, |\frac{-y}{8}|\} \\ &= \frac{1}{8}\max\{|x|, |y|\} \\ &= \frac{1}{8}p(x, y) \\ &= (\frac{1}{4})(\frac{1}{2})p(x, y). \end{aligned} \tag{3.15}$$

Now take ln for both sides of (3.15) we get

$$\begin{aligned} \ln(p(fx, fy)) &= \ln((\frac{1}{4})(\frac{1}{2})p(x, y)) \\ &= -ln(4) + \ln(\frac{1}{2}p(x, y)) \\ &\leq -ln(4) + \ln\Big(\frac{1}{2}p(x, y) + \frac{1}{11}p(x, fx) + \frac{1}{11}p(y, fy) + \frac{1}{11}p(x, fy) \\ &\quad + \frac{1}{11}p(y, fx) + \frac{1}{11}\frac{p(x, fx)p(y, fy)}{1 + p(x, y)}\Big). \end{aligned}$$

Hence,

$$ln(4) + F(p(x, y)) \leq F(\mathcal{H}_f(x, y)).$$

Therefore, all conditions of Theorem 3.2 are satisfied and we deduce that f has a unique fixed point $z = 0 \in \bigcap_{i=1}^{4} V_i$ and $p(z, z) = 0$ holds true.

Another consequence of Theorem 3.2 is the following.

Theorem 3.6. *Under the assumptions of Theorem 3.2. The function f satisfies the orbital condition (2.2). In particular, there exist $z \in E$ such that $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$; also, f has the property (P).*

Proof. By Theorem 3.2, the set of fixed points for f is not empty. We will prove that f satisfies condition (2.2) of Theorem 2.6. Let $x \in Y$ be arbitrary. Putting $x = x$ and $y = fx$ in condition (3.1) of Theorem 3.2, we have

$$\begin{aligned} \lambda + F\left(p\left(fx, f^2x\right)\right) &\leq F\left(\mathcal{H}_f\left(x, fx\right)\right) \\ &\leq F\left(\begin{array}{c} ap\left(x, fx\right) + bp\left(x, fx\right) + cp\left(fx, f^2x\right) \\ +dp\left(x, f^2x\right) + ep\left(fx, fx\right) + l\frac{p(x,fx).p\left(fx,f^2x\right)}{1+p(x,fx)} \end{array}\right), \end{aligned}$$

By (P4) and repeating the same process as in proof Theorem 3.2, we get that

$$\begin{aligned} &\lambda + F\left(p\left(fx, f^2x\right)\right) \\ &\leq F\left(\begin{array}{c} (a+b+d)\,p\left(x, fx\right) + (c+d+l)\,p\left(fx, f^2x\right) \\ +(e-d)\,p\left(fx, fx\right) \end{array}\right) \end{aligned} \tag{3.16}$$

by symmetry we have,

$$\begin{aligned} &\lambda + F\left(p\left(fx, f^2x\right)\right) \\ &\leq F\left(\begin{array}{c}(a+c+e)\,p\left(x, fx\right) + (b+e+l)\,p\left(fx, f^2x\right) \\ +(d-e)\,p\left(fx, fx\right)\end{array}\right). \end{aligned} \tag{3.17}$$

Using the same argument as in the proof of Theorem 3.2, we deduce that

$$p\left(fx, f^2x\right) \leq \beta p\left(x, fx\right),$$

where $\beta = \frac{2a+b+c+d+e}{2-(b+c+d+e+2l)} < 1$, which is a consequence of (3.3). Thus, f satisfies the orbital condition. By Theorem 2.6, there exists $z \in E$ such that $p\left(z, z\right) = 0$ and $p\left(fz, z\right) = p\left(fz, fz\right)$. So, by Lemma 2.8, f has the property (P). □

4. Application to integral equations

In this section, we will give an application to some integral equation to show the usability of the main result. Consider the integral equation

$$u(t) = h(u(t)) + \int_0^t H(t,r)\zeta(r, u(r))\ dr, \quad \text{for all } t \in [0,1], \tag{4.1}$$

where, $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}$, $H : [0,1] \times [0,1] \to \mathbb{R}$ and $h : \mathbb{R} \to [0, \infty)$ are functions.

Let $X = C([0,1])$ be the set of all real continuous functions on $[0,1]$, endowed with the partial metric

$$p(u,v) = \max\{\sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |v(t)|\}, \text{ for all } u, v \in X.$$

Clearly, (X, p) is a 0-complete partial metric space.

Let $\kappa, \eta \in X$, $\kappa_0, \eta_0 \in \mathbb{R}$ such that for all $t \in [0,1]$ we have

$$\kappa_0 \leq \kappa(t) \leq \eta(t) \leq \eta_0, \tag{4.2}$$

$$\kappa(t) \leq h(u(t)) + \int_0^t H(t,r)\zeta(r, \eta(r))\ dr, \tag{4.3}$$

and

$$\eta(t) \geq h(u(t)) + \int_0^t H(t,r)\zeta(r, \kappa(r))\ dr. \tag{4.4}$$

Let for all $r \in [0,1]$, $\zeta(r, \cdot)$ and $h(.)$ are decreasing functions, that is,

$$x, y \in \mathbb{R}, x \geq y \text{ implies } \zeta(r,x) \leq \zeta(r,y). \tag{4.5}$$

and

$$h(x) \leq h(y). \tag{4.6}$$

Assume that,

$$\max_{t\in[0,1]} \int_0^1 H(t,s)ds < e^{-\lambda}, \text{for some } \lambda \in (0,\infty). \tag{4.7}$$

and

$$\sup_{r\in[0,1]} |\zeta(r,u(r))| \leq \sup_{r\in[0,1]} |u(r)|. \tag{4.8}$$

Define a mapping $f : X \to X$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))dr; \quad t \in [0,1]. \tag{4.9}$$

Also, suppose that for all $x, y \in \mathbb{R}$ with ($x \leq \eta_0$ and $y \geq \kappa_0$) or ($x \geq \kappa_0$ and $y \leq \eta_0$) we have,

$$|h(u(t))| \leq \tfrac{1}{8}e^{-\lambda} \max\{\sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |f(u(t))|\}. \tag{4.10}$$

Theorem 4.1. *Under the assumptions* (4.2)-(4.10)*, the integral equation* (4.1) *has a solution z such that $z \in C([0,1])$ with $\kappa(t) \leq z(t) \leq \eta(t)$ for all $t \in [0,1]$.*

Proof. Define the closed subsets of X, U_1 and U_2 by

$$U_1 = \{u \in X : u \leq \eta\}$$

and

$$U_2 = \{u \in X : u \geq \kappa\}.$$

Also define the mapping $f : U_1 \cup U_2 \to U_1 \cup U_2$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr, \text{ for all } t \in [0,1].$$

Now we prove that,

$$f(U_1) \subseteq U_2 \text{ and } f(U_2) \subseteq U_1. \tag{4.11}$$

Suppose, $u \in U_1$, that is,

$$u(r) \leq \eta(r), \text{ for all } r \in [0,1].$$

Using condition (4.5) and (4.6) we obtain that

$$\zeta(r,u(r)) \geq \zeta(r,\eta(r)), \text{ for all } r \in [0,1].$$

and

$$h(u(r)) \geq h(\eta(r)), \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.3) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr \geq h(\eta(t)) + \int_0^t H(t,r)\zeta(r,\eta(r))\ dr = \eta(t) \geq \kappa(t),$$

for all $t \in [0,1]$. Then we have $f(u(t)) \in U_2$. Similarly, let $u \in U_2$, that is,

$$u(r) \geq \kappa(r), \text{ for all } r \in [0,1].$$

Using condition (4.5) and (4.6) we obtain that

$$\zeta(r,u(r)) \leq \zeta(r,\kappa(r)), \text{ for all } r \in [0,1].$$

and

$$h(u(r)) \leq h(\kappa(r)), \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.4) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr \leq h(\kappa(t)) + \int_0^t H(t,r)\zeta(r,\kappa(r))\ dr = \kappa(t) \leq \eta(t),$$

for all $t \in [0,1]$. Then we have $f(u(t)) \in U_1$. Also, we deduce that (4.11) holds.

Let, $x \in U_1$ and $y \in U_2$. Then from (4.9), for all $t \in [0,1]$, we have

$$\begin{aligned}
|f(x(t))| &= |h(x(t)) + \int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\leq |h(x(t))| + |\int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\leq |h(x(t))| + \int_0^t |H(t,r)||\zeta(r,x(r))|\ dr \\
&\leq |h(x(t))| + \int_0^t |H(t,r)| \max\{\sup_{r\in[0,1]} |\zeta(r,x(r))|, \sup_{r\in[0,1]} |\zeta(r,y(r))|)\ dr \\
&\leq |h(x(t))| + \max_{t\in[0,1]} \int_0^t H(t,r)p(x,y)dr \\
&\leq |h(x(t))| + \frac{1}{8}e^{-\lambda}p(x,y) \\
&\leq \frac{1}{8}e^{-\lambda}p(x,fx) + \frac{1}{8}e^{-\lambda}p(x,y) \\
&= e^{-\lambda}(\frac{1}{8}p(x,fx) + \frac{1}{8}p(x,y)).
\end{aligned}$$

Thus,

$$\sup_{t\in[0,1]} |f(x(t))| \leq e^{-\lambda}(\frac{1}{8}p(x,fx) + \frac{1}{8}p(x,y)). \tag{4.12}$$

Similarly, we have

$$\sup_{t\in[0,1]} |f(y(t))| \leq e^{-\lambda}(\frac{1}{8}p(y,fy) + \frac{1}{8}p(x,y)). \tag{4.13}$$

Hence, from (4.12) and (4.13) we have

$$\begin{aligned}\max\{\sup_{t\in[0,1]} |f(x(t))|, \sup_{t\in[0,1]} |f(y(t))|\} &\leq e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy))\\ &\leq e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy)\\ &+ \frac{1}{8}p(x,fy) + \frac{1}{8}p(y,fx)).\end{aligned}$$

Therefore,

$$p(fx,fy) \leq e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy) + \frac{1}{8}p(x,fy) + \frac{1}{8}p(y,fx))$$

and so,

$$\ln(p(fx,fy)) \leq -\lambda + ln\Big(\frac{1}{8}p(x,y)+\frac{1}{8}p(x,fx)+\frac{1}{8}p(y,fy)+\frac{1}{8}p(x,fy)+\frac{1}{8}p(y,fx)\Big),$$

which implies that $\lambda + F(p(fx,fy)) \leq F(\mathcal{H}_f(x,y))$ is satisfied for $F(\alpha) = \ln(\alpha)$ for all $\alpha \in X$ with $a = b = c = d = e = \frac{1}{8}$ and $l = 0$. Hence, all conditions of Theorem 3.2 holds and f has a fixed point z such that $z \in C([0,1])$ with $\kappa \leq z(t) \leq \eta$ for all $t \in [0,1]\}$. That is, $z \in U_1 \cap U_2$ is a solution to (4.1). □

Conflict of interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conclusion

We conclude that every F-rational cyclic contraction mapping $f : \bigcup_{i=1}^{m} V_i \to \bigcup_{i=1}^{m} V_i$ defined on a 0-complete partial metric space (X,p) has a unique fixed point $z \in \bigcap_{i=1}^{m} V_i$ and for any $x_0 \in \bigcup_{i=1}^{m} V_i$, the sequence $x_n = f^n x_0$ converges to z in topology $\tau(p)$, where V_i is nonempty closed subset of X for each $i = 1, \cdots, m$.

References

[1] M. Abbas, B. Ali and S. Romaguera, *Fixed and periodic points of generalized contractions in metric spaces*, Fixed Point Theory Appl., 2013, 2013:243.

[2] T. Abdeljawad, E. Karapinar, K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces,* Appl. Math. Lett., 24 (2011), 1900–1904.

[3] T. Abdeljawad, J. O. Alzabut, A. Mukheimer, Y. Zaidan, *Banach contraction principle for cyclical mappings on partial metric spaces*, Fixed Point Theory Appl., 2012:154 (2012).

[4] O. Acar and I. Altun, *A fixed point theorem for multivalued mappings with δ-distance*, Abstr. Appl. Anal., Volume 2014, Article ID 497092, 5 pages.

[5] O. Acar, G. Durmaz and G. Minak, *Generalized multivalued $F-$contractions on complete metric spaces*, Bulletin of the Iranian Mathematical Society, 40(2014), 1469-1478.

[6] J. Ahmad, A. Al-Rawashdeh and A. Azam, *Some new fixed point theorems for generalized contractions in complete metric spaces*, Fixed Point Theory and Applications, 2015, 2015:80.

[7] M. Arshad, S. U. Khan, J. Ahmad, *Fixed point results for F-contractions involving some new rational expressions*, JP Journal of Fixed Point Theory and Appl. Volume 11, Number 1, 2016, Pages 79-97.

[8] H. Aydi, M. Abbas, C. Vetro, *Partial Hausdorff metric and Nadlers fixed point theorem on partial metric spaces*, Topology Appl., (2012) 159, 3234-3242.

[9] N. Hussain, J. Ahmad and A. Azam, *On Suzuki-Wardowski type fixed point theorems*, J. Nonlinear Sci. Appl., 8 (2015), 1095-1111.

[10] A. Hussain, M. Arshad, S. U. Khan, *$\tau-$Generalization of fixed point results for $F-$contraction*, Bangmod Int. J. Math. & Comp. Sci., Vol. 1, No. 1, 2015, Pages 127-137.

[11] G. S. Jeong, B. E. Rhoades, *Maps for which $F(T)$ =$F(T^n)$,* Fixed Point Theory Appl., 6(2005), 87–131.

[12] E. Karapinar, I. M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett., 24(2011), 1894–1899.

[13] W. A. Kirk, P. S. Srinavasan, P. Veeramani, *Fixed points for mappings satisfying cylical contractive conditions*, Fixed Point Theory, 4 (2003), 79–89.

[14] S.G. Matthews, *Partial metric topology Proc. 8th Summer Conference on General Topology and Applications*, Ann. New York Acad. Sci., vol. 728, 1994, pp. 183–197.

[15] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl., 2014, 2014:210.

[16] S. Romaguera, *A. Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl., (2010), Article ID 493298, 6 pages.

[17] N. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory Appl. 2013, Article ID, 277 (2013). doi:10.1186/1687-1812-2013-277.

[18] M. Sgroi and C. Vetro, *Multi-valued $F-$contractions and the solution of certain functional and integral equations*, Filomat, 27:7 (2013), 1259-1268.

[19] D. Wardowski, *Fixed point of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012, Article ID 94 (2012).

Accepted: 3.11.2017

CLASS OF ADMISSIBLE PERTURBATIONS OF SPECIAL EXPRESSIONS INVOLVING COMPLETELY MONOTONIC FUNCTIONS

Jerico B. Bacani

Julius Fergy T. Rabago
Department of Mathematics and Computer Science
College of Science
University of the Philippines Baguio
Gov. Pack Rd., Baguio City 2600
Philippines
jicderivative@yahoo.com
jfrabago@gmail.com

Abstract. In this article, a class $\mathbf{M_2}$ of admissible perturbations of the special expression $\mathbf{M}_0 = \sum_{k=0}^{r} c_k t^{\alpha_k} D_t^{\rho_k}$ in the weighted space $\mathcal{L}^2_\omega([1,\infty))$ will be presented. It will be shown that the operator $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$, where ω belongs to the family of completely monotonic functions, is an admissible perturbation of $\mathbf{M}_0$ in the non-weighted space $\mathcal{L}^2([1,\infty))$, and eventually preserves the essential spectrum and nullity of $\mathbf{M}_0$ in that space. Our discussion will be limited only to special expressions with $\alpha_1 < \rho_1$.

Keywords: special expression, admissible perturbations, $\mathcal{L}^2$-space, weighted $\mathcal{L}^2$-space, essential spectrum, nullity, completely monotonic functions.

1. Introduction

In 2014, a new class of admissible perturbations in the weighted space $\mathcal{L}^2_\omega([1,\infty))$ of the *special expression* $\mathbf{M}_0$ was identified and studied by J. B. Bacani in [4]. We recall that this differential operator is of the form

$$\mathbf{M}_0 = \sum_{k=0}^{r} c_k t^{\alpha_k} D_t^{\rho_k} \tag{1}$$

where $c_k \in \mathbb{C}$ and $D_t = \frac{d}{dt}$ with

i) $\rho_k \in \mathbb{N}$ for every k, such that

$$0 = \rho_0 < \rho_1 \cdots < \rho_r = n \tag{2}$$

ii) $\alpha_k \in \mathbb{R}$ for every k, satisfying

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_1 \leqslant \rho_1, \tag{3}$$

and

$$1 \geqslant \frac{\alpha_k - \alpha_{k-1}}{\rho_k - \rho_{k-1}} \geqslant \frac{\alpha_{k+1} - \alpha_k}{\rho_{k+1} - \rho_k}, \tag{4}$$

for $k = 1, \ldots, r-1$ if $r > 1$.

The perturbation, which the author denoted it by $\mathbf{M_1}$, satisfies the following conditions:

AP_1) For every $i > l$,

$$\sup_{t \in \mathbf{I}} \left| \frac{a_i(t)}{t^{i-l} a_l(t)} \right| \quad \text{exists,}$$

where $a_l(t) \in C^l(\mathbf{I}), \mathbf{I} = [1, \infty), l = 0, 1, \ldots, n-1$; and

AP_2) There exist auxiliary functions $b_l(t)$ such that

$$\left| \frac{a_l(t)}{b_l(t)} \right| \quad \text{is} \quad \text{bounded,}$$

where $0 < b_l(t) \in C^\infty([1, \infty))$ for all l and $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$ as $t \to \infty$.

The perturbation was shown to be different from the one presented in [3] and the one published in [2].

In the current work, another class of admissible perturbations of the special expression $\mathbf{M}_0$ in the weighted space, which we denote by $\mathbf{M_2}$, will be presented. It will be shown that the operator $\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}}$, where ω belongs to the family of completely monotonic functions, is an admissible perturbation of $\mathbf{M}_0$ in the non-weighted space $\mathcal{L}^2([1, \infty))$, and eventually preserves the essential spectrum of $\mathbf{M}_0$ in that space. As in [4], our discussion will be limited only to special expressions with $\alpha_1 < \rho_1$. We point out that this new class of perturbation is indeed different from what has been presented in [4] by giving an example. For other related works in the study of special expression $\mathbf{M}_0$, we offer the following articles: [5, 7, 9, 10] and [11].

In this section, we provide further discussion about special expressions, as well as some basic definitions and notations used in this paper.

The special expression $\mathbf{M}_0$, its essential part, and the polygonal path it generates are defined as follows (cf. [10]).

Definition 1.1. *A* special expression $\mathbf{M}_0$ *is a differential expression in* $\mathcal{L}^2([1, \infty))$ *of the form* (1) *where* ρ_k *and* α_k *satisfy* (2), (3), *and* (4). *We denote* $\sigma_1 < \sigma_2 < \cdots < \sigma_{s-1}$ *those indices* $k (k = 1, \ldots, r-1)$ *for which the strong inequality holds in* (3) *and with* $\sigma_0 = 0$ *and* $\sigma_s = r$. *The coefficients* c_k *are arbitrary complex constants and for* $k = \sigma_1, \ldots, r$,

$$c_\sigma \in \mathbf{C} \setminus \{0\}, \qquad \text{and} \qquad \sum_{\substack{\rho_\delta + \rho_x = 2\sigma \\ \sigma_i \leqslant \delta, x \leqslant \sigma_{i+1}}} (-1)^{\rho_\delta + \sigma} c_\delta c_x \geqslant 0$$

where $\sigma = \rho_{\sigma_i}, \ldots, \rho_{\sigma_{i+2}}, i = 1, \ldots, s-1$. *The indices* $\sigma_1, \ldots, \sigma_s$ *are called* kink indices. *The essential part of* $\mathbf{M}_0$ *is given by*

$$\mathbf{M}_{0,0} = \sum_{k=0}^{\sigma_1} c_k t^{\alpha_k} D_t^{\rho_k}.$$

We can give a graphical interpretation of the special expression $\mathbf{M}_0$ in $\mathbb{R}^2$ by plotting and joining by a line the points (ρ_k, α_k) and $(\rho_{k+1}, \alpha_{k+1})$ for $k = 0, 1, \ldots, r$. The resulting graph is called the *polygonal path generated by* $\mathbf{M}_0$. If we let $m_{\sigma_i} (i = 1, \ldots, s)$ be the slope of the line connecting the points $(\rho_{\sigma_{i-1}}, \alpha_{\sigma_{i-1}})$ and $(\rho_{\sigma_{i-1}}, \alpha_{\sigma_{i-1}})$, then

$$1 \geqslant m_{\sigma_1} \geqslant m_{\sigma_2} \geqslant \cdots \geqslant m_{\sigma_s}.$$

Hence, the polygonal path generated by $\mathbf{M}_0$ lies on or below the bisectrix. Furthermore, the polygonal path generated by $\mathbf{M}_0$ corresponds to the graph of the function $\gamma : [0, n] \to \mathbb{R}$ defined by

$$\gamma(k) = \frac{1}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \left\{ (k - \rho_{\sigma_i})\alpha_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - k)\alpha_{\sigma_i} \right\} \tag{5}$$

for $k \in [\rho_{\sigma_i}, \rho_{\sigma_{i+1}}], i = 0, 1, 2, \ldots, s-1$.

In [11], Schultze evaluated the essential spectrum and nullity of $\mathbf{M}_0$ and showed that the essential spectrum and nullity of the essential part of $\mathbf{M}_0$ and the essential spectrum and nullity of $\mathbf{M}_0$ are indeed equal. In short, he had the following results for $\alpha_1 < \rho_1$:

Theorem 1.2. *Let* $\mathbf{M}_0$ *be a special expression. Then, for* $\alpha_1 < \rho_1$,

$$\sigma_e(\mathbf{M}_0) = \sigma_e(\mathbf{M}_{0,0}) = \left\{ \sum_{k=0}^{\sigma_1} c_k z^{\rho_k} : Re(z) = 0 \right\}, \tag{6}$$

and for every $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$,

$$\operatorname{nul}(\mathbf{M}_0 - x) = \operatorname{nul}(\mathbf{M}_{0,0} - x) + \sum_{i=1}^{s-1} \# \left\{ z | \sum_{k=\sigma_i}^{\sigma_i+1} c_k z^{\rho_k} = x, Re\, z < 0 \right\}, \tag{7}$$

where

$$\operatorname{nul}(\mathbf{M}_{0,0} - x) = \# \left\{ z | \sum_{k=0}^{\sigma_1} c_k z^{\rho_k} = x, Re\, z < 0 \right\}.$$

Also, in the usual $\mathcal{L}^2$-space, a different class of perturbations, called admissible perturbations $\mathbf{M}$ of special expressions was determined by Mumpar-Victoria [9] in her paper, and was shown to preserve essential spectrum and nullity of special expressions.

Definition 1.3. *Let* $\mathbf{M}$ *be a differential expression of the form*

$$\mathbf{M} = \sum_{l=0}^{n-1} a_l(t) D_t^l. \tag{8}$$

We say that $\mathbf{M}$ *is an admissible perturbation of the special expression* $\mathbf{M}_0$ *if there exists a* B *such that the coefficients* $a_l(t)$ *satisfy the following*

$$\sup_{[x,x+1]\subset \mathbf{I}} \int_x^{x+1} \left| \frac{a_l(t)}{b_l(t)} \right|^2 dt < B \tag{9}$$

where $b_l(t) \in C^l(\mathbf{I})$ *for* $l = 0, 1, \ldots, n-1$ *and* $0 < b_l(t)$ *is an auxiliary function in* $C^\infty(\mathbf{I})$ *satisfying*

$$b_l(t) = o(t^{\gamma(l+1)}) \quad \textit{and} \quad b_l(t) = o(t^{\gamma(l)}) \tag{10}$$

as $t \longrightarrow \infty$.

For the invariance of nullity, we can only admit a somewhat less general class of perturbations consisting of expressions (8) *satisfying*

$$\sup_{[x,x+1]\subset \mathbf{I}} \int_x^{x+1} \left| \frac{a_j^{(j-l)}(t)}{b_l(t)} \right|^2 dt < \tilde{B} \tag{11}$$

for $l = 0, \ldots, n-1$ *and* $j = l, \ldots, n-1$.

Theorem 1.4. *Let* $\mathbf{M}_0$ *be a special expression and* $\mathbf{M}$ *be an admissible perturbation of* $\mathbf{M}_0$ *of the form* (8) *satisfying* (9) *and* (10). *Then*

$$\sigma_e(\mathbf{M}_0 + \mathbf{M}) = \sigma_e(\mathbf{M}_0).$$

In addition, if $\mathbf{M}$ *satisfies* (11), *then,* $\operatorname{nul}(\mathbf{M}_0 + \mathbf{M} - x) = \operatorname{nul}(\mathbf{M}_0 - x)$ *for every* $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$.

2. Completely monotonic functions and the weighted $\mathcal{L}^2$-space

A function f belongs to the class of completely monotonic (c.m.) functions on $\mathbf{I}$ if it possesses derivatives $f^{(n)}(x)$ for $n = 1, 2, \ldots$, and $(-1)^k f^{(k)}(x) \geqslant 0$ for all $k = 0, 1, 2, \ldots$ on $\mathbf{I}$. The well-known Bernstein's Theorem (see [12], p. 161) states that a necessary and sufficient condition for f to be a c.m. function on $(0, \infty)$ is that

$$f(x) = \int_0^\infty e^{-xt} d\chi(t), \tag{12}$$

where $\chi(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. From this, one can easily infer that a non-identically zero c.m. function $f(x)$ cannot vanish for any positive x. A more precise characterization of c.m. functions, also known as their *Bernstein representation*, is given as follows.

Theorem 2.1 (Bernstein)**.** *If $f(x)$ is a c.m. function on $(0,\infty)$, then it is the Laplace transform of a unique Radon measure μ on $[0,\infty)$; that is,*

$$f(x) = \int_0^\infty e^{-xt}\mu(dt), \tag{13}$$

for all $x > 0$. Conversely, if μ is a Radon measure on $[0,\infty)$ such that the above integral is convergent for $x > 0$, then it defines a c.m. function.

The cumulative distribution function for measure μ on $[0,\infty)$ is denoted by $\chi_\mu(t) = \mu[0,t]$. It is often written as $\chi(t)$, as in (12), for simplicity.

Completely monotonic functions appear naturally in various fields, such as in probability theory, numerical analysis, physics and potential theory. A thorough discussion of the main properties of these functions is given in Chapter IV of [12]. Readers may also want to see [8] for a good survey of some properties of completely monotonic functions.

The following elementary functions are examples of c.m. functions:

$$e^{-ax}, \quad \frac{1}{(\beta+\mu x)^\nu}, \quad \text{and} \quad \ln\left(b+\frac{c}{x}\right),$$

where $\min\{a,\beta,\mu,\nu\} \geqslant 0$ with β and μ are not both zero, $b \geqslant 1$, and $c > 0$.

Other examples of elementary c.m. functions are

$$e^{\frac{a}{x}},\ a > 0, \quad \text{and} \quad \frac{\ln(1+x)}{x}.$$

Obviously, given any two c.m. functions f and g, their linear combination and their product are also c.m. (cf. [8]).

The results of Schultze in [11] were generalized in the weighted $\mathcal{L}^2$-space by Agapito (cf. [1]). The weighted space is defined as follows:

Definition 2.2. *Suppose the function $\omega : [1,\infty) \longrightarrow (0,\infty)$ is measurable. The* space $\mathcal{L}^2_\omega(\mathbf{I})$ of weighted square integrable functions *over $\mathbf{I} = [1,\infty)$ is defined by*

$$\mathcal{L}^2_\omega(\mathbf{I}) := \left\{ f : \mathbf{I} \to \mathbb{C} | f \text{ is measurable and } \int_{\mathbf{I}} |f(t)|^2\omega(t)dt < \infty \right\},$$

with inner product given by

$$\langle f, g\rangle_\omega := \int_{\mathbf{I}} f(t)\overline{g(t)}\omega(t)dt$$

for any functions $f, g \in \mathcal{L}^2_\omega(\mathbf{I})$. $\mathcal{L}^2_\omega(\mathbf{I})$ is commonly known as the weighted $\mathcal{L}^2$-space *with ω as the weight function.*

Remark 2.3. It is easy to show that space $\mathcal{L}^2_\omega(\mathbf{I})$ together with the inner product $\langle\cdot,\cdot\rangle_\omega$ is a Hilbert space. Furthermore, one can show that $\mathcal{L}^2(\mathbf{I})$ is equivalent to $\mathcal{L}^2_\omega(\mathbf{I})$ under the isometry $W : \mathcal{L}^2_\omega(\mathbf{I}) \to \mathcal{L}_2(\mathbf{I})$ given by $Wf = \omega^{\frac{1}{2}} f$.

In [4], Bacani considered weight functions $\omega : [1,\infty) \longrightarrow (0,\infty)$ that satisfy the following conditions in the study of a class of admissible perturbations:

(A1) $\omega \in C^{\infty}([1,\infty))$.

(A2) $\dfrac{x^k\omega^{(k)}}{\omega} = O(1)$ for $k = 0, 1, \ldots, n$.

The next Lemma was also used in [4].

Lemma 2.4. *If $\omega : [1,\infty) \longrightarrow (0,\infty)$ satisfies* (A1) *or* (A2) *then so does ω^{α}, for any $\alpha \in \mathbb{R}$.*

In this current work, we consider weight functions ω belonging to the class of completely monotonic functions but with the restriction that $\omega > 0$ for all $x \in (0,\infty)$.

Clearly, any c.m. function ω satisfies condition (A1) and we claim that it also satisfies (A2). We formally state this result in the following Lemma.

Lemma 2.5. *If ω is a c.m. function then it satisfies* (A1) *and* (A2).

Proof. Let ω be a c.m. function. It is clear that (A1) is already satisfied. Hence, we only need to prove (A2). First, we note that the following inequality holds:

$$f_k(x) = x^k e^{-\frac{xk}{k+1}} \leqslant (k+1)ke^{-k}, \quad \forall x \in (0,\infty), k \geqslant 1.$$

Indeed, by taking the derivative of $f_k(x)$ with respect to x, we get

$$f_k'(x) = kx^{k-1}e^{-\frac{xk}{k+1}} - \frac{k}{k+1}x^k e^{-\frac{xk}{k+1}} = kx^{k-1}e^{-\frac{xk}{k+1}}\left(1 - \frac{x}{k+1}\right).$$

Evidently, the above derivative is positive if $x < k+1$ and negative if $x > k+1$. Thus $f_k(x)$ is increasing on $(0, k+1)$ and decreasing on $(k+1,\infty)$. Therefore, the global maximum is achieved at $x = k+1$ with $f_k(k+1) = (k+1)ke^{-k}$.

Now, in view of the above inequality, we find that

$$y^k e^{-y} \leqslant (k+1)^k e^{-k} e^{-\frac{y}{k+1}}, \quad k \geqslant 1, y > 0.$$

From (13), we obtain

$$\begin{aligned} |x^k\omega^{(k)}(x)| &= x^k \int_0^{\infty} e^{-xt}t^k\mu(dt) = \int_0^{\infty} e^{-xt}(xt)^k\mu(dt) \\ &\leqslant \int_0^{\infty} (k+1)^k e^{-k} e^{-\frac{xt}{k+1}}\mu(dt) \\ &= (k+1)^k e^{-k}\left[\omega\left(\frac{x}{k+1}\right) - \mu(\{0\})\right]. \end{aligned}$$

Note that $\lim_{x\to\infty}\omega(x) = \mu(\{0\})$. So, as x approaches infinity, we see that $\lim_{x\to\infty} x^k\omega^{(k)}(x) = 0$, for all $n \geqslant 1$. Moreover, dividing both sides of the above inequality by ω, we obtain

$$\frac{|x^k\omega^{(k)}(x)|}{\omega(x)} \leqslant (k+1)^k e^{-k}\frac{\left[\omega\left(\frac{x}{k+1}\right) - \mu(\{0\})\right]}{\omega(x)}.$$

Using the result stated in [6, Proposition D, p. 145], we see that the quotient $\omega(\frac{x}{k+1})/\omega(x)$ is actually bounded. This, in turn, shows that $|x^k\omega^{(k)}(x)|/\omega(x)$ is also bounded as desired. This proves the lemma. □

Some of useful properties of "big $O's$" are the following: $f = O(f)$, $O(f)\cdot O(g) = O(f\cdot g)$, and for $n > 0$, $O(x^{-n}) = O(1)$. Using these identities and by (A2), we have

$$\omega^{(k)}(x)/\omega(x) = x^{-k}\cdot O(1) = O(x^{-k})\cdot O(1) = O(x^{-k}) = O(1).$$

It can also be seen easily from the previous lemma that $0 \leqslant \lim_{x\to\infty}\omega^{(k)}(x)/\omega(x) \leqslant \lim_{x\to\infty} Cx^{-k} = 0$. Finally, by Squeeze Theorem, we get $\lim_{x\to\infty}\omega^{(k)}(x)/\omega(x) = 0$, or equivalently, $\omega^{(k)}(x)/\omega(x) = o(1) = O(1)$.

3. Main result

A new class of admissible perturbations of special expressions in the weighted space has been identified and it is of the form

$$\mathbf{M_2} = \sum_{l=0}^{n-1} a_l(t)D_t^l \tag{14}$$

satisfying the following conditions:

(AP1) For every $l = 0, 1, \ldots, n-1$, $a_l(t)$ is completely monotonic; and

(AP2) There exist auxiliary functions $b_l(t)$ such that

$$\sup_{l\leqslant i\leqslant n-1}\left|\frac{a_i(t)}{b_l(t)}\right| < \infty,$$

where $0 < b_l(t) \in C^\infty([1,\infty))$ for all l and $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$ as $t \longrightarrow \infty$.

Now we present the main result of our study.

Theorem 3.1. *Let $\mathbf{M}_0$ be a special expression of the form* (1) *with $\alpha_1 < \rho_1$. Let $\mathbf{M_2}$, which is of the form* (14) *that satisfies* (AP1) *and* (AP2)*, be an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2_\omega([1,\infty))$. If $\omega > 0$ is a c.m. function, then $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2([1,\infty))$. In addition, $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ preserves the essential spectrum and nullity of $\mathbf{M}_0$ in $\mathcal{L}^2([1,\infty))$.*

Proof. We follow [4] for the proof. Consider the admissible perturbation $\mathbf{M_2}$ of the form (14) of $\mathbf{M}_0$ presented above. Let $\mathbf{I} = [1, \infty)$ and for $l = 0, 1, \ldots, n-1$, $a_l(t)$ is completely monotonic. Also, assume that $\omega > 0$ is a c.m. function in $\mathbf{I}$.

Now, note that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an operator that can be transformed, via Leibniz's rule, as follows (cf. [4]):

$$\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}} = \omega^{\frac{1}{2}}\sum_{l=0}^{n-1} a_l(\omega^{-\frac{1}{2}}y)^{(l)} = \sum_{l=0}^{n-1}(\mathbf{R}_l(t))\,y^{(l)},$$

where

$$\mathbf{R}_l(t) = \sum_{i=l}^{n-1}\binom{i}{l}a_i\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}.$$

To show that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2(\mathbf{I})$, we need to show that Definition 1.3 is satisfied. Equivalently, we need to present that there exists a $B > 0$ such that

$$\sup_{[x,x+1]\subset\mathbf{I}}\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)}\right|^2 dt < B,$$

where $\mathbf{R}_l(t) \in C^1(\mathbf{I})$ for $l = 0, 1, \ldots, n-1$ and $0 < \mathbf{S}_l(t)$ is an auxiliary function in $C^\infty(\mathbf{I})$ satisfying

$$\mathbf{S}_l(t) = o(t^{\gamma(l+1)}) \quad \text{and} \quad \mathbf{S}_l(t) = o(t^{\gamma(l)})$$

as $t \longrightarrow \infty$.

Since $a_l(t)$ is c.m. for $l = 0, 1, \ldots, n-1$, $a_l(t) \in C^\infty((0,\infty))$. In particular, $a_l(t) \in C^l(\mathbf{I})$ for $l = 0, 1, \ldots, n-1$. Note that ω is c.m. function. So, by using Lemma 2.5, we can claim that $\omega \in C^\infty(\mathbf{I})$. Also, by using Lemma 2.4, one can show that $\omega^{\frac{1}{2}}$ and $\omega^{-\frac{1}{2}}$ are elements of $C^\infty(\mathbf{I})$. It follows that $\omega^{\frac{1}{2}}, \omega^{-\frac{1}{2}} \in C^l(\mathbf{I})$ for $l = 0, 1, \ldots, n-1$. Therefore, $\mathbf{R}^l(t) \in C^l(\mathbf{I})$ for $l = 0, 1, \ldots, n-1$.

Now let $\mathbf{S}_l(t) = \omega^{\frac{1}{2}}b_l(t)\omega^{-\frac{1}{2}} = b_l(t)$. Since $0 < b_l(t) \in C^\infty(\mathbf{I})$, so is $\mathbf{S}_l(t)$. Clearly, $\mathbf{S}_l(t) = o(t^{\gamma(l+1)})$ and $\mathbf{S}_l(t) = o(t^{\gamma(l)})$ because $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$. Thus, there exists an auxiliary function $\mathbf{S}_l(t)$.

We now proceed on evaluating the integral $\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)}\right|^2 dt$, where $1 \leqslant x < \infty$, as follows:

$$\begin{aligned}\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)}\right|^2 dt &= \int_x^{x+1}\left|\frac{\sum_{i=l}^{n-1}\binom{i}{l}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)}\right|^2 dt \\ &\leqslant J^2\int_x^{x+1}\left|\frac{\sum_{i=l}^{n-1}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)}\right|^2 dt,\end{aligned}$$

$$\left(\text{where } J = \sup_{l\leqslant i\leqslant n-1}\left\{\binom{i}{l}\right\}\right)$$

$$\leqslant J^2\int_x^{x+1}\left(\sum_{i=l}^{n-1}\left|\frac{a_i(t)}{b_l(t)}\frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}}\right|\right)^2 dt,$$

(by Δ inequality)

$$\leqslant J^2\int_x^{x+1}\left(\sum_{i=l}^{n-1}\left|\frac{a(t)}{b_l(t)}\frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}}\right|\right)^2 dt,$$

$$\left(\text{where } a(t) = \sup_{l\leqslant i\leqslant n-1}\{a_i(t)\}\right)$$

$$\leqslant J^2\int_x^{x+1}\left(O(1)\sum_{i=l}^{n-1}1\right)^2\left|\frac{a(t)}{b_l(t)}\right|^2 dt,$$

(by Lemma 2.4 and 2.5)

$$\leqslant J^2K^2(n-l)^2\int_x^{x+1}\left|\frac{a(t)}{b_l(t)}\right|^2 dt, \quad \text{for some constant } K,$$

$$\leqslant J^2K^2(n-l)^2\int_x^{x+1}M^2dt, \quad \text{for some constant } M$$

$$= J^2K^2(n-l)^2M^2.$$

Letting $B = J^2K^2(n-l)^2M^2$, we see that

$$\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)}\right|^2 dt \leqslant B.$$

Consequently, taking the supremum of both sides over the interval $[x, x+1] \subset [1,\infty)$, we have

$$\sup_{[x,x+1]\subset\mathbf{I}}\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)}\right|^2 dt < B,$$

where $B = J^2K^2(n-l)^2M^2$.

For the invariance of nullity we need to show that equation (11) is satisfed. Let $J = \sup_{l\leqslant j\leqslant n-1}\{\binom{i}{j}\}, a(t) = \sup_{l\leqslant j\leqslant n-1}\{a_j(t)\}$, and M be some non-negative constant. Hence, for $l = 0,\ldots,n-1$ and $j = l,\ldots,n-1$, we have the following

$$\left|\frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)}\right|^2 = \left|\frac{1}{b_l(t)}\sum_{i=j}^{n-1}\binom{i}{j}\left(a_i^{(j-l)}\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-j)} + a_i(\omega^{\frac{1}{2}})^{(j-l)}(\omega^{-\frac{1}{2}})^{(i-j)}\right.\right.$$

$$
\begin{aligned}
&+ a_i \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)} \Big) \Big|^2 \\
\leqslant\ & J^2 \left| \frac{a(t)}{b_l(t)} \sum_{i=j}^{n-1} \Big(\omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-j)} + (\omega^{\frac{1}{2}})^{(j-l)} (\omega^{-\frac{1}{2}})^{(i-j)} \right. \\
&\left. + \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)} \Big) \right|^2 \\
\leqslant\ & 9J^2 \left| \frac{a(t)}{b_l(t)} \sum_{i=j}^{n-1} \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-j)} \right|^2 \\
\leqslant\ & 9J^2 \left| \frac{a(t)}{b_l(t)} \right|^2 \left(\sum_{i=j}^{n-1} \left| \frac{(\omega^{-\frac{1}{2}})^{(i-j)}}{\omega^{-\frac{1}{2}}} \right| \right)^2 \\
=\ & 9J^2 \left| \frac{a(t)}{b_l(t)} \right|^2 \left(O(1) \sum_{i=j}^{n-1} 1 \right)^2 \\
=\ & 9J^2 K^2 (n-l)^2 \left| \frac{a(t)}{b_l(t)} \right|^2 \\
\leqslant\ & 9J^2 K^2 (n-l)^2 M^2.
\end{aligned}
$$

Hence,

$$
\left| \frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)} \right|^2 \leqslant 9J^2 K^2 (n-l)^2 M^2. \tag{15}
$$

Let $\tilde{C} = 9J^2K^2(n-l)^2M^2$. Integrating equation (15) over the interval $[x, x+1]$ and using (AP2) we obtain

$$
\int_x^{x+1} \left| \frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)} \right| dt \leqslant \int_x^{x+1} \tilde{C} dt = \tilde{C}.
$$

Now, taking the supremum of both sides over the interval $[x, x+1] \subset\ [1, \infty)$, we have

$$
\sup_{[x,x+1] \subset \mathbf{I}} \int_x^{x+1} \left| \frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)} \right| dt\ \leqslant\ \tilde{C}.
$$

Thus, we have shown that

$$
\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}} = \sum_{l=0}^{n-1} (\mathbf{R}_l(t))\, y^{(l)},
$$

which is obviously of the form (1.3), is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2([1, \infty))$. Moreover, as what we have shown previously, $\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}}$ satisfies

(9) and (10). In addition, it also satisfies equation (11). We now conclude that, using Theorem (1.4), the essential spectrum and the nullity of $\mathbf{M}_0$ are preserved under this kind of admissible perturbation in $\mathcal{L}^2$-space, In short, we have proven the following claim:

$$\sigma_e(\mathbf{M}_0 + \omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}) = \sigma_e(\mathbf{M}_0)$$

and

$$\text{nul}(\mathbf{M}_0 + \mathbf{M_2} - x) = \text{nul}(\mathbf{M}_0 - x)$$

for every $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$. □

In the next section we provide an example of the admissible perturbation presented above. We point out that our example does not satisfy condition (AP_1) of [4]. This means that the class of admissible perturbation being studied in the present paper is indeed different from the one presented in [4].

4. Example

Consider the following special expression

$$\mathbf{M}_0 y = y + \frac{1}{2}t^{\frac{1}{2}}y' + t^{\frac{3}{4}}y'' \tag{16}$$

and the weight function $\omega(t) = \left(1 + x^{-\frac{1}{2}}\right)^2$ which is a c.m. function.

We will show that the differential expression $\mathbf{M_2}$ defined by

$$\mathbf{M_2}y = e^{-2t}y + \ln(1 + t^{-1})y',$$

is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2_\omega(\mathbf{I})$, i.e., it should satisfy (AP1) and (AP2). We first show that $a_0(t) = e^{-2t}$ and $a_1(t) = \ln(1 + t^{-1})$ are c.m. functions. Note that

$$a_0^{(k)}(t) = (-1)^k 2^k e^{-2t}, \quad \forall k = 0, 1, 2, \ldots$$

and

$$a_1^{(k)}(t) = (-1)^k \frac{(k-1)!}{t^k(t+1)^k} \sum_{j=0}^{k-1} \binom{n}{j} t^j, \quad \forall k = 1, 2, \ldots.$$

It follows that

$$0 \leqslant (-1)^k a_0^{(k)}(t) = 2^k e^{-2t} < \infty, \quad \forall k \geqslant 0, t \in \mathbf{I},$$

and $0 \leqslant a_1(t) = \ln(1 + t^{-1}) < \infty$, and for $k \geqslant 1$, we have

$$0 \leqslant (-1)^k a_1^{(k)}(t) = \frac{(k-1)!}{t^k(t+1)^k} \sum_{j=0}^{k-1} \binom{n}{j} t^j < \infty, \quad \forall t \in \mathbf{I}.$$

Thus, $a_0(t)$ and $a_1(t)$ are c.m. functions and condition (AP1) is satisfied.

Now, $b_l(t)$ must be of the following forms (for $l = 0, 1$):

$$b_0(y) = o(t^{\gamma(1)}) = o(t^{\frac{1}{2}}) \quad \text{and} \quad b_0(t) = o(t^{\gamma(0)}) = o(1), \tag{17}$$

$$b_1(y) = o(t^{\gamma(2)}) = o(t^{\frac{3}{4}}) \quad \text{and} \quad b_1(t) = o(t^{\gamma(1)}) = o(t^{\frac{1}{2}}), \tag{18}$$

and, in addition,

$$\left|\frac{\sup\{e^{-2t}, \ln(1+t^{-1})\}}{b_l(t)}\right| = \left|\frac{\ln(1+t^{-1})}{b_l(t)}\right| = 1 < \infty, \quad \forall l = 0, 1.$$

Let $b_0(t) = b_1(t) = t^{-1}$. Clearly, (17) and (18) are satisfied. Also, we have, by L'Hôpital's rule, that

$$\lim_{t\to\infty} \frac{\ln(1+t^{-1})}{t^{-1}} = \lim_{t\to\infty} \frac{\frac{1}{-t^{-2}(1+t^{-1})}}{-t^{-2}} = \lim_{t\to\infty} \frac{1}{1+t^{-1}} = 1 < \infty.$$

Hence, condition (AP2) is satisfied. So, we have shown that $\mathbf{M_2}y$ is an admissible perturbation of $\mathbf{M}_0 y$ in $\mathcal{L}^2_\omega(\mathbf{I})$.

Now we proceed on showing that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}y$ is an admissible perturbation of $\mathbf{M}_0 y$ in $\mathcal{L}^2(\mathbf{I})$. Considering $\mathbf{M_2}y = e^{-2t}y + \ln(1+t^{-1})y'$, we have the following:

$$\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}y = \sum_{l=0}^{n-1}\left(\sum_{i=l}^{1}\binom{i}{l}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}\right)y^{(l)}.$$

Let $\mathbf{R}_l(t) = \sum_{i=l}^{1}\binom{i}{l}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}$, where $l = 0, 1$. So,

$$\mathbf{R}_0(t) = \sum_{i=0}^{1}\binom{i}{0}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i)} = e^{-2t} + \frac{\ln(1+t^{-1})}{2t+2t^{\frac{3}{2}}};$$

$$\mathbf{R}_1(t) = \sum_{i=1}^{1}\binom{i}{1}a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-1)} = \ln(1+t^{-1}).$$

It can be verified that $\mathbf{R}_0(t)$ and $\mathbf{R}_1(t)$ are c.m. functions. Hence, $\mathbf{R}_l(t)$ satisfies (AP1) for $l = 0, 1$. We can let $\mathbf{S}_l(t) = b_l(t) = t^{-1}$ for $l = 0, 1$ so that

$$\mathbf{S}_0(y) = o(t^{\frac{1}{2}}) \text{ and } \mathbf{S}_0(t) = o(1), \quad \text{and} \quad \mathbf{S}_1(y) = o(t^{\frac{3}{4}}) \text{ and } \mathbf{S}_1(t) = o(t^{\frac{1}{2}}),$$

and

$$\left|\frac{\sup\left\{e^{-2t} + \frac{\ln(1+t^{-1})}{2t+2t^{\frac{3}{2}}}, \ln(1+t^{-1})\right\}}{\mathbf{S}_l(t)}\right| = \left|\frac{\ln(1+t^{-1})}{t^{-1}}\right| = 1 < \infty, \quad \forall l = 0, 1.$$

We conclude that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2(\mathbf{I})$. In addition, $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ preserves the essential spectrum and nullity of $\mathbf{M}_0$ in $\mathcal{L}^2([1,\infty))$.

Remark 4.1. We point out that the functions $a_0(t) = e^{-2t}$ and $a_1(t) = \ln(1 + t^{-1})$ do not satisfy (AP_1), since the statement

$$\left|\frac{a_1(t)}{t^{1-0}a_0(t)}\right| = \left|\frac{\ln(1+t^{-1})}{te^{-2t}}\right| \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty$$

implies that the supremum

$$\sup_{t\in\mathbf{I}}\left|\frac{a_1(t)}{t^{1-0}a_0(t)}\right| = \sup_{t\in\mathbf{I}}\left|\frac{\ln(1+t^{-1})}{te^{-2t}}\right|$$

does not exist.

Acknowledgment

This work was funded by the UP System Enhanced Creative Work and Research Grant (ECWRG-2015-1-006). The first author would like to thank UP Baguio for the research load credit given in writing the manuscript. Both authors would like to thank the referees for their valuable comments.

References

[1] J.C. Agapito, *On relatively compact perturbations of special expressions in* $\mathcal{L}^2_\omega(\mathrm{I})$, Ph. D. Thesis, Department of Mathematics, University of the Philippines, 1993.

[2] J.C. Agapito, J.B. Bacani and M.P. Roque, *On invariance of nullities of special expressions under admissible perturbations in weighted space*, Matimyas Matematika, 29 (2006), 1–8.

[3] J.B. Bacani, *On admissible perturbations of special expressions in weighted* L_2*-space*, Master's Thesis, Department of Mathematics and Computer Science, University of the Philippines, 2004.

[4] J.B. Bacani, *Another class of admissible perturbations of special expressions*, Int. J. Math. Anal., 8 (1) (2014), 1–8.

[5] E. Balslev and T.W. Gamelin, *The essential spectrum of a class of ordinary differential operators*, Pacific J. Math., 14(3) (1964), 755–776.

[6] C. O'Cinneide, *A property of completely monotonic functions*, J. Austral. Math. Soc. (Series A), 42 (1987), 143–146.

[7] J.A. Collera, *Admissible perturbations of differential expressions with exponentially decaying coefficients preserving the nullities*, Int. J. Math. Anal., 7 (57) (2013), 2803–2810.

[8] K.S. Miller and S.G. Samko, *Completely monotonic functions*, Integral Transform. Spec. Funct., 12 (4) (2001), 389–402.

[9] M.R. Mumpar-Victoria, *Perturbations of a class of ordinary differential expressions preserving the essential spectrum and the nullities*, Science Diliman, 11(1) (1999), 25–33.

[10] M.P. Roque, *Spectral properties of differential expressions with exponentially and logarithmically growing coefficients,* in Functional Analysis and Global Analysis: Proceedings of the Conference Held in Manila, Philippines, October 20-26, 1996, Springer-Verlag Singapore, 1997.

[11] B. Schultze, *Spectral properties of not necessarily self-adjoint linear operators*, Advances in Mathematics, 83 (1990), 75–95.

[12] D.V. Widder, *The Laplace Transforms*, Princeton University Press, 1941.

Accepted: 2.12.2017

THE TOPOLOGICAL INDICES OF THE CAYLEY GRAPHS OF DIHEDRAL GROUP D_{2n} AND THE GENERALIZED QUATERNION GROUP Q_{2^n}

S. Shokrolahi Yancheshmeh*
Department of Mathematics
Ahvaz Branch
Islamic Azad University
Ahvaz
Iran
shokrolahisara@yahoo.com

R. Modabbernia
Department of Mathematics
Shoushtar Branch
Islamic Azad University
Shoushtar
Iran

M. Jahandideh
Departemt of Mathematics
Mahshahr Branch
Islamic Azad University
Mahshahr
Iran

Abstract. A topological index of a simple connected graph Γ is a numeric quantity related to the structure of the graph Γ. The set of all automorphisms of Γ under the composition of mapping forms a group which is denoted by $\mathrm{Aut}(\Gamma)$. Let G be a group, and let $S \subset G$ be a set of group elements such that the identity element $1 \notin S$. The Cayley graph associated with (G, S) is defined as the directed graph with vertex set G and edge set E such that $e = xy$ is an edge of E if $(x^{-1}y) \in S$ for every vertices x, y in G. In this paper we define the Cayley graph of the Dihedral group D_{2n} and the Cayley graph of the generalized quaternion group Q_{2^n} on the specified subsets of these groups, and compute the Wiener, Szeged and PI indices of these graphs.

Keywords: Cayley graph, Dihedral group, generalized Quaternion group, topological index.

1. Introduction

Let G be a group with identity element 1 and let S be a nonempty subset of G such that, $1 \notin S$. The Cayley graph of G relative to S is denoted by $\Gamma = \Gamma(G, S)$ and it is a directed graph with vertex set G and edge set E consisting of the

∗. Corresponding author

ordered pairs (x,y) such that $(x^{-1}y) \in S$. It is obvious that the Cayley graph $\Gamma(G,S)$ of a group G is undirected iff $S = S^{-1}$, i.e, $(x,y) \in E$ iff $(y,x) \in E$ in this case $\Gamma(G,S)$ is called the Cayley graph of G relative to S. The Cayley graph $\Gamma(G,S)$ is connected iff the set S is a generating set of the group G. The Cayley graph has been studied in [10, 11]. Let Γ (V,E) be a graph with vertex set V and edge set E, An automorphism θ of Γ is a bijective function on V which preserves the edges of Γ, i.e, e=uv is an edge of Γ iff $e^\theta = u^\theta v^\theta$ is an edge of Γ too. The set of all automorphisms of Γ forms a group $Aut(\Gamma)$ under combination of functions. $Aut(\Gamma)$ acts transitively on V if for two arbitrary vertices u and v there is an automorphisms $\theta \in Aut(\Gamma)$ such that $u^\theta = v$, then in this case Γ is said to be a vertex-transitive graph. In graph theory it has been shown that the Cayley graph (directed or undirected) of a group is always vertex-transitive.

The topological index of Γ is a numerical quantity which is constant under any arbitrary automorphism of Γ. The Wiener index is the oldest topological index of the simple connected graph, it first was introduced by a chemist named H.Wiener for molecular graphs while he was studying properties of chemical compounds [2, 3, 5, 9]. The Wiener index of a connected graph Γ is denoted by $W(\Gamma)$ and it is defined by:

$$W(\Gamma) = \sum_{\{u,v\}\subseteq V} d(u,v) = \frac{1}{2}\sum_{u\in V} d(u).$$

Where d(u,v) is the distance between two vertices u and v, and $d(u)$ is the distances between u with other vertices of Γ.

The Szeged index [4, 8] is another invariant of a graph Γ which is related to the distribution of the vertices of the graph and denoted by $Sz(\Gamma)$ and it is defined as follows:

Let $e = uv$ be an edge of the graph Γ, the sets $N_u(e|\Gamma)$ and $N_v(e|\Gamma)$ are defined as: $N_u(e|\Gamma) = \{x \in V | d(x,u) < d(x,v)\}$, $N_v(e|\Gamma) = \{x \in V | d(x,v) < d(x,u)\}$. Then $N_u(e|\Gamma)$ is the set of all vertices of Γ which are closer to u than v, $N_v(e|\Gamma)$ is defined similarly. The cardinalities of $N_u(e|\Gamma)$ and $N_v(e\Gamma)$ are denoted by $n_v(e|\Gamma)$ and $n_v(e|\Gamma)$ respectively.

Now, the Szeged index of Γ which is denoted by $Sz(\Gamma)$ is defined as:

$$Sz(\Gamma) = \sum_{e=uv\in E} n_u(e|\Gamma).n_v(e|\Gamma).$$

The Szeged index is closely related to the Wiener index such that in the case that the graph Γ is a tree, they are the same. Thus the Szeged index concerned with the distribution of the vertices of the graph, there is a topological index named Padmakar-Ivan index [6, 7] which is related to the distribution of the edges of the graph. The Padmakar-Ivan index of graph Γ is denoted by $PI(\Gamma)$ and is defined as follows:

$$PI(\Gamma) = \sum_{e=uv\in E} (n_{eu}(e|\Gamma) + n_{ev}(e|\Gamma)).$$

Where $n_{eu}(e|\Gamma)$ (resp. $n_{ev}(e|\Gamma)$) is the number of edges of the subgraph of Γ which has the vertax set $N_u(e|\Gamma)$(resp. $N_v(e|\Gamma)$).

The Dihedral group D_{2n} is the symmetry group of an n-sided regular polygonal which it has the following presentation $D_{2n} =< a, b \ |a^2 = b^n = I, (ab)^2 = I >$. Considering subset $S_1 = \{a, ab, b^{n-1}, b\}$ of D_{2n}, we can define the Cayley graph $\mathrm{Cay}(D_{2n}, S_1)$ in the cases that n is odd or even.

The generalized Quaternion Q_{2^n} is the non-abelian group of order 2^n and it has the following presentation $Q_{2^n} =< a, b|a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} >$. For some integer $n \geq 3$. The ordinary Quaternion group corresponds to the case n=3. Now, we regard the subset $S_2 = \{a, a^{2^{n-1}-1}, b, a^{2^{n-2}}\}$ of Q_{2^n} and define the Cayley graph $Cay(Q_{2^n}, S_2)$

It is obvious that $S_1 = S_1^{-1}$ and $S_2 = S_2^{-1}$ also S_1 and S_2 are generating sets of groups (D_{2n}, (Q_{2^n} respectively, so the Cayley graphs $\mathrm{Cay}(D_{2n}, S_1)$ and $\mathrm{Cay}(Q_{2^n}, S_2)$ both are undirected connected graphs. Here we try to compute the Wiener and Szeged indices of the Cayley graph $\mathrm{Cay}(D_{2n}, S_1)$.

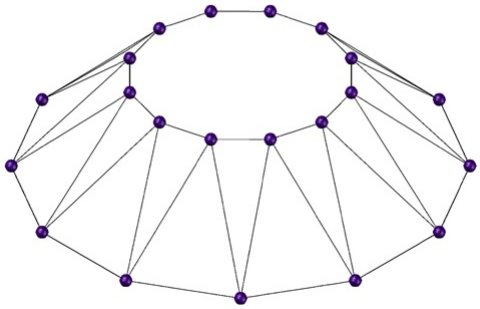


Figure 1: The $Cay(D_{2n}, S_1)$

2. Computation of Wiener, Szeged and PI indices of Cay(D_{2n}, S_1)

In this paper the following lemma [1] are frequently used.

Lemma 1. *Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. suppose Aut(G) has orbits $\Delta_i = \Delta_i(u_i)$, $1 \leq i \leq k$, on V, where u_i is orbit representative. Then $W(G) = \frac{1}{2}\sum_{i=1}^{k} |\Delta_i| d(u_i)$. Where $d(u_i) = \sum_{x \in V} d(u_i, x)$. As a result of this lemma, when Aut(G) acts transitively on V, i.e, G is a vertex transitive graph, then $W(G) = \frac{1}{2}|V|d(u)$, for some $u \in V$.*

Lemma 2. *Let G=(V,E) be a simple connected graph with vertex set V and edge set E. If Aut(G) on E has orbits $E_1, E_2, ..., E_r$ with representatives $e_1, e_2, ..., e_r$,*

where $e_i = u_i v_i \in E$ *then*:

$$Sz(G) = \sum_{i=1}^{r} |E_i|(n_{u_i}(e_i|G).n_{v_i}(e_i|G)), PI(G) = \sum_{i=1}^{r} |E_i|(n_{e_i u_i}(e_i|G)+n_{e_i v_i}(e_i|G)).$$

Proposition 1. *The Wiener index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is* $W(\Gamma) = \frac{n^2(n+1)}{2}$

Proof. Because Γ is a vertex transitive graph by the lemma 1. the Wiener index of Γ is:

$$W(G) = \frac{1}{2}|D_{2n}|d(x). \tag{1}$$

Where x is an arbitrary vertex of Γ. We have $D_{2n} = \{1, b, b^2, ..., b^{n-1}, a, ab, ab^2, ..., ab^{n-1}\}$. Suppose that n is even, without less of generality we may choose $x = 1$ and calculate d(1) as follows:

$$d(1) = \sum_{u \in V} d(1, u) = \sum_{i=1}^{n-1} d(1, b^i) + \sum_{i=1}^{n} d(1, ab^i). \tag{2}$$

Because the vertices $1, b, b^2, ..., b^{n-1}$ form a cycle so, we have $d(1, b^i) = i, 1 \leq i \leq \frac{n}{2}$, $d(1, b^i) = n - i, \frac{n}{2} + 1 \leq i \leq n - 1$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n-1} d(1, b^i) &= \sum_{i=1}^{\frac{n}{2}} d(1, b^i) + \sum_{i=\frac{n}{2}+1}^{n-1} d(1, b^i) = \sum_{i=1}^{\frac{n}{2}} i + \sum_{i=\frac{n}{2}+1}^{n-1} n - i \\ &= \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) + \frac{1}{2}(\frac{n}{2})(\frac{n}{2} - 1) = \frac{n^2}{4}. \end{aligned} \tag{3}$$

Also the vertices $a, ab, ab^2, ..., ab^{n-1}$ form a cycle we have: $d(1, ab^i) = i$, $1 \leq i \leq \frac{n}{2}$, $d(1, ab^i) = n - i + 1$, $\frac{n}{2} + 1 \leq i \leq n$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n} d(1, ab^i) &= \sum_{i=1}^{\frac{n}{2}} d(1, ab^i) + \sum_{i=\frac{n}{2}+1}^{n} d(1, ab^i) = \sum_{i=1}^{\frac{n}{2}} i \\ &+ \sum_{i=\frac{n}{2}+1}^{n} n - i + 1 = \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) + \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) = (\frac{n}{2})(\frac{n}{2} + 1). \end{aligned} \tag{4}$$

Considering (3), (4) and replacing in (2), $d(1) = \frac{n}{2}(n+1)$. Now, suppose that n is odd, $d(1, b^i) = i$, $1 \leq i \leq \frac{n-1}{2}$, $d(1, b^i) = n - i$, $\frac{n+1}{2} + 1 \leq i \leq n - 1$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n-1} d(1, b^i) &= \sum_{i=1}^{\frac{n-1}{2}} d(1, b^i) + \sum_{i=\frac{n+1}{2}}^{n-1} d(1, b^i) = \sum_{i=1}^{\frac{n-1}{2}} i + \sum_{i=\frac{n+1}{2}}^{n-1} n - i \\ &= \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) + \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) = \frac{(n-1)(n+1)}{4} \end{aligned} \tag{5}$$

Also, $d(1, ab^i) = i$, $1 \leq i \leq \frac{n-1}{2}$, $d(1, ab^i) = n-i+1$, $\frac{n+1}{2}+1 \leq i \leq n$. Therefore:

$$\begin{aligned}\sum_{i=1}^{n} d(1, ab^i) &= \sum_{i=1}^{\frac{n-1}{2}} d(1, ab^i) + \sum_{i=\frac{n+1}{2}}^{n} d(1, ab^i) = \sum_{i=1}^{\frac{n-1}{2}} i + \sum_{i=\frac{n+1}{2}}^{n} n-i+1 \\ &= \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) + \frac{1}{2}(\frac{n+1}{2})(\frac{n+3}{2}) \\ &= \frac{n+1}{4}(\frac{n-1}{2})(\frac{n+3}{2}) = \frac{n+1}{4}(n+1).\end{aligned} \tag{6}$$

Considering (5),(6) and replacing in (2) we have:

$$d(1) = \frac{n}{2}(n+1). \tag{7}$$

We see that in two cases n is even or odd the same result is obtained, so replacing (7) in (1) the proof is completed. □

Proposition 2. *The Szeged index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is:*

$$Sz(\Gamma) = \begin{cases} n(\frac{5}{2}n^2 - 4n + 2), & n \ \ is \ \ even \\ n(\frac{5}{2}n^2 - 3n + \frac{5}{2}), & n \ \ is \ \ odd. \end{cases}$$

Proof. Let E be the edge set of the graph Γ. If we consider the action of $Aut(\Gamma)$ on the E, it is obvious that it has two orbits, the first is the set of edges which are the sides of the upper and lower polygons and the second is the edges that are located between these polygons, which are denoted by E_1 and E_2 respectively. $Aut(\Gamma)$ acts transitively on each orbit and the set E breaks into E_1 and E_2 whose union is E. By the lemma 2, we have:

$$Sz(G) = \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma))$$

Where $e_i = u_i v_i$ is an arbitrary edge in E_i. We choose edge $e_1 = u_1 v_1 \in E_1$ as a representative of E_1, and suppose that n is even, $(\frac{n}{2} - 1)$ vertices on upper polygon and $(\frac{n}{2})$ vertices on lower polygon form the set of vertices which are closer to u_1 than v_1. $(N_{u_1}(e_1|\Gamma))$. So is the set $(N_{v_1}(e_1|\Gamma))$. Also if the edge $e_2 = u_1 v_2 \in E_2$ is a representative 0f E_2 then $(\frac{n}{2})$ vertices on upper polygon and $(\frac{n}{2})$ vertices on lower polygon form the set of vertices which are closer to u_1 than v_2, $(N_{u_1}(e_2|\Gamma))$. So is the set $(N_{v_2}(e_2|\Gamma))$. Therefore we have:

$$|N_{u_1}(e_1|\Gamma| = |N_{v_1}(e_1|\Gamma)| = n-1, \ |N_{u_1}(e_2|\Gamma| = |N_{v_2}(e_2|\Gamma)| = \frac{n}{2}.$$

It is clear that $\rho(u) = 4$ for every vertex, therefore $|E| = 4n$, and in two cases n is even or odd, $|E_1| = |E_2| = 2n$, so, we have:

$$\begin{aligned} Sz(G) &= \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma)) \\ &= |E_1|((n_{u_1}(e_1|\Gamma).n_{v_1}(e_1|\Gamma)) + |E_2|(n_{u_1}(e_2|\Gamma).n_{v_2}(e_2|\Gamma)) \\ &= 2n(n-1)(n-1) + 2n(\frac{n}{2})(\frac{n}{2}) = 2n(n-1)^2 + \frac{n^2}{2} = n(\frac{5}{2}n^2 - 4n + 2). \end{aligned}$$

Now, when n is odd and $e_1 = u_1v_1$ and $e_2 = u_1v_2$ are representative of E_1 and E_2, respectively considering, $(\frac{n}{2} - 1)$ vertices on upper polygon and $(\frac{n}{2} - 1)$ vertices on lower polygon form the set $N_{u_1}(e_1|\Gamma)$. So is the set $N_{v_1}(e_1|\Gamma)$. Also $(\frac{n+1}{2})$ vertices on upper polygon and $(\frac{n+1}{2})$ vertices on lower polygon form the set $N_{u_1}(e_2|\Gamma)$. So is the set $N_{v_2}(e_2|\Gamma)$. Therefore we have:

$$|N_{u_1}(e_1|\Gamma| = |N_{v_1}(e_1|\Gamma)| = n - 1,\ |N_{u_1}(e_2|\Gamma| = |N_{v_2}(e_2|\Gamma)| = \frac{n+1}{2}.$$

The Szeged index in this case is obtained as follows:

$$\begin{aligned} Sz(\Gamma) &= \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma)) \\ &= |E_1|((n_{u_1}(e_1|\Gamma).n_{v_1}(e_1|\Gamma)) + |E_2|(n_{u_1}(e_2|\Gamma).n_{v_2}(e_2|\Gamma)) \\ &= 2n(n-1)(n-1) + 2n(\frac{n+1}{2})(\frac{n+1}{2}) \\ &= 2n(n-1)^2 + 2n(\frac{n+1}{2})^2 = n(\frac{5}{2}n^2 - 3n + \frac{5}{2}). \end{aligned}$$

□

Proposition 3. *The PI index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is:*

$$PI(\Gamma) = \begin{cases} 2n(5n-12), & n \ \ is \ \ even \\ 2n(5n-11), & n \ \ is \ \ odd. \end{cases}$$

Proof. Similar to the proof of previous proposition, the action of $Aut(\Gamma)$ on the E has two orbits E_1, E_2 and by the lemma 2, the PI index of Γ can be obtained by the formula as follows:

$$PI(\Gamma) = \sum_{i=1}^{2} |E_i|(n_{e_iu_i}(e_i|\Gamma) + n_{e_iv_i}(e_i|\Gamma)). \tag{8}$$

Where $e_i = u_iv_i$ is an arbitrary edge in E_i. When n is even by considering, $n_{u_1}(e_1|\Gamma)$, $n_{v_1}(e_1|\Gamma)$, $n_{u_1}(e_2|\Gamma)$, $n_{v_2}(e_2|\Gamma)$ in the proof of proposition 2.2 we have:

$$(n_{e_1u_1}(e_1|\Gamma) + n_{e_1v_1}(e_1|\Gamma)) = (2n-5), (n_{e_2u_1}(e_2|\Gamma) + n_{e_2v_2}(e_2|\Gamma)) = \frac{n}{2} - 1.$$

So, by (8) $PI(\Gamma) = 2n(2(2n-5)) + 2n(2(\frac{n}{2}-1)) = 2n(5n-12)$. Similarly in the case that n is odd we have:

$$(n_{e_1u_1}(e_1|G) + n_{e_1v_1}(e_1|\Gamma)) = (2n-5),\ (n_{e_2u_1}(e_2|G) + n_{e_2v_2}(e_2|\Gamma)) = \frac{n-1}{2}.$$

Therefore $PI(\Gamma) = 2n(2(2n-5)) + 2n(2(\frac{n-1}{2})) = 2n(5n-11)$. □

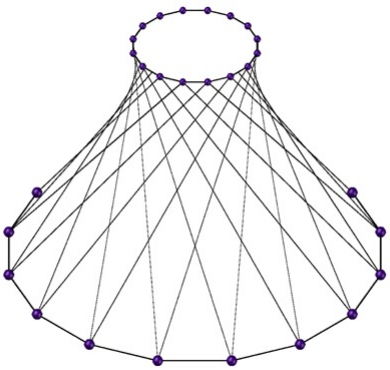


Figure 2: The Cay (Q_{2^n}, S_2)

3. Computation of Wiener, Szeged and PI indices of Cay(Q_{2^n}, S_2)

Proposition 4. *The Wiener index of the graph $H = Cay(Q_{2^n}, S_2)$ is $W(H) = 2^{n-1}(2^n(2^{n-4}+1)-2)$.*

Proof. As what was said in introduction, the Cayley graph is a vertex-transitive graph so by the lemma1 we have:

$$W(H) = \frac{1}{2}|Q_{2^n}|d(x). \tag{9}$$

Where x is an arbitrary vertex of H. We have $Q_{2^n} = \{1, a, a^2, ..., a^{2^{n-1}-1}, b, ab, a^2b, ..., a^{2^{n-1}-1}b\}$. Now let $x = 1$ and calculate d(1), by Fig. 2 we have:

$$d(1) = \sum_{u\in V} d(1,u) = \sum_{i=1}^{2^{n-1}-1} d(1,a^i) + \sum_{i=1}^{2^{n-1}} d(1,a^ib). \tag{10}$$

Because the vertices $1, a, a^2, ..., a^{2^{n-1}-1}$ form a cycle we have $d(1,a^i)$=$d(1,a^{2^{n-1}-i})$, $1 \le i \le 2^{n-1}$,

$$\sum_{i=1}^{2^{n-1}-1} d(1,a^i) = 2(\sum_{i=1}^{2^{n-2}-1} d(1,a^i)) + d(1,a^{2^{n-2}}), \tag{11}$$

$$d(1,a^i)=i, 1\le i\le 2^{n-3}+1,$$
$$d(1,a^i)=2^{n-2}-i+2, 2^{n-3}+2\le i\le 2^{n-2},$$
$$d(1,a^{2^{n-2}})=2.$$

So:

$$\sum_{i=1}^{2^{n-2}-1} d(1,a^i)=\sum_{i=1}^{2^{n-3}+1} d(1,a^i)+\sum_{i=2^{n-3}+2}^{2^{n-2}-1} d(1,a^i)$$
$$\sum_{i=1}^{2^{n-3}+1} i+\sum_{i=2^{n-3}+2}^{2^{n-2}-1} 2^{n-2}-i+2 \tag{12}$$
$$=\frac{(2^{n-3}+1)(2^{n-3}+2)}{2}+\frac{(2^{n-3})(2^{n-3}+1)}{2}-3$$
$$=(2^{n-3}+1)(2^{n-3}+1)-3=(2^{n-3}+1)^2-3.$$

Considering (11),(12) we have:

$$\sum_{i=1}^{2^{n-1}-1} d(1,a^i)=2((2^{n-3}+1)^2-3)+2=2(2^{n-3}+1)^2-4. \tag{13}$$

Also, the vertices $b, ab, a^2b, ..., a^{2^{n-1}-1}b$ form a cycle so we have $d(1,a^ib) = d(1,a^{2^{n-2}+i}b), 1\le i\le 2^{n-2}$,

$$\sum_{i=1}^{2^{n-1}} d(1,a^ib)=2(\sum_{i=1}^{2^{n-2}-2} d(1,a^ib)). \tag{14}$$

$$d(1,a^ib)=i+1, 1\le i\le 2^{n-3},$$
$$d(1,a^ib)=2^{n-2}-i+1, 2^{n-3}+1\le i\le 2^{n-2},$$
$$\sum_{i=1}^{2^{n-2}} d(1,a^ib)=\sum_{i=1}^{2^{n-3}} d(1,a^ib)+\sum_{i=2^{n-3}+1}^{2^{n-2}} d(1,a^ib),$$
$$=\sum_{i=1}^{2^{n-3}}(i+1)+\sum_{i=2^{n-3}+1}^{2^{n-2}}(2^{n-2}-i+1)=\sum_{i=1}^{2^{n-3}}(i+1)+\sum_{i=2^{n-3}+1}^{2^{n-2}} i$$
$$=\frac{(2^{n-3}+1)(2^{n-3}+2)}{2}-1+\frac{(2^{n-3})(2^{n-3}+1)}{2}$$
$$=(2^{n-3}+1)(2^{n-4}+1)-1+(2^{n-4})(2^{n-3}+1) \tag{15}$$
$$=(2^{n-3}+1)(2^{n-4}+1+2^{n-4})-1$$
$$=(2^{n-3}+1)(2^{n-3}+1)-1=(2^{n-3}+1)^2-1=(2^{n-2})(2^{n-4}+1).$$

Now replacing (15) in (14) we have:

$$\sum_{i=1}^{2^{n-1}} d(1, a^i b) = (2^{n-1})(2^{n-4}+1). \tag{16}$$

Therefore considering (16) and (13) and replacing in (10) we see:

$$\begin{aligned} d(1) &= 2(2^{n-3}+1)^2) - 4 + (2^{n-1})(2^{n-4}+1) \\ &= (2^{n-1})(2^{n-4}+1) - 2 + (2^{n-1})(2^{n-4}+1) \\ &= (2^{n-1})(2^{n-3}+1) - 2 = (2^n)(2^{n-4}+1) - 2. \end{aligned} \tag{17}$$

So by replacing (17) in (9) the proof is done. □

Proposition 5. *The Szeged index of the graph $H = Cay(Q_{2^n}, S_2)$ is:*

$$Sz(H) = 2^{3n-1}.$$

Proof. We denote the vertex set and the edge set of H by V, E respectively. Note that there is no vertex in V which it's distances from the nodes of the edge $e \in E$ be the same, for any arbitrary edge $e = uv$ of E. So we have $|N_u(e|H)| + |N_v(e|H)| = |V|$. Also, because the shape of the graph is symmetric, Fig. 2, so the number of vertices which are closer to u than v are the same to the vertices which are closer to v than u, for any edge $e = uv \in E$, therefore:

$$|N_u(e|H)| = |N_v(e|H)| = \frac{1}{2}|V| = \frac{1}{2}|Q_{2^n}| = 2^{n-1}.$$

By the Fig. 2, it is clear that:

$$|E| = 2^{n-1} + 2^{n-1} + 4.(2^{n-2}) = 2^{n+1}. \tag{18}$$

So $Sz(H) = \sum_{e=uv\in E}(n_u(e|H).n_v(e|H)) = (2^{n-1})(2^{n-1})|E| = 2^{3n-1}$. □

Proposition 6. *The PI index of the graph $H = Cay(Q_{2^n}, S_2)$ is:*

$$PI(\Gamma) = \begin{cases} 2^n(7.2^{n-2} - 9), & n > 3 \\ 96, & n = 3. \end{cases}$$

Proof. We have $PI(H) = \sum_{e=uv\in E}(n_{eu}(e|H) + n_{ev}(e|H))$. In the case n=3, we see for any edge $e = uv$ of E, $n_{eu}(e|H) = n_e v(e|H) = 3$, so by (18) and the formula of PI index it is done. According to the Fig. 2, it is clear that the action of Aut(H) on the edge set E of H has two orbits E_1,E_2. First, the set of the edges which are the sides of the upper and lower polygons, second the rest of the edges. we let the edges $e_1 = u_1v_1$ and $e_2 = u_1v_2$ as representative of E_1 and E_2, respectively. By Fig. 2 we have:

$$\begin{aligned} n_{e_1u_1}(e_1|H) &= n_{e_1v_1}(e_1|H) \\ &= 2^{n-3} - 1 + 2^{n-3} - 1 + 2^{n-1} + 2^{n-3} - 2 + (2^{n-3} - 1).4 + 2 = 2^n - 6. \end{aligned}$$

And $n_{e_2u_1}(e_2|H) = n_{e_2v_2}(e_2|H) = 2^{n-2} + 2^{n-2} - 2 + 2^{n-2} - 1 = 3(2^{n-2}) - 3$. So, By lemma.2,

$$\begin{aligned} PI(H) &= \sum_{i=1}^{2} |E_i|(n_{e_iu_i}(e_i|H + n_{e_iv_i}(e_i|H)) \\ &= |E_1|(n_{e_1u_1}(e_1|H) + n_{e_1v_1}(e_1|H)) + |E_2|(n_{e_2u_1}(e_2|H) + n_{e_2v_2}(e_2|H)) \\ &2^n(2^n - 6) + 2^n(3.2^{n-2} - 3) = 2^n(7.2^{n-2} - 9). \end{aligned}$$

□

References

[1] M. R. Darafsheh, *Computation of topological indices of some graphs*, Acta Appl. Math., 110 (2010), 1225-1235.

[2] R. C. Entringer, *Distance in graphs: Trees*, J. Combin. Math. Combin. Comput., 24 (1997), 65-84.

[3] I. Gutman, S. Klavzar, B. Mohr (Eds.), *Fifty years of the Wiener index*, Match. Commun. Math. Comput. Chem., 35 (1997), 1-259.

[4] I. Gutman, *A formula for the Wiener number of trees and its extension to cyclic Graphs.*, Graph theory Notes NY, 27 (1994), 9-15.

[5] H. Hosoya, *Topological Index. A Newly Proposed Quantity Characterizing the Topological Nature of Structural Isomers of Saturated Hydrocarbons*, Bull. Chem. Soc. Japan, 44 (1971), 2332-2339.

[6] P. V. Khadikar, *Distribution of some sequences of points on elliptic curves,* On a Novel structural descriptor PI, Nat. Acad. Sci. Lett., 23 (2000), 113-118.

[7] P. V. Khadikar. S. Karmakar, V. k. Agrawal, *Relationship and relative correction potential of the Wiener, Szeged and PI Indices*, Nat. Acad. Sci. Lett., 23 (2000), 165-170.

[8] S. Klavzar, A. Rajapakse, I. Gutman, *The Szeged and the Wiener index of graphs*, Applied Mathematics Letter, 9 (1996), 45-49.

[9] H. Wiener, *Structural Determination of Paraffin Boiling Points*, J. Am. Chem. Soc., 69 (1947), 17-20.

[10] N. Biggs, *Algebraic graph theory*, Second Edition, Cambridge Mathematical Library, Cambridge University Press, 1993.

[11] C. Godsil, Royle,G. *Algebraic graph theory*, Graduate Texts in Mathematics, Vol 207, Springer, 2001.

Accepted: 9.12.2017

GOING BEYOND THE STANDARD MODEL

B.G. Sidharth
International Institute for Applicable Mathematics and Information Sciences
Hyderabad (India)
and
Udine (Italy)
B.M. Birla Science Centre
Adarsh Nagar
Hyderabad - 500 063 (India)
birlasc@gmail.com

Abstract. In this communication we argue that we can account for the shortcomings of the Standard Model by including noncommutative geometry leading to a non-zero (electron) neutrino mass.

Keywords: particle physics, standard model, Dirac matrices.

It is well known that the standard model of particle physics is as of now the most complete theory and yet there are frantic efforts to go beyond the standard model to overcome its shortcomings. Some of these are:

1. It fails to deliver the mass to the neutrino which thus remains a massless particle in this theory.

2. This apart it does not include gravity, which is otherwise one of the four fundamental interactions.

3. There is the hierarchy problem viz., the wide range of masses for the elementary particles or even for the quarks.

4. It appears that other as yet undiscovered particles exist which could change the picture, for example in supersymmetry in which the particles have their supersymmetric counterparts.

5. The standard model has no place for dark matter, which on the other hand has not yet been definitely found. Nor is there place for dark energy.

6. Finally one has to explain the 18 odd arbitrary constants which creep into the theory.

There are however obvious shortcomings which can be addressed in a relatively simple manner which could enable us to go beyond the standard model.

Let us start with the standard model Lagrangian

$$
\begin{aligned}
LGWS = & \sum_f (\bar{\Psi}_f(\imath\gamma^\mu\partial_\mu - m_f)\Psi_f - eQ_f\Psi_f\gamma^\mu\Psi_f A_\mu) + \\
& + \frac{g}{\sqrt{2}}\sum_\imath(\bar{a}_L^\imath\gamma^\mu b_L^\imath W_\mu^+ + \bar{b}_L^\imath\gamma^\mu a_L^\imath W_\mu^-) + \\
& + \frac{g}{2C_w}\sum_f \bar{\Psi}_f\gamma^\mu(I_f^3 - 2S_w^2Q_f - I_f^3\gamma_5)\Psi_f Z_\mu + \\
& - \frac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - \imath e(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 - \frac{1}{2}|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + \\
& - \imath e(W_\mu^+ + A_\nu - W_\nu^+ A_\mu) + \imath g' c_w(W_\mu^+ Z_\nu - W_\nu^+ Z_\mu|^2 + \\
& - \frac{1}{4}|\partial_\mu Z_\nu - \partial_\nu Z_\mu + \imath g' c_w(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 + \\
& - \frac{1}{2}M_\eta^2\eta^2 - \frac{gM_\eta^2}{8M_W}\eta^3 - \frac{g'^2M_\eta^2}{32M_W}\eta^4 + |M_W W_\mu^+ + \frac{g}{2}\eta W_\mu^+|^2 + \\
& + \frac{1}{2}|\partial_\mu\eta + \imath M_Z Z_\mu + \frac{\imath g}{2C_w}\eta Z_\mu|^2 - \sum_f \frac{g}{2}\frac{m_F}{M_W}\bar{\Psi}_f\Psi_f\eta,
\end{aligned}
\tag{1}
$$

which includes the Dirac Lagrangian amongst other things.

We would now like to point out that all this has been on the basis of the usual point spacetime which is what may be called commutative. But in recent years several authors including in particular the present author have worked with a noncommutative spacetime which originates back to Snyder in the late forties itself. (This was in an attempt to overcome the divergences).

We first observe that it was Dirac [1] who pointed out two intriguing features of his equation: 1. The Compton wavelength and 2. Zitterbewegung.

For the former, his intuition was that we can never make measurements at space or time points. We need to observe over an interval to get a meaningful definition of momentum for example. This interval was the Compton region [2]. Next, his solution was rapidly oscillatory, what is called Zitterbewegung. This oscillatory behaviour disappears on averaging over spacetime intervals over the Compton region. Once this is done while meaningful physics appears, we are left with not points but minimum intervals.

This leads to a noncommutative geometry. One model for this is that of Snyder [3]. Applied at the Compton wavelength this leads to the so called Snyder-Sidharth dispersion relation, the geometry being given by [4]

$$
[x_\imath, x_j] = \beta_{\imath j}\cdot l^2. \tag{2}
$$

As described in details in reference [5] this leads to a modification in the Dirac and also the Klein-Gordon equation. This is because (2) in particular it leads to the following energy momentum relation ([4])

$$
E^2 - p^2 - m^2 + \alpha l^2 p^4 = 0, \tag{3}
$$

where α is a scalar constant, $\sim 10^{-3}$ [6, 7]. Though the value of α is of no consequence for the present work, it may be mentioned that α gives the Schwinger term. If we work with this energy momentum relation (3) and follow the usual process we get as in the usual Dirac theory

$$\{\gamma^{\mu} p_{\mu} - m\}\, \psi \equiv \{\gamma^{\circ} p^{\circ} + \Gamma\}\, \psi = 0. \tag{4}$$

We now include the extra term in the energy momentum relation (3). It can be easily shown that this leads to

$$p_0^2 - \left(\Gamma\Gamma + \{\Gamma\beta + \beta\Gamma\} + \beta^2 \alpha l^2 p^4\right) \psi = 0. \tag{5}$$

Whence the modified Dirac equation

$$\left\{\gamma^{\circ} p^{\circ} + \Gamma + \gamma^5 \alpha l p^2\right\}.\psi = 0. \tag{6}$$

The Modified Dirac equation contains an extra term. The extra term gives a slight mass for the neutrino which is roughly of the correct order viz., $10^{-8} m_e$, m_e being the mass of the electron. The behaviour too is that of the neutrino [5, 8].

To sum up the introduction of the noncommutative geometry described in (2) leads to a Dirac like equation (6) and a Lagrangian that leads to the electron neutrino mass.

It must be pointed out that the modified Lagrangian differs from the usual Lagrangian in that the γ^0 matrix is now replaced by a new matrix

$$\gamma^{0'} = \gamma^0 + \gamma^0 \cdot \gamma^5 l p^2$$

that includes the term giving rise to the neutrino mass. We can verify that the modified Lagrangian gives back the modified Dirac equation (6). Further as has been discussed in detail the extra term arising out of the noncommutative geometry is the direct result of the dark energy which thus also features in the modified standard model Lagrangian. This apart this argument has been shown to lead to a mass spectrum for elementary particles that includes all the elementary particles, giving the masses with about 5% or less error [4].

References

[1] P.A.M. Dirac, *The principles of quantum mechanics*, Clarendon Press, Oxford, 1958.

[2] B.G. Sidharth, Das Abhishek, Int. J. Mod. Phys. A., 32 (2017), 175-173.

[3] H.S. Snyder, *Physical review*, 72 (1947), 68-71.

[4] B.G. Sidharth, *The thermodynamic universe*, World Scientific, Singapore, 2008.

[5] B.G. Sidharth, Int. J. Mod. Phys. E, 19 (2010), 1-8.

[6] B.G. Sidharth, A. Das, R. Arka, Electron J. Theor. Phys., 34 (2015), 139-152.

[7] B.G. Sidharth, A. Das, R. Arka, Int. J. Th. Phys., 55 (2016), 801-808.

[8] B.G. Sidharth, *New advances in physics*, 11 (2017), 5-97.

Accepted: 5.01.2018

ON THE JOINT (m,q)-PARTIAL ISOMETRIES AND THE JOINT m- INVERTIBLE TUPLES OF COMMUTING OPERATORS ON A HILBERT SPACE

Ould Ahmed Mahmoud Sid Ahmed
Mathematics Department
College of Science
Jouf University
Aljouf 2014
Saudi Arabia
sidahmed@ju.edu.sa

Abstract. The study of tuples of commuting operators was the subject of intensive study by many authors. Our aim in this work is to consider a generalization of the notions of m-partial isometries and (m,q)-partial isometries (resp. m- left inverse and m-right inverse) of a single operator done in [23] and [21] (resp. in [14],[19], [22]) to the multivariable operators. We study some of the basic properties of these tuples of commuting operators. A commuting d-tuple of operators $\mathbf{T}=(T_1,\ldots,T_d)$ acting on a Hilbert space $\mathcal{H}$ is called a joint $(m;(q_1,\ldots,,q_d))$-partial isometry, if

$$\mathbf{T}^q\left(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\right)=0.$$

Keywords: m-isometric tuple, partial isometry, m-left inverse, m-right inverse, joint spectrum, joint approximate spectrum.

1. Introduction and terminologies

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. For $T\in\mathcal{B}(\mathcal{H})$ we shall write $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)^{\perp}$ for the null space, the range of T and the orthogonal complement of $\mathcal{N}(T)$ respectively. $I=I_{\mathcal{H}}$ being the identity operator. In what follows $\mathbb{N},\mathbb{Z}_+$ and $\mathbb{C}$ stands the sets of positive integers, nonnegative integers and complex numbers respectively. Denote by $\overline{\lambda}$ the complex conjugate of a complex number λ in $\mathbb{C}$. We shall henceforth shorten $\lambda I_{\mathcal{H}}-T$ by $\lambda-T$. The spectrum, the point spectrum, the approximate point spectrum of an operator T are denoted by $\sigma(T),\sigma_p(T)$ and $\sigma_{ap}(T)$ respectively. T^* means the adjoint of T.

The study of tuples of commuting operators was the subject of intensive study by many authors as in [4], [9], [10], [11], [12], [13], [25] , [26] and the references therein.

Our aim in this paper is to extend the notions of m-partial isometries ([23]) and (m,q)-partial isometries ([21]) for single variable operators to the tuples of commuting operators defined on a complex Hilbert space.

For $d \in \mathbb{N}$, let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a tuple of commuting bounded linear operators. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ denote tuples of nonnegative integers multi-indices) and set $|\alpha| := \sum_{1 \le j \le d} |\alpha_j|$, $\alpha! := \alpha_1! \cdots \alpha_d!$. Further, define $\mathbf{T}^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_d^{\alpha_d}$ where $T_j^{\alpha_j}$ denotes the product of T_j times itself α_j times.

One of the most important subclasses, of the algebra of all bounded linear operators acting on a Hilbert space, the class of partial isometries operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an isometry if $T^*T = I$ and partial isometry if $TT^*T = T$. In recent years this classes has been generalized, in some sense, to the larger sets of operators so-called m-isometries and m-partial isometries. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-isometric for some integer $m \ge 1$ if it satisfies the operator equation

$$\sum_{0 \le k \le m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0. \tag{1.1}$$

It is immediate that T is m-isometric if and only if

$$\sum_{0 \le k \le m} (-1)^k \binom{m}{k} \|T^{m-k} x\|^2 = 0, \tag{1.2}$$

for all $x \in \mathcal{H}$. Major work on m-isometries has been done in a long paper consisting of three parts by Agler and Stankus ([1], [2], [3]) and have since then attracted the attention of several other authors (see for example [6], [7], [8], [15]). More recently a generalization of these operators to m-partial isometries has been studied in the paper of A. Saddi and the present author in [23] and by the present author in [21].

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a m-partial isometry for some integer $m \ge 1$ (see [23]) if

$$T\left(T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + \binom{m}{2} T^{*m-2}T^{m-2} - \cdots + (-1)^m I\right) = 0. \tag{1.3}$$

and it is a (m,q)-partial isometry for $m \in \mathbb{N}$ and $q \in \mathbb{Z}_+$ (see ([21]) if

$$T^q\left(T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + \binom{m}{2} T^{*m-2}T^{m-2} - \cdots + (-1)^m I\right) = 0. \tag{1.4}$$

Gleason and Richter [16] extend the notion of m-isometric operators to the case of commuting d-tuples of bounded linear operators on a Hilbert space. The defining equation for an m-isometric tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ reads:

$$\sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^\alpha = 0 \tag{1.5}$$

or equivalently

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^{\alpha}x\|^2=0 \text{ for all } x\in\mathcal{H}. \tag{1.6}$$

Recently, P. H. W. Hoffmann and M. Mackey in [20] introduced the concept of (m,p)-isometric tuples on normed space. A tuple of commuting linear operators $\mathbf{T}:=(T_1,\ldots,T_d)$ with $T_j : X\longrightarrow X$ (normed space) is called an (m,p)-isometry (or an (m,p)-isometric tuple) if, and only if, for given $m\in\mathbb{N}$ and $p\in(0,\infty)$,

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^{\alpha}x\|^p=0 \text{ for all } \quad x\in X. \tag{1.7}$$

Definition 1.1. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a tuple of operators.*

(1) *If $T_iT_j=T_jT_i \quad 1\leq i,\ j\leq d$, we say that $\mathbf{T}$ is a commuting tuple.*

(2) *If $T_iT_j=T_jT_i$, $T_iT_j^*=T_j^*T_i, 1\leq i\neq j\leq d$, we say that $\mathbf{T}$ is a doubly commuting tuple.*

Definition 1.2 ([18])**.** *A commuting tuple $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ is called:*

(1) *matricially quasinormal if T_i commutes with $T_j^*T_k$ for all $i,j,k\in\{1,2,\ldots,d\}$.*

(2) *jointly quasinormal if T_i commutes with $T_j^*T_j$ for all $i,j\in\{1,\ldots,d\}$ and*

(3) *spherically quasinormal if T_j commutes with $|\mathbf{T}|:=\left(\sum_{1\leq j\leq d}T_j^*T_j\right)^{\frac{1}{2}}$ for all $j=1,\ldots,d$.*

If $\mathcal{M}$ is a common invariant subspace of $\mathcal{H}$ for each $T_j\in\mathcal{B}(\mathcal{H})$, then $\mathbf{T}_{|\mathcal{M}}=(T_{1|\mathcal{M}},T_{2|\mathcal{M}},\ldots,T_{d|\mathcal{M}})$ denote an d-tuple of compressions of $\mathcal{M}$.

The contents of this paper are the following. Introduction and terminologies are described in the first part. The second part is devoted to the study of some basic properties of the class of $(m;(q_1,\ldots,q_d))$-partial isometries tuples. Several spectral properties of some $(m;(q_1,\ldots,q_d))$-partial isometries are obtained in section three; concerning the joint point spectrum,the joint approximate spectrum and the spectral radius. In the fourth section we present some results concerning the m-left inverses and the m-right inverses for tuples of operators.

2. The joint $(m;(q_1,\ldots,q_d))$-partial isometries tuples of commuting operators

In this Section, we introduce and study some basic properties of a joint $(m;(q_1,\ldots,q_d))$-partial isometry operators tuples. All of these results are fairly straightforward generalizations of the corresponding single variable results that was proved in [21] and [23].

Definition 2.1. *Given $m \in \mathbb{N}$ and $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$. An commuting d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is called a joint $(m; (q_1, \ldots, q_d))$-partial isometry (or joint $(m; (q_1, \ldots, q_d))$-partial isometric d-tuple) if and only if*

$$\mathbf{T}^q \left(\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^{\alpha} \right) = 0.$$

Remark 2.1. (1) When $d = 1$ and $q \in \mathbb{N}$, the Definition 2.1 coincides with Definition 2.1 in [21].

(2) Every m-isometric d-tuple of commuting operators on $\mathcal{H}$ is a joint $(m; (q_1, \ldots, q_d))$- partial isometry d-tuple.

(3) Every joint $(m; (q_1, \ldots, q_d))$-partial isometry of commuting operators $\mathbf{T} = (T_1, \ldots, T_d)$ such that $\mathbf{T}$ is entry-wise invertible, $\mathbf{T}$ is an m-isometric d-tuple.

Remark 2.2. Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a commuting operator 2-tuple, we have that

(i) $\mathbf{T}$ is a joint $(1; (1, 1))$-partial isometry pair if

$$T_1 T_2 \Big(I - T_1^* T_1 - T_2^* T_2 \Big) = 0.$$

(ii) $\mathbf{T}$ is a joint $(2; (1, 1))$-partial isometry pair if

$$T_1 T_2 \Big(I - 2T_1^* T_1 - 2T_2^* T_2 + T_1^{*2} T_1^2 + T_2^{*2} T_2^2 + 2T_1^* T_2^* T_1 T_2 \Big) = 0.$$

Remark 2.3. Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting operator d-tuple. Then $\mathbf{T}$ is a joint $(1; (1, \ldots, 1))$-partial isometry if and only if

$$T_1 ... T_d \Big(I - T_1^* T_1 - T_2^* T_2 - \cdots - T_d^* T_d \Big) = 0.$$

Example 2.1. Consider $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ and let $\mathbf{T} = (\frac{1}{\sqrt{d}} T, \frac{1}{\sqrt{d}} T, \ldots, \frac{1}{\sqrt{d}} T) \in \mathcal{B}(\mathbb{C}^3)^d$. It is easy to see that $\mathbf{T}$ is a joint $(1; (1, \ldots, 1))$-partial isometry d-tuple.

Remark 2.4. If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a doubly commuting d-tuple of operators on $\mathcal{H}$. Then $\mathbf{T}$ is a joint $(1; (1, \ldots, 1)$-partial isometry if and only if $\mathbf{T}^* := (T_1^*, T_2^*, \ldots, T_d^*)$ is so.

The following example of a joint $(m; (q_1, \ldots, q_d))$-partial isometry is adopted form [20].

Example 2.2. Let $S \in \mathcal{B}(\mathcal{H})$ be a (m, q_1)-partial isometry operator,$d \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in (\mathbb{C}^d,\ \|.\|_2)$ with

$$\|\lambda\|_2^2 = \sum_{1\leq j\leq d} |\lambda_j|^2 = 1.$$

Then the operator tuple $\mathbf{T} = (T_1, \ldots, T_d)$ with $T_j = \lambda_j S$ for $j = 1, \ldots, d$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple.

In fact,it is clair that $T_iT_j = T_jT_i$ for all $1 \leq i;\ j \leq d$. Further, by the multinomial expansion, we get

$$\begin{aligned}\left(|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_d|^2\right)^k &= \sum_{\alpha_1+\alpha_2+\cdots+\alpha_d=k} \binom{k}{\alpha_1, \alpha_2, \ldots, \alpha_d} \prod_{1\leq i\leq d} |\lambda_i|^{2\alpha_i} \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathbf{T}^q \sum_{0\leq j\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \\ &= \mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{1\leq j\leq d} |\lambda|^{2\alpha_j} S^{*|\alpha|} S^{|\alpha|} \\ &= \prod_{1\leq j\leq d} \lambda_j^{q_j} S^{|q|} \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} S^{*k} S^k \\ &= \prod_{1\leq j\leq d} \lambda_j^{q_j} S^{|q|-q_1} \underbrace{S^{q_1} \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} S^{*k} S^k}_{=0} = 0.\end{aligned}$$

Consequently, $\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple as required.

The following example shows that the question about joint $(m; (q_1, \ldots, q_d))$-partial isometry for d-tuple is non trivial. There exists a d-tuple of commuting operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ such that each T_j is a (m, q_j)-partial isometry for $j = 1, \ldots, d$, but $\mathbf{T} = (T_1, \ldots, T_d)$ is not a joint $(m; (q_1, \ldots, q_d))$-partial isometry.

Example 2.3. Let us consider $\mathcal{H} = \mathbb{C}^3$ and define $T_1 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is straightforward that T_1 and T_2 commute. Moreover,T_1 and T_2 are joint $(2; 1)$-partial isometry but (T_1, T_2) is not a joint $(2; (1, 1))$-partial isometry.

Lemma 2.1. *Let $\mathbb{S}_d$ be the group of permutation on d symbols $\{1,2,\dots,d\}$ and let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a d-tuple of commuting operators. If $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry , then for every $\sigma\in\mathbb{S}_d$, $\mathbf{T}_\sigma:=(T_{\sigma(1)},T_{\sigma(2)},\dots,T_{\sigma(d)})$ is a joint $(m;(q_{\sigma(1)},\dots,q_{\sigma(d)}))$-partial isometry.*

Proof. The proof follows from the condition that $\prod_{1\le j\le d}T_j=\prod_{1\le j\le d}T_{\sigma(j)}$ and the identity

$$\prod_{1\le j\le d}T_j^{q_j}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\prod_{1\le j\le d}T_j^{*\alpha_j}\prod_{1\le j\le d}T_j^{\alpha_j}=0.$$

□

Theorem 2.1. *Let $m\in\mathbb{N}$ and $q=(q_1,\dots,q_d)\in\mathbb{Z}_+^d$. Let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for T_j for all $j=1,2,\dots,d$. Then the following properties are equivalent.*

(1) *$\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry.*

(2)
$$\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^\alpha\mathbf{T}^{*q}x\|^2=0,\ \text{for all } x\in\mathcal{H}.$$

Proof. First, assume that $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry. We have that for all $x\in\mathcal{H}$

$$\begin{aligned}&\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^{*q}x=0\\&\implies\langle\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^{*q}x,\ x\rangle=0\\&\implies\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^\alpha\mathbf{T}^{*q}x\|^2=0.\end{aligned}$$

Thus, (2) holds.

To prove the converse, assume that the equality in (2) is holds. It follows that,

$$\begin{aligned}&\langle\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^{*q}x,\ x\rangle=0,\ \forall\ x\in\mathcal{H}\\&\implies\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^{*q}x=0,\forall\ x\in\mathcal{H}.\end{aligned}$$

Hence,

$$\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha=0\ \text{ on }\ \overline{\mathcal{R}(\mathbf{T}^{*q})}=\mathcal{N}(\mathbf{T}^q)^\perp.$$

As $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j $(1 \leq j \leq d)$, we have that

$$\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} = 0 \text{ on } \mathcal{N}(\mathbf{T}^q)$$

and hence,

$$\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} = 0.$$

□

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let $m \in \mathbb{N}$ and $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$. Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j, $1 \leq j \leq d$. Then the following properties are equivalent*

1. *$\mathbf{T}$ is a joint $(m; q_1, \ldots, q_d))$-partial isometry.*
2. *$\mathbf{T}|_{\mathcal{N}(\mathbf{T}^q)^\perp} := \left(\mathbf{T}_1|_{\mathcal{N}(\mathbf{T}^q)^\perp}, T_2|_{\mathcal{N}(\mathbf{T}^q)^\perp}, \ldots, T_d|_{\mathcal{N}(\mathbf{T}^q)^\perp}\right)$ is an m-isometric tuple.*

Remark 2.5. It is easy to see that every joint $(m; (1, \ldots, 1))$-partial isometry d-tuple of commuting operators is a joint $(m; (q_1, \ldots, q_d))$-partial isometry.

In the following theorem we show that by imposing certain conditions on a joint $(m; (q_1, \ldots, q_d))$-partial isometry of operators it becomes a joint $(m; (1, \ldots, 1))$-partial isometry.

Theorem 2.2. *If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such is a joint $(m; (q_1, \ldots, q_d))$-partial isometry and $\mathcal{N}(T_j) = \mathcal{N}(T_j^2)$ for each j, $1 \leq j \leq d$, then $\mathbf{T}$ is a joint $(m; (1, \ldots, 1))$-partial isometry.*

Proof. By the assumption we have for $j = 1, \ldots, d$ that $\mathcal{N}(T_j) = \mathcal{N}(T_j^n)$ for all positive integer n. It follows that

$$\mathbf{T}^q \left(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \right) = 0$$

implies

$$\prod_{1\leq j\leq d} T_j \left(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \right) = 0.$$

□

The following proposition generalized ([23], Proposition 3.1)

Proposition 2.1. *If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a jointly quasinormal and a joint $(m;(1,\ldots,1))$-partial isometry, then $\mathbf{T}$ is a joint $(1;(1,\ldots,1))$-partial isometry.*

Proof. Since $\mathbf{T} = (T_1, \ldots, T_d)$ is a jointly quasinormal and a joint $(m;(1,\ldots,1))$-partial isometry, it follows that

$$\prod_{1\leq j\leq d} T_j\Big(I - \sum_{1\leq j\leq d} T_j^*T_j\Big)^m = 0.$$

A straightforward computation using this last equation yields that

$$\prod_{1\leq j\leq d} T_j\Big(I - \sum_{1\leq j\leq d} T_j^*T_j\Big) = 0.$$

The proof is complete. □

Definition 2.2. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d)$ are two commuting d-tuples of operators on a common Hilbert space $\mathcal{H}$. We said that $\mathbf{S}$ is unitary equivalent to $\mathbf{T}$ if there exists an unitary operator $V \in \mathcal{B}(\mathcal{H})$ such that*

$$\mathbf{S} = (S_1, \ldots, S_d) = (V^*T_1V, V^*T_2V, \ldots, V^*T_dV).$$

Proposition 2.2. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ are two commuting d-tuple of operators such that $\mathbf{S}$ is unitary equivalent to $\mathbf{T}$, then $\mathbf{T}$ is a joint $(m,(q_1,\ldots,q_d)$-partial isometry if and only if $\mathbf{S}$ is a joint $(m,(q_1,\ldots,q_d)$-partial isometry.*

Proof. Suppose that $\mathbf{S}$ and $\mathbf{T}$ are unitary equivalent,that is there exists a unitary operator $V \in \mathcal{B}(\mathcal{H})$ such that $S_j = V^*T_jV, (1 \leq j \leq d)$. Since $T_iT_j = T_jT_i$; it follows that

$$(V^*T_jV)(V^*T_iV) = (V^*T_iV)(V^*T_jV) \text{ for all } 1 \leq i,j \leq d.$$

Using the observations above, we get the following identity

$$\begin{aligned}
&\mathbf{S}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{S}^{*\alpha}\mathbf{S}^{\alpha} \\
&= V^*\mathbf{T}^qV \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} V^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}V \\
&= V^*\Big(\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} V^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\Big)V.
\end{aligned}$$

□

In the proof of the following theorem, we need the following formula

Remark 2.6. For $n, d, k_1, k_2, \ldots, k_d \in \mathbb{N}$ with $k_1 + ... + k_d = n$, $n \geq 1$ and $d \geq 2$, we have

$$\binom{n}{k_1 \cdots k_d} = \sum_{1 \leq j \leq d} \binom{n-1}{k_1 \cdots k_j - 1 \cdots k_d}.$$

It will known that if $\mathbf{T} = (T_1, \ldots, T_d)$ is an m-isometric tuple then $\mathbf{T}$ is an n-isometric tuple for $n \geq m$. This result is not true for joint $(m; (q_1, \ldots, q_d))$-partial isometric tuple. Since it was shown in [21] that a (m, q)-partial isometry operator need not be a $(m+1, q)$-partial isometry and vice versa.

Theorem 2.3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple of commuting operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $1 \leq j \leq d$. Then $\mathbf{T}$ is a $(m+n; (q_1, \ldots, q_d))$-partial isometry d-tuple for $n \in \mathbb{N}$.*

Proof. To prove that $\mathbf{T}$ is a joint $(m+n; (q_1, \ldots, q_d))$-partial isometry, it suffices to prove that $\mathbf{T}$ is a joint $(m+1; (q_1, \ldots, q_d))$-partial isometry.

Indeed, for $x \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= \|\mathbf{T}^{*q} x\|^2 + \sum_{1 \leq k \leq m} (-1)^k \Big[\binom{m}{k} + \binom{m}{k-1} \Big] \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 - (-1)^m \\
&\sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|-k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- (-1)^m \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= - \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!(\alpha_1 + ... + \alpha_d)}{\alpha_1!.\alpha_2 .... \alpha_d!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- (-1)^m \sum_{|\alpha|=m+1} \frac{m!(\alpha_1 + ... + \alpha_d)}{\alpha_1!.\alpha_2 .... \alpha_d!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= - \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} (-1)^k \binom{m}{k}
\end{aligned}$$

$$\begin{aligned}
&\frac{k!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}\|T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j\mathbf{T}^{*q}x\|^2\\
&-(-1)^m\sum_{1\le j\le d}\sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}\|T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j\mathbf{T}^{*q}x\|^2\\
&=-\sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k}(-1)^k\binom{m}{k}\frac{k!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2\\
&-(-1)^m\sum_{1\le j\le d}\sum_{|\alpha|=m}\frac{m!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2\\
&=-\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2=0.
\end{aligned}$$

This completes the proof. □

Proposition 2.3. *Let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j=1,2,\dots,d$. Assume that $\mathbf{T}$ satisfies*

$$\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T_j\mathbf{T}^{\alpha}\mathbf{T}^{*q}x\|^2=0 \ \text{ for all } \ x\in\mathcal{H}. \tag{2.1}$$

Then $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry if and only if $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry.

Proof. If $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry, then $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry by Theorem 2.3.

Conversely, assume that $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry and satisfies the equation (2.1). It follows that

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j\mathbf{T}^{*q}=0$$

and hence

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j=0 \ \text{ on } \ \overline{\mathcal{R}\big(\mathbf{T}^{*q}\big)}=\mathcal{N}\big(\mathbf{T}^q\big)^{\perp}.$$

On the other hand, since $T_j\big(\mathcal{N}(\mathbf{T}^q)\big)\subset\mathcal{N}(\mathbf{T}^q)$ and $T_j^*\big(\mathcal{N}(\mathbf{T}^q)\big)\subset\mathcal{N}(\mathbf{T}^q)$, we have $T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j\,\mathcal{N}\big(\mathbf{T}^q\big)\subseteq\mathcal{N}\big(\mathbf{T}^q\big)$. Thus,

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0 \ \text{ on } \ \mathcal{N}\big(\mathbf{T}^q\big).$$

From the above equalities we conclude that

$$\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j = 0 \text{ on } \mathcal{H}=\mathcal{N}(\mathbf{T}^q)\oplus\mathcal{N}(\mathbf{T}^q)^{\perp}.$$

Since $\mathbf{T}$ is a joint $(m+1,q)$-isometry, we get

$$\begin{aligned}0 &= \mathbf{T}^q \sum_{0\leq k\leq m+1}(-1)^k\binom{m+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &- \mathbf{T}^q \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{T}^q \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j = 0.\end{aligned}$$

Consequently, $\mathbf{T}$ is a joint $(m,(q_1,\ldots,q_d))$-isometry. □

Proposition 2.4. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a joint $(m;(q_1,\ldots,q_d))$-partial isometry d-tuple. The following statements hold:*

(i) $\mathbf{T}$ is a joint $(m+1;(q_1,\ldots,q_d))$-partial isometry d-tuple if and if $\mathbf{T}$ satisfies

$$\mathbf{T}^q\Big(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\Big)=0. \tag{2.2}$$

(ii) If $\mathcal{N}(T_j)=\mathcal{N}(T_j^2)$ for each $j;1\leq j\leq d$, then $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d))$-partial isometry d-tuple if and only if $\mathbf{T}$ satisfies (2.1)

Proof. (i) Since $\mathbf{T}$ is a joint $(m,(q_1,\cdots,q_d))$-partial isometry d-tuple, we have

$$\begin{aligned}&\mathbf{T}^q \sum_{0\leq k\leq m+1}(-1)^k\binom{m+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= -\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j.\end{aligned}$$

Thus, $\mathbf{T}$ is a joint $(m+1,(q_1,\ldots,q_d))$-partial isometry d-tuple if and only if

$$\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0, \tag{2.3}$$

which is equivalent to the equation (2.2).

(ii) Suppose that $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d))$-partial isometry d-tuple. Then

$$\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0,$$

and the equation (2.1) is satisfied.

Conversely, suppose that (2.1) is fulfilled. Under the conditions $\mathcal{N}(T_j)=\mathcal{N}(T_j^2)$ for each j; $1\leq j\leq d$, we see easily that

$$\mathcal{N}(\mathbf{T}^q)=\mathcal{N}(\prod_{1\leq j\leq d}T_j)=\bigcap_{1\leq j\leq d}\mathcal{N}(T_j),$$

and therefore we have (2.3). Now, from (i), $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d)$-partial isometry d-tuple.

□

3. Spectral properties of joint $(m;(q_1,\ldots,q_d))$-partial isometries tuples

Spectral properties of commuting d-tuples of operators received important attention during last decades. For more details, the interested reader is referred to [9], [10], [11], [12], [13], [24] and the references therein.

First, we recapitulate very briefly the following definitions.

Definition 3.1. *Let $\mathbf{T}=(T_1,\ldots,T_d)$ be a d-tuple of operators on a complex Hilbert space $\mathcal{H}$.*

1. *A point $\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{C}^d$ is called a joint point eigenvalue of $\mathbf{T}$ if there exists a non zero vector $x\in\mathcal{H}$ such that*

$$(T_j-\lambda_j)x=0 \text{ for } j=1,\ldots,d.$$

Or equivalently if there exists a non-zero vector $x\in\mathcal{H}$ such that $x\in\bigcap_{1\leq j\leq d}\mathcal{N}(T_j-\lambda_j)$,i.e.;

$$\sigma_p(\mathbf{T})=\{\lambda\in\mathbb{C}^d:\bigcap_{1\leq j\leq d}\mathcal{N}(T_j-\lambda_j)\neq\{0\}\}.$$

2. *The joint point spectrum, denoted by $\sigma_p(\mathbf{T})$ of $\mathbf{T}$ is the set of all joint eigenvalues of $\mathbf{T}$.*

Definition 3.2. *For a commuting d-tuple* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$. *A number* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ *is in the joint approximate point spectrum* $\sigma_{ap}(\mathbf{T})$ *if and only if there exists a sequence of unit vector* $(x_n)_n$ *such that*

$$(T_j - \lambda_j)x_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \ \text{ for every } \ \ j = 1, ..., d.$$

Lemma 3.1. *(*[16]*) Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting tuples of bounded operators. Then*

$$\sigma_{ap}(\mathbf{T}) = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \ \exists\, (x_n)_n \subset \mathcal{H},$$

$$\|x_n\| = 1, \ \ / \ \lim_{n \longrightarrow \infty} \sum_{1 \leq j \leq d} \|(T_j - \lambda_j)x_n\| = 0 \Big\}.$$

Definition 3.3 ([26])**.** *The Taylor spectrum of commuting d-tuple* $= (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *is the set of all complex d-tuple* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ *with the property that the translated d-tuple* $(T_1 - \lambda_1, \ldots, T_d - \lambda_d)$ *is note invertible. The symbol* $\sigma(\mathbf{T})$ *will stand for the Taylor spectrum of* $\mathbf{T}$.

Remark 3.1 ([26])**.** Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be an d-tuple of commuting operators on $\mathcal{H}$. $(\lambda_1, \ldots, \lambda_d) \notin \sigma(\mathbf{T})$ if there exist operators $U_1, \ldots, U_d, V_1, \ldots, V_d \in \mathcal{B}(\mathcal{H})$) such that

$$\sum_{1 \leq k \leq d} U_k(T_k - \lambda_k I) = I \text{ and } \ \sum_{1 \leq k \leq d} (T_k - \lambda_k I)V_k = I.$$

The spectral radius of $\mathbf{T}$ is

$$r(\mathbf{T}) = \max\{\|\lambda\|_2, \ \ \lambda \in \sigma(\mathbf{T})\}$$

where $\|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2 \right)^{\frac{1}{2}}$.

In the following results we examine some spectral properties of a joint $(m; (q_1, \ldots, q_d))$-partial isometries.That extend the case of single variable m-partial isometries studied in [23].

We put

$$\mathbb{B}(\mathbb{C}^d) := \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \ / \ \|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2 \right)^{\frac{1}{2}} < 1 \ \}$$

and

$$\partial\mathbb{B}(\mathbb{C}^d) := \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \ / \ \|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2 \right)^{\frac{1}{2}} = 1 \ \}.$$

In ([16], Lemma 3.2), the authors proved that if $\mathbf{T}$ is a m-isometric tuple, then the joint approximate point spectrum of $\mathbf{T}$ is in the boundary of the unit ball $\mathbb{B}(\mathbb{C}^d)$. This is not true for a joint $(m; (q_1, \ldots, q_d))$-partial isometry tuple as shown in the following example.

Example 3.1. Let $\mathbf{T} = (T, 0, ..., 0) \in \mathcal{B}(\mathbb{C}^2)^d$, where T is the matrix operator $T = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}$ with $|a|^2 = \frac{1+\sqrt{5}}{2}$. It is easily to show that $\mathbf{T}$ is a joint $(2; (1, 0, 0, \ldots, 0))$-partial isometry and further $\sigma(\mathbf{T}) = \{0, a\} \times \{0\} \times ... \times \{0\}$.

However, if in addition assume that T_j reduces $\mathcal{N}(\mathbf{T}^q)$ for $1 \leq j \leq d$, we obtain the following result.

Theorem 3.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a joint $(m; (q_1, \ldots, q_d))$-partial isometry of d-tuple of operators such that $\mathcal{N}(\mathbf{T^q})$ is a reducing subspace for each T_j $(1 \leq j \leq d)$. Then $\sigma_{ap}(\mathbf{T}) \subset \partial\mathbb{B}(\mathbb{C}^d) \cup \big[0\big]$ where*

$$\big[0\big] := \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \prod_{1\leq k\leq d} \lambda_k = 0\}.$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_{n\geq 1} \subset \mathcal{H}$, with $||x_n|| = 1$ such that $(T_j - \lambda_j I)x_n \longrightarrow 0$ for all $j = 1, 2, \ldots, d$. Since for $\alpha_j > 1$,

$$T_j^{\alpha_j} - \lambda_j^{\alpha_j} = (T_j - \lambda_j) \sum_{1\leq k\leq \alpha_j} \lambda_j^{k-1} T_j^{\alpha_j - k}.$$

By induction, for $\alpha \in \mathbb{Z}_+^d$, we have

$$(T^\alpha - \lambda^\alpha I) = \sum_{1\leq k\leq d} \Big(\prod_{i\leq k} \lambda_i^{\alpha_i}\Big)\Big(T_j^{\alpha_j} - \lambda_j^{\alpha_i}\Big) \prod_{i>k} T_i^{\alpha_i}.$$

Since, $\mathcal{R}(\mathbf{T}^q) \subset \mathcal{N}(\mathbf{T}^q)^\perp$ we have from Corollary 2.1 that, for all $n \geq 1$

$$\begin{aligned}
0 &= \lambda^q \langle \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \mathbf{T}^q x_n \, , x_n\rangle \\
&= \lambda^q \langle \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha (\mathbf{T}^q - \lambda^q) x_n, \; x_n\rangle \\
&+ \lambda^{2q} \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} ||\mathbf{T}^\alpha x_n||^2 \\
&= \lambda^q \langle \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha (\mathbf{T}^q - \lambda^q x_n | x_n\rangle \\
&+ \lambda^{2q} \Bigg\{ \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \Big(||(\mathbf{T}^\alpha - \lambda^\alpha) x_n||^2 \\
&+ 2Re\langle (\mathbf{T}^\alpha - \lambda^\alpha) x_n | \lambda^\alpha x_n\rangle + |\lambda^\alpha|^2 \Big) \Bigg\}
\end{aligned}$$

as $\big(\mathbf{T}^\alpha - \lambda^\alpha I\big)x_n \to 0$ as $n \longrightarrow \infty$ for all $\alpha \in \mathbb{Z}_+^d$ we obtain that

$$0 = \lambda^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^2 = \lambda^q (1 - \|\lambda\|_2^2)^m,$$

where $\|\lambda\|_2 = \left(\sum_{1\leq k\leq d}|\lambda_k|^2\right)^{\frac{1}{2}}$. We deduce that $\lambda^q = 0$ or $\|\lambda\|_2 = 1$. This implies that

$$\lambda \in \{(\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d : \prod_{1\leq k\leq d} \lambda_k = 0\} \text{ or } \lambda \in \partial\mathbb{B}(\mathbb{C}^d).$$

□

Corollary 3.1. *If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j $(1 \leq j \leq d)$, then $r(\mathbf{T}) = 1$.*

Proof. It is known (see for example [24]) that the convex envelopes of all spectra coincide. Thus from Theorem 3.1 we have that the approximate point spectrum of $\mathbf{T} = (T_1, \ldots, T_d)$ is contained in the boundary of the unit ball, it follows that $r(\mathbf{T}) = 1$. □

We have also, the following properties.

Proposition 3.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for T_j $(1 \leq j \leq d)$. The following properties hold.*

1. *If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})\backslash[0]$, then $1 \in \sigma_{ap}(\sum_{1\leq j\leq d} \lambda_j T_j^*)$.*
2. *If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{p}(\mathbf{T})\backslash[0]$, then $1 \in \sigma_{p}(\sum_{1\leq j\leq d} \lambda_j T_j^*)$.*
3. *Eigenvectors of $\mathbf{T}$ corresponding to two joint eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ such that $\sum_{1\leq j\leq d} \lambda_j \overline{\mu}_j \neq 1$ are orthogonal.*

Proof. 1. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})\backslash\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \prod_{1\leq k\leq d} \lambda_k = 0\}$, choose a sequence $(x_n)_n \subset \mathcal{H}$, such that $\|x_n\| = 1$ and $(T_j - \lambda_j)x_n \longrightarrow 0$ for all $j = 1, 2, \ldots, d$. Following similar arguments it is easy to see that for all $\alpha_j \geq 0$. $(T_j^{\alpha_j} - \lambda_j^{\alpha_j})x_n \longrightarrow 0$ as $n \longrightarrow \infty$ and

$$(T^\alpha - \lambda^\alpha)x_n \longrightarrow 0.$$

On the other hand

$$\begin{aligned}\mathbf{T}^{*\alpha}\mathbf{T}^\alpha(\mathbf{T}^q - \lambda^q)x_n &= \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}\mathbf{T}^\alpha x_n \\ &= \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}\big(\mathbf{T}^\alpha - \lambda^\alpha\big)x_n + \lambda^q\mathbf{T}^{*\alpha}\lambda^\alpha x_n \longrightarrow 0.\end{aligned}$$

Since $\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry, we observe that

$$\lambda^q \sum_{1\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\lambda\mathbf{T}^*)^\alpha x_n \longrightarrow 0.$$

and hence,

$$\lambda^q\Big(\mathbf{I}-\sum_{1\leq j\leq d}\lambda_j T_j^*\Big)^m x_n\longrightarrow 0.$$

Using the fact that $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_{ap}(\mathbf{T})\backslash[0]$, we get

$$\Big(I-\sum_{1\leq j\leq d}\lambda_j T_j^*\Big)^m x_n\longrightarrow 0.$$

We deduce that $(I_{\mathcal{H}}-\sum_{1\leq j\leq d}\lambda_j T_j^*)$ is not bounded below and hence $1\in\sigma_{ap}(\sum_{1\leq j\leq d}\lambda_j T_j^*)$,

and the proof of this implication is over.

2. Let $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_p(\mathbf{T})\backslash[0]$, there exists a non zero vector $x\in\mathcal{H}$ such that

$$T_j x=\lambda_j x \text{ for } j=1,2,..,d.$$

By using a similar argument as in 1 we show $(I_{\mathcal{H}}-\sum_{1\leq j\leq d}\lambda_j T_j^*)x=0$. From which it follows that $1\in\sigma_p(\sum_{1\leq j\leq d}\lambda_j T_j^*)$.

3. Let $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\mu=(\mu_1,\ldots,\mu_d)$ be eigenvalues of $\mathbf{T}$ such that $\sum_{1\leq j\leq d}\lambda_j\overline{\mu}_j\neq 1$. Assume that $T_j x=\lambda_j x$ and $T_j y=\mu_j y$ for $j=1,\ldots,d$. Since $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j=1,\ldots,d$, we have $\mathcal{R}(\mathbf{T}^q)\subseteq\mathcal{N}(\mathbf{T}^q)^{\perp}$. Moreover since $\mathbf{T}$ is a $(m;(q_1,\ldots,q_d)$-partial isometry tuple,it follows from Corollary 2.1 that

$$\begin{aligned}
0 &= \Big\langle\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\mathbf{T}^q x,\ y\Big\rangle\\
&= \lambda^q\Big\langle\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}x,\ y\Big\rangle\\
&= \lambda^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\big(\lambda.\overline{\mu}\big)^{\alpha}\langle x,\ y\rangle\\
&= \Big(1-\sum_{1\leq j\leq d}\lambda_j\overline{\mu}_j\Big)^m\langle x,\ y\rangle.
\end{aligned}$$

where $\lambda.\overline{\mu}=(\lambda_1\overline{\mu}_1,\lambda_2\overline{\mu}_2,,...,\lambda_d\overline{\mu}_d)$.

Since $1-\sum_{1\leq j\leq d}\lambda_j\overline{\mu}_j\neq 0$, we obtain that $\langle x\mid y\rangle=0$. □

Lemma 3.2. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be an joint $(m;(q_1,\ldots,q_d))$-partial isometry such that $\mathcal{N}(\mathbf{T^q})$ is a reducing subspace for T_j, $j=1,\ldots,d$. Let $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{ap}(\mathbf{T})$ such that $\sum_{1\leq j\leq d}\lambda_j\overline{\mu_j}\neq 1$. If $(x_n)_n$ and $(y_n)_n$ are two sequences of unit vectors in $\mathcal{H}$ such that*

$$\|(T_j-\lambda_j)x_n\|\longrightarrow 0 \text{ and } \|(T_j-\mu_j)y_n\|\longrightarrow 0 \ (as\ n\longrightarrow\infty) \text{ for all } j=1,2,...,d,$$

then we have

$$\langle x_n|\ y_n\rangle\longrightarrow 0 \ (as\ n\longrightarrow\infty). \tag{3.1}$$

Proof. Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sequences of unit vectors in $\mathcal{H}$ such that

$$\|(T_j-\lambda_j)x_n\| \longrightarrow 0 \text{ and } \|(T_j-\mu_j)y_n\| \longrightarrow 0 \ \ (\text{as } \ n \longrightarrow \infty) \ \text{ for all } \ j=1,2,...,d.$$

Then for all $\alpha_j, q_j \in \mathbb{N}$, we have $\lim_{n\to\infty}(T_j^{\alpha_j+q_j}-\lambda_j^{\alpha_j+q_j})x_n = 0$ and $\lim_{n\to\infty}(T_j^{\alpha_j}-\mu_j^{\alpha_j})y_n = 0$. On the other hand, we also have

$$\lim_{n\to\infty}(\mathbf{T}^{\alpha+q}-\lambda^{\alpha+q})x_n = 0 \quad \text{and} \quad \lim_{n\to\infty}(\mathbf{T}^{\alpha}-\mu^{\alpha})y_n = 0.$$

Since $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometric tuple such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j=1,\dots,d$, it follows that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\langle\Big(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\mathbf{T}^{q}x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\big(\mathbf{T}^{\alpha+q}-\lambda^{\alpha+q}+\lambda^{\alpha+q}\big)x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle\Big(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\lambda^{\alpha+q}\big)x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\mathbf{T}^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\big(\mathbf{T}^{\alpha}-\mu^{\alpha}+\mu^{\alpha}\big)y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\mu^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^{k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\big(\overline{\lambda}\mu\big)^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \Big(1-\sum_{1\le j\le d}\overline{\lambda_j}\mu_j\Big)^{m}y_n\rangle\\
&= \Big(1-\sum_{1\le j\le d}\lambda_j\overline{\mu_j}\Big)^{m}\lim_{n\to\infty}\langle x_n \mid y_n\rangle.
\end{aligned}$$

By the assumption $\sum_{1\le j\le d}\lambda_j\overline{\mu_j} \neq 1$, we obtain that $\lim_{n\to\infty}\langle x_n \mid y_n\rangle = 0$. □

4. The joint m-left inverse and joint m-right inverse of commuting tuples of operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a left invertible if there is an operator $S \in \mathcal{B}(\mathcal{H})$ such that $ST = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$, denotes the identity operator. The operator S is called a left inverse of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

a right invertible if there is an operator $R \in \mathcal{B}(\mathcal{H})$ such that $TR = I_{\mathcal{H}}$. The operator R is called a right inverse of T.

The left and right m-invertibility of operator have been introduced by the present author in [22] and by B. P. Duggal and V. Müller in [14].

Given a positive integer m. A bounded linear operator T is called a m-left invertible (resp. m-right invertible) if there exists a bounded linear operator S such that

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^kT^k = 0\Big(\text{resp.}\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^kS^k = 0\Big).$$

The m-invertibility have been extensively studied in the recent paper [19] by C. Gu. In [17] the author extends the notions of m-left and m-right invertibility respectively to the notions of m-left generalized inverse and m-right generalized inverse on Banach spaces.

The following definition generalize the definition of m-left invertibility and m-right invertibility of a single operator to tuples of commuting operators.

Definition 4.1. *Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators on $\mathcal{H}$,we say that $\mathbf{T}$ is a joint m-left invertible (resp. joint m-right invertible) operator for some integer $m \geq 1$, if there exists a commuting d-tuple of operators $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ such that*

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = 0$$

$$\Big(\textit{resp.}\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^\alpha\mathbf{S}^\alpha = 0\Big).$$

$\mathbf{S}$ is called a joint m-left (resp. m-right) inverse of $\mathbf{T}$.

We say that $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is m-invertible d-tuple of commuting operators if it has both a m-left inverse and a m-right inverse.

An interesting example of a m-left invertible commuting tuple of operators is that of an m-isometric tuple of operators.

Remark 4.1. It is clear that $\mathbf{S}$ is a m-left inverse of $\mathbf{T}$ if and only if $\mathbf{T}^*$ is a m-left inverse of $\mathbf{S}^*$.

Example 4.1. Let $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then the pair $\mathbf{T} = (T_1, T_1)$ is m-invertible tuple with m-inverse $\mathbf{S} = (S_1, S_2)$ in $\mathcal{B}(\mathbb{C}^2)^2$.

Remark 4.2. 1. If $\mathbf{S} = (S_1, S_2)$ and $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be commuting pairs of operators then $\mathbf{S}$ is a joint left inverse of $\mathbf{T}$ if $S_1T_1 + S_2T_2 = I$ and it is a joint 2-left inverse of $\mathbf{T}$ if

$$S_1^2T_1^2 + S_2^2T_2^2 + 2S_1S_2T_1T_2 - 2\big(S_1T_1 + S_2T_2\big) + I = 0. \tag{4.1}$$

2. $\mathbf{S} = (S_1, \ldots, S_d)$ is a joint left inverse (or joint 1-left inverse) of $\mathbf{T} = (T_1, T_2, ..., T_d)$ if

$$S_1T_1 + S_2T_2 + ... + S_dT_d = I_{\mathcal{H}} \tag{4.2}$$

and it is a joint 2-left inverse of $\mathbf{T}$ if

$$I_{\mathcal{H}} - 2\sum_{1\leq j\leq d} S_jT_j + \sum_{1\leq j\leq d} S_j^2T_j^2 + 2\sum_{1\leq j<k\leq d} S_jS_kT_jT_k = 0. \tag{4.3}$$

For $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$, set

$$\beta_m(\mathbf{S}, \mathbf{T}) = \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha.$$

Lemma 4.1. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be commuting d-tuples of operators, then the following identity holds for $m \in \mathbb{N}$:*

$$\beta_{m+1}(\mathbf{S}, \mathbf{T}) = -\beta_m(\mathbf{S}, \mathbf{T}) + \sum_{1\leq j\leq d} S_j\beta_m(\mathbf{S}, \mathbf{T})T_j.$$

Proof.

$$\beta_{m+1}(\mathbf{S}, \mathbf{T}) = (-1)^{m+1}I_{\mathcal{H}} + \sum_{1\leq k\leq m} (-1)^k\Big[\binom{m}{k} + \binom{m}{k-1}\Big] \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+ \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = -\sum_{0\leq k\leq m} (-1)^k\binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+ \sum_{0\leq k\leq m-1} (-1)^k\binom{m}{k} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$\sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$= -\beta_m(\mathbf{S}, \mathbf{T}) + \sum_{0\leq k\leq m-1} (-1)^k\binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!(\alpha_1 + \cdots + \alpha_d)}{\alpha_1!.\cdots.\alpha_d!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+ \sum_{|\alpha|=m+1} \frac{m!(\alpha_1 + \cdots + \alpha_d)}{\alpha_1!.\alpha_2....\alpha_d!}\mathbf{S}^\alpha\mathbf{T}^\alpha = -\beta_m(\mathbf{S}, \mathbf{T})$$

$$+ \sum_{1\leq j\leq d}\sum_{0\leq k\leq m-1} (-1)^k\binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!\alpha_j}{\alpha_1!.\alpha_2!\cdots\alpha_d!}$$

$$\Big(S_jS^{\alpha_1}\cdots S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}\cdots S_d^{\alpha_d}T^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_d^{\alpha_d}T_j\Big)$$

$$+ \sum_{1\leq j\leq d}\sum_{|\alpha|=m+1} \frac{m!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}S_jS^{\alpha_1}...S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}...S_d^{\alpha_d}T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j$$

$$
\begin{aligned}
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&+ \sum_{1\le j\le d}\sum_{|\alpha|=m}\frac{m!}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k1}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}S_j\beta_m(\mathbf{S},\mathbf{T})T_j.
\end{aligned}
$$

Which completed the proof. □

Remark 4.3. From Lemma 4.1, it is clear that if $\mathbf{S}$ is a joint m-left inverse (resp. m-right inverse) of $\mathbf{T}$, then $\mathbf{S}$ is an joint m-left inverse (resp. m-right inverse) of $\mathbf{T}$ for all integer $n \ge m$.

For $k, n \in \mathbb{N}$ denote the (descending Pochhammer) symbol by $n^{(k)}$, i.e.

$$
n^{(k)} = \begin{cases} 0, \text{ if } n = 0 \\ 0 \text{ if } n > 0 \text{ and } k > n \\ \binom{n}{k}k! \text{ if } n > 0 \text{ and } k \le n. \end{cases}
$$

Proposition 4.1. *Let $\mathbf{S} = (S_1, \dots, S_d) \in \mathcal{B}(\mathcal{H})^d$ and $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ are commuting operators. Then the following properties hold:*

1.

$$
\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \sum_{0\le k\le n}\frac{n^{(k)}}{k!}\beta_k(\mathbf{S},\mathbf{T}), \text{ for all } n = 0, 1, \dots.
$$

2. If $\mathbf{S}$ is a joint m-left inverse of $\mathbf{T}$, then

$$
\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \sum_{0\le k\le m-1}\frac{n^{(k)}}{k!}\beta_k(\mathbf{S},\mathbf{T}), \text{ for all } n = 0, 1, \dots.
$$

Proof. 1. We prove the statement by indication on n. For $n = 0, 1$ the statement is true. Suppose that the statement is true for n.

Form the identity

$$
\beta_{n+1}(\mathbf{S},\mathbf{T}) = \sum_{0\le k\le n+1}(-1)^{n+1-k}\binom{n+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha
$$

it follows that

$$
\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \beta_{n+1}(\mathbf{S},\mathbf{T}) - \sum_{0\le k\le n}(-1)^{n+1-k}\binom{n+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha.
$$

By the assumption and similar calculation as in [5] we obtained

$$\begin{aligned}
&\sum_{|\alpha|=n+1} \frac{(n+1)!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha} \\
&= \beta_{n+1}(\mathbf{S},\mathbf{T}) - \sum_{0\leq k\leq n}(-1)^{n+1-k}\binom{n+1}{k}\sum_{0\leq j\leq k}\frac{k^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T}) \\
&= \beta_{n+1}(\mathbf{S},\mathbf{T}) - \sum_{0\leq j\leq n}\beta_j(\mathbf{S},\mathbf{T})\Big(\sum_{j\leq k\leq n}(-1)^{n+1-k}\binom{n+1}{k}\frac{k^{(j)}}{j!}\Big) \\
&= \beta_{n+1}(\mathbf{S},\mathbf{T}) - \sum_{0\leq k\leq n}\frac{(n+1)^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T})\Big(\underbrace{\sum_{0\leq r\leq n-j}(-1)^{n+1-j-r}\binom{n+1-j}{r}}_{=-1}\Big) \\
&= \sum_{0\leq j\leq n+1}\frac{(n+1)^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T}).
\end{aligned}$$

2. The result follows immediately from the fact that if $\mathbf{S}$ is a m-left inverse of $\mathbf{T}$ then $\beta_k(\mathbf{S},\mathbf{T}) = 0$ for all $k \geq m$ (see Lemma 4.1). □

The following proposition is a generalization of [[22], Lemmas 3.1 and 3.2].

Proposition 4.2. *Let $\mathbf{T} = (T_1,\ldots,T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. The following statements hold.*

(1) *If $\mathbf{T}$ possesses a joint 2-left inverse $\mathbf{S} = (S_1,\ldots,S_d)$, then*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha} = (1-n)I_{\mathcal{H}} + n\Big(\sum_{1\leq j\leq d}S_jT_j\Big),\ \ \forall\, n\in\mathbb{N}. \tag{4.4}$$

(2) *If $\mathbf{T}$ possesses a joint 2-right inverse $\mathbf{R} = (R_1,\ldots,R_d)$, then*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{T}^{\alpha}\mathbf{R}^{\alpha} = (1-n)I_{\mathcal{H}} + n\Big(\sum_{1\leq j\leq d}T_jR_j\Big),\ \ \forall\, n\in\mathbb{N}. \tag{4.5}$$

Proof. We shall prove equality (4.4) by induction on n. For $n = 0$ or $n = 1$ it is clear. Assume that (4.4) is true for n and prove it for $n+1$. Indeed, a simple calculation shows that

$$\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha} = \sum_{\alpha_1+\cdots+\alpha_d=n+1}\frac{(n+1)n!}{\alpha_1!\cdots\alpha_d!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha}$$

$$
\begin{aligned}
&= \sum_{\alpha_1+\cdots+\alpha_d=n+1} \frac{(\alpha_1+\cdots+\alpha_d)n!}{\alpha_1!\cdots\alpha_d!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha} \\
&= \sum_{1\le k\le d}\Bigg(\sum_{\alpha_1+\cdots+\alpha_k-1+\cdots+\alpha_d=n} \frac{n!}{\alpha_1!\cdots(\alpha_k-1)!\cdots\alpha_d!} S_k S_1^{\alpha_1}\cdots S_k^{\alpha_k-1}\cdots S_d^{\alpha_d} \\
&\cdot\ T_1^{\alpha_1}\cdots T_k^{\alpha_k-1}\cdots T_d^{\alpha_d}T_k\Bigg) \\
&= \sum_{1\le k\le d} S_k\Bigg(\sum_{|\beta|=n}\frac{n!}{\beta!}\mathbf{S}^{\beta}\mathbf{T}^{\beta}\Bigg)T_k.
\end{aligned}
$$

Since $\mathbf{T}$ possesses a joint 2-left inverse tuple $\mathbf{S}$, it follow from the induction hypothesis and (4.3) that

$$
\begin{aligned}
&\sum_{|\beta|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha} \\
&= \sum_{1\le k\le d} S_k\Bigg(\sum_{|\beta|=n}\frac{n!}{\beta!}\mathbf{S}^{\beta}\mathbf{T}^{\beta}\Bigg)T_k \\
&= \sum_{1\le k\le d} S_k\Bigg((1-n)I_{\mathcal{H}} + n\sum_{1\le j\le d} S_jT_j\Bigg)T_k \\
&= (1-n)\sum_{1\le k\le d} S_kT_k + n\sum_{1\le j,\,k\le d} S_kS_jT_kT_j \\
&= (1-n)\sum_{1\le k\le d} S_kT_k + n\Bigg(\sum_{1\le k\le d} S_k^2T_k^2 + 2\sum_{1\le j<k\le d} S_jS_kT_jT_k\Bigg) \\
&= (1-n)\sum_{1\le k\le d} S_kT_k + n\Bigg(-I_{\mathcal{H}} + 2\sum_{1\le j\le d} S_jT_j\Bigg) \\
&= -nI_{\mathcal{H}} + (n+1)\Bigg(\sum_{1\le k\le d} S_kT_k\Bigg),
\end{aligned}
$$

so that (4.4) holds for $n+1$. Exchanging $\mathbf{S}=\mathbf{R}$ and $\mathbf{T}$ and similar to the above proof, we can prove that (4.5) holds.

□

Theorem 4.1. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple such that $\mathbf{T}$ possesses a joint m-left inverse $\mathbf{S}=(S_1,\ldots,S_d)\in\mathcal{B}(\mathcal{H})^d$, then the following statements hold:*

(1) $[0]\not\subset\sigma_{ap}(\mathbf{T})$,

(2) If $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_{ap}(\mathbf{T})$, then $1\in\sigma_{ap}(\sum_{1\le j\le d}\lambda_jS_j)$,

(3) *If $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_p(\mathbf{T})$, then $1\in\sigma_p(\sum_{1\le j\le d}\lambda_jS_j)$.*

Proof. (1) Suppose contrary to our claim that $[0] \subset \sigma_{ap}(\mathbf{T})$ and let $\lambda = (\lambda_1, ..., \lambda_d) \in [0]$. Then there exists a sequence $(x_n)_n \in \mathcal{H}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, 2, ..., d.$$

For $\alpha_j \geq 1$ we deduce that

$$(T_j^{\alpha_j} - \lambda_j^{\alpha_j})x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, 2, ..., d$$

which mean that $(\mathbf{T}^\alpha - \lambda^\alpha)x_n \longrightarrow 0$ and hence,$(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha)x_n \longrightarrow 0$.

Now, we get

$$\begin{aligned}
&\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0\\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0\\
&\Longrightarrow (-1)^m x_n + \sum_{1\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\prod_{1\leq j\leq d}\lambda_j^{\alpha_j}\Big)\mathbf{S}^\alpha x_n \longrightarrow 0\\
&\Longrightarrow x_n \longrightarrow 0 \text{ as } n \longrightarrow 0 \text{ (since } \lambda \in [0]),
\end{aligned}$$

which is impossible.

(2) Let $\lambda = (\lambda_1, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_n \in \mathcal{H}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, \ldots, d.$$

$$\begin{aligned}
&\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0\\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0\\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\prod_{1\leq j\leq d}\lambda_j^{\alpha_j}\Big)\mathbf{S}^\alpha x_n \longrightarrow 0\\
&\Longrightarrow \Big(I_{\mathcal{H}} - \sum_{1\leq j\leq d}\lambda_j S_j\Big)^m x_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

We deduce that $(I - \sum_{1\leq j\leq d}\lambda_j S_j)$ is not bounded from below. Consequently, $1 \in \sigma_{ap}(\sum_{1\leq j\leq d}\lambda_j S_j)$ as required.

(3) The argument is similar to one given in (2). This achieves the proof of the Theorem. □

The proof of the following theorem is similar to the proof of Theorem 4.1, so we omit it.

Theorem 4.2. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting tuple of operators such that* $\mathbf{T}$ *possesses a joint* m*-right inverse* $\mathbf{R} = (R_1, \ldots, R_d) \in \mathcal{B}(\mathcal{H})^d$*, then the following statements hold:*

(1) $[0] \not\subset \sigma_{ap}(\mathbf{R})$.
(2) *If* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{R})$*, then* $1 \in \sigma_{ap}(\sum_{1 \leq j \leq d} \lambda_j T_j)$.
(3) *If* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(\mathbf{R})$*, then* $1 \in \sigma_p(\sum_{1 \leq j \leq d} \lambda_j T_j)$.

Remark 4.4. When $d = 1$, the Theorem 4.2 is proved in ([22], Lemma 3.4).

Acknowledgments

The author would like to express his warm thanks to the referee for comments and many valuable suggestions.

References

[1] J. Agler and M. Stankus, *m-isometric transformations of Hilbert Space, I*, Integral Equations and Operator Theory, 21 (1995), 383-429.

[2] J. Agler, M. Stankus, *m-isometric transformations of Hilbert Space, II*, Integral Equations and Operator Theory, 23 (1) (1995), 1–48.

[3] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space, III*, Integral Equations and Operator Theory, 24 (4) (1996), 379–421.

[4] C. G. Ambrozie, M. Engliš and V. Müller, *Operator tuples and analytic models over general domains in* $\mathbb{C}^n$, J. Operator Theory, 47 (2002), 287–302.

[5] F. Bayart, *m-isometries on Banach spaces*, Math. Nachr., 284 (2011), 2141–2147.

[6] T. Bermúdez, A. Martinón, J. A. Noda, *Products of m-isometries*, Linear Algebra and its Applications, 438 (2013), 80–86.

[7] T. Bermúdez, A. Martinón, V. Müller and J. A. Noda, *Perturbation of m-isometries by nilpotent operators*, Abstr. Appl. Anal., vol.2014. Article ID 745479, 6 pages.

[8] T. Bermúdez, A. Martinón and E. Negrín, *Weighted shift operators which are m-isometries*, Integral Equations Operator Theory, 68 (2010), no. 3, 301– 312.

[9] M. Chõ, I.H. Jeon, I.B. Jung, J.I. Lee, K. Tanahashi, *Joint spectra of n-tuples of generalized Aluthge transformations*, Rev. Roumaine Math. Pures Appl., 46 (6) (2001), 725–730.

[10] M. Chō,V. Müller, *Spectral commutativity of multioperators*, Functional Analysis Approximation and Computation 4:1 (2012), 21-25.

[11] M. Chō and W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J., 21(2):251–258, 1992.

[12] M. Chō, I. H. Jeon, J. I. Lee, *Joint spectra of doubly commuting n-tuples of operators and their Aluthge transforms*, Nihonkai Math. J., 11 (1) (2000), 87–96.

[13] M. Chō, R. E. Curto, T. Huruya, *n-Tuples of operators satisfying* $\sigma_T(AB) = \sigma_T(BA)$, Linear Algebra Appl., 341 (2002), 291–298.

[14] B. P. Duggal and V. Müller, *Tensor product of left n-invertible operators*, Studia Math., 215 (2013), 113–125.

[15] B.P. Duggal, Tensor product of n-isometries, Linear Alg. Appl., 437(2012), 307-318.

[16] J. Gleason and S. Richter, *m-Isometric Commuting Tuples of Operators on a Hilbert Space*, Integral equation and operator theory, Vol. 56, No. 2 (2006), 181-196.

[17] H. Ezzahraoui, *On m-generalized invertible operators on Banach spaces*, Ann. Funct. Anal. Volume 7, Number, 4 (2016), 609–621.

[18] J. Gleason, *Quasinormality of Toeplitz with Analytic symbols*, Houston Journal of Mathematics. Volume 32, No. 1, (2006), 293–298.

[19] C. Gu, *Structures of left n-invertible operators and their applications*, Studia Mathematica, 226 (3), (2015).

[20] P. H. W. Hoffmann and M. Mackey, (m, p) *and* (m, ∞)*-isometric operator tuples on normed spaces*, Asian-Eur. J. Math., Vol. 8, No. 2 (2015).

[21] O. A. Mahmoud Sid Ahmed, *Generalization of m-Partial isometries on a Hilbert space*, International Journal of Pure and Applied Mathematics Volume 104 No. 4 (2015), 599–619.

[22] O. A. Mahmoud Sid Ahmed, *Some properties of m-isometries and m-invertible operators on Banach spaces*, Acta Math. Sci. Ser. B Engl. Ed., 32 (2012), 520–530.

[23] A. Saddi and O. A. Mahmoud Sid Ahmed, *m-partial isometries on Hilbert spaces*, Intern. J. Funct. Anal. Operators Theory Appl., 2 (2010), No. 1, 67–83.

[24] Z. Slodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, Studia Math., 50 (1974), 127-148.

[25] R. Soltani, B. Khani Robati, K. Hedayatian, *Hypercyclic tuples of the adjoint of the weighted composition operators*, Turk. J. Math., 36 (2012) , 452–462.

[26] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal., 6 (1970), 172-191.

Accepted: 5.01.2018

SEPARATION AXIOMS IN TOPOLOGICAL ORDERED SPACES

S. Shanthi
Department of Mathematics
Arignar Anna Govt. Arts College
Namakkal -637 001
Tamilnadu, India
shanthiwini2005@yahoo.co.in

N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India
nrajesh_topology@yahoo.co.in

Abstract. In this paper, we introduce and study some new type of separation axioms in topological ordered spaces via ω-open sets.

Keywords: ω-open sets, topological ordered space.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of semiopen sets introduced by Levine in 1963. A topological ordered space $(X, \tau, \leq)$ is a topological space (X, τ) equipped with a partial order $\leq$ (that is, reflexive, transitive and antisymetric). A subset A of $(X, \tau, \leq)$ is said to be increasing (resp. decreasing) if $A = \{x \in X : a \leq x \text{ for } a \in A\}$ (resp. $A = \{x \in X : x \leq a \text{ for some } a \in A\}$), that is, if $A = \bigcup_{a \in A} [a, \rightarrow]$ (resp. $A = \bigcup_{a \in A} [\rightarrow, a]$), where $[a, \rightarrow] = \{x \in X : a \leq x\}$ (resp. $[\rightarrow, a] = \{x \in X : x \leq a\}$). In this paper, we introduce and study some new type of separation axioms in topological ordered spaces via ω-open sets.

2. Preliminaries

For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively.

Definition 2.1. *A subset S of a topological space (X, τ) is said to be semiopen [3] if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$.*

Definition 2.2 ([9])**.** *A subset A of a topological space (X, τ) is said to be an ω-closed set if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in (X, τ). The complement of an ω-closed set is called an ω-open set. The family of all ω-open (ω-closed) subsets of (X, τ) is denoted by τ_ω (resp. $\omega C(X)$).*

Definition 2.3 ([10])**.** *The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior of A and is denoted by $\omega\,\mathrm{Int}(A)$.*

Definition 2.4 ([7])**.** *A subset $M(x)$ of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set S such that $x \in S \subset M(x)$.*

Definition 2.5 ([10])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-irresolute if $f^{-1}(U) \in \tau_\omega$ for every $U \in \sigma_\omega$.*

Definition 2.6. *A topological space (X, τ) is said to be:*

1. *ω-T_1 [6] if for every $x, y \in X$, $x \neq y$, there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \notin V$ and $y \in V$, $x \notin U$.*

2. *ω-T_2 [6] if for every $x, y \in X$, $x \neq y$, there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \notin V$ and $y \in V$, $x \notin U$ and $U \cap V = \emptyset$.*

3. *ω-regular [10] if for any closed set F in X and $a \in X \backslash F$, there exist disjoint ω-open sets U and V in X containing a and F, respectively.*

Definition 2.7 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) T_1-ordered if for each pair of elements $a \not\leq b$ (that is, a is not related to b) in X there exists a decreasing (resp. increasing) open set U containing b (resp. a) such that $a \notin U$ (resp. $b \notin U$). $(X, \tau, \leq)$ is said to be T_1-ordered if it is both lower and upper T_1-ordered.*

Definition 2.8 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be T_2-ordered if for each pair of elements $a \not\leq b$ in x there exist disjoint open sets U and V in X containing a and b respectively, U is increasing and V is decreasing.*

Definition 2.9 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) regularly ordered if for each increasing (resp. decreasing) closed set F in X and $a \notin F$ there exist disjoint open sets U and V containing a and F respectively, U is decreasing (resp. increasing) and V is increasing (resp. decreasing). $(X, \tau, \leq)$ is said to be regularly ordered if it is both lower and upper regularly ordered.*

3. On ω-T_1 and ω-T_2 ordered spaces

Definition 3.1. *Let $(X, \tau, \leq)$ be a topological ordered space and A a subset of X. Define:*
$I^{\omega}(A) = \cap\{F : F$ is an increasing ω-closed subset of X containing $A\}$,
$D^{\omega}(A) = \cap\{K : K$ is a decreasing ω-closed subset of X containing $A\}$,
$I^{o\omega}(A) = \cup\{G : G$ is an increasing ω-open subset of X contained in $A\}$,
$D^{o\omega}(A) = \cup\{H : H$ is an decreasing ω-open subset of X contained in $A\}$,
Cleraly, $I^{\omega}(A)$ (resp. $D^{\omega}(A)$) is the smallest increasing (resp. decreasing) ω-closed subset of X containing A and $I^{o\omega}(A)$ (resp.$D^{o\omega}(A)$) is the largest increasing (resp. decreasing) ω-open subset of X contained in A.

Proposition 3.2. *For any subset A of a topological ordered space $(X, \tau, \leq)$, the following hold:*

1. $X\backslash I^{\omega}(A) = D^{o\omega}(X\backslash A)$.
2. $X\backslash D^{\omega}(A) = I^{o\omega}(X\backslash A)$.
3. $X\backslash I^{o\omega}(A) = D^{\omega}(X\backslash A)$.
4. $X\backslash D^{o\omega}(A) = I^{\omega}(X\backslash A)$.

Proof. We shall prove (1) only, (2), (3) and (4) can be proved in a similar manner.

(1) Since $I^{\omega}(A)$ is an ω-closed increasing set containing A, $X\backslash I^{\omega}(A)$ is an ω-open decreasing set such that $X\backslash I^{\omega}(A) \subset X\backslash A$. Let U be an another ω-open decreasing set such that $U \subset X\backslash A$. Then $X\backslash U$ is an ω-closed increasing set such that $X\backslash U \supset A$. It follows that $I^{\omega}(A) \subset X\backslash U$. That is $U \subset X\backslash I^{\omega}(A)$. Thus, $X\backslash I^{\omega}(A)$ is the largest ω-open decreasing set such that $X\backslash I^{\omega}(A) \subset X\backslash A$. That is $X\backslash I^{\omega}(A) = D^{o\omega}(X\backslash A)$. □

Lemma 3.3. *Let $(X, \tau, \leq)$ be a topological ordered space and A a subset of X. Then $x \in I^{\omega}(A)$ (resp. $x \in D^{\omega}(A)$) if and only if for every decreasing (resp. increasing) ω-open subset U of X containing x, $U \cap A \neq \emptyset$.*

Proof. Let U be a decreasing ω-open subset of X containing x such that $U \cap A = \emptyset$. Then $X\backslash U$ is an increasing ω-closed subset of X containing A. Therefore, $I^{\omega}(A) \subset X\backslash U$. Thus $x \notin I^{\omega}(A)$. Conversely, if $x \notin I^{\omega}(A)$. Then $X\backslash I^{\omega}(A)$ is a decreasing ω-open subset of X containing x, but disjoint from A. The case of $D^{\omega}(A)$ can be deal similarly. □

Definition 3.4. *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) ω-T_1-ordered if for each pair of elements $a \not\leq b$ (that is, a is not related to b) in X there exists a decreasing (resp. increasing) ω-open set U containing b (resp. a) such that $a \notin U$ (resp. $b \notin U$). $(X, \tau, \leq)$ is said to be ω-T_1-ordered if it is both lower and upper ω-T_1-ordered.*

Clearly every T_1-ordered space is ω-T_1-ordered and every ω-T_1-ordered space is ω-T_1 but the converses are not true, in general.

Example 3.5. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_1 space but not T_1-ordered.

Example 3.6. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_1 space but not ω-T_1 ordered.

Theorem 3.7. *For a topological ordered space $(X, \tau, \leq)$, the following statements are equivalent:*

1. $(X, \tau, \leq)$ is lower (resp. upper) ω-T_1-ordered,
2. for each pair $a \not\leq b$ of X, there exists an ω-open set U containing a (resp. b) such that $x \not\leq b$ (resp. $a \not\leq x$) for all $x \in U$,
3. for each $x \in X$, $[\leftarrow, x]$ (resp. $[x, \rightarrow]$) is ω-closed,
4. when the net $\{x_\alpha\}_{\alpha \in A}$ ω-converges to a and $x_\alpha \leq b$ (resp. $b \leq x_\alpha$) for all $\alpha \in A$, then $a \leq b$ (resp. $b \leq a$).

Proof. We shall prove the theorem for lower ω-T_1-ordered spaces only. (1) $\Rightarrow$ (2): Let $a \not\leq b$. Then by hypothesis, there exists an increasing ω-open set U containing a such that $b \notin U$. If $x \in U$ and $x \leq b$, then $b \in U$, a contradiction.
(2) $\Rightarrow$ (3): Let $y \in X \backslash [\leftarrow, x]$. Then $y \not\leq x$. Then there exists an ω-open set U containing y such that $u \not\leq x$ for all $u \in U$. That is, $y \in U \subset X \backslash [\leftarrow, x]$. Hence $[\leftarrow, x]$ is ω-closed.
(3) $\Rightarrow$ (1): Obvious.
(1) $\Rightarrow$ (4): Let $\{x_\alpha\}_{\alpha \in A}$ be a net in X ω-converges to a and $x_\alpha \leq b$ for all $\alpha \in A$. If possible $a \not\leq b$, then by hypothesis, $X \backslash [\leftarrow, x]$ is an ω-open set containing a. Then there exists $\lambda \in A$ such that $x_\alpha \in X \backslash [\leftarrow, x]$ for all $\alpha \leq \lambda$. That is $x_\alpha \not\leq b$ for all $\alpha \geq \lambda$, which is a contradiction.
(4) $\Rightarrow$ (1): Let X be not lower ω-T_1-ordered. Then there exists a pair $a \not\leq b$ in X such that for every ω-open set U containing a, there exists $x_u \in U$ with $x_u \leq b$. Let $\mathcal{U}$ be the collection of all ω-open sets containing a, then $\mathcal{U}$ is directed by inclusion and the net $\{x_u\}_{u \in U}$ ω-converges to a. By hypothesis $a \leq b$, which is a contradiction. □

Corollary 3.8. *If $(X, \tau, \leq)$ is lower (upper) ω-T_1-ordered and $\tau \leq \tau^\star$, then $(X, \tau^\star, \leq)$ is also lower (upper) ω-T_1-ordered.*

Theorem 3.9. *A topological ordered space $(X, \tau, \leq)$ is ω-T_1-ordered if and only if for each $x \in X$, there exists an ω-T_1-ordered ω-open set in X which is both increasing and decreasing containing x.*

Proof. If X is ω-T_1-ordered, then X is the required set for each $x \in X$. Conversely, let $a \not\leq b$ in X. Then, by hypothesis, there exist ω-T_1-ordered ω-open sets U_1 and U_2 in X containing a and b, respectively, where U_1 and U_2 are both increasing and decreasing sets. If $b \notin U_1$ and $a \notin U_2$, then there is nothing to prove. But if $b \in U_1$ (resp. $a \in U_2$), then there exist ω-open sets V and W in U_1 (resp. U_2) containing a and b, respectively, V is increasing and W is decreasing in U_1 (resp. U_2) also $b \notin V$ and $a \notin W$. Since U_1 (resp. U_2) is both increasing and decreasing and it also an ω-open subset of X, V is an increasing and W is a decreasing ω-open subsets of X. Hence $(X, \tau, \leq)$ is ω-T_1-ordered. □

Theorem 3.10. *Let f be an order preserving (that is, $x \leq y$ in X if and only if $f(x) \leq^\star f(y)$ in $X^\star$) ω-irresolute function from a topological ordered space $(X, \tau, \leq)$ to a topological ordered space $(X^\star, \tau^\star, \leq^\star)$. If $(X^\star, \tau^\star, \leq^\star)$ is ω-T_1-ordered, then $(X, \tau, \leq)$ is also ω-T_1-ordered.*

Proof. Let $a \not\leq b$ in X. Then $f(a) \not\leq^\star f(b)$ in X. Then there exists an increasing ω-open set $U^\star$ in $X^\star$ containing $f(a)$ but $f(b) \notin U^\star$. Since f is order preserving and ω-irresolute, $f^{-1}(U^\star) = U$ is an increasing ω-open subset of X. Also, $a \in U$ and $b \notin U$. Hence $(X, \tau, \leq)$ is lower ω-T_1-ordered. Similarly, we can prove $(X, \tau, \leq)$ is upper ω-T_1-ordered. □

Definition 3.11. *A topological ordered space $(X, \tau, \leq)$ is called a C_ω-space if whenever F is an ω-closed subset of X. $i(F)$ and $d(F)$ are also ω-closed subsets of X, where $i(F) = \cup\{[x, \rightarrow] : x \in F\}$ and $d(F) = \cup\{[\leftarrow, x] : x \in F\}$.*

Definition 3.12 ([5])**.** *Let f be a function from $(X, \tau, \leq)$ onto $(X, \tau^\star, \leq^\star)$. Then $\leq^\star$ is called a quotient order of $\leq$ induced by f if $x^\star \leq^\star y^\star$ for $x^\star, y^\star \in X^\star$ if and only if there exist $x \in f^{-1}(x^\star)$, $y \in f^{-1}(y^\star)$ such that $x \leq y$.*

Theorem 3.13. *Suppose $(X, \tau, \leq)$ is a C_ω-space, ω-T_1-ordered space and f an ω-irresolute $\omega^\star$-closed function of $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$, where $\leq^\star$ is the quotient order induced by f. Then $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_1-ordered space.*

Proof. We will first show that $(X^\star, \tau^\star, \leq^\star)$ is a C_ω-space. Let $F^\star$ be an ω-closed subset of $X^\star$. Then $f^{-1}(F^\star)$ is an ω-closed subset of X. Then $i_{X^\star}(F^\star) = f(i_X(f^{-1}(F^\star)))$, $d_{X^\star}(F^\star) = f(d_X(f^{-1}(F^\star)))$. Hence $i_{X^\star}(F^\star)$ and $d_{X^\star}(F^\star)$ are ω-closed sets in $X^\star$ as f is ω-irresolute and ω-closed function. Now, to prove the theorem it is sufficient to show that every ω-T_1, C_ω-space is ω-T_1-ordered. If $(X, \tau, \leq)$ is an ω-T_1 C_ω-space, then $\{x\}$ is ω-closed for all $x \in X$. Since $(X, \tau, \leq)$ is an C_ω-space, $[x, \rightarrow]$ and $[\leftarrow, x]$ are ω-closed subsets of X for all x. Hence $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_1-ordered space. □

Definition 3.14. *A topological ordered space $(X, \tau, \leq)$ is said to be ω-T_2-ordered if for each pair of elements $a \not\leq b$ in x there exist disjoint ω-open sets U and V in X containing a and b, respectively, U is increasing and V is decreasing.*

Example 3.15. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_2 space but not T_2-ordered.

Example 3.16. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_2 space but not ω-T_2 ordered.

Theorem 3.17. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2-ordered if and only if for each $x \in X$, there exists an increasing (resp. decreasing) ω-clopen subset of X containing x which is ω-T_2-ordered.*

Proof. If X is ω-T_2-ordered, then X is the required increasing (resp. decreasing) ω-clopen subset of X for all $x \in X$. Conversely, let $x \not\leq y$ in X. By hypothesis, there exists an increasing ω-clopen set V in X containing x. If $y \in V$, then nothing to prove. If $y \notin V$. Then $X \backslash V$ is a decreasing ω-clopen subset of X containing y. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. Dually, we can prove the theorem for decreasing ω-clopen subsets of X. □

Theorem 3.18. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2-ordered if and only if for each pair of points $x \not\leq y$ in X, there exists an increasing (resp. decreasing) ω-irresolute function f of a space $(X, \tau, \leq)$ into a ω-T_2-ordered space $(X^\star, \tau^\star, \leq^\star)$ such that $f(x) \neq^\star f(y)$ (resp. $f(y) \not\leq^\star f(x)$).*

Proof. If $(X, \tau, \leq)$ is ω-T_2-ordered then the identity mapping is the required function. Conversely, let $x \not\leq y$ in X. By hypothesis, there exists an increasing ω-irresolute mapping f of a space X into an ω-T_2-ordered space $(X^\star, \tau^\star, \leq^\star)$ with $f(x) \neq^\star f(y)$. Therefore, there exist disjoint ω-open sets $U^\star$ and $V^\star$ in $X^\star$ containing $f(x)$ and $f(y)$, respectively, where $U^\star$ is increasing and $V^\star$ is decreasing. Since f is increasing ω-irresolute, $f^{-1}(U^\star)$ is an increasing ω-open set containing x and $f^{-1}(V^\star)$ is a decreasing ω-open set containing y. Also $f^{-1}(U^\star) \cap f^{-1}(V^\star) = \emptyset$. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. Analogously we can prove the theorem for decreasing functions. □

Theorem 3.19. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2 if and only if for each $x \in X$ the intersection of all increasing (resp. decreasing) ω-closed i-ω-neighbourhoods (resp. d-ω-neighbourhoods) of x is $[x^\star, \rightarrow]$ (resp. $[\leftarrow, x]$).*

Proof. Let $(X, \tau, \leq)$ be an ω-T_2-ordered space and $x \in X$. If $G^\star = \cap\{F : F$ is an increasing ω-closed i-ω-neighbourhood of $x\}$. Clearly, $[x^\star, \rightarrow] \subset G^\star$. Let $y \notin [x^\star, \rightarrow]$. Then $x \not\leq y$. Then there exist disjoint ω-open sets U and V containing x and y, respectively such that U is increasing and V is decreasing. Hence $x \in U \subset X \backslash V$. Therefore, $X \backslash V$ is an increasing ω-closed i-ω-neighbourhood of x and $y \notin X \backslash V$. Hence $G^\star = [x^\star, \rightarrow]$. Similarly, we can show that intersection of all decreasing ω-closed i-ω-neighbourhoods of x is $[\leftarrow, x]$. Conversely, let $x \leq y$ in X. Then $y \notin [x^\star, \rightarrow]$. By hypothesis, there exists an increasing ω-closed i-ω-neighbourhood F of x such that $y \notin F$. Then $y \in X \backslash F$, $X \backslash F$ is

a decreasing ω-open set. Also. there exists an increasing ω-open set U such that $x \in U \subset F$. Clearly, $U\ (X \backslash F) = \emptyset$. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. By dual argument we can prove that ω-T_2-ordered if intersection of all decreasing ω-closed d-ω-neighbourhoods of x is $[x^\star, \rightarrow]$ for all $x \in X$. □

Theorem 3.20. *Suppose $(X, \tau, \leq)$ is an ω-T_2-ordered space and f an one-to-one $\omega^\star$-open and $\omega^\star$-closed mapping of $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$, where $\leq^\star$ is the quotient order induced by f. Then $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_2-ordered.*

Proof. Let $x^\star \in X^\star$ and $\mathcal{F}^\star = \cap\{F^\star : F^\star$ is an increasing ω-closed i-ω-neighborhood of $x^\star\}$. Then $[x^\star, \rightarrow] \subset \mathcal{F}^\star$. Suppose $y^\star \notin [x^\star, \rightarrow]$, that is $x^\star \leq^\star y^\star$ does not hold. Thus $x \not\leq y$ for $x, y \in X$ such that $f(x) = x^\star$ and $f(y) = y^\star$. Therefore, there exists an increasing ω-closed i-ω-neighborhood F of x such that $y \notin F$. Since f is one to one $\omega^\star$-open and $\omega^\star$-closed, $f(F)$ is an increasing ω-closed i-ω-neighborhood of $x^\star$ not containing $y^\star$, that is $\mathcal{F}^\star = [x^\star, \rightarrow]$. Hence by Theorem 3.19 $(X^\star, \tau^\star, \leq^\star)$ is ω-T_2-ordered. □

4. Properties of ω-regularly ordered spaces

Definition 4.1. *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) ω-regularly ordered if for each increasing (resp. decreasing) closed set F in X and $a \notin F$ there exist disjoint ω-open sets U and V containing a and F respectively, U is decreasing (resp. increasing) and V is increasing (resp. decreasing). $(X, \tau, \leq)$ is said to be ω-regularly ordered if it is both lower and upper ω-regularly ordered.*

Remark 4.2. *Clearly, every regularly ordered space is ω-regularly ordered. However, the converse need not be true as can be true as can be seen from the following example.*

Example 4.3. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is ω-regularly ordered, but not regularly ordered.

Example 4.4. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $b \leq b, c \leq c$. Then $(X, \tau, \leq)$ is ω-regular but not ω-regularly ordered.

Theorem 4.5. *Every upper (resp. lower) ω-regularly ordered, upper (resp. lower) T_1-ordered space is ω-T_2-ordered.*

Proof. Let $(X, \tau, \leq)$ be an upper ω-regularly ordered, upper T_1-ordered space and $a \not\leq b$ in X. Then $[a, \rightarrow]$ is an increasing closed subset of X not containing b. Thus there exist disjoint ω-open sets U, V in X such that $[a, \rightarrow] \subset U$ and $b \in V, U$ is increasing and V is decreasing. Therefore, U and V are disjoint ω-open sets containing a and b respectively, U is increasing and V is decreasing. Hence, $(X, \tau, \leq)$ is ω-T_2-ordered.

Similarly, every lower ω-regularly ordered, lower T_1-ordered space is ω-T_2-ordered. □

Definition 4.6. *A subset A of topological ordered space $(X, \tau, \leq)$ is said to be i-ω-neighborhood (resp. d-ω-neighborhood) of $B \subset X$, if there exists an increasing (resp. decreasing) ω-open set G in X such that $B \subset G \subset A$.*

Theorem 4.7. *For a topological ordered space $(X, \tau, \leq)$, the following are equivalent:*

1. *$(X, \tau, \leq)$ is lower (resp. upper) ω-regularly ordered.*
2. *For each $x \in X$ and each increasing (resp. decreasing) open set U in X containing x, there exists an increasing (resp. decreasing) ω-open set V such that $x \in V \subset I^{\omega}(V) \subset U$ (resp. $x \in V \subset D^{\omega}(V) \subset U$).*
3. *For every decreasing (resp. increasing) closed set F, the intersection of all decreasing (resp. increasing) ω-closed d-ω-neighborhoods (resp. i-ω-neighborhoods) of F is exactly F.*
4. *For every nonempty set A and an increasing (resp. decreasing) open subset B of X such that $A \cap B \neq \emptyset$, there exists an increasing (resp. decreasing) ω-open set V in X such that $A \cap V \neq \emptyset$ and $I^{\omega}(V) \subset B$ (resp. $D^{\omega}(V) \subset B$).*
5. *For every nonempty set A and a decreasing (resp. increasing) closed set B such that $A \cap B = \emptyset$, there exist disjoint ω-open sets U and V with $A \cap U \neq \emptyset$, U is increasing (resp. decreasing) and $B \subset V$, V is decreasing (resp. increasing) in X.*
6. *When a net $\{X_\alpha\}_{\alpha \in \Lambda}$ is residually in each increasing (resp. decreasing) ω-open set containing a and a net $\{y_\alpha\}_{\alpha \in \Lambda}$ is residually contained in each decreasing (resp. increasing) ω-open set containing a decreasing (resp. increasing) closed set F, $x_\alpha \leq y_\alpha$ for $\alpha \in \Lambda$, then $a \in F$.*

Proof. We shall prove the theorem for lower ω-regularly ordered spaces only.

(1)⇒(2): Let U be an increasing open subset of X containing x. Then, $X \backslash U$ is a decreasing closed subset of X such that $x \notin X \backslash U$. Thus there exist disjoint ω-open sets V and W containing x and $X \backslash U$ respectively, V is increasing and W is decreasing in X. Since, $V \subset X \backslash W, X \backslash W$ is increasing ω-closed, therefore, $I^{\omega}(V) \subset X \backslash W \subset U$. Hence $x \in V \subset I^{\omega}(V) \subset X \backslash W \subset U$.

(2)⇒(3): Let F be a decreasing closed subset of X and $F^{\star} = \cap\{K :: \text{is a decreasing } \omega\text{-closed, } d\text{-}\omega\text{-neighborhood of } F\}$, then $F \subset F^{\star}$. Let $x \notin F$. Then $x \in X \backslash F, X \backslash F$ is an increasing ω-open set. Thus, there exists an increasing ω-open set U such that $x \notin U \subset I^{\omega}(U) \subset X \backslash F$. Therefore $F \subset X \backslash I^{\omega}(U) \subset X \backslash U$, which implies that $X \backslash U$ is a decreasing ω-closed d-ω-neighborhood of F not containing x. That is $x \notin F^{\star}$. Hence $F = F^{\star}$.

(3)⇒(4): Let A be a nonempty set of B an increasing open subset of X such that $A \cap B \neq \emptyset$. Then, there exists $x \in X$ such that $x \in A \cap B$. By hypothesis, there exists a decreasing ω-closed d-ω-neighborhood K of $X\backslash B$ such that $x \notin K$. Define, $U = X\backslash K$. Then U is an increasing ω-open set such that $U \cap A \neq \emptyset$. Since K is a d-ω-neighborhood of $X\backslash B$, there exists a decreasing ω-open subset W of X such that $X\backslash B \subset W \subset K$. That is, $U = X\backslash K \subset X\backslash W \subset B$. Then $I^{\omega}(U) \subset X\backslash W \subset B$.

(4)⇒(5): Let A be a nonempty set and let B a decreasing closed subset of X such that $A \cap B = \emptyset$. Then, $X\backslash B$ is an increasing open subset of X such that $A \cap (X\backslash B) \neq \emptyset$. Thus by hypothesis, there exists an incresing ω-open set U in X such that $A \cap U \neq \emptyset$ and $I^{\omega}(U) \subset X\backslash B$. Define $X\backslash I^{\omega}(U) = V$. Then, V is a decreasing ω-open subset of X containing B and disjoint from U.

(5)⇒(6): Let a and F be as given in (5) of the theorem with $a \notin F$. Then by hypothesis, there exist disjoint ω-open sets U, V in X such that $a \in U$, U is increasing and $F \subset V$, V is decreasing in X. Thus there exists $\lambda \in \Lambda$ such that $x_\alpha \in U$ and $y_\alpha \in V$ for all $\alpha \geq \lambda$. Then, $x_\alpha \not\leq y_\alpha$ for all $\alpha \geq \lambda$, otherwise $U \cap V \neq \emptyset$, which is contradiction. Hence $a \in F$.

(6)⇒(1): Suppose $(X, \tau, \leq)$ is not lower ω-regularly ordered. Then, there exists a decreasing closed set F in X and $x \in X\backslash F$ such that every increasing ω-open set containing x intersects every decreasing ω-open set containing F. Let $\mathcal{U}$ denote the family of all increasing ω-open subsets of X containing x and $\mathcal{V}$ be the family of all decreasing ω-open subsets of X containing F. Then both $\mathcal{U}$ and $\mathcal{V}$ are ordered and directed by inclusion. The product $\mathcal{U} \times \mathcal{V}$ may be ordered by agreeing that $(U_1, V_1) \leq (U_2, V_2)$ if and only if $U_2 \subset U_1$ and $V_2 \subset V_1$. For each $(U, V) \in \mathcal{U} \times \mathcal{V}$, an element $x_{(U,V)}$ may be selected in $U \cap V$. The net $\{x_{(U,V)}\}_{(U,V)\in\mathcal{U}\times\mathcal{V}}$ is residually contained in each increasing ω-open subset of X containing x and in each decreasing ω-open subset of X containing F. Hence (6) does not hold. □

Definition 4.8 ([4]). *Let $(X, \tau, \leq)$ be a topological ordered space and $Y \subset X$. then, $(Y, \tau_y, \leq_y)$ with the induced order and induced topology is said to be τ-compatibly ordered if and only if for each τ_y-closed set F, increasing (resp. decreasing) in Y, there exists a τ-closed set $F^\star$, increasing (resp. decreasing) in X, such that $F = F^\star \cap Y$.*

Definition 4.9 ([1]). *Let F be a mapping of $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$. Then f is called dual isotomic if $f(x) \leq^\star f(y)$ implies $x \leq y$.*

Definition 4.10. *A mapping $f : (X, \tau, \leq) \to (X^\star, \tau^\star, \leq^\star)$ is called weakly order continuous if for each increasing (resp. decreasing τ-open, $\tau^\star$-closed set $U^\star$, $f^{-1}(U^\star)$ is increasing (resp. decreasing) τ-open, τ-closed, respectively in X.*

Theorem 4.11. *Let $(X, \tau, \leq)$ be an ω-regularly ordered space and f a dual isotonic, $\omega^\star$-open and weakly order continuous mapping from $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$. Then $(X^\star, \tau^\star, \leq^\star)$, is also ω-regularly ordered.*

Proof. Let $F^\star$ be an increasing closed subset of $X^\star$ and $a \notin F^\star$. then $f^{-1}(F^\star)$ is an increasing closed subset of X and $f^{-1}(a) \notin f^{-1}(F^\star)$. Thus by hypothesis, there exist disjoint ω-open sets U and V such that $f^{-1}(a) \in U$, U is decreasing and $f^{-1}(F^\star) \subset V$, V is increasing. Since f is dual isotonic and $\omega^\star$-open, $f(U) = U^\star$ and $f(V) = V^\star$ are disjoint ω-open subset of $X^\star$ such that $a \in U^\star$. $U^\star$ is decreasing and $F^\star \subset V$, $V^\star$ is increasing. Hence, $(X^\star, \tau^\star, \leq^\star)$ is upper ω-regularly ordered. Similarly $(X^\star, \tau^\star, \leq^\star)$ is lower ω-regularly ordered. □

References

[1] S. P. Arya and Kavita Gupta, *New separation axioms in topological ordered spaces*, Indian J. Pure Appl. Math., 22 (1991), 461-468.

[2] D. C. J. Burgess and S. D. Mccartan, *Order-continuous functions and order-connected spaces,* Proc. Camb. Phil. Soc., 38 (1970), 27-31.

[3] N. Levine, *Semiopen sets and semicontinuity in topological spaces*, Amer. Math. Monthly, 70 (1963), 36-41.

[4] S. D. Mccartan, *Separation axioms for topological ordered spaces*, Proc. Camb. Phil. Soc., 64 (1961), 965-973.

[5] T. Miwa, *On images of topological ordered spaces under some quotient mappings*, Math. J. Okayama Univ, 18 (1975/76), 99-104.

[6] H. Maki and N. Rajesh, *Characterizations of ω-like closed sets and separation axioms in topological spaces* (under preparation).

[7] L. Nachbin, *Topology and order*, D. Van. Nostrand Inc, pinceton, New Jeresey, 1965.

[8] Shanthi Leela and G. Balasubramanian, *New separation axioms in ordered topological spaces*, Indian J. Pure Appl. Math., 33(7) (2002), 1011-1016.

[9] P. Sundaram and M. Sheik John, *Weakly closed sets and weakly continuous maps in topological spaces*, Proc. 82nd session of the Indian Science Congress, Calcutta, 1995 (P-49).

[10] M. Sheik John, *A study on generalizations of closed sets and continuous maps in topological spaces*, Ph. D. Thesis, Bharathiyar University, Coimbatore, India (2002).

Accepted: 21.01.2018

CONFORMAL ANTI-INVARIANT SUBMERSIONS FROM KENMOTSU MANIFOLDS ONTO RIEMANNIAN MANIFOLDS

Sushil Kumar*
Department of Mathematics and Astronomy
University of Lucknow
Lucknow-India
sushilmath20@gmail.com

Rajendra Prasad
Department of Mathematics and Astronomy
University of Lucknow
Lucknow
India
rp.manpur@rediffmail.com

Abstract. In this paper, we define conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds. Further we obtain some results on such submersions from Kenmotsu manifolds into Riemannian manifolds admitting vertical or horizontal structural vector fields. Among the results we find necessary and sufficient conditions for conformal anti-invariant submersions to be totally geodesic. Moreover, we derive decomposition theorems by using the existence of conformal anti-invariant submersions. Finally, we give some examples of conformal anti-invariant submersions such that characteristic vector field ξ is horizontal or vertical.

Keywords: Riemannian submersion, conformal submersion, anti-invariant submersion, conformal anti-invariant submersion.

1. Introduction

The theory of Riemannian submersion between Riemannian manifolds was introduced by O'Neill [13] in 1966 and Gray [9] in 1967. It was useful if one should study such submersions between manifolds with differentiable structures. When Watson was studying almost Hermitian submersions between almost Hermitian manifolds [16] in 1976, he found that the base manifold and each fiber have the same kind of structure as the total space in most of the cases. We note that almost Hermitian submersions have been extended to the almost contact manifolds [6] in 1985. We know that Riemannian submersions are related with mathematical physics and have their applications in the Yang-Mills theory [15], supergravity and superstring theories ([12], [13]), Kaluza-Klein theory ([5], [11]) etc.

*. Corresponding author

On the other hand, as a generalization of Riemannian submersion, horizontally conformal submersion are introduced [3] and defined as follows: Consider two Riemannian manifolds M of dimension m and N of dimension n with Riemannian metrics g_M and g_N respectivly. A smooth map $f : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(U, V) = g_N(f_*U, f_*V), \tag{1.1}$$

for every $U, V \in \Gamma(\ker f_*)^\perp$. It is known that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Let f is a smooth map between Riemannian manifolds and $x \in M$. Then, f is called horizontally weakly conformal map at x if either (i) $f_{*x} = 0$ or (ii) f_{*x} maps the horizontal space $\mathcal{H} = (\ker f_*)^\perp$ conformally onto $T_{f(x)}N$, *i.e.*, f_{*x} is surjective and f_* satisfies the equation (1.1) for U, V vectors tangent to $\mathcal{H}_x$. We call the point x a critical point if it satisfies the type (i) and we call the point x regular point if it satisfies the type (ii). The square root $\lambda(x) = \sqrt{\wedge(x)}$ is called dilation, where number $\wedge(x)$ is called the square dilation which is necessarily non-negative . If horizontally weakly conformal map f is said to be horizontally homothetic, then the gradient of their dilation λ is vertical, i.e., $\mathcal{H}(grad\lambda) = 0$ at regular points. A horizontally weakly conformal map f is called horizontally conformal submersion if f has no critical points [3]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. The horizontal conformal maps were introduced independently by Fuglede in 1978 [8] and by Ishihara in 1979 [12]. From the above argument, one can determine that the notion of horizontal conformal maps is a generalization of the idea of Riemannian submersions.

We denote the kernel space of f_* by $\ker f_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker f_*)^\perp$ to $\ker f_*$. Then the tangent bundle of M has the following decomposition

$$TM = (\ker f_*) \oplus (\ker f_*)^\perp. \tag{1.2}$$

We also denote the *range* of f_* by $rangef_*$ and consider the orthogonal complementary space $(rangef_*)^\perp$ to $rangef_*$ in the tangent bundle TN of N. Thus the tangent bundle TN of N has the following decomposition

$$TN = (rangef_*) \oplus (rangef_*)^\perp. \tag{1.3}$$

We know that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties. The concept of conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds was studied by Akyol and Sahin [1].

In this paper, we study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds. The paper is organized as follows. In the second section, we collect main notions and formulae which need for this paper.

In section 3, we introduce conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds, investigates the geometry of leaves of the horizontal distribution and the vertical distribution. In this section we also find necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic.

In section 4, we consider conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is horizontal vector field. Finally in section 5, we give some examples of conformal anti-invariant submersions such that the characteristic vector field ξ is vertical or horizontal.

2. Preliminaries

Let $\widetilde{M}$ be an almost contact metric manifold. So there exist on $\widetilde{M}$, a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and g is Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \eta(\xi) = 1, \tag{2.2}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for any vector fields X and Y on $\widetilde{M}$ and I is the identity tensor field [2]. An almost contact metric manifold $\widetilde{M}$ is also denoted by $(\widetilde{M}, \phi, \xi, \eta, g)$.

An almost contact metric manifold $\widetilde{M}$ is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for X and Y on $\widetilde{M}$, where ∇ is the Riemannian connection of the Riemannian metric g. From above equation, we have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Definition 1 ([3]). *Let M and N are two Riemannian manifolds with Riemannian metrics g_M and g_N, respectively. If f is a differentiable map from (M, g_M) to (N, g_N), then f is called semi-conformal or horizontally weakly conformal at p if either*

(i) $df_p = 0$, or

(ii) df_p maps the horizontal space $\mathcal{H}_p = (\ker(df_p))^{\perp}$ conformally onto $T_{f(p)}N$ i.e., df_p is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_N(df_*U, df_*V) = \Lambda(p) g_M(U, V), \qquad (U, V \in \mathcal{H}_p), \tag{2.6}$$

where $p \in M$.

Watson introduced the fundamental tensors of a submersion in [13]. It is known that the fundamental tensor play similar role to that of the second fundamental form of an immersion. Defined O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.7}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.8}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [7], and ∇ is Levi-Civita connection on M. On the other hand, from equations (2.7) and (2.8), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.9}$$

$$\nabla_X U = \mathcal{H}\nabla_X U + \mathcal{T}_X U, \tag{2.10}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{2.11}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{2.12}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If U is basic, then $\mathcal{A}_X U = \mathcal{H}\nabla_X U$.

It is seen that for $p \in M$, $X \in \mathcal{V}_p$ and $U \in \mathcal{H}_p$ the linear operators $\mathcal{A}_U$, $\mathcal{T}_X : T_pM \to T_pM$, are skew-symmetric, that is

$$g(\mathcal{A}_U E, F) = -g(E, \mathcal{A}_U F) \text{ and } g(\mathcal{T}_X E, F) = -g(E, \mathcal{T}_X F), \tag{2.13}$$

for each $E, F \in T_pM$. We have also defined the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{\mathcal{V}\times\mathcal{V}}$ is precisely the second fundamental form of the fibers of f. Since $\mathcal{T}_V$ is skew-symmetric we get: f has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$. For the special case when f is horizontally conformal we have the following:

Proposition 1 ([10], (2.1.2))**.** *Let f be a horizontal conformal submersion between Riemannian manifolds (M, g_M) and (N, g_N) with dilation λ and U, V be horizontal vectors, then*

$$\mathcal{A}_U V = \frac{1}{2}\{\mathcal{V}[U, V] - \lambda^2 g_M(U, V) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.14}$$

We know that the skew-symmetric part of $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

Let $f : (M, g_M) \to (N, g_N)$ is a smooth map between Riemannian manifolds. Then the differential of f_* of f can be observed a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the pullback bundle which has fibers $(f^{-1}TN)_p = T_{f(p)}N$ has a connection ∇ induced from the Riemannian connection and ∇^M the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(U, V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V), \tag{2.15}$$

for vector fields $U, V \in \Gamma(TM)$, where ∇^f is the pullback connection. We know that the second fundamental form is symmetric.

Next, we find necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recall that a differentiable map f between two Riemannian manifolds is called totally geodesic if $(\nabla f_*)(V, W) = 0$, for all $V, W \in \Gamma(TM)$.

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

We know that the followings from [14]. Let $B = M \times N$ be a manifold with Riemannian metric g_B and assume that the canonical foliations D_M and D_N intersect perpendicularly everywhere. Then g_B is the metric tensor of

(i) a twisted product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is totally umbilical foliation,

(ii) a warped product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel,

We note in this case, from [4] we have $\nabla_X U = X(\ln F)U$, for $X \in \Gamma(TM)$ and $U \in \Gamma(TN)$, where ∇ is the Riemannian connection on $M \times N$.

(iii) a usual product of Riemannian manifolds if and only if D_M and D_N are totally geodesic foliations.

Next, we explain a decomposition theorem related to the concept of twisted product manifold. However, we first define the adjoint map of a map. Let $f : (M, g_M) \to (N, g_N)$ be a map between Riemannian manifolds (M, g_M) and (N, g_N). Then the adjoint map ${}^*f_*$ of f_* is characterized $g_M(X, {}^*f_{*p}Y) = g_N(f_{*p}X, Y)$ by $X \in T_pM$, $Y \in T_{f(p)}N$ and $p \in M$. Considering f^h_* at each $p \in M$ as a linear transformation

$$f^h_{*p} : ((\ker f_*)^{\perp}_{(p)}, g_{M(p)((\ker f_*)^{\perp}_p)}) \to (rangef_{*(q)}, g_{N(q)(rangef_*)(q)}),$$

we will denote the adjoint $f^h_{*(p)}$ by ${}^*f^h_{*(p)}$. Let $f^h_{*(p)}$ be the adjoint of $f^h_{*(p)} : (T_pM, g_{M(p)}) \to (T_{(q)}N, g_{N(q)})$. The linear transformation $({}^*f_{*p})^h : (rangef_{*(p)}) \to (\ker f_*)^{\perp}_{(p)}$ defined $({}^*f_{*(p)})^h Y =^* f^h_{*(p)} Y$, where $Y \in (ranrgef_{*(p)}), q = f(p)$, is an isomorphism and $(f^h_{*(p)})^{-1} = ({}^*f_{*p})^h =^* f^h_{*(p)}$.

Lastly, we recollection the subsequent lemma from [3].

Lemma 1. *Let (M, g_M) and (N, g_N) are two Riemannian manifolds. If $f : M \to N$ horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields U, V and vertical vector fields X, Y, we have*

$$\begin{aligned}(i)\nabla df(U,V) &= U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U,V)df(gradln\lambda);\\ (ii)\nabla df(X,Y) &= -df(\mathcal{A}^{\mathcal{V}}_X Y);\\ (iii)\nabla df(U,X) &= -df(\nabla^M_U X) = df((\mathcal{A}^{\mathcal{H}})^*_U X).\end{aligned}$$

where $(\mathcal{A}^{\mathcal{H}})^*_X$ is the adjoint of $(\mathcal{A}^{\mathcal{H}}_X)$ characterized by $\langle(\mathcal{A}^{\mathcal{H}})^*_U E, F\rangle = \langle E, \mathcal{A}^{\mathcal{H}}_U F\rangle$, $(E, F \in \Gamma(TM))$.

3. Conformal anti-invariant submersions

In this section we are going to introduce and study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is vertical vector field.

Definition 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. A horizontally conformal submersion $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ with dilation λ is a called conformal anti-invariant submersion if the distribution $\ker f_*$ is anti-invariant with respect to ϕ i.e., $\phi(\ker f_*) \subseteq (\ker f_*)^{\perp}$. We have $\phi(\ker f_*)^{\perp} \cap \ker f_* \neq \{0\}$. We denote the complementary orthonormal distribution to $\phi(\ker f_*)$ of μ in $(\ker f_*)^{\perp}$. Then we have*

$$(\ker f_*)^{\perp} = \phi(\ker f_*) \oplus \mu. \tag{3.1}$$

For any $U \in \Gamma(\ker f_*)^{\perp}$, we have

$$\phi U = BU + CU, \tag{3.2}$$

where $BU \in \Gamma(\ker f_*)$ and $CU \in \Gamma(\mu)$. On the additional fact, since $f_*(\Gamma(\ker f_*)^{\perp}) = TN$ and f is a conformal submersion, for every $X \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equation (3.2) we get $\frac{1}{\lambda^2} g_N(f_*\phi U, f_* CX) = 0$, which denotes that

$$TN = f_*(\phi(\ker f_*)) \oplus f_*(\mu). \tag{3.3}$$

Lemma 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then*

$$g_M(CV, \phi X) = 0, \tag{3.4}$$

and

$$g_M(\nabla^M_U CV, \phi X) = -g_M(CV, \phi \mathcal{A}_U X) + \eta(X) g_M(CU, CV), \tag{3.5}$$

for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, since $BV \in \Gamma(\ker f_*)$ and $\phi X \in (\Gamma(\ker f_*)^{\perp})$, using equations (3.2) and (2.3), we have $g_M(CV, \phi X) = 0$. Now, using equations (2.3), (2.11) and (3.4), we get $g_M(\nabla_U CV, \phi X) = -g_M(CV, \nabla_U \phi X), = -g_M(CV, \phi \mathcal{A}_U X) + \eta(X) g_M(CU, CV)$, since $\phi \mathcal{V} \nabla_U X \in \Gamma(\phi(\ker f_*))$. □

Theorem 1. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ *is integrable,*

(ii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$
$- g_M(\mathcal{H}gradln\lambda, CV) g_M(U, \phi X) + g_M(\mathcal{H}gradln\lambda, CU) g_M(V, \phi X)$
$- 2 g_M(\mathcal{H}gradln\lambda, \phi X) g_M(CU, V)$, *for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.*

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, since $\phi V \in (\Gamma \ker f_* \oplus \mu)$ and $\phi X \in (\Gamma(\ker f_*)^{\perp})$. Using equations (2.1), (2.3), (2.4) and (3.2), we get

$$\begin{aligned} g_M([U, V], X) &= g_M(\phi \nabla_U V, \phi X) + \eta(X) \eta(\nabla_U V) \\ &\quad - g_M(\phi \nabla_V U, \phi X) - \eta(X) \eta(\nabla_V U), \\ &= g_M(\nabla_U \phi V, \phi X) - g_M(\nabla_V \phi U, \phi X) - g_M([U, V], \xi) \eta(X), \\ &= g_M(\nabla_U BV, \phi X) + g_M(\nabla_U CV, \phi X) - g_M(\nabla_V BU, \phi X) \\ &\quad - g_M(\nabla_V CU, \phi X) - g_M([U, V], \xi) \eta(X). \end{aligned}$$

Since f is a conformal submersion, using equations (2.11) and (2.12), we get

$$\begin{aligned} g_M([U, V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \frac{1}{\lambda^2} g_N(f_* \nabla_U CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(f_* \nabla_V CU, f_* \phi X) - g_M([U, V], \xi) \eta(X). \end{aligned}$$

Using equations (2.5), (2.15), (3.4) and lemma 1(i), we get

$$\begin{aligned} g_M([U, V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H}gradln\lambda, CV) g_M(U, \phi X) \\ &\quad + g_M(\mathcal{H}gradln\lambda, CU) g_M(V, \phi X) \\ &\quad - 2 g_M(\mathcal{H}gradln\lambda, \phi X) g_M(CU, V) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. □

Theorem 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^{\perp}$ is integrable,

(ii) f is horizontally homothetic,

(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using Theorem (1), we get

$$\begin{aligned} g_M([U,V],X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U,\phi X) g_M(\mathcal{H}gradln\lambda, CV) \\ &\quad + g_M(V,\phi X) g_M(\mathcal{H}gradln\lambda, CU) - 2g_M(CU,V) g_M(\mathcal{H}gradln\lambda, \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X). \end{aligned}$$

Now, using conditions (i) and (ii), we get (iii)

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X).$$

Similarly, one can obtain the other assertions. □

Remark 1. Let f be a conformal anti-invariant submersion is conformal Lagrangian submersion, if $\phi(\ker f_*) = (\ker f_*)^{\perp}$. Then equation (3.3), we have $TN = f_*(\phi(\ker f_*)^{\perp})$.

Corollary 1. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ is integrable,

(ii) $\mathcal{A}_U \phi V = \mathcal{A}_V \phi U$,

(iii) $(\nabla f_*)(V, \phi U) = (\nabla f_*)(U, \phi V)$, for $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, since $\phi X \in (\Gamma(\ker f_*)^{\perp})$ and $\phi V \in \Gamma(\phi(\ker f_*))$. From Theorem (1), we have

$$\begin{aligned} g_M([U,V],X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H}gradln\lambda, CV) g_M(U,\phi X) \\ &\quad + g_M(\mathcal{H}gradln\lambda, CU) g_M(V,\phi X) - 2g_M(\mathcal{H}gradln\lambda, \phi X) g_M(CU,V) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X). \end{aligned}$$

Since f conformal Lagrangian submersion, we have

$$g_M([U,V],X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X),$$

which implies $(i) \Leftrightarrow (ii)$. On the further using definition (2) and equation (2.11), we get

$$\begin{aligned} g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{A}_V BU, \phi X), \\ &= \frac{1}{\lambda^2} g_N(f_*\mathcal{A}_U BV, f_*\phi X) - \frac{1}{\lambda^2} g_N(f_*\mathcal{A}_V BU, f_*\phi X), \\ &= \frac{1}{\lambda^2} g_N(f_*(\nabla_U BV), f_*\phi X) - \frac{1}{\lambda^2} g_N(f_*(\nabla_V BU), f_*\phi X). \end{aligned}$$

Now, using equation (2.15) we have

$$\begin{aligned}
&\frac{1}{\lambda^2}g_N(f_*(\nabla_U BV), f_*\phi X) - \frac{1}{\lambda^2}g_N(f_*(\nabla_V BU), f_*\phi X)\\
&= \frac{1}{\lambda^2}g_N(-(\nabla f_*)(U, BV) + \nabla^f_U f_* BV, f_*\phi X)\\
&- \frac{1}{\lambda^2}g_N(-(\nabla f_*)(V, BU) + \nabla^f_V f_* BU, f_*\phi X),\\
&= \frac{1}{\lambda^2}[g_N((\nabla f_*)(V, BU) - (\nabla f_*)(U, BV), f_*\phi X)],
\end{aligned}$$

which proves that $(ii) \Leftrightarrow (iii)$. □

Theorem 3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $-\frac{1}{\lambda^2}g_N(\nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV, \phi X)$
$-g_M(U, \phi X)g_M(\mathcal{H}gradln\lambda, CV) + g_M(U, CV)g_M(\mathcal{H}gradln\lambda, \phi X)$
$-\eta(X)g_M(U, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4), (2.11), (2.12) and (3.2), we have

$$\begin{aligned}
g_M(\nabla_U V, X) &= g_M(\nabla_U \phi V, \phi X) + \eta(X)\eta(\nabla_U V),\\
&= g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X) + \eta(X)\eta(\nabla_U V).
\end{aligned}$$

Since f is conformal submersion, using equation (2.15), lemma $1(i)$, definition (2) and equation (3.4), we get

$$\begin{aligned}
g_M(\nabla_U V, X) = {} & g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U, \phi X)\\
& - \eta(X)g_M(U, V) + g_M(\mathcal{H}gradln\lambda, \phi X)g_M(U, CV)\\
& + \frac{1}{\lambda^2}g_N(\nabla^f_U f_* CV, f_*\phi X),
\end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. □

Theorem 4. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) f is horizontally homothetic,
(iii) $g_M(\mathcal{A}_U BV, \phi X) - \eta(X)g_M(U, V) = -\frac{1}{\lambda^2}g_N(\nabla^f_U f_* CV, f_*\phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using Theorem (3), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U, \phi X) - \eta(X)g_M(U, V) + g_M(\mathcal{H}gradln\lambda, \phi X)g_M(U, CV) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_* \phi X).$$

Using conditions (i) and (ii), we get (iii)

$$g_M(\mathcal{A}_U BV, \phi X) - \eta(X)g_M(U, V) = -\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_* \phi X).$$

Similarly, one can obtain the other assertions. □

Corollary 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ defines a totally geodesic foliation on M,
(ii) $g_M(\mathcal{A}_U \phi V, \phi X) = \eta(X)g_M(U, V)$,
(iii) $-\frac{1}{\lambda^2} g_N((\nabla f_*)(U, \phi V), f_* \phi X) = \eta(X)g_M(U, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, from definition (2), $\phi V \in \Gamma(\phi(\ker f_*))$ and $\phi X \in \Gamma((\ker f_*)^{\perp})$. Using Theorem (3), we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U, \phi X) \\ &\quad - \eta(X)g_M(U, V) + g_M(\mathcal{H}gradln\lambda, \phi X)g_M(U, CV) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - \eta(X)g_M(U, V) \\ &= g_M(\mathcal{A}_U \phi V, \phi X) - \eta(X)g_M(U, V), \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$.

On the further needed, using equation (2.11), we get

$$g_M(\mathcal{A}_U BV, \phi X) = g_M(\nabla_U BV, \phi X).$$

Since f is conformal submersion, we get

$$g_M(\mathcal{A}_U BV, \phi X) = \frac{1}{\lambda^2} g_N(f_* \nabla_U BV, f_* \phi X).$$

Using equation (2.15), we get

$$\begin{aligned} g_M(\mathcal{A}_U BV, \phi X) &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, BV), f_* \phi X) \\ &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, \phi V), f_* \phi X), \end{aligned}$$

which shows $(ii) \Leftrightarrow (iii)$. □

Theorem 5. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on* M,

(ii) $-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)=g_M(\mathcal{T}_X\phi Y,BU)$
$+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)$, *for* $X,Y\in\Gamma(\ker f_*)$ *and* $U\in(\Gamma(\ker f_*)^{\perp})$.

Proof. For $X,Y\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4), (2.10) and (3.2), we get

$$\begin{aligned} g_M(\nabla_XY,U) &= g_M(\phi\nabla_XY,\phi U)+\eta(\nabla_XY)\eta(U),\\ &= g_M(\mathcal{T}_X\phi Y,BU)+g_M(\mathcal{H}\nabla_X\phi Y,CU). \end{aligned}$$

Since ∇ is torsion free and $[X,\phi Y]\in\Gamma(\ker f_*)$, we get

$$g_M(\nabla_XY,U)=g_M(\mathcal{T}_X\phi Y,BU)+g_M(\nabla_{\phi Y}X,CU),$$

using equations (2.3) and (2.4), we get

$$g_M(\nabla_XY,U)=g_M(\mathcal{T}_X\phi Y,BU)+g_M(\nabla_{\phi Y}\phi X,\phi CU),$$

here we have used μ is invariant. Since f is conformal submersion, using equation (2.15) and Lemma 1(i), we get

$$\begin{aligned} g_M(\nabla_XY,U) &= g_M(\mathcal{T}_X\phi Y,BU)-\frac{1}{\lambda}g_M(\mathcal{H}gradln\lambda,\phi Y)g_N(f_*\phi X,f_*\phi CU)\\ &\quad-\frac{1}{\lambda}g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*\phi Y,f_*\phi CU)\\ &\quad+\frac{1}{\lambda}g_M(\phi Y,\phi X)g_N(f_*\mathcal{H}gradln\lambda,f_*\phi CU)\\ &\quad+\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU). \end{aligned}$$

Next, using definition (2) and equation (3.4), we have

$$\begin{aligned} g_M(\nabla_XY,U) &= g_M(\mathcal{T}_X\phi Y,BU)+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)\\ &\quad+\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU), \end{aligned}$$

which shows $(i)\Leftrightarrow(ii)$. □

Theorem 6. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then any two of the followings conditions imply the third:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on* M,

(ii) λ *is constant on* $\Gamma(\mu)$,

(iii) $\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)=-g_M(\mathcal{T}_X\phi Y,\phi U)$,

for $X,Y\in\Gamma(\ker f_*)$ *and* $U\in(\Gamma(\ker f_*)^{\perp})$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, from Theorem (5), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Now, using conditions (i) and (iii), we have$g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) = 0$. From above equation λ is constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

Corollary 3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the following statements are equivalent to each other:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) $\mathcal{T}_X \phi Y = 0$, for $X, Y \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^{\perp}$, from Theorem (5), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, \phi U))$, which proves $(i) \Leftrightarrow (ii)$. □

Theorem 7. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is a totally geodesic map if and only if*

$$\begin{aligned} -\nabla^f_V f_* W &= f_*(\phi \mathcal{A}_V \phi W_1 + \phi \mathcal{V} \nabla_V B W_2 + \phi \mathcal{A}_V C W_2 + C \mathcal{H} \nabla_V \phi W_1 \\ &\quad + C \mathcal{A}_V B W_2 + C \mathcal{H} \nabla_V C W_2 - \eta(W_1) V + g_M(V, W_2) \xi), \end{aligned}$$

for any $V \in (\Gamma(\ker f_*)^{\perp}), W \in \Gamma(TM)$, where $W = W_1 + W_2$, $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Proof. Taking equation (2.15) and using equations (2.1), and (2.4), we get

$$(\nabla f_*)(V, W) = \nabla^f_V f_* W + f_*(\phi \nabla_V \phi W - \eta(W) V - \eta(\nabla_V W) \xi),$$

for any $V \in (\Gamma(\ker f_*)^{\perp}), W \in \Gamma(TM)$.

Now using equations (2.15) and (3.2), we get

$$\begin{aligned} (\nabla f_*)(V, W) = \nabla^f_V f_* W + f_*(&\phi \mathcal{A}_V \phi W_1 + B \mathcal{H} \nabla_V \phi W_1 + C \mathcal{H} \nabla_V \phi W_1 \\ &+ B \mathcal{A}_V B W_2 + C \mathcal{A}_V B W_2 + \phi \mathcal{V} \nabla_V B W_2 + \phi \mathcal{A}_V C W_2 + B \mathcal{H} \nabla_V C W_2 \\ &+ C \mathcal{H} \nabla_V C W_2 + \eta(W_1) B V - \eta(W_1) V + g_M(V, W_2) \xi), \end{aligned}$$

for $W = W_1 + W_2 \in \Gamma(TM)$, where $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Thus taking into account the vertical terms, we get

$$(\nabla f_*)(V,W) = \nabla^f_V f_*W + f_*\{(\phi(\mathcal{A}_V\phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) + C(\mathcal{H}\nabla_V\phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) - \eta(W_1)V) + g_M(V,W_2)\xi\}.$$

Thus

$$\begin{aligned}(\nabla f_*)(V,W) &= 0 \Leftrightarrow \\ -\nabla^f_V f_*W &= f_*(\phi(\mathcal{A}_V\phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) + g_M(V,W_2)\xi \\ &\quad + C(\mathcal{H}\nabla_V\phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) - \eta(W_1)V).\end{aligned}$$

Therefore, we obtain the result. □

Definition 3. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi\ker f_*,\mu)-$totally geodesic map provided $(\nabla f_*)(\phi X,U)=0$, for $X\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^\perp)$.*

Theorem 8. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi\ker f_*,\mu)-$totally geodesic map if and if f is horizontally homohtetic map.*

Proof. For $X\in\Gamma(\ker f_*)$ and $U\in\Gamma(\mu)$, using lemma $1(i)$, we get

$$(\nabla f_*)(\phi X,U) = \phi X(ln\lambda)f_*(U) + U(ln\lambda)f_*(\phi X) - g_M(\phi X,U)f_*(gradln\lambda),$$

From above equation, if f is a horizontally homothetic, then $(\nabla f_*)(\phi X,U)=0$.

Conversely, if $(\nabla f_*)(\phi X,U)=0$, we find

$$\phi X(ln\lambda)f_*(U) + U(ln\lambda)f_*(\phi X) = 0. \tag{3.6}$$

Taking inner product in above equation with $f_*(\phi X)$ and since f is conformal submersion, we have

$$g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*U,f_*\phi X) + g_M(\mathcal{H}gradln\lambda,U)g_N(f_*\phi X,f_*\phi X) = 0.$$

Above equation shows that λ is a constant $\Gamma(\mu)$.

On the other hand taking inner product in equation (3.6) with f_*X, we get

$$g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*U,f_*\phi U) + g_M(\mathcal{H}gradln\lambda,U)g_N(f_*\phi X,f_*U) = 0.$$

From above equation shows that λ is a constant on $\Gamma(\phi(\ker f_*))$. Thus λ is a constant on $\Gamma((\ker f_*)^\perp)$. □

Theorem 9. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion. Then f is totally geodesic map if and only if*

(i) $\phi\mathcal{T}_X\phi Y - \eta(Y)X + g_M(X,Y)\xi) = 0$ and $\mathcal{H}\nabla_Y\phi X \in \Gamma(\phi \ker f_*)$,
(ii) f is horizontally homothetic map,
(iii) $\widehat{\nabla}_X BU = -\mathcal{T}_X CU$ and $\mathcal{T}_X BU + \mathcal{H}\nabla_X CU \in \Gamma(\phi \ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$, using equations (2.1), (2.4) and (2.15), we get

$$(\nabla f_*)(X,Y) = f_*(\phi(\nabla_X\phi Y) - \eta(Y)X + \eta(X)\eta(Y)\xi - \eta(\nabla_X Y)\xi).$$

Now, using equations (2.10) and (3.2), we get

$$(\nabla f_*)(X,Y) = f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y - \eta(Y)X + g_M(X,Y)\xi).$$

Thus shows that $\phi\mathcal{T}_X\phi Y - \eta(Y)X + g_M(X,Y)\xi) = 0$ and $\mathcal{H}\nabla_X\phi Y \in \Gamma(\phi \ker f_*)$.

On the other hund lemma $1(i)$, we get

$$(\nabla f_*)(U,V) = U(ln\lambda)f_*(V) + V(ln\lambda)f_*(U) - g_M(U,V)f_*(gradln\lambda),$$

for $U, V \in \Gamma(\mu)$. It is obvious that if f is horizontally homothetic, it follows that $(\nabla f_*)(U,V) = 0$. Conversely, if $(\nabla f_*)(U,V) = 0$, taking $V = \phi U$ in above equation, we have

$$U(ln\lambda)f_*(\phi U) + \phi U(ln\lambda)f_*(U) = 0. \tag{3.7}$$

Taking inner product in equation (3.7) with $f_*\phi U$, we get

$$g_M(\mathcal{H}gradln\lambda, U)g_N(f_*\phi U, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*U, f_*\phi U) = 0.$$

From above equation λ is constant on $\Gamma(\mu)$. On the other hand, for $X, Y \in \Gamma(\ker f_*)$, from lemma $1(i)$, we get

$$(\nabla f_*)(\phi X, \phi Y) = \phi X(ln\lambda)f_*(\phi Y) + \phi Y(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi Y)f_*(gradln\lambda).$$

Again if f is horizontally homothetic, then $(\nabla f_*)(\phi X, \phi Y) = 0$. Conversely, if $(\nabla f_*)(\phi X, \phi Y) = 0$, putting $X = Y$ in above equation, we get

$$2\phi X(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi X)f_*(gradln\lambda) = 0.$$

Taking inner product in above equation with $f_*\phi X$ and since f is conformal submersion, we have $g_M(\phi X, \phi X)g_M(gradln\lambda, \phi X) = 0$.

From above equation, λ is constant on $\Gamma(\phi \ker f_*)$. Thus λ is constant on $\Gamma((\ker f_*)^\perp)$.

Now, for $X \in \Gamma(\ker f_*)$ and $U \in \Gamma((\ker f_*)^\perp)$, using equations (2.1), (2.4) and (2.15), we get $(\nabla f_*)(X,U) = f_*(\phi(\nabla_X\phi U) - \eta(\nabla_X U)\xi)$. Now, again using equations (2.10) and (3.2), we get

$$(\nabla f_*)(X,U) = f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU).$$

Thus $(\nabla f_*)(X,U) = 0 \Leftrightarrow f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU) = 0$. Therefore, we obtain the result. □

From Theorems (3) and (5) in terms of the second fundamental forms of such submersions.

Theorem 10. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion. Then M is a locally product manifold of the form $M_{(\ker f_*)^\perp}\times M_{(\ker f_*)}$ if*

$$\begin{aligned}(3.8)\qquad &-\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV,f_*\phi X)=g_M(\mathcal{A}_UBV,\phi X)\\ &-g_M(\mathcal{H}gradln\lambda,CV)g_M(U,\phi X)+g_M(\mathcal{H}gradln\lambda,\phi X)g_M(U,CV),\end{aligned}$$

and

$$\begin{aligned}(3.9)\qquad &-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)\\ &=\ g_M(\mathcal{T}_X\phi Y,BU)+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU),\end{aligned}$$

for $X,Y\in\Gamma(\ker f_*)$ and $U,V,W\in(\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)^\perp}$ and $M_{(\ker f_*)}$ are integral manifolds of the distributions $\Gamma(\ker f_*)$ and $(\Gamma(\ker f_*)^\perp)$. Conversely, if M is a locally product manifold of the form $M_{(\ker f_*)^\perp}\times M_{(\ker f_*)}$ then we have

$$\begin{aligned}&-\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV,f_*\phi X)\\ &=g_M(\mathcal{H}gradln\lambda,\phi X)g_M(U,CV)-g_M(\mathcal{H}gradln\lambda,CV)g_M(U,\phi X),\end{aligned}$$

and $-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)=g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)$.

Again, from Corollaries (2) and (3), we have the following theorem.

Theorem 11. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is a locally product manifold if and only if $g_M(\mathcal{A}_U\phi V,\phi X)=\eta(X)g_M(U,V)$ and $\mathcal{T}_X\phi Y=0$, for $X,Y\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^\perp)$.*

Theorem 12. *Let f be a conformal anti-invariant submersion from a Kenmotsu manifolds (M,ϕ,ξ,η,g_M) to a Riemannian manifold (N,g_N). Then M is a locally twisted product manifold of the form $M_{(\ker f_*)}\times M_{(\ker f_*)^\perp}$ if and only if*

$$\begin{aligned}(3.10)\qquad &-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)\\ &=g_M(\mathcal{T}_X\phi Y,BU)+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU),\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad g_M(U,V)H=&-B\mathcal{A}_UBV+CV(ln\lambda)BU-B(\mathcal{H}gradln\lambda)g_M(U,CV)\\ &-\phi^*f_*(\nabla^f_U f_*CV),\end{aligned}$$

for $X,Y\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)}$ and $M_{(\ker f_*)^\perp}$ are integral manifolds of the distributions $(\ker f_*)^\perp$ and $(\ker f_*)$ and H is the mean curvature vector field of $M_{(\ker f_*)^\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4), (2.10) and (3.2), we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_X \phi Y, CU)$. Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_{\phi Y} X, CU).$$

Using equations (2.3), (2.4) and (2.12), we have

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\nabla_{\phi Y} \phi X, \phi CU).$$

Since f is conformal submersion, using equation (2.15) and lemma $1(i)$, we find

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) - \frac{1}{\lambda^2} g_M(\mathcal{H}grad ln\lambda, \phi Y) g_N(f_* \phi X, f_* \phi CU) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H}grad ln\lambda, \phi X) g_N(f_* \phi Y, f_* \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_M(\phi X, \phi Y) g_N(f_* \mathcal{H}grad ln\lambda, f_* \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Next, using definition (2) and equation (3.2), we obtain

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi X, \phi Y) g_M(\mathcal{H}grad ln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Thus shows that $M_{(\ker f_*)}$ is totally geodesic if and only if

$$\begin{aligned} &- \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) \\ &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi X, \phi Y) g_M(\mathcal{H}grad ln\lambda, \phi CU). \end{aligned}$$

On the other hand for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4), (2.11), (2.12) and (3.2), we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X).$$

Since f is conformal submersion, using equation (2.15) and lemma $1(i)$, we obtain that

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - \frac{1}{\lambda^2} g_M(\mathcal{H}grad ln\lambda, U) g_N(f_* CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H}grad ln\lambda, CV) g_N(f_* U, f_* \phi X) \\ &\quad + \frac{1}{\lambda^2} g_M(U, CV) g_N(f_* \mathcal{H}grad ln\lambda, f_* \phi X) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_* \phi X) + \eta(X) \eta(\nabla_U V). \end{aligned}$$

Moreover, using definition (2) and (3.4), we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U, \phi X) \\ &\quad + \eta(X)\eta(\nabla_U V) + g_M(U, CV)g_M(\mathcal{H}gradln\lambda, \phi X) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_*\phi X) + \eta(X)\eta(\nabla_U V). \end{aligned}$$

Then, we have

$$\begin{aligned} g_M(U,V)H &= -B\mathcal{A}_U BV + CV(ln\lambda)BU - B(\mathcal{H}gradln\lambda)g_M(U, CV) \\ &\quad -\phi f_*(\nabla^f_U f_* CV) - g_M(U,V)\xi + \eta(U)\eta(V)\xi, \end{aligned}$$

which proves. □

Theorem 13. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion with rank$(\ker f_*) > 1$. If M is a locally warped product manifold of the form $M_{(\ker f_*)^\perp} \times_\lambda M_{(\ker f_*)}$, then either f is horizontally homothetic submersion or the fibers are one dimensional.*

Proof. Since f is a conformal submersion, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using, equations (2.3), (3.4) and lemma $1(i)$, we obtain

$$-U(ln\lambda)g_M(\phi X, \phi Y) = \phi Y g_M(U, \phi X).$$

For $U \in \Gamma(\mu)$, we get $-U(ln\lambda)g_M(\phi X, \phi Y) = 0$.
From above equation, we find that λ is a constant on $\Gamma(\mu)$.
For $U = \phi X \in \Gamma(\phi(\ker f_*))$, we have

$$\phi X(ln\lambda)g_M(\phi X, \phi Y) = \phi Y(ln\lambda)g_M(\phi X, \phi X). \tag{3.12}$$

Interchanging the roles of Y and X in equation (3.12), we get

$$\phi Y(ln\lambda)g_M(\phi X, \phi X) = \phi X(ln\lambda)g_M(\phi Y, \phi Y). \tag{3.13}$$

From equations (3.12) and (3.13), we get

$$\phi X(ln\lambda) \parallel \phi X \parallel^2 \parallel \phi Y \parallel^2 = \phi X(ln\lambda)(g_M(\phi X, \phi Y))^2. \tag{3.14}$$

From (3.14), either λ is a constant on $\Gamma(\phi(\ker f_*))$ or $\Gamma(\phi(\ker f_*))$ is 1-dimensional. □

4. Conformal anti-invariant submersions admitting horizontal structure vector field

In this section, we study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ

is horizontal vector field. Using definition (2), we have $(\ker f_*)^{\perp} = \phi(\ker f_*) \oplus \mu$, where $\mu = \phi(\mu) \oplus < \xi >$. Thus, for any $U \in \Gamma(\ker f_*)^{\perp}$, we have

$$\phi U = BU + CU, \tag{4.1}$$

where $BU \in \Gamma(\ker f_*)$ and $CU \in \Gamma\phi(\mu)$.

Now, we suppose that X is vertical and U is horizontal vector fields. Using equations (2.3), (4.1), (2.4) and (2.10), we have

$$g_M(CU, \phi X) = 0. \tag{4.2}$$

$$g_M(\nabla_U CV, \phi X) = -g_M(CU, \phi \mathcal{A}_V X). \tag{4.3}$$

Since f is conformal submersion, using equation (4.2), we have

$$\frac{1}{\lambda^2} g_N(f_* CV, f_* \phi X) = 0, \tag{4.4}$$

for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$.

Theorem 14. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then followings are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ is integrable,
(ii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$
$-g_M(\mathcal{H}gradln\lambda, CV) g_M(U, \phi X)$
$+g_M(\mathcal{H}gradln\lambda, CU) g_M(V, \phi X) - 2g_M(\mathcal{H}gradln\lambda, \phi X) g_M(CU, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, since $\phi X \in \Gamma(\ker f_*)^{\perp}$ and $\phi V \in (\Gamma \ker f_* \oplus \mu)$. Using equations (2.3), (2.4) and (4.1), we have

$$\begin{aligned} g_M([U, V], X) &= g_M(\nabla_U BV, \phi X) - g_M(\nabla_V BU, \phi X) + g_M(\nabla_U CV, \phi X) \\ &\quad - g_M(\nabla_V CU, \phi X). \end{aligned}$$

Since f is a conformal submersion, using equation (2.10), we get

$$\begin{aligned} g_M([U, V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \frac{1}{\lambda^2} g_N(f_* \nabla_U CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(f_* \nabla_V CU, f_* \phi X). \end{aligned}$$

Thus using equation (2.15), (4.2) and lemma 1(i), we have

$$\begin{aligned} g_M([U, V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H}gradln\lambda, CV) \\ &\quad + g_M(V, \phi X) g_M(\mathcal{H}gradln\lambda, CU) - 2g_M(CU, V) g_M(\mathcal{H}gradln\lambda, \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

Theorem 15. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^{\perp}$ is integrable,
(ii) f is horizontally homothetic,
(iii) $\frac{1}{\lambda^2}g_N(\nabla^f_V f_*CU-\nabla^f_U f_*CV,f_*\phi X)=g_M(\mathcal{A}_UBV-\mathcal{A}_VBU,\phi X)$, for $X\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^{\perp})$.

Proof. For $X\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^{\perp})$, using Theorem (14), we get

$$\begin{aligned} & g_M([U,V],X)\\ = \;& g_M(\mathcal{A}_UBV-\mathcal{A}_VBU,\phi X)-g_M(U,\phi X)g_M(\mathcal{H}gradln\lambda,CV)\\ & +g_M(V,\phi X)g_M(\mathcal{H}gradln\lambda,CU)-2g_M(CU,V)g_M(\mathcal{H}gradln\lambda,\phi X)\\ & -\frac{1}{\lambda^2}g_N(\nabla^f_V f_*CU-\nabla^f_U f_*CV,f_*\phi X). \end{aligned}$$

Since $(\ker f_*)^{\perp}$ is integrable and f is horizontally homothetic, we get

$$\frac{1}{\lambda^2}g_N(\nabla^f_V f_*CU-\nabla^f_U f_*CV,f_*\phi X)=g_M(\mathcal{A}_UBV-\mathcal{A}_VBU,\phi X),$$

using conditions (i) and (ii), we get (iii). Similarly, one can obtain the other assertions. □

Remark 2. Let $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion. If $\phi(\ker f_*)\oplus<\xi>=(\ker f_*)^{\perp}$, then $C=0$ from equation (4.1).

Corollary 4. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal Lagrangian submersion. Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ is integrable,
(ii) $\mathcal{A}_U\phi V=\mathcal{A}_V\phi U$,
(iii) $(\nabla f_*)(V,\phi U)=(\nabla f_*)(U,\phi V)$, for $X\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^{\perp})$.

Proof. For $X\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^{\perp})$, since $\phi X\in\Gamma(\ker f_*)^{\perp}$ and $\phi V\in\Gamma(\phi\ker f_*)$, from Theorem (14), we have

$$\begin{aligned} g_M([U,V],X) = \;& g_M(\mathcal{A}_UBV-\mathcal{A}_VBU,\phi X)-g_M(U,\phi X)g_M(\mathcal{H}gradln\lambda,CV)\\ & +g_M(V,\phi X)g_M(\mathcal{H}gradln\lambda,CU)-2g_M(CU,V)g_M(\mathcal{H}gradln\lambda,\phi X)\\ & -\frac{1}{\lambda^2}g_N(\nabla^f_V f_*CU-\nabla^f_U f_*CV,f_*\phi X). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we get

$$g_M([U,V],X)=g_M(\mathcal{A}_UBV-\mathcal{A}_VBU,\phi X)=0,$$

which proves $(i) \Leftrightarrow (ii)$.

On the other hand, since f is a conformal submersion, using equations (2.3), (2.11) and (2.15), we get

$$\begin{aligned} &g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) \\ &= \frac{1}{\lambda^2}\{g_N((\nabla f_*)(U, BV), f_*\phi X) - g_N((\nabla f_*)(V, BU), f_*\phi X)\}, \end{aligned}$$

which proves $(ii) \Leftrightarrow (iii)$. □

Theorem 16. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ defines a totally geodesic foliation on M.
$(ii) -\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV, \phi X)$
$-g_M(U, \phi X) g_M(\mathcal{H} grad ln\lambda, CV)$
$+g_M(U, CV) g_M(\mathcal{H} grad ln\lambda, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4), (4.1), (2.10) and (2.11), we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X),$$

Since f is a conformal submersion, using equation (2.15) and lemma $1(i)$, we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} grad ln\lambda, CV) \\ &\quad + g_M(U, CV) g_M(\mathcal{H} grad ln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X), \end{aligned}$$

which shows that $(i) \Leftrightarrow (ii)$. □

Theorem 17. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

$(i)(\ker f_*)^{\perp}$ defines a totally geodesic folation on M,
(ii) f is horizontally homothetic,
$(iii) -\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X) = g_M(A_U BV, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, from Theorem (16), we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} grad ln\lambda, CV) \\ &\quad + g_M(U, CV) g_M(\mathcal{H} grad ln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X), \end{aligned}$$

Since $(\ker f_*)^\perp$ defines a totally geodesic foliation on M and f is horizontally homothetic, we have $-\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV, f_*\phi X) = g_M(A_U BV, \phi X)$, which haves any two conditions imply the three. Similarly, one can obtain the other assertions. □

Corollary 5. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then the following statements are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $\mathcal{A}_U\phi V = 0$,
(iii) $(\nabla f_*)(U, \phi V) = 0$, for $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi V \in \Gamma(\phi \ker f_*)$ and $\phi X \in (\Gamma(\ker f_*)^\perp)$. From theorem (16), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(V, \phi X)g_M(\mathcal{H}gradln\lambda, CV) + g_M(U, CV)g_M(\mathcal{H}gradln\lambda, \phi X) + \frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV, f_*\phi X),$$

Since f is a conformal Lagrangian submersion, we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X), \tag{4.5}$$

which proves $(i) \Leftrightarrow (ii)$. On the other hand, using equation (4.5) and (2.11), since f is a conformal Lagrangian submersion and using equation (2.15), we get $g_M(\mathcal{A}_U BV, \phi X) = \frac{1}{\lambda^2}g_N((\nabla f_*)(U, BV) + \eta(V)f_*U, f_*\phi X)$ which proves $(ii) \Leftrightarrow (iii)$. □

Theorem 18. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,
(ii) $-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) = g_M(\mathcal{T}_Y\phi X, BU)$
$+g_M(\phi Y, \phi X)g_M(\mathcal{H}gradln\lambda, \phi CU) - \eta(U)g_M(X, Y)$, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4) and (4.1), we have

$$g_M(\nabla_X Y, U) = g_M(T_X\phi Y, BU) + g_M(\nabla_{\phi Y}\phi X, \phi CU) + \eta(U)\eta(\nabla_X Y).$$

Since f is a conformal submersion, using equation (2.15) and lemma $1(i)$, we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X\phi Y, BU) + g_M(\phi Y, \phi X)g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) - \eta(U)g_M(X, Y), \end{aligned}$$

If $(\ker f_*)$ is a totally geodesic foliation on M, then we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

Theorem 19. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) λ is constant on $\Gamma(\mu)$,

(iii) $-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) = g_M(\mathcal{T}_X \phi Y, \phi U) - \eta(U) g_M(X, Y)$, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, from Theorem (18), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(T_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y), \end{aligned}$$

Now, if we have (i) and (iii), then we have

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

Corollary 6. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on M.*

(ii) $\mathcal{T}_X \phi Y = 0$, *for $X, Y \in \Gamma(\ker f_*)$.*

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$, from Theorem (18), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, \phi U) - \eta(U) g_M(X, Y)$, which proves $(i) \Leftrightarrow (ii)$. □

Theorem 20. *Let f be a conformal anti-invariant submersion from a Kenmotsu manifolds $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is a totally geodesic map if and only if*

$$\begin{aligned} (4.6) -\nabla^f_V f_* W &= f_*(\phi \mathcal{A}_V \phi W_1 + \phi \mathcal{V} \nabla_V B W_2 + \phi \mathcal{A}_V C W_2 + C \mathcal{H} \nabla_V \phi W_1 \\ &\quad + C \mathcal{A}_V C W_2 + C \mathcal{H} \nabla_V C W_2 + \eta(W_2) CV - \eta(\nabla_V W) \xi), \end{aligned}$$

for any $V \in \Gamma(\ker f_*)^\perp, W \in \Gamma(TM)$, where $W = W_1 + W_2$, $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^\perp)$.

Proof. For any $V \in \Gamma(\ker f_*)^{\perp}, W \in \Gamma(TM)$, using equations (2.1), (2.4), (2.15) and (4.1), we get

$$\begin{aligned}(\nabla f_*)(V,W) &= \nabla^f_V f_* W + f_*(\phi\mathcal{A}_V\phi W_1 + B\mathcal{H}_V\phi W_1 + C\mathcal{H}_V\phi W_1 \\ &\quad + B\mathcal{A}_V BW_2 + C\mathcal{A}_V BW_2 + \phi\mathcal{V}\nabla_V BW_2 + \phi\mathcal{A}_V CW_2 \\ &\quad + B\mathcal{H}\nabla_V CW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_2)\phi V - \eta(\nabla_V W)\xi).\end{aligned}$$

If $(\nabla f_*)(V,W) = 0$, then we have

$$\begin{aligned}-\nabla^f_V f_* W &= f_*(\phi A_V\phi W_1 + \phi\mathcal{V}\nabla_V BW_2 + \phi\mathcal{A}_V CW_2 + C\mathcal{H}_V\phi W_1 \\ &\quad + C\mathcal{A}_V BW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_2)\phi V - \eta(\nabla_V W)\xi).\end{aligned}$$

Thus $(\nabla f_*)(V,W) = 0 \Leftrightarrow$ we get equation (4.6). □

Theorem 21. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then f is called a $(\phi\ker f_*,\mu)-$totally geodesic map if and if f is horizontally homohtetic map.*

Proof. For $U \in \Gamma(\ker f_*)$ and $X \in (\Gamma\mu)$, using lemma $1(i)$, we have

$$(\nabla f_*)(\phi U, X) = \phi U(ln\lambda) f_*(X) + X(ln\lambda) f_*(\phi U) - g_M(\phi U, X) f_*(\mathcal{H}gradln\lambda).$$

From above equation, if f is a horizontally homothetic then $(\nabla f_*)(\phi U, X) = 0$.

Conversely, if $(\nabla f_*)(\phi U, X) = 0$, we obtain

$$\phi U(ln\lambda) f_*(X) + X(ln\lambda) f_*(\phi U) = 0. \tag{4.7}$$

Taking inner prouct above equation with $f_*(\phi U)$ and since f is a conformal submersion, we write

$$g_M(\mathcal{H}gradln\lambda, \phi U) g_N(f_*X, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, X) g_N(f_*\phi U, f_*\phi U) = 0.$$

Above equation implies that λ is constant on $\Gamma(\mu)$. On the other hand, again taking inner product with f_*X, we get

$$g_M(\mathcal{H}gradln\lambda, \phi U) g_N(f_*X, f_*X) + g_M(\mathcal{H}gradln\lambda, X) g_N(f_*\phi U, f_*X) = 0,$$

From above equation, it follows that λ is constant on $\Gamma(\phi(\ker f_*))$. Thus λ is constant on $(\Gamma(\ker f_*)^{\perp})$. □

Theorem 22. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f : (M,\phi,\xi,\eta,g_M) \to (N,g_N)$ be a conformal anti-invariant submersion. Then f is totally geodesic map if and only if*

(i) $\mathcal{T}_X\phi Y = 0$ and $C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi \in \Gamma(\phi\ker f_*)$,
(ii) f is horizontally homothetic map,
(iii) $\widehat{\nabla}_X BV + \mathcal{T}_X CV = 0$,
and $\mathcal{T}_X BV + \mathcal{H}\nabla_X CV \in \Gamma(\phi\ker f_*)$, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^{\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$, using equations (2.1), (2.4), (4.1) and (2.10), we get $(\nabla f_*)(X,Y) = f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi)$. If $(\nabla f_*)(X,Y) = 0$, then we have $f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi) = 0$, which prove $\mathcal{T}_X\phi Y = 0$ and $C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi \in \Gamma(\phi\ker f_*)$.

On the other hand, from lemma 1(i) we have $(\nabla f_*)(V,W) = V(ln\lambda)f_*(W) + W(ln\lambda)f_*(V) - g_M(V,W)f_*(\mathcal{H}gradln\lambda)$, for any $V, W \in \Gamma(\mu)$. It is obvious that if f is horizontal homothetic, it follows that $(\nabla f_*)(V,W) = 0$. Conversely, if $(\nabla f_*)(V,W) = 0$, taking $W = \phi V$ in above equation, we have

$$V(ln\lambda)f_*(\phi V) + \phi V(ln\lambda)f_*(V) = 0. \tag{4.8}$$

Taking inner product in equation (4.8) with $f_*\phi V$, we get

$$g_M(\mathcal{H}gradln\lambda, V)g_M(\phi V, \phi V) + g_M(\mathcal{H}gradln\lambda, \phi V)g_M(V, \phi V) = 0.$$

From above equation, λ is constant on $\Gamma(\mu)$. On the other hand, from lemma 1(i) we have $(\nabla f_*)(\phi X, \phi Y) = \phi X(ln\lambda)f_*(\phi Y) + \phi X(ln\lambda)f_*(\phi Y) - g_M(\phi Y, \phi X)f_*(\mathcal{H}gradln\lambda)$,

Again if f is horizontal homothetic, it follows that $(\nabla f_*)(\phi X, \phi Y) = 0$. Conversely, if $(\nabla f_*)(\phi X, \phi Y) = 0$, taking $X = Y$ in above equation, we have $2\phi X(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi X)f_*(\mathcal{H}gradln\lambda) = 0$.

Taking inner product above equation with $f_*\phi X$ and since f is conformal submersion, we get $g_M(\phi X, \phi X)g_M(\mathcal{H}gradln\lambda, \phi X) = 0$. From above equation, λ is constant on $\Gamma(\ker f_*)^\perp$. Thus λ is constant on $\Gamma(\phi\ker f_*)$.

Now, for $U \in \Gamma(\ker f_*)^\perp$ and $X \in \Gamma(\ker f_*)$, from equations (2.15), (2.1), (2.4), (4.1), (2.9) and (2.10), we get

$$\begin{aligned}(\nabla f_*)(X,V) &= f_*(B\mathcal{T}_XBV + C\mathcal{T}_XBV + \phi\widehat{\nabla}_XBV + B\mathcal{H}\nabla_XCV \\ &\quad + C\mathcal{H}\nabla_XCV + \phi\mathcal{T}_XCV + \eta(V)BX + \eta(V)X).\end{aligned}$$

Thus $(\nabla f_*)(X,V) = 0 \Leftrightarrow f_*(C\mathcal{T}_XBV + \phi\widehat{\nabla}_XBV + C\mathcal{H}\nabla_XCV + \phi\mathcal{T}_XCV)$. □

Theorem 23. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion. Then M is a locally twisted product manifold of the form $M_{(\ker f_*)} \times M_{(\ker f_*)^\perp}$ if and only if*

$$\begin{aligned}&-\frac{1}{\lambda^2}g_N(\nabla_{f_*\phi Y}f_*\phi X, f_*\phi CV) \\ &= g_M(\mathcal{T}_X\phi Y, BV) + g_M(\phi Y, \phi V)g_M(\mathcal{H}gradln\lambda, \phi X) - g_M(X,Y)\eta(V),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}g_M(V,W)H &= -B\mathcal{A}_VBW + CW(ln\lambda)BV - B\mathcal{H}gradln\lambda g_M(V, CW) \\ &\quad - \phi f^*(\nabla^f_V f_*CW),\end{aligned} \tag{4.10}$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in (\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)}$ and $M_{(\ker f_*)^\perp}$ are integral manifolds of the distributions $(\ker f_*)^\perp$ and $(\ker f_*)$ and H is the mean curvature vector field of $M_{(\ker f_*)^\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $V \in (\Gamma(\ker f_*)^{\perp})$, from equations (2.3), (2.4) and (2.10), we have $g_M(\nabla_X Y, V) = g_M(\phi\nabla_X Y, \phi V)+\eta(V)\eta(\nabla_X Y) = g_M(\mathcal{T}_X\phi Y, BV) + g_M(\mathcal{H}\nabla_X\phi Y, CV) - g_M(X, Y)\eta(V)$.

Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get $g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X\phi Y, BV) + g_M(\mathcal{H}\nabla_{\phi Y} X, CV) - g_M(X, Y)\eta(V)$.

Using equation (2.3), (2.4) and (2.11), we get $g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X\phi Y, BV)+ g_M(\nabla_{\phi Y}\phi X, \phi CV) - g_M(X, Y)\eta(V)$. Since f is a conformal submersion, using equations (2.15), (4.2) and lemma $1(i)$, we have

$$\begin{aligned} g_M(\nabla_X Y, V) = {} & g_M(\mathcal{T}_X\phi Y, BV) + g_M(\phi X, \phi Y) g_M(\mathcal{H}gradln\lambda, \phi CV) \\ & + \frac{1}{\lambda^2} g_N(\nabla_{f_*\phi Y} f_*\phi X, f_*\phi CV) - g_M(X, Y)\eta(V), \end{aligned}$$

Thus it follows that $M_{(\ker f_*)}$ is totally geodesic if and only if equation (4.9) is satisfied.

On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in (\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4) and (2.11), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{A}_V BW, \phi X) + g_M(\mathcal{H}\nabla_V CW, \phi X) \\ &\quad +\eta(Y) g_M(X, \phi V) \end{aligned}$$

Since f is a conformal submersion, using equation (2.15) and lemma $1(i)$, we have

$$\begin{aligned} g_M(\nabla_V W, X) = {} & g_M(\mathcal{A}_V BW, \phi X) - \frac{1}{\lambda^2} g_M(\mathcal{H}gradln\lambda, V) g_N(f_*CW, f_*\phi X) \\ & - \frac{1}{\lambda^2} g_M(\mathcal{H}gradln\lambda, CW) g_N(f_*V, f_*\phi X) \\ & + \frac{1}{\lambda^2} g_M(V, CW) g_N(f_*\mathcal{H}gradln\lambda, f_*\phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_V f_*CW, f_*\phi X). \end{aligned}$$

Moreover, using equation (4.2), we get

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{A}_V BW, \phi X) - g_M(\mathcal{H}gradln\lambda, W) g_M(V, \phi X) \\ &\quad + g_M(\mathcal{H}gradln\lambda, \phi X) g_M(V, CW) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_V f_*CW, f_*\phi X) + \eta(W) g_M(V, \phi X) \end{aligned}$$

Thus $M_{(\ker f_*)^{\perp}}$ is totally umbilical $\Leftrightarrow$ equation (4.10) satisfied. □

Similarly, from Theorem (15), we deduce the following result.

Theorem 24. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion with rank$(\ker f_*) > 1$. If M is a locally warped product manifold of the form $M_{(\ker f_*)^{\perp}} \times_\lambda M_{(\ker f_*)}$, then either f is horizontally homothetic submersion or the fibers are one dimensional.*

5. Example

Note that given an Euclidean space $(x_1, ...x_{2m}, x_{2m+1})$ with coordinates we can canonically choose an almost contact structure ϕ on R^{2m+1} as follows:

$$\phi(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + ... + a_{2m-1}\frac{\partial}{\partial x_{2m-1}} + a_{2m}\frac{\partial}{\partial x_{2m}} + a_{2m+1}\frac{\partial}{\partial x_{2m+1}})$$
$$= (-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + ... - a_{2m}\frac{\partial}{\partial x_{2m-1}} + a_{2m-1}\frac{\partial}{\partial x_{2m}}),$$

where $\xi = \frac{\partial}{\partial x_{2m+1}}$ and $a_1, a_2, ..., a_{2m}, a_{2m+1}$ are $C^\infty-$real valued functions in R. Let $\eta = dx_{2m+1}$ and $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_{2m}}, \frac{\partial}{\partial x_{2m+1}})$ is orthogonal basis of vector fields on R^{2m+1}.

Example 1. Define a map $f : R^5 \to R^2$ by $f(x_1, ..., x_5)=(e^{x_2}\cos x_4, e^{x_2}\sin x_4)$. Then we have

$$\ker f_* =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_2}$.

Example 2. Define a map $f : R^5 \to R^3$ by

$$f(x_1, ..., x_5) = (e^{x_1}\sin x_3, e^{x_1}\cos x_3, e^{x_1}\sin x_5)$$

Then we have

$$\ker f_* =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_1}$.

References

[1] M.A. Akyol and B. Sahin, *Conformal anti-invariant submersions from almost Hermitian manifolds*, Turk J. Math., 40 (2016), 43-70.

[2] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., 509 (1976), Berlin-Heidelberg-New York.

[3] P. Baird and J.C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, London Mathematical Society Monographs, 29, Oxford University Press, The Clarendon Press, Oxford, (2003).

[4] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans Amer Math. Soc., 145 (1969), 1-49.

[5] J.P. Bourguignon and H.B. Lawson, *A Mathematician's visit to Kaluza-Klein theory*, Rend. Semin. Mat. Torino Fasc. Spec., (1989), 143-163.

[6] D. Chinea, *Almost contact metric submersions*, Rend. Circ. Mat. Palermo, 34(1) (1985), 89–104.

[7] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian submersions and related topics*, World Scientific Publishing Co., 2004.

[8] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), 28 (1978), 107–144.

[9] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions,* Journal of Applied Mathematics and Mechanics, 16 (1967), 715–737.

[10] D. Gromoll, W. Klingenberg and W. Meyer, *Riemannsche Geometrie im Groβen*, Lecture Notes in Mathematics, 55, Springer, 1975.

[11] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Class. Quantum Gravity, 4 (1987), 1317-1325.

[12] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto. Univ., 19 (1979), 215–229.

[13] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J., 13 (1966), 458–469.

[14] R. Ponge, H. Reckziegel, *Twisted products in pseudo-Riemannian geometry*, Geom Dedicata, 48 (1993), 15-25.

[15] B. Watson and G, $G^{|}-$Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., 57 (1983), 324-349, Teubner, Leipzig.

[16] B. Watson, *Almost Hermitian submersions*, J. Differential Geometry, 11(1), (1976), 147–165.

Accepted: 1.02.2018

THE EFFECT OF METHODS OF OPERATION RESEARCH IN OBTAINING THE BEST RESULTS IN THE TRADE

Ahmed Atallah Alsaraireh*
The University of Jordan
Department of Computer Information Systems
Aqaba
Jordan
ah_8545@yahoo.com
a.alsarairah@ju.edu.jo

Mohammad Almasarweh
The University of Jordan
Department of Business Administration
Aqaba
Jordan
m.almasarweh@ju.edu.jo

Mahmoud Barakat Alnawaiseh
The University of Jordan
Department of Business Administration
Aqaba
Jordan
m.alnawaiseh@ju.edu.jo

S. Al Wadi
The University of Jordan
Department of Risk Management and Insurance
Aqaba
Jordan
s.alwadi@ju.edu.jo

Vandana Bhama
Indian Institute of Technology
Department of Management Studies
India
vdrai1988@gmail.com

Abstract. This study aims to determine the most effective method in operation research regarding the potential to reduce the cost in a minimum time and achieving more profit. In this study, three methods were used : Simplex method, Simplex method and transportation problems and Simplex method, transportation problems, and critical path method. A sample of 10 traders, who work in the same field and import the same commodities, participated in the study. Three models were applied to evaluate the result and compare between them, actual result from these models and randomly

∗. Corresponsing author

results. After that the researchers applying the survey to find the degree of traders satisfaction for these models and results.

The Statistical Package (SPSS) software was used for data analysis. The results of this study indicated that a three methods were a better. Also the results of questioner are linked to the satisfaction of traders on the three methods.

Keywords: operation research methods, traders satisfaction, mathematical models.

1. Introduction

The commerce field is developing and growing rapidly , and increasing the aware competency between all organizations in the world, and striving to be more successes, survival and achieving to a competitive advantages. Therefore, all organizations seek to minimize the cost and time, and increase their margin profit. For these reasons, this study aims to link this important filed with mathematical approaches including operation research, like some: Simplex method, Transportation problems, and Critical path method. For more details about the application of the operation research in a new fields refer to (V.N. Mishra, L.N. Mishra, 2012), (L.N. Mishra, 2017) and (Deepmala, 2014).

2. Literature review

Operations research as a science has been used to help solve decision problems using mathematical and statistical models for a long time, and it has been develop in many scientific fields such as : Mathematics, Engineering and Management. It is one of the areas that has contributed to solving many of mathematical problems and management problems: simplex method, Transportation problems and the network.

The first book of operations research appeared in 1946 As "Methods of Research Operations" for Morris and Campbells, the most important discoveries were in this. In (Dantezig, 1949) developed a method of problem solving in the Simplex Method, which had the greatest effect in obtaining results in a clear mathematical style, also in (Dantezig, 1951) discussed a Proof of the Equivalence of the Programming Problem and the Game Problem. In 1952, the Society of Research published its first journal of operations research and the publication of another journal, which helped to develop this area and its use in decision making. Then Taylor, Fayol, Gilbert, and Mayo used scientific methods and applied the principles of assignment problems.

In (Walker and Kelly, 1959) developed a critical method that helped resolve many of the problems that created the way for solution in terms of distance and time.

(Dantzig, 1963), discussed a Linear Programming and Extensions, and (Giant, 1966) used the principle of drawing to solve various problems in many projects but this mothod is not enough to solve all problems, for example, we cannot use this method for three variables or more. In (Khachian, 1979) proposed a new method of solving the linear program, but theoretically only. In

(Karmarkar, 1984) developed a new polynomial-time algorithm for linear programming and introduced an algebraic method with high results but the rest of the simplex is the easiest. In (Lucey, 1994) discussed a quantitative techique in the operation research, also (Kolman and Robert, 1995) introduced a new application about the linear programming and applied a different branch in operation research.

The applications continued to be widely period-intensiveuntil (Hillier, 2001), (Winston, 2004), (Khobragade, 2005), and (Hamdy, 2007) presented a new application in operation research, particularly in Semplex and transportation problems, and they discussed an alternative approach about simplex method.

Later, (Hashamdar, 2011), (Khobragade, 2012), (Vaidya, 2012), introduced new information to find an alternative approach to the simplex, method Optimum solution to the simplex method, and optimize analytical condition in structural analysis. in the recent years, (Kedia, 2013), (Vaidya, 2014), and (Jervin, 2015) discussed a New Variant of Simplex to solve a game problem using a quick simplex method. In (Vaidya and Kasturiwale, 2016) discussed a new approach while solving two phase simplex method, and they discussed this subject with respect to a number of iteration.

Since the applications remained separate and all studies discussed the evolution of simplex method, this study attempts to investigate the application of more than one mathematical model in problem solving and work on diffusion.

3. The problem of the study and questions

Due to the previous studies were based on separate mathematical models and did not apply some of models on the same problem. Therefore, it is necessary to form and apply several mathematical models in order to compare the results and determine the most appropriate model of the solution.

Question 1: Is there a difference between the actual results and the expected results?

Question 2: do the best results depend on the model of operation research?

Question 3: which is the best model?

Question 4: are the traders significantly satisfied ?

4. The importance of study

This study aims to encourage traders to use mathematical applications in the process of comparison between random application and mathematical application.

5. Methodology and procedures

The Study population: It refers to all traders who were importing the same commodities.

The Study Sample: The traders who were importing the same commodities in Aqaba-Jordan.

Study Tool: This study prepared and applied mathematical models to compare results.

1. Collect the actual data from traders to apply the models.
2. Calculate the results of the models for all traders.
3. Choose the best result of each trader.

6. Models of operation research

1. Simplex method .
2. Simplex method and transportation problems.
3. Simplex method, transportation problems, and critical path method.

7. The results

This study aims to investigate if the mathematical models are effect of a results on a commerce problem. So we will present the results in two parts:

7.1 Part 1

the researchers here discussed the answers of the following questions:

Question 1: Is there a difference between the actual results and expected results?

Question 2: do the best results depend on the model of operation research?

Question 3: which is the best model?

The following table represents the values of minimum costs, maximum profits, and minimum time in two columns . The first column represents the actual results that take from the traders, while The second column represents the expected values of operation research models.

as shown by previous tables, there is a difference in results between the real results and expected results. The models were better than the real results, moreover the best result depends on the type of model. But there exist a difference in the result of model, because there is a different way in transportation problems. Also it is noticeable that the last model is the best one.

Table 1: Results of Models

	The actual results	The results, If the trader applied the three models		
	without models	model 1	model 2	model 3
Trader1	Max. profit(10000), Min. cost(15000), Min. time(25 days)	Max. profit(10500), Min. cost(14400), Min. time(24 days)	Max. profit(11200), Min. cost(14100), Min. time(23 days)	Max. profit(12000), Min.cost(13500), Min. time(20 days)
Trader2	Max. profit(13000), Min. cost(11000), Min. time(30 days)	Max. profit(13500), Min. cost(10700), Min. time(26 days)	Max. profit(13600), Min. cost(9000), Min. time(23 days)	Max. profit(14000), Min. cost(9600), Min. time(19 days)
Trader3	Max. profit(18000), Min. cost(12000), Min. time(29 days)	Max. profit(17000), Min. cost(11000), Min. time(28 days)	Max. profit(10000), Min. cost(10300), Min.time(25days)	Max. profit(10000), Min. cost(10000), Min. time(20 days)
Trader 4	Max. profit(8000), Min. cost(6000), Min. time(15 days)	Max. profit(8300), Min. cost(5800), Min.time(13 days)	Max. profit(9000), Min. cost(5200), Min.time(11 days)	Max. profit(12000), Min. cost(4500), Min. time(10 days)
Trader 5	Max. profit(9000), Min. cost(8000), Min. time(15 days)	Max. profit(9200), Min. cost,(7600), Min. time(14 days)	Max. profit,(9600), Min. cost(7300), Min. time(13 days)	Max. profit(10000), Min. cost(7000), Min. time(11 days)
Trader6	Max. profit(16000), Min. cost(10000), Min. time(22 days)	Max. profit(17000), Min. cost(9500), Min. time,(19 days)	Max. profit(17500), Min. cost(9200), Min. time(17 days)	Max. profit(18000), Min. cost(8800), Min. time(16 days)
Trader7	Max. profit(14000), Min. cost(4000), Min. time(18 days)	Max. profit(14500), Min. cost(3950), Min. time(18 days)	Max. profit(14700), Min. cost(3800), Min.time(16 days)	Max. profit(14900), Min. cost(3700), Min.time(15 days)
Trader8	Max. profit(23000), Min. cost(7000), Min. time(17 days)	Max. profit(23400), Min. cost(6700), Min. time(15 days)	Max. profit(24000), Min. cost(6500), Min. time(14 days)	Max. profit(24600), Min. cost(6200), Min. time(13 days)
Trader9	Max. profit(7000), Min. cost(5000), Min. time(12 days)	Max. profit(7300), Min. cost(4900), Min. time(11 days)	Max. profit(7500), Min. cost(4700), Min. time(10 days)	Max. profit(8000), Min. cost(4300), Min. time(9 days)
Trader 10	Max. profit(20000), Min. cost(11000), Min. time(30 days)	Max. profit(20100), Min. cost(10900), Min. time,(27 days)	Max. profit(20500), Min. cost,(10600), Min. time,(24 days)	Max. profit(21000), Min. cost(10000), Min. time(22 days)

7.2 Part 2.

In this part we discuss the answer of the following question:

Question 4: are the traders significantly satisfied ?

This part was concerned with the enumeration of the variables that called for the application of the mathematical models that were enumerated in this study and were divided into two axes so that each axis contained a set of questions.

This questionnaire was introduced after the mathematical models were applied to the data collected from the study sample and observation results.

7.3 Question axes:

The first axis: Evaluation of models in terms of the importance, quality and features available. This axis has been covered in the following questions Q1, Q2, ..., Q7.

The second axis: the spread of mathematical models and to the attempt to apply and comprehensiveness and level of empowerment. This axis has been covered in Q8, Q9, ..., Q12.

Table 2: Results

		Frequency	Percent	Valid Percent	Cumulative Percent
Q1	Agree	6	30	30	30
	Strongly agree	14*	70	70	100
	Total	20	100	100	-
Q2	Agree	8	40	40	40
	Strongly agree	12*	60	60	100
	Total	20	100	100	-
Q3	Disagree	1	5	5	3
	Agree	6	30	30	35
	Strongly agree	13*	65	65	100
	Total	20	100	100	
Q4	Disagree	1	5	5	5
	Agree	9	45	45	50
	Strongly agree	10*	50	50	100
	Total	20	100	100	-
Q5	Agree	11*	55	55	55
	Strongly agree	9	45	45	100
	Total	20	100	100	-
Q6	Disagree	1	5	5	5
	Agree	8	40	40	45
	Strongly agree	11*	55	55	100
	Total	20	10	100	-
Q7	Disagree	1	5	5	5
	Agree	13*	65	65	70
	Strongly agree	6	30	30	100
	Total	20	100	100	-
Q8	Disagree	3	15	15	15
	Agree	10*	50	50	65
	Strongly agree	7	35	35	100
	Total	20	100	100	-
Q9	Disagree	2	10	10	10
	Agree	7	35	35	45
	Strongly agree	11*	55	55	100
	Total	20	100	100	-
Q10	Disagree	1	5	5	5
	Agree	6	30	30	35
	Strongly agree	13*	65	65	100
	Total	20	100	100	-
Q11	Agree	6	30	30	30
	Strongly agree	14*	70	70	10
	Total	20	100	100	-
Q12	Agree	6	30	30	30
	Strongly agree	14*	70	70	100
	Total	20	100	100	-

8. The questionnaire results

The following table represents the frequency distribution of all the questions asked and the higher frequency and percentages for each frequency were determined.

The previous table presents the distribution of data pertaining to the percentage and frequency of all questions that used in study, and we notice that the maximum percentage of the results was at strongly agree and agree items.

Table 3: My caption

	Q	Measure	Strongly Agree	Agree	Disagree	mean	st. dev.	Result
First Axis	Q1	Frequency %	14 70	6 30	- -	2.7	0.470162	Strongly agree
First Axis	Q2	Frequency %	12 60	8 40	- -	2.6	0.50262	Strongly agree
First Axis	Q3	Frequency %	13 65	6 30	1 5	2.6	0.598243	Strongly agree
First Axis	Q4	Frequenc y%	10 50	9 45	1 5	2.45	0.60481	Strongly agree
First Axis	Q5	Frequency %	9 45	11 55	- -	2.45	0.51042	Strongly agree
First Axis	Q6	Frequency %	11 55	8 40	1 5	2.5	0.60698	Strongly agree
First Axis	Q7	Frequency %	6 30	13 65	1 5	2.25	0.55012	Agree
	First Result	Frequency %	75 53.5	61 43.5	4 3	2.51	0.25163	Strongly agree
Second Axis	Q8	Frequency %	7 35	10 50	3 15	2.2	0.55012	Agree
Second Axis	Q9	Frequency %	11 55	7 35	2 10	2.45	0.69585	Strongly agree
Second Axis	Q10	Frequency %	13 65	6 30	1 5	2.6	0.68633	Strongly agree
Second Axis	Q11	Frequency %	14 70	6 30	- -	2.7	0.47016	Strongly agree
Second Axis	Q12	Frequency %	14 70	6 30	- -	2.7	0.47016	Strongly agree
	Second Result	Frequency %	59 59	35 35	6 6	2.53	0.22734	Strongly agree

Table 4: Likart Scale

Weighted average	Degree of approval
1 - 1.66	Disagree
1.67 - 2.33	Agree
2.34 - 3	Strongly Agree

Table(3) presents all the results and comparisons required to answer the axes of the questionnaire that was applied and relying on the Likart scale, provides results essential to make judgments.

Note that the result of first axis 2.51 was strictly agree and was the result of second axis 2.53 that mean agree. Therefore, we can say that the results of the study indicate that the models are important and have advantages in the application and high quality in improving the results and that the models are comprehensive solutions to trade problems, specifically financial.

9. Conclusion

This study revealed that there is a difference between the actual values and expected values when we applied the mathematical models. The result of models was better than the actual values, also we note that the traders' satisfaction was significant.

References

[1] G. Dantzig, *A proof of the equivalence of the programming problem and the game problem,* in T.C. Koopmans (ed.), Activity Analysis of Production and Allocation, Wiley, New York, 1951.

[2] James E. Kelley, Jr, Morgan R. Walker, *Critical-path planning and scheduling*, Papers presented at the December 1-3, eastern joint IRE-AIEE-ACM computer conference, 160-173, December 01-03, 1959, Boston.

[3] G. Dantzig, *Linear programming and extensions*, Princeton University Press, Princeton, New Jersey, 1963.

[4] L.G. Khachian, *Polynomial algorithm for linear programming*, Doklady Akademia Nauk USSR, Mathematica, 244 (1979), 1093-1096.

[5] N. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica, 4 (1984), 373-395.

[6] T. Lucey, *Quantitative Techniques*, 2nd edition, D. P. Publications Ltd., 1992.

[7] Kolman and Robert, *Elementary linear programming with application*, 2nd edition, Academic Press Inc, United States, 1995.

[8] Hillier, *Introduction to operations research*, 7th edition, McGraw-Hill Science/Engineering/Math; 7 edition, 2001.

[9] Wayne Winston, *Operations research: applications and algorithms*, (4th edition), Duxbury, 2004.

[10] A. Hamdy, *Operations research an introduction*, 8th edition. Prentice-Hall, Inc. Upper Saddle River, NJ, USA 2007.

[11] N.W. Khobragade, P.G. Khot, *Alternative approach to the simplex method-II*, Acta Ciencia Indica, India, 2005.

[12] H. Hashamdar, Z. Ibrahim, M. Jameel, A. Karbakhsh, Z. Ismail and M. Kobraei, *Use of the simplex method to optimize analytical condition in structural analysis*, International Journal of Physical Sciences, 6(2011), 691-697.

[13] N.W. Khobragade, P.G. Khot, *An alternative approach to the simplex method-1*, Bulletin of Pure and Applied Sciences, 23E(2012), 35-40.

[14] N.V. Vaidya, N.W. Khobragade, *Optimum solution to the simplex method - an alternative approach*, Int. J Latest Trend Math. 2(2012), 99-105.

[15] R.G. Kedia, *A new variant of simplex method*, international Journal of Engineering and Management Research, 3 (2013), 2013.

[16] N.V. Vaidya, N.N. Kasturiwale, *Solving game problems using a quick simplex algorithm a new method*, Int. J Latest Trend Math, 2014, 165-182.

[17] Jervin Zen Lobo, *Two square determinant approach for simplex method*, IOSR Journal of Mathematics, 2015, 2278-5728.

[18] N.V. Vaidya and N.N. Kasturiwale, British Journal of Mathematics and Computer Science, 16 (2016), 1-15, Article no.BJMCS.24440.

[19] V.N. Mishra, L.N. Mishra, *Trigonometric approximation of signals (functions) in Lp*$(p \geq 1)$ *norm,* International Journal of Contemporary Mathematical Sciences, 7 (2012), 909-918.

[20] L.N. Mishra, *On existence and behavior of solutions to some nonlinear integral equations with Applications*, Ph.D. Thesis, National Institute of Technology, Silchar 788 010, Assam, India, 2017.

[21] Deepmala, *A study on fixed point theorems for nonlinear contractions and its applications,* Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur 492 010, Chhatisgarh, India, 2014.

Accepted: 2.02.2018

SOME NEW k-FRACTIONAL INTEGRAL INEQUALITIES CONTAINING MULTIPLE PARAMETERS VIA GENERALIZED (s,m)-PREINVEXITY

Yao Zhang
Department of Mathematics
College of Science
China Three Gorges University
Yichang, 443002
P. R. China
yaozhangctgu@163.com

Tingsong Du*
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang, 443002
P. R. China
tingsongdu@ctgu.edu.cn

Hao Wang
Department of Mathematics
College of Science
China Three Gorges University
Yichang, 443002
P. R. China
haowangctgu@163.com

Abstract. We establish some new k-fractional integral inequalities for differentiable functions based on generalized (s,m)-preinvexity. We also prove Hadamard-type inequalities involving products of two generalized (s,m)-preinvex functions. These inequalities include some previously known results as special cases.

Keywords: Hadamard-type inequalities; generalized (s,m)-preinvex functions; k-fractional integrals.

1. Introduction

The following double inequality is notable in the literature as the Hermite-Hadamard inequality.

Theorem 1.1. *Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of real numbers and $a, b \in I$ along with $a < b$. The following double*

∗. Corresponding author

inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

A large number of generalizations and refinements on the inequality (1.1) have been presented, for example, see [7, 8, 10, 13, 16, 17, 19, 20, 21, 27, 28] and the references therein.

In 2013, Sarikaya et al. established the following Hadamard-type inequalities by utilizing Riemann-Liouville fractional integrals.

Theorem 1.2 ([30]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function along with $0 \leq a < b$ and let $f \in L^1[a,b]$. Suppose that f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}[J^{\mu}_{a^+}f(b) + J^{\mu}_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where the symbols $J^{\mu}_{a^+}f$ and $J^{\mu}_{b^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu > 0$ defined by

$$J^{\mu}_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\mathrm{d}t, \quad a < x$$

and

$$J^{\mu}_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)\mathrm{d}t, \quad x < b.$$

Here, $\Gamma(\mu)$ is the gamma function and its definition is $\Gamma(\mu) = \int_0^{\infty} e^{-t}t^{\mu-1}\mathrm{d}t$. It is to be noted that $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral reduces to the classical integral.

In 2016, Sarikaya and Yildirim presented another form with respect to Riemann-Liouville fractional Hadamard-type inequalities as follows.

Theorem 1.3 ([31]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\Big[J^{\mu}_{(\frac{a+b}{2})^+}f(b) + J^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\mu > 0$.

Due to the extensive application of Riemann-Liouville fractional integrals, there have been many studies involving this integral operator, for example, see [14, 15, 22, 26, 33] and the references therein.

In 2012, Mubeen and Habibullah presented the following k-fractional integrals.

Definition 1.1 ([24]). *Let $f \in L^1[a,b]$, then Riemann-Liouville k-fractional integrals ${}_kJ^{\mu}_{a^+}f(x)$ and ${}_kJ^{\mu}_{b^-}f(x)$ of order $\mu > 0$ are given as*

$$ {}_kJ^{\mu}_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)\mathrm{d}t, \quad (0 \le a < x < b) $$

and

$$ {}_kJ^{\mu}_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)\mathrm{d}t, \quad (0 \le a < x < b), $$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k-gamma function defined by $\Gamma_k(\mu) = \int_0^{\infty} t^{\mu-1}e^{-\frac{t^k}{k}}\mathrm{d}t$. Furthermore, $\Gamma_k(\mu+k) = \mu\Gamma_k(\mu)$ and ${}_kJ^0_{a^+}f(x) = {}_kJ^0_{b^-}f(x) = f(x)$.

In the case of $k = 1$, the k-fractional integrals reduces to Riemann-Liouville fractional integrals. For some recent results related to the k-fractional integral inequalities see [1, 2, 5, 29, 32].

In 2016, Farid et al. popularized Theorem 1.3 to the form of k-fractional integrals.

Theorem 1.4 ([11]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$ f\Big(\frac{a+b}{2}\Big) \le \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b) + {}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big] \le \frac{f(a)+f(b)}{2} \tag{1.4} $$

with $\mu, k > 0$.

The main aim of this article is to establish some new k-fractional integral inequalities related to generalized (s,m)-preinvex functions. The obtained k-fractional integral inequalities can be viewed as the extension of the results of [6, 11, 18, 23] and [25].

To end this section, let us recall some special functions and basic definitions as follows.

(1) The beta function:

$$ \beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x,y > 0, $$

(2) The hypergeometric function:

$$ {}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad c > b > 0, |z| < 1. $$

Definition 1.2 ([12]). *A function $f : [0,\infty) \to \mathbb{R}$ is named s-convex in the second sense with $s \in (0,1]$, if*

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in [0,\infty)$ and $\alpha, \beta \ge 0$ along with $\alpha + \beta = 1$.

Definition 1.3 ([9]). *A set $K \subseteq \mathbb{R}^n$ is named m-invex with respect to the mapping $\eta : K \times K \times (0,1] \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + \lambda\eta(y,x,m) \in K$ holds for all $x, y \in K$ and $\lambda \in [0,1]$.*

Definition 1.4 ([9]). *Let $K \subseteq \mathbb{R}^n$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For some fixed $s, m \in (0,1]$, f is said to be generalized (s,m)-preinvex, if*

$$f\big(mx + t\eta(y,x,m)\big) \le m(1-t)^s f(x) + t^s f(y)$$

is valid for all $x, y \in K$ and $t \in [0,1]$.

Definition 1.5 ([3]). *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to the mapping $\eta : K \times K \to \mathbb{R}^n$. For every $x, y \in K$, the η-path P_{xv} joining the points x and $v = x + \eta(y,x)$ is defined by*

$$P_{xv} = \big\{z | z = x + t\eta(y,x), t \in [0,1]\big\}.$$

Definition 1.6. *Let $K \subseteq \mathbb{R}^n$ be an m-invex set with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For every $u, v \in K$ and $m \in (0,1]$, the η_m-path P_{vw} joining the points mv and $w = mv + \eta(u,v,m)$ is defined by*

$$P_{vw} = \big\{z | z = mv + \lambda\eta(u,v,m), \lambda \in [0,1]\big\}.$$

Remark 1.1. If $\eta(u,v,m)$ with $m = 1$ reduces to $\eta(u,v)$, then Definition 1.6 reduces to Definition 1.5.

2. k-Fractional inequalities involving differentiable functions

Throughout this section, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}^*$ be the set of all positive integers and let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R} \setminus \{0\}$ for some fixed $m \in (0,1]$, $a, b \in K$ with $a < b$. Assume that $f : K \to \mathbb{R}$ is a differentiable function such that f' is integrable on the η_m-path $P_{vw} : w = mv + \eta(u,v,m)$ for arbitrary $u, v \in [a,b]$. Before stating the results we define the following notations:

$$\begin{aligned} &\mathcal{H}_{\eta_m}(\mu, k; n, x) \\ &:= \frac{n+1}{2}\left[\frac{\eta^{\frac{\mu}{k}}(x,a,m) f\big(ma + \eta(x,a,m)\big) + \eta^{\frac{\mu}{k}}(b,x,m) f(mx)}{\eta(b,a,m)}\right. \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&+\frac{\eta^{\frac{\mu}{k}}(x,a,m)f(ma)+\eta^{\frac{\mu}{k}}(b,x,m)f\big(mx+\eta(b,x,m)\big)}{\eta(b,a,m)}\Bigg]\\
&-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{2\eta(b,a,m)}\\
&\times\Bigg[{}_kJ^{\mu}_{(ma)^+}f\bigg(ma+\frac{1}{n+1}\eta\big(x,a,m\big)\bigg)\\
&+{}_kJ^{\mu}_{(ma+\eta(x,a,m))^-}f\bigg(ma+\frac{n}{n+1}\eta\big(x,a,m\big)\bigg)\\
&+{}_kJ^{\mu}_{(mx)^+}f\bigg(mx+\frac{1}{n+1}\eta\big(b,x,m\big)\bigg)\\
&+{}_kJ^{\mu}_{(mx+\eta(b,x,m))^-}f\bigg(mx+\frac{n}{n+1}\eta\big(b,x,m\big)\bigg)\Bigg].
\end{aligned}
$$

Especially if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, equation (2.1) reduces to

$$
\begin{aligned}
&\mathcal{H}(\mu,k;n,x)\\
&:=\frac{n+1}{2}\bigg[\frac{(x-a)^{\frac{\mu}{k}}+(b-x)^{\frac{\mu}{k}}}{b-a}f(x)+\frac{(x-a)^{\frac{\mu}{k}}f(a)+(b-x)^{\frac{\mu}{k}}f(b)}{b-a}\bigg]\\
&-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{2(b-a)}\bigg[{}_kJ^{\mu}_{a^+}f\bigg(\frac{n}{n+1}a+\frac{1}{n+1}x\bigg)+{}_kJ^{\mu}_{x^-}f\bigg(\frac{1}{n+1}a+\frac{n}{n+1}x\bigg)\\
&+{}_kJ^{\mu}_{x^+}f\bigg(\frac{n}{n+1}x+\frac{1}{n+1}b\bigg)+{}_kJ^{\mu}_{b^-}f\bigg(\frac{1}{n+1}x+\frac{n}{n+1}b\bigg)\bigg].
\end{aligned}
$$

We need the succeeding lemma.

Lemma 2.1. *The following k-fractional integral identity along with $x\in(a,b)$, $n\in\mathbb{N}^*$, $\mu>0$ and $k>0$ holds:*

$$
\begin{aligned}
&\mathcal{H}_{\eta_m}(\mu,k;n,x)\\
&=\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{2\eta(b,a,m)}\bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(ma+\frac{n+t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t\\
&\quad-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(ma+\frac{1-t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t\bigg\}\\
&\quad-\frac{\eta^{\frac{\mu}{k}+1}(b,x,m)}{2\eta(b,a,m)}\bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(mx+\frac{1-t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t\\
&\quad-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(mx+\frac{n+t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t\bigg\}.
\end{aligned}
\tag{2.2}
$$

Proof. By integration by parts and changing the variable, we can state

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\bigg(ma+\frac{n+t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t\\
&=\frac{(n+1)t^{\frac{\mu}{k}} f\big(ma+\frac{n+t}{n+1}\eta(x,a,m)\big)}{\eta(x,a,m)}\bigg|_0^1-\int_0^1\frac{\mu(n+1)t^{\frac{\mu}{k}-1}f\big(ma+\frac{n+t}{n+1}\eta(x,a,m)\big)}{k\eta(x,a,m)}\mathrm{d}t\\
&=\frac{(n+1)f\big(ma+\eta(x,a,m)\big)}{\eta(x,a,m)}\\
&\quad-\frac{\mu(n+1)^{\frac{\mu}{k}+1}}{k\eta^{\frac{\mu}{k}+1}(x,a,m)}\int_{ma+\frac{n}{n+1}\eta(x,a,m)}^{ma+\eta(x,a,m)}\bigg[u-\bigg(ma+\frac{n}{n+1}\eta\big(x,a,m\big)\bigg)\bigg]^{\frac{\mu}{k}-1}f(u)\mathrm{d}u\\
&=\frac{(n+1)f\big(ma+\eta(x,a,m)\big)}{\eta(x,a,m)}\\
&\quad-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(x,a,m)}{}_kJ^{\mu}_{(ma+\eta(x,a,m))^-}f\bigg(ma+\frac{n}{n+1}\eta\big(x,a,m\big)\bigg).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\bigg(ma+\frac{1-t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t\\
&=-\frac{(n+1)f(ma)}{\eta(x,a,m)}+\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(x,a,m)}{}_kJ^{\mu}_{(ma)^+}f\bigg(ma+\frac{1}{n+1}\eta\big(x,a,m\big)\bigg),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\bigg(mx+\frac{1-t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t\\
&=-\frac{(n+1)f(mx)}{\eta(b,x,m)}+\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(b,x,m)}{}_kJ^{\mu}_{(mx)^+}f\bigg(mx+\frac{1}{n+1}\eta\big(b,x,m\big)\bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\bigg(mx+\frac{n+t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t\\
&=\frac{(n+1)f\big(mx+\eta(b,x,m)\big)}{\eta(b,x,m)}\\
&\quad-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(b,x,m)}{}_kJ^{\mu}_{(mx+\eta(b,x,m))^-}f\bigg(mx+\frac{n}{n+1}\eta\big(b,x,m\big)\bigg).
\end{aligned}$$

After suitable rearrangements we obtain the desired result. This ends the proof.

Corollary 2.1. *In Lemma 2.1, if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, we have*

$$
\begin{aligned}
&\mathcal{H}(\mu,k;n,x)\\
&=\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(\frac{n+t}{n+1}x+\frac{1-t}{n+1}a\bigg)\mathrm{d}t\\
&-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(\frac{1-t}{n+1}x+\frac{n+t}{n+1}a\bigg)\mathrm{d}t\Bigg\}\\
&-\frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(\frac{n+t}{n+1}x+\frac{1-t}{n+1}b\bigg)\mathrm{d}t\\
&-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(\frac{1-t}{n+1}x+\frac{n+t}{n+1}b\bigg)\mathrm{d}t\Bigg\}.
\end{aligned}
\tag{2.3}
$$

Remark 2.1. (i) In Lemma 2.1, if $\eta(u,v,m)$ with $m=1$ reduces to $\eta(u,v)$ for $u,v\in[a,b]$, putting $k=1$ along with $n=1$, we have Lemma 2.8 in [25].

(ii) In Corollary 2.1,

(a) putting $k=1$, we have Lemma 2.5 in [4],

(b) putting $k=1=n$, we have Lemma 1 in [23]. Further, putting $\mu=1$, we have Lemma 1 in [18].

Utilizing Lemma 2.1, the following theorem can be obtained.

Theorem 2.1. *Suppose that $|f'|^q$ for $q\geq 1$ is generalized (s,m)-preinvex, then for $x\in(a,b)$, $n\in\mathbb{N}^*$, $\mu>0$ and $k>0$, the following k-fractional integral inequality holds:*

$$
\begin{aligned}
\big|\mathcal{H}_{\eta_m}(\mu,k;n,x)\big|\leq&\bigg(\frac{1}{\frac{\mu}{k}+1}\bigg)^{1-\frac{1}{q}}\Bigg\{\frac{\big|\eta^{\frac{\mu}{k}+1}(x,a,m)\big|}{2|\eta(b,a,m)|}\Bigg[\Big(m\Phi_1(\mu,k,n,s)\big|f'(a)\big|^q\\
&+\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\\
&+\Big(m\Phi_2(\mu,k,n,s)\big|f'(a)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\Bigg]\\
&+\frac{\big|\eta^{\frac{\mu}{k}+1}(b,x,m)\big|}{2|\eta(b,a,m)|}\Bigg[\Big(m\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\\
&+\Big(m\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_2(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\Bigg]\Bigg\},
\end{aligned}
\tag{2.4}
$$

where

$$
\Phi_1(\mu,k,n,s)=\int_0^1 t^{\frac{\mu}{k}}\bigg(\frac{1-t}{n+1}\bigg)^s\mathrm{d}t=\frac{\beta(\frac{\mu}{k}+1,s+1)}{(n+1)^s}
$$

and

$$\Phi_2(\mu,k,n,s)=\int_0^1 t^{\frac{\mu}{k}}\left(\frac{n+t}{n+1}\right)^s \mathrm{d}t=\begin{cases}\dfrac{{}_2F_1[-s,1;\frac{\mu}{k}+2;\frac{1}{2}]}{\frac{\mu}{k}+1}, & n=1,\\ \dfrac{n^s{}_2F_1[-s,\frac{\mu}{k}+1;\frac{\mu}{k}+2;-\frac{1}{n}]}{(\frac{\mu}{k}+1)(n+1)^s}, & n>1.\end{cases}$$

Proof. Using Lemma 2.1, the power-mean inequality and the generalized (s,m)-preinvexity of $|f'|^q$, we get

$$\begin{aligned}\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right| &\le \frac{\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu}{k}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[(I_1)^{\frac{1}{q}}+(I_2)^{\frac{1}{q}}\right]\\ &\quad+\frac{\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu}{k}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[(I_3)^{\frac{1}{q}}+(I_4)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(ma+\frac{n+t}{n+1}\eta\big(x,a,m\big)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{1-t}{n+1}\right)^s\left|f'(a)\right|^q+\left(\frac{n+t}{n+1}\right)^s\left|f'(x)\right|^q\right]\mathrm{d}t,\\ I_2&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(ma+\frac{1-t}{n+1}\eta\big(x,a,m\big)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{n+t}{n+1}\right)^s\left|f'(a)\right|^q+\left(\frac{1-t}{n+1}\right)^s\left|f'(x)\right|^q\right]\mathrm{d}t,\\ I_3&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(mx+\frac{1-t}{n+1}\eta\big(b,x,m\big)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{n+t}{n+1}\right)^s\left|f'(x)\right|^q+\left(\frac{1-t}{n+1}\right)^s\left|f'(b)\right|^q\right]\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}I_4&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(mx+\frac{n+t}{n+1}\eta\big(b,x,m\big)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{1-t}{n+1}\right)^s\left|f'(x)\right|^q+\left(\frac{n+t}{n+1}\right)^s\left|f'(b)\right|^q\right]\mathrm{d}t.\end{aligned}$$

Hence the proof is completed.

Corollary 2.2. *In Theorem 2.1,*

(*i*) *if we put $q=1$, we have:*

$$\begin{aligned}&\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right|\\ &\le\frac{\Phi_1(\mu,k,n,s)+\Phi_2(\mu,k,n,s)}{2|\eta(b,a,m)|}\Big[\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|\Big(m\left|f'(a)\right|+\left|f'(x)\right|\Big)\\ &\quad+\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|\Big(m\left|f'(x)\right|+\left|f'(b)\right|\Big)\Big].\end{aligned}$$

Especially if $\eta(u,v,m)$ with $m=1$ reduces to $\eta(u,v)$ for $u,v\in[a,b]$, and choosing $k=1=n$, we get Theorem 3.1 *proved by Noor et al. in* [25],

(ii) if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, we have:

$$
\begin{aligned}
&\big|\mathcal{H}(\mu,k;n,x)\big|\\
&\le\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\Bigg\{\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg[\Big(\Phi_1(\mu,k,n,s)\big|f'(a)\big|^q+\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\\
&\quad+\Big(\Phi_2(\mu,k,n,s)\big|f'(a)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\Bigg]\\
&\quad+\frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg[\Big(\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\\
&\quad+\Big(\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_2(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}
$$

Especially if we choose $k=1=n$ along with $s=1$, we get Theorem 3 *established by Mihai and Mitroi in* [23]. *Further, if we take $\mu=1$, we obtain Theorem* 3 *presented by Latif in* [18],

(iii) if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, and putting $x=\frac{a+b}{2}$, $n=1$, we obtain:

$$
\begin{aligned}
&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1}\mathcal{H}\left(\mu,k;1,\frac{a+b}{2}\right)\right|\\
&=\left|\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\frac{2\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{a^+}f\left(\frac{3a+b}{4}\right)\right.\right.\\
&\quad\left.\left.+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f\left(\frac{3a+b}{4}\right)+{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right)+{}_kJ^{\mu}_{b^-}f\left(\frac{a+3b}{4}\right)\right]\right|\\
&\le\frac{b-a}{8}\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\Bigg\{\left[\Phi_1(\mu,k,1,s)\big|f'(a)\big|^q+\Phi_2(\mu,k,1,s)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{\frac{1}{q}}\\
&\quad+\left[\Phi_2(\mu,k,1,s)\big|f'(a)\big|^q+\Phi_1(\mu,k,1,s)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{\frac{1}{q}}\\
&\quad+\left[\Phi_2(\mu,k,1,s)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Phi_1(\mu,k,1,s)\big|f'(b)\big|^q\right]^{\frac{1}{q}}\\
&\quad+\left[\Phi_1(\mu,k,1,s)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Phi_2(\mu,k,1,s)\big|f'(b)\big|^q\right]^{\frac{1}{q}}\Bigg\}\\
&\le\frac{b-a}{8}\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\big(1+2^{1-\frac{s}{q}}\big)\Big(\Phi_1^{\frac{1}{q}}(\mu,k,1,s)+\Phi_2^{\frac{1}{q}}(\mu,k,1,s)\Big)\Big[\big|f'(a)\big|+\big|f'(b)\big|\Big].
\end{aligned}
$$

The second inequality is obtained by utilizing the s-convexity of $|f'|^q$ and the fact that

$$\sum_{i=1}^{n}(u_i+v_i)^\theta \le \sum_{i=1}^{n}(u_i)^\theta + \sum_{i=1}^{n}(v_i)^\theta,\ u_i, v_i \ge 0,\ 1 \le i \le n,\ 0 \le \theta \le 1.$$

If $|f'|^q$ for $q>1$ is also generalized (s,m)-preinvex, we obtain the following result.

Theorem 2.2. *Assume that $|f'|^q$ for $q>1$ is generalized (s,m)-preinvex with $\frac{1}{p}+\frac{1}{q}=1$, then for $x \in (a,b)$, $n \in \mathbb{N}^*$, $\mu>0$ and $k>0$, the following k-fractional integral inequality holds:*

$$\begin{aligned}
&\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right| \\
&\le \left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}} \left\{ \frac{\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|}{2|\eta(b,a,m)|}\left[\left(m\Psi_1\left|f'(a)\right|^q + \Psi_2\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right.\right. \\
&\quad \left. + \left(m\Psi_2\left|f'(a)\right|^q + \Psi_1\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad + \frac{\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|}{2|\eta(b,a,m)|}\left[\left(m\Psi_2\left|f'(x)\right|^q + \Psi_1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.\left. + \left(m\Psi_1\left|f'(x)\right|^q + \Psi_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\},
\end{aligned} \tag{2.5}$$

where

$$\Psi_1 = \int_0^1 \left(\frac{1-t}{n+1}\right)^s \mathrm{d}t = \frac{1}{(s+1)(n+1)^s},$$
$$\Psi_2 = \int_0^1 \left(\frac{n+t}{n+1}\right)^s \mathrm{d}t = \frac{(n+1)^{s+1}-n^{s+1}}{(s+1)(n+1)^s}.$$

Proof. From Lemma 2.1, utilizing the Hölder inequality and the generalized (s,m)-preinvexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right| &\le \frac{\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu p}{k}}\mathrm{d}t\right)^{\frac{1}{p}}\left[(J_1)^{\frac{1}{q}}+(J_2)^{\frac{1}{q}}\right] \\
&\quad + \frac{\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu p}{k}}\mathrm{d}t\right)^{\frac{1}{p}}\left[(J_3)^{\frac{1}{q}}+(J_4)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^1 \left|f'\left(ma+\frac{n+t}{n+1}\eta(x,a,m)\right)\right|^q \mathrm{d}t \\
&\le \int_0^1 m\left(\frac{1-t}{n+1}\right)^s \left|f'(a)\right|^q + \left(\frac{n+t}{n+1}\right)^s \left|f'(x)\right|^q \mathrm{d}t, \\
J_2 &= \int_0^1 \left|f'\left(ma+\frac{1-t}{n+1}\eta(x,a,m)\right)\right|^q \mathrm{d}t
\end{aligned}$$

$$\leq \int_0^1 m\left(\frac{n+t}{n+1}\right)^s \left|f'(a)\right|^q + \left(\frac{1-t}{n+1}\right)^s \left|f'(x)\right|^q \mathrm{d}t,$$

$$J_3 = \int_0^1 \left|f'\left(mx + \frac{1-t}{n+1}\eta\big(b,x,m\big)\right)\right|^q \mathrm{d}t$$

$$\leq \int_0^1 m\left(\frac{n+t}{n+1}\right)^s \left|f'(x)\right|^q + \left(\frac{1-t}{n+1}\right)^s \left|f'(b)\right|^q \mathrm{d}t$$

and

$$J_4 = \int_0^1 \left|f'\left(mx + \frac{n+t}{n+1}\eta\big(b,x,m\big)\right)\right|^q \mathrm{d}t$$

$$\leq \int_0^1 m\left(\frac{1-t}{n+1}\right)^s \left|f'(x)\right|^q + \left(\frac{n+t}{n+1}\right)^s \left|f'(b)\right|^q \mathrm{d}t.$$

This completes the proof.

Corollary 2.3. *In Theorem 2.2,*

(i) if $\eta(u,v,m) = u - mv$ with $m = 1$ for $u, v \in [a,b]$, we have:

$$\begin{aligned}\left|\mathcal{H}(\mu,k;n,x)\right| \leq \left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}} \Bigg\{ &\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg[\Big(\Psi_1\left|f'(a)\right|^q + \Psi_2\left|f'(x)\right|^q\Big)^{\frac{1}{q}} \\ &+ \Big(\Psi_2\left|f'(a)\right|^q + \Psi_1\left|f'(x)\right|^q\Big)^{\frac{1}{q}}\Bigg] \\ &+ \frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\Bigg[\Big(\Psi_2\left|f'(x)\right|^q + \Psi_1\left|f'(b)\right|^q\Big)^{\frac{1}{q}} \\ &+ \Big(\Psi_1\left|f'(x)\right|^q + \Psi_2\left|f'(b)\right|^q\Big)^{\frac{1}{q}}\Bigg]\Bigg\}.\end{aligned}$$

Especially if we choose $k = 1 = n$ along with $s = 1$, we get Theorem 2 established by Mihai and Mitroi in [23]. Further, if we take $\mu = 1$, we obtain Theorem 2 presented by Latif in [18],

(ii) if $\eta(u,v,m) = u - mv$ with $m = 1$ for $u, v \in [a,b]$, putting $x = \frac{a+b}{2}$, $n = 1$, and utilizing similar arguments as in (iii) of Corollary 2.3, we have:

$$\begin{aligned}&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1} \mathcal{H}\left(\mu,k;1,\frac{a+b}{2}\right)\right| \\ &= \left|\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{2^{\frac{2\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{a^+} f\left(\frac{3a+b}{4}\right)\right.\right. \\ &\quad \left.\left. + {}_kJ^{\mu}_{(\frac{a+b}{2})^-} f\left(\frac{3a+b}{4}\right) + {}_kJ^{\mu}_{(\frac{a+b}{2})^+} f\left(\frac{a+3b}{4}\right) + {}_kJ^{\mu}_{b^-} f\left(\frac{a+3b}{4}\right)\right]\right|\end{aligned}$$

$$\leq \frac{b-a}{8}\Big(\frac{1}{\frac{\mu p}{k}+1}\Big)^{\frac{1}{p}}\Bigg\{\Big[\frac{1}{2^s(s+1)}|f'(a)|^q+\frac{2^{s+1}-1}{2^s(s+1)}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q\Big]^{\frac{1}{q}}$$
$$+\Big[\frac{2^{s+1}-1}{2^s(s+1)}|f'(a)|^q+\frac{1}{2^s(s+1)}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q\Big]^{\frac{1}{q}}$$
$$+\Big[\frac{2^{s+1}-1}{2^s(s+1)}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{1}{2^s(s+1)}|f'(b)|^q\Big]^{\frac{1}{q}}$$
$$+\Big[\frac{1}{2^s(s+1)}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{2^{s+1}-1}{2^s(s+1)}|f'(b)|^q\Big]^{\frac{1}{q}}\Bigg\}$$
$$\leq \frac{b-a}{8}\Big(\frac{1}{\frac{\mu p}{k}+1}\Big)^{\frac{1}{p}}\big(1+2^{1-\frac{s}{q}}\big)\Bigg\{\Big[\frac{1}{2^s(s+1)}\Big]^{\frac{1}{q}}+\Big[\frac{2^{s+1}-1}{2^s(s+1)}\Big]^{\frac{1}{q}}\Bigg\}\Big[|f'(a)|+|f'(b)|\Big].$$

Remark 2.2. In Theorem 2.2, if $\eta(u,v,m)$ with $m=1$ reduces to $\eta(u,v)$ for $u,v\in[a,b]$, and choosing $k=1=n$, we get Theorem 3.3 proved by Noor et al. in [25].

3. k-Fractional inequalities for products of two functions

We next establish k-fractional integral inequality involving products of two generalized (s,m)-preinvex functions.

Theorem 3.1. *Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\to\mathbb{R}\setminus\{0\}$ for some fixed $m\in(0,1]$, $a,b\in K$ with $0\leq a<b$. If $f,g: K\to(0,+\infty)$ are generalized (s_1,m)-preinvex and generalized (s_2,m)-preinvex, respectively, then the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}\Big[{}_kJ^{\mu}_{(ma)^+}(fg)\big(ma+\eta(b,a,m)\big)+{}_kJ^{\mu}_{(ma+\eta(b,a,m))^-}(fg)(ma)\Big]\\ &\leq\Big[\frac{\mu}{2(\mu+ks_1+ks_2)}+\frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}\Big]\Big[m^2f(a)g(a)+f(b)g(b)\Big]\\ &+\Big[\frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}+\frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}\Big]\Big[mf(a)g(b)+mf(b)g(a)\Big].\end{aligned}\tag{3.1}$$

Proof. Since f is generalized (s_1,m)-preinvex and g is generalized (s_2,m)-preinvex, we get

$$\begin{aligned}&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma)^+}(fg)\big(ma+\eta(b,a,m)\big)\\ &=\frac{\mu}{2k\eta^{\frac{\mu}{k}}(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}\big(ma+\eta(b,a,m)-u\big)^{\frac{\mu}{k}-1}f(u)g(u)\mathrm{d}u\\ &=\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}f\big(ma+t\eta(b,a,m)\big)g\big(ma+t\eta(b,a,m)\big)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mu}{2k}\int_0^1 (1-t)^{\frac{\mu}{k}-1}\Big[m(1-t)^{s_1}f(a)+t^{s_1}f(b)\Big]\Big[m(1-t)^{s_2}g(a)+t^{s_2}g(b)\Big]\mathrm{d}t\\
&= \frac{\mu}{2k}\int_0^1 \Big[m^2(1-t)^{\frac{\mu}{k}+s_1+s_2-1}f(a)g(a)+mt^{s_2}(1-t)^{\frac{\mu}{k}+s_1-1}f(a)g(b)\\
&\quad + mt^{s_1}(1-t)^{\frac{\mu}{k}+s_2-1}f(b)g(a)+t^{s_1+s_2}(1-t)^{\frac{\mu}{k}-1}f(b)g(b)\Big]\mathrm{d}t\\
&= \frac{\mu}{2(\mu+ks_1+ks_2)}m^2f(a)g(a)+\frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}mf(a)g(b)\\
&\quad + \frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}mf(b)g(a)+\frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}f(b)g(b).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma+\eta(b,a,m))^-}(fg)(ma)\\
&\le \frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}m^2f(a)g(a)+\frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}mf(a)g(b)\\
&\quad + \frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}mf(b)g(a)+\frac{\mu}{2(\mu+ks_1+ks_2)}f(b)g(b).
\end{aligned}
$$

By adding both sides of the above inequalities we can obtain the desired result. This completes the proof.

Corollary 3.1. *In Theorem 3.1, if the mapping $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$ and $s_1=s_2=s$, we obtain*

$$
\begin{aligned}
&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a)}\Big[{}_kJ^{\mu}_{a^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\eta(b,a))^-}(fg)(a)\Big]\\
&\le \left[\frac{\mu}{2(\mu+2ks)}+\frac{\mu\beta(2s+1,\frac{\mu}{k})}{2k}\right]\Big[f(a)g(a)+f(b)g(b)\Big]\\
&\quad + \left[\frac{\mu\beta(s+1,\frac{\mu}{k}+s)}{k}\right]\Big[f(a)g(b)+f(b)g(a)\Big].
\end{aligned}
$$

Especially if $\eta(b,a)=b-a$ and $k=1=s$, we get

$$
\begin{aligned}
&\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\Big[J^{\mu}_{a^+}f(b)g(b)+J^{\mu}_{b^-}f(a)g(a)\Big]\\
&\le \frac{\mu^2+\mu+2}{2(\mu+1)(\mu+2)}\Big[f(a)g(a)+f(b)g(b)\Big]+\frac{\mu}{(\mu+1)(\mu+2)}\Big[f(a)g(b)+f(b)g(a)\Big],
\end{aligned}
$$

which is Theorem 2.1 established by Chen in [6].

Corollary 3.2. *In Theorem* 3.1*, if the mapping $\eta(b,a,m)=b-ma$ with $m=1$, $s_1=1=s_2$ and $g(x)=1$, we obtain*

$$
\frac{\Gamma_k(\mu+k)}{2(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{a^+}f(b)+{}_kJ^{\mu}_{b^-}f(a)\Big]\le\frac{f(a)+f(b)}{2}.
$$

Especially if we take $k = 1$, we get

$$\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\Big[J^{\mu}_{a^+}f(b)+J^{\mu}_{b^-}f(a)\Big]\le\frac{f(a)+f(b)}{2},$$

which is the right hand side of the inequality (1.2).

Another k-fractional integral inequality involving products of two generalized (s,m)-preinvex functions is obtained as follows.

Theorem 3.2. *With the same assumptions in Theorem 3.1, we have*

$$\begin{aligned}(3.2)\quad &\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}\Big[{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)\\&\quad+{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)\Big]\\&\le\Upsilon_1\big[m^2f(a)g(a)+f(b)g(b)\big]+\Upsilon_2\big[mf(a)g(b)+mf(b)g(a)\big],\end{aligned}$$

where

$$\Upsilon_1=\frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}+\frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}$$

and

$$\Upsilon_2=\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}.$$

Proof. Since f is generalized (s_1,m)-preinvex and g is generalized (s_2,m)-preinvex, we have

$$\begin{aligned}&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)\\&=\frac{\mu2^{\frac{\mu}{k}-1}}{k\eta^{\frac{\mu}{k}}(b,a,m)}\int_{ma+\frac{1}{2}\eta(b,a,m)}^{ma+\eta(b,a,m)}\big(ma+\eta(b,a,m)-u\big)^{\frac{\mu}{k}-1}f(u)g(u)\mathrm{d}u\\&=\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}f\bigg(ma+\frac{1+t}{2}\eta(b,a,m)\bigg)g\bigg(ma+\frac{1+t}{2}\eta(b,a,m)\bigg)\mathrm{d}t\\&\le\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}\bigg[m\bigg(\frac{1-t}{2}\bigg)^{s_1}f(a)+\bigg(\frac{1+t}{2}\bigg)^{s_1}f(b)\bigg]\\&\times\bigg[m\bigg(\frac{1-t}{2}\bigg)^{s_2}g(a)+\bigg(\frac{1+t}{2}\bigg)^{s_2}g(b)\bigg]\mathrm{d}t\\&=\frac{\mu}{2k}\int_0^1\bigg[\frac{m^2(1-t)^{\frac{\mu}{k}+s_1+s_2-1}}{2^{s_1+s_2}}f(a)g(a)+\frac{m(1-t)^{\frac{\mu}{k}+s_1-1}(1+t)^{s_2}}{2^{s_1+s_2}}f(a)g(b)\\&+\frac{m(1-t)^{\frac{\mu}{k}+s_2-1}(1+t)^{s_1}}{2^{s_1+s_2}}f(b)g(a)+\frac{(1-t)^{\frac{\mu}{k}-1}(1+t)^{s_1+s_2}}{2^{s_1+s_2}}f(b)g(b)\bigg]\mathrm{d}t\end{aligned}$$

$$= \frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}m^2f(a)g(a)+\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}mf(a)g(b)$$
$$+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}mf(b)g(a)+\frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}f(b)g(b).$$

Similarly we get

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)$$
$$\leq \frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}m^2f(a)g(a)+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}mf(a)g(b)$$
$$+\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}mf(b)g(a)+\frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}f(b)g(b).$$

By adding both sides of the above inequalities we can obtain the desired result. This ends the proof.

Corollary 3.3. *In Theorem* 3.2*, if the mapping* $\eta(b,a,m)$ *with* $m=1$ *reduces to* $\eta(b,a)$ *and* $s_1=s_2=s$*, we obtain*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a)}\Big[{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^-}(fg)(a)\Big]$$
$$\leq\left[\frac{\mu}{2^{2s+1}(\mu+2ks)}+\frac{{}_2F_1[-2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}\right]\Big[f(a)g(a)+f(b)g(b)\Big]$$
$$+\left[\frac{{}_2F_1[-s,\frac{\mu}{k}+s;\frac{\mu}{k}+s+1;\frac{1}{2}]\mu}{2^s(\mu+ks)}\right]\Big[f(a)g(b)+f(b)g(a)\Big].$$

Especially if $\eta(b,a)=b-a$ *and* $s=1$ *, we get*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}(fg)(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}(fg)(a)\Big]$$
$$\leq\left[\frac{1}{2}-\frac{\mu}{2(\mu+k)}+\frac{\mu}{4(\mu+2k)}\right]\Big[f(a)g(a)+f(b)g(b)\Big]$$
$$+\left[\frac{\mu}{2(\mu+k)}-\frac{\mu}{4(\mu+2k)}\right]\Big[f(a)g(b)+f(b)g(a)\Big].$$

Corollary 3.4. *In Theorem* 3.2*, if the mapping* $\eta(b,a,m)=b-ma$ *with* $m=1$*,* $s_1=s_2=s$ *and* $g(x)=1$*, we obtain*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big]$$
$$\leq\left[\frac{\mu}{2^{2s+1}(\mu+2ks)}+\frac{{}_2F_1[-2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}+\frac{{}_2F_1[-s,\frac{\mu}{k}+s;\frac{\mu}{k}+s+1;\frac{1}{2}]\mu}{2^s(\mu+ks)}\right]$$
$$\times\Big[f(a)+f(b)\Big].$$

Especially for $s = 1$ *, we get*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b) + {}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

which is the right hand side of the inequality (1.4).

Acknowledgment

This work was supported by the National Natural Science Foundation of China (No. 61374028) and sponsored by Research Fund for Excellent Dissertation of China Three Gorges University (No. 2018SSPY132 and No. 2018SSPY134).

References

[1] P. Agarwal, J. Tariboon, S.K. Ntouyas, *Some generalized Riemann-Liouville k-fractional integral inequalities*, J. Inequal. Appl., 2016 (2016), Article Number 122, 13 pages.

[2] A. Ali, G. Gulshan, R. Hussain, A. Latif, M. Muddassar, *Generalized inequalities of the type of Hermite-Hadamard-Fejer with quasi-convex functions by way of k-fractional derivatives*, J. Comput. Anal. Appl., 22 (7) (2017), 1208-1219.

[3] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal., 60 (8) (2005), 1473-1484.

[4] M.U. Awan, M.A. Noor, M.V. Mihai, K.I. Noor, *Fractional Hermite-Hadamard inequalities for differentiable s-Godunova-Levin functions*, Filomat, 30 (12) (2016), 3235-3241.

[5] M.U. Awan, M.A. Noor, M.V. Mihai, K.I. Noor, *On bounds involving k-Appell's hypergeometric functions*, J. Inequal. Appl., 2017 (2017), Article Number 118, 15 pages.

[6] F.X. Chen, *A note on Heimite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., 33 (2014), 299-306.

[7] F.X. Chen, Y.M. Feng, *New inequalities of Hermite-Hadamard type for functions whose first derivatives absolute values are s-convex*, Ital. J. Pure Appl. Math., 32 (2014), 213-222.

[8] T.S. Du, J.G. Liao, L.Z. Chen, M.U. Awan, *Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized* (α, m)*-preinvex functions*, J. Inequal. Appl., 2016 (2016), Article Number 306, 24 pages.

[9] T.S. Du, J.G. Liao, Y.J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions*, J. Nonlinear Sci. Appl., 9 (5) (2016), 3112-3126.

[10] S. Erden, M.Z. Sarikaya, *On generalized some inequalities for convex functions*, Ital. J. Pure Appl. Math., 38 (2017), 455-468.

[11] G. Farid, A.U. Rehman, M. Zahra, *On Hadamard-type inequalities for k-fractional integrals*, Konuralp J. Math., 4 (2) (2016), 79-86.

[12] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., 48 (1) (1994), 100-111.

[13] D.Y. Hwang, S.S. Dragomir, *Extensions of the Hermite-Hadamard inequality for r-preinvex functions on an invex set*, Bull. Aust. Math. Soc., 95 (3) (2017), 412-423.

[14] S.R. Hwang, K.L. Tseng, K.C. Hsu, *New inequalities for fractional integrals and their applications*, Turkish J. Math., 40 (3) (2016), 471-486.

[15] M. Iqbal, M.I. Bhatti, K. Nazeer, *Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc., 52 (3) (2015), 707-716.

[16] A. Kashuri, R. Liko, *Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions*, Proyecciones, 36 (1) (2017), 45-80.

[17] M.A. Khan, T. Ali, S.S. Dragomir, M.Z. Sarikaya, *Hermite-Hadamard type inequalities for conformable fractional integrals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., (2017), 1-16.

[18] M.A. Latif, *Inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications*, Arab J. Math. Sci., 21 (1) (2015), 84-97.

[19] M.A. Latif, S.S. Dragomir, *New inequalities of Hermite-Hadamard and Fejér type via preinvexity*, J. Comput. Anal. Appl., 19 (4) (2015), 725-739.

[20] M.A. Latif, *On some new inequalities of Hermite-Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value*, Ukrainian Math. J., 67 (10) (2016), 1552-1571.

[21] Y.J. Li, T.S. Du, B. Yu, *Some new integral inequalities of Hadamard-Simpson type for extended (s,m)-preinvex functions*, Ital. J. Pure Appl. Math., 36 (2016), 583-600.

[22] M. Matłoka, *Some inequalities of Hadamard type for mappings whose second derivatives are h-convex via fractional integrals*, J. Fract. Calc. Appl., 6 (1) (2015), 110-119.

[23] M.V. Mihai, F.C. Mitroi, *Hermite-Hadamard type inequalities obtained via Riemann-Liouville fractional calculus*, Acta Math. Univ. Comenian. (N. S.), 83 (2) (2014), 209-215.

[24] S. Mubeen, G.M. Habibullah, *k-fractional integrals and application*, Int. J. Contemp. Math. Sciences, 7 (2) (2012), 89-94.

[25] M.A. Noor, K.I. Noor, M.V. Mihai, M.U. Awan, *Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 78 (3) (2016), 163-174.

[26] C. Peng, C. Zhou, T.S. Du, *Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity*, Ital. J. Pure Appl. Math., 38 (2017), 345-367.

[27] S. Qaisar, C.J. He, S. Hussain, *On new inequalities of Hermite-Hadamard type for generalized convex functions*, Ital. J. Pure Appl. Math., 33 (2014), 139-148.

[28] S. Qaisar, M. Iqbal, M. Muddassar, *New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals*, J. Comput. Anal. Appl., 20 (7) (2016), 1318-1328.

[29] M.Z. Sarikaya, A. Karaca, *On the k-Riemann-Liouville fractional integral and applications*, Int. J. Stat. Math., 1 (3) (2014), 33-43.

[30] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., 57 (2013), 2403-2407.

[31] M.Z. Sarikaya, H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes, 17 (2) (2016), 1049-1059.

[32] E. Set, M. Tomar, M.Z. Sarikaya, *On generalized Grüss type inequalities for k-fractional integrals*, Appl. Math. Comput., 269 (2015), 29-34.

[33] E. Set, İ. İşcan, H.H. Kara, *Hermite-Hadamard-Fejer type inequalities for s-convex function in the second sense via fractional integrals*, Filomat, 30 (12) (2016), 3131-3138.

Accepted: 6.02.2018

SOME OPERATOR α-GEOMETRIC MEAN INEQUALITIES

Jianming Xue
Oxbridge College
Kunming University of Science and Technology
Kunming, Yunnan 650106
P. R. China
xuejianming104@163.com

Abstract. In this paper, we refine an operator α-geometric mean inequality as follows: let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$, then for each $\alpha \in [0,1]$,

$$(\Phi(A) \sharp_\alpha \Phi(B))^2 \leq \left(\frac{K(h)}{K^{2r}(h')}\right)^2 \Phi^2(A \sharp_\alpha B),$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.
Keywords: operator inequalities, α-geometric mean, positive linear maps.

1. Introduction

Throughout this paper, $\|\cdot\|$ is the operator norm and I denotes the identity operator. $A \geq 0$ ($A > 0$) implies that A is positive (strictly positive) operator. Φ is a positive unital linear map if $\Phi(A) \geq 0$ with $A \geq 0$ and $\Phi(I) = I$. For $A, B > 0$ and $\alpha \in [0,1]$, the α-geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}},$$

when $\alpha = \frac{1}{2}$, $A \sharp_{\frac{1}{2}} B = A \sharp B$ is said to be the geometric mean.

Seo [1] gave the following α-geometric mean inequality: let Φ be a positive unital linear map. If $0 < m_1 \leq A, B \leq M_1$ for some numbers $m_1 \leq M_1$. Then for $\alpha \in [0,1]$,

$$\Phi(A) \sharp_\alpha \Phi(B) \leq K(m, M, \alpha)^{-1} \Phi(A \sharp_\alpha B),$$

where $m = \frac{m_1}{M_1}$, $M = \frac{M_1}{m_1}$ and the generalized Kantorovich constant $K(m, M, \alpha)$ ([2, Definition 2.2]) is defined by

$$K(m, M, \alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha - 1)(M - m)} \left(\frac{\alpha - 1}{\alpha} \frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in R$.

Fu [3] squared operator α-geometric mean inequality: let Φ be a positive unital linear map. If $0 < m \leq A, B \leq M$ for some numbers $m \leq M$. Then for $\alpha \in [0,1]$

$$(\Phi(A) \sharp_\alpha \Phi(B))^2 \leq K^2(h)\, \Phi^2(A \sharp_\alpha B), \tag{1.1}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

A great number of results on operator inequalities have been given in the literature, for example, see [4-8] and the references therein.

In this paper, we will get a stronger result than (1.1) and apply it to obtain an operator α-geometric mean inequality to the power of $2p$ $(p \geq 2)$.

2. Main results

In this section, the main results of this paper will be given. To do this, the following lemmas are necessary.

Lemma 1 ([9]). *Let $A, B > 0$. Then*

$$\|AB\| \leq \frac{1}{4} \|A + B\|^2. \tag{2.1}$$

Lemma 2 ([10]). *Let $A > 0$. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.2}$$

Lemma 3 ([11]). *Let $A, B > 0$. Then for $1 \leq r < \infty$,*

$$\|A^r + B^r\| \leq \|(A+B)^r\|. \tag{2.3}$$

Lemma 4 ([12]). *Let $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$. Then for each $\alpha \in [0,1]$,*

$$K^r(h')(A \sharp_\alpha B) \leq A \nabla_\alpha B, \tag{2.4}$$

where $K(h') = \frac{(h'+1)^2}{4h'}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Lemma 5. *Let $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$. Then for each $\alpha \in [0,1]$,*

$$K^r(h')(A^{-1} \sharp_\alpha B^{-1}) \leq A^{-1} \nabla_\alpha B^{-1}, \tag{2.5}$$

where $K(h') = \frac{(h'+1)^2}{4h'}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof. If $0 < m \leq A \leq m' < M' \leq B \leq M$, it follows that

$$0 < \frac{1}{M} \leq B^{-1} \leq \frac{1}{M'} < \frac{1}{m'} \leq A^{-1} \leq \frac{1}{m}.$$

By $h' = \frac{M'}{m'} = \frac{\frac{1}{m'}}{\frac{1}{M'}}$ and (2.4), we have

$$K^r\left(h'\right)\left(A^{-1}\sharp_\alpha B^{-1}\right) \leq A^{-1}\nabla_\alpha B^{-1}.$$

If $0 < m \leq B \leq m' < M' \leq A \leq M$, similarly, (2.5) holds.

This completes the proof. □

Theorem 1. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$, then for each $\alpha \in [0,1]$,*

$$\left(\Phi\left(A\right)\sharp_\alpha \Phi\left(B\right)\right)^2 \leq \left(\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\right)^2 \Phi^2\left(A\sharp_\alpha B\right), \tag{2.6}$$

where $K\left(h\right) = \frac{(h+1)^2}{4h}$, $K\left(h'\right) = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\left\{\alpha, 1-\alpha\right\}$.

Proof. The inequality (2.6) is equivalent to

$$\left\|\left(\Phi\left(A\right)\sharp_\alpha \Phi\left(B\right)\right)\Phi^{-1}\left(A\sharp_\alpha B\right)\right\| \leq \frac{K\left(h\right)}{K^{2r}\left(h'\right)}.$$

It is easy to see that

$$\left(1-\alpha\right)\left(A + MmA^{-1}\right) \leq \left(1-\alpha\right)\left(M+m\right) \tag{2.7}$$

and

$$\alpha\left(B + MmB^{-1}\right) \leq \alpha\left(M+m\right). \tag{2.8}$$

Summing up inequalities (2.7) and (2.8), we get

$$A\nabla_\alpha B + Mm\left(A^{-1}\nabla_\alpha B^{-1}\right) \leq M+m$$

and hence

$$\Phi\left(A\nabla_\alpha B\right) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right) \leq M+m. \tag{2.9}$$

Compute

$$\begin{aligned}
&\left\|\Phi\left(A\right)\sharp_\alpha \Phi\left(B\right) MmK^{2r}\left(h'\right)\Phi^{-1}\left(A\sharp_\alpha B\right)\right\| \\
&\leq \frac{1}{4}\left\|K^r\left(h'\right)\Phi\left(A\right)\sharp_\alpha \Phi\left(B\right) + MmK^r\left(h'\right)\Phi^{-1}\left(A\sharp_\alpha B\right)\right\|^2 \quad (\text{by}\,(2.1)) \\
&\leq \frac{1}{4}\left\|K^r\left(h'\right)\Phi\left(A\right)\sharp_\alpha \Phi\left(B\right) + MmK^r\left(h'\right)\Phi\left(A^{-1}\sharp_\alpha B^{-1}\right)\right\|^2 \quad (\text{by}\,(2.2)) \\
&\leq \frac{1}{4}\left\|\Phi\left(A\right)\nabla_\alpha \Phi\left(B\right) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right)\right\|^2 \quad (\text{by}\,(2.4),(2.5)) \\
&\leq \frac{1}{4}\left\|\Phi\left(A\nabla_\alpha B\right) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right)\right\|^2 \\
&\leq \frac{1}{4}\left(M+m\right)^2. \quad (\text{by}\,(2.9))
\end{aligned}$$

That is

$$\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)\Phi^{-1}\left(A\sharp_{\alpha}B\right)\right\| \leq \frac{\left(M+m\right)^{2}}{4MmK^{2r}\left(h'\right)} = \frac{K\left(h\right)}{K^{2r}\left(h'\right)}.$$

Thus, (2.6) holds.

This completes the proof. □

Remark 1. Since $h' > 1$, then

$$\frac{K\left(h\right)}{K^{2r}\left(h'\right)} < K\left(h\right).$$

Thus, inequality (2.6) is tighter than (1.1).

Theorem 2. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$ and $2 \leq p < \infty$, then for each $\alpha \in [0,1]$,*

$$\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2p} \leq \frac{1}{16}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{p}\Phi^{2p}\left(A\sharp_{\alpha}B\right), \tag{2.10}$$

where $K\left(h\right) = \frac{\left(h+1\right)^{2}}{4h}$, $K\left(h'\right) = \frac{\left(h'+1\right)^{2}}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\left\{\alpha, 1-\alpha\right\}$.

Proof. The inequality (2.10) is equivalent to

$$\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)\right\| \leq \frac{1}{4}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{\frac{p}{2}}. \tag{2.11}$$

By the operator reverse monotonicity of inequality (2.6), we have

$$\Phi^{-2}\left(A\sharp_{\alpha}B\right) \leq \left(\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\right)^{2}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{-2}. \tag{2.12}$$

Since $0 < m \leq A, B \leq M$, it follows that

$$m \leq \Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right) \leq M$$

and hence

$$\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2} + M^{2}m^{2}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{-2} \leq M^{2}+m^{2}. \tag{2.13}$$

Compute

$$\begin{aligned}
&\left\|(\Phi(A)\sharp_\alpha\Phi(B))^p M^p m^p \Phi^{-p}(A\sharp_\alpha B)\right\| \\
&\le \frac{1}{4}\left\|\left(\frac{K(h)}{K^{2r}(h')}\right)^{\frac{p}{2}}(\Phi(A)\sharp_\alpha\Phi(B))^p + \left(\frac{M^2m^2}{\frac{K(h)}{K^{2r}(h')}}\right)^{\frac{p}{2}}\Phi^{-p}(A\sharp_\alpha B)\right\|^2 \quad (\text{by}\,(2.1)) \\
&\le \frac{1}{4}\left\|\frac{K(h)}{K^{2r}(h')}(\Phi(A)\sharp_\alpha\Phi(B))^2 + \frac{M^2m^2}{\frac{K(h)}{K^{2r}(h')}}\Phi^{-2}(A\sharp_\alpha B)\right\|^p \quad (\text{by}\,(2.3)) \\
&\le \frac{1}{4}\left\|\frac{K(h)}{K^{2r}(h')}(\Phi(A)\sharp_\alpha\Phi(B))^2 + \frac{K(h)}{K^{2r}(h')}M^2m^2(\Phi(A)\sharp_\alpha\Phi(B))^{-2}\right\|^p \quad (\text{by}\,(2.12)) \\
&\le \frac{1}{4}\left(\frac{K(h)}{K^{2r}(h')}(M^2+m^2)\right)^p. \quad (\text{by}\,(2.13))
\end{aligned}$$

That is

$$\left\|(\Phi(A)\sharp_\alpha\Phi(B))^p\Phi^{-p}(A\sharp_\alpha B)\right\| \le \frac{1}{4}\left(\frac{K^2(h)\left(M^2+m^2\right)^2}{K^{4r}(h')M^2m^2}\right)^{\frac{p}{2}}.$$

Thus, (2.10) holds.

This completes the proof. □

Lemma 6 ([13]). *For any bounded operator X,*

$$|X| \le tI \Leftrightarrow \|X\| \le t \Leftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \ge 0 \quad (t \ge 0). \tag{2.14}$$

Theorem 3. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \le A \le m' < M' \le B \le M$ or $0 < m \le B \le m' < M' \le A \le M$ and $2 \le p < \infty$, then for each $\alpha \in [0,1]$,*

$$\begin{aligned}
&(\Phi(A)\sharp_\alpha\Phi(B))^p\Phi^{-p}(A\sharp_\alpha B) + \Phi^{-p}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^p \\
&\le \frac{1}{2}\left(\frac{K^2(h)\left(M^2+m^2\right)^2}{K^{4r}(h')M^2m^2}\right)^{\frac{p}{2}},
\end{aligned} \tag{2.15}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof. Put $t = \frac{1}{2}(\frac{K^2(h)(M^2+m^2)^2}{K^{4r}(h')M^2m^2})^{\frac{p}{2}}$, $X_1 = (\Phi(A)\sharp_\alpha\Phi(B))^p\Phi^{-p}(A\sharp_\alpha B)$, $X_2 = \Phi^{-p}(A\sharp_\alpha B)(\Phi(A)\sharp_\alpha\Phi(B))^p$ and $X = X_1 + X_2$. By (2.11) and (2.14), we have

$$\begin{bmatrix} tI & X_1 \\ X_2 & tI \end{bmatrix} \ge 0 \tag{2.16}$$

and

$$\begin{bmatrix} tI & X_2 \\ X_1 & tI \end{bmatrix} \geq 0. \tag{2.17}$$

Summing up (2.16) and (2.17), we have

$$\begin{bmatrix} 2tI & X \\ X & 2tI \end{bmatrix} \geq 0.$$

Since X is self-adjoint, (2.15) follows from the maximal characterization of geometric mean.

This completes the proof. □

Acknowledgments

This work is supported by Scientific Research Fund of Yunnan Provincial Education Department (No. 2014Y645).

References

[1] Y. Seo, *Reverses of Ando's inequality for positive linear maps,* Math. Inequal. Appl., 14 (2011), 905-910.

[2] T. Furuta, J. Mićić, J.E. Pečarić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[3] X. Fu, *An operator α-geometric mean inequality*, J. Math. Inequal., 9 (2015), 947-950.

[4] J. Xue, *Some refinements of operator reverse AM-GM mean inequalities,* J. Inequal. Appl., 2017 (2017).

[5] J. Xue, *On reverse weighted arithmetic-geometric mean inequalities for two positive operators*, Ital. J. Pure Appl. Math., 37 (2017), 113-116.

[6] P. Zhang, *More operator inequalities for positive linear maps,* Banach J. Math. Anal., 9 (2015), 166-172.

[7] M. Lin, *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. Appl., 402 (2013), 127-132.

[8] L. Zou, *An arithmetic-geometric mean inequality for singular values and its applications*, Linear Algebra Appl., 528 (2017), 25-32.

[9] R. Bhatia, F. Kittaneh, *Notes on matrix arithmetic-geometric mean inequalities,* Linear Algebra Appl., 308 (2000), 203-211.

[10] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[11] T. Ando, X. Zhan, *Norm inequalities related to operator monotone functions*, Math. Ann., 315 (1999), 771-780.

[12] H. Zuo, G. Shi, M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., 5 (2011), 551-556.

[13] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

Accepted: 7.02.2018

THE HOMO SEPARATION ANALYSIS METHOD FOR SOLVING THE PARTIAL DIFFERENTIAL EQUATION

M. Zuriqat
Department of Mathematics
Al al-Bayt University
P.O. Box 130095, Al Mafraq
Jordan
d.alzuriqat@yahoo.com

Abstract. In this work,the homo separation analysis method (HSAM) is applied to obtain the exact solution for linear and nonlinear partial differential equation. The proposed algorithm presents a procedure of constructing the set of base functions and gives the one-order deformation equation in a simple form. This analytical method is a combination of the homotopy analysis method (HAM) with the separation of variables method. The exact solution is constructed by choosing an initial guess in addition to only one term of the series obtained by HAM. This work verifies the validity and the potential of the HSAM for the study of nonlinear partial differential equation.

Keywords: homotopy analysis method, separation of variables, partial differential equations, analytical method, Black-scholes equation.

1. Introduction

The study of nonlinear partial differential equation is of crucial importance in all areas of physics and engineering, as well as in other disciplines. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an exact solution to a given nonlinear problem. The importance of obtaining the exact solutions of nonlinear PDE in mathematics is stell a significant problem that needs new methods. Several numerical and analytical methods have been developed and successfully employed to solve linear and nonlinear PDE. Such methods include variational iteration method [4, 12], Adomian decomposition method [5, 10, 21], differential transform method [2, 6, 17], the homopety perturbation method [15], the exp method [13], Legendre polynomial method [8, 9] and the homotopy analysis method [18, 19, 22]. Some of these methods use specific transformations and others give the solution as a series which converges to the exact solution. Recently, a lot of attention has been focused on the studies to getting exact solution for linear and nonlinear PDEs. Zhang and others [3, 14] give the exact solution for some specific nonlinear PDE. However, Yang [7] used the modified homotopy perturbation method to obtain the exact solution of the Fokker-Plank equation. Furthermore, Karbalaie et al.[1] used homotopy perturbation method with sepration of variables to find

exact solution of PDE. The HAM yields rapidly convergent series solutions by using few iterations for both linear and nonlinear differential equations. The HAM was successfully applied to solve many nonlinear problems such as Riccati differential equation of fractional order [11], fractional KdV-Burgers-Kuramoto equation [16] and systems of fractional algebraic-differential equations [20]. In this paper, we developed a symbolic algorithm to find the exact solution of nonlinear PDE by using the construct of mth-order deformation equation of HAM and the technique of sepration of variables. We present an elegant fast approach by designing and utilizing a proper initial guess which satisify the initial condition of PDE as follows

$$u_0(x,t) = u(x,0)c_1(t) + \frac{\partial}{\partial x}u(x,0)c_2(t),$$

where $u(x,0)$ is the initial condition of the PDE. The Initial guess $u_0(x,t)$ has the form of sepration of variables, as an initial condition for HAM. By using this method, the other of the PDE to be solved is reduced into an ODE or system of ODEs. The organization of this paper is as follows: in section 2, we present the basic idea of HAM and the construct of homo sepration analysis method (HSAM). In Section 3, four examples are solved to illustrated the applicability of the considered method. Finally, relevant conclusions are drawn in section 4.

2. Homo sepration analysis method

The homotopy analysis method based on the concept of the homotopy, a fundamental concept in topology and differential geometry. In this section, the algorithm of this method is briefly illustrated. To achieve our gool, we consider the nonlinear partial differential equation

$$u_t = F(x,t,u,u_x,u_{xx},u_{xt}), \qquad t \geq 0, \tag{2.1}$$

subject to the initial condition

$$u(x,0) = f(x). \tag{2.2}$$

The so-called zero-order deformation equation of the equation (2.1) can be constructed as follows

$$\begin{aligned}(1-q)L[\phi(x,t,q) - u_0(x,t)] &= qh[\frac{\partial}{\partial t}\phi(x,t,q) - F(x,t,\phi(x,t,q),\\ &\frac{\partial}{\partial x}\phi(x,t,q), \frac{\partial^2}{\partial x^2}\phi(x,t,q), \frac{\partial^2}{\partial x \partial t}\phi(x,t,q))],\end{aligned} \tag{2.3}$$

where $q \in [0,1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $\phi(x,t,q)$ is unknown function and $u_0(x,t)$ is an initial guess of $u(x,t)$ which satisfy the initial condition. Obviously, when $q = 0$

$$\phi(x,t,0) = u_0(x,t), \tag{2.4}$$

and when $q = 1$, we have

$$\phi(x,t,1) = u(x,t). \tag{2.5}$$

Expanding $\phi(x,t,q)$ in Taylor series with respect to q, we get

$$\phi(x,t,q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}. \tag{2.7}$$

If the initial guess $u_0(x,t)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (2.6) converges at $q = 1$, one has

$$u(x,t) = \phi(x,t,1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{2.8}$$

Define the vector

$$\overrightarrow{u}_m(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_m(x,t)\}.$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q, then setting $q = 0$ and dividing them by $m!$, finally using (2.7), we have the so-called high-order deformation equations

$$L[u_m(x,t) - \chi_m\ u_{m-1}(x,t)] = h\ \Re_m(\overrightarrow{u}_{m-1}(x,t)), \tag{2.9}$$

subject to the initial conditions

$$u_m(x,0) = 0,$$

where

$$\begin{aligned} \Re_m(\overrightarrow{u}_{m-1}(x,t)) &= \frac{\partial}{\partial t}u_{m-1}(x,t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}F(x,t,\phi(x,t,q), \\ &\quad \frac{\partial}{\partial x}\phi(x,t,q), \frac{\partial^2}{\partial x^2}\phi(x,t,q), \frac{\partial^2}{\partial x \partial t}\phi(x,t,q))|_{q=0}, \end{aligned} \tag{2.10}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$$

Select the auxiliary linear operator $L = \frac{\partial}{\partial t}$, and $h = -1$, then the mth-order deformation equation (2.9) can be written in the form

$$\frac{\partial}{\partial t}(u_m(x,t) - \chi_m\ u_{m-1}(x,t)) + \Re_m(\overrightarrow{u}_{m-1}(x,t)) = 0. \tag{2.11}$$

For special case take $m = 1$ in equation (2.11) and using the relations (2.4),(2.10), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = \frac{\partial}{\partial t}u_0(x,t) - F(x,t,u_0(x,t),$$

$$\frac{\partial}{\partial x}u_0(x,t), \frac{\partial^2}{\partial x^2}u_0(x,t), \frac{\partial^2}{\partial x \partial t}u_0(x,t)) \equiv 0, \tag{2.12}$$

By utilizing the results of equation (2.8), we approximate the analytical solution $u(x,t)$, by the truncated series:

$$u(x,t) = u_0 + u_1 + ... + u_{n-1} = \sum_{i=0}^{n-1} u_i, \tag{2.13}$$

For simplicity, we assume that $u_n(x,t) = 0$, when $n > 1$, which means that the exact solution in equation (2.13) is

$$u(x,t) = u_0(x,t).$$

To illustrate our basic idea, we consider the initial approximation of equation (2.1) as follows

$$\begin{aligned} u(x,t) &= u_0(x,t) = u(x,0)c_1(t) + \frac{\partial}{\partial x}u(x,0)c_2(t), \\ . &= f(x)c_1(t) + f'(x)c_2(t). \end{aligned} \tag{2.14}$$

Our gool in this method is finding $c_1(t)$ and $c_2(t)$. Since equation (2.14) satisfies the initial condition, we get

$$c_1(0) = 1,\ c_2(0) = 0. \tag{2.15}$$

By substituting equation (2.14) into equation (2.12), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}u_1(x,t) &= f(x)c_1'(t) + f'(x)c_2'(t) - F(x,t,f(x)c_1(t) + f'(x)c_2(t), \\ & f'(x)c_1(t) + f''(x)c_2(t), f''(x)c_1(t) + f'''(x)c_2'(t), \end{aligned} \tag{2.16}$$

$$f'(x)c_1'(t) + f''(x)c_2'(t)) \equiv 0.$$

The partial differential equation (2.16) transform into an ordinary differential equation or a system of ordinary differential equations. The exact solution of the partial differential equation is found when the target unknowns $c_1(t)$ and $c_2(t)$ are computed, by utlizing (2.16) and the initial conditions (2.15).

3. Numerical results

In this work, we carefully propose the HSAM, a reliable modification of the HAM, that gives the exact solution of the linear and non linear partial differential equation. To demonstrate the effectiveness of the method, we consider here the following four examples.

Example 3.1. Consider the following nonhomogenous partial differential equation

$$u_t = -x^2 e^t u_{xx} + (x+2)u_x + tx, \tag{3.1}$$

with the initial condition

$$u(x,0) = x + 2.$$

Using the relation (2.14), then we get the initial approximation

$$u(x,t) = u_0(x,t) = (x+2)c_1(t) + c_2(t),$$

and by using the relation (2.16), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = x(c_1'(t) - c_1(t) - t) + (c_2'(t) + 2c_1'(t) - 2c_1(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) - c_1(t) - t = 0,\ c_1(0) = 1, \tag{3.2}$$

$$c_2'(t) + 2c_1'(t) - 2c_1(t) = 0,\ c_2(0) = 0. \tag{3.3}$$

Solving the equations (3.2) and (3.3), by using the ODEs properties, we obtain

$$c_1(t) = 2e^t - t - 1, \quad c_2(t) = -t^2,$$

and the exact solution is

$$u(x,t) = (x+2)(2e^t - t - 1) - t^2.$$

Example 3.2. Consider the following Black-scholes equation

$$u_t = u_{xx} + (k-1)u_x - ku, \tag{3.4}$$

with the initial condition

$$u(x,0) = e^x - 1. \tag{3.5}$$

The initial approximation has the form

$$u(x,t) = u_0(x,t) = (e^x - 1)c_1(t) + e^x c_2(t),$$

and by using the relation (2.16), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = e^x(c_1'(t) + c_2'(t)) - (c_1'(t) + kc_1(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) + kc_1(t) = 0,\ c_1(0) = 1, \tag{3.6}$$

$$c_1'(t) + c_2'(t) = 0,\ c_2(0) = 0. \tag{3.7}$$

Solving the equations (3.6) and (3.7), by using the ODEs properties, we obtain

$$c_1(t) = e^{-kt},\quad c_2(t) = 1 - e^{-kt},$$

and the exact solution is

$$u(x,t) = (e^x - 1)e^{-kt} + e^x(1 - e^{-kt}).$$

Example 3.3. Consider the following nonlinear partial differential equation

$$u_t = u_{xx} + u_x(u + u_{xx}), \tag{3.8}$$

subject to the initial condition

$$u(x,0) = \sin x. \tag{3.9}$$

The initial approximation has the form

$$u(x,t) = u_0(x,t) = \sin x\ c_1(t) + \cos x\ c_2(t),$$

then

$$\frac{\partial}{\partial t}u_1(x,t) = \sin x(c_1'(t) + c_1(t)) + \cos x(c_2'(t) + c_2(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) + c_1(t) = 0,\ c_1(0) = 1, \tag{3.10}$$

$$c_2'(t) + c_2(t) = 0,\ c_2(0) = 0. \tag{3.11}$$

Solving the equations (3.10) and (3.11), by using the ODEs properties, we obtain

$$c_1(t) = e^{-t},\quad c_2(t) = 0,$$

and the exact solution is

$$u(x,t) = e^{-t}\sin x.$$

Example 3.4. Consider the following nonlinear partial differential equation

$$u_t = u^2 - 4u\ u_x + 2u_{xt} - \frac{1}{8}u\ , \tag{3.12}$$

subject to the initial condition

$$u(x,0) = e^{\frac{1}{4}x}. \tag{3.13}$$

Choose the initial approximation

$$\begin{aligned} u(x,t) &= u_0(x,t) = e^{\frac{1}{4}x}\ c_1(t) + \frac{1}{4}e^{\frac{1}{4}x}\ c_2(t) \\ &= \frac{1}{4}e^{\frac{1}{4}x}(4c_1(t) + c_2(t)), \end{aligned}$$

then

$$\frac{\partial}{\partial t}u_1(x,t) = \frac{1}{32}e^{\frac{1}{4}x}\ [4(4c_1'(t) + c_2'(t)) + (4c_1(t) + c_2(t))] \equiv 0.$$

We obtain the ordinary differential equation

$$4y'(t) + y(t) = 0,\ y(0) = 4, \tag{3.14}$$

where

$$y(t) = 4c_1(t) + c_2(t)$$

Solving the equations (3.14), by using the ODEs properties, we obtain

$$4c_1(t) + c_2(t) = 4e^{-\frac{1}{4}t},$$

and the exact solution is

$$u(x,t) = e^{\frac{1}{4}(x-t)}.$$

4. Conclusions

The fundamental goal of this work is to propose a simple method for the solution of PDEs. A combined form of the HAM with sepration of variables is effectively used to handle linear and nonlinear partial differential equations. The main advantage of the method is its fast and gives exact solution for our problem. In this research work, it was demonstrated through different examples how this new method can be used for solving linear and nonlinear PDE. When compared with the existing published methods, it is easy to notice that the new method has many advantages. It is straightforward, easy to understand and requiring much less computations to perform a limited number of steps of the simple procedure that can be applied to find the exact solution of a wide range of types of PDEs Finally, the recent appearance of nonlinear partial differential equations as models in some fields such as models in science and engineering makes it is necessary to investigate the method of solutions for such equations. and we hope that this work is a step in this direction.

References

[1] A. Karbalaie, H. H. Muhammed, M. Shabni and M. M. Montazeri, *Exact solution of partial differential equation using Homo-Sepration of Variables*, International Journal of Nonlinear Science, 17 (2014), 84-90.

[2] A. Freihat and S. Momani, *Application of multistep generalized differential transform method for the solutions of the fractional-order Chua's system*, Discrete Dynamics in Nature and Society, Art. ID 427393, (2012), pp. 1-12.

[3] A. M. Wazwaz, *The tanh-coth and the sech methods for exact solutions of the Jaulent–Miodek equation*, Phys. Lett. A, 366 (2007), 85–90.

[4] D. D. Ganji, M. Jannatabadi and E. Mohseni, *Application of He's variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM*, J. Comput. Appl. Math., 207 (2007), 35–45.

[5] D. H. Shou and J. H. He, *Beyond Adomian method: The variational iteration method for solving heat-like and wave-like equations with variable coefficients*, Phys. Lett. A, 372 (3) (2008), 233-237.

[6] E. Abuteen, A. Freihat, M. Al-Smadi, H. Khalil and R. Ali Khan, *A new reliable algorithm using the generalized differential transform method for the numeric analytic solution of fractional-order Liu chaotic and hyperchaotic systems*, Journal of Mathematics and Statistics, 75 (9) (2013), 263-276.

[7] G. Yang, R. Chen and L. Yao, *On exact solutions to partial differential equations by the modified homotopy perturbation method*, Acta Mathematicae Applicatae Sinica, 28 (2012), 91-98.

[8] H. Khalil, R. A. Khan, M. A-Smadi and A. Freihat, *A generalized algorithm based on Legendre polynomials for numerical solutions of coupled system of fractional order differential equations*, Journal of Fractional Calculus and Applications, 6 (2), (2015), 123-143.

[9] H. Khalil, R. A. Khan, M. A-Smadi, A. Freihat and N. Shawagfeh, *New operational matrix for shifted Legendre polynomials and Fractional differential equations with variable coefficients*, Punjab University Journal of Mathematics, 47 (2015), 81-103.

[10] J. B. Chen and X. G. Geng, *Decomposition to the modified Jaulent–Miodek hierarchy*, Chaos Solitons Fractals, 30 (2006), 797–803.

[11] J. Cang, Y. Tan, H. Xu and S. Liao, *Series solutions of non-linear Riccati differential equations with fractional order*, Chaos Solitons and Fractals, 40 (2009), 1-9.

[12] J. H. He and X. H. Wu, *Construction of solitary solution and compacton-like solution by variational iteration method*, Chaos. Soliton. Fract., 29 (2006), 108-111.

[13] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos. Solitons and Fractals, 30 (2006), 700-708.

[14] J. L. Zhang, M. L. Wang and X. R. Li, *The subsidiary elliptic-like equation and the exact solutions of the higher-order nonlinear Schrödinger equation*, Chaos Solitons Fractals, 33 (2007), 1450–1457.

[15] L. Jin, *Homotopy Perturbation Method for Solving Partial Differential Equations with Variable Coefficients*, Int. J. Contemp. Math. Sciences, 28 (2008), 1395-1407.

[16] L. Song and H. Zhang, *Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation*, Physics Letters A, 367 (2007), 88-94.

[17] M. J. Jang, C. L. Chen and Y. C.Liu, *Two-dimensional differential transform for partial differential equations*, Applied Mathematics and Computation, 121 (2001), 261-270.

[18] M. M. Rashidi, G. Domairry and S. Dinarvand, *The Homotopy Analysis Method for Explicit Analytical Solutions of Jaulent–Miodek Equations*, Numerical Methods for Partial Differential Equations, 25 (2) (2008), 430-439.

[19] M. Zurigat, A. A. Freihat and A. Handam, *The multi-step homotopy analysis method for solving the Jaulent-Miodek equations*, Proyecciones Journal of Mathematics, 34 (2015), 45-54.

[20] M. Zurigat, S. Momani and A. Alawneh, *Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method*, Computer and Mathematics with Applications, 59 (2010), 1227-1235.

[21] S. Ghosh and A. Roy, *An adaptation of adomian decomposition for numeric–analytic integration of strongly nonlinear and chaotic oscillators*, Comput. Method Appl. Mech. Eng., 196 (4–6) (2007), 1133–1153.

[22] V.G. Gupta and S. Gupta, *Application of homotopy analysis method for solving nonlinear cauchy problem*, Surveys in Mathematics and its Applications, 7 (2012), 105-116.

Accepted: 16.02.2018

GLOBAL DYNAMICS OF AN SIVS EPIDEMIC MODEL WITH BILINEAR INCIDENCE RATE

Mahmood Parsamanesh
Department of Mathematics
Faculty of Sciences
University of Zabol
Zabol
Iran
m.parsamanesh@uoz.ac.ir

Abstract. An SIS type epidemic model with variable population size is considered. The model includes a temporary vaccination program to prevent individuals from infection and to eradicate the disease. If $\mathcal{R}_0 < 1$, the disease-free equilibrium is locally and globally asymptotically stable i.e. the disease will be wiped out from population. When $\mathcal{R}_0 > 1$, the endemic equilibrium is locally asymptotically stable employing a result in stability of the second additive compound matrix. In addition, by using a geometric approach it is shown that this equilibrium is also globally asymptotically stable. So in this case, the disease will persist in population permanently. Also, a briefly discussion is made on the minimum amount of vaccination which is necessary to eradicate the disease. Finally, some numerical examples are given to confirm the obtained results.

Keywords: SIS epidemic model, vaccination, asymptotic stability, compound matrix method, geometric approach.

1. Introduction

The spread and control of infectious diseases in a population have been an important issue in recent years. The behavior of Population can be studied by using mathematical models and computer simulation. Many epidemic models have been introduced by authors for various type of diseases. The susceptible-infected-susceptible (SIS) epidemic models are one of the well known type of epidemic models. This type of models is appropriate for some infections in which individuals don't obtain permanent immunity after recovery. Vaccination is known as an efficient strategy to give immunity to the individuals and thus may also be included in the SIS epidemic models by considering a separate compartment for vaccinated individuals in the model formulation. These models may be deterministic [13,16] or stochastic [23,6], with constant [13] or variable [16,17] population size, and with standard [16,17] or bilinear incidence [10,5]. The organization of this paper is as the following: The formulation of the model and some basic properties such that, the boundedness of solutions, the basic reproduction number, and the equilibria of the model, will be given in the next section. The asymptotic stabilities of the disease-free equilibrium and the

endemic state are studied in sections 3 and 4, respectively. A brief discussion on the effect of vaccination is done in section 5. Eventually after section 6 for some numerical examples, we summarize the results presented in the paper.

2. Model description

We consider the following deterministic SIS epidemic model with vaccination and bilinear incidence established by Li and Ma [10]:

$$\begin{cases} S' = (1-q)\Lambda - \beta SI - (\mu+\varphi)S + \gamma I + \theta V, \\ I' = \beta SI - (\mu+\gamma+\alpha)I, \\ V' = q\Lambda + \varphi S - (\mu+\theta)V. \end{cases} \tag{2.1}$$

Variables $S(t), I(t)$ and $V(t)$ denote respectively the number of susceptible, infectious and vaccinated individuals at time time t. The parameters of the model are as follows:

Λ: Number of new individuals added into the population per unit of time,
q: Fraction of new individuals that are vaccinated,
β: Contact rate,
γ: Recovery rate,
μ: Natural death rate,
α: Disease-related death rate,
φ: Vaccination rate for susceptible individuals,
θ: Rate of loosing immunity in vaccinated individuals.

All parameter values are assumed to be non-negative and Λ and μ are positive. The force of infection βSI represents the number of new infected individuals per unit time and is of bilinear form.

From system (2.1) it can be seen that the total population size N is not constant and is expressed by the following equation:

$$N' = \Lambda - \mu N - \alpha I. \tag{2.2}$$

We see that

$$N' \leq \Lambda - \mu N,$$

and thus

$$\limsup_{t\to\infty} N \leq \frac{\Lambda}{\mu}.$$

Therefore total population N and as a result I, S and V are also bounded. We can see that the feasible region $\Gamma = \{(S,I,V) \in \mathbb{R}^3_+ : \ S+I+V \leq \frac{\Lambda}{\mu}\}$ is a positively invariant set of system (2.1) i.e. the solutions remain in Γ with initial vector $(S(0), I(0), V(0))$ in Γ. System (2.1) has two equilibria: A disease-free equilibrium when $I = 0$;

$$E^0 = (S^0, I^0, V^0) = \left(\frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\theta+\varphi)}, 0, \frac{\Lambda(\mu q+\varphi)}{\mu(\mu+\theta+\varphi)} \right),$$

and an endemic equilibrium when $I > 0$;

$$E^* = (S^*, I^*, V^*),$$

with

$$\begin{aligned} S^* &= \frac{\mu+\gamma+\alpha}{\beta}, \\ I^* &= \frac{\beta\Lambda[\mu(1-q)+\theta] - \mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}{\beta(\mu+\alpha)(\mu+\theta)}, \\ V^* &= \frac{q\Lambda + \varphi(\mu+\gamma+\alpha)/\beta}{\mu+\theta}. \end{aligned}$$

We use the next generation matrix method developed in [21], to find the basic reproduction number of the model. Let $\mathbf{y} = I$, where $\mathbf{y}$ indicates all infected states such as exposed and infectious individuals. Thus the second equation in system (2.1) can be written as

$$\frac{d\mathbf{y}}{dt} = \mathcal{F} - \mathcal{W},$$

with $\mathcal{F} = \beta SI$ and $\mathcal{W} = (\mu+\gamma+\alpha)I$.

Also let

$$F = \frac{\partial \mathcal{F}}{\partial \mathbf{y}}\Big|_{E^0} = \beta S^0 \quad \text{and} \quad W = \frac{\partial \mathcal{W}}{\partial \mathbf{y}}\Big|_{E^0} = \mu+\gamma+\alpha.$$

Therefore the basic reproduction number of the model is obtained by

$$\mathcal{R}_0 = \rho(FW^{-1}) = \frac{\beta S^0}{\mu+\gamma+\alpha} = \frac{\beta\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}. \tag{2.3}$$

Notice that

$$I^* = \frac{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}{\beta(\mu+\alpha)(\mu+\theta)}(\mathcal{R}_0 - 1),$$

and so the endemic equilibrium E^* exists if $\mathcal{R}_0 > 1$. Hence we can state the following:

Lemma 2.1. *When $\mathcal{R}_0 \leq 1$ system (2.1) has only the disease-free equilibrium E^0 and if $\mathcal{R}_0 > 1$ it also has a unique endemic equilibrium E^*.*

3. Stability of the disease-free equilibrium

In this section we consider the local and global stability of the disease-free equilibrium. Firstly it can be seen easily that eigenvalues of the Jacobian matrix of system (2.1) at E^0 are

$$\begin{aligned} \lambda_1 &= (\mu+\gamma+\alpha)(\mathcal{R}_0 - 1), \\ \lambda_2 &= -\mu, \\ \lambda_3 &= -(\mu+\varphi+\theta), \end{aligned}$$

thus we can obtain the following theorem:

Theorem 3.1. *The disease-free equilibrium E^0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

Our next task is to prove the global stability of E^0. This task has been discussed also by many authors especially [3,11,15,12] and various methods have been established.

Theorem 3.2. *The disease-free equilibrium E^0 of system (2.1) is globally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. Consider the Lyapunov function $L(t) = I(t)$. Now, we consider two below cases:

Case(a): If $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} \leq 1$, we see that $\mathcal{R}_0 < 1$ and we have

$$\beta S - (\mu + \gamma + \alpha) \leq \beta \left(\frac{\Lambda}{\mu}\right) - (\mu + \gamma + \alpha) \leq 0. \tag{3.1}$$

Case (b): If $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} > 1$ and $\mathcal{R}_0 \leq 1$. $\mathcal{R}_0 \leq 1$ implies $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} \leq \frac{\mu+\varphi+\theta}{\mu(1-q)+\theta}$ and therefore $S^0 \leq \frac{\mu+\gamma+\alpha}{\beta}$. So the region

$$\Omega = \left\{(S, I, V) \in \Gamma : S > \frac{\mu + \gamma + \alpha}{\beta}, I > 0\right\}$$

is not an invariant set because contains no equilibrium when $\mathcal{R}_0 \leq 1$. Hence, any solution with initial value in Ω lies in $\Gamma\backslash\Omega$ after a finite period of time. For any $(S, I, V) \in \Gamma\backslash\Omega$ we have $S \leq \frac{\mu+\gamma+\alpha}{\beta}$ and thus

$$\beta S - (\mu + \gamma + \alpha) \leq 0. \tag{3.2}$$

From (3.1) and (3.2) we see

$$L' = I[\beta S - (\mu + \gamma + \alpha)] \leq 0. \tag{3.3}$$

Therefore, from the LaSalle's invariance principle [14], we find that every solution of the model (2.1) with initial values in Γ, ultimately approaches E^0, when $\mathcal{R}_0 \leq 1$. □

4. Stability of the endemic equilibrium

In this section we study the local and global asymptotic stability of the endemic state E^*. We firstly attempt to explore the uniform persistence of (2.1) when $\mathcal{R}_0 > 1$. Suppose that (X, d) is a locally compact metric space and H is a closed subset of X with boundary ∂H and interior $\overset{\circ}{H}$. We state the following definition about uniform persistence which can be found in [8]:

Definition 4.1. A semi-dynamical system $\Phi_t(x) : H \times \mathbb{R}^+ \to H$ defined on H is said to be uniform persistence if there exists some $\eta > 0$ such that

$$\liminf_{t\to\infty} \mathrm{d}(\Phi_t(x), \partial H) > \eta,$$

for all $x \in H$.

Definition 4.2. A subset Σ of H is said to be a uniform repeller if and only if there exists an $\eta > 0$ such that for all $x \in H\backslash\Sigma$, $\liminf_{t\to\infty} \mathrm{d}(\Phi_t(x), \Sigma) > \eta$.

The definitions (4.1) and (4.2) state that a semi-dynamical system defined on a closed subset of a locally compact metric space is uniform persistence if the boundary of such subset is uniform repeller. The following result about persistence has been proved by Fonda in [7]:

Lemma 4.3. *Let F be a compact subset of X such that $X\backslash F$ is positively invariant. F is uniform repeller if and only if there exists a neighborhood U of F and a continuous function $P : X \longrightarrow \mathbb{R}^+$ satisfying:*
(i) $P(x) = 0$ if and only if $x \in F$,
(ii) For any $x \in U$, there exists a T_x such that $P(\Phi_{T_x}(x)) > P(x)$.

Now, consider the dynamical system (2.1) on positively invariant region Γ. Also, we let

$$H = \Gamma = \{(S, I, V) \in \mathbb{R}^3_+ : \ S + I + V \leq \frac{\Lambda}{\mu}\},$$

$$\Sigma = \partial\Gamma = \{(S, I, V) \in \Gamma : \ I = 0\},$$

and hence

$$H\backslash\Sigma = \overset{\circ}{\Gamma} = \{(S, I, V) \in \Gamma : \ I > 0\}.$$

Thus to show uniform persistence of system (2.1), we must show that Σ is uniform repeller when $\mathcal{R}_0 > 1$.

Theorem 4.4. *System* (2.1) *is uniform persistence if $\mathcal{R}_0 > 1$.*

Proof. Obviously, Σ is a compact set and $H\backslash\Sigma$ is positively invariant when $\mathcal{R}_0 > 1$. Let define $P : \Gamma \longrightarrow \mathbb{R}_+$ by $P(S, I, V) = I$, and $U = \{(S, I, V) \in \Gamma : P(S, I, V) < \zeta\}$, where $\zeta > 0$ is chosen so small that

$$\frac{\beta\Lambda[\mu(1-q)+\theta]}{(\mu+\gamma+\alpha)[\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta]} > 1. \tag{4.1}$$

The condition (i) is clearly satisfied. Assume that the condition (ii) doesn't hold, i.e. there exists a $\widetilde{x} \in U$ such that $P(\Phi_t(\widetilde{x})) < P(\widetilde{x}) < \zeta$ for all $t > 0$. This implies $I < \zeta$ and thus we see from system (2.1),

$$\begin{cases} S' \geq (1-q)\Lambda - \beta\zeta S - (\mu+\varphi)S + \theta V, \\ V' = q\Lambda + \varphi S - (\mu+\theta)V. \end{cases} \tag{4.2}$$

Hence, $\liminf_{t\to\infty} S(t;\widetilde{x}) \geq \frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+(\mu+\theta)\beta\zeta}$, where $S(t;\widetilde{x})$ denotes the solution $S(t)$ with initial value $\widetilde{x}$. Thus there exists some T_S such that for any $t > T_S$

$$S(t;\widetilde{x}) \geq \frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta}. \tag{4.3}$$

Now consider the function $W(t) = I(t)$. We see

$$\begin{aligned} W' = I' &= I[\beta S - (\mu+\gamma+\alpha)] \\ &\geq I\left[\beta\frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta} - (\mu+\gamma+\alpha)\right] \\ &= I(\mu+\gamma+\alpha)\left[\frac{\beta\Lambda[\mu(1-q)+\theta]}{(\mu+\gamma+\alpha)[\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta]} - 1\right]. \end{aligned} \tag{4.4}$$

Therefore from (4.1) and (4.4) it is concluded that $W(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. But this result contradicts the boundedness of $W(t)$. Hence the condition (ii) must be also satisfied and Σ is uniformly repeller. This completes the proof. □

In the following, we consider the stability of the endemic equilibrium employing the approach used in [1] and properties of compound matrices. The (local asymptotic) stability of the E^* is equivalent to stability of the corresponding Jacobian matrix of system (2.1) at E^*:

$$J^* = J(E^*) = \begin{pmatrix} -\beta I^* - (\mu+\varphi) & -(\mu+\alpha) & \theta \\ \beta I^* & 0 & 0 \\ \varphi & 0 & -(\mu+\theta) \end{pmatrix}.$$

We can easily see that $tr(J^*) < 0$ and $det(J^*) < 0$. Assume that λ_1, λ_2 and λ_3 are eigenvalues of J^* such that $\Re(\lambda_1) \leq \Re(\lambda_2) \leq \Re(\lambda_3)$, where $\Re(.)$ denotes the real part of a complex number. Then, $\lambda_1\lambda_2\lambda_3 < 0$ yields either $\Re(\lambda_j) < 0$ for $j = 1, 2, 3$ or $\Re(\lambda_1) < 0 < \Re(\lambda_2) \leq \Re(\lambda_3)$.

On the other hand, the second additive compound matrix $J^{[2]}(E^*)$ of the Jacobian matrix J^* (for example, by a strict formula in appendix of [19]) can be calculated as

$$J^{[2]}(E^*) = \begin{pmatrix} -\beta I^* - (\mu+\varphi) & 0 & -\theta \\ 0 & -\beta I^* - (2\mu+\varphi+\theta) & -(\mu+\alpha) \\ \varphi & \beta I^* & -(\mu+\theta) \end{pmatrix}$$

Notice that eigenvalues of $J^{[2]}(E^*)$ are $\lambda_1+\lambda_2$, $\lambda_1+\lambda_3$ and $\lambda_2+\lambda_3$. Besides, $tr(J^*) = \lambda_1+\lambda_2+\lambda_3 < 0$ implies $\Re(\lambda_1+\lambda_2) < 0$ and $\Re(\lambda_1+\lambda_3) < 0$. One can show that $det(J^{[2]}(E^*)) < 0$, and thus we must have $\Re(\lambda_2+\lambda_3) < 0$. All eigenvalues of $J^{[2]}(E^*)$ have negative real part and as a result it is stable. Any n by n real matrix M is stable if and only if $(-1)^n det(M) > 0$ and $M^{[2]}$ is stable [19], thus the Jacobian matrix J^* is stable and we have the following theorem:

Theorem 4.5. *The endemic equilibrium E^* of system (2.1) is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

To analyze the global stability of E^* we use a geometric approach developed by Li and Muldowney [18]. This method has been used to deal with the global stability of endemic equilibrium in many epidemic models [4,1,20,22,2]. Here we briefly explain the method:

Consider the autonomous equation $x' = f(x)$, where $f : D \longrightarrow \mathbb{R}^n$ is a C^1 function on an open set $D \subset \mathbb{R}^n$. The solution of the differential equation with initial value x_0 is denoted by $x(t; x_0)$. Assume that two following conditions hold:

($\mathcal{G}_1$) The system has a unique equilibrium $\overline{x}$ in D;

($\mathcal{G}_2$) There exists a compact absorbing set $K \subset D$.

Let $Q(x)$ be a $M \times M$, $M = \binom{n}{2}$, matrix-valued function which is C^1 on D. Moreover, assume that $Q^{-1}(x)$ exists and is continuous for $x \in K$. Define a quantity $\overline{q}_2$ as

$$\overline{q}_2 = \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t} \int_0^t \psi\left(B(x(r; x_0))\right) dr,$$

where,

$$B = Q_f Q^{-1} + QJ^{[2]}Q^{-1}.$$

Here, $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix J, Q_f is obtained by $(q_{ij})_f = \left(\frac{\partial q_{ij}}{\partial x}\right)^{\top} .f(x)$, and $\psi(B)$ is the *Lozinskiĭ* measure of B with respect to a vector norm $|.|$ in $\mathbb{R}^M$, defined by $\psi(B) = \lim_{h\to 0^+} \frac{|I+hB|-1}{h}$. The following result has been proved in [18].

Lemma 4.6. *Suppose that D is simply connected and conditions $\mathcal{G}_1$ and $\mathcal{G}_2$ are satisfied. The unique equilibrium $\overline{x}$ of system $x' = f(x)$ is globally asymptotically stable in D if $\overline{q}_2 < 0$.*

We are now in position to prove the global stability of the endemic equilibrium.

Theorem 4.7. *The endemic equilibrium E^* of system (2.1) is globally asymptotic stable if $\mathcal{R}_0 > 1$ and the following condition holds:*

$$\mu > \max\Big\{\theta - \varphi, \gamma - \theta + \varphi, \gamma - \theta + \alpha\Big\}.$$

Proof. The uniform persistence of system (2.1) in the bounded region Γ, stated in Theorem 4.4, implies that there exists a compact set K in $\overset{\circ}{\Gamma}$ which is absorbing for solutions of system(2.1) (see [9]).

The Jacobian matrix of system (2.1) is

$$J = J(S, I, V) = \begin{pmatrix} -\beta I - (\mu + \varphi) & -\beta S + \gamma & \theta \\ \beta I & \beta S - (\mu + \gamma + \alpha) & 0 \\ \varphi & 0 & -(\mu + \theta) \end{pmatrix},$$

and its corresponding second additive compound matrix is

$$J^{[2]} = \begin{pmatrix} -\beta I + \beta S & & \\ -(2\mu+\varphi+\gamma+\alpha) & 0 & -\theta \\ 0 & -\beta I-(2\mu+\varphi+\theta) & -\beta S+\gamma \\ -\varphi & \beta I & \beta S-(2\mu+\gamma+\alpha+\theta) \end{pmatrix}.$$

Set the function $Q = Q(S, I, V) = \frac{S}{I}\mathbb{I}_3$, where $\mathbb{I}_3$ is the identity matrix. We obtain $Q_f = \left(\frac{S'}{I} - \frac{SI'}{I^2}\right)\mathbb{I}_3$ and thus $Q_f Q^{-1} = \left(\frac{S'}{S} - \frac{I'}{I}\right)\mathbb{I}_3$. On the other hand obviously $QJ^{[2]}Q^{-1} = J^{[2]}$. Therefore the matrix $B = Q_f Q^{-1} + QJ^{[2]}Q^{-1}$ can be written in block form as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

in which,

$B_{11} = \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha)$,
$B_{12} = (0-\theta), \qquad B_{21} = (0-\varphi)^\top$,
$B_{22} = \begin{pmatrix} \frac{S'}{S} - \frac{I'}{I} - \beta I - (2\mu+\varphi+\theta) & -\beta S+\gamma \\ \beta I & \frac{S'}{S} - \frac{I'}{I} + \beta S - (2\mu+\gamma+\alpha+\theta) \end{pmatrix}$.

From [18], if we select the norm $\Big|(u,v,w)\Big| = \max\Big\{\Big|u\Big|, \Big|v\Big|+\Big|w\Big|\Big\}$ for $(u,v,w) \in \mathbb{R}^3$, then we have

$$\psi(B) = \sup\{g_1, g_2\},$$

where $g_1 = \psi(B_{11}) + \Big|B_{12}\Big|$ and $g_2 = \psi(B_{22}) + \Big|B_{21}\Big|$ and ψ denotes the Lozinskiĭ measure with respect to the defined norm. Thus, we have $\psi(B_{11}) = \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha), \mid B_{12} \mid = \theta, \mid B_{21} \mid = \varphi$ and

$$\begin{aligned}\psi(B_{22}) &= \max\left\{\frac{S'}{S} - \frac{I'}{I} - (2\mu+\varphi+\theta), \frac{S'}{S} - \frac{I'}{I} - (2\mu+\gamma+\alpha+\theta)\right\} \\ &= \frac{S'}{S} - \frac{I'}{I} - (2\mu+\theta) + \max\{-\varphi, -\alpha\}\end{aligned}$$

From second equation of system (2.1) we have $\beta S = \frac{I'}{I} + (\mu+\gamma+\alpha)$, thus

$$\begin{aligned}g_1 &= \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha) + \theta \\ &= \frac{S'}{S} - \beta I - \mu - \varphi + \theta \le \frac{S'}{S} - \mu - \varphi + \theta,\end{aligned}$$

and

$$\begin{aligned} g_2 &= \frac{S'}{S} - \frac{I'}{I} - (2\mu + \theta) + \max\{-\varphi, -\alpha\} + \varphi \\ &= \frac{S'}{S} - \beta S - \mu - \theta + \gamma + \alpha + \varphi + \max\{-\varphi, -\alpha\} \\ &= \frac{S'}{S} - \beta S - \mu - \theta + \gamma + \max\{\alpha, \varphi\}. \\ &\leq \frac{S'}{S} - \mu - \theta + \gamma + \max\{\alpha, \varphi\}. \end{aligned}$$

Therefore

$$\begin{aligned} \psi(B) = \sup\{g_1, g_2\} &\leq \frac{S'}{S} - \mu + \max\{-\varphi + \theta, -\theta + \gamma + \varphi, -\theta + \gamma + \alpha\} \\ &= \frac{S'}{S} - \mu - \min\{\varphi - \theta, \theta - \gamma - \varphi, \theta - \gamma - \alpha\}. \end{aligned}$$

By the assumption in the state of theorem we get

$$\psi(B) = \sup\{g_1, g_2\} \leq \frac{S'}{S} - \eta,$$

in which $\eta = \beta + \mu + \min\{\varphi - \theta, \theta - \gamma - \varphi, \theta - \gamma - \alpha\} > 0$.

Therefore for any solution of system (2.1) with $(S(0), I(0), V(0)) \in K$, we have

$$\frac{1}{t}\int_0^t \psi(B)dr \leq \frac{1}{t}\int_0^t \left(\frac{S'}{S} - \eta\right) dr = \frac{1}{t}\ln\frac{S(t)}{S(0)} - \eta,$$

which implies

$$\overline{q}_2 = \limsup_{t\to\infty} \sup \frac{1}{t}\int_0^t \psi(B)dr < -\frac{1}{2}\eta < 0,$$

and according to the Lemma 4.6, E^* is globally asymptotically stable. □

5. The impact of vaccination

In this section we discuss briefly the model in absence of vaccination. If we consider model (2.1) without vaccination i.e. we omit the compartment V, we have an SIS epidemic model and the vaccination-free basic reproduction number becomes

$$\widetilde{\mathcal{R}}_0 = \frac{\beta\Lambda}{\mu(\mu + \gamma + \alpha)},$$

and therefore

$$\mathcal{R}_0 = \left(1 - \frac{\mu q + \varphi}{\mu + \theta + \varphi}\right)\widetilde{\mathcal{R}}_0.$$

This shows that $\mathcal{R}_0 \leq \widetilde{\mathcal{R}}_0$ and thus disease will extinct sooner in present of vaccination. For under study population if $\widetilde{\mathcal{R}}_0 < 1$ the disease will die out,

but when $\widetilde{\mathcal{R}}_0 > 1$ it will persist in population and thus vaccination must be performed such that $\mathcal{R}_0 \leq 1$. To this be satisfied we must have

$$\varphi + \mu q \widetilde{\mathcal{R}}_0 \geq (\mu + \theta)(\widetilde{\mathcal{R}}_0 - 1).$$

This shows that to eradicate the disease at least a fraction

$$q^*_{vac} = \frac{1}{\mu \widetilde{\mathcal{R}}_0} \left[(\mu + \theta)(\widetilde{\mathcal{R}}_0 - 1) - \varphi \right]. \tag{5.1}$$

of new members must be vaccinated. From (5.1) it can be concluded that greater values for φ and less values for θ yield less values for q^*_{vac}. This shows that increase in the proportion of susceptible individuals who protected by vaccination (φ), and also in the period time of loosing immunity $\left(\frac{1}{\theta}\right)$, accelerates the disease eradication. however, it must be noted that if $q^*_{vac} > 1$, the vaccination can not overcome the disease even if all new members are vaccinated.

6. An example

In this section, using numerical approach the theoretical results obtained in preceding sections will be examined. First, a bifurcation diagram of compartment $I(t)$ in terms of various values of parameter q is presented. Then, some solutions of the model in two cases $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ are given.

Example 6.1. Let values of parameters in model (2.1) are as $\Lambda = 0.2, \beta = 0.6, \mu = 0.1, \varphi = 0.2, \alpha = 0.2, \gamma = 0.3, \theta = 0.2$, and $q \in (0, 1)$. Assume also that the unit of population size is one million individuals and initial values are $S(0) = 0.8, I(0) = 0.4$ and $V(0) = 0.5$.

A bifurcation diagram of number of infected individuals (variable $I(t)$) in terms of fraction of vaccinated new members (parameter q) is presented in Figure 1. For values $q < 0.5$ obviously final values of $I(t)$ have positive values and the endemic equilibrium E^* is stable, while for $q > 0.5$ the disease-free equilibrium E^0 is stable. This can be also concluded from relation (5.1) where $q^*_{vac} = 0.5$ is optimal value for which the disease will be extinct from population.

Now let $q = 0.2$ and $q = 0.8$. For these values for q and same values for other parameters as before, we have $\mathcal{R}_0 = 1.12 > 1$ and $\mathcal{R}_0 = 0.88 < 1$, respectively. Then, according to Theorem 4.6 and Theorem 3.2 disease persists and will be wiped out in these cases, respectively. Solutions of the model for these two values of parameter q are shown in Figure 2.

7. Summary

In this paper, we studied an SIS epidemic model that includes a vaccination program. The vaccination consist of new members and susceptible individuals. Although vaccination effect is perfect i.e. no vaccinated individual becomes infectious, its immunity is lost gradually. The basic reproduction number $\mathcal{R}_0$, and

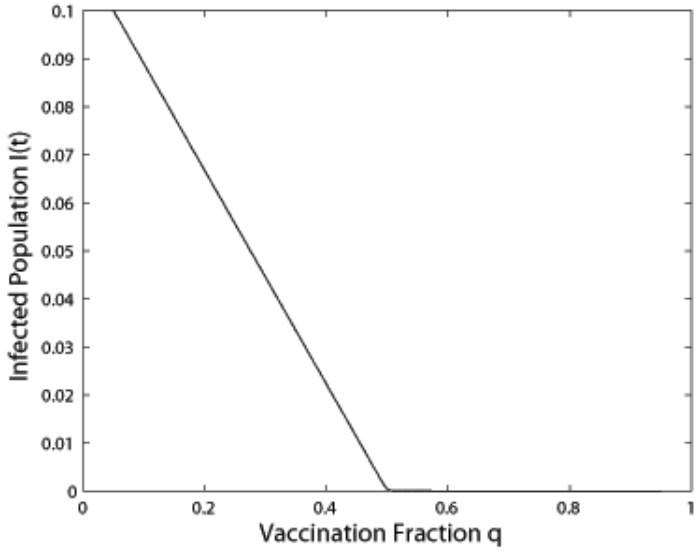


Figure 1: A diagram for final values of infected population $I(t)$ in terms of various values of fraction q.

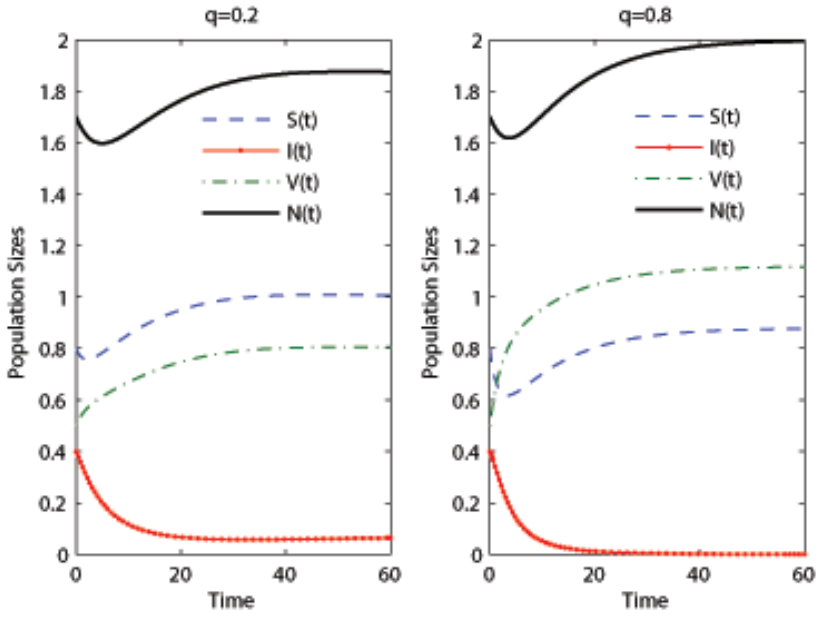


Figure 2: Solutions of model (2.1) for values $q = 0.2$ and $q = 0.8$ and other parameter values as in Example 6.1.

equilibria of the model were found. The dynamics of the model were determined by threshold $\mathcal{R}_0$; if $\mathcal{R}_0 < 1$, it was proved that the disease-free equilibrium is locally as well as globally asymptotically stable. In this case the disease dies out and disease extinction occurs. while the disease will persist in population permanently if $\mathcal{R}_0 > 1$. Indeed it was proved that the endemic equilibrium is locally asymptotically stable by means of the second additive compound matrix method and is globally asymptotically stable using a geometric approach. Impact of vaccination was briefly discussed and an optimal fraction of new members who must be vaccinated to disease dies out, was found. Finally, the theoretical results were discussed also numerically in some examples.

References

[1] Julien Arino, C. Connell McCluskey and Pauline van den Driessche, *Global results for an epidemic model with vaccination that exhibits backward bifurcation*, SIAM Journal on Applied Mathematics, 64 (2003), 260-276.

[2] Bruno Buonomo and Cruz Vargas-De-León, *Global stability for an HIV-1 infection model including an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, 385 (2012), 709-720.

[3] Carlos Castillo-Chávez, Zhilan Feng, and Wenzhang Huang, *On the computation of R_0 and its role on, Mathematical approaches for emerging and reemerging infectious diseases: an introduction*, 1-229, 2002.

[4] Meng Fan, Michael Y Li, and Ke Wang, *Global stability of an SEIS epidemic model with recruitment and a varying total population size*, Mathematical Biosciences, 170 (2001), 199-208.

[5] Rahman Farnoosh and Mahmood Parsamanesh, *Disease extinction and persistence in a discrete-time SIS epidemic model with vaccination and varying population size*, Filomat, 31 (2017), 4735-4747.

[6] Rahman Farnoosh and Mahmood Parsamanesh, *Stochastic differential equation systems for an SIS epidemic model with vaccination and immigration*, Communications in Statistics-Theory and Methods, 46 (2017), 8723-8736.

[7] Alessandro Fonda, *Uniformly persistent semidynamical systems*, Proceedings of the American Mathematical Society, pages 111-116, 1988.

[8] HI Freedman, Shigui Ruan, and Moxun Tang, *Uniform persistence and flows near a closed positively invariant set*, Journal of Dynamics and Differential Equations, 6 (1994), 583-600.

[9] Vivian Hutson and Klaus Schmitt, *Permanence and the dynamics of biological systems*, Mathematical Biosciences, 111 (1992), 1-71.

[10] Li Jianquan and Ma Zhien, *Global analysis of SIS epidemic models with variable total population size*, Mathematical and Computer Modelling, 39 (2004), 1231-1242.

[11] Jean Claude Kamgang and Gauthier Sallet, *Global asymptotic stability for the disease free equilibrium for epidemiological models*, Comptes Rendus Mathematique, 341 (2005), 433-438.

[12] Jean Claude Kamgang and Gauthier Sallet, *Computation of threshold conditions for epidemiological models and global stability of the disease-free equilibrium (DFE)*, Mathematical Biosciences, 213 (2008), 1-12.

[13] Christopher M Kribs-Zaleta and Jorge X Velasco-Hernandez, *A simple vaccination model with multiple endemic states*, Mathematical biosciences, 164 (2000), 183-201.

[14] Joseph La Salle and Solomon Lefschetz, *Stability by Liapunov's Direct Method with Applications by Joseph L Salle and Solomon Lefschetz,* volume 4, Elsevier, 2012.

[15] István G Laukó, *Stability of disease free sets in epidemic models*, Mathematical and computer modelling, 43 (2006), 1357-1366.

[16] Jianquan Li and Zhien Ma, *Qualitative analyses of SIS epidemic model with vaccination and varying total population size*, Mathematical and Computer Modelling, 35 (2002), 1235-1243.

[17] Jianquan Li, Zhien Ma, and Yicang Zhou, *Global analysis of SIS epidemic model with a simple vaccination and multiple endemic equilibria*, Acta Mathematica Scientia, 26 (2006), 83-93.

[18] Michael Y Li and James S Muldowney, *A geometric approach to global-stability problems*, SIAM Journal on Mathematical Analysis, 27 (1996), 1070-1083.

[19] Michael Y Li and Liancheng Wang, *A criterion for stability of matrices*, Journal of mathematical analysis and applications, 225 (1998), 249-264.

[20] Chengjun Sun and Ying-Hen Hsieh, *Global analysis of an SEIR model with varying population size and vaccination*, Applied Mathematical Modelling, 34 (2010), 2685-2697.

[21] Pauline Van den Driessche and James Watmough, *Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission*, Mathematical Biosciences, 180 (2002), 29-48.

[22] Wei Yang, Chengjun Sun, and Julien Arino, *Global analysis for a general epidemiological model with vaccination and varying population*, Journal of Mathematical Analysis and Applications, 372 (2010), 208-223.

[23] Yanan Zhao and Daqing Jiang, *The threshold of a stochastic SIS epidemic model with vaccination*, Applied Mathematics and Computation, 243 (2014), 718-727.

Accepted: 21.02.2018

USING MULTI-SCALE AUTO CONVOLUTION MOMENTS TO GET IMAGE AFFINE INVARIANT FEATURES

Fengwen Zhai*
School of Electronic and Information Engineering
Lanzhou Jiaotong University,
Lanzhou, Gansu, 730070
China
930843420@qq.com

Jianwu Dang
School of Electronic and Information Engineering
Lanzhou Jiaotong University,
Lanzhou, Gansu, 730070
China

Yangping Wang
School of Electronic and Information Engineering
Lanzhou Jiaotong University,
Lanzhou, Gansu, 730070
China

Jing Jin
School of Electronic and Information Engineering
Lanzhou Jiaotong University,
Lanzhou, Gansu, 730070
China

Abstract. This paper includes two important works. First of all, the complete mathematical proof procedure of Multi-Scale Auto convolution was summarized and the simplified geometric proof the procedure was proposed. Secondly, Multi-Scale Auto convolution moments were adopted to describe images' maximally stable extremal regions to get affine invariant features of images. In the second job, the Multi-Scale Auto convolution moments of the image features were calculated on each feature's MSER region to form image features' descriptors, and then the image feature matching was performed. In order to verify the validity of the second job, the proposed algorithm were compared with the SIFT algorithm and MSER_SURE algorithm. Simulation experiments show that, for affine transformed images, the feature matching accuracy of the second job is much higher than the classical SIFT algorithm and the MSER_SURE algorithm, which indicates that using Multi-Scale Auto convolution moments on the MSER regions could get effective affine invariant image features.

Keywords: multi-scale auto convolution, maximally stable extremal regions, feature matching, affine invariant, pattern recognition.

∗. Corresponding author

1. Introduction

Images provide a variety of information of colors, intensities, textures, edges and shapes. To understand and recognize the images, it is necessary to acquire invariant features of digital images. The image features mainly fall into three categories: region-based features, texture-based features, and point features. The point features are widely used in the field of image retrieval, target recognition and tracking, image three-dimensional reconstruction, image super-resolution reconstruction, image mosaic and image registration. David Lowe proposed the scale-invariant feature transform (SIFT) algorithm in 1999 and further improved in 2004. SIFT features remain invariant to rotation, scaling, and brightness changes, and maintain a certain degree of stability against affine transformation and noises [1]. Morel et al. proposed the ASIFT algorithm in 2009[2]. ASIFT uses the discrete image shooting angle changes to simulate the continuous shooting angle changes. The matching effect of ASIFT greatly exceeds the SIFT algorithm in affine invariance aspect, but the time efficiency is lower than SIFT. In 2011, Ethan Rublee et al. proposed ORB algorithm[3] on ICCV. The ORB algorithm is based on the FAST algorithm and the BRIEF algorithm, but the ORB algorithm could not get scale invariant and affine invariant features. Image moments could keep some invariant characteristics. A lot of researchers have concentrated on the study of adopting moments to describe image features [4, 5, 6]. But the before moments could only get scale or rotational or transformational invariances while affine invariance is also important for real applications[7, 8, 9, 10]. In 2002, Heikkil J and Esa et al. proposed Multi-scale Auto convolution (MSA)[11] moments which are affine invariant when applied to pattern recognition. And in the later years, Esa et al. polished the MSA moments algorithm from different aspects [12, 13, 14, 15]. Due to the strong affine invariance of MSA moments, there are more and more researchers applied MSA moments into image processing. In 2014, Shao C et al. improved the MSA moments classification ability by taking the cosine value of the angle between two most prominent points within image feature's neighborhood as the feature's probability density value[16]. In 2015, Li et al. proposed a feature point matching method based on the geometric relation constraint and MSA moments [17]. In the same year, Zhang et al. Combined the MSA moments with the SIFT algorithm [18] and improved the match accuracy of the SIFT algorithm. However, it was found that the proof process of MSA moment calculation method is complicated and has a lot of jumping in the reference[12], which is inconvenient for readers to understand and to simulation. At the same time, MSA Moment has strict mathematical features, if the MSA moments were introduced into feature point matching, the neighborhood window similar to SIFT algorithm and ORB algorithm can not be used directly. In view of the above two problems, firstly, this paper summarizes the complete proof of the MSA moments and then proposes a more easy-to-understand calculation method through spatial geometry analysis. Secondly, the regions of the Maximally Stable Extremal

Regions(MSER)[19] were selected as the simple background neighborhoods of the image features, so that the MSA moments can be effectively applied to the description of the image features, which was defined as MSER_MSA for convenient. Thirdly, the MSER_MSA algorithm was compared with the SIFT algorithm and the MSER_SURF[20] algorithm to prove the effectiveness of the MSER_MSA algorithm for large-scale affine transformed image pairs.

The remaining part of this paper is structured as follows. The second part introduces the basic principle of the MSA moments and the MSER algorithm. The third part summarizes the complete proof procedure and proposes the easy-to-understand geometric proof of the MSA moments. The fourth part introduces the MSER_MSA algorithm to detail. The fifth part compares the proposed MSER_MSA algorithm with the SIFT and MSER_SURF algorithms. At last, the conclusion part concludes with a summary of the effectiveness of the proposed MSER_MSA algorithm and the next work direction.

2. MSA Moments and MSER algorithm

In 2005, Esa et al. Proposed MSA moments. To understand MSA moments, references [12] gives three definitions.

Definition 1. $\mathbf{X}$ is the base point, and the $\mathbf{\Gamma}$ is the affine transform matrix, $\mathbf{X}'$ corresponds to $\mathbf{X}$

$$\mathbf{X}' = \mathbf{\Gamma}(\mathbf{X}) = \mathbf{T}\mathbf{X} + \mathbf{t} \tag{1}$$

Among them $\mathbf{t}, \mathbf{x} \in \mathbb{R}^2$, and T is a non-singular real matrix, the inverse transform recorded as

$$\mathbf{Y}' = \Gamma^{-1}(\mathbf{Y}) = \mathbf{T}^{-1}\mathbf{Y} - \mathbf{T}^{-1}\mathbf{t} \tag{2}$$

Definition 2. Let $f(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}$, and $f(\mathbf{x}) \geq 0$ is a grayscale function defined on a two-dimensional grayscale image. Applying the affine transformation to the image, a new image grayscale function can be obtained.

$$f'(\mathbf{x}) = f(\Gamma^{-1}(\mathbf{x})) = f(\mathbf{T^{-1}x} - \mathbf{T^{-1}t}) \tag{3}$$

Definition 3. For the feature $\mathbf{A}$ extracted from the image, the feature $\mathbf{A}$ is called the affine invariant feature when $f \circ (\mathbf{A})$ and $f' \circ (\mathbf{A})$ is the same.

Randomly take three non-collinear three points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ in the image as the basis for the two arbitrary variables α and β, then random variable $\mathbf{u}$ could be gotten form Eq. (4)

$$\mathbf{u} = \alpha(\mathbf{x_1} - \mathbf{x_0}) + \beta(\mathbf{x_2} - \mathbf{x_0}) + \mathbf{x_0} \tag{4}$$

To transform the point $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ using $\Gamma = \Gamma\{\mathbf{T}, \mathbf{t}\}$, then $\mathbf{x}_0{}' = \mathbf{T}\mathbf{x}_0 + \mathbf{t}$, $\mathbf{x_1}' = \mathbf{T}\mathbf{x_1} + \mathbf{t}$, $\mathbf{x}_2{}' = \mathbf{T}\mathbf{x}_2 + \mathbf{t}$. Assume $\mathbf{u}' = \alpha(\mathbf{x}_1{}' - \mathbf{x}_0{}') + \beta(\mathbf{x}_2{}' - \mathbf{x}_0{}') + \mathbf{x}_0{}'$,

then

$$
\begin{aligned}
\mathbf{u}' =& \alpha(\mathbf{x}_1{}' - \mathbf{x}_0{}') + \beta(\mathbf{x}_2{}' - \mathbf{x}_0{}') + \mathbf{x}_0{}' \\
=& \alpha((\mathbf{T}\mathbf{x}_1 + \mathbf{t}) - (\mathbf{T}\mathbf{x}_0 + \mathbf{t})) + \beta((\mathbf{T}\mathbf{x}_2 + \mathbf{t}) - (\mathbf{T}\mathbf{x}_0 + \mathbf{t})) + \mathbf{T}\mathbf{x}_0 + \mathbf{t} \\
=& \alpha(\mathbf{T}\mathbf{x}_1 - \mathbf{T}\mathbf{x}_0) + \beta(\mathbf{T}\mathbf{x}_2 - \mathbf{T}\mathbf{x}_0) + \mathbf{T}\mathbf{x}_0 + \mathbf{t} \\
=& \mathbf{T}\mathbf{u} + \mathbf{t}
\end{aligned} \tag{5}
$$

From Eq.(5), to transform the basis point $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ by $\Gamma = \Gamma\{\mathbf{T}, \mathbf{t}\}$, then we can obtain three new base point $\mathbf{x}_0{}'$, $\mathbf{x}_1{}'$, $\mathbf{x}_2{}'$. The same affine transformation $\Gamma = \Gamma\{\mathbf{T}, \mathbf{t}\}$ is performed on point $\mathbf{u}$ to get a new point $\mathbf{u}'$. The linear relationship between $\mathbf{u}'$ and $\mathbf{x}_0{}'$, $\mathbf{x}_1{}'$, $\mathbf{x}_2{}'$ and the linear relationship between $\mathbf{u}$ and$\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ remain unchanged. And from Eq. (3), Eq.(6) can be obtained.

$$
f'(\mathbf{u}') = f(\Gamma^{-1}(\mathbf{u}')) = f(\mathbf{T^{-1}}\mathbf{u}' - \mathbf{T^{-1}}\mathbf{t}) = f(\mathbf{T^{-1}}(\mathbf{T}\mathbf{u} + \mathbf{t}) - \mathbf{T^{-1}}\mathbf{t})) = f(\mathbf{u}) \tag{6}
$$

In summary, we can obtain an affine invariant feature $f(\mathbf{U})$, where $\mathbf{u}$ is a random variable of U. The MSA moment is the mathematical expectation of $f(\mathbf{U})$.

$$
F(\alpha, \beta) = E[f(U_{\alpha,\beta})] \tag{7}
$$

The reference[12] gives the calculation formula of $F(\alpha, \beta)$ as equation (8)

$$
F(\alpha, \beta) = \left[\int_{\mathbb{R}^2} \hat{f}(-\xi)\hat{f}(\alpha\xi)\hat{f}(\beta\xi)\hat{f}(\gamma\xi)d\xi \right] / \hat{f}(0)^3 \tag{8}
$$

2.1 The basic principle of MSER algorithm

MSER's full name is Maximally Stable Extremal Regions, which was put forward by J. Matatas in 2002[19]. MSER algorithm uses three definitions: Region, Extremal Region and Maximally Stable Extremal Region.

Definition 4. The region is a part of an image, for any two points **A** and **B**, **A** can reach **B** along the adjacent pixel within this part, that is, there is a connected path between any two points.

Definition 5. The definition of Extreme region is related to the grayscale threshold. After the grayscale threshold is set, the image is divided into two part according to the threshold, the pixels smaller than the threshold are set to black, the pixels larger than the threshold are set to white. The region in the image is made up of black pixels. And the region can become the extreme region if no more pixels can be found to enlarge the current region.

For example, threshold segmentation is performed on an image. Then select a series of grayscale thresholds. For example, we take $i = 0, 1, \cdots, 255$. When the threshold is i, the gray value of each pixel in the image is compared with the threshold. Points below this threshold are "black spots", otherwise, "white

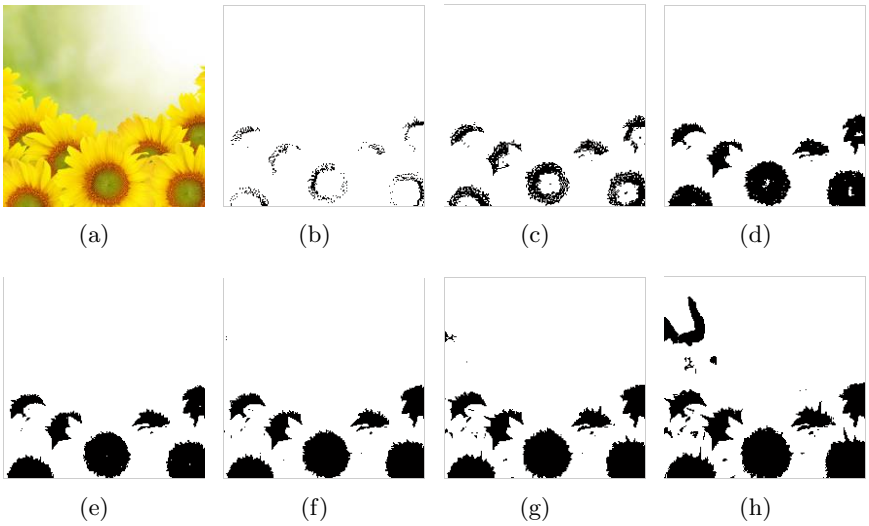


Figure 1: Extreme Region illustration figures. (a) is the original image. (b),(c),(d),(e),(f),(g),(h) are the Extreme Regions that appear in the image when the grayscale thresholds are set to 80, 100, 120, 140, 160, 180, 200.

spots". This will get a picture of a frame. When all the images are arranged in order of threshold, you can see: The first picture is all white (all pixels have a threshold of no less than 0). Then the black spots appear gradually on the image, and some black spots gradually fuse together to form a small area to become the connected areas. The last picture is all black as shown in Fig.1. In these figures, all the black connected areas are Extreme Regions. Even an isolated black spot is also an Extreme Region. The determination of Extreme Region is closely related to this gray threshold. It is unable to determine a region or point is Extreme Region or not when leaving this grayscale threshold.

Definition 6. During the process of obtaining Extreme Region, some Extreme Regions increase slightly with the increasing of the set gray threshold. Such a group of Extreme Regions is a nested relationship from small to large. We use $Q_1, Q_2, \cdots, Q_{i-1}, Q_i$ to represent this series of mutually exclusive Extreme Region sequences. The condition of Q_i^* could be Maximally Stable Extremal Region if and only if the minimum value of $q(i)$ is achieved when $i = i^*$.

$$q(i) = \frac{|Q_{i+\xi} - Q_{i-\xi}|}{|Q_i|} \tag{9}$$

Because Q is a collection of pixels, the absolute value represents the cardinality of the collection. We can think of it as the area of this Extreme Region. As can be seen from the above explanation, the maximum stable Extreme Region

is the Maximally Stable Extremal Region when the gray threshold changes. So we find all the Extreme Regions and then use the formula (9) to determine the final Maximally Stable Extremal Regions which is abbreviated as MSER.

The MSER regions can be processed as simple background image fragments. Esa et al. have experimentally proved that the MSA moments should be applied to image pattern recognition with simple backgrounds.

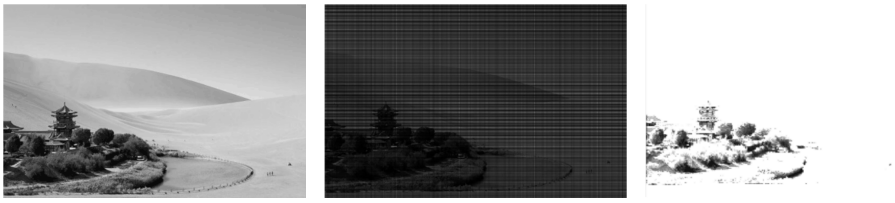
3. The MSA moment calculation method

3.1 The summarized proof for MSA moment calculation

The proof process of MSA moment calculation includes convolution calculation. As we all know that convolution calculation in the time domain is very time-consuming. It has been tested that the convolution of three $5 * 5$ matrices requires nearly 1 minute to run in our experimental environment. Therefore, Esa et al. converts the convolution computation of time domain into frequency domain product[21], and gives the proof process of MSA moment. However, the proof process of MSA moment calculation is complicated and inconsistent in the reference [12], which is inconvenient for readers to understand and to simulate. Firstly, this paper summarizes the detailed proof process of the MSA moment calculation method as follows.

$$
\begin{aligned}
(1)\ F(\alpha,\beta) &= \int_{D^2} f(\mathbf{u})(p_\alpha * p_\beta * p_\gamma)(u)d\mathbf{u} \\
(2)\ &= \int_{D^2} \overline{\overline{f(\mathbf{u})}}(p_\alpha * p_\beta * p_\gamma)(\mathbf{u})d\mathbf{u} \\
(3)\ &= \int_{D^2} \overline{\mathbb{F}(\overline{f(\mathbf{u})})}\mathbb{F}((p_\alpha * p_\beta * p_\gamma)(\mathbf{u}))d\xi \\
(4)\ &= \int_{D^2} \overline{\mathbb{F}(\overline{f(\mathbf{u})})}\mathbb{F}(p_\alpha)\mathbb{F}(p_\beta)\mathbb{F}(p_\gamma)d\xi \\
(5)\ &= \int_{D^2} \overline{\mathbb{F}(\overline{f(\mathbf{u})})}\frac{1}{\alpha\beta\gamma\|f_{R^2}\|_{L^1}{}^3}\mathbb{F}(f(\frac{\mathbf{u}}{\alpha}))\mathbb{F}(f(\frac{\mathbf{u}}{\beta}))\mathbb{F}(f(\frac{\mathbf{u}}{\gamma}))d\xi \\
(6)\ &= \frac{1}{\alpha\beta\gamma\|f_{R^2}\|_{L^1}{}^3}\int_{D^2} \overline{\mathbb{F}(\overline{f(\mathbf{u})})}\mathbb{F}(f(\frac{\mathbf{u}}{\alpha}))\mathbb{F}(f(\frac{\mathbf{u}}{\beta}))\mathbb{F}(f(\frac{\mathbf{u}}{\gamma}))d\xi \\
(7)\ &= \frac{\alpha\beta\gamma}{\alpha\beta\gamma\|f_{R^2}\|_{L^1}{}^3}\int_{D^2} F(-\xi)F(\alpha\xi))F(\beta\xi)F(\gamma\xi)d\xi \\
(8)\ &= \frac{1}{\|f_{R^2}\|_{L^1}{}^3}\int_{D^2} F(-\xi)F(\alpha\xi))F(\beta\xi)F(\gamma\xi)d\xi \\
(9)\ &= \frac{1}{F(0)^3}\int_{D^2} F(-\xi)F(\alpha\xi))F(\beta\xi)F(\gamma\xi)d\xi
\end{aligned}
$$

Among them, D^2 is the integral domain of variable $\mathbf{u}$, and $\overline{f(x)}$ is the conjugate of $f(x)$. Step (2) to (3) is based on the Plancherel formula $\int_{R^2} f(x)\overline{g(x)} = \int_{R^2} \mathbb{F}(f(x))\overline{\mathbb{F}(g(x))}$. The Fourier transform convolution theorem was used



(a) Original image $\mathbf{I}$ (b) $\alpha\mathbf{I}$ image, $\alpha = 2$ (c) $\alpha\mathbf{I}$ image, $\alpha = 0.5$

Figure 2: The comparison figure between images $\mathbf{I}$ and $\alpha\mathbf{I}$.

from step (3) to step (4). step (5) is calculated by substituting $p_\alpha(u)$, $p_\beta(u)$ and $p_\gamma(u)$ into the formula, where $p_\alpha(u) = f(\frac{u}{\alpha})/(\|f_{R^2}\|_{L^1} * \alpha)$. step (6) to step (7) is based on the similarity principle of the Fourier transform. Step (8) to step (9) is based on $\|f_{R^2}\|_{L^1} = F(0)$.

3.2 Geometric Proof of MSA Moment Calculation Method

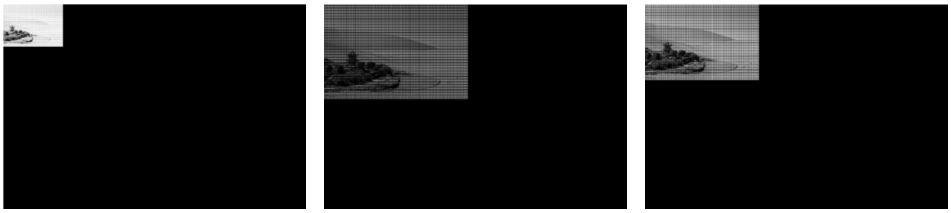
In Eq. (10), the proof of MSA moment calculation method is very rigorous. The functions and variables in the mathematical derivation from (5) to (9) have no specific physical meanings, which is hard for reader to understand. After detailed reasoning and simulating, this paper proofed the MSA moment calculation method directly through geometric analysis. The proof begins at step(4) in Eq.(10), as shown in Eq. (11)

$$F(\alpha,\beta) = \int_{D^2} \overline{\mathbb{F}(\overline{f(\mathbf{u})})}\mathbb{F}(p_\alpha)\mathbb{F}(p_\beta)\mathbb{F}(p_\gamma)d\xi \tag{10}$$

Firstly, the variables and functions in the Eq.(11) are described here. D^2 is the integral field of variable $\mathbf{u}$, $\mathbf{u} = \alpha\mathbf{x_1} + \beta\mathbf{x_2} + \gamma\mathbf{x_0}$, let the size of image I be $m \times n$, then $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \in [1:m, 1:n]$, so D^2 is $[1:(|\alpha|+|\beta|+|\gamma|)\times m, 1:(|\alpha|+|\beta|+|\gamma|)\times n]$. Where p_α represents the probability density of $\alpha\mathbf{x_1}$, p_β represents the probability density of $\beta\mathbf{x}_2$, p_γ represents the probability density of $\gamma\mathbf{x}_0$. The range or the integral field of $\alpha\mathbf{x_1}$, $\beta\mathbf{x}_2$, $\gamma\mathbf{x}_0$ is D^2, $f(\mathbf{x})$ represents the gray value of the point $\mathbf{x}$ in the image $\mathbf{I}$.

Secondly, the proposed geometric proof is given below. $\alpha\mathbf{I}$ represents α times expanded image of $\mathbf{I}$. Let $f_\alpha(\mathbf{z})$ represent the gray value of the point $\mathbf{z}$ in image $\alpha\mathbf{I}$. When $|\alpha| > 1$, $\mathbf{I}$ increases to $\alpha\mathbf{I}$ by inserting 0. When $|\alpha| < 1$, by taking the "enrichment" method, $\mathbf{I}$ will be shrunk to α times of image $\mathbf{I}$. The "enrichment" means to sum up the pixels' gray value within a small region. So $\|f_\alpha(\mathbf{z})\|_{L^1} = \|f(\mathbf{x})\|_{L^1}$. Fig. 2 shows the relationship between $\mathbf{I}$ and $\alpha\mathbf{I}$.

The following demonstrates that the density of variable $\alpha\mathbf{x_1}$ on image $\alpha\mathbf{I}$ is equal to the density on D^2. Because $\mathbf{x_1} \in [1:m, 1:n]$, so $\alpha\mathbf{x_1} \in [\alpha:\alpha m, \alpha:\alpha n]$. Even if the integration domain of $\alpha\mathbf{x_1}$ is D^2, $\alpha\mathbf{x_1}$ can only take values in



(a) Gray value of $\alpha\mathbf{x_1}$ on D^2 (b) Gray value of $\beta\mathbf{x_2}$ on D^2 (c) Gray value of $\gamma\mathbf{x_3}$ on D^2

Figure 3: the intensities of $\alpha\mathbf{x_1}$, $\beta\mathbf{x_2}$, $\gamma\mathbf{x_0}$ with D^2, $\alpha = 0.7$, $\beta = 1.5$.

sub-domains in $[\alpha : \alpha m, \alpha : \alpha n]$, as shown in Fig.3, and equation (12).

$$p_{\alpha\mathbf{X_1}}^{D_2}(\mathbf{u}) = \begin{cases} p_{\alpha\mathbf{X_1}}{}^{\alpha\mathbf{I}}(\mathbf{u}), & \mathbf{u} \in \alpha\mathbf{I} \\ 0, & \mathbf{u} \in D^2 \end{cases} \tag{11}$$

In the equation (12), $p_{\alpha\mathbf{X_1}}^{D_2}(\mathbf{u}_1)$ represents the density of the variable $\alpha\mathbf{x_1}$ on the integral domain D^2, and $p_{\alpha\mathbf{X_1}}^{\alpha\mathbf{I}}(\mathbf{z})$ represents the density of the variable $\alpha\mathbf{x_1}$ on the integral domain $\alpha\mathbf{I}$. Because $\alpha\mathbf{x_1}$, $\beta\mathbf{x_2}$, $\gamma\mathbf{x_0}$ are independent to each other, so $\int_{D^2} p_{\alpha\mathbf{X_1}}{}^{D^2}(\mathbf{u}_1) = \int_{\alpha\mathbf{I}} p_{\alpha\mathbf{X_1}}{}^{\alpha\mathbf{I}}(\mathbf{z}_1) = 1$. Similarly, $\int_{D^2} p_{\beta\mathbf{X_2}}{}^{D^2}(\mathbf{u}_2) = \int_{\beta\mathbf{I}} p_{\beta\mathbf{X2}}{}^{\beta\mathbf{I}}(\mathbf{z}_2) = 1$, $\int_{D^2} p_{\gamma\mathbf{X_0}}{}^{D^2}(\mathbf{u}_0) = \int_{\gamma\mathbf{I}} p_{\gamma\mathbf{X_0}}{}^{\gamma\mathbf{I}}(\mathbf{z}_0) = 1$. In the equation (11), p_α represents $p_{\alpha\mathbf{X_1}}^{D_2}(\mathbf{u}_1)$, p_β represents $p_{\beta\mathbf{X_2}}^{D^2}(\mathbf{u}_2)$, p_γ represents $p_{\gamma\mathbf{X_0}}^{D^2}(\mathbf{z}_0)$. During the actual calculation, $\mathbb{F}(p_\alpha)$ can be calculated according to equation (12). The specific calculation process is to stretch the original image $\mathbf{I}$ to $\alpha\mathbf{I}$, then perform Fourier transform in the way of zero-padding, as shown in Eq.(13).

$$\mathbb{F}(p_\alpha) = \mathbb{F}(p_{\alpha\mathbf{X_1}}^{\alpha\mathbf{I}}(\mathbf{z}), (|\alpha| + |\beta| + |\gamma|) * m, (|\alpha| + |\beta| + |\gamma|) * n) \tag{12}$$

In Eq. (13), $p_{\alpha\mathbf{X_1}}^{\alpha\mathbf{I}}(\mathbf{z}) = f_\alpha(\mathbf{z})/\|f_\alpha(\mathbf{z})\|_{L^1}$. The calculation methods of $\mathbb{F}(p_\beta)$, $\mathbb{F}(p_\gamma)$ are similar to $\mathbb{F}(p_\alpha)$

Through the above geometric analysis, the calculation of the MSA moment can be directly carried out according to Eq.(11). So, the deduction from step (5) to step (9) of the Eq.(10) could be omitted. The method presented in this paper is more intuitive than the mathematical proof method in [8], and is easy to understand and to realize. At the same time, the simulation results show that the calculation results of the proposed method are consistent with the results of the reference [12].

4. The MSA moments of characteristic points are computed on MSER region

In order to apply the scale invariance and affine invariance of MSA moments into the description of image features, MSER method here is used as feature point neighborhood extraction method, and the 31 orders' MSA moments are calculated as the feature descriptors of the most stable extremal regions. The

31 orders were recommended by reference [12]. For conveniently expressing, the method proposed in this paper was named as MSER_MSA, and the MSER_MSA algorithm was designed as follows:

Algorithm 1 MSER_MSA.

Input:

Reference image **I**, floating image **J**;

Output:

Matching feature pairs;

1: To perform Gauss smoothing on image **I** and image **J**;
2: The most stable extremal regions of **I** and **J** are extracted respectively, and the external ellipses are obtained;
3: The centroids of the external ellipses are taken as the feature points of the image;
4: The 31 orders' MSA moments of each feature point in its ellipse neighborhood are calculated and used as the feature descriptor of each feature point;
5: By comparing the feature descriptors, the features of image **I** and image **J** are matched.

5. Experiment

The experimental simulation platform of this paper is configured as follows: Windows XP operating system, 3.30GHz CPU, 3.0G memory, MATLAB 2015. In the experiments, the proposed MSER_MSA algorithm was compared with the SIFT algorithm and the classical MSER_SURF method. The floating images in Fig.4 were obtained by share transforming the reference images. The floating images in Fig.5 were obtained by share and scale transforming the reference images.

As can be seen that from Fig.4 and Fig.5, SIFT algorithm's matching effect of affine transformation is poor, and the MSER_SURF algorithm has better matching effect for the image with smaller affine transformation scale, but the matching effect of MSER_SURF is sensitive to scale changes, and can not obtain correct matching for the images with relatively large affine changes and scale changes. The MSER_MSA performed stably for scale change and affine change, and the correct matching number is only slightly less for image pairs with large scale and affine changes.

Fig.6 is the matching accuracy comparison chart of SIFT, MSER_SURF and MSER_MSA. The image samples came from 4 images. Each image generates a group of samples. The first one in each group is the reference image, and the rest 3 samples within each group are obtained by affine transforming the reference image, and the transforming scales are progressively increased in each group. From Fig.6, it can be seen that the accuracy rates of SIFT,

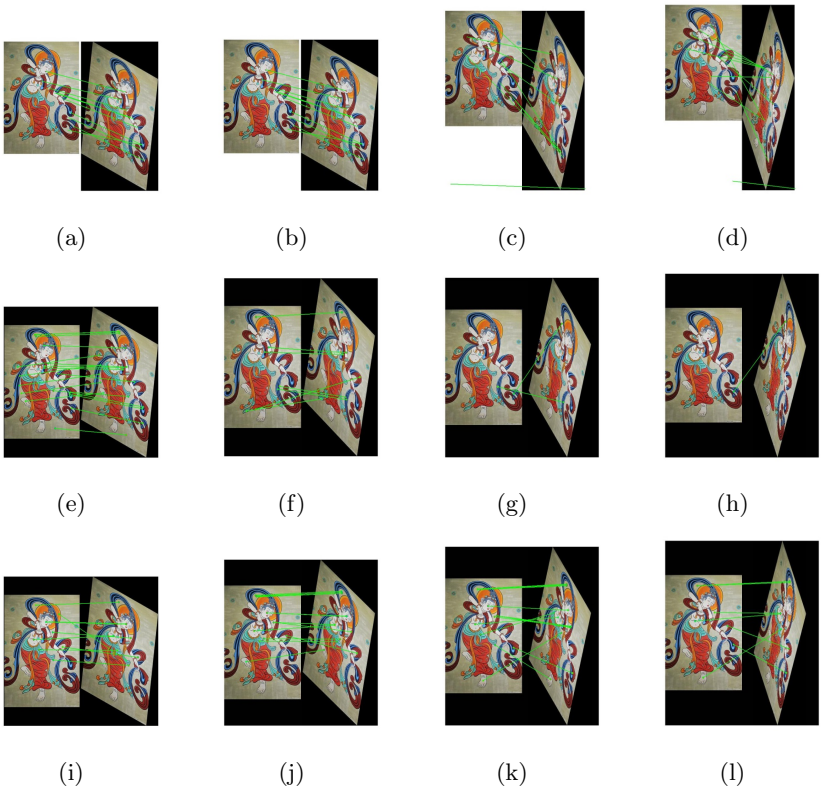


Figure 4: The feature matching results of SIFT, MSER_SURF and MSER_MSA of images with share transforming. (a),(b),(c) and (d) are the matching results of SIFT. (e),(f),(g) and (h) are the matching results of MSER_SURF. (i),(j),(k) and (l) are the matching results of MSER_MSA.

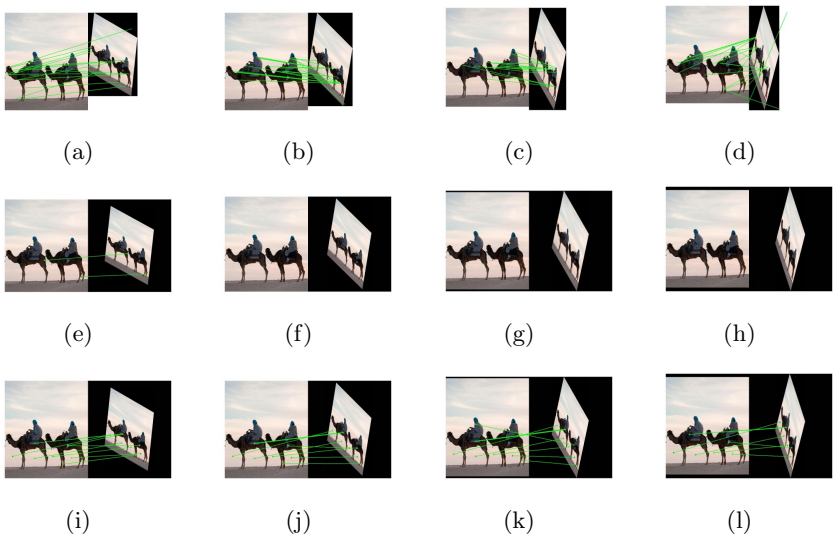


Figure 5: The feature matching results of SIFT, MSER_SURF and MSER_MSA of images with share and scale transforming. (a),(b),(c) and (d) are the matching results of SIFT. (e),(f),(g) and (h) are the matching results of MSER_SURF. (i),(j),(k) and (l) are the matching results of MSER_MSA.

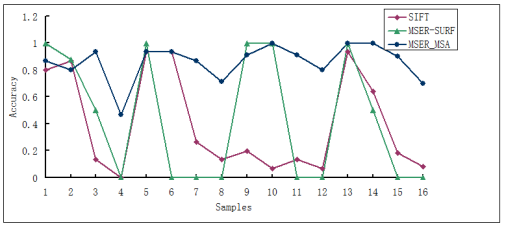


Figure 6: Accuracy comparison chart of SIFT, MSER_SURF and MSER_MSA

MSER_SURF and MSER_MSA are not far-off, but for large scaled transforming image pairs, the accuracy rates of SIFT and MSER_SURF are significantly lower than MSER_MSA.

Since the MSA algorithm involves Fourier transforms, the time efficiency is slightly lower. At the same time, the MSA algorithm relies on the segmentation effect of MSER algorithm. So the matching effect of artificial experimental images is better than that of realistic images.

6. Conclusion

MSA moments have scale invariant and affine invariant characteristics, and have achieved good recognition results in the field of image pattern recognition. However, there is not much research on the application of MSA moments in filed of image feature describing. After many experiments and analysis, it was found that if we want to apply MSA moments to the description of image features, the simple background neighborhoods of each feature points must be obtained firstly. Therefore, this article combines the MSER method with the MSA moments. The maximum stable extreme regions are taken as the neighborhoods of image feature points. And the 31 orders' MSA moments of each feature point are calculated as the feature descriptor of each feature point. Experimental results show that the proposed MSER_MSA algorithm can make full use of the invariant characteristics of MSA moments. However, it was also found that the MSA moments are sensitive for images with larger illumination changes. At the same time, the time efficiency of the MSA moment is low. Therefore, the next work of this study will focus on solving the illumination sensitivity of MSA moments and improving the time efficiency of MSA moments.

References

[1] Zhai F, Dang J, Wang Y, et al., *Application of Shape Context and Belief Propagation in Removing SIFT Mismatches*, Journal of Computer-Aided Design & Computer Graphics, 28(2016), 443-449.

[2] Morel J M, Yu G., *ASIFT: A New Framework for Fully Affine Invariant Image Comparison*, Siam Journal on Imaging Sciences, 2 (2011), 438-469.

[3] Rublee E, Rabaud V, Konolige K, et al., *ORB: An efficient alternative to SIFT or SURF*, IEEE International Conference on Computer Vision, IEEE, 2012, 2564-2571.

[4] MingKuei Hu, *Visual pattern recognition by moment invariants*, Information Theory Ire Transactions on, 8 (1962), 179-187.

[5] Zhai Feng-Wen, Muhammad Asim Azim, WANG Yang-PING et al., *Amendment of Zernike moment fast conputation*, Journal of Jilin University(Engineering and Technology Edition), 44 (2014):1860-1866.

[6] Dae San Kim, Seung-Hwan Yang, *A recursive formula for power moments of 2-dimensional kloosterman sums associated with general linear groups*, Italian journal of Pure and Applied Mathematics, 34 (2015), 7-16.

[7] Huang Bo, Zhao Ji-Yin, Zheng Rui-Rui et al., *Affine invariant pattern recognition based on multi-scale autoconvolution normalized histograms*, CTA Electronic Sinica, 39 (2011), 64-69.

[8] Huang Bo, Zhao Xiao-Hui, Chao Xin-Qin et al., *Histogram affine invariants extraction method based on two-scale auto-convolution*, Journal on Communications, 32 (2012), 140-146.

[9] Yang Jian-Wei, Li Pei-Yao, *Affine Invariant Feature Extraction Based on Fractional Order Moment*, Acta Automatica Sinica, 41 (2015), 2147-2154.

[10] Miao Jun, Chun Jun, Zhang Gui-Meri, *An Affine Invariant Line Descriptor and Line Matching*, Acta Electronic Sinica, 43 (2015), 2505-2512.

[11] Heikkil J., *Multi-Scale Autoconvolution for Affine Invariant Pattern Recognition*, 16Th International Conference on Pattern Recognition, IEEE Computer Society, 2002, 10119.

[12] Rahtu E, Salo M, Heikkila J., *Affine invariant pattern recognition using multiscale autoconvolution*, IEEE Transactions on Pattern Analysis & Machine Intelligence, 27(2005), 908-18.

[13] Kannala J, Rahtu E, Heikkila J., *Affine registration with multi-scale autoconvolution*, IEEE International Conference on Image Processing, IEEE, 2005, 1064-7.

[14] Rahtu E, Salo M, Heikkilä J., *Multiscale Autoconvolution Histograms for Affine Invariant Pattern Recognition*, British Machine Vision Conference 2006, Edinburgh, Uk, September. DBLP, 2006:1059-1068.

[15] Rahtu E, Salo M, Heikkil J, et al., *Generalized affine moment invariants for object recognition*, International Conference on Pattern Recognition, IEEE Computer Society, 2006, 634-637.

[16] Shao C, Ding Q, Luo H., *An improved multi-scale auto convolution transform*, Proceedings of SPIE - The International Society for Optical Engineering, 9301 (2014), 120-127.

[17] Li Hui-Hui, Hua Li, Yang Ning et al., *Multi-target association algorithm for remote sensing images based on MSA features and simulated annealing optimization*, Journal of Jilin University (Engineering and Technology Edition), 45 (2015), 1353-1359.

[18] Zhang Xiongmei,Yi Zhaoxiang, Cai Xingfu et al., *SAR Image Registration Method Based on Improved SIFT*, Computer Engineering, 41(2015), 223-226.

[19] Donoser M, Bischof H., *Efficient Maximally Stable Extremal Region (MSER) Tracking*, IEEE Computer Society Conference on Computer Vision and Pattern Recognition, IEEE Computer Society, 2006, 553-560.

[20] Tao L, Jing X, Sun S, et al., *Combining SURF with MSER for image matching*, IEEE International Conference on Granular Computing, IEEE, 2014, 286-290.

[21] B.G. Sidharth, *A new integral transform*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 15-18.

Accepted: 26.02.2018

ON THE DOUBLE FROBENIUS GROUP OF THE FORM 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

J. Moori
Department of Mathematical Sciences
North West University
P Bag X2046, Mmabatho
South Africa, 2735
jamshid.moori@nwu.ac.za

P. Perumal
Department of Mathematics
Durban University of Technology
P.O.Box 953, Durban
South Africa, 4000
preggyp@dut.ac.za

Abstract. Let G be a finite group. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H. If G admits $\overline{H}$ as a group of automorphisms such that $C_G(N) = \{1_G\}$ and GN is also a Frobenius group with kernel G and complement N, then $\overline{G} = GNH$ is called a double Frobenius group (or 2-Frobenius group). The group $\overline{G} = GNH$ is a product of subgroups $G \leq \overline{G}$, $N \leq \overline{G}$, $H \leq \overline{G}$ with $G \trianglelefteq \overline{G}$, $GN \trianglelefteq \overline{G}$ and $\overline{G} = G{:}NH = GN{:}H$. In this article we shall construct a double Frobenius group of the form $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$, where $G \cong 2^{2r}$, $N \cong \mathbb{Z}_{2^r-1}$ and $H \cong \mathbb{Z}_2$, where $r \in \mathbb{N}, r \geq 2$. The construction is a general one that gives examples of double Frobenius groups for particular values of n. In addition to the general construction of the group $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$, we calculate in general the conjugacy classes, Fischer matrices and character table of the group. One example $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, (the case $r = 2$) is demonstrated.

Keywords: double Frobenius group, Fischer matrices, character table

1. Introduction

The case where a Frobenius group $\overline{H} = NH$ acts by automorphisms on a group G has received some study in recent years. In this situation various properties (parameters) of G are found to be close to the corresponding properties of $C_G(H)$ and H. These properties include the order, rank, exponent, nilpotency class and Fitting height of G. Khukhro in [7], Khukhro and Makarenko in [8] and Khukhro, Makarenko and Shumyatsky in [6] have obtained some results in this regard. We are interested in the case where a Frobenius group $\overline{H}$ with kernel N and complement H acts as a group of automorphisms on a group G such that $C_G(N) = \{1_G\}$ and GN is also a Frobenius group. In recent years graphs associated with finite groups have received much attention. In particular the

prime graph or Gruenberg-Kegel graph has been the subject of most attention in interest and research. Both Frobenius and double Frobenius groups appear in the study of the prime graphs of finite groups. As a result of one of the key classification theorems of the prime graphs of finite groups (the Gruenberg-Kegel Theorem), we have that: if G is a finite solvable group (Frobenius groups may be solvable, double Frobenius groups are always solvable) with a disconnected prime graph, then G is either Frobenius or double Frobenius.

2. Preliminary results

We will briefly describe here some results on Frobenius and double Frobenius groups, but first include some material on the method of coset analysis and Fischer matrices.

2.1 Coset analysis

We will briefly discuss the method of *coset analysis* which is used to determine the conjugacy classes of group extensions $\overline{G} = N.G$ where N is an abelian normal subgroup of $\overline{G}$. The technique is used for both split and non-split extensions. The technique was first used by Moori in [11]. More details can be found in [12] and [11].

Let $\overline{G} = N.G$, where $N \lhd \overline{G}$ and $\overline{G}/N \cong G$, be a finite group extension.

1. For each $g \in G$ let $\overline{g} \in \overline{G}$ map to g under the natural epimorphism $\pi : \overline{G} \to G$ and let $g_1 = N\overline{g}_1, g_2 = N\overline{g}_2, \ldots, g_r = N\overline{g}_r$ be representatives for the conjugacy classes of $G \cong \overline{G}/N$. Therefore, $\overline{g}_i \in \overline{G}$, $\forall i$, and by convention we take $\overline{g}_1 = 1_{\overline{G}}$.

2. The method of coset analysis constructs for each conjugacy class $[g_i]_G$, $1 \leq i \leq r$, a number of conjugacy classes of $\overline{G}$. For each $1 \leq i \leq r$, we let $g_{i1}, g_{i2}, \ldots, g_{ic(g_i)}$ be the corresponding representatives of these classes. That is, each conjugacy class of $\overline{G}$ corresponds uniquely to a conjugacy class of G.

3. We use the notation $U = \pi(\overline{U})$ for any subset $\overline{U} \subseteq \overline{G}$. Therefore, $\pi^{-1}([g_i]_G) = \bigcup_{j=1}^{c(g_i)} [g_{ij}]_{\overline{G}}$ for any $1 \leq i \leq r$. We also assume that $\pi(g_{ij}) = g_i$ and by convention we take $g_{11} = 1_{\overline{G}}$.

4. For fixed $i \in \{1, 2, \ldots, r\}$, act N by conjugation on the coset $N\overline{g}_i$ and let the resulting orbits be $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$. If N is abelian (for both a split and non-split extensions), then
$$|Q_{i1}| = |Q_{i2}| = \ldots = |Q_{ik_i}| = \frac{|N|}{k}.$$

5. Act $\overline{G}$ on $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$ and suppose f_{ij} orbits fuse together to form a new orbit Δ_{ij}, and let the total number of new resulting orbits in this

action be $c(g_i)$ (that is $1 \leq j \leq c(g_i)$). Then $\overline{G}$ has a conjugacy class $[g_{ij}]_{\overline{G}}$ that contains Δ_{ij} and $|[g_{ij}]_{\overline{G}}| = |[g_i]| \times |\Delta_{ij}|$. We repeat steps 4 and 5 above for all $i \in \{1, 2, \ldots, r\}$.

2.2 Fischer matrices

If $\overline{G} = N.G$ is a group extension, then $\overline{G}$ acts on the classes of N and on the $Irr(N)$. By Brauer's Theorem (see [5], theorem 6.32 and corollary 6.33) the number of orbits of both these actions are the same. Let $\theta_1, \theta_2, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$ and let $\overline{H}_k$ and H_k denote the corresponding inertia and inertia factor groups of θ_k. Bernd Fischer showed that the character table of $\overline{G}$ can be constructed by using the character tables of the inertia factor groups H_k together with some matrices called Fischer-Clifford matrices (see [4]). We define here the Fischer matrices which are used in the construction of the character table of any group extension $\overline{G} = N.G,\ N \lhd \overline{G}$

1. For each $[g_i]_G$ (conjugacy class of G), there corresponds a Fischer matrix which we denote by M_i.

2. $[g_{ij}]_{\overline{G}} \bigcap \overline{H}_k = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_{\overline{H}_k}$, where $g_{ijkn} \in \overline{H}_k$ and by $c(g_{ijkn})$ we mean the number of $\overline{H}_k$-conjugacy classes that form a partition for $[g_{ij}]_{\overline{G}}$. Since $g_{11} = 1_{\overline{G}}$, we have $g_{11k1} = 1_{\overline{G}}$ and thus $c(g_{11k1}) = 1$ for all $1 \leq k \leq t$.

3. $[g_i]_G \bigcap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_{H_k}$, where $g_{ikm} \in H_k$ and by $c(g_{ik})$ we mean the number of H_k-conjugacy classes that form a partition for $[g_i]_G$. Since $g_1 = 1_G$, we have $g_{1k1} = 1_G$ and thus $c(g_{1k1}) = 1$ for all $1 \leq k \leq t$. Also, $\pi(g_{ijkn}) = g_{ikm}$ for some $m = f(j, n)$.

4. The top of the columns of the Fischer matrix M_i are labeled by the representatives of $[g_{ij}]_{\overline{G}}$, $1 \leq j \leq c(g_i)$ obtained by coset analysis and below each g_{ij} we put $|C_{\overline{G}}(g_{ij})|$.

5. The bottom of the columns of M_i are labeled by some weights m_{ij} defined by
$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N| \frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}.$$

6. To label the rows of M_i we define the set J_i to be
$$J_i = \{(k, m) |\ 1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik})\}.$$

7. Then each row of M_i is indexed by a pair $(k, m) \in J_i$.

8. For a fixed $1 \leq k \leq t$, we let M_{ik} be a submatrix of M_i with rows corresponding to the pairs $(k, 1), (k, 2), \ldots, (k, r_k)$.

9. Let
$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|}\widetilde{\psi_k}(g_{ijkn})$$
for which $\pi(g_{ijkn}) = g_{ikm})$ and where $\widetilde{\psi_k}$ is the extension of θ_k to $\overline{H}_k$.

10. For each i, corresponding to the conjugacy class $[g_i]_G$, we define the Fischer matrix $M_i = \big(a_{ij}^{(k,m)}\big)$, where $1 \le k \le t,\ 1 \le m \le c(g_{ik}),\ 1 \le j \le c(g_i)$.

11. The Fischer matrix M_i is now given by
$$M_i = \big(a_{ij}^{(k,m)}\big) = \left(\begin{array}{c} M_{i1} \\ \hline M_{i2} \\ \hline \vdots \\ \hline \vdots \\ \hline M_{it} \end{array}\right)$$

12. In the Fischer matrix M_i, each M_{ik} is the submatrix corresponding to the inertia group $\overline{H}_k$ and it's inertia factor H_k.

The Fischer matrices have numerous properties that help in their computations. For details on the Fischer matrices and their properties, see [1], [2], [13], [15], [17], [14] and [18].

2.3 Frobenius group

Definition 2.1. *We define a group $\overline{H} = N{:}H$ to be a Frobenius group if it has a proper subgroup $H \neq \{1_{\overline{H}}\}$ such that $H \cap H^x = \{1_{\overline{H}}\}$ for all $x \in \overline{H} - H$. The subgroup H is called the complement and the non-trivial subgroup N is called the kernel.*

2.4 Properties of Frobenius Groups and their characters

We list here some properties of Frobenius groups and their ordinary characters. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H.

1. The kernel N is unique and nilpotent and $N \lhd \overline{H}$.
2. $C_{\overline{H}}(n) \le N\ \forall\ 1_{\overline{H}} \neq n \in N$ and $C_{\overline{H}}(h) \le H\ \forall\ 1_{\overline{H}} \neq h \in H$.
3. $|H|\big|\big(|N| - 1\big)$.
4. The action of H on N is fixed point free. Hence, the lengths of the non-trivial orbits of the action of H on N is $|H|$ by the orbit stabilizer theorem. Therefore $|N| = 1 + m|H|$ where m is the number of non-trivial orbits of the action of H on N.

5. The Sylow p-subgroups of H are generalized quaternion or cyclic if $p = 2$ and cyclic if $p \neq 2$.
6. $\overline{H}$ has a trivial center.
7. $c(\overline{H}) = \frac{c(N)-1}{|H|} + c(H)$ where $c(\overline{H})$, $c(N)$ and $c(H)$ are the number of conjugacy classes of $\overline{H}$, N and H respectively.
8. If $\phi_1 \neq \phi \in Irr(N)$ where ϕ_1 is the trivial character of N, then $\phi^{\overline{H}} \in Irr(\overline{H})$.
9. If $\psi \in Irr(\overline{H})$, then either $N \subset ker\psi$ or $\psi = \phi^{\overline{H}}$ for some irreducible character $\phi \neq \phi_1$ of N, thus the irreducible characters of $\overline{H}$ are of two types; those with kernel containing N and those induced from irreducible characters of N.
10. If $\psi \in Irr(\overline{H})$, such that $ker\psi \not\supset N$ and ρ is the regular representation of H, then $\psi|_H = n\rho$ where $n \in \mathbb{N}$ and we have that $|H|\big|\phi(1_{\overline{H}})$. Furthermore, if $\phi_1 \neq \phi \in Irr(N)$ is a linear character, then $|H| = \phi(1_{\overline{H}})$.
11. If $\phi_1 \neq \phi \in Irr(N)$, then ϕ has inertia group $I_{\overline{H}}(\phi) = N$.
12. $\overline{H}$ has $\alpha = \frac{c(N)-1}{|H|}$ distinct irreducible characters of the form $\phi^{\overline{H}}, \phi_1 \neq \phi \in Irr(N)$ and hence, $c(H) = n$ irreducible characters $\psi \in Irr(\overline{H})$, with $N \subset ker\psi$

Proofs of the results mentioned here can be found in [16].

2.5 Double Frobenius group

Definition 2.2. *Let G be a finite group. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H. If G admits $\overline{H}$ as a Frobenius group of automorphisms with $C_G(N) = \{1_G\}$ such that GN is also a Frobenius group with kernel G and complement N, then $\overline{G} = GNH$ is called a double Frobenius group.*

2.6 Properties of double Frobenius groups

We list here some properties of double Frobenius groups. Let $\overline{G} = GNH$ be a double Frobenius group with Frobenius subgroups GN and NH.

1. $G \trianglelefteq \overline{G}$ and $GN \trianglelefteq \overline{G}$.
2. Double Frobenius groups are solvable.
3. In the double Frobenius group $\overline{G} = GNH$, N is cyclic and of odd order and H is cyclic.
4. The center of a double Frobenius group is trivial.

Before describing our construction we mention the following two results which we make reference to.

Lemma 2.3 ([10]). *Let $(G,+)$ be an elementary abelian group of order p^n for some prime p. There is a cyclic fixed point free automorphism group of order k on G if and only if $k \mid p^n - 1$.*

Proof. See [10][Corollary 5.4]. □

Proposition 2.4 ([10]). *Let Φ be a fixed point free automorphism group on the additive group $(N,+)$. Then the semi-direct product $G = \Phi{:}N$ is a Frobenius group with complement Φ and kernel N.*

Proof. See [10][Proposition 7.3]. □

3. Constructing the group $\overline{G} = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

From our definition, a double Frobenius group is the result of a group action of a Frobenius group $\overline{H}$ on a finite group G. Our construction therefore begins here. We look for a Frobenius group $\overline{H}$. Frobenius groups play an important role in finite group theory as point stabilizers of Zassenhaus groups (doubly transitive permutation groups in which some non-identity element fixes two points but none fixes three). They also appear frequently as maximal subgroups of the finite simple groups.

We aim to construct double Frobenius groups using $PSL(n,q)$ with $n = 2$ and q even. For q even, $PSL(2,q)$ has maximal subgroups which are Dihedral groups of order $2(q-1)$ or $2(q+1)$, (see King [9]).

Since q is even, $q-1$ is odd and hence, the maximal subgroup of order $2(q-1)$ is a Frobenius group due to the fact that the Dihedral group D_{2m} is Frobenius if m is odd. Let $q = 2^r,\ 2 \leq r \in \mathbb{N}$, then the Frobenius group $D_{2(q-1)}$ has the form $\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$. Now for q even, $PSL(2,q) \cong SL(2,q) \leq GL(2,q)$. The natural action of $GL(2,q)$ on the elementary abelian group of order q^2 implies that q^2:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$) $\leq q^2{:}GL(2,q)$.

Therefore, $(2^r)^2$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$) $\leq (2^r)^2{:}GL(2,2^r)$. Since $2^r - 1$ divides $2^{2r} - 1$ (because $2^{2r} - 1 = 2^r - 1 \times 2^r + 1$), if 2^{2r}:$\mathbb{Z}_{2^r-1}$ is a Frobenius group, then by Proposition 2.4 and Lemma 2.3, 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$) is a double Frobenius group.

4. Fischer matrices and character table of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

In this section we shall determine the conjugacy classes, Fischer matrices and character table of the double Frobenius group 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$). We give a general description of the conjugacy classes, Fischer matrices and character table of the group.

Let $\overline{G} = GNH$ be a double Frobenius group with GN and NH Frobenius groups. Consider now the double Frobenius group 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$). Let $\overline{G} = GNH = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$). Then, $G \cong 2^{2r},\ \ N \cong \mathbb{Z}_{2^r-1}$ and $H \cong \mathbb{Z}_2$.

5. The group $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$

We will first determine the conjugacy classes of the Frobenius group $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$.

5.1 Conjugacy classes of $\overline{H}$

We know that $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$ is a Frobenius group with kernel $N = \langle a \rangle \cong \mathbb{Z}_{2^r-1}$ and complement $H = \langle b \rangle \cong \mathbb{Z}_2$. Since H acts on N with fixed point free action, we have that the number of non-trivial orbits of H on N is given by $\alpha = \frac{|N|-1}{|H|}$ and the length of each orbit is given by $|H|$. Therefore, here the orbits of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ have lengths 1 and 2. Using the method of coset analysis, briefly described in Section 2.1, we analyse the coset Nh for each $h \in H$ and find the values of k where k is the order of the stabilizer in N of h. The values of k can be determined from the action of H on N. Since this action is fixed point free, $k = 2^r - 1$ for $h = 1_H$ and $k = 1$ for $h = b$.

For $h = 1_H$, $k = 2^r - 1$, $f_1 = 1$ and $f_i = 2 \ \forall i \in \{2, 3, \ldots, \alpha + 1\}$.

For $h = 1_H$, $k = 2^r - 1$, $f_1 = 1$:

$$\left|C_{\overline{H}}(x)\right| = \frac{(2^r - 1) \times 2}{1} = |\overline{H}|.$$

So for $f_1 = 1$, we have the identity class of $\overline{H}$.

For $h = 1_H$, $k = 2^r - 1$, $f_i = 2$:

$$\left|C_{\overline{H}}(x)\right| = \frac{(2^r - 1) \times 2}{2} = |N|.$$

So for $f_i = 2$, we have a class of $\overline{H}$ containing x with $\circ(x) = 2^r - 1$. The size of the conjugacy class is

$$\left|[x]_{\overline{H}}\right| = \frac{|\overline{H}|}{\left|C_{\overline{H}}(x)\right|} = \frac{(2^r - 1) \times 2}{2^r - 1} = 2.$$

Note that there are $\frac{2^r-2}{2} = 2^{r-1} - 1$ such classes in NH.

For $h = b$ we have $k = 1$, $f = 1$:

$$\left|C_{\overline{H}}(x)\right| = \frac{1 \times 2}{1} = 2.$$

Therefore,

$$\left|[x]_{\overline{H}}\right| = \frac{|\overline{H}|}{\left|C_{\overline{H}}(x)\right|} = \frac{(2^r - 1) \times 2}{2} = 2^r - 1.$$

So, for the coset Nb there is a unique involutary class of $\overline{H}$ containing h.

From the above we deduce that there are $1 + (2^{r-1} - 1) + 1 = 2^{r-1} + 1$ classes in $\overline{H}$. This number can be confirmed by (6) in section 2.4, since $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$ and we have that

$$c(\overline{H}) = c(H) + \alpha = c(H) + \frac{c(N) - 1}{|H|} = 2 + \frac{2^r - 2}{2} = 2^{r-1} + 1.$$

The full list of conjugacy classes based on coset analysis is given in Table 1.

Table 1: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$\circ(\overline{h})$	$\left\lvert C_{NH}(\overline{h})\right\rvert$
1_H	1	1	$(2^r-1)2$
	$(2^r-1)A_1$	c_1	2^r-1
	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$
	$(2^r-1)A_\alpha$	c_α	2^r-1
b	$2A$	2	2

Note: Table notes
1. $(2^r-1)A_i$ is the conjugacy class containing elements of order 2^r-1.
2. $(2A)$ is the conjugacy class containing elements of order 2 using the notation of the ATLAS.
3. Also here $H = \langle b\rangle$ and c_i for $1 = 1, 2, ..., \alpha$ divides the order of $N = 2^r - 1$.

5.2 Character Table of $\overline{H}$

We obtain here a general description of the character table of the Frobenius group $\overline{H}$. To construct the character table we will use the results (7-11) listed in section 2.4 above and the following:

1. The irreducible characters of $\overline{H}$ with kernel containing N are χ_1 and χ_2 of degree 1 and those induced from non-trivial irreducible characters of N are $\chi_3, \chi_4, \ldots, \chi_{2+\alpha}$ of degree 2 (see table below).

2. $$\psi^{\overline{H}}(a_j) = \left|C_{\overline{H}}(a_j)\right| \sum_{i=1}^{m} \frac{\phi(x_i)}{\left|C_N(x_i)\right|},$$

   where $\phi \in Irr(N)$, $[a_j]$ is the conjugacy class of $\overline{H}$ containing a_j and $x_1, x_2, \ldots, x_m$ are class representatives for the classes of N that fuse to $[a_j]$. Since $\left|C_{\overline{H}}(a_j)\right| = 2^r-1$ and $\left|C_N(x_i)\right| = |N| = 2^r-1$, $\phi^{\overline{H}}(a_j) = \sum_{i=1}^{m} \phi(x_i)$. The sum on the right hand side are the orbit sums of the action of H on N.

3. We know that $\mathbb{Z}_2$ acts fixed point free on $\mathbb{Z}_{2^r-1}$, the number of non-trivial orbits is given by $\alpha = \frac{|N|-1}{|H|} = 2^{r-1} - 1$ and the length of each orbit is given by $|H| = 2$. Also, since $\overline{H}$ is dihedral, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ is given by $ba^ib^{-1} = a^{-i}$.

   The orbits are: $\Theta_j = \{a^i, a^{-i}\}$ for $j = 1, 2, \ldots, \alpha$ and $i = 1, 2, \ldots, \alpha$.

4. Now let $p_j = \sum_{i=1}^{2} \phi(x_i) = \left[c^j(i, 1)\right]$, for $j = 2, 3, \ldots, \alpha$, then $p_j = \phi(x_i) + \phi(x_i^{-1})$.

So $p_j = \phi(x_i) + \overline{\phi(x_i)} = 2t$, where t is the real part of $exp(\frac{2\pi i}{2^r-1})$.

The following table gives the partial character table of $\overline{H}$, namely the values of the induced characters of degree 2, $\chi_3, \chi_4, \ldots \chi_{2+\alpha}$, on classes $\big((2^r-1)A_i\ ,\ i = 1, 2, \ldots, \alpha\big)$, where $p_j = \sum_{j=1}^{2} t_j = 2t_j$ and $t_j = \exp\big(\frac{2\pi i}{2^r-1}\big)$.

	$(2^r-1)A_1$	$(2^r-1)A_2$	$(2^r-1)A_3$	.....	$(2^r-1)A_\alpha$
χ_3	p_1	p_2	p_3	.....	p_α
χ_4	p_2	p_3	p_4	.....	$p_{\alpha-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{2+\alpha}$	p_α	$p_{\alpha-1}$	$p_{\alpha-2}$	.....	p_1

We now produce the character tables of $\mathbb{Z}_2$, $\mathbb{Z}_{2^r-1}$ and the Frobenius group $\overline{H}$ in Tables 2, 3 and 4 respectively.

Table 2: Character Table of $\mathbb{Z}_2$

Classes	e	b
χ_1	1	1
χ_2	1	-1

Table 3: Character Table of $\mathbb{Z}_{2^r-1}$

Classes	e	a	a^2	...	...	a^{2^r-2}
ϕ_1	1	1	1	1	1	1
ϕ_2	1	p	p^2	...	...	p^{2^r-2}
$\vdots$						$\vdots$
$\vdots$						$\vdots$
ϕ_{2^r-1}	1	p^{2^r-2}	p^{2^r-3}	...	...	p

Note:Table note: $p = e^{\frac{2\pi i}{2^r-1}}$

Remark 5.1. For $n \in PSL(2,q) \cong SL(2,q)$, q even $(q = 2^r)$, let $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{F}_q^*$ and $\circ(\lambda) = q-1$. Then there exists an involution $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ for $x, y, z, t \in \mathbb{F}_q$ such that $bnb^{-1} = n^{-1}$ and $\langle n,\ b\rangle \cong \mathbb{Z}^{2^r-1} : \mathbb{Z}_2$.

For $b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the following conditions apply:

Table 4: Character Table of $\overline{H} = NH$

(g)	(1)	$(2^r-1)A_1 \ldots\ldots (2^r-1)A_\alpha$	$(2A)$
$\lvert C_{\overline{H}}(g)\rvert$	$(2^r-1)2$	$2^r-1 \ldots\ldots 2^r-1$	2
χ_1	1	$1 \ldots\ldots 1$	1
χ_2	1	$1 \ldots\ldots 1$	-1
χ_3	2	$p_1 \ldots\ldots p_\alpha$	0
χ_4	2	$p_2 \ldots\ldots p_{\alpha-1}$	0
$\vdots$	$\vdots$	$\vdots \ldots\ldots \vdots$	$\vdots$
$\chi_{2+\alpha}$	2	$p_\alpha \ldots\ldots p_1$	0

Note: Table note: $p_j = \sum_{j=1}^{2} t_j = 2t_j$ where $t_j = \exp\left(\frac{2\pi i}{2^r-1}\right)$.

- $xt - yz = 1 \quad \ldots\ldots\ldots (1)$
- $b^2 = \begin{pmatrix} x^2+yz & xy+yt \\ xz+zt & yz+t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots (2)$
- $bnb^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \quad \ldots\ldots\ldots (3)$

From equation (2) above , we have that:

- $x^2 + yz = 1 \quad \ldots\ldots\ldots (4)$
- $xy + yt = 0 \quad \ldots\ldots\ldots (5)$
- $xz + zt = 0 \quad \ldots\ldots\ldots (6)$
- $zy + t^2 = 1 \quad \ldots\ldots\ldots (7)$

From equation (5) above, we have that $y(x + t) = 0$ and that $y = 0$ or $x + t = 0$. Similarly from equation (6), we have $z(x + t) = 0$ and $z = 0$ or $x + t = 0$. From these two equations we have the following six cases to consider.

- Case 1: $y = 0$ and $x + t = 0$.
- Case 2: $z = 0$ and $x + t = 0$.
- Case 3: $y = z = 0$.
- Case 4: $y = 0$.
- Case 5: $z = 0$.
- Case 6: $x + t = 0$.

We now consider each case:

1. Case 1: $y = 0$ and $x + t = 0$. If $x + t = 0$ then $x = -t = t$ since in prime field $\mathbb{F}_2, -1 \equiv 1$. Now $y = 0$ implies by equation (4) that $x^2 = 1$ and $x = 1$. This gives us $b = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.

   Now $bnb^{-1} = n^{-1}$ implies that $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$.

   So, $\begin{pmatrix} \lambda & 0 \\ z\lambda + z\lambda^{-1} & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. Therefore, we must have $z\lambda + z\lambda^{-1} = 0$ and

   $z(\lambda + \lambda^{-1}) = 0$ implies $z = 0$ since $\lambda + \lambda^{-1} = 0$ implies that $\circ(\lambda) = 2$ which we can not have.

   Hence, this gives $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

2. Case 2: $z = 0$ and $x + t = 0$. Just as in Case 1 above we arrive at the conclusion that $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

3. Case 3: If $y = z = 0$, then as in Cases 1 and 2 above, we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

4. Case 4: If $y = 0$ then equation (4) implies that $x = 1$. So b has the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. This case is similar to Case 1 above and we get: $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

5. Case 5: $z = 0$ gives us a similar conclusion as in Case 4 above where we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

6. Case 6: If $x + t = 0$, then $x = t$. There are two cases to consider here: (i)$x = t \neq 0$ and (ii)$x = t = 0$.

   If $x = t \neq 0$, then b has the form $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$. Then for $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\lambda \neq 0$, $bnb^{-1} = n^{-1}$ gives:

$$\begin{pmatrix} x & y \\ z & x \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ z & x \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

   So,

$$\begin{pmatrix} x^2\lambda + zy\lambda^{-1} & xy\lambda + xy\lambda^{-1} \\ xz\lambda + xz\lambda^{-1} & yz\lambda + x^2\lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

So, this gives us the following: $xy\lambda + xy\lambda^{-1} = 0$ ..... (8). Then $xy(\lambda + \lambda^{-1}) = 0$ and $\lambda + \lambda^{-1} = 0$ or $xy = 0$. If $\lambda + \lambda^{-1} = 0$, then $\lambda = \lambda^{-1}$ and $\circ(\lambda) = 2$ which is not possible since λ is an element in a cyclic group of odd order. Therefore, $xy = 0$ and since $x \neq 0$, $y = 0$. Similarly using the equation $xz\lambda + xz\lambda^{-1} = 0$, we arrive at the conclusion $z = 0$. So for this first case $x = t \neq 0$ implies that $y = z = 0$ and equation (1) gives $x = 1$. Hence, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction. For the second case, if $x + t = 0$ and $x = t = 0$, then equations (1) and (4) implies that $yz = 1$. Thus, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$.

Then $z = y^{-1}$ and $b = \begin{pmatrix} 0 & \lambda^i \\ \lambda^{-i} & 0 \end{pmatrix}$ for $\lambda^i \in \mathbb{F}_q^*$.

Before describing the conjugacy classes of the double Frobenius group 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$), we make the following Note.

Note 5.2. 1. $|PSL(2,q)| = q^3 - q$ if q is even.

2. If q is even, then $PSL(2,q) \cong SL(2,q)$.

3. The group $SL(2,q)$, $q = 2^t$, $t \geq 1$ has $q+1$ distinct conjugacy classes. These classes are described in the Table 5.

Table 5: Conjugacy Classes of $SL(2,q)$, q even.

Class	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
Rep. of Class	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & r + r^q \end{pmatrix}$
No. of Classes	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert C_{SL(2,2^t)}(g)\rvert$	$q^3 - q$	q	$q - 1$	$q + 1$
$\lvert C_g\rvert$	1	$q^2 - 1$	$q(q+1)$	$q(q-1)$

Note: Table notes;
1. $\alpha \in \mathbb{F}_q^*$, $\alpha = \epsilon^k$, $k \neq 0$
2. $\mathbb{F}_{q^2}^* = \langle\theta\rangle$ and $r = \theta^{(q-1)j}$ for $j = 1, 2, \ldots, \frac{q}{2}$.

6. Conjugacy classes of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

We determine now the conjugacy classes of the double Frobenius group $\overline{G} = GNH = G{:}NH$ where $N = \langle a\rangle \cong \mathbb{Z}_{2^r-1}$ and $H = \langle b\rangle \cong \mathbb{Z}_2$.

To determine the conjugacy classes of $\overline{G}$, we consider the cosets $\overline{h}G$ where $\overline{h} \in NH = \overline{H}$.

The coset $1G$:

Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^{2r}$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}, C_{\overline{H}}(\overline{h}) = \overline{H}$ on G.

Now let $nh \in \overline{H} = NH$ for $1_{\overline{H}} \neq n \in N,\ 1_{\overline{H}} \neq h \in H$. Then for $g \in G$, $g^{nh} = nhgh^{-1}n^{-1} = n(hgh^{-1})n^{-1} = n(g^h)n^{-1} = (g^h)^n$.

Therefore to act $\overline{H}$ on G, we act $h \in H$ first and then act $n \in N$.

Now $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2 \leq PSL(2,2^r) \cong SL(2,2^r) \leq GL(2,2^r)$ and $H = \langle b \rangle$ where b is an involution in $PSL(2,2^r)$. The group $SL(2,q)$, q even, $(q = 2^r\ , r \geq 1)$ has $q+1$ distinct conjugacy classes. Of these $q+1$ classes there is only one class of involutions. The size of this class is q^2-1 and for any involution $b \in SL(2,q)$, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$, where $x, z \in \mathbb{F}_q^*$. See the Note 5.2.

Now $G \cong V_2(\mathbb{F}_q)$, the vector space of dimension two over the field of $q = 2^r$ elements. So $G = \{0,\ \lambda^i e_1,\ \lambda^j e_2,\ (\lambda^i e_1 + \lambda^j e_2)\}$ for $i,\ j = \{0, 1, 2, \ldots, q-2\}$, where $\lambda^i,\ \lambda^j \in \mathbb{F}_q^*$, and $\{e_1,\ e_2\}$ is a basis of G, with $e_1^2 = 1,\ e_2^2 = 1$. So in the action of $\overline{H}$ on G, we look at the action of $b = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$ on the elements of $G \cong 2^{2r} = \{0,\ \lambda^i e_1,\ \lambda^j e_2,\ (\lambda^i e_1 + \lambda^j e_2)\}$ followed by the action of N on these orbits (the orbits of b on G).

Now H acts on the 2^{2r} elements of G fixing (including the identity) 2^r elements and permuting the remaining $2^{2r} - 2^r$ elements in orbits of length two. So the elements of G are now in $\frac{2^{2r}-2^r}{2}$ orbits of length two plus the 2^r fixed points. Acting N on these 2^r fixed points and the $\frac{2^{2r}-2^r}{2}$ orbits gives the following:

1. Each of the $(2^r - 1)$ (identity excluded) fixed points fuses with $\frac{2^r-2}{2} = 2^{r-1} - 1$ of the two cycles to form an orbit of size $(2^r - 1)$. There are $(2^r - 1)$ of these orbits.

2. The remaining $2^{2r} - \{(2^r - 1)(2^r - 1) + 1\} = 2(2^r) - 2$ elements fuse to form an orbit of size $2(2^r - 1)$.

So the action of $C_{\overline{H}}(\overline{h}) = \overline{H}$ on G, gives the following: one orbit of length one, one orbit of length $2(2^r - 1)$ and $(2^r - 1)$ orbits of length $(2^r - 1)$.

Therefore we have: $k = 2^{2r}$ and $f_1 = 1,\ f_i = 2^r - 1$ for $i = 2, 3, \ldots, 2^r - 1$ and $f_{2^r+1} = 2(2^r - 1)$.

$k = 2^{2r},\ f_1 = 1$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^{2r} \times (2^r - 1) \times 2}{1} = |\overline{G}|,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^{2r} \times (2^r - 1) \times 2} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$\underline{k = 2^{2r},\ f_i = 2^r - 1}$ for $i = 2, 3, \ldots, 2^r - 1$.

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r},$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^{2r}} = 2^r - 1.$$

$\underline{k = 2^{2r},\ f_i = 2(2^r - 1) :}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times (2^r - 1)} = 2^{2r},$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^{2r}} = 2 \times (2^r - 1).$$

Therefore the identity coset $1G$ produces the following conjugacy classes of $\overline{G} = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$): the identity conjugacy class, one class of size $2(2^r - 1)$ and $(2^r - 1)$ classes of size $(2^r - 1)$. The order of the non -identity elements in all the classes from the identity coset is two.

The coset $\underline{bG :}$

When G acts on the coset bG, it partitions the coset into 2^r orbits of size 2^r. The 2^r orbits of the action of G on the coset bG consists of the orbit containing b, and $(2^r - 1)$ remaining orbits each containing $b\lambda^i e_1$ for $i = 0, 1, \ldots, q-2$ where $q = 2^r$. The orbit containing b also contains $b\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q - 2$. Each of the orbits containing $b\lambda^i e_1$ also contain $b\lambda^i e_2$ for $i = 0, 1, \ldots, q - 2$ and a two cycle $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i,\ j = 0, 1, \ldots, q - 2$.

We now act the centralizer of $b \in H$, $C_{\overline{H}}(b)$ on these 2^r orbits. Since $C_{\overline{H}}(b) = \langle b \rangle$, the action of the centralizer is just the action of $\langle b \rangle$.

When b acts on G, it fixes zero and each $\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q - 2$ and permutes the remaining $2^{2r} - 2^r$ elements of G into $\frac{2^{2r}-2^r}{2}$ orbits of length two. These $\frac{2^{2r}-2^r}{2}$ orbits are of the form $(\lambda^i e_1\ ,\ \lambda^i e_2)$ for $i = 0, 1, \ldots, q - 2$ and $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i,\ j = 0, 1, \ldots, q - 2$.

This implies that when $\langle b \rangle = C_{\overline{H}}(b)$ acts on the 2^r orbits, it permutes the elements in each of the 2^r orbits.

$\underline{k = 2^r,\ f_i = 1}$ for $i = 1, 2, \ldots, 2^r$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{2^r \times 2}{1} = 2 \times 2^r,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^r} = 2^r \times (2^r - 1).$$

Therefore, the coset bG produces 2^r conjugacy classes of $\overline{G}$ each of size $2^r(2^r - 1)$. We determine now the orders of the elements in these 2^r conjugacy

classes. Using the result for element orders in the group $\overline{G} = G{:}\overline{H}$, where G is an elementary 2-group, we have that $\omega = gg^b$ for $g \in G,\ b \in H$. Now if $\omega = 1$, then $g^{-1} = g = g^b$ since $\circ(g) = 2$. Therefore, if b fixes $g \in G$, then $\circ(\overline{g}) = 2$ for $gb = \overline{g} \in \overline{G}$. If $\omega \neq 1$, then $\circ(\overline{g}) = 4$. Since the conjugacy class containing b is the class that has the fixed points, from the discussion above the order of the elements in this class is two (as would be expected since the class has the element b in it). The order of the elements in the remaining $(2^r - 1)$ conjugacy classes is four.

The coset a_iG : for $i = 1, 2, \ldots, m$ where m is the number of non-trivial orbits of the action of H on N. Now the action of G on the coset a_iG for $i = 1, 2, \ldots, m$ produces a single orbit of length 2^{2r}. The centralizer of $a_i \in N$, $C_{\overline{H}}(a_i) = \langle a_i \rangle$ acts fixed point free on the orbit permuting the elements in the orbit. Therefore, for each $a_i \in N$ for $i = 1, 2, \ldots, m$, there is a single conjugacy class of $\overline{G}$. There are m such classes.

$k = 1,\ f = 1$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times (2^r - 1)}{1} = 2^r - 1,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r}.$$

The coset a_iG, produces a single conjugacy class of $\overline{G}$ of size $2(2^{2r})$. The order of the elements in the class is $2^r - 1$.

Note 6.1. From the discussion above we can determine the number of conjugacy classes of $\overline{G}$. The classes produced by the identity coset are the identity class, $(2^r - 1)$ classes of size $(2^r - 1)$ and one class of size $2(2^r - 1)$. The coset bG produces 2^r classes of size $2^r(2^r - 1)$. Each of the cosets a_iG for $i = 1, 2, \ldots, m$ where $m = \frac{|N|-1}{H} = \frac{2^r-2}{2} = 2^{r-1} - 1$ is the number of non-trivial orbits of H on N produces a single conjugacy class of size $2(2^{2r})$. There are m such classes.

$$c(\overline{G}) = \left(1 + 2^r - 1 + 1\right) + \left(2^r\right) + \left(2^{r-1} - 1\right) = 2 \times 2^r + 2^{r-1} = 2^{r+1} + 2^{r-1}.$$

The full list of conjugacy classes based on coset analysis is given in Table 6.

The following Proposition and Remark will be used to construct the Fischer matrices of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

Proposition 6.2 ([15]). *If G is elementary abelian and $M = Im(\phi_g)$ where ϕ_g is an endomorphism of G defined by $\phi_g : x \mapsto xgx^{-1}g^{-1}$ for $x \in G$, then $[G : M] = k$ where k is the number of elements of G fixed by a class representative $\overline{h} \in \overline{H}$ where $\overline{G} = G : \overline{H}$.*

Proof. The orbits $Q_1, Q_2, \ldots, Q_k$ of G acting on $\overline{h}G$ are the same as the orbits $D_1, D_2, \ldots, D_k$ of M acting on $\overline{h}G$ by left multiplication. Also the orbits $D_1, D_2, \ldots, D_k$ can be identified with the elements of G/M. Then it follows that $G/M = [G : M] = k$. □

Table 6: Conjugacy Classes of $\overline{G} = G{:}(NH)$

$\overline{H} = NH$	$\overline{G} = G{:}(NH)$	$\circ(\overline{g})$	$\lvert[g]\rvert$	$\lvert C_{\overline{G}}(\overline{g})\rvert$
1	1	1	1	$2^{2r}(2^r-1)2$
	$(2A)_1$	2	2^r-1	$2(2^{2r})$
	$(2A)_2$	2	2^r-1	$2(2^{2r})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(2A)_{2^r-1}$	2	2^r-1	$2(2^{2r})$
	$(2A)_{2^r+1}$	2	$2(2^r-1)$	2^{2r}
b	$(2B)$	2	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_1$	4	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_2$	4	$2^r(2^r-1)$	$2(2^r)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(4A)_{2^r-1}$	4	$2^r(2^r-1)$	$2(2^r)$
a_1	$(2^r-1)A_1$	c_1	$2(2^{2r})$	2^r-1
a_2	$(2^r-1)A_2$	c_2	$2(2^{2r})$	2^r-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$(2^r-1)A_m$	c_α	$2(2^{2r})$	2^r-1

Remark 6.3. If G is an elementary abelian p-group, then from coset analysis for the group $\overline{G} = G{:}\overline{H}$, we obtain $k = p^m$ for $0 \leq m \leq n$, where $|G| = p^n$ and k is the number of elements of G fixed by a class representative $\overline{h}$ of $\overline{H}$. Suppose for some class representative $\overline{h} \in \overline{H}$, we have the orbits $Q_1, Q_2, \ldots, Q_k$ of the action of G on $\overline{h}G$. Then for $h \in C_{\overline{H}}(\overline{h})$, suppose that acting h on the orbits $Q_1, Q_2, \ldots, Q_k$, we get $f_1 = f_2 = \ldots = f_k = 1$ and that the entries of the first column of $M(\overline{h})$ are 1. Then in this case, the Fischer matrix $M(\overline{h})$ coincides with the character table of the abelian group G/M of order $k = p^m$.

7. Fischer matrices of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

In this section we will give a general description of the number of Fischer matrices and their form for the double Frobenius group $\overline{G} = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$). For

each conjugacy class of $\overline{H}$ there is a corresponding Fischer matrix. Therefore there are $2^{r-1}+1$ such matrices. The action of $\overline{H}$ on G produces 2^r+1 orbits. The Fischer matrix $M(1_{\overline{H}})$ corresponding to the identity coset is therefore a $\big((2^r+1)\times(2^r+1)\big)$ matrix. Since the action of $\overline{H}$ of G has 2^r+1 orbits, by Brauer's Lemma the action of $\overline{H}$ on $Irr(G)$ has 2^r+1 orbits also. The lengths of the orbits are 1, (2^r-1) and $2(2^r-1)$. The number of orbits of each length is 1, (2^r-1) and 1 respectively. We can show that the orbits of the action of $\overline{H}$ on $Irr(G)$ also has lengths 1, (2^r-1) and $2(2^r-1)$ and that the number of orbits is also 1, (2^r-1) and 1 respectively. Now when $\overline{H}$ acts on $Irr(G)$, the possibilities for orbit lengths are: 1, 2, (2^r-1) and $2(2^r-1)$. Let the number of orbits of length one be a, the number of orbits of length two be b, the number of orbits of length 2^r-1 be c and the number of orbits of length $2(2^r-1)$ be d. Then

$$a+b+c+d=2^r+1 \ \cdots\cdots(1)$$

and

$$a+2b+(2^r-1)c+2(2^r-1)d=2^{2r} \ \cdots\cdots(2).$$

where $a,b,c,d\in\mathbb{N}$.

We find values for a,b,c,d. Note first that we can assume that $r\geq 2$ in equations 1 and 2 above since $r=0$ and $r=1$ give trivial cases for the double Frobenius group $2^{2r}:(\mathbb{Z}_{2^r-1}:\mathbb{Z}_2)$.

We know that $a\geq 1$ since the action of $\overline{H}$ on $Irr(G)$ fixes the identity character. We claim that $a=1$. So suppose that $a>1$. Then equation (2) implies that $(a-1)+2b+c(2^r-1)+2d(2^r-1)=2^{2r}-1 \ \cdots\cdots(3)$. So, $(a-1)+2b+c(2^r-1)+2d(2^r-1)=(2^r-1)(2^r+1) \ \cdots\cdots(4)$. Since 2^r-1 divides the right hand side of equation (4), it must divide the left hand side also. Therefore, 2^r-1 divides $(a-1)+2b$. So $(2^r-1)\alpha=a-1+b$ for some $\alpha\in\mathbb{N}$. From this equation we get $a+b=\alpha 2^r-\alpha+1$. Substituting this into equation (1) above gives $c+d=2^r(1-\alpha)+\alpha \ \cdots\cdots(5)$. Since $r\geq 2$, and $a,b,c,d\in\mathbb{N}$, equation (5) will only be true if $\alpha=0$ or $\alpha=1$.

$\underline{\alpha=0}$**:**

Then $a+b=1$ and since $a\neq 0$, $b=0$ and $a=1$. This is a contradiction since $a>1$.

$\underline{\alpha=1}$**:**

Then $a+b=2^r$ and $c+d=1$. Therefore there are two cases to consider:(i) $c=0$ and $d=1$ (ii) $c=1$ and $d=0$.

If $c=0$ and $d=1$, then equation (2) implies that $a+2b+2(2^r-1)=2^{2r}$ and hence $2^r+b+2(2^r-1)=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-2+2^r+2.2^r \ \cdots\cdots(6)$. Now 2^r divides the left hand side of equation (6) and must divide the right hand side also. Therefore, 2^r divides $b-2$ and $b=2^r\gamma+2$ for some $\gamma\in\mathbb{N}$. This is a contradiction since $b<2^r$.

If $d=0$ and $c=1$, then equation(2) implies that $a+2b+2^r-1=2^{2r}$ and hence $2^r+b+2^r-1=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-1+2.2^r \ \cdots\cdots(7)$. Now 2^r divides the left hand side of equation(7) and

must divide the right hand side also. Therefore, 2^r divides $b-1$ and $b = 2^r\delta + 1$ for some $\delta \in \mathbb{N}$. This is a contradiction since $b < 2^r$.

Therefore, $a = 1$ as claimed. With $a = 1$, from equation (1) we now have that $b+c+d = 2^r$ and from equation (2) we have that $2b+(2^r-1)c+2(2^r-1)d = (2^r-1)(2^r+1)$. Since 2^r-1 divides the right hand side of this equation, it must divide the left hand side. This implies that $2^r - 1$ divides $2b$. Since $2^r - 1$ is odd, $2^r - 1$ must divide b. Thus, we have $(2^r - 1)\epsilon = b$ for some $\epsilon \in \mathbb{N}$. From equation (1), $a + b + c + d = 2^r + 1$ which implies that $b + c + d = 2^r$ and hence that $b \leq 2^r$. So, $(2^r - 1)\epsilon \leq 2^r$ and since $r \geq 2$, this inequality is true only if $\epsilon = 0$ or $\epsilon = 1$.

If $\epsilon = 1$, then $b = 2^r - 1$ and equation (1) now implies that $c + d = 1$. There are two cases to consider.

$\underline{c = 0\ , d = 1}$**:**
Equation (2) now implies that $2(2^r - 1) + 0 + 2(2^r - 1) = (2^r + 1)(2^r - 1)$, which gives us $4 = 2^r + 1$ which is false.

$\underline{c = 1\ , d = 0}$**:**
Equation (2) now implies that $2(2^r - 1) + 2^r - 1 = (2^r + 1)(2^r - 1)$ and $3 = 2^r + 1$ which is false for $r \geq 2$.

Thus we must have $\epsilon = 0$ and hence $b = 0$.

With $a = 1$ and $b = 0$, equation (1) and equation (2) now give $c + d = 2^r$ and $c + 2d = 2^r + 1$ respectively. Solving gives us $d = 1$ and $c = 2^r - 1$.

Therefore $a = 1 = d$, $b = 0$ and $c = 2^r$. Therefore, when $\overline{H}$ acts on $Irr(G)$, there is one orbit of length one, $2^r - 1$ orbits of length $2^r - 1$ and one orbit of length $2(2^r - 1)$. This is the same number of orbits and orbit lengths as the action of $\overline{H}$ on G.

7.1 The Inertia groups and inertia factor groups of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

Using the results of the section above, we can give a general description of the inertia groups and inertia factor groups for the double Frobenius group 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$).

Now denote the inertia groups by $\overline{P}$ and the inertia factor groups by P. Since the action of $\overline{H}$ on the $Irr(G)$ has orbit lengths $1,\ (2^r-1),\ 2(2^r-1)$, the inertia groups are:
$\overline{P}_1 = \overline{G} = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$) , $\overline{P}_2 = \overline{P}_3 = \cdots = \overline{P}_{2^r} = G$:$H = 2^{2r}$:$\mathbb{Z}_2$, $\overline{P}_{2^r+1} = G = 2^{2r}$.

The inertia factor groups are:

$$P_1 = \overline{H} = \mathbb{Z}_{2^r-1} : \mathbb{Z}_2\ ,\ P_2 = P_3 = \cdots = P_{2^r} = H = \mathbb{Z}_2\ ,\ P_{2^r+1} = \{1_{\overline{H}}\}.$$

7.2 Fischer matrices

There are $(2^r + 1)$ Fischer matrices, namely, $M(1_{\overline{H}})$, $M(b)$ and $M(a_i)$ for $i = 1, 2, \cdots\cdots, m$ where $m = 2^{r-1} - 1$ is the number of non-trivial orbits of the action of H on N.

$\underline{M(1_{\overline{H}}):}$

This is a $\big((2^r+1)\times(2^r+1)\big)$ matrix. The first row of the matrix is a row of 1's. The first column of the matrix consists of the (2^r+1) entries: $\big(1,\ 2^r-1,\ 2^r-1,\ \cdots\cdots\cdots,2^r-1,\ 2(2^r-1)\big)^t$. There are (2^r-1) entries of (2^r-1). The last column of the matrix consists of the entries $\big(1,\ -1,\ -1,\cdots\cdots\cdots,2^r-2\big)^t$. There are (2^r-1) entries of -1. The last row of the matrix consists of the entries $\big(2(2^r-1),\ -2,\ -2,\cdots\cdots\cdots,2^r-2\big)$. The remainder of the matrix is a $\big(2^r-1\times 2^r-1\big)$ block whose rows are just a permutation of the entries $\big(2^r-1,\ -1,\ -1,\cdots\cdots\cdots,-1\big)$. This is the block denoted by $X\ X\ X$ in the Fischer matrix shown below.

$$M(1_{\overline{H}})=\left(\begin{array}{c|cccc|c}
1 & 1 & 1 & \ldots\ldots & 1 & 1\\
\hline
2^r-1 & & & & & -1\\
2^r-1 & & & & & -1\\
\cdots & & & & & \cdots\\
\cdots & X & X & X & X & \cdots\\
\cdots & & & & & \cdots\\
2^r-1 & & & & & -1\\
\hline
2(2^r-1) & -2 & -2 & \ldots\ldots & -2 & 2^r-2
\end{array}\right)$$

$\underline{M(b):}$

This is a $\big(2^r\times 2^r\big)$ matrix. By Remark 6.3, the Fischer matrix corresponding to $b\in\overline{H}$ coincides with the character table of the elementary abelian group of order $k=2^m$ where k is number of fixed points of the action of b on G.

$$M(b)=\left(\begin{array}{c} Character\ Table\ of\\ elementary\ abelian\ group\\ of\ order\ 2^m \end{array}\right)$$

$\underline{M(a_i):}$

These are just singleton matrices with entry 1 of which there are $m=2^{r-1}-1$ in number.

8. Example-the group $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

In this section we apply the theory developed in section 4 to the group $\overline{G}=2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2$. Let $\overline{G}=GNH$ where $G\cong 2^{2r}$, $N\cong\mathbb{Z}_{2^r-1}$, $H\cong\mathbb{Z}_2$ and $NH\cong\overline{H}$. Let $r=2$, then $G\cong 2^4$, $N\cong\mathbb{Z}_3=\langle a\rangle$, $H\cong\mathbb{Z}_2=\langle b\rangle$ and $NH\cong\overline{H}\cong\mathbb{Z}_3{:}\mathbb{Z}_2$. We know that $\mathbb{Z}_3{:}\mathbb{Z}_2\leq PSL(2,4)\cong SL(2,4)$. We note also that $PSL(2,4)$ has a single class of involutions and a single class of elements of order three. Now $o(b)=2,\ b=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $a=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}$ where $\langle\lambda\rangle=\mathbb{F}_4^*,\ \circ(a)=3$.

$G = \{0,\ e_1,\ e_2,\ (e_1+e_2),\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2,\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2),\ \lambda e_1+e_2,\ \lambda^2 e_1+e_2,\ e_1+\lambda e_2,\ e_1+\lambda^2 e_2,\ \lambda^2 e_1+\lambda e_2,\ \lambda e_1+\lambda^2 e_2\}$ and $GF(4) = \{0,\ 1,\ \lambda,\ \lambda^2\}$.

Now, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $be_1 = e_2$ and $be_2 = e_1$.

8.1 Conjugacy classes of $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$)

To calculate the conjugacy classes of $\overline{G}$, we need the conjugacy classes of the group $\overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$. The conjugacy classes of $\overline{H}$ are represented in Table 7.

Conjugacy classes of : $\overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$.

Table 7: Conjugacy Classes of $\mathbb{Z}_3$:$\mathbb{Z}_2$

classes of $\mathbb{Z}_3$:$\mathbb{Z}_2$	$[1]$	$[b]$	$[a]$
$\lvert C_G(g)\rvert$	6	2	3
$\circ(g)$	1	2	3
$\lvert[g]\rvert$	1	3	2

To calculate the conjugacy classes of $\overline{G}$ we use the method of coset analysis.

Conjugacy classes of : $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$).

To determine the conjugacy classes of $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$), we consider the cosets $\overline{h}G$ where $\overline{h} \in \overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$.

The coset $1G$: Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^4$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}$, $C_{\overline{H}}(\overline{1_{\overline{H}}}) = \overline{H}$ on G.

To act $\overline{H}$ on G, we first act $b \in \mathbb{Z}_2$ and then act $a \in \mathbb{Z}_3$.

The action of b on G:

In this action, b fixes $\{0,\ (e_1+e_2),\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2)\}$ and permutes the remaining 12 elements in the following 2 cycles:

$\{e_1,\ e_2\},\ \{\lambda e_1,\ \lambda e_2\},\ \{\lambda^2 e_1,\ \lambda^2 e_2\},\ \{(\lambda e_1+e_2),\ (e_1+\lambda e_2)\},\ \{(\lambda^2 e_1+e_2),\ (e_1+\lambda^2 e_2\},\ \{(\lambda^2 e_1+\lambda e_2),\ (\lambda e_1+\lambda^2 e_2\}$.

Now acting $a \in \mathbb{Z}_3$ on the 4 fixed points of b and the six 2 cycles we get:

a fixes the zero vector of G and when it acts on the orbits of b, each fixed point $\lambda^i(e_1+e_2)$ for $i = 0,\ 1,\ 2$ fuses with a 2 cycle to form an orbit of size three as follows:

$$\Theta_1 = \{(e_1+e_2),\ (\lambda^2 e_1+\lambda e_2),\ (\lambda e_1+\lambda^2 e_2\},$$

$$\Theta_2 = \{\lambda(e_1+e_2),\ (\lambda^2 e_1+e_2),\ (e_1+\lambda^2 e_2)\},$$

$$\Theta_3 = \{\lambda^2(e_1 + e_2),\ (\lambda e_1 + e_2),\ (e_1 + \lambda e_2)\}.$$

The remaining orbits come together under the action of a to form the orbit Θ_4 of size six.

$$\Theta_4 = \{e_1,\ e_2,\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2\}.$$

Thus, the identity coset produces 5 orbits (conjugacy classes) of $\overline{G}$, viz, the singleton orbit containing the identity, three orbits of size three containing the remaining three fixed points, one in each orbit, and an orbit of size six.

Therefore we have:$k = 2^4$ and $f_1 = 1,\ f_2 = 3,\ f_3 = 3,\ f_4 = 3,\ f_5 = 6$.

$\underline{k = 2^4,\ f_1 = 1:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^4 \times 6}{1} = |\overline{G}|,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{2^4 \times 6} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$\underline{k = 2^4,\ f_i = 3:}$ for $i = 2, 3, 4$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^4 \times 6}{3} = 32,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{32} = 3.$$

This will give us three conjugacy classes of $\overline{G}$ of size three. The order of the elements in all three classes is two.

$\underline{k = 2^4,\ f_5 = 6:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_5} = \frac{2^4 \times 6}{6} = 16,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{16} = 6.$$

This gives us a fifth conjugacy class from the identity coset of $\overline{G}$ of size six. The order of the elements in this class is two.

The coset $\underline{bG:}$

First we act G on the coset bG. The action of G on the coset bG partitions the coset into four orbits of size four. The orbits are:

$$\Delta_1 = \{b,\ b(e_1 + e_2),\ b\lambda(e_1 + e_2),\ b\lambda^2(e_1 + e_2)\},$$

$$\Delta_2 = \{be_1,\ be_2,\ b(\lambda e_1 + \lambda^2 e_2),\ b(\lambda^2 e_1 + \lambda e_2)\},$$

$$\Delta_3 = \{b\lambda e_1,\ b\lambda e_2,\ b(\lambda^2 e_1 + e_2),\ b(e_1 + \lambda^2 e_2)\},$$

$$\Delta_4 = \{b\lambda^2 e_1,\ b\lambda^2 e_2,\ b(e_1 + \lambda e_2),\ b(\lambda e_1 + e_2)\}.$$

We also note that in the orbit Δ_1 the g entry of the element bg where $g \in G$ is the fixed point of the action of b on G, and in the orbits Δ_i for $i = 2, 3, 4$, the g entries of the element bg where $g \in G$ are the entries of the two cycles of the action of b on G. See the action of b on G above.

Next, we act the centralizer of b on the four orbits Δ_i for $i = 1, 2, 3, 4$. Now $C_{\overline{H}}(b) = \langle b \rangle$. Therefore the action of the centralizer is the same as the action of b. From the comment above, when b acts on the four orbits Δ_i for $i = 1, 2, 3, 4$, it permutes the elements in each orbit. Therefore for the coset bG we have $k = 4$ and $f_i = 1$ for $i = 1, 2, 3, 4$.

$\underline{k = 4,\ f_i = 1}$ for $i = 1, 2, 3, 4$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{4 \times 2}{1} = 8,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{8} = 12.$$

Therefore, the coset bG produces four conjugacy classes of $\overline{G}$ each of size twelve. The order of the elements in the first of these classes is two (the class containing b) and the order of the elements in the remaining three classes is four.

The coset $\underline{aG}$:

Finally we act G on the coset aG. When G acts on the coset aG, it simply permutes the 16 elements in the coset producing an orbit of length sixteen. Next we act the centralizer of a on this orbit. But $C_{\overline{H}}(a) = \langle a \rangle$. Therefore the action of the centralizer is the same as the action of $\langle a \rangle$. When $\langle a \rangle$ acts on the orbit of length sixteen it permutes the elements in the orbit. Therefore, here $k = 1,\ f = 1$.

$\underline{k = 1,\ f = 1}$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times 3}{1} = 3,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{3} = 32.$$

The coset aG, produces a single conjugacy class of $\overline{G}$ of size thirty two. The order of the elements in the class is three.

The full list of the conjugacy classes of $\overline{G}$ is described in Table 8.

Table 8: Conjugacy Classes of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

$\overline{H} = NH$	$\overline{G} = G : (NH)$	$\circ(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$	*power map*(π^2)	*power map*(π^3)
1	$1A$	1	96	$(1A)$	$(1A)$
	$(2A)$	2	32	$(1A)$	$(2A)$
	$(2B)$	2	32	$(1A)$	$(2B)$
	$(2C)$	2	32	$(1A)$	$(2C)$
	$(2D)$	2	16	$(1A)$	$(2D)$
b	$(2E)$	2	8	$(1A)$	$(2E)$
	$(4A)$	4	8	$(2A)$	$(4A)$
	$(4B)$	4	8	$(2B)$	$(4B)$
	$(4C)$	4	8	$(2C)$	$(4C)$
a	$(3A)$	3	3	$(3A)$	$(1A)$

8.2 Fischer matrices of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.

We construct the Fischer matrices of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, for each conjugacy class of $\overline{H} = \mathbb{Z}_3 : \mathbb{Z}_2$. From the previous sections we know that there are three conjugacy classes of $\overline{H}$ and therefore three Fischer matrices of $\overline{G}$. For the Fischer matrix corresponding to the identity class of $\mathbb{Z}_3{:}\mathbb{Z}_2$ we look at the action of $\overline{H}$ on $G = 2^4$. There are five orbits of lengths 1, 3, 3, 3, 6. The Fischer matrix corresponding to the identity class is $M(1_{\overline{H}})$ which is a (5×5) matrix. Since the action of $\overline{H}$ on G has five orbits of lengths 1, 3, 3, 3 and 6, we know that the action of $\overline{H}$ on $Irr(G)$ also produces five orbits of lengths 1, 3, 3, 3 and 6 as described in Section 7.

From Section 7.1 we have the following inertia and inertia factor groups. The inertia groups are:

$$\overline{P}_1 = \overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \overline{P}_4 = G{:}H = 2^4{:}\mathbb{Z}_2\ ,\ \overline{P}_5 = G = 2^4.$$

The corresponding inertia factor groups are:

$$P_1 = NH = \mathbb{Z}_3{:}\mathbb{Z}_2\ ,\ P_2 = P_3 = P_4 = H = \mathbb{Z}_2\ ,\ P_5 = \{1_{\overline{H}}\}.$$

The Fischer matrices are constructed using the theory of the Fischer Clifford matrices. We refer the reader to [1], [2], [13], [15], [17] and [18] for details.

$\underline{M(1_{\overline{H}}) :}$

$$M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & 3 & -1 \\ 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

The matrix corresponding to $b \in \mathbb{Z}_2$ is a 4×4 matrix $M(b)$ given by:

$$M(b) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Finally the third Fischer matrix is a (1×1) matrix with the singleton entry 1. This matrix $M(a)$ is given by :$M(a) = (1)$.

8.3 Character table of $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$)

We can now construct the character table of $\overline{G}$ using the Fischer matrices above and the character tables of the inertia factor groups $P_1 = \overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$ and $P_2 = P_3 = P_4 = H = \mathbb{Z}_2$.

We divide the character table of $\overline{G}$ into blocks as shown in the matrix below. Each block A_i, B_i, C_i for $i = 1, 2, 3, 4, 5$ corresponds to an inertia group $\overline{P}_i$. Also the A_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the identity coset $1G$, the B_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the coset bG and the C_i blocks for $i = 1, 2, 3, 4, 5$ come the conjugacy classes produced by the coset aG.

$$\left(\begin{array}{c|c|c} A_1 & B_1 & C_1 \\ \hline A_2 & B_2 & C_2 \\ \hline A_3 & B_3 & C_3 \\ \hline A_4 & B_4 & C_4 \\ \hline A_5 & B_5 & C_5 \end{array}\right)$$

First we need the character tables of $\overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$ and $H = \mathbb{Z}_2$

Character table $\overline{H}$:

$(\overline{h})$	(1)	(b)	(a)
$\lvert C_{\overline{H}}(\overline{h})\rvert$	6	2	3
$\circ(\overline{h})$	1	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Character table H :

(h)	(1)	(b)
$\lvert C_H(h)\rvert$	2	2
$\circ(h)$	1	2
χ_1	1	1
χ_2	1	-1

We now calculate the characters of $\overline{G}$, which fall into five blocks ($A_i,\ for\ i = 1,2,3,4,5$) with inertia groups $\overline{P}_1 = \overline{G},\ \overline{P}_2 = G{:}H,\ \overline{P}_3 = G{:}H,\ \overline{P}_4 = G{:}H,\ \overline{P}_5 = G$ by using the Fischer matrices and inertia factor groups $P_1 = \overline{H},\ P_2 = \mathbb{Z}_2,\ P_3 = \mathbb{Z}_2,\ P_4 = \mathbb{Z}_2,\ P_5 = \{1_{\overline{G}}\}$.

We complete the character table of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ by multiplying rows of $M(g)$ for $g \in \{1_{\overline{H}},\ b,\ a\}$ with sections of the character tables of the inertia factor groups corresponding to each $g \in \{1_{\overline{H}},\ b,\ a\}$.

The first block of table above A_1 is the block corresponding to conjugacy classes from the identity $1_{\overline{H}}$. To obtain this block, we multiply the first column $C_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1st$ column of $\overline{H}$ by $M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1st$ row of $M(1_{\overline{H}})$.

We get:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

For the A_2 block, we multiply $C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1st$ column of $\mathbb{Z}_2$ by $M_2 = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix} = 2nd$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_3 block, we multiply $C_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1st$ column of $\mathbb{Z}_2$ by $M_3 = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix} = 3rd$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \end{pmatrix}.$$

For the A_4 block, we multiply $C_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1st$ column of $\mathbb{Z}_2$ by $M_4 = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix} = 4th$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 & -1 \end{pmatrix}.$$

For the A_5 block, we multiply $C_5 = \begin{pmatrix} 1 \end{pmatrix} = 1st$ column of $\{1_{\overline{G}}\}$ by $M_5 = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix} = 5th$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

For the next block, the B_i block for $i = 1,2,3,4,5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(b)$. To complete the B_1 block of the table, we

multiply $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$=$2nd$ column of $\overline{H}$ by $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$= $1st$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the B_2 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$= $2nd$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

For the B_3 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$= $3rd$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

For the B_4 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$= $4th$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

For the B_5 block of the table, we will have a row of zeros since $P_5 \bigcap [b] = \emptyset$ and hence, $M_5(b)$ will not exist.

To complete the C_i block for $i = 1, 2, 3, 4, 5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(a) = \{1_{\overline{G}}\}$. To complete the C_1 block of the table, we multiply $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$=$3rd$ column of $\overline{H}$ by $\begin{pmatrix} 1 \end{pmatrix}$= $1st$ row of $M(a)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

For the C_i blocks for $i = 2, 3, 4, 5$, we have zeros since $P_i \bigcap [a] = \emptyset$ for $i = 2, 3, 4, 5$ and therefore M_i for $i = 2, 3, 4, 5$ does not exist.

Character table of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$:

$(\overline{g})$	$(1A)$	$(2A)$	$(2B)$	$(2C)$	$(2D)$	$(2E)$	$(4A)$	$(4B)$	$(4C)$	$(3A)$
$\|[(\overline{g})]\|$	1	3	3	3	6	12	12	12	12	32
$\|C_{\overline{G}}(\overline{g})\|$	96	32	32	32	16	8	8	8	8	3
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1	-1	1
χ_3	2	2	2	2	2	0	0	0	0	-1
χ_4	3	3	-1	-1	-1	1	1	-1	-1	0
χ_5	3	3	-1	-1	-1	-1	-1	1	1	0
χ_6	3	-1	3	-1	-1	1	-1	1	-1	0
χ_7	3	-1	3	-1	-1	-1	1	-1	1	0
χ_8	3	-1	-1	3	-1	1	-1	-1	1	0
χ_9	3	-1	-1	3	-1	-1	1	1	-1	0
χ_{10}	6	-2	-2	-2	2	0	0	0	0	0

Acknowledgements

The second author thanks his supervisor (first author) for his unwavering guidance and support. The financial support from the National Research Foundation (NRF) of South Africa, the North West University and the Durban University of Technology is also acknowledged.

References

[1] F. Ali and J. Moori, *The Fischer-Clifford matrices of a maximal subgroup of $\acute{F}i_{24}$*, Representation Theory, 7 (2003), 300-321.

[2] F. Ali, *The Fischer-Clifford matrices of a maximal subgroup of the Sporadic simple group of Held*, Algebra Colloquium, 14 (2007), 135-142.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Altas of Finite Groups*, Clarendon Press, Oxford University Press, Eynsham, 1985.

[4] B.Fischer, *Clifford - matrices*, Progr. Math., 95, Michler G.O. and Ringel C. M(eds), Birkhauser, Basel, (1991), 1-16.

[5] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, Sans Francisco, London, 1967.

[6] E.I. Khukhro, N.Yu. Makarenko, P. Shumyatsky, *Frobenius groups of automorphisms and their fixed points*, arXiv:1010.0343v1, 2010.

[7] E.I. Khukhro, *Fitting height of a finite group with a Frobenius group of automorphisms*, J. Algebra, 366 (2012), 1-11.

[8] E.I. Khukhro, N.Yu. Makarenko, *Finite groups and Lie rings with a metacyclic Frobenius group of automorphisms*, J. Algebra, 386 (2013), 77-103.

[9] O. King, *The subgroup structure of finite classical groups in terms of geometric configurations*, British Combinatorial Conference, Surveys in Combinatorics , Cambridge University Press, 2005, 29-56.

[10] P. Mayr, *Finite fixed point free automorphism Groups*, Diploma Thesis, University of Linz, 1998.

[11] J. Moori, *On the Groups G^+ and $\overline{G}$ of the forms $2^{10} : M_{22}$ and $2^{10} : \overline{M}_{22}$*, PhD thesis, University of Birmingham, 1975.

[12] J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Math. Soc., 2 (1981), 61-67.

[13] J. Moori and T. Seretlo, *On the Fischer-Clifford matrices of a maximal subgroup of the Lyons group L_y*, Bull. Iranian Math. Soc., 39 (2013), 1037-1052.

[14] J. Moori, P. Perumal, *On the Frobenius group of the form 29^2:$SL(2,5)$*, submitted.

[15] Z.E. Mpono, *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, Pietermaritzburg, 1998.

[16] P. Perumal, *On The Theory Of The Frobenius Groups*, MSc Thesis, University of Kwa-Zulu Natal, Pietermaritzburg, 2012.

[17] A. L. Prins and R. L. Fray, *The Fischer-Clifford matrices of an extension group of the form 2^7:(2^5:S_6)*, Int. J. Group Theory, 3 (2014), 21-39.

[18] N.S. Whitney, *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Kwazulu Natal, Pietermaritzburg, 1993.

Accepted: 8.03.2018

FINITE GROUPS WHOSE ALL PROPER SUBGROUPS ARE GPST-GROUPS

Pengfei Guo*
College of Mathematics and Statistics
Hainan Normal University
Haikou, 571158
P. R. China
guopf999@163.com

Yue Yang
College of Mathematics and Statistics
Hainan Normal University
Haikou, 571158
P. R. China
1441568839@qq.com

Abstract. A set $\mathcal{W} = \{W_1, \ldots, W_t\}$ of nilpotent Hall subgroups of G is a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all i, j. A finite group G is called a GPST-group if G has a complete Wielandt set $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$. In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups, i.e., they are precisely minimal non-PST-groups.

Keywords: Wielandt set, GPST-group, supersoluble group, power automorphism, permutable subgroup.

1. Introduction

All groups considered in this paper are finite and our notation is standard.

Let Σ be an abstract group theoretical property, for example, nilpotency, supersolubility, solubility, etc. If all proper subgroups of a group G have the property Σ but G does not have it, then G is called a minimal non-Σ-group.

The structures of minimal non-Σ-groups have been studied for various classes of groups Σ, and many classical results about this topic have been obtained. For instance, Miller and Moreno [8], Schmidt [12], and Doerk [5] analyzed the structures of minimal non-abelian groups, minimal non-nilpotent groups, and minimal non-supersoluble groups, respectively. However, the complete classifications of minimal non-nilpotent groups and minimal non-supersoluble groups were given by Ballester-Bolinches, Esteban-Romero and Robinson [4], Ballester-Bolinches and Esteban-Romero [2], respectively.

∗. Corresponding author

On the other hand, Robinson [10] characterized minimal non-T-groups (T-groups are groups in which normality is a transitive relation, i. e., if the normality of H in K and of K in G always imply that H is normal in G). A subgroup H of G is said to be s-permutable in G if H permutes all Sylow subgroups of G. Agrawal [1] studied PST-groups, i. e., groups in which Sylow permutability is a transitive relation. A group is a soluble PST-group if and only if it has an abelian Hall subgroup L of odd order such that G/L is nilpotent, and every element of G induces a power automorphism in L. Robinson [11] also gave a complete classification of minimal non-PST-groups.

The aim in the present work is to determine the structure of a kind of minimal non-Σ-groups. Guo and Skiba [6] called a set $\mathcal{W} = \{W_1, \ldots, W_t\}$ of nilpotent Hall subgroups of G a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all i, j, and characterized the structure of a group G which has a complete Wielandt set $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$. The specific result is as follows.

Theorem A [6, Theorem A]**.** A group G has a complete Wielandt set of subgroups $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$ if and only if $G = D \rtimes M$ is a supersoluble group where $D = G^{\mathcal{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G.

In view of the structure of a group described in Theorem A is very close to soluble PST-group but weaker than soluble PST-group, so Guo and Skiba [6] called this group a generalized PST-group or GPST-group for short.

In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups. Our main result is as follows:

Theorem 1.1. *Let p and q be distinct prime divisors of the order of a group G. Then G is a minimal non-GPST-group if and only if G is one of the following types:*

(I) *$G = P \rtimes Q$, where $P = \langle a, b\rangle$ is an elementary abelian p-group of order p^2, and $Q = \langle x\rangle$ is cyclic of order q^r. Define $a^x = a^i$, $b^x = b^{i^j}$, $p \equiv 1 (\mathrm{mod}\ q^f)$, and $r \geq 1$, where i is the least positive primitive q^f-th root of unity modulo p, $j = 1 + kq^{f-1}$, with $0 < k < q$ and $r \geq f$;*

(II) *$G = P \rtimes Q$, where $Q = \langle x\rangle$ is cyclic of order $q^r > 1$, with $q \nmid p - 1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle x^q\rangle$ in Q;*

(III) *$G = P \rtimes Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle x\rangle$ is cyclic of order $q^r > 1$, x induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and x centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$;*

(IV) *$G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle x\rangle$ is cyclic of order q^r, q^f is the highest power of q dividing*

$p-1$ and $r > f \geq 1$. Define $a_j^x = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^x = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

Coincidentally, by comparing with main result in [11], minimal non-GPST-groups are precisely minimal non-PST-groups. In addition, Ballester-Bolinches and Esteban-Romero [3] introduced an interesting definition. Let p be a prime. A group G is said to be a $\mathcal{Y}_p$-group if, for all p-subgroups H and S of G such that $H \leq S$, H is S-permutable in $N_G(S)$. They also gave that: a group G is a soluble PST-group if and only if G is a $\mathcal{Y}_p$-group for all primes p [3, Theorem 4]. Hence the classification of minimal non-$\mathcal{Y}_p$-group in [4, Theorem 2] may be regarded as a local approach to the classification of minimal non-PST-groups. Our result is given naturally.

Corollary 1.2. *Let G be a group. Then the following conditions are equivalent:*

(i) *G is a minimal non-PST-group.*

(ii) *G is a minimal non-GPST-group.*

(iii) *G is a minimal non-$\mathcal{Y}_p$-group for every prime divisor p of the order of G.*

2. Preliminary results

We collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([7])**.** *Let $\{P_1, P_2, \ldots, P_r\}$ be a Sylow basis of a soluble group G. Then the following statements are equivalent:*

(a) *Every subgroup of P_i permutes every subgroup of P_j for $i \neq j$.*

(b) *The nilpotent residual $G^{\mathcal{N}}$ of G is an abelian Hall subgroup of G, and every element of G induces a power automorphism in $G^{\mathcal{N}}$.*

Lemma 2.2. *Let G be a minimal non-GPST-group. Then there exists a normal non-cyclic Sylow p-subgroup P of G and a non-normal cyclic Sylow q-subgroup Q of G with $p \neq q$ such that $|G| = p^a q^b$ for positive integers a and b.*

Proof. Since every proper subgroup of G is a GPST-group, G is supersoluble or minimal non-supersoluble by Theorem A. By a result of Doerk [5], G is soluble and G has a nontrivial normal Sylow p-subgroup $P = O_p(G) \neq 1$, for some prime p. Let $G = P \rtimes H$ and $\mathcal{W} = \{P, H_1, \ldots, H_t\}$ of nilpotent Hall subgroups of G be a complete Wielandt set, where H is a p'-group of G. If $t \geq 2$, then $PH_1, PH_2, \ldots, PH_t$ are GPST-groups. Thus $H_1, H_2, \ldots, H_t$ permute every maximal subgroup of P whether or not P is cyclic. Since the normality of P and the fact that H is a GPST-group again, G is a GPST-group, a contradiction. Hence $t = 1$. Similar arguments as above, if $|\pi(H)| \geq 2$, then G is a GPST-group, a contradiction. So $H = Q \in \mathrm{Syl}_q(G)$ with $q \neq p$ a prime. If Q is non-cyclic, then $\langle x, P\rangle \neq G$ for every element x of Q. The minimality of G implies that $\langle x, P\rangle$ is a GPST-group. By applying Theorem A, every maximal subgroup of P is normal in G. By induction again, every subgroup of

P permutes every subgroup of $\langle x\rangle$. By Lemma 2.1, P is abelian and x induces a power automorphism on P. So G is a GPST-group, a contradiction. This induces that $H = \langle x\rangle$, where $|x| = q^b > 1$. Clearly, P is non-cyclic by the definition of GPST-group. The proof of Lemma 2.2 is complete. □

Lemma 2.3 ([14],Lemma 5)**.** *Suppose $G = P\langle x\rangle$, P is a normal p-subgroup of G and x is a q-element. If all maximal subgroups of Sylow subgroups of G are normal in G, then x induces a power automorphism on $P/\Phi(P)$.*

Lemma 2.4 ([9],13.4.3)**.** *Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that $a^\alpha = a^l$ for all a in A. If α is nontrivial and has order prime to p, then α is fixed-point-free.*

Lemma 2.5 ([5])**.** *Let G be a minimal non-supersoluble group. Then*

(1) *G has a unique normal Sylow p-subgroup P;*

(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $P/\Phi(P)$ is non-cyclic;*

(3) *If $p \neq 2$, then the exponent of P is p;*

(4) *If P is non-abelian and $p = 2$, then the exponent of P is 4;*

(5) *If P is abelian, then the exponent of P is p.*

3. The proof of Theorem 1.1

Proof. If G is a minimal non-GPST-group, then we may assume $G = PQ$ by Lemma 2.2, where P is a non-cyclic normal Sylow p-subgroup of G and $Q = \langle x\rangle$ is a non-normal Sylow q-subgroup of G of order q^r. Since all Sylow q-subgroups are conjugate in G, we only consider the case that Q acts on P. So we inverstigate the following two cases.

(1) Assume that G is supersoluble and $d(P) = k$, where $d(P)$ is the rank of P.

Let $1 \trianglelefteq \cdots \trianglelefteq R \trianglelefteq P \trianglelefteq \cdots \trianglelefteq G$ be an arbitrary chief series of G. By Maschke's Theorem [9, Theorem 8.1.2], there exists a normal subgroup N of G contained in P such that $P/\Phi(P) = R/\Phi(P) \times N/\Phi(P)$, where $|N/\Phi(P)| = p$. Clearly, $N \not\leq R$ and $1 \trianglelefteq N \trianglelefteq P \trianglelefteq G$ is a normal series of G. By applying Schreier's Refinement Theorem [9, Theorem 3.1.2], P has another maximal subgroup $K \neq R$ such that K is normal in G. Therefore, P has at least two maximal subgroups R and K which are normal in G.

Now we prove $k = 2$. If $k \geq 3$, then we can let $P/\Phi(P) = \langle \bar{a_1}\rangle \times \langle \bar{a_2}\rangle \times \cdots \times \langle \bar{a_k}\rangle$ where $a_1, a_2, \ldots, a_{k-1} \in R$, $a_2, a_3, \ldots, a_k \in K$. Since $R\langle x\rangle$ is a GPST-group, every maximal subgroup of R is normal in G. By Lemma 2.3, $(y\Phi(R))^x = y^l\Phi(R)$ for every $y \in R$, where l is a positive integer. Thus, $(y\Phi(P))^x = y^l\Phi(P)$ for every $y \in R$. Similarly, $(z\Phi(P))^x = z^m\Phi(P)$ for every $z \in K$, where m is a positive integer. Furthermore, $a_2^l\Phi(P) = (a_2\Phi(P))^x = a_2^m\Phi(P)$, and so $l \equiv m (\mathrm{mod}\ p)$. Hence, $(a_i\Phi(P))^x = a_i^l\Phi(P)$ for $i = 1, 2, \ldots, k$. It is easy to

see that every maximal subgroup of P is normal in G. By Theorem A, G is a GPST-group. This contradiction implies $k = 2$.

Now we let $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \bar{a_1} \rangle \times \langle \bar{a_2} \rangle$, where $a_1 \in R, a_2 \in K$, $\bar{a_1}^x = \bar{a_1}^{k_1}$ and $\bar{a_2}^x = \bar{a_2}^{k_2}$. If $k_1 = k_2$, then every maximal subgroup of P is normal in G, and so G is a GPST-group, a contradiction. Hence, $k_1 \neq k_2$. Furthermore, we have that P has only two maximal subgroups which are normal in G. Clearly, at least one action of which x acts on R and K is nontrivial. Without loss of generality, we may assume that x induces an automorphism α on R. Since every subgroup of $R\langle x \rangle$ is a GPST-group and by induction, it follows from Theorem A that every subgroup of R permutes every subgroup of $\langle x \rangle$. By Lemma 2.1, R is abelian and α is a power automorphism on R. By Lemma 2.4, α is fixed-point-free. Hence, we have either $K \cap R = 1$ if $K\langle x \rangle = K \times \langle x \rangle$ or $K\langle x \rangle \neq K \times \langle x \rangle$. If $K \cap R = 1$ and $K\langle x \rangle = K \times \langle x \rangle$, then $P = \langle a, b \rangle$ is an elementary abelian group of order p^2. We can easily have that G is of type (I) with $f = 1$ and $k = q - 1$.

If $K\langle x \rangle \neq K \times \langle x \rangle$, similar arguments as above, K is abelian, and x induces a power automorphism in K. Thus, $\Phi(P) = R \cap K \leq Z(P)$. If $|P : Z(P)| \leq p$, then P is abelian.

We prove that P is elementary abelian. Let $\Omega_1(P)$ be the group generated by all elements of order p in P and assume that $\Omega_1(P) \neq P$. Then $\langle \Omega_1(P), x \rangle \neq G$ and it is a GPST-group. Therefore x induces a power automorphism in $\Omega_1(P)$, i. e., there is a positive integer t, relatively prime to p, such that $a^x = a^t$ for all $a \in \Omega_1(P)$. Let β be the automorphism of P induced by x and let γ be the automorphism of P in which $a \mapsto a^t$. Then $\beta\gamma^{-1}$ is an automorphism of P fixing each element of order p and $\beta\gamma^{-1}$ has order equal to a power of p, say p^d. Obviously $\beta\gamma = \gamma\beta$, so $\beta^{p^d} = \gamma^{p^d} \in \langle \gamma \rangle$. But β has order prime to p, so $\beta \in \langle \gamma \rangle$ and β is a power automorphism of P, a contradiction.

Assume that $P = \langle a \rangle \times \langle b \rangle$ is elementary abelian. Let q^f be the order of the automorphism of P induced by x, $a^x = a^i$ and $b^x = b^s$, where i and s are two distinct primitive q^f-th roots of unity modulo p. Then $0 < f \leq r$ and $p \equiv 1 (\mathrm{mod}\ q^f)$. Since $P\langle x^q \rangle \neq G$, x^q induces a power automorphism in P and $i^q = s^q$. So i and s both have order q^f. Then $s = i^j$ for some integer $j \not\equiv 1 (\mathrm{mod}\ q^f)$. Now $i^q = s^q = s^{jq}$, so $j \equiv 1 (\mathrm{mod}\ q^{f-1})$, and we can assume that $j = 1 + kq^{f-1}$ where $0 < k < q$. Hence G is again of type (I).

If $|P : Z(P)| = p^2$, then $\Phi(P) = R \cap K = Z(P)$, and so P is minimal non-abelian and $|P'| = p$. Let $P_1 = \langle a, P' \rangle$ and $P_2 = \langle b, P' \rangle$. Then P_1Q and P_2Q are GPST-groups. By hypothesis, x induces power automorphisms in P_1 and P_2, say $g \mapsto g^{n_1}$ and $g \mapsto y^{n_2}$ respectively. By Lemma 2.4, these two power automorphisms are fixed-point-free. However, they must agree on P', so $n_1 \equiv n_2 (\mathrm{mod}\ p)$ and we can assume $n_1 = n_2$. Since $[a, b]^{n_1} = [a, b]^x = [a^{n_1}, b^{n_1}]$, $n_1^2 \equiv n_1 (\mathrm{mod}\ p)$ and $n_1 \equiv 1 (\mathrm{mod}\ p)$, a contradiction.

(2) Assume that G is minimal non-supersoluble.

Let M be a maximal subgroup of G such that $Q \leq M$. Then $M = P_3Q$, where P_3 is a Sylow p-subgroup of M. By $[P_3, Q] \leq P \cap P_3Q = P_3$, we have

$N_G(P_3) \geq P_3Q = M$. Since $N_P(P_3) > P_3$, P_3 is normal in G. By Lemma 2.5 and the maximality of M, $P_3 = \Phi(P)$ is the Sylow p-subgroup of M.

Case 1. If G is also a minimal non-nilpotent group, then by applying [4, Theorem 3], G is of either type (II) or type (III).

Case 2. If G is not a minimal non-nilpotent group and P is abelian, then by applying [2, Theorem 9, 10], we assume that $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle x\rangle$ is cyclic of order q^r, q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^x = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^x = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

For a maximal subgroup $P\langle x^q\rangle$ of G and any element a_k of P, by computation, $a_k^{x^q} = a_k^i$. So G is of type (IV).

Case 3. Assume that G is not a minimal non-nilpotent group and P is non-abelian. By applying [2, Theorem 9, 10], we may assume that $G = PQ$ such that $P = \langle a_0, a_1\rangle$ is an extraspecial group of order p^3 with exponent p, $Q = \langle x\rangle$ is a cyclic group of order 2^r with 2^f the largest power of 2 dividing $p-1$ and $r > f \geq 1$, and $a_0^x = a_1$ and $a_1^x = a_0^i y$, where $y \in \langle [a_0, a_1]\rangle$ and i is a primitive 2^f-th root of unity modulo p.

Since every subgroup of $P\langle x^2\rangle$ is a GPST-group, every subgroup of P permutes every subgroup of $\langle x^2\rangle$ by induction. If $P\langle x^2\rangle \neq P \times \langle x^2\rangle$, then P is abelian by Lemma 2.1, a contradiction. Hence $P\langle x^2\rangle = P \times \langle x^2\rangle$ and $a_0^{x^2} = a_1^x = a_0^i y = a_0$, which implies that $x = 1$ and $i \equiv 1 (\mathrm{mod}\ p)$, a contradiction. Therefore, G is not of the type as above.

Conversely, it is easy to check that all groups satisfied types (I)—(IV) are minimal non-GPST-groups. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11661031).

References

[1] R.K. Agrawal, *Finite groups whose subnormal subgroups permute with all Sylow subgroups*, Proc. Amer. Math. Soc., 47(1) (1975), 77-83.

[2] A. Ballester-Bolinches, R. Esteban-Romero, *On minimal non-supersoluble groups*, Rev. Mat. Iberoam., 23(1) (2007), 127-142.

[3] A. Ballester-Bolinches, R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups*, J. Algebra, 251(2) (2002), 727-738.

[4] A. Ballester-Bolinches, R. Esteban-Romero, D.J.S. Robinson, *On finite minimal non-nilpotent groups*, Proc. Amer. Math. Soc., 133(12) (2005), 3455-3462.

[5] K. Doerk, *Minimal nicht überauflösbare, endliche Gruppen*, Math. Z., 91 (1966), 198-205.

[6] W.B. Guo, A.N. Skiba, *Finite groups with permutable complete Wielandt sets of subgroups*, J. Group Theory, 18 (2015), 191-200.

[7] B. Huppert, *Zur sylowstruktur auflösbarer gruppen*, Arch. Math., 12 (1961), 161-169.

[8] G.A. Miller, H.C. Moreno, *Nonabelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc., 4 (1903), 398-404.

[9] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[10] D.J.S. Robinson, *Groups which are minimal with respect to normality being intransitive*, Pacific J. Math., 31(3) (1969), 777-785.

[11] D.J.S. Robinson, *Minimality and Sylow-permutability in locally finite groups*, Ukrainian Math. J., 54(6) (2002), 1038-1048.

[12] O.J. Schmidt, *Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind*, Mat. Sbornik, 31 (1924), 366-372.

[13] A.N. Skiba, *Some characterizations of finite σ-soluble $P\sigma T$-groups*, J. Algebra, 495 (2018), 114-129.

[14] G.L. Walls, *Groups with maximal subgroups of Sylow subgroups normal*, Israel J. Math., 43 (1982), 166-168.

[15] X.L. Yi, A.N. Skiba, *Some new characterizations of PST-groups*, J. Algebra, 399 (2014), 39-54.

Accepted: 13.03.2018

INDUSTRIAL DATA FORECASTING USING DISCRETE WAVELET TRANSFORM

S. Al Wadi*
The University of Jordan
Department of Risk Management and Insurance
Aqaba
Jordan
sadam_alwadi@yahoo.co.uk

Ahmed Atallah Alsaraireh
The University of Jordan
Department of Computer Information Systems
Aqaba
Jordan
ah_8545@yahoo.com
a.alsarairah@ju.edu.jo

Abstract. Since the industrial data plays significant element in any economic growth and these data have many factors that effect on its behavior. Therefore, in this article events of productivity of the Extractive Industry in Jordan will be forecasted using some of traditional model which is (ARIMA model) compound with Orthogonal wavelet transform (OWT) in order to improve the forecasting accuracy. First, the series of dataset will be decomposed by OWT's then the smooth's series will be predicted using ARIMA model, OWT+ ARIMA model in order to improve the forecasting accuracy. As a results the compound model (OWT+ ARIMA) is better than the ARIMA model directly in forecasting accuracy.
Keywords: operation research methods, traders satisfaction, mathematical models.

1. Introduction

Industry stock market tendencies are a challenging task. Numerous factors influence industry stock market performance, including political events, general economic conditions, and trader expectations. Though stock and futures traders rely heavily on various types of intelligent systems to make trading decisions to date their success has been limited [1]. Even financial experts find it difficult to make accurate predictions, because industrial stock market trends tend to be nonlinear, uncertain, and non-stationary. No consensus exists among experts as to the effectiveness of forecasting industrial time series. More specific, Some of the important factors influencing industrial productivity are: Technological Development, Quality of Human Resources, Availability of Finance, Managerial

∗. Corresponding author

Talent, Government Policy and Natural Factors. Consequently, Many models that has suggested in order to studying and predicting the industrial data such as [9,10] has discussed the volatility of the industrial data and provide a significant model for forecasting. In 2008, [11] investigates the interactive relationships between oil price shocks and Chinese stock market using multivariate vector auto-regression. [12] have discussed the performance of the Singular Spectrum Analysis (SSA) technique is assessed by applying it to 24 series measuring the monthly seasonally unadjusted industrial production for important sectors of the German, French and UK economies. The results are compared with those obtained using the HoltWinters and ARIMA models. Statistical selection procedures are used in a variety of applications to select the best of a finite set of alternatives. Best" is defined with respect to the (largest or smallest) mean, where the mean is inferred with statistical sampling, as in simulation optimization. Many sequential selection procedures are proposed to select a good design when the number of alternatives is large, see Alrefaei and Almomani [13], Almomani and Alrefaei [14], Almomani and Abdul Rahman [15], Almomani and Ababneh [16], Almomani and Alrefaei [17], Al-Salem et al. [18], Almomani et al. [19].

One of the most important statistical procedure is the forecasting accuracy. Therefore, many attempts have been made to forecast financial markets such as [20-25]. One of the traditional forecasted model is ARIMA model. However, the main disadvantage of ARIMA model is the enormous difficulty of interpreting the results. This study diverges from previous attempts at forecasting stock prices by proposing a method that uses the Wavelet transforms (WT) combined with ARIMA, this process create a transparent architecture. WT is a relatively new field in signal processing [2]. Wavelets are mathematical functions that decompose data into different frequency components, after which each component is studied with a resolution matched to its scale, where a scale denotes a time horizon [3]. WT is closely related to the volatile and time varying characteristics of the real-world time series and is not limited by the stationarity assumption [4]. WT decomposes a process into different scales, making it useful in distinguishing seasonality, revealing structural breaks and volatility clusters, and identifying local and global dynamic properties of a process at specific timescales [5]. WT has been shown to be particularly useful in analyzing, modeling, and predicting the behavior of financial instruments as diverse as stocks and exchange rates [6,7,26,27].

Moreover, WT has applications in forecasting in many fields such as in electronic forecasting [28], finance and economic [29, 31], also it used with short and long memory forecasting [30], and other fields [32, 33]. This study applies WT using the OWT functions which are (Haar, Daubechies, Coiflet and symmelt) to decompose the industry time series then combine the approximation coefficients with ARIMA model in order to make improve the forecasting accuracy then select the best WT function in forecasting. Then finally we compare the fore-

casting using the combined method with the forecasting using ARIMA model directly OWT+ ARIMA model.

2. Methodology

Table 1: shows the research framework

Industry Time Series Data
OWT (Haar, Daubechies, Symmelet, and Coiflet) functions
.....
Smoothed coefficients
Improving the forecasting accuracy by using ARIMA model for the transformed data.
Selection the best model in decomposition and forecasting

2.1 Wavelet transform

WT is based on Fourier transform (FT) which shows any function as the sum of the sine and cosine functions. WT should satisfies the following condition [8]:

$$C_{\varphi} = \int_0^{\infty} \frac{|\varphi(f)|}{f} df < \infty, \tag{1}$$

where $\varphi(f)$ is the FT and a function of frequency f, of $\varphi(t)$. WT is a mathematical function that can be used in many fields such. WT was introduced to solve problems associated with the FT as they occur specially with non-stationary dataset or with signals that are localized in time, space, or frequency. There are two types of WT which are Father WT which describes the smooth frequency components of a signal and mother WT which describes the detailed components. Mathematically, the following equations represent the father WT and mother WT respectively, with $j = 1, ..., J$ in the J-level WT decomposition [6]:

$$\phi_{j,k} = 2^{-j/2}\phi\left(t - 2^j k/2^j\right),\ \varphi_{j,k} = 2^{-j/2}\varphi\left(t - 2^j k/2^j\right), \tag{2}$$

father WT and mother WT should satisfy the following conditions:

$$\int \phi(t)\, dt = 1 \text{ and } \int \phi(t)\, dt = 0. \tag{3}$$

Time series data (f(t)) is an input represented by WT and can be built up as a sequence of projections onto father WT and mother WT indexed by both $k, k = 0, 1, 2, ...$ and by $S = 2j, j = 1, 2, 3, ...J$. Mathematically:

$$S_{j,k} = \int \phi_j, kf(t)dt, \text{ and } d_{j,k} = \int \varphi_j, kf(t)dt. \tag{4}$$

The orthogonal wavelet series approximation to f(t) is defined by:

$$F(t) = \sum S_j, k\phi_j, k(t) + \sum d_j, k\phi_j, k(t) + \sum d_j - 1, k\phi_j - 1, k(t) + ... + \sum d_1, k\phi_1, k(t) \tag{5}$$

$$S_j(t)) = \sum S_j, k\phi_j, k(t), \quad D_j(t) =) \sum d_j, k\phi_j, k(t). \tag{6}$$

2.2 ARIMA model

ARMA is a suitable model for the stationary time series data, although most of the software uses least square estimation which requires stationary. To overcome this problem and to allow ARMA model to handle non-stationary data, the researchers investigate a special class for the non-stationary data. This model is called Auto-regressive Integrated Moving Average (ARIMA). This idea is to separate a non-stationary series one or more times until the time series becomes stationary, and then find the fit model. ARIMA model has got very high attention in the scientific world. This model is popularized by George Box and Gwilym Jenkins in 1970s [25]. There are a huge number of ARIMA models; generally there are ARIMA (p, q, d) where: P: order of autoregressive part (AR), d: degree of first differentiation (I) and q: order of the first moving part (MA). Note that, if there is no differencing been done (d = 0), then ARMA model can be got from ARIMA model. The general mathematical ARIMA model can be defined as [25]

$$W_t = \mu + \frac{\beta(\nu)}{\varepsilon(\nu)} a_t.$$

Where:

t: Indexes time.

W_t: The response series Y_t or a difference of the response series.

μ: The mean term.

ν: The backshift operator; that is,$\nu X_t = X_{t-1}$

$\beta(\nu)$: The autoregressive operator, represented as a polynomial in the backshift operator:

$$\varepsilon(\nu) = 1 - \varepsilon_1(\nu) - ... - \varepsilon_p \nu^p$$

$\varepsilon(\nu)$: The moving-average operator, represented as a polynomial in the backshift operator:

$$\beta(\nu) = 1 - \beta_1(\nu) - ... - \beta_p \nu^p.$$

a_t: The independent disturbance, also called the random error.

2.3 Mathematical criteria

The author is used some criteria in order to make fair comparison between ARIMA and ARIMA-WT can be presented in this section. Some types of accuracy criteria have used; Root means squared error (RMSE), Percentage root

mean absolute percentage error (MAPE) and mean absolute error (MAE) for the mathematical formulas refer to [9].

3. Dataset, analysis and discussion

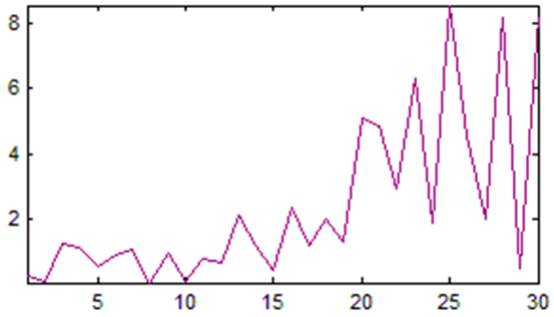


Figure 1: Original Data Set

The daily price index of ASE for a specific period of time has been selected as the statistical population, more than 4000 observations were accumulated for each variable from related databases in the mentioned period. Figure 1 shows the diagram of the dataset. Regarding the target of this paper the following table will show the results of the forecasting accuracy using ARIMA model directly and ARIMA+WT.

Table 2: forecasting results

	ARIMA directly	ARIMA+ Haar	ARIMA+ Daubechies	ARIMA + Coiflet	ARIMA+ symmelt
RMSE	1.56	0.6	0.3	0.4	0.7
MAPE	52	28.6	20	22	29
MAE	1.34	0.4	0.1	0.35	0.9

For the sake of fair comparison the same number of data set is selected. The suitable forecasted model for forecasting the sample data is the fitted Daubechies WT- ARIMA (2,0,2) with RSME equal to 0.3 as presented in Table 2. While the Fitted ARIMA model directly is ARIMA (1,0,2) with RMSE 1.56 which means that the forecasting accuracy has improved by combining OWT+ARIMA model. Also, the best forecasted model is Daubechies + ARIMA model. Moreover, to insure the results the authors have used MAPE and MAE which was the best for ARIMA + Daubechies also since these values were 20 and 0.1 respectively. While these values were more that ARIMA+ Daubechies based on other functions.

4. Conclusion

ARIMA model is the most general way of forecasting since there is no need for any assumptions and it is not limited to specific type of pattern. These models can be fitted to any set of time series data (stationary or non-stationary) by estimating the parameters p, d, and q to be suitable with the required dataset. In this study, firstly, the industry stock price is modeled using wavelet method. Secondly, we compared ARIMA+ OWT with ARIMA directly model in content of forecasting accuracy. Thirdly, we tested the accuracy of these models by using RMSE, MAPE and MSE assessing functions. Final the forecasting accuracy have improved using the suggested model. Indeed, we found ARIMA+OWT is suitable model in content of industry sector specially Daubechies WT with ARIMA model.

References

[1] Y.S. Abu-Mostafa, A.F. Atiya, *Introduction to financial forecasting*, Applied Intelligence, 1996, 205-213.

[2] A. Cohen, I. Daubechies, P. Vial, *Wavelets on the interval and fast wavelet transform*, Applied and Computational Harmonic, 1993, 5481.

[3] J.B. Ramsey, Z. Zhang, *The analysis of foreign exchange data using wave form dictionaries*, Journal of Empirical Finance, 1997, 341372.

[4] A. Popoola, K. Ahmad, *Testing the suitability of wavelet preprocessing for TSK fuzzy models*, in: Proceeding of FUZZ-IEEE: International Conference Fuzzy System Network, 2006, 13051309.

[5] R. Gencay, F. Selcuk, B. Whitcher, *Differentiating intraday seasonalities through wavelet multi-scaling*, PhysicaA, 2001, 543556.

[6] J.B. Ramsey, *The contribution of wavelets to the analysis of economic and financial data*, Philosophical Transactions of the Royal Society of London Series A-Mathematical Physical and Engineering Sciences, 1999, 25932606.

[7] K. Papagiannaki, N. Taft, Z.-L. Zhang, C.Diot, *Long-term forecasting of internet backbone traffic*, IEEE Transactions on Neural Networks, 2005, 11101124.

[8] R. Gencay, F. Selcuk, B. Whitcher, *An introduction to wavelets and other filtering methods in finance and economics*, Academic Press, NewYork, 2002.

[9] Firas Muhammad Al-Rawashdi, S. Alwadi and Mohammad Hasan Saleh, *Wavelet methods in forecasting for insurance companies listed in amman stock exchange*, European Journal of Economics, finance and administrative sciences, 82 (2015), 54-60.

[10] S. Al Wadi, *Improving volatility risk forecasting accuracy in industry sector*, International Journal of Mathematics and Mathematical Sciences, 2017.

[11] R.G. Cong, Y.M. Wei, J.L. Jiao, and Y. Fan, *Relationships between oil price shocks and stock market: An empirical analysis from China*, Energy Policy, 36 (2008), 3544-3553.

[12] H. Hassani, S. Heravi, A. Zhigljavsky, *Forecasting european industrial production with singular spectrum analysis*, International journal of forecasting, 25 (2009), 103-118.

[13] M.H. Alrefaei, M.H. Almomani, *Subset selection of best simulated systems*, Journal of the Franklin Institute, 344 (2007), 495-506.

[14] M.H. Almomani, M.H. Alrefaei, *A three-stage procedure for selecting a good enough simulated system*, Journal of Applied Probability and Statistics, 6 (2012) (2012), 13-27.

[15] M.H. Almomani, R. Abdul Rahman, *Selecting a good stochastic system for the large number of alternatives*, Communications in Statistics-Simulation and Computation, 41 (2012), 222-237.

[16] M.H. Almomani, F. Ababneh, *The expected opportunity cost and selecting the optimal subset*, Applied Mathematical Sciences, 9 (2015), 6507-6519.

[17] M.H. Almomani, M.H. Alrefaei, *Ordinal optimization with computing budget allocation for selecting an optimal subset*, Asia-Pacific Journal of Operational Research, 33 (2016), DOI: 10.1142/S0217595916500093.

[18] M. Al-Salem, M.H. Almomani, M.H. Alrefaei, Ali Diabat, *On the optimal computing budget allocation problem for large scale simulation optimization*, Simulation Modelling Practice and Theory, 71 (2017), 149-159.

[19] M.H. Almomani, M.H. Alrefaei, S. Al Mansour, *A combined statistical selection procedure measured by expected opportunity cost*, Arabian Journal for Science and Engineering, 2017.

[20] S. Al Wadi, *Improving volatility risk forecasting accuracy in industry sector*, International Journal of Mathematics and Mathematical Sciences, 2017.

[21] A.M. Awajan, M.T. Ismail, S. Al Wadi, *A hybrid EMD-MA for forecasting stock market index*, Italian Journal of Pure and Applied Mathematics, 38 (2017), 1-20.

[22] A.M. Awajan, M.T. Ismail, S.A. Wadi, *Forecasting time series using EMD-HW bagging*, International Journal of Statistics and Economics, 18 (2017), 9-21.

[23] J.J. Jaber, N. Ismail, S. Al Wadi, M.H. Saleh, *Forecasting of volatility risk for Jordanian banking sector*, Far East Journal of Mathematical Sciences, 101 (2017), 1491-1507.

[24] S. Al Wadi, F. Ababneh, H. Alwadi, M.T. Ismail, *Maximum overlap discrete wavelet methods in modeling banking data*, Far East Journal of Applied Mathematics, 84 (2013).

[25] M.T.I.S. Al Wadia, M. Tahir Ismail, *Selecting wavelet transforms model in forecasting financial time series data based on ARIMA model*, Applied Mathematical Sciences, 5 (2011), 315-326.

[26] S. Al Wadi, M.T. Ismail, S.A.A. Karim, *A comparison between the Daubechies wavelet transformation and the fast Fourier transformation in analyzing insurance time series data*, Far East J. Appl. Math, 45 (2010), 53-63.

[27] S. Al Wadi, M.T. Ismail, S.A. Abdul Kari, *Discovering Structure breaks in Amman stocks market*, Journal of Applied Sciences, 11 (2011), 1273-1278.

[28] S. Li, P. Wang, L. Goel, *Electric load forecasting using wavelet transform and extreme learning machine*, In ESANN, 2014.

[29] L. Bai, S. Yan, X. Zheng, B.M. Chen, *Market turning points forecasting using wavelet analysis*, Physica A: Statistical Mechanics and its Applications, 437 (2015), 184-197.

[30] O. Renaud, F. Murtagh, J.L. Starck, *Wavelet-based forecasting of short and long memory time series*, Université de Genève/Faculté des sciences économiques et sociales, 2002.

[31] J.B. Ramsey, *The contribution of wavelets to the analysis of economic and financial data*, Wavelets: The Key to Intermittent Information, Volume Wavelets: the key to intermittent information, 221-236.

[32] C.A.G. Santos, P.K.M.M. Freire, G.B.L. Silva, R.M. Silva, *Discrete wavelet transform coupled with ANN for daily discharge forecasting into Três Marias reservoir*, Proceedings of the International Association of Hydrological Sciences, 364 (2014), 100-105.

[33] S. Schlüter, C. Deuschle, *Using wavelets for time series forecasting: Does it pay off?*, IWQW discussion paper series, 2010.

Accepted: 14.03.2018

THREE-DIMENSIONAL AIR QUALITY ASSESSMENT SIMULATIONS INSIDE SKY TRAIN PLATFORM WITH AIRFLOW OBSTACLES ON HEAVY TRAFFIC ROAD

Kewalee Suebyat

Nopparat Pochai

Department of Mathematics
Faculty of Science
King Mongkut's Institute of Technology Ladkrabang
Bangkok 10520
Thailand
and
Centre of Excellence in Mathematics
Commission on Higher Education (CHE)
Si Ayutthaya Road, Bangkok 10400
Thailand
kew26_tan@hotmail.com
nop_math@yahoo.com

Abstract. Air pollutant levels in Bangkok are generally high in street tunnels. They are particularly elevated in almost closed street tunnels such as an area the Bangkok sky train platform with high traffic volume where dispersion is limited. This area has no air quality measurement stations even though there is a high percentage of people living around this vicinity. We are interested to conduct a research the Bangkok sky train platform due to the traffic density and enormous polluted areas. Therefore, we proposed a numerical modeling of air pollution concentration in sky train platform with airflow obstacles on heavy traffic road as an approximated solution of the three-dimensional advection-diffusion equation by using the finite difference methods. Our research presentation is based on how air pollution model depends on the flow of air pollution and wind directions including the governing equation of the corresponding three-dimensional advection-diffusion equation is presented. This also includes the initial condition and boundary conditions of traffic and polluted areas. In order to illustrate the performance of the model, the numerical experiments are presented. The comparison between the two methods and the simulations of air pollution control are proposed. The three-dimensional advection-diffusion equation is solved by using the Forward Time, Centered Space (FTCS) and Forward Time, Backward Space (FTBS) schemes. The results obtained indicate that the FTCS method provides a better result than FTBS method. Furthermore, the proposed experimental variations of the boundary condition in the entrance gate do affect the air pollutant concentration of each floor.

Keywords: air pollutant concentration, finite difference techniques, air quality, heavy traffic, sky train platform, tunnel.

1. Introduction

Currently, Thailand is facing a rapid growth in both agriculture and industry resulting in Bangkok being the center of prosperity in all aspects with rapid population increase in a blink of an eye, followed by a high demand for travel and transportation. This creates an intensifying traffic congestion and air pollution that derives from the rapid increase of cars and vehicles. Air pollution is one of the main and biggest problems and Bangkok has reached a critical level with hazardous substances in some areas because pollution is created by human beings and natural phenomena damaging the environment and human well-being. Not only is air pollution hazardous locally but it is one of the world's biggest killers and issue because people faced health problems such as asthma, bronchitis, cancer, etc. If people are constantly exposed to high levels of dust, they may suffer from illnesses such as silicosis or asbestosis. So it can be said that air pollution from traffic is tremendously serious especially a Bangkok sky train platform than any other areas. The volume of carbon monoxide and nitrogen oxides in this area is higher than the standard volume. This issue should be realized for the study interest and further research to find solutions to reduce pollution.

In [1], the Kriging method for regression analysis can be used to analytically relate the mass emission rate of carbon monoxide and nitrogen dioxide at the Bangkok Mass Transit System (BTS). The results indicate that the concentration of carbon monoxide exceed the Bangkok standard volume and the concentration of nitrogen dioxide does not exceed the Bangkok standard volume. Undeniably, the air pollutant concentration was related to the traffic flow pattern, traffic characteristic, street geometries, and human activities. The traffic flow is the main pollution source in many urban areas. It causes more ambient air pollutants such as carbon monoxide (CO), sulfur dioxide (SO2), nitrogen dioxide (NO2), nitrogen oxides (NOx), volatile organic compounds (VOCs), ozone (O3), particulate matter (PM10), benzene, heavy metals, and respirable particulate matter (PM2.5 and PM10) [2, 3, 4]. Therefore, 1D Lighthill-Whitham-Richards traffic model and advection-diffusion-reaction pollution model for estimating the pollution emission rate due to traffic flow in big cities are proposed in [3]. The modeling of oxidation and hydrolysis of sulfur and nitrogen oxide used the convection-diffusion-reactions equation for higher order accurate solutions. The technique of Lax and Wendroff is introduced in [5]. A three-dimensional advection-diffusion equation of air pollutant is applied to a street tunnel configuration by using the FTCS finite difference method with air flow in x and y directions in [6]. In [7], a one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media [8] is solved using explicit finite difference method for three dispersion problems: (i) solute dispersion along the

steady flow, (ii) temporarily dependent solute dispersion along the uniform flow, and (iii) solute dispersion is temporarily dependent on the steady flow through inhomogeneous medium to find solutions. A study of vehicle exhaust dispersion within different street canyons models in urban ventilated by cross-wind by using the advanced computational and mathematical models. The pollutant concentrations are estimated for the street canyon models, which may include simplified photochemistry and particle deposition-suspension algorithms. After application of Box model to the street canyon. [9] can be calculated for CO, NOx, SPM, and PM10 by this box model. In [10], the effects of the variations of atmospheric stability classes and wind velocities on the three-dimensional air-quality models are observed. The fractional step method is used to solve the dispersion model for advection-diffusion equation.

In this research, we are interested in traffic density of the area the Bangkok sky train platform. We proposed the numerical modeling of air pollutant concentration in sky train platform with airflow obstacles on heavy traffic road. The estimated three-dimensional advection-diffusion equation is used by the finite difference method. In our research, we indicate that the air pollution modeling depends on air pollution flows and wind directions. In the second section, the governing equation corresponding to the model is the three-dimensional advection-diffusion equation including the initial condition and boundary conditions. The third section is numerical techniques. The finite difference technique introduced two methods for calculating the air pollutant concentration. The three-dimensional advection-diffusion equation is solved by using the Forward Time, Centered Space (FTCS) and Forward Time, Backward Space (FTBS) schemes. In order to illustrate the performance of the model, in section 4 the numerical experiments are presented. This is the comparisons between the two methods and the simulations of the proposed air pollution control. Finally, the discussion and conclusion are presented in section 5.

2. Governing equation

The street tunnel configuration is shown in Figure 1. That is, the street is flanked by buildings on both sides, including the top area is also closed. The bottom floor is the street floor, next up is the ticket floor and the top floor is the platform floor. For both sides of the street are the section of buildings. In this research, we assume that there are wind inflow in x- and y-directions and there are the obstacles as columns. The columns are on both sides of the street tunnel. The air pollutant concentrations are emitted from the entrance gate and the right side gap as Figure 2(a). We consider the wind inflow along x- and y-directions as Figure 2(b). Then the consider domain becomes : $\Omega = \{(x, y, z); 0 \leq x \leq L, 0 \leq y \leq W, 0 \leq z \leq H\}$, where W is the width (m), L is the length (m) and H is the height (m) of the street tunnel.



Figure 1: The street tunnel configuration.

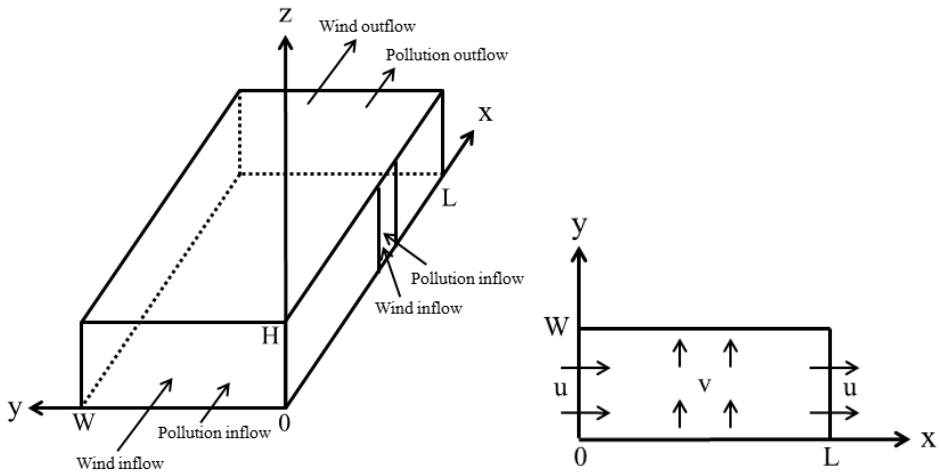


Figure 2: (a) The direction of pollution and wind flows. (b) The wind directions for street tunnel.

The air pollutant concentrations can be described by the three-dimensional advection-diffusion equation as follows:

$$\frac{\partial C}{\partial t} + V \cdot \nabla C = \nabla \cdot \left(\bar{K} \otimes \nabla C \right), \tag{1}$$

where $C = C(x, y, z, t)$ is the air pollutant concentration at point (x, y, z) in Cartesion coordinates at time t (kg/m^3), $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$, and $\otimes$ is matrix multiplication. The vector V is the wind velocity field (m/sec), $\bar{K}$ is the eddy-diffusivity or dispersion tensor (m^2/sec).

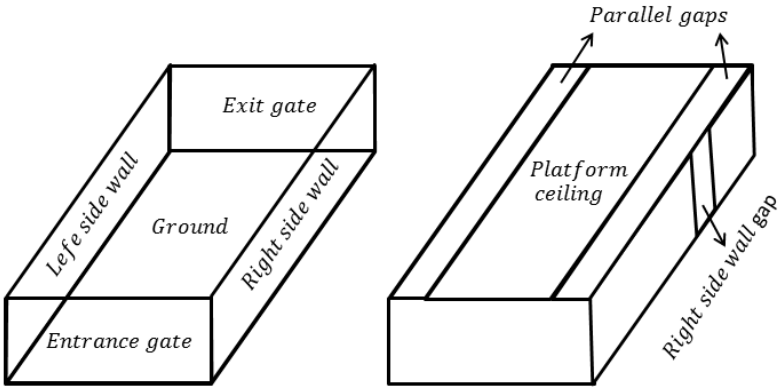


Figure 3: The components of the street tunnel.

The three-dimensional advection-diffusion equation in Eq. (1), can be written as

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = k_x\frac{\partial^2 C}{\partial x^2} + k_y\frac{\partial^2 C}{\partial y^2} + k_z\frac{\partial^2 C}{\partial z^2}, \tag{2}$$

where u, v, and w are the constant wind velocity (m/sec) in x, y, and z-directions, respectively, k_x, k_y, and k_z are the constant diffusion coefficient (m^2/sec) in x, y, and z-directions, respectively.

The assumptions of Eq. (2) are defined that the wind inflow in x- and y-directions are the horizontal direction and in z-direction is the vertical direction. Consequently, the three-dimensional advection-diffusion equation in Eq. (2), can be written as:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = k_h\frac{\partial^2 C}{\partial x^2} + k_h\frac{\partial^2 C}{\partial y^2} + k_v\frac{\partial^2 C}{\partial z^2}, \tag{3}$$

where k_h is the constant dispersion coefficient in the horizontal direction (m^2/sec) and k_v is the constant dispersion coefficient in the vertical direction (m^2/sec).

We consider the components of the street tunnel shown in Figure 3. Figure 4 shows the model of the problem with A is the right parallel gap size along the ceiling, B is the left parallel gap size along the ceiling and $G - F$ is the right side wall gap of the beside consider domain. (D_{cn}, E_{rn}) is a center point of the columns for $cn = 1, 2, ..., ncl$ and $rn = 1, 2$, where ncl is the number of the columns. The initial condition, there is no initial pollutant $C(x, y, z, 0) = 0$, for all $(x, y, z) \in \Omega$. For the boundary conditions are assumed that

Entrance gate : $C(0, y, z, t) = c_1$.
Margin of entrance gate : $\frac{\partial C}{\partial x}(0, y, z, t) = c_2, y = 0, W, z = 0, H$.
Exit gate : $\frac{\partial C}{\partial x}(L, y, z, t) = c_3$.
Left side wall : $\frac{\partial C}{\partial y}(x, W, z, t) = c_4$.
Right side wall gap : $C(x, 0, z, t) = c_5, F \leq x \leq G$.

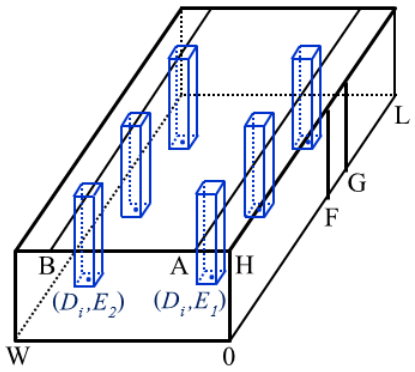


Figure 4: Model of the problem.

Right side wall : $\frac{\partial C}{\partial y}(x,0,z,t) = c_6$, otherwise.

Ground : $\frac{\partial C}{\partial z}(x,y,0,t) = c_7$.

Platform ceiling : $\frac{\partial C}{\partial z}(x,y,H,t) = c_8, A < y < B$.

Parallel gaps : $\frac{\partial C}{\partial z}(x,y,H,t) = c_9$, otherwise.

Center columns : $C(D_{cn}, E_{rn}, z, t) = 0, 0 \leq z \leq H$.

Front and back columns : $\frac{\partial C}{\partial x}(D_{cn} - 1, y, z, t) = \frac{\partial C}{\partial x}(D_{cn} + 1, y, z, t) = c_1 0, E_{rn} - 1 \leq y \leq E_{rn} + 1$, for all $t > 0$.

Left and right columns: $\frac{\partial C}{\partial y}(x, E_{rn} - 1, z, t) = \frac{\partial C}{\partial y}(x, E_{rn} + 1, z, t) = c_1 1, D_{cn} - 1 \leq x \leq D_{cn} + 1$, for all $t > 0$.
Where c_1 and c_5 are the air pollutant concentration inflow in $x-$ and $y-$ directions, respectively, $c_2, c_3, c_4, c_6, c_7, c_8, c_9, c_{10}$, and c_{11} are the rate of change of air pollutant concentration in each the boundary conditions.

3. Numerical techniques

We use the finite difference methods to compute a numerical approximation to the solutions of a three-dimensional advection-diffusion equation. The solution domain of the problem over time $0 \leq t \leq T$ is covered by a mesh of grid spacing: $x_i = i\Delta x, i = 0, 1, 2, ..., M; y_j = j\Delta y, j = 0, 1, 2, ..., N; z_k = k\Delta z, k = 0, 1, 2, ..., P; t_n = n\Delta t, n = 0, 1, 2, ..., Q$; parallel to the space and time coordinate axes, respectively. Approximation the solution of the air pollutant concentration $C^n_{i,j,k}$ to $C(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ are calculated at the point of intersection of these lines, namely, $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ which is referred to as the (i, j, k, n) grid point. The constant spatial and temporal grid-spacing are $\Delta x = \frac{L}{M}, \Delta y = \frac{W}{N}, \Delta z = \frac{H}{P}, \Delta t = \frac{T}{Q}$, respectively. In this research, we distinguish two difference methods as following:

3.1 Forward time central space scheme

The first method, we use an explicit forward difference estimate for the time derivative (FT), and central difference approximations for the space derivative (CS), so the acronym FTCS. Consequently, the three-dimensional advection-diffusion equation in Eq. (3) becomes

$$\frac{C_{i,j,k}^{n+1}-C_{i,j,k}^{n}}{\Delta t}+u\left(\frac{C_{i+1,j,k}^{n}-C_{i-1,j,k}^{n}}{2\Delta x}\right)+v\left(\frac{C_{i,j+1,k}^{n}-C_{i,j-1,k}^{n}}{2\Delta y}\right)$$

$$=D_h\left(\frac{C_{i+1,j,k}^{n}-2C_{i,j,k}^{n}+C_{i-1,j,k}^{n}}{(\Delta x)^2}\right)+D_h\left(\frac{C_{i,j+1,k}^{n}-2C_{i,j,k}^{n}+C_{i,j-1,k}^{n}}{(\Delta y)^2}\right)$$

$$+D_v\left(\frac{C_{i,j,k+1}^{n}-2C_{i,j,k}^{n}+C_{i,j,k-1}^{n}}{(\Delta z)^2}\right). \tag{4}$$

Rearrangement and simplification of Eq. (4),

$$\begin{aligned} C_{i,j,k}^{n+1} &= \left(s_x+\frac{r_x}{2}\right)C_{i-1,j,k}^{n}+\left(s_y+\frac{r_y}{2}\right)C_{i,j-1,k}^{n}+(s_z)\,C_{i,j,k-1}^{n} \\ &+ \left(s_x-\frac{r_x}{2}\right)C_{i+1,j,k}^{n}+\left(s_y-\frac{r_y}{2}\right)C_{i,j+1,k}^{n}+(s_z)\,C_{i,j,k+1}^{n} \\ &+ (1-2s_x-2s_y-2s_z)\,C_{i,j,k}^{n}, \end{aligned} \tag{5}$$

in which $r_x=\frac{u\Delta t}{\Delta x}, r_y=\frac{v\Delta t}{\Delta y}, s_x=\frac{D_h\Delta t}{(\Delta x)^2}, s_y=\frac{D_h\Delta t}{(\Delta y)^2}$ and $s_z=\frac{D_v\Delta t}{(\Delta z)^2}$.

The finite difference scheme for the left end and the right end of the fictitious points are following:

$$C_{-1,j,k}^{n}=\frac{4C_{0,j,k}^{n}-C_{1,j,k}^{n}-2c_2\Delta x}{3}, \tag{6}$$

$$C_{i,-1,k}^{n}=\frac{4C_{i,0,k}^{n}-C_{i,1,k}^{n}-2c_6\Delta y}{3}, \tag{7}$$

$$C_{i,j,-1}^{n}=\frac{4C_{i,j,0}^{n}-C_{i,j,1}^{n}-2c_7\Delta z}{3}, \tag{8}$$

$$C_{M+1,j,k}^{n}=\frac{4C_{M,j,k}^{n}-C_{M-1,j,k}^{n}+2c_3\Delta x}{3}, \tag{9}$$

$$C_{i,N+1,k}^{n}=\frac{4C_{i,N,k}^{n}-C_{i,N-1,k}^{n}+2c_4\Delta y}{3}, \tag{10}$$

$$C_{i,j,P+1}^{n}=\frac{4C_{i,j,P}^{n}-C_{i,j,P-1}^{n}+2c_8\Delta z}{3}. \tag{11}$$

3.2 Forward time backward space scheme

The second method, we calculated by using an explicit forward difference estimate for the time derivative (FT), and backward difference approximations for the space derivative (BS), so the acronym FTBS. The approximate solution of a

three-dimensional advection-diffusion equation in Eq. (3) use the FTCS scheme satisfies

$$\frac{C_{i,j,k}^{n+1}-C_{i,j,k}^{n}}{\Delta t}+u\left(\frac{C_{i,j,k}^{n}-C_{i-1,j,k}^{n}}{2\Delta x}\right)+v\left(\frac{C_{i,j,k}^{n}-C_{i,j-1,k}^{n}}{2\Delta y}\right)$$

$$=D_h\left(\frac{C_{i,j,k}^{n}-2C_{i-1,j,k}^{n}+C_{i-2,j,k}^{n}}{(\Delta x)^2}\right)+D_h\left(\frac{C_{i,j,k}^{n}-2C_{i,j-1,k}^{n}+C_{i,j-2,k}^{n}}{(\Delta y)^2}\right)$$

$$+D_v\left(\frac{C_{i,j,k}^{n}-2C_{i,j,k-1}^{n}+C_{i,j,k-2}^{n}}{(\Delta z)^2}\right). \tag{12}$$

Rearrangement and simplification of Eq. (12),

$$\begin{aligned} C_{i,j,k}^{n+1} &= (s_x)\,C_{i-2,j,k}^{n}+(s_y)\,C_{i,j-2,k}^{n}+(s_z)\,C_{i,j,k-2}^{n} \\ &+ (r_x-2s_x)\,C_{i-1,j,k}^{n}+(r_y-2s_y)\,C_{i,j-1,k}^{n}-(2s_z)\,C_{i,j,k-1}^{n} \\ &+ (1+s_x+s_y+s_z-r_x-r_y)\,C_{i,j,k}^{n}. \end{aligned} \tag{13}$$

The finite difference scheme for the left end of the fictitious points are following:

$$C_{-1,j,k}^{n}=\frac{4C_{0,j,k}^{n}-C_{1,j,k}^{n}-2c_2\Delta x}{3}, \tag{14}$$

$$C_{-2,j,k}^{n}=\frac{13C_{0,j,k}^{n}-4C_{1,j,k}^{n}-14c_2\Delta x}{9}, \tag{15}$$

$$C_{i,-1,k}^{n}=\frac{4C_{i,0,k}^{n}-C_{i,1,k}^{n}-2c_6\Delta y}{3}, \tag{16}$$

$$C_{i,-2,k}^{n}=\frac{13C_{i,0,k}^{n}-4C_{i,1,k}^{n}-14c_6\Delta y}{9}, \tag{17}$$

$$C_{i,j,-1}^{n}=\frac{4C_{i,j,0}^{n}-C_{i,j,1}^{n}-2c_7\Delta z}{3}, \tag{18}$$

$$C_{i,j,-2}^{n}=\frac{13C_{i,j,0}^{n}-4C_{i,j,1}^{n}-14c_7\Delta z}{9}. \tag{19}$$

4. Numerical experiments

4.1 Comparison between FTCS and FTBS solutions in sky train platform on a single

In this section, we describe about a comparision of some numerical methods for solving the three-dimensional advection-diffusion equation. There are two methods. These are forward time central space (FTCS) and forward time backward space (FTBS). We consider the domain as a single layer as shown in Figure 4 that the length(L), width(W) and height(H) of tunnel are 198, 21 and 28 meters, respectively. Then, the problem domain is $\Omega=\{(x,y,z);0\le x\le 198,0\le y\le 21,0\le z\le 28\}$. We assume that $\Delta x=\Delta y=\Delta z=2$ m, $\Delta t=0.06$ sec,

$T = 2$ min, $u = 0.5$ m/sec, $v = 0$ m/sec, $D_h = D_v = 0.001$ m^2/sec, $c_1 = 0.5$, $c_3 = c_9 = -0.01$, $c_2 = c_4 = c_5 = c_6 = c_7 = c_8 = c_{10} = c_{11} = 0$, $A = 4$, $B = 17$, and $ncl = 9$. Figures 6 and 7 are solved by various methods. These are FTCS and FTBS methods, respectively. The solutions of air pollutant concentration by using the FTCS method in Eq. (5) are shown in Figure 6. This figure show about contour and surface plots of air pollutant concentration levels after 2 minutes passed. You can be seen that the maximum value of air pollutant concentration is 0.6 kg/m^3. Furthermore, the contour and surface plots of air pollutant concentration levels after 2 minutes passed for solutions of air pollutant concentration by using the FTBS method in Eq. (13) are shown in Figure 7. The maximum value of air pollutant concentration in Figure 7 is 0.5 kg/m^3.

The approximations of air pollutant concentration for FTCS and FTBS methods are compared in Figure 11. We choose $\Delta x = \Delta y = \Delta z = 2$ m, $\Delta t = 0.06$ sec, $T = 30$ sec, $u = 0.1$ m/sec, $v = 0$ m/sec, $D_h = D_v = 0.001$ m^2/sec, $c_1 = 0.5$, $c_3 = c_9 = -0.01$, $c_2 = c_4 = c_5 = c_6 = c_7 = c_8 = c_{10} = c_{11} = 0$, $A = 4$, $B = 17$ and $ncl = 9$. It can be seen that the trend of results from both methods in the same way. How do you know which one is better? The idea is to find out the method whether we will change the grid-spacing, the solution is stable. Table. 1 shows the stable of FTCS and FTBS approximate solutions. You can be seen that if we choose $\lambda = \gamma = 0.03$, and $\lambda = 0.06$ then the solution of the FTBS method are unstable but the FTCS method is stable. Consequently, the FTCS method gives better than the FTBS method.

4.2 Numerical simulations of air pollutant assessment in sky train platform on triple layers

In this section, explicit forward time and central space (FTCS) scheme have been presented. We distinguish three scenarios of released air pollutant phenomenons demonstrated by using the finite difference in Eq. (5). In all scenarios, the air pollutant concentration is flowing along the x and y-directions, these are constants or functions. In addition, there are two parallel gaps along the ceiling, the rate of change are decreased at the parallel gaps. Moreover, both sides were flanked by buildings. All of the building walls are non-absorbing air pollution materials, there is no rate of change.

For three scenarios, we consider the length, width and height of tunnel are 198, 21 and 28 meters, respectively. Then, the problem domain is $\Omega = \{(x, y, z); 0 \leq x \leq 198, 0 \leq y \leq 21, 0 \leq z \leq 28\}$, when $0 \leq z < 9, 9 \leq z < 22, 22 \leq z \leq 28$ are street floor, ticket floor and platform floor, respectively, see in Figure 5. We consider c_1 in the boundary condition in the entrance gate of each floor, we distinguish 3 scenarios in the consider domain as follows:

4.2.1 Scenario A : Air pollutant flowing into the street floor

If we consider BTS station area, we will see that the street floor has a lot of cars. This causes heavy traffic, as a result air pollution is higher than other

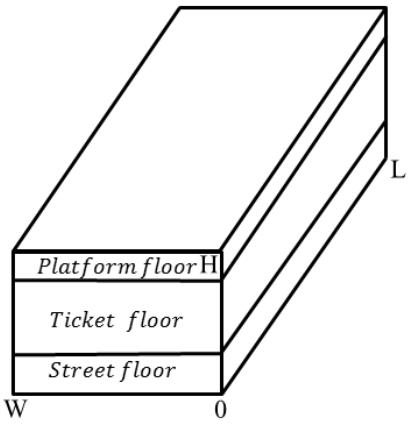


Figure 5: The problem domain.

floors. Therefore, we assumed the air pollutant concentration at the entrance gate of the street floor is constant. However, ticket floor and platform floor are assumed that the pollution cannot be reached. So, we assumed that there is no rate of change of air pollutant concentration at the entrance gate of the ticket and platform floors. Therefore, c_1 in the boundary condition of all 3 floors as follows:

Street floor : $c_1 = 0.5$.
Ticket floor : $c_1 = \frac{\partial C}{\partial x} = 0$.
Platform floor : $c_1 = \frac{\partial C}{\partial x} = 0$.

The problem domain is $\Omega = \{(x, y, z); 0 \leq x \leq 198, 0 \leq y \leq 21, 0 \leq z < 9\}$. The grid spacing: $\Delta x = \Delta y = \Delta z = 1$ m, $\Delta t = 0.06$ sec and for the time $T = 2$ min. We assume: $c_3 = c_9 = -0.01$, $c_5 = 0.2$, $c_2 = c_4 = c_6 = c_7 = c_8 = c_{10} = c_{11} = 0$, $A = 4$, $B = 17$, $F = 125$, $G = 135$ and $ncl = 9$. The wind velocity and diffusion coefficient are taken to be $u = 2.7778, v = u/20$ m/sec and $D_h = 0.1592, D_v = 0.05$ m^2/sec. Therefore, the results of ScenarioA for three different floors are shown in Figure 8. That is, in Figures 8(a), 8(c) and 8(e) show the contour plot of the air pollutant concentration levels for street, ticket and platform floors, respectively. Meanwhile, the surface plot of the air pollutant concentration levels for street, ticket and platform floors are shown in Figures 8(b), 8(d) and 8(f), respectively. It can be seen from Figure 8 that the air pollutant concentration in the platform floor is very low. It comes from only the pollution on the right side wall gap. So, the air pollution on platform floor is less than 0.2 kg/m^3. The air pollutant concentration of Scenario A with the different floors are shown in Figure 12(a).

4.2.2 Scenario B : Air pollutant flowing into every floors

In reality, we noticed that the air pollutant concentration depends on the height of the tunnel, so if the height increases, the air pollutant concentration will be

less. Thus, the air pollutant concentration at the entrance gate for the street floor, ticket floor and platform floor can be described as different decreasing functions varied with the height of the tunnel. Therefore, c_1 in the boundary condition of all 3 floors as follows:

Street floor : $c_1 = 0.5 - 0.02z$.

Ticket floor : $c_1 = 0.32 - 0.015z$.

Platform floor : $c_1 = 0.01 - 0.005z$.

The problem domain is $\Omega = \{(x, y, z); 0 \leq x \leq 198, 0 \leq y \leq 21, 9 \leq z < 22\}$. The grid spacing: $\Delta x = \Delta y = \Delta z = 1$ m, $\Delta t = 0.06$ sec and for the time $T = 2$ min. We assume: $c_3 = c_9 = -0.01$, $c_5 = 0.2$, $c_2 = c_4 = c_6 = c_7 = c_8 = c_{10} = c_{11} = 0$, $A = 4$, $B = 17$, $F = 125$, $G = 135$ and $ncl = 9$. We choose $u = 2.7778, v = u/20$ m/sec, $D_h = 0.1592, D_v = 0.05$ m^2/sec. So, the results of Scenario B for three different floors are shown in Figure 9. That is, in Figures 9(a), 9(c) and 9(e) show the contour plot of the air pollutant concentration levels for street, ticket and platform floors, respectively. Meanwhile, the surface plot of the air pollutant concentration levels for street, ticket and platform floors are shown in Figures 9(b), 9(d) and 9(f), respectively. As the results, the air pollution continues to be released as the decreasing functions but gradually decreases. Therefore, the air pollution in three floors of this scenario is higher especially, the air pollutant concentration on the street floor is as high as 0.7 kg/m^3. The air pollutant concentration of Scenario B with the different floors are shown in Figure 12(b).

4.2.3 Scenario C : Air pollutant flowing through the street floor and their gaps

The air pollutant concentration on the street floor is assumed to be a constant. Furthermore, for more realism, we can see that the air pollutant concentration from the previous floor impact on the next floor. So we will define the pollution of the next floor by applying the principle of average. That is the air pollutant concentration at the entrance of the next floor is the average of air pollutant concentration of gaps on the previous floor. Therefore, c_1 in the boundary condition of all 3 floors as follows:

Street floor : $c_1 = 0.5$.

Ticket floor : $c_1 = c_{avS}$.

Platform floor : $c_1 = c_{avT}$,

where c_{avS} and c_{avT} are the average of air pollutant concentration of gaps on the street floor and ticket floor, respectively.

The problem domain is $\Omega = \{(x, y, z); 0 \leq x \leq 198, 0 \leq y \leq 21, 22 \leq z \leq 28\}$. The grid spacing: $\Delta x = \Delta y = \Delta z = 1$ m, $\Delta t = 0.06$ sec and for the time $T = 2$ min. We assume: $c_3 = c_9 = -0.01$, $c_5 = 0.2$, $c_2 = c_4 = c_6 = c_7 = c_8 = c_{10} = c_{11} = 0$, $A = 4$, $B = 17$, $F = 125$, $G = 135$ and $ncl = 9$. We choose $u = 2.7778, v = u/20$ m/sec, $D_h = 0.1592, D_v = 0.05$ m^2/sec. So, the results of Scenario C for three different floors are shown in Figure 10. That is, in Figures

10(a), 10(c) and 10(e) show the contour plot of the air pollutant concentration levels for street, ticket and platform floors, respectively. Meanwhile, the surface plot of the air pollutant concentration levels for street, ticket and platform floors are shown in Figures 10(b), 10(d) and 10(f), respectively. It can also be obtained by Figure 13 that the air pollutant concentration gradually decreases because we bring the value of the previous floor to the next floor. So, the pollution on the platform floor is least the pollution than other floors as 0.5 kg/m^3. The air pollutant concentration of Scenario C with the different floors are shown in Figure 12(c).

5. Discussion and conclusion

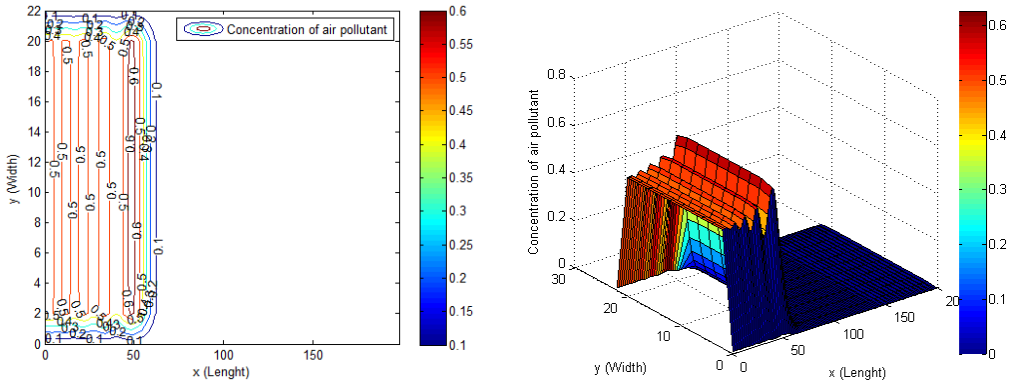
In this research, the air pollutant concentration model is presented. The finite difference methods such as FTCS and FTBS methods can be used to estimate the air pollutant concentration. Also, it is appealing that the grid spacing is different so FTCS method has been chosen because when making comparisons between FTCS and FTBS methods in some cases, the solution for FTBS method is unstable while the solution for FTCS method is stable. Hence, FTCS method provides a better result than FTBS method.

Furthermore, we proposed three scenarios for estimating the air pollutant concentration as follows; Scenario A: there is no rate of change of air pollutant concentration at the entrance gate of the ticket and platform floors. Based on the results, the air pollution in the platform floor is low because it only emits pollution at the wall gap at the right side. Scenario B: the air pollutant concentration at the entrance of three floors can be described by different decreasing functions depended on the height of the tunnel. As the results, the air pollution continues to be released as the decreasing functions but gradually decreases. Therefore, the air pollution in this scenario is high when compared to other scenarios. Scenario C: the air pollution at the entrance of the next floor is the average of air pollutant concentration of gaps on the previous floor. The results of this scenario show that the pollution gradually decreases because we used the value of the previous floor to the next floor which significantly reduces air pollution.

Summary of the three scenarios: Scenario A is a simple model that is an economical method to use. It requires a few of collected data. Scenario B is a realistic numerical simulation. The approximated air pollutant concentration depends on the height but requires a lot of field data. Scenario C is a fairly good model. The simulation needs to average the pollutant concentration level. It is used as the input air pollutant concentration to the above floor. However, the suitable model depends on the provided field data.

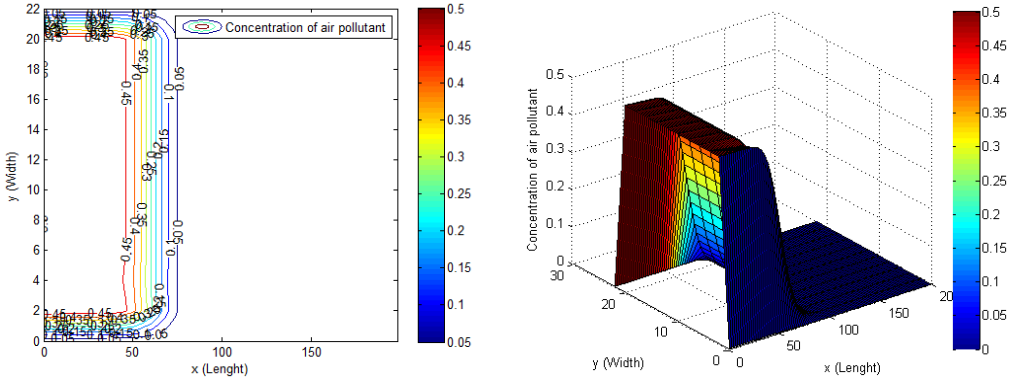
Table 1: The stable of FTCS and FTBS approximate solutions.

$\lambda = \frac{\Delta t}{\Delta x}$	$\gamma = \frac{\Delta t}{(\Delta x^2)}$	Δx	Δy	Δz	Δt	$FTCS$	$FTBS$
0.02	0.0067	3.0	3.0	3.0	0.06	stable	stable
	0.0267	1.5	1.5	1.5	0.03	stable	stable
0.03	0.015	2.0	2.0	2.0	0.06	stable	stable
	0.03	1.0	1.0	1.0	0.03	stable	unstable
0.06	0.06	1.0	1.0	1.0	0.06	stable	unstable
	0.12	0.5	0.5	0.5	0.03	stable	unstable



(a) (b)

Figure 6: Contour and surface plot of air pollutant concentration levels after the past 2 minutes computed by FTCS method.



(a) (b)

Figure 7: Contour and surface plot of air pollutant concentration levels after the past 2 minutes computed by FTBS method.

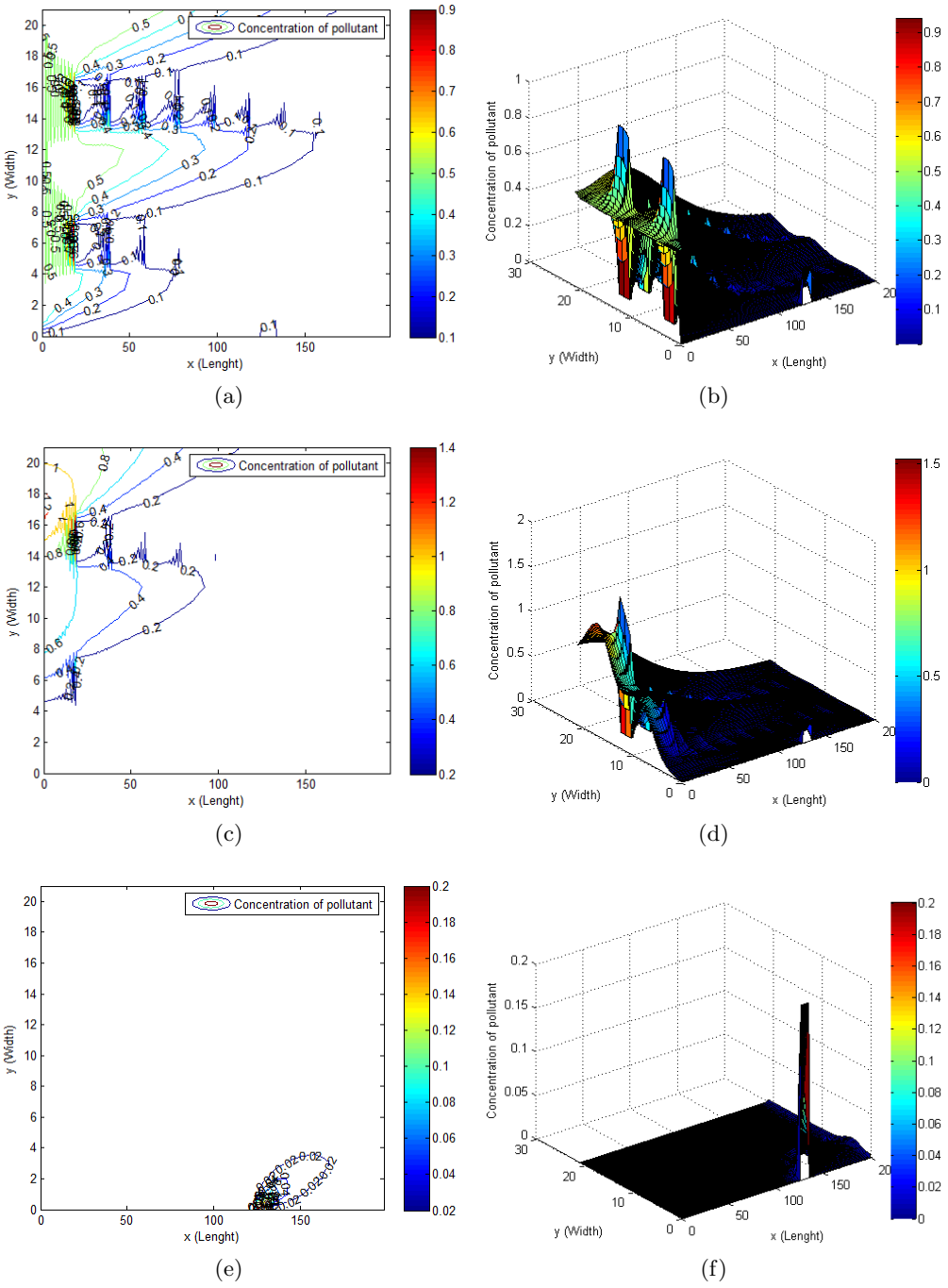


Figure 8: Contour and surface plot of air pollutant concentration levels after the past 2 minutes for the respective streets, tickets and platform floors. (Scenario A)

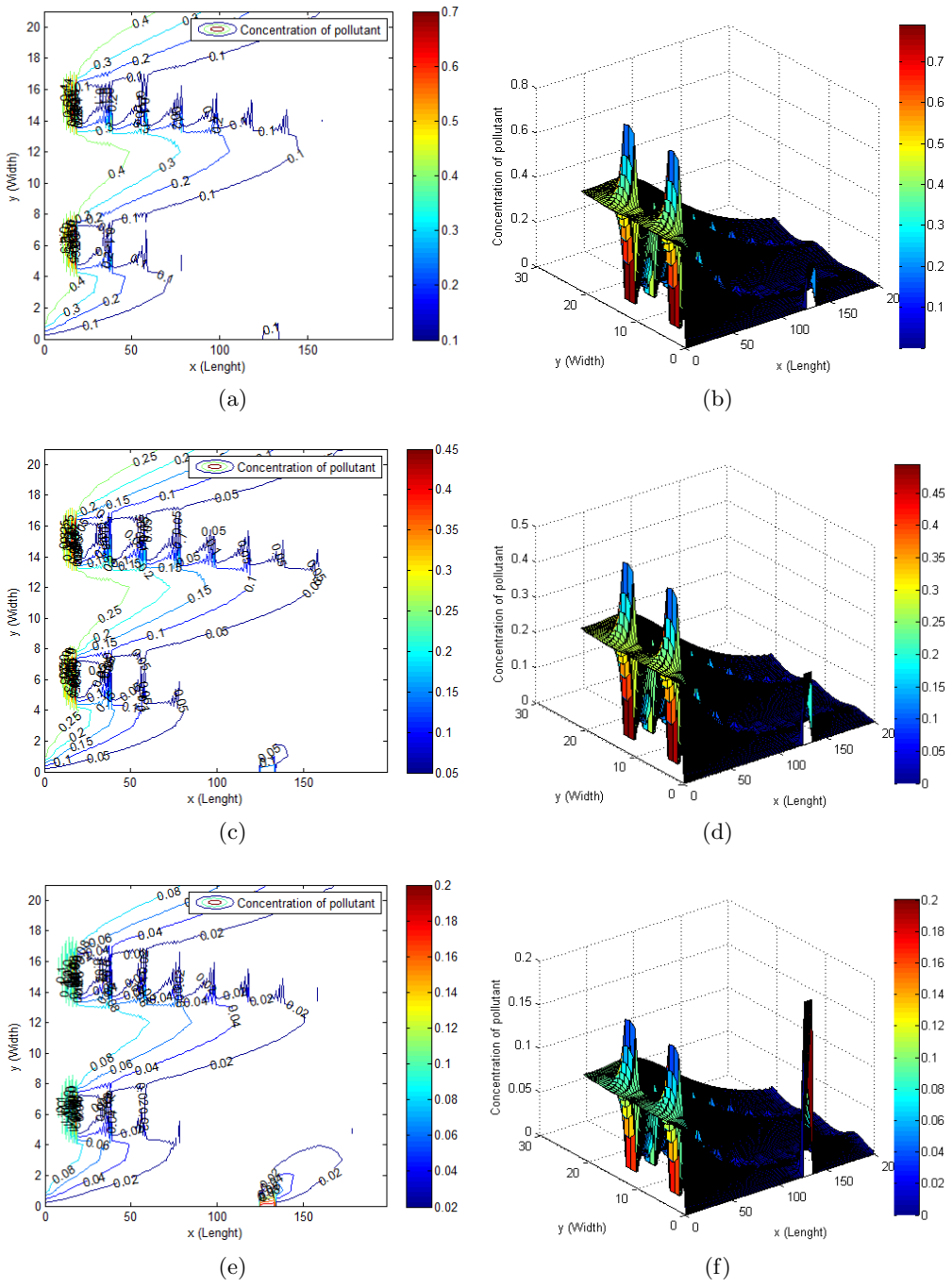


Figure 9: Contour and surface plot of air pollutant concentration levels after the past 2 minutes for the respective streets, tickets and platform floors. (Scenario B)

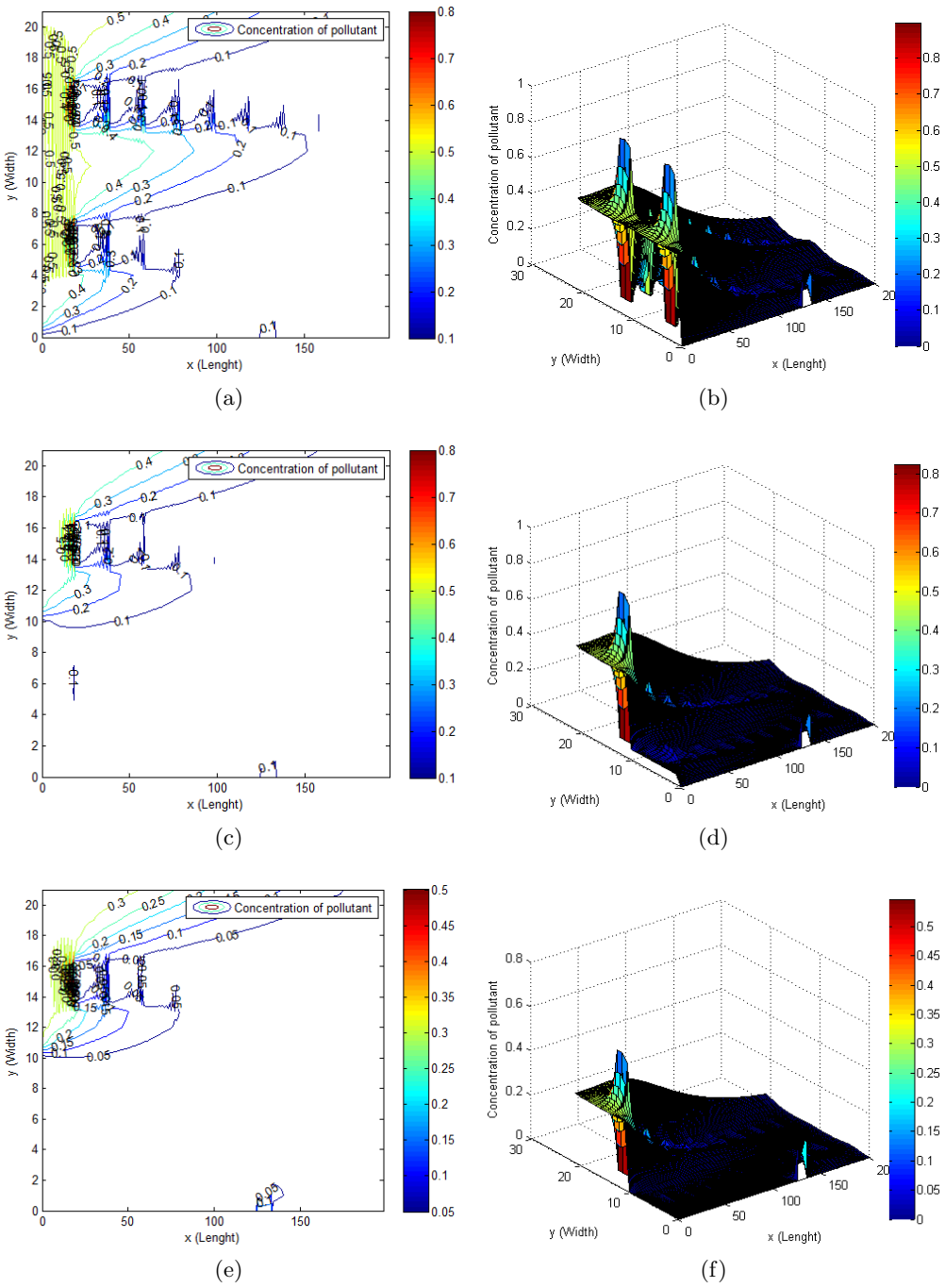


Figure 10: Contour and surface plot of air pollutant concentration levels after the past 2 minutes for the respective streets, tickets and platform floors. (Scenario C)

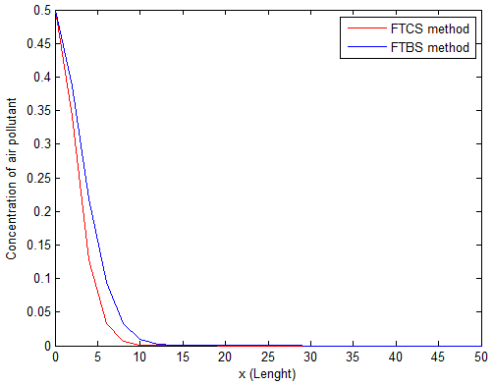
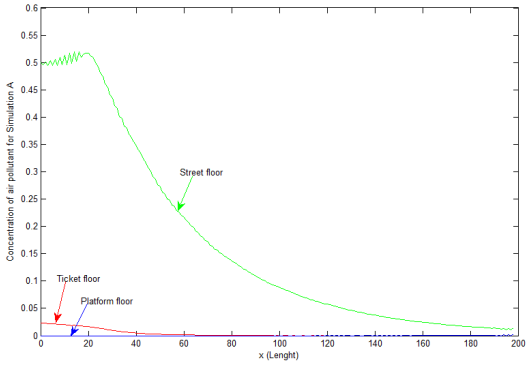
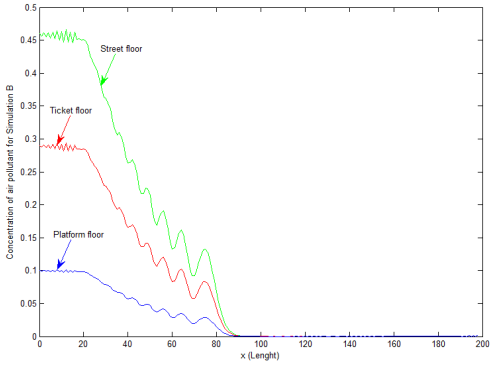


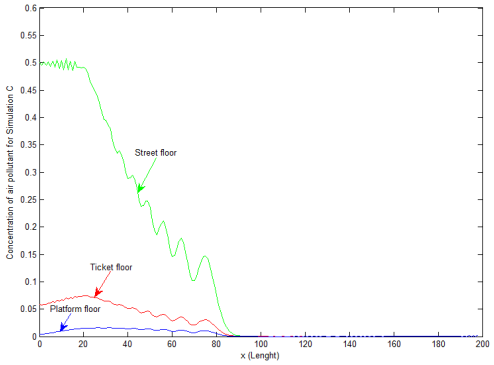
Figure 11: Comparison of air pollutant concentration between FTCS and FTBS methods after the past 30 seconds.



(a)



(b)



(c)

Figure 12: The air pollutant concentration with the different floors of (a) Scenario A. (b) Scenario B. (c) Scenario C.

Acknowledgements

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The authors greatly appreciate valuable comments received from the reviewers.

References

[1] U. Charusombat, *Air pollution distribution under an elevated train station (A case study of Silom station in downtown Bangkok)*, Faculty of Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1994.

[2] S.T. Leong, S. Muttamara and P. Laortanakul, *Air pollution and traffic measurements in Bangkok streets*, Asian J. Energy Environ., 3 (2002), 185-213.

[3] L.J. Alvarez-Vazquez, N. Garcia-Chan, A. Martinez and M.E. Vzquez-Mendez, *Numerical simulation of air pollution due to traffic flow in urban networks*, Journal of Computational and Applied Mathematics, 326 (2007), 44-61.

[4] M. Kampa and E. Castanas, *Human health effects of air pollution*, Environmental Pollution, 151 (2008), 362-367.

[5] N. Sanin and G. Montero, *A finite difference model for air pollution simulation*, Advances in Engineering Software, 38 (2007), 358-365.

[6] M. Thongmoon, *Numerical experiment of air pollutant concentration in the street tunnel*, International Mathematical Forum, 10 (2010), 449-465.

[7] A. Kumar, D. K. Jaiswal, and A. Kumar, *Analytical solutions to one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media*, Journal of Hydrology, 380 (2010), 330-337.

[8] S. Savovic and A. Djordjevich, *Finite difference solution of the one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media*, International Journal of Heat and Mass Transfer, 55 (2012), 4291-4294.

[9] R. S. Kanakiya, S. K. Singh, and P. M. Mehta, *Urban canyon modelling: a need for the design of future Indian cities*, International Research Journal of Environment Sciences, 4(7) (2015), 86-95.

[10] S. A. Konglok and N. M. Pochai, *Numerical computations of three-dimensional air-Quality model with variations on atmospheric stability classes and wind velocities using fractional step method*, IAENG International Journal of Applied Mathematics, 46(1) (2016), 112-120.

[11] L. Sun, K. C. Wong, P. Wei, S. Ye, H. Huang, F. Yang, D. Westerdahl, K. K. Louie, W. Y. Luk, and Z. Ning, *Development and application of a next generation air sensor network for the Hong Kong marathon 2015 air quality monitoring*, Lecture Notes in Engineering and Computer Science: Sensors, 16 (2016), 211.

Accepted: 26.03.2018

A CONSTRUCTION OF CONGRUENCE-SIMPLE SEMIRINGS

Barbora Batíková
Department of Mathematics
CULS, Kamýcká 129
165 21 Praha 6-Suchdol
Czech Republic
batikova@tf.czu.cz

Tomáš Kepka
Department of Algebra
MFF UK, Sokolovská 83
186 75 Praha 8
Czech Republic
kepka@karlin.mff.cuni.cz

Petr Němec*
Department of Mathematics
CULS, Kamýcká 129
165 21 Praha 6-Suchdol
Czech Republic
nemec@tf.czu.cz

Abstract. A construction of congruence-simple semirings is presented.

The congruence-simple semirings of positive rational (real) numbers are fairly familiar, but the other (congruence-)simple semirings are regarded as somewhat apocryphal. It is easy to show that simple semirings split into three basic classes: the additively cancellative semirings, the additively nil-semirings and the additively idempotent ones. The first class includes all simple rings and many subsemirings of ordered rings. The second class includes many congruence-simple semigroups equipped with constant addition, but what remains is quite enigmatic so far ([1]). Now, we come to the third class, the additively idempotent simple semirings. These semirings (at least in the finite case) are of interest because of possible applications in cryptology (see e.g. [9]) and they are constructed as endomorphism semirings of semilattices (see [2], [6], [7] and [8]). The present note continues this line of research. Finally, notice that few pieces of information on simple semirings and general semirings are available in [4] or [5].

∗. Corresponding author

1. Preliminaries

Let $A = A(*)$ be a groupoid. An element $a \in A$ is called *left (right) neutral* if $a * x = x$ ($x * a = x$) for all $x \in A$, and *left (right) absorbing* if $a * x = a$ ($x * a = a$) for all $x \in A$. If $A = A(+)$ then $0_A \in A$ ($o_A \in A$) means that 0_A (o_A) is (the unique) left and right neutral (absorbing) element of $A(+)$ and $0_A \notin A$ ($o_A \notin A$) denotes the fact that $A(+)$ has no (left and right) neutral (absorbing) element. Similarly, if $A = A(\cdot)$ then $1_A \in A$ means that 1_A is (the unique) left and right neutral element of $A(\cdot)$.

A semilattice is a commutative and idempotent semigroup. If M ($= M(+)$) is a semilattice then a basic order $\leq$ is defined on M by $x \leq y$ iff $x + y = y$. Now, an element $w \in M$ is the smallest (greatest, resp.) element of the ordered set $M(\leq)$ if and only if $w = 0_M$ ($w = o_M$, resp.).

A non-empty subset I of M is an *ideal* if $M + I = I$. This ideal is said to be *prime* if $M \setminus I$ is a subsemilattice of M. For every $x \in N = M \setminus \{o_M\}$ (here and throughout the paper $N = M$ if $o_M \notin M$ and similarly for $M \setminus \{0_M\}$), the set $A_x = \{\, y \in M \mid y \not\leq x \,\}$ is called *principal prime ideal determined by the element* x. Clearly, a subset I of M is a prime ideal if and only if the set $V = M \setminus I$ is a proper subsemilattice of M such that $x \in V$ whenever $x \leq y \in V$. This prime ideal is principal if and only if $o_V \in V$.

A *semiring* is a non-empty set equipped with two associative binary operations that are usually written as addition and multiplication. The addition is commutative and the multiplication distributes over the addition. Given a semiring S, a *(left S-)semimodule* (${}_SM =$) M is a commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ such that $(a+b)x = ax + bx$, $a(x + y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If S is a semiring then $R = \underline{R}(S) = \{\, a \in S \mid Sa = \{a\} \,\}$ denotes the set of right multiplicatively absorbing elements. If $a \in \underline{R}(S)$ then $a + a = aa + aa = (a + a)a = a$ and $a(b + b) = ab + ab = ab$ for every $b \in S$. Consequently, the semiring S is additively idempotent, provided that the right semimodule $\underline{R}(S)_S$ is faithful, i.e., for all $a, b \in S$, $a \neq b$, there is at least one $x \in \underline{R}(S)$ with $xa \neq xb$.

Let S be a semiring. A non-empty subset I of S is a *left (right) ideal* of S if $SI \cup (I + I) \subseteq I$ ($IS \cup (I + I) \subseteq I$). A left (right) ideal I is called *minimal* if $|I| \geq 2$ and $J = I$ whenever J is a left (right) ideal with $|J| \geq 2$ and $J \subseteq I$. A non-empty subset I of S is an *ideal* if $SI \cup IS \cup (I + I) \subseteq I$ and it is a *bi-ideal* if $SI \cup IS \cup (I + S) \subseteq I$. In the latter case, the relation $(I \times I) \cup \mathrm{id}_S$ is a congruence of the semiring S. Finally, S is called

– *simple* (more precisely: *congruence–simple*) if S has just two congruence relations (then these are id_S and $S \times S$ and $|S| \geq 2$);
– *(bi–)ideal–simple* if $S = I$ whenever I is an (bi–)ideal of S with $|I| \geq 2$.

Throughout the paper, all semirings and semimodules are assumed to be additively idempotent. It means that the respective additive semigroups $M(+)$ are semilattices.

2. Auxiliary results (a)

Throughout this section, let S be an (additively idempotent) semiring such that $|R| \geq 2$.

2.1 Lemma. (i) *If $0_S \in S$ then $0_R \in R$, $R0_S = \{0_R\}$ and $S0_S \leq 0_R$.*
(ii) *If $o_S \in S$ then $o_R \in R$, $Ro_S = \{o_R\}$ and $o_R \leq So_S$.*

Proof. $a0_S+b = a0_S+ab = a(0_S+b) = ab$ and $ao_S+b = ao_S+ab = a(o_S+b) = ao_S$ for all $a \in S$ and $b \in R$. □

2.2 Lemma. *Assume that either $R + S = S$ or the right semimodule R_S is faithful. Then:*
(i) *If $0_S \in S$ then $0_S = 0_R \in R$.*
(ii) *If $0_R \in R$ then $0_R = 0_S \in S$.*

Proof. (i) By 2.1(i), $ab0_S = 0_R = a0_S$ for all $a \in R$ and $b \in S$, and hence $b0_S = 0_S$, provided that R_S is faithful. If $R+S = S$ then $0_R+S = 0_R+R+S = R + S = S$.
(ii) $a(b + 0_R) = ab + a0_R = ab + 0_R = ab$ for all $a \in R$ and $b \in S$. □

2.3 Lemma. *Assume that either $R \cap (S + a) \neq \emptyset$ for every $a \in S$ or the right semimodule R_S is faithful. Then:*
(i) *If $o_S \in S$ then $o_S = o_R \in R$.*
(ii) *If $o_R \in R$ then $o_R = o_S \in S$.*

Proof. Using 2.1(ii), we proceed similarly as in the proof of 2.2. □

2.4 Lemma. *If the semiring S is simple then the right semimodule R_S is faithful.*

Proof. Take into account that $ab = b \neq c = ac$ for all $a \in S$ and $b, c \in R$, $b \neq c$. □

2.5 Lemma. (i) *R is the smallest (right) ideal of S.*
(ii) *$R + S$ is the smallest bi-ideal of R.*

Proof. It is easy. □

2.6 Corollary. (i) *The semiring S is ideal-simple iff $R = S$.*
(ii) *The semiring S is bi-ideal-simple iff $R + S = S$.* □

2.7 Lemma. *If the semiring S is simple then $R + S = S$ and the right semimodule R_S is faithful and simple.*

Proof. By 2.4 and 2.6(ii), R_S is faithful and $R+S=S$. If α is a congruence of R_S then ϱ is a congruence of S, where $(a,b)\in\varrho$ iff $(ca,cb)\in\alpha$ for every $c\in R$. Of course, $\varrho\cap(R\times R)=\alpha$. □

2.8 Lemma. *Let the right semimodule R_S be faithful and simple. If $\varrho\neq\mathrm{id}_S$ is a congruence of the semiring S then $R\times R\subseteq\varrho$.*

Proof. Let $(a,b)\in\varrho$, $a,b\in S$, $a\neq b$. Since R_S is faithful, $ca\neq cb$ for at least one $c\in R$, so that $\alpha=\varrho\cap(R\times R)\neq\mathrm{id}_R$. Clearly, α is a congruence of R_S, and therefore $\alpha=R\times R$. □

2.9 Lemma. *Assume that $R+S=S$. The following conditions are equivalent:*

(i) *$\varrho=S\times S$, whenever ϱ is a congruence of S such that $R\times R\subseteq\varrho$.*

(ii) *For every $a\in S$ there is $b\in R$ with $a\leq b$.*

Proof. (i) implies (ii). Define a relation σ on S by $(a,b)\in\sigma$ iff $(a+R)\cap(b+R)\neq\emptyset$. Then σ is a congruence of S and $R\times R\subseteq\sigma$. Thus $\sigma=S\times S$.
(ii) implies (i). If $a,b\in S$ then $a\leq c$, $b\leq d$ and $e\leq a$, $f\leq b$ for some $c,d,e,f\in R$. Now, $(a,c)=(a+e,a+c)\in\varrho$, $(b,d)=(b+f,b+d)\in\varrho$, $(c,d)\in\varrho$ and, finally, $(a,b)\in\varrho$. □

2.10 Proposition. *The semiring S is simple iff the right semimodule R_S is both faithful and simple and the ideal R is both upwards and downwards cofinal in S.*

Proof. The direct implication follows from 2.4, 2.7 and 2.9, while the converse one from 2.8 and 2.9. □

2.11 Lemma. *Assume that $ac=bc$ for all $a,b\in R$, $c\in S$ and put $\mu(c)=ac$. Then μ is a homomorphism of the semiring S onto the semiring R and $\mu|R=\mathrm{id}_R$. Further, the left semimodule ${}_SR$ is not faithful, and if the right semimodule R_S is faithful then $S=R$ and $\mu=\mathrm{id}_R$.*

Proof. Easy to check. □

2.12 Lemma. (i) *If the right semimodule R_S is faithful and $S\neq R$ then $ac\neq bc$ for some $a,b\in R$ and $c\in S$.*
(ii) *If the left semimodule ${}_RS$ is faithful then for all $a,b\in R$, $a\neq b$, there is $c\in S$ with $ac\neq bc$.*
(iii) *If the right semimodule R_S is simple and the left semimodule ${}_RS$ is not faithful then $ac=bc$ for all $a,b\in R$ and $c\in S$.*

Proof. Use 2.11. □

3. Auxiliary results (b)

Let S be a non-trivial semiring. A (left S-)semimodule M ($=_S M$) will be called characteristic if the following conditions are satisfied:

(a) $|M| \geq 3$;

(b) $0_M \in M$ and $S0_M = \{0_M\}$;

(c) $o_M \in M$ and $So_M = \{o_M\}$;

(d) M is faithful;

(e) There is a mapping $\underline{\varepsilon} : N = M \setminus \{o_M\} \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x \in N$, $y, z \in M$, $y \leq x$, $z \not\leq x$.

Throughout this section, let M be a characteristic semimodule and $L = M \setminus \{0_M\}$.

3.1 Lemma. (i) *$\underline{\varepsilon}$ is an injective mapping of N into R.*
(ii) *$|R| \geq 2$ and $R = \{\, a \in S \mid aM = \{0_M, o_M\} \,\}$.*
(iii) *$o_S \in S$, $o_R \in R$ and $o_S = o_R = \underline{\varepsilon}(0_M)$.*
(iv) *$o_S L = \{o_M\}$.*

Proof. (i) We have $a\underline{\varepsilon}(x)y = \underline{\varepsilon}(x)y$ for all $a \in S$, $x \in N$ and $y \in M$ (use (e), (b) and (c)). Since M is faithful, we get $\underline{\varepsilon}(x) \in R$. If $\underline{\varepsilon}(x_1) = \underline{\varepsilon}(x_2)$ then $0_M = \underline{\varepsilon}(x_1)x_1 = \underline{\varepsilon}(x_2)x_1$ and $x_1 \leq x_2$. By symmetry, we obtain $x_1 = x_2$.
(ii) Since $|N| \geq 2$ due to (a), we get $|R| \geq 2$ due to (i). If $a \in R$ and $x \in M$ are such that $ax \in N$ then $ax = \underline{\varepsilon}(ax)ax = 0_M$. Thus $aM = \{0_M, o_M\}$. Conversely, if $a \in S$ is such that $aM = \{0_M, o_M\}$ then $bax = ax$ for all $b \in S$ and $x \in M$. Since M is faithful, we get $ba = a$ and $a \in R$.
(iii) $(\underline{\varepsilon}(0_M) + a)x = \underline{\varepsilon}(0_M)x + ax = o_M + ax = o_M = \underline{\varepsilon}(0_M)x$ for every $x \in L$. Of course, $(\underline{\varepsilon}(0_M) + a)0_M = 0_M = \underline{\varepsilon}(0_M)0_M$. Thus $\underline{\varepsilon}(0_M) + a = \underline{\varepsilon}(0_M)$ and $\underline{\varepsilon}(0_M) = o_S$.
(iv) If $x \in L$ then $o_S x = \underline{\varepsilon}(0_M)x = o_M$. □

3.2 Lemma. *The following conditions are equivalent for $a \in S$:*

(i) $a = 0_S \in S$.

(ii) $a = 0_R \in R$.

(iii) $a + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ *for every* $x \in M \setminus \{o_M\}$.

(iv) $aN = \{0_M\}$.

Moreover, if $o_N \in N$ then these conditions are equivalent to:

(v) $ao_N = 0_M$.

Proof. (i) implies (iii) and (ii) implies (iii) trivially, and, evidently, (iv) is equivalent to (v). Now, we show that (iii) implies (iv). Indeed, $0_M = \underline{\varepsilon}(x)x = (\underline{\varepsilon}(x) + a)x = \underline{\varepsilon}(x)x + ax = ax$ for every $x \in N$. On the other hand, if (iv) is true then $a \in R$ by 3.1(ii) and $(a + b)x = ax + bx = 0_M + bx = bx$ for all $b \in S$ and $x \in N$. Then $(a+b)y = by$ for every $y \in M$, $a+b = b$ and $a = 0_S = 0_R$. □

3.3 Lemma. *Let $0_S \in S$. Then:*
(i) $0_S = 0_R \in R$.
(ii) $N + N = N$.
(iii) $0_S = \underline{\varepsilon}(w)$ *for* $w \in N$ *iff* w *is the greatest element of* N.

Proof. (i) See 3.2.
(ii) If $x, y \in N$ then $0_S(x+y) = 0_Sx+0_Sy = 0_M$ by 3.2(iv). Hence $x+y \neq o_M$.
(iii) If $0_S = \underline{\varepsilon}(w)$ then $\underline{\varepsilon}(w)x = 0_Sx = 0_M$ and $x \leq w$ for every $x \in N$. Conversely, if w is the greatest element of N then $\underline{\varepsilon}(w)N = \{0_M\}$ and $\underline{\varepsilon}(w) = 0_S$ by 3.2. □

3.4 Lemma. *The following conditions are equivalent:*

(i) $0_S \in S$ *and* $0_S \in \underline{\varepsilon}(N)$.

(ii) $0_R \in R$ *and* $0_R \in \underline{\varepsilon}(N)$.

(iii) *The set N has the greatest element (if w is that element then $\underline{\varepsilon}(w) = 0_S = 0_R$).*

Proof. See 3.2 and 3.3. □

3.5 Lemma. (i) *If $a \in S$ and $x \in M$ then $ax = 0_M$ iff $x \in N$ and $a \leq \underline{\varepsilon}(x)$.*
(ii) *If $x, y \in N$ then $x \geq y$ iff $\underline{\varepsilon}(x) \leq \underline{\varepsilon}(y)$.*

Proof. It is easy. □

3.6 Lemma. *The semimodule M is simple.*

Proof. Let α be a congruence of M. If $(0_M, o_M) \in \alpha$ then $\alpha = M \times M$. If $(x, o_M) \in \alpha$ for some $x \in N$ then $(0_M, o_M) = (\underline{\varepsilon}(x)x, \underline{\varepsilon}(x)o_M) \in \alpha$. If $x, y \in N$ are such that $x \not\leq y$ and $(x, y) \in \alpha$ then $(0_M, o_M) = (\underline{\varepsilon}(y)y, \underline{\varepsilon}(y)x) \in \alpha$. □

3.7 Lemma. *The right semimodule R_S is faithful.*

Proof. If $a, b \in S$, $a \neq b$, then $ax \neq bx$ for at least one $x \in M$ and we can assume that $bx \not\leq ax$. Then $ax \in N$, $c = \underline{\varepsilon}(ax) \in R$ and $cax = 0_M \neq o_M = cbx$. Thus $ca \neq cb$. □

3.8 Lemma. *Let $a \in R$ and $A = \{ x \in M \mid ax = 0_M \}$. Then:*
(i) *$0_M \in A$, $o_M \notin A$ and A is a subsemilattice of $M(+)$.*
(ii) *If $v \in M$ then $a = \underline{\varepsilon}(v)$ iff $v = o_A \in A$.*

Proof. (i) This is obvious.
(ii) If $a = \underline{\varepsilon}(v)$ then $v \in A$ and, if $y \not\leq v$, then $ay = o_M$ implies $y \notin A$. Thus $v = o_A \in A$. Conversely, if $v = o_A \in A$ then $ax = \underline{\varepsilon}(v)x$ for every $x \in M$ (use 3.1(ii)). Since ${}_SM$ is faithful, we get $a = \underline{\varepsilon}(v)$. □

3.9 Lemma. *Assume that every infinite strictly increasing sequence of elements from M (or N, $K = N \setminus \{0_M\}$) is upwards cofinal in N (K). Put $R_1 = R \setminus \{0_S\}$ ($R_1 = R$ if $0_S \notin S$). Then $R_1 \subseteq \underline{\varepsilon}(N) \subseteq R$.*

Proof. First, observe that if $x_1 < x_2 < x_3 < \dots$ is a sequence of elements from M then $x_i \in K$ for every $i \geq 2$. Next, if $a \in R_1$ then $A = \{ x \in M \mid ax = 0_M \} \subsetneqq N$ by 3.2(iv). Consequently, using our assumption, we get $o_A \in A$ and $a = \underline{\varepsilon}(o_A)$ by 3.8. □

3.10 Lemma. *If every infinite strictly decreasing sequence of elements from R is downwards cofinal in R_1 then every infinite strictly increasing sequence of elements from M is upwards cofinal in $N \setminus \{o_N\}$.*

Proof. If $x_1 < x_2 < x_3 < \dots$ is a sequence of elements from M then $x_i \in N$ for every $i \geq 1$ and $\underline{\varepsilon}(x_1) > \underline{\varepsilon}(x_2) > \underline{\varepsilon}(x_3) > \dots$ by 3.5(ii) and 3.1(i). If, moreover, $x \in N \setminus \{o_N\}$ then $\underline{\varepsilon}(x) \neq 0_S$, and hence $\underline{\varepsilon}(x_j) \leq \underline{\varepsilon}(x)$ and $x \leq x_j$ for some $j \geq 1$. □

3.11 Lemma. *If every infinite strictly increasing sequence of elements from R is upwards cofinal in $R_2 = R \setminus \{o_S\}$ then every infinite strictly decreasing sequence of elements from M is downwards cofinal in $L = M \setminus \{0_M\}$.*

Proof. If $x_1 > x_2 > x_3 > \dots$ is a sequence of elements from M then $x_i \in K = M \setminus \{0_M, o_M\}$ for every $i \geq 2$ and $\underline{\varepsilon}(x_2) < \underline{\varepsilon}(x_3) < \dots$ by 3.5(ii) and 3.1(i). If, moreover, $x \in L$ then $\underline{\varepsilon}(x) \neq o_S$, and hence $\underline{\varepsilon}(x) \leq \underline{\varepsilon}(x_j)$ and $x_j \leq x$ for some $j \geq 1$. □

3.12 Lemma. *Assume that $R_1 \subseteq \underline{\varepsilon}(N)$. Then:*
(i) *If every infinite strictly decreasing sequence of elements from M is downwards cofinal in K then every infinite strictly increasing sequence of elements from R is upwards cofinal in R_2.*
(ii) *If every infinite strictly increasing sequence of elements from M is upwards cofinal in K then every infinite strictly decreasing sequence of elements from R is downwards cofinal in R_1.*

Proof. (i) Let $a_1 < a_2 < a_3 < \dots$ be a sequence of elements from R. Then $a_i \in R_3 = R \setminus \{0_S, o_S\}$ for every $i \geq 2$ and $a_i = \underline{\varepsilon}(x_i)$, where $x_i \in K$. We have $x_2 > x_3 > x_4 > \dots$ and if $a \in R_3$ then $a = \underline{\varepsilon}(x)$, $x \in K$, $x \geq x_j$ and $a \leq a_j$ for some $j \geq 2$.
(ii) Let $a_1 > a_2 > a_3 > \dots$ be a sequence of elements from R. Then $a_i \in R_1$ for every $i \geq 1$ and $a_i = \underline{\varepsilon}(x_i)$, $x_i \in N$, $x_1 < x_2 < x_3 < \dots$. If $a \in R_1$ then $a = \underline{\varepsilon}(x)$, $x \in N$, $x \leq x_j$ and $a_j \leq a$ for some $j \geq 1$. □

For $x \in M$, define a relation ν_x on S by $(a, b) \in \nu_x$ iff $acx = bcx$ for every $c \in S$.

3.13 Lemma. (i) *ν_x is a congruence of the semiring S.*
(ii) *$\nu_x \cap (R \times R)$ is a congruence of the right semimodule R_S.*
(iii) *$R \times R \subseteq \nu_x$ iff $Sx \subseteq \{0_M, o_M\}$.*
(iv) *If $Sx \subseteq \{0_M, o_M\}$ then $\nu_x = S \times S$.*
(v) *$\nu_x = S \times S$ iff $R \times R \subseteq \nu_x$.*
(vi) *If $x \in K$ then $\nu_x = S \times S$ iff $Sx = \{0_M, o_M\}$.*
(vii) *$0_M, o_M \in P(M) = \{\, x \in M \mid Sx \subseteq \{0_M, o_M\} \,\}$ and $P(M)$ is a subsemimodule of M.*

Proof. (i) and (ii). Easy to check.
(iii) If $R \times R \subseteq \nu_x$ then $acx = o_S cx = \underline{\varepsilon}(0_M)cx$ for all $a \in R$ and $c \in S$. If $cx \neq 0_M$ then $acx = o_M$. If, moreover, $cx \in N$ then $o_M = \underline{\varepsilon}(cx)cx = 0_M$, a contradiction. Thus $cx = o_M$ and we see that $Sx \subseteq \{0_M, o_M\}$.
(iv) and (v). Use (iii).
(vi) If $x \in K$ then $\underline{\varepsilon}(x)x = 0_M$ and $o_S x = \underline{\varepsilon}(0_M)x = o_M$.
(vii) Easy to check. □

3.14 Remark. Assume that the right semimodule R_S is simple (cf. 2.10). If $x \in M$ is such that $Sx \not\subseteq \{0_M, o_M\}$ then $R \times R \not\subseteq \nu_x$, and hence $x \in K$ and $\nu_x \cap (R \times R) = \mathrm{id}_R$. Now, if $a, b \in R$ are such that $a < b$ then $acx = 0_M$ and $bcx = o_M$ for some $c \in S$ (see 3.1(ii)). Consequently, $cx \in K$, $ac \leq \underline{\varepsilon}(x)$ (see 3.5(i)) and $\underline{\varepsilon}(x) = ac + \underline{\varepsilon}(x) < bc + \underline{\varepsilon}(x) \in R$.

3.15 Remark. Assume that the left R-semimodule ${}_R S$ is not faithful. Then there are $a, b \in R$ such that $a < b$ and $ac = bc$ for every $c \in S$.
(i) Let $a = \underline{\varepsilon}(x)$ and $b = \underline{\varepsilon}(y)$, $x, y \in N$. Then $y < x$ and $\underline{\varepsilon}(x)cz = acz = bcz = \underline{\varepsilon}(y)cz$ for every $z \in M$. In particular, if $cz \leq x$ then $cz \leq y$. Since $y < x$, it follows that $x \notin SM$. In fact, $x \neq c_1x_1 + \dots + c_nx_n$ for all $n \geq 1$, $c_i \in S$, $x_i \in M$, and the same is true for x' such that $y < x' \leq x$.
(ii) Let $a = 0_S$ and $b = \underline{\varepsilon}(y)$, $y \in N$. Then $y \in K$ and $0_S cz = \underline{\varepsilon}(y)cz$ for every $z \in M$. In particular, if $cz \in N$ then $cz \leq y$. Consequently, for all $n \geq 1$, $c_i \in S$ and $x_i \in M$, either $x = c_1x_1 + \dots + c_nx_n \leq y$ or $x = o_M$.
(iii) If the semiring S is simple then $ce = de$ for all $c, d, e \in S$. Consequently, $|S| = 2$ and $S = R$.

Consider the following two conditions:

(f1) For all $a, b \in R$, $a \neq b$ ($a < b$) and $x \in K$ there is at least one $c \in S$ with $acx \neq bcx$ (then $acx = 0_M$ and $bcx = o_M$).

(f2) $K \subseteq Sx$ for every $x \in K$.

3.16 Lemma. (f2) *implies* (f1).

Proof. Since ${}_SM$ is faithful, there is $y \in M$ such that $ay \neq by$. Clearly, $y \in K$, and if (f2) is true then $y = cx$, $c \in S$. Thus $acx \neq bcx$. □

3.17 Lemma. *Assume that the right semimodule R_S is simple. Then* (f1) *is equivalent to $Sx \cap K \neq \emptyset$ for every $x \in K$.*

Proof. If (f1) is true and $x \in K$ then $\nu_x \cap (R \times R) = \mathrm{id}_R$, and hence $Sx \not\subseteq \{0_M, o_M\}$ by 3.13(iii). Conversely, if $Sx \cap K \neq \emptyset$ then $R \times R \not\subseteq \nu_x$ and, R_S being simple, we get $\nu_x \cap (R \times R) = \mathrm{id}_R$. □

3.18 Lemma. *Let the set K contain the smallest element, say w, and let $Sw \not\subseteq \{0_M, o_M, w\}$. If $c \in S$ is such that $cw \notin \{0_M, o_M, w\}$ then $\underline{\varepsilon}(w)c = o_S$, $\underline{\varepsilon}(w) \neq o_S \neq c$ and $c \notin R$.*

Proof. Since $cw \in K$, we have $c \notin R$, and hence $c \neq o_S$ (use 3.1(ii)). Since $w \neq 0_M$, we have $\underline{\varepsilon}(w) \neq o_S$. Now, if $y \in L = M \setminus \{0_M\}$ then $w \leq y$, and hence $o_M = \underline{\varepsilon}(w)cw \leq \underline{\varepsilon}(w)cy$ (we have $w < cw$). Consequently, $\underline{\varepsilon}(w)cx = o_S x$ for every $x \in M$ and we get $\underline{\varepsilon}(w)c = o_S$. □

3.19 Lemma. *Let $|R| \geq 3$ and let the set K contain the smallest element. If the condition* (f1) *is satisfied then $o_S = ac$ for some $a \in R_2 = R \setminus \{o_S\}$ and $c \in S \setminus R$.*

Proof. Let w be the smallest element in K. Since $|R| \geq 3$, there are $a, b \in R$ such that $a \neq b$ and $aw = bw$ (use 3.1(ii)). Now, by (f1), there is $c \in S$ such that $acw \neq bcw$. Then $cw \notin \{0_M, o_M, w\}$ and 3.18 applies. □

3.20 Remark. If $|R| = 2$ and (f1) is true then $|S| = 3$, $1_S \in S$ and $o_S \neq 0_S = ac$ for all $a \in R \setminus \{o_S\} = \{0_S\}$ and $c \in S \setminus R = \{1\}$.

3.21 Lemma. *$o_S \in R_2 + R_2$ in each of the following cases:*

1. *The set K has at least one minimal element but no smallest element.*
2. *The set R_2 has at least one maximal element but no greatest element.*
3. *There are $u, v \in K$ such that, for every $x \in K$, either $x \not\leq u$ or $x \not\leq v$.*

Proof. (i) Let $w \in K$ be minimal. Since w is not the smallest element of K, there is $v \in K$ with $w \not\leq v$. If $x \in K$ is such that $x \leq v$ then $x \not\leq w$, and hence $(\underline{\varepsilon}(w) + \underline{\varepsilon}(v))x = o_M$. If $x \not\leq v$ then $(\underline{\varepsilon}(w) + \underline{\varepsilon}(v))x = o_M$ as well. Thus $\underline{\varepsilon}(w) + \underline{\varepsilon}(v) = \underline{\varepsilon}(0_M) = o_S$.
(ii) If a is maximal in R_2 then $b \not\leq a$ for some $b \in R_2$, and so $a + b = o_S$. □

3.22 Remark. Let $b \in S \setminus R$ be such that the set bM is finite. Due to 3.1(ii), we have $bM = \{0_M, w_1, \ldots, w_n, o_M\}$, where $n \geq 1$ and $w_i \in K$. Put $w = w_1 + \cdots + w_n$. There are $u_1, \ldots, u_n \in K$ such that $w_i = bu_i$, Then $w = bu$, $u = u_1 + \cdots + u_n$.
(i) If $n = 1$ or $N + N = N$ then $w \in K$.
(ii) Assume that $w \in K$. Then $w = w_j$ for some $1 \leq j \leq n$. If $a = \underline{\varepsilon}(w)$ then $a \in R_2$, $abx = 0_M$ for $x \in M$, $bx \in N$, and $aby = o_M$ for $y \in M$, $by = o_M$.
(iii) Let $w \in K$ and $bN \subseteq N$. Then $bN \leq w$, $abN \leq aw = 0_M$ and $ab = 0_S \in S$ by 3.2(iv). Since $b \notin R$, we have $b \neq 0_S$. On the other hand, $a = 0_S$ iff w is the greatest element of N.
(iv) If $w \in K$, w is not the greatest element of N and $bN \subseteq N$ then $a \neq 0_S \neq b$ and $ab = 0_S$.
(v) Assume that $bL \subseteq L$, $L = M \setminus \{0_M\}$. If $v \in K$ is such that $v < w_i$ for every $i = 1, \ldots, n$ and if $\overline{a} = \underline{\varepsilon}(v)$ then $\overline{a} \neq o_S$ and $\overline{a}bx = o_M$ for every $x \in L$. Thus $\overline{a}b = o_S$. Of course, $b \notin R$, and so $b \neq o_S$ as well.

3.23 Remark. Let $v \in K$, $a = \underline{\varepsilon}(v)$ and $b \in S$. We have $a \in R_2$.
(i) If $bN \leq v$ then $abN = \{0_M\}$, and hence $ab = 0_S \in S$ by 3.2. Conversely, if $ab = 0_S \in S$ then $bN \leq v$.
(ii) Let $ab = 0_S \in S$. If $x, y \in N$ are such that $bx \not\leq by \neq 0_M$ then $b \neq 0_S$, $\underline{\varepsilon}(by)bx = o_M$, $\underline{\varepsilon}(by)b \neq 0_s, o_S$ and $\underline{\varepsilon}(by) \neq o_S$. Of course, $a = \underline{\varepsilon}(v) < \underline{\varepsilon}(by)$, $by < v$ and $\underline{\varepsilon}(by) \in R_3 = R \setminus \{0_S, o_S\}$. If $a \neq 0_S$ (equivalently, $v \neq o_N$) then $(a, \underline{\varepsilon}(by)) \in R_3 \times R_3$ and $(ab, \underline{\varepsilon}(by)b) = (0_S, \underline{\varepsilon}(by)b) \notin R_3 \times R_3$. In particular, the relation $\alpha_3 = (R_3 \times R_3) \cup \mathrm{id}_R$ is not a congruence of the right semimodule R_S. Similarly, $\alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, where $R_1 = R \setminus \{0_S\}$, is not a congruence of R_S.
(iii) Let $ab = 0_S \neq b$ and let $bx \leq by$ for all $x, y \in N$ such that $by \neq 0_M$. Now, $bx_1 = by_1$ whenever $bx_1 \neq 0_M \neq by_1$, $x_1, y_1 \in N$. Since $ab = 0_S \neq b$, we have $b \notin R$, and hence there is $w \in K$ such that $bM = \{0_M, w, o_M\}$, $bN = \{0_M, w\}$ and $\underline{\varepsilon}(w)b = 0_S$ (cf. 3.22(iv)).

If $bw = 0_M$ then $(b + \underline{\varepsilon}(w))z = \underline{\varepsilon}(w)z$ for every $z \in M$, so that $b \leq \underline{\varepsilon}(w)$, $b^2 = 0_S \neq \underline{\varepsilon}(w)$ and $w \neq o_N$. We have $bN \leq v$, and so $w \leq v$, $a = \underline{\varepsilon}(v) \leq \underline{\varepsilon}(w)$.

If $bw \neq 0_M$ then $bw = w$, and so $bz = b^2z$ for every $z \in M$ and we get $b^2 = b$.

3.24 Remark. Assume that the condition (f1) is satisfied.
(i) If $u, v \in N$ are such that $u < v$ then $\underline{\varepsilon}(v) < \underline{\varepsilon}(u)$, and hence for every $x \in K$ there is $c \in S$ with $cx \leq v$ and $cx \not\leq u$. In particular, if $u = 0_M$ and $v \neq o_M$ then $cx \in K$. If v is minimal in K (and $u = 0_M$) then we get $cx = v$.
(ii) If v is minimal in K then $v \in Sx$ for every $x \in K$.

(iii) If $K \subseteq Sv$ for a minimal element $v \in K$ then (f2) is true.
(iv) Let A denote the subsemimodule of ${}_SM$ generated by $0_M, o_M$ and all minimal elements from K. Then $A \subseteq Sx$ for every $x \in K$. Thus (f2) is true, provided that $A = M$.
(v) (cf. 3.19) Assume that the set K contains the smallest element w. If $a, b \in R$ are such that $a < b$ then $acw = 0_M$ and $bcw = o_M$ for some $c \in S$. Since $cw \neq o_M$, we have $c \neq o_S$. Furthermore, $bcw = o_M$ implies $bcL = \{o_M\}$, where $L = M \setminus \{0_M\}$, and hence $bc = o_S$. Of course, $c \notin R$ follows from 3.1(ii). If $|R| \geq 3$ then we can choose $b \neq o_S$.
(vi) Assume that $o_N \in N$. If $a, b \in R$ are such that $a < b$ then $aco_N = 0_M$ and $bco_N = o_M$ for some $c \in S$. Since $co_N \neq 0_M$, we have $c \neq 0_S$. Furthermore, $aco_N = 0_M$ implies $acN = \{0_M\}$ and $ac = 0_S$ (see 3.2(iv)). If $|R| \geq 3$ then we can choose $a \neq 0_S$.

4. Auxiliary results (c)

Let S be a semiring and M ($=_S M$) a characteristic (left S-)semimodule (see the preceding section). Furthermore, let α be a congruence of the semimodule R_S such that $\alpha \neq \mathrm{id}_R$ and $\alpha \neq R \times R$. We put $A = \{\, a \in R \mid (a, o_S) \notin \alpha \,\}$ and $B = R \setminus A$. Thus $B = \{\, b \in R \mid (b, o_S) \in \alpha \,\}$ is just the block of α containing the absorbing element o_S and we have $A \neq \emptyset$. If $0_S \in S$ then $0_S \in A$.

4.1 Lemma. *Assume that the set A has no maximal element. Then:*
(i) *There is an infinite strictly increasing sequence $a_1 < a_2 < a_3 < \dots$ of elements from A.*
(ii) *If $b \in B$ and $i \geq 1$ then $b \not\leq a_i$.*

Proof. If $b \leq a_i$ then $(a_i, o_S) = (a_i + b, a_i + o_S) \in \alpha$, a contradiction. □

4.2 Lemma. *Let $a_0 \in R \setminus \underline{\varepsilon}(N)$. Then:*
(i) $0_M \in C = \{\, x \in M \mid a_0 x = 0_M \,\}$ *and $C(+)$ is a subsemilattice of $M(+)$.*
(ii) *The set C has no maximal element.*
(iii) $a_0 < \underline{\varepsilon}(x)$ *for every $x \in C$.*
(iv) *If $C = N$ then $a_0 = 0_S \in S$.*

Proof. If $v \in C$ is maximal in C then $v = o_C \in C$, $\underline{\varepsilon}(v)x = 0_M = a_0 x$ for $x \leq v$ and $\underline{\varepsilon}(v)y = o_M$ for $y \not\leq v$. But $y \not\leq v$ means $y \notin C$, $a_0 y \neq 0_M$ and $a_0 y = o_M$ by 3.1(ii). Thus $\underline{\varepsilon}(v)z = a_0 z$ for every $z \in M$ and $\underline{\varepsilon}(v) = a_0$, a contradiction By 3.5(i), we get $a_0 < \underline{\varepsilon}(x)$ for every $x \in C$. If $C = N$ then $a_0 N = \{0_M\}$ and $a_0 = 0_S$ by 3.2. □

4.3 Lemma. *Let $a_0 \in R \setminus \underline{\varepsilon}(N)$. Then:*
(ii) *There is an infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from C.*
(ii) *If $u \in M \setminus C$ then $u \not\leq x_i$ for every $i \geq 1$.*

(iii) $\underline{\varepsilon}(x_1) > \underline{\varepsilon}(x_2) > \underline{\varepsilon}(x_3) > \dots$ *is an infinite strictly decreasing sequence of elements from* R *such that* $\underline{\varepsilon}(x_i) > a_0$ *for every* $i \geq 1$.

Proof. Use 4.2 and 3.5(ii). □

4.4 Lemma. *If* a_0 *is maximal in* A *then* $(a_0, a) \notin \alpha$ *for every* $a \in R$ *such that* $a_0 < a$.

Proof. We have $a \notin A$, $(a, o_S) \in \alpha$, and hence $(a_0, a) \notin \alpha$. □

4.5 Lemma. *Let* $a, b \in R$, $c \in S$ *and* $x \in M$ *be such that* $(a, b) \in \alpha$, $a < b$ *and* $acx \neq bcx$. *Then* $x \in K$, $\underline{\varepsilon}(x) < bc + \underline{\varepsilon}(x) = \overline{a} \in R$ *and* $(\underline{\varepsilon}(x), \overline{a}) \in \alpha$.

Proof. Since $acx \neq bcx$, we have $acx = 0_M$ and $bcx = o_M$ by 3.1(ii). Consequently, $x, cx \in K$ and $(ac+\underline{\varepsilon}(x))y \leq (ac+\underline{\varepsilon}(x))x = acx+\underline{\varepsilon}(x)x = 0_M$ for $y \leq x$. On the other hand, if $z \not\leq x$ then $(ac + \underline{\varepsilon}(x))z = o_M$. Then $ac + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ and $ac \leq \underline{\varepsilon}(x)$. Furthermore, $(\underline{\varepsilon}(x), \overline{a}) = (ac+\underline{\varepsilon}(x), bc+\underline{\varepsilon}(x)) \in \alpha$ and $\underline{\varepsilon}(x) \leq \overline{a}$. Since $\overline{a}x = bcx + \underline{\varepsilon}(x)x = o_M \neq 0_M = \underline{\varepsilon}(x)x$, we conclude that $\underline{\varepsilon}(x) < \overline{a}$. □

4.6 Lemma. *Assume that* (f1) *is true. Then for every* $x \in K$ *there is* $\overline{a} \in R$ *with* $\underline{\varepsilon}(x) < \overline{a}$ *and* $(\underline{\varepsilon}(x), \overline{a}) \in \alpha$.

Proof. Since $\alpha \neq \mathrm{id}_R$, there are $a, b \in R$ with $a < b$ and $(a, b) \in \alpha$. By (f1), $acx \neq bcx$ for some $c \in S$. The rest follows from 4.5. □

4.7 Lemma. *Assume that* (f1) *is true. If* a_0 *is maximal in* A *then* $a_0 \notin \underline{\varepsilon}(N)$.

Proof. Since $a_0 \in A$, we have $a_0 \neq o_S = \underline{\varepsilon}(0_M)$. The rest is clear from 4.4 and 4.6. □

4.8 Lemma. *Aassume that* (f1) *is true and* $A \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$. *If* $a \in A \setminus \{0_S\}$ *then:*
(i) *There is an infinite strictly increasing sequence* $a = a_1 < a_2 < a_3 < \dots$ *of elements from* A *such that all these elements belong to the same block of the congruence* α.
(ii) *If* $b \in B$ *then* $b \not\leq a_i$ *for every* $i \geq 1$.
(iii) *There is an infinite strictly decreasing sequence* $x_1 > x_2 > x_3 > \dots$ *of elements from* K *such that* $a_i = \underline{\varepsilon}(x_i)$ *for every* $i \geq 1$.
(iv) *If* $x \in N$ *is such that* $x \geq x_j$ *for some* $j \geq 1$ *(equivalently,* $\underline{\varepsilon}(x) \leq a_j$*) then* $\underline{\varepsilon}(x) \in A$.

Proof. We have $a_1 = a = \underline{\varepsilon}(x_1)$ for some $x_1 \in K$. By 4.6, there is $a_2 \in R$ with $a_1 < a_2$ and $(a_1, a_2) \in \alpha$. Clearly, $a_2 \in A \setminus \{0_S\}$, and so $a_2 = \underline{\varepsilon}(x_2)$ for some $x_2 \in K$. Now, $x_1 > x_2$ and we can proceed in this way on. □

4.9 Lemma. *Assume that* (f1) *is true and* $A \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$. *If the set* A *has at least one maximal element then* $0_S \in S$, $A = \{0_S\}$ *and* $B = R \setminus \{0_S\}$.

Proof. If a_0 is maximal in A then $a_0 \notin \underline{\varepsilon}(N)$ by 4.7, and hence $a_0 = 0_S \in S$. Since 0_S is maximal in A, we have $A = \{0_S\}$ and $B = R \setminus \{0_S\}$. □

5. Auxiliary results (d)

In this section, let S be a semiring such that $R = \underline{R}(S) \neq \emptyset$.

5.1 Lemma. Assume that $0_S \in S$ and the right semimodule R_S is faithful. Then $0_S = 0_R \in R$ and the following conditions are equivalent:

(i) $\alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, where $R_1 = R \setminus \{0_S\}$, is a congruence of R_S.

(ii) $ab \neq 0_S$ for all $a \in R_1$ and $b \in S_1 = S \setminus \{0_S\}$.

(iii) $cd \neq 0_S$ for all $c, d \in S_1$.

(iv) Either $|S| = 1$ or S_1 is a subsemiring of S.

Proof. Proceeding similarly as in the proof of 2.2(i), we show that $0_S = 0_R \in R$. Now, if α_1 is a congruence of R_S and if $ab = 0_S$ for some $a \in R_1$ and $b \in S_1$ then $(0_S, a'b) = (ab, a'b) \in \alpha_1$ for every $a' \in R_1$ and it follows that $a'b = 0_S$. Thus $R_1 b = \{0_S\}$. Since $0_S b \leq ab = 0_S$, we have $Rb = \{0_S\}$. Of course, $R0_S \leq Rb$, and therefore $Rb = \{0_S\} = R0_S$. Since R_S is faithful, we conclude that $b = 0_S$, a contradiction. We have proved that $ab \neq 0_S$. If $cd = 0_S$ for some $c, d \in S_1$ then $ec \neq e0_S = 0_S$ for some $e \in R$ and we have $ec \in R_1$ and $ecd = e0_S = 0_S$. The rest is clear. □

5.2 Lemma. Assume that $o_S \in S$ and the right semimodule R_S is faithful. Then $o_S = o_R \in R$ and the following conditions are equivalent:

(i) $\alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, where $R_2 = R \setminus \{o_S\}$, is a congruence of R_S.

(ii) $a + b \neq o_S \neq ac$ for all $a, b \in R_2$ and $c \in S_2 = S \setminus \{o_S\}$.

(iii) Either $|S| = 1$ or S_2 is a subsemiring of S.

Proof. We can proceed similarly as in the proof of 5.1. □

5.3 Remark. Assume that $o_S \in S$, the semimodule R_S is faithful and $o_S \notin R_2 S_2$. Assume, furthermore, that the set R_2 has no maximal element and that α_2 is not a congruence of R_S. Then $\alpha_2 \neq \mathrm{id}_R$, R is infinite and, by 5.2, $a + b = o_S$ for some $a, b \in R_2$. Moreover, there is an infinite strictly increasing sequence $a = a_1 < a_2 < a_3 < \dots$ of elements from R_2. Since $a + b = o_S$, we have $b \not\leq a_i$ for every $i \geq 1$.

5.4 Remark. Assume that $o_S \in S$ ($0_S \in S$) and $o_S \in R$ ($0_S \in R$) (cf. 2.1, 2.2, 2.3 and 2.4). Put $A = \{\, a \in S \mid o_S a = o_S \,\}$ ($A = \{\, a \in S \mid 0_S a = 0_S \,\}$) and $B = S \setminus A$. Clearly, $o_S \in A$ ($0_S \in A$), A is a subsemiring of S, $A + S = A$ ($B + S = B$) and $SB \subseteq B$. Further, put $\beta = (A \times A) \cup (B \times B)$.
(i) $\beta = \mathrm{id}_S$ iff $|S| = 1, 2$.
(ii) $\beta = S \times S$ iff o_S (0_S) is multiplicatively absorbing (equivalently, $|R| = 1$).
(iii) If $S_2 = S \setminus \{o_S\}$ is a subsemiring of S then β is a congruence of the semiring S.
(iv) If $|S| \geq 3$, $|R| \geq 2$ and S is a simple semiring then S_2 is not a subsemiring of S (cf. 5.2 and 2.10).

5.5 Proposition. *Assume that $|R| \geq 3$ and the right semimodule R_S is both faithful and simple* (cf. 2.10). *Then:*
(i) *If $0_S \in S$ then $0_S \in R$ and $ab = 0_S$ for some $a \in R_1$ and $b \in S_1$.*
(ii) *If $o_S \in S$ then $o_S \in R$ and either $a + b = o_S$ for some $a, b \in R_2$ or $cd = o_S$ for some $c \in R_2$ and $d \in S_2$.*

Proof. Use 5.1 and 5.2. □

6. Main results

Let S be a semiring and M ($=_S M$) be a characteristic (left S-) semimodule (in particular, $|M| \geq 3$ and $|S| \geq |R| \geq 2$). Put $N = M \setminus \{o_M\}$, $L = M \setminus \{0_M\}$, $K = M \setminus \{0_M, o_M\}$, $R_1 = R \setminus \{0_S\}$, $R_2 = R \setminus \{o_S\}$ and $R_3 = R \setminus \{0_S, o_S\}$. Consider the following four conditions:

(g1) Any infinite strictly increasing sequence of elements from K is upwards cofinal in K.

(g2) Any infinite strictly decreasing sequence of elements from K is downwards cofinal in K.

(h1) Any infinite strictly increasing sequence of elements from R_3 is upwards cofinal in R_3.

(h2) Any infinite strictly decreasing sequence of elements from R_3 is downwards cofinal in R_3.

6.1 Lemma. (i) *If (g1) is true then (h2) is true.*
(ii) *If (g1) and (g2) are true then (h1) is true.*
(iii) *If (h1) is true then (g2) is true.*

Proof. See 3.9, 3.10, 3.11 and 3.12. □

6.2 Lemma. (i) *If (g1) is true then $R_1 \subseteq \underline{\varepsilon}(N) \subseteq R$.*
(ii) *If (h2) is true and $o_N \notin N$ then (g1) is true.*

(iii) *If (h2) is true and $o_N \in N$ then every infinite strictly increasing sequence of elements from M is upwards cofinal in $N \setminus \{o_N\}$.*

Proof. See 3.9, 3.10, 3.11 and 3.12. □

6.3 Lemma. *Assume that (g1),(g2) or (h1),(h2) are true. Then:*
(i) *If the set K has no minimal element then the ordered set $L(\leq)$ is a lattice and $R_2 + R_2 = R_2$.*
(ii) *If the set K has a minimal element, but no smallest one, then $|M| \geq 4$ and $R_2 + R_2 \neq R_2$ (i.e., $o_S \in R_2 + R_2$).*
(iii) *If the set K has the smallest element then the set R_2 has the greatest element and $R_2 + R_2 = R_2$.*

Proof. (i) Given $x \in L$, there is an infinite strictly decreasing sequence $x = x_1 > x_2 > x_3 > \dots$ of elements from L. If (h1) is true then (g2) is true by 6.1(iii). If (g2) is true and $y \in L$ then $y \geq x_i$ for some $i \geq 1$. Thus the element $x_i \in L$ is a lower bound of elements x and y. Now, put $A = \{\, z \in L \mid z \leq x, z \leq y \,\}$. The set A is non-empty and if $o_A \in A$ then $o_A = \inf(x, y)$. On the other hand, if $o_A \notin A$ and (g1) is true then $N \subseteq A$, and therefore either $x \leq y$ or $y \leq x$, a contradiction with $o_A \notin A$. Finally, if $o_A \notin A$ and (g1) is not true then (h2) is true, $o_N \in N$ and $N \setminus \{o_N\} \subseteq A$ (use 6.2(ii),(iii)). But then $\{x, y\} \subseteq \{o_N, o_M\}$ and we get a contradiction again.

It remains to show that $R_2 + R_2 = R_2$. For, let $a, b \in R_2$ be such that $a + b = o_S$. If $a = \underline{\varepsilon}(u)$ and $b = \underline{\varepsilon}(v)$ for some $u, v \in N$ then $u, v \in K$ and $z \leq u$, $z \leq v$ for some $z \in K$ (since L is a lattice) and $a = \underline{\varepsilon}(u) \leq \underline{\varepsilon}(z)$, $b = \underline{\varepsilon}(v) \leq \underline{\varepsilon}(z)$. Thus $o_S = a + b \leq \underline{\varepsilon}(z) \in R_2$, a contradiction. Therefore, we can assume that $a \notin \underline{\varepsilon}(N)$.

Since $a+b = o_S$ and $a, b \in R_2$, we see that $a, b \in R_1$. According to 6.2(i), the condition (g1) is not true, and hence (h1), (h2) are true and $o_N \in N$ (see 6.2(ii)). By 3.4, $0_S = \underline{\varepsilon}(o_N)$ and, by 3.8, $o_A \notin A = \{\, x \in M \mid ax = 0_M \,\}$. Using 6.2(iii), we conclude that $A = N \setminus \{o_N\}$. Consequently, $(a+b)x = 0_M + bx = bx$ for every $x \in N \setminus \{o_N\}$. Of course, $(a+b)o_M = o_M = bo_M$ and $(a+b)o_N = o_M + bo_N = o_M$. Since $b \neq 0_S = \underline{\varepsilon}(o_N)$, we have $bN \neq \{0_M\}$, and so $bo_N = o_M$. We have proved that $(a + b)y = by$ for every $y \in M$ and, since ${}_SM$ is faithful, it means that $o_S = a + b = b \neq o_S$, the final contradiction.
(ii) Clearly, $|M| \geq 4$ and we use 3.21.
(iii) Let w be the smallest element of K. If $\underline{\varepsilon}(w)$ is the greatest element of R_2 then $R_2 + R_2 = R_2$ trivially. If not then $a \not\leq \underline{\varepsilon}(w)$ for some $a \in R_2$. Then $a \notin \underline{\varepsilon}(N)$ and, since ${}_SM$ is faithful, there is $v \in M$ with $av \not\leq \underline{\varepsilon}(w)v$. Clearly, $v \in K$, $w \leq v$, $av \neq 0_M$, and hence $av = o_M$ (see 3.1(ii)). Since $\underline{\varepsilon}(w)v \neq av = o_M$, we have $\underline{\varepsilon}(w)v = 0_M$, and so $v \leq w$. Thus $w = v$ and $aw = o_M$. We have $a \neq 0_S$ (otherwise $aw \leq \underline{\varepsilon}(w)v$) and it follows from 6.2(i) that (g1) is not true. Using 6.2(ii), we see that $o_N \in N$ and (h2) is true. Besides, it follows from 3.8 and 6.2(ii) that $a(N \setminus \{o_N\}) = \{0_M\}$. Since $aw = o_M$, we get $w = o_N$ and,

w being the smallest element in K, we get $|M| = 3$, $|R| = 2$, $R_2 = \{0_S\}$ and $a = 0_S = \underline{\varepsilon}(w)$, a contradiction. □

6.4 Remark. Let (h2) be true and let $o_N \in N$. If $a \in R \setminus \underline{\varepsilon}(N)$ then a is the smallest element in R_1.

6.5 Lemma. *Assume that (g1),(g2) or (h1),(h2) are true. Then:*
(i) *If the set R_2 has no maximal element then $R_2 + R_2 = R_2$.*
(ii) *If $v \in K$ is minimal in K then $\underline{\varepsilon}(v)$ is maximal in R_2.*
(iii) *If $a \in R_2$ is maximal in R_2 then $a = \underline{\varepsilon}(w)$, where w is minimal in K.*
(iv) *If $a \in R_2$ is maximal, but not the greatest element of R_2 then $R_2 + R_2 \neq R_2$.*
(v) *If $w \in K$ is the smallest element of K then $\underline{\varepsilon}(w)$ is the greatest element of R_2.*
(vi) *If $a \in R_2$ is the greatest element of R_2 then $R_2 + R_2 = R_2$ and $a = \underline{\varepsilon}(w)$, where w is the smallest element of K.*

Proof. (i) By 6.1(ii), the condition (h1) is true and the equality follows easily.
(ii) Let $a \in R$ be such that $\underline{\varepsilon}(v) \leq a$. If $a = \underline{\varepsilon}(u)$, $u \in N$, then $u \leq v$ and either $u = v$ and $a = \underline{\varepsilon}(v)$ or $u = 0_M$ and $a = o_S$. On the other hand, if $a \notin \underline{\varepsilon}(N)$ then $a \neq 0_S$ (otherwise $a = 0_S = \underline{\varepsilon}(v)$) and it follows from 6.2(i),(ii) and 6.4 that a is the smallest element of R_1. Since $\underline{\varepsilon}(v) < a$, we get $\underline{\varepsilon}(v) = 0_S$, $v = o_N$, $|M| = 3$ and (g1) is true. Then $a \in \underline{\varepsilon}(N)$ by 6.2(i), a contradiction.
(iii) If $a \notin \underline{\varepsilon}(N)$ then either $a = 0_S$, $|R| = 2$, $|N| = 3$, a contradiction, or a is the smallest element of R_1, $|R| = 3$, $|M| \leq 3$, a contradiction again. Thus $a \in \underline{\varepsilon}(N)$.
(iv) This is obvious.
(v) By (ii), $\underline{\varepsilon}(w)$ is maximal in R_2. If $a \not\leq \underline{\varepsilon}(w)$, $a \in R_2$, then $a \notin \underline{\varepsilon}(N)$, a contradiction with 6.2(i) and 6.4.
(vi) Clearly, $R_2 + R_2 = R_2$. If $a \notin \underline{\varepsilon}(N)$ then either $a = 0_S$, $|M| = 3$ and $a = \underline{\varepsilon}(o_N)$, a contradiction, or a is the smallest element of R_1, so that $|R| = 3$, $R = \{0_S, a, o_S\}$, $|M| = 3$, (g1) is true and $a \in \underline{\varepsilon}(N)$, a contradiction again. □

In the reamining part of this section, assume that the condition (f1) (see the definition before 3.16) is satisfied and either the conditions (g1),(g2) are true or the conditions (h1),(h2) are true (then, due to 6.1, the conditions (h1), (h2), (g2) are true anyway).

6.6 Theorem. Assume that $0_s \notin S$ (then $o_N \notin N$ – see 3.4). The semiring S is simple if and only if the following two conditions are satisfied:

1. $S = R + S$.
2. Either $o_S \in R_2S_2$ ($o_S \in S_2S_2$, resp.) or the set K contains at least one minimal element (equivalently, the set R_2 has a maximal element).

Proof. (i) Let S be simple. Then $S = R + S$ follows from 2.7 and the right semimodule R_S is simple and faithful (2.4, 2.7 and/or 3.7). Since $0_S \notin S$, we

have $o_N \notin N$ by 3.4, $|N| \geq 3$, $|R| \geq 3$ and $\alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R \neq \mathrm{id}_R, R \times R$. By 5.2, $o_S \in (R_2 + R_2) \cup R_2S_2$. If K has no minimal element then $o_S \in R_2S_2$ follows from 6.3(i).
(ii) Assume that the conditions (1) and (2) are satisfied. With respect to 2.10, we have to check that the right semimodule R_S is simple. For, let $\alpha \neq \mathrm{id}_R, R \times R$ be a congruence of R_S and $A = \{\, a \in R \,|\, (a, o_S) \notin \alpha \,\}$.

Let $a_0 \in A \setminus \underline{\varepsilon}(N)$. The condition (h2) is true, and hence $a_0 = 0_S \in S$ by 4.3(iii), a contradiction with $0_S \notin S$. It follows that $A \subseteq \underline{\varepsilon}(N)$. By 4.9, the set A has no maximal element.

Let $a, b \in A$. By 4.8(i), there are sequences $a = a_1 < a_2 < a_3 < \dots$ and $b = b_1 < b_2 < b_3 < \dots$ of elements from A such that $(a, a_i) \in \alpha$ and $(b, b_i) \in \alpha$ for every $i \geq 1$. Since (h1) is true, $a \leq b_j$ and $b \leq a_k$ for some $j, k \geq 1$. From this, $(a, b) \in \alpha$ and it means that the set A is contained in a block of α. Furthermore, it follows from 4.8(ii) that $R \setminus A = \{o_S\}$. Consequently, $\alpha = \alpha_2$ and, using (2) and 5.2, we conclude that the set K contains at least one minimal element. We have $R_2 + R_2 = R_2$ (since α_2 is a congruence of R_S), and therefore K has the smallest element by 6.3(ii), a contradiction with 3.19 and 5.2. □

6.7 Theorem. *Assume that $0_S \in \underline{\varepsilon}(N)$ (equivalently, $o_N \in N$). Then the semiring S is simple if and only if the condition* 6.6(2) *is satisfied.*

Proof. (i) Let S be simple. If $|R| \geq 3$ then $\alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$ is not a congruence of R_S and 6.6(2) follows from 5.2 and 6.3(i) (see the proof of 6.6). On the other hand, if $|R| = 2$ then $|M| = 3$, $|S| = 3$, $S = \{0_S, 1_S, o_S\}$ and S is not simple ($\{(0_S, 0_S), (0_S, 1_S), (1_S, 0_S), (1_S, 1_S), (o_S, o_S)\}$ is a congruence of S).
(ii) Let the condition 6.6(2) be satisfied. With respect to 2.10, we have to show that R_S is simple. For, let $\alpha \neq \mathrm{id}_R, R \times R$ be a congruence of R_S and $A = \{\, a \in R \,|\, (a, o_S) \notin \alpha \,\}$.

We have $0_S \in A \cap \underline{\varepsilon}(N)$. If $a_0 \in A \setminus \underline{\varepsilon}(N)$ then $a_0 = 0_S$ follows from 4.3(iii) and we see that $A \subseteq \underline{\varepsilon}(N)$. By 4.7, the set A has no maximal element. By 4.6, $(0_S, \overline{a}) \in \alpha$ for at least one $\overline{a} \in R_1$; of course $\overline{a} \in A$. Proceeding similarly as in the proof of 6.6, we show that the set $A \setminus \{0_S\}$ is contained in a block of α. Consequently, the set A is contained in a block of α, $R \setminus A = \{o_S\}$ (see 4.8(ii)) and $\alpha = \alpha_2$. Now, again, we proceed similarly as in the proof of 6.6 to gain the final contradiction. □

6.8 Theorem. Assume that $0_S \in S \setminus \underline{\varepsilon}(N)$ (equivalently, $0_S \in S$ and $o_N \notin N$). Then the semiring S is simple if and only if the conditions 6.6(2) is satisfied and, moreover, the following condition is satisfied as well:

1. $0_S \in R_1S_1 (0_S \in S_1S_1$, resp.).

Proof. (i) Let S be simple. Since $o_N \notin N$, we have $|N| \geq 3$, $|R| \geq 3$ and $\alpha_1, \alpha_2 \neq \mathrm{id}_R, R \times R$ (see 5.1 and 5.2). Since R_S is simple by 2.10, neither α_1 nor α_2 is a congruence of R_S and it remains to use 5.1, 5.2 and 6.3(i).

(ii) Let the conditions (1) and 6.6(2) be satisfied. In view of 2.10, we have to check that the right S-semimodule R_S is simple. For, let $\alpha \neq \mathrm{id}_R, R \times R$ be a congruence of R_S and $A = \{\, a \in R \,|\, (a, o_S) \notin \alpha\}$. Then $|R| \geq 3$, $0_S \in A$, $o_S \notin A$ and it follows from 4.2 and 4.3 that $A' = A \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$. If $A' = \emptyset$ then $A = \{0_S\}$ and $\alpha = \alpha_1$, a contradiction with (1) and 5.1. It means that $A' \neq \emptyset$. Proceeding similarly as in the proof of 6.6, we find that $A' \times A' \subseteq \alpha$ and $R \setminus A = \{o_S\}$. Thus $A' = R_3$ and $R_3 \times R_3 \subseteq \alpha$.

Assume, for a moment, that $ab = o_S$ for some $a \in R_2$ and $b \in S_2$. If $a = 0_S$ then $Sb = \{o_S\} = So_S$ and $b = o_S$, R_S being faithful, a contradiction. Thus $a \in R_3$ and, since $b \neq o_S$, there is $v \in K$ with $bv \neq o_M$. We have $bv \in N$, $\underline{\varepsilon}(bv)bv = 0_M$ and $\underline{\varepsilon}(bv)b \neq o_S$. If $\underline{\varepsilon}(bv) = o_S$ (i.e., $bv = 0_M$) then $a \leq \underline{\varepsilon}(bv)$ and $o_S = ab \leq \underline{\varepsilon}(bv)b \neq o_S$, a contradiction. Thus $\underline{\varepsilon}(bv) \neq o_S$ and $bv \neq 0_M$. On the other hand, $\underline{\varepsilon}(bv) \neq 0_S$ (since $0_S \notin \underline{\varepsilon}(N)$), and therefore $\underline{\varepsilon}(bv) \in R_3$, $(a, \underline{\varepsilon}(bv)) \in \alpha_3 = (R_3 \times R_3) \cup \mathrm{id}_R$ and, finally, $(ab, \underline{\varepsilon}(bv)b) = (o_S, \underline{\varepsilon}(bv)b) \notin \alpha_3$. We see that α_3 is not a congruence of R_S and, since $\alpha_3 \subseteq \alpha$, we come to the equality $\alpha = \alpha_2$. But this is a contradiction with $ab = o_S$ and 5.2.

We have proved that $o_S \notin R_2S_2$. Now, according to 6.6(2), the set K has at least one minimal element. If $\overline{a}, \overline{b} \in R_2$ are such that $\overline{a} + \overline{b} = o_S$ then $\overline{a}, \overline{b} \in R_3$, $(\overline{a}, \overline{b}) \in \alpha$ and $(\overline{a}, o_S) = (\overline{a} + \overline{a}, \overline{a} + \overline{b}) \in \alpha$, a contradiction. Thus $o_S \notin R_2 + R_2$ and. using 6.3(ii), we conclude that the set K has the smallest element, a contradiction with 3.19. □

6.9 Remark. Assume that $|R| = 2$. Then $|M| = 3$, $2 \leq |S| \leq 3$ and both S_1 and S_2 are subsemirings of S. In fact, if $|S| = 2$ then $S = R$ is simple, but the condition (f1) is not true. If $|S| = 3$ then $S = \{0_S, 1_S, o_S\}$ is not simple.

6.10 Theorem. The semiring S is simple if and only if the following three conditions are satisfied:

1. $S = R + S$.
2. $S_2 = S \setminus \{o_S\}$ is not a subsemiring of S.
3. If $0_S \in S$ then $S_1 = S \setminus \{0_S\}$ is not a subsemiring of S.

Proof. (i) Let S be simple. Then $S = R + S$ follows from 2.7. The right semimodule R_S is simple and faithful. Consequently, $|R| \geq 3$ anyway and $\alpha_2 \neq \mathrm{id}_R, R \times R$ ($\alpha_1 \neq \mathrm{id}_R, R \times R$ provided that $0_S \in S$). By 5.2 and 5.1, neither S_2 nor S_1 is a subsemiring of S.
(ii) Let the conditions (1), (2) and (3) be fulfilled. It follows from (1) and (2) thar either $a + b = o_S$ for some $a, b \in R_2$ or $cd = o_S$ for some $c \in R_2$ and $d \in S_2$. From (1) and (3) follows that if $0_S \in S$ then $ef = 0_S$ for some $e \in R_1$ and $f \in S_1$ (see 5.2 and 5.1). If the set K has no minimal element then $R_2 + R_2 = R_2$ by 6.3(i), and hence $cd = o_S$ and we use either 6.6 or 6.7 or 6.8 to show that S is simple. On the other hand, if the set K has a minimal element then the condition 6.6(2) is satisfied and we use 6.6, 6.7 or 6.8. □

7. Constructions

Let M $(= M(+))$ be a semilattice containing at least three elements and such that $0_M \in M$ and $o_M \in M$. The set $\underline{E}$ $(= \underline{E}_{0,1})$ of all endomorphisms f of M such that $f(0_M) = 0_M$ amd $f(o_M) = o_M$ is a unitary subsemiring of the full endomorphism semiring of M. The set $(\underline{D}_{0,1} =)\underline{D} = \{ f \in \underline{E} \,|\, |f(M)| \leq 2 \}$ $(= \{ f \in \underline{E} \,|\, f(M) = \{0_M, o_M\} \})$ is an ideal of the semiring $\underline{E}$ and there is a one-to-one correspondence between endomorphisms $q \in \underline{D}$ and prime ideals of the semilattice M; namely, $q \leftrightarrow A_q = \{ x \in M \,|\, q(x) = o_M \}$. If $f \in \underline{E}$ then $A_{qf} = \{ x \in M \,|\, f(x) \in A_q \}$ and $fq = q$. Consequently, $\underline{D} = \underline{R}(\underline{E})$ and, in fact, if E is a subsemiring of $\underline{E}$ such that $\underline{D} \subseteq E$ then $\underline{R}(E) = \underline{D}$.

For $x \in N = M \setminus \{o_M\}$ define $q_x \in \underline{D}$ by $q_x(y) = 0_M$ for $y \leq x$ and $q_x(z) = o_M$ for $z \not\leq x$. Then $A_q = \{ z \in M \,|\, z \not\leq x \}$ and the endomorphisms q_x correspond to principal prime ideals of the semilattice M. We put $(\underline{B}_{0,1} =)$ $\underline{B} = \{ q_x \,|\, x \in N \}$ and $(\underline{C}_{0,1} =)$ $\underline{C} = \{ q_{x_1} + \cdots + q_{x_n} \,|\, n \geq 1, x_i \in N \}$. Clearly, $\underline{C}$ is just the subsemiring of $\underline{D}$ generated by the set $\underline{B}$. The following three lemmas are obvious:

7.1 Lemma. *$\underline{C} = \underline{B}$ iff the ordered set $M(\leq)$ is a lattice.* □

7.2 Lemma. *$\underline{D} = \underline{B}$ iff $o_A \in A$ for every (proper) subsemilattice A of M.* □

7.3 Lemma. *Assume that every infinite strictly increasing sequence of element from M is upwards cofinal in N. Then $\underline{D} = \underline{B}$ iff either $o_N \in N$ or $N + N \neq N$.* □

In the remaining part of this section, let E be a subsemiring of $\underline{E}$ such that $\underline{B} \subseteq E$. The following three lemmas are obvious.

7.4 Lemma. (i) $\underline{C} \subseteq E$.
(ii) $\underline{R}(E) = E \cap \underline{D}$.
(iii) $q_{0_M} = o_E \in \underline{R}(E)$. □

7.5 Lemma. (i) *If $0_E \in E$ then $0_E \in \underline{R}(E)$ and $N + N = N$.*
(ii) *If $x \in N$ then $q_x = 0_E$ iff $x = o_N \in N$.*
(iii) *If $N + N = N$ then the mapping ξ, where $\xi(N) = \{0_M\}$ and $\xi(o_M) = o_M$, belongs to $\underline{D}$. If $o_N \in N$ then $\xi = q_{o_N} \in \underline{B}$.*
(iv) *If $N + N = N$ and $\xi \in E$ then $\xi = 0_E$.* □

7.6 Lemma. *Assume that every infinite strictly increasing sequence of elements from M is upwards cofinal in N. Then:*
(i) If $N + N \neq N$ then $o_N \notin N$, $0_E \notin E$ and $\underline{R}(E) = \underline{B} = \underline{D}$.
(ii) If $o_N \in N$ then $o_N \in N$, $0_E = q_{o_N}$ and $\underline{R}(E) = \underline{B} = \underline{D}$.
(iii) If $N + N = N$ and $o_N \notin N$ then $\xi \in \underline{D}$ (see 7.5(iii)) and $0_E \in E$ iff $\xi \in E$ (then $0_E = \xi$).
(iv) If $N + N = N$, $o_N \notin N$ and $\xi \in E$ then $\xi \notin \underline{B}$ and $\underline{R}(E) = \underline{D} = \underline{B} \cup \{\xi\}$.
(v) If $N + N = N$, $o_N \notin N$ and $\xi \notin E$ then $\xi \notin \underline{B} = \underline{R}(E)$ and $\underline{D} = \underline{B} \cup \{\xi\}$. □

The semilattice M becomes a left E-semimodule via the natural action of endomorphisms and we see readily that the conditions 3(a),...,(e) are fulfilled. In our case, $\underline{\varepsilon}(x) = q_x$, $x \in M \setminus \{o_M\}$ and ${}_EM$ is a characteristic semimodule.

In what follows, we restrict ourselves to the case when every infinite strictly increasing (decreasing, resp.) sequence of elements from M is upwards (downwards, resp.) cofinal in $N = M \setminus \{o_M\}$ ($L = M \setminus \{0_M\}$, resp.). This means that the conditions 6(g1) and 6(g2) are fulfilled. We still have to introduce two additional conditions.

(F1) For all $x \in N$ and $y, z \in K = M \setminus \{0_M, o_M\}$, $x < y$, there is at least one $f \in E$ such that $f(z) \leq y$ and $f(z) \not\leq x$.

(F2) For all $x, y \in K$ there is at least one $f \in E$ such that $f(x) = y$.

Notice that (F2) implies (F1) and if (F2) is true then $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M are all (pair-wise different) subsemimodules of ${}_EM$.

7.7 Assume that $N + N \neq N$ (i.e., $o_M \in N + N$). Using (g1), we see that there is no infinite strictly increasing sequence of elements in M. By 7.6, the set N has at least two maximal elements, $0_E \notin E$ and $\underline{R}(E) = \underline{B}$ ($= \underline{D}$). The ordered set $M(\leq)$ is a lattice. Besides, (F1) is equivalent to 3(f1).

7.7.1 Theorem. *Assume that the set L (or K) has at least one minimal element. Then:*
(i) *There is no infinite strictly decreasing sequence of elements in M.*
(ii) *If* (F1) *is true then the semiring E is simple if and only if the set $\underline{B}$ is downwards cofinal in E (i.e., $E \subseteq \underline{E} + \underline{B}$).*

Proof. (i) This follows from (g2).
(ii) Since $N + N \neq N$, we have $|M| \geq 4$. Now, (F1) implies (f1) and it remains to use 6.6. □

7.7.2 Theorem. Assume that the set L has no minimal element and (F1) is true. The following conditions are equivalent:

(i) The semiring E is simple.

(ii) $\underline{B}$ is downwards cofinal in E (i.e., $E \subseteq \underline{E} + \underline{B}$) and there are $w \in K$ and $f \in E$ such that $f(K) \neq \{o_M\}$ and $f(x) \not\leq w$ for every $x \in K$.

(iii) $\underline{B}$ is downwards cofinal in E and there are $f, g \in E$ such that $f(K) \neq \{o_M\} \neq g(K)$ and $fg(K) = \{o_M\}$.

Proof. (i) implies (ii). We use 6.6. By 6.6(1), $E = E + \underline{B}$ and we take into account 6.6(2), where $o_E = q_{0_M}$.
(ii) implies (iii). We have $q_w f = q_{0_M} = o_E$, $q_w \neq o_E \neq f$.
(iii) implies (i). Use 6.6 again. □

7.8 Assume that $o_N \in N$. Then $N + N = N$, there is no infinite strictly increasing sequence of elements in M, $M(\leq)$ is a lattice, $0_E = q_{o_N} \in E$ and $\underline{R}(E) = \underline{B} = \underline{D}$ (see 7.1 and 7.6). Clearly, (F1) is equivalent to 3(f1).

7.8.1 Theorem. *Assume that the set L (or K) has at least one minimal element. Then:*
(i) *There is no infinite strictly decreasing sequence of elements in M.*
(ii) *If (F1) is true then the semiring E is simple.*

Proof. Use 6.7. □

7.8.2 Theorem. *Assume that the set L has no minimal element and (F1) is true. The following conditions are equivalent:*

(i) *The semiring E is simple.*

(ii) *There are $w \in K$ and $f \in E$ such that $f(K) \neq \{o_M\}$ and $f(x) \not\leq w$ for every $x \in K$.*

(iii) *There are $f, g \in E$ such that $f(K) \neq \{o_M\} \neq g(K)$ and $fg(K) = \{o_M\}$.*

Proof. Use 6.7. □

7.9 Assume that $o_N \notin N$, $N + N = N$ (then N has no maximal element) and $\xi \in E$ (by 7.6(iii), we have $\xi \in E$ iff $0_E \in E$, and then $0_E = \xi$). By 7.6(iv), $\underline{R}(E) = \underline{D} = \underline{B} \cup \{\xi\}$ and $\xi = 0_E \notin \underline{B}$. Now, (F1) is equivalent to 3(f1) (use the fact that $o_N \notin N = N + N$).

7.9.1 Theorem. *Assume that the set L (or K) has at least one minimal element. Then:*
(i) *There is no infinite strictly decreasing sequence of elements in M.*
(ii) *If (F1) is true then the semiring E is simple if and only if the following two equivalent conditions are satisfied:*

(ii1) *There are $w \in K$ and $f \in E$ such that $\{0_M\} \neq f(K) \leq w$.*

(ii2) *There are $f, g \in E$ such that $f(K) \neq \{0_M\} \neq g(K)$ and $fg(K) = \{0_M\}$.*

Proof. Use 6.8. □

7.9.2 Theorem. *Assume that the set L has no minimal element and (F1) is true. The following conditions are equivalent:*

(i) *The semiring E is simple.*

(ii) *There are $w_1, w_2 \in K$ and $f_1, f_2 \in E$ such that $f_1(K) \neq \{o_M\}$, $f_2(K) \neq \{0_M\}$, $f_1(K) \not\leq w_1$ and $f_2(K) \leq w_2$ for every $x \in K$.*

(iii) *There are* $f_1, f_2, g_1, g_2 \in E$ *such that* $f_1(K) \neq \{o_M\} \neq g_1(K)$, $f_2(K) \neq \{0_M\} \neq g_2(K)$, $f_1g_1(K) = \{o_M\}$ *and* $f_2g_2(K) = \{0_M\}$

Proof. Use 6.8. □

7.10 Assume that $o_N \notin N$, $N + N = N$ (then N has no maximal element) and $\xi \notin E$. Then $0_E \notin E$ by 7.6(iii) and, by 7.6(v), $\underline{R}(E) = \underline{B}$. Now, (F1) is equivalent to 3(f1).

7.10.1 Theorem. *Assume that the set* L *(or* K*) has at least one minimal element. Then:*
(i) *There is no infinite strictly decreasing sequence of elements in* M.
(ii) *If (F1) is true then the semiring* E *is simple if and only if* $E \subseteq \underline{E} + \underline{B}$.

Proof. Use 6.6. □

7.10.2 Theorem. *Assume that the set* L *has no minimal element and (F1) is true. The following conditions are equivalent:*

(i) *The semiring* E *is simple.*

(ii) $E \subseteq \underline{E} + \underline{B}$ *and there are* $w \in K$ *and* $f \in E$ *such that* $f(K) \neq \{o_M\}$ *and* $f(x) \not\leq w$ *for every* $x \in K$.

(iii) $E \subseteq \underline{E} + \underline{B}$ *and there are* $f, g \in E$ *such that* $f(K) \neq \{o_M\} \neq g(K)$ *and* $fg(K) = \{o_M\}$.

Proof. Use 6.6. □

7.11 Remark. (cf. 3.22) (i) Assume that there is an endomorphism $f \in E$ such that $f(M)$ is finite, $f(N) \subseteq N$ and $|f(M)| \geq 3$.

We have $f(M) = \{0_M, w_1, \ldots, w_n, o_M\}$, $n \geq 1$, $w_i \in K$. Assume that $w = w_1 + \cdots + w_n \in K$ (e.g., $n = 1$ or $N + N = N$) and $w \neq o_N$. Then $q_w f(N) \subseteq q_w(\{0_M, w_1, \ldots, w_n\}) = \{0_M\}$. Consequently, $q_w f = \xi = 0_E \in E$. If $o_N \in N$ then $q_w f = q_{o_N} = 0_E$. Of course, since $w \neq o_N$, we have $q_w \neq 0_E$. Since $f \notin \underline{R}(E)$, we have $f \neq 0_E$ as well.
(ii) Assume that $N + N = N$ and let $u, v \in K$ be such that $u < v$. Put $f(K) = \{u\}$, $g(K) = \{v\}$, $f(0_M) = 0_M = g(0_M)$ and $f(o_M) = o_M = g(o_M)$. Clearly, $f, g \in \underline{E}$, $f(M) = \{0_M, u, o_M\}$, $g(M) = \{0_M, v, o_M\}$, $f(N) = \{0_M, u\}$, $g(N) = \{0_M, v\}$, $q_u f = \xi$, $f \neq \xi \neq q_u$, $q_u g = q_{0_M}$ and $q_u \neq q_{0_M} \neq g$.

References

[1] R. El Bashir, J. Hurt, A. Jančařík and T. Kepka, *Simple commutative semirings,* J. Algebra, 263 (2001), 277-306.

[2] R. El Bashir and T. Kepka, *Congruence-simple semirings,* Semigroup Forum, 75 (2007), 588-608.

[3] V. Flaška, *One very particular example of a congruence-simple semiring,* Europ. J. Comb., 30 (2009), 759-763.

[4] J. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science,* Pitman Monographs, 54, Longman, Harlow, 1992.

[5] V. Hebisch and H. J. Weinert, *Halbringe - Algebraische Theorie und Anwendungen in der Informatik,* Teubner, Stuttgart, 1993.

[6] J. Ježek and T. Kepka, *The semiring of 1-preserving endomorphisms of a semilattice,* Czech. Math. J., 59 (2009), 999-1003.

[7] J. Ježek, T. Kepka and M. Maróti, *The endomorphism semiring of a semilattice,* Semigroup Forum, 78 (2009), 21-26.

[8] A. Kendziorra, J. Zumbrägel, *Finite simple additively idempotent semirings,* J. Algebra, 388 (2013), 43-64.

[9] G. Maze, C. Monico and J. Rosenthal, *Public key cryptography based on semigroup actions,* Adv. Math. Commun., 1 (2007), 489-507.

[10] S. S. Mitchell and P. B. Fenoglio, *Congruence-free commutative semirings,* Semigroup Forum, 37 (1988), 79-91.

[11] C. Monico, *On finite congruence-simple semirings,* J. Algebra, 271 (2004), 846-854.

[12] J. Zumbrägel, *Classification of finite congruence-simple semirings with zero,* J. Algebra Appl., 7 (2008), 363-377.

Accepted: 3.05.2018

ON THE CONFORMAL CURVATURE TENSOR OF ϵ-KENMOTSU MANIFOLDS

Abdul Haseeb[*]
Department of Mathematics
Faculty of Science
Jazan University, Jazan-2097
Kingdom of Saudi Arabia
malikhaseeb80@gmail.com
haseeb@jazanu.edu.sa

Mobin Ahmad
Department of Mathematics
Integral University, Kursi Road
Lucknow-226026, India
mobinahmad@rediffmail.com

Sheeba Rizvi
Department of Mathematics
Integral University, Kursi Road
Lucknow-226026, India
sheeba.rizvi7@gmail.com

Abstract. The conformal curvature tensor under certain curvature conditions has been studied for an ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection. Finally, we give an example of a 3-dimensional ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

Keywords: ϵ-Kenmotsu manifolds, semi-symmetric non-metric connection, η-Einstein manifold, conformal curvature tensor.

1. Introduction

In 1972, K. Kenmotsu [14] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by various authors such as J. B. Jun et al. [13], G. Pathak and U. C. De [17], M. M. Tripathi and N. Nakkar [19], A. Yildiz et al. [23] and many others. In 1993, A. Bejancu and K. L. Duggal [4] introduced the concept of (ϵ)-Sasakian manifolds, which later on showed by X. Xufeng and C. Xiaoli [20] that the manifolds are real hypersurfaces of indefinite Kahlerian manifolds. An (ϵ)-almost paracontact manifolds were introduced by M. M. Tripathi et al. [18], while the concept of (ϵ)-Kenmotsu manifolds was introduced

∗. Corresponding author

by U. C. De and A. Sarkar [8] who showed that the existence of new structure on indefinite metrics influences the curvatures.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [9]. In 1930, E. Bartolotti [3] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [12] introduced semi-symmetric metric connection in a Riemannian manifold and this was studied systematically by K. Yano [21].

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a semi-symmetric connection if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a non-zero 1-form associated with a vector field ξ and is defined by $\eta(X) = g(X,\xi)$.

Further, a semi-symmetric connection is called a semi-symmetric non-metric connection [1], if

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) &= \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \\ &= -\eta(Y)g(X,Z) - \eta(Z)g(X,Y),\end{aligned} \tag{1.2}$$

where $X, Y, Z \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M.

Let M be an n-dimensional ϵ-Kenmotsu manifold and ∇ be the Levi-Civita connection on M, the semi-symmetric non-metric connection $\bar{\nabla}$ on M is given by [1]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{1.3}$$

The semi-symmetric non-metric connection have been studied by several authors such as N. S. Agashe and M. R. Chafle [1], L. S. Das et al. [5], U. C. De and S. C. Biswas [6], U. C. De and D. Kamilya [7], S. K. Pandey et al. [16], Ajit Barman [2] and many others. Motivated by the above studies, in this paper we study certain curvature conditions of an ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

The paper is organized as follows : In Section 2, we give a brief introduction of an ϵ-Kenmotsu manifold. In Section 3, we obtain the relation between the curvature tensor of an ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection and the Levi-Civita connection. In Section 4, we study quasi-conformally flat, ξ-conformally flat, pseudoconformally flat and ϕ-conformally flat ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection and it is shown that in each case the manifold is an η-Einstein manifold. Section 5 is devoted to study ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the curvature condition $\bar{S} \cdot \bar{C} = 0$.

2. Preliminaries

An n-dimensional smooth manifold (M, g) is said to be an ϵ-almost contact metric manifold [8], if it admits a $(1,1)$ tensor field ϕ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$g(\xi, \xi) = \epsilon, \tag{2.3}$$

$$\eta(X) = \epsilon g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y) \tag{2.5}$$

for all vector fields X, Y on M, where ϵ is 1 or -1 according as ξ is space like or time like vector field and rank ϕ is $(n-1)$. If

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.6}$$

for every $X, Y \in \chi(M)$, then we say that $M(\phi, \xi, \eta, g, \epsilon)$ is an almost contact metric manifold. Also, we have

$$\phi\xi = 0, \ \eta(\phi X) = 0. \tag{2.7}$$

If an ϵ-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.8}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an ϵ-Kenmotsu manifold [8].
An ϵ-almost contact metric manifold is an ϵ-Kenmotsu manifold, if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.9}$$

Further, in an ϵ-Kenmotsu manifold, the following relations hold [8, 10, 11]:

$$(\nabla_X \eta)Y = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.12}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.13}$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.14}$$

$$S(X, \xi) = -(n - 1)\eta(X), \tag{2.15}$$

$$Q\xi = -\epsilon(n - 1)\xi, \tag{2.16}$$

where $g(QX, Y) = S(X, Y)$.

$$S(\phi X, \phi Y) = S(X, Y) + \epsilon(n - 1)\eta(X)\eta(Y). \tag{2.17}$$

We note that if $\epsilon = 1$ and the structure vector field ξ is space like, then an ϵ-Kenmotsu manifold is usual Kenmotsu manifold.

An ϵ-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form [22]

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.18}$$

where a and b are scalar functions of ϵ.

3. Curvature tensor in an ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection

Let M be an n-dimensional ϵ-Kenmotsu manifold. The curvature tensor $\bar{R}$ with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

By virtue of (1.1) and (3.1), we have

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & + ((\nabla_X\eta)Z)Y - ((\nabla_Y\eta)Z)X,\end{aligned} \tag{3.2}$$

where

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of the connection ∇. Using (2.10) in (3.2) we get

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + g(X, Z)Y - g(Y, Z)X \\ & + (1 + \epsilon)[\eta(Y)X - \eta(X)Y]\eta(Z).\end{aligned} \tag{3.3}$$

Now contracting X in (3.3), we get

$$\bar{S}(Y, Z) = S(Y, Z) + (1 - n)g(Y, Z) + (1 + \epsilon)(n - 1)\eta(Y)\eta(Z), \tag{3.4}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M.
This gives

$$\bar{Q}Y = QY + (1-n)Y + (1+\epsilon)(n-1)\eta(Y)\xi. \tag{3.5}$$

Contracting again Y and Z in (3.4), it follows that

$$\bar{r} = r + n(1-n) + (1+\epsilon)(n-1), \tag{3.6}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.

Lemma 3.1. *Let M be an n-dimensional ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection, then*

$$\bar{R}(X,Y)\xi = 0, \tag{3.7}$$

$$\bar{R}(\xi,X)Y = -(1+\epsilon)[g(X,Y)\xi - \eta(X)\eta(Y)\xi], \tag{3.8}$$

$$\bar{S}(Y,\xi) = 0, \tag{3.9}$$

$$\bar{Q}\xi = 0. \tag{3.10}$$

Proof. By replacing $Z = \xi$ in (3.3) and using (2.4) and (2.11), we get (3.7). (3.8) follows from (3.3) and (2.12). Taking $Z = \xi$ in (3.4) and using (2.2), (2.4) and (2.15), we get (3.9). From (3.5), (3.16) and (2.2), we get (3.10). □

Definition 3.2. *The conformal curvature tensor $\bar{C}$ of type $(1,3)$ in an n-dimensional ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is given by* ([15], [22])

$$\begin{aligned}\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ - g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.11}$$

where $\bar{R}$, $\bar{S}$, $\bar{Q}$ and $\bar{r}$ are the Riemannain curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature with respect to the semi-symmetric non-metric connection, respectively. The Ricci tensor $\bar{S}$ and the Ricci operator $\bar{Q}$ are related by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$.

By using (3.3)-(3.6) in (3.11), we obtain

$$\begin{aligned}\bar{C}(X,Y)Z &= C(X,Y)Z + \frac{2n+1+\epsilon}{n-2}(g(Y,Z)X - g(X,Z)Y) \\ &- \frac{1+\epsilon}{n-2}(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ &- \frac{(1+\epsilon)(n-1)}{n-2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\end{aligned}$$

is the conformal curvature tensor with respect to the Levi-Civita connection ∇.

The equation (3.12) is the relation between the conformal curvature tensors with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇.

4. Flatness conditions in ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection

Definition 4.1. *An ϵ-Kenmotsu manifold M is said to be:*
(i) quasi-conformally flat with respect to the semi-symmetric non-metric connection, if

$$g(\bar{C}(X,Y)Z,\phi W) = 0, \qquad X,Y,Z,W \in \chi(M); \tag{4.1}$$

(ii) ξ-conformally flat with respect to the semi-symmetric non-metric connection, if

$$\bar{C}(X,Y)\xi = 0, \qquad X,Y \in \chi(M); \tag{4.2}$$

(iii) pseudoconformally flat with respect to the semi-symmetric non-metric connection, if

$$g(\bar{C}(\phi X,Y)Z,\phi W) = 0, \qquad X,Y,Z,W \in \chi(M); \quad \textit{and} \tag{4.3}$$

(iv) ϕ-conformally flat with respect to the semi-symmetric non-metric connection, if

$$\phi^2\bar{C}(\phi X,\phi Y)\phi Z = 0, \qquad X,Y,Z \in \chi(M). \tag{4.4}$$

Firstly, we consider quasi-conformally flat ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From the equations (3.11)

and (4.1), we have

$$\begin{aligned} g(\bar{R}(X,Y)Z,\phi W) &= \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,\phi W) - \bar{S}(X,Z)g(Y,\phi W) \\ &+ g(Y,Z)g(\bar{Q}X,\phi W) - g(X,Z)g(\bar{Q}Y,\phi W)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)] \end{aligned} \tag{4.5}$$

which by taking $Y = Z = \xi$ and using (2.4), (3.7), (3.9) and (3.10) reduces to

$$\bar{S}(X,\phi W) = \frac{\bar{r}}{(n-1)}g(X,\phi W). \tag{4.6}$$

Replacing W by ϕW and using (2.1), (4.6) yields

$$\bar{S}(X,W) = \frac{\bar{r}}{(n-1)}g(X,W) - \frac{\epsilon\bar{r}}{(n-1)}\eta(X)\eta(W). \tag{4.7}$$

In view of (3.4) and (3.6), (4.7) takes the form

$$S(X,W) = (\epsilon + \frac{r}{n-1})g(X,W) - (n + \frac{\epsilon r}{n-1})\eta(X)\eta(W). \tag{4.8}$$

Thus we have the following theorem:

Theorem 4.2. *An n-dimensional quasi-conformally flat ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Secondly, we consider ξ-conformally flat ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.2), we can write

$$\begin{aligned} g[\bar{R}(X,Y)\xi &- \frac{1}{(n-2)}(\bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y + g(Y,\xi)\bar{Q}X - g(X,\xi)\bar{Q}Y) \\ &+ \frac{\bar{r}}{(n-1)(n-2)}(g(Y,\xi)X - g(X,\xi)Y), W] = 0 \end{aligned} \tag{4.9}$$

which by using (2.4), (3.7) and (3.9) reduces to

$$\begin{aligned} &\eta(X)\bar{S}(Y,W) - \eta(Y)\bar{S}(X,W) \\ &+ \frac{\bar{r}}{(n-1)}(\eta(Y)g(X,W) - \eta(X)g(Y,W)) = 0. \end{aligned} \tag{4.10}$$

Putting $Y = \xi$ and then using (2.2), (2.4) and (3.9) in (4.10), we get

$$\bar{S}(X,W) = \frac{\bar{r}}{(n-1)}g(X,W) - \frac{\epsilon\bar{r}}{(n-1)}\eta(X)\eta(W). \tag{4.11}$$

In view of (3.4) and (3.6), (4.11) takes the form

$$S(X,W) = (\epsilon + \frac{r}{n-1})g(X,W) - (n + \frac{\epsilon r}{n-1})\eta(X)\eta(W). \tag{4.12}$$

Thus we have the following theorem:

Theorem 4.3. *An n-dimensional ξ-conformally flat ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Next, taking $Z = \xi$ in (3.12) and using (2.2) and (2.4), we have

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{2n\epsilon}{n-2}(\eta(Y)X - \eta(X)Y). \tag{4.13}$$

Since $\eta(X)Y - \eta(Y)X = R(X,Y)\xi \neq 0$, in an ϵ-Kenmotsu manifold, in general, then we have the following theorem:

Theorem 4.4. *In an ϵ-Kenmotsu manifold ξ-conformally flatness with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are not equivalent.*

Thirdly, we consider pseudoconformally flat ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.3), we can write

$$\begin{aligned} g(\bar{R}(\phi X,Y)Z,\phi W) &= \frac{1}{(n-2)}[\bar{S}(Y,Z)g(\phi X,\phi W) \\ &- \bar{S}(\phi X,Z)g(Y,\phi W) + \bar{S}(\phi X,\phi W)g(Y,Z) - \bar{S}(Y,\phi W)g(\phi X,Z)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(\phi X,\phi W) - g(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{4.14}$$

In view of (3.3), (3.4) and (3.6), (4.14) takes the form

$$\begin{aligned} g(R(\phi X,Y)Z,\phi W) &= \frac{1}{(n-2)}[S(Y,Z)g(\phi X,\phi W) \\ &- 2(n-1)g(Y,Z)g(\phi X,\phi W) + (1+\epsilon)\eta(Y)\eta(Z)g(\phi X,\phi W) \\ &- S(\phi X,Z)g(Y,\phi W) + 2(n-1)g(\phi X,Z)g(Y,\phi W) \\ &+ g(Y,Z)S(\phi X,\phi W) - g(\phi X,Z)S(Y,\phi W)] \\ &- \frac{r-(n-1)(2n-3-\epsilon)}{(n-1)(n-2)}[g(Y,Z)g(\phi X,\phi W) - g(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{4.15}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.15) and sum up with respect to i, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,Y)Z,\phi e_i) &= \frac{1}{(n-2)}\sum_{i=1}^{n-1}[S(Y,Z)g(\phi e_i,\phi e_i) \\ &- 2(n-1)g(Y,Z)g(\phi e_i,\phi e_i) + (1+\epsilon)\eta(Y)\eta(Z)g(\phi e_i,\phi e_i) \\ &- S(\phi e_i,Z)g(Y,\phi e_i) + 2(n-1)g(\phi e_i,Z)g(Y,\phi e_i) \\ &+ g(Y,Z)S(\phi e_i,\phi e_i) - g(\phi e_i,Z)S(Y,\phi e_i)] \\ &- \frac{r-(n-1)(2n-3-\epsilon)}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g(Y,Z)g(\phi e_i,\phi e_i) - g(\phi e_i,Z)g(Y,\phi e_i)]. \end{aligned} \tag{4.16}$$

It is easy to verify that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y,Z) + g(Y,Z) - \epsilon\eta(Y)\eta(Z), \tag{4.17}$$

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{4.18}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{4.19}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y,Z) + \epsilon(n-1)\eta(Y)\eta(Z), \tag{4.20}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{4.21}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{4.22}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y,Z) - \eta(Y)\eta(Z), \tag{4.23}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.24}$$

By virtue of (4.17), (4.19), (4.20), (4.22) and (4.23), (4.16) yields

$$S(Y,Z) = [\frac{r}{n-1} - n + 3 - \epsilon(n-2)]g(Y,Z) - [\frac{r}{n-1} - n + 2(1+\epsilon)]\eta(Y)\eta(Z). \tag{4.25}$$

Thus we have the following theorem:

Theorem 4.5. *An n-dimensional pseudoconformally flat ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Lastly, we consider ϕ-conformally flat ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (4.4), we have

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{4.26}$$

Using (3.11), (4.26) can be written as

$$\begin{aligned} g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{(n-2)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) \\ &-\bar{S}(\phi X, \phi Z)g(\phi Y, \phi W) + \bar{S}(\phi X, \phi W)g(\phi Y, \phi Z) - \bar{S}(\phi Y, \phi W)g(\phi X, \phi Z)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{4.27}$$

In view of (3.3), (3.4) and (3.6), (4.27) takes the form

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{(n-2)}[S(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-S(\phi X,\phi Z)g(\phi Y,\phi W)+S(\phi X,\phi W)g(\phi Y,\phi Z)-S(\phi Y,\phi W)g(\phi X,\phi Z)] \\ &-\frac{r+(n-1)(1+\epsilon)}{(n-1)(n-2)}[g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{4.28}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.28) and sum up with respect to i, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \frac{1}{(n-2)}\sum_{i=1}^{n-1}[S(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &-S(\phi e_i,\phi Z)g(\phi Y,\phi e_i)+S(\phi e_i,\phi e_i)g(\phi Y,\phi Z)-S(\phi Y,\phi e_i)g(\phi e_i,\phi Z)]. \\ &-\frac{r+(n-1)(1+\epsilon)}{(n-2)(n-2)}\sum_{i=1}^{n-1}[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)-g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \end{aligned} \tag{4.29}$$

By virtue of (4.18), (4.19), (4.21), (4.22) and (4.24), (4.29) can be written as

$$\begin{aligned} S(\phi Y,\phi Z)+g(\phi Y,\phi Z) &= \frac{1}{(n-2)}[(n-3)S(\phi Y,\phi Z) \\ &+(r+n-1)g(\phi Y,\phi Z)]-\frac{r+(n-1)(1+\epsilon)}{(n-1)}g(\phi Y,\phi Z) \end{aligned} \tag{4.30}$$

from which it follows that

$$S(\phi Y,\phi Z) = (n-2)[\frac{r+n-1}{n-2}-\frac{r+(n-1)(1+\epsilon)}{n-1}-1]g(\phi Y,\phi Z). \tag{4.31}$$

Using (2.5) and (2.17), (4.31) yields

$$\begin{aligned} S(Y,Z) &= [\frac{r}{n-1}-(n-2)\epsilon-n+3]g(Y,Z) \\ &-\epsilon[\frac{r}{n-1}-(n-2)\epsilon+2]\eta(Y)\eta(Z). \end{aligned} \tag{4.32}$$

Thus we have the following theorem:

Theorem 4.6. *An n-dimensional ϕ-conformally flat ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

5. ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition $\bar{S}\cdot\bar{C}=0$

In this section we investigate ϵ-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition $\bar{S}\cdot\bar{C}=0$, where $\bar{S}$ is

the Ricci tensor with respect to the semi-symmetric non-metric connection of type $(0,2)$. Let the manifold satisfies the condition

$$(\bar{S}(X,Y)\cdot\bar{C})(U,V)W=0, \tag{5.1}$$

where $X,Y,U,V,W\in\chi(M)$. The above equation implies

$$\begin{aligned}&(X_{\wedge\bar{S}}Y)\bar{C}(U,V)W+\bar{C}((X_{\wedge\bar{S}}Y)U,V)W\\&+\bar{C}(U,(X_{\wedge\bar{S}}Y)V)W+\bar{C}(U,V)(X_{\wedge\bar{S}}Y)W=0,\end{aligned} \tag{5.2}$$

where the endomorphism $X\wedge_{\bar{S}}Y$ is defined by

$$(X\wedge_{\bar{S}}Y)Z=\bar{S}(Y,Z)X-\bar{S}(X,Z)Y. \tag{5.3}$$

Therefore in view of (5.3), (5.2) takes the form

$$\begin{aligned}&\bar{S}(Y,\bar{C}(U,V)W)X-\bar{S}(X,\bar{C}(U,V)W)Y+\bar{S}(Y,U)\bar{C}(X,V)W\\&-\bar{S}(X,U)\bar{C}(Y,V)W+\bar{S}(Y,V)\bar{C}(U,X)W-\bar{S}(X,V)\bar{C}(U,Y)W\\&+\bar{S}(Y,W)\bar{C}(U,V)X-\bar{S}(X,W)\bar{C}(U,V)Y=0.\end{aligned} \tag{5.4}$$

Taking $X=\xi$ in (5.4) and using (3.9), we obtain

$$\begin{aligned}&\bar{S}(Y,\bar{C}(U,V)W)\xi+\bar{S}(Y,U)\bar{C}(\xi,V)W\\&+\bar{S}(Y,V)\bar{C}(U,\xi)W+\bar{S}(Y,W)\bar{C}(U,V)\xi=0\end{aligned} \tag{5.5}$$

which by taking $U=W=\xi$ and then using (3.9) reduces to

$$\bar{S}(Y,\bar{C}(\xi,V)\xi)\xi+\bar{S}(Y,V)\bar{C}(\xi,\xi)\xi=0. \tag{5.6}$$

Using (3.11) in (5.6), we obtain

$$\bar{S}^2(Y,V)=\frac{\bar{r}}{n-1}\bar{S}(Y,V). \tag{5.7}$$

In view of (3.4)-(3.6) and (3.9), (5.7) yields

$$\begin{aligned}S^2(Y,V)=&(\frac{r}{n-1}+n-1+\epsilon)S(X,Y)-(r+n\epsilon-\epsilon)g(Y,V)\\&+(1+\epsilon)(r+n^2-n)\eta(Y)\eta(V).\end{aligned} \tag{5.8}$$

Thus we can state the following theorem:

Theorem 5.1. *If an n-dimensional ϵ-Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ satisfying the condition $\bar{S}\cdot\bar{C}=0$. Then*

$$\begin{aligned}S^2(Y,V)=&(\frac{r}{n-1}+n-1+\epsilon)S(X,Y)-(r+n\epsilon-\epsilon)g(Y,V)\\&+(1+\epsilon)(r+n^2-n)\eta(Y)\eta(V).\end{aligned}$$

Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are standard coordinates of R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = \epsilon z \frac{\partial}{\partial x}, \quad e_2 = \epsilon z \frac{\partial}{\partial y}, \quad e_3 = -\epsilon z \frac{\partial}{\partial z} = \xi$$

which are linearly independent at each point of M and hence form a basis of M. Define an indefinite metric g on M^3 as

$$g(e_1, e_1) = \quad g(e_2, e_2) = \quad g(e_3, e_3) = \epsilon,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined as $\eta(X) = \epsilon g(X, e_3)$ for all $X \in \chi(M)$, and let ϕ be the $(1, 1)$ tensor field on M defined as

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0,$$

$$g(X, \xi) = \epsilon\eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Then for $\xi = e_3$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the indefinite metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \epsilon e_2, \quad [e_1, e_3] = \epsilon e_1.$$

The Riemannian connection ∇ of the indefinite metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -\epsilon e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \epsilon e_1, \quad \nabla_{e_2} e_1 = 0, \tag{5.9}$$

$$\nabla_{e_2} e_2 = -\epsilon e_3, \quad \nabla_{e_2} e_3 = \epsilon e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Thus from (5.9), it follows that the manifold satisfies $\nabla_X \xi = \epsilon(X - \eta(X)\xi)$ for $\xi = e_3$. Hence the manifold is an indefinite Kenmotsu manifold.

From the equation (5.9) and the expression of curvature tensor $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, it can be easily verified that

$$R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_3)e_1 = e_3, \quad R(e_2, e_3)e_1 = 0, \tag{5.10}$$

$$R(e_1,e_2)e_2=-e_1, \qquad R(e_1,e_3)e_2=0, \qquad R(e_2,e_3)e_2=e_3,$$

$$R(e_1,e_2)e_3=0, \qquad R(e_1,e_3)e_3=-e_1, \quad R(e_2,e_3)e_3=-e_2.$$

By using (1.3) in (5.9), we obtain

$$\bar{\nabla}_{e_1}e_1=-\epsilon e_3, \quad \bar{\nabla}_{e_2}e_1=0, \quad \bar{\nabla}_{e_3}e_1=0, \quad \bar{\nabla}_{e_1}e_2=0, \quad \bar{\nabla}_{e_2}e_2=-\epsilon e_3, \tag{5.11}$$

$$\bar{\nabla}_{e_3}e_2=0, \quad \bar{\nabla}_{e_1}e_3=(1+\epsilon)e_1, \quad \bar{\nabla}_{e_2}e_3=(1+\epsilon)e_2, \quad \bar{\nabla}_{e_3}e_3=e_3.$$

From (3.3) and (5.10), we can easily obtain the following components of the curvature tensor with respect to the semi-symmetric non-metric connection as

$$\bar{R}(e_1,e_2)e_1=(1+\epsilon)e_2, \quad \bar{R}(e_1,e_3)e_1=(1+\epsilon)e_3, \quad \bar{R}(e_2,e_3)e_1=0, \tag{5.12}$$

$$\bar{R}(e_1,e_2)e_2=-(1+\epsilon)e_1, \quad \bar{R}(e_1,e_3)e_2=0, \quad \bar{R}(e_2,e_3)e_2=(1+\epsilon)e_3,$$

$$\bar{R}(e_1,e_2)e_3=0, \quad \bar{R}(e_1,e_3)e_3=0, \quad \bar{R}(e_2,e_3)e_3=0.$$

By using the above expressions, we get the Ricci tensors and the scalar curvatures as follows:

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=-2\epsilon,$$

$$\bar{S}(e_1,e_1)=\bar{S}(e_2,e_2)=-(1+\epsilon), \quad \bar{S}(e_3,e_3)=-2(1+\epsilon),$$

$$r=-6\epsilon, \quad \bar{r}=-4(1+\epsilon).$$

By using the equations (1.1) and (5.11), the torsion tensors are given by

$$\overline{T}(e_1,e_1)=0, \quad \overline{T}(e_1,e_3)=e_1, \quad \overline{T}(e_1,e_2)=0,$$

$$\overline{T}(e_2,e_2)=0, \quad \overline{T}(e_2,e_3)=e_2, \quad \overline{T}(e_3,e_3)=0.$$

In view of (1.2), we find

$$(\bar{\nabla}_{e_1}g)(e_1,e_3)=-\epsilon, \quad (\bar{\nabla}_{e_2}g)(e_2,e_3)=-\epsilon, \quad (\bar{\nabla}_{e_3}g)(e_1,e_2)=0.$$

Hence M is a 3-dimensional ϵ-Kenmotsu manifold with respect to the semi symmetric non-metric connection.

Acknowledgement

The authors are thankful to the referee for his/her valuable suggestions for the improvement of the paper.

References

[1] N.S. Agashe, M.R. Chafle, *A semi-symmetric non-metric connection on a Riemannian manifold,* Indian J. Pure Appl. Math., 23 (1992), 399-409.

[2] A. Barman, *On LP-Sasakian manifolds admitting a semi-symmetric non-metric connection*, Kyungpook Math. J., 58 (2018), 105-116.

[3] E. Bartolotti, *Sulla geometria della varieta a connection affine*, Ann. di Mat., 4(8) (1930), 53-101.

[4] A. Bejancu, K.L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Int. J. Math. Math. Sci., 16 (1993), 545-556.

[5] L.S. Das, M. Ahmad, A. Haseeb, *On semi-invariant submanifolds of a nearly Sasakian manifold admitting a semi-symmetric non-metric connection*, Journal of Applied Analysis, 17(1) (2011), 119-130.

[6] U.C. De, S.C. Biswas, *On a type of semi-symmetric non-metric connection on a Riemannian manifold,* Istanbul Univ. Fen Fak. Mat. Dergisi, 55-56(1996-1997), 237-243.

[7] U.C. De, D. Kamilya, *On a type of semi-symmetric non-metric connection on a Riemannian manifold,* Istanbul Univ. Fen Fak. Mat. Dergisi, 53(1994), 37-41.

[8] U.C. De, A. Sarkar, *On ϵ-Kenmotsu manifolds*, Hardonic J., 32 (2009), 231-242.

[9] A. Friedmann, J.A. Schouten, *Uber die Geometric der halbsymmetrischen Ubertragung*, Math. Z., 21 (1924), 211-223.

[10] A. Haseeb, *Some results on projective curvature tensor in an ϵ-Kenmotsu manifold*, Palestine J. Math., 6(Special Issue:II)(2017), 196-203.

[11] A. Haseeb, M.A. Khan, M.D. Siddiqi, *Some more results on an ϵ-Kenmotsu manifold with a semi-symmetric metric connection*, Acta Math. Univ. Comenianae, 85(1) (2016), 9-20.

[12] H.A. Hayden, *Subspaces of a space with torsion,* Proc. London Math. Soc., 34 (1932), 27-50.

[13] J.B. Jun, U.C. De, G. Pathak, *On Kenmotsu manifolds*, J. Korean Math. Soc., 42(3) (2005), 435-445.

[14] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (1972), 93-103.

[15] C. Özgür, *ϕ-conformally flat Lorentzian para-Sasakian manifolds*, Radovi Matematički, 12 (2003), 99-106.

[16] S.K. Pandey, G. Pandey, K. Tiwari, R.N. Singh, *On a semi-symmetric non-metric connection in an indefinite para-Sasakian manifold*, Journal of Mathematics and Computer Science, 12(2014), 159-172.

[17] G. Pathak, U.C. De, *On a semi-symmetric connection in a Kenmotsu manifold,* Bull. Calcutta Math. Soc., 94(4)(2002), 319-324.

[18] M.M. Tripathi, E. Kilic, S.Y. Perktas, S. Keles, *Indefinite almost paracontact metric manifolds,* Int. J. Math. Math. Sci., (2010), Article ID: 846195, 19 pages.

[19] M.M. Tripathi, N. Nakkar, *On a semi-symmetric non-metric connection in a Kenmotsu manifold,* Bull. Calcutta Math. Soc., 16(2001), 323-330.

[20] X. Xufeng, C. Xiaoli, *Two theorems on ϵ-Sasakian manifolds*, Int. J. Math. Math. Sci., 21(2)(1998), 249-254.

[21] K. Yano, *On semi-symmetric metric connections*, Revue Roumaine de Math. Pures Appl., 15(1970), 1579-1586.

[22] K. Yano, M. Kon, *Structures on Manifolds*, Series in Pure Math., Vol. 3, World Sci., 1984.

[23] A. Yildiz, U.C. De, B.E. Acet, *On Kenmotsu manifolds satisfying certain curvature conditions*, SUT J. Math., 45(2009), 89-101.

Accepted: 3.05.2018

EXPONENTIAL STABILITY OF NONLINEAR SYSTEMS VIA ALTERNATE CONTROL

Xingkai Hu*
Faculty of Civil Engineering and Mechanics
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China
and
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P. R. China
huxingkai84@163.com

Linru Nie
Faculty of Civil Engineering and Mechanics
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China
and
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P. R. China

Abstract. In this paper, the exponential stability of nonlinear systems via alternate control is considered. Our result avoids solving linear matrix inequalities. A numerical example is given to show effectiveness of the result.

Keywords: exponential stability, nonlinear systems, alternate control.

1. Introduction

Throughout this paper, let $\|x\|$ be the Euclidean norm of the vector x. $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and A^T are the largest, the smallest eigenvalue and the transpose of a real symmetric matrix A, respectively. The symmetrical positive definite matrix A is represented by $A > 0$. I represents the proper dimension identity matrix. $f\left(x\left(t_0^-\right)\right)$ is defined by $f\left(x\left(t_0^-\right)\right) = \lim_{t\to t_0^-} f(x(t))$.

*. Corresponding author

Recently, Feng et al.[2] considered nonlinear systems via alternate control as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + B_1 x(t), & mT \le t < mT + \tau, \\ \dot{x}(t) = Ax(t) + f(x(t)) + B_2 x(t), & mT + \tau \le t < (m+1)T, \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ denotes the state vector, $f : R^n \to R^n$ is said to be a continuous nonlinear function if $f(0) = 0$, there exists a constant $l \ge 0$ such that $\|f(x)\| \le l\|x\|$, $A, B_1, B_2 \in R^{n\times n}$ are constant matrices, $T > 0$ denotes the control period, $\tau \in (0, T)$ is a constant.

Meanwhile, Feng et al.[2] presented three conditions to guarantee system (1.1) to be exponentially stable. Two conditions are to find $g_1 > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $g_2 \in R$ and $0 < P \in R^{n\times n}$ satisfying

$$PA + A^T P + PB_1 + B_1^T P + \epsilon_1 P^2 + \epsilon_1^{-1} L + g_1 P \le 0 \tag{1.2}$$

and

$$PA + A^T P + PB_2 + B_2^T P + \epsilon_2 P^2 + \epsilon_2^{-1} L - g_2 P \le 0. \tag{1.3}$$

Inequalities (1.2) and (1.3) are equivalent to the linear matrix inequalities as follows:

$$\begin{bmatrix} PA + A^T P + PB_1 + B_1^T P + \epsilon_1^{-1} L + g_1 P & -P \\ -P & -\epsilon_1^{-1} I \end{bmatrix} \le 0$$

and

$$\begin{bmatrix} PA + A^T P + PB_2 + B_2^T P + \epsilon_2^{-1} L - g_2 P & -P \\ -P & -\epsilon_2^{-1} I \end{bmatrix} \le 0.$$

Although linear matrix inequalities can be solved in polynomial-time, the computation amount of solving linear matrix inequalities is not very small [1]. Therefore, our purpose is to find $h_1 < 0$, $h_2 \in R$ and $0 < P \in R^{n\times n}$ avoiding solving linear matrix inequalities such that the system (1.1) is exponentially stable. For more information on this topic and its applications have been presented in the literature, for instance, see [5-7, 9, 10].

2. Main result

We need two lemmas which play a major role in the proof of theorem.

Lemma 2.1 ([4]). *Let $x, y \in R^n$. Then*

$$\left|x^T y\right| \le \|x\| \|y\|.$$

Lemma 2.2 ([4]). *Let $A \in R^{n\times n}$ be a symmetric matrix. Then for all $x \in R^n$,*

$$\lambda_{\min}(A)\, x^T x \le x^T A x \le \lambda_{\max}(A)\, x^T x.$$

Theorem 2.1. *Let $0 < P \in R^{n\times n}$ such that the following two conditions are satisfied:*

(1) $h_1 < 0$,

(2) $h_1\tau + h_2(T-\tau) < 0$,

where $\beta_1 = \lambda_{\max}(P^{-1}(PA + A^TP + PB_1 + B_1^TP))$, $\beta_2 = \lambda_{\max}(P)$, $\beta_3 = \lambda_{\min}(P)$, $\beta_4 = \lambda_{\max}(P^{-1}(PA + A^TP + PB_2 + B_2^TP))$, $h_1 = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, $h_2 = \beta_4 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$. Then, system (1.1) is exponentially stable at origin.

Proof. Let us construct the following Lyapunov function:

$$V(x(t)) = x^T(t)\,Px(t).$$

Let $t \in [mT, mT+\tau)$, by Lemma 2.1 and 2.2, we have

$$\begin{aligned}
D^+(V(x(t))) &= 2x^T(t)\,P(Ax(t) + f(x(t)) + B_1x(t)) \\
&= x^T(t)\left(PA + A^TP + PB_1 + B_1^TP\right)x(t) + 2x^T(t)\,Pf(x(t)) \\
&\le \beta_1 x^T(t)\,Px(t) + 2\sqrt{x^T(t)\,Px(t)\,f^T(x(t))\,Pf(x(t))} \\
&\le \beta_1 x^T(t)\,Px(t) + 2\sqrt{x^T(t)\,Px(t)\,\beta_2 f^T(x(t))\,f(x(t))} \\
&\le \beta_1 x^T(t)\,Px(t) + 2\sqrt{x^T(t)\,Px(t)\,\beta_2 l^2 x^T(t)\,x(t)} \\
&\le \beta_1 x^T(t)\,Px(t) + 2l\sqrt{x^T(t)\,Px(t)\,\frac{\beta_2}{\beta_3}x^T(t)\,Px(t)} \\
&= h_1 V(x(t)),
\end{aligned}$$

which means

$$V(x(t)) \le V\left(x(mT)^-\right)e^{h_1(t-mT)}. \tag{2.1}$$

Similarly, let $t \in [mT+\tau, (m+1)T)$, we have

$$\begin{aligned}
D^+(V(x(t))) &= 2x^T(t)\,P(Ax(t) + f(x(t)) + B_2x(t)) \\
&= x^T(t)\left(PA + A^TP + PB_2 + B_2^TP\right)x(t) + 2x^T(t)\,Pf(x(t)) \\
&\le \beta_4 x^T(t)\,Px(t) + 2\sqrt{x^T(t)\,Px(t)\,f^T(x(t))\,Pf(x(t))} \\
&\le h_2 V(x(t)),
\end{aligned}$$

which infers

$$V(x(t)) \le V\left(x(mT+\tau)^-\right)e^{h_2(t-mT-\tau)}. \tag{2.2}$$

When $m = 0$, let $t \in [0, \tau)$, from (2.1), we have

$$V(x(t)) \le V(x(0))\,e^{h_1 t},$$

so

$$V\left(x\left(\tau^-\right)\right) \le V(x(0))\,e^{h_1\tau}. \tag{2.3}$$

Let $t \in [\tau, T)$, by (2.2) and (2.3), we have

$$\begin{aligned} V(x(t)) &\leq V(x(\tau^-))\, e^{h_2(t-\tau)} \\ &\leq V(x(0))\, e^{h_1\tau + h_2(t-\tau)}, \end{aligned}$$

so

$$V\left(x\left(T^-\right)\right) \leq V(x(0))\, e^{h_1\tau + h_2(T-\tau)}. \tag{2.4}$$

When $m = 1$, let $t \in [T, T+\tau)$, by (2.1) and (2.4), we have

$$\begin{aligned} V(x(t)) &\leq V(x(T^-))\, e^{h_1(t-T)} \\ &\leq V(x(0))\, e^{h_1\tau + h_2(T-\tau) + h_1(t-T)}, \end{aligned}$$

so

$$V\left(x\left((T+\tau)^-\right)\right) \leq V(x(0))\, e^{2h_1\tau + h_2(T-\tau)}. \tag{2.5}$$

Let $t \in [T+\tau, 2T)$, by (2.2) and (2.5), we have

$$\begin{aligned} V(x(t)) &\leq V\left(x\left((T+\tau)^-\right)\right) e^{h_2(t-T-\tau)} \\ &\leq V(x(0))\, e^{2h_1\tau + h_2(T-\tau) + h_2(t-T-\tau)}. \end{aligned}$$

By induction, when $m = k,\ k = 0, 1, \cdots$, let $t \in [kT, kT+\tau)$, we have

$$V(x(t)) \leq V(x(0))\, e^{kh_1\tau + kh_2(T-\tau) + h_1(t-kT)}, \tag{2.6}$$

so

$$V\left(x\left((kT+\tau)^-\right)\right) \leq V(x(0))\, e^{(k+1)h_1\tau + kh_2(T-\tau)}. \tag{2.7}$$

Let $t \in [kT+\tau, (k+1)T)$, by (2.2) and (2.7), we have

$$\begin{aligned} V(x(t)) &\leq V\left(x\left((kT+\tau)^-\right)\right) e^{h_2(t-kT-\tau)} \\ &\leq V(x(0))\, e^{(k+1)h_1\tau + kh_2(T-\tau) + h_2(t-kT-\tau)}. \end{aligned} \tag{2.8}$$

By (2.6), we have

$$\begin{aligned} V(x(t)) &\leq V(x(0))\, e^{kh_1\tau + kh_2(T-\tau)} \\ &< V(x(0))\, e^{\frac{t-\tau}{T}(h_1\tau + h_2(T-\tau))} \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}, \end{aligned} \tag{2.9}$$

where $t \in [kT, kT+\tau)$.

By (2.8), we know

Case 1. When $h_2 > 0$, we have

$$\begin{aligned} V(x(t)) &< V(x(0))\, e^{(k+1)h_1\tau + (k+1)h_2(T-\tau)} \\ &< V(x(0))\, e^{\frac{t}{T}(h_1\tau + h_2(T-\tau))} \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \end{aligned} \tag{2.10}$$

Case 2. When $h_2 \leq 0$, we have

$$\begin{aligned} V(x(t)) &\leq V(x(0))\, e^{(k+1)h_1\tau + kh_2(T-\tau)} \\ &< V(x(0))\, e^{kh_1\tau + kh_2(T-\tau)} \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \end{aligned} \tag{2.11}$$

So, for any $h_2 \in R$, by (2.10) and (2.11), we have

$$V(x(t)) < V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}, \tag{2.12}$$

where $t \in [kT + \tau, (k+1)T)$.
For all $t > 0$, we can conclude from (2.9) and (2.12) that

$$V(x(t)) < V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \tag{2.13}$$

By Lemma 2.2 and (2.13), we have

$$\begin{aligned} \lambda_{\min}(P)\, \|x(t)\|^2 &\leq V(x(t)) \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))} \\ &\leq \|x(0)\|^2 \lambda_{\max}(P)\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \end{aligned}$$

That is

$$\|x(t)\| < \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\, \|x(0)\|\, e^{\frac{t-T}{2T}(h_1\tau + h_2(T-\tau))}.$$

This completes the proof. □

Remark 2.1. [2, Theorem 1] and [3, Theorem 1] need to solve the linear matrix inequalities, however, Theorem 2.1 avoids solving them.

3. A numerical example

In this section, we study the control of Chua's oscillator via employing above theoretical result.

Example 3.1. The Chua's system [8] is given as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\eta x_2. \end{cases} \tag{3.1}$$

where α and η are two parameters,

$$g(x_1) = bx_1 + 0.5(a-b)(|x_1 + 1| - |x_1 - 1|),$$

where a and b are two given constants satisfying $a < b < 0$.

In order to use the above result, we rewrite system (3.1) as follows:

$$\dot{x}(t) = Ax + f(x),$$

where

$$A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}, f(x) = \begin{bmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.$$

By calculations, we have

$$\begin{aligned} ||f(x)||^2 &= 0.25\alpha^2(a-b)^2[(x_1+1)^2 \\ &+ (x_1-1)^2 - 2|(x_1+1)(x_1-1)|] \\ &= 0.5\alpha^2(a-b)^2(x_1^2+1-|x_1^2-1|) \\ &= \begin{cases} \alpha^2(a-b)^2, & x_1^2 > 1 \\ \alpha^2(a-b)^2x^2, & x_1^2 \le 1 \end{cases} \\ &\le \alpha^2(a-b)^2x_1^2 \\ &\le \alpha^2(a-b)^2(x_1^2+x_2^2+x_3^2). \end{aligned}$$

So, we choose $l^2 = \alpha^2(a-b)^2$.

In the initial condition $x(0) = (22, -2, -15)^T$, the Chua's system exhibits chaotic phenomenon when

$$\alpha = 9.2156,\ \eta = 15.9946,\ a = -1.24905,\ b = -0.75735,$$

as shown in Figure 1.

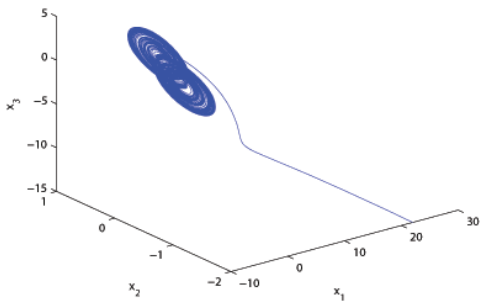


Figure 1: The chaotic phenomenon of (3.1) .

At the same time, for convenience of calculations, we choose $P = I$, $B_1 = diag(-12, -13, -14)$, $B_2 = diag(-12, -13, -12)$, $T = 1$ and $\tau = 0.5$. Small calculations show that $\beta_1 = -9.9296$, $\beta_2 = \beta_3 = 1$, $\beta_4 = -8.3946$, $l = 4.5313$, $h_1 = -0.8670$, $h_2 = 0.6680$ and $h_1\tau + h_2(T-\tau) = -0.0995 < 0$. Thus, in the

initial condition $x(0) = (22, -2, -15)^T$, system (3.1) is exponentially stable by Theorem 2.1, as shown in Figure 2.

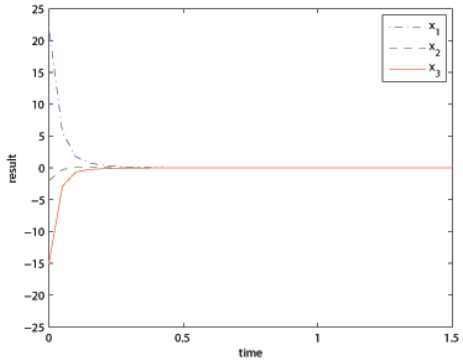


Figure 2: Time response curves of (3.1) via alternate control.

References

[1] S. Boyd, L. EI Ghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequalities in system and control theory*, SIAM, Philadelphia, 1994.

[2] Y. Feng, C. Li, T. Huang, W. Zhao, *Alternate control systems*, Adv. Differ. Equ., 305 (2014).

[3] X. Hu, H. Wu, Y. Feng, J. Xiong, *Alternate-continuous-control systems with double-impulse*, Adv. Differ. Equ., 298 (2017).

[4] R.A. Horn, C.R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[5] T. Huang, C. Li, W. Yu, G. Chen, *Synchronization of delayed chaotic systems with parameter mismatches by using intermittent linear state feedback*, Nonlinearity, 22 (2009), 569-584.

[6] X. Li, S. Song, *Stabilization of delay systems: delay-dependent impulsive control*, IEEE Trans. Autom. Control, 62 (2017), 406-411.

[7] X. Li, J. Wu, *Stability of nonlinear differential systems with state-dependent delayed impulses*, Automatica, 64 (2016), 63-69.

[8] L.P. Shilnikov, *Chua's circuit: rigorous results and furture problems*, Int. J. Bifur. Chaos Appl. Sci. Engrg., 4 (1994), 489-519.

[9] X. Yang, Z. Yang, X. Nie, *Exponential synchronization of discontinuous chaotic systems via delayed impulsive control and its application to secure*

communication, Commun. Nonlinear Sci. Numer. Simul., 19 (2014), 1529-1543.

[10] L. Zou, Y. Peng, Y. Feng, Z. Tu, *Impulsive control of nonlinear systems with impulse time window and bounded gain error*, Nonlinear Anal. Model. Control, 23 (2018), 40-49.

Accepted: 9.05.2018

ROUGH APPROXIMATIONS IN KU-ALGEBRAS

Moin Akhtar Ansari*

Ali N.A. Koam
Department of Mathematics
College of Science, Post Box 2097
New Campus, Jazan University
Jazan, KSA
maansari@jazanu.edu.sa
akoum@jazanu.edu.sa

Abstract. In this paper, the concept of roughness in KU-algebras is introduced. We study the lower and upper approximations of KU-subalgebras and KU-ideals and proved that the lower/upper approximation of KU-subalgebra (resp., KU-ideals) is a KU-subalgebra (resp., KU-ideals). A connection between rough sets and KU-Algebras with their weak and strong ideals have also been taken under consideration and some related results have been shown.

Keywords: KU-subalgebras, KU-ideals, lower approximations, upper approximations, definable.

1. Introduction

The notion of rough sets was introduced by Pawlak in his paper [21]. The theory of rough sets has emerged as another major powerfull mathematical approach for managing and handling different types of uncertainty in information systems that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning, pattern recognition and knowledge discovery. In connection with algebraic structures, Biswas and Nanda [4] gave the notion of rough subgroups, and Kuroki [13] introduced rough ideals in semigroups. Xiao and Zhang [27] introduced rough prime ideals and rough fuzzy prime ideals in semigroups. Ameri et al. [3] applied rough set theory to hyper BCK-algebra. Dudek et al. [7] and Ma [15] applied rough set theory to BCI-algebras. Jun et al. considered roughness in BCK-algebra [10], lattice implication algebras [11] and BCC-algebra [12]. Mao and Zhou [16] studied the rough set theory in Pseudo-BCK-algebra. Torkzadeh and Ghorbani [26] studied rough filters in B-Algebras.

∗. Corresponsing author

Prabpayak and Leerawat [22] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties in [22, 23].

The concept of fuzzy sets was introduced by Zadeh [29]. There are several authors who considered KU-algebras in terms of different types of fuzzy sets, for instance, Mostafa et al. [17] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals, also see [18, 19]. Akram et al. [1] studied interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras. In [2, 28], the author applied the concept of cubic sets to KU-algebras. Davvaz et al. [6] introduced neutrosophic ideals of neutrosophic KU-algebras. Gulistan et al. [8, 9] studied the generalized version fuzzy KU-ideals of KU-algebras. Muhiuddin [20] studied bipolar fuzzy KU-subalgebras/ideals of KU-algebras. Senapati et al. [24, 25] introduced T-fuzzy KU-subalgebras and intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra. As a work in computer science Chen et al.[5] worked on data mining framework based on rough set theory to improve location selection decisions as a case study of a restaurant chain whereas Karimi [14] studied rough sets and Gray sets.

2. Preliminaries

In this section we shall define some basic concepts including KU-algebras, KU-subalgebras, KU-ideals and shall provide examples based on them.

Definition 1 ([23])**.** *By a KU-algebra we mean an algebra* $(X, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X$,

$(ku1) : (x * y) * [(y * z) * (x * z)] = 0,$
$(ku2) : x * 0 = 0,$
$(ku3) : 0 * x = x,$
$(ku4) : x * y = 0 = y * x$ *implies* $x = y.$

In what follows, let $(X, *, 0)$ denote a KU-algebra unless otherwise specified. For brevity we also call X a KU-algebra. In X we can define a binary relation $\leq$ by : $x \leq y$ if and only if $y * x = 0$.

Definition 2 ([23])**.** $(X, *, 0)$ *is a KU-algebra if and only if it satisfies:*

$(ku5) : (y * z) * (x * z) \leq (x * y),$
$(ku6) : 0 \leq x,$
$(ku7) : x \leq y,\ y \leq x$ *implies* $x = y,$
$(ku8) : x \leq y$ *if and only if* $y * x = 0.$

Definition 3. *In a KU-algebra, the following identities are true [17]:*

(1) $z * z = 0,$
(2) $z * (x * z) = 0,$
(3) $x \leq y$ *imply* $y * z \leq x * z,$
(4) $z * (y * x) = y * (z * x)$, *for all* $x, y, z \in X,$

(5) $y * [(y * x) * x] = 0$.

Example 1 ([17])**.** Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by the following table

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to see that X is KU-algebra.

Definition 4 ([23])**.** *A subset S of KU-algebra X is called a KU-subalgebra of X if $x * y \in S$, whenever $x, y \in S$.*

Definition 5 ([23])**.** *A non-empty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:*

(1) $0 \in A$,

(2) $x * (y * z) \in A$, $y \in A$ *implies* $x * z \in A$, *for all* $x, y, z \in X$.

Example 2. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which $*$ is defined by the following table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	2	4	5
2	0	0	0	1	4	5
3	0	0	0	0	4	5
4	0	0	0	1	0	5
5	0	0	0	0	0	0

Clearly $(X, *, 0)$ is a KU-algebra. It is easy to show that $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ are KU-ideals of X.

Definition 6. *Let A be a nonempty subset of a KU-algebra X and $0 \in A$. Then,*

(1) *A is called a weak KU-ideal of X if $y * x \in A$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$;*

(2) *A is called a strong KU-ideal of X if $(y * x) \cap A \neq \emptyset$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$.*

3. Roughness in KU-algebras

Let V be a set and E an equivalence relation on V and let $P(V)$ denote the power set of V. For all $x \in V$, let $[x]_E$ denote the equivalence class of x with respect to E. Define the functions $\underline{E}, \overline{E} : P(V) \to P(V)$ as follows: $\forall\, S \in P(V)$,

$$\underline{E}(S) = \{x \in V : [x]_E \subseteq S\}$$

and

$$\overline{E}(S) = \{x \in V : [x]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an approximation space. Let S be a subset of V. Then S is said to be definable if $\underline{E}(S) = \overline{E}(S)$ and rough otherwise. $\underline{E}(S)$ is called the lower approximation of S while $\overline{E}(S)$ is called the upper approximation.

Throughout this section X is a KU-algebra. Let A be a KU-ideal of X. Define a relation Θ on X by $(x, y) \in \Theta$ if and only if $x * y \in A$ and $y * x \in A$. Then Θ is an equivalence relation on X related to a KU-ideal A of X. Moreover satisfies $(x, y) \in \Theta$ and $(u, v) \in \Theta$ imply $(x * u, y * v) \in \Theta$.

Hence Θ is a congruence relation on X. Let A_x denote the equivalence class of x with respect to the equivalence relation Θ related to a KU-ideal A of X, and X/A denote the collection of all equivalence classes, that is, $X/A = \{A_x : x \in X\}$. Then $A_0 = A$. If $A_x * A_y$ is defined as the class containing $x * y$, that is, $A_x * A_y = A_{x*y}$, then $(X/A, *, A_0)$ is a KU-algebra. Let Θ be an equivalence relation on X related to a KU-ideal A of X. For any nonempty subset S of X, the lower and upper approximation of S are denoted by $\underline{\Theta}(A, S)$ and $\overline{\Theta}(A, S)$ respectively, that is,

$$\underline{\Theta}(A, S) = \{x \in X : A_x \subseteq S\}$$

and

$$\overline{\Theta}(A, S) = \{x \in x : A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\Theta}(A, S)$ and $\overline{\Theta}(A, S)$ are denoted by $\underline{\Theta}(A)$ and $\overline{\Theta}(A)$, respectively.

Definition 7 ([21]). *Given an approximation space (U, Θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, Θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.*

Definition 8 ([21]). *Let (U, Θ) be an approximation space and X be a nonempty subset of U.*

(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.

(ii) If $\overline{Apr}(X) = \emptyset$, then X is called empty interior.

(iii) If $Apr(X) = U$, then X is called empty exterior.

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ be a KU-algebra with the Cayley's table as follows (see [28]).

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	0	0	0	0

Let $A = \{0, 1\}$ be a KU-ideal of X ($A \lhd X$) and let Θ be an equivalence relation on X related to A. Then $A_0 = A_1 = A$, $A_2 = \{2\}$, $A_3 = \{3\}$, and $A_4 = \{4\}$.

Hence

$$\begin{aligned}
\underline{\Theta}(A,\{0,1\}) &= \{0,1\} \lhd X \\
\underline{\Theta}(A,\{0,2\}) &= \{2\} \\
\underline{\Theta}(A,\{0,3\}) &= \{3\} \\
\underline{\Theta}(A,\{0,1,2,3\}) &= \{0,1,2,3\} \lhd X
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Theta}(A,\{0,1\}) &= \{0,1\} \lhd X \\
\overline{\Theta}(A,\{0\}) &= \{0,1\} \\
\overline{\Theta}(A,\{2\}) &= \{0,2\} \\
\overline{\Theta}(A,\{1,2,3\}) &= \{0,1,2,3\} \lhd X \\
\overline{\Theta}(A,\{0,2,3\}) &= \{0,1,2,3\} \lhd X \\
\overline{\Theta}(A,\{1,2,3,4\}) &= \{0,1,2,3,4\} \lhd X.
\end{aligned}$$

In Example 3, we know that there exists a non-KU-ideal S of X such that their lower and upper approximation are KU-ideals of X. Also we choose some non-KU-ideals S of X such that their lower and upper approximation are KU-ideals of X.

Proposition 1. *Let Θ and Ξ be equivalence relations on X related to KU-ideals A and B of X, respectively. If S and T are nonempty subsets of X. Then*

(1) $\underline{\Theta}(A,S) \subseteq S \subseteq \overline{\Theta}(A,S)$;
(2) $\underline{\Theta}(A,\emptyset) = \emptyset = \overline{\Theta}(A,\emptyset)$
(3) $\overline{\Theta}(A,S \cup T) = \overline{\Theta}(A,S) \cup \overline{\Theta}(A,T)$;
(4) $\underline{\Theta}(A,S \cap T) = \underline{\Theta}(A,S) \cap \underline{\Theta}(A,T)$;
(5) $S \subseteq T$ *implies* $\underline{\Theta}(A,S) \subseteq \underline{\Theta}(A,T)$ *and* $\overline{\Theta}(A,S) \subseteq \overline{\Theta}(A,T)$;
(6) $\underline{\Theta}(A,S) \cup \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A,S \cup T)$;
(7) $\overline{\Theta}(A,S \cap T) \subseteq \overline{\Theta}(A,S) \cap \overline{\Theta}(A,T)$;
(8) $\Theta \subseteq \Xi$ *and* $A \subseteq B$ *implies* $\underline{\Xi}(B,S) \subseteq \overline{\Theta}(A,S)$ *and* $\overline{\Theta}(A,S) \subseteq \overline{\Xi}(B,S)$.

Proof. (1) If $x \in \underline{\Theta}(A,S)$, then $x \in A_x \subseteq S$. Hence $\underline{\Theta}(A,S) \subseteq S$. Next, if $x \in S$, then, since $x \in A_x$, we have $A_x \cap S \neq \phi$, and so $x \in \overline{\Theta}(A,S)$. Thus $S \subseteq \overline{\Theta}(A,S)$.

(2) is straightforward.

(3) Note that

$$\begin{aligned}
x \in \overline{\Theta}(A,S \cup T) &\iff A_x \cap (S \cup T) \neq \phi \\
&\iff (A_x \cap S) \cup (A_x \cap T) \neq \phi \\
&\iff A_x \cap S \neq \phi \ \text{ or } A_x \cap T \neq \phi \\
&\iff x \in \overline{\Theta}(A,S) \text{ or } a \in \overline{\Theta}(A,T) \\
&\iff x \in \overline{\Theta}(A,S) \cup \overline{\Theta}(A,T).
\end{aligned}$$

Thus

$$\overline{\Theta}(A, S \cup T) = \overline{\Theta}(A, S) \cup \overline{\Theta}(A, T).$$

(4) Note that

$$\begin{aligned} x \in \underline{\Theta}(A, S \cap T) &\iff A_x \subseteq S \cap T \\ &\iff A_x \subseteq S \ \text{ and } \ A_x \subseteq T \\ &\iff x \in \underline{\Theta}(A, S) \text{ and } x \in \underline{\Theta}(A, T) \\ &\iff x \in \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T). \end{aligned}$$

Thus

$$\underline{\Theta}(A, S \cap T) = \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T).$$

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, by (3) we have

$$\underline{\Theta}(A, S) = \underline{\Theta}(A, S \cap T) = \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T).$$

This implies that $\underline{\Theta}(A, S) \subseteq \underline{\Theta}(A, T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, by (2) we have

$$\overline{\Theta}(A, T) = \overline{\Theta}(A, S \cup T) = \overline{\Theta}(A, S) \cup \overline{\Theta}(A, T).$$

This implies that $\overline{\Theta}(A, S) \subseteq \overline{\Theta}(A, T)$.

(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by (4) we have

$$\underline{\Theta}(A, S) \subseteq \underline{\Theta}(A, S \cup T) \quad \text{and} \quad \underline{\Theta}(A, T) \subseteq \underline{\Theta}(A, S \cup T).$$

This implies $\underline{\Theta}(A, S) \cup \underline{\Theta}(A, T) \subseteq \underline{\Theta}(A, S \cup T)$.

(7) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by (4) we have

$$\overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, S) \quad \text{and} \quad \overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, T).$$

This implies $\overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, S) \cap \overline{\Theta}(A, T)$.

(8) Since $\Theta \subseteq \Xi$. If $x \in \underline{\Xi}(B, S)$, then $B_x \subseteq S$. But $\Theta \subseteq \Xi$, then $A_x \subseteq B_x \subseteq S$, that is, $A_x \subseteq S$. Thus $x \in \underline{\Theta}(A, S)$. Hence

$$\underline{\Xi}(B, S) \subseteq \underline{\Theta}(A, S).$$

Now let x be any element of $\overline{\Theta}(S)$. So $A_x \cap S \neq \phi$, then there exists $y \in A_y \cap S$ such that $y \in A_y$ and $y \in S$. Hence $(y, x) \in \Theta$, that is $y * x \in A$. Since $A \subseteq B$, it follows that $y * x \in B$ and $x * y \in B$ so that $(y, x) \in \Xi$, that is, $y \in B_x$. Therefore $y \in B_x \cap S$, which means that $x \in \Xi(B, S)$. This completes the proof. □

Proposition 2. *Let Θ be an equivalence relation on X related to a KU-ideal A of X. If S is a nonempty subset of X. Then*

(1) $\underline{\Theta}(A, \underline{\Theta}(A, S)) = \underline{\Theta}(A, S)$;
(2) $\overline{\Theta}(A, \overline{\Theta}(A, S)) = \overline{\Theta}(A, S)$;
(3) $\overline{\Theta}(A, \underline{\Theta}(A, S)) = \underline{\Theta}(A, S)$;
(4) $\underline{\Theta}(A, \overline{\Theta}(A, S)) = \overline{\Theta}(A, S)$;
(5) $\underline{\Theta}(A, S) = (\overline{\Theta}(A, S^c))^c$;
(6) $\overline{\Theta}(A, S) = (\underline{\Theta}(A, S^c))^c$;
(7) $\underline{\Theta}(A, A_x) = X = \overline{\Theta}(A, A_x)$, *for all* $x \in X$.

Proof. The proof is straightforward. □

Proposition 3. *Let Θ be an equivalence relation on X related to a KU-ideal A of X. If S is a nonempty subset of X. Then*
(1) $\overline{\Theta}(A,S) * \overline{\Theta}(A,T) \subseteq \overline{\Theta}(A, S*T)$;
(2) If Θ is congruence relation, then $\underline{\Theta}(A,S) * \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A, S*T)$.

Proof. (1) Let c be any element of $\overline{\Theta}(A,S) * \overline{\Theta}(A,T)$. Then $c = p*q$ with $p \in \overline{\Theta}(A,S)$ and $q \in \overline{\Theta}(A,T)$. Thus there exist elements $x, y \in S$ such that

$$x \in A_p \cap S \quad \text{and} \quad y \in A_q \cap T.$$

Thus $x \in A_p$, $y \in A_q$, $x \in S$, and $y \in T$. Since Θ is a congruence on S, it follows that

$$x * y \in A_p * A_q \in A_{p*q}.$$

On the other hand, since $x * y \in S * T$. We have $x * y \in A_{p*q} \cap S * T$, and so $c = p * q \in \overline{\Theta}(A, S*T)$. Thus we have

$$\overline{\Theta}(A,S) * \overline{\Theta}(A,T) \subseteq \overline{\Theta}(A, S*T).$$

(2) Assume that Θ is complete, let c be any element of $\underline{\Theta}(A,S) * \underline{\Theta}(A,T)$. Then $c = p*q$ with $p \in \underline{\Theta}(A,S)$ and $q \in \underline{\Theta}(A,T)$. It follows that $A_p \subseteq S$ and $A_q \subseteq T$. Since Θ is a congruence relation on S, we have

$$A_{p*q} = A_p * A_q \subseteq S * T.$$

So $c = p*q \in \underline{\Theta}(A, S*T)$. Thus

$$\underline{\Theta}(A,S) * \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A, S*T).$$

This completes the proof. □

Proposition 4. *Let Θ and Ξ be equivalence relations on X related to KU-ideals A and B of X, respectively. If S and T are nonempty subsets of X. Then*
(1) $\overline{\Theta \cap \Xi}(A \cap B, S) \subseteq \overline{\Theta}(A,S) \cap \overline{\Xi}(B,S)$;
(2) $\underline{\Theta \cap \Xi}(A \cap B, S) \supseteq \underline{\Theta}(A,S) \cap \underline{\Xi}(B,S)$.

Proof. (1) Note that $\Theta \cap \Xi$ is also a congruence relation on S. Let $c \in \overline{\Theta \cap \Xi}(A \cap B, S)$, then $[A \cap B]_c \cap S \neq \phi$. Then there exists an element $x \in [A \cap B]_c \cap S$. Since $(x,c) \in \Theta \cap \Xi$, we have

$$(x,c) \in \Theta \text{ and } (x,c) \in \Xi.$$

Thus we have $x \in A_c$ and $x \in B_c$. Since $x \in S$, we have $x \in A_c$, $x \in S$ and $x \in B_c$, $x \in S$. This implies that

$$x \in A_c \cap S \quad \text{and} \quad x \in B_c \cap S$$

$$A_c \cap S \neq \phi \quad \text{and} \quad B_c \cap S \neq \phi.$$

So $c \in \overline{\Theta}(A,S)$ and $c \in \overline{\Xi}(B,S)$, hence $c \in \overline{\Theta}(A,S) \cap \overline{\Xi}(B,S)$. Thus we obtain

$$\overline{\Theta \cap \Xi}(A \cap B, S) \subseteq \overline{\Theta}(A,S) \cap \overline{\Xi}(B,S).$$

(2) Since $\Theta \cap \Xi \subseteq \Theta$ and $\Theta \cap \Xi \subseteq \Xi$, which implies that

$$\underline{\Theta}(A,S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S) \quad \text{and} \quad \underline{\Xi}(B,S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S)$$

$$\Longrightarrow \underline{\Theta}(A,S) \cap \underline{\Xi}(B,S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S).$$

This completes the proof. □

Theorem 1. *Let (X, Θ) be an approximation space. Then*
(1) *for every $S \subseteq X$, $\underline{\Theta}(A,S)$ and $\overline{\Theta}(A,S)$ are definable sets,*
(2) *for every $x \in X$, A_x is definable set.*

Proof. (1) By Proposition 2 (1) and (3), we have

$$\underline{\Theta}(A, \underline{\Theta}(A,S)) = \underline{\Theta}(A,S) = \overline{\Theta}(A, \underline{\Theta}(A,S)).$$

Hence $\underline{\Theta}(A,S)$ is definable. On the other hand by Proposition 2 (2) and (4), we have

$$\overline{\Theta}(A, \overline{\Theta}(A,S)) = \overline{\Theta}(A,S) = \underline{\Theta}(A, \overline{\Theta}(A,S)).$$

Therefore $\overline{\Theta}(A,S)$ is a definable set.

(2) By Proposition 2 (7) the proof is clear. □

Definition 9. *A nonempty subset S of X is called an upper (resp. a lower) rough KU-subalgebra of X if the upper (resp. nonempty lower) approximation of S is a KU-subalgebra of X. If S is both an upper and a lower rough KU-subalgebra of X, we say that S is a rough KU-subalgebra of X.*

Theorem 2. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a KU-subalgebra of X, then*
(1) *$\overline{\Theta}(A,S)$ is a KU-subalgebra of X.*
(2) *$\underline{\Theta}(A,S)$ is a KU-subalgebra of X.*

Proof. (1) Let $x, y \in \overline{\Theta}(A,S)$. Then

$$A_x \cap S \neq \emptyset \text{ and } A_y \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a,x) \in \Theta$ and $(b,y) \in \Theta$. Since Θ is a congruence relation on X, we have $(a*b, x*y) \in \Theta$. Hence $a*b \in A_{x*y}$. Since S is a KU-subalgebra of X, we get $a*b \in S$, and therefore $a*b \in A_{x*y} \cap S$, that is, $A_{x*y} \cap S \neq \emptyset$. This shows that $x*y \in \overline{\Theta}(A,S)$, and consequently $\overline{\Theta}(A,S)$ is a KU-subalgebra of X.

(2) Let $x, y \in \underline{\Theta}(A,S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a KU-subalgebra of X, it follows that

$$A_{x*y} = A_x * A_y \subseteq S$$

so that $x*y \in \underline{\Theta}(A,S)$. Hence $\underline{\Theta}(A,S)$ is a KU-subalgebra of X. □

The following example shows that the converse of Theorem 2(1) may not be true.

Example 4. Let $X = \{0, 1, 2, 3, 4\}$ be a KU-algebra with the Cayley's table as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

Let $A = \{0, 1, 2\}$ be a KU-ideal of X ($A \lhd X$) and let Θ be an equivalence relation on X related to A. Then $A_0 = A_1 = A_2 = A$, $A_3 = \{3\}$, and $A_4 = \{4\}$. Note that $S = \{1, 3\}$ is not a KU-subalgebra of X, but $\overline{\Theta}(A, S) = \{0, 1, 2, 3\}$ is KU-subalgebra of X.

Definition 10. *A nonempty subset S of X is called an upper (resp. a lower) rough KU-ideal of X if the upper (resp. nonempty lower) approximation of S is a KU-ideal of X. If S is both an upper and a lower rough KU-ideal of X, we say that S is a rough KU-ideal of X.*

Theorem 3. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a KU-ideal of X containing A, then*

(1) *$\overline{\Theta}(A, S)$ is a KU-ideal of X.*

(2) *$\underline{\Theta}(A, S)$ is a KU-ideal of X.*

Proof. (1) Let S be a KU-ideal of X containing A. Obviously $0 \in \overline{\Theta}(A, S)$. Let $x, y, z \in X$ be such that $y \in \overline{\Theta}(A, S)$ and $x * (y * z) \in \overline{\Theta}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{x*(y*z)} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{x*(y*z)}$. Hence $(a, y) \in \Theta$ and $(b, (x*(y*z))) \in \Theta$, which implies $y*a \in A \subseteq S$ and $(x*(y*z))*b \in A \subseteq S$. Since $a, b \in S$ and S is a KU-ideal, we get

$$y \in S \text{ and } x * (y * z) \in S,$$

it follows from Definition 5(2) that $x * z \in S$. Note that $x * z \in A_{x*z}$, thus $x * z \in A_{x*z} \cap S$, that is, $A_{x*z} \cap S \neq \emptyset$. Hence $x * z \in \overline{\Theta}(A, S)$ and therefore $\overline{\Theta}(A, S)$ is a KU-ideal of X.

(2) Let S be a KU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\Theta}(A, S)$. Let $x, y, z \in X$ be such that $y \in \underline{\Theta}(A, S)$ and $x * (y * z) \in \underline{\Theta}(A, S)$. Then

$$A_y \in S \text{ and } A_x * (A_y * A_z) = A_{x*(y*z)} \subseteq S.$$

Let $w \in A_{x*z} = A_x * A_z$. Then $w = A_x * A_z$ for some $a \in A_x$ and $c \in A_z$. From $a \in A_x$ and $c \in A_z$, we have $(a, x) \in \Theta$ and $(c, z) \in \Theta$. Taking $b \in A_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(a * (b * c), x * (y * z)) \in \Theta \text{ and so } a * (b * c) \in A_{x*(y*z)} \subseteq S.$$

Since S is a KU-ideal of X, it follows from Definition 5(2) that $w \in a * c \in S$, so that $A_{x*z} \subseteq S$. Hence $x * z \in \underline{\Theta}(A, S)$ and therefore $\underline{\Theta}(A, S)$ is a KU-ideal of X. □

Theorem 4. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a weak KU-ideal of X containing A, then*
(1) *$\overline{\Theta}(A, S)$ is a weak KU-ideal of X.*
(2) *$\underline{\Theta}(A, S)$ is a weak KU-ideal of X.*

Proof. (1) Let S be a weak KU-ideal of X containing A. Obviously $0 \in \overline{\Theta}(A, S)$. Let $x, y \in X$ be such that $y \in \overline{\Theta}(A, S)$ and $y * x \in \overline{\Theta}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{y*x} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{y*x}$. Hence $(a, y) \in \Theta$ and $(b, (y * x)) \in \Theta$, which implies

$$y * a \in A \subseteq S \text{ and } (y * x) * b \in A \subseteq S.$$

Since $a, b \in S$ and S is a weak KU-ideal, we get $y \in S$ and $y * x \in S$, it follows from Definition 6(1) that $x \in S$. Note that $x \in A_x$, thus $x \in A_x \cap S$, that is, $A_x \cap S \neq \emptyset$. Hence $x \in \overline{\Theta}(A, S)$ and therefore $\overline{\Theta}(A, S)$ is a weak KU-ideal of X.

(2) Let S be a weak KU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\Theta}(A, S)$. Let $x, y \in X$ be such that $y \in \underline{\Theta}(A, S)$ and $y * x \in \underline{\Theta}(A, S)$. Then

$$A_y \in S \text{ and } A_y * A_x = A_{y*x} \subseteq S.$$

Let $w \in A_x$. Then $w = A_x$ for some $a \in A_x$. From $a \in A_x$, we have $(a, x) \in \Theta$. Taking $b \in A_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(b * a, y * x) \in \Theta \text{ and } b * a \in A_{y*x} \subseteq S.$$

Since S is a weak KU-ideal of X, it follows from Definition 6(1) that $w = a \in S$, so that $A_x \subseteq S$. Hence $x \in \underline{\Theta}(A, S)$ and therefore $\underline{\Theta}(A, S)$ is a weak KU-ideal of X. □

Theorem 5. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a strong KU-ideal of X containing A, then*
(1) *$\overline{\Theta}(A, S)$ is a strong KU-ideal of X.*
(2) *$\underline{\Theta}(A, S)$ is a strong KU-ideal of X.*

Proof. (1) Let $x, y \in X$ be such that

$$(y * x) \cap \overline{\Theta}(A, S) \neq \emptyset \text{ and } y \in \overline{\Theta}(A, S).$$

Then $A_y \cap S \neq \emptyset$ and so there exist $z \in X$ such that $z = y * x$ and $z \in \overline{\Theta}(A, S)$. Hence $A_z \cap S \neq \emptyset$ and so there exist $c, d \in X$ such that

$$c \in A_z \cap S \text{ and } d \in A_y \cap S.$$

Hence $c\Theta z$ and $d\Theta y$ where $c, d \in S$. Thus we $z * c \in A \subseteq S$ and $y * d \in A \subseteq S$. Since S is a strong KU-ideal and $c, d \in S$, we have $z \in S$ and $y \in S$. Thus we have proved that $(y * x) \cap A \neq \emptyset$ and $y \in A$. Since S is a strong KU-ideal, we have $x \in S$ and so $A_x \cap S \neq \emptyset$ which means that $\overline{\Theta}(A, S)$ is a strong KU-ideal of S.

(2) Let $x, y \in X$ be such that

$$(y * x) \cap \underline{\Theta}(A, S) \neq \emptyset \text{ and } y \in \underline{\Theta}(A, S).$$

Let $a \in A_x$ and $b \in A_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on X, $b * a\Theta y * x$. Since $(y * x) \cap \underline{\Theta}(A, S) \neq \emptyset$, then there exist $t \in X$ such that $t \in y * x$ and $t \in \underline{\Theta}(A, S)$. Now, $t \in b * a\Theta y * x$ implies that there exist $z \in b * a$ such that $z\Theta t$ and so $A_t = A_z \subseteq S$. Hence $z \in S$ and so $(b * a) \cap S \neq \emptyset$. On the other hand, we have $b \in A_y \subseteq S$. Since S is a strong KU-ideal of X, then we have $a \in S$ which implies $A_x \subseteq S$ that means $x \in \underline{\Theta}(A, S)$. Therefore, $\underline{\Theta}(A, S)$ is a strong KU-ideal of S. □

4. Conclusion

Roughness is one of the important method to tackle the uncertainty and vagueness in information system. It is widely used in database management system more basically in data mining and big data. In recent years roughness has been applied and used successfully in a number of challenging fields such as pure and applied algebras, computer science, engineering, medical science and soft computing method in biology and and computer science.

This paper connects KU-algebras (KU-ideals) with roughness through definitions, examples and results based on lower and upper approximations of this logical algebras. We expect that this connection may invoke some new directions with other related and similar concepts in logical algebras and different types of KU-algebras including soft KU-algebras and hyper soft KU-algebras togather with some reasoning and logic. The next step in the direction of this work could be the results based on rough set approach to handle medical related data.

References

[1] M. Akram, N. Yaqoob and J. Kavikumar, *Interval-valued* $(\widetilde{\theta}, \widetilde{\delta})$*-fuzzy KU-ideals of KU-algebras*, Int. J. Pure Appl. Math., 92 (2014), 335-349.

[2] M. Akram, N. Yaqoob and M. Gulistan, *Cubic KU-subalgebras,* Int. J. Pure Appl. Math., 89 (2013), 659-665.

[3] R. Ameri, R. Moradian and R. A. Borzooei, *Rough set theory applied to hyper BCK-algebra*, Ratio Math., 26 (2014), 3-20.

[4] R. Biswas and S. Nanda, *Rough groups and rough subgroups,* Bull. Polish Acad. Sci. Math., 42 (1994), 251-254.

[5] Chen, L.F. and Tsai, C.T. *Data mining framework based on rough set theory to improve location selection decisions: a case study of a restaurant chain*, Tourism Management, 53 (2016), 197-206.

[6] B. Davvaz, S.M. Mostafa and F.F. Kareem, *Neutrosophic ideals of neutrosophic KU-algebras*, Gazi University Journal of Science, 30 (2017), 463-472.

[7] W. Dudek, Y.B. Jun and H.S. Kim, *Rough set theory applied to BCI-algebras*, Quasigroups and Relt. Syst., 9 (2002), 45-54.

[8] M. Gulistan, M. Shahzad and S. Ahmed, *On* (α, β)*-fuzzy KU-ideals of KU-algebras*, Afrika Matematika, 26 (2015), 651-661.

[9] M. Gulistan, M. Shahzad and N. Yaqoob, *On* $(\in, \in \vee q_k)$*-fuzzy KU-ideals of KU-algebras,* Acta Universitatis Apulensis, 39 (2014), 75-83.

[10] Y.B. Jun, *Roughness of ideals in BCK-algebras*, Sci. Math. Japon., 7 (2002), 115-119.

[11] Y.B. Jun and X. Yang, *Roughness of filters in lattice implication algebras*, Bull. Pol. Acad. Sci. Math., 52 (2004), 341-352.

[12] Y.B. Jun and K.H. Kim, *Rough set theory applied to BCC-algebras*, Int. Math. Forum, 41 (2007), 2023-2029.

[13] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci., 100 (1997), 139-163.

[14] T. Karimi, *Rough sets and gray sets*, Agah publishers, 2, 2015.

[15] X. Ma, *Applications of rough soft sets in BCI-algebras and decision making*, J. Intell. Fuzzy Syst., 29 (2015), 1079-1085.

[16] X. Mao and H. Zhou, *The application of rough set theory in Pseudo-BCK-algebra*, J. Math. Res. Appl., 36 (2016), 23-35.

[17] S.M. Mostafa, M.A. Abd-Elnaby and M.M.M. Yousef, *Fuzzy ideals of KU-Algebras*, Int. Math. Forum., 6 (2011), 3139-3149.

[18] S.M. Mostafa, M.A. Abd-Elnaby and O.R. Elgendy, *Interval-Valued Fuzzy KU-Ideals in KU-Algebras,* Int. Math. Forum., 6 (2011), 3151-3159.

[19] S.M. Mostafa, A.E. Radwan, F.A. Ibrahem and F.F. Kareem, *Interval value fuzzy n-fold KU-ideals of KU-algebras*, J. Math. Comput. Sci., 5 (2015), 246.

[20] G. Muhiuddin, *Bipolar fuzzy KU-subalgebras/ideals of KU-algebras*, Ann. Fuzzy Math. Inform., 8 (2014), 339-504.

[21] Z. Pawlak, *Rough sets*, Int. J. Comput. Inform. Sci., 11 (1982), 341-356.

[22] C. Prabpayak and U. Leerawat, *On ideals and congurences in KU-algebras*, Scientia Magna J., 5 (2009), 54-57.

[23] C. Prabpayak and U. Leerawat, *On isomorphisms of KU-algebras*, Scientia Magna J., 5 (2009), 25-31.

[24] T. Senapati, *T-fuzzy KU-subalgebras of KU-algebras*, Ann. Fuzzy Math. Inform., 10 (2015), 261-270.

[25] T. Senapati and K. P. Shum, *Atanassov's intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra*, J. Intell. Fuzzy Syst., 30 (2016), 1169-1180.

[26] L. Torkzadeh and S. Ghorbani, *Rough filters in B-Algebras*, Int. J. Math. Math. Sci., 2011.

[27] Q. M. Xiao and Z. L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci., 176 (2006), 725-733.

[28] N. Yaqoob, S.M. Mostafa and M.A. Ansari, *On cubic KU-ideals of KU-algebras*, ISRN Algebra, 2013.

[29] L.A. Zadeh, *Fuzzy sets*, Inform. Control., 8 (1965), 338-353.

Accepted: 11.05.2018

ON HYPERIDEALS OF ORDERED SEMIHYPERGROUPS

ZE GU
School of Mathematics and Statistics
Zhaoqing University
Zhaoqing, Guangdong, 526061
P.R. China
guze528@sina.com

Abstract. Prime, weakly prime and semiprime hyperideals in ordered semihypergroups were studied by Kehayopulu. In this paper, we introduce the concepts of weakly semiprime and irreducible hyperideals in ordered semihypergroups. The relationship between the five classes of hyperideals is established. Finally, we characterize semisimple ordered semihypergroups and intra-regular ordered semihypergroups in terms of these hyperideals.

Keywords: ordered semihypergroups, hyperideals, semisimple ordered semihypergroups, intra-regular ordered semihypergroups.

1. Introduction and preliminaries

The algebraic hyperstructure theory was first introduced in 1934 by Marty [18]. Hyperstructures have many applications in several branches of both pure and applied sciences (see [4-6,8,9,11-13,23]). Recently, Heidari and Davvaz applied the hyperstructure theory to ordered semigroups and introduced the concept of ordered semihypergroups (see [16]), which is a generalization of the concept of ordered semigroups. Furthermore, the ordered semihypergroup theory was enriched by the work of many researchers, for example [2,3,10,14,15,22,23]. In particular, the hyperideal theory on semihypergroups and ordered semihypergroups can be seen in [1-3,17-19,22,23]. Recently, N. Kehayopulu (see [18-19]) introduced prime, weakly prime and semiprime hyperideals in ordered semihyergroups and studied ordered semihypergroups with these hyperideals. Motivated by the previous work on hyperideals of (ordered) semihypergroups, we attempt in the present paper to study hyperideals of ordered semihypergroups in detail. In this article, we introduce the notions of weakly semiprime and irreducible hyperideals in ordered semihypergroups, and moreover establish the relationship between the five classes of hyperideals. Finally, semisimple ordered semihypergroups and intra-regular ordered semihypergroups are characterized in terms of these hyperideals. Partial results which are consistent with the conclusions in [19] are reorganized and proved. We recall first some basic notions of ordered semihypergroup.

A *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a *hyperoperation* or *hypercomposition*, that is a mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S. If $x \in S$ and A, B are nonempty subsets of S, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $\circ$ is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in S$.

An *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is *compatible* with the hyperoperation $\circ$, meaning that for any $a, b, x \in S$, $a \leq b$ implies that $a \circ x \leq b \circ x$ and $x \circ a \leq x \circ b$. Here, let $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Let S be an ordered semihypergroup and A be a nonempty subset of S. We say that A is a *hyperideal* of S if (1) $S \circ A \subseteq A$, $A \circ S \subseteq A$ and (2) $a \in A, b \in S$ and $b \leq a$ imply that $b \in A$. For $\emptyset \neq H \subseteq S$, we denote

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For convenience, we write $(a]$ instead of $(\{a\}]$ for $H = \{a\}$. We denote by $I(a)$ the hyperideal of S generated by a. On can easily obtain that

$$I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S].$$

2. Hyperideals of ordered semihypergroups

In this section, we recall prime, weakly prime and semiprime hyperideals and introduce the concepts of weakly semiprime and irreducible hyperideals. Moreover, we study the relationship between the five classes of hyperideals.

Let $(S, \circ, \leq)$ be an ordered semihypergroup and I be a hyperideal of S. I is called *prime* if for any $A, B \subseteq S$, $A \circ B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ (equivalent to for all $a, b \in S$, $a \circ b \subseteq I$ implies $a \in I$ or $b \in I$); I is called *semiprime* if for any $A \subseteq S$, $A \circ A \subseteq I$ implies $A \subseteq I$ (equivalent to for any $a \in S$, $a \circ a \subseteq I$ implies $a \in I$); I is called *weakly prime* if for all hyperideals A, B of S, $A \circ B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$;

Definition 2.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup and I be a hyperideal of S. I is called weakly semiprime if for any hyperideal A of S, $A \circ A \subseteq I$ implies $A \subseteq I$; I is called irreducible if for all hyperideals I_1, I_2 of S, $I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$.

Remark 2.2. It is easy to see that a prime hyperideal is semiprime and weakly prime, and a semiprime hyperideal is weakly semiprime.

For further studying the relationship between the five classes of hyperideals, we need the following result which can be easily obtained.

Lemma 2.3. *Let S be an ordered semihypergroup. Then*
(1) $A \subseteq (A], ((A]] = (A]$ *for all* $A \subseteq S$;
(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B], ((A] \circ (B]] = (A \circ B]$ *for all* $A, B \subseteq S$;

(4) $(T] = T$ *for every hyperideal* T *of* S;

(5) *If* A, B *are hyperideals of* S, *then* $(A \circ B), A \cap B$ *and* $A \cup B$ *are hyperideals of* S;

(6) $(S \circ A \circ S]$ *is a hyperideal of* S *for all* $A \subseteq S$, *in particular, we write* $(S \circ \{a\} \circ S]$ *as* $(S \circ a \circ S]$.

Lemma 2.4. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is the intersection of all irreducible hyperideals of* S *containing* I.

Proof. Let $\{I_\alpha \mid \alpha \in \Gamma\}$ be the set of all irreducible hyperideals of S containing I. Obviously, $I \subseteq \bigcap_{\alpha\in\Gamma} I_\alpha$. It suffices to prove that $\bigcap_{\alpha\in\Gamma} I_\alpha - I = \emptyset$. Suppose that there exists $a \in \bigcap_{\alpha\in\Gamma} I_\alpha - I$. Let $\Omega = \{H \text{ is a hyperideal} \mid I \subseteq H, a \notin H\}$. Then $\Omega \neq \emptyset$ from $I \in \Omega$, and Ω is partially ordered under inclusion. By Zorn's Lemma, there exists a maximal element $M \in \Omega$. Next we show that M is irreducible. Let A, B be two hyperideals such that $A \cap B = M$. Then $A \in \Omega$ or $B \in \Omega$. Thus $A = M$ or $B = M$. Otherwise, $M \subsetneqq A$ and $M \subsetneqq B$ which contradicts the maximality of M. Hence $I = \bigcap_{\alpha\in\Gamma} I_\alpha$. □

Theorem 2.5. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is prime if and only if it is semiprime and weakly prime. In particular, if* S *is commutative, then the prime and weakly prime hyperideals concide.*

Proof. ($\Rightarrow$) It is obtained from Remark 2.2.

($\Leftarrow$) Let $a, b \in S$ and $a \circ b \subseteq I$. Then $(b \circ S \circ a] \circ (b \circ S \circ a] \subseteq (b \circ S \circ a \circ b \circ S \circ a] \subseteq (S \circ (a \circ b) \circ S] \subseteq (S \circ I \circ S] \subseteq (I] = I$. Since I is semiprime, $(b \circ S \circ a] \subseteq I$. Thus $(S \circ b \circ S] \circ (S \circ a \circ S] \subseteq (S \circ b \circ S \circ S \circ a \circ S] \subseteq (S \circ (b \circ S \circ a) \circ S] \subseteq (S \circ I \circ S] \subseteq I$. Since I is weakly prime, we have $(S \circ b \circ S] \subseteq I$ or $(S \circ a \circ S] \subseteq I$. Suppose that $(S \circ a \circ S] \subseteq I$. Then $(I^2(a)] \circ I(a) = (I^2(a)] \circ (I(a)] \subseteq (I^3(a)] = ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]^3] \subseteq (((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)^2] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]] \subseteq ((S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]] \subseteq (((S \circ a \cup S \circ a \circ S) \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)]] \subseteq ((S \circ a \circ S]] \subseteq (I] = I$. Since I is weakly prime, we have $(I^2(a)] \subseteq I$ or $I(a) \subseteq I$. If $I(a) \subseteq I$, then $a \in I(a) \subseteq I$. If $(I^2(a)] \subseteq I$, then $I^2(a) \subseteq I$. Since I is weakly prime, $I(a) \subseteq I$ and so $a \in I$. By symmetries, we obtain $b \in I$ if $(S \circ b \circ S] \subseteq I$. Therefore, I is a prime hyperideal of S.

Let S be commutative, I weakly prime and $a, b \in S$ such that $a \circ b \subseteq I$. Then $I(a) \circ I(b) = (a \cup S \circ a] \circ (b \cup S \circ b] \subseteq (a \circ b \cup S \circ a \circ b] \subseteq (I \cup S \circ I] \subseteq (I] = I$. Since I is weakly prime, we have $a \in I(a) \subseteq I$ or $b \in I(b) \subseteq I$. □

Theorem 2.6. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is weakly prime if and only if it is weakly semiprime and irreducible.*

Proof. ($\Rightarrow$) From Remark 2.2, we know that I is weakly semiprime. Next we show that I is irreducible. Let I_1, I_2 be two hyperideals such that $I_1 \cap I_2 = I$. Then $I \subseteq I_1$ and $I \subseteq I_2$. Moreover, $I_1 \circ I_2 \subseteq I_1 \cap I_2 = I$. Since I is weakly prime, $I_1 \subseteq I$ or $I_2 \subseteq I$. Hence $I_1 = I$ or $I_2 = I$.

($\Leftarrow$) Let A, B be two hyperideals such that $A \circ B \subseteq I$. Then $(A \cap B) \circ (A \cap B) \subseteq A \circ B \subseteq I$. Since I is weakly semiprime, $A \cap B \subseteq I$. Thus $I = I \cup (A \cap B) = (I \cup A) \cap (I \cup B)$. Since I is irreducible, we have $I \cup A = I$ or $I \cup B = I$. Hence $A \subseteq I$ or $B \subseteq I$. Consequently, I is weakly prime. □

Combining Theorem 2.5 and 2.6, we have the following result.

Corollary 2.7. *Let S be an ordered semihypergroup and I a hyperideal of S. Then I is prime if and only if it is semiprime and irreducible.*

3. Semisimple and intra-regular ordered semihypergroups

In this section, we mainly characterize semisimple ordered semihypergroups and intra-regular semihypergroups in terms of the five classes hyperideals introduced in the previous section.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. S is called *semisimple* if $a \in (S \circ a \circ S \circ a \circ S]$ for every $a \in S$; S is called *intra-regular* if $a \in (S \circ a \circ a \circ S]$ for every $a \in S$.

Theorem 3.1. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is semisimple;*
(2) *$A \cap B = (A \circ B]$ for all hyperideals A, B of S;*
(3) *$(A^2] = A$ for every hyperideal A of S;*
(4) *Every hyperideal of S is weakly semiprime.*

Proof. (1)$\Rightarrow$(2) Let A, B be hyperideals of S. If $a \in A \cap B$, then $a \in (S \circ a \circ S \circ a \circ S] = ((S \circ a \circ S) \circ (a \circ S)] \subseteq (A \circ B]$. Thus $A \cap B \subseteq (A \circ B]$. On the other hand, $(A \circ B] \subseteq (A] \cap (B] = A \cap B$. Hence $A \cap B = (A \circ B]$.

(2)$\Rightarrow$(3) It is obvious.

(3)$\Rightarrow$(4) Let A and I be hyperideals of S such that $A^2 \subseteq I$. Then $(A^2] \subseteq (I] = I$. By hypothesis, $A = (A^2] \subseteq I$.

(4)$\Rightarrow$(1) By the proof of Theorem 2.5, we have $(I^3(a)] \subseteq (S \circ a \circ S]$. Following the method, we can obtain $(I^5(a)] \subseteq (S \circ a \circ S \circ a \circ S]$. Moreover, $((I^2(a)]^2] \circ ((I^2(a)]^2] \subseteq (I^4(a)] \circ (I^4(a)] \subseteq (I^4(a)] \circ (I(a)] \subseteq (I^5(a)]$ and $((I^2(a)]^2]$ is a hyperideal. Since $(S \circ a \circ S \circ a \circ S]$ is weakly semiprime, $((I^2(a)]^2] \subseteq (S \circ a \circ S \circ a \circ S]$. Continuing the process twice, we have $a \in I(a) \subseteq (S \circ a \circ S \circ a \circ S]$. □

Theorem 3.2. *Let S be an ordered semihypergroup. Then S is intra-regular if and only if every hyperideal of S is semiprime.*

Proof. ($\Rightarrow$) Let I be a hyperideal of S and $a \in S$ such that $a^2 \subseteq I$. Then $a \in (S \circ a^2 \circ S] \subseteq (S \circ I \circ S] \subseteq (I] = I$.

($\Leftarrow$) Let $a \in S$. Then $a^2 \circ a^2 = a \circ a^2 \circ a \subseteq S \circ a^2 \circ S \subseteq (S \circ a^2 \circ S]$. Since $(S \circ a^2 \circ S]$ is a hyperideal of S, by hypothesis, $(S \circ a^2 \circ S]$ is semiprime. Thus $a^2 \subseteq (S \circ a^2 \circ S]$. In the same way, we have $a \in (S \circ a^2 \circ S]$. □

Theorem 3.3. *Let S be an ordered semihypergroup and Θ be the set of all hyperideals of S. Then I is weakly prime for every $I \in \Theta$ if and only if S is semisimple and Θ is a chain.*

Proof. ($\Rightarrow$) From Theorem 3.1, we know that S is semisimple. Let $A, B \in \Theta$. Since $(A \circ B] \in \Theta$ and $A \circ B \subseteq (A \circ B]$, by hypothesis, we have $A \subseteq (A \circ B]$ or $B \subseteq (A \circ B]$. Moreover, $(A \circ B] = A \cap B$ from Theorem 3.1. Hence $A \subseteq B$ or $B \subset A$. Consequently, Θ is a chain under inclusion.

($\Leftarrow$) From Theorem 2.6 and 3.1, it suffice to verify that I is irreducible for any $I \in \Theta$. Let $A, B \in \Theta$ and $A \cap B = I$. Since Θ is a chain, $A \subset B$ or $B \subset A$. Thus $A = I$ or $B = I$. Hence I is irreducible. □

From Theorem 3.1, 3.2 and 3.3, we can easily obtain the following result.

Theorem 3.4. *Let S be an ordered semihypergroup and Θ be the set of all hyperideals of S. Then I is prime for every $I \in \Theta$ if and only if S is intra-regular and Θ is a chain.*

We end this paper with an example by illustrating the previous results.

Example 3.5. Let $S = \{a, b, c, d, e\}$ with the hyperoperation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e
a	{b,c}	{a}	{a}	{a}	{a}
b	{a}	{b,c}	{b,c}	{b,c}	{b,c}
c	{a}	{b,c}	{b,c}	{b,c}	{b,c}
d	{a}	{b,c}	{b,d}	{d,e}	{d,e}
e	{a}	{b,c}	{c}	{d,e}	{e}

$$\leq := \{(a,a), (b,b), (c,c), (c,b), (d,d), (e,e), (e,d)\}.$$

One can check that $(S, \circ, \leq)$ is an ordered semihypergoup (see [19]). The hyperideals of S are S and $I = \{a, b, c\}$. Obviously, the set of all hyperideals forms a chain. One can also check that S is intra-regular. Thus, by Theorem 3.4, the hyperideals of S are prime. We give an independent proof as following: indeed, $S \backslash I = \{d, e\}$ and $d \circ d = d \circ e = e \circ e = \{d, e\} \nsubseteq I$, $e \circ e = \{e\} \nsubseteq I$. By Theorem 2.5 and 2.6, we know that the ideals of S are weakly prime, semiprime, weakly semiprime and irreducible. Moreover, by Theorem 3.3, S is semisimple.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11701504), the Young Innovative Talent Project of Department of Education of Guangdong Province (No. 2016KQNCX180) and the University Natural Science Project of Anhui Province (No. KJ2018A0329).

References

[1] T. Changphas, B. Davvaz, *Hyperideal theory in ordered semihypergroups*, Xanthi: International Congress on Algebraic Hyperstructures and its Applications, 51-54, 2014.

[2] T. Changphas, B. Davvaz, *Properties of hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math., 33 (2014), 425-432, .

[3] T. Changphas, B. Davvaz, *Bi-hyperideals and quasi-hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math., 35 (2015), 493-508.

[4] P. Corsini, *Sur les semi-hypergroupes*, Atti Soc. Pelorit. Sci. Fis. Math. Nat., 26 (1980), 363-372.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Tricesimo: Aviani Editore, 1993.

[6] P. Corsini, V. Leoreanu-Fotea, *Applications of Hyperstructure Theory. Advances in Mathematics*, Dordrecht: Kluwer Academic Publishers, 2003.

[7] P. Corsini, M. Shabir, T. Mahmood, *Semisimple semihypergroups in terms of hyperideals and fuzzy hyperideals*, Iranian Journal of Fuzzy Systems, 8 (2011), 95-111.

[8] B. Davvaz, *Polygroup Theory and Related Systems*, Hackensack: World Scientific Publishing Co. Pte. Ltd., 2013.

[9] B. Davvaz, *Semihypergroup Theory*, Academic Press, Elsevier, 2016.

[10] B. Davvaz, P. Corsini, T. Changphas, *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder*, European Journal of Combinatorics, 44 (2015), 208-217.

[11] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, Florida: International Academic Press, 2007.

[12] M. De Salvo, D. Freni, G. Lo Faro, *Fully simple semihypergroups*, J. Algebra, 399 (2014), 358-377.

[13] M. De Salvo, D. Freni, D. Fasino et al., *Fully simple semihypergroups, transitive digraphs, and sequence A000712*, J. Algebra, 415 (2014), 65-87.

[14] Z. Gu, X.L. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra, 450 (2016), 384-397.

[15] Z. Gu, X.L. Tang, *Characterizations of (strongly) ordered regular relations on ordered semihypergroups*, J. Algebra, 465 (2016), 100-110.

[16] D. Heidari, B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 73 (2011), 85-96.

[17] K. Hila, B. Davvaz, K. Naka, *On quasi-hyperideals in semihypergroups*, Comm. Algebra, 39 (2011), 4183-4194.

[18] N. Kehayopulu, *Left regular and intra-regular ordered hypersemigroups in terms of semiprime and fuzzy semiprime subsets*, Sci. Math. Jpn., 80 (2017), 295-305.

[19] N. Kehayopulu, *On ordered hypersemigroups with idempotent ideals, prime or weakly prime ideals*, European Journal of Pure and Applied Mathematics, 11 (2018), 10-22.

[20] F. Marty, *Sur une generalization de la notion de groupe*, Stockholm: Proceedings of the 8th Congress Math. Scandinaves, 45-49, 1934.

[21] C.G. Massouros, *On connections between vector spaces and hypercompositional structures*, Ital. J. Pure Appl. Math., 34 (2015), 133-150.

[22] B. Pibaljommee, B. Davvaz, *Characterizations of (fuzzy) bi-hyperideals in ordered semihypergroups*, J. Intell. Fuzzy Systems, 28 (2015), 2141-2148.

[23] J. Tang, A. Khan, Y.F. Luo, *Characterizations of semisimple ordered semihypergroups in terms of fuzzy hyperideals*, J. Intell. Fuzzy Systems, 30 (2016), 1735-1753.

Accepted: 16.05.2018

BAYESIAN ESTIMATION AND PREDICTION BASED ON EXPONENTIAL RESIDUAL TYPE II CENSORED LIFE DATA

Ghassan K. Abufoudeh*

Raed R. Abu Awwad

Department of Mathematics
Faculty of Arts and Sciences
University of Petra
Amman
Jordan
ghassan_math@yahoo.com

Abstract. In this paper, we consider statistical inference problems for the residual life data come from exponential model based on type II censored data. Maximum likelihood and Bayesian approaches are used to estimate the scale parameter for exponential model also we construct symmetric credible intervals. Further, we propose to estimate the posterior predictive density of the future ordered observations and then obtain the corresponding predictors and we obtain the predictive survival function to compute the predictive interval for the missing order statistics. Numerical comparisons are conducted to assess the performance of the estimators of the parameter as well as the predictors of future ordered data.

Keywords: residual life data, exponential distribution, type II censored data, Maximum likelihood estimation, Bayes estimation, Bayes prediction.

1. Introduction

Exponential distribution plays an important role in lifetime data analysis and is a commonly used distribution in reliability engineering. In the last few years, many researchers have developed inference procedures for exponential model. Sarhan (2003) obtained the empirical Bayes estimators of exponential model. Janeen (2004) discussed the empirical Bayes estimators of the exponential distribution parameter based on record values. Yimin and Weian (2010) estimated the scale parameter for two parameters exponential distribution using empirical Bayes procedure under the type I censoring life test. Chen and Lio (2010) considered the parameter estimation for exponential distribution under progressive type I interval censoring. Also, several authers studied Bayesian inference and prediction. Kundu (2008) studied the Bayesian inference of the Weibull

∗. Corresponding author

parameters when the data are progressively censored under the squared error loss function. Kundu and Howlader (2010) described the Bayesian inference and prediction of the inverse Weibull distribution for type II censored data, they obtained the Bayes estimatiors based on the square error loss function. Pradham and Kundu (2011) studied the Bayesian estimation and prediction of the two parameter Gamma distribution, they considered the Bayes estimation under square error loss function with the assumption that the scale parameter has a Gamma prior and the shape parameter has any log-cocave prior. Al-Hussaini (1999) studied the Bayesian prediction problem for a large class of lifetime distribution. For more details, one can refer to Balakrishnan et al. (2005).

The main aim of this paper is to compute the MLE and Bayes estimate of the unknown parameter based on residual exponential life data under square error loss function and predicte the residual life time for the missing items which is important especially in actuarial, medical and engineering sciences.

Suppose that n items are kept under observation until failure. These items could be some systems, components, or computer chips in reliability study experiments, or they could be patients put under certain drug or clinical conditions and their lifetimes $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ follow the exponential distribution with the probability density function (pdf)

$$f(x;\theta) = \begin{cases} \theta.e^{-\theta x}, & \text{if } x > 0, \theta > 0 \\ 0, & \text{if } x \leq 0, \end{cases} \tag{1}$$

and cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\theta x}, x > 0, \theta > 0. \tag{2}$$

Here $\theta > 0$ is the scale parameter.

The residual life random variable at age t is defined as $Y = X - t \mid X > t,\ t > 0$. These residual life data and its ordering can be effectively applied in reliability theory and play an important role, especially if one observes only the residual lifetime. This type of data do arise naturally in survival actuarial studies.

The cdf of the residual life variable can be obtained as

$$G(y) = \Pr(X - t < y \mid X > t) = \frac{F(t+y) - F(t)}{1 - F(t)}\ , \quad y > 0, \tag{3}$$

and the corresponding pdf is

$$g(y) = \frac{f(t+y)}{1 - F(t)}, \quad y > 0. \tag{4}$$

Now, for the exponential distribution we get

$$G(y) = 1 - e^{-\theta y},\ y > 0,\ \theta > 0, \tag{5}$$

and

$$g(y) = \theta e^{-\theta y},\ y > 0, \theta > 0. \tag{6}$$

Now, for some reason or other, one may terminate the experiment at the $r-th$ failure, that is, at time $X_{r:n}$, we obtain type II censored sample. Here r is fixed, while $X_{r:n}$, the duration of the experiment is random. The likelihood function in this case is

$$L(\theta \mid \underset{\sim}{X}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} g\left(x_i \mid \theta\right) \left[1 - G(x_r;\theta)\right]^{n-r}\ ,\ x_1 < x_2 < ... < x_r. \tag{7}$$

Consider we have complete data for the residual life times $Y_1, Y_2, ..., Y_n$, where $Y_i = X_i - t \mid X_i > t$, $i = 1, 2, ..., n$ are the residual lifetime random variables.

2. Maximum likelihood estimation

In this section, we derive the maximum likelihood estimator (MLE) of the parameter θ of the exponential model based on type II censored data using the residual life observations. Let $Y_{1:n} < Y_{2:n} < ... < Y_{r:n}$ be a type II censored sample of size r $(1 < r < n)$. On using Eq.(5), Eq.(6) and Eq.(7), the likelihood function is given by

$$L\left(\theta \mid data\right) = \frac{n!}{(n-r)!} \theta^r e^{-\theta[y_r(n-r) + \sum_{i=1}^{r} y_i]}.$$

By differentiating the natural logarithm of the likelihood function with respect to θ and equating the resulting term to zero, we get

$$\theta = \frac{r}{y_r\left(n-r\right) + \sum_{i=1}^{r} y_i}. \tag{8}$$

The MLE of θ, say $\overset{\wedge}{\theta}$, is obtained as a solution of Eq.(8). For more details about the existence of these MLEs and uniqueness, see Balakrishnan and Kateri (2008).

3. Bayes estimation

For the Bayesian inference and life testing plan, we need to assume some prior distributions to obtain the Bayes estimates (BEs) of θ and the corresponding credible intervals based on type II censored data, the natural choice of the scale parameter θ has a conjugate gamma prior $Gamma\left(a, b\right)$, with pdf

$$\pi_1\left(\theta \mid a, b\right) = \begin{cases} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, & \text{if } \theta > 0 \\ 0, & \text{if } \theta \leq 0, \end{cases}$$

with the hyper-parameters $a > 0$, $b > 0$ and $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$.

For more details, see Kundu (2008), and Kundu and Raqab (2012). For computing the Bayes estimates, we assume mainly a squared error loss (SEL) function only.

Based on a Type II censored data $Y_{1:n} < Y_{2:n} < ... < Y_{r:n}, (1 \leq r \leq n)$ and the prior of θ, we obtain the conditional posterior of θ using Eq.(5) and Eq.(6) as

$$\begin{aligned} \pi(\theta \mid data) &\propto \prod_{i=1}^{r} g(y_i \mid \theta)\left[1 - G(y_r;\theta)\right]^{n-r} \pi_1(\theta) \\ &\propto \theta^{r+a-1} e^{-\theta\left(b + y_r(n-r) + \sum_{i=1}^{r} y_i\right)}. \end{aligned} \tag{9}$$

The conditional posterior distribution of θ given data, $\pi(\theta \mid data)$ is

$$Gamma\left(r + a, b + y_r(n-r) + \sum_{i=1}^{r} y_i\right). \tag{10}$$

Therefore, the BE of θ, say $\hat{\theta}_{Bayes}$, is

$$\hat{\theta}_{Bayes} = \frac{r + a}{b + y_r(n-r) + \sum_{i=1}^{r} y_i}. \tag{11}$$

Also, the $(1-\beta)100\%$ Bayesian credible interval for θ is given by (θ_L, θ_U) such that

$$\int_{\theta_L}^{\theta_U} \pi(\theta \mid data)\, d\theta = 1 - \beta.$$

So,

$$\Pr(\lambda_L < \theta < \infty) = 1 - \frac{\beta}{2} \text{ and } \Pr(\lambda_U < \theta < \infty) = \frac{\beta}{2}. \tag{12}$$

Using Eq.(10) and the incomplete gamma function which is defined as $\Gamma(a, c) = \int_c^\infty x^{a-1}e^{-x}dx$, $a > 0, c > 0$, we obtain

$$\Gamma(r + a, u\theta_L) = \left(1 - \frac{\beta}{2}\right)\Gamma(r + a) \text{ and } \Gamma(r + a, u\theta_U) = \frac{\beta}{2}\Gamma(r + a), \tag{13}$$

where $u = b + y_r(n-r) + \sum_{i=1}^{r} y_i$.

Now, by using a suitable numerical method to solve equations (13), we obtain (θ_L, θ_U) .

4. Bayes prediction

The Bayes prediction of the unknown observation from the future sample based on current a vailable sample, known as informative sample, is an important feature in Bayes analysis. We mainly consider the estimation of posterior predictive

density of a future observation based on the current data. The objective is to provide an estimate of the future observation of an residual experiment based on the results obtained from an informative experiment.

Let $y_{1:n} < y_{2:n} < ... < y_{r:n}$ be the observed sample known as informative sample and $y_{r+1:n} < y_{r+2:n} < ... < y_{n:n}$ be the unobserved future sample. Our goal is to predict $Y_{s:n}$, $r < s \leq n$. The posterior predictive density of $Y_{s:n}$ given the observed data $\underset{\sim}{Y} = (y_{1:n}, y_{2:n}, ..., y_{r:n})$ is defined as

$$\pi_{Y_s}(y \mid data) = \int_0^\infty g_{Y_s|data}(y \mid \theta)\, \pi(\theta \mid data)\, d\theta \ , \ y_s > y_r,$$

where $g_{Y_s|data}(y \mid \theta)$ is the conditional density of Y_s given θ and data $\underset{\sim}{Y}$, see for example Chen, Shao and Ibrahim (2000).

Using the Markovian property of the conditional order statistics, see David and Nagaraja (2003), the conditional pdf of $Y_{s:n}$ given $\underset{\sim}{Y}$ is just the conditional pdf of $Y_{s:n}$ given $Y_{r:n}$, $(r+1 \leq s \leq n)$ that is,

$$\begin{aligned} g_{Y_s|data}(y_s \mid \alpha, \lambda) &= g_{Y_s|Y_r}(y_s \mid \alpha, \lambda) = \frac{g_{r;s:n}(y_r, y_s)}{g_{r:n}(y_r)} \\ &= c\ \theta\ \left[1 - e^{-\theta(y-y_r)}\right]^{s-r-1} e^{-\theta(y-y_r)(n-s+1)}, \end{aligned} \tag{14}$$

where $g_{r;s:n}(y_r, y_s)$ is the joint pdf of the $r-th$ and $s-th$ order statistics from a sample of size n from the parent distribution $G(.)$. One can observe that the conditional density of $Y_{s:n}$ given $Y_{r:n}$ is just the marginal density of $(s-r)-th$ order statistics from a sample of size $(n-r)$ from the left truncated distribution of $G(.)$ at y_r. By using the binomial expansian, we have

$$g_{Y_s|data}(y \mid \alpha, \lambda) = c\ \theta \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i\, e^{-\theta(y-y_r)[n+i-s+1]}, \tag{15}$$

where $c = \frac{(n-r)!}{(s-r-1)!(n-s)!}$. So, the posterior predictive density of $Y_{s:n}$ at any point $y > y_r$ is

$$\begin{aligned} \pi_{Y_s}(y \mid data) &= c \int_0^\infty \theta \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i e^{-\theta(y-y_r)[n+i-s+1]}\ \pi(\theta \mid data) d\theta \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \int_0^\infty \theta^{r+a} e^{-\theta[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^r y_i+b]} d\theta \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \frac{\Gamma(r+a+1)}{[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^r y_i+b]^{r+a+1}}. \end{aligned}$$

The Bayes predictor (BP) of $Y_{s:n}$ under SEL function is

$$\begin{aligned} Y_{s:n}^{BP} &= \int_{y_r}^{\infty} y\ \pi_{Y_s}(y \mid data)\, dy \\ &= c\int_{y_r}^{\infty} y \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \\ &\quad \frac{\Gamma(r+a+1)}{[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^{r} y_i + b]^{r+a+1}} dy. \end{aligned}$$

By using the transformation $v = (y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^{r} y_i + b$, we get

$$\begin{aligned} (16)\quad Y_{s:n}^{BP} &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \frac{1}{(n-s+i+1)^2} \\ &\quad \left[\frac{b+\sum_{i=1}^{r} y_i + y_r(n-r) + y_r(n+i-s+1)(r+a-1)}{(r+a-1)}\right]. \end{aligned}$$

Another important problem is to construct a two sided predictive interval of the order statistics Y_s. For this, we need to obtain the predictive survival function of Y_s which is defined as

$$\begin{aligned} (17)\quad S_{Y_s|data}(y \mid \theta) &= \Pr(Y > y) = \int_{y}^{\infty} g_{Y_s|data}(z \mid \theta)\, dz \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} e^{-\theta(y-y_r)(n-s+i+1)}. \end{aligned}$$

Under the SEL function, the predictive survival function of Y_s is

$$\begin{aligned} &S_{Y_s|data}^{P}(y \mid \alpha, \lambda) \\ &= \int_{0}^{\infty} c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} e^{-\theta(y-y_r)(n-s+i+1)} \\ &\quad \pi(\theta \mid data)\, d\theta. \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} \\ &\quad \frac{[y_r(n-r)+\sum_{i=1}^{r} y_i + b]^{r+a}}{[(y-y_r)(n-s+i+1)+y_r(n-r)+\sum_{i=1}^{r} y_i + b]^{r+a}}. \end{aligned} \tag{18}$$

Now, the $(1-\beta)\,100\%$ predictive interval of Y_s can be found by solving the following non -linear equations (19) for the lower bound (L) and upper bound (U) using a suitable numerical technique

$$\hat{S}{}_{Y_s|data}^{P}(L) = 1 - \frac{\beta}{2} \text{ and } \hat{S}{}_{Y_s|data}^{P}(U) = \frac{\beta}{2}. \tag{19}$$

5. Simulations and data analysis

5.1 Simulations

We report some numerical experiments performed to evaluate the behavior of the proposed methods for different sampling schemes and different priors based on type II censored data. We have assumed $\theta = 2$ to generate exponential residual life data at $t = 0.2$, $t = 0.5$, and $t = 0.9$. To compute the BEs under SEL function, we have assumed $\pi_1(\theta)$, the prior of θ, has gamma density function with shape and scale parameters a and b respectively. For the computations of BEs, we consider types of prior for θ: first prior is the non-informative prior, $i.e$ $a = b = 0$, we call this prior as $Prior$ 0, second prior is the informative prior, her we use three cases namely $(Prior\ 1:\ a = 1, b = 2)$, $(Prior\ 2:\ a = 1, b = 3)$, and $(Prior\ 3:\ a = 1, b = 4)$ .From Eq.8 and Eq.11, we notice that the MLE and BE are equals under $Prior$ 0. In each sampling schemes, we compute the MLE and the BE of θ under SEL function and 95% credible intervals of θ based on 10,000 samples. We report the average Bayes estimates, mean squared errors (MSEs), coverage percentages lengths and average credible intervals lenghts for θ based on 10,000 replications.

Table 1: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.2$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1846	1.6744	1.4269	1.2260
	(0.5848)	(0.2436)	(0.3878)	(0.5426)
Scheme 2: $n = 25, r = 15$	2.0850	1.7407	1.6311	1.4420
	(0.3933)	(0.2101)	(0.2472)	(0.3695)
Scheme 3: $n = 25, r = 20$	2.1844	1.8432	1.6445	1.5803
	(0.2926)	(0.1352)	(0.1905)	(0.2320)
Scheme 4: $n = 40, r = 10$	2.3275	1.6942	1.3856	1.2412
	(0.9030)	(0.2552)	(0.4404)	(0.6294)
Scheme 5: $n = 40, r = 20$	2.1413	1.7811	1.6714	1.5249
	(0.2658)	(0.1757)	(0.1827)	(0.2585)
Scheme 6: $n = 40, r = 30$	1.9813	1.8752	1.7477	1.6773
	(0.1200)	(0.0865)	(0.1297)	(0.1109)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

From tables 1,2 and 3, it is evident that the MLEs and Bayes estimates of θ improve when more information is available (r gets large), this observations is valid for priors ($Prior\ 1, Prior\ 2\ and\ Prior\ 3$) when $t = 0.2$, $t = 0.5$, and $t = 0.9$ and the Bayes estimates of θ under informative priors get better than the MLEs of θ. The Bayes estimates of θ under informative priors perform less better when the variance gets small. From table 4, we observe that the average credible intervals lenghts for θ become smaller when r increases and n is fixed for all priors when $t = 0.2$, $t = 0.5$, and $t = 0.9$.

For computing the predictors, based on exponential residual data we have obtained the point predictors and 95% predictive intervals (PIs) at $t = 0.2$, $t = 0.5$, and $t = 0.5$ for the missing order statistics Y_s, $r < s \leq n$ using $Prior$ 1. From table 5 and 6, we observe that the predicted values for the missing order statistics $Y_{s:n}$ are quite close to each other and fall in their PIs for all schemes.

Table 2: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.5$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1529	1.6004	1.4559	1.2141
	(0.5083)	(0.3032)	(0.3646)	(0.4532)
Scheme 2: $n = 25, r = 15$	2.1741	1.8137	1.5602	1.4170
	(0.3508)	(0.1919)	(0.2752)	(0.3230)
Scheme 3: $n = 25, r = 20$	2.1056	1.7677	1.6984	1.5445
	(0.2423)	(0.1604)	(0.1718)	(0.2066)
Scheme 4: $n = 40, r = 10$	2.1689	1.6156	1.4082	1.2768
	(0.5192)	(0.2661)	(0.4266)	(0.4884)
Scheme 5: $n = 40, r = 20$	2.1633	1.7927	1.6738	1.5455
	(0.2758)	(0.1742)	(0.1789)	(0.2161)
Scheme 6: $n = 40, r = 30$	2.0712	1.9185	1.7328	1.6936
	(0.1819)	(0.1016)	(0.1340)	(0.1353)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

Table 3: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.9$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1661	1.6435	1.4305	1.2974
	(0.4285)	(0.2821)	(0.3883)	(0.4123)
Scheme 2: $n = 25, r = 15$	2.2071	1.8032	1.5639	1.4454
	(0.3329)	(0.1496)	(0.2441)	(0.3094)
Scheme 3: $n = 25, r = 20$	2.0603	1.8125	1.7065	1.5159
	(0.1848)	(0.1243)	(0.1397)	(0.1809)
Scheme 4: $n = 40, r = 10$	2.1492	1.6915	1.4051	1.2488
	(0.5191)	(0.2422)	(0.4252)	(0.5043)
Scheme 5: $n = 40, r = 20$	2.1784	1.8448	1.6361	1.5587
	(0.2374)	(0.1534)	(0.1853)	(0.2117)
Scheme 6: $n = 40, r = 30$	2.0036	1.8822	1.7688	1.6280
	(0.1430)	(0.1093)	(0.1250)	(0.1316)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

Table 4: Average credible intervals lengths of θ based on residual type II censored data.

Schemes		*Prior*1	*Prior*2	*Prior*3
$n = 25, r = 10$	$t = 0.2$	2.7801 (0.94)	2.8156 (0.95)	2.8116 (0.94)
	$t = 0.5$	2.8632 (0.95)	2.8016 (0.91)	2.8068 (0.93)
	$t = 0.9$	2.8630 (0.92)	2.8713 (0.95)	2.7282 (0.95)
$n = 25, r = 15$	$t = 0.2$	2.1333 (0.95)	2.1384 (0.95)	2.1870 (0.97)
	$t = 0.5$	2.1600 (0.92)	2.2269 (0.93)	2.1665 (0.96)
	$t = 0.9$	2.1862 (0.95)	2.1852 (0.94)	2.1661 (0.93)
$n = 25, r = 20$	$t = 0.2$	1.8204 (0.96)	1.8177 (0.96)	1.8049 (0.95)
	$t = 0.5$	1.8020 (0.95)	1.8675 (0.92)	1.8371 (0.93)
	$t = 0.9$	1.8312 (0.94)	1.8265 (0.93)	1.8231 (0.92)
$n = 40, r = 10$	$t = 0.2$	2.7981 (0.95)	1.8483 (0.94)	2.8470 (0.93)
	$t = 0.5$	2.8390 (0.94)	2.7808 (0.96)	2.7822 (0.94)
	$t = 0.9$	2.7631 (0.92)	2.8666 (0.96)	2.8301 (0.95)
$n = 40, r = 20$	$t = 0.2$	1.8381 (0.94)	1.8337 (0.94)	1.8256 (0.94)
	$t = 0.5$	1.8494 (0.96)	1.8459 (0.94)	1.8910 (0.95)
	$t = 0.9$	1.8298 (0.94)	1.9649 (0.95)	1.8303 (0.93)
$n = 40, r = 30$	$t = 0.2$	1.4673 (0.95)	1.4863 (0.94)	1.4298 (0.96)
	$t = 0.5$	1.4764 (0.94)	1.4638 (0.96)	1.4657 (0.97)
	$t = 0.9$	1.4839 (0.95)	1.4300 (0.95)	1.4722 (0.94)

Note: The first entry represents the average lenght of the credible intervals, while the corresponding coverage percentages is given between the parentheses.

Table 5: Point predictors and PIs for the missing order statistics $\mathrm{Y}_{s:n}$, $r+1 \leq s \leq n$.

Schemes	*Predicted Values* (95% *PI*)	*Predicted Values* (95% *PI*)	*Predicted Values* (95% *PI*)
$n = 25, r = 10$	$t = 0.2$	$t = 0.5$	$t = 0.9$
$Y_{11:n}$	0.2370 (0.1982, 0.3556)	0.2927 (0.2731, 0.6551)	0.2504 (0.2109, 0.3713)
$Y_{14:n}$	0.3751 (0.2379, 0.6571)	0.4508 (0.2937, 0.7737)	0.3912 (0.2513, 0.6787)
$Y_{17:n}$	0.5551 (0.3119, 1.0245)	0.6569 (0.3784, 1.1944)	0.5748 (0.3268, 1.0533)
$Y_{19:n}$	0.7147 (0.3804, 1.3505)	0.8397 (0.4569, 1.5677)	0.7375 (0.3967, 1.3857)
$Y_{21:n}$	0.9333 (0.4731, 1.8044)	1.0899 (0.5630, 2.0875)	0.9603 (0.4911, 1.8485)
$Y_{23:n}$	1.2809 (0.6124, 2.5571)	1.4880 (0.7225, 2.9494)	1.3148 (0.6332, 2.6159)
$Y_{25:n}$	2.1749 (0.9016, 4.7886)	2.5117 (1.0537, 5.5046)	2.2262 (0.9280, 4.8910)
$n = 25, r = 15$			
$Y_{16:n}$	0.4719 (0.4175, 0.6329)	0.5385 (0.4729, 0.7324)	0.5369 (0.4703, 0.7341)
$Y_{18:n}$	0.6035 (0.4508, 0.9153)	0.6970 (0.5131, 1.0727)	0.6980 (0.5111, 1.0801)
$Y_{20:n}$	0.7759 (0.5173, 1.2597)	0.9047 (0.5932, 1.4878)	0.9093 (0.5925, 1.5021)
$Y_{21:n}$	0.8873 (0.5632, 1.4822)	1.0390 (0.6485, 1.7559)	1.0458 (0.6488, 1.7748)
$Y_{23:n}$	1.2122 (0.6954, 2.1528)	1.4306 (0.8077, 2.5640)	1.4439 (0.8106, 2.5964)
$Y_{24:n}$	1.4908 (0.7997, 2.7692)	1.7662 (0.9334, 3.3068)	1.7852 (0.9385, 3.3517)
$Y_{25:n}$	2.0478 (0.9738, 4.1686)	2.4374 (1.1433, 4.9930)	2.4677 (1.1518, 5.0663)
$n = 25, r = 20$			
$Y_{21:n}$	1.0750 (0.9405, 1.4664)	0.7690 (0.6696, 1.0582)	1.1499 (0.9983, 1.5910)
$Y_{22:n}$	1.2472 (0.9721, 1.8594)	0.8963 (0.6930, 1.3486)	1.3440 (1.0339, 2.0339)
$Y_{23:n}$	1.4769 (1.0380, 2.3601)	1.0660 (0.7417, 1.7186)	1.6028 (1.1082, 2.5981)
$Y_{24:n}$	1.8214 (1.1470, 3.1327)	1.3205 (0.8222, 2.2894)	1.9910 (1.2310, 3.4687)
$Y_{25:n}$	2.5104 (1.3436, 4.8671)	1.8296 (0.9675, 3.5710)	2.7675 (1.4526, 5.4233)

Note: The first entry represents the point predictor, while the corresponding PIs is given between the parentheses.

Table 6: Point predictors and PIs for the missing order statistics $Y_{s:n}$, $r+1 \leq s \leq n$.

Schemes	*Predicted Values* (95% *PI*)	*Predicted Values* (95% *PI*)	*Predicted Values* (95% *PI*)
$n = 40, r = 10$	$t = 0.2$	$t = 0.5$	$t = 0.9$
$Y_{11:n}$	0.1817 (0.1573, 0.2561)	0.1434 (0.1228, 0.2065)	0.1836 (0.1591, 0.2581)
$Y_{14:n}$	0.2620 (0.1808, 0.4285)	0.2114 (0.1427, 0.3523)	0.2640 (0.1827, 0.4307)
$Y_{17:n}$	0.3519 (0.2197, 0.6059)	0.2875 (0.1756, 0.5023)	0.3544 (0.2216, 0.6083)
$Y_{20:n}$	0.4542 (0.2675, 0.8040)	0.3438 (0.2018, 0.6117)	0.4564 (0.2695, 0.8067)
$Y_{23:n}$	0.5726 (0.3243, 1.0327)	0.4056 (0.2312, 0.7311)	0.5750 (0.3264, 1.0355)
$Y_{26:n}$	0.7134 (0.3923, 1.3044)	0.4742 (0.2641, 0.8633)	0.7160 (0.3944, 1.3077)
$Y_{29:n}$	0.8868 (0.4757, 1.6414)	0.5511 (0.3012, 1.0116)	0.8895 (0.4780, 1.6451)
$Y_{32:n}$	1.1130 (0.5829, 2.0854)	0.6386 (0.3434, 1.1810)	1.1162 (0.5853, 2.0897)
$Y_{35:n}$	1.4384 (0.7323, 2.7372)	0.7400 (0.3922, 1.3784)	1.4418 (0.7349, 2.7423)
$Y_{38:n}$	2.0249 (0.9818, 3.9715)	0.8611 (0.4497, 1.6152)	2.0291 (0.9842, 3.9781)
$Y_{40:n}$	3.1481 (1.3623, 6.7199)	1.0105 (0.5198, 1.9111)	3.1537 (1.3657, 6.7298)
$n = 40, r = 20$			
$Y_{21:n}$	0.3643 (0.3366, 0.4450)	0.4397 (0.4066, 0.5359)	0.4822 (0.4444, 0.5924)
$Y_{23:n}$	0.4257 (0.3530, 0.5695)	0.5129 (0.4262, 0.6843)	0.5661 (0.4668, 0.7622)
$Y_{25:n}$	0.4947 (0.3823, 0.6951)	0.5951 (0.4612, 0.8341)	0.6602 (0.5069, 0.9337)
$Y_{27:n}$	0.5731 (0.4203, 0.8337)	0.6887 (0.5064, 0.9994)	0.76730 (0.5586, 1.1228)
$Y_{29:n}$	0.6642 (0.4665, 0.9928)	0.7973 (0.5616, 1.1891)	0.8915 (0.6218, 1.3401)
$Y_{31:n}$	0.7726 (0.5227, 1.1825)	0.9266 (0.6285, 1.4153)	1.0395 (0.6984, 1.5989)
$Y_{33:n}$	0.9068 (0.5921, 1.4192)	1.0865 (0.7113, 1.6975)	1.2226 (0.7931, 1.9219)
$Y_{35:n}$	1.0826 (0.6813, 1.7355)	1.2962 (0.8177, 2.0747)	1.4626 (0.9149, 2.3536)
$Y_{37:n}$	1.3383 (0.8054, 2.2130)	1.6011 (0.9657, 2.6441)	1.8115 (1.0841, 3.0053)
$Y_{40:n}$	2.3799 (1.2058, 4.5515)	2.8430 (1.4431, 5.4325)	3.2331 (1.6307, 6.1969)
$n = 40, r = 30$			
$Y_{31:n}$	0.6022 (0.5554, 0.7362)	0.9975 (0.9352, 1.1758)	0.8688 (0.8042, 1.0535)
$Y_{33:n}$	0.7155 (0.5856, 0.9644)	1.1483 (0.9754, 1.4796)	1.0251 (0.8459, 1.3684)
$Y_{35:n}$	0.8641 (0.6470, 1.2395)	1.3461 (1.0571, 1.8458)	1.2305 (0.9305, 1.7479)
$Y_{37:n}$	1.0800 (0.7434, 1.6425)	1.6335 (1.1855, 2.3823)	1.5279 (1.0636, 2.3039)
$Y_{40:n}$	1.9598 (1.0722, 3.6069)	2.8046 (1.6231, 4.9973)	2.7417 (1.5171, 5.0142)

Note: The first entry represents the point predictor, while the corresponding PIs is given between the parentheses.

5.2 Data analysis

We analyze the residual real lifetimes data (in minutes) to breakdown of an insulating fluid at voltage 40 kv is usually assumed to be exponential in engineering theory. The real lifetimes data has been recently considered by Nelson (1982). The sample consists of the smallest nine observations of tweleve times to breakdown as follows: $1, 1, 2, 3, 12, 25, 46, 56, 68$. Before we analyze the data, we divide all the points by 100.

The MLE of θ based on the above data is determined to be 1.4887. The Bayes estimate of θ using $Prior1$ is 1.1273. Also, the 95% cerdible interval of θ is $(0.9737, 2.0889)$. Now, we consider the prediction of the $10-th, 11-th$ and $12-th$ order statistics which are missing. The predicted values and the 95% PI of the $10-th, 11-th$ and $12-th$ order statistics are presented in Table 7. It is

observed that all predicted values with respect to SEL function are all ordered and fall in their corresponding PI.

Table 7: Point predictors and PIs for the real data.

$n = 12, r = 9$	$Y_{10:n}$	$Y_{11:n}$	$Y_{12:n}$
$Predicted\ Values$	103.40	319.23	411.35
$(95\%\ PI)$	$(79.63, 118.81)$	$(304.42, 335.28)$	$(394.90, 431.83)$

References

[1] E.K. Al-Hussaini, *Predicting observable from a general class of distributions*, Journal of Statistical Planning and Inference, 79 (1999), 79-81.

[2] N. Balakrishnan, C.T. Lin, P.S. Chan, *A comparison of two simple prediction intervals for exponential distribution*, IEEE T. Reliab., 54 (2005), 27-33.

[3] N. Balakrishnan, M. Kateri, *On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data*, Statistics & Probability Letters, 78 (2008), 2971-2975.

[4] D.G. Chen, Y.L. Lio, *Parameter estimation for generalized exponential distribution under progressive Type I interval censoring*, Computational Statistics and Data Analysis, 54 (2010), 1581-1591.

[5] M.H. Chen, Q.M. Shao, J.G. Ibrahim, *Monte Carlo Methods in Bayesian Computation*, Springer-Verlag, New York, 2000.

[6] H.A. David, H.N. Nagaraja, *Order Statistics*, 3rd Edition, Wiley, New York, 2003.

[7] Z.F. Janeen, *Empirical Bayes analysis of record statistics based on LINEX and quadratic loss functions*, Comput. Math. Appl., 47 (2004), 947-954.

[8] M.P. Kaminskiy, V.V. Krivtsov, *A Simple Proccedure for Bayesian Estimation of the Weibull Distribution*, IEEE Transactions on Reliability Analysis, 54 (2005), 612-616.

[9] D. Kundu, *Bayesian Inference and Life Testing Plan for the Weibull Distribution in Presence of Progressive Censoring*, Technometrics, 50 (2008), 144-154.

[10] D. Kundu, H. Howlader, *Bayesian Inference and Prediction of the Inverse Weibull Distribution for Type-II Censored Data*, Computational Statistics and Data Analysis, 54 (2010),1547-1558.

[11] D. Kundu, M.Z. Raqab, *Bayesian Inference and Prediction for a Type-II censored Weibull Distribution*, Journal of Statistical Planning and Inference, 142 (2012), 41-47.

[12] W. Nelson, *Applied Life Data Analysis*, John Wiley and Sons, New York, 1982.

[13] B. Pradhan, D. Kundu, *Bayes Estimation and Prediction of the Two-Parameter Gamma Distribution*, Journal of Statistical Computation and Simulation, 81 (2011), 1187-1198.

[14] A.A. Sarhan, *Empirical Bayes estimetes in exponential reliability model*, Appl. Math. Comput., 135 (2003), 319-332.

[15] S. Yimin, Y. Weian, *The EB Estimation of Scale-parameter for Two parameter Exponential Distribution Under the Type-I Censoring Life Test*, Journal of Physical Sciences, 14 (2010), 25-30.

Accepted: 17.05.2018

ON COMPUTING DIFFERENTIAL TRANSFORM OF NONLINEAR NON-AUTONOMOUS FUNCTIONS AND ITS APPLICATIONS

Essam R. El-Zahar*
Department of Mathematics
Faculty of Sciences and Humanities
Prince Sattam Bin Abdulaziz University
Alkharj, 11942
KSA
and
Department of Basic Engineering Science
Faculty of Engineering
Shebin El-Kom, 32511
Menofia University
Egypt
essam_zahar2006@yahoo.com

Abdelhalim Ebaid
Department of Mathematics
Faculty of Science
Tabuk University
P.O. Box 741, Tabuk 71491
KSA

Abstract. Although being powerful, the differential transform method yet suffers from a drawback which is how to compute the differential transform of nonlinear non-autonomous functions that can limit its applicability. In order to overcome this defect, we introduce in this paper, new general formulas and their related recurrence relations for computing the differential transform of any analytic nonlinear non-autonomous function with one or multi-variable. Several test examples for different types of nonlinear differential and integro-differential equations are solved to demonstrate the applicability and validity of the present method. The obtained results declare that the suggested approach not only effective but also a straightforward and powerful for solving differential and integro-differential equations with complex nonlinearities.

Keywords: differential transform method, nonlinear non-autonomous functions, nonlinear differential and integro-differential equations.

1. Introduction

The Differential Transform Method (DTM) which is based on Taylor series expansion was first introduced by Zhou [1] and has been successfully applied to a wide class of nonlinear problems arising in mathematical sciences and engineer-

∗. Corresponding author

ing. The main advantage of DTM is that it can be applied directly to nonlinear differential equations with no need for linearization, discretization, or perturbation. Additionally, DTM does not generate secular terms (noise terms) and does not need to analytical integrations as other semi-analytical numerical methods such as HPM, HAM, ADM or VIM [2-7] and so DTM is an attractive tool for solving differential equations. Although this method has been proved to be an efficient tool for handling nonlinear differential and integro-differential equations, the nonlinear functions used in these equations are restricted to certain kinds of nonlinearities, e.g., polynomials and products with derivatives. For other types of nonlinear functions, Chang and Chang [8] construct a new algorithm based on obtaining a differential equation satisfied by this nonlinear function and then applying DTM to this obtained differential equation. Although their treatment was found effective for some forms of nonlinearity [9,10] it significantly increases the computational budget, especially if there are two or more nonlinear functions involved in the differential equation being investigated [11]. Moreover, in the case of complex nonlinearities, it may be quite difficult to obtain the differential equations satisfied by these nonlinear functions. To overcome this difficulty, a new formula has been derived [11-14] to calculate the differential transform of nonlinear autonomous one variable functions $f(y)$. Unfortunately, for nonlinear non-autonomous multi-variable functions $f\left(t, y_j(t)\right)$, $j = 1, 2, .., m$, no related formula has been given to calculate their transform functions. In order to overcome this defect, new general formulas and their related recurrence relations are deduced in this paper for computing the differential transform of any analytic nonlinear non-autonomous function with one or multi-variable. The proposed method deals directly with the nonlinear non-autonomous function in its form without any special kinds of transformations or algebraic manipulations. Also, there is no need to compute the differential transform of other functions to obtain the required one. For autonomous function, as a special case of the current study, these formulas and recurrence relations have the same mathematical structure as the Adomian polynomials but with constants instead of variable components. The applicability and validity of the present method are demonstrated through solving several test examples including nonlinear differential and integro-differential equations of different types.

2. New differential transform formulas

The basic definitions , fundamental theorems , convergence , error analysis of DTM and its applicability for various kinds of differential and integro-differential equations are given in [1, 15-31]. If a differential equation contains an analytic nonlinear non-autonomous function $f\left(t, y(t)\right)$ then the differential transform $F(n)$ of the function $f\left(t, y(t)\right)$ can be computed from the following theorems, where we assume that $f\left(t, y(t)\right) \neq f_0\left(t\right) f_1\left(y(t)\right)$.

Theorem 1. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f\left(t, y(t)\right)$ at a point t_0 can be computed from the formula*

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{n} Y(i)\lambda^i\right)\right]_{\lambda=0} \tag{1}$$

where $Y(i)$ is the differential transform of $y(t)$.

Proof. The differential transform$F(n)$ of $f\left(t, y(t)\right)$ at a point t_0 is defined as

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{dt^n} f\left(t, y(t)\right)\right]_{t=t_0}. \tag{2}$$

And since $y(t)$ can be expressed as

$$y(t) = \sum_{i=0}^{\infty} Y(i)(t - t_0)^i, \tag{3}$$

where $Y(i)$ is the differential transform of $y(t)$ about t_0 , then we have

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{dt^n} f\left(t, \sum_{i=0}^{\infty} Y(i)(t - t_0)^i\right)\right]_{t=t_0}. \tag{4}$$

Now, let $t - t_0 = \lambda$, then Eq. 4 becomes

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{\infty} Y(i)\lambda^i\right)\right]_{\lambda=0}.$$

And since $F(n)$ is a function of t_0 and $\{Y(i)\}_{i=0}^{n}$ only, then

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{n} Y(i)\lambda^i\right)\right]_{\lambda=0}.$$

By this way, the proof of Theorem 1 is completed.

Theorem 2. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f\left(t, y(t)\right)$ at a point t_0 , satisfies the analytic recurrence relation*

$$F(n) = \frac{1}{n}\left(\frac{\partial}{\partial t_0} F(n-1) + \sum_{i=0}^{n-1}(i+1)Y(i+1)\frac{\partial}{\partial Y(i)} F(n-1)\right),\ n \geq 1 \tag{5}$$

where $F(0) = f\left(t_0,\ Y(0)\right)$.

Proof. Since we have

$$f^{(n)}(t, y(t)) = \frac{\partial}{\partial t} f^{(n-1)}\left(t, y(t)\right) + \frac{\partial}{\partial y(t)} f^{(n-1)}\left(t, y(t)\right)\frac{dy(t)}{dt},$$

then

$$F(n) = \frac{1}{n!}\left[\frac{\partial}{\partial t} f^{(n-1)}\left(t, \sum_{i=0}^{\infty} Y(i)(t-t_0)^i\right)\right.$$
$$\left. + \sum_{i=0}^{\infty} \frac{\partial}{\partial Y(i)} f^{(n-1)}\left(t, \sum_{i=0}^{\infty} Y(i)(t-t_0)^i\right)(i+1)Y(i+1)\right]_{t=t_0}$$
$$= \frac{1}{n!}\left[(n-1)!\frac{\partial}{\partial t_0}F(n-1) + (n-1)!\sum_{i=0}^{\infty}(i+1)\,Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right]$$

and since $F(n-1)$is a function of $t_0$and$\{Y(i)\}_{i=0}^{n-1}$, then

$$F(n) = \frac{1}{n}\left(\frac{\partial}{\partial t_0}F(n-1) + \sum_{i=0}^{n-1}(i+1)Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right),\ n \geq 1.$$

By this way, the proof of Theorem 2 is completed.

Thus by Theorems 1 and 2 we have implemented a new algorithm for computing the one-dimensional differential transform of any analytic nonlinear non-autonomous function $f\,(t, y(t))$.

As a special case of the present study, i.e., for autonomous functions, the formula (1) and recurrence relation (5) are reduced to, respectively

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} Y(i)\,\lambda^i\right)\right]_{\lambda=0}, \tag{6}$$

$$F(n) = \frac{1}{n}\left(\sum_{i=0}^{n-1}(i+1)\,Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right),\ n \geq 1, \tag{7}$$

where the special formula (6) is the present formula in [11-14]. In fact if a system of one-dimensional differential equations contains m coupled analytic nonlinearity function $f\,(t, y_j(t))$, $j = 1, 2, .., m$, where we assume that $f\,(t, y_j(t)) \neq f_0\,(t)\,f_1\,(y_1(t))\,f_2\,(y_2(t))\,...f_m\,(y_m(t))$, then Theorems 1 and 2 can be easily extended to multi-variable function and satisfy the following algorithms.

Corollary 1. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f\,(t, y_j(t))$, $j = 1, 2, .., m$ at a point t_0 can be defined by*

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(t_0 + \lambda, \sum_{i=0}^{n} Y_j(i)\,\lambda^i\right)\right]_{\lambda=0}, j = 1, 2, .., m, \tag{8}$$

where $Y_j(i)$ is the differential transform of $y_j(t)$.

Corollary 2. *The differential transform $F(n)$ of any analytic nonlinear non-autonomous function $f(t, y_j(t))$, $j = 1, 2, .., m$ at a point t_0 , satisfies the analytic recurrence relation*

$$F(n)=\frac{1}{n}\left(\frac{\partial}{\partial t_0}F(n-1)+\sum_{j=1}^{m}\sum_{i=0}^{n-1}(i+1)Y_j(i+1)\frac{\partial}{\partial Y_j(i)}F(n-1)\right), n\geq 1, \tag{9}$$

where $F(0) = f(t_0,\ Y_j(0))$.
Moreover and as a special case of the present study, i.e., for autonomous function, the definition (8) and recurrence relation (9) are reduced, respectively, to

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} Y_j(i)\,\lambda^i\right)\right]_{\lambda=0}, j = 1, 2, .., m, \tag{10}$$

$$F(n) = \frac{1}{n}\left(\sum_{j=1}^{m}\sum_{i=0}^{n-1}(i+1)Y_j(i+1)\frac{\partial}{\partial Y_j(i)}F(n-1)\right), \tag{11}$$

which have the same mathematical structure as the Adomian polynomials in [32] but with constants instead of variable components.

Table 1. Some fundamental operations of DTM.

Original function	Transformed function
$u(t) = \beta\,(\,v(t) \pm w(t)\,)$	$U(k) = \beta\,V(k) \pm \beta\,W(k)$
$u(t) = v(t)\ w(t)$	$U(k) = \sum_{\ell=0}^{k} V(\ell)\,W(k-\ell)$
$u(t) = \frac{d^m v(t)}{dt^m}$	$U(k) = \frac{(k+m)!}{k!}V(k+m)$
$u(t) = (\beta + t)^m$	$U(k) = H[m,k]\frac{m!}{k!(m-k)!}(\beta + t_0)^{m-k}$, $H[m,k] = \begin{cases}1, & \text{if } m \geq k\\ 0, & \text{if } m < k\end{cases}$
$u(t) = e^{\lambda t}$	$U(k) = \frac{\lambda^k}{k!}e^{\lambda\, t_0}$
$u(t) = \sin(\omega t + \beta)$	$U(k) = \frac{\omega^k}{k!}\sin(\omega t_0 + \beta + \frac{k\pi}{2})$
$u(t) = \cos(\omega t + \beta)$	$U(k) = \frac{\omega^k}{k!}\cos(\omega t_0 + \beta + \frac{k\pi}{2})$
$u(t) = \int_{t_0}^{t} v(t)dt$	$U(k) = \frac{V(k-1)}{k}, k \geq 1, U(0) = 0$
$u(t) = \int_{t_0}^{t} v(t)w(t)$	$U(k) = \frac{1}{k}\sum_{l}^{k-1} V(l)W(k-l-1), k \geq 1, k \geq 1, U(0) = 0.$

3. Applications

In this section, we have solved different types of differential and integro-differential problems with different forms of nonlinear non-autonomous functions that the formula in the literature [11-14] was found not applicable.

Example 1. Consider the nonlinear initial-value problem

$$y'(t) - y(t) = \ln\left(t + y(t)\right), t \in [1,2], y(t) \geq 0, y(1) = 0. \tag{12}$$

Using the basic properties of DTM, Table.1, and taking the transform of equations in (12) result in

$$(k+1)Y(k+1) - Y(k) = F(k), Y(0) = 0, k = 0, 1, 2, ..., \tag{13}$$

where$F(k)$is the differential transform of the nonlinear term $\ln(t+y(t))$. $F(k)$is computed using the present method and given by

$$\begin{aligned}
&F(0) = \ln(1+Y(0)), \;\; F(1) = \frac{1+Y(1)}{1+Y(0)}, \\
&F(2) = \frac{Y(2)}{1+Y(0)} - \frac{1}{2}\frac{(1+Y(1))^2}{(1+Y(0))^2}, \\
&F(3) = \frac{Y(3)}{1+Y(0)} - \frac{Y(2)\,(1+Y(1))}{(1+Y(0))^2} + \frac{(1+Y(1))^3}{3\,(1+Y(0))^3} \\
&F(4) = \frac{Y(4)}{1+Y(0)} - \frac{Y(3)\,(1+Y(1))}{(1+Y(0))^2} + \frac{Y(2)\,(1+Y(1))^2}{(1+Y(0))^3} \\
&\qquad - \frac{Y(2)^2}{2\,(1+Y(0))^2} - \frac{(1+Y(1))^4}{4\,(1+Y(0))^4}.
\end{aligned} \tag{14}$$

Therefore, a combination of the recurrence relation(13) and the computed $F(k)$ in (14) with the solution formula (13) results in the series solution $\tilde{y}(t) = \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 + \frac{1}{24}(t-1)^4 + \frac{1}{120}(t-1)^5 + ...$ For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t) = \mathrm{e}^{t-1} - t$, which is the exact solution.

Table 2: Numerical results for Example 2.

	$\mid y(t_i) - \tilde{y}(t_i) \mid$		
t_i	$N = 5$	$N = 10$	$N = 15$
0.0	0.0000+e00	0.0000+e00	0.0000+e00
0.2	7.3689e-006	8.2100e-010	9.6316e-011
0.4	4.5510e-004	1.3793e-006	2.3291e-008
0.6	4.8955e-003	1.1150e-004	3.0230e-006
0.8	2.5675e-002	2.4110e-003	2.6524e-004
1.0	9.0957e-002	2.5516e-002	8.3823e-003

Example 2. Consider the nonlinear initial-value problem

$$y'(t) + \varepsilon y(t)^2 = \varepsilon \sin(\, ty(t)\,), t \in [0,1]\, y(0) = 1. \tag{15}$$

Taking the differential transform of equations in (15) results in

$$(k+1)Y(k+1)+\varepsilon\sum_{\ell=0}^{k}Y(\ell)\,Y(k-\ell)=\varepsilon F(k), Y(0)=1, k=0,1,2,..., \tag{16}$$

where $F(k)$ is the differential transform of the nonlinear term $\sin(ty(t))$.$F(k)$ is computed using the present method and given by

$$\begin{aligned} &F(0)=0,\ \ F(1)=Y(0),\ \ F(2)=Y(1),\ \ F(3)=\frac{-Y(0)^3}{6}+Y(2),\\ &F(4)=y(3)-\frac{1}{2}Y(0)^2y(1),\\ &F(5)=\frac{1}{120}Y(0)^5-\frac{1}{2}Y(0)Y(1)^2-\frac{1}{2}Y(0)^2Y(2)+Y(4). \end{aligned} \tag{17}$$

Using (16) and (17), the series solution is obtained and given at $\varepsilon=0.1$ by

$$\tilde{y}(t)=1-\frac{1}{10}t+\frac{3}{50}t^2-\frac{23}{3000}t^3-\frac{119}{60000}t^4+\frac{247}{300000}t^5-\frac{2233}{4500000}t^6+....$$

The presented results are compared with those obtained using MATLAB built-in solver ode45 in Table 2. The ode45 solver integrates ODEs using explicit 4th & 5th Runge-Kutta (4, 5) formula [33]. In order to guarantee a good numerical reference, ode45 is configured using an absolute error of $10^{-12}$and relative error of 10^{-8}.

Example 3. Consider the nonlinear first order Volterra integro-differential equation

$$\begin{aligned} &y'(t)=\cos t-\frac{t^2}{2}+1+\int_0^t \sin^{-1}(1-\tau+y(\tau))\,d\tau,\\ &t\in[0,1]\,, y\,(0)=-1, y'\,(0)=2. \end{aligned} \tag{18}$$

The differential transform of equations in (18) are

$$\begin{aligned} &(k+1)Y(k+1)=\frac{1}{k!}\cos\left(\frac{k\pi}{2}\right)-\frac{\delta(k-2)}{2}+\delta(k)+\frac{F(k-1)}{k},\\ &Y(0)=-1\,, Y(1)=2\,,\ k\geq 1, \end{aligned}$$

where $F(k)$ is the differential transform of the nonlinear term $\sin^{-1}(1-\tau+y(\tau))$. By applying the present method to the nonlinear function $f(\tau,y(\tau))=\sin^{-1}(1-\tau+y(\tau))$ at $\tau_0=0$, $Y(0)=-1$ and $Y(1)=2$, and by using the principal values

of the square root and inverse trigonometric functions, we get

$$
\begin{aligned}
&F(0)=0,\ F(1)=1,\ \ F(2)=Y(2),\ \ F(3)=Y(3)+\frac{1}{6},\\
&F(4)=Y(4)+\frac{1}{2}Y(2),\\
&F(5)=Y(5)+\frac{3}{40}+\frac{1}{2}Y(3)+\frac{1}{2}Y(2)^2,\\
&F(6)=\frac{3}{8}Y(2)+\frac{1}{2}Y(4)+Y(2)Y(3)+Y(6)+\frac{1}{6}Y(2)^3\\
&F(7)=\frac{1}{112}\left(56Y(3)+84\right)Y(2)^2+Y(2)Y(4)+\frac{3}{8}Y(3)\\
&\qquad+\frac{1}{2}Y(5)+Y(7)+\frac{1}{2}Y(3)^2+\frac{5}{112}
\end{aligned}
\tag{19}
$$

Hence, the series solution is obtained and given by

$$\tilde{y}(t)=-1+2t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7+\frac{1}{362880}t^9+...$$

For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t)=\sin(t)+t-1$, which is the exact solution.

Example 4. Consider the nonlinear second order Volterra integro-differential equation

$$y''(t)-2y(t)y'(t)=-t+\int_0^t\frac{\sec^2\tau}{1+y(\tau)^2}\,d\tau, t\in[0,1]\,, y(0)=0, y'(0)=1. \tag{20}$$

The differential transform of equations in (20) are $(k+2)(k+1)Y(k+2)=2\sum_{\ell=0}^{k}(\ell+1)\ Y(\ell+1)\,Y(k-\ell)-\delta(k-1)+\frac{F(k-1)}{k}\,,\ Y(0)=0\,,\ Y(1)=1\,,\ Y(2)=0\,,\ k\geq 1$,where $F(k)$ is the differential transform of the nonlinear term $\frac{\sec^2\tau}{1+y(\tau)^2}$ obtained using the present method at $\tau_0=0$ and given by

$$
\begin{aligned}
&F(0)=1,\ F(1)=0,\ \ F(2)=0,\ \ F(3)=0,\\
&F(4)=\frac{2}{3}-2Y(3),\ Y(5)=-2Y(4),\\
&F(6)=\frac{-13}{45}-2Y(5)+2Y(3)-Y(3)^2,\\
&F(6)=(2-2Y(3))Y(4)-2Y(6),\\
&F(7)=\frac{17}{35}+5Y(3)^2-\frac{10}{3}Y(3)-2Y(3)Y(5)+2Y(5)-2Y(7)-Y(4)^2
\end{aligned}
\tag{21}
$$

Hence, the approximate series solution is obtained and given by

$$\tilde{y}(t)=t+\frac{1}{3}t^3+\frac{2}{15}t^5+\frac{17}{315}t^7+\frac{62}{2835}t^9+...$$

For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t) = \tan t$, which is the exact solution.

Example 5. Consider the nonlinear Volterra integro-differential equation with proportional delay

$$y'\left(\frac{t}{2}\right) = \frac{1}{2} - t\sin t + \int_0^t \frac{y\,(3\tau)^2 \sin t}{(3\tau+1)^2}\,d\tau, y(0) = 1, y'(0) = 1. \tag{22}$$

The differential transform of equations in (22) are

$$(k+1)Y(k+1)\left(\frac{1}{2}\right)^{k+1} = \frac{1}{2}\delta(k) - \sum_{\ell=0}^{k}\frac{1}{\ell!}\sin\left(\frac{\ell\pi}{2}\right)\delta(k-\ell-1)$$
$$+\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\sin\left(\frac{\ell\pi}{2}\right)\frac{F(k-\ell-1)}{k-\ell}\ , Y(0) = 1\,,\ Y(1) = 1\,,\ k \geq 1,$$

where $F(k)$ is the differential transform of the nonlinear term $\frac{w(\tau)^2}{(3\tau+1)^2}$, $w\,(\tau) = \sum_{k=0}^{\infty} 3^k Y(k)\tau^k$, obtained using the present method and given by

$$\begin{aligned} &F(0) = 1,\ F(1) = 0,\ \ F(2) = 18Y(2),\ \ F(3) = -54Y(2) + 54Y(3),\\ &F(4) = 81Y(2)^2 + 162Y(2) - 162Y(3) + 162Y(4),\\ &F(5) = -486Y(2)^2 + (486Y(3) - 486)Y(2) - 486Y(4) + 486Y(5) + 486Y(3). \end{aligned} \tag{23}$$

Hence, the series solution is obtained and given by

$$\tilde{y}(t) = t + 1$$

which is the exact solution.

Example 6. Consider the following nonlinear non-autonomous initial-value ODE system

$$\begin{aligned} y_1'(t) &= -y_1(t) + t + \ln\left(y_1(t) - \frac{1}{t + y_2(t)}\right)\\ y_2'(t) &= -y_2(t) - 1 + \frac{4}{y_1(t)} - \ln\left(t + y_2(t)\right)\\ y_1(0) &= 2\ , \quad y_2(0) = 1. \end{aligned} \tag{24}$$

Applying the differential transform to (24), results in

$$\begin{aligned} (k+1)\,Y_1(k+1) &= \delta(K-1) - Y_1(k) + F_1(k)\\ (k+1)\,Y_2(k+1) &= -Y_2(k) - \delta(K) + F_2(k)\\ Y_1(0) &= 2\ , \quad Y_2(0) = 1. \end{aligned} \tag{25}$$

where $F_1(k)$ and $F_2(k)$ are the differential transform of the nonlinear functions $f_1 = \ln\left(y_1(t) - \frac{1}{t+y_2(t)}\right)$ and $f_2 = \frac{4}{y_1(t)} - \ln(t + y_2(t))$, respectively. $F_1(k)$ and $F_2(k)$ are computed using the present method and given by

$$
\begin{aligned}
&F_1(0) = 0, \;\; F_1(1) = Y_1(1) + 1 + Y_2(1), \\
&F_1(2) = Y_1(2) - (1 + Y_2(1))^2 + Y_2(2) - \frac{1}{2}(Y_1(1) + 1 + Y_2(1))^2 \\
&F_1(3) = Y_1(3) + (1 + Y_2(1))^3 - 2(1 + Y_2(1))Y_2(2) + Y_2(3) \\
&\qquad - (Y_1(2) - 1 - 2Y_2(1) - Y_2(1)^2 + Y_2(2))(Y_1(1) \\
&\qquad + 1 + Y_2(1)) + \frac{1}{3}(Y_1(1) + 1 + Y_2(1))^3,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&F_2(0) = 2, \;\; F_2(1) = -Y_1(1) - Y_2(1) - 1 \\
&F_2(2) = \frac{1}{2}Y_1(1)^2 - Y_1(2) - Y_2(2) + \frac{1}{2}(1 + Y_2(1))^2 \\
&F_2(3) = -\frac{1}{4}Y_1(1)^3 + Y_1(2)Y_1(1) - Y_1(3) - Y_2(3) \\
&\qquad + (1 + Y_2(1))\, Y_2(2) - \frac{1}{3}(1 + Y_2(1))^3 .
\end{aligned} \tag{27}
$$

Hence, the series solutions are obtained and given as

$$\tilde{y}_1(t) = 2 - 2t + t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 - \frac{1}{60}t^5 + ...$$

$$\tilde{y}_2(t) = 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + ...$$

For sufficiently large number of terms, the closed form of the obtained series solutions are $\tilde{y}_1(t) = 2e^{-t}$, $\tilde{y}_2(t) = e^t - t$, which are the exact solutions.

4. Conclusions

In this paper, new general formulas and recurrence relations have been derived for computing the differential transform of any analytic nonlinear non-autonomous functions with one or multi-variable. As a special case of the present study, i.e., for autonomous function, the current formulas and recurrence relations have the same mathematical structure as the Adomian polynomials but with constants instead of variable components. The main advantage of the present method is that it can deal directly with nonlinear non-autonomous functions in their forms without any special transformation or algebraic manipulations and can be directly implemented in any symbolic language such as MATHEMATICA or MAPLE. The suggested modified DTM has been successfully applied on different types of differential and integro-differential equations with nonlinear non-autonomous functions that no related formula in the literature has been given to calculate their transform functions. The method is applied

directly to the nonlinear equations with no need to analytical integration which is essential for other semi-analytical numerical methods such as HPM, HAM, ADM or VIM. Moreover, the obtained series solutions declare that the suggested method is a straight forward even in solving different types of differential and integro-differential equations with complex nonlinearities. Finally, the authors believe that the present study should be extended to include similar differential and intego-differential equations in the applied sciences, which increases its applicability.

References

[1] J.K. Zhou, *Differential transformation and its applications for electrical circuits*, Huazhong University Press, Wuhan–China 1986.

[2] H. Hosseinzadeh, H. Jafari, M.R. Gholami, D.D. Ganji, *Solving a class of boundary value problems in structural engineering and fluid mechanics using homotopy perturbation and adomian decomposition methods*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 687-698.

[3] A. Ebaid, R. Rach, E. El-Zahar, *A new analytical solution of the hyperbolic Kepler equation using the Adomian decomposition method,* Acta Astronautica, 138 (2017), 1-9.

[4] H. Ghaneai, M.M. Hosseini, *Solving differential-algebraic equations through variational iteration method with an auxiliary parameter*, Applied Mathematical Modelling, 40 (2016), 3991-4001.

[5] Y. Zhao, Z. Lin, Z. Liu, S. Liao, *The improved homotopy analysis method for the Thomas–Fermi equation*, Applied Mathematics and Computation, 218(2012), 8363-8369.

[6] S. Al-Shara, F. Awawdeh, S. Abbasbandy, *An automatic scheme on the homotopy analysis method for solving nonlinear algebraic equations*, Italian journal of pure and applied mathematics, 37 (2017), 5-14.

[7] J. Biazar, M.A. Asadi, F. Salehi, *Rational Homotopy Perturbation Method for solving stiff systems of ordinary differential equations*, Applied Mathematical Modelling, 39(2015), 1291-1299.

[8] S.H.Chang, I.L. Chang, *A new algorithm for calculating one-dimensional differential transform of nonlinear functions*, Applied Mathematics and Computation, 195 (2008), 799-808.

[9] A.E.H. Ebaid, *Approximate periodic solutions for the non-linear relativistic harmonic oscillator via differential transformation method*, Communications in Nonlinear Science and Numerical Simulation, 15(2010), 1921-1927.

[10] A. E. Ebaid, *A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities*, Communications in Nonlinear Science and Numerical Simulation, 16 (2011), 528-536.

[11] A. E. Ebaid, *On a new differential transformation method for solving nonlinear differential equations*, Asian-European Journal of Mathematics, 6 (2013), 1350057.

[12] S.H. Behiry, *Differential transform method for nonlinear initial-value problems by adomian polynomials*, Journal of Applied & Computational Mathematics, 2012.

[13] H.S. Nik, F. Soleymani, *A Taylor-type numerical method for solving nonlinear ordinary differential equations*, Alexandria Engineering Journal, 52 (2013), 543-550.

[14] H. Fatoorehchi, H. Abolghasemi, *Improving the differential transform method: a novel technique to obtain the differential transforms of nonlinearities by the Adomian polynomials*, Applied Mathematical Modelling, 37 (2013), 6008-6017.

[15] V.S. Erturk, Z.M. Odibat, S. Momani, *Application of multi-step differential transform method for the analytical and numerical solutions of the density dependent Nagumo telegraph equation*, Romanian Journal of Physics, 57 (2012), 1065-1078.

[16] M. Abdulkawi, *Solution of Cauchy type singular integral equations of the first kind by using differential transform method*, Applied Mathematical Modelling, 39 (2015), 2107-2118.

[17] E.R. EL-Zahar, *Approximate analytical solutions of singularly perturbed fourth order boundary value problems using differential transform method*, Journal of King Saud University (Science), 25 (2013), 257–265.

[18] E.R. El-Zahar, *Applications of adaptive multi step differential transform method to singular perturbation problems arising in science and engineering*, Applied Mathematics and Information Sciences, 9 (2015), 223-232.

[19] E.R. El-Zahar, *Piecewise approximate analytical solutions of high order singular perturbation problems with a discontinuous source term*, International Journal of Differential Equations, 2016, Article ID 1015634, 12 pages, 2016. doi:10.1155/2016/1015634.

[20] A. Gökdoğan, M. Merdan, A. Yildirim, *Adaptive multi-step differential transformation method to solving nonlinear differential equations*, Mathematical and Computer Modelling, 55 (2012), 761-769.

[21] A. Arikoglu, I. Ozkol, *Solutions of integral and integro-differential equation systems by using differential transform method*, Computers & Mathematics with Applications, 56 (2008), 2411-2417.

[22] M. Al-Smadi, A. Freihat, O.A. Arqub, N. Shawagfeh, *A novel multistep generalized differential transform method for solving fractional-order Lü chaotic and hyperchaotic systems*, Journal of Computational Analysis & Applications, 19 (2015), 713-724.

[23] K. Moaddy, A. Freihat, M. Al-Smadi, E. Abuteen, I. Hashim, *Numerical investigation for handling fractional-order Rabinovich–Fabrikant model using the multistep approach*, Soft Computing, 1-10, 2016.

[24] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani, R. Ali Khan, *Numerical multistep approach for solving fractional partial differential equations*, International Journal of Computational Methods, 14 (2017), 1750029.

[25] A. El-Ajou, O.A. Arqub, M. Al-Smadi, *A general form of the generalized Taylor's formula with some applications*, Applied Mathematics and Computation, 256 (2015), 851-859.

[26] E.R. El-Zahar, H.M. Habib, M.M. Rashidi, I.M. El-Desoky, *A Comparison of Explicit Semi-Analytical Numerical Integration Methods for Solving Stiff ODE Systems*, American Journal of Applied Sciences, 12 (2015), 304.

[27] Z.M. Odibat, S. Kumar, N. Shawagfeh, A. Alsaedi, T. Hayat, *A study on the convergence conditions of generalized differential transform method*, Mathematical Methods in the Applied Sciences, 40 (2017), 40-48.

[28] C. Bervillier, *Status of the differential transformation method*, Applied Mathematics and Computation, 218 (2012), 10158-10170.

[29] R. Barrio, *Performance of the Taylor series method for ODEs/DAEs*, Applied Mathematics and Computation, 163 (2005), 525-545.

[30] E. R. EL-Zahar, *An Adaptive Step-Size Taylor Series Based Method and Application to Nonlinear Biochemical Reaction Model*, Trends in Applied Sciences Research, 7 (2012), 901-912.

[31] M. Thongmoon, S. Pusjuso, *The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations*, Nonlinear Analysis: Hybrid Systems, 4 (2010), 425-431.

[32] J.S. Duan, *Convenient analytic recurrence algorithms for the Adomian polynomials*, Applied Mathematics and Computation, 217 (2011), 6337-6348.

[33] J.R. Dormand, P.J. Prince, *A family of embedded Runge-Kutta formulae*, Journal of computational and applied mathematics, 6 (1980), 19-26.

Accepted: 17.05.2018

NAGSC: NESTEROV'S ACCELERATED GRADIENT METHODS FOR SPARSE CODING

Liang Liu
Chongqing University of Posts and Telecommunications
Chongqing, 400065
China
liuliang@cqupt.edu.cn

Ling Zhang
Chongqing Industry Polytechnic College
Chongqing, 401120
China
466695057@qq.com

Xiangguang Dai
Key Laboratory of Intelligent Information Processing and Control of Chongqing Municipal Institutions of Higher Education
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China

Yuming Feng*
Chongqing Engineering Research Center of Internet of Things and Intelligent Control Technology
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
Chinayumingfeng25928@163.com

Abstract. This paper proposes efficient algorithms for Sparse Coding. Firstly, Sparse Coding is divided into two sub-convex problems including L1 and L2 problems. Secondly, we transform the nonsmooth L1 problem into two smooth sub-problems, and alternatively optimize them by Nesterov's Accelerated Gradient methods (NAG). Thirdly, we apply NAG to optimize L2 problem. Finally, L1 and L2 problems are iteratively solved until convergence. Experiments show that our proposed algorithms are effective to optimize L1, L2 and learn over-complete bases.

Keywords: sparse coding, nonsmooth, nonconvex, accelerated gradient.

1. Introduction

Sparse Coding (SC) can be viewed as an unsupervised method to find representations for the original data. Given the unlabeled image data, SC learns bases to capture the intrinsic features, and the learned bases resemble the receptive

*. Corresponding author

fields of visual neurons [1, 2]. If the number of bases is smaller than the dimension of the original data, SC can be applied to dimensional reduction such as Principal Component Analysis [3], Locality Preserving Projections [4] and Linear Discriminant Analysis [5]; otherwise, SC can be used to learn a set of over-complete bases. Based on these properties, SC is widely applied to pattern recognition [6, 7], clustering [8, 9] and signal processing [10, 11].

SC is to learn sets of basis vectors so that an input vector can be represented by the combination of these basis vectors. Given an input vector $\overrightarrow{\xi} \in R^m$, SC can find the basis vectors $\overrightarrow{b_1}, \cdots, \overrightarrow{b_r} \in R^m$ and the coefficient vector $\overrightarrow{s} \in R^r$ such that the $\overrightarrow{\xi}$ can be represented by

$$\overrightarrow{\xi} \approx \sum_j \overrightarrow{b_j} s_j.$$

Generally, SC hopes to learn over-complete sets of basis vectors $\overrightarrow{b_1}, \cdots, \overrightarrow{b_r} \in R^m$ to represent a input vector $\overrightarrow{\xi} \in R^m$ (i.e. $r > m$). Given n input vectors $\overrightarrow{\xi^1}, \cdots, \overrightarrow{\xi^n}$ and their corresponding coefficient vectors $\overrightarrow{s^1}, \cdots, \overrightarrow{s^n}$, the SC model can be mathematically defined by

$$\begin{aligned} (1) \qquad \min_{b_j, s^i} \quad & \sum_{i=1}^{n} \frac{1}{2} \parallel \overrightarrow{\xi}^i - \sum_{j=1}^{r} \overrightarrow{b}_j s_j^i \parallel^2 + \beta \sum_{i=1}^{n} \sum_{j=1}^{r} \parallel s_j^i \parallel_1 \\ \text{subject to} \quad & \parallel \overrightarrow{b_j} \parallel^2 \leq c, \forall j = 1, \cdots, r. \end{aligned}$$

Suppose that $V = [\overrightarrow{\xi_1}, \cdots, \overrightarrow{\xi_n}] \in R^{m \times n}$ is an input matrix, $W = [\overrightarrow{b_1}, \cdots, \overrightarrow{b_r}] \in R^{m \times r}$ is a basis matrix and $H = [\overrightarrow{s^1}, \cdots, \overrightarrow{s^n}] \in R^{r \times n}$ is a coefficient matrix, then problem (1) can be transformed into the following matrix form:

$$\begin{aligned} (2) \qquad \min_{W,H} \quad & F(W,H) = \frac{1}{2} \|V - WH\|_F^2 + \beta \sum_{i=1}^{n} \parallel H_i \parallel_1 \\ \text{subject to} \quad & \sum_i W_{i,j}^2 \leq c, \forall j = 1, \cdots, r. \end{aligned}$$

In this paper, we propose efficient algorithms to optimize SC. Firstly, problem (2) is divided into two sub-problems including L1 and L2 problems. Secondly, we transform the nonsmooth L1 problem into two smooth sub-problems, and alternatively optimize them by Nesterov's Accelerated Gradient methods (NAG) [12]. Thirdly, we apply NAG to optimize L2 problem. Finally, L1 and L2 problems are alternatively solved until convergence. According to the block-coordinate-descent method [13], we divide problem (2) into two convex problems as follows:

$$\begin{aligned} (3) \qquad \min_{W} \quad & F(W) = \frac{1}{2} \|V - WH\|_F^2 \\ \text{subject to} \quad & \sum_i W_{i,j}^2 \leq c, \forall j = 1, \cdots, r \end{aligned}$$

and

$$\min_H F(H) = \frac{1}{2}\|V - WH\|_F^2 + \beta \sum_{i=1}^{n} \| H_i \|_1 . \tag{4}$$

For each sub-problem, we construct three sequences and update them recursively. These sequences can accelerate the optimization of each sub-problem. Therefore, this scheme leads to the convergence rate at $O(\frac{1}{k^2})$.

2. L1 optimization

Suppose that $P = [p_{ij}] = \max(H, 0)$, $Q = [q_{ij}] = \max(-H, 0)$, then $H = P - Q$. Problem (4) can be transformed as follows:

$$\begin{aligned} \min_{P,Q} \quad & F(P,Q) = \frac{1}{2}\|V - W(P-Q)\|_F^2 + \beta \sum_{i,j} p_{ij} + \beta \sum_{i,j} q_{ij} \\ \text{subject to} \quad & P \geq 0, Q \geq 0. \end{aligned} \tag{5}$$

To minimize (5), the common approach is the block-coordinate-descent method, where P and Q are alternatively minimized until convergence. Hence, we divide (5) into two the following optimization problems:

$$\begin{aligned} \min_{P} \quad & F(P) = \frac{1}{2}\|V + WQ - WP\|_F^2 + \beta \sum_{i,j} p_{ij} \\ \text{subject to} \quad & P \geq 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \min_{Q} \quad & F(Q) = \frac{1}{2}\|V - WP - (-W)(Q)\|_F^2 + \beta \sum_{i,j} q_{ij} \\ \text{subject to} \quad & Q \geq 0. \end{aligned} \tag{7}$$

According to (6) and (7), it is clear that both of them have the same matrix form. In the following, we only consider how to optimize (6), then (7) can be solved accordingly. Based on Lemma 1 and Lemma 2, NAG can solve (6) efficiently.

Lemma 1 ([14])**.** The objective function $F(P)$ is convex.

Lemma 2 ([14])**.** The gradient $\nabla F(P)$ is Lipschitz continuous, and the Lipschitz constant is $\| W^T W \|$.

Recent researches show that NAG are suitable for convex optimization problems and can obtain the convergence rate at $O(\frac{1}{k^2})$. To use NAG effectively, the optimization problem should be convex and its gradient is Lipschitz continuous.

In particular, three sequences are constructed by NAG, and they are alternatively updated in each iteration. At the iteration number $k \geq 1$, the updating rules for optimizing (6) are given as follows:

$$(8) \quad Y_k = \arg\min_{Y \geq 0} \{F(P_k) + < \nabla F(P_k), Y - P_k > + \frac{1}{2} L \parallel Y - P_k \parallel_F^2\},$$

$$(9) \quad Z_k = \arg\min_{Z \geq 0} (\frac{L}{2} \parallel Z - Z_{k-1} \parallel_F^2 + \tau_k(< \nabla F(P_k), Z >)),$$

$$(10) \quad P_{k+1} = \alpha_k Z_k + (1 - \alpha_k) Y_k,$$

where $\alpha_k = \frac{2}{k+3}$, $\tau_k = \frac{k+1}{2}$ and $< \cdot, \cdot >$ is the sum of the element-wise multiplication of two matrices. In the following, we apply KKT conditions to solve constrained optimization problems (8) and (9). By the Lagrange multiplier method, (8) is re-written to the following form:

$$(11) \quad L(Y, \lambda) = F(P_k) + < \nabla F(P_k), Y - P_k > + \frac{1}{2} L \parallel Y - P_k \parallel_F^2 - < \lambda, Y >,$$

where $\lambda \in R^{r \times n}$. Let (Y_k, λ_k) be the optimal solution of (11). The KKT conditions are summarized as

$$(12) \quad Y_k \geq 0, \lambda_k \geq 0, < \lambda_k, Y_k >= 0,$$

$$(13) \quad \nabla F(P_k) + L(Y - P_k) - \lambda_k = 0.$$

However, λ_k is unknown. According to (13), we have

$$(14) \quad Y_k = P_k - \frac{1}{L} \nabla F(P_k) + \frac{1}{L} \lambda_k.$$

It is clear that only $\lambda_k > 0$ and $Y_k = 0$ can satisfy (12). Hence, we obtain

$$(15) \quad \lambda_k = \max(\nabla F(P) - L P_k, 0).$$

Next, we substitute (15) into (14). The optimal solution of Y_k can be simplified by the following term.

$$(16) \quad Y_k = \max(P - \frac{1}{L} \nabla F(P_k), 0).$$

Similarly, the optimal solution of Z_k can be obtained as follows:

$$(17) \quad Z_k = \max(Z_k - \frac{\tau_k}{L} \nabla F(P_k), 0).$$

Above all, the final updating rules for (6) can be summarized as the following sequences:

$$(18) \quad Y_k = \max(P_k - \frac{1}{L} \nabla F(P_k), 0),$$

$$(19) \quad Z_k = \max(Z_k - \frac{k+1}{2L} \nabla F(P_k), 0),$$

$$(20) \quad P_{k+1} = \alpha_k Z_k + (1 - \alpha_k) Y_k,$$

where $L = \| W^T W \|$ and $\nabla F(P_k) = W^T W P_k - (W^T V + W^T W Q) + \beta E$, wherein all the elements in matrix $E \in R^{r \times n}$ are 1. Similarly, the final updating rules for (7) can be summarized as the following sequences:

$$Y_k = \max(Q_k - \frac{1}{L}\nabla F(Q_k), 0), \tag{21}$$

$$Z_k = \max(Z_k - \frac{k+1}{2L}\nabla F(Q_k), 0), \tag{22}$$

$$Q_{k+1} = \alpha_k Z_k + (1 - \alpha_k) Y_k, \tag{23}$$

where $\nabla F(Q_k) = W^T W Q_k - (-W^T V + W^T W P) + \beta E$. We summarize above updating rules (18) to (23) in Algorithm 1.

Algorithm 1 NAG for L1 (NAGL1)

Input: V, W, H, β, *outer*
Output: H
Initialization: *inner*, $L = \frac{1}{\|W^T W\|}$, $k \leftarrow 0$, $i \leftarrow 0$
for $i = 0$ to *outer* **do**
1. $P = max(H, 0)$
2. $Q = max(-H, 0)$
3. $Z = P$
4. $WTV = W^T V + W^T W Q$
for $k = 0$ to *inner* **do**
5. $F(P) = W^T W P - WTV + \beta$
6. $Y = \max(P - L\nabla F(P), 0)$
7. $Z = \max(Z - \frac{k+1}{2} L \nabla F(P), 0)$
8. $P = \frac{2}{k+3} Z + (1 - \frac{2}{k+3}) Y$
end for
9. $Z = Q$
10. $WTV = -W^T V + W^T W P$
for $k = 0$ to *inner* **do**
11. $F(Q) = W^T W Q - WTV + \beta$
12. $Y = \max(Q - L\nabla F(Q), 0)$
13. $Z = \max(Z - \frac{k+1}{2} L \nabla F(Q), 0)$
14. $Q = \frac{2}{k+3} Z + (1 - \frac{2}{k+3}) Y$
15. $k = k + 1$
end for
16. $H = P - Q$
17. $i = i + 1$
end for

Algorithm 1 accepts input V, W, H, β and *outer* and outputs H. W and H can be obtained from BCD. *inner* is the iteration number of each sub-problem and *outer* is the iteration number of Algorithm 1. In the following Theorem 1

and Theorem 2, Algorithm 1 can be demonstrated to achieve the convergence rate at $O(\frac{1}{k^2})$.

Theorem 1. Supposed that the two sequences $\{Y_k\}_{k=0}^{\infty}$ and $\{P_k\}_{k=0}^{\infty}$ are generated by (19) and (20), then we have

$$F(Y_k) - F(P^*) \leq \frac{2L \parallel P^* - P_k \parallel_F^2}{(k+1)(k+2)},$$

where P^* is an optimal solution to (6).

Proof. According to Proposition 2.1 in [15] and Lemma 1 in [12], we obtain

$$F(Y_k) - F(P^*) \leq \frac{1}{A_k}(\phi_0(P) - F(P^*)) \tag{24}$$

and

$$A_k F(Y_k) \leq \min_X \{\frac{1}{2} L \parallel X - P_k \parallel_F^2 + \sum_{i=0}^{k} \tau_i < \nabla F(P_i), X - P_i >\}, \tag{25}$$

where $A_k = \sum_0^k \tau_i = \frac{(k+1)(k+2)}{4}$ and $\phi_k(X) = \frac{1}{2} L \parallel X - P_k \parallel_F^2 + \sum_{i=0}^{k} \tau_i < \nabla F(P_i), X - P_i >$. We get

$$\begin{aligned}
&F(Y_k) - (1 - \frac{1}{A_k}) F(P^*) \leq \frac{1}{A_k} \phi_0(P^*) \\
&= \frac{1}{A_k} \{L \parallel P^* - P_k \parallel_F^2 + \tau_0 [F(P_0) + < \nabla F(P_0), P^* - P_0 >]\} \\
&\leq \frac{1}{A_k} \frac{L}{2} \parallel P^* - P_k \parallel_F^2 + \frac{1}{A_k} \tau_0 F(P^*) \\
&\leq \frac{1}{A_k} \frac{L}{2} \parallel P^* - P_k \parallel_F^2 + \frac{1}{A_k} F(P^*).
\end{aligned}$$

According to simple algebra, $F(Y_k) - F(P^*) \leq \frac{2L \| P^* - P_k \|_F^2}{(k+1)(k+2)}$.

Theorem 2. Supposed that the two sequences $\{Y_k\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ are generated by (22) and (23), then we have

$$F(Y_k) - F(Q^*) \leq \frac{2L \parallel Q^* - Q_k \parallel_F^2}{(k+1)(k+2)},$$

where Q^* is an optimal solution to (7).

3. Constrained quadratic programming problem optimization

At the iteration number $k \geq 1$, the updating rules for optimizing (3) are

$$Y_k = \underset{\|Y_{(i)}\|_2^2 \leq c}{\arg\min}\{F(W_k) + < \nabla F(W_k), Y - W_k > + \frac{1}{2} L \parallel Y - W_k \parallel_F^2\}, \tag{26}$$

$$Z_k = \underset{\|Z_{(i)}\|_2^2 \leq c}{\arg\min}(\frac{L}{2} \parallel Z - Z_{k-1} \parallel_F^2) + \tau_k(< \nabla F(W_k), Z >)), \tag{27}$$

$$W_{k+1} = \alpha_k Z_k + (1 - \alpha_k) Y_k, \tag{28}$$

where $i = 1, 2, \cdots, r$, $Y_{(i)}$ is the i-th column of Y and $Z_{(i)}$ is the i-th column of Z. By the Lagrange multiplier method, (26) can be transformed as follows:

$$\begin{aligned} L(Y, \lambda) = {} & F(W_k) + < \nabla F(W_k), Y - W_k > + \frac{1}{2} L \parallel Y - W_k \parallel_F^2 \\ & - \frac{1}{2} \sum_i^r \lambda_i (c - \parallel Y_{(i)}, \|_2^2), \end{aligned} \tag{29}$$

where $\lambda = [\lambda_1, \cdots, \lambda_r]^T \in R^r$. Let (Y, λ) be the optimal solution of (26). The KKT conditions are as follows:

$$\parallel Y_{(i)} \parallel_2^2 \leq c, \lambda_i \geq 0, \lambda_i(\parallel Y_{(i)} \parallel_2^2 - c) = 0, \tag{30}$$

$$\nabla F(W_k)_{(i)} + L(Y_{(i)} - W_{k(i)}) + \sum_i^r \lambda_i Y_{(i)} = 0, \tag{31}$$

where $\nabla F(W_k)_{(i)}$ is the i-th column of $\nabla F(W_k)_{(i)}$ and $W_{k(i)}$ is the i-th of W_k. According to (31), we have

$$Y_{(i)} = \frac{L W_{k(i)} - \nabla F(W_k)_{(i)}}{L + \lambda_i}. \tag{32}$$

However, λ_i is an unknown variable. We find that only $\lambda_i > 0$ and $\parallel Y_{(i)} \parallel_2^2 - c = 0$ can satisfy (30). We substitute (32) into $\parallel Y_{(i)} \parallel_2^2 - c = 0$, and obtain

$$\lambda_i = \max(\frac{\parallel L W_{k(i)} - \nabla F(W_k)_{(i)} \parallel_2}{\sqrt{c}} - L, 0). \tag{33}$$

Similarly to (27), we get

$$Z_{(i)} = \frac{L Z_{k(i)} - \tau_k \nabla F(W_k)_{(i)}}{L + \gamma_i} \tag{34}$$

and

$$\gamma_i = \max(\frac{\parallel L Z_{k(i)} - \tau_k \nabla F(W_k)_{(i)} \parallel_2^2}{\sqrt{c}} - L, 0). \tag{35}$$

According to above analysis, the final updating rules for (3) can be summarized as follows:

$$\lambda_i = \max\left(\frac{\| LW_{k(i)} - \nabla F(W_k)_{(i)} \|_2^2}{\sqrt{c}} - L, 0\right), \tag{36}$$

$$Y_{k(i)} = \frac{LW_{k(i)} - \nabla F(W_k)_{(i)}}{L + \lambda_i}, \tag{37}$$

$$\gamma_i = \max\left(\frac{\| LZ_{k(i)} - \tau_k \nabla F(W_k)_{(i)} \|_2^2}{\sqrt{c}} - L, 0\right), \tag{38}$$

$$Z_{k(i)} = \frac{LZ_{k(i)} - \tau_k \nabla F(W_k)_{(i)}}{L + \gamma_i}, \tag{39}$$

$$W_{k+1} = \frac{2}{k+3} Z_k + (1 - \frac{2}{k+3}) Y_k, \tag{40}$$

where $L = \| HH^T \|$ and $\nabla F(W) = WHH^T - VH^T$. We summarize above updating rules in Algorithm 2.

Algorithm 2 NAG for CQP (NAGCQP)

Input: V, W, H, c, *outer*
Output: W
Initialization: $Z \leftarrow W$, $L = \| HH^T \|$, $c = \sqrt{c}$
for $k = 0$ to *outer* **do**
1. $\nabla F(W) = WHH^T - VH^T$
for $i = 0$ to r **do**
2. $\lambda = \max(\frac{\|LW_{(i)} - \nabla F(W)_{(i)}\|_2}{c} - L, 0)$
3. $Y_{(i)} = \frac{LW_{(i)} - \nabla F(W)_{(i)}}{L+\lambda}$
4. $\gamma = \max(\frac{\|LZ_{(i)} - \frac{k+1}{2} \nabla F(W)_{(i)}\|_2}{c} - L, 0)$
5. $Z_{(i)} = \frac{LZ_{(i)} - \frac{k+1}{2} \nabla F(W)_{(i)}}{L+\gamma}$
end for
6. $W = \frac{2}{k+3} Z + (1 - \frac{2}{k+3}) Y$.
end for $W = W^{k+1}$

Algorithm 2 accepts input V, W, c and *outer* and outputs W. W and H can be obtained from BCD. *outer* is the iteration number of Algorithm 2. To save space, we do not prove that Algorithm 2 has the convergence rate of $O(\frac{1}{k^2})$.

4. Experiments

In this section, three experiments are presented to evaluate the performances of our proposed algorithms for the COIL20 dataset. The dataset is from Columbia University which can be downloaded on the web site: `http://www1.cs.columbia.edu/CAVE/software/softlib/coil-20.php`. There are 1440 16×16 gray scale images in total, and this dataset includes 20 different classes.

Firstly, Algorithm 1 is evaluated to optimize the L1 function. We compared NAGL1, Feature-sign [16] and MexLasso [17] in the running time, the sparse degree and the objective value. Let $inner = 20$, $outer = 100$ and $\beta = 1$. Table 1 shows the results of different dimensions r by different algorithms. When the number of dimensions is smaller, MexLasso achieves the shorter times, the sparser values and the smaller objective values. With the increase of dimensions, NAGL1 achieves the best performance. However, Feature-sign needs more time to converge than NAGL1 and MexLasso.

Table 1. The running time, the sparseness and the objective value in a time limit of 1500 seconds.

	Time			Sparseness			Objective value		
	r=100	r=500	r=1000	r=100	r=500	r=1000	r=100	r=500	r=1000
NAGL1	4.679	77.937	246.046	0.111	0.127	0.160	129931	96544	57259
MexLasso	1.238	54.228	524.050	0.111	0.127	0.160	129931	96544	57259
Feature-sign	25.797	1024.693	-	0.155	0.088	-	130102	96546	-

Secondly, Algorithm 2 is evaluated to optimize the constrained quadratic programming function. We compared NAGCQP and LagDual [16] in the running time and the objective value. Let $outer = 100$ and $c = 10$. Table 1 shows the results of different dimensions r by different algorithms. When the number of dimensions is smaller, NAGCQP performs as well as LagDual. With the increase of dimensions, LagDual performs better than NAGCQP.

Table 2. The running time and the objective value in a time limit of 1500 seconds.

	Time			Objective value		
	r=100	r=500	r=1000	r=100	r=500	r=1000
NAGCQP	0.596	4.259	10.962	125890	7057	4234
LagDual	0.041	0.487	2.387	125890	7057	2186

Thirdly, our proposed algorithms NAGL1 and NAGCQP have a fast convergence speed. However, it doesn't mean that our proposed algorithm NAGSC (details in Algorithm 3) can learn over-complete bases. NAGSC and ESC [16] are compared to learn over-complete bases. We run each algorithm until the relative error is less than 10^{-6} (i.e., $|F_{new} - F_{old}|/F_{old} < 10^{-6}$). Let $m = 196$, $r = 256$, $n = 10000$, $c = 1$, $\beta = 0.4$, $inner = 10$, $outer = 100$ and $iter = 50$. Figure 1 shows the learned over-complete bases of natural images by each algorithm. NAGSC and ESC take 22.4518 minutes and 55.7725 minutes in learning 256 bases, respectively.

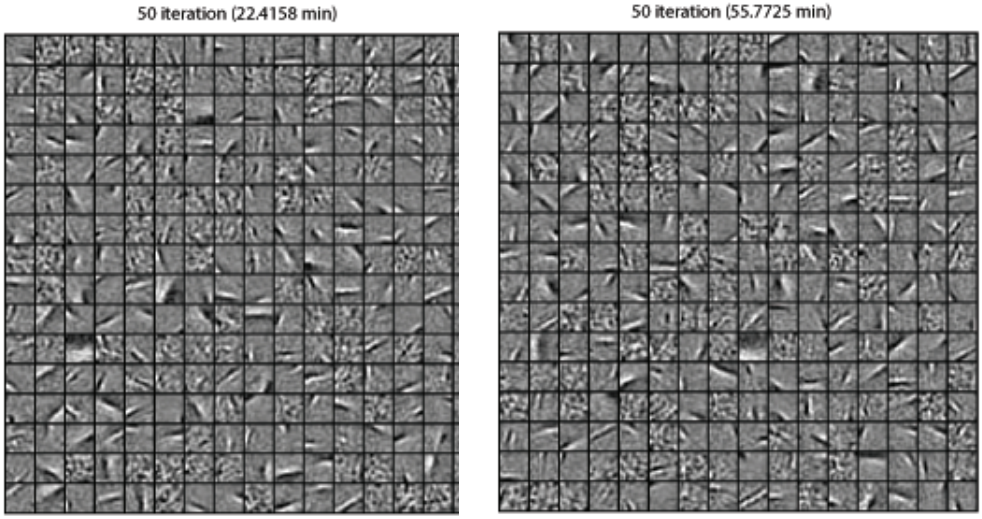
5. Conclusion and future work

In this paper, Sparse Coding is formulated by L1 and L2 problems and we present algorithms for each sub-problem: NAG for solving L1 and L2 problems.

Algorithm 3 NAGSC

Input: V, W, H, c, β, $outer$, $iter$
Output: W, H
for $k = 0$ to $iter$ **do**
1. W=NAGLCQP(V, W, H, c, $outer$);
2. H=NAGL1(V, W, H, β, $outer$);

end for



NAGSC ESC

Figure 1. Learned over-complete natural image bases

Experiments show that our proposed algorithms learn over-complete bases more quickly.

Two topics should be discussed in future work:

- There are many different sparse coding models, hence, the corresponding NAG should be discussed to optimize them.
- The Lipshcitz constant L in algorithm 1 and 2 is fixed, therefore, a variable Lipshcitz constant L should be discussed.

Acknowledgments

This work is supported by the Research Foundation of Chongqing Municipal Education Commission (KJ1710253), Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education ([2017]3), Foundation of Chongqing Development and Reform Commission (2017[1007]), Foundation of

Wanzhou Development and Reform Commission ([2017]32), and Foundation of Chongqing Three Gorges University.

References

[1] B. A. Olshausen, D. J. Field, *Emergence of simple-cell receptive field properties by learning a sparse code for natural images*, Nature, 381 (1996), 607-609.

[2] B. A. Olshausen, D. J. Field, *Sparse coding with an overcomplete basis set: a strategy employed by v1?*, Vision Research, 37 (1997), 3311-3325.

[3] M. Turk, A. P. Pentland, *Face recognition using eigenface*, IEEE Conf. Computer Vision and Pattern Recognition, 118 (1991), 586-591.

[4] X. He, P. Niyogi, *Locality preserving projections*, Advances in Neural Information Processing Systems, 16 (2003), 186-197.

[5] P. N. Belhumeur, J. P. Hespanha, D. J. Kriegman, *Eigenfaces vs. fisherfaces: recognition using class specific linear projection*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 19 (2002), 711-720.

[6] K. Labusch, E. Barth, T. Martinetz, *Simple method for high-performance digit recognition based on sparse coding*, IEEE Trans Neural Netw, 19 (2008), 1985-1989.

[7] B. He, D. Xu, N. Rui, M. V. Heeswijk, Q. Yu, Y. Miche, A. Lendasse, *Fast face recognition via sparse coding and extreme learning machine*, Cognitive Computation, 6 (2014), 264-277.

[8] M. Zheng, J. Bu, C. Chen, C. Wang, L. Zhang, G. Qiu, D. Cai, *Graph regularized sparse coding for image representation*, IEEE Transactions on Image Processing A Publication of the IEEE Signal Processing Society, 20 (2011), 1327-1336.

[9] Z. Shu, C. Zhao, P. Huang, *Constrained sparse concept coding algorithm with application to image representation*, Ksii Transactions on Internet and Information Systems, 8 (2014), 3211-3230.

[10] L. Andrs, P. Zsolt, S. Gbor, *Efficient sparse coding in early sensory processing: Lessons from signal recovery*, Plos Computational Biology, 8 (2012), e1002372.

[11] Y. Zhao, Z. Liu, Y. Wang, H. Wu, S. Ding, *Sparse coding algorithm with negentropy and weighted 1-norm for signal reconstruction*, Entropy, 19 (2017), 599.

[12] Y. Nesterov, *Smooth minimization of non-smooth functions*, Mathematical Programming, 103 (2005), 127-152.

[13] D. P. Bertsekas, *Nonlinear programming*, Athena scientific Belmont, 1999.

[14] N. Guan, D. Tao, Z. Luo, B. Yuan, *Nenmf: an optimal gradient method for nonnegative matrix factorization*, IEEE Transactions on Signal Processing, 60 (2012), 2882-2898.

[15] M. Baes, *Estimate sequence methods: extensions and approximations*, Institute for Operations Research, 2009.

[16] H. Lee, A. Battle, R. Raina, A. Y. Ng, *Efficient sparse coding algorithms. nips*, Proc of Nips, 19 (2007), 801-808.

[17] J. Mairal, F. Bach, J. Ponce, G. Sapiro, *Online learning for matrix factorization and sparse coding*, Journal of Machine Learning Research, 11 (2009), 19-60.

Accepted: 28.05.2018

CONTRA WEAKLY-θ_I-PRECONTINUOUS FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

Manisha Shrivastava*
Department of Mathematics
Govt. J.Y. Chhattisgarh College
Raipur, Chhattisgarh
India-492001
shrivastavamanisha9@gmail.com

Takashi Noiri
2949–1 Shiokita-cho,
Hinagu, Yatsushiro–shi, Kumomoto–ken
869–5142 JAPAN
t.noiri@nifty.com

Purushottam Jha
Department of Mathematics
Govt. D.K.P.G. College
Baloda Bazar-493332
Raipur, Chhattisgarh
India–494661
purush.jha@gmail.com

Abstract. The present authors [23] introduced and studied the notion of weakly θ_I-preopen sets in ideal topological spaces. In this paper, we apply this set to introduce and study a new class of functions called contra weakly θ_I-precontinuous functions in ideal topological spaces. Some characterizations and several basic properties of this class of functions are obtained. Further, we introduce the notions of contra-θ_I-precontinuous, contra-θ_I-α-continuous, contra-θ_I-semicontinuous and contra-θ_I-β-continuous functions in ideal topological spaces and also establish relationships among these new classes of functions.

Keywords: weakly θ_I-preopen sets, weakly θ_I-precontinuous, contra weakly-θ_I-precontinuous, contra-θ_I-precon-tinuous, contra-θ_I-α-continuous, contra-θ_I-semicontinuous and contra-θ_I-β-continuous.

1. Introduction

The concept of continuous functions is one of the important and basic topic in the theory of classical point set topology and several branches of mathematics. General Topologists have introduced and investigated many different generalizations of continuous functions. Dontchev [6] introduced a new class of functions called contra-continuous functions. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-

∗. Corresponding author

continuous [6] if the preimage of every open set of Y is closed in X. Subsequently, some generalized forms of this class of functions namely contra-supercontinuous functions [15], *contra-precontinuous* functions [16] etc., in topological spaces and *contra-pre-I*-continuous functions [22] etc. in ideal topological spaces were propounded and studied. An ideal I in a topological space has been considered since 1930 by Kuratowski [13] and Vaidyanathaswamy [24] (c.f.[25]). After several decades, Jankovic and Hamlett [12] investigated the topological ideal which is the generalization of general topology. Jankovic and Hamlett (c.f.[8], [9]) also introduced the notion of I-open sets in topological spaces. Abd El-Monsef et al. [2] further investigated I-open sets and I-continuous functions. Some generalized forms of I-open sets are introduced in [3], [7] (see [11]) and other papers. Al-Omari and Noiri [5] introduced the notions of θ_I-preopen, θ_I-semiopen, θ_I-α-*open*, θ_I-β-open sets by using θ_I-*open* sets and derived the decomposition of θ_I-precontinuous functions and θ_I-β-continuous functions in ideal topological spaces. In addition to this, the present authors [23] defined and investigated the notion of weakly θ_I-preopen sets and established the decomposition of weakly θ_I-*precontinuity*.

The purpose of this paper is to give a new class of functions called contra weakly-θ_I-precontinuous functions via weakly θ_I-preopen sets in ideal topological spaces and study their fundamental properties and characterizations. We also devise the concept of contra-θ_I-*precontinuous*, contra-θ_I-α-*continuous*, contra-θ_I-semicontinuous and contra-θ_I-β-continuous functions in ideal topological spaces and establish their interrelationships.

2. Preliminaries

An ideal on a topological space (X, τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions:

1. $V \in I$ and $U \subset V$ implies $U \in I$,
2. $V \in I$ and $U \in I$ implies $V \cup U \in I$.

The pair (X, τ, I) of a topological space (X, τ) and an ideal I on X is called an ideal topological space or simply an ideal space. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the collection of all subsets of X, then the set operator $(.) : P(X) \to P(X)$ is called the local function [24] of A with respect to τ and ideal I, is defined as follows: For a subset $A \subseteq X$, the set $A^*(I, \tau) = \{x \in X: (U \cap A) \notin I$, for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$ [12]. We simply write A^* instead of $A^*(I, \tau)$ in case there is no chance of confusion [24] (c.f. [12], [13]). It is well known that $Cl^*(A) = A \cup A^*(I, \tau)$ defines a Kuratowski closure operator for a topology $\tau^*(I, \tau)$ called the $*$-topology which is finer than τ. The topology τ^* is generated by the base $\beta(I, \tau) = \{U \setminus I: U \in \tau$ and $I \in I\}$. In general $\beta(I, \tau)$ is not always a topology as shown in [12]. X^* is often a proper subset of X.

Throughout this paper, for a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. The family of all weakly θ_I-*preopen* (resp. θ_I-*preopen*, θ_I-semiopen, θ_I-α-open, θ_I-β-open) sets of the space (X, τ, I) will be denoted by $W\theta_IPO(X)$ (resp. $\theta_IPO(X)$, $\theta_ISO(X)$, $\theta_I\alpha O(X)$, $\theta_I\beta O(X)$) .

We start with recalling following definitions and results, which are necessary for this study in the sequel.

Definition 1. A subset A of topological space (X, τ) is said to be preopen [18] (resp. semiopen [14], α-*open* [19] [22], β-open [1]) if $A \subset Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(Int(A)))$, $A \subset Cl(Int(Cl(A)))$).

Lemma 1 ([12])**.** *Let (X, τ, I) be an ideal topological space and A, B be any two subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $A^* \subseteq B^*$;*
2. *If $A^* = Cl(A^*) \subseteq Cl(A)$;*
3. *$(A^*)^* \subseteq A^*$;*
4. *$(A \cup B)^* = A^* \cup B^*$.*
5. *If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$.*

Lemma 2 ([5])**.** *Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties hold.*

1. *If A is open, then $Cl(A) = Cl_{\theta_I}(A) = Cl_\theta(A)$.*
2. *If A is closed, then $Int(A) = Int_{\theta_I}(A) = Int_\theta(A)$.*

Lemma 3 ([10])**.** *For a subset A of a topological space (X, τ), the following properties hold:*

1. *$sCl(A) = A \cup Int(Cl(A))$,*
2. *If A is open then $sCl(A) = Int(Cl(A))$.*

Definition 2. A subset A of an ideal topological space (X, τ, I) is said to be θ_I-*preopen* [5] (resp. θ_I-semiopen [5], θ_I-β-open [5], θ_I-α-open [5], weakly θ_I-preopen [23]) if $A \subseteq Int(Cl_{\theta_I}(A)))$ (resp. $A \subseteq Cl(Int_{\theta_I}(A))$, $A \subseteq Cl(Int(Cl_{\theta_I}(A)))$, $A \subseteq Int(Cl(Int_{\theta_I}(A)))$, $A \subseteq sCl(Int(Cl_{\theta_I}(A)))$).

Definition 3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(1) precontinuous [18] if preimage of every open set in Y is preopen in X.

(2) semicontinuous [14] if the inverse image of each open set in Y is semiopen in X.

(3) β-continuous [1] if the inverse image of each open set in Y is β-open in X.

(4) α-continuous [19] if the inverse image of each open set in Y is α-open in X.

Definition 4. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) weakly *θ_I-precontinuous* [23] if the preimage of every open set in Y is weakly θ_I-preopen in X.

(2) weakly *θ_I-preirresolute* [23] if the preimage of every weakly θ_I-preopen set in Y is a weakly *θ_I-preopen* set in X.

Definition 5. A space (X, τ) is said to be extremally disconnected [26] if the closure of every open set in X is open.

Theorem 1 ([23])**.** *If a topological space (X, τ) is extremally disconnected and $A \in \theta_I SO(X)$, then $A \in \theta_I \alpha O(X)$.*

Theorem 2 ([23])**.** *If a topological space (X, τ) is extremally disconnected and $A \in \theta_I \beta O(X)$, then $A \in W\theta_I PO(X)$.*

3. Contra weakly-θ_I-precontinuous functions

Definition 6. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be contra weakly-θ_I-precontinuous if the preimage of every closed set in Y is weakly *θ_I-preopen* in X.

Example 1. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b, c\}, \{c, d\}, \{b, c, d\}, \{c\}\}$ and $I = P(X)$ then (X, τ, I) is an ideal topological space. $C(X) = \{X, \phi, \{a, d\}, \{a, b\}, \{a\}, \{a, b, d\}\}$. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{1, 2\}, \{1, 2, 3\}, \{2\}\}$ and $C(Y) = \{Y, \phi, \{3, 4\}, \{4\}, \{1, 3, 4\}\}$. Then (Y, σ) be a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function defined as $f(a) = 3$, $f(b) = 2$, $f(c) = 4$, $f(d) = 1$. Then f is contra weakly-*θ_I-precontinuous*, since the preimage of every closed set in Y is weakly θ_I-preopen in X.

Definition 7. Let A be any subset of an ideal topological space (X, τ, I).

(1) The intersection of all weakly *θ_I-preclosed* sets containing A is called the weakly *θ_I-pre-closure* of A and it is denoted by $W\theta_I\text{-}PCl(A)$.

(2) The union of all weakly *θ_I-preopen* sets contained in A is called the weakly *θ_I-pre-interior* of A and it is denoted by $W\theta_I\text{-}PInt(A)$.

Remark 1. Let A be any subset of an ideal topological space (X, τ, I), then the following properties hold:

(1) A is weakly *θ_I-preclosed* if and only if $W\theta_I\text{-}PCl(A) = A$.

(2) A is weakly *θ_I-preopen* if and only if $W\theta_I\text{-}PInt(A) = A$.

Theorem 3. *Let A be any subset of an ideal topological space (X, τ, I) and $x \in X$. Then $x \in W\theta_I\text{-}PCl(A)$ if and only if $A \cap U \neq \phi$ for each $U \in W\theta_I PO(X, x)$.*

Proof. *Necessity.* Let $x \notin W\theta_I\text{-}PCl(A)$, then suppose $U = X \setminus W\theta_I\text{-}PCl(A)$ be a weakly θ_I-*preopen* set containing x in X such that $U \cap W\theta_I\text{-}PCl(A) = \phi$. Hence the result.

Sufficiency. Let U be a weakly θ_I-*preopen* set containing x in X such that $U \cap W\theta_I\text{-}PCl(A) = \phi$. Then $X \setminus U$ is a weakly θ_I-*preclosed* set containing A. Since $W\theta_I\text{-}PCl(A)$ is the smallest weakly θ_I-*preclosed* set containing A, $W\theta_I\text{-}PCl(A) \subseteq X \setminus U$. Therefore $x \notin W\theta_I\text{-}PCl(A)$. □

Definition 8. Let A be any subset of a space (X, τ). The set $\bigcap\{U : U \in \tau, A \subset U\}$ is called the kernel of A [20] and is denoted by $ker(A)$. In [17], the kernel of A is called the $\wedge$-set.

Lemma 4 ([15], [17])**.** *The following properties hold for subsets A and B of a space (X, τ) :*

(1) *$x \in ker(A)$ if and only if $A \cap F \neq \phi$ for any $F \in C(X, x)$.*

(2) *$A \subset ker(A)$ and $A = ker(A)$ if A is open in X.*

(3) *$A \subset B$, then $ker(A) \subset ker(B)$.*

Theorem 4. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function from an ideal topological space (X, τ, I) to (Y, σ). Then the following statements are equivalent:*

(1) *f is contra weakly-θ_I-precontinuous,*

(2) *The inverse image of each open set in Y is weakly θ_I-preclosed in X,*

(3) *For each $x \in X$ and each closed subset G of Y containing $f(x)$, there exists a weakly θ_I-preopen set U in X such that $x \in U$ and $f(U) \subset G$.*

(4) *$f(W\theta_I\text{-}PCl(M)) \subset ker(f(M))$ for every subset M of X,*

(5) *$W\theta_I\text{-}PCl(f^{-1}(N)) \subset f^{-1}(ker(N))$ for every subset N of Y.*

Proof. (1) $\Leftrightarrow$ (2) is obvious.

(1) $\Rightarrow$ (3). Let $x \in X$ and G be any closed subset of Y containing $f(x)$. Since f is contra *weakly-θ_I*-precontinuous, $f^{-1}(G)$ is weakly θ_I-preopen in X. Let $U = f^{-1}(G)$. Then $x \in U = f^{-1}(G)$. Thus $f(U) \subset G$.

(3) $\Rightarrow$ (4). Let M be any subset of X. Let $y \in Y$ be any element such that $y \notin ker(f(M))$. Then by Lemma 4, there exists a closed set V containing y in Y such that $f(M) \cap V = \phi$; hence $M \cap f^{-1}(V) = \phi$. By (3) for any $x \in f^{-1}(\{y\})$, there exists a weakly θ_I-*preopen* set W containing 'x' such that $f(W) \subset V$. Therefore $f(M \cap W) \subset f(M) \cap f(W) \subset f(M) \cap V = \phi$ and hence $M \cap W = \phi$. Thus by Theorem 3, $x \notin W\theta_I\text{-}PCl(M)$ for any $x \in f^{-1}(\{y\})$. Thus $f^{-1}(\{y\}) \cap W\theta_I\text{-}PCl(M) = \phi$ and hence $y \notin f(W\theta_I\text{-}PCl(M))$. Hence we

obtain $f(W\theta_I - PCl(M)) \subset ker(f(M)$ for every subset M of X.

(4) $\Rightarrow$ (5). Let N be any subset of Y, then $f^{-1}(N)$ is a subset of X. By (4) and by Lemma 4, we have $f(W\theta_I\text{-}PCl(f^{-1}(N))) \subset ker(f(f^{-1}(N)) \subset ker(N)$. Therefore $W\theta_I\text{-}PCl(f^{-1}(N)) \subset f^{-1}(ker(N))$.

(5) $\Rightarrow$ (1). Let U be an open subset of Y, then by Lemma 4, $ker(U) = U$. By (6), $W\theta_I\text{-}PCl(f^{-1}(U)) \subset f^{-1}(ker(U)) = f^{-1}(U)$ and thus we get $W\theta_I\text{-}PCl(f^{-1}(U)) \subset f^{-1}(U)$. We already have $f^{-1}(U) \subset W\theta_I\text{-}PCl(f^{-1}(U))$. Therefore we get $f^{-1}(U) = W\theta_I\text{-}PCl(f^{-1}(U))$. Thus U is weakly θ_I-*preclosed* in X. Hence f is contra weakly-θ_I-precontinuous. □

Remark 2. The concepts of weakly θ_I-*precontinuous* functions and contra-weakly θ_I-*precontinuous* functions are independent.

Example 2. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{b\}\}$ and $I = P(X)$ then (X, τ, I) is an ideal topological space. Let $Y = \{p, q, r\}$ and $\sigma = \{Y, \phi, \{q, r\}\}$. Then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function defined as $f(a) = q$, $f(b) = r$, $f(c) = f(d) = p$. Then f is weakly θ_I-*precontinuous*. But it is not contra weakly-θ_I-*precontinuous*, since the preimage of a closed set $\{p\}$ in Y is $\{\{c\}, \{d\}\}$, which is not weakly θ_I-*preopen* in X.

Example 3. Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \phi, \{1\}, \{1, 2\}, \{1, 2, 4\}\}$ and $I = P(X)$ then (X, τ, I) is an ideal topological space. Let $Y = \{a, b, c, d\}$, $\sigma = \{Y, \phi, \{b\}, \{b, c\}, \{b, c, d\}\}$. Then (Y, σ) is a topological space. Suppose a function $f : (X, \tau, I) \to (Y, \sigma)$ is defined as $f(1) = a$, $f(2) = c$, $f(3) = d$, $f(4) = b$. Then it is contra weakly-θ_I-*precontinuous* but it is not weakly θ_I-precontinuous, since the preimage of an open set $\{b, c\}$ in Y is $\{2, 4\}$, which is not weakly θ_I-preopen in X.

Remark 3. The following example shows that composition of any contra weakly-θ_I-*precontinuous* functions need not be contra weakly-θ_I-*precontinuous* in general.

Example 4. Let $X = Y = Z = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{b\}, \{a, b, c\}\}$ and $I_1 = \{\phi, \{b\}\}$, $\sigma = \{Y, \phi, \{b\}\}$, $I_2 = P(X)$ then (X, τ, I_1) and (Y,σ,I_2) are ideal topological spaces. Let $\eta = \{Z, \phi, \{a\}, \{a, b, c\}\}$, then ($Z$, η) is a topological space. Let $f : (X, \tau, I_1) \to (Y, \sigma, I_2)$ be a function defined as $f(a) = c$, $f(b) = d$, $f(c) = b$ and $f(d) = a$. Then f is contra weakly-θ_I-*precontinuous*. Let $g : (Y, \sigma, I_2) \to (Z, \eta)$ be a function defined as $g(a) = b$, $g(b) = c$, $g(c) = d$ and $g(d) = a$. Then g is contra weakly-θ_I-*precontinuous*. But their composition $g \circ f$: $(X, \tau, I_1) \to (Z, \eta)$ is not contra weakly-θ_I-*precontinuous*, since the preimage of a closed set $\{b, c, d\}$ in (Z, η) i.e. $(g \circ f)^{-1}(\{b, c, d\}) = f^{-1}(g^{-1}(\{b, c, d\})) = f^{-1}(\{a, b, c\}) = \{a, c, d\}$, which is not weakly θ_I-*preopen* in (X, τ, I_1).

Theorem 5. *If a function $f : (X, \tau, I) \to (Y, \sigma)$ is contra weakly θ_I-precontinuous and Y is a regular space, then f is weakly θ_I-precontinuous.*

Proof. Let (X, τ, I) be any ideal topological space and let $x \in X$ be any point. Suppose P is an open set in Y with $f(x) \in P$. Since Y is regular, then there exists an open set Q in Y with $f(x) \in Q$ such that $Cl(Q) \subset P$. Since f is contra weakly θ_I-*precontinuous*, then by Theorem 4, there exists a weakly θ_I-preopen set M containing 'x' such that $f(M) \subset Cl(Q)$. Then $f(M) \subset Cl(Q) \subset P$. It follows from Theorem 18 of [23] that f is weakly θ_I-*precontinuous*. □

Definition 9. An ideal topological space (X, τ, I) is said to be weakly θ_I-preconnected if X is not the union of two disjoint non-empty weakly θ_I-preopen subsets of X.

Theorem 6. *If a function $f : (X, \tau, I) \to (Y, \sigma)$ is a surjection and contra weakly-θ_I-precontinuous and X is weakly θ_I-preconnected, then Y is connected.*

Proof. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a contra weakly-θ_I-precontinuous function from a weakly θ_I-*preconnected* space X onto Y. On contrary, assume that space Y is disconnected. For, let $Y = A \cup B$ be the separation of Y into two disjoint non-empty clopen subsets of Y. Since f is onto contra weakly-θ_I-precontinuous function, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty weakly θ_I-*preopen* sets in X and also $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ forms a separation of X, which is contrary to the assumption that X is weakly θ_I-*preconnected*. Hence Y is connected. □

Theorem 7. *Let (X, τ, I) be any ideal topological space, which is weakly θ_I-preconnected and (Y, σ) be a T_1-space. If $f : (X, \tau, I) \to (Y, \sigma)$ is a contra weakly θ_I-precontinuous function, then f is constant.*

Proof. Suppose the space X is weakly θ_I-*preconnected*. Since Y is a T_1-space, $\lambda = \{f^{-1}(\{y\}) : y \in Y\}$ is the disjoint weakly θ_I-*preopen* partition of X. If $| \lambda | \geq 2$ (where $| \lambda |$ denotes the cardinality of the set λ), then X is the union of two disjoint non-empty weakly θ_I-preopen sets. But X is weakly θ_I-*preconnected* and we have $| \lambda | = 1$. It shows that the function f is constant. □

Definition 10 ([20])**.** A space (X, τ) is said to be locally indiscrete if every open set in X is closed in X.

Theorem 8. *If a function $f : (X, \tau, I) \to (Y, \sigma)$ is weakly θ_I-precontinuous and the space (Y, σ) is locally indiscrete, then f is contra weakly-θ_I-precontinuous.*

Proof. Let V be any open set in (Y, σ). Since Y is locally indiscrete, V is closed in Y. Since f is weakly θ_I-precontinuous, $f^{-1}(V)$ is weakly θ_I-preclosed in (X, τ, I). Hence, by Theorem 4 f is contra weakly-θ_I-precontinuous. □

Definition 11. An ideal topological space (X, τ, I) is said to be weakly θ_I-pre-T_2 if for each pair of distinct points 'x' and 'y' in X, there exist weakly θ_I-*preopen* sets U and V containing 'x' and 'y', respectively, such that $U \cap V = \phi$.

Note - A closed neighbourhood of a point 'x' is a closed set that contains an open set containing the point 'x'.

Definition 12 ([4]). A Urysohn space (also called T-$2\frac{1}{2}$-space or T_e-space) is a space in which any two distinct points can be separated by closed neighbourhoods.

Theorem 9. *Let (X, τ, I) be any ideal topological space. For each pair of points x_1, $x_2 \in X$ with $x_1 \neq x_2$, there exists a function f from space X into an Urysohn space Y such that $f(x_1) \neq f(x_2)$ and function f is contra weakly-θ_I-precontinuous at x_1 and x_2, then the space X is weakly θ_I-pre-T_2.*

Proof. Let x_1, $x_2 \in X$ with $x_1 \neq x_2$ be any pair of points and suppose $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then by assumption, $y_1 \neq y_2$. Since Y is an Urysohn space, there exist two open sets U and V in Y such that $y_1 \in U$, $y_2 \in V$ and $Cl(U) \cap Cl(V) = \phi$. Since f is contra weakly θ_I-*precontinuous*, by Theorem 4, there exist two weakly θ_I-*preopen* sets P and Q containing x_1 and x_2, respectively, such that $f(P) \subset Cl(U)$ and $f(Q) \subset Cl(V)$. Since $Cl(U) \cap Cl(V) = \phi$, $P \cap Q = \phi$. Hence X is weakly θ_I-pre-T_2. □

Corollary 1. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a contra weakly-θ_I-precontinuous injection and (Y, σ) is Urysohn, then (X, τ, I) is weakly θ_I-pre-T_2.*

Theorem 10. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \eta)$ are any two functions, then following hold:*

1. *If f is a contra weakly-θ_I-precontinuous function and g is a continuous function, then their composition $g \circ f$ is contra weakly-θ_I-precontinuous.*
2. *If f is weakly-θ_I-preirresolute and g is weakly-θ_I-precontinuous, then their composition $g \circ f$ is contra weakly-θ_I-precontinuous.*

Proof. 1. Let F be any closed set in (Z, η). Since g is continuous, $g^{-1}(F)$ is closed in (Y, σ, J). Since f is contra weakly-θ_I-*precontinuous*, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is weakly-θ_I-*preopen* in (X, τ, I). Therefore their composition $g \circ f$ is contra weakly-θ_I-*precontinuous*.

2. Let F be any closed set in (Z, η). Since g is weakly-θ_I-precontinuous, $g^{-1}(F)$ is weakly-θ_I-*preopen* in (Y, σ, J). Since f is weakly-θ_I-*preirresolute*, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is weakly-θ_I-*preopen* in (X, τ, I). Therefore their composition $g \circ f$ is contra weakly-θ_I-*precontinuous*. □

4. Contra-θ_I-precontinuous, contra-θ_I-α-continuous, contra-θ_I-semicontinuous and contra-θ_I-β-continuous functions

Here, we introduce the notions of contra-θ_I-precontinuous, contra-θ_I-α-*continuous*, contra-θ_I-*semicontinuous* and contra-θ_I-β-*continuous* functions and elaborate their relationship with contra weakly-θ_I-precontinuous functions and their interrelationships.

Definition 13. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be θ_I-α-continuous if the preimage of every open set in Y is θ_I-α-open in X.

Definition 14. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be θ_I-semicontinuous if the preimage of every open set in Y is θ_I-semiopen in X.

Definition 15. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be contra-θ_I-precontinuous ((resp. contra-θ_I-α-*continuous*, contra-θ_I-*semicontinuous*, contra-θ_I-β-*continuous*) if the preimage of every closed set in Y is θ_I-*preopen* (resp. θ_I-α-*open*, θ_I-*semiopen*, θ_I-β-*open*) in X.

Theorem 11. (1) *Every contra-θ_I-precontinuous function is contra weakly-θ_I-precontinuous.*

(2) *Every contra-precontinuous function is contra-θ_I-precontinuous and therefore contra weakly-θ_I-precontinuous.*

(3) *Every contra-θ_I-α-continuous function is contra-θ_I-precontinuous and contra-θ_I-semicontinuous.*

(4) *Every contra weakly-θ_I-precontinuous function is contra-θ_I-β-continuous.*

(5) *Every contra-θ_I-semicontinuous function is contra-θ_I-β-continuous.*

Proof. (1) Suppose a function $f : (X, \tau, I) \to (Y, \sigma)$ is contra-θ_I-*precontinuous* and A is any closed set in Y, then $f^{-1}(A)$ is θ_I-preopen in X and $f^{-1}(A)$ is also weakly θ_I-*preopen* in X and therefore the function f is contra weakly-θ_I-*precontinuous*, since every θ_I-*preopen* set is weakly θ_I-*preopen*. For, if A is θ_I-*preopen* set in X. By using the definition of a θ_I-*preopen* set, we have $A \subseteq Int(Cl_{\theta_I}(A)) \subseteq sCl(Int(Cl_{\theta_I}(A)))$. This shows that A is weakly θ_I-*preopen*. This shows that A is weakly θ_I-*preopen*.

(2) Suppose a function $f : (X, \tau, I) \to (Y, \sigma)$ is contra-precontinuous and A is any closed set in Y, then $f^{-1}(A)$ is preopen in X and therefore $f^{-1}(A)$ is θ_I-*preopen* in X, it follows that the function f is contra-θ_I-*precontinuous*, therefore f is contra weakly-θ_I-*precontinuous*, since every preopen set is θ_I-*preopen* and therefore weakly θ_I-*preopen*. For, if A is any preopen set in X, $A \subseteq Int(Cl(A)) \subseteq Int(Cl_{\theta_I}(A)))$ and therefore A is θ_I-*preopen* and hence A is weakly θ_I-*preopen*.

(3) Suppose a function $f : (X, \tau, I) \to (Y, \sigma)$ is contra-θ_I-α-*continuous* and A is any closed set in Y, then $f^{-1}(A)$ is θ_I-α-*open* in X and also $f^{-1}(A)$ is θ_I-*preopen*, consequently function f is contra-θ_I-*precontinuous*. For, if A is any θ_I-α-*open* set in X, $A \subseteq Int(Cl(Int_{\theta_I}(A)))$. By using Lemma 3, $A \subseteq Int(Cl(Int_{\theta_I}(A))) \subseteq Int(Cl(Int(A))) \subseteq Int(Cl(Int(Cl_{\theta_I}(A)))) = sCl(Int(Cl_{\theta_I}(A)))$. Hence A is weakly θ_I-*preopen*.

(4) Suppose $f : (X, \tau, I) \to (Y, \sigma)$ is any contra weakly-θ_I-*precontinuous* function and A is any closed set in Y, then $f^{-1}(A)$ is weaklyθ_I-*preopen* in X and $f^{-1}(A)$ is also θ_I-β-*open* in X, since every weaklyθ_I-*preopen* set is θ_I-β-open. For, if A is weaklyθ_I-preopen then we have $A \subseteq sCl(Int(Cl_{\theta_I}(A)))$ $\subseteq Cl(Int(Cl_{\theta_I}(A))))$. This implies that A is θ_I-β-*open*. Hence f is contra-θ_I-β-*continuous*.

(5) Suppose a function $f : (X, \tau, I) \to (Y, \sigma)$ is contra-θ_I-*semicontinuous* and A is any closed set in Y, then $f^{-1}(A)$ is θ_I-*semiopen* in X and $f^{-1}(A)$ is also θ_I-β-*open* in X, since every θ_I-*semiopen* set is θ_I-β-open. For, if A be any θ_I-*semiopen* set in X, $A \subseteq Cl(Int_{\theta_I}(A))$. Let us assume that $Int_{\theta_I}(A)$ $\subseteq Int(A) \subseteq Int(Cl_{\theta_I}(A)))$, $A \subseteq Cl(Int_{\theta_I}(A)) \subseteq Cl(Int(Cl_{\theta_I}(A)))$, which implies that A is θ_I-β-open. Thus f is contra-θ_I-β-*continuous*.

□

Remark 4. The converses (1)-(5) of Theorem 12 are not true, as shown by the following examples.

Example 5. In Example 1, $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b, c\}, \{c, d\}, \{b, c, d\}, \{c\}\}$ and $I = P(X)$ then (X, τ, I) is an ideal topological space. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{1, 2\}, \{1, 2, 3\}, \{2\}\}$. Then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ is a function defined as $f(a) = 3$, $f(b) = 2$, $f(c) = 4$, $f(d) = 1$. Then f is contra weakly θ_I-precontinuous, since the preimage of every closed set in Y is weakly θ_I-*preopen* in X. But f is not contra θ_I-*precontinuous*, since the preimage of a closed set $\{3, 4\}$ in Y is $\{a, c\}$, which is not θ_I-*preopen* in X.

Example 6. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$ then (X, τ, I) is an ideal topological space. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{1, 3\}, \{1, 3, 4\}\}$. Then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function defined as $f(a) = 2$, $f(b) = 3$, $f(c) = 1$, $f(d) = 4$. Then f is contra weakly-θ_I-*precontinuous*, since the preimage of every closed set in Y is weakly-θ_I-*preopen* in X. But f is neither contra θ_I-α-*continuous* nor *contra-precontinuous*, since the preimage of a closed set $\{2, 4\}$ in Y is $\{a, b\}$, which is neither θ_I-α-*open* nor preopen in X.

Example 7. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{d\}, \{b\}, \{b, d\}\}$ and $I = \{\phi, \{b\}, \{d\}, \{b, d\}\}$ then (X, τ, I) is an ideal topological space. Let $Y = \{p, q, r, s\}$, $\sigma = \{Y, \phi, \{p\}, \{p, q, r\}\}$. Then (Y, σ) be a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be function defined as $f(a) = q$, $f(b) = p$, $f(c) = r$, $f(d) = s$. Then f is contra-θ_I-β-*continuous*, since the preimage of every closed set in Y is θ_I-β-*open* in X. But it is not contra weakly-θ_I-*precontinuous*, since the preimage of a closed set $\{q, r, s\}$ in Y is $\{a, c, d\}$, which is not weakly θ_I-*preopen* in X.

Example 8. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b\}\}$ and $I = \{\phi, \{b, c\}\}$ then (X, τ, I) is an ideal topological space. Let $Y = \{p, q, r, s\}$, $\sigma = \{Y, \phi, \{p\}, \{p, q, r\}, \{p, r\}\}$. Then (Y, σ) is a topological space. Let f :

$(X, \tau, I) \to (Y, \sigma)$ be a function defined as $f(a) = r$, $f(b) = q$, $f(c) = s$, $f(d) = p$. Then f is contra- θ_I-β-*continuous*, since the preimage of every closed set in Y is θ_I-β-open in X. But it is not contra θ_I-semicontinuous, since the preimage of a closed set $\{s\}$ in Y is $\{c\}$, which is not θ_I-semiopen in X.

Theorem 12. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be any contra θ_I-semicontinuous function from an ideal topological space (X, τ, I) to any space (Y, σ). If the space (X, τ, I) is extremally disconnected, then f is contra θ_I-α-continuous.*

Proof. Let $f : (X, \tau, I) \to (Y, \sigma)$ be any contra-θ_I-semicontinuous function from the space X to Y. If F is any closed set in Y, then $f^{-1}(F)$ is θ_I-semiopen in X. Since X is extremally disconnected, by Theorem 1, $f^{-1}(F)$ is θ_I-α-*open* in X. Hence f is contra-θ_I-α-*continuous*. □

Theorem 13. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be any contra-θ_I-β-continuous function from an ideal topological space (X, τ, I) to any space (Y, σ). If the space (X, τ, I) is extremally disconnected, then f is contra weakly-θ_I-precontinuous.*

Proof. This follows directly from Theorem 2. □

References

[1] M.E. Abd-El Monsef, S.N. El-Deeb, R.A. Mahmoud, *β-open sets and β-continuous mapping*, Bull. Fac. Sci. Assiut Univ.A, 12 (1983), 77-90.

[2] M.E. Abd-El Monsef, E.F. Lashien and A.A. Nasef, *On I-open sets and I-continuous functions*, Kyungpook Math. J., 32(1) (1992), 21-30.

[3] M. Akdağ, *θ-I-open sets*, Kochi J.Math., 3 (2008), 217-229.

[4] P. Alexandroff, P. Urysohn, *Memoire sur les Espaces Topologiques Compacts*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect.I., 14 (1929), 196.

[5] A. Al-Omari, T. Noiri, *Decompositions of continuity in ideal topological spaces*, Anal. St. Univ. "Al.I. Cuza" Iasi (S.N.) Mat., 60 (2014), 37-49.

[6] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Int. J. Math. Math. Sci., 19(2) (1996), 303-310.

[7] J. Dontchev, *On pre-I-open sets and a decomposition of I-continuity*, Banyan Math. J., 2 (1996).

[8] T.R. Hamlett, D.S. Janković, *Ideals in General Topology and Applications* (Mid-dletown,CT,1988), Lecture Notes in Pure and Appl.Math., 123, Dekker, New York, 1990, 115-125.

[9] T.R. Hamlett, D.S. Janković, *Compatible extensions of ideals*, Boll.Un.Mat.Ital., 7 (1992), 453-465.

[10] E. Hatir, T. Noiri , *Strong C-sets and decompositions of continuity*, Acta Math. Hungar., 94(4) (2002), 281-287.

[11] E. Hatir, T. Noiri, *On β-I-open sets and a decomposition of almost-I-continuity*, Bull. Malays. Math. Sci. Soc. (2), 29 (2006), 119-124.

[12] D.S. Janković, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97(4) (1990), 295-310.

[13] K. Kuratowski, *Topology,* Vol.I, Academic Press, New York, 1966.

[14] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70 (1963), 36-41.

[15] S. Jafari and T. Noiri, *Contra-super-continuous functions*, Ann. Univ. Sci. Budapest, 42 (1999), 27-34.

[16] S. Jafari and T. Noiri, *Contra-precontinuous functions*, Bull. Malays. Math. Sci. Soc.(2), 25 (2002), 115-128.

[17] H. Maki, *The special issue in commemoration of Prof. Jazusada IKEDA's retirement,* 1. Oct., 1986, 139-146.

[18] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt., 53 (1982), 47-53.

[19] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math.Hungar., 41 (1983), 213218.

[20] M. Mršević, *On pairwise R_0 and R_1 bitopological spaces*, Bull. Math. Soc. Sci. Math. R.S. Roumance, 30(78) (1986), 141-148.

[21] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math, 15 (1965), 961-70.

[22] T. Noiri, S. Jafari, K. Viswanathan and J. Jayasudha, *Contra-pre-I-continuous functions*, Int. Journal of Math. Analysis, 7(8)(2013), 349-359.

[23] M. Shrivastava, T. Noiri and P. Jha, *Weakly θ_I-Preopen Sets and Decomposition of Continuity*, Ital. J. Pure Appl. Math. (accepted).

[24] R. Vaidyanathaswamy, *The localization theory in set topology*, Proc. Indian Acad. Sci., 20 (1945), 51-61.

[25] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, 1960.

[26] S. Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

Accepted: 6.06.2018

NONEXISTENCE OF GLOBAL SOLUTIONS TO A FRACTIONAL NONLINEAR ULTRA-PARABOLIC SYSTEM

Lamairia Abd Elhakim
Department of Mathematics and Informatics
LAMIS Laboratory
University of Tebessa
Algeria
and
Department of Mathematics
University of Annaba
Algeria
hakim24039@gmail.com

Haouam Kamel*
Department of Mathematics and Informatics
LAMIS Laboratory
University of Tebessa
Algeria
haouam@yahoo.fr

Rebiai Belgacem
Department of Mathematics and Informatics
LAMIS Laboratory
University of Tebessa
Algeria
brebiai@gmail.com

Abstract. In this work, we study the sufficient conditions for that ensure the nonexistence of global solutions to a Cauchy problem for a fractional nonlinear ultra-parabolic system. The Blowing-up solutions is also presented. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.

Keywords: fractional derivatives, nonlinear ultra-parabolic equations, nonexistence, test-function, blowing-up solutions.

1. Introduction

The main objective of this work is to improve the results of Kerbal and Kirane [6] by considering fractional in time and space for the nonlinear ultra-parabolic

*. Corresponding author

system

$$D_{0|t_1}^{\alpha_1}(u-u_2)+D_{0|t_2}^{\alpha_2}(u-u_1)+(-\Delta)^{\frac{\alpha}{2}}(|u|)=k_1|u|^{p_1}|v|^{q_1},\ k_1=\text{const.} \tag{1.1}$$

$$D_{0|t_1}^{\beta_1}(v-v_2)+D_{0|t_2}^{\beta_2}(v-v_1)+(-\Delta)^{\frac{\beta}{2}}(|v|)=k_2|u|^{p_2}|v|^{q_2},\ k_2=\text{const.} \tag{1.2}$$

posed for $(t_1,t_2,x)\in Q=\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^N$, and supplemented with the initial conditions

$$u(t_1,0,x)=u_1(t_1,x),u(0,t_2,x)=u_2(t_2,x), \tag{1.3}$$

$$v(t_1,0,x)=v_1(t_1,x),v(0,t_2,x)=v_2(t_2,x). \tag{1.4}$$

Here $p_1\geq 0,p_2>1,q_1>1,q_2\geq 0$, $0<\alpha_1,\alpha_2<1$, $0<\beta_1,\beta_2<1$, $1\leq\alpha,\beta\leq 2$ are constants and $D_{0|t_j}^{\alpha_j}$, $D_{0|t_j}^{\beta_j}$, $j=1,2$ are the fractional derivatives in the sense of Riemann-Liouville. The operator $D_{0|t}^{\alpha}$ is defined, for an absolutely continuous function $g:\mathbb{R}_+\longrightarrow\mathbb{R}$, by $\left(D_{0|t}^{\alpha}\right)g(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{g(\tau)}{(t-\tau)^{\alpha}}d\tau$, and $\Gamma(\alpha)=\int_0^{\infty}r^{\alpha-1}e^{-r}dr$ is the Euler gamma function. The fractional power of the Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ $(1\leq\alpha\leq 2)$ stands for diffusion in media with impurities and is defined as $(-\Delta)^{\frac{\alpha}{2}}v(x)=\mathcal{F}^{-1}\left(|\zeta|^{\alpha}\mathcal{F}(v)(\zeta)\right)(x)$, where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. In addition, it satisfies the following condition $\forall v;\mathbb{R}^N\longrightarrow\mathbb{R}$ we have $(-\Delta)^{\frac{\alpha}{2}}v\in L^{\frac{p}{p-m}}\left(\mathbb{R}^N\right)$ and the operator $D_{0|t}^{\alpha}$ counts for the abnormal diffusion, a recently very much studied topic in probability, physics, chemistry, biology, image processing, etc, see for instance [7,8] and their references. Classical multi-time or ultraparabolic problems have received a special interest and attention by authors due to their application in real life problems, see for example [2,5,9,13], while the fractional analogs are in their preliminary steps.

2. Preliminaries

The right-sided Riemann-Liouville derivatives of order $0<\alpha<1$ for an absolutely continuous function $g:\mathbb{R}_+\longrightarrow\mathbb{R}$, is defined by:

$$\left(D_{t|T}^{\alpha}\right)g(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T\frac{g(\tau)}{(\tau-t)^{\alpha}}d\tau.$$

Note that for a differentiable function g, the left-sided Caputo derivatives of order α is defined as:

$$D_{0|t}^{\alpha}(g-g(0))(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{g'(\tau)}{(\tau-t)^{\alpha}}d\tau.$$

Finally, taking into account the following integration by parts formula:

$$\int_0^T f(t)D_{0|t}^{\alpha}g(t)\,dt=\int_0^T D_{t|T}^{\alpha}f(t)\,g(t)\,dt,$$

where $f,\ g \in \mathrm{C}^{\infty}(J,\mathbb{R}),\ J \subset \mathbb{R}$.

We also need some preparatory lemmas based on the function

$$\phi(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^{\lambda}, & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{2.1}$$

where $\lambda \ge 2$.

We define the regular function $0 \le \psi \le 1$:

$$\psi(\xi) = \begin{cases} 1, & \text{if } \ 0 \le \xi \le 1, \\ \text{decreasing}, & \text{if } \ 1 \le \xi \le 2, \\ 0, & \text{if } \quad \xi \ge 2, \end{cases} \tag{2.2}$$

which will be used hereafter.

3. Results

We consider the system with a two-dimensional fractional time (1.1)-(1.2) and let us set

$$I_0 = \int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi dP + \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi dP,$$

$$J_0 = \int_Q v_2 D^{\beta_1}_{t_1|T} \varphi dP + \int_Q v_1 D^{\beta_2}_{t_2|T} \varphi dP.$$

Definition 1. *Let $Q_T = (0,T) \times (0,T) \times \mathbb{R}^N,\ 0 < T < +\infty$, and we say that $(u,v) \in \left(L^1_{loc}(Q_T)\right)^2$ is a local weak solution to problem (1) on Q_T, if $u^{p_i} v^{q_i} \in L^1_{loc}(Q_T),\ i = 1,2$, and it is such that*

$$\begin{aligned} &\int_Q k_1 |u|^{p_1} |v|^{q_1} \varphi dP + I_0 \\ &= \int_Q u D^{\alpha_1}_{t_1|T} \varphi dP + \int_Q u D^{\alpha_2}_{t_2|T} \varphi dP + \int_Q |u| \left(-\Delta\right)^{\frac{\alpha}{2}} \varphi dP, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\int_Q k_2 |u|^{p_2} |v|^{q_2} \varphi dP + J_0 \\ &= \int_Q v D^{\beta_1}_{t_1|T} \varphi dP + \int_Q v D^{\beta_2}_{t_2|T} \varphi dP + \int_Q |v| \left(-\Delta\right)^{\frac{\beta}{2}} \varphi dP, \end{aligned} \tag{3.2}$$

for any test function $\varphi \in C^{\infty}$, such that $\varphi(T,t_2,x) = \varphi(t_1,T,x) = 0$, and $P = (t_1,t_2,x)$.

Now, set

$$\sigma_1 = -\frac{q_1\beta_1 - (N+2)(p_2q_1 - 1) + p_2q_1\alpha_1}{p_2q_1 - 1},$$

$$\sigma_2 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha_1}{p_2q_1-1},$$

$$\sigma_3 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha_1}{p_2q_1-1},$$

$$\sigma_4 = -\frac{q_1\beta_1 - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_5 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_6 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_7 = -\frac{q_1\beta_1 - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1},$$

$$\sigma_8 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1},$$

$$\sigma_9 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1}.$$

Theorem 1. *Let $p_1 \geq 0$, $q_2 \geq 0$, $p_2 > 1$, $q_1 > 1$. Let $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, suth that $u_0 \geq 0$, $v_0 \geq 0$, and assume that,*

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu dP > 0, \quad \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu dP > 0,$$

$$\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^\mu dP > 0, \quad \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^\mu dP > 0,$$

then solutions to system (1.1)-(1.2) *blow-up whenever*

$$\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} \leq 0.$$

Proof. Assume that the solution is nontrivial and global. Next, replacing φ by φ^μ in (3.1) and then using Holder's inequality to estimate the I_u and I_v (As we shall see later), we obtain the following estimates:

$$\int_Q u|D^{\alpha_1}_{t_1|T}\varphi^\mu| \leq \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}}$$

$$\cdot \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}, \tag{3.3}$$

$$\int_Q u|D^{\alpha_2}_{t_2|T}\varphi^\mu| \leq \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}}$$

$$\cdot \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}, \tag{3.4}$$

$$\int_Q u|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi^{\mu}| \leq \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^{\mu}\right)^{\frac{1}{p_2}}$$

$$\cdot\left(\int_Q k_2^{-\frac{1}{p_2-1}}|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi^{\mu}|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}. \tag{3.5}$$

Similarly, we have

$$\int_Q v|D^{\beta_1}_{t_1|T}\varphi^{\mu}| \leq \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^{\mu}\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_1}_{t_1|T}\varphi^{\mu}|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}} \tag{3.6}$$

$$\int_Q v|D^{\beta_2}_{t_2|T}\varphi^{\mu}| \leq \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^{\mu}\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_2}_{t_2|T}\varphi^{\mu}|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}}, \tag{3.7}$$

$$\int_Q v|\left(-\Delta\right)^{\frac{\beta}{2}}\varphi^{\mu}| \leq \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^{\mu}\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|\left(-\Delta\right)^{\frac{\beta}{2}}\varphi^{\mu}|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}} \tag{3.8}$$

If we set

$$I_u = \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^{\mu}\right)^{\frac{1}{p_2}},\ I_v = \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^{\mu}\right)^{\frac{1}{q_1}},$$

$$A(p_2) = \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_1}_{t_1|T}\varphi^{\mu}|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$A(q_1) = \left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_1}_{t_1|T}\varphi^{\mu}|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$B(p_2) = \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_2}_{t_2|T}\varphi^{\mu}|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$B(q_1) = \left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_2}_{t_2|T}\varphi^{\mu}|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$C(p_2) = \left(\mu\int_Q k_2^{-\frac{1}{p_2-1}}|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$C(q_1) = \left(\mu\int_Q k_1^{-\frac{1}{q_1-1}}|\left(-\Delta\right)^{\frac{\beta}{2}}\varphi|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$I_0^\mu = \int_Q u_2 D_{t_1|T}^{\alpha_1} \varphi^\mu + \int_Q u_1 D_{t_2|T}^{\alpha_2} \varphi^\mu,$$
$$J_0^\mu = \int_Q v_2 D_{t_1|T}^{\beta_1} \varphi^\mu + \int_Q v_1 D_{t_2|T}^{\beta_2} \varphi^\mu,$$

then, using estimates (3.2), (3.7), we can write (3.1) as

$$I_v + I_0^\mu \leq I_u^{\frac{1}{p_2}} \left(A(p_2) + B(p_2) + C(p_2)\right),$$
$$I_u + J_0^\mu \leq I_v^{\frac{1}{q_1}} \left(A(q_1) + B(q_1) + C(q_1)\right).$$

Since $I_0^\mu \ , \ J_0^\mu > 0$ we have

$$I_v \leq I_u^{\frac{1}{p_2}} \left(A(p_2) + B(p_2) + C(p_2)\right), \tag{3.9}$$
$$I_u \leq I_v^{\frac{1}{q_1}} \left(A(q_1) + B(q_1) + C(q_1)\right). \tag{3.10}$$

Now, from (3.8) and (3.9), we have

$$I_v + I_0^\mu \leq I_v^{\frac{1}{p_2 q_1}} \left(A^{\frac{1}{p_2}}(q_1) + B^{\frac{1}{p_2}}(q_1) + C^{\frac{1}{p_2}}(q_1)\right) \left(A(p_2) + B(p_2) + C(p_2)\right).$$

Then Young's inequality implies

$$\begin{aligned} I_v + I_0^\mu \leq K \Bigg[& \left(A^{\frac{1}{p_2}}(q_1) A(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(B^{\frac{1}{p_2}}(q_1) A(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(C^{\frac{1}{p_2}}(q_1) A(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} \\ & + \left(A^{\frac{1}{p_2}}(q_1) B(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(B^{\frac{1}{p_2}}(q_1) B(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(C^{\frac{1}{p_2}}(q_1) B(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} \\ & + \left(A^{\frac{1}{p_2}}(q_1) C(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(B^{\frac{1}{p_2}}(q_1) C(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} + \left(C^{\frac{1}{p_2}}(q_1) C(p_2)\right)^{\frac{p_2 q_1}{p_2 q_1 - 1}} \Bigg], \end{aligned}$$

for some positive constant K. We choose the test function $\varphi(t_1, t_2, x)$ in the form

$$\varphi(t_1, t_2, x) = \varphi_1(t_1)\, \varphi_2(t_2)\, \varphi_3(x),$$

where $\varphi_1(t_1) = (1 - t_1/T)_+^\lambda$, [1] $\varphi_2(t_2) = (1 - t_2/T)_+^\lambda$ and $\varphi_3(x) = \psi\left(|x|^2/T^2\right)$, and let us now pass to the new variables

$$\tau_1 = T^{-1} t_1, \ \tau_2 = T^{-1} t_2, \ \ y = T^{-1} x;$$

we have,

$$A(p_2) = C T^{-\alpha_1 + (N+2)\left(1 - \frac{1}{p_2}\right)},$$
$$A(q_1) = C T^{-\beta_1 + (N+2)\left(1 - \frac{1}{q_1}\right)},$$

1. $(1 - t_1/T)_+^\lambda = \sup\left\{0, (1 - t_1/T)^\lambda\right\}$

$$B(p_2) = CT^{-\alpha_2+(N+2)\left(1-\frac{1}{p_2}\right)},$$

$$B(q_1) = CT^{-\beta_2+(N+2)\left(1-\frac{1}{q_1}\right)},$$

$$C(p_2) = CT^{-\alpha+(N+2)\left(1-\frac{1}{p_2}\right)},$$

$$C(q_1) = CT^{-\beta+(N+2)\left(1-\frac{1}{q_1}\right)}.$$

For some positive constant C. Hence, we obtain

$$I_v + I_0^{\mu} \leq K\left[T^{\sigma_1} + T^{\sigma_2} + \cdots + T^{\sigma_9}\right]. \tag{3.11}$$

Similarly, we obtain for I_u the estimate

$$I_u + J_0^{\mu} \leq K\left[T^{\delta_1} + T^{\delta_2} + \cdots + T^{\delta_9}\right]. \tag{3.12}$$

Finally, passing to the limit as $T \to +\infty$, we observe that:

Either $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} < 0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction. Or $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} = 0$ and in this case, following the similar analysis used in [6], we prove a contradiction. □

References

[1] P. Biler, T. Funaki, WA. Woyczynski, *Fractal burgers equations*, Journal of Differential Equations, 148(1) (1998), 9-46.

[2] K. Deng, H. A. Levine, *The role of critical exponents in blow-up theorems, The sequel*, J. Math. Anal. Appl., 243 (2000), 85-126.

[3] K. M. Furati, M. Kirane, *Necessary conditions for the existence of global solutions to systems of fractional differential equations*, Fractional Calculus and Applied Analysis, 11(3) (2008), 281-298.

[4] K. Haouam, M. Sfaxi, *Critical exponent for nonlinear hyperbolic system with spatio-temporal fractional derivatives,* I. J. of App. Math., 6 (2011), 661-871.

[5] N. Ju, *The maximum principle and the global attractor for the dissipative 2D quasigeostrophic equations*, Communications in Pure and Applied Analysis, (2005), 161-181.

[6] S. Kerbal, M. Kirane, *Nonexistence results for the Cauchy problem for nonlinear ultraparabolic equations*, Abstract and Applied Analysis, 2011 (2011), 1-10.

[7] M. Kirane, Y. Laskri, N.-E. Tatar, *Critical exponents of Fujita type for certain evolution equations and systems with Spatio-Temporal Fractional derivatives*, J. Math. Anal. Appl., 312 (2005), 488-501.

[8] E. Lanconelli, A. Pascucci, S. Polidoro, *Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance,* in Nonlinear Problems in Mathematical Physics and Related Topics, II, vol. 2 of Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, NY, USA, (2002), 243-265.

[9] E. Mitidieri, S. N. Pokhozhaev, *A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*, Proc. Steklov Inst. Math., 234(3) (2001), 1-362.

[10] B. Rebiai, K. Haouam, *Nonexistence of global solutions to a nonlinear fractional reaction-diffusion system*, IAENG International Journal of Applied Mathematics, 45(4) (2015), 259-262.

[11] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Theory and Applications, Gordon and Breach Science Publishers, Yverdon, 1993.

[12] A. S. Tersenov, *Ultraparabolic equations and unsteady heat transfer*, J. Evolution Equ., 5(2) (2005), 277-289.

[13] W. Walter, *Parabolic differential equations and inequalities with several time variables*, Mathematische Zeitschrift, 191(2) (1986), 319-323.

Accepted: 14.06.2018

A NEW INTUITIONISTIC FUZZY DIVERGENCE MEASURE AND ITS APPLICATIONS TO HANDLE FAULT DIAGNOSIS OF TURBINE

Rakesh Kumar*
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005, India
rakeshmaths@yahoo.in

Om Parkash
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005, India
omparkash777@yahoo.co.in

Abstract. The literature of probability, fuzzy and intuitionistic fuzzy divergence measures provides the applications of a variety of divergence measures to different disciplines dealing with real life problems. Many such divergence measures have been generated through different approaches but still there is a scope that better ones can be developed which will provide applications to variety of disciplines. The present communication involving the development of a new intuitionistic measure of divergence for fuzzy distributions is a motivation in this direction. The newly proposed mathematical model is helpful for the study of fault diagnosis of turbine. In the present paper, we have provided an algorithm which can handle the main faults in the turbine along with useful information for future trends and verified the results numerically.

Keywords: fuzzy sets, intuitionistic fuzzy sets, intuitionistic fuzzy divergence measure/cross-entropy, fault diagnosis, turbine, vibration fault.

1. Introduction

This is to be emphasized that while dealing with the phenomenon of steam turbine failure, many faults are responsible for the cause of vibration of turbine and consequently delivers complex relation between vibration and fault types of turbine. The occurrence of such turbine faults compels us to provide immediate attention, cause of fault and timely diagnose for its repair so as to avoid more accidents and much financial losses. This is a real life problem dealing with diagnose of the vibration fault of steam turbine, the various diagnosis method are available in the literature. These methods are outstanding contributions of (Ye, 2009; Salahshoor, Khoshro, & Kordestani, 2011; Zhao, Liu, & He, 2012; Yang, Lee, Junker, & Ghezel-Ayagh, 2010; Kyriazis, & Mathioudakis; 2009).

∗. Corresponding author

It has been observed that while dealing with turbine faults problems, there may be a variety of reasons for the same symptom of the fault and as a consequence of the phenomenon the system becomes more complex and difficult for its analysis and precision. The subjectivity of the experiment suggests that the fuzzy set theory introduced by (Zadeh, 1965) can play a vital role in handling such a type of uncertain information and finds widespread applications. This is to add that fuzzy set theory was extended to intuitionistic fuzzy set theory by (Atanassov, 1986), who characterize a membership function and non-membership function so as to provide mathematical background in handling imperfect situations where proper decision cannot be made. Some work related with IFSs theory has been provided by (He & He, 2016; He, He & Chen, 2015; Li & Ren, 2015). This theory is an alternative approach which finds tremendous applications in various fields including image processing, pattern recognition, fault diagnose and medical science. Motivated by the well known distance measures in probability spaces due to (Kullback, & Leibler, 1951; Bhandari, & Pal, 1993) investigated and introduced a new fuzzy divergence measure. Some other pioneer, who contributed towards the fuzzy (Parkash & Sharma, 2005; Parkash & Kumar, 2016; He, He, Wang & Chen, 2015; He, He, Lee, Kim, Zhang & Yang, 2017; Fei & Li, 2016; Li, 2016, Li & Liu, 2015; Dey, Pradhan, Pal & Pal, 2015; Dey & Pal, 2016; Dey, Pal & Pal, 2016 etc.).

The present communication deals with the investigations and enlargement of a new intuitionistic fuzzy divergence measure responsible for detecting the fault diagnosis of turbine. In section 2, some drawbacks in the existing IF-divergence measures have been pointed out. To overcome the drawbacks, a new IF-divergence measure has been proposed and studied its elegant properties in the form of theorems. These results have presented in section 3 of the paper. In section 4, application of a new IF-divergence Measure to handle fault diagnosis of turbine is discussed and also numerical example has been presented to illustrate the procedure of proposed method based on fault diagnosis of turbine.

2. Preliminaries and review of existing intuitionistic fuzzy divergence measures

In this section, fundamental knowledge concerning about IFS and intuitionistic fuzzy divergence measure is introduced.

2.1 Intuitionistic fuzzy sets

Definition 2.1. *An intuitionistic fuzzy set A defined on the universe of discourse $X = (x_1, x_2, ..., x_n)$ given by the expression (Atanassov, 1986)*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{2.1}$$

where the function $\mu_A(x) : X \to [0, 1]$ and $\nu_A(x) : X \to [0, 1]$ denote the membership degree and the non-membership degree respectively with the condition

that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \ for \ every \ x \in X. \tag{2.2}$$

Now, let IFS(X) denote the family of all intuitionistic fuzzy sets in the finite universe $X = (x_1, x_2, ..., x_n)$ and $A, B \in IFS(X)$ given by $A = \{\langle x, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x, \mu_B(x_i), \nu_B(x_i)\rangle | x_i \in X\}$.Then some set operations can be defined as follows:

Complement of A, $A^C = \{\langle x, \nu_A(x_i), \mu_A(x_i)\rangle | x_i \in X\}$. Intersection of A and B, $A \cap B = \{\langle x, \min\{\mu_A(x_i), \mu_B(x_i)\}, \max\{\nu_A(x_i), \nu_B(x_i)\}\rangle | x_i \in X\}$. Union of A and B, $A \cup B = \{\langle x, \max\{\mu_A(x_i), \mu_B(x_i)\}, \min\{\nu_A(x_i), \nu_B(x_i)\}\rangle | x_i \in X\}$. Inclusion Relation, $A \subseteq B$ if and only if $\mu_A(x_i) \leq \mu_B(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i)$, $\forall x_i \in X$.

Definition 2.2. *Let* $A = \{\langle x, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ *and* $B = \{\langle x, \mu_B(x_i), \nu_B(x_i)\rangle | x_i \in X\}$ *be two intuitionistic fuzzy sets in* X. *A mapping* $D : IFS(X) \times IFS(X) \to \mathbb{R}$ *is a divergence measure for intuitionistic fuzzy sets if it satisfies the following axioms (Montes, Pal, Jains, & Montes, 2015):*

(a) $D(A, B) = D(B, A)$.

(b) $D(A, B) = 0 \ if \ and \ only \ if \ A = B$.

(c) $D(A \cap C, B \cap C) \leq D(A, B) \ for \ every \ C \in IFS(X)$.

(d) $D(A \cup B, B \cup C) \leq D(A, B) \ for \ every \ C \in IFS(X)$.

Again, the non negativity of the divergence measure is not required in the above axioms. However, it is trivially deduced from (b) and (c) (or (b) and (d)).

A variety of divergence measures for intuitionistic fuzzy sets have been introduced by various researchers with their own merits, demerits and limitations. We, now review the following existing divergence measures for IFSs to make a further meaningful study.

$$\begin{aligned} D_{VS}(A, B) = \sum_{i=1}^{n} \Bigg(& \mu_A(x_i) \ln \left(\frac{2\mu_A(x_i)}{\mu_A(x_i) + \mu_B(x_i)} \right) \\ & + \left(\nu_A(x_i) \ln \left(\frac{2\nu_A(x_i)}{\nu_A(x_i) + \nu_B(x_i)} \right) \right) \Bigg) \end{aligned} \tag{2.3}$$

which is (Vlachos & Sergiadis, 2007) measures of intuitionistic fuzzy sets. (Vlachos & Sergiadis, 2007) also defined the symmetric version of measure (2.3) given by

$$\begin{aligned} & D_{VS}^{sym}(A, B) = D_{VS}(A, B) + D_{VS}(B, A) \\ & = \sum_{i=1}^{n} \left(\mu_A(x_i) \ln \left(\frac{2\mu_A(x_i)}{\mu_A(x_i) + \mu_B(x_i)} \right) + \left(\nu_A(x_i) \ln \left(\frac{2\nu_A(x_i)}{\nu_A(x_i) + \nu_B(x_i)} \right) \right) \right. \\ & \left. + \ \mu_B(x_i) \ln \left(\frac{2\mu_B(x_i)}{\mu_A(x_i) + \mu_B(x_i)} \right) + \left(\nu_B(x_i) \ln \left(\frac{2\nu_B(x_i)}{\nu_A(x_i) + \nu_B(x_i)} \right) \right) \right) \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
D_{ZY}(A,B) = &\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\\
&\ln\left(\frac{2(\mu_A(x_i)+1-\nu_A(x_i))}{(\mu_A(x_i+1-\nu_A(x_i))+(\mu_B(x_i+1-\nu_B(x_i))}\right)\\
&+\sum_{i=1}^{n}\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\\
&\ln\left(\frac{2(\nu_A(x_i)+1-\mu_A(x_i))}{(\nu_A(x_i)+1-\mu_A(x_i))+(\nu_B(x_i)+1-\mu_B(x_i))}\right)
\end{aligned}
\tag{2.5}
$$

which is (Zhang and Jiang, 2008) measures of intuitionistic fuzzy sets.

$$
\begin{aligned}
D_{WY}(A,B) = &\sum_{i=1}^{n}\left(\mu_A(x_i)\ln\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)\right.\\
&\left.+\left(\nu_A(x_i)\ln\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right)+\pi_A(x_i)\ln\left(\frac{2\pi_A(x_i)}{\pi_A(x_i)+\pi_B(x_i)}\right)\right)
\end{aligned}
\tag{2.6}
$$

which is (Wei & Yei, 2010; Hung, 2012) measures of intuitionistic fuzzy sets. The symmetric discrimination of measures (2.6) is given by

$$
\begin{aligned}
&D^{sym}_{WY}(A,B) = D_{WY}(A,B)+D_{WY}(B,A)\\
&=\sum_{i=1}^{n}\left(\mu_A(x_i)\ln\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)+\left(\nu_A(x_i)\ln\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right)\right.\\
&+\mu_B(x_i)\ln\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)+\left(\nu_B(x_i)\ln\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right)\Big)\\
&\left.+\pi_A(x_i)\ln\left(\frac{2\pi_A(x_i)}{\pi_A(x_i)+\pi_B(x_i)}\right)+\left(\pi_B(x_i)\ln\left(\frac{2\pi_B(x_i)}{\pi_A(x_i)+\pi_B(x_i)}\right)\right)\right)
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
D_J(A,B) = &\sum_{i=1}^{n}\left(\pi_A(x_i)\ln\left(\frac{2\pi_A(x_i)}{\pi_A(x_i)+\pi_B(x_i)}\right)\right.\\
&\left.+\left(\Delta_A(x_i)\ln\left(\frac{2\Delta_A(x_i)}{\Delta_A(x_i)+\Delta_B(x_i)}\right)\right)\right)
\end{aligned}
\tag{2.8}
$$

which is (Junjun, Dengbao, & Cuicui, 2013) measures of intuitionistic fuzzy sets. where $\Delta_A(x_i) = |\mu_A(x_i - \nu_A(x_i))|$, denotes that how close the membership and non membership degrees are. The symmetric divergence measure of (2.8) are defined as follows

$$
D^{sym}_J(A,B) = D_J(A,B)+D_J(B,A). \tag{2.9}
$$

The above listed measures have some drawbacks introduced by (Vlachos & Sergiadis, 2007; Zhang & Jiang, 2008; Junjun, Dengbao, & Cuicui, 2013) and thus needs a modification as defined in (Verma and Sharma, 2012). The present communication is a step in this direction of removing these drawbacks. Such a new proposal has been investigated and introduced in the section 3.

3. A new intuitionistic fuzzy divergence measure

We now propose the intuitionistic fuzzy divergence measure of A against B as given by the following mathematical expression

$$
\begin{aligned}
& D(A,B) \\
& = -\log_2 \left[\frac{1}{2} \left\{ 1 + \frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \right) \left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2} \right)} \right. \right. \\
& \left. \left. + \sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \right) \left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2} \right)} \right) \right\} \Bigg],
\end{aligned} \tag{3.1}
$$

where the functions $\mu_A(x), \mu_B(x) \& \nu_A(x), \nu_B(x)$ denote the membership degree and the non-membership degree respectively. The measure defined above overcomes all the drawbacks, i.e. it satisfies all the axioms defined in definition 2.2. To prove the validity of proposed measure, we now study the following properties.

3.1 Properties of the proposed intuitionistic fuzzy divergence measure

Theorem 3.1. *Let $A, B, C \in IFS(X)$, then the proposed measure $D(A,B)$ given by equation (3.1) satisfies the following properties are given as follows:*

(a) $D(A,B) = D(B,A)$ and $0 \leq D(A,B) \leq 1$.

(b) $D(A,B) = 0$ *if and only if* $A = B$.

(c) $D(A \cap B, B \cap C) \leq D(A,B)$ *for every* $C \in IFS(X)$.

(d) $D(A \cup B, B \cup C) \leq D(A,B)$ *for every* $C \in IFS(X)$.

(e) $D(A,B) = D(A^C, B^C)$.

(f) $D(A, B^C) = D(A^C, B)$.

(g) $D(A, A \cup B) = D(A \cap B, B) \leq D(A,B)$ for $A \subseteq B$ and $B \subseteq A$.

(h) $D(A \cap B, A \cup B) = D(A,B)$.

Proof. From equation (3.1), it is understood that $D(A,B) = D(B,A)$. Further for all $x_i \in X$, we have

$$\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}$$
$$\leq \frac{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}+\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}}{2}$$
$$\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}$$
$$\leq \frac{\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}+\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}}{2}$$

On adding the above two equations and take summations on both sides, we get

$$\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.$$
$$\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)$$
$$\leq \sum_{i=1}^{n}\left(\frac{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}+\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}}{2}\right.$$
$$\left.+\frac{\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}+\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}}{2}\right)$$
$$\Rightarrow 0 \leq \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.$$
$$\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right) \leq 1$$
$$\Rightarrow \frac{1}{2} \leq \frac{1}{2}\left[1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.\right.$$
$$\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right] \leq 1$$
$$\Rightarrow 0 \leq -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right)\right.\right.$$
$$\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right] \leq 1$$

$$\Rightarrow 0 \leq D(A,B) \leq 1.$$

(b) It is obvious that $D(A,B) = 0$ if and only if $A = B$.

(c) We have

$$\begin{aligned}
&\left[\sqrt{\min\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right)}\right. \\
&\left. - \sqrt{\min\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right)}\right]^2 \\
&\leq \left(\sqrt{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}} - \sqrt{\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}}\right)^2 \\
&\Rightarrow \begin{bmatrix} \min\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right) \\ +\min\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right) \\ -2\sqrt{\min\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right)} \\ \times\sqrt{\min\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}, \frac{\mu_C(x_i)+1-\nu_C(x_i)}{2}\right)} \end{bmatrix} \\
&\leq \begin{bmatrix} \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \\ \frac{\mu_B(x_i)+1-\nu_B(x_i)}{2} \\ -2\sqrt{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}} \\ \times\sqrt{\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}} \end{bmatrix}.
\end{aligned} \tag{3.2}$$

Similarly, we have

$$\Rightarrow \begin{bmatrix} \max\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}, \frac{\nu_C(x_i)+1-\mu_C(x_i)}{2}\right) \\ +\max\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}, \frac{\nu_C(x_i)+1-\mu_C(x_i)}{2}\right) \\ -2\sqrt{\max\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}, \frac{\nu_C(x_i)+1-\mu_C(x_i)}{2}\right)} \\ \times\sqrt{\max\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}, \frac{\nu_C(x_i)+1-\mu_C(x_i)}{2}\right)} \end{bmatrix}$$

$$\leq \left[\begin{array}{l} \dfrac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \\ \dfrac{\nu_B(x_i)+1-\mu_B(x_i)}{2} \\ -2\sqrt{\dfrac{\nu_A(x_i)+1-\mu_A(x_i)}{2}} \\ \times\sqrt{\dfrac{\nu_B(x_i)+1-\mu_B(x_i)}{2}} \end{array}\right] \tag{3.3}$$

Adding equations (3.2) and (3.3), yields $D(A\cap B, B\cap C)\leq D(A,B)$.

(d) The proof is on similar lines as in part (c).

(e) Consider

$$\begin{aligned} &D(A,B) \\ &= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.\right.\right. \\ &\left.\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right)\right] \\ &= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right.\right.\right. \\ &\left.\left.\left.+\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right)\right)\right] = D(A^C,B^C). \end{aligned}$$

(f) Consider

$$\begin{aligned} &D(A,B^C) \\ &= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right.\right.\right. \\ &\left.\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right)\right)\right] \\ &= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right.\right.\right. \\ &\left.\left.\left.+\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right)\right)\right] = D(A^C,B) \end{aligned}$$

(g) Consider $D(A, A\cup B)$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_{A\cup B}(x_i)+1-\nu_{A\cup B}(x_i)}{2}\right)}\right.\right.\right.$$

$$+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_{A\cup B}(x_i)+1-\mu_{A\cup B}(x_i)}{2}\right)}\Bigg)\Bigg)\Bigg]$$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\min\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2},\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.\right.$$

(3.4)

$$\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\min\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2},\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right)\right]$$

Similarly,$D(A, A\cup B)$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\min\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2},\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.\right.$$

(3.5)

$$\left.\left.+\sqrt{\min\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2},\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right)\right]$$

For $A\subseteq B$, (3.4) and (3.5) gives, $D(A, A\cup B) = D(A\cap B, B) = D(A, B)$.

Again, for $B\subseteq A$, (3.4) and (3.5) gives, $D(A, A\cup B) = D(A\cap B, B) = 0 \leq D(A, B)$.

(h) Proceeding on similar lines as above, we can prove the required result.

Consider

$$D(A\cap B, A\cup B)$$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_{A\cap B}(x_i)+1-\nu_{A\cap B}(x_i)}{2}\right)\left(\frac{\mu_{A\cup B}(x_i)+1-\nu_{A\cup B}(x_i)}{2}\right)}\right.\right.\right.$$

$$\left.\left.\left.+\sqrt{\left(\frac{\nu_{A\cap B}(x_i)+1-\mu_{A\cap B}(x_i)}{2}\right)\left(\frac{\nu_{A\cup B}(x_i)+1-\mu_{A\cup B}(x_i)}{2}\right)}\right)\right)\right]$$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\min\left(\frac{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}}{\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}},\right)\max\left(\frac{\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}}{\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}},\right)}\right.\right.\right.$$

$$\left.\left.\left.+\sqrt{\min\left(\frac{\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}}{\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}},\right)\max\left(\frac{\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}}{\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}},\right)}\right)\right)\right]$$

$$= -\log_2\left[\frac{1}{2}\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}\right.\right.\right.$$

$$\left.\left.\left.+\sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2},\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right)\right] = D(A, B)$$

Hence the result.

With the study of above properties, we claim that the intuitionistic fuzzy divergence measure proposed in (3.1) is an appropriate measure of divergence. In the next section, we provide the application of the proposed measure to fault diagnosis.

4. Applications of the intuitionistic fuzzy divergence measure to fault-diagnosis of turbine

To apply the proposed model to measure the fault diagnose in steam turbine, we need the following algorithm.

4.1 Fault-diagnosis algorithm

Assume that there exist m fault patterns (knowledge of fault samples), which are represented by IFS $F_i(i = 1, 2, ..., m)$, and there is a testing sample to be recognized which is represented by a IFS F_t. Then diagnosis result F_k should be nearest one to F_t that is

$$D(F_t, F_k) = Min_{1 \leq i \leq m}\{D(F_t, F_i)\}.$$

where $D(F_t, F_i)$ expresses the discrimination degree of the IFS F_t from F_i. The value of $D(F_t, F_i)$ is calculated by using equation (3.1). Then, we decide that testing sample F_t should belong to the fault pattern F_k, where $k = Min_{1 \leq i \leq m}\{D(F_t, F_i)\}$.

The proposed divergence measure of IFSs can realize the classification and identification of the fault by comparing the intuitionistic fuzzy divergence values between a diagnosing sample and the knowledge of system faults. The minimum value of the intuitionistic fuzzy divergence will detect and confirm the type of the fault. The fault-diagnosis process using the intuitionistic fuzzy divergence measure is shown in Fig.-1. The following numerical verification will prove the authenticity of the algorithm mentioned above.

Numerical verification

During our investigation, we have taken the ten kinds of familiar fault types in rotating machines, such as unbalance, offset center, and oil-membrane oscillation, are used as the knowledge of fault samples. The vibration frequency of the turbine is divided into nine different frequency ranges representing the failure patterns known as IFSs as shown in Table-1. (Ye, Qiao, & Wei, 2005). Next, we have established the knowledge database of fault types and then calculated the intuitionistic fuzzy cross entropy between a fault-testing sample and fault knowledge samples. Ten fault knowledge samples in table-1 are expressed by IFSs:

$F_1 = [0.00, 0.00]/x_1 + [0.00, 0.00]/x_2 + [0.00, 0.00]/x_3 + [0.00, 0.00]/x_4$

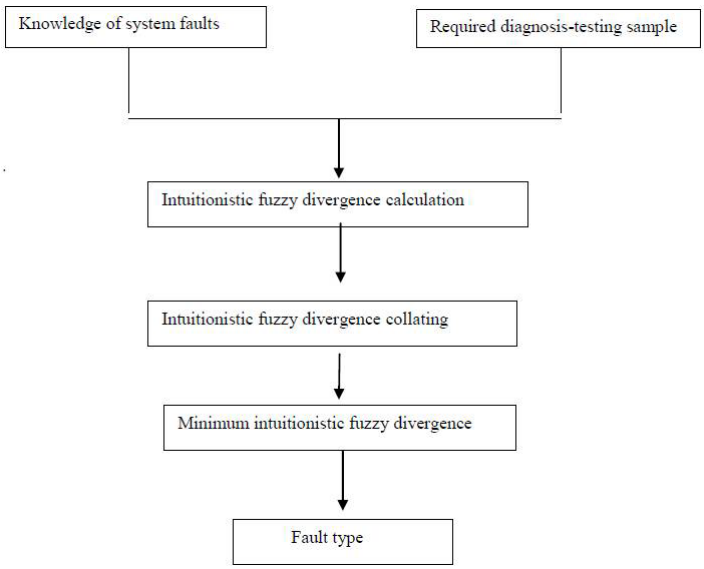


Figure 1: Block diagram of fault diagnosis using the intuitionistic fuzzy cross-entropy

$+[0.85,1.00]/x_5+[0.04,0.06]/x_6+[0.04,0.07]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$
$F_2 = [0.00,0.00]/x_1 + [0.28,0.31]/x_2 + [0.09,0.12]/x_3 + [0.55,0.70]/x_4$
$+[0.00,0.00]/x_5+[0.00,0.00]/x_6+[0.00,0.00]/x_7+[0.00,0.00]/x_8+[0.08,0.13]/x_9$
$F_3 = [0.00,0.00]/x_1 + [0.00,0.00]/x_2 + [0.00,0.00]/x_3 + [0.00,0.00]/x_4$
$+[0.30,0.58]/x_5+[0.40,0.62]/x_6+[0.08,0.13]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$
$F_4 = [0.09,0.11]/x_1 + [0.78,0.82]/x_2 + [0.00,0.00]/x_3 + [0.08,0.11]/x_4$
$+[0.00,0.00]/x_5+[0.00,0.00]/x_6+[0.00,0.00]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$
$F_5 = [0.09,0.12]/x_1 + [0.09,0.11]/x_2 + [0.08,0.12]/x_3 + [0.09,0.12]/x_4$
$+[0.18,0.21]/x_5+[0.08,0.13]/x_6+[0.08,0.13]/x_7+[0.08,0.12]/x_8+[0.08,0.12]/x_9$
$F_6 = [0.00,0.00]/x_1 + [0.00,0.00]/x_2 + [0.00,0.00]/x_3 + [0.00,0.00]/x_4$
$+[0.18,0.22]/x_5+[0.12,0.17]/x_6+[0.37,0.45]/x_7+[0.00,0.00]/x_8+[0.22,0.28]/x_9$
$F_7 = [0.00,0.00]/x_1 + [0.00,0.00]/x_2 + [0.08,0.12]/x_3 + [0.86,0.93]/x_4$
$+[0.00,0.00]/x_5+[0.00,0.00]/x_6+[0.00,0.00]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$
$F_8 = [0.00,0.00]/x_1 + [0.27,0.32]/x_2 + [0.08,0.12]/x_3 + [0.54,0.62]/x_4$
$+[0.00,0.00]/x_5+[0.00,0.00]/x_6+[0.00,0.00]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$
$F_9 = [0.85,0.93]/x_1 + [0.00,0.00]/x_2 + [0.00,0.00]/x_3 + [0.00,0.00]/x_4$
$+[0.85,1.00]/x_5+[0.04,0.06]/x_6+[0.04,0.07]/x_7+[0.08,0.12]/x_8+[0.00,0.00]/x_9$
$F_{10} = [0.00,0.00]/x_1 + [0.00,0.00]/x_2 + [0.00,0.00]/x_3 + [0.00,0.00]/x_4$
$+[0.85,1.00]/x_5+[0.77,0.83]/x_6+[0.19,0.23]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$

Suppose that the IFSs of fault-testing samples are as follows:

$F_{t1} = [0.00,0.00]/x_1 + [0.00,0.00]/x_2 + [0.10,0.10]/x_3 + [0.90,0.90]/x_4$
$+[0.00,0.00]/x_5+[0.77,0.83]/x_6+[0.21,0.35]/x_7+[0.00,0.00]/x_8+[0.00,0.00]/x_9$

$F_{t2} = [0.39, 0.39]/x_1 + [0.07, 0.07]/x_2 + [0.10, 0.10]/x_3 + [0.06, 0.06]/x_4$
$+[0.00, 0.00]/x_5+[0.13, 0.13]/x_6+[0.00, 0.00]/x_7+[0.00, 0.00]/x_8+[0.35, 0.35]/x_9$
The cross-entropy values of vague sets are calculated by use of equation (3.1) as follows:
$D(F_{t1}, F_1) = 0.003200, D(F_{t1}, F_2) = 0.082967, D(F_{t1}, F_3) = 0.001733, D(F_{t1}, F_4)$
$= 0.000288, D(F_{t1}, F_5) = 0.000205, D(F_{t1}, F_6) = 0.000117, D(F_{t1}, F_7) = 0.000279,$
$D(F_{t1}, F_8) = 0.000328, D(F_{t1}, F_9) = 0.000548, D(F_{t1}, F_{10}) = 0.082573;$
$D(F_{t2}, F_1) = 0.003081, D(F_{t2}, F_2) = 0.082835, D(F_{t2}, F_3) = 0.001582, D(F_{t2}, F_4)$
$= 0.000048, D(F_{t2}, F_5) = 0.000132, D(F_{t2}, F_6) = 0.000155, D(F_{t2}, F_7) = 0.00018,$
$D(F_{t2}, F_8) = 0.000131, D(F_{t2}, F_9) = 0.000102, D(F_{t2}, F_{10}) = 0.000082.$

Diagnosis result 1: The fault-diagnosis result is as follows:
$F_6 \to F_3 \to F_5 \to F_7 \to F_4 \to F_8 \to F_9 \to F_1 \to F_{10} \to F_2.$

From the above diagnosis order, we observe the sequence of vibration fault of turbine. This sequence has discovered that firstly antithrust bearing is damage, next offset center, and then oil membrane oscillation, and so on, which results in the vibration of turbine.

Diagnosis result 2: The fault-diagnosis result is as follows:
$F_4 \to F_{10} \to F_9 \to F_8 \to F_5 \to F_6 \to F_7 \to F_3 \to F_1 \to F_2.$

From the above sequence our observations proves that the fault causing the vibration of turbine has been resulted firstly from radial impact friction of rotor, next non uniform bearing stiffness, and then looseness of bearing block, and so on.

From the above diagnostic scheme, we claim that the proposed diagnosis method is effective and provides reasonable information in detecting a fault during vibration of turbine.

Conclusions

In this paper, we have proposed a new intuitionistic fuzzy divergence measure and proved some elegant properties of the intuitionistic fuzzy divergence measure in detail. Furthermore, based on the proposed divergence measure for IFSs, an algorithm to deal with fault diagnosis of turbine under fuzzy environments is described. On the basis of diagnosis results for the turbine, the proposed method cannot only diagnose the main fault of the turbine; it can also detect useful information for multi-fault analysis. Finally, a numerical example is provided to illustrate the fault diagnosis of turbine and proves the authenticity of our model providing useful information in detecting the fault under consideration. This work can be extended to Interval valued intuitionistic fuzzy divergence measures.

Fault samples	Frequency range (f: operating frequency)								
	0.01-0.39f	0.40-0.49f	0.50f	0.51-0.99f	f	2f	3-5f	Odd times of f	High freq $> 5f$
Pneumatic force couple	(0.00, 0.00)	(0.28, 0.31)	(0.09, 0.12)	(0.55, 0.70)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.08, 0.13)
Unbalance	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.85, 1.00)	(0.04, 0.06)	(0.04, 0.07)	(0.00, 0.00)	(0.00, 0.00)
Offset center	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.30, 0.58)	(0.40, 0.62)	(0.08, 0.13)	(0.00, 0.00)	(0.00, 0.00)
Radial impact friction of rotor	(0.09, 0.12)	(0.09, 0.11)	(0.08, 0.12)	(0.09, 0.12)	(0.18, 0.21)	(0.08, 0.13)	(0.08, 0.13)	(0.08, 0.12)	(0.08, 0.12)
Oil-membrane oscillation	(0.09, 0.11)	(0.78, 0.82)	(0.00, 0.00)	(0.08, 0.11)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)
Damage of antithrust bearing	(0.00, 0.00)	(0.00, 0.00)	(0.08, 0.12)	(0.86, 0.93)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)
Symbiosis looseness	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.18, 0.22)	(0.12, 0.17)	(0.37, 0.45)	(0.00, 0.00)	(0.22, 0.28)
Surge	(0.00, 0.00)	(0.27, 0.32)	(0.08, 0.12)	(0.54, 0.62)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)
Looseness of bearing block	(0.85, 0.93)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.08, 0.12)	(0.00, 0.00)
Non-uniform bearing stiffness	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.00, 0.00)	(0.77, 0.83)	(0.19, 0.23)	(0.00, 0.00)	(0.00, 0.00)

Acknowledgement

The authors are thankful to University Grants Commission (UGC), New Delhi for providing the financial assistance for the preparation of the manuscript.

References

[1] K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[2] D. Bhandari, N.R. Pal, *Some new information measure for fuzzy sets*, Information Science, 67 (1993), 209-228.

[3] A. Dey, A. Pal, *Prim's algorithm for solving minimum spanning tree problem in fuzzy environment*, Annals of Fuzzy Mathematics and Informatics, 12 (2016), 419-430.

[4] A. Dey, A. Pal, T. Pal, *Interval type 2 fuzzy set in fuzzy shortest path problem*, Mathematics, 4 (2016), 62.

[5] A. Dey, R. Pradhan, A. Pal, T. Pal, *The fuzzy robust graph coloring problem*. In Proceedings of the 3rd International Conference on Frontiers of Intelligent Computing: Theory and Applications (FICTA), pp. 805-813, Springer, 2015.

[6] W. Fei, D.F. Li, *Bilinear programming approach to solve interval bimatrix games in tourism planning management*, International Journal of Fuzzy Systems, 18 (2016), 504-510.

[7] Y. He, Z. He, *Extensions of Atanassov's intuitionistic fuzzy interaction Bonferroni means and Their Application to Multiple attribute decision making*, IEEE Transactions on Fuzzy Systems, 24 (2016), 558-573.

[8] Y. He, Z. He, H. Chen, *Intuitionistic Fuzzy Interaction Bonferroni Means and Its Application to Multiple Attribute Decision Making*, IEEE Transactions on Cybernetics, 45 (2015), 116-128.

[9] Y. He, Z. He, D.H. Lee, K.J. Kim, L. Zhang, X. Yang, *Robust Fuzzy Programming Method for MRO Problems Considering Location Effect*, Dispersion Effect and Model Uncertainty, Computers Industrial Engineering, 105 (2017), 76-83.

[10] Y. He, Z. He, G. Wang, H. Chen, *Hesitant Fuzzy Power Bonferroni Means and their Application to Multiple Attribute Decision Making*, IEEE Transactions on Fuzzy Systems, 23 (2015), 1655-1668.

[11] K.C. Hung, *Medical pattern recognition: applying an improved intuitionistic fuzzy cross-entropy approach*, Advances in fuzzy systems, Article ID, 863549, 2012.

[12] M. Junjun, Y. Dengbao, W. Cuicui, *A novel cross-entropy and entropy measures of IFSs and their applications*, Knowledge-Based Systems, 48 (2013), 37-45.

[13] S. Kullback, R.A. Leibler, *On information and sufficiency*, The Annals Mathematical Statistics, 22 (1951), 79-86.

[14] A. Kyriazis, K. Mathioudakis, *Gas turbine fault diagnosis using fuzzy-based decision fusion*, J. Propulsion Power, 25 (2009), 335-343.

[15] D.F. Li, *Decision and Game Theory in Management with Intuitionistic Fuzzy Sets*, Germany: Springer, Heidelberg, 2014.

[16] D.F. Li, *Linear Programming Models and Methods of Matrix Games with Payoffs of Triangular Fuzzy Numbers*, Berlin, Springer, Heidelberg, 2016.

[17] D.F. Li, J.C. Liu, *A parameterized non-linear programming approach to solve matrix games with payoffs of I-fuzzy numbers*, IEEE Transactions on Fuzzy Systems, 23 (2015), 885-896.

[18] D.F. Li, H.P. Ren, *Multi-attribute decision making method considering the amount and reliability of intuitionistic fuzzy information*, Journal of Intelligent Fuzzy Systems, 28 (2015), 1877-1883.

[19] I. Montes, N.R. Pal, V. Janis, S. Montes, *Divergence measures for intuitionistic fuzzy sets*, IEEE Transactions on Fuzzy Systems, 23 (2015), 444-456.

[20] O. Parkash, R. Kumar, *New parametric and non - parametric measures of cross entropy*, Canadian Journal of Pure Applied Sciences, 10 (2016), 3921-3925.

[21] O. Parkash, P.K. Sharma, *Some new measures of fuzzy directed divergence and their generalization*, Journal of the Korean Society of Mathematical Education Series B 12 (2005), 307-315.

[22] K. Salahshoor, M.S. Khoshro, M. Kordestani, *Fault detection and diagnosis of an industrial steam turbine using a distributive configuration of adaptive neuro-fuzzy inference systems*, Simulation Modell. Pract. Theory, 19 (2011), 1280-1293.

[23] R. Verma Sharma, B.D. Sharma, *On generalized intuitionistic fuzzy divergence (relative information) and their properties*, Journal of Uncertain Systems, 6 (2012), 308-320.

[24] K. Vlachos, G.D. Sergiadis, *Intuitionistic fuzzy information-applications to pattern recognition*, Pattern Recognition Letters, 28 (2007), 197-206.

[25] P. Wei, J. Ye, *Improved intuitionistic fuzzy cross-entropy and its application to pattern recognition,* International Conference on Intelligent Systems and Knowledge Engineering, 114-116, 2010.

[26] W. Yang, K.Y. Lee, S.T. Junker, H. Ghezel-Ayagh, *Fuzzy fault diagnosis and accommodation system for hybrid fuel-cell/gas-turbine power plant*, IEEE Trans. Energy Convers., 25 (2010), 1187-1194.

[27] J. Ye, X.L. Qiao, H.L. Wei, *Fault diagnosis of turbine on similarity measures between vague sets*, In Proceedings of the Asia pacific symposium on safety (Part B, pp. 1358-1362), Shaoxing, Zhejiang, China, 2005.

[28] J. Ye, *Fault diagnosis of turbine based on fuzzy cross entropy of vague sets*, Expert Systems with Applications, 36 (2009), 8103-8106.

[29] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[30] Q.S. Zhang, S.Y. Jiang, *A note on information entropy measures for vague sets and its applications*, Information Sciences, 178 (2008), 4184-4191.

[31] X. Zhao, Y. Liu, X. He, *Fault diagnosis of gas turbine based on fuzzy matrix and the principle of maximum membership degree*, Energy Proc., 16 (2012), 1448-1454.

Accepted: 1.07.2018

FUZZY PROTECTION METHOD FOR FLOOD ATTACKS IN SOFTWARE DEFINED NETWORKING (SDN)

Mohammad Hadi Zahedi*
Department of Computer Engineering
K.N. Toosi University of Technology
Tehran
Iran
mhadi_zahedi@yahoo.com

Abbas Ali Rezaee
Department of Computer Engineering
Payame-Noor University
Tehran
Iran

Zeinab Dehghan
Department of Computer Engineering
Payame-Noor University
Tehran
Iran

Abstract. Flood attacks (FA) are a type of distributed denial of service (DDoS) attacks. In FA, an attacker sends massive floods of packets to consume all resources. Hierarchical architecture and numerous weaknesses in the structure of communication protocols in conventional networks lead to the fact that firewalls are incapable to provide an integrated and effective mechanism against these attacks. With the emergence of Software Defined Networking (SDN), there are new prospects for solving structural and security problems in conventional networks. This study investigates some ideas for protecting against distributed FA using SDN. Later, by analyzing the strengths and weaknesses of these ideas, a heterogeneous method is proposed based on a combination of conventional service provider and the Software Defined controller. In the proposed method, the attack detection and the fuzzy decision modules are located in the service provider and the controller (i.e. SDN), respectively. In order to simulate the heterogeneous method the MiniNet emulator is applied in combination with the Pox controller. Afterwards, the simulated model is evaluated. Results show that besides protecting against attacks in conventional networks, the proposed method provides other benefits including the extent of computational load and the response time in comparision to other Software Defined methods.

Keywords: fuzzy system, software defined networks, DDoS attacks, flood attacks, OpenFlow.

∗. Corresponding author

1. Introduction

One of the most important measures of information security is the accessibility of resources, which means that we have to make sure of the correct functioning of information processing devices and the communication channels. In other words, authorized people should have access to resources whenever needed. Denials of service (DoS) attacks are amongst the threats which seriously challenge access to information. Using these attacks, attackers try to hog a major portion of the accessible resources in order to interrupt service [1-2]. FA is amongst the most dangerous denial of service attacks, which bombards the network due to weaknesses in communication protocols. For instance, SynFlood attacks which occur in transfer plane account for more than 90 percent of DoS attacks [3]. The hierarchical structure, weakness in communication protocols, and the lack of a centralized command center in current networks causes serious challenges to protect against these attacks [4, 5].

Transfer control protocol (stocktickerTCP) weakneses lead to challenges listed above. One is the weakness of three-stage handshaking operation. This problem is due to the lack of authentication for the communication initiator by the service provider. Under normal conditions, after receiving the SYN packet, the service provider allocates some memory to follow the communication session and waits for the communication to start. Utilizing this weakness, attackers start sending floods of invalid packets with the sign SYN=1 and fill up the service provider's memory with invalid requests. When the memory is clogged, the service provider has to abandon new requests [6] and so it cannot provide anymore service.

User datagram protocol (UDP) also has its own probems. This protocol starts a communication without connection, which means that the connection is a one-way communication in one direction from source to destination without considering the status of the receiver [7]. In UDP attacks, packets are sent from random ports to the victim machine to saturate the bandwidth of the target machine and prevent service provisioning for the users. In this state Firewalls are not able to alleviate these attacks [8].

With the emergence of Software Defined networks (SDN's), new prospects for solving the structural problems of conventional networks have appeared [9]. In this paper, a novel method is proposed based on the cooperation between the service provider and the SDN controller for protecting against flood attacks in conventional networks.

The rest of this paper is organized as follows: Section 2 provides a background to better understand the concepts of the study. Section 3 investigates the related works and research. Section 4 introduces the proposed heterogeneous defense method and Section 5 evaluates the performance of the proposed method. Section 6 discusses the conclusion.

2. Software Defined Networks (SDN): state of the art

Defense against FA attacks has been studied for several decades. Some studies [10, 11] have investigated a number of methods for protecting against these attacks. They use firewalls or attack filtering devices. Based on previous studies, firewalls and attack filtering devices do not provide effective approaches to protect against attacks where weaknesses in protocols are the basis for attack [12]. Because of management complexities and weakness in communication protocols, the architecture of conventional networks cannot provide effective and integrated approaches for protecting against FA.

In the architecture of SDN, data and the control section have been separated. The architecture of the horizontal network is more intelligent and controllable. One of the big ideas in SDN's is the fact that a device, called a controller, is directly linked to all the devices in a domain; it is aware of the network architecture and it plans the network from a central point. An SDN controller changes the planning model of the network from distributed mode to the centralized mode [13]. The centralized planning of the network is a valuable characteristic which can be used for different types of security policy. Using SDN's, there is no need for technical managers to control and manage all the devices on the network through terminals. The controller acts as a middleware which separate components of the network's physical layer, such as routers, switches, firewall, and load balancer from the software layer. In fact, the controller monitors the network from a central point using both information obtained and flexible instruction. In Fig. 1, the structure of SDN's is illustrated. It consists of three major components including the controller, virtual switches, and overlapping networks. Moreover, SDN's are based on the OpenFlow protocol [14].

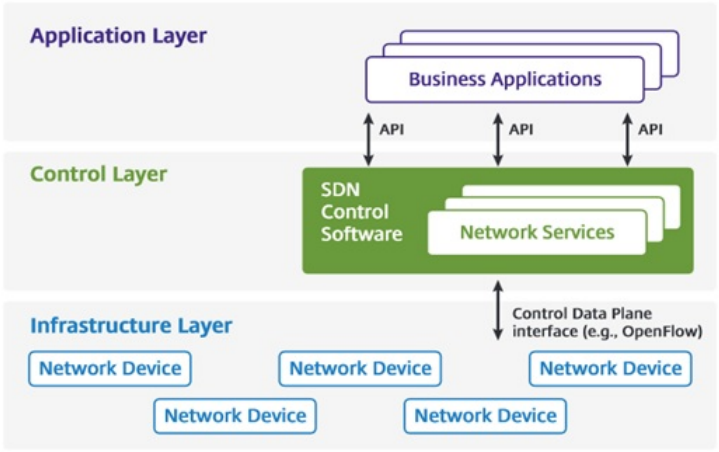


Figure 1: Structure of a SDN [14]

2.1 OpenFlow protocol

OpenFlow protocol is the first standard communication interface which is defined between the control and the data transfer [3]. Using this protocol, high-level programming languages can be used for creating low-level instructions and transferring them to routers and switches. As shown in Fig. 2, this protocol will first define the central controller. Then, with a secure connection to the controller, it enables the control of the network. In SDN's, the central controller maintains all the network rules and issues instruction through the OpenFlow protocol as needed [15].

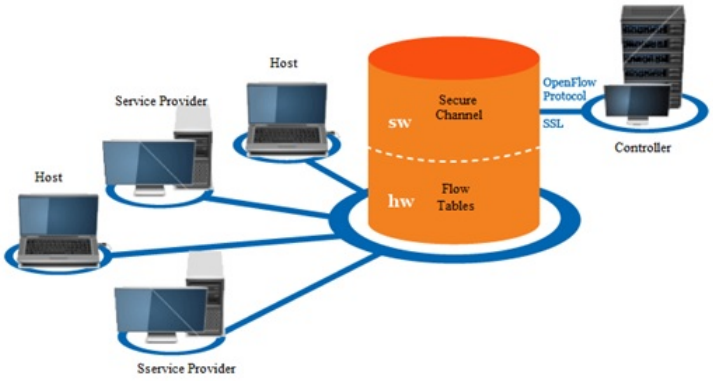


Figure 2: Schematic Representation of OpenFlow Protocol [16]

One of the issues in OpenFlow protocol is the flow-based nature of the exchange policies. A flow table is a list of all the flow entries which includes matching fields, counter, and instruction. After being received, the arriving packets are compared to the matching fields and if there is a match, the packet will enter the exchange process. Otherwise, it is referred to the controller, which decides what will happen. Nowadays, in large-scale datacenters there is a huge interest in OpenFlow. Huge companies design switches based on this protocol and many of them support this protocol. Google has used this protocol in its datacenters [17].

3. Related works

In the literature, there are various methods that protect against distributed denial of service (DDoS) attacks [18-19]. There are some studies regarding the methods to protect against DDoS attacks using SDN [20-21]. In [20] the random hopping plan of the service provider using OpenFlow in SDN's has been proposed. In this plan, using static IPs for service providers in the network has been proposed as a challenge and it is argued that using active static IPs in a domain or network, will lead to multiple scanning and denial of service attacks.

In order to resolve these issues, the authors proposed a method where real IPs are replaced with virtual IPs and the domain name is retrievable through the system while the real IPs are only visible to authorized individuals. This method prevents the identification of active sources on the network. In [21], a strategy is presented for security in the controller core in order to detect the security issues of flows in switches. This strategy is called FortNox and it has been developed based on Nox controller [5]. In this method, the security core is added to the controller, which controls the passing flows and monitors the conflicts and problems in flows. In this method, digital signature is used for authenticating programs. It has a 7-second overhead for checking the flows against the Nox controller.

In [22], a method is introduced for validating the source address based on Nox controller. In this method, each packet located outside the network that intends to enter the network is directed towards this controller. The process of validating the source of the packet is carried out by the controller. If the source is reliable, it permits the traffic to move along. Otherwise, the packet is stopped. This method can prevent IP spoofing operations in FA. In [23], a method based on Nox controller is suggested for detecting DDoS attacks. In this method, an algorithm is proposed for detecting attacks. It uses unsupervised neural networks and the characteristics of arriving traffic to detect DoS attacks. In this work, there is no clear discussion on attacks responds method.

In [12], the method of protecting webservers from SynFlood attacks based on Pox controller is presented. In this method, a controller called OPERETTA is introduced which validates each Syn packet and after validation, it permits the connection. In this technique, two scenarios for responding to FA has been discussed which include a central and a local scenario. In [24], a method for protecting against distributed traffic attacks in cloud environments based on the Floodlight controller is introduced. This controller consists of two modules of detection and defense. The detection module identifies attacks in the cloud and informs the defense module. Therfore the necessary measures are taken. This method is the first method proposed for protecting against attacks in cloud environment using SDN's.

In studies [12 , 23,24], detection and defense module is located centrally in the controller. One of the main problems with these methods is the increased load exerted on the controller during the FA. When the attacks start, a huge number of packets with spoofed sources flood the service provider. The controller must analyze all the packets and take necessary measures. In such cases, an additional computational load is exerted on the controller and it is possible that due to this extra load, the controller fails to manage and control the network requests [25]. The second problem with the above-mentioned methods occurs during the transfer of normal flows of the network to the controller. Authenticating the packets by the controller can create a delay in responding to authorized flows. Since the majority theories of SDN's are flow-based, the

additional delay in setting up an authorized flow is one of the main concerns .In Table 1; various methods proposed by collegues are summarized.

Method	Proposed Year	Controller	Establishment
SOM	2010	NOX	Centralized controller
FortNox	2012	NOX	Centralized controller
DaMask	2015	Floodlight	Centralized controller
OPERETTA	2015	POX	Centralized controller

Table 1. Published methods

4. Proposed method

The proposed method is a heterogeneous defense method based on the cooperation of conventional network service provider and the Software Defined controller to protect against FA. In the presented method, the detection and defense components are separated in order to decrease the computational load and prevent delay in authorized flows sent to the controller. The detection and defense modules with numerous analyses are seperated. During these analyses, it is noticed that attack detection in the targeted service provider which are more accurate and presents less false-positives [26]. In Fig. 3, the structure of the proposed heterogeneous method is demonstrated.

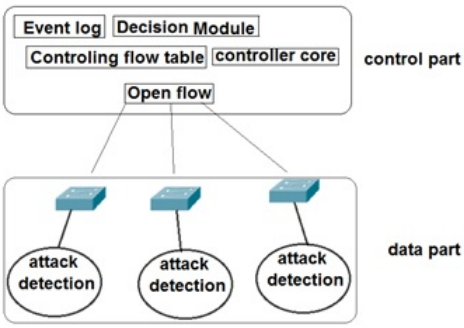


Figure 3: The structure of the proposed heterogeneous method

In heterogeneous method, the attack detection module is located in service provider and the decision module is located in the controller. It applied fuzzy methods to make decisions [27].

In the introduced method, Open vSwitch is used in order to integrate and connect the conventional service provider to the Software Defined controller. Open vSwitch is an open source software switch, licensed under Apache License 2.0. Besides supporting SFlow and NetFlow protocols, it also supports every version of OpenFlow protocol (1.0 to 1.5) [28]. In Fig. 4, the proposed algorithm

is shown in three phases. The closed-movement phase in the proposed method includes the following phases: 1- Connection request, 2- Event analysis and report, 3. Decision and response.

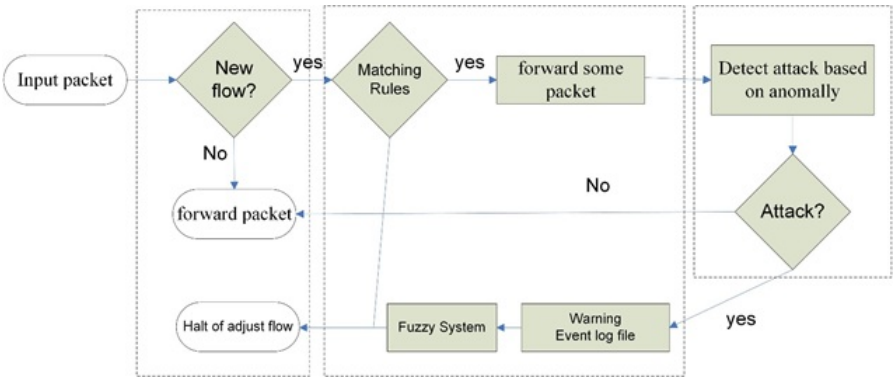


Figure 4: The architecture of proposed system

4.1 Connection request phase

In this phase, the packet enters the OpenFlow switch. Then, the switch checks the flow tables. If the arriving packet is present in the flow tables, the permission for the flow to move towards the destination is issued. Otherwise (if arriving packet belongs to a new flow), it is sent to the controller so that the necessary actions can be determined Fig. 5.

```
OpenFlow Protocol
▽ Header
    Version: 0x01
    Type: Packet In (AM) (10)  //Packet Arrival
    Length: 72
    Transaction ID: 0
▽ Packet In
    Buffer ID: 4294967295
    Frame Total Length: 54
    Frame Recv Port: 4
    Reason Sent: No matching flow (0)  //The Received Packets is a New and it is Not Switches Flow Tables
  ▽ Frame Data: 000000000004000000000006080045000284b2d40001906...
    ▷ Ethernet II, Src: 00:00:00_00:00:06 (00:00:00:00:00:06), Dst: 00:00:00_00:00:04 (00:00:00:00:00:04)
    ▷ Internet Protocol Version 4, Src: 10.0.0.4 (10.0.0.4), Dst: 10.0.0.1 (10.0.0.1)
    ▽ Transmission Control Protocol, Src Port: ftp-data (20), Dst Port: 30 (30), Seq: 13456, Len: 0
        Source port: ftp-data (20)
        Destination port: 30 (30)
        [Stream index: 2]
        Sequence number: 13456    (relative sequence number)
        Header length: 20 bytes
      ▽ Flags: 0x002 (SYN)
```

Figure 5: An Example of a Packet Sent to the Controller

Once the packet reaches the controller, preliminary investigations such as identifying the destination address and type of protocol are carried out. In case the packet is determined to be based on the current rules of the network, the

controller creates a temporary flow table and permits some packets to continue towards the service provider Fig. 6.

```
                // Creating Event For Listening To Arriving Packet
def launch ():
    core.openflow.addListenerByName("PacketIn", _handle_PacketIn)

                // Parsing Packet
def _handle_PacketIn (event):
    packet = event.parsed
    dst_port = table.get(packet.dst)

                // The Expiration Time For Temporary Flow Tables
    msg = of.ofp_get_n_packet_flow_mod()
    msg.match.dl_src = packet.src
    msg.match.dl_dst = packet.dst
    msg.actions.append(of.ofp_action_output(port = dst_port))
    event.connection.send(msg)
```

Figure 6: The Preliminary Analysis of the Packet and the Permission for Multiple Packets in the Flow

4.2 Event analysis and report phase

In this phase, packets are forwarded towards the service provider based on the temporary flow tables. Once the packets enter, the attack detection module starts there analysis. Different methods have been proposed for detecting DDoS attacks. Pprevious studies have shown that anomaly-based methods provide better performance compared to signature-based methods in order to detect attacks [24]. On the other hand, in anomaly based detecting methods, the presence of a few packets is enough for detecting attacks, while in signature-based methods all the packets related to that flow are required [24]. Therefore, considering the benefits of the anomaly-based method and the current limitations (the low number of packets arriving at the detection module) this detection technique is a suitable approach to be used in the heterogeneous method. The parameters in [29], such as packet type entropy, packet rate, and the number of packets for attack detection are applied.

4.3 Decision phase

After receiving the event report by the controller, the decision module comes online and updates the event report file to send the necessary decision to the defense component. The decision determines the type of response to the attack and can be carried out by the fuzzy controller. Under this condition the parameters of trust in the service provider and the sensitivity of the provided service

are considred. This phase makes a decision firstly and then responses to the attack subsequently.

4.3.1 Decision making step

Considering the fact that the parameters of trust and sensitivity can be represented in a fuzzy manner, these two parameters are used for fuzzy decision making and reporting the extent of the damage.

- **The parameter of trust in the service provider**

As shown in Fig. 4, after detecting the attack by the service provider, the event report is sent to the controller where it will be placed in the attack event file. By reading this file, the number of attacks on a service provider is calculated. Whenever an attack occurs on the service provider, it will be added to the event log file and the controller's trust score for the service provider is lowered. The trust score can be expressed using lingual variables including high, medium, and low trust.

- **The parameter of the service sensitivity**

The parameter for the service sensitivity can vary from a network to another. This parameter should be given to the controller by the network administrator. For instance, an attack on the web service provider, email service provider, or special ports such as 53, 80, and 443 will be more sensitive than non-reserved ports. The service sensitivity score can also be expressed using lingual variables including high, medium, and low sensitivity.

In the decision making step, using the different values of trust and sensitivity, the damage can be calculated and reported to the defense component. The inflicted damage can invoke different responses to the attack. For example, if the damage estimation is high, in response, the controller must choose a method which not only alleviates the attack but also does not lead to false positives in the network. In order to simulate the proposed fuzzy model, the fuzzy environment of the MATLAB emulator, triangular membership function, and center of gravity defuzzification are used. Using trust inputs, sensitivity, and "if... then" rules entered into the fuzzy system, the fuzzy inference system (FIS) is constructed. The set of utilized rules can be seen in Fig. 7.

The output of the fuzzy inference system can be turned into FCL language by MATLAB and using frameworks such as PyFuzzy, it can be implemented into Python and integrated into the controller. Fig. 8 shows the fuzzy decision behavior model.

4.3.2 Response to attack step

After detection of the damage, decisions will be sent to the response to attack component. A simple method is to prevent the construction of flow tables in the

Inputs		Confidence		
		low	middle	high
Sensitivity	low	middle	high	high
	middle	low	low	high
	high	low	low	middle

Figure 7: Fuzzy rules

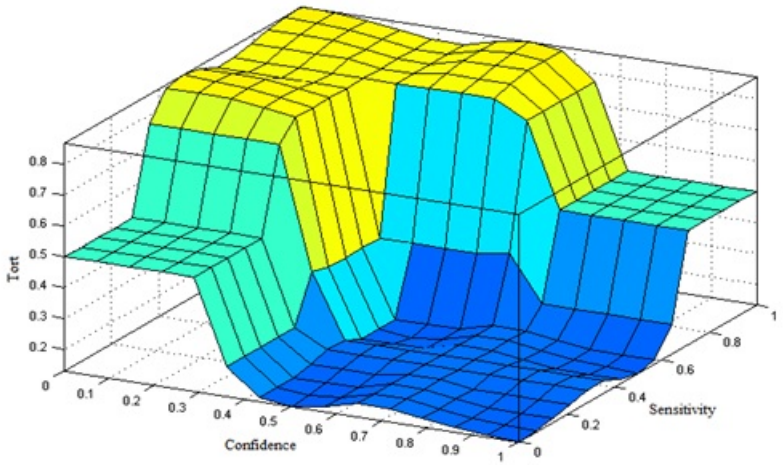


Figure 8: Fuzzy decision behavior model

attack source. In other words, after receiving the report, the Packet-In event related to the attack packets must be halted and the formation of flow tables for similar packets is prevented. Fig. 9 shows the code fragment related to halting the attack event. After the event report in this code fragment, controller's instruction is sent to the switch connected to the attacker and the flow passing through is halted.

5. Performance evaluation of the proposed algorithm

The proposed method is analyzed and evaluated in the MiniNet simulator which is one of the most popular Software Defined simulators by applying POX controller [11, 20, 27] on a Linux OS. MiniNet uses architecture of several virtual hosts that can construct virtual links, which can be connected to OpenvSwitch. In the MiniNet environment, every virtual host has a separate OS kernel and each is able to run its programs and commands independently.

```
def block_handler (event):

        packet = event.parsed   // Detecting The Connection Rule
        tcpp = event.parsed.find('tcp')
        udpp = event.parsed.find('udp')

        if tcpp:
        if tcpp.srcport in block_ports or tcpp.dstport in block_ports:
            core.getLogger ("SYN Receive")
            core.getLogger("DDos Blocker Start...").info("Blocked TCP-Attack Sourece
            event.halt = True     // Halting The Event Related To TCP
        elif udpp:
                if udpp.srcport in block_ports or udpp.dstport in block_ports:
            core.getLogger("DDos Blocker Start...").info("Blocked UDP-Attack Sourece
            event.halt = True     // Halting The Event Related To UDP
```

Figure 9: Halting the Attack Flow

In order to set up the Mininet emulator, Ubuntu 12.10 32bit OS with Quad Processor Q6600 CPU using 4 GB of memory is used. For running the Pox controller a laptop computer with an Intel 2Core i684-1.4 GHz CPU is used. In order to simulate the FA, Hping3 application is used. This tool is one of the most powerful programs for sending customized packets towards a determined destination. This tool is generally used for penetration tests. Hping3 provides the possibility to send a flood of packets by spoofing the source's IP towards the destination. In order to analyze and measure the rate of in-transit packets, WireShark tool is used. For measuring the processes which consumed resources the HTop tool [30] is used. Also for creating legitimate requests for receiving information iPerf [31] is utilized.

The network architecture consists of a number of Linux systems, which are connected to seven OpenFlow switches. In this architecture, a web server and the users are working. Also, there are a number of infected systems which try to attack and interrupt the normal flow of services in the network as shown in Fig. 10.

Since the heterogeneous defense method is a technique based on the cooperation of a service provider in conventional networks and a Software Defined controller, it should be able to not only pass the attacks in conventional networks but also provide a number of benefits over the Software Defined centralized defense method. Therefore, in order to evaluate the efficiency of the heterogeneous defense method, two comparisons are made. In the first comparison in Section 5.1, the superiority of the heterogeneous defense method over traditional methods is evaluated. Then, in Section 5.2, the heterogeneous defense method with defense methods concentrated in the controller is compared.

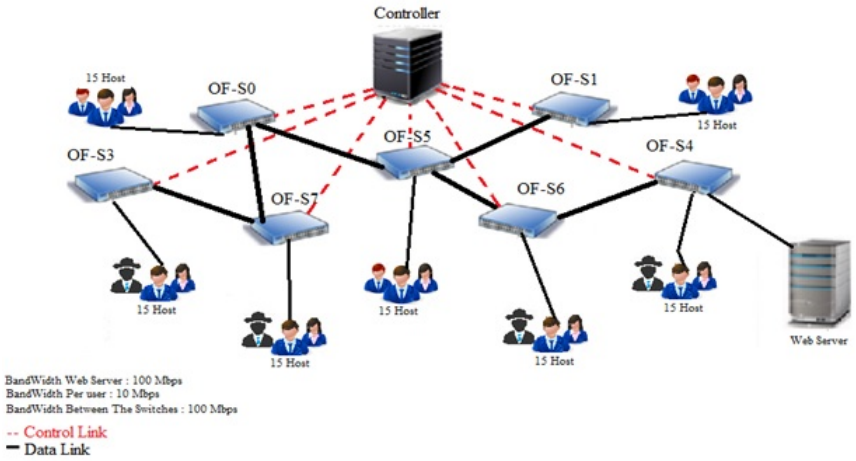


Figure 10: Network Architecture and Controller for Evaluating the Heterogeneous Defense Method

5.1 Comparing the proposed method with traditional methods

In this section, comparison of the heterogeneous defense method with traditional methods of protecting against attacks (SynCookies and the firewall present in the service provider) is done. In order to do the evaluation, criteria such as response time for authorized users and the processing load exerted on the service provider is used.

- **Service provider's response time**

In order to measure the response time of the service provider for authorized users in the traditional method and the heterogeneous method an experiment with a number of different estimated rates of SynFlood packets on the web service provider is carried out. In the first step after the attack, traditional methods are used (SynCookies and the firewall present in the service provider) to pass the attacks and results are recorded. Then, by activating the heterogeneous defense system defens against the attacks are tried and the results were also recorded.

As obtained from Fig. 11, in traditional method, by increasing the rate of attacks, a significant amount of time is spent for responding to unauthorized requests and the response time for authorized users is increased. In the heterogeneous defense method, after detecting the attack, the controller is informed of the situation. Then, the controller identifies the source and destination of the attacks and sends decisions to the switch connected to the attacker to cut off or adjust the flow. Afterwards, the attacks are halted and the response

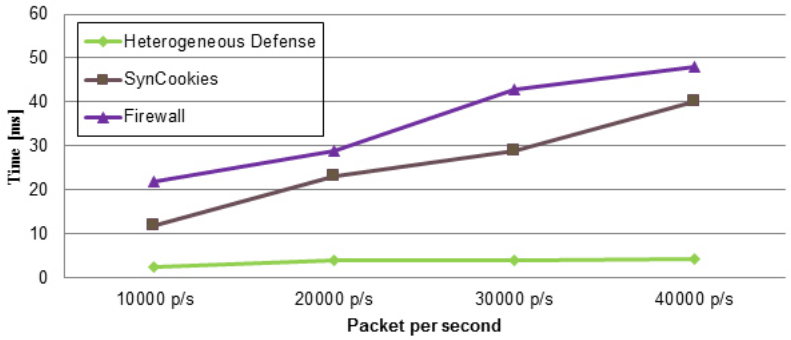


Figure 11: Average Response Time of the Service Provider for Authorized Requests over Packet/second

time is revived. In the proposed method, the controller does not allow attack flow towards connection links and the resources of the service provider from the attacker switch.

- **The usage of central processor**

In Fig. 12, the usage level of the central processor during the protection against attacks using the traditional and the heterogeneous methods is provided. As can be seen, in traditional defense methods, a high amount of processing load is exerted on the central processor of the service provider during the defense against the attacks. By increasing the rate of the attacks, the usage of the processor is also increased. Meanwhile in the heterogeneous method, due to the separation of the defense system, the only load exerted on the service provider due to calculations of the attack detection module, which can be seen in Fig. 12.

The results of experiments show that traditional method in comparision with the heterogeneous method provides protection of resources, access for the authorized users and prevents interference with network resources.

5.2 Comparing the proposed method with methods concentrated in the controller in thwarting flood attacks

For evaluation, criteria such as the computational load on the controller and the delay in responding to authorized flows are used. As previously mentioned in Section 3 , [12] , [23], and [24] provide methods for detecting and thwarting denial of service attacks based on Software Defined networks. In all proposed schemes, the detection and defense modules are concentrated on the controller. However, in the heterogeneous defense method, the detection module is in the service provider and the decision module is in the controller. The strategy proposed in

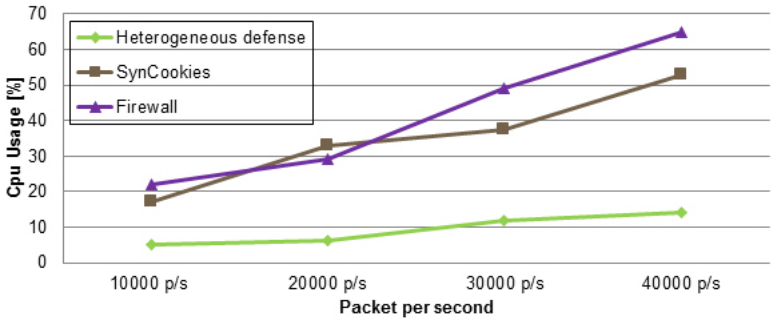


Figure 12: Average Processing Load on the Service Provider in Traditional and Heterogeneous Defense Methods

[24] is designed for cloud environments and [23] only discusses attack detection methods. Therefore [12] , provides a method for thwarting attacks in traditional networks.

In this scheme, the OPERETTA controller is proposed. The OPERETTA is an evolved version of the Pox controller for thwarting SynFlood attacks in conventional networks. In this scheme, each TCP packet which intends to reach the service provider is validated by the controller. If it passes the validation process, it is allowed to go to the service provider. Among the important problems in the OPERETTA controller, the high delay in delivering validated packets to the service provider can be mentioned. Here, the proposed heterogeneous defense method is compared with the centralized method of detection and defense based on Pox and OPERETTA controller. Table.2 shows the specifications of the testing environment for each controller.

Detection and Defense Method	Detection and Defense Centralized in Pox Controller	Detection and Defense Centralized in OPERETTA controller	Proposed Deterogeneous Dethod
Detectionand defense system	Integrated	Integrated	Based on cooperation
Simulator	MiniNet	MiniNet	MiniNet
Controller core type	Pox	OPERETTA	Pox
Controller hardware	Intel i3-2365m-1.4Ghz-Ram 4GB	Intel i3-2365m-1.4Ghz-Ram 4GB	Intel 2core i686 1.4 Ghz Ram 4GB
Experimental OS	Kubuntu 13.10 64bit	Kubuntu 13.10 64bit	Ubuntu 12.10 32bit
Ability to thwart attacks	SynFlood	SynFlood	SynFlood-UdpFlood

Figure 13: Specifications of the Testing Environment

Evaluating the processing load in controller

In order to evaluate the processing load on the controller in the heterogeneous method, a flood attack consisting of Syn packets with a number of different rates is carried out on the web service provider as shown in Fig. 10. Then, using the Htop tool, the results for the usage of the central processor on the controller process were measured. The obtained results were compared to OPERETTA centralized controller and the standard centralized method in Pox, extracted from [12]. The results of this comparison can be seen in Fig. 13.

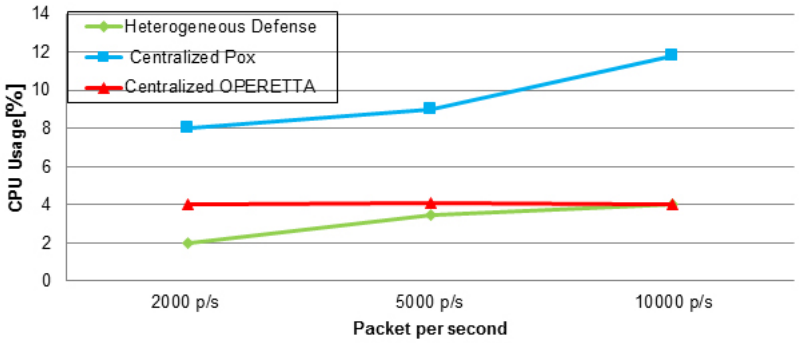


Figure 14: Average Usage of Central Processor over Packet per second

As the results show, the heterogeneous defense scheme provides a lower processing rate in the range of 2000 and 5000 packets. Compared to other methods in the range of 10000 packets, it is almost equal to the OPERETTA controller. However, as mentioned in Table. 2, the processor of the heterogeneous controller has a lower specification compared to the processor of OPERETTA controller and Pox.

6. Conclusions

Denial of service flood attacks is amongst the most common and powerful attacks which abuse the computational resources and the bandwidth of the network. The simplicity of creating denial of service tools for carrying out flood attacks and the problems of thwarting them due to the weaknesses in the structure of communication protocols have turned these attacks into the most common and most significant attacks ever. In this study, traditional networks, SDN's and the strategies of these networks for thwarting flood attacks are discussed. Moreover, their strengths and weaknesses of all methods are discussed. Then, a heterogeneous approach based on the cooperation of the service provider is presented. Also Software Defined controller, which is able to thwart flood attacks in the network, is introduced. Results show that the proposed method has a lower computational load and response time compared to other methods centralized in the controller. There will never be an ultimate defense method which can

protect all the devices against these attacks. For the future work, concentration should be on mechanism based on voting to detect attacks in a heterogeneous manner. Each service provider will have a vote and based on the votes, the controller will implement security measures in the network.

References

[1] Larson, Dave, *Distributed denial of service attacks–holding back the flood*, Network Security, 3 (2016), 5-7.

[2] *Alert (TA14-017A) UDP-based Amplification Attacks*, US-CERT, 8.

[3] Moore, D., G. Voelker and S. Savage, *Inferring Internet Denial-of-Service Activity In 10th USENIX Security Symposium*, W ashington DC, 200 (2001).

[4] Goransson, Paul and Chuck Black, *Software Defined Networks: A Comprehensive Approach*, Elsevier, 2014.

[5] McCauley, Murphy, *About pox*, URL: http://www. noxrepo. org/pox/about-pox/. Online (2013).

[6] C. Meadows, *A formal framework and evaluation method for network denial of service,* in Computer Security Foundations Workshop, 1999, Proceedings of the 12th IEEE, 1999, 4-13.

[7] U. D. Protocol, *RFC 768 J. Postel ISI 28 August 1980,* Isi, 1980.

[8] UDP flood attack, https://en.wikipedia.org/wiki/UDP_flood_attack.

[9] N. McKeown, T. Anderson, H. Balakrishnan, G. Parulkar, L. Peterson, J. Rexford, et al., *OpenFlow: enabling innovation in campus networks,* ACM SIGCOMM Computer Communication Review, 38 (2008), 69-74.

[10] T. Peng, C. Leckie and K. Ramamohanarao, *Survey of network-based defense mechanisms countering the DoS and DDoS problems,* ACM Computing Surveys (CSUR), 39 (2007), 2007.

[11] S. T. Zargar, J. Joshi and D. Tipper, *A survey of defense mechanisms against distributed denial of service (DDoS) flooding attacks," Communications Surveys & Tutorials*, IEEE, 15 (2013), 2046-2069.

[12] S. Fichera, L. Galluccio, S. C. Grancagnolo, G. Morabito and S. Palazzo, *Operetta: An OPEnflow-based REmedy to mitigate TCP SYNFLOOD Attacks against web servers*, Computer Networks, 92 (2015), 89-100.

[13] E. Borcoci, *Software Defined Networking and Architectures,* in Fifth International Conference on Advances in Future Internet (AFIN 2013), 2013.

[14] A. Hakiri, A. Gokhale, P. Berthou, D. C. Schmidt and T. Gayraud, *Software Defined networking: challenges and research opportunities for future internet,* Computer Networks, 75 (2014), 453-471.

[15] T. N. Subedi, K. K. Nguyen and M. Cheriet, *OpenFlow-based in-network Layer-2 adaptive multipath aggregation in data centers,* Computer Communications, 61 (2015), 58-69.

[16] A. Tavakoli, *Exploring a centralized/distributed hybrid routing protocol for low power wireless networks and large-scale datacenters,* University of California, Berkeley, 2009.

[17] S. Sondur, *Software Defined Networking for Beginners.*

[18] Gulisano, Vincenzo, Mar Callau-Zori, Zhang Fu, Ricardo Jiménez-Peris, Marina Papatriantafilou and Marta Patiño-Martínez, *STONE: A streaming DDoS defense framework*, Expert Systems with Applications 42, no. 24 (2015): 9620-9633.

[19] Yan, Qiao and F. Richard Yu, *Distributed denial of service attacks in software-defined networking with cloud computing*, IEEE Communications Magazine 53, 4 (2015), 52-59.

[20] J. H. Jafarian, E. Al-Shaer and Q. Duan, *Openflow random host mutation: transparent moving target defense using Software Defined networking,* in Proceedings of the first workshop on Hot topics in Software Defined networks, 2012, 127-132.

[21] P. Porras, S. Shin, V. Yegneswaran, M. Fong, M. Tyson and G. Gu, *A security enforcement kernel for OpenFlow networks,* in Proceedings of the first workshop on Hot topics in Software Defined networks, 2012, 121-126.

[22] G. Yao, J. Bi and P. Xiao, *Source address validation solution with OpenFlow/NOX architecture,* in Network Protocols (ICNP), 2011 19th IEEE International Conference on, 2011, 7-12.

[23] R. Braga, E. Mota and A. Passito, *Lightweight DDoS flooding attack detection using NOX/OpenFlow,* in Local Computer Networks (LCN), 2010 IEEE 35th Conference on, 2010, 408-415.

[24] B. Wang, Y. Zheng, W. Lou and Y. T. Hou, *DDoS attack protection in the era of cloud computing and Software Defined Networking,* Computer Networks, 81 (2015), 308-319.

[25] A. Tootoonchian, S. Gorbunov, Y. Ganjali, M. Casado and R. Sherwood, *On controller performance in Software Defined networks,* in USENIX Workshop on Hot Topics in Management of Internet, Cloud, and Enterprise Networks and Services (Hot-ICE), 2012.

[26] U. Tariq, Y. Malik, B. Abdulrazak and M. Hong, *Collaborative peer to peer defense mechanism for ddos attacks,* Procedia Computer Science, 5 (2011), 157-164.

[27] L. Feinstein, D. Schnackenberg, R. Balupari and D. Kindred, *Statistical approaches to DDoS attack detection and response,* in DARPA Information Survivability Conference and Exposition, 2003, Proceedings, 2003, 303-314.

[28] Production Quality, Multilayer Open Virtual Switch http://openvswitch.org.

[29] K. Lee, J. Kim, K. H. Kwon, Y. Han and S. Kim, *DDoS attack detection method using cluster analysis,* Expert Systems with Applications, 34 (2008), 1659-1665.

[30] *Htop Documentation http://hisham.hm/htop/*

[31] *Iperf Documentation, http://iperf.fr/*

[32] S. H. Yeganeh, A. Tootoonchian and Y. Ganjali, *On scalability of Software Defined networking,* Communications Magazine, IEEE, 51 (2013), 136-141.

ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS
INFORMATION FOR AUTHORS

- Before an article can be published in the Italian Journal of Pure and Applied Mathematics, the author is required to contribute with a small fee, which has to be calculated with the following formula:

  fee = EUR (20 + 4n)

  where **n** is the number of pages of the article to be published.

- The above amount has to be payed through an international credit transfer in the following bank account:

  Bank name: **CREDI FRIULI - CREDITO COOPERATIVO FRIULI**
  IBAN code: **IT 96 U 0708512304035210033938**
  BIC code: **CCRTIT2TK00 = SWIFT CODE**
  Account owner: **FORUM EDITRICE UNIVERSITARIA UDINESE SRL**
  VIA LARGA 38
  33100 UDINE (ITALY)

- All bank commissions **must be payed** by the author, adding them to the previous calculated net amount

- Include the following **mandatory** causal in the credit transfer transaction:

  CONTRIBUTO PUBBLICAZIONE ARTICOLO SULL'ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

- Please, include also **First Name**, **Last Name** and **Paper Title** in the credit transfer transaction.

- After the transaction ends successfully, the author is requested to send an e-mail to the following addresses:

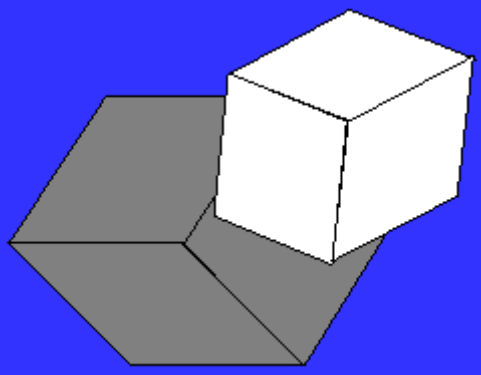
  azamparo@forumeditrice.it
  piergiuliocorsini@gmail.com

  This e-mail should contain the author's personal information (Last name, First Name, Postemail Address, City and State, PDF copy of the bank transfer), in order to allow Forum Editrice to create an invoice for the author himself.

- Payments, orders or generic fees will not be accepted if they refer to Research Institutes, Universities or any other public and private organizations).

- Finally, when the payment has been done and the e-mail has been received, Forum Editrice will issue an invoice receipt in PDF format and will send it by e-mail to the author.

IJPAM – Italian Journal of Pure and Applied Mathematics Issue n° 40-2018



Publisher
Forum Editrice Universitaria Udinese Srl
Via Larga 38 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

Rivista semestrale: Autorizzazione Tribunale di Udine n. 8/98 del 19.3.98 - Direttore responsabile: Piergiulio Corsini

ISSN 2239-0227